

B. Sc. DEGREE PROGRAMME

(2019 Admission onwards)

COPLEMENTARY MATHEMATICS

SECOND SEMESTER

COPLEMENTARY : MTS2C02 - MATHEMATICS-2

4 hours/week

3 Credits

75 Marks [Int:15+Ext:60]

SECOND SEMESTER

MTS2 C02:MATHEMATICS-2

4 hours/week

3 Credits

75 Marks[Int.15 + Ext. 60]

Text (1)	Calculus I (2/e) : Jerrold Marsden & Alan Weinstein <i>Springer-Verlag New York Inc(1985) ISBN 0-387-90974-5</i>
Text (2)	Calculus II (2/e) : Jerrold Marsden & Alan Weinstein <i>Springer-Verlag New York Inc(1985) ISBN 0-387-90975-3</i>
Text(3)	Advanced Engineering Mathematics(6/e) : Dennis G Zill Jones & Bartlett Learning, LLC(2018)ISBN: 978-1-284-10590-2

Module I Text (1) & (2) 18 hrs

5.1: Polar coordinates and Trigonometry – Cartesian and polar coordinates
(*Only representation of points in polar coordinates, relationship between Cartesian and polar coordinates, converting from one system to another and regions represented by inequalities in polar system are required*)

5.3 : Inverse functions-inverse function test, inverse function rule

5.6: Graphing in polar coordinates- *Checking symmetry of graphs given in polar equation, drawings, tangents to graph in polar coordinates*

8.3: Hyperbolic functions- hyperbolic sine, cosine, tan etc., derivatives, anti differentiation formulas

8.4: Inverse hyperbolic functions- inverse hyperbolic functions (*their derivatives and anti derivatives*)

10.3: Arc length and surface area- Length of curves, Area of surface of revolution about x and y axes

10.4: Parametric curves- parametric equations of lines and circles, tangents to parametric curves, length of a parametric curve, speed

10.5: Length and area in polar coordinates- arc length and area in polar coordinates , *Area between two curves in polar coordinates*

Module II	Text (2)	20 hrs
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11.3: Improper integrals- integrals over unbounded intervals, comparison test, integrals of unbounded functions

11.4: Limit of sequences and Newton's method- $\varepsilon - N$ definition, limit of powers, comparison test, Newton's method

11.5: Numerical Integration- Riemann Sum, Trapezoidal Rule, Simpson's Rule

12.1: The sum of an infinite series- convergence of series, properties of limit of sequences (*statements only*), geometric series, algebraic rules for series, the i^{th} term test

12.2: The comparison test and alternating series- comparison test, ratio comparison test, alternating series, alternating series test, absolute and conditional convergence

12.3: The integral and ratio test-integral test, p-series, ratio test, root test

12.4: Power series – ratio test for power series, root test, differentiation and integration of power series, algebraic operation on power series

12.5: Taylor's formula- Taylor and Maclaurian series, *Taylor's formula with remainder in integral form*, *Taylor's formula with remainder in derivative form*, convergence of Taylor series, Taylor series test, some important Taylor and Maclaurian series

Module III	Text(3)	12 hrs
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7.6: Vector spaces – *definition, examples, subspaces, basis, dimension, span*

7.7: Gram-Schmidt Orthogonalization Process- *orthonormal bases for $\mathbb{R}^n$* , construction of orthonormal basis of $\mathbb{R}^n$

8.2: Systems of Linear Algebraic Equations- General form, solving systems, augmented matrix, Elementary row operations, Elimination Methods- *Gaussian elimination, Gauss-Jordan elimination, row echelon form, reduced row echelon form, inconsistent system*, networks, homogeneous system, *over and underdetermined system*

8.3: Rank of a Matrix- *definition*, row space, rank by row reduction, rank and linear system, *consistency of linear system*

8.4: Determinants- *definition, cofactor (quick introduction)*

8.5: Properties of determinant- *properties, evaluation of determinant by row reducing to triangular form*

Module IV **Text(3)** **14 hrs**

8.6: Inverse of a Matrix – finding inverse, properties of inverse, adjoint method, row operations method, using inverse to solve a linear system

8.8: The eigenvalue problem- *Definition, finding eigenvalues and eigenvectors*, complex eigenvalues, eigenvalues and singular matrices, eigenvalues of inverse

8.9: Powers of Matrices- *Cayley Hamilton theorem*, finding the inverse

8.10: Orthogonal Matrices- symmetric matrices and eigenvalues, inner product, *criterion for orthogonal matrix*, construction of orthogonal matrix

8.12 Diagonalization- diagonalizable matrix -*sufficient conditions*, orthogonal diagonalizability of *symmetric matrix*, Quadratic Forms

8.13: LU Factorization- *definition*, Finding an LU- factorization, Doolittle method, solving linear systems (*by LU factorization*), relationship to determinants

References:

1	Soo T Tan: <i>Calculus Brooks/Cole, Cengage Learning(2010)ISBN 0-534-46579-X</i>
2	Gilbert Strang: <i>Calculus Wellesley Cambridge Press(1991)ISBN:0-9614088-2-0</i>
3	Ron Larson. Bruce Edwards: <i>Calculus(11/e) Cengage Learning(2018) ISBN: 978-1-337-27534-7</i>
4	Robert A Adams & Christopher Essex : <i>Calculus Single Variable (8/e) Pearson Education Canada (2013) ISBN: 0321877403</i>
5	Joel Hass, Christopher Heil & Maurice D. Weir : <i>Thomas' Calculus(14/e) Pearson (2018) ISBN 0134438981</i>
6	Peter V O'Neil: <i>Advanced Engineering Mathematics(7/e) Cengage Learning(2012)ISBN: 978-1-111-42741-2</i>
7	Erwin Kreyszig : <i>Advanced Engineering Mathematics(10/e) John Wiley & Sons(2011) ISBN: 978-0-470-45836-5</i>
8	Glyn James: <i>Advanced Modern Engineering Mathematics(4/e) Pearson Education Limited(2011) ISBN: 978-0-273-71923-6</i>

MODULE 1

Text Book 1- Calculus I (2/e) : Jerrold Marsden & Alan Weinstein Springer-Verlag New York Inc(1985)
ISBN 0-387-90974-5

Text Book 2- Calculus II (2/e) : Jerrold Marsden & Alan Weinstein Springer-Verlag
New York Inc(1985) ISBN 0-387-90975-3

Sections 5.3, 5.3, 5.6 Text 1

8.3, 8.4 Text 2

10.3, 10.4, 10.5 Text 2

Trigonometric Functions

The derivative of $\sin x$ is $\cos x$ and of $\cos x$ is $-\sin x$; everything else follows from this.

Many problems involving angles, circles, and periodic motion lead to trigonometric functions. In this chapter, we study the calculus of these functions, and we apply our knowledge to solve new problems.

The chapter begins with a review of trigonometry. Well-prepared students may skim this material and move on quickly to the second section. Students who do not feel prepared or who failed Orientation Quiz C at the beginning of the book should study this review material carefully.

5.1 Polar Coordinates and Trigonometry

Trigonometric functions provide the link between polar and cartesian coordinates.

This section contains a review of trigonometry, with an emphasis on the topics which are most important for calculus. The derivatives of the trigonometric functions will be calculated in the next section.

The circumference C and area A of a circle of radius r are given by

$$C = 2\pi r, \quad A = \pi r^2$$

(see Fig. 5.1.1), where π is an irrational number whose value is approximately 3.14159 ¹

¹ For details on the fascinating history of π , see P. Beckman, *A History of π* , Golem Press, 1970. To establish deeper properties of π such as its irrationality (discovered by Lambert and Legendre around 1780), a careful and critical examination of the definition of π is needed. The first explicit expression for π was given by Viète (1540–1603) as

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}}} \cdots$$

which is obtained by inscribing regular polygons in a circle. Euler's famous expression $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ is discussed in Example 3, Section 12.5. For an elementary proof of the irrationality of π , see M. Spivak, *Calculus*, Benjamin, 1967.

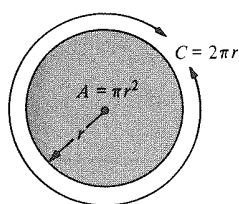


Figure 5.1.1. The circumference and area of a circle.

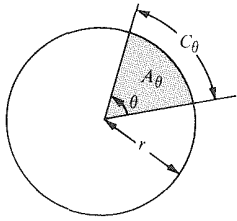


Figure 5.1.2. The length C_θ and area A_θ are proportional to θ .

If two rays are drawn from the center of the circle, both the length and area of the part of the circle between the rays are proportional to the angle between the rays. Thus, if we measure angles in degrees, the length C_θ and area A_θ between rays making an angle θ (see Fig. 5.1.2) are determined by the relations

$$\frac{C_\theta}{2\pi r} = \frac{\theta}{360}, \quad \frac{A_\theta}{\pi r^2} = \frac{\theta}{360} \quad (\theta \text{ in degrees}),$$

since a full circle corresponds to an angle of 360 degrees.

These formulas become simpler if we adopt the *radian* unit of measure, in which the total angular measure of a circle is defined to be 2π . Then our previous formulas become

$$\frac{C_\theta}{2\pi r} = \frac{\theta}{2\pi}, \quad \frac{A_\theta}{\pi r^2} = \frac{\theta}{2\pi} \quad (\theta \text{ in radians})$$

or simply

$$C_\theta = r\theta, \quad A_\theta = \frac{1}{2}r^2\theta.$$

The formulas of calculus are also simpler when angles are measured in radians rather than degrees. Unless explicit mention is made of degrees, all angles in this book will be expressed in radians. If you use a calculator to do computations with angles measured in radians, be sure that it is operating in the radian mode.

Example 1 An arc of length 10 meters on a circle of radius 4 meters subtends what angle at the center of the circle? How much area is enclosed in this part of the circle?

Solution In the formula $C_\theta = r\theta$, we have $C_\theta = 10$ and $r = 4$, so $\theta = \frac{10}{4} = 2\frac{1}{2}$ (radians). The area enclosed is $A_\theta = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 16 \cdot \frac{5}{2} = 20$ square meters. $\blacktriangle$

Conversions between degrees and radians are made by multiplying or dividing by the factor $360/2\pi = 180/\pi \approx 57.296$ degrees per radian.

Degrees and Radians

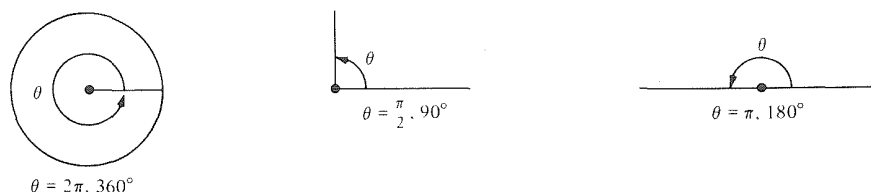
To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi} \approx 57^\circ 18' \approx 57.296^\circ$.
To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ} \approx 0.01745$.

The following table gives some important angles in degrees and radians:

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

The measures of right angles and straight angles are shown in Fig. 5.1.3.

Figure 5.1.3. A complete circle, a right angle, and a straight angle in degrees and radians.



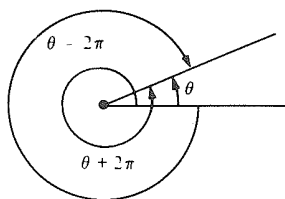


Figure 5.1.4. θ , $\theta - 2\pi$, and $\theta + 2\pi$ measure the same geometric angle.

Negative numbers and numbers larger than 2π (or 360°) can also be used to represent angles. The convention is that θ and $\theta + 2\pi$ represent the same geometric angle; hence so do $\theta + 4\pi, \theta + 6\pi, \dots$ as well as $\theta - 2\pi, \theta - 4\pi, \dots$ (see Fig. 5.1.4). The angle $-\theta$ equals $2\pi - \theta$ and is thus the “mirror image” of θ (see Fig. 5.1.5). Note, also, that rays making angles of θ and $\theta + \pi$ with a given ray point in opposite directions along the same straight line (see Fig. 5.1.6).

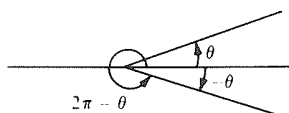


Figure 5.1.5. The angle $-\theta$, or $2\pi - \theta$, is the mirror image of θ .

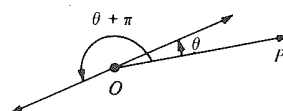


Figure 5.1.6. The rays making angles of θ and $\theta + \pi$ with OP point in opposite directions.

- Example 2** (a) Convert to radians: $36^\circ, 160^\circ, 280^\circ, -300^\circ, 460^\circ$.
 (b) Convert to degrees: $5\pi/18, 2.6, 6.27, 0.2, -9.23$.

Solution (a) $36^\circ \rightarrow 36 \times 0.01745 = 0.6282$ radian;
 $160^\circ \rightarrow 160 \times 0.01745 = 2.792$ radians;
 $280^\circ \rightarrow 280 \times 0.01745 = 4.886$ radians;
 $-300^\circ \rightarrow -300 \times 0.01745 = -5.235$ radians, or
 $-300 \times \pi/180 = -5\pi/3$ radians;
 $460^\circ \rightarrow 460^\circ - 360^\circ = 100^\circ \rightarrow 100 \times 0.01745 = 1.745$ radians.
 (b) $5\pi/18 \rightarrow 5\pi/18 \times 180/\pi = 50^\circ$;
 $2.6 \rightarrow 2.6 \times 57.296 = 148.97^\circ$;
 $6.27 \rightarrow 6.27 \times 57.296 = 359.25^\circ$;
 $0.2 \rightarrow 0.2 \times 57.296 = 11.46^\circ$.
 $-9.23 \rightarrow -9.23 \times 180/\pi = -528.84^\circ \rightarrow 720^\circ - 528.84^\circ = 191.16^\circ$. ▲

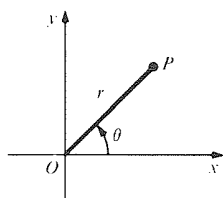


Figure 5.1.7. The polar coordinates (r, θ) of a point P .

Cartesian coordinates (x, y) represent points in the plane by their distances from two perpendicular lines. In the *polar coordinate* representation, a point P is associated with each pair (r, θ) of numbers in the following way.² First, a ray is drawn through the origin making an angle of θ with the positive x axis. Then one travels a distance r along this ray, if r is positive. (See Fig. 5.1.7.) If r is negative, one travels a distance $-r$ along the ray traced in the opposite

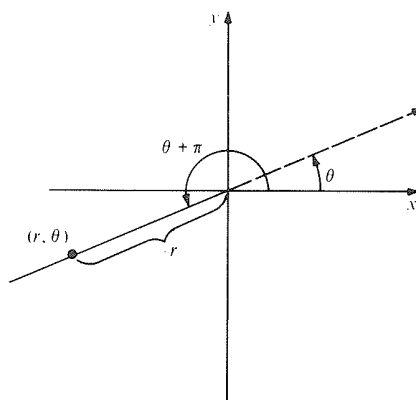


Figure 5.1.8. Plotting (r, θ) for negative r .

² Polar coordinates were first used successfully by Newton (1671) and Jacques Bernoulli (1691). The definitive treatment of polar coordinates in their modern form was given by Leonhard Euler in his 1748 textbook *Introductio in analysin infinitorum*. See C. B. Boyer, “The foremost textbook of modern times,” *American Mathematical Monthly* **58** (1951): 223–226.

direction. One arrives at the point P ; we call (r, θ) its *polar coordinates*. Notice that the resulting point is the same as the one with polar coordinates $(-r, \theta + \pi)$ (see Fig. 5.1.8) and that the pair $(r, \theta + 2\pi n)$ represents the same point as (r, θ) , for any integer n .

Example 3 Plot the points P_1 , P_2 , P_3 , and P_4 whose polar coordinates are $(5, \pi/6)$, $(-5, \pi/6)$, $(5, -\pi/6)$, and $(-5, -\pi/6)$, respectively.

Solution (See Fig. 5.1.9.) The point $(-5, -\pi/6)$ is obtained by rotating $\pi/6 = 30^\circ$ clockwise to give an angle of $-\pi/6$ and then moving 5 units backwards on

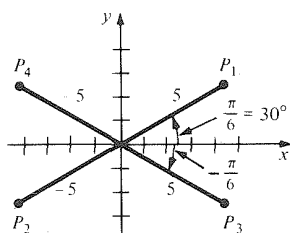


Figure 5.1.9. Some points in polar coordinates.

this line to the point P_4 shown. The other points are plotted in a similar way. ▲

Example 4 Describe the set of points P whose polar coordinates (r, θ) satisfy $0 \leq r \leq 2$ and $0 \leq \theta < \pi$.

Solution Since $0 \leq r \leq 2$, we can range from the origin to 2 units from the origin. Our angle with the x axis varies from 0 to π , but not including π . Thus we are confined to the region in Fig. 5.1.10. The negative x axis is dashed since it is not included in the region. ▲

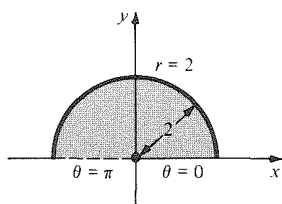


Figure 5.1.10. The region $0 \leq r \leq 2, 0 \leq \theta < \pi$.

If θ is a real number, we define $\cos \theta$ to be x and $\sin \theta$ to be y , where (x, y) are the cartesian coordinates of the point P on the circle of radius one whose polar coordinates are $(1, \theta)$. (See Fig. 5.1.11.) If an angle ϕ° is given in degrees, $\sin \phi^\circ$ or $\cos \phi^\circ$ means $\sin \theta$ or $\cos \theta$, where θ is the same angle measured in radians. Thus $\sin 45^\circ = \sin(\pi/4)$, $\cos 60^\circ = \cos(\pi/3)$, and so on.

The sine and cosine functions can also be defined in terms of ratios of sides of right triangles. (See Fig. 5.1.12.) By definition, $\cos \theta = |OA'|$, and by similar triangles,

$$\cos \theta = |OA'| = \left| \frac{OA'}{1} \right| = \left| \frac{OA'}{OB'} \right| = \frac{|OA|}{|OB|} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}.$$

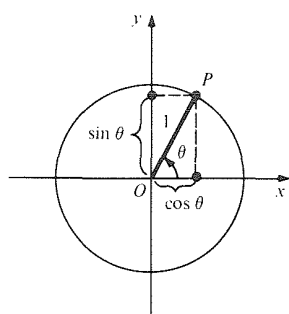


Figure 5.1.11. The cosine and sine of θ are the x and y coordinates of the point P .

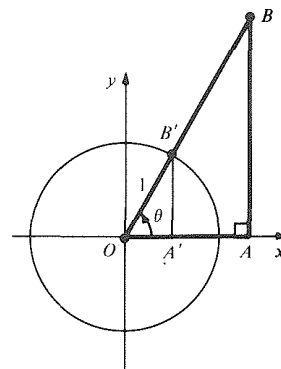


Figure 5.1.12. The triangles OAB and $OA'B'$ are similar; $\cos \theta = |OA|/|OB|$ and $\sin \theta = |AB|/|OB|$.

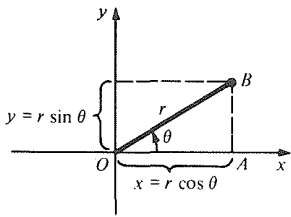


Figure 5.1.13. Converting polar to cartesian coordinates.

In the same way, we see that

$$\sin \theta = \frac{|AB|}{|OB|} = \frac{\text{side opposite to } \theta}{\text{hypotenuse}}.$$

It follows (see Fig. 5.1.13) that if the point B has cartesian coordinates (x, y) and polar coordinates (r, θ) , then $\cos \theta = |OA|/|OB| = x/r$ and $\sin \theta = |AB|/|OB| = y/r$, so we obtain the following relations.

Cartesian and Polar Coordinates

$x = r \cos \theta$ and $y = r \sin \theta$,
where (x, y) are cartesian coordinates and (r, θ) are polar coordinates.

Example 5 Convert from cartesian to polar coordinates: $(2, -4)$; and from polar to cartesian coordinates: $(6, -\pi/8)$.

Solution

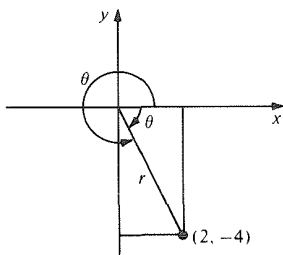


Figure 5.1.14. Find the polar coordinates of $(2, -4)$.

We plot $(2, -4)$ as in Fig. 5.1.14. Then $r = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$ and $\cos \theta = 2/(2\sqrt{5}) = 1/\sqrt{5} \approx 0.447214$, so from tables or a calculator³, $\theta = 1.107$; but we must take $\theta = -1.107$ (or 5.176) since we are in the fourth quadrant. Thus the polar coordinates of $(2, -4)$ are $(2\sqrt{5}, -1.107)$.

The cartesian coordinates of the point with polar coordinates $(6, -\pi/8)$ are

$$x = r \cos \theta = 6 \cos(-\pi/8) = (6)(0.92388) = 5.5433$$

and

$$y = r \sin \theta = 6 \sin(-\pi/8) = (6)(-0.38268) = -2.2961.$$

That is, $(5.5433, -2.2961)$. This point is also in the fourth quadrant as it should be. ▲

Example 6 (a) Show that $\sin \theta = \cos(\pi/2 - \theta)$ for $0 \leq \theta \leq \pi/2$. (b) Show that $\sin$ is an odd function: $\sin(-\theta) = -\sin \theta$ (assume that $0 \leq \theta \leq \pi/2$).

Solution

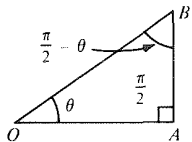


Figure 5.1.15. If $\theta = \angle BOA$, then $\angle OBA = \pi/2 - \theta$.

(a) In Fig. 5.1.15, the angle OBA is $\pi/2 - \theta$ since the three angles must add up to π by plane geometry. Therefore $\sin \theta = (\text{opposite/hypotenuse}) = |AB|/|OB|$ and $\cos(\pi/2 - \theta) = (\text{adjacent/hypotenuse}) = |AB|/|OB| = \sin \theta$.

(b) Referring to Fig. 5.1.16, we see that if θ is switched to $-\theta$, this changes the sign of $y = \sin \theta$. Hence $\sin(-\theta) = -\sin \theta$. ▲

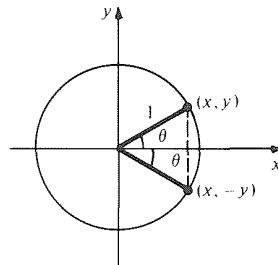


Figure 5.1.16. $\sin \theta$ is an odd function.

³ Many calculators are equipped with a $\cos^{-1}$ (arc cos) function which computes the angle whose cosine is given. If your calculator does not have such a function, you can use the cosine function together with the method of bisection (see Example 7, Section 3.1). The inverse cosine (and other trigonometric) functions are discussed in Section 5.4.

The other trigonometric functions can be defined in terms of the sine and cosine:

$$\text{Tangent: } \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{|AB|}{|OA|} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

$$\text{Cotangent: } \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} = \frac{|OA|}{|AB|}.$$

$$\text{Secant: } \sec \theta = \frac{1}{\cos \theta} = \frac{|OB|}{|OA|}.$$

$$\text{Cosecant: } \csc \theta = \frac{1}{\sin \theta} = \frac{|OB|}{|AB|}.$$

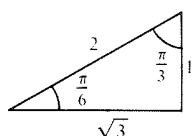
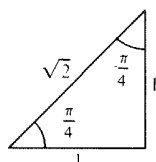


Figure 5.1.17. Two basic examples.

Some frequently used values of the trigonometric functions can be read off the right triangles shown in Fig. 5.1.17. For example, $\cos(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/4) = 1/\sqrt{2}$, $\tan(\pi/4) = 1$, $\cos(\pi/6) = \sqrt{3}/2$, and $\tan(\pi/3) = \sqrt{3}$. (The proof that the $1, 2, \sqrt{3}$ triangle has angles $\pi/3, \pi/6, \pi/2$ is an exercise in euclidean geometry; see Fig. 5.1.18.)

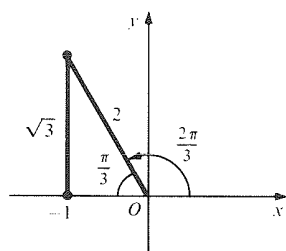
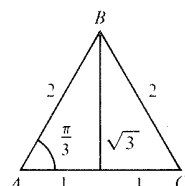


Figure 5.1.19. Illustrating the sine and cosine of $2\pi/3$.

Figure 5.1.18. The angles of an equilateral triangle are all equal to $\pi/3$.



Special care should be taken with functions of angles which are not between 0 and $\pi/2$ —that is, angles not in the first quadrant—to ensure that their signs are correct. For instance, we notice in Fig. 5.1.19 that $\sin(2\pi/3) = \sqrt{3}/2$ and $\cos(2\pi/3) = -1/2$.

The following table gives some commonly used values of $\sin$, $\cos$, and $\tan$:

θ°	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm \infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\pm \infty$	0

Over the centuries, large tables of values of the trigonometric functions have been compiled. The first such table, compiled by Hipparchus and Ptolemy, appeared in Ptolemy's *Almagest*. Today these values are also on many pocket calculators. Since angles as well as some lengths can be directly measured (as in surveying), the trigonometric relations can then enable us to compute lengths which may be inaccessible (see Example 7).

Calculator Discussion

You may be curious about how pocket calculators compute their values of $\sin \theta$ and $\cos \theta$. Some analytic expressions are available, such as

$$\sin \theta = \theta - \frac{\theta^3}{3 \cdot 2} + \frac{\theta^5}{5 \cdot 4 \cdot 3 \cdot 2} - \frac{\theta^7}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \cdots$$

(as will be proved in Section 12.5), but using these is inefficient and inaccurate. Instead a rational function of θ is fitted to many known values of $\sin \theta$ (or $\cos \theta$, $\tan \theta$, and so on) and this rational function is used to calculate approximate values at the remaining points. Thus when θ is entered and $\sin \theta$ pressed, a program in the calculator calculates the value of this rational function.

If you experiment with your calculator—for example, by calculating $\tan \theta$ for θ near $\pi/2$ —you might discover some inaccuracies in this method.⁴ ▲

Example 7 (a) In Fig. 5.1.20, find x . (b) A tree 50 meters away subtends an angle of 53° as seen by an observer. How tall is the tree?

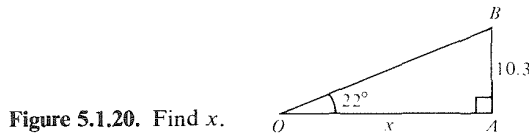


Figure 5.1.20. Find x .

Solution (a) We find that $\tan 22^\circ = 10.3/x$, so $x = 10.3/\tan 22^\circ$. From tables or a calculator, $\tan 22^\circ \approx 0.404026$, so $x \approx 25.4934$. (b) Refer to Fig. 5.1.21. $|AB| = |OA|\tan 53^\circ = 50 \tan 53^\circ$. Using tables or a calculator, this becomes $50(1.3270) = 66.35$ meters. ▲

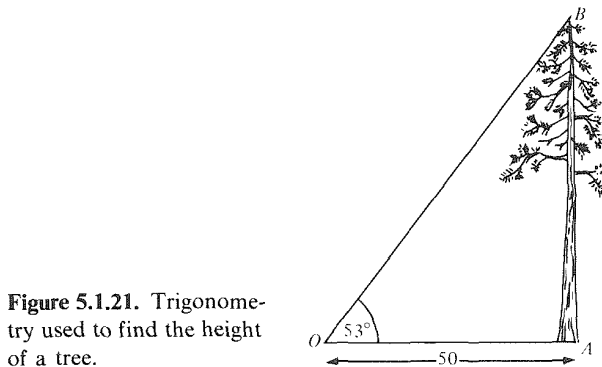


Figure 5.1.21. Trigonometry used to find the height of a tree.

From the definition of $\sin$ and $\cos$, the point P with cartesian coordinates $x = \cos \theta$ and $y = \sin \theta$ lies on the unit circle $x^2 + y^2 = 1$. Therefore, for any value of θ ,

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (1)$$

This is an example of *trigonometric identity*—a relationship among the trigonometric functions which is valid for all θ .

⁴ For more details on how calculators do those computations, see “Calculator Function Approximation” by C. W. Schelin, *Am. Math. Monthly* Vol. 90 (1983), 317–325.

Relationship (1) is, in essence, a statement of Pythagoras' theorem for a right triangle ($OA'B'$ in Fig. 5.1.12). For a general triangle, the correct relationship between the three sides is given by the *law of cosines*: with notation as in Fig. 5.1.22, we have

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (2)$$

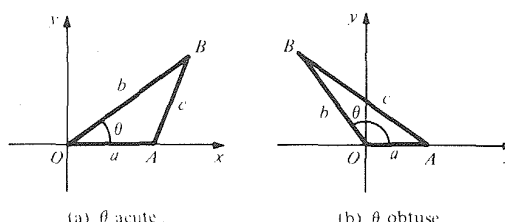


Figure 5.1.22. Proving the law of cosines.

To prove equation (2), note that the (x, y) coordinates of B are $x = b \cos \theta$ and $y = b \sin \theta$; those of A are $x = a, y = 0$. By the distance formula and equation (1),

$$\begin{aligned} c^2 &= (b \cos \theta - a)^2 + (b \sin \theta)^2 = b^2 \cos^2 \theta - 2ab \cos \theta + a^2 + b^2 \sin^2 \theta \\ &= b^2 (\cos^2 \theta + \sin^2 \theta) + a^2 - 2ab \cos \theta = b^2 + a^2 - 2ab \cos \theta, \end{aligned}$$

so equation (2) is proved.

In Fig. 5.1.22 we situated the triangles in a particular way, but this was just a device to prove equation (2); since any triangle can be moved into this special position, the formula holds in the general situation of Fig. 5.1.23.

Note that when $\theta = \pi/2$, $\cos \theta = 0$ and so equation (2) reduces to Pythagoras' theorem: $c^2 = a^2 + b^2$.

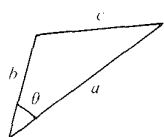


Figure 5.1.23. Data for the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos \theta$.

Example 8 In Fig. 5.1.24, find x .

Solution By the law of cosines

$$\begin{aligned} x^2 &= (20.2)^2 + (13.4)^2 - 2(20.2)(13.4)\cos(10.3^\circ) \\ &= (408.04) + (179.56) - 532.64 = 54.96. \end{aligned}$$

Taking square roots, we find $x \approx 7.41$. $\blacktriangle$

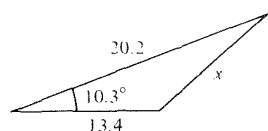


Figure 5.1.24. Find x .

Now consider the situation in Fig. 5.1.25. By the distance formula,

$$\begin{aligned} |PQ|^2 &= (\cos \phi - \cos \theta)^2 + (\sin \phi - \sin \theta)^2 \\ &= \cos^2 \phi - 2 \cos \phi \cos \theta + \cos^2 \theta + \sin^2 \phi - 2 \sin \phi \sin \theta + \sin^2 \theta \\ &= 2 - 2 \cos \phi \cos \theta - 2 \sin \phi \sin \theta. \end{aligned}$$

On the other hand, by the law of cosines (2) applied to $\triangle OPQ$,

$$|PQ|^2 = 1^2 + 1^2 - 2 \cos(\phi - \theta),$$

since $\phi - \theta$ is the angle at the vertex O . Comparing our two expressions for $|PQ|^2$ gives the identity

$$\cos(\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta,$$

which is valid for all ϕ and θ . If we replace θ by $-\theta$ and recall that $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$, this identity yields

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi. \quad (3)$$

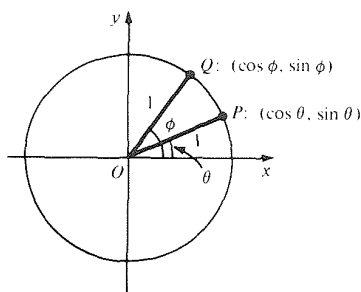


Figure 5.1.25. Geometry for the proof of the addition formulas.

Now if we write $\pi/2 - \phi$ for ϕ and recall that $\cos(\pi/2 - \phi) = \sin \phi$ and $\sin(\pi/2 - \phi) = \cos \phi$, the same identity gives

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi. \quad (4)$$

Identities (3) and (4), called the *addition formulas* for sine and cosine, will be essential for calculus. From these basic identities, we can also derive many others by algebraic manipulation.

For integral calculus, two of the most important consequences of (3) are the *double-angle formulas*. Setting $\theta = \phi$ in (3) gives

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta.$$

Thus

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta). \quad (5)$$

Similarly,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1$$

gives

$$\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1). \quad (6)$$

- Example 9** (a) Prove the product formula: $\cos \theta \cos \phi = \frac{1}{2}[\cos(\theta - \phi) + \cos(\theta + \phi)]$.
 (b) Prove that $1 + \tan^2 \theta = \sec^2 \theta$.

Solution (a) Add the identity for $\cos(\theta - \phi)$ to that for $\cos(\theta + \phi)$:

$$\begin{aligned} \cos(\theta - \phi) + \cos(\theta + \phi) &= \cos \phi \cos \theta + \sin \phi \sin \theta + \cos \theta \cos \phi - \sin \theta \sin \phi \\ &= 2 \cos \theta \cos \phi. \end{aligned}$$

Dividing by 2 gives the product formula.

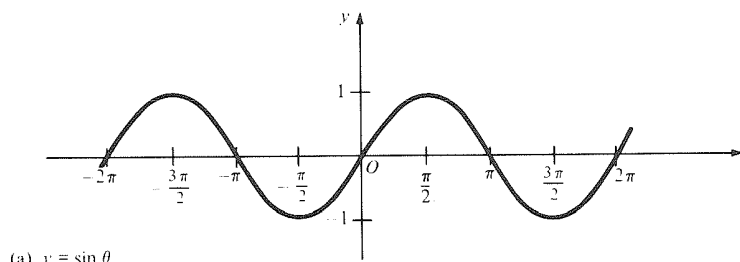
(b) Divide both sides of $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$; then, using $\tan \theta = \sin \theta / \cos \theta$ and $1 / \cos \theta = \sec \theta$, we get $\tan^2 \theta + 1 = \sec^2 \theta$ as required. $\blacktriangle$

Some of the most important trigonometric identities are listed on the inside front cover of the book for handy reference. They are all useful, but you can get by quite well by memorizing only (1) through (4) above and deriving the rest when you need them.

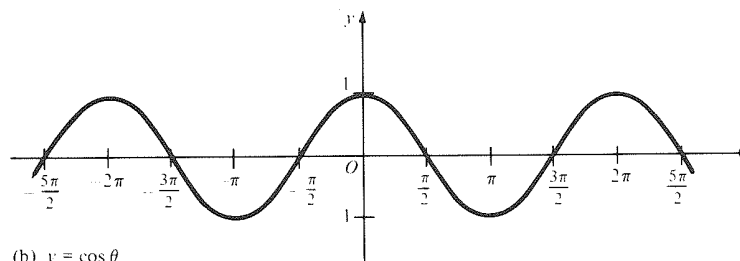
From the available values of the trigonometric functions, one can accurately draw their graphs. The calculus of these functions, studied in the next section, confirms that these graphs are correct, so there are no maxima, minima, or inflection points other than those in plain view in Fig. 5.1.26 on the following page.

Perhaps the most important fact about these functions is their *periodicity*: When a function f satisfies $f(\theta + \tau) = f(\theta)$ for all θ and a given positive τ , f is said to be *periodic* with *period* τ . The reciprocal $1/\tau$ is called the *frequency*. All the functions in Fig. 5.1.26 are periodic with period 2π ; this enables us to draw the entire graph by repeating the segment over an interval of length 2π . The trigonometric functions also have $4\pi, 6\pi, 8\pi, \dots$ as additional periods, but the least period of a periodic function is always unique. Note that the functions $\tan$ and $\cot$ have π as their least period, while 2π is the least period of the other four trigonometric functions.

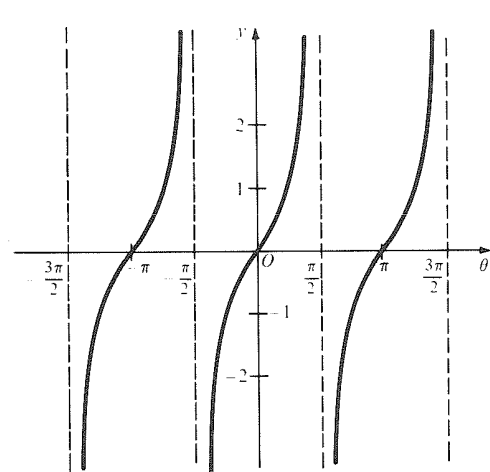
Figure 5.1.26. Graphs of the trigonometric functions.



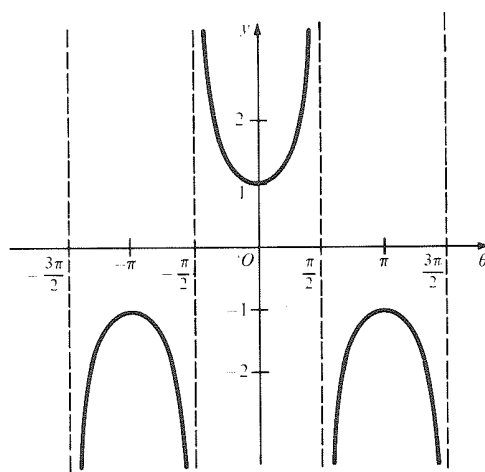
(a) $y = \sin \theta$



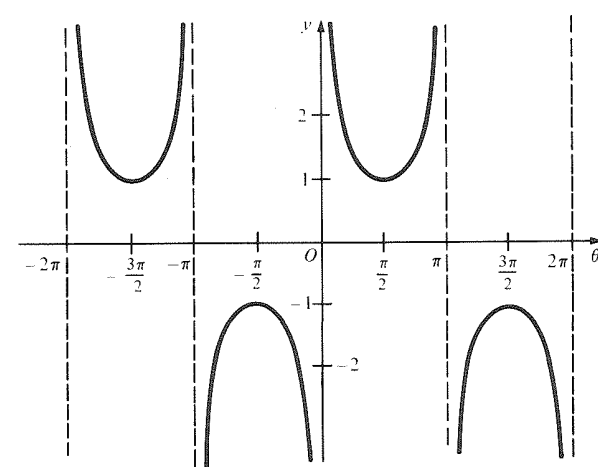
(b) $y = \cos \theta$



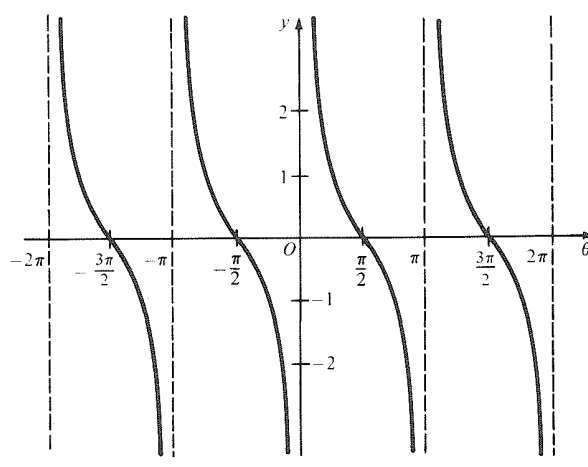
(c) $y = \tan \theta$



(d) $y = \sec \theta$



(e) $y = \csc \theta$



(f) $y = \cot \theta$

- Example 10** (a) Sketch the graph $y = \cos 2x$. What is the (least) period of this function?
 (b) Sketch the graph of $y = 3 \cos 5\theta$.

Solution (a) We obtain $y = \cos 2x$ by taking the graph of $y = \cos x$ and compressing the graph horizontally by a factor of 2 (see Fig. 5.1.27). The function repeats every π units on the x axis, so it is periodic with (least) period π .

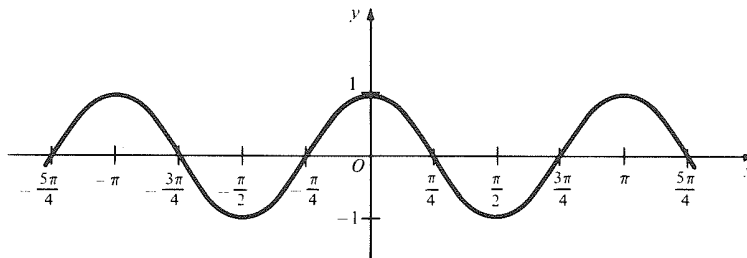


Figure 5.1.27. The graph of $y = \cos 2x$.

- (b) We obtain $y = 3 \cos 5\theta$ by compressing the graph of $y = \cos \theta$ horizontally by a factor of 5 and stretching it vertically by a factor of 3 (see Fig. 5.1.28). ▲

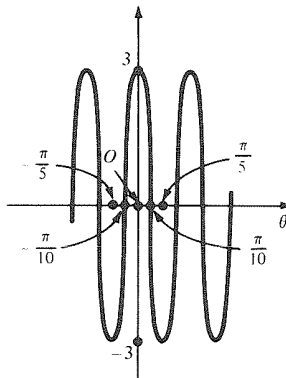


Figure 5.1.28. The graph of $y = 3 \cos 5\theta$.

- Example 11** Where are the inflection points of $\tan \theta$? For which values of θ do you expect $\tan \theta$ to be a differentiable function?

Solution Recall that an inflection point is a point where the second derivative changes sign—that is, a point between different types of concavity. On the graph of $\tan \theta$, notice that the graph is concave upward on $(0, \pi/2)$ and downward on $(-\pi/2, 0)$. Hence 0 is an inflection point, as are $\pi, -\pi$, and so forth. The general inflection point is $n\pi$, where $n = 0, \pm 1, \pm 2, \dots$. (But $\pi/2$ is not an inflection point because $\tan \theta$ is not defined there.)

From the graph, we expect that $\tan \theta$ will be a differentiable function of θ except at $\pm \pi/2, \pm 3\pi/2, \dots$. ▲

Exercises for Section 5.1

- If an arc of a circle with radius 10 meters subtends an angle of 22° , how long is the arc? How much area is enclosed in this part of the circle?
- An arc of radius 15 feet subtends an angle of 2.1 radians. How long is the arc? How much area is enclosed in this part of the circle?
- An arc of radius 18 meters has length 5 meters. What angle does it subtend? How much area is enclosed in this part of the circle?
- An arc of length 110 meters subtends an angle of 24° . What is the radius of the arc? How much area is enclosed in this part of the circle?
- Convert to radians: $29^\circ, 54^\circ, 255^\circ, 130^\circ, 320^\circ$.
- Convert to degrees: $5, \pi/7, 3.2, 2\pi/9, \frac{1}{2}, 0.7$.
- Simplify so that $0 \leq \theta < 2\pi$ or $0 \leq \theta^\circ < 360^\circ$.
 (a) Radians: $7\pi/3, 16\pi/5, 15\pi$.
 (b) Degrees: $520^\circ, 1745^\circ, 385^\circ$.
- Simplify so that $0 \leq \theta < 2\pi$ or $0 \leq \theta^\circ < 360^\circ$.
 (a) Radians: $5\pi/2, 48\pi/11, 13\pi + 1$.
 (b) Degrees: $470^\circ, 604^\circ, 75^\circ, 999^\circ$.

9. Plot the following points given in polar coordinates: $(3, \pi/2)$, $(5, -\pi/4)$, $(1, 2\pi/3)$, $(-3, \pi/2)$.
 10. Plot the following points given in polar coordinates: $(6, 3\pi/2)$, $(-2, \pi/6)$, $(7, -2\pi/3)$, $(1, \pi/2)$, $(4, -\pi/6)$.

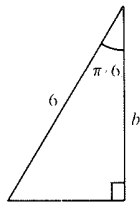
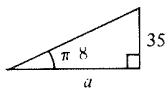
In Exercises 11–14, sketch the set of points whose polar coordinates (r, θ) satisfy the given conditions.

11. $-1 \leq r \leq 2$; $\pi/3 \leq \theta < \pi/2$.
 12. $0 < r < 4$; $-\pi/6 < \theta < \pi/6$.
 13. $2 \leq r < 3$; $-\pi/2 \leq \theta \leq \pi$.
 14. $-2 \leq r \leq -1$; $-\pi/4 < \theta < 0$.
 15. Find the polar coordinates of $(x, y) = (5, -2)$.
 16. Find the cartesian coordinates of $(r, \theta) = (2, \pi/6)$.
 17. Convert from cartesian to polar coordinates:
 (a) $(1, 0)$, (b) $(3, 4)$, (c) $(\sqrt{3}, 1)$, (d) $(\sqrt{3}, -1)$,
 (e) $(-\sqrt{3}, 1)$.
 18. Convert from polar to cartesian coordinates:
 (a) $(0, \pi/8)$, (b) $(1, 0)$, (c) $(2, \pi/4)$, (d) $(8, 3\pi/2)$,
 (e) $(2, \pi)$.
 19. Convert from cartesian to polar coordinates:
 (a) $(1, -1)$, (b) $(0, 2)$, (c) $(\frac{1}{2}, 7)$, (d) $(-12, -5)$,
 (e) $(-3, 8)$, (f) $(\frac{3}{4}, \frac{3}{4})$.
 20. Convert from cartesian to polar coordinates:
 (a) $(-4, 4)$, (b) $(1, 15)$, (c) $(19, -3)$, (d) $(-5, -6)$,
 (e) $(0.3, 0.9)$, (f) $(-\frac{3}{2}, \frac{1}{2})$.
 21. Convert from polar to cartesian coordinates:
 (a) $(6, \pi/2)$, (b) $(-12, 3\pi/4)$, (c) $(4, -\pi)$,
 (d) $(2, 13\pi/2)$, (e) $(8, -2\pi/3)$, (f) $(-1, 2)$.
 22. Convert from polar to cartesian coordinates:
 (a) $(-1, -1)$, (b) $(1, \pi)$, (c) $(10, 2.7)$,
 (d) $(5, 7\pi/2)$, (e) $(8, 7\pi)$, (f) $(4, -3\pi)$.
 23. Show that $\tan \theta = \cot(\pi/2 - \theta)$ assuming that $0 \leq \theta \leq \pi/2$.
 24. Show that $\sec \theta = \csc(\pi/2 - \theta)$ for $0 \leq \theta \leq \pi/2$.
 25. Show that $\cos \theta = \cos(-\theta)$ for $0 \leq \theta \leq \pi$.
 26. Show that $\tan \theta = -\tan(-\theta)$ for $0 \leq \theta \leq \pi$, $\theta \neq \pi/2$.

Refer to Fig. 5.1.29 for Exercises 27–30.

27. Find a .

28. Find b .



29. Find c .

30. Find d .

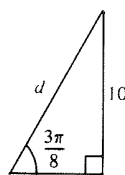
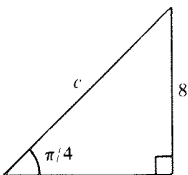


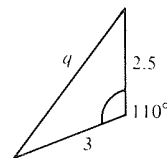
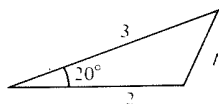
Figure 5.1.29. Find a, b, c, d .

31. An airplane flying at 5000 feet has an angle of elevation of 25° from observer A . Observer B sees that airplane directly overhead. How far apart are A and B ?
 32. A leaning tower tilts at 9° from the vertical directly away from an observer who is 500 meters away from its base. If the observer sees the top of the tower at an angle of elevation of 22° , how high is the tower?
 33. A mountain 3000 meters away subtends an angle of 17° at an observer. How tall is the mountain?
 34. A pedestrian 100 meters from the outdoor elevator at the Fairhill Hotel at noon sees the elevator at an angle of 10° . The elevator, steadily rising, makes an angle of 20° after 30 seconds has elapsed. How fast is the elevator rising? When will it make an angle of 30° ?

Refer to Fig. 5.1.30 for Exercises 35–38.

35. Find p .

36. Find q .



37. Find r .

38. Find s .

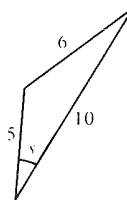
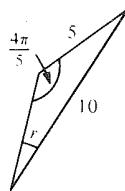


Figure 5.1.30. Find p, q, r , and s .

39. Prove that $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$.
 40. Prove that $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$.
 41. Prove that $\sin \theta \sin \phi = \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2}$.
 42. Express $\sin(3\theta)$ in terms of $\sin \theta$ and $\cos \theta$.
 Simplify the expressions in Exercises 43–50.
 43. $\sin(\theta + \frac{\pi}{2})$
 44. $\sin(\frac{3\pi}{2} + \theta)$
 45. $\cos(\frac{3\pi}{2} - \theta)$
 46. $\tan(\theta + \frac{7}{2}\pi)$
 47. $\sec(6\pi + \theta)$
 48. $\sin(\theta - \frac{9}{2}\pi)$
 49. $\cos(\theta + \frac{\pi}{2})\sin(\phi - \frac{3\pi}{2})$.
 50. $\frac{\sin(\theta + \frac{5}{2}\pi)}{\cos(\frac{\pi}{2} - \theta)}$.

Compute the quantities in Exercises 51–54 by using trigonometric identities, without using tables or a calculator.

51. $\cos 7\frac{1}{2}^\circ$ 52. $\tan 22\frac{1}{2}^\circ$
 53. $\sec \frac{\pi}{12}$ 54. $\sin \frac{\pi}{12}$

Derive the identities in Exercises 55–60, making use of the table of identities on the inside front cover of the book.

55. $\sec \theta + \tan \theta = \frac{1 + \sin \theta}{1 - 2 \sin^2(\theta/2)}$
 56. $8 \cos \theta + 8 \cos 2\theta = -9 + 16(\cos \theta + \frac{1}{4})^2$
 57. $\sec^2 \frac{\theta}{2} = \frac{2 \sec \theta}{\sec \theta + 1}$
 58. $\csc^2 \frac{\theta}{2} = \frac{2 \sec \theta}{\sec \theta - 1}$
 59. $\csc \theta \csc \phi = \frac{2 \sec(\theta + \phi) \sec(\theta - \phi)}{\sec(\theta + \phi) - \sec(\theta - \phi)}$
 60. $\tan \theta \tan \phi = \frac{\sec(\theta + \phi) - \sec(\theta - \phi)}{\sec(\theta + \phi) + \sec(\theta - \phi)}$

Sketch the graph of the functions in Exercises 61–68.

61. $2 \cos 3\theta$ 62. $\cos(3\theta + \frac{\pi}{6})$
 63. $\tan \frac{3\theta}{2}$ 64. $\tan \frac{\theta}{2}$
 65. $4 \sin 2x \cos 2x$ 66. $\sin x + \cos x$
 67. $\sin 3\theta + 1$ 68. $\csc 2\theta$

69. Locate the inflection points of $\cot \theta$ by inspecting the graph.
 70. Locate the maximum and minimum points of $\sin \theta$ by inspecting the graph.
 71. For what θ do you expect $\sec \theta$ and $\cot \theta$ to be differentiable?

72. Where is $\sec \theta$ concave upward?

73. Light travels at velocity v_1 in a certain medium, enters a second medium at angle of incidence θ_1 (measured from the normal to the surface), and refracts at angle θ_2 while travelling a different velocity v_2 (in the second medium). According to Snell's law, $v_1/v_2 = \sin \theta_1/\sin \theta_2$.

- (a) Light enters at 60° and refracts at 30° . The first medium is air ($v_1 = 3 \times 10^{10}$ centimeters per second). Find the velocity in the second medium.
 (b) Show that if $v_1 = v_2$, then the light travels in a straight line.
 (c) The speed halves in passing from one medium to another, the angle of incidence being 45° . Calculate the angle of refraction.

74. Scientists and engineers often use the approximations $\sin \theta \approx \theta$ and $\cos \theta \approx 1$, valid for θ near zero. Experiment with your hand calculator to determine a region of validity for θ that guarantees eight-place accuracy for these approximations.

Exercises 75–78 concern the *law of sines*.

75. Using the notation in Fig. 5.1.31, prove that

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

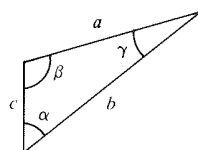


Figure 5.1.31. Law of sines.

76. Find a in Fig. 5.1.32.

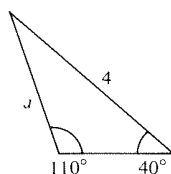


Figure 5.1.32. Find a .

77. Show that the common value of $(\sin \alpha)/a$, $(\sin \beta)/b$, and $(\sin \gamma)/c$ is the reciprocal of twice the radius of the circumscribed circle.
 78. (a) A parallelogram is formed with acute angle θ and sides l, L . Find a formula for its area. (b) A parallelogram is formed with an acute angle θ , a base of length L , and a diagonal opposite θ of length d . Find a formula for its area.
 79. Show that $A \cos \omega t + B \sin \omega t = \alpha \cos(\omega t - \theta)$, where (α, θ) are the polar coordinates of (A, B) .
 80. Use Exercise 79 to write $f(t) = \cos t + \sqrt{3} \sin t$ as $\alpha \cos(t - \theta)$ for some α and θ . Use this to graph f .
 81. Light of wavelength λ is diffracted through a single slit of width a . This light then passes through a lens and falls on a screen. A point P is on the screen, making an angle θ with the lens axis. The intensity I at the point P on the screen is

$$I = I_0 \left[\frac{\sin([\pi a \sin \theta]/\lambda)}{[\pi a \sin \theta]/\lambda} \right]^2, \quad 0 < \theta < \pi,$$

where I_0 is the intensity when P is on the lens axis.

- (a) Show that the intensity is zero for $\sin \theta = \lambda/a$.
 (b) Find all values of θ for which $\theta > 0$ and $I = 0$.
 (c) Verify that I is approximately I_0 when $[\pi a \sin \theta]/\lambda$ is close enough to zero. (Use $\sin \theta \approx \theta$; see Exercise 74.)
 (d) In practice, $\lambda = 5 \times 10^{-5}$ centimeters, and $a = 10^{-2}$ centimeters. Check by means of a calculator or table that $\sin \theta = \lambda/a$ is approximately $\theta = \lambda/a$.
 82. The current I in a circuit is given by the formula $I(t) = 20 \sin(311t) + 40 \cos(311t)$. Let $r = (20^2 + 40^2)^{1/2}$ and define the angle θ by $\cos \theta = 20/r$, $\sin \theta = 40/r$.
 (a) Verify by use of the sum formula for the sine function that $I(t) = r \sin(311t + \theta)$.
 (b) Show that the *peak current* is r (the maximum value of I).
 (c) Find the *period* and *frequency*.
 (d) Determine the *phase shift* (in radians), that is, the value of t which makes $311t + \theta = 0$.

83. The instantaneous power input to an AC circuit is $p = vi$, where v is the instantaneous potential difference between the circuit terminals and i is the instantaneous current. If the circuit is a pure resistor, then $v = V \sin(\omega t)$ and $i = I \sin(\omega t)$.
- (a) Verify by means of trigonometric identities that

$$p = \frac{VI}{2} - \frac{VI}{2} \cos(2\omega t).$$

- (b) Draw a graph of p as a function of t .

84. Two points are located on one bank of the Colorado River, 500 meters apart. A point on the opposite bank makes angles of 88° and 80° with the line joining the two points. Find the lengths of two cables to be stretched across the river connecting the points.

5.2 Differentiation of the Trigonometric Functions

Differentiation rules for sine and cosine follow from arguments using limits and the addition formulas.

In this section, we will derive differentiation formulas for the trigonometric functions. In the course of doing so, we will use many of the basic properties of limits and derivatives introduced in the first two chapters.

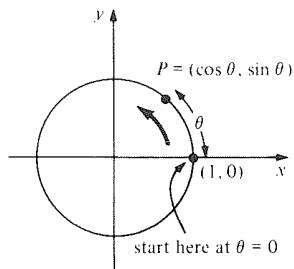


Figure 5.2.1. The point P moves at unit speed around the circle.

The unit circle $x^2 + y^2 = 1$ can be described by the parametric equations $x = \cos \theta$, $y = \sin \theta$. As θ increases, the point $(x, y) = (\cos \theta, \sin \theta)$ moves along the circle in a counterclockwise direction (see Fig. 5.2.1).

The length of arc on the circle between the point $(1, 0)$ (corresponding to $\theta = 0$) and the point $(\cos \theta, \sin \theta)$ equals the angle θ subtended by the arc. If we think of θ as time, the point $(\cos \theta, \sin \theta)$ travels a distance θ in time θ , so it is moving with unit speed around the circle. At $\theta = 0$, the tangent line to the circle is vertical, so the velocity of the point is 1 in the vertical direction; thus, we expect that

$$\left. \frac{dx}{d\theta} \right|_{\theta=0} = 0 \quad \text{and} \quad \left. \frac{dy}{d\theta} \right|_{\theta=0} = 1.$$

That is,

$$\left. \frac{d \cos \theta}{d\theta} \right|_{\theta=0} = 0, \quad (1)$$

$$\left. \frac{d \sin \theta}{d\theta} \right|_{\theta=0} = 1. \quad (2)$$

We will now derive (1) and (2) using limits.

According to the definition of the derivative, formulas (1) and (2) amount to the following statements about limits:

$$\lim_{\Delta\theta \rightarrow 0} \frac{\cos \Delta\theta - 1}{\Delta\theta} = 0 \quad (3)$$

and

$$\lim_{\Delta\theta \rightarrow 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1. \quad (4)$$

To prove (3) and (4), we use the geometry in Fig. 5.2.2 and shall denote $\Delta\theta$ by the letter ϕ for simplicity of notation. When $0 < \phi < \pi/2$, we have

$$\text{area triangle } OCB = \frac{1}{2} |OC| \cdot |AB| = \frac{1}{2} \sin \phi,$$

$$\text{area triangle } OCB < \text{area sector } OCB = \frac{1}{2} \phi,$$

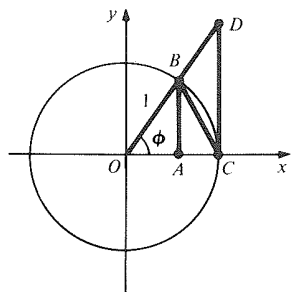


Figure 5.2.2. Geometry used to determine $\cos'0$ and $\sin'0$.

65. Suppose that $f'(x) = 1/x$. (We do not yet have such a function at our disposal, but we will see in Chapter 6 that there is one.) Show that $\int \tan \theta \, d\theta = -f(\cos \theta) + C$.
66. Show that $\int \tan \theta \, d\theta = f(\sec \theta) + C$, where f is a function such that $f'(x) = 1/x$. (The apparent conflict with Exercise 65 will be resolved in the next chapter.)
67. Show that $(d/d\theta)\cos \theta = -\sin \theta$ can be derived from $(d/d\theta)\sin \theta = \cos \theta$ by using the identity $\cos \theta = \sin(\pi/2 - \theta)$ and the chain rule.
68. (a) Evaluate $\lim_{\Delta\theta \rightarrow 0} (\tan \Delta\theta)/\Delta\theta$ using the methods of the beginning of this section.
 (b) Use part (a) and the addition formula for the tangent (see the inside front cover) to prove that $\tan' \theta = \sec^2 \theta$, $-\pi/2 < \theta < \pi/2$, that is, prove that

$$\lim_{\Delta\theta \rightarrow 0} \frac{\tan(\theta + \Delta\theta) - \tan \theta}{\Delta\theta} = \sec^2 \theta.$$

- ★69. Suppose that $\phi(x)$ is a function “appearing from the blue” with the property that

$$\frac{d\phi}{dx} = \frac{1}{\cos x}.$$

Calculate: (a) $\frac{d}{dx}(\phi(3x)\cos x)$, (b) $\int_0^1 \frac{1}{\cos x} \, dx$,

$$(c) \frac{d^2}{dx^2}(\phi(2x)\sin 2x).$$

- ★70. Let ψ be a function such that

$$\frac{d\psi}{dx} = \phi,$$

where ϕ is described in Exercise 69. Prove that

$$\begin{aligned} \frac{d}{dx} \left(\psi(x) \sin x + \frac{d}{dx} \left(\psi(x) \cos x - \frac{x^2}{2} \right) \right) \\ = -\phi(x) \sin x \end{aligned}$$

- ★71. Give a geometric “proof” that $\sin' = \cos$ and $\cos' = -\sin$ by these steps:

- (a) Consider the parametric curve $(\cos \theta, \sin \theta)$ as in the beginning of this section. Argue that $(\cos' \theta)^2 + (\sin' \theta)^2 = 1$, since the point moves at unit speed around the circle (see Exercise 34, Section 2.4).
 (b) Use the relation $\cos^2 \theta + \sin^2 \theta = 1$ to show that $\sin \theta \sin' \theta + \cos \theta \cos' \theta = 0$.
 (c) Conclude from (a) and (b) that $(\sin' \theta)^2 = \cos^2 \theta$ and $(\cos' \theta)^2 = \sin^2 \theta$.
 (d) Give a geometric argument to get the correct signs in the square roots of the relations in (c).

5.3 Inverse Functions

The derivative of an inverse function is the reciprocal of its derivative.

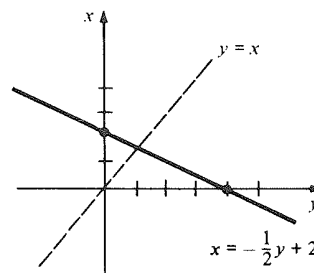
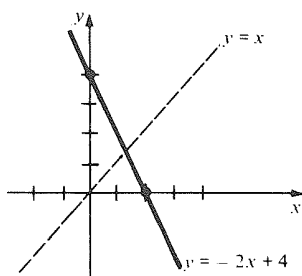
Sometimes two variable quantities are related in such a way that either one may be considered as a function of the other. The relationship between the quantities may be expressed by either of two functions, which are called *inverses* of one another. In this section, we will learn when a given function has an inverse, and we will see a useful relationship between the derivative of a function and the derivative of its inverse.

A simple example of a function with an inverse is the linear function $y = f(x) = mx + b$ with $m \neq 0$. We can solve for x in terms of y to get $x = (1/m)y - b/m$. Considering the expression $(1/m)y - b/m$ as a function $g(y)$ of y , we find that $y = f(x)$ whenever $x = g(y)$, and vice versa.

The graphs of f and g are very simply related. We interchange the role of the x and y axes by flipping the graph over the diagonal line $y = x$ (which bisects the right angle between the axes) or by viewing the graph through the back of the page held so that the x axis is vertical. (See Fig. 5.3.1 for a specific example.) Notice that the (constant) slopes of the two graphs are reciprocals of one another.

Whenever two functions, f and g , have the property that $y = f(x)$ whenever $x = g(y)$, and vice versa, we say that f and g are *inverses* of one another; the graphs of f and g are then related by flipping the x and y axes, as in Fig. 5.3.1. If we are given a formula for $y = f(x)$, we can try to find the inverse function by solving for x in terms of y .

Figure 5.3.1. The functions $f(x) = -2x + 4$ and $g(y) = -\frac{1}{2}y + 2$ are inverses to one another.



Example 1 Find an inverse function for $f(x) = x^3$. Graph f and its inverse.

Solution Solving the relation $y = x^3$ for x in terms of y , we find $x = \sqrt[3]{y}$, so the cuberoot function $g(y) = \sqrt[3]{y}$ is the inverse function to the cubing function $f(x) = x^3$. The graphs of f and g are shown in parts (a) and (b) of Fig. 5.3.2. In part (c), we have illustrated the important fact that the names of variables

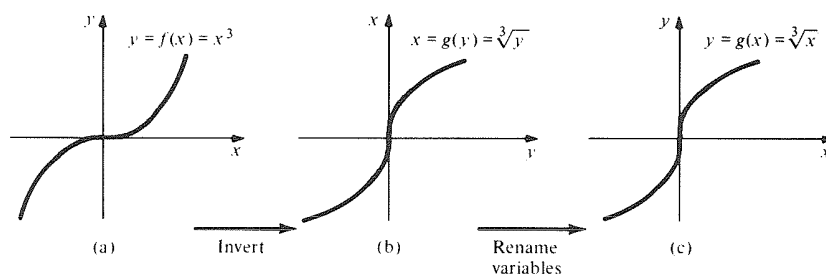


Figure 5.3.2. The function $y = x^3$ and its inverse $x = \sqrt[3]{y}$.

used with a function are arbitrary; since we like to have y as a function of x , we may write $g(x) = \sqrt[3]{x}$, and so the graph $y = \sqrt[3]{x}$ is another acceptable picture of the cuberoot function. ▲

Not every function has an inverse. For instance, if $y = f(x) = x^2$, we may solve for x in terms of y to get $x = \pm\sqrt{y}$, but this does not give x as a function defined for all y , for two reasons:

- (1) if $y < 0$, the square root $\sqrt{y}$ is not defined;
- (2) if $y > 0$, the two choices of positive or negative sign give two different values for x .

We can also see the difficulty geometrically. If we interchange the axes by flipping the parabola $y = x^2$ (Fig. 5.3.3), the resulting “horizontal parabola” is

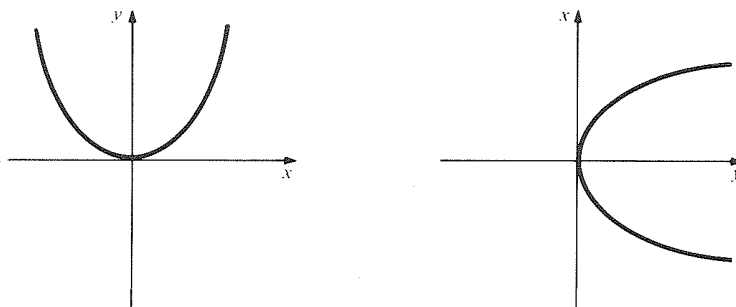


Figure 5.3.3. Flipping the parabola $y = f(x) = x^2$ produces a curve which is not the graph of a function.

not the graph of a function defined on the real numbers, since it intersects some vertical lines twice and others not at all.

This situation is similar to one we encountered in Section 2.3, where a

curve like $x^2 + y^2 = 1$ defined y as an implicit function of x only if we looked at *part* of the curve. In the present case, we can obtain an invertible function if we restrict $f(x) = x^2$ to the domain $[0, \infty)$. (The choice $(-\infty, 0]$ would do as well.) Then the inverse function $g(y) = \sqrt{y}$ is well defined with domain $[0, \infty)$ (see Fig. 5.3.4). The domain $(-\infty, 0]$ for f would have led to the other square root sign $-\sqrt{y}$ for g .

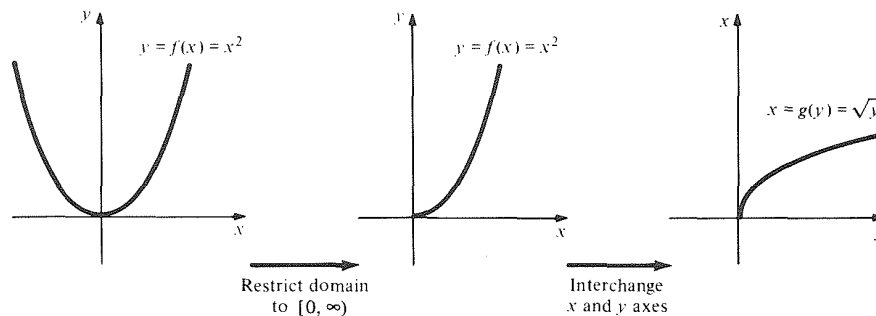


Figure 5.3.4. The function $f(x) = x^2$ and its inverse $g(y) = \sqrt{y}$.

The inverse to a function f , when it exists, is sometimes denoted by f^{-1} and read “ f inverse”. A function with an inverse is said to be *invertible*.

Warning Notice from this example that the inverse f^{-1} is not in general the same as $1/f$.

We summarize our work to this point in the following box.

Inverse Functions

The inverse function to a function f is a function g for which $g(y) = x$ when $y = f(x)$, and vice versa.

The inverse function to f is denoted by f^{-1} .

To find a formula for f^{-1} , try to solve the equation $y = f(x)$ for x in terms of y . If the solution is unique, set $f^{-1}(y) = x$.

The graph of f^{-1} is obtained from that of f by flipping the figure to interchange the horizontal and vertical axes.

It may be necessary to restrict the domain of f before there is an inverse function.

Example 2 Let $f(x) = x^2 + 2x + 3$. Restrict f to a suitable interval so that it has an inverse. Find the inverse function and sketch its graph.

Solution We may solve the equation $y = x^2 + 2x + 3$ for x in terms of y by the quadratic formula:

$$x^2 + 2x + 3 - y = 0$$

gives $x = [-2 \pm \sqrt{4 - 4(3 - y)}]/2 = -1 \pm \sqrt{y - 2}$. If we choose $y \geq 2$ and the $+$ sign, we get $x = -1 + \sqrt{y - 2}$ for the inverse function. The restriction $y \geq 2$ corresponds to $x \geq -1$ (see Fig. 5.3.5). (The answer $x = -1 - \sqrt{y - 2}$ for $y \geq 2$ and $x \leq -1$ is also acceptable—this is represented by the dashed portion of the graph. ▲

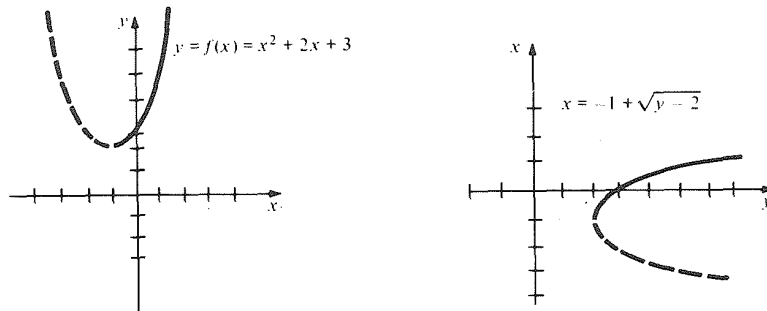


Figure 5.3.5. Restricting the domain of $f(x)$ to $[-1, \infty)$ gives a function with an inverse defined on $[2, \infty)$.

Example 3 Sketch the graph of the inverse function for each function in Fig. 5.3.6.

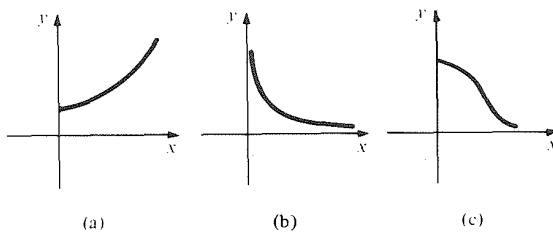


Figure 5.3.6. Sketch the graph of the inverse.

Solution The graphs which we obtain by viewing the graphs in Fig. 5.3.6 from the reverse side of the page, are shown in Fig. 5.3.7. ▲

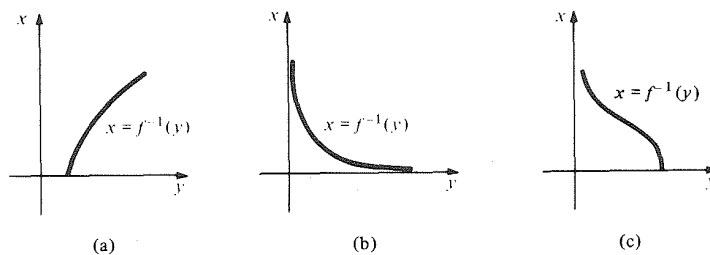


Figure 5.3.7. Graphs of the inverse functions (compare Fig. 5.3.6).

There is a simple geometric test for invertibility; a function is invertible if each horizontal line meets the graph in at most one point.

Example 4 Determine whether or not each function in Fig. 5.3.8 is invertible on its domain.

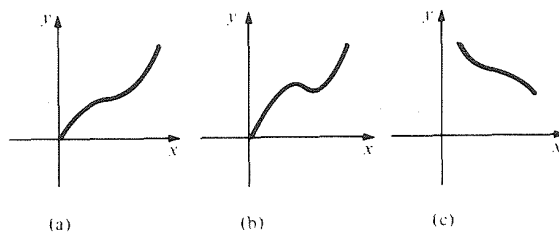


Figure 5.3.8. Is there an inverse?

Solution Applying the test just mentioned, we find that the functions (a) and (c) are invertible while (b) is not. ▲

A function may be invertible even though we cannot find an explicit formula for the inverse function. This fact gives us a way of obtaining “new func-

tions.” The following is a useful calculus test for finding intervals on which a function is invertible. In the next section, we shall use it to obtain inverses for the trigonometric functions.

Inverse Function Test

Suppose that f is continuous on $[a, b]$ and that f is increasing at each point of (a, b) . (For instance, this holds if $f'(x) > 0$ for each x in (a, b) .) Then f is invertible on $[a, b]$, and the inverse f^{-1} is defined on the interval $[f(a), f(b)]$.

If f is decreasing rather than increasing at each point of (a, b) , then f is still invertible; in this case, the domain of f^{-1} is $[f(b), f(a)]$.

To justify this test, we apply the increasing function test from Section 3.2 to conclude that f is increasing on $[a, b]$; that is, if $a \leq x_1 < x_2 \leq b$, then $f(x_1) < f(x_2)$. In particular, $f(a) < f(b)$. If y is any number in $(f(a), f(b))$, then by the intermediate value theorem (first version, Section 3.1), there is an x in (a, b) such that $f(x) = y$. If $y = f(a)$ or $f(b)$, we can choose $x = a$ or $x = b$. Since f is increasing on $[a, b]$, for any y in $[f(a), f(b)]$ there can only be one x such that $y = f(x)$. Thus, by definition, f is invertible on $[a, b]$ and the domain of f^{-1} is the range $[f(a), f(b)]$ of values of f on $[a, b]$. The proof of the second assertion in the inverse function test is similar.

We can allow open or infinite intervals in the inverse function test. For instance, if f is continuous and increasing on $[a, \infty)$, and if $\lim_{x \rightarrow \infty} f(x) = \infty$, then f has an inverse defined on $[f(a), \infty)$.

Example 5 Verify that $f(x) = x^2 + x$ has an inverse if f is defined on $[-\frac{1}{2}, \infty)$.

Solution Since f is differentiable on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$ and hence on $[-\frac{1}{2}, \infty)$. But $f'(x) = 2x + 1 > 0$ for $x > -\frac{1}{2}$. Thus f is increasing. Also, $\lim_{x \rightarrow \infty} f(x) = \infty$. Hence the inverse function test guarantees that f has an inverse defined on $[f(-\frac{1}{2}), \infty)$. ▲

Example 6 Let $f(x) = x^5 + x$.

- Show that f has an inverse on $[-2, 2]$. What is the domain of this inverse?
- Show that f has an inverse on $(-\infty, \infty)$.
- What is $f^{-1}(2)$?

▮(d) Numerically calculate $f^{-1}(3)$ to two decimal places of accuracy.

Solution (a) $f'(x) = 5x^4 + 1 > 0$, so by the inverse function test, f is invertible on $[-2, 2]$. The domain of the inverse is $[f(-2), f(2)]$, which is $[-34, 34]$.
 (b) Since $f'(x) > 0$ for all x in $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$. Now f takes arbitrarily large positive and negative values as x varies over $(-\infty, \infty)$; it takes all values in between by the intermediate value theorem, so the domain of f^{-1} is $(-\infty, \infty)$. There is no simple formula for $f^{-1}(y)$, the solution of $x^5 + x = y$, but we can calculate $f^{-1}(y)$ for any specific values of y to any desired degree of accuracy. (This is really no worse than the situation for $\sqrt{x}$. If the inverse function to $x^5 + x$ had as many applications as the square-root function, we would learn about it in high school, tables would be readily available for it, calculators would calculate it at the touch of a key, and there would be a standard notation like $\mathcal{S}y$ for the solution of $x^5 + x = y$, just as $\sqrt[5]{y}$ is the standard notation for the solution of $x^5 = y$.)

(c) Since $f(1) = 1^5 + 1 = 2$, $f^{-1}(2)$ must equal 1.

(d) To calculate $f^{-1}(3)$ —that is, to seek an x such that $x^5 + x = 3$ —we use the method of bisection described in Example 7, Section 3.1. Since $f(1) = 2 < 3$ and $f(2) = 34 > 3$, x must lie between 1 and 2. We can squeeze toward the correct answer by calculating:

$$\begin{array}{ll} f(1.5) = 9.09375 & \text{so } 1 < x < 1.5, \\ f(1.25) = 4.30176 & \text{so } 1 < x < 1.25, \\ f(1.1) = 2.71051 & \text{so } 1.1 < x < 1.25, \\ f(1.15) = 3.16135 & \text{so } 1.1 < x < 1.15, \\ f(1.14) = 3.06541 & \text{so } 1.1 < x < 1.14, \\ f(1.13) = 2.97244 & \text{so } 1.13 < x < 1.14, \\ f(1.135) = 3.01856 & \text{so } 1.13 < x < 1.135. \end{array}$$

Thus, to two decimal places, $x \approx 1.13$. (About 10 minutes of further experimentation gave $f(1.132997566) = 3.000000002$ and $f(1.132997565) = 2.999999991$. What does this tell you about $f^{-1}(3)$?) ▲



Calculator Discussion

Recall (see Section R.6) that a function f may be thought of as an operation key on a calculator. The inverse function should be another key, which we can label f^{-1} . According to the definition, if we feed in any x , then push f to get $y = f(x)$, then push f^{-1} , we get back $x = f^{-1}(y)$, the number we started with. Likewise if we feed in a number y and first push f^{-1} and then f , we get y back again. By Fig. 5.3.4, $y = x^2$ and $x = \sqrt{y}$ (for $x \geq 0$, $y \geq 0$) are inverse functions. Try it out numerically, by pushing $x = 3.0248759$, then the x^2 key, then the $\sqrt{x}$ key. Try it also in the reverse order. (The answer may not come out exactly right because of roundoff errors.) ▲

As the preceding calculator discussion suggests, there is a close relation between inverse functions and composition of functions as discussed in connection with the chain rule in Section 2.2. If f and g are inverse functions, then $g(y)$ is that number x for which $f(x) = y$, so $f(g(y)) = y$; i.e., $f \circ g$ is the *identity function* which takes each y to itself: $(f \circ g)(y) = y$. Similarly, $g(f(x)) = x$ for all x , so $g \circ f$ is the identity function as well.

If we assume that f^{-1} and f are differentiable, we can apply the chain rule to the equation

$$f^{-1}(f(x)) = x$$

to obtain $(f^{-1})'(f(x)) \cdot f'(x) = 1$, which gives

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Writing y for $f(x)$, so that $x = f^{-1}(y)$, we obtain the formula

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Since the expression $(f^{-1})'$ is awkward, we sometimes revert to the notation $g(y)$ for the inverse function and write

$$g'(y) = \frac{1}{f'(x)}.$$

Notice that although dy/dx is not an ordinary fraction, the rule

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

is valid as long as $dy/dx \neq 0$. (Maybe the “reciprocal” notation f^{-1} is not so bad after all!)

Inverse Function Rule

To differentiate the inverse function $g = f^{-1}$ at y , take the reciprocal of the derivative of the given function at $x = f^{-1}(y)$:

$$g'(y) = \frac{1}{f'(g(y))} \quad \text{if } g = f^{-1},$$

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

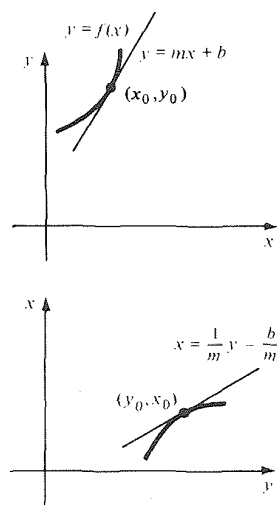


Figure 5.3.9. The inverse of the tangent line is the tangent line of the inverse.

Notice that our earlier observation that the slopes of inverse linear functions are reciprocals is just a special case of the inverse function rule. Figure 5.3.9 illustrates how the general rule is related to this special case.

Assuming that the inverse f^{-1} is continuous (see Exercise 41), we can prove that f^{-1} is differentiable whenever $f' \neq 0$ in the following way. Recall that $f'(x_0) = dy/dx = \lim_{\Delta x \rightarrow 0} (\Delta y/\Delta x)$, where Δx and Δy denote changes in x and y . On the other hand,

$$g'(y_0) = \frac{dx}{dy} = \lim_{\Delta y \rightarrow 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim_{\Delta y \rightarrow 0} (\Delta y/\Delta x)},$$

by the reciprocal rule for limits. But $\Delta x \rightarrow 0$ when $\Delta y \rightarrow 0$, since g is continuous, so

$$g'(y_0) = \frac{1}{\lim_{\Delta y \rightarrow 0} (\Delta y/\Delta x)} = \frac{1}{dy/dx}.$$

Example 7 Use the inverse function rule to compute the derivative of $\sqrt{x}$. Evaluate the derivative at $x = 2$.

Solution Let us write $g(y) = \sqrt{y}$. This is the inverse function of $f(x) = x^2$. Since $f'(x) = 2x$,

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}},$$

so $(d/dy)(\sqrt{y}) = 1/(2\sqrt{y})$. We may substitute any letter for y in this result, including x , so we get the formula

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

When $x = 2$, the derivative is $1/(2\sqrt{2})$. $\blacktriangle$

Example 7 reproduces the rule for differentiating $x^{1/2}$ that we learned in Section 2.3. In fact, one can similarly use inverse functions to obtain an alternative proof of the rule for differentiating fractional powers: $(d/dx)x^{p/q} = (p/q)x^{(p/q)-1}$.

Example 8 Find $(f^{-1})'(2)$, where f is the function in Example 6.

Solution We know that $f^{-1}(2) = 1$, so $(f^{-1})'(2) = 1/f'(1)$. But $f'(x) = 5x^4 + 1$, so $f'(1) = 6$, and hence $(f^{-1})'(2) = \frac{1}{6}$. ▲

The important point in Example 8 is that we differentiated f^{-1} without having an explicit formula for it. We will exploit this idea in the next section.

Supplement to Section 5.3 Inverse Functions and Yogurt

If a yogurt culture is added to a quart of boiled milk and set aside for 4 hours, then the sourness of the resulting yogurt depends upon the temperature at which the mixture was kept. By performing a series of experiments, we can plot the graph of a function $S = f(T)$, where T is temperature and S is the sourness measured by the amount of lactic acid in grams in the completed yogurt. (See Fig. 5.3.10.) If T is too low, the culture is dormant; if T is too high, the culture is killed.

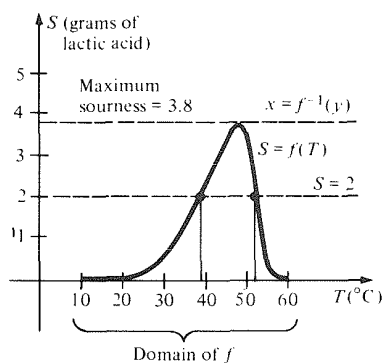


Figure 5.3.10. The sourness of yogurt as a function of fermentation temperature.

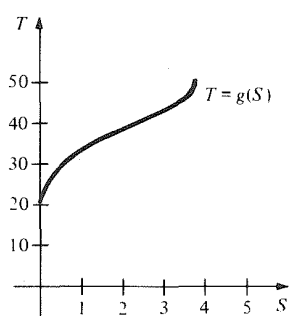


Figure 5.3.11. The graph of T as a function of S .

In making yogurt to suit one's taste, one might desire a certain degree of sourness and wish to know what temperature to use. (Remember that we are fixing all other variables, including the time of fermentation.) To find the temperature which gives $S = 2$, for instance, one may draw the horizontal line $S = 2$, see if it intersects the graph of f , and read off the value of T (Fig. 5.3.10).

From the graph, we see that there are two possible values of T : 38°C and 52°C . Similarly, there are two possible temperatures to achieve any value of S strictly between zero and the maximum value 3.8 of f . If, however, we restrict the allowable temperatures to the interval $[20, 47]$, then we will get a unique value of T for each S in $[0, 3.8]$. The new function $T = g(S)$, which assigns to each the sourness value the proper temperature for producing it, is the inverse function to f (Fig. 5.3.11).

Exercises for Section 5.3

Find the inverse for each of the functions in Exercises 1–6 on the given interval.

1. $f(x) = 2x + 5$ on $[-4, 4]$.
2. $f(x) = -\frac{1}{3}x + 2$ on $(-\infty, \infty)$.
3. $f(x) = x^5$ on $(-\infty, \infty)$.
4. $f(x) = x^8$ on $(0, 1]$.

5. $h(t) = \frac{t-10}{t+3}$ on $[-1, 1]$.

6. $a(s) = \frac{2s+5}{-s+1}$ on $[-\frac{1}{2}, \frac{1}{2}]$.

7. Find the inverse function for the function $f(x) = (ax+b)/(cx+d)$ on its domain. What must you assume about a , b , c , and d ?

8. Find an inverse function g for $f(x) = x^2 + 2x + 1$ on some interval containing zero. What is $g(9)$? What is $g(x)$?
9. Sketch the graph of the inverse of each of the functions in Fig. 5.3.12.

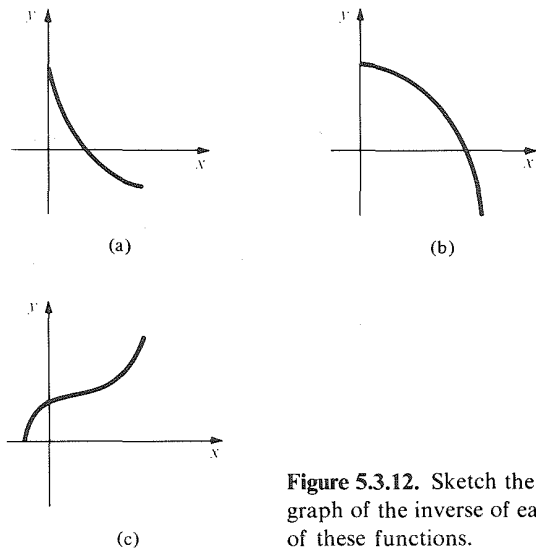


Figure 5.3.12. Sketch the graph of the inverse of each of these functions.

10. Determine whether each function in Fig. 5.3.13 has an inverse. Sketch the inverse if there is one.

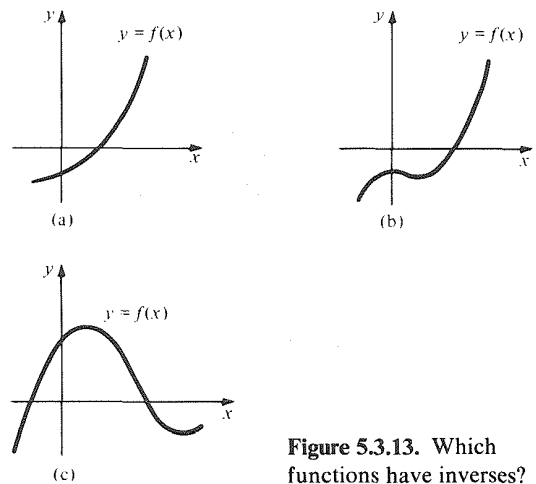


Figure 5.3.13. Which functions have inverses?

11. Draw a graph of $f(x) = (3x + 1)/(2x - 2)$ and a graph of its inverse function.
12. Draw a graph of $f(x) = (x - 1)/(x + 1)$ and a graph of its inverse function.
13. Find the largest possible intervals on which $f(x) = 1/(x^2 - 1)$ is invertible. Sketch the graphs of the inverse functions.
14. Sketch a graph of $f(x) = x/(1 + x^2)$ and find an interval on which f is invertible.

15. Let $f(x) = x^3 - 4x^2 + 1$.
- Find an interval containing 1 on which f is invertible. Denote the inverse by g .
 - Compute $g(-7)$ and $g'(-7)$.
 - What is the domain of g ?
 - Compute $g(-5)$ and $g'(-5)$.
16. Let $f(x) = x^5 + x$. (a) Find $f^{-1}(246)$.
(b) Find $f^{-1}(4)$, correct to at least two decimal places.
17. Show that $f(x) = \frac{1}{3}x^3 - x$ is not invertible on any open interval containing 1.
18. Find intervals on which $f(x) = x^5 - x$ is invertible.
19. Show that if n is odd, $f(x) = x^n$ is invertible on $(-\infty, \infty)$. What is the domain of the inverse function?
20. Discuss the invertibility of $f(x) = x^n$ for n even.
21. Show that $f(x) = -x^3 - 2x + 1$ is invertible on $[-1, 2]$. What is the domain of the inverse?
22. (a) Show that $f(x) = x^3 - 2x + 1$ is invertible on $[2, 4]$. What is the domain of the inverse function?
(b) Find the largest possible intervals on which f is invertible.
23. Verify the inverse function rule for $y = x^3$.
24. Verify the inverse function rule for the function $y = (ax + b)/(cx + d)$ by finding dy/dx and dx/dy directly. (See Exercise 7.)
25. If $f(x) = x^3 + 2x + 1$, show that f has an inverse on $[0, 2]$. Find the derivative of the inverse function at $y = 4$.
26. Find $g'(0)$, where g is the inverse function to $f(x) = x^9 + x^5 + x$.
27. Let $y = x^3 + 2$. Find dx/dy when $y = 3$.
28. If $f(x) = x^5 + x$, find the derivative of the inverse function when $y = 34$.

For each function f in Exercises 29–32, find the derivative of the inverse function g at the points indicated.

29. $f(x) = 3x + 5$; find $g'(2)$, $g'(\frac{3}{4})$.
30. $f(x) = x^5 + x^3 + 2x$; find $g'(0)$, $g'(4)$.
31. $f(x) = \frac{1}{12}x^3 - x$ on $[-1, 1]$; find $g'(0)$, $g'(\frac{11}{12})$.
32. $f(x) = \sqrt{x-3}$ on $[3, \infty)$; find $g'(4)$, $g'(8)$.
33. Enter the number 2.6 on your calculator, then push the x^2 key followed by the $\sqrt{x}$ key. Is there any roundoff error? Try the $\sqrt{x}$ key, then the x^2 key. Also try a sequence such as $x^2, \sqrt{x}, x^2, \sqrt{x}, x^2, \sqrt{x}$. Do you get the original number? Try these experiments with different starting numbers.
34. If we think of a French–English dictionary as defining a function from the set of French words to the set of English words (does it really?), how is the inverse function defined? Discuss.
35. Suppose that $f(x)$ is the number of pounds of beans you can buy for x dollars. Let $g(y)$ be the inverse function. What does $g(y)$ represent?

36. Let $f_1(x) = x$, $f_2(x) = 1/x$, $f_3(x) = 1 - x$, $f_4(x) = 1/(1 - x)$, $f_5(x) = (x - 1)/x$, and $f_6(x) = x/(x - 1)$.

- (a) Show that the composition of any two functions in this list is again in the list. Complete the "composition table" below. For example $(f_2 \circ f_3)(x) = f_2(1 - x) = 1/(1 - x) = f_4(x)$.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2						
f_3						
f_4						
f_5						
f_6						

- (b) Show that the inverse of any function in the preceding list is again in the list. Which of the functions equal their own inverses?
37. (a) Find the domain of

$$f(x) = \sqrt{(2x + 5)/(3x + 7)}.$$

- (b) By solving for the x in the equation $y = \sqrt{(2x + 5)/(3x + 7)}$, find a formula for the inverse function $g(y)$.

- (c) Using the inverse function rule, find a formula for $f'(x)$.

- (d) Check your answer by using the chain rule.

38. Suppose that f is concave upward and increasing on $[a, b]$.

- (a) By drawing a graph, guess whether f^{-1} is concave upward or downward on the interval $[f(a), f(b)]$.

- (b) What if f is concave upward and decreasing on $[a, b]$?

39. Show that if the inverse function to f on S is g , with domain T , then the inverse function to g on T is f , with domain S . Thus, the inverse of the inverse function is the original function, that is, $(f^{-1})^{-1} = f$.

- ★40. Under what conditions on a , b , c , and d is the function $f(x) = (ax + b)/(cx + d)$ equal to its own inverse function?

- ★41. Suppose that $f'(x_0) > 0$ and $f(x_0) = y_0$. If g is the inverse of f with $g(y_0) = x_0$, show that g is continuous at y_0 by filling in the details of the following argument:

- (a) For Δx sufficiently small,

$$\frac{1}{2} f'(x_0) > \Delta y / \Delta x > \frac{1}{2} f'(x_0).$$

- (b) As $\Delta y \rightarrow 0$, $\Delta x \rightarrow 0$ as well.

- (c) Let $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then $\Delta x = g(y_0 + \Delta y) - g(y_0)$.

5.4 The Inverse Trigonometric Functions

The derivatives of the inverse trigonometric functions have algebraic formulas.

In the previous section, we discussed the general concept of inverse function and developed a formula for differentiating the inverse. Now we will apply this formula to study the inverses of the sine, cosine, and the other trigonometric functions.

We begin with the function $y = \sin x$, using the inverse function test to locate an interval on which $\sin x$ has an inverse. Since $\sin' x = \cos x > 0$ on $(-\pi/2, \pi/2)$, $\sin x$ is increasing on this interval, so $\sin x$ has an inverse on the interval $[-\pi/2, \pi/2]$. The inverse is denoted $\sin^{-1}y$.⁴ We obtain the graph of $\sin^{-1}y$ by interchanging the x and y coordinates. (See Fig. 5.4.1.)

The values of $\sin^{-1}y$ may be obtained from a table for $\sin x$. (Many pocket calculators can evaluate the inverse trigonometric functions as well as the trigonometric functions.)

The Inverse Sine Function

$x = \sin^{-1}y$ means that $\sin x = y$ and $-\pi/2 \leq x \leq \pi/2$. The number $\sin^{-1}y$ is expressed in radians unless a degree sign is explicitly shown.

⁴ Although the notation $\sin^2 y$ is commonly used to mean $(\sin y)^2$, $\sin^{-1}y$ does not mean $(\sin y)^{-1} = 1/\sin y$. Sometimes the notation $\arcsin y$ is used for the inverse sine function to avoid confusion.

5.6 Graphing in Polar Coordinates

Periodic functions are graphed as closed curves in polar coordinates.

The graph in polar coordinates of a function f consists of all those points in the plane whose polar coordinates (r, θ) satisfy the relation $r = f(\theta)$. Such graphs are especially useful when f is built up from trigonometric functions, since the entire graph is drawn when we let θ vary from 0 to 2π .

Various properties of f may appear as symmetries of the graph. For instance, if $f(\theta) = f(-\theta)$, then its graph is symmetric in the x axis; if $f(\pi - \theta) = f(\theta)$, it is symmetric in the y axis; and if $f(\theta) = f(\pi + \theta)$, it is symmetric in the origin (see Fig. 5.6.1).

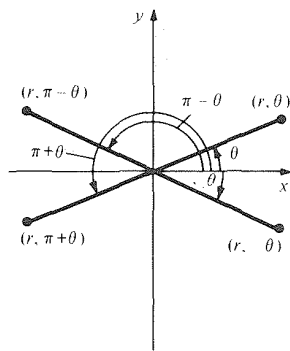


Figure 5.6.1. Symmetry in x axis: $f(\theta) = f(-\theta)$;
symmetry in y axis: $f(\pi - \theta) = f(\theta)$;
symmetry in origin: $f(\theta) = f(\pi + \theta)$.

Example 1 Plot the graph of $r = \cos 2\theta$ in the xy plane and discuss its symmetry.

Solution We know from the *cartesian* graph $y = \cos 2x$ (Fig. 5.1.27) that as θ increases from 0 to $\pi/4$, $r = \cos 2\theta$ decreases from 1 to 0. As θ continues from $\pi/4$ to $\pi/2$, $r = \cos 2\theta$ becomes negative and decreases to -1 . Thus (r, θ) traces out the path in Fig. 5.6.2.

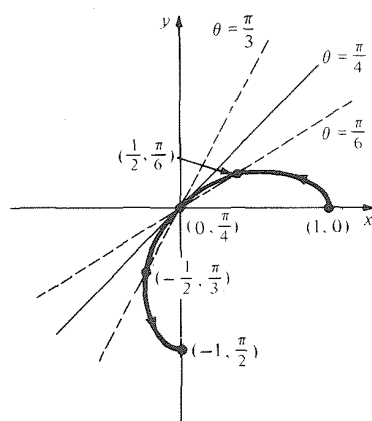


Figure 5.6.2. The graph
 $r = \cos 2\theta$ for $0 \leq \theta \leq \pi/2$.

We can complete the path as θ goes through all values between 0 and 2π , sweeping out the four petals in Fig. 5.6.3, or else we may use symmetry in the x and y axes. In fact, $f(-\theta) = \cos(-2\theta) = \cos 2\theta = f(\theta)$, so we have symmetry in the x axis. Also,

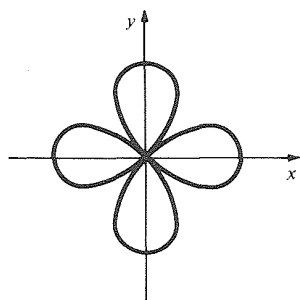


Figure 5.6.3. The full graph $r = \cos 2\theta$, the “four-petaled rose.”

$$\begin{aligned} f(\pi - \theta) &= \cos[2(\pi - \theta)] \\ &= \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta \\ &= \cos 2\theta = f(\theta), \end{aligned}$$

which gives symmetry in the y axis. Finally,

$$\begin{aligned} f\left(\theta + \frac{\pi}{2}\right) &= \cos 2\left(\theta + \frac{\pi}{2}\right) = \cos(2\theta + \pi) \\ &= \cos 2\theta \cos \pi - \sin 2\theta \sin \pi \\ &= -\cos 2\theta = -f(\theta). \end{aligned}$$

Thus, the graph is unchanged when reflected in the x axis and the y axis. When we rotate by 90° , $r = f(\theta)$ reflects through the origin; that is, r changes to $-r$. ▲

Example 2 Sketch the graph of $r = f(\theta) = \cos 3\theta$.

Solution The graph is symmetric in the x axis and, moreover, $f(\theta + \pi/3) = -f(\theta)$. This means that we need only sketch the graph for $0 \leq \theta \leq \pi/3$ and obtain the rest by reflection and rotations. Thus, we expect a three- or six-petaled rose. As θ varies from 0 to $\pi/3$, 3θ increases from 0 to π , and $\cos 3\theta$ decreases from 1 to -1 . Hence, we get the graph in Fig. 5.6.4.

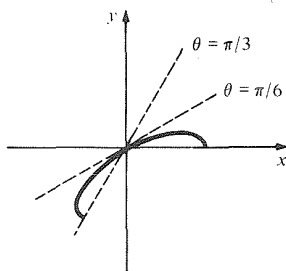


Figure 5.6.4. Beginning the graph of $r = \cos 3\theta$.

Reflect across the x axis to complete the petal and then rotate by $\pi/3$ and reflect through the origin (see Fig. 5.6.5). ▲

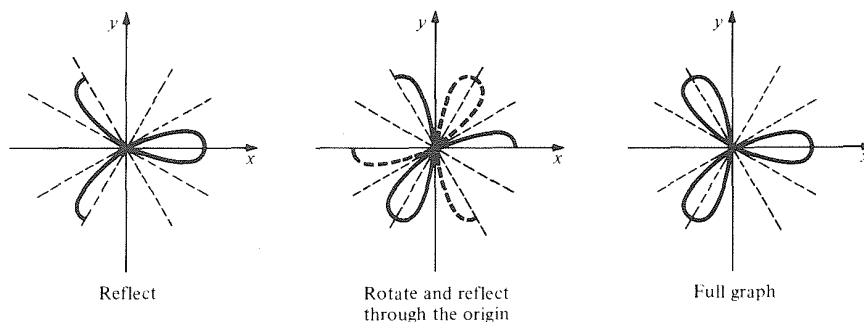


Figure 5.6.5. The graphing of $r = \cos 3\theta$.

Example 3 If $f(\theta + \pi/2) = f(\theta)$, what does this tell you about the graph of $r = f(\theta)$?

Solution This means that the graph will have the same appearance if it is rotated by 90° , since replacing θ by $\theta + \pi/2$ means that we rotate through an angle $\pi/2$. ▲

Figure 5.6.6 shows two other graphs in polar coordinates with striking symmetry. (The curve in (b) is discussed in Example 7 below.)

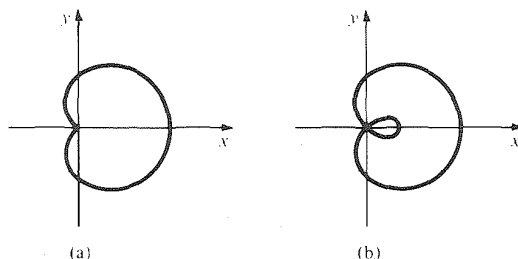


Figure 5.6.6.

(a) $r = 1 + \cos \theta$

("cardioid");

(b) $r = 1 + 2 \cos \theta$

("limaçon").

Example 4 Convert the relation $r = 1 + 2 \cos \theta$ to cartesian coordinates.

Solution We substitute $r = \sqrt{x^2 + y^2}$ and $\cos \theta = x/r = x/\sqrt{x^2 + y^2}$ to get

$$\sqrt{x^2 + y^2} = 1 + \frac{2x}{\sqrt{x^2 + y^2}}.$$

That is, $x^2 + y^2 - \sqrt{x^2 + y^2} - 2x = 0$. $\blacktriangle$

Calculus can help us to draw graphs in polar coordinates by telling us the slope of tangent lines (see Fig. 5.6.7).

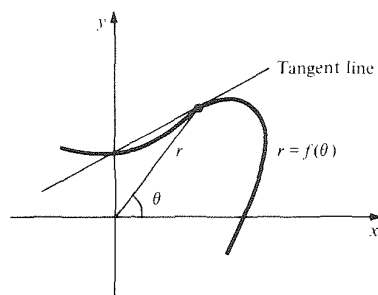


Figure 5.6.7. What is the slope of the tangent line?

This slope at a point (r, θ) is not $f'(\theta)$, since $f'(\theta)$ is the rate of change of r with respect to θ , while the slope is the rate of change of y with respect to x . To calculate dy/dx , we write

$$x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta.$$

This is a parametric curve with θ as the parameter. According to the formula $dy/dx = (dy/d\theta)/(dx/d\theta)$ from Section 2.4, with t replaced by θ ,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

Dividing numerator and denominator by $\cos \theta$ gives

$$\frac{dy}{dx} = \frac{f'(\theta) \tan \theta + f(\theta)}{f'(\theta) - f(\theta) \tan \theta}. \quad (1)$$

Tangents to Graphs in Polar Coordinates

The slope of the tangent to the graph of $r = f(\theta)$ at (r, θ) is

$$\frac{(\tan \theta) dr/d\theta + r}{dr/d\theta - r \tan \theta}.$$

- Example 5** (a) Find the slope of the line tangent to the graph of $r = 3 \cos^2 \theta$ at $\theta = \pi/6$.
 (b) Find the slope of the line tangent to the graph of $r = \cos 3\theta$ at $(r, \theta) = (-1, \pi/3)$.

Solution (a) Here, $f(\theta) = 3 \cos^2 \theta$, so $dr/d\theta = f'(\theta) = -12 \cos \theta \sin \theta$ (by the chain rule). Now $f(\pi/6) = 3 \cos^2(\pi/6) = \frac{3}{4}$ and $f'(\pi/6) = -12 \cos(\pi/6) \sin(\pi/6) = -12 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = -3\sqrt{3}$. Thus formula (1) gives

$$\frac{dy}{dx} = \frac{-3\sqrt{3}(1/\sqrt{3}) + 3/4}{-3\sqrt{3} - (3/4)(1/\sqrt{3})} = \frac{3\sqrt{3}}{13},$$

so the slope of the tangent line is $3\sqrt{3}/13$.

(b) Here, $f(\theta) = \cos 3\theta$, so the slope is, by formula (1),

$$\begin{aligned} \frac{f'(\theta) \tan \theta + f(\theta)}{f'(\theta) - f(\theta) \tan \theta} &= \frac{-3 \sin 3\theta \tan \theta + \cos 3\theta}{-3 \sin 3\theta - \cos 3\theta \tan \theta} \\ &= \frac{1 - 3 \tan 3\theta \tan \theta}{-\tan \theta - 3 \tan 3\theta}. \end{aligned}$$

Hence, at $\theta = \pi/3$, the slope is $1/-1.732 \approx -0.577$. ▲

Calculus can aid us in other ways. A local maximum of $f(\theta)$ will be a point on the graph where the distance from the origin is a local maximum, as in Fig. 5.6.8. The methods of Chapter 3 can be used to locate these local maxima (as well as the local minima).

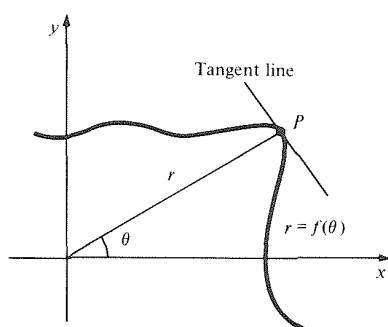


Figure 5.6.8. The point P corresponds to a local maximum point of $f(\theta)$.

- Example 6** Calculate the slope of the line tangent to $r = f(\theta)$ at (r, θ) if f has a local maximum there. Interpret geometrically.

Solution At a local maximum, $f'(\theta) = 0$. Plugging this into formula (1) gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(\theta)}{-f(\theta) \tan \theta} \\ &= -\frac{1}{\tan \theta}. \end{aligned}$$

Since dy/dx is the negative reciprocal of $\tan \theta$, the tangent line is perpendicular to the line from the origin to (r, θ) . (See Fig. 5.6.8.) ▲

Example 7 Find the maxima and minima of $f(\theta) = 1 + 2 \cos \theta$. Sketch the graph of $r = 1 + 2 \cos \theta$ in the xy plane.

Solution Here $dr/d\theta = -2 \sin \theta$, which vanishes if $\theta = 0, \pi$. Also, $d^2r/d\theta^2 = -2 \cos \theta$, which is -2 at $\theta = 0$, $+2$ at $\theta = \pi$. Hence $r = 3$, $\theta = 0$ is a local maximum and $r = -1$, $\theta = \pi$ is a local minimum. The tangent lines are vertical there. The curve passes through $r = 0$ when $\theta = \pm 2\pi/3$. The curve is symmetric in the x axis and can be plotted as in Fig. 5.6.9. ▲

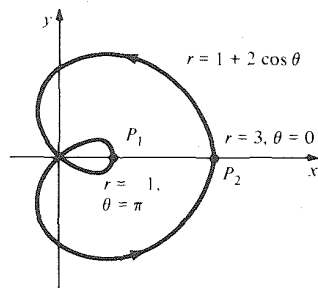


Figure 5.6.9. The maxima and minima of $1 + 2 \cos \theta$ correspond to the points P_1 and P_2 .

Exercises for Section 5.6

In Exercises 1–10, sketch the graph of the given function in polar coordinates. Also convert the given equation to cartesian coordinates.

1. $r = \cos \theta$
2. $r = 2 \sin \theta$
3. $r = 1 - \sin \theta$
4. $r = \sin\left(\frac{\theta}{2}\right) + 1$
5. $r = 3$
6. $\sin \theta = 1$
7. $r^2 + 2r \cos \theta + 1 = 0$
8. $r^2 \sin 2\theta = \frac{1}{2}$
9. $r = \sin 3\theta$
10. $r = \cos \theta - \sin \theta$

11. Describe the graph of the equation $r = \text{constant}$.
12. What is the equation of a line through the origin?
13. If $f(\pi/2 - \theta) = f(\theta)$, what does this tell you about the graph of $r = f(\theta)$?
14. If $f(\theta + \pi) = -f(\theta)$, what does this tell you about the graph of $r = f(\theta)$?

Convert the relations in Exercises 15–22 to polar coordinates.

15. $x^2 + y^2 = 1$
16. $xy = 1$
17. $x^2 + xy + y^2 = 1$
18. $y = x^5 + x^3 + 2$
19. $y = \frac{1}{1 - x^2}$
20. $y = \frac{x}{1 + x}$
21. $y = x + 1$
22. $y = \frac{1}{x} + x^2$

In Exercises 23–32, find the slope of the tangent line at the indicated point.

23. $r = \tan \theta$; $\theta = \frac{2\pi}{3}$.
24. $r = \tan \theta$; $\theta = \frac{\pi}{4}$.
25. $r = 2 \sin 5\theta$; $\theta = \frac{\pi}{2}$.
26. $r = 1 + 2 \sin 2\theta$; $\theta = 0$.
27. $r = \cos 4\theta$; $\theta = \frac{\pi}{3}$.
28. $r = 2 - \sin \theta$; $\theta = \frac{\pi}{6}$.
29. $r = 3 \sin \theta + \cos(\theta^2)$; $\theta = 0$.
30. $r = \sin 3\theta \cos 2\theta$; $\theta = 0$.
31. $r = \theta^2 + 1$; $\theta = 5$.
32. $r = \sec \theta + 2\theta^3$; $\theta = \frac{\pi}{6}$.

Find the maximum and minimum values of r for the functions in Exercises 33–38. Sketch the graphs in the xy plane.

33. $r = \cos 4\theta$
34. $r = 2 \sin 5\theta$
35. $r = \tan \theta$
36. $r = 2 - \sin \theta$
37. $r = 1 + 2 \sin 2\theta$
38. $r = \theta^2 + 1$
39. Find the maximum and minimum values of $r = \sin 3\theta \cos 2\theta$. Sketch its graph in the xy plane.
- ★40. Sketch the graph of $r = \sin^3 3\theta$.

Supplement to Chapter 5

Length of days

We outline an application of calculus to a phenomenon which requires no specialized equipment or knowledge for its observation—the setting of the sun.

Using spherical trigonometry or vector methods, one can derive a formula relating the following variables:

A = angle of elevation of the sun above the horizon;
 l = latitude of a place on the earth's surface;
 α = inclination of the earth's axis (23.5° or 0.41 radian);
 T = time of year, measured in days from the first day of summer in the northern hemisphere (June 21);
 t = time of day, measured in hours from noon.⁵

The formula reads:

$$\sin A = \cos l \sqrt{1 - \sin^2 \alpha \cos^2 \left(\frac{2\pi T}{365} \right)} \cos \left(\frac{2\pi t}{24} \right) + \sin l \sin \alpha \cos \left(\frac{2\pi T}{365} \right). \quad (1)$$

We will derive (1) in the supplement to Chapter 14. For now, we will simply assume the formula and find some of its consequences.⁶

At the time S of sunset, $A = 0$. That is,

$$\cos \left(\frac{2\pi S}{24} \right) = -\tan l \frac{\sin \alpha \cos(2\pi T/365)}{\sqrt{1 - \sin^2 \alpha \cos^2(2\pi T/365)}}. \quad (2)$$

Solving for S , and remembering that $S \geq 0$ since sunset occurs after noon, we get

$$S = \frac{12}{\pi} \cos^{-1} \left[-\tan l \frac{\sin \alpha \cos(2\pi T/365)}{\sqrt{1 - \sin^2 \alpha \cos^2(2\pi T/365)}} \right]. \quad (3)$$

For example, let us compute when the sun sets on July 1 at 39° latitude. We have $l = 39^\circ$, $\alpha = 23.5^\circ$, and $T = 11$. Substituting these values in (3), we find $\tan l = 0.8098$, $2\pi T/365 = 0.1894$ (that is, 10.85°), $\cos(2\pi T/365) = 0.9821$, and $\sin \alpha = 0.3987$. Therefore, $S = (12/\pi) \cos^{-1}(-0.3447) = (12/\pi)(1.9227) = 7.344$. Thus $S = 7.344$ (hours after noon); that is, the sun sets at 7:20:38 (if noon is at 12:00).

For a fixed point on the earth, S may be considered a function of T . Differentiating (3) and simplifying, we find

$$\frac{dS}{dT} = -\frac{24}{365} \tan l \left\{ \frac{\sin \alpha \sin(2\pi T/365)}{[1 - \sin^2 \alpha \cos^2(2\pi T/365)] \sqrt{1 - \sin^2 \alpha \cos^2(2\pi T/365)} \sec^2 l} \right\}. \quad (4)$$

The critical points of S occur when $2\pi T/365 = 0, \pi$, or 2π ; that is, $T = 0$, $365/2$, or $T = 365$ —the first day of summer and the first day of winter. For the northern hemisphere, $\tan l$ is positive. By the first derivative test, $T = 0$ (or 365) is a local maximum and $T = 365/2$ a minimum. Thus we get the graph shown in Fig. 5.S.1.

⁵ By noon we mean the moment at which the sun is highest in the sky. To find out when noon occurs in your area, look in a newspaper for the times of sunrise and sunset, and take the midpoint of these times. It will probably not be 12:00, but it should change only very slowly from day to day (except when daylight savings time comes or goes).

⁶ If $\pi/2 - \alpha < |l| < \pi/2$ (inside the polar circles), there will be some values of t for which the right-hand side of formula (1) does not lie in the interval $[-1, 1]$. On the days corresponding to these values of t , the sun will never set ("midnight sun").

If $l = \pm \pi/2$, then $\tan l = \infty$, and the right-hand side does not make sense at all. This reflects the fact that, at the poles, it is either light all day or dark all day, depending upon the season.

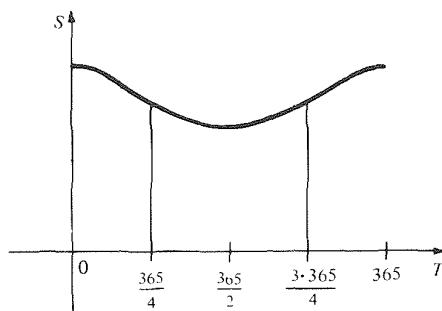


Figure 5.S.1. Sunset time as a function of the date. The sun sets late in summer, early in winter.

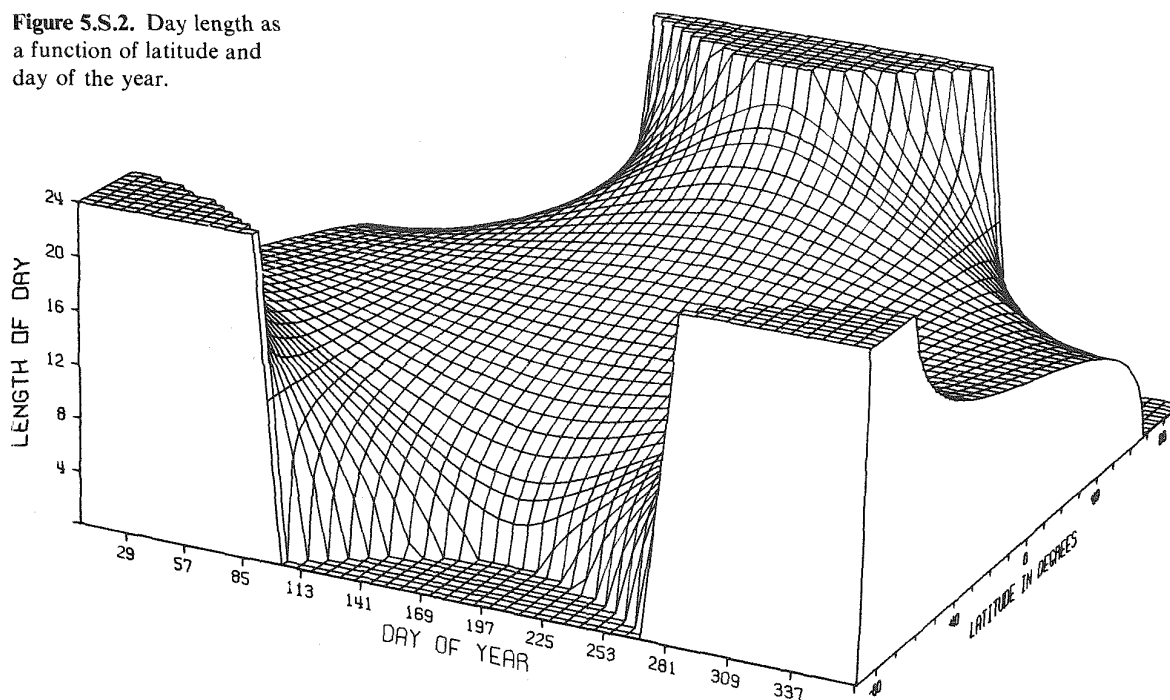
Using the first-order approximation, we can also determine how much the sunset time changes from one day to the next. If we set $\Delta T = 1$, we have

$$\Delta S \approx \frac{dS}{dT} \Delta T = \frac{dS}{dT}, \quad (5)$$

and dS/dT is given by (4). Thus, if the number of days after June 21 is inserted, along with the latitude l , formula (4) gives an approximation for the number of minutes later (or earlier) the sun will set the following evening. Note that the *difference* between sunset times on two days is the same whether we measure time in minutes from noon or by the clock ("standard time").

Formula (3) also tells us how long the days are as a function of latitude and day of the year. Plotting the formula on a computer (taking careful account of the polar regions) gives Fig. 5.S.2.⁷ Graphs such as the one shown in Fig. 5.S.1 result if l is fixed and only T varies.

Figure 5.S.2. Day length as a function of latitude and day of the year.



Example Use (4) and (5) to compute how many minutes earlier the sun will set on July 2 at latitude 39° than on July 1.

Solution Substituting $T = 11$ into (4), we obtain $\Delta S \approx -0.0055$; that is, the sun should set 0.0055 hour, or 20 seconds, earlier on July 2 than on July 1. Computing the time of sunset on July 2 by formula (3), with $T = 12$, we obtain $S = 7.338$, or

⁷Note that, in this figure, the date is measured from December 21st rather than June 21st.

7:20:17, which is 21 seconds earlier than the time computed by (4) and (5) for sunset on July 2. The error in the first-order approximation to ΔS is thus about 1 part in 20, or 5%. ▲

Differentiating formula (4), we find that the extreme values of dS/dT occur when $2\pi T/365 = \pi/2$ or $3\pi/2$. (These are the inflection points in Fig. 5.6.1.) When $2\pi T/365 = \pi/2$ (the first day of fall), dS/dT is equal to $-(24/365)\tan l \sin \alpha$; at this time, the days are getting shorter most rapidly.⁸ When $2\pi T/365 = 3\pi/2$ (the first day of spring), the days are lengthening most rapidly, with $dS/dT = (24/365)\tan l \sin \alpha$.

It is interesting to note how this maximal rate, $(24/365)\tan l \sin \alpha$, depends on latitude. Near the equator, $\tan l$ is very small, so the rate is near zero, corresponding to the fact that seasons don't make much difference near the equator. Near the poles, $\tan l$ is very large, so the rate is enormous. This large rate corresponds to the sudden switch from nearly 6 months of sunlight to nearly 6 months of darkness. At the poles, the rate is "infinite." (Of course, in reality the change isn't quite sudden because of the sun's diameter, the fact that the earth isn't a perfect sphere, refraction by the atmosphere, and so forth.)

The reader who wishes to explore these topics further should read the supplements to Section 9.5 and Chapter 14 and try the following exercises.

Exercises for the Supplement to Chapter 5

1. According to the *Los Angeles Times* for July 12, 1975, the sun set at 8:06 P.M. The latitude of Los Angeles is 33.57° North. Guess what time the July 13 paper gave for sunset? What about July 14?
2. Determine your latitude (approximately) by measuring the times of sunrise and sunset.
3. Calculate d^2S/dT^2 to confirm that the inflection points of S occur at the first days of spring and fall.
4. Derive formula (4) by differentiating (3). (It may help you to use the chain rule with $u = \sin \alpha \cos(2\pi T/365)$ as the intermediate variable.)
5. At latitude l on the earth, on what day of the year is the day 13 hours long? Sketch the relation between T and l in this case.
6. Near springtime in the temperate zone (near 45°), show that the sunsets are getting later at a rate of about 1.6 minutes a day (or 11 minutes a week).
7. Planet VCH revolves about its sun once every 590 VCH "days." Each VCH "day" = 19 earth hours = 1140 earth minutes. Planet VCH's axis is inclined at 31° . What time is sunset, at a latitude of 12° , 16 VCH days after the first day of summer? (Assume that each VCH "day" is divided into 1440 "minutes.")
8. For which values of T does the sun never set if $l > 90^\circ - \alpha$ (that is, near the North Pole)? Discuss.
9. (a) When the sun rises at the equator on June 21, how fast is its angle of elevation changing? (b) Using the linear approximation, estimate how long it takes for the sun to rise 5° above the horizon.
10. Let L be the latitude at which the sun has the highest noon elevation on day T . (a) Find a formula for L in terms of T . (b) Graph L as a function of T .

⁸ One of the authors was stimulated to do these calculations by the observation that he was most aware of the shortening of the days at the beginning of the school year. This calculation provides one explanation for the observation; perhaps the reader can think of others.

Review Exercises for Chapter 5

Perform the conversions in Exercises 1–6.

- 66° to radians.
- $\pi/10$ radians to degrees.
- $(4, \pi/2)$ from polar to cartesian coordinates.
- $(3, 6)$ from cartesian to polar coordinates.
- The equation $y = x^2$ from cartesian to polar coordinates.
- The equation $r^2 = \cos 2\theta$ from polar to cartesian coordinates.

7. Prove that

$$\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}.$$

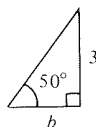
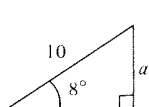
8. Prove that

$$\tan \theta = \frac{\sin(\theta + \phi) + \sin(\theta - \phi)}{\cos(\theta + \phi) + \cos(\theta - \phi)}.$$

Refer to Fig. 5.R.1 for Exercises 9–12.

9. Find a .

10. Find b .



11. Find c .

12. Find d .

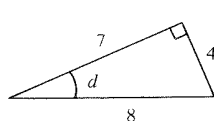
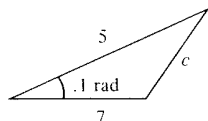


Figure 5.R.1. Find a , b , c , and d .

- Side BC of the equilateral triangle ABC is trisected by points P and Q . What is the angle between AP and AQ ?
- A 100-meter building and a 200-meter building stand 500 meters apart. Where between the buildings should an observer stand so that the taller building subtends twice the angle of the shorter one?

Differentiate the functions in Exercises 15–34.

- $y = -3 \sin 2x$.
- $y = 8 \tan 10x$.
- $y = x + x \sin 3x$.
- $y = x^2 \cos x^2$.
- $f(\theta) = \theta^2 + \frac{\theta}{\sin \theta}$.
- $g(x) = \sec x + \left[\frac{1}{x \cos(x+1)} \right]$.
- $h(y) = y^3 + 2y \tan(y^3) + 1$.
- $x(\theta) = \left[\sin\left(\frac{\theta}{2}\right) \right]^{4/7} + \theta^9 + 11\sqrt{\theta} + 4$.
- $y(x) = \cos(x^8 - 7x^4 - 10)$.
- $f(y) = (2y^3 - 3 \csc \sqrt{y})^{1/3}$.

- $f(x) = \sec^{-1}[(x + \sin x)^2]$.
- $f(x) = \cot^{-1}(20 - \sqrt[4]{x})$.
- $r(\theta) = 6 \cos^3(\theta^2 + 1) + 1$.
- $r(\theta) = \frac{7 \sin a\theta}{\sin b\theta + \cos c\theta}$; a, b, c constants.
- $\sin^{-1}(\sqrt{x})$.
- $\tan^{-1}(\sqrt{1+x^2})$.
- $\tan(\sin \sqrt{x})$.
- $\tan(\cos x + \csc \sqrt{x})$.
- $\sin^{-1}(\sqrt{x} + \cos 3x)$.
- $\frac{\sin \sqrt{x}}{\cos^{-1}(\sqrt{x} + 1)}$.
- Let $y = x^2 + \sin(2x + 1)$ and $x = t^3 + 1$. Find dy/dx and dy/dt .
- Let $g = 1/r^2 + (r^2 + 4)^{1/3}$ and $r = \sin 2\theta$. Find dg/dr and $dg/d\theta$.
- Let $h = x \sin^{-1}(x + 1)$ and $x = y - y^3$. Find dh/dx and dh/dy .
- Let $f = x^{3/5} + \sqrt{2x^4 + x^2 - 6}$ and $x = y + \sin y$. Find df/dx and df/dy .
- Let $f = \tan^{-1}(2x^3)$ and $x = a + bt$, where a and b are constants. Find df/dx and df/dt .
- Let $y = \sin^{-1}(u^2)$ and $u = \frac{\cos x + 1}{x^2 + 1}$. Find dy/du and dy/dx .

Find the antiderivatives in Exercises 41–50.

- $\int \sin 3x \, dx$.
- $\int (x^{3/2} + \cos 2x) \, dx$.
- $\int (4 \cos 4x - 4 \sin 4x) \, dx$.
- $\int (\cos x + \sin 2x + \cos 3x) \, dx$.
- $\int (3x^2 \sin x^3 + 2x) \, dx$.
- $\int \left[\left(\frac{4}{x^2} \right) \sin \left(1 + \frac{1}{x} \right) \right] \, dx$.
- $\int \sin(u + 1) \, du$.
- $\int (x^3 + 3 \sec^2 x) \, dx$.
- $\int \left[\frac{1}{(4 + y^2)} \right] \, dy$.
- $\int \left[\frac{1}{(s^2 + 1)} - 2 \cos 5s \right] \, ds$.

Find the definite integrals in Exercises 51–54.

- $\int_0^\pi \left(\sin \frac{\theta}{2} + \sin 2\theta \right) d\theta$.
- $\int_{-\pi}^{3\pi} \cos^2 u \, du$ [Hint: Use a trigonometric identity.]
- $\int_{-1}^1 \left[\frac{1}{\sqrt{4-x^2}} \right] \, dx$.
- $\int_2^3 \left[\frac{4}{(5+6x^2)} \right] \, dx$.

55. (a) Verify that

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

- (b) Differentiate $f(x) = \cos^{-1} x + \sin^{-1} x$ to conclude that f is constant. What is the constant?
- (c) Find $\int \cos^{-1} x \, dx$.
- (d) Find $\int \sin^{-1} 3x \, dx$.
56. (a) If F and G are antiderivatives for f and g , show that $F(x)G(x) + C$ is the antiderivative of $f(x)G(x) + F(x)g(x)$.
- (b) Find the antiderivative of $x \sin x - \cos x$.
- (c) Find the antiderivative of $x \sin(x+3) - \cos(x+3)$.
57. Let $f(x) = x^3 - 3x + 7$.
- (a) Find an interval containing zero on which f is invertible.
- (b) Denote the inverse by g . What is the domain of g ?
- (c) Calculate $g'(7)$.
58. (a) Show that the function $f(x) = \sin x + x$ has an inverse g defined on the whole real line.
- (b) Find $g'(0)$.
- (c) Find $g'(2\pi)$.
- (d) Find $g'\left(1 + \frac{\pi}{2}\right)$.
59. Let f be a function such that $f'(x) = 1/x$. (We will find such a function in the next chapter.) Show that the inverse function to f is equal to its own derivative.
60. Find a formula for the inverse function to $y = \sin^2 x$ on $[0, \pi/2]$. Where is this function differentiable?
61. A balloon is released from the ground 10 meters from the base of a 30-meter lamp post. The balloon rises steadily at 2 meters per second. How fast is the shadow of the balloon moving away from the base of the lamp after 4 seconds?
62. Three runners are going around a track which is an equilateral triangle with sides 50 meters long. If the runners are equally spread and all running counterclockwise at 20 kilometers per hour, at what rate is the distance between a pair of them changing when they are:
- (a) leaving the vertices?
- (b) arriving at the vertices?
- (c) at the midpoints of the sides?
63. A pocket watch is swung counterclockwise on the end of its chain in a vertical circle; it undergoes circular motion, but not uniform, and the tension T in the chain is given by $T = m[(v^2/R) + g \cos \theta]$, where θ is the angle from the downward direction. Suppose the length is $R = 0.5$ meter, the watch mass is $m = 0.1$ kilograms, and the tangential velocity is $v = f(\theta)$; $g = 9.8$ meters per second per second.

- (a) At what points on the circular path do you expect $dv/d\theta = 0$?
- (b) Compute $dT/d\theta$ when $dv/d\theta = 0$.
- (c) If the speed v is low enough at the highest point on the path, then the chain will become slack. Find the critical speed v_c below which the chain becomes slack.
64. A wheel of unit radius rolls along the x axis uniformly, rotating one half-turn per second. A point P on the circumference at time t (in seconds) has coordinates (x, y) given by $x = \pi t - \sin \pi t$, $y = 1 - \cos \pi t$.
- (a) Find the velocities dx/dt , dy/dt and the accelerations d^2x/dt^2 , d^2y/dt^2 .
- (b) Find the speed $\sqrt{(dx/dt)^2 + (dy/dt)^2}$ when y is a maximum.
65. Refer to Fig. 5.R.2. A girl at point G on the riverbank wishes to reach point B on the opposite side of the river as quickly as possible. She starts off in a rowboat which she can row at 4 kilometers per hour, and she can run at 16 kilometers per hour. What path should she take? (Ignore any current in the river.)

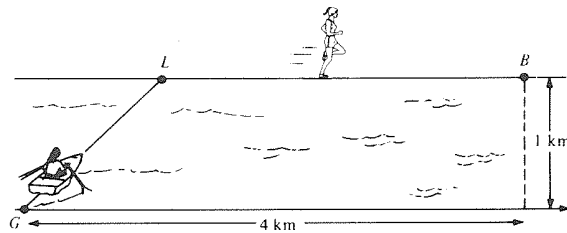


Figure 5.R.2. Find the best path from G to B .

66. The angle of deviation δ of a light ray entering a prism of Snell's index n and apex angle A is given by
- $$\delta = \sin^{-1}(n \sin \rho) + \sin^{-1}(n \sin(A - \rho)) - A$$
- where ρ depends on the angle of incidence ϕ of the light ray. (See Fig. 5.R.3.)

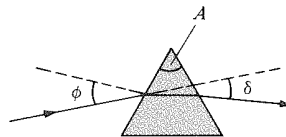


Figure 5.R.3. Light passing through a prism.

- (a) Find $d\delta/d\rho$.
- (b) Show that $d\delta/d\rho = 0$ occurs for $\rho = A/2$. This is a minimum value for δ .
- (c) By Snell's law, $n = \sin \phi / \sin \rho$. Verify that $n = \sin[(A + \delta)/2] / \sin(A/2)$ when $\rho = A/2$.

67. A man is driving on the freeway at 50 miles per hour. He sees the sign in Fig. 5.R.4. How far from the sign is θ maximum? How fast is θ changing at this time?

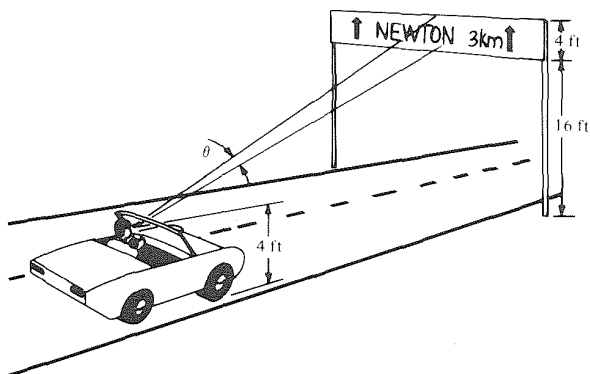


Figure 5.R.4. At what distance is θ biggest?

68. Two lighthouses on a straight coastline are 10 kilometers apart. A ship sees the two lights, and the lines of sight make an angle of 120° with one another. How far from the shore could the ship possibly be?
69. Where is $f(x) = 1 + 2 \sin x \cos x$ concave upward? Concave downward? Find its inflection points and sketch its graph.
70. Sketch the graph of $y = x + \sin x$ on $[-2\pi, 2\pi]$.
71. Prove that $f(x) = x - 1 - \cos x$ is increasing on $[0, \infty)$. What inequality can you deduce?
72. Suppose that the graph of $r = f(\theta)$ is symmetric in the line $x = y$. What does this imply about f ? Sketch the graphs in the xy plane of the functions in Exercises 73–76 given in polar coordinates.
73. $r = \cos 6\theta$.
74. $r = 1 + 3 \cos \theta$.
75. $r = \sin \theta + \cos \theta$.
76. $r = \theta^2 \cos^2 \theta$. (Use your calculator to locate the zeros of $dr/d\theta$.)

In Exercises 77 and 78, find a formula for the tangent line to the graph at the indicated point.

77. $r = \cos 4\theta$; $\theta = \pi/4$.

78. $r = \frac{1}{1 - \sin^2 \theta}$; $\theta = \pi/2$.

79. (a) Using a calculator, try to determine whether $\tan^{-1}[\tan(\pi - 10^{-20})]$ is in the interval $(0, 1)$.
 (b) Do part (a) without using a calculator.
 (c) Do some other calculator experiments with trigonometric functions. How else can you “fool” your calculator (or vice versa!)?

80. On a calculator, put any angle in radians on the display and successively press the buttons “sin” and “cos,” alternatively, until you see the numbers 0.76816 and 0.69481 appear on the display.
 (a) Try to explain this phenomenon from the graphs of $\sin x$ and $\cos x$, using composition of functions.

- (b) Can you guess the solutions x, y of the equations $\sin(\cos x) = x$, $\cos(\sin y) = y$?

- ★(c) Using the mean value and intermediate value theorems, show that the equations in (b) have exactly one solution.

- ★81. If f is differentiable with a differentiable inverse, and $g(x) = f^{-1}(\sqrt{x})$, what is $g'(x)$?

- ★82. Consider water waves impinging on a breakwater which has two gaps as in Fig. 5.R.5. With the notation in the figure, analyze the maximum and minimum points for wave amplitude along the shore. The two wave forms emanating from P and Q can be described at any point R as $\alpha \cos(k\rho - \omega t)$, where ρ is the distance from the source P or Q ; k, ω, α are constant. The net wave is described by their sum; the amplitudes do *not* add; ignore complications such as reflections of waves off the beach.⁹

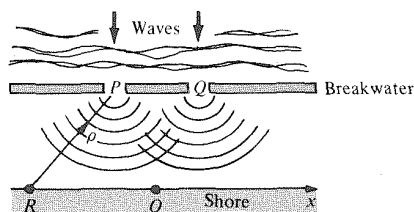


Figure 5.R.5. Find the wave pattern on the shore.

- ★83. Find a formula for the second derivative of an inverse function.
- ★84. Prove that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

is differentiable for all x , but that $f'(x)$ is not continuous at zero.

- ★85. Show that the function

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

is twice differentiable but that the second derivative is not continuous.

⁹ We recommend the book *Waves and Beaches* by W. Bascom (Anchor Books, 1965; revised, 1980) as a fascinating study of the mathematics, physics, engineering, and aesthetics of water waves.

39. A credit card company advertises: "Your interest rate on the unpaid balance is 17% compounded continuously, but federal law requires us to state that your annual interest rate is 18.53%." Explain.
40. If a credit card charges an interest rate of 21% compounded continuously, what is the actual annual percentage rate?
41. A certain calculus textbook sells according to this formula: $S(t) = 2000 - 1000e^{-0.3t}$, where t is the time in years and $S(t)$ is the number of books sold.
- Find $S'(t)$.
 - Find $\lim_{t \rightarrow \infty} S(t)$ and discuss.
 - Graph S .
42. A foolish king, on losing a famous bet, agrees to pay a wizard 1 cent on the first day of the month, 2 cents on the second day, 4 cents on the third, and so on, each day doubling the sum. How much is paid on the thirtieth day?
43. The author of a certain calculus textbook is awake writing in the stillness of 2 A.M. A sound disturbs him. He discovers that the toilet tank fills up fast at first, then slows down as the water

is being shut off. Examining the insides of the tank and contemplating for a moment, he thinks that maybe during shutoff the rate of flow of water into the tank is proportional to the height left to go; that is, $dx/dt = c(h - x)$, where x = height of water, h = desired height of water, and c = a constant (depending on the mechanism). Show that $x = h - Ke^{-ct}$. What is K ?

Looking at this formula for x , he says "That explains why my tank is always filling!" and goes to bed.

44. (a) Verify that the solution of $dy/dt = p(t)y$ is $y = y_0 \exp P(t)$, where $P(t)$ is the antiderivative of $p(t)$ with $P(0) = 0$.
 (b) Solve $dy/dt = ty$; $y = 1$ when $t = 0$.
- ★45. (a) Show that a solution of $t(da/dt) = a + h$ is

$$a(t) = t \int_1^t \frac{h(s)}{s^2} ds + tC.$$

- (b) Solve $t(da/dt) = a + e^{-1/t}$, $a(1) = 1$.
- ★46. Develop a general formula for the *doubling time* of a population in terms of its growth rate.
- ★47. Develop a general formula for the half-life of the amplitude of a damped spring (Exercise 28, Section 8.1).

8.3 The Hyperbolic Functions

The points $(\cos t, \sin t)$ lie on a circle, and $(\cosh t, \sinh t)$ lie on a hyperbola.

The hyperbolic functions are certain combinations of exponential functions which satisfy identities very similar to those for the trigonometric functions. We shall see in the next section that the inverse hyperbolic functions are important in integration.

A good way to introduce the hyperbolic functions is through a differential equation which they solve. Recall that $\sin t$ and $\cos t$ are solutions of the equation $d^2x/dt^2 + x = 0$. Now we switch the sign and consider $d^2x/dt^2 - x = 0$. (This corresponds to a negative spring constant!)

We already know one solution to this equation: $x = e^t$. Another is e^{-t} , because when we differentiate e^{-t} twice we bring down, via the chain rule, two minus signs and so recover e^{-t} again. The combination

$$x = Ae^t + Be^{-t}$$

is also a solution, as is readily verified.

If we wish to find a solution analogous to the sine function, with $x = 0$ and $dx/dt = 1$ when $t = 0$, we must pick A and B so that

$$0 = A + B,$$

$$1 = A - B,$$

so $A = \frac{1}{2}$ and $B = -\frac{1}{2}$, giving $x = (e^t - e^{-t})/2$.

Similarly, if we wish to find a solution analogous to the cosine function, we should pick A and B such that $x = 1$ and $dx/dt = 0$ when $t = 0$; that is,

$$1 = A + B,$$

$$0 = A - B,$$

so $A = B = \frac{1}{2}$, giving $x = (e^t + e^{-t})/2$.

This reasoning leads to the following definitions.

Hyperbolic Sine and Cosine

The *hyperbolic sine function*, written $\sinh t$, is defined by

$$\sinh t = \frac{e^t - e^{-t}}{2}. \quad (1)$$

The *hyperbolic cosine function*, written $\cosh t$, is defined by

$$\cosh t = \frac{e^t + e^{-t}}{2}. \quad (2)$$

(See Fig. 8.3.1.)

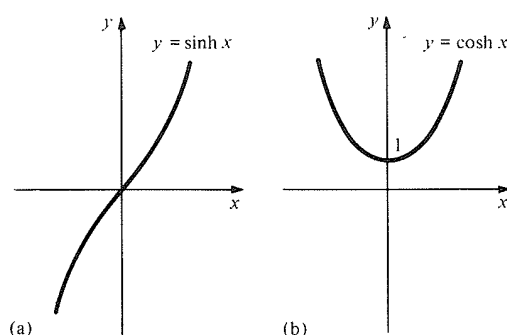


Figure 8.3.1. The graphs of $y = \sinh x$ and $y = \cosh x$.

The usual trigonometric functions $\sin t$ and $\cos t$ are called the *circular functions* because $(x, y) = (\cos t, \sin t)$ parametrizes the circle $x^2 + y^2 = 1$. The functions $\sinh t$ and $\cosh t$ are called *hyperbolic functions* because $(x, y) = (\cosh t, \sinh t)$ parametrizes one branch of the hyperbola $x^2 - y^2 = 1$; that is, for any t , we have the identity

$$\cosh^2 t - \sinh^2 t = 1. \quad (3)$$

(See Fig. 8.3.2.)

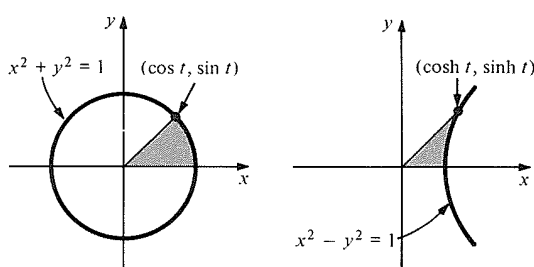


Figure 8.3.2. The points $(\cos t, \sin t)$ lie on a circle, while $(\cosh t, \sinh t)$ lie on a hyperbola.

To prove formula (3), we square formulas (1) and (2):

$$\cosh^2 t = \frac{1}{4}(e^t + e^{-t})^2 = \frac{1}{4}(e^{2t} + 2 + e^{-2t})$$

and

$$\sinh^2 t = \frac{1}{4}(e^t - e^{-t})^2 = \frac{1}{4}(e^{2t} - 2 + e^{-2t}).$$

Subtracting gives formula (3).

Example 1 Show that $e^x = \cosh x + \sinh x$.

Solution By definition,

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Adding, $\cosh x + \sinh x = e^x/2 + e^{-x}/2 + e^x/2 - e^{-x}/2 = e^x$. $\blacktriangle$

Similarly, $e^{-x} = \cosh x - \sinh x$.

The other hyperbolic functions can be introduced by analogy with the trigonometric functions:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad (4)$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

Various general identities can be proved exactly as we proved formula (3); for instance, the addition formulas are:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y, \quad (5a)$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y. \quad (5b)$$

Example 2 Prove identity (5a).

Solution By definition,

$$\sinh(x + y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^x e^y - e^{-x} e^{-y}}{2}.$$

Now we plug in $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$ to get

$$\begin{aligned} \sinh(x + y) &= \frac{1}{2} [(\cosh x + \sinh x)(\cosh y + \sinh y) \\ &\quad - (\cosh x - \sinh x)(\cosh y - \sinh y)]. \end{aligned}$$

Expanding,

$$\begin{aligned} \sinh(x + y) &= \frac{1}{2} (\cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y + \sinh x \sinh y \\ &\quad - \cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y - \sinh x \sinh y) \\ &= \cosh x \sinh y + \sinh x \cosh y. \quad \blacktriangle \end{aligned}$$

Notice that in formula (5b) for $\cosh(x + y)$ there is no minus sign. This is one of several differences in signs between rules for the hyperbolic and circular functions. Another is in the following:

$$\frac{d}{dx} \sinh x = \cosh x, \quad (6a)$$

$$\frac{d}{dx} \cosh x = \sinh x. \quad (6b)$$

Example 3 Prove formula (6a).

Solution By definition, $\sinh x = (e^x - e^{-x})/2$, so

$$\frac{d}{dx} \sinh x = \frac{e^x - (-1)e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x. \quad \blacktriangle$$

We also note that

$$\sinh(-x) = -\sinh x \quad (\sinh \text{ is odd})$$

and

$$\cosh(-x) = \cosh x \quad (\cosh \text{ is even}).$$

From formulas (5a) and (5b) we get the half-angle formulas:

$$\sinh^2 x = \frac{\cosh 2x - 1}{2} \quad \text{and} \quad \cosh^2 x = \frac{\cosh 2x + 1}{2}. \quad (8)$$

Example 4 Prove that $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$.

Solution Since $\tanh x = \sinh x / \cosh x$, the quotient rule gives

$$(d/dx)\tanh x = (\cosh x \cdot \cosh x - \sinh x \cdot \sinh x) / \cosh^2 x = 1 - \tanh^2 x.$$

From $\cosh^2 x - \sinh^2 x = 1$ we get $1 - \sinh^2 x / \cosh^2 x = 1 / \cosh^2 x$; that is, $1 - \tanh^2 x = \operatorname{sech}^2 x$. $\blacktriangle$

Hyperbolic Functions and Their Derivatives

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\frac{d}{dx} \sinh x = \cosh x,$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\frac{d}{dx} \cosh x = \sinh x,$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x},$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x,$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x},$$

$$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x,$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x},$$

$$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x,$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x},$$

$$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x.$$

Example 5 Differentiate: (a) $\sinh(3x + x^3)$; (b) $\cos^{-1}(\tanh x)$; (c) $3x / (\cosh x + \sinh 3x)$.

Solution (a)

$$\frac{d}{dx} \sinh(3x + x^3) = \frac{d}{du} \sinh u \cdot \frac{du}{dx},$$

where $u = 3x + x^3$. We compute

$$\frac{d}{du} \sinh u = \cosh u \quad \text{and} \quad \frac{du}{dx} = 3 + 3x^2.$$

Expressing everything in terms of x , we have

$$\frac{d}{dx} \sinh(3x + x^3) = \cosh(3x + x^3) \cdot 3(1 + x^2).$$

$$(b) \quad \frac{d}{dx} \cos^{-1}(\tanh x) = \frac{d}{du} \cos^{-1} u \cdot (du/dx), \quad \text{where } u = \tanh x.$$

We find

$$\frac{d}{du} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-\tanh^2 x}} = \frac{-1}{\operatorname{sech} x}$$

(from the identity $1 - \tanh^2 x = \operatorname{sech}^2 x$ proved in Example 4), and

$$\frac{du}{dx} = \operatorname{sech}^2 x$$

(also from Example 4). Hence

$$\frac{d}{dx} \cos^{-1}(\tanh x) = -\operatorname{sech} x.$$

$$(c) \quad \frac{d}{dx} \left(\frac{3x}{\cosh x + \sinh 3x} \right) = \frac{3(\cosh x + \sinh 3x) - 3x(\sinh x + 3\cosh 3x)}{(\cosh x + \sinh 3x)^2}$$

(by the quotient rule). $\blacktriangle$

Let us return now to the equation $d^2x/dt^2 - \omega^2x = 0$. Its solution can be summarized as follows.

The Equation $d^2x/dt^2 - \omega^2x = 0$

The solution of

$$\frac{d^2x}{dt^2} - \omega^2x = 0 \tag{9}$$

is

$$x = x_0 \cosh \omega t + \frac{v_0}{\omega} \sinh \omega t, \tag{10}$$

where $x = x_0$ and $dx/dt = v_0$ when $t = 0$.

That formula (10) gives a solution of equation (9) is easy to see:

$$\frac{dx}{dt} = \omega x_0 \sinh \omega t + v_0 \cosh \omega t,$$

using formula (6) and the chain rule. Differentiating again, we get

$$\frac{d^2x}{dt^2} = \omega^2 x_0 \cosh \omega t + \omega v_0 \sinh \omega t = \omega^2 x,$$

so equation (9) is verified.

One may prove that (10) gives the only solution of equation (9) just as in the case of the spring equation (Exercise 54).

Example 7 Solve for $f(t)$: $f'' - 3f = 0$, $f(0) = 1$, $f'(0) = -2$.

Solution We use formula (10) with $\omega^2 = 3$ (so $\omega = \sqrt{3}$), $x_0 = f(0) = 1$, $v_0 = f'(0) = -2$, and with $f(t)$ in place of x . Thus our solution is

$$f(t) = \cosh \sqrt{3} t - \frac{2}{\sqrt{3}} \sinh \sqrt{3} t. \quad \blacktriangle$$

Example 8 Prove that $\cosh x$ has a minimum value of 1 at $x = 0$.

Solution $(d/dx)\cosh x = \sinh x$ vanishes only if $x = 0$ since $e^x = e^{-x}$ exactly when $e^{2x} = 1$; that is, $x = 0$. Also, $(d^2/dx^2)\cosh x = \cosh x$, which is 1 at $x = 0$, so $\cosh x$ has a minimum at $x = 0$, by the second derivative test. $\blacktriangle$

The kind of reasoning in the preceding example, together with the techniques of graphing, enable us to graph all the hyperbolic functions. These are shown in Fig. 8.3.3.

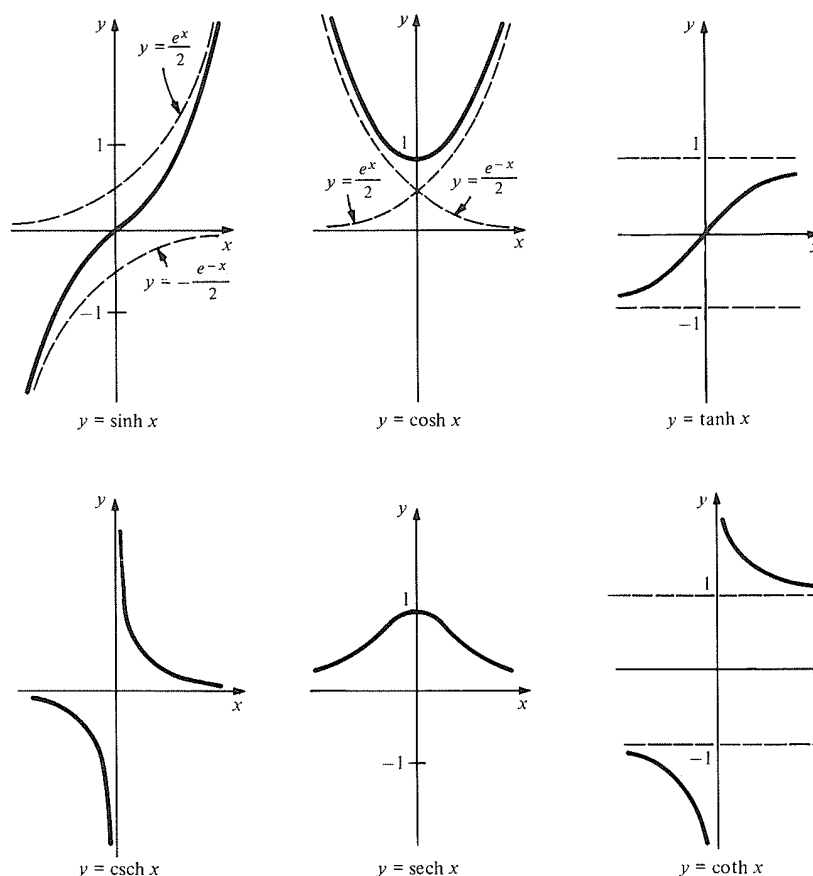


Figure 8.3.3. Graphs of the hyperbolic functions.

Finally, the differentiation formulas for the hyperbolic functions lead to integration formulas.

Antidifferentiation Formulas for Hyperbolic Functions

$$\begin{aligned} \int \cosh x \, dx &= \sinh x + C, & \int \operatorname{csch}^2 x \, dx &= -\coth x + C, \\ \int \sinh x \, dx &= \cosh x + C, & \int \operatorname{sech} x \tanh x \, dx &= -\operatorname{sech} x + C, \\ \int \operatorname{sech}^2 x \, dx &= \tanh x + C, & \int \operatorname{csch} x \coth x \, dx &= -\operatorname{csch} x + C. \end{aligned}$$

Example 9 Compute the integrals (a) $\int (\sinh 3x + x^3) dx$, (b) $\int \tanh x dx$, (c) $\int \cosh^2 x dx$,
 (d) $\int \frac{\sinh x}{1 + \cosh^2 x} dx$.

Solution (a) $\int (\sinh 3x + x^3) dx = \int \sinh 3x dx + \int x^3 dx = \frac{1}{3} \cosh 3x + \frac{x^4}{4} + C.$

(b) $\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \ln|\cosh x| + C = \ln(\cosh x) + C$

(since $\cosh x \geq 1$, $|\cosh x| = \cosh x$).

(c) Here we use the half-angle formula (8):

$$\int \cosh^2 x dx = \int \frac{\cosh 2x + 1}{2} dx = \frac{1}{4} \sinh 2x + \frac{x}{2} + C.$$

(d) $\int \frac{\sinh x}{1 + \cosh^2 x} dx = \int \frac{du}{1 + u^2} \quad (u = \cosh x)$
 $= \tan^{-1} u + c = \tan^{-1}(\cosh x) + C. \blacktriangle$

Supplement to Section 8.3: Unstable Equilibrium Points

In Section 8.1, we studied approximations to the differential equation of motion $m(d^2x/dt^2) = f(x)$, where the force function $f(x)$ satisfied the equilibrium condition $f(x_0) = 0$ at some position x_0 . The linearized equation was

$$m \frac{d^2x}{dt^2} = f'(x_0)(x - x_0)$$

or, setting $y = x - x_0$, the displacement from equilibrium,

$$m \frac{d^2y}{dt^2} = f'(x_0)y.$$

If $f'(x_0) < 0$, this is the spring equation which has oscillating solutions given by trigonometric functions.

Now we can use the hyperbolic functions to study the case $f'(x_0) > 0$. The general solution is $y = A \cosh(\sqrt{f'(x_0)} t) + B \sinh(\sqrt{f'(x_0)} t)$, with A and B depending on the initial values of y and dy/dt , as in formula (10). We can also write this solution as $y = (A + B)e^{\sqrt{f'(x_0)}t} + (A - B)e^{-\sqrt{f'(x_0)}t}$. No matter how small the initial values, unless they are chosen so carefully that $A + B = 0$, the solution will approach $+\infty$ or $-\infty$ as $t \rightarrow \infty$. We say that the point x_0 is an *unstable equilibrium* position. In contrast to the stable case treated in Section 8.1, the linearization is not useful for all t , since most solutions eventually leave the region where the linear approximation is valid. Still, we can correctly conclude that solutions starting arbitrarily close to x_0 do not usually stay close, and that there are special solutions which approach x_0 as $t \rightarrow \infty$.⁴

⁴ This analysis is useful in more advanced studies of differential equations. See, for instance, *Elementary Differential Equations and Boundary Value Problems*, by W. Boyce and R. DiPrima, Chapter 9, Third Edition, Wiley (1977) and *Differential Equations and Their Applications*, by M. Braun, Chapter 4, Third Edition, Springer-Verlag (1983).

For instance, let us find the unstable equilibrium point for the pendulum equation

$$m \frac{d^2\theta}{dt^2} = -m \frac{g}{l} \sin \theta = f(\theta).$$

First note that $f(\theta) = 0$ at $\theta = 0$ and π , corresponding to the bottom and top of pendulum swing (see Fig. 8.1.6). At $\theta = \pi$, $f'(\theta) = -(mg/l)\cos\theta = -(mg/l) \cdot (-1) = mg/l > 0$, so the top position is an unstable equilibrium point. At that point, the linearized equation is given by $(d^2/dt^2)(\theta - \theta_0) = (mg/l)(\theta - \theta_0)$, with solutions $\theta - \theta_0 = Ce^{\sqrt{mg/l}t} + De^{-\sqrt{mg/l}t}$. For most initial conditions, C will not equal zero, and so the pendulum will move away from the equilibrium point. If the initial conditions are chosen just right, we will have $C = 0$, and the pendulum will gradually approach the top position as $t \rightarrow \infty$, but it will never arrive.

A very different application of hyperbolic functions, to the shape of a hanging cable, is given in Example 6, Section 8.5.

Exercises for Section 8.3

Prove the identities in Exercises 1–8.

- $\tanh^2 x + \operatorname{sech}^2 x = 1$.
- $\coth^2 x = 1 + \operatorname{csch}^2 x$.
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
- $\sinh^2 x = (\cosh 2x - 1)/2$.
- $\frac{d}{dx} \cosh x = \sinh x$.
- $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$.
- $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$.
- $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$.

Differentiate the functions in Exercises 9–24.

- $\sinh(x^3 + x^2 + 2)$
- $\tan^{-1}(\cosh x)$
- $\sinh x \sinh 5x$
- $\frac{\sinh x}{1 + \cosh x}$
- $\sinh(\cos(8x))$
- $\cos(\sinh(x^2))$
- $\sinh^2 x + \cosh^2 x$
- $\sinh^4 x + \cosh^4 x$
- $\coth 3x$
- $(\tanh x)(\operatorname{sech} x)$
- $\exp(\tanh 2x)$
- $\sin^{-1}(\tanh x)$
- $\frac{\cosh x}{1 + \tanh x}$
- $(\operatorname{csch} 2x)(1 + \tan x)$
- $(\cosh x) \left(\int \frac{dx}{1 + \tanh^2 x} \right)$
- $(\sinh x) \left(\int \frac{dx}{1 + \operatorname{sech}^2 x} \right)$

Solve the differential equations in Exercises 25–28.

- $y'' - 9y = 0$, $y(0) = 0$, $y'(0) = 1$.
- $f'' - 81f = 0$, $f(0) = 1$, $f'(0) = -1$.
- $g'' - 3g = 0$, $g(0) = 2$, $g'(0) = 0$.
- $h'' - 9h = 0$, $h(0) = 2$, $h'(0) = 4$.
- Find the solution of the equation $d^2x/dt^2 - 9x = 0$, for which $x = 1$ and $dx/dt = 1$ at $t = 0$.
- Solve $d^2y/dt^2 - 25y = 0$, where $y = 1$ and $dy/dt = -1$ when $t = 0$.
- Find $f(t)$ if $f'' = 36f$, and $f(0) = 2$, $f'(0) = 0$.
- Find $g(t)$ if $g''(t) = 25g(t)$, and $g(0) = 0$, $g'(0) = 1$.

Sketch the graphs of the functions in Exercises 33–36.

- $y = 3 + \sinh x$
- $y = (\cosh x) - 1$
- $y = \tanh 3x$
- $y = 3 \cosh 2x$

Compute the integrals in Exercises 37–46.

- $\int \cosh 3x \, dx$
- $\int [\operatorname{csch}^2(2x) + (3/x)] \, dx$
- $\int \coth x \, dx$
- $\int x \tanh(x^2) \, dx$
- $\int \sinh^2 x \, dx$
- $\int \cosh^2 9y \, dy$
- $\int e^x \sinh x \, dx$
- $\int e^{2t} \cosh 2t \, dt$
- $\int \cosh^2 x \sinh x \, dx$
- $\int \frac{\sinh x}{\cosh^3 x} \, dx$

Compute dy/dx in Exercises 47–50.

$$47. \frac{\sinh(x+y)}{xy} = 1$$

$$48. x + \cosh(xy) = 3$$

$$49. \tanh(3xy) + \sinh y = 1$$

$$50. \coth(x-y) - 3y = 6$$

- ★51. (a) Find the *unstable* equilibrium point for the equation of motion $d^2x/dt^2 = x^2 - 1$.
 (b) Write down the linearized equation of motion at this point.
- ★52. An atom of mass m in a linear molecule is subjected to forces of attraction by its neighbors given by

$$f(x) = -\frac{k_1}{(x-x_1)^2} + \frac{k_2}{(x-x_2)^2},$$

where $k_2 > k_1 > 0$, $0 < x_1 < x < x_2$. Show that $x_0 = (x_1 + \alpha x_2)/(1 + \alpha)$, where $\alpha = \sqrt{k_1/k_2}$, is an unstable equilibrium. Write the linearized equations at this point.

- ★53. Prove the identity $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.
- ★54. Prove that the equation $d^2x/dt^2 - \omega^2x = 0$, with $x = x_0$ and $dx/dt = v_0$ when $t = 0$, has a unique solution given by $x = x_0 \cosh \omega t + (v_0/\omega) \sinh \omega t$. [Hint: Study Exercises 31 and 32 of Section 8.1.] Why doesn't the energy method of p. 372 work in this case?

8.4 The Inverse Hyperbolic Functions

The inverse hyperbolic functions occur in several basic integration formulas.

We now study the inverses of the hyperbolic functions using the methods of Section 5.3. As with the inverse trigonometric functions, this yields some interesting integration formulas for algebraic functions.

We turn first to the inverse $\sinh$ function. Since $(d/dx)\sinh x = \cosh x$ is positive (Example 8, Section 8.3), $\sinh x$ is increasing. The range of $\sinh x$ is in fact $(-\infty, \infty)$ since $\sinh x \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$. Thus, from the inverse function test in Section 5.3, we know that $y = \sinh x$ has an inverse function defined on the whole real line, denoted $\sinh^{-1}y$ by analogy with the notation for the inverse trigonometric functions. From the general formula

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)} \quad (\text{where } y = f(x))$$

for the derivative of an inverse function, we get

$$\frac{d}{dy} \sinh^{-1}y = \frac{1}{(d/dx)(\sinh x)} = \frac{1}{\cosh x}.$$

From $\cosh^2x - \sinh^2x = 1$, we get

$$\frac{d}{dy} \sinh^{-1}y = \frac{1}{\cosh x} = \frac{1}{\sqrt{1 + \sinh^2x}} = \frac{1}{\sqrt{1 + y^2}}. \quad (1)$$

The positive square root is taken because $\cosh x$ is always positive.

Example 1 Calculate (a) $\frac{d}{dx} \sinh^{-1}(3x)$ and (b) $\frac{d}{dx} [\sinh^{-1}(3 \tanh 3x)]$.

Solution (a) Let $u = 3x$, so $\sinh^{-1}3x = \sinh^{-1}u$. By the chain rule,

$$\frac{d}{dx} \sinh^{-1}(3x) = \left(\frac{d}{du} \sinh^{-1}u \right) \frac{du}{dx}.$$

By formula (1) with y replaced by u , we get

$$\frac{d}{dx} \sinh^{-1}(3x) = \frac{1}{\sqrt{1 + u^2}} 3 = \frac{3}{\sqrt{1 + 9x^2}}.$$

(b) By the formula $(d/dx)\sinh^{-1}x = 1/\sqrt{1+x^2}$, and the chain rule,

$$\begin{aligned}\frac{d}{dx}\sinh^{-1}(3\tanh 3x) &= \frac{1}{\sqrt{1+9\tanh^2 3x}} \cdot 3 \frac{d}{dx}\tanh 3x \\ &= \frac{1}{\sqrt{1+9\tanh^2 3x}} \cdot 3 \cdot 3 \cdot \operatorname{sech}^2 3x \\ &= \frac{9\operatorname{sech}^2 3x}{\sqrt{1+9\tanh^2 3x}}. \blacktriangle\end{aligned}$$

There is an explicit formula for $\sinh^{-1}y$ obtained by solving the equation

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

for x . Multiplying through by $2e^x$ and gathering terms on the left-hand side of the equation gives $2e^{2x} - 2e^x y - 1 = 0$. Hence

$$(e^x)^2 - 2e^x y - 1 = 0,$$

and so, by the quadratic formula,

$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}.$$

Since e^x is positive, we must select the positive square root. Thus, $e^x = y + \sqrt{y^2 + 1}$, and so $x = \sinh^{-1}y$ is given by

$$\sinh^{-1}y = \ln(y + \sqrt{y^2 + 1}). \quad (2)$$

The basic properties of $\sinh^{-1}$ are summarized in the following display.

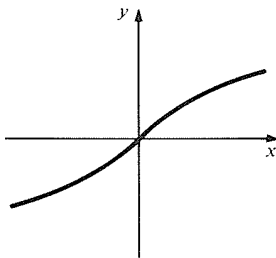


Figure 8.4.1. The graph of the $y = \sinh^{-1}x$.

Inverse Hyperbolic Sine Function

1. $\sinh^{-1}x$ is the inverse function of $\sinh x$; $\sinh^{-1}x$ is defined and is increasing for all x (Fig. 8.4.1); by definition: $\sinh^{-1}x = y$ is that number such that $\sinh y = x$.
2. $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$.
3. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x + C = \ln(x + \sqrt{1+x^2}) + C$.

Example 2 Find $\sinh^{-1}5$ numerically by using logarithms.

Solution By (2), $\sinh^{-1}5 = \ln(5 + \sqrt{5^2 + 1}) = \ln(5 + \sqrt{26}) = \ln(10.100) \approx 2.31$. $\blacktriangle$

Example 3 Verify the formula $\int \frac{dx}{\sqrt{1+x^2}} = \ln(x + \sqrt{1+x^2}) + C$.

Solution

$$\begin{aligned}\frac{d}{dx}\ln(x + \sqrt{1+x^2}) &= \frac{1}{(x + \sqrt{1+x^2})} \left(1 + \frac{x}{\sqrt{1+x^2}}\right) \quad (\text{by the chain rule}) \\ &= \left(\frac{1}{x + \sqrt{1+x^2}}\right) \left(\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Thus the antiderivative for $1/\sqrt{1+x^2}$ is $\ln(x + \sqrt{1+x^2}) + C$. $\blacktriangle$

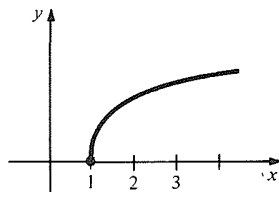


Figure 8.4.2. The graph of $y = \cosh^{-1}x$.

In a similar fashion we can investigate $\cosh^{-1}x$. Since $\cosh x$ is increasing on $[0, \infty)$ and has range $[1, \infty)$, $\cosh^{-1}x$ will be increasing, will be defined on $[1, \infty)$, and will have range $[0, \infty)$. Its graph can be obtained from that of $\cosh x$ by the usual method of looking through the page from the other side (Fig. 8.4.2).

By the same method that we obtained formula (1), we find

$$\frac{d}{dx} \cosh^{-1}x = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1). \quad (3)$$

Example 4 Find $\frac{d}{dx} \cosh^{-1}(\sqrt{x^2 + 1})$, $x \neq 0$.

Solution Let $u = \sqrt{x^2 + 1}$. Then, by the chain rule,

$$\begin{aligned} \frac{d}{dx} \cosh^{-1}(\sqrt{x^2 + 1}) &= \left(\frac{d}{du} \cosh^{-1}u \right) \cdot \left(\frac{du}{dx} \right) \\ &= \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{x}{\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2}} \cdot \frac{x}{\sqrt{x^2 + 1}} = \frac{x}{|x|} \cdot \frac{1}{\sqrt{x^2 + 1}}. \end{aligned}$$

Therefore,

$$\frac{d}{dx} \cosh^{-1}(\sqrt{x^2 + 1}) = \begin{cases} \frac{1}{\sqrt{x^2 + 1}} & \text{if } x > 0, \\ \frac{-1}{\sqrt{x^2 + 1}} & \text{if } x < 0. \blacktriangle \end{cases}$$

Similarly, we can consider $\tanh^{-1}x$ (see Fig. 8.4.3) and get, for $-1 < x < 1$,

$$\frac{d}{dx} \tanh^{-1}x = \frac{1}{1 - x^2}. \quad (4)$$

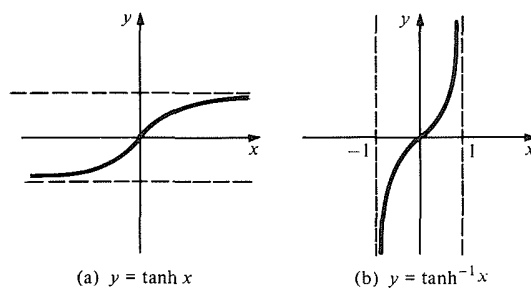


Figure 8.4.3. The graphs of $y = \tanh x$ and $y = \tanh^{-1}x$.

Example 5 Prove that $\tanh^{-1}x = \frac{1}{2} \ln[(1+x)/(1-x)]$, $-1 < x < 1$.

Solution Let $y = \tanh^{-1}x$, so

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(e^y - e^{-y})}{(e^y + e^{-y})}.$$

Thus $x(e^y + e^{-y}) = e^y - e^{-y}$. Multiplying through by e^y and gathering terms on the left:

$$(x-1)e^{2y} + x + 1 = 0$$

$$e^{2y} = \frac{1+x}{1-x},$$

$$2y = \ln\left(\frac{1+x}{1-x}\right),$$

$$y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),$$

as required. $\blacktriangle$

Example 6 Show directly that the antiderivative of $\frac{1}{1-x^2}$ for $|x| < 1$ is $\frac{1}{2} \ln \frac{1+x}{1-x} + C$ by noticing that

$$\frac{1}{1-x^2} = \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right).$$

Solution Since

$$\frac{1}{1-x^2} = \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right),$$

an antiderivative is

$$-\frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad |x| \neq 1.$$

If $|x| < 1$, $(1+x)/(1-x) > 0$, so the absolute value signs can be removed. $\blacktriangle$

The remaining inverse functions are investigated in a similar way; the results are summarized in Fig. 8.4.4 and the box on the next page.

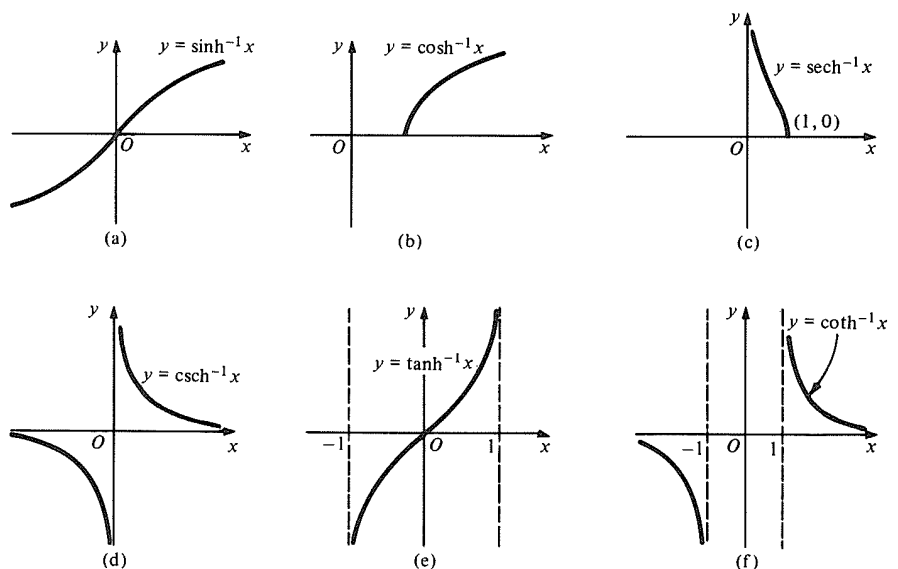


Figure 8.4.4. Graphs of the inverse hyperbolic functions.

The Inverse Hyperbolic Functions

$$\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1});$$

$$\frac{d}{dx} \sinh^{-1}x = \frac{1}{\sqrt{x^2 + 1}};$$

$$\cosh^{-1}x = \ln(x + \sqrt{x^2 - 1});$$

$$\frac{d}{dx} \cosh^{-1}x = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1;$$

$$\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right);$$

$$\frac{d}{dx} \tanh^{-1}x = \frac{1}{1-x^2}, \quad |x| < 1;$$

$$\coth^{-1}x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right);$$

$$\frac{d}{dx} \coth^{-1}x = \frac{1}{1-x^2}, \quad |x| > 1;$$

$$\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right);$$

$$\frac{d}{dx} \operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1;$$

$$\operatorname{csch}^{-1}x = \begin{cases} \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), & x > 0; \\ -\ln\left(\frac{1+\sqrt{1+x^2}}{-x}\right), & x < 0; \end{cases} \quad \frac{d}{dx} \operatorname{csch}^{-1}x = \begin{cases} \frac{-1}{x\sqrt{1+x^2}}, & x > 0, \\ \frac{1}{x\sqrt{1+x^2}}, & x < 0; \end{cases}$$

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1}x + C = \ln(x + \sqrt{x^2 + 1}) + C;$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1}x + C = \ln(x + \sqrt{x^2 - 1}) + C, \quad x > 1;$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C, \quad (|x| \neq 1),$$

$$= \begin{cases} \tanh^{-1}x + C = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) + C, & |x| < 1, \\ \coth^{-1}x + C = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) + C, & |x| > 1; \end{cases}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1}x + C = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C, \quad 0 < x < 1;$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{csch}^{-1}x + C = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) + C, \quad x > 0.$$

Example 7 Find $\int \frac{dx}{\sqrt{3x^2 - 1}}$.

Solution Going through the list of integration formulas for hyperbolic functions, we find $\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1}x + C$. Note that $\sqrt{3x^2 - 1} = \sqrt{(\sqrt{3}x)^2 - 1}$. Using the technique of integration by substitution with $u = \sqrt{3}x$ and $du = \sqrt{3}dx$,

we have

$$\begin{aligned}\int \frac{dx}{\sqrt{3x^2 - 1}} &= \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2 - 1}} = \frac{1}{\sqrt{3}} \cosh^{-1} u + C \\ &= \frac{1}{\sqrt{3}} \cosh^{-1}(\sqrt{3}x) + C. \blacktriangle\end{aligned}$$

Example 8 Find $\int \frac{dx}{\sqrt{4 + x^2}}$.

Solution

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + 4}} &= \frac{1}{2} \int \frac{dx}{\left[\sqrt{1 + (x/2)^2}\right]} = \frac{1}{2} \int \frac{2 du}{\sqrt{1 + u^2}} \quad \left(u = \frac{x}{2}\right) \\ &= \sinh^{-1} u + C = \sinh^{-1}\left(\frac{x}{2}\right) + C. \blacktriangle\end{aligned}$$

Exercises for Section 8.4

Calculate the derivatives of the functions in Exercises 1–12.

1. $\cosh^{-1}(x^2 + 2)$
2. $\sinh^{-1}(x^3 - 2)$
3. $\sinh^{-1}(3x + \cos x)$
4. $\cosh^{-1}(x^2 - \tan x)$
5. $x \tanh^{-1}(x^2 - 1)$
6. $x^2 \coth^{-1}(x + 1)$
7. $\frac{x + \cosh^{-1} x}{\sinh^{-1} x + x}$
8. $\frac{1 + \sinh^{-1} x}{1 - \cosh^{-1} x}$
9. $\exp(1 + \sinh^{-1} x)$
10. $\exp(3 + \cosh^{-1} x)$
11. $\sinh^{-1}[\cos(3x)]$
12. $\cosh^{-1}[2 + \sin(x^2)]$

In Exercises 13–16 calculate the indicated values numerically using logarithms.

13. $\tanh^{-1}(0.5)$
14. $\coth^{-1}(1.3)$
15. $\operatorname{sech}^{-1}(0.3)$
16. $\operatorname{csch}^{-1}(1.2)$

Derive the identities in Exercises 17–20.

17. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $|x| > 1$
18. $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$, $|x| > 1$
19. $\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x}$, $0 < x < 1$
20. $\operatorname{csch}^{-1} x = \ln \frac{1 + \sqrt{1 + x^2}}{x}$, $x > 0$.

Derive the identities in Exercises 21–24.

21. $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}$.
22. $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$.

$$23. \frac{d}{dx} \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1 - x^2}}.$$

$$24. \frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}.$$

In Exercises 25–28 verify the given integration formulas by differentiation.

$$25. \int \frac{dx}{\sqrt{x^2 - 1}} = \ln(x + \sqrt{x^2 - 1}) + C, \quad |x| > 1.$$

$$26. \int \frac{dx}{1 - x^2} = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) + C, \quad |x| > 1.$$

$$27. \int \frac{dx}{x\sqrt{1 - x^2}} = -\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + C, \quad 0 < x < 1.$$

$$28. \int \frac{dx}{x\sqrt{1 + x^2}} = -\ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right) + C, \quad x > 0.$$

Calculate the integrals in Exercises 29–36.

$$29. \int \frac{dx}{1 - 4x^2}$$

$$30. \int \frac{dx}{4x^2 + 1}$$

$$31. \int \frac{dx}{\sqrt{4x^2 + 1}}$$

$$32. \int \frac{dx}{x\sqrt{1 - 4x^2}}$$

$$33. \int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx$$

$$34. \int \frac{e^x}{e^x \sqrt{1 - e^{2x}}} dx$$

$$35. \int \frac{e^x}{1 - e^{2x}} dx$$

$$36. \int \frac{\tan x dx}{\sqrt{1 + \cos^2 x}}$$

★37. Is the function $\cosh^{-1}(\sqrt{x^2 + 1})$ differentiable at all x ?

15. *Two-stage mixing.* Suppose the pollutants from the lake in Example 4 empty into a second smaller lake rather than the stream. At $t = 0$, the smaller lake contains 10^7 liters of pure water. The second lake has an exit stream carrying the same volume of fluid as entered. Find the amount of pollution in the second lake as a function of time.
16. Repeat Exercise 15 with the size and flow rates of the lake in Example 4 replaced by those in Exercise 14.
17. The terminal speed of a person in free fall in air is about 64 meters per second (see Example 5). How long does it take to reach 90% of terminal speed? How far has the person fallen in this time? ($g = 9.8$ meters per second²).
18. The terminal speed of a person falling with a parachute is about 6.3 meters per second (see Example 5). How long does it take to reach 90% of terminal speed? How far has the person fallen in this time? ($g = 9.8$ meters per second²).
19. *Falling object with drag resistance.* Redo Example 5 assuming that the resistance is proportional to the square of the velocity.
20. An object of mass 10 kilograms is dropped from a balloon. The force of air resistance is $0.07v$, where v is the velocity. What is the object's velocity as a function of time? How far has the object traveled before it is within 10% of its terminal velocity?
21. What is the acceleration of the rocket in Example 6 just before burnout?
22. How high is the rocket in Example 6 at burnout?
23. Sketch the direction field for the equation $dy/dx = y + 2x$ and solve the equation.
24. Sketch the direction field for the equation $dy/dx = -3y + x$ and solve the equation.
- ★25. Assuming that $P(x)$ and $Q(x)$ are continuous functions of x , prove that the problem $y' = P(x)y + Q(x)$, $y(0) = y_0$ has *exactly* one solution.
26. Express the solution of the equation $y' = xy + 1$, $y(0) = 1$ in terms of an integral.
- ★27. *Bernoulli's equation.* This equation has the form $dy/dx = P(x)y + Q(x)y^n$, $n = 2, 3, 4, \dots$.
 - (a) Show that the equation satisfied by $w = y^{1-n}$ is linear in w .
 - (b) Use (a) to solve the equation $x(dy/dx) = x^4y^3 - y$.
- ★28. *Riccati equation.* This equation is $dy/dx = P(x) + Q(x)y + R(x)y^2$.
 - (a) Let $y_1(x)$ be a known solution (found by inspection). Show that the general solution is $y(x) = y_1(x) + w(x)$, where w satisfies the Bernoulli equation (see Exercise 27) $dw/dx = [Q(x) + 2R(x)y_1(x)]w + R(x)w^2$.
 - (b) Use (a) to solve the Riccati equation $y' = y/x + x^3y^2 - x^5$, taking $y_1(x) = x$.
- ★29. Redo the rocket propulsion Example 6 adding air resistance proportional to velocity.
- ★30. Solve the equation $dy/dt = -\lambda y + r$, where λ and r are constants. Write a two page report on how this equation was used to study the Van Meegeren art forgeries which were done during World War II. (See M. Braun, *Differential Equations and their Applications*, Third Edition, Springer-Verlag, New York (1983), Section 1.3).

Review Exercises for Chapter 8

Solve the differential equations with the given conditions in Exercises 1–22.

1. $\frac{dy}{dt} = 3y$, $y(0) = 1$.
2. $\frac{dy}{dt} = y$, $y(0) = 1$.
3. $\frac{d^2y}{dt^2} + 3y = 0$, $y(0) = 0$, $y'(0) = 1$.
4. $\frac{d^2y}{dt^2} + 9y = 0$, $y(0) = 2$; $y'(0) = 0$.
5. $\frac{dy}{dt} = 3y + 1$, $y(0) = 1$.
6. $\frac{dy}{dt} = (\cos t)y + \cos t$, $y(0) = 0$.
7. $\frac{dy}{dt} = t^3y^2$, $y(0) = 1$.
8. $\frac{dy}{dt} + 10y = 0$, $y(0) = 1$.
9. $f' = 4f$; $f(0) = 1$.
10. $f' = -4f$; $f(0) = 1$.
11. $f'' = 4f$; $f(0) = 1$, $f'(0) = 1$.
12. $f'' = -4f$; $f(0) = 1$, $f'(0) = 1$.
13. $\frac{d^2x}{dt^2} + x = 0$; $x = 1$ when $t = 0$, $x = 0$ when $t = \pi/4$.
14. $\frac{d^2x}{dt^2} + 6x = 0$; $x = 1$ when $t = 0$, $x = 6$ when $t = 1$.
15. $\frac{d^2x}{dt^2} - 9x = 0$, $x(0) = 0$, $x'(0) = 1$.
16. $\frac{d^2x}{dt^2} + 16x = 0$, $x(0) = 1$, $x'(0) = -1$.
17. $\frac{dy}{dx} = e^{x+y}$, $y(0) = 1$.
18. $\frac{dy}{dt} = 8e^{3y}$, $y(0) = 1$.
19. $\frac{dx}{dt} = -4x$; $x = 1$ when $t = 0$.

20. $\frac{dy}{dx} = y/\ln y$, $y(1) = e$.
21. $\frac{dy}{dt} = \frac{y}{t-1} + \frac{1}{t-1}$, $y(0) = 0$.
22. $\frac{dx}{dt} = \frac{x}{8-t} + \frac{3}{8-t}$, $x(0) = 1$.
23. Solve for $g(t)$: $3(d^2/dt^2)g(t) = -7g(t)$, $g(0) = 1$, $g'(0) = -2$. Find the amplitude and phase of $g(t)$. Sketch.
24. Solve for $z = f(t)$: $d^2z/dt^2 + 5z = 0$, $f(0) = -3$, $f'(0) = 4$. Find the amplitude and phase of $f(t)$. Sketch.
25. Sketch the graph of y as a function of x if $-d^2y/dx^2 = 2y$; $y = \frac{1}{2}$, and $dy/dx = \frac{1}{2}$ when $x = 0$.
26. Sketch the solution of $d^2x/dt^2 + 9x = 0$, where $x(0) = 1$, $x'(0) = 0$.
27. Sketch the graph of the solution of $y' = -8y$, $y = 1$ when $x = 0$.
28. Sketch the graph of $y = f(x)$ if $f' = 2f + 3$, and $f(0) = 0$.
29. Sketch the graph of the solution to $dx/dt = -x + 3$, $x(0) = 0$ and compute $\lim_{t \rightarrow \infty} x(t)$.
30. Sketch the graph of the solution to $dx/dt = -2x + 2$, $x(0) = 0$ and compute $\lim_{t \rightarrow \infty} x(t)$.
31. Solve $d^2x/dt^2 = dx/dt$, $x(0) = 1$, $x'(0) = 1$ by letting $y = dx/dt$.
32. Solve $d^2x/dt^2 = [1/(1+t)]dx/dt$, $x(0) = 1$, $x'(0) = 1$ by letting $y = dx/dt$.
33. Solve $d^2y/dx^2 + dy/dx = x$, $y(0) = 0$, $y'(0) = 1$. [Hint: Let $w = dy/dx$.]
34. Solve $y'' + 3yy' = 0$ for $y(x)$ if $y(0) = 1$, $y'(0) = 2$. [Hint: $yy' = (y^2/2)'$.]
35. Solve $d^2y/dx^2 = 25y$, $y(0) = 0$, $y(1) = 1$.
36. Solve $d^2y/dx^2 = 36y$, $y(0) = 1$, $y(1) = 0$.
- Differentiate the functions in Exercises 37–44.
37. $\sinh(3x^2)$
38. $\tanh(x^3 + x)$
39. $\cosh^{-1}(x^2 + 1)$
40. $\tanh^{-1}(x^4 - 1)$
41. $(\sinh^{-1}x)(\cosh 3x)$
42. $(\cosh^{-1}3x)(\tanh x^2)$
43. $\exp(1 - \cosh^{-1}(3x))$
44. $3(\cosh^{-1}(5x^2) + 1)$

Calculate the integrals in Exercise 45–50.

45. $\int \frac{\cosh x}{1 + \sinh^2 x} dx$
46. $\int \operatorname{sech}^2 x \tanh x \sqrt{2 + \tanh^2 x} dx$
47. $\int \frac{dx}{9 - x^2}$
48. $\int \frac{dx}{\sqrt{x^2 + 4}}$
49. $\int x \sinh x dx$
50. $\int x \cosh x dx$

51. A weight of 5 grams hangs on a spring with spring constant $k = 2.1$. Find the displacement $x(t)$ of the mass if $x(0) = 1$ and $x'(0) = 0$.

52. A weight hanging on a spring oscillates with a frequency of two cycles per second. Find the displacement $x(t)$ of the mass if $x(0) = 1$ and $x'(0) = 0$.
53. An observer sees a weight of 10 grams on a spring undergoing the motion $x(t) = 10 \sin(8t)$.
(a) What is the spring constant?
(b) What force acts on the weight at $t = \pi/16$?
54. A 3-foot metal rod is suspended horizontally from a spring, as shown in Fig. 8.R.1. The rod bobs up and down around the equilibrium point, 5 feet from the ground, and an amplitude of 1 foot and a frequency of two bobs per second. What is the maximum length of its shadow? How fast is its shadow changing length when the rod passes the middle of its bob?

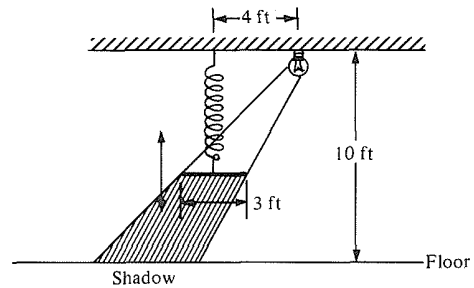


Figure 8.R.1. Study the movement of the shadow of the bobbing rod.

55. *Simple Harmonic Motion with Damping.* Consider the equation $x'' + 2\beta x' + \omega^2 x = 0$, where $0 < \beta < \omega$. (a) Show that $y = e^{\beta t}x$ satisfies a harmonic oscillator equation. (b) Show that the solution is of the form $x = e^{-\beta t}(A \cos \omega_1 t + B \sin \omega_1 t)$ where $\omega_1 = \sqrt{\omega^2 - \beta^2}$, and A and B are constants. (c) Solve $x'' + 2x' + 4x = 0$; $x(0) = 1$, $x'(0) = 0$, and sketch.
56. *Forced oscillations.*
(a) Show that a solution of the differential equation $x'' + 2\beta x' + \omega^2 x = f_0 \cos \omega_0 t$ is
- $$x_1(t) = \frac{f_0}{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 \beta^2} [2\omega_0 \beta \sin \omega_0 t + (\omega^2 - \omega_0^2) \cos \omega_0 t].$$
- (b) Show that the general solution is $x(t) = x_1(t) + x_0(t)$ where x_0 is the solution found in Exercise 55.
- (c) *Resonance.* Show that the “amplitude” $f_0/[(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 \beta^2]$ of the solution is largest when ω_0 is near ω (the natural frequency) for β (the friction constant) small, by maximizing the amplitude for fixed f_0 , ω ,

β and variable ω_0 . (This is the phenomenon responsible for the Tacoma bridge disaster . . . somewhat simplified of course; see Section 12.7 for further information.)

57. If a population doubles every 10 years and is now 100,000, how long will it take to reach 10 million?
58. The half-life of a certain radioactive substance is 15,500 years. What percentage will have decayed after 50,000 years?
59. A certain radioactive substance decreases at a rate of 0.00128% per year. What is its half-life?
60. (a) The population of the United States in 1900 was about 76 million; in 1910, about 92 million. Assume that the population growth is uniform, so $f(t) = e^{\gamma t}f(0)$, t in years after 1900. (i) Show that $\gamma \approx 0.0191$. (ii) What would have been a reasonable prediction for the population in 1960? In 1970? (iii) At this rate, how long does it take for the population to double?
(b) The actual U.S. population in 1960 was about 179 million and in 1970 about 203 million. (i) By what fraction did the "growth rate factor" (" γ ") change between 1900–1910 and 1960–1970? (ii) Compare the percentage increase in population from 1900 to 1910 with the percentage increase from 1960 to 1970.
61. If an object cools from 100°C to 80°C in an environment of 18°C in 8 minutes, how long will it take to cool from 100° to 50°C ?
62. "Suppose the pharaohs had built nuclear energy plants. They might have elected to store the resulting radioactive wastes inside the pyramids they built. Not a bad solution, considering how well the pyramids have lasted. But plutonium-239 stored in the oldest of them—some 4600 years ago—would today still exhibit 88 percent of its initial radioactivity." (see G. Hardin, "The Fallibility Factor," *Skeptical* 14 (1976): 12.)
(a) What is the half-life of plutonium?
(b) How long will it take for plutonium stored today to have only 1% of its present radioactivity? How long for $\frac{1}{1000}$?
63. (a) The oil consumption rate satisfies the equation $C(t) = C_0 e^{rt}$, where C_0 is the consumption rate at $t = 0$ (number of barrels per year) and r is a constant. If the consumption rate is $C_0 = 2.5 \times 10^{10}$ barrels per year in 1976 and $r = 0.06$, how long will it take before 2×10^{12} barrels (the total world's supply) are used up?
(b) As the fuel is almost used up, the prices will probably skyrocket and other sources of energy will be turned to. Let $S(t)$ be the supply left at time t . Assume that $dS/dt = -\alpha S$, where α is a constant (the panic factor). Find $S(t)$.
64. (a) A bank advertises "5% interest on savings—but you earn more because it is compounded continuously." The formula for computing the amount $M(t)$ of money in an account t days after $M(0)$ dollars is deposited (and left untouched) is $M(t) = M(0)e^{0.05t/365}$. What is the percentage increase on an amount $M(0)$ left untouched for 1 year?
(b) A bank wants to compute its interest by the method in part (a), but it wants to give only \$5 interest on each \$100 that is left untouched for 1 year. How must it change the formula for that to occur?
65. A certain electric circuit is governed by the equation $LdI/dt + RI = E$, where E , R , and L are constants. Graph the solution if $I_0 < E/R$.
66. If a savings account containing P dollars grows at a rate $dP/dt = rP + W$ (interest with continuous deposits), find P in terms of its value P_0 at $t = 0$.
67. (a) Sketch the direction field for the equation $y' = -9x/y$. Solve the equation exactly.
(b) Find the orthogonal trajectories for the solutions in (a).
68. Consider the predator–prey model in Example 5, Section 8.5. Solve this explicitly if $r = 0$ (i.e., ignore deaths of the prey).
69. Suppose your car radiator holds 4 gallons of fluid two thirds of which is water and one third is old antifreeze. The mixture begins flowing out at a rate of $\frac{1}{2}$ gallon per minute while fresh water is added at the same rate. How long does it take for the mixture to be 95% fresh water? Is it faster to wait until the radiator has drained before adding fresh water?
70. An object falling freely with air resistance has a terminal speed of 20 meters per sec. Find a formula for its velocity as a function of time.
71. The current I in a certain electric circuit is governed by $dI/dt = -3I + 2\sin(\pi t)$, and $I = 1$ at $t = 0$. Find the solution.
72. In Example 6, Section 8.6, suppose that the burnout mass is 10% of the initial mass M_0 , the burnout time is 3 minutes, and the rocket thrust is $3M_0 g$. Calculate the acceleration of the rocket just before burnout in terms of M_0 and g .
73. Sketch the direction field for the equation $y' = 3y + 4$ and solve it.
74. Sketch the direction field for the equation $y' = -4y + 1$ and solve it.
75. Let $x(t)$ be the solution of $dx/dt = x^2 - 5x + 4$, $x(0) = 3$. Find $\lim_{t \rightarrow \infty} x(t)$.
76. Let x satisfy $dx/dt = x^3 - 4x^2 + 3x$, $x(0) = 2$. Find $\lim_{t \rightarrow \infty} x(t)$.
77. Test the accuracy of the Euler method by using a ten-step Euler method on the problem of finding $y(1)$ if $y' = y$ and $y(0) = 1$. Compare your answer with the exact solution and with a twenty-step Euler method.

78. Solve $y' = \csc y$ approximately for $0 \leq x \leq 1$ with $y(0) = 1$ using a ten-step Euler method.
 79. Numerically solve for $y(2)$ if $y' = y^2$ and $y(0) = 1$, using a twenty-step Euler method. Do you detect some numerical trouble? What do you think is going wrong?
 80. Solve for $y(1)$ if $y' = \cos(x + y)$ and $y(0) = 0$, using a ten-step Euler method.
 81. Solve $y' = ay + b$, given constants a and b , by
 (a) introducing $w = y + b/a$ and a differential equation for w ;
 (b) treating it as a separable equation; and
 (c) treating it as a linear equation.
 Are your answers the same?
 ★82. Let $x(t)$ be the solution of $dx/dt = -x + 3 \sin \pi t$, $x(0) = 1$. Find α , ω and θ such that $\lim_{t \rightarrow \infty} [x(t) - \alpha \cos(\omega t + \theta)] = 0$.
 ★83. *Simple pendulum.* The equation of motion for a simple pendulum (see Fig. 8.R.2) is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta.$$

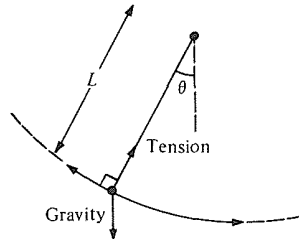


Figure 8.R.2. A simple pendulum.

- (a) Let $w(t) = d\theta/dt$. Show that

$$\frac{d}{dt} \left(\frac{w^2}{2} \right) = \frac{g}{L} \frac{d}{dt} \cos \theta,$$

and so

$$\frac{w^2}{2} = \frac{g}{L} (\cos \theta - \cos \theta_0),$$

where $w = 0$ when $\theta = \theta_0$ (the maximum value of θ).

- (b) Conclude that θ is implicitly determined by

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \sqrt{\frac{2g}{L}} t.$$

- (c) Show that the period of oscillation is

$$\begin{aligned} T &= 2 \sqrt{\frac{2L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \\ &= 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \end{aligned}$$

where $k = \sin(\theta_0/2)$. [Hint: Write $\cos \theta = 1 - 2 \sin^2(\theta/2)$.] The last integral is called an *elliptic integral of the first kind*, and cannot be evaluated explicitly.

- (d) How does the answer in (c) compare with the prediction from linearized oscillations for θ_0 small?

- ★84. A photographer dips a thermometer into a developing solution to determine its temperature in degrees Centigrade. The temperature $\theta(t)$ registered by the thermometer satisfies a differential equation $d\theta/dt = -k(\theta - \bar{\theta})$, where $\bar{\theta}$ is the true temperature of the solution and k is a constant. How can the photographer determine when θ is correct to within 0.1°C ?

- ★85. (a) Find $y = f(x)$ so that

$$\int_a^b f(x) dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

for all a and b .

- (b) What geometric problem leads to the problem (a).

10.3 Arc Length and Surface Area

Integration can be used to find the length of graphs in the plane and the area of surfaces of revolution.

In Sections 4.6, 9.1, and 9.2, we developed formulas for areas under and between graphs and for volumes of solids of revolution. In this section we continue applying integration to geometry and obtain formulas for lengths and areas.

The length of a piece of curve in the plane is sometimes called the *arc length* of the curve. As we did with areas and volumes, we assume that the length exists and will try to express it as an integral. For now, we confine our attention to curves which are graphs of functions; general curves are considered in the next section.

We shall begin with an argument involving infinitesimals to derive the formula for arc length. Following this, a different derivation will be given using step functions. The second method is the “honest” one, but it is also more technical.

We consider a curve that is a graph $y = f(x)$ from $x = a$ to $x = b$, as in Fig. 10.3.1. The curve may be thought of as being composed of infinitely

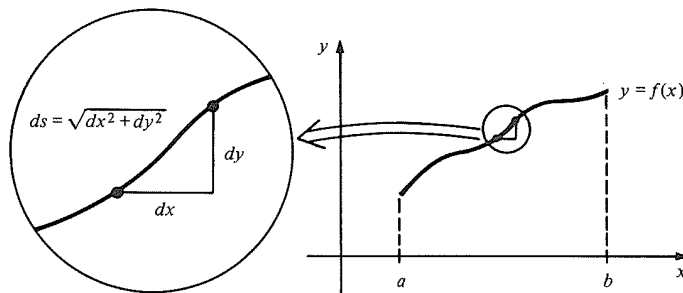


Figure 10.3.1. An “infinitesimal segment” of the graph of f .

many infinitesimally short segments. By the theorem of Pythagoras, the length ds of each segment is equal to $\sqrt{dx^2 + dy^2}$. But $dy/dx = f'(x)$, so $dy = f'(x)dx$ and $ds = \sqrt{dx^2 + [f'(x)]^2 dx^2} = \sqrt{1 + [f'(x)]^2} dx$. To get the total length, we add up all the infinitesimal lengths: $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2} dx$.

Length of Curves

Suppose that the function f is continuous on $[a, b]$, and that the derivative f' exists and is continuous (except possibly at finitely many points) on $[a, b]$. Then the length of the graph of f on $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (1)$$

Let us check that formula (1) gives the right result for the length of an arc of a circle.

Example 1 Use integration to find the length of the graph of $f(x) = \sqrt{1-x^2}$ on $[0, b]$, where $0 < b < 1$. Then find the length geometrically and compare the results.

Solution By formula (1), the length is $\int_0^b \sqrt{1 + [f'(x)]^2} dx$, where $f(x) = \sqrt{1-x^2}$. We have

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}, \quad [f'(x)]^2 = \frac{x^2}{1-x^2}, \quad 1 + [f'(x)]^2 = \frac{1}{1-x^2}.$$

Hence

$$L = \int_0^b \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(b) - \sin^{-1}(0) = \sin^{-1}(b).$$

Examining Fig. 10.3.2, we see that $\sin^{-1}(b)$ is equal to θ , the angle intercepted

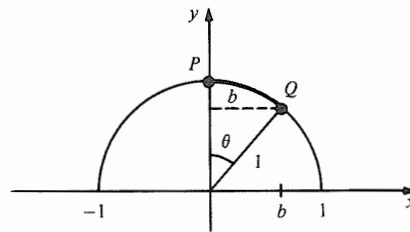


Figure 10.3.2. The length of the arc PQ is $\theta = \sin^{-1}b$.

by the arc whose length we are computing. By the definition of radian measure, the length of the arc is equal to the angle $\theta = \sin^{-1}(b)$, which agrees with our calculation by means of the integral. $\blacktriangle$

Example 2 Find the length of the graph of $f(x) = (x-1)^{3/2} + 2$ on $[1, 2]$.

Solution We are given $f(x) = (x-1)^{3/2} + 2$ on $[1, 2]$. Since $f'(x) = \frac{3}{2}(x-1)^{1/2}$, the length of the graph is

$$\begin{aligned} \int_a^b \sqrt{1 + [f'(x)]^2} dx &= \int_1^2 \sqrt{1 + \frac{9}{4}(x-1)} dx = \frac{1}{2} \int_1^2 \sqrt{9x-5} dx \\ &= \frac{1}{18} \int_4^{13} u^{1/2} du \quad (\text{where } u = 9x-5) \\ &= \frac{1}{27} (13^{3/2} - 8) \approx 1.44. \quad \blacktriangle \end{aligned}$$

Due to the square root, the integral in formula (1) is often difficult or even impossible to evaluate by elementary means. Of course, we can always approximate the result numerically (see Section 11.5 for specific examples). The following example shows how a simple-looking function can lead to a complicated integral for arc length.

Example 3 Find the length of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

Solution We substitute $f(x) = x^2$ and $f'(x) = 2x$ into formula (1):

$$L = \int_0^1 \sqrt{1 + (2x)^2} dx = 2 \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2 + x^2} dx.$$

Now substitute $x = \frac{1}{2} \tan \theta$ and $\sqrt{(1/2)^2 + x^2} = \frac{1}{2} \sec \theta$:

$$\int \sqrt{\left(\frac{1}{2}\right)^2 + x^2} dx = \int \left(\frac{1}{2} \sec \theta\right) \left(\frac{1}{2} \sec^2 \theta d\theta\right) = \frac{1}{4} \int \sec^3 \theta d\theta.$$

We evaluate the integral of $\sec^3 \theta$ using the following trickery³:

$$\begin{aligned} \int \sec^3 \theta d\theta &= \int \sec \theta \sec^2 \theta d\theta = \int \sec \theta (\tan^2 \theta + 1) d\theta \\ &= \int \sec \theta \tan^2 \theta d\theta + \int \sec \theta d\theta \\ &= \int (\sec \theta \tan \theta) \tan \theta d\theta + \ln|\sec \theta + \tan \theta|. \end{aligned}$$

(see Example 10, Section 10.2.) Now integrate by parts:

$$\begin{aligned} \int (\sec \theta \tan \theta) \tan \theta d\theta &= \int \frac{d}{d\theta} (\sec \theta) \tan \theta d\theta \\ &= \sec \theta \tan \theta - \int \sec \theta \sec^2 \theta d\theta \\ &= \sec \theta \tan \theta - \int \sec^3 \theta d\theta. \end{aligned}$$

Substituting this formula into the last expression for $\int \sec^3 \theta d\theta$ gives

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \ln|\sec \theta + \tan \theta|;$$

so

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|) + C.$$

Since $2x = \tan \theta$ and $\sec \theta = 2 \cdot \sqrt{(1/2)^2 + x^2} = \sqrt{1 + 4x^2}$, we can express the integral $\int \sec^3 \theta d\theta$ in terms of x as

$$x\sqrt{1 + 4x^2} + \frac{1}{2} \ln|2x + \sqrt{1 + 4x^2}| + C.$$

[One may also evaluate the integral $\int \sqrt{(1/2)^2 + x^2} dx$ using integral formula (43) from the endpapers.] Substitution into the formula for L gives

$$\begin{aligned} L &= \frac{1}{2} \left(x\sqrt{1 + 4x^2} + \frac{1}{2} \ln|2x + \sqrt{1 + 4x^2}| \right) \Big|_0^1 \\ &= \frac{1}{2} \left[\sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5}) \right] \approx 1.479. \quad \blacktriangle \end{aligned}$$

Example 4 Express the length of the graph of $f(x) = \sqrt{1 - k^2 x^2}$ on $[0, b]$ as an integral.

Solution We get

$$f'(x) = -\frac{k^2 x}{\sqrt{1 - k^2 x^2}}, \quad \text{so} \quad \sqrt{1 + [f'(x)]^2} = \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2 x^2}},$$

³ We can also write

$$\int \sec^3 \theta d\theta = \int \frac{\cos \theta}{\cos^4 \theta} d\theta = \int \frac{\cos \theta}{(1 - \sin^2 \theta)^2} d\theta = \int \frac{du}{(1 - u^2)^2} \quad (u = \sin \theta).$$

The last integral may now be evaluated by partial fractions.

Thus,

$$L = \int_0^b \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2x^2}} dx.$$

It turns out that the antiderivative

$$\int \sqrt{\frac{1 + (k^4 - k^2)x^2}{1 - k^2x^2}} dx$$

cannot be expressed (unless $k^2 = 0$ or 1) in terms of algebraic, trigonometric, or exponential functions. It is a new kind of function called an *elliptic* function. (See Review Exercises 85 and 92 for more examples of such functions.) ▲

We now turn to the derivation of formula (1) using step functions.

Our first principle for arc length is that the length of a straight line segment is equal to the distance between its endpoints. Thus, if $f(x) = mx + q$ on $[a, b]$, the endpoints (see Fig. 10.3.3) of the graph are $(a, ma + q)$ and

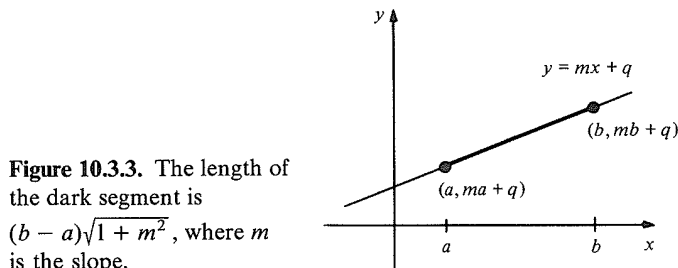


Figure 10.3.3. The length of the dark segment is $(b - a)\sqrt{1 + m^2}$, where m is the slope.

$(b, mb + q)$, and the distance between them is

$$\begin{aligned} \sqrt{(a - b)^2 + [(ma + q) - (mb + q)]^2} &= \sqrt{(a - b)^2 + m^2(a - b)^2} \\ &= (b - a)\sqrt{1 + m^2}. \end{aligned}$$

(Since $a < b$, the square root of $(a - b)^2$ is $b - a$.)

Our strategy, as in Chapter 9, will be to interpret the arc length for a simple curve as an integral and then use the same formula for general curves. In the case of the straight line segment, $f(x) = mx + q$, whose length between $x = a$ and $x = b$ is $(b - a)\sqrt{1 + m^2}$, we can interpret m as the derivative $f'(x)$, so that

$$\text{Length} = (b - a)\sqrt{1 + m^2} = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Since the formula for the length is an integral of f' , rather than of f , it is natural to look next at the functions for which f' is a step function. If f' is constant on an interval, f is linear on that interval; thus the functions with which we will be dealing are the *piecewise linear* (also called *ramp*, or *polygonal*) functions.

To obtain a piecewise linear function, we choose a partition of the interval $[a, b]$, say, $(x_0, x_1, \dots, x_n)$ and specify the values $(y_0, y_1, \dots, y_n)$ of

the function f at these points. For each $i = 1, 2, \dots, n$, we then connect the point (x_{i-1}, y_{i-1}) to the point (x_i, y_i) by a straight line segment (see Fig. 10.3.4).

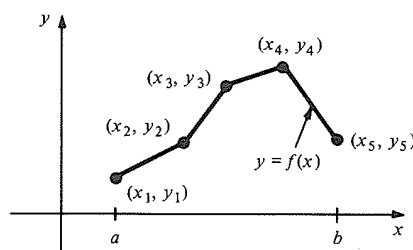


Figure 10.3.4. The graph of a piecewise linear function.

The function $f(x)$ is differentiable on each of the intervals (x_{i-1}, x_i) , where its derivative is constant and equal to the slope $(y_i - y_{i-1})/(x_i - x_{i-1})$. Thus the function $\sqrt{1 + [f'(x)]^2}$ is a step function $[a, b]$, with value

$$k_i = \sqrt{1 + \left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2}$$

on (x_{i-1}, x_i) .⁴ Therefore,

$$\begin{aligned} \int_a^b \sqrt{1 + [f'(x)]^2} \, dx &= \sum_{i=1}^n k_i \Delta x_i \\ &= \sum_{i=1}^n \sqrt{1 + \left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2} (x_i - x_{i-1}) \\ &= \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}. \end{aligned}$$

Note that the i th term in this sum, $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$, is just the length of the segment of the graph of f between (x_{i-1}, y_{i-1}) and (x_i, y_i) .

Now we invoke a second principle of arc length: if n curves are placed end to end, the length of the total curve is the sum of the lengths of the pieces. Using this principle, we see that the preceding sum is just the length of the graph of f on $[a, b]$. So we have now shown, for piecewise linear functions, that the length of the graph of f on $[a, b]$ equals the integral $\int_a^b \sqrt{1 + [f'(x)]^2} \, dx$.

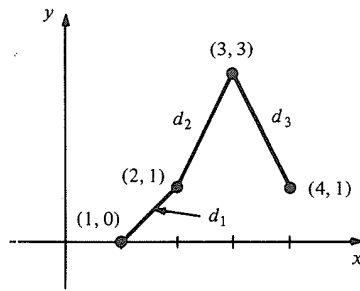
Example 5 Let the graph of f consist of straight line segments joining $(1, 0)$ to $(2, 1)$ to $(3, 3)$ to $(4, 1)$. Verify that the length of the graph, as computed directly, is given by the formula $\int_a^b \sqrt{1 + [f'(x)]^2} \, dx$.

Solution The graph is sketched in Fig. 10.3.5. The length is

$$d_1 + d_2 + d_3 = \sqrt{1 + 1^2} + \sqrt{1 + 2^2} + \sqrt{1 + (-2)^2} = \sqrt{2} + 2\sqrt{5}.$$

⁴ Actually, $\sqrt{1 + [f'(x)]^2}$ is not defined at the points $x_0, x_1, \dots, x_n$, but this does not matter when we take its integral, since the integral is not affected by changing the value of the integrand at isolated points.

Figure 10.3.5. The length of this graph is $d_1 + d_2 + d_3$.



On the other hand,

$$f'(x) = \begin{cases} 1 & \text{on } (1, 2) \\ 2 & \text{on } (2, 3) \\ -2 & \text{on } (3, 4) \end{cases}$$

(and is not defined at $x = 1, 2, 3, 4$). Thus, by the definition of the integral of a step function (see Section 4.3),

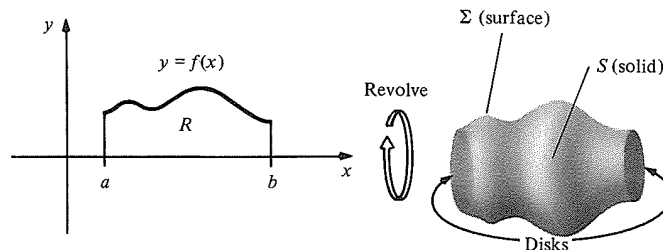
$$\begin{aligned} \int_1^4 \sqrt{1 + [f'(x)]^2} \, dx &= (\sqrt{1 + 1^2}) \cdot 1 + (\sqrt{1 + 2^2}) \cdot 1 + [\sqrt{1 + (-2)^2}] \cdot 1 \\ &= \sqrt{2} + 2\sqrt{5} \end{aligned}$$

which agrees with the preceding answer. $\blacktriangle$

Justifying the passage from step functions to general functions is more complicated than in the case of area, since we cannot, in any straightforward way, squeeze a general curve between polygons as far as length is concerned. Nevertheless, it is plausible that any reasonable graph can be approximated by a piecewise linear function, so formula (1) should carry over. These considerations lead to the technical conditions stated in conjunction with formula (1).

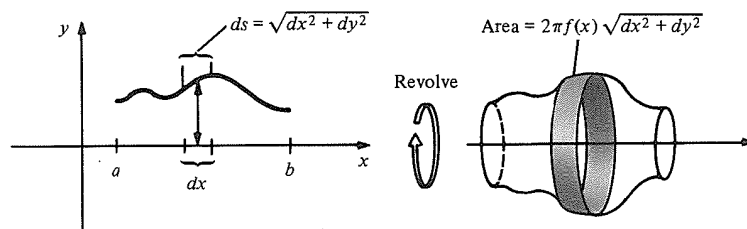
If we revolve the region R under the graph of $f(x)$ (assumed non-negative) on $[a, b]$, about the x axis, we obtain a solid of revolution S . In Section 9.1 we saw how to express the volume of such a solid as an integral. Suppose now that instead of revolving the region, we revolve the graph $y = f(x)$ itself. We obtain a curved surface Σ , called a *surface of revolution*, which forms part of the boundary of S . (The remainder of the boundary consists of the disks at the ends of the solid, which have radii $f(a)$ and $f(b)$; see Fig. 10.3.6.) Our next goal is to obtain a formula for the *area* of the surface Σ . Again we give the argument using infinitesimals first.

Figure 10.3.6. The boundary of the solid of revolution S consists of the surface of revolution Σ obtained by revolving the graph, together with two disks.



Referring to Figure 10.3.7, we may think of a smooth surface of revolution as being composed of infinitely many infinitesimal bands, as in Fig.

Figure 10.3.7. The surface of revolution may be considered as composed of infinitely many infinitesimal frustums.



10.3.7. The area of each band is equal to its circumference $2\pi f(x)$ times its width $ds = \sqrt{dx^2 + dy^2}$, so the total area is

$$\int_a^b 2\pi f(x) \sqrt{dx^2 + dy^2} = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

since $dy = f'(x) dx$.

Area of a Surface of Revolution about the x Axis

The area of the surface obtained by revolving the graph of $f(x) (\geq 0)$ on $[a, b]$ about the x axis is

$$A = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (2)$$

We now check that formula (2) gives the correct area for a sphere.

Example 6 Find the area of the spherical surface of radius r obtained by revolving the graph of $y = \sqrt{r^2 - x^2}$ on $[-r, r]$ about the x axis.

Solution As in Example 1, we have $\sqrt{1 + [f'(x)]^2} = r/\sqrt{r^2 - x^2}$, so the area is

$$\int_{-r}^r 2\pi \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} dx = 2\pi r \int_{-r}^r dx = 2\pi r \cdot 2r = 4\pi r^2$$

which is the usual value for the area of a sphere. ▲

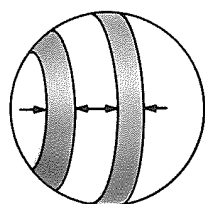


Figure 10.3.8. Bands of equal width have equal area.

If, instead of the entire sphere, we take the band obtained by restricting x to $[a, b]$ ($-r \leq a \leq b \leq r$), the area is $2\pi \int_a^b r dx = 2\pi r(b - a)$. Thus the area obtained by slicing a sphere by two parallel planes and taking the middle piece is equal to $2\pi r$ times the distance between the planes, regardless of where the two planes are located (see Fig. 10.3.8). Why doesn't the "longer" band around the middle have more area?

As with arc length, the factor $\sqrt{1 + [f'(x)]^2}$ in the integrand sometimes makes it impossible to evaluate the surface area integrals by any means other than numerical methods (see Section 11.5). To get a problem which can be solved, we must choose f carefully.

Example 7 Find the area of the surface obtained by revolving the graph of x^3 on $[0, 1]$ about the x axis.

Solution We find that $f'(x) = 3x^2$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + 9x^4}$, so

$$\begin{aligned} A &= 2\pi \int_0^1 \sqrt{1 + 9x^4} x^3 dx = \frac{\pi}{2} \int_0^1 \sqrt{1 + 9u} du \quad (u = x^4, du = 4x^3 dx) \\ &= \frac{\pi}{18} \int_0^9 (1 + v)^{1/2} dv \quad \left(u = \frac{1}{9}v, du = \frac{1}{9}dv\right) \\ &= \frac{\pi}{18} \left[\frac{2}{3} (1 + v)^{3/2} \right]_0^9 = \frac{\pi}{27} (10^{3/2} - 1) \approx 3.56. \blacktriangle \end{aligned}$$

By a method similar to that for deriving equation (2), we can derive a formula for the area obtained by revolving the graph $y = f(x)$ about the y axis for $a \leq x \leq b$. Referring to Figure 10.3.9, the area of the shaded band is

$$\text{Width} \times \text{Circumference} = ds \cdot 2\pi x = 2\pi x \sqrt{dx^2 + dy^2}$$

Thus the surface area is $\int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} dx$.

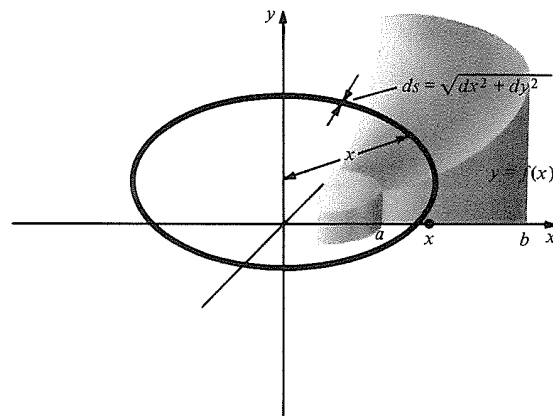


Figure 10.3.9. Rotating $y = f(x)$ about the y axis.

Area of a Surface of Revolution about the y Axis

The area of the surface obtained by revolving the graph of $f(x) (\geq 0)$ on $[a, b]$ about the y axis is

$$A = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2} dx. \quad (3)$$

Example 8 Find the area of the surface obtained by revolving the graph $y = x^2$ about the y axis for $1 \leq x \leq 2$.

Solution If $f(x) = x^2$, $f'(x) = 2x$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + 4x^2}$. Then

$$\begin{aligned} A &= 2\pi \int_1^2 x \sqrt{1 + 4x^2} dx = \frac{\pi}{4} \int_5^{17} u^{1/2} du \quad (u = 1 + 4x^2, du = 8x dx) \\ &= \frac{\pi}{4} \left(\frac{2}{3} u^{3/2} \right) \Big|_5^{17} = \frac{\pi}{6} (17^{3/2} - 5^{3/2}) \approx 30.85. \blacktriangle \end{aligned}$$

Finally we sketch how one derives formula (2) using step functions. The derivation of formula (3) is similar (see Exercise 41).

If $f(x) = k$, a constant, the surface is a cylinder of radius k and height $b - a$. Unrolling the cylinder, we obtain a rectangle with dimensions $2\pi k$ and $b - a$ (see Fig. 10.3.10), whose area is $2\pi k(b - a)$, so we can say that the area of the cylinder is $2\pi k(b - a)$.

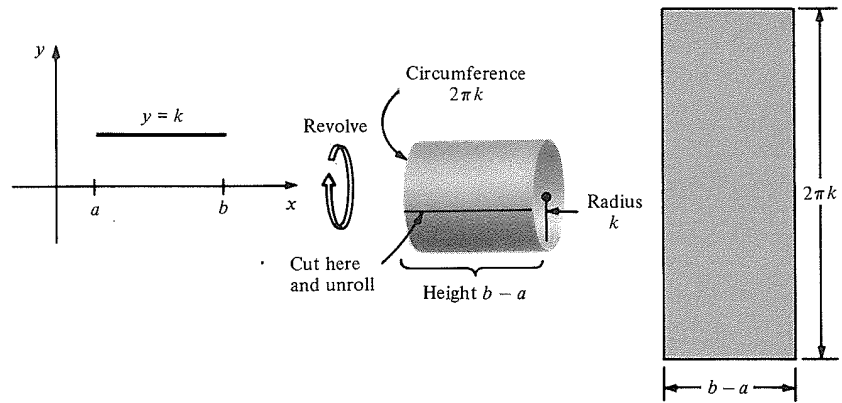


Figure 10.3.10. The area of the shaded cylinder is $2\pi k(b - a)$.

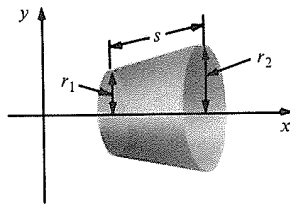


Figure 10.3.11. A frustum of a cone.

Next we look at the case where $f(x) = mx + q$, a linear function. The surface of revolution, as shown in Fig. 10.3.11, is a frustum of a cone—that is, the surface obtained from a right circular cone by cutting it with two planes perpendicular to the axis. To find the area of this surface, we may slit the frustum along a line and unroll it into the plane, as in Fig. 10.3.12, obtaining a circular sector of radius r and angle θ with a concentric sector of radius $r - s$ removed. By the definition of radian measure, we have $\theta r = 2\pi r_2$ and $\theta(r - s) = 2\pi r_1$, so $\theta s = 2\pi(r_2 - r_1)$, or $\theta = 2\pi[(r_2 - r_1)/s]$; from this we find that $r = r_2 s / (r_2 - r_1)$. The area of the figure is

$$\begin{aligned} \frac{\theta}{2\pi} [\pi r^2 - \pi(r - s)^2] &= \frac{\theta}{2} [r^2 - (r^2 - 2rs + s^2)] = \frac{\theta}{2} (2rs - s^2) \\ &= \theta s \left(r - \frac{s}{2} \right) = 2\pi(r_2 - r_1) \left(\frac{r_2 s}{r_2 - r_1} - \frac{s}{2} \right) \\ &= 2\pi s \left(r_2 - \frac{r_2 - r_1}{2} \right) = \pi s(r_1 + r_2). \end{aligned}$$

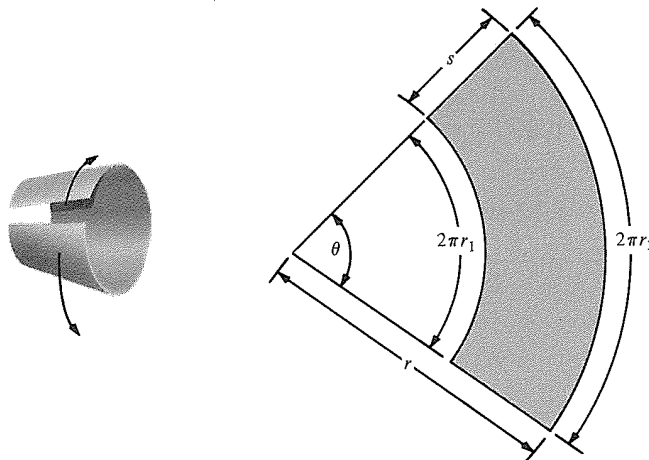


Figure 10.3.12. The area of the frustum, found by cutting and unrolling it, is $\pi s(r_1 + r_2)$.

(Notice that the proof breaks down in the case $r_1 = r_2 = k$, a cylinder, since r is then “infinite.” Nevertheless, the resulting formula $2\pi ks$ for the area is still correct!)

We now wish to express $\pi s(r_1 + r_2)$ as an integral involving the function $f(x) = mx + q$. We have $r_1 = ma + q$, $r_2 = mb + q$, and $s = \sqrt{1 + m^2}(b - a)$ (see Fig. 10.3.3), so the area is

$$\begin{aligned} & \pi\sqrt{1 + m^2}(b - a)[m(b + a) + 2q] \\ &= \pi\sqrt{1 + m^2}[m(b^2 - a^2) + 2q(b - a)] \\ &= 2\pi\sqrt{1 + m^2}\left[m\frac{b^2 - a^2}{2} + q(b - a)\right] \\ &= 2\pi\sqrt{1 + m^2}\left(m\frac{x^2}{2} + qx\right)\Big|_a^b. \end{aligned}$$

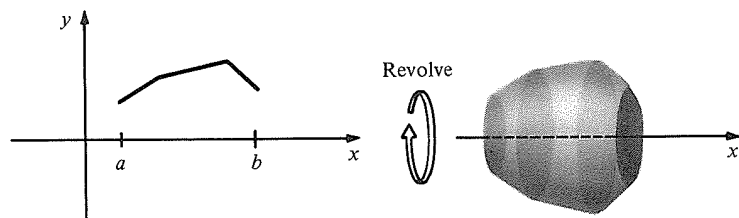
Since $m(x^2/2) + qx$ is the antiderivative of $mx + q$, we have

$$\begin{aligned} & 2\pi\sqrt{1 + m^2}\left(m\frac{x^2}{2} + qx\right)\Big|_a^b \\ &= 2\pi\int_a^b \sqrt{1 + m^2}(mx + q) dx \\ &= 2\pi\int_a^b \left[\sqrt{1 + f'(x)^2}\right] f(x) dx, \end{aligned}$$

so we have succeeded in expressing the surface area as an integral.

Now we are ready to work with general surfaces. If $f(x)$ is *piecewise* linear on $[a, b]$, the surface obtained is a “conoid,” produced by pasting together a finite sequence of frustums of cones, as in Fig. 10.3.13.

Figure 10.3.13. The surface obtained by revolving the graph of a piecewise linear function is a “conoid” consisting of several frustums pasted together.



The area of the conoid is the sum of the areas of the component frustums. Since the area of each frustum is given by the integral of the function $2\pi\sqrt{1 + [f'(x)]^2} f(x)$ over the appropriate interval, the additivity of the integral implies that the area of the conoid is given by the same formula:

$$A = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx.$$

We now assert, as we did for arc length, that this formula is true for general functions f . [To do this rigorously, we would need a precise definition of surface area, which is rather complicated to give (much more complicated, even, than for arc length).]

Example 9 The polygonal line joining the points (2, 0), (4, 4), (7, 5), and (8, 3) is revolved about the x axis. Find the area of the resulting surface of revolution.

Solution The function f whose graph is the given polygon is

$$f(x) = \begin{cases} 2(x-2), & 2 \leq x \leq 4, \\ \frac{1}{3}(x-4) + 4, & 4 \leq x \leq 7, \\ -2(x-7) + 5, & 7 \leq x \leq 8. \end{cases}$$

Then we have

$$f'(x) = \begin{cases} 2, & 2 < x < 4, \\ \frac{1}{3}, & 4 < x < 7, \\ -2, & 7 < x < 8. \end{cases}$$

$$\begin{aligned} \text{Thus, } A &= 2\pi \int_2^8 f(x) \sqrt{1 + f'(x)^2} \, dx \\ &= 2\pi \left[\int_2^4 f(x) \sqrt{1 + f'(x)^2} \, dx + \int_4^7 f(x) \sqrt{1 + f'(x)^2} \, dx \right. \\ &\quad \left. + \int_7^8 f(x) \sqrt{1 + f'(x)^2} \, dx \right] \\ &= 2\pi \left\{ \int_2^4 [2(x-2)] \sqrt{1+4} \, dx + \int_4^7 \left[\frac{1}{3}(x-4) + 4 \right] \sqrt{1 + \frac{1}{9}} \, dx \right. \\ &\quad \left. + \int_7^8 [-2(x-7) + 5] \sqrt{1+4} \, dx \right\}. \end{aligned}$$

Using $\int (x-a) \, dx = \frac{1}{2}(x-a)^2 + C$, we find

$$\begin{aligned} A &= 2\pi \left\{ \sqrt{5} \left[(x-2)^2 \right]_2^4 + \frac{\sqrt{10}}{3} \left[\frac{1}{6}(x-4)^2 + 4x \right]_4^7 + \sqrt{5} \left[-(x-7)^2 + 5x \right]_7^8 \right\} \\ &= 2\pi \left(4\sqrt{5} + \frac{9}{2}\sqrt{10} + 4\sqrt{5} \right) \approx 201.8. \quad \blacktriangle \end{aligned}$$

Exercises for Section 10.3

- Find the length of the graph of the function $f(x) = x^4/8 + 1/4x^2$ on $[1, 3]$.
- Find the length of the graph of the function $f(x) = (x^4 - 12x + 3)/6x$ on $[2, 4]$.
- Find the length of the graph of the function $y = [x^3 + (3/x)]/6$ on $1 \leq x \leq 3$.
- Find the length of the graph of the function $y = \sqrt{x}(4x-3)/6$ on $1 \leq x \leq 9$.
- Express the length of the graph of x'' on $[a, b]$ as an integral. (Do not evaluate.)
- Express the length of the graph of $f(x) = \sin x$ on $[0, 2\pi]$ as an integral. (Do not evaluate.)
- Express the length of the graph of $f(x) = x \cos x$ on $[0, 1]$ as an integral. (Do not evaluate.)
- Express the length of the graph $y = e^{-x}$ on $[-1, 1]$ as an integral. (Do not evaluate.)

In Exercises 9–12, let the graph of f consist of straight line segments joining the given points. Verify that the

length of the graph as computed directly is equal to that given by the arc length formula.

- (0, 0) to (1, 2) to (2, 1) to (5, 0).
- (1, 1) to (2, 2) to (3, 0).
- (-1, -1) to (0, 1) to (1, 2) to (2, -2).
- (-2, 2) to (-1, -3) to (3, 1).

Find the area of the surfaces obtained by revolving the curves in Exercises 13–20.

- The graph of $\sqrt{x+1}$ on $[0, 2]$ about the x axis.
- The graph of $y = [x^3 + (3/x)]/6$, $1 \leq x \leq 3$ about the x axis.
- The graph of $y = \sqrt{x}(4x-3)/6$, $1 \leq x \leq 9$ about the x axis.
- The graph of e^x on $[0, 1]$ about the x axis.
- The graph of $y = \cos x$ on $[-\pi/2, \pi/2]$ about the x axis.
- The parabola $y = \sqrt{x}$ on $[4, 5]$ about the x axis.
- The graph of $x^{1/3}$ on $[1, 3]$ about the y axis.
- The graph of $y = \ln x$ on $[2, 3]$ about the y axis.

In Exercises 21–24, the polygon joining the given points is revolved about the x axis. Find the area of the resulting surface of revolution.

21. $(0, 0)$ to $(1, 1)$ to $(2, 0)$.
22. $(1, 0)$ to $(3, 2)$ to $(4, 0)$.
23. $(2, 1)$ to $(3, 2)$ to $(4, 1)$ to $(5, 3)$.
24. $(4, 0)$ to $(5, 2)$ to $(6, 1)$ to $(8, 0)$.
25. Find the length of the graph of $a(x + b)^{3/2} + c$ on $[0, 1]$, where a , b , and c are constants. What is the effect of changing the value of c ?
26. Find the length of the graph of $y = x^2$ on $[0, b]$.
27. Express the length of the graph of $f(x) = 2x^3$ on $[-1, 2]$ as an integral. Evaluate numerically to within 1.0 by finding upper and lower sums. Compare your results with a string-and-ruler measurement.
28. Find the length, accurate to within 1 centimeter, of the curve in Fig. 10.3.14.

Figure 10.3.14. Find the length of this curve.

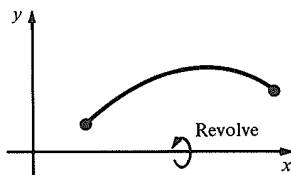


For each of the functions and intervals in Exercises 29–32, express as an integral: (a) the length of the curve; (b) the area of the surface obtained by revolving the curve about the x axis. (Do not evaluate the integrals.)

29. $\tan x + 2x$ on $[0, \pi/2]$
30. $x^3 + 2x - 1$ on $[1, 3]$
31. $1/x + x$ on $[1, 2]$
32. $e^x + x^3$ on $[0, 1]$

33. Find the area, accurate to within 5 square centimeters, of the surface obtained by revolving the curve in Fig. 10.3.15 around the x axis.

Figure 10.3.15. Find the area of the surface obtained by revolving this curve.



34. Use upper and lower sums to find the area, accurate to within 1 unit, of the surface obtained by revolving the graph of x^4 on $[0, 1]$ about the x axis.
35. Prove that the length of the graph of $f(x) = \cos(\sqrt{3}x)$ on $[0, 2\pi]$ is less than or equal to 4π .
36. Suppose that $f(x) > g(x)$ for all x in $[a, b]$. Does this imply that the length of the graph of f on $[a, b]$ is greater than or equal to that for g ? Justify your answer by a proof or an example.
37. Show that the length of the graph of $\sin x$ on $[0.1, 1]$ is less than the length of the graph of $1 + x^4$ on $[0.1, 1]$.
38. Suppose that the function f on $[a, b]$ has an inverse function g defined on $[a, \beta]$. Assume that $0 < a < b$ and $0 < \alpha < \beta$.

- (a) Find a formula, in terms of f , for the area of the surface obtained by revolving the graph of g on $[\alpha, \beta]$ about the x axis.
- (b) Show that this formula is consistent with the one in formula (3) for the area of the surface obtained by revolving the graph of f on $[a, b]$ about the y axis.

39. Write an integral representing the area of the surface obtained by revolving the graph of $1/(1 + x^2)$ about the x axis. Do not evaluate the integral, but show that it is less than $2\sqrt{5}\pi^2$ no matter how long an interval is taken.

40. Craftsman Cabinet Company was preparing a bid on a job that required epoxy coating of several tank interiors. The tanks were constructed from steel cylinders C feet in circumference and height H feet, with spherical steel caps welded to each end (see Fig. 10.3.16). Specifications required a $\frac{1}{8}$ -inch coating. The 20-year-old estimator quickly figured the cylindrical part as HC square feet. For the spherical cap he stretched a tape measure over the cap to obtain S ft.

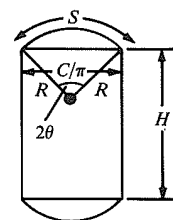


Figure 10.3.16. A cross section of a tank requiring an epoxy coating on its interior.

- (a) Write equations which can be used to find the surface area of the steel cap in terms of S and C . [Hint: Revolve $y = \sqrt{R^2 - x^2}$ about the y axis, $0 \leq x \leq C/2\pi$.]
- (b) Write equations for finding the surface area of the tank in terms of S , C , and H .
- (c) Determine the cost for six tanks with $H = 16$ feet, $C = 37.7$ feet, $S = 13.2$ feet, given that the coating costs \$2.10 per square foot.

41. (a) Calculate the area of the frustum shown in Fig. 10.3.17 using geometry alone. (b) Derive formula (3) using step functions.

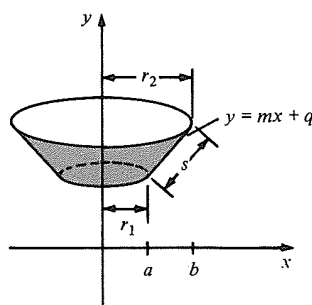


Figure 10.3.17. A line segment revolved around the y axis becomes a frustum of a cone.

10.4 Parametric Curves

Arc lengths may be found by integral calculus for curves which are not graphs of functions.

We begin this section with a study of the differential calculus of parametric curves, a topic which was introduced in Section 2.4. The arc length of a parametric curve is then expressed as an integral.

Recall from Section 2.4 that a *parametric curve* in the xy plane is specified by a pair of functions: $x = f(t)$, $y = g(t)$. The variable t , called the *parameter* of the curve, may be thought of as time; the pair $(f(t), g(t))$ then describes the path in the plane of a moving point. Many physical situations, such as the motion of the Earth about the sun and a car moving on a twisting highway, can be conveniently idealized as parametric curves.

Example 1 (a) Describe the motion of the point (x, y) if $x = \cos t$ and $y = \sin t$, for t in $[0, 2\pi]$. (b) Describe the motion of the point (t, t^3) for t in $(-\infty, \infty)$.

Solution (a) At $t = 0$, the point is at $(1, 0)$. Since $\cos^2 t + \sin^2 t = 1$, the point (x, y) satisfies $x^2 + y^2 = 1$, so it moves on the unit circle. As t increases from zero, $x = \cos t$ decreases and $y = \sin t$ increases, so the point moves in a counter-clockwise direction. Finally, since $(\cos(2\pi), \sin(2\pi)) = (1, 0)$, the point makes a full rotation after 2π units of time (see Fig. 10.4.1).
 (b) We have $x^3 = t^3 = y$, so the point is on the curve $y = x^3$. As t increases so does x , and the point moves from left to right (see Fig. 10.4.2). ▲

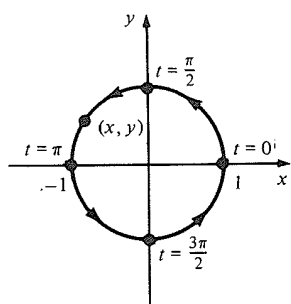


Figure 10.4.1. The point $(\cos t, \sin t)$ moves in a circle.

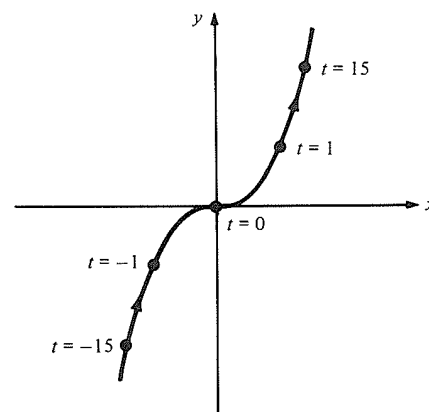


Figure 10.4.2. The motion of the point (t, t^3) .

Example 1(b) illustrates a general fact: Any curve $y = f(x)$ which is the graph of a function can be described parametrically: we set $x = t$ and $y = f(t)$. However, parametric equations can describe curves which are not the graphs of functions, like the circle in Example 1(a).

The equations

$$x = at + b, \quad y = ct + d$$

describe a straight line. To show this, we *eliminate the parameter* t in the following way. If $a \neq 0$, solve the first equation for t , getting $t = (x - b)/a$. Substituting this into the second equation gives $y = c[(x - b)/a] + d$; that is, $y = (c/a)x + (ad - bc)/a$, which is a straight line with slope c/a . If $a = 0$, we have $x = b$ and $y = ct + d$. If $c \neq 0$, then y takes all values as t varies and b is fixed, so we have the vertical line $x = b$ (which is not the graph of a function). If $c = 0$ as well as $a = 0$, then $x = b$ and $y = d$, so the graph is a “stationary” point (b, d) .

Similarly, we can see that

$$x = r \cos t + x_0, \quad y = r \sin t + y_0$$

describes a circle by writing

$$\frac{x - x_0}{r} = \cos t, \quad \frac{y - y_0}{r} = \sin t.$$

Therefore,

$$\left(\frac{x - x_0}{r}\right)^2 + \left(\frac{y - y_0}{r}\right)^2 = \cos^2 t + \sin^2 t = 1$$

or $(x - x_0)^2 + (y - y_0)^2 = r^2$, which is the equation of a circle with radius r and center (x_0, y_0) . As t varies from 0 to 2π , the point (x, y) moves once around the circle.

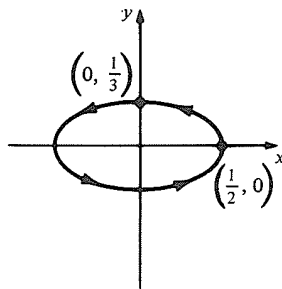


Figure 10.4.3. The parametric curve $x = \frac{1}{2} \cos t$, $y = \frac{1}{3} \sin t$ is an ellipse.

Parametric Equations of Lines and Circles

Straight line

$$\begin{aligned} x &= at + b, & -\infty < t < \infty; \\ y &= ct + d & a \text{ and } c \text{ not both zero; the line passes} \\ & & \text{through } (b, d) \text{ with slope } c/a. \end{aligned}$$

Circle

$$\begin{aligned} x &= r \cos t + x_0, & 0 \leq t \leq 2\pi; \\ y &= r \sin t + y_0, & r > 0, r = \text{radius}, (x_0, y_0) = \text{center}. \end{aligned}$$

Other curves can be written conveniently in parametric form as well. For example, $4x^2 + 9y^2 = 1$ (an ellipse) can be written as $x = \frac{1}{2} \cos t$, $y = \frac{1}{3} \sin t$. As t goes from 0 to 2π , the point moves once around the ellipse (see Fig. 10.4.3). General properties of ellipses are studied in Section 14.1.

The same geometric curve can often be represented parametrically in more than one way. For example, the line $x = at + b$, $y = ct + d$ can also be represented by

$$x = t, \quad y = \frac{ct}{a} + \frac{ad - bc}{a}$$

or by

$$x = t^3, \quad y = \frac{ct^3}{a} + \frac{ad - bc}{a}.$$

(If we used t^2 , we would get only half of the line since $t^2 \geq 0$ for all t .)

In Section 2.4 we saw that the tangent line to a parametric curve $(x, y) = (f(t), g(t))$ at the point $(f(t_0), g(t_0))$ has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t_0)}{f'(t_0)}$$

If $f'(t_0) = 0$ and $g'(t_0) \neq 0$, the tangent line is vertical; if $f'(t_0)$ and $g'(t_0)$ are both zero, the tangent line is not defined. Since the tangent line passes through $(f(t_0), g(t_0))$, we may write its equation in point-slope form:

$$y = \frac{g'(t_0)}{f'(t_0)}[x - f(t_0)] + g(t_0). \quad (1)$$

Example 2 Find the equation of the tangent line when $t = 1$ for the curve $x = t^4 + 2\sqrt{t}$, $y = \sin(t\pi)$.

Solution When $t = 1$, $x = 3$ and $y = \sin \pi = 0$. Furthermore, $dx/dt = 4t^3 + 1/\sqrt{t}$, which equals 5 when $t = 1$; $dy/dt = \pi \cos(t\pi)$, which equals $-\pi$ when $t = 1$. Thus the equation of the tangent line is, by formula (1),

$$y = -\frac{\pi}{5}(x - 3) + 0 \quad \text{or} \quad y = -\frac{\pi}{5}x + \frac{3\pi}{5}. \quad \blacktriangle$$

If a curve is given parametrically, it is natural to express its tangent line parametrically as well. To do this, we transform equation (1) to the form

$$\frac{y - g(t_0)}{g'(t_0)} = \frac{x - f(t_0)}{f'(t_0)}.$$

We can set both sides of this equation equal to t , obtaining

$$x = tf'(t_0) + f(t_0), \quad y = tg'(t_0) + g(t_0). \quad (2)$$

Equation (2) is the parametric equation for a line with slope $g'(t_0)/f'(t_0)$ if $f'(t_0) \neq 0$. If $f'(t_0) = 0$ but $g'(t_0) \neq 0$, equations (2) describe a vertical line. If $f'(t_0)$ and $g'(t_0)$ are both zero, equations (2) describe a stationary point.

It is convenient to make one more transformation of equations (2), so that the tangent line passes through (x_0, y_0) at the same time t_0 as the curve, rather than at $t = 0$. Substituting $t - t_0$ for t , we obtain the formulas

$$x = f'(t_0)(t - t_0) + f(t_0), \quad y = g'(t_0)(t - t_0) + g(t_0). \quad (3)$$

Notice that the functions in formulas (3) which define the tangent line to a curve are exactly the *linear approximations* to the functions defining the curve itself. If we think of $(x, y) = (f(t), g(t))$ as the position of a moving particle, then the tangent line at t_0 is the path which the particle would follow if, at time t_0 , all constraining forces were suddenly removed and the particle were allowed to move freely in a straight line. (See Fig. 10.4.4.)

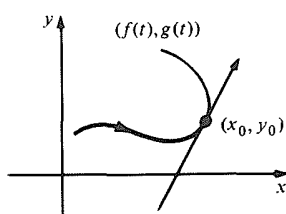


Figure 10.4.4. If the forces constraining a particle to the curve $(f(t), g(t))$ are removed at t_0 , then the particle will follow the tangent line at t_0 .

Example 3 A child is whirling an object on a string, letting out string at a constant rate, so that the object follows the path $x = (1 + t)\cos t$, $y = (1 + t)\sin t$.

- Sketch the path for $0 \leq t \leq 4\pi$.
- At $t = 4\pi$ the string breaks, so that the object follows its tangent line. Where is the object at $t = 5\pi$?

Solution (a) By plotting some points and thinking of (x, y) as moving in an ever enlarging circle, we obtain the sketch in Fig. 10.4.5.

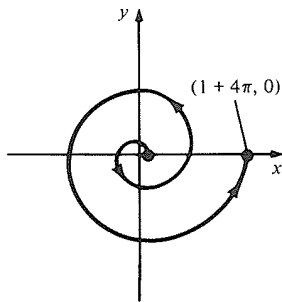


Figure 10.4.5. The curve $((1+t)\cos t, (1+t)\sin t)$ for t in $[0, 4\pi]$.

(b) We differentiate:

$$f'(t) = \frac{dx}{dt} = (1+t)(-\sin t) + \cos t, \quad \text{and}$$

$$g'(t) = \frac{dy}{dt} = (1+t)\cos t + \sin t.$$

When $t_0 = 4\pi$, we have

$$f(t_0) = (1+4\pi)\cos 4\pi = 1+4\pi, \quad \text{and} \quad g(t_0) = (1+4\pi)\sin 4\pi = 0,$$

$$f'(t_0) = (1+4\pi) \cdot 0 + 1 = 1, \quad \text{and} \quad g'(t_0) = (1+4\pi) \cdot 1 + 0 = 1+4\pi.$$

By formulas (3), the equations of the tangent line are

$$x = t - 4\pi + (1+4\pi), \quad y = (1+4\pi)(t - 4\pi) + 0.$$

When $t = 5\pi$, the object, which is now following the tangent line, is at $x = 1 + 5\pi \approx 16.71$, $y = (1 + 4\pi)\pi \approx 42.62$. ▲

Tangents to Parametric Curves

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of a curve C . If f and g are differentiable at t_0 , and $f'(t_0)$ and $g'(t_0)$ are not both zero, then the tangent line to C at t_0 is defined by the parametric equations:

$$x = f'(t_0)(t - t_0) + f(t_0), \quad y = g'(t_0)(t - t_0) + g(t_0).$$

If $f'(t_0) \neq 0$, this line has slope $g'(t_0)/f'(t_0)$, and its equation can be written as

$$y = \frac{g'(t_0)}{f'(t_0)} [x - f(t_0)] + g(t_0).$$

If $f'(t_0) = 0$ and $g'(t_0) \neq 0$, the line is vertical; its equation is

$$x = f(t_0).$$

Example 4 Consider the curve $x = t^3 - t$, $y = t^2$.

- Plot the points corresponding to $t = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
- Using these points, together with the behavior of the functions t^2 and $t^3 - t$, sketch the entire curve.
- Find the slope of the tangent line at the points corresponding to $t = 1$ and $t = -1$.
- Eliminate the parameter t to obtain an equation in x and y for the curve.

Solution (a) We begin by making a table:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$x = t^3 - t$	-6	0	$\frac{3}{8}$	0	$-\frac{3}{8}$	0	6
$y = t^2$	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

These points are plotted in Fig. 10.4.6. The number next to each point is the corresponding value of t . Notice that the point $(0, 1)$ occurs for $t = -1$ and $t = 1$.

(b) We plot x and y against t in Fig. 10.4.7. From the graph of x against t , we conclude that as t goes from $-\infty$ to ∞ , the point comes in from the left, reverses direction for a while, and then goes out to the right. From the graph

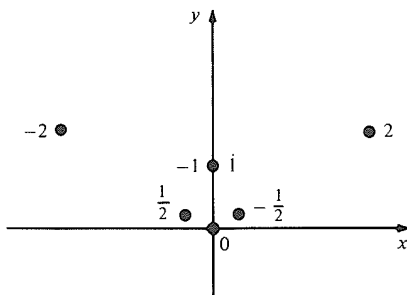


Figure 10.4.6. Some points on the curve $(t^3 - t, t^2)$.

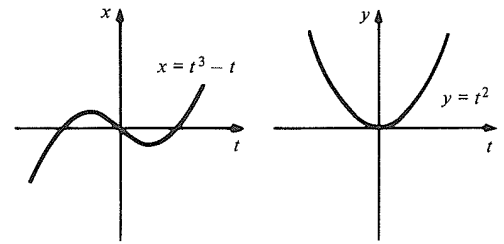


Figure 10.4.7. The graphs of x and y plotted separately against t .

of y against t , we see that the point descends for $t < 0$, reaches the bottom at $y = 0$ when $t = 0$, and then ascends for $t > 0$. Putting this information together with the points we have plotted, we sketch the curve in Fig. 10.4.8.

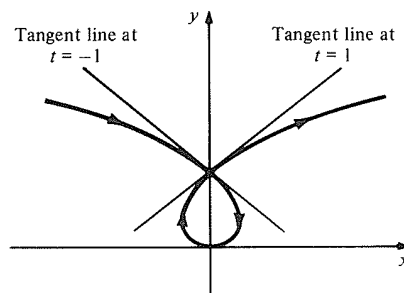


Figure 10.4.8. The parametric curve $(t^3 - t, t^2)$.

(c) The slope of the tangent line at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}.$$

When $t = -1$, the slope is -1 ; when $t = 1$, the slope is 1 . (See Fig. 10.4.8.)

(d) We can eliminate t by solving the second equation for t to get $t = \pm\sqrt{y}$ and substituting in the first to get $x = \pm(y^{3/2} - y^{1/2})$. To obtain an equation without fractional powers, we square both sides. The result is $x^2 = y(y - 1)^2$, or $x^2 = y^3 - 2y^2 + y$. In this form, it is not so easy to predict the behavior of the curve, particularly at the “double point” $(0, 1)$. ▲

Example 5 (a) Sketch the curve $x = t^3$, $y = t^2$. (b) Find the equation of the tangent line at $t = 1$. (c) What happens at $t = 0$?

Solution (a) Eliminating the parameter t , we have $y = x^{2/3}$. The graph has a cusp at the origin, as in Fig. 10.4.9. (Cusps were discussed in Section 3.4.)
 (b) When $t = 1$, we have $x = t^3 = 1$, $y = t^2 = 1$, $dx/dt = 3t^2 = 3$, and $dy/dt = 2t = 2$, so the tangent line is given by

$$x = 3(t - 1) + 1, \quad y = 2(t - 1) + 1.$$

It has slope $\frac{2}{3}$. (You can also see this by differentiating $y = x^{2/3}$ and setting $x = 1$.)

(c) When $t = 0$, we have $dx/dt = 0$ and $dy/dt = 0$, so the tangent line is not defined. ▲

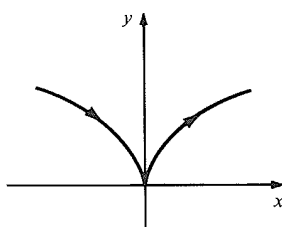


Figure 10.4.9. The curve (t^3, t^2) has a cusp at the origin.

Example 6 Consider the curve $x = \cos 3t$, $y = \sin t$. Find the points where the tangent is horizontal and those where it is vertical. Use this information to sketch the curve.

Solution The tangent line is vertical when $dx/dt = 0$ and horizontal when $dy/dt = 0$. (If both are zero, there is no tangent line.)

We have $dx/dt = -3 \sin 3t$, which is zero when $t = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$ (the curve repeats itself when t reaches 2π); $dy/dt = \cos t$, which is zero when $t = \pi/2$ or $3\pi/2$. We make a table:

t	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$x = \cos 3t$	1	-1	0	1	-1	1	0	-1
$y = \sin t$	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$
Tangent	vert	vert	hor	vert	vert	vert	hor	vert

Using the fact that $\sqrt{3}/2 \approx 0.866$, we sketch this information in Fig. 10.4.10. Connecting these points in the proper order with a smooth curve, we obtain Fig. 10.4.11. This curve is an example of a *Lissajous figure* (see Review Exercise 93 and 94 at the end of this chapter). ▲

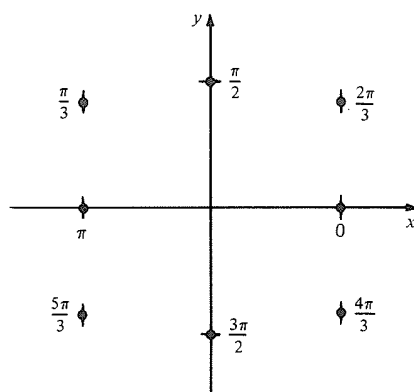


Figure 10.4.10. Points on the curve $(\cos 3t, \sin t)$ with horizontal and vertical tangent.

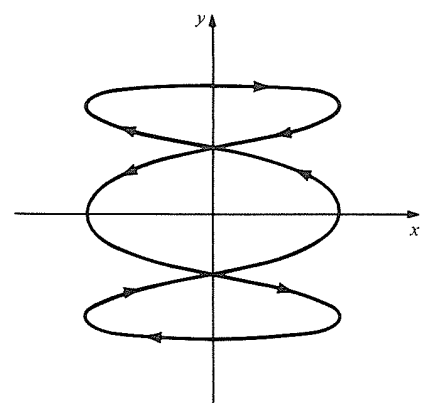


Figure 10.4.11. The curve $(\cos 3t, \sin t)$ is an example of a Lissajous figure.

What is the length of the curve given by $(x, y) = (f(t), g(t))$ for $a \leq t \leq b$? To get a formula in terms of f and g , we begin by considering the case in which the point $(f(t), g(t))$ moves along the graph of a function $y = h(x)$; that is, $g(t) = h(f(t))$.

If $f(a) = \alpha$ and $f(b) = \beta$, the length of the curve is $\int_{\alpha}^{\beta} \sqrt{1 + [h'(x)]^2} dx$ by formula (1) of Section 10.3. If we change variables from x to t in this integral, we have $dx = f'(t) dt$, so the length is

$$\int_a^b \sqrt{1 + [h'(f(t))]^2} f'(t) dt.$$

To eliminate the function h from this formula, we may apply the chain rule to

$g(t) = h(f(t))$, getting $g'(t) = h'(f(t)) \cdot f'(t)$. Solving for $h'(f(t))$ and substituting in the integral gives

$$\int_a^b \sqrt{1 + \left[\frac{g'(t)}{f'(t)} \right]^2} f'(t) dt$$

or

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (4)$$

Formula (4) involves only the information contained in the parametrization. Since we can break up any reasonably behaved parametric curve into segments, each of which is the graph of a function or a vertical line (for which we see that equation (4) gives the correct length, since $f'(t) \equiv 0$), we conclude that equation (4) ought to be valid for any parametric curve.

Equation (4) may be derived using infinitesimals in the following way. Refer to Fig. 10.4.12 and note that $ds^2 = dx^2 + dy^2$. Thus

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Integrating from $t = a$ to $t = b$ reproduces formula (4).

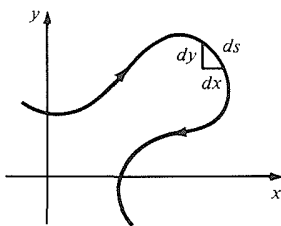


Figure 10.4.12. Finding the length of a parametric curve.

Length of a Parametric Curve

Suppose that a parametric curve C is given by continuous functions $x = f(t)$, $y = g(t)$, for $a \leq t \leq b$, and that $f'(t)$ and $g'(t)$ exist and are continuous, except possibly for finitely many points. Then the length of C is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Example 7 Find the length of the circle of radius 2 which is given by the parametric equations $x = 2 \cos t + 3$, $y = 2 \sin t + 4$, $0 \leq t \leq 2\pi$.

Solution We find $f'(t) = dx/dt = -2 \sin t$ and $g'(t) = dy/dt = 2 \cos t$, so

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{4 \sin^2 t + 4 \cos^2 t} dt \\ &= \int_0^{2\pi} 2 \sqrt{\sin^2 t + \cos^2 t} dt = \int_0^{2\pi} 2 dt = 4\pi \end{aligned}$$

(which equals 2π times the radius). $\blacktriangle$

Example 8 Find the length of (a) $x = t^8$, $y = t^4$ on $[1, 3]$ and (b) $x = t \sin t$, $y = t \cos t$ on $[0, 4\pi]$.

Solution (a) We are given $x = f(t) = t^8$ and $y = g(t) = t^4$ on $[1, 3]$. The length is

$$L = \int_1^3 \sqrt{(8t^7)^2 + (4t^3)^2} dt = \int_1^3 4t^3 \sqrt{(2t^4)^2 + 1} dt.$$

Letting $u = 2t^4$, we have the length

$$L = \frac{1}{2} \int_2^{162} \sqrt{u^2 + 1} \, du.$$

Making the substitution $u = \tan \theta$, $du = \sec^2 \theta \, d\theta$, we get

$$\int \sqrt{u^2 + 1} \, du = \int \sec^3 \theta \, d\theta = I.$$

(see Fig. 10.4.13). Integrating by parts,

$$I = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta \, d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta.$$

Since

$$\int \sec \theta \, d\theta = \int \sec \theta \cdot \frac{\tan \theta + \sec \theta}{\tan \theta + \sec \theta} \, d\theta = \ln |\tan \theta + \sec \theta|,$$

we get $I = \sec \theta \tan \theta - I + \ln |\tan \theta + \sec \theta| + C$. Thus

$$I = \int \sec^3 \theta \, d\theta = \frac{\sec \theta \tan \theta + \ln |\tan \theta + \sec \theta|}{2} + C.$$

(Compare Example 3, Section 10.3.) Putting everything together in terms of u ,

$$L = \frac{1}{4} \left[\sqrt{u^2 + 1} \cdot u + \ln |u + \sqrt{u^2 + 1}| \right]_2^{162} \approx 6561.1.$$

(b) If $x = t \sin t$ and $y = t \cos t$, $dx/dt = \sin t + t \cos t$ and $dy/dt = \cos t - t \sin t$. Therefore,

$$\begin{aligned} \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 &= \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t + \cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t \\ &= 1 + t^2. \end{aligned}$$

Thus, using (a), the length is

$$\begin{aligned} \int_0^{4\pi} \sqrt{1 + t^2} \, dt &= \frac{1}{2} \left[t\sqrt{1 + t^2} + \ln |t + \sqrt{1 + t^2}| \right]_0^{4\pi} \\ &= \frac{1}{2} \left[4\pi\sqrt{1 + 16\pi^2} + \ln(4\pi + \sqrt{1 + 16\pi^2}) \right] \approx 80.8 \quad \blacktriangle \end{aligned}$$

Example 9 Show that if $x = f(t)$ and $y = g(t)$ is any curve with $(f(0), g(0)) = (0, 0)$ and $(f(1), g(1)) = (0, a)$, then the length of the curve for $0 \leq t \leq 1$ is at least equal to a . What can you say if the length is exactly equal to a ?

Solution It is evident that $[g'(t)]^2 \leq [f'(t)]^2 + [g'(t)]^2$, so

$$g'(t) \leq \sqrt{[f'(t)]^2 + [g'(t)]^2}.$$

Integrating from 0 to 1, we have

$$\int_0^1 g'(t) \, dt \leq \int_0^1 \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt.$$

By the fundamental theorem of calculus, the left-hand side is equal to $g(1) - g(0) = a - 0 = a$; the right-hand side is the length L of the curve, so we have $a \leq L$. If $a = L$, the integrands must be equal; that is, $g'(t) = \sqrt{f'(t)^2 + g'(t)^2}$, which is possible only if $f'(t)$ is identically zero; that is, $f(t)$ is constant. Since $f(0) = f(1) = 0$, we must have $f(t)$ identically zero; that is, the point (x, y) stays on the y axis.

We have shown that the shortest curve between the points $(0, 0)$ and $(0, a)$ is the straight line segment which joins them. $\blacktriangle$

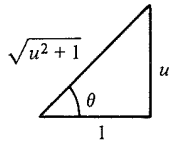


Figure 10.4.13. If $\tan \theta = u$, $\sqrt{u^2 + 1} = \sec \theta$.

Given a point moving according to $x = f(t)$, $y = g(t)$, the integral

$$D(t) = \int_a^t \sqrt{[f'(s)]^2 + [g'(s)]^2} ds$$

is the distance (along the curve) travelled by the point between time a and time t . The derivative $D'(t)$ should then represent the speed of the point along the curve. By the fundamental theorem of calculus (alternative version), we have

$$D'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2}.$$

Speed

Let a point move according to the equations $x = f(t)$, $y = g(t)$. Then the speed of the point at time t is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Suppose that an object is constrained to move along the curve $x = f(t)$, $y = g(t)$ and that at time t_0 the constraining forces are removed, so the particle continues along the tangent line

$$x = f'(t_0)(t - t_0) + f(t_0), \quad y = g'(t_0)(t - t_0) + g(t_0).$$

At time $t_0 + \Delta t$, the particle is at $(f'(t_0)\Delta t + f(t_0), g'(t_0)\Delta t + g(t_0))$, which is at distance $\sqrt{f'(t_0)^2 + g'(t_0)^2} \Delta t$ from $(f(t_0), g(t_0))$. Thus the distance travelled in time Δt after the force is removed is equal to Δt times the speed at t_0 , so we have another justification of our formula for the speed.

Example 10 A particle moves around the elliptical track $4x^2 + y^2 = 4$ according to the equations $x = \cos t$, $y = 2 \sin t$. When is the speed greatest? Where is it least?

Solution The speed is

$$\sqrt{\left[\frac{d(\cos t)}{dt}\right]^2 + \left[\frac{d(2 \sin t)}{dt}\right]^2} = \sqrt{\sin^2 t + 4 \cos^2 t} = \sqrt{1 + 3 \cos^2 t}.$$

Without any further calculus, we observe that the speed is greatest when $\cos t = \pm 1$; that is, $t = 0, \pi, 2\pi$, and so forth. The speed is least when $\cos t = 0$; that is, $t = \pi/2, 3\pi/2, 5\pi/2$, and so on. ▲

Example 11

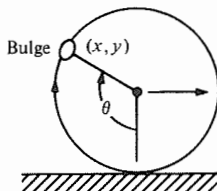


Figure 10.4.14. Investigate how a bulge on a tire moves.

The position (x, y) of a bulge in a bicycle tire as it rolls down the street can be parametrized by the angle θ shown in Fig. 10.4.14. Let the radius of the tire be a . It can be verified by methods of plane trigonometry that $x = a\theta - a \sin \theta$, $y = a - a \cos \theta$. (This curve is called a *cycloid*.)

- Find the distance travelled by the bulge for $0 \leq \theta \leq 2\pi$, using the identity $1 - \cos \theta = 2 \sin^2(\theta/2)$. This distance is greater than $2\pi a$ (distance the tire rolls).
- Draw a figure for one arch of the cycloid, and superimpose the circle of radius a with center at $(\pi a, a)$, together with the line segment $0 \leq x \leq 2\pi a$ on the x axis. Show that the three enclosed areas are each πa^2 .

Solution (a) The distance d is the arc length of the cycloid for $0 \leq \theta \leq 2\pi$. Thus,

$$\begin{aligned} d &= \int_0^{2\pi} \sqrt{(a - a \cos \theta)^2 + (a \sin \theta)^2} d\theta \\ &= a \int_0^{2\pi} \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = a \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos \theta} d\theta \\ &= a \sqrt{2} \int_0^{2\pi} \sqrt{2} \sin\left(\frac{\theta}{2}\right) d\theta = -4a \cos(\theta/2) \Big|_0^{2\pi} = -4a(-1 - 1) = 8a. \end{aligned}$$

(b) Refer to Fig. 10.4.15. The total area beneath the arch is

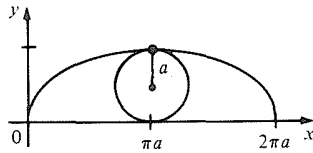


Figure 10.4.15. One arch of the cycloid.

$$\begin{aligned} A &= \int_0^{2\pi} y dx = \int_0^{2\pi} y \frac{dx}{d\theta} d\theta \\ &= \int_0^{2\pi} a^2 (1 - \cos \theta)(1 - \cos \theta) d\theta = a^2 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\ &= a^2 \left[(\theta - 2 \sin \theta) \Big|_0^{2\pi} + \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) d\theta \right] \\ &= a^2 \left[2\pi + \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi} \right] = 3\pi a^2. \end{aligned}$$

The area of the circle is πa^2 , so by symmetry each of the other two congruent regions also has area πa^2 . $\blacktriangle$

Exercises for Section 10.4

For the parametric curves in Exercises 1–4, sketch the curve and find an equation in x and y by eliminating the parameter.

1. $x = 4t - 1$, $y = t + 2$.
2. $x = 2t + 1$, $y = t^2$.
3. $x = \cos \theta + 1$, $y = \sin \theta$.
4. $x = \sin \theta$, $y = \cos \theta - 3$.

Find a parametric representation for each of the curves in Exercises 5–12.

5. $2x^2 + y^2 = 1$.
6. $16x^2 + 9y^2 = 1$.
7. $4xy = 1$.
8. $y = 3x - 2$.
9. $y = x^3 + 1$.
10. $3x^2 - y^2 = 1$.
11. $y = \cos(2x)$.
12. $y^2 = x + x^2$.

Find the equation of the tangent line to each of the curves in Exercises 13–16 at the given point.

13. $x = \frac{1}{2}t^2 + t$, $y = t^{2/3}$; $t_0 = 1$.
14. $x = 1/t$, $y = \sqrt{t+1}$; $t_0 = 2$.
15. $x = \cos^2(t/2)$, $y = \frac{1}{2}\sin t$; $t_0 = \pi/2$.
16. $x = \theta - \sin \theta$, $y = 1 - \cos \theta$; $\theta_0 = \pi/4$.

17. A bead is sliding on a wire, having position $x = (2 - 3t)^2$, $y = 2 - 3t$ at time t . If the bead flies off the wire at time $t = 1$, where is it when $t = 3$?

18. A piece of mud on a bicycle tire is following the cycloid $x = 6t - 3 \sin 2t$, $y = 3 - 3 \cos 2t$. At time $t = \pi/2$, the mud becomes detached from the tire. Along what line is it moving? (Ignore gravity.)

Sketch each of the parametric curves in Exercises 19–22, find an equation in x and y by eliminating the parameter, and find the points where the tangent line is horizontal or vertical.

19. $x = t^2$, $y = \cos t$. (What happens at $t = 0$?)
20. $x = \pi/2 - s$, $y = 2 \sin 2s$.
21. $x = \cos 2t$, $y = \sin t$.
22. $(x, y) = (\cos t, \sin 2t)$.

23. Find the length of $x = t^2$, $y = t^3$ on $[0, 1]$.
24. Find the length of the curve given by $x = \frac{1}{2} \sin 2t$, $y = 3 + \cos^2 t$ on $[0, \pi]$.
25. Find the length of the curve (t^2, t^4) on $0 \leq t \leq 1$.
26. Find the length of the parametric curve $(e^t(\cos t)\sqrt{t^3+1}, 2e^t(\cos t)\sqrt{t^3+1})$ on $[0, 1]$.
27. Show that if $x = a \cos t + b$ and $y = a \sin t + d$: (a) the speed is constant; (b) the length of the curve on $[t_0, t_1]$ is equal to the speed times the elapsed time $(t_1 - t_0)$.
28. An object moves from left to right along the curve $y = x^{3/2}$ at constant speed. If the point is at $(0, 0)$ at noon and at $(1, 1)$ at 1:00 P.M., where is it at 1:30 P.M.?
29. Consider the parametrized curve $x = 2 \cos \theta$, $y = \theta - \sin \theta$.
(a) Find the equation of the tangent line at $\theta = \pi/2$.
(b) Sketch the curve.
(c) Express the length of the curve on $[0, \pi]$ as an integral.

30. Show that if

$$\frac{dx}{dt} \frac{d^2x}{dt^2} = - \frac{dy}{dt} \frac{d^2y}{dt^2},$$

then the speed of the curve $x = f(t)$, $y = g(t)$ is constant.

31. A particle travels a path in space with speed $s(t) = \sin^2(\pi t) + \tan^4(\pi t) \sec^2(\pi t)$. Find the distance $\int_0^{10} s(t) dt$ travelled in the first ten seconds.
32. A car loaded with skiers climbs a hill to a ski resort, constantly changing gears due to variations in the incline. Assume, for simplicity, that the motion of the auto is planar: $x = x(t)$, and $y = y(t)$, $0 \leq t \leq T$. Let $s(t)$ be the distance travelled along the road at time t (Fig. 10.4.16).

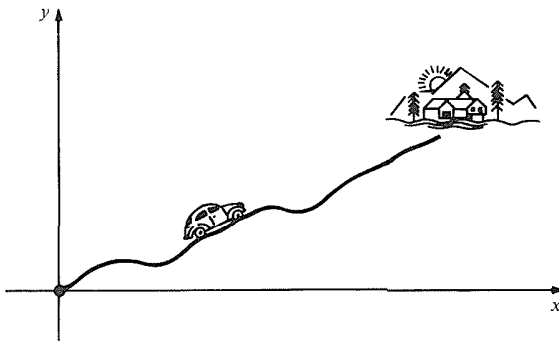


Figure 10.4.16. A car on its way to a ski cabin.

- (a) The value $s(10)$ is the difference in the odometer readings from $t = 0$ to $t = 10$. Explain.
- (b) The value $s'(t)$ is the speedometer reading at time t . Explain.
- (c) The value $y'(t)$ is the rate of change in altitude, while $x'(t)$ is the rate of horizontal approach to the resort. Explain.
- (d) What is the average rate of vertical ascent? What is the average speed for the trip?
33. A child walks with speed k from the center of a merry-go-round to its edge, while the equipment rotates counterclockwise with constant angular speed ω . The motion of the child relative to the ground is $x = kt \cos \omega t$, $y = kt \sin \omega t$.
- (a) Find the velocities $\dot{x} = dx/dt$, $\dot{y} = dy/dt$.
- (b) Determine the speed.
- (c) The child experiences a Coriolis force opposite to the direction of rotation, tangent to the edge of the merry-go-round. The magnitude of this force is the mass m of the child times the factor $\sqrt{\ddot{x}(0)^2 + \ddot{y}(0)^2}$, where $\ddot{x} = d^2x/dt^2$. Find this force.
- ★34. (a) Find a parametric curve $x = f(t)$, $y = g(t)$ passing through the points $(1, 1)$, $(2, 2)$, $(4, 2)$, $(5, 1)$, $(3, 0)$, and $(1, 1)$ such that the functions f and g are both piecewise linear and the curve is a polygon whose vertices are the given points in the given order.
- (b) Compute the length of this curve by formula (4) and then by elementary geometry. Compare the results.
- (c) What is the area of the surface obtained by revolving the given curve about the y axis?
- ★35. At each point (x_0, y_0) of the parabola $y = x^2$, the tangent line is drawn and a point is marked on this line at a distance of 1 unit from (x_0, y_0) to the right of (x_0, y_0) .
- (a) Describe the collection of points thus obtained as a parametrized curve.
- (b) Describe the collection of points thus obtained in terms of a relation between x and y .
- ★36. If $x = t$ and $y = g(t)$, show that the points where the speed is maximized are points of inflection of $y = g(x)$.
- ★37.⁵ (a) Looking at a map of the United States, estimate the length of the coastline of Maine.
- (b) Estimate the same length by looking at a map of Maine.
- (c) Suppose that you used detailed local maps to compute the length of the coastline of Maine. How would the results compare with that obtained in part (b)?
- (d) What is the "true" length of the coastline of Maine?
- (e) What length for the coastline can you find given in an atlas or almanac?
- ★38. On a movie set, an auto races down a street. A follow-spot lights the action from 20 meters away, keeping a constant distance from the auto in order to maintain the same reflected light intensity for the camera. The follow-spot location (x, y) is the pursuit curve
- $$x = t - 20 \operatorname{sech}\left(\frac{t}{20}\right), \quad y = 20 \operatorname{sech}\left(\frac{t}{20}\right),$$
- called a *tractrix*. Graph it.

⁵ For further information on the ideas in this exercise, see B. Mandelbrot, *Fractals: Form, Chance and Dimension*, Freeman, New York (1977).

10.5 Length and Area in Polar Coordinates

Some length and area problems are most easily solved in polar coordinates.

The formula $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$ for the length of a parametric curve can be applied to the curve $r = f(\theta)$ in polar coordinates if we take the parameter to be θ in place of t . We write:

$$x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Suppose that θ runs from α to β (see Fig. 10.5.1). By formula (4) of Section 10.4, the length is

$$\int_{\alpha}^{\beta} \sqrt{[f'(\theta) \cos \theta - f(\theta) \sin \theta]^2 + [f'(\theta) \sin \theta + f(\theta) \cos \theta]^2} d\theta$$

which simplifies to

$$\int_{\alpha}^{\beta} \sqrt{[f'(\theta)]^2 + [f(\theta)]^2} d\theta.$$

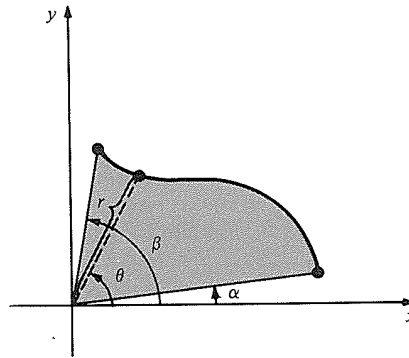


Figure 10.5.1. The length of the curve is

$$\int_{\alpha}^{\beta} \sqrt{(dr/d\theta)^2 + r^2} d\theta.$$

Arc Length in Polar Coordinates

The length of the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, is given by

$$L = \int_{\alpha}^{\beta} \sqrt{f'(\theta)^2 + f(\theta)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta. \quad (1)$$

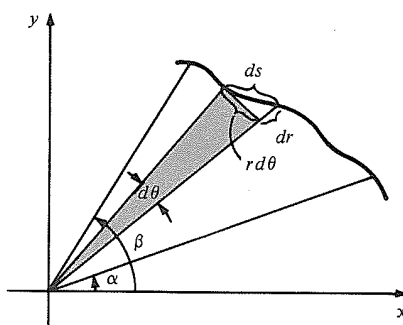
One can obtain the same formula by an infinitesimal argument, following Fig. 10.5.2. By Pythagoras' theorem, $ds^2 = dr^2 + (r d\theta)^2$, or $ds = \sqrt{dr^2 + r^2 d\theta^2}$. If we use $dr = f'(\theta) d\theta$, this becomes

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 d\theta^2 + r^2 d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta,$$

so

$$L = \int_{\alpha}^{\beta} ds = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta.$$

Figure 10.5.2. The infinitesimal element of arc length ds equals $\sqrt{dr^2 + r^2 d\theta^2}$.



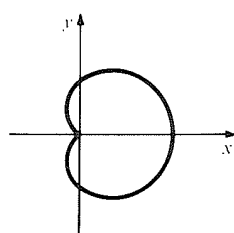
Example 1 Find the length of the curve $r = 1 - \cos \theta$, $0 \leq \theta \leq 2\pi$.

Solution We find $dr/d\theta = \sin \theta$, so by equation (1),

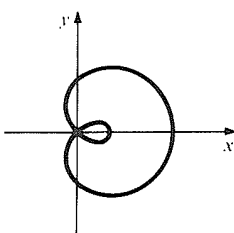
$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\sin^2 \theta + (1 - \cos \theta)^2} d\theta = \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} d\theta \\ &= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos \theta} d\theta = \sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2 \frac{\theta}{2}} d\theta \\ &= 2 \int_0^{2\pi} \sin \frac{\theta}{2} d\theta = 4 \int_0^{\pi} \sin u du \quad \left(u = \frac{\theta}{2}\right) \\ &= 4(-\cos u)|_0^{\pi} = 8. \quad \blacktriangle \end{aligned}$$

Example 2 Find the length of the cardioid $r = 1 + \cos \theta$ ($0 \leq \theta \leq 2\pi$).

Solution (This curve is sketched in the accompanying figure.) The length is



(a)



(b)

(a) $r = 1 + \cos \theta$
("cardioid");

(b) $r = 1 + 2 \cos \theta$
("limaçon").

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2 \cos \theta} d\theta. \end{aligned}$$

This can be simplified, by the half-angle formula $\cos^2(\theta/2) = (1 + \cos \theta)/2$, to

$$L = \int_0^{2\pi} 2 \cos \frac{\theta}{2} d\theta = 0.$$

Something is wrong here! We forgot that $\cos(\theta/2)$ can be negative, while the square root $\sqrt{2 + 2 \cos \theta}$ must be positive; i.e.,

$$\sqrt{2 + 2 \cos \theta} = \sqrt{4 \cos^2 \frac{\theta}{2}} = 2 \left| \cos \frac{\theta}{2} \right|.$$

The correct evaluation of L is as follows:

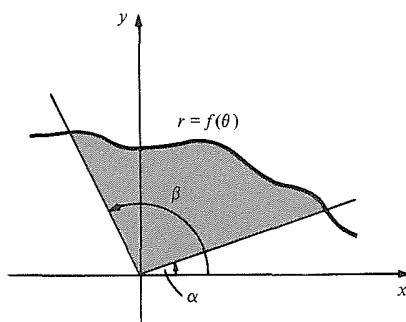
$$L = \int_0^{2\pi} 2 \left| \cos \frac{\theta}{2} \right| d\theta = \int_0^{\pi} 2 \cos \frac{\theta}{2} d\theta - \int_{\pi}^{2\pi} 2 \cos \frac{\theta}{2} d\theta$$

since $\cos(\theta/2) > 0$ on $(0, \pi)$ and $\cos(\theta/2) < 0$ on $(\pi, 2\pi)$. Thus

$$L = 4 \sin \frac{\theta}{2} \Big|_0^{\pi} - 4 \sin \frac{\theta}{2} \Big|_{\pi}^{2\pi} = 4(1 - 0) - 4(0 - 1) = 8. \quad \blacktriangle$$

The curve expressed in polar coordinates by the equation $r = f(\theta)$, together with the rays $\theta = \alpha$ and $\theta = \beta$, encloses a region of the type shown (shaded) in Fig. 10.5.3. We call this the region *inside* the graph of f on $[\alpha, \beta]$.

Figure 10.5.3. The region inside the graph $r = f(\theta)$ on $[\alpha, \beta]$ is shaded.



We wish to find a formula for the area of such a region as an integral involving the function f . We begin with the simplest case, in which f is a constant function $f(\theta) = k$. The region inside the curve $r = k$ on $[\alpha, \beta]$ is then a circular sector with radius k and angle $\beta - \alpha$ (see Fig. 10.5.4). The area is $(\beta - \alpha)/2\pi$ times the area πk^2 of a circle of radius k , or $\frac{1}{2}k^2(\beta - \alpha)$. We can express this as the integral $\int_{\alpha}^{\beta} \frac{1}{2}f(\theta)^2 d\theta$.

If f is a step function, with $f(\theta) = k_i$ on (θ_{i-1}, θ_i) , then the region inside the graph of f is of the type shown in Fig. 10.5.5. Its area is equal to the sum of the areas of the individual sectors, or

$$\sum_{i=1}^n \frac{1}{2} k_i^2 \Delta\theta_i = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 d\theta.$$

By approximating f with step functions, we conclude that the same formula holds for general f .

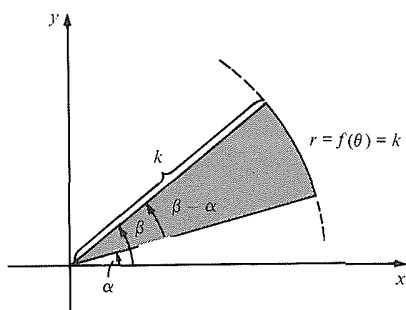


Figure 10.5.4. The area of the sector is $\frac{1}{2}k^2(\beta - \alpha)$.

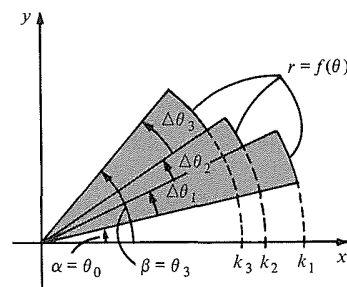


Figure 10.5.5. The area of the shaded region is $\sum \frac{1}{2}k_i^2 \Delta\theta_i$.

Area in Polar Coordinates

The area of the region enclosed by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

This formula can also be obtained by an infinitesimal argument. Indeed, the area dA of the shaded triangle in Fig. 10.5.2 is $\frac{1}{2}(\text{base}) \times (\text{height})$

$= \frac{1}{2}(r d\theta)r = \frac{1}{2}r^2 d\theta$, so the area inside the curve is

$$\int_{\alpha}^{\beta} dA = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

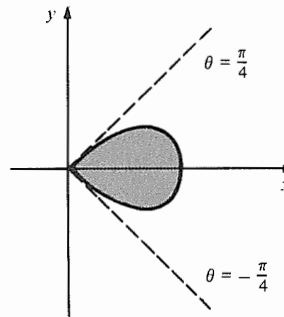
which agrees with the formula in the preceding box.

Example 3 Find the area enclosed by one petal of the four-petaled rose $r = \cos 2\theta$ (see Fig. 5.6.3).

Solution The petal shown in Fig. 10.5.6 is enclosed by the arc $r = \cos 2\theta$ and the rays $\theta = -\pi/4$ and $\theta = \pi/4$. Notice that the rays do not actually appear in the boundary of the figure, since the radius $r = \cos(\pm \pi/2)$ is zero there. The area is given by $\frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta$. By the half-angle formula this is

$$\frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1 + \cos 4\theta}{2} d\theta = \frac{1}{4} \left(\theta + \frac{\sin 4\theta}{4} \right) \Big|_{-\pi/4}^{\pi/4} = \frac{\pi}{8} \cdot \blacktriangle$$

Figure 10.5.6. One leaf of the four-petaled rose $r = \cos 2\theta$.



Example 4 Find the area enclosed by the cardioid $r = 1 + \cos \theta$ (see Fig. 5.6.6).

Solution The area enclosed is defined by $r = 1 + \cos \theta$ and the full range $0 \leq \theta \leq 2\pi$, so

$$A = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta.$$

Again using the half-angle formula,

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\cos \theta + \frac{\cos 2\theta}{2} \right) d\theta \\ &= \frac{1}{2} \left[\frac{3\theta}{2} + 2\sin \theta + \frac{\sin 2\theta}{4} \right] \Big|_0^{2\pi} = \frac{3\pi}{2} \cdot \blacktriangle \end{aligned}$$

Example 5 Find a formula for the area between two curves in polar coordinates.

Solution Suppose $r = f(\theta)$ and $r = g(\theta)$ are the two curves with $f(\theta) \geq g(\theta) > 0$. We are required to find a formula for the shaded area in Fig. 10.5.7. The area is just the difference between the areas for f and g ; that is,

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)^2 - g(\theta)^2] d\theta. \blacktriangle$$

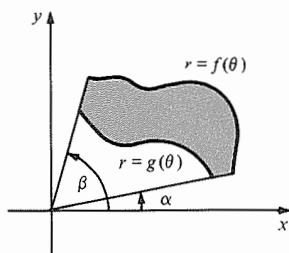
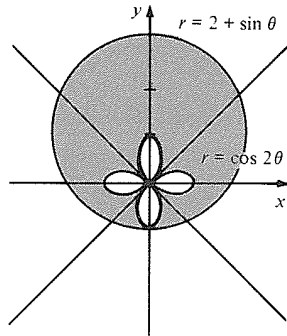


Figure 10.5.7. The area of the shaded region is $\frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)^2 - g(\theta)^2] d\theta$.

Example 6 Sketch and find the area of the region between the curves $r = \cos 2\theta$ and $r = 2 + \sin \theta$, $0 \leq \theta \leq 2\pi$.

Solution The curves are sketched in Fig. 10.5.8. To do this, we plotted points for θ at multiples of $\pi/4$ and then noted whether r was increasing or decreasing on each of the intervals between these θ values. To find the shaded area, we must be careful because of the sign changes of $g(\theta) = \cos 2\theta$. The inner loops are described in the following way by positive functions:



$$g(\theta) = \begin{cases} \cos 2\theta, & 0 \leq \theta \leq \frac{\pi}{4} \\ -\cos 2\theta, & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ \cos 2\theta, & \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4} \\ -\cos 2\theta, & \frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4} \\ \cos 2\theta, & \frac{7\pi}{4} \leq \theta \leq 2\pi \end{cases}$$

In fact, we are lucky because in the formula in Example 5, $g(\theta)$ is squared anyway, so the shaded area is simply

$$\begin{aligned} & \frac{1}{2} \int_0^{2\pi} [(2 + \sin \theta)^2 - \cos^2 2\theta] d\theta \\ &= \frac{1}{2} \left[\int_0^{2\pi} 4 d\theta + \int_0^{2\pi} 4 \sin \theta d\theta + \int_0^{2\pi} \sin^2 \theta d\theta - \int_0^{2\pi} \cos^2 2\theta d\theta \right] \\ &= \frac{1}{2} [8\pi + 0 + \pi - \pi] = 4\pi. \blacktriangle \end{aligned}$$

Exercises for Section 10.5

Find the length of the curves in Exercises 1–4.

- $r = 3(1 + \sin \theta)$; $0 \leq \theta \leq 2\pi$.
- $r = 1/(\cos \theta + \sin \theta)$; $0 \leq \theta \leq \pi/2$.
- $r = 4\theta^2$; $0 \leq \theta \leq 3$.
- $r = 8\theta^2$; $0 \leq \theta \leq 1$.

Sketch and find the area of the region bounded by the curves in Exercises 5–10.

- $r = 3 \sin \theta$; $0 \leq \theta \leq \pi$.
- $r = 2(1 + \sin \theta)$; $0 \leq \theta \leq 2\pi$.
- $r = \theta$; $0 \leq \theta \leq 3\pi/2$.
- $r = \theta \cos(\theta^3)$; $0 \leq \theta \leq \pi/4$.
- $r = 4 + \sin \theta$; $0 \leq \theta \leq 2\pi$.
- $r = \theta + \sin 4\theta$; $\pi/4 \leq \theta \leq \pi$. [Hint: Find the critical points of r .]

- Check the arc length formula in polar coordinates for a circle.
- Check the area formula in polar coordinates for a segment of a circle and a whole circle.

In Exercises 13–16, sketch and find the length (as an integral) of the graph of $r = f(\theta)$, $\alpha \leq \theta \leq \beta$. (The answer may be in the form of an integral.) Then find the area of the region bounded by this graph and the rays $\theta = \alpha$ and $\theta = \beta$.

- $r = \tan(\theta/2)$; $-\pi/2 \leq \theta \leq \pi/2$.
- $r = \theta + \sin(\theta^2)$; $-\pi/4 \leq \theta \leq 3\pi/4$.
- $r = \sec \theta + 2$; $0 \leq \theta \leq \pi/4$.
- $r = 2e^{3\theta}$; $\ln 2 \leq \theta \leq \ln 3$.

In Exercises 17–20, find the length of and areas bounded by the following curves between the rays indicated. Express the areas as numbers but leave the length as integrals.

- $r = \theta(1 + \cos \theta)$; $\theta = 0$, $\theta = \pi/2$.
- $r = 1/\theta$; $\theta = 1$, $\theta = \pi$.
- $f(\theta) = \sqrt{1 + 2 \sin 2\theta}$; $\theta = 0$, $\theta = \pi/2$.
- $f(\theta) = \theta^2 - (\pi/2)\theta + 4$; $\theta = 0$, $\theta = \pi/2$.

In Exercises 21–24, sketch and find the area of each of the regions between each of the following pairs of curves ($0 \leq \theta \leq 2\pi$). Then find the length of the curves which bound the regions.

- $r = \cos \theta$, $r = \sqrt{3} \sin \theta$.
- $r = 3$, $r = 2(1 + \cos \theta)$.
- $r = 2 \cos \theta$, $r = 1 + \cos \theta$.
- $r = 1$, $r = 1 + \cos \theta$.

- The curve $r = e^\theta$ is called a *logarithmic spiral*. Find the length of the loop of the logarithmic spiral for θ in $[2n\pi, 2(n+1)\pi]$.

26. Suppose that the distance from the origin to $(x, y) = (f(t), g(t))$ attains its maximum value at $t = t_0$. Show that the tangent line at t_0 is perpendicular to the line from the origin to the point $(f(t_0), g(t_0))$.
- ★27. An elliptical orbit is parametrized by $x = a \cos \theta$, $y = b \sin \theta$, $0 \leq \theta \leq 2\pi$. This parametrization is 2π -periodic. In Chapter 18 we shall show that for any T -periodic parametrization of a continuously differentiable closed curve $x = x(t)$, $y = y(t)$ which is a *simple* (never crosses itself),

$$\text{area enclosed} = \int_0^T \frac{1}{2} [x(t)\dot{y}(t) - \dot{x}(t)y(t)] dt,$$

where $\dot{x}(t) = dx/dt$ and $\dot{y}(t) = dy/dt$. (See also Review Exercise 95 for this Chapter.)

- (a) Use this formula to verify that the area enclosed by an ellipse of semiaxes a and b is πab .
- (b) Apply the formula to the case of a curve $x(t) = r \cos t$, $y(t) = r \sin t$, where $r = r(t)$, showing that the area enclosed is $\frac{1}{2} \int_0^T r^2 dt$.

Review Exercises for Chapter 10

Evaluate the integrals in Exercises 1–50.

1. $\int 3 \sin^2 x \cos x \, dx$
2. $\int \sin^2 2x \cos^3 2x \, dx$
3. $\int \sin 3x \cos 5x \, dx$
4. $\int \cos 4x \sin 6x \, dx$
5. $\int \frac{x^3}{\sqrt{1-x^2}} \, dx \quad (|x| < 1)$
6. $\int \frac{dx}{(x^2+2)^2}$
7. $\int \frac{\sqrt{x^2-16}}{x} \, dx \quad (x > 4)$
8. $\int \frac{dx}{\sqrt{x^2-16}} \quad (x > 4)$
9. $\int \frac{dx}{x^2+x+2}$
10. $\int \frac{dx}{\sqrt{x^2+x+2}}$
11. $\int \frac{dx}{x^3+x^2}$
12. $\int \frac{dx}{x^3-27}$
13. $\int \frac{x^3}{(x^2+1)^2} \, dx$
14. $\int \frac{x^2}{(x+1)^3} \, dx$
15. $\int \frac{dx}{x^2+4x+5}$
17. $\int \sin \sqrt{x} \, dx$
19. $\int \frac{dx}{(1-\cos ax)^2}$
21. $\int \frac{\sin^2 x}{\cos x} \, dx$
23. $\int \frac{\tan^{-1} x}{1+x^2} \, dx$
25. $\int \frac{x}{x^3-9} \, dx$
27. $\int e^{\sqrt{x}} \, dx$
29. $\int \frac{dx}{1+e^x}$
31. $\int \left(\frac{x}{x^2-1} \right)^3 \, dx$
33. $\int \sin 3x \cos 2x \, dx$
35. $\int \frac{x}{x^2+1} \, dx$
37. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$
39. $\int_2^3 \frac{x}{x^2+1} \, dx$
41. $\int \frac{x}{x^2+3} \, dx$
43. $\int x^3 \ln x \, dx$
45. $\int_1^2 \frac{(\ln 3x+5)^3}{x} \, dx$
47. $\int_0^1 \sinh^2 x \, dx$
49. $\int_0^{2\pi} \frac{\sin \theta}{1+\cos \theta + \cos^2 \theta} d\theta$
50. $\int_0^{\pi/100} \sec^2 100x \, dx$
26. $\int (x+5) \ln x \, dx$
28. $\int x^3 \sqrt{1-x^2} \, dx$
30. $\int \frac{x}{(x-3)^8} \, dx$
32. $\int \sqrt{x^2+2x+3} \, dx$
34. $\int \sin^2 3x \cos^4 3x \, dx$
36. $\int \frac{x}{(x^2+1)^2} \, dx$
38. $\int (e^x+1)^3 e^x \, dx$
40. $\int_0^{\pi/2} \sin x e^{\cos x} \, dx$
42. $\int \frac{x}{\sqrt{x+1}} \, dx$
44. $\int \sqrt{1+\sin x} \cdot \cos x \, dx$
46. $\int_e^{e^4} \frac{\ln t^2}{t^2} \, dt$
48. $\int_{\pi/8}^{\pi/4} \frac{\sin \theta}{\sqrt{1-\cos^2 \theta}} d\theta$

In Exercises 51–54, find the length of the given graph.

51. $y = 3x^{3/2}$, $0 \leq x \leq 9$
52. $y = (x+1)^{3/2} + 1$, $0 \leq x \leq 2$
53. $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 2$
54. $y = \frac{x^4}{4} + \frac{1}{8x^2}$, $1 \leq x \leq 2$

In Exercises 55–58, find the area of the surface obtained by revolving the given graph about the given axis.

55. $y = x^2$, $0 \leq x \leq 1$, about the y axis.
56. $y = \sqrt{x}$, $0 \leq x \leq 1$, about the x axis.
57. $y = \log_{10} x$, $10 \leq x \leq 100$, about the y axis.
58. $y = 2^x$, $3 \leq x \leq 4$, about the x axis.

For each of the pairs of parametric equations in Exercises 59–64, sketch the curve and find an equation in x and y by eliminating the parameter.

59. $x = t^2; y = t - 1$ 60. $x = 2t + 5; y = t^3$
 61. $x = 3t; y = 2t + 1$ 62. $x = t; y = t$
 63. $x = 0; y = t^4$ 64. $x = t^2\sqrt{t^2 - 1}; y = t^2$

65. Find the equation of the tangent line to the curve $x = t^4, y = 1 + t^3$ at $t = 1$.
 66. Find the equation of the tangent line to the parametric curve $x = 3 \cos t, y = \sin t$ at $t = \pi/4$.
 67. Find the arc length of $x = t^2, y = 2t^4$ from $t = 0$ to $t = 2$.
 68. Find the length of $x = e^t \sin t, y = e^t \cos t$ from $t = 0$ to $t = \pi/2$.

Find the arc length (as an integral if necessary) and area enclosed by each of the graphs given in polar coordinates in Exercises 69–74.

69. $r = \theta^2; 0 < \theta \leq \frac{\pi}{2}$.
 70. $r = \frac{1}{\cos \theta}; 0 < \theta \leq \frac{\pi}{4}$.
 71. $r = \frac{1}{2} + \cos 2\theta; 0 \leq \theta \leq \pi$.
 72. $r = 2|\cos \theta|; 0 \leq \theta \leq 2\pi$.
 73. $r = 3 \cos^4 \frac{\theta}{4}; 0 \leq \theta \leq \pi$.
 74. $r = \frac{1}{2} \sin^2 \frac{\theta}{2}; \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$.

If f is a function on $[0, 2\pi]$, then the numbers

$$\left. \begin{aligned} a_m &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx \\ b_m &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx \, dx \end{aligned} \right\} \quad (m = 0, 1, 2, \dots)$$

are called the *Fourier coefficients* of f . Find all the Fourier coefficients of each of the functions in Exercises 75–82.

75. $\sin 2x$ 76. $\sin 5x$
 77. $\cos 3x$ 78. $\cos 8x$
 79. $3 \cos 4x$ 80. $2 \cos 8x + \sin 7x + \cos 9x$
 81. $\sin^2 x$ 82. $\cos^3 x$

83. The solution of the *logistic equation* of population biology, $dN/dt = (k_1 N - k_2)N$, $N(0) = N_0$, requires the evaluation of the definite integral

$$\int_{N_0}^{N(t)} \frac{du}{(k_1 u - k_2)u}.$$

- (a) Evaluate by means of partial fraction methods and compare your answer with Exercise 19, Section 8.5.
 (b) The integral is just the time t . Solve for $N(t)$ in terms of t , using exponentials.
 (c) Find $\lim_{t \rightarrow \infty} N(t)$ when it exists.
 84. Kepler's second law of planetary motion says that *the radial segment drawn from the sun to a planet sweeps out equal areas in equal times*. Locate the origin $(0, 0)$ at the sun and introduce polar coordinates (r, θ) for the planet location. Assume the *angular momentum* of the planet (of mass m) about the sun is constant; $mr^2\dot{\theta} = mk$,

$k = \text{constant}$, and $\dot{\theta} = d\theta/dt$. Establish Kepler's second law by showing $\int_s^{s+h} r^2 \dot{\theta} \, dt$ is the same for all times s ; thus the area swept out is the same for all time intervals of length h .

85. An elliptical satellite circuits the earth in a circular orbit. The angle ϕ between its major axis and the direction to the earth's center oscillates between $+\phi_m$ and $-\phi_m$ (*librations* of the earth satellite). It is assumed that $0 < \phi_m < \pi/2$, so that the satellite does not tumble end over end. The time T for one complete cycle of this oscillation is given by

$$T = \frac{4}{\pi} \int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos 2\phi - \cos 2\phi_m}}.$$

Change variables in the integral via the formulas

$$\sin \phi = \sin \phi_m \sin \beta \quad (\text{which defines } \beta),$$

$$\cos 2\phi = 1 - 2 \sin^2 \phi,$$

$$\cos 2\phi_m = 1 - 2 \sin^2 \phi_m,$$

to obtain the *elliptic integral representation*

$$T = \frac{4}{\pi \sqrt{2}} \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}$$

for the period of libration T , where $k^2 = \sin^2 \phi_m$.

- ★86. An engineer is studying the impact of an infinite bar by a short round-headed bar, making a maximum indentation α_1 . Applying Hertz' theory of impact, she obtains the equation $\frac{1}{2} \rho c_0 \Omega \alpha' = k(\alpha_1^{3/2} - \alpha^{3/2})$ for the indentation α at time t . The symbols ρ, c_0, Ω, k are constants. The equation is solved by an initial integration to get

$$t = \frac{\rho c_0 \Omega}{2k\sqrt{\alpha_1}} \int_0^{\alpha/\alpha_1} \frac{du}{1 - u^{3/2}}.$$

- (a) Evaluate the integral by making the substitution $v = \sqrt{u}$, followed by the method of partial fractions.
 (b) Substitute $s = (4tk\sqrt{\alpha_1})/(3\rho c_0 \Omega)$ to obtain

$$9s = \frac{2\pi}{\sqrt{3}} + 2 \ln \left| \frac{1+y+y^2}{(1-y)^2} \right| - 4\sqrt{3} \tan^{-1} \left(\frac{2y+1}{\sqrt{3}} \right),$$

where $y = \sqrt{\alpha/\alpha_1}$.

- ★87. Find a general formula for $\int dx/\sqrt{ax^2 + bx + c}$; $a \neq 0$. There will be two cases, depending upon the sign of a .
 ★88. Let $f(x) = x^n$, $0 < a \leq x \leq b$. For which rational values of n can you evaluate the integral occurring in the formula for:
 (a) The area under the graph of f ?
 (b) The length of the graph of f ?
 (c) The volume of the surface obtained by revolving the region under the graph of f about the x axis? The y axis?

- (d) The area of the surface of revolution obtained by revolving the graph of f about the x axis? The y axis?

Evaluate these integrals.

- ★89. Same as Exercise 88, but with $f(x) = 1 + x^n$.
 ★90. Same as Exercise 88, but with $f(x) = (1 + x^2)^n$.
 ★91. (a) Find the formula for the area of the surface obtained by revolving the graph of $r = f(\theta)$ about the x axis, $\alpha \leq \theta \leq \beta$.
 (b) Find the area of the surface obtained by revolving $r = \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$ about the x axis (express as an integral if necessary).
 ★92. Consider the integral

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

- (a) Show that, for $k = 0$ and $k = 1$, this integral can be evaluated in terms of trigonometric and exponential functions and their inverses.
 (b) Show that, for any k , the integral may be transformed to one of the form

$$\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

(This integrand occurs in the sunshine formula—see the supplement to Section 9.5.)

- (c) Show that the integral

$$\int \sqrt{1-k^2\sin^2\theta} \, d\theta$$

(which also occurs in the sunshine formula) arises when one tries to find the arc length of an ellipse $x^2/a^2 + y^2/b^2 = 1$. Express k in terms of a and b .

Due to the result of part (c), the integrals in parts (a), (b), and (c) are called *elliptic integrals*.

- ★93. Consider the parametric curve given by $x = \cos mt$, $y = \sin nt$, when m and n are integers. Such a curve is called a *Lissajous figure* (see Example 6, Section 10.4).
 (a) Plot the curve for $m = 1$ and $n = 1, 2, 3, 4$.
 (b) Describe the general behavior of the curve if $m = 1$, for any value of n . Does it matter whether n is even or odd?
 (c) Plot the curve for $m = 2$ and $n = 1, 2, 3, 4, 5$.
 (d) Plot the curve for $m = 3$ and $n = 4, 5$.
 ★94. (Lissajous figures continued). The path $x = x(t)$, $y = y(t)$ of movement of the *tri-suspension pendulum* of Fig. 10.R.1 produces a Lissajous figure of the general form $x = A_1\cos(\omega_1 t + \theta_1)$, $y = A_2\sin(\omega_2 t + \theta_0)$.

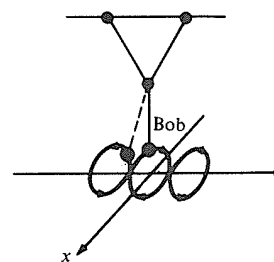


Figure 10.R.1. The bob on this pendulum traces out a Lissajous figure.

- (a) Draw the Lissajous figures for $\omega_1 = \omega_2 = 1$, $A_1 = A_2$, for some sample values of θ_1, θ_2 . The figures should come out to be straight lines, circles, ellipses.
 (b) When $\omega_1 = 1$, $\omega_2 = 3$, $A_1 = A_2 = 1$, the bob retraces its path, but has two self-intersections. Verify this using the results of Exercise 93. Conjecture what happens when ω_2/ω_1 is the ratio of integers.
 (c) When $\omega_2/\omega_1 = \pi$, $A_1 = A_2 = 1$, the bob does not retrace its path, and has infinitely many self-intersections. Verify this, graphically. Conjecture what happens when ω_2/ω_1 is irrational (not the quotient of integers).

- ★95. Consider the curve $r = f(\theta)$ for $0 \leq \theta \leq 2\pi$ as a parametric curve: $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. Assuming that $f(\theta) > 0$ for all θ in $[0, 2\pi]$ and that $f(2\pi) = f(0)$, show that the area enclosed by the curve is given by

$$-\int_0^{2\pi} y \frac{dx}{dt} dt \quad (A)$$

as well as by $\int_0^{2\pi} x(dy/dt)dt$ and by the more symmetric formula

$$\frac{1}{2} \int_0^{2\pi} \left[x \frac{dy}{dt} - y \frac{dx}{dt} \right] dt.$$

[Hint: Substitute the definitions of x and y into (A), integrate by parts, and use the formula for area in polar coordinates.] These formulas are in fact valid for any closed parametric curve. (See Section 18.4.)

- ★96. (a) If r is a non-repeated root of $Q(x)$, show that the portion of the partial fraction expansion of $P(x)/Q(x)$ corresponding to the factor $x - r$ is $A/(x - r)$ where $A = P(r)/Q'(r)$. (b) Use (a) to calculate $\int [(x^2 + 2)/(x^3 - 6x + 11x - 6)] dx$.

MODULE 2

Text Book 2- Calculus II (2/e) : Jerrold Marsden &
Alan Weinstein Springer-Verlag
New York Inc(1985) ISBN 0-387-90975-3

Sections 11.3 – 11.5
12.1 – 12.5

★42. Prove l'Hôpital's rule for $x_0 = \infty$ as follows:

- (i) Let f and g be differentiable on (a, ∞) with $g'(x) \neq 0$ and $g'(x) \neq 0$ for all $x > a$. Use Cauchy's mean value theorem to prove that for every $\varepsilon > 0$, there is an $M > a$ such that for $y > x > M$,

$$\left| \frac{f(y) - f(x)}{g(y) - g(x)} - l \right| < \varepsilon$$

- (ii) Write

$$\frac{f(x)}{g(x)} = \lim_{y \rightarrow \infty} \frac{f(x) - f(y)}{g(x) - g(y)}$$

and choose y sufficiently large,

to conclude that

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon.$$

- (iii) Complete the proof using (ii).

★43. (a) Find $\lim_{a \rightarrow 0} \frac{1}{a} \ln \left(\frac{e^a - 1}{a} \right)$

(b) Find $\lim_{a \rightarrow \infty} \frac{1}{a} \ln \left(\frac{e^a - 1}{a} \right)$

- (c) Are your results consistent with the computations of Exercise 30, Section 9.4?

11.3 Improper Integrals

The area of an unbounded region is defined by a limiting process.

The definite integral $\int_a^b f(x) dx$ of a function f which is non-negative on the interval $[a, b]$ equals the area of the region under the graph of f between a and b . If we let b go to infinity, the region becomes unbounded, as in Fig. 11.3.1. One's first inclination upon seeing such unbounded regions may be to assert that their areas are infinite; however, examples suggest otherwise.

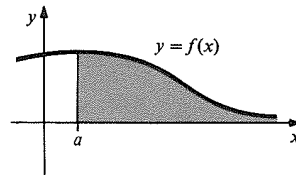


Figure 11.3.1. The region under the graph of f on $[a, \infty)$ is unbounded.

Example 1 Find $\int_1^b \frac{1}{x^4} dx$. What happens as b goes to infinity?

Solution We have

$$\int_1^b \frac{dx}{x^4} = \int_1^b x^{-4} dx = \left. \frac{x^{-3}}{-3} \right|_1^b = \frac{1/b^3 - 1}{-3} = \frac{1 - 1/b^3}{3}.$$

As b becomes larger and larger, this integral always remains less than $\frac{1}{3}$; furthermore, we have

$$\lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^4} = \lim_{b \rightarrow \infty} \frac{1 - 1/b^3}{3} = \frac{1}{3}. \blacktriangle$$

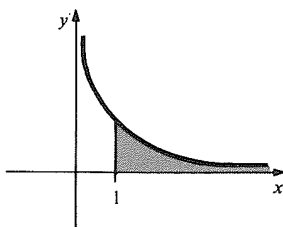


Figure 11.3.2. The region under the graph of $1/x^4$ on $[1, \infty)$ has finite area. It is $\int_1^\infty (dx/x^4) = \frac{1}{3}$.

Example 1 suggests that $\frac{1}{3}$ is the area of the unbounded region consisting of those points (x, y) such that $1 \leq x$ and $0 \leq y \leq 1/x^4$. (See Fig. 11.3.2.) In accordance with our notation for finite intervals, we denote this area by $\int_1^\infty (dx/x^4)$. Guided by this example, we define integrals over unbounded intervals as limits of integrals over finite intervals. The general definition follows.

Integrals over Unbounded Intervals

Suppose that for a fixed, f is integrable on $[a, b]$ for all $b > a$. If the limit $\lim_{b \rightarrow \infty} \int_a^b f(x) dx$ exists, we say that the *improper integral* $\int_a^\infty f(x) dx$ is *convergent*, and we define its value by

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

Similarly, if, for fixed b , f is integrable on $[a, b]$ for all $a < b$, we define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx,$$

if the limit exists.

Finally, if f is integrable on $[a, b]$ for all $a < b$, we define

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx,$$

if the improper integrals on the right-hand side are *both* convergent.

If an improper integral is not convergent, it is called *divergent*.

Example 2 For which values of the exponent r is $\int_1^\infty x^r dx$ convergent?

Solution We have

$$\lim_{b \rightarrow \infty} \int_1^b x^r dx = \lim_{b \rightarrow \infty} \left. \frac{x^{r+1}}{r+1} \right|_1^b = \lim_{b \rightarrow \infty} \frac{b^{r+1} - 1}{r+1} \quad (r \neq -1).$$

If $r+1 > 0$ (that is, $r > -1$), the limit $\lim_{b \rightarrow \infty} b^{r+1}$ does not exist and the integral is divergent. If $r+1 < 0$ (that is, $r < -1$), we have $\lim_{b \rightarrow \infty} b^{r+1} = 0$ and the integral is convergent—its value is $-1/(r+1)$. Finally, if $r = -1$ we have $\int_1^b x^{-1} dx = \ln b$, which does not converge as $b \rightarrow \infty$. We conclude that $\int_1^\infty x^r dx$ is convergent just for $r < -1$. ▲

Example 3 Find $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

Solution We write $\int_{-\infty}^\infty (dx/(1+x^2)) = \int_{-\infty}^0 (dx/(1+x^2)) + \int_0^\infty (dx/(1+x^2))$. To evaluate these integrals, we use the formula $\int (dx/(1+x^2)) = \tan^{-1}x$. Then

$$\begin{aligned} \int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a \rightarrow -\infty} (\tan^{-1}0 - \tan^{-1}a) \\ &= 0 - \lim_{a \rightarrow -\infty} \tan^{-1}a = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}. \end{aligned}$$

(See Fig. 5.4.5 for the horizontal asymptotes of $y = \tan^{-1}x$.) Similarly, we have

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} (\tan^{-1}b - \tan^{-1}0) = \frac{\pi}{2},$$

$$\text{so } \int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \quad \blacktriangle$$

Sometimes we wish to know that an improper integral converges, even though we cannot find its value explicitly. The following test is quite effective for this situation.

Comparison Test

Suppose that f and g are functions such that

- (i) $|f(x)| \leq g(x)$ for all $x \geq a$ and
- (ii) $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ exist for every $b > a$.

Then

- (1) If $\int_a^\infty g(x) dx$ is convergent, so is $\int_a^\infty f(x) dx$, and
- (2) if $\int_a^\infty f(x) dx$ is divergent, so is $\int_a^\infty g(x) dx$.

Similar statements hold for integrals of the type

$$\int_{-\infty}^b f(x) dx \quad \text{and} \quad \int_{-\infty}^\infty f(x) dx.$$

Here we shall explain the idea behind the comparison test. A detailed proof is given at the end of this section.

If $f(x)$ and $g(x)$ are both positive functions (Fig. 11.3.3(a)), then the region under the graph of f is contained in the region under the graph of g , so

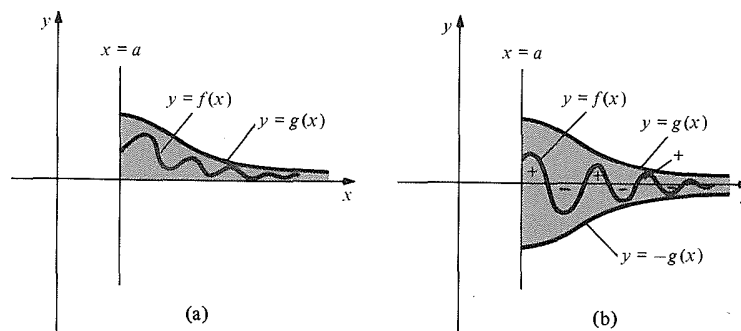


Figure 11.3.3. Illustrating the comparison test.

the integral $\int_a^b f(x) dx$ increases and remains bounded as $b \rightarrow \infty$. We expect, therefore, that it should converge to some limit. In the general case (Fig. 11.3.3(b)), the sums of the plus areas and the minus areas are both bounded by $\int_a^\infty g(x) dx$, and the cancellations can only help the integral to converge.

Note that in the event of convergence, the comparison test only gives the inequality $-\int_a^\infty g(x) dx \leq \int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx$, but it does not give us the value of $\int_a^\infty f(x) dx$.

Example 4 Show that $\int_0^\infty \frac{dx}{\sqrt{1+x^8}}$ is convergent, by comparison with $1/x^4$.

Solution We have $1/\sqrt{1+x^8} < 1/\sqrt{x^8} = 1/x^4$, so it is tempting to compare with $\int_0^\infty (dx/x^4)$. Unfortunately, the latter integral is not defined because $1/x^4$ is unbounded near zero. However, we can break the original integral in two parts:

$$\int_0^\infty \frac{dx}{\sqrt{1+x^8}} = \int_0^1 \frac{dx}{\sqrt{1+x^8}} + \int_1^\infty \frac{dx}{\sqrt{1+x^8}}$$

The first integral on the right-hand side exists because $1/\sqrt{1+x^8}$ is continuous on $[0, 1]$. The second integral is convergent by the comparison test, taking $g(x) = 1/x^4$ and $f(x) = 1/\sqrt{1+x^8}$. Thus $\int_0^\infty (dx/\sqrt{1+x^8})$ is convergent. ▲

Example 5 Show that $\int_0^{\infty} \frac{\sin x}{(1+x)^2} dx$ converges (without attempting to evaluate).

Solution We may apply the comparison test by choosing $g(x) = 1/(1+x)^2$ and $f(x) = (\sin x)/(1+x)^2$, since $|\sin x| \leq 1$. To show that $\int_0^{\infty} (dx/(1+x)^2)$ is convergent, we can compare $1/(1+x)^2$ with $1/x^2$ on $[1, \infty)$, as in Example 4, or we can evaluate the integral explicitly:

$$\begin{aligned} \int_0^{\infty} \frac{dx}{(1+x)^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(1+x)^2} = \lim_{b \rightarrow \infty} \left[\frac{-1}{(1+x)} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[1 - \frac{1}{1+b} \right] = 1. \blacktriangle \end{aligned}$$

Example 6 Show that $\int_1^{\infty} \frac{dx}{\sqrt{1+x^2}}$ is divergent.

Solution We use the comparison test in the reverse direction, comparing $1/\sqrt{1+x^2}$ with $1/x$. In fact, for $x \geq 1$, we have $1/\sqrt{1+x^2} \geq 1/\sqrt{x^2+x^2} = 1/\sqrt{2} x$. But $\int_1^b (dx/\sqrt{2} x) = (1/\sqrt{2}) \ln b$, and this diverges as $b \rightarrow \infty$. Therefore, by statement (2) in the comparison test, the given integral diverges. $\blacktriangle$

We shall now discuss the second type of improper integral. If the graph of a function f has a vertical asymptote at one endpoint of the interval $[a, b]$, then the integral $\int_a^b f(x) dx$ is not defined in the usual sense, since the function f is not bounded on the interval $[a, b]$. As with integrals of the form $\int_a^{\infty} f(x) dx$, we are dealing with areas of unbounded regions in the plane—this time the unboundedness is in the vertical rather than the horizontal direction. Following our earlier procedure, we can define the integrals of unbounded functions as limits, which are again called improper integrals.

Integrals of Unbounded Functions

Suppose that the graph of f has $x_0 = b$ as a vertical asymptote and that for a fixed, f is integrable on $[a, q]$ for all q in $[a, b)$. If the limit $\lim_{q \rightarrow b-} \int_a^q f(x) dx$ exists, we shall say that the *improper integral* $\int_a^b f(x) dx$ is *convergent*, and we define

$$\int_a^b f(x) dx = \lim_{q \rightarrow b-} \int_a^q f(x) dx.$$

Similarly, if $x = a$ is a vertical asymptote, we define

$$\int_a^b f(x) dx = \lim_{p \rightarrow a+} \int_p^b f(x) dx,$$

if the limit exists. (See Fig. 11.3.4.)

If both $x = a$ and $x = b$ are vertical asymptotes, or if there are vertical asymptotes in the interior (a, b) , we may break up $[a, b]$ into subintervals such that the integral of f on each subinterval is of the type considered in the preceding definition. If each part is convergent, we may add the results to get $\int_a^b f(x) dx$. The comparison test may be used to test each for convergence. (See Example 9 below.)

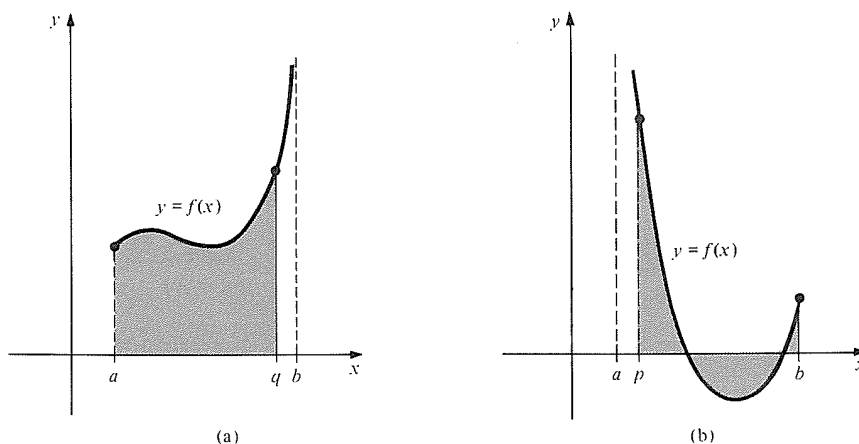


Figure 11.3.4. Improper integrals defined by
(a) the limit $\lim_{q \rightarrow b-} \int_a^q f(x) dx$ and
(b) the limit $\lim_{p \rightarrow a+} \int_p^b f(x) dx$.

Example 7 For which values of r is $\int_0^1 x^r dx$ convergent?

Solution If $r \geq 0$, x^r is continuous on $[0, 1]$ and the integral exists in the ordinary sense. If $r < 0$, we have $\lim_{x \rightarrow 0+} x^r = \infty$, so we must take a limit. We have

$$\lim_{p \rightarrow 0+} \int_p^1 x^r dx = \lim_{p \rightarrow 0+} \left. \frac{x^{r+1}}{r+1} \right|_p^1 = \frac{1}{r+1} \left(1 - \lim_{p \rightarrow 0+} p^{r+1} \right),$$

provided $r \neq -1$. If $r+1 > 0$ (that is, $r > -1$), we have $\lim_{p \rightarrow 0+} p^{r+1} = 0$, so the integral is convergent and equals $1/(r+1)$. If $r+1 < 0$ (that is, $r < -1$), $\lim_{p \rightarrow 0+} p^{r+1} = \infty$, so the integral is divergent. Finally, if $r+1 = 0$, we have $\lim_{p \rightarrow 0+} \int_p^1 x^r dx = \lim_{p \rightarrow 0+} (0 - \ln p) = \infty$. Thus the integral $\int_0^1 x^r dx$ converges just for $r > -1$. (Compare with Example 2.) $\blacktriangle$

Example 8 Find $\int_0^1 \ln x dx$.

Solution We know that $\int \ln x dx = x \ln x - x + C$, so

$$\begin{aligned} \int_0^1 \ln x dx &= \lim_{p \rightarrow 0+} (1 \ln 1 - 1 - p \ln p + p) \\ &= 0 - 1 - 0 + 0 = -1 \end{aligned}$$

($\lim_{p \rightarrow 0+} p \ln p = 0$ by Example 6, Section 11.2). $\blacktriangle$

Example 9 Show that the improper integral $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$ is convergent.

Solution This integral is improper at both ends; we may write it as $I_1 + I_2$, where

$$I_1 = \int_0^1 (e^{-x}/\sqrt{x}) dx \quad \text{and} \quad I_2 = \int_1^\infty (e^{-x}/\sqrt{x}) dx$$

and then we apply the comparison test to each term. On $[0, 1]$, we have $e^{-x} \leq 1$, so $e^{-x}/\sqrt{x} \leq 1/\sqrt{x}$. Since $\int_0^1 (dx/\sqrt{x})$ is convergent (Example 7), so is I_1 . On $[1, \infty)$, we have $1/\sqrt{x} \leq 1$, so $e^{-x}/\sqrt{x} \leq e^{-x}$; but $\int_1^\infty e^{-x} dx$ is convergent because

$$\int_1^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} (e^{-1} - e^{-b}) = e^{-1}$$

Thus I_2 is also convergent and so $\int_0^\infty (e^{-x}/\sqrt{x}) dx$ is convergent. $\blacktriangle$

Improper integrals arise in arc length problems for graphs with vertical tangents.

Example 10 Find the length of the curve $y = \sqrt{1 - x^2}$ for x in $[-1, 1]$. Interpret your result geometrically.

Solution By formula (1), Section 10.3, the arc length is

$$\begin{aligned}\int_{-1}^1 \sqrt{1 + (dy/dx)^2} dx &= \int_{-1}^1 \sqrt{1 + (-x/\sqrt{1-x^2})^2} dx \\ &= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}.\end{aligned}$$

The integral is improper at both ends, since

$$\lim_{x \rightarrow -1+} \frac{1}{\sqrt{1-x^2}} = \lim_{x \rightarrow 1-} \frac{1}{\sqrt{1-x^2}} = \infty.$$

We break it up as

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{p \rightarrow -1+} \int_p^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{q \rightarrow 1-} \int_0^q \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{p \rightarrow -1+} (\sin^{-1} 0 - \sin^{-1} p) + \lim_{q \rightarrow 1-} (\sin^{-1} q - \sin^{-1} 0) \\ &= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi.\end{aligned}$$

Geometrically, the curve whose arc length we have just found is a semicircle of radius 1, so we recover the fact that the circumference of a circle of radius 1 is 2π . ▲

Example 11 Luke Skywalker has just been knocked out in his spaceship by his archenemy, Captain Tralfamadore. The evil captain has set the controls to send the spaceship into the sun! His perverted mind insists on a slow death, so he sets the controls so that the ship makes a constant angle of 30° with the sun (Fig. 11.3.5). What path will Luke's ship follow? How long does Luke have to wake up if he is 10 million miles from the sun and his ship travels at a constant velocity of a million miles per hour?

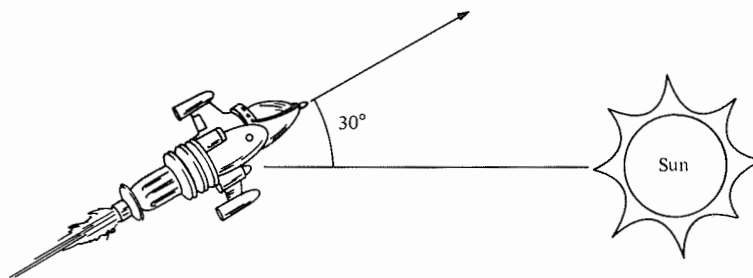


Figure 11.3.5. Luke Skywalker's ill-fated ship.

Solution We use polar coordinates to describe a curve $(r(t), \theta(t))$ such that the radius makes a constant *angle* α with the tangent ($\alpha = 30^\circ$ in the problem). To find $dr/d\theta$, we observe, from Fig. 11.3.6(a), that

$$\Delta r \approx \frac{r \Delta \theta}{\tan \alpha} \quad \text{so} \quad \frac{dr}{d\theta} = \frac{r}{\tan \alpha}. \quad (1)$$

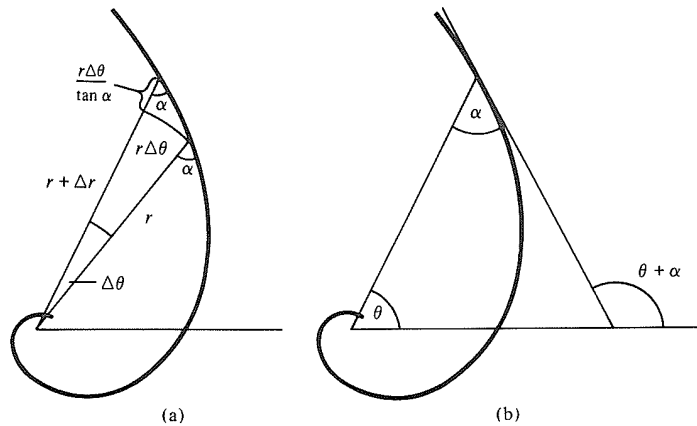


Figure 11.3.6. The geometry of Luke's path.

We can derive formula (1) rigorously, but also more laboriously, by calculating the slope of the tangent line in polar coordinates and setting it equal to $\tan(\theta + \alpha)$ as in Fig. 11.3.6(b). This approach gives

$$\frac{\tan \theta (dr/d\theta) + r}{dr/d\theta - r \tan \theta} = \tan(\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha},$$

so that again

$$\frac{dr}{d\theta} = \frac{r}{\tan \alpha}.$$

The solution of equation (1) is³

$$r(\theta) = r(0)e^{\theta/\tan \alpha}. \quad (2)$$

For this solution to be valid, we must regard θ as a continuous variable ranging from $-\infty$ to ∞ , not as being between zero and 2π . As $\theta \rightarrow \infty$, $r(\theta) \rightarrow \infty$ and as $\theta \rightarrow -\infty$, $r(\theta) \rightarrow 0$, so the curve spirals outward as θ increases and inward as θ decreases (if $0 < \alpha < \pi/2$). This answers the first question: Luke follows the logarithmic spiral given by equation (2), where $\theta = 0$ is chosen as the starting point.

From Section 10.6, the distance Luke has to travel is the arc length of equation (2) from $\theta = 0$ to $\theta = -\infty$, namely, the improper integral

$$\begin{aligned} \int_{-\infty}^0 \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta &= \int_{-\infty}^0 \sqrt{\frac{r^2}{\tan^2 \alpha} + r^2} d\theta = \int_{-\infty}^0 r \sqrt{\cot^2 \alpha + 1} d\theta \\ &= \int_{-\infty}^0 r(0) e^{\theta/\tan \alpha} \frac{1}{\sin \alpha} d\theta \\ &= \frac{r(0)}{\cos \alpha} e^{\theta/\tan \alpha} \Big|_{-\infty}^0 = \frac{r(0)}{\cos \alpha}. \end{aligned}$$

With velocity = 10^6 , $r(0) = 10^7$, and $\cos \alpha = \cos 30^\circ = \sqrt{3}/2$, the time needed to travel the distance is

$$\text{time} = \frac{\text{distance}}{\text{velocity}} = \frac{10^7}{\sqrt{3}/2} \times \frac{1}{10^6} = \frac{20}{\sqrt{3}} \approx 11.547 \text{ hours}$$

Thus Luke has less than 11.547 hours to wake up. ▲

³ See Section 8.2. If you have not read Chapter 8, you may simply check directly that equation (2) is a solution of (1).

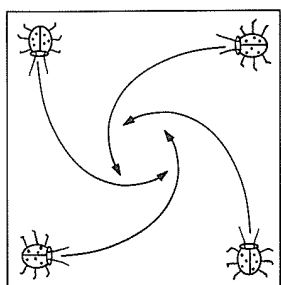


Figure 11.3.7. These love bugs follow logarithmic spirals.

The logarithmic spiral turns up in another interesting situation. Place four love bugs at the corners of a square (Fig. 11.3.7). Each bug, being in love, walks directly toward the bug in front of it, at constant top bug speed. The result is that the bugs all spiral in to the center of the square following logarithmic spirals. The time required for the bugs to reach the center can be calculated as in Example 11 (see Exercise 46).

We conclude this section with a proof of the comparison test. The proof is based on the following principle:

Let F be a function defined on $[a, \infty)$ such that

- (i) *F is nondecreasing; i.e., $F(x_1) \leq F(x_2)$ whenever $x_1 < x_2$;*
- (ii) *F is bounded above: there is a number M such that $F(x) \leq M$ for all x .*

Then $\lim_{x \rightarrow \infty} F(x)$ exists and is at most M .

The principle is quite plausible, since the graph of F never descends and never crosses the line $y = M$, so that we expect it to have a horizontal asymptote as $x \rightarrow \infty$. (See Fig. 11.3.8).

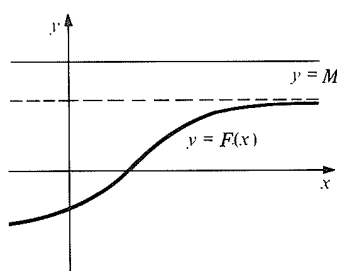


Figure 11.3.8. The graph of a nondecreasing function lying below the line $y = M$ has a horizontal asymptote.

A rigorous proof of the principle requires a careful study of the real numbers,⁴ so we shall simply take the principle for granted, just as we did for some basic facts in Chapter 3. A similar principle holds for nonincreasing functions which are bounded below.

Now we are ready to prove statement (1) in the comparison test as stated in the box on p. 530. (Statement (2) follows from (1), for if $\int_a^\infty g(x) dx$ converged, so would $\int_a^\infty f(x) dx$.)

Let

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

be the positive and negative parts of f , respectively. (See Fig. 11.3.9.)

Notice that $f = f_1 + f_2$. Let $F_1(x) = \int_a^x f_1(t) dt$ and $F_2(x) = \int_a^x f_2(t) dt$. Since f_1 is always non-negative, $F_1(x)$ is increasing. Moreover, by the assumptions of the comparison test,

$$F_1(x) \leq \int_a^x |f(t)| dt \leq \int_a^x g(t) dt \leq \int_a^\infty g(t) dt,$$

so F_1 is bounded above by $\int_a^\infty g(t) dt$. Thus, F_1 has a limit as $x \rightarrow \infty$. Likewise,

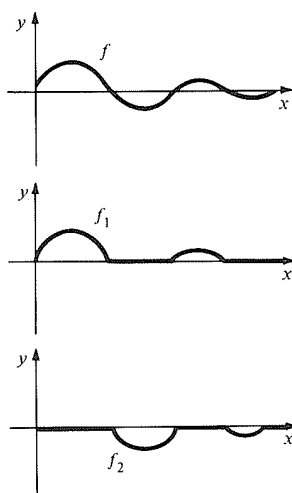


Figure 11.3.9. f_1 and f_2 are the positive and negative parts of f .

⁴ See the theoretical references listed in the preface.

F_2 has a limit since F_2 is decreasing and bounded below. Since

$$\int_a^x f(t) dt = F_1(x) + F_2(x),$$

it, too, has a limit as $x \rightarrow \infty$. ■

Exercises for Section 11.3

Evaluate the improper integrals in Exercises 1–8.

1. $\int_1^\infty \frac{3}{x^2} dx$
2. $\int_2^\infty \frac{dx}{x^2}$
3. $\int_1^\infty e^{-5x} dx$
4. $\int_0^\infty e^{-3x} dx$
5. $\int_2^\infty \frac{dx}{x^2 - 1}$
6. $\int_0^\infty \frac{1}{(1+x)^2} dx$
7. $\int_{-\infty}^\infty \frac{dx}{4+x^2}$
8. $\int_{-\infty}^\infty \frac{dx}{9+x^2}$

Show, using the comparison test, that the integrals in Exercises 9–12 are convergent.

9. $\int_0^\infty \frac{dx}{3+x^3}$
10. $\int_0^\infty \frac{\sin x dx}{\sqrt{1+x^4}}$
11. $\int_0^\infty \frac{e^{-x}}{1+x} dx$
12. $\int_1^\infty \frac{e^{-x}}{1+\ln x} dx$

Show, using the comparison test, that the integrals in Exercises 13–16 are divergent.

13. $\int_0^\infty \frac{dx}{\sqrt{2+x^2}}$
14. $\int_0^\infty \frac{dx}{8+x+1/x}$
15. $\int_1^\infty \frac{(2+\sin x) dx}{1+x}$
16. $\int_1^\infty \frac{(3-\cos x) dx}{\sqrt{1+x^2}}$

Evaluate the improper integrals in Exercises 17–20.

17. $\int_0^{10} \frac{dx}{x^{2/3}}$
18. $\int_0^1 \frac{dx}{x^{3/4}}$
19. $\int_0^1 \frac{dx}{\sqrt{1-x}}$
20. $\int_0^1 \frac{dx}{(1-x)^{2/3}}$

Using the comparison test, determine the convergence of the improper integrals in Exercises 21–24.

21. $\int_{-1}^1 \frac{dx}{x^2+x}$
22. $\int_{-1}^1 \frac{dx}{(x^2+x)^{1/3}}$
23. $\int_{-\infty}^\infty e^{-|x|} dx$
24. $\int_0^\infty \frac{dx}{(1+x^3)^{1/3}}$

Determine the convergence or divergence of the integrals in Exercises 25–40.

25. $\int_{-1}^\infty \frac{\tan^{-1} x}{(2+x)^3} dx$
26. $\int_0^\infty \frac{\sin x}{1+x^2} dx$
27. $\int_{-\infty}^\infty \frac{x}{(x^2+1)^{3/2}} dx$
28. $\int_2^\infty \frac{1}{t^2-1} dt$
29. $\int_1^\infty \frac{1}{(5x^2+1)^{2/3}} dx$
30. $\int_1^\infty \frac{1}{x^2} \left(1 - \frac{1}{x}\right) dx$

31. $\int_1^\infty \frac{1}{(4x-3)^{1/3}} dx$
32. $\int_0^\infty \left[\cos x + \frac{1}{(x+1)^2} \right] dx$
33. $\int_{-\infty}^{-2} \left(\frac{1}{x^{6/5}} - \frac{1}{x^{4/3}} \right) dx$
34. $\int_0^1 \frac{e^{-t}}{\sqrt[3]{t^2}} dt$
35. $\int_1^2 \frac{1}{\sqrt{t-1}} dt$
36. $\int_{-\infty}^\infty \frac{\cos(x^2+1)}{x^2} dx$. [Hint: Use the comparison test on a small interval.]
37. $\int_{-\infty}^2 \left(\frac{1}{x^{5/3}} - \frac{1}{x^{4/3}} \right) dx$
38. $\int_{-4}^{10} \left[\frac{1}{(x+4)^{2/3}} + \frac{1}{(x-10)^{2/3}} \right] dx$
39. $\int_2^\infty \frac{dx}{x \ln x}$
40. $\int_1^\infty e^{-x} \ln x dx$
41. Consider the spirals defined in polar coordinates by the parametric equations $\theta = t$, $r = t^{-k}$. For which values of k does the spiral have finite arc length for $\pi/2 \leq t < \infty$? (Use the comparison test.)
42. Does the spiral $\theta = t$, $r = e^{-\sqrt{t}}$ have finite arc length for $\pi \leq t < \infty$?
43. Find the area under the graph of the function $f(x) = (3x+5)/(x^3-1)$ from $x=2$ to $x=\infty$.
44. Find the area between the graphs $y = x^{-4/3}$ and $y = x^{-5/3}$ on $[1, \infty)$.
45. In Example 11, suppose that Luke's airshoes melt down when he is 10^6 miles from the sun. Now how long does he have to wake up?
46. Let α in Fig. 11.3.7 be 60° (α is defined in Example 11). Find the time required for the bugs to reach the center in terms of their speed and their initial distance from the center.
47. The region under curve $y = e^{-x}$ is rotated about the x axis to form a solid of revolution. Find the volume obtained by discarding the portion on $-\infty < x \leq 10$ (after slicing the solid at $x = 10$).

48. Determine the lateral surface area of the surface of revolution obtained by revolving $y = e^{-x}$, $0 \leq x < \infty$, about the x axis.
49. Show that $\lim_{A \rightarrow 0^+} [\int_{-3}^A (dx/x) + \int_A^2 (dx/x)]$ exists and determine its value.
50. Discuss the following "calculations":

$$(a) \int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_{-1}^1 = -1 + (-1) = -2;$$

$$(b) \int_{\pi/2}^{5\pi/2} \frac{\cos x}{(1 + \sin x)^3} dx \\ = -\frac{1}{2} \cdot \frac{1}{(1 + \sin x)^2} \Big|_{\pi/2}^{5\pi/2} = 0.$$

51. You can simulate the logarithmic spiral yourself as follows: Stand in an open field containing a lone tree and lock your neck muscles so that your head is pointed at a fixed angle α to your body. Walk forward in such a way that you are always looking at the tree. Prove that you will walk along a logarithmic spiral.
52. The probability P that a phonograph needle will last in excess of 150 hours is given by the formula $P = \int_{150}^{\infty} \frac{1}{100} e^{-t/100} dt$. Find the value of P .
53. The probability p that the score on a reading comprehension test is no greater than the value a is

$$p = \int_{-\infty}^a \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\tau-\mu)^2/2\sigma^2} d\tau;$$

σ, μ are constants.

- (a) Let $x = (\tau - \mu)/\sigma$ and $x_1 = (a - \mu)/\sigma$. Show that

$$p = \int_{-\infty}^{x_1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

- (b) Show that $\int_{-\infty}^{\infty} e^{-x^2/2} dx < \infty$.

- ★54. Pearson and Lee studied the inheritance of physical characteristics in families in 1903. One law that resulted from these studies is

$$P = \int_{-\infty}^{(\tau-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

for the probability P that a mother's height is not greater than τ inches. The estimated values of μ and σ are $\mu = 62.484$ inches, $\sigma^2 = 5.7140$ square inches.

- (a) Determine the value of P by appeal to integral tables for

$$\int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

using $\tau = 63$ inches. Look in a mathematical table under *probability functions* or *normal distribution*.

- (b) According to the study, how many mothers out of 100 are likely to have height not exceeding 63 inches?

★55. (a) Evaluate $\int_0^{\infty} \frac{du}{u^{1/2} + u^{3/2}}.$

- (b) For what p and q is

$$\int_0^{\infty} \frac{dx}{x^p + x^q}$$

convergent?

- ★56. Consider the surface of revolution obtained by revolving the graph of $f(x) = 1/x$ on the interval $[1, \infty)$ about the x axis.

- (a) Show that the area of this surface is infinite.
- (b) Show that the volume of the solid of revolution bounded by this surface is finite.
- (c) The results of parts (a) and (b) suggest that one could fill the solid with a finite amount of paint, but it would take an infinite amount of paint to paint the surface. Explain this paradox.

Next consider the surface of revolution obtained by revolving the curve $y = 1/x^r$ for x in $[1, \infty)$ about the x axis.

- (d) For which values of r does this surface have finite area?
- (e) For which values of r does the solid surrounded by this surface have finite volume? Compute the volume for these values of r .

- ★57. Show that if $0 < f'(x) < 1/x^2$ for all x in $[0, \infty)$, then $\lim_{x \rightarrow \infty} f(x)$ exists.

11.4 Limits of Sequences and Newton's Method

Solutions of equations can often be found as the limits of sequences.

This section begins with a discussion of sequences and their limits. The topic will be taken up again in Section 12.1 when we study infinite series. A *sequence* is just an "infinite list" of numbers: $a_1, a_2, a_3, \dots$, with one a_n for each natural number n . A number l is called the *limit* of this sequence if, roughly speaking, a_n comes and remains arbitrarily close to l as n increases.

Perhaps the most familiar example of a sequence with a limit is that of an infinite decimal expansion. Consider, for instance, the equation

$$\frac{1}{3} = 0.333 \dots \quad (1)$$

in which the dots on the right-hand side are taken to stand for “infinitely many 3’s.” We can interpret equation (1) without recourse to any metaphysical notion of infinity: the finite decimals 0.3, 0.33, 0.333, and so on are approximations to $\frac{1}{3}$, and we can make the approximation as good as we wish by taking enough 3’s. Our sequence $a_1, a_2, \dots$ is defined in this case by $a_n = 0.33 \dots 3$, with n 3’s (here the three dots stand for only finitely many 3’s). In other words,

$$a_n = \frac{3}{10} + \frac{3}{100} + \dots + \frac{3}{10^n}. \quad (2)$$

We can estimate the difference between a_n and $\frac{1}{3}$ by using some algebra. Multiplying equation (2) by 10 gives

$$10a_n = 3 + \frac{3}{10} + \dots + \frac{3}{10^{n-1}}, \quad (3)$$

and subtracting equation (2) from equation (3) gives

$$9a_n = 3 - \frac{3}{10^n},$$

$$a_n = \frac{1}{3} - \frac{1}{3} \left(\frac{1}{10^n} \right).$$

Finally,

$$\frac{1}{3} - a_n = \frac{1}{3} \left(\frac{1}{10^n} \right). \quad (4)$$

As n is taken larger and larger, the denominator 10^n becomes larger and larger, and so the difference $\frac{1}{3} - a_n$ becomes smaller and smaller. In fact, if n is chosen large enough, we can make $\frac{1}{3} - a_n$ as small as we please. (See Fig. 11.4.1.)

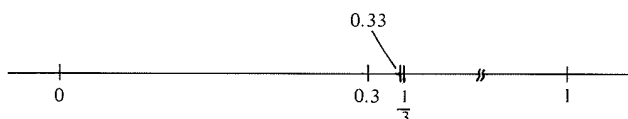


Figure 11.4.1. The decimal approximations to $\frac{1}{3}$ form a sequence converging to $\frac{1}{3}$.

Example 1 How large must n be for the error $\frac{1}{3} - a_n$ to be less than 1 part in 1 million?

Solution By equation (4), we must have

$$\frac{1}{3} \left(\frac{1}{10^n} \right) < 10^{-6}$$

or $10^{-n} < 3 \cdot 10^{-6}$. It suffices to have $n \geq 6$, so the finite decimal 0.333333 approximates $\frac{1}{3}$ to within 1 part in a million. So do the longer decimals 0.3333333, 0.33333333, and so on. ▲

There is nothing special about the number 10^{-6} in Example 1. Given *any* positive number ϵ , we will always be able to make $\frac{1}{3} - a_n = \frac{1}{3}(1/10^n)$ less than ϵ by letting n be sufficiently large. We express this fact by saying that $\frac{1}{3}$ is the

limit of the numbers

$$a_n = \frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}$$

as n becomes arbitrarily large, or

$$\lim_{n \rightarrow \infty} \left(\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} \right) = \frac{1}{3}.$$

We may think of a sequence $a_1, a_2, a_3, \dots$ as a function whose domain consists of the natural numbers $1, 2, 3, \dots$ (Occasionally, we allow the domain to start at zero or some other integer.) Thus we may represent a sequence graphically in two ways—either by plotting the points $a_1, a_2, \dots$ on a number line or by plotting the pairs (n, a_n) in the plane.

Example 2 (a) Write the first six terms of the sequence $a_n = n/(n+1)$, $n = 1, 2, 3, \dots$. Represent the sequence graphically in two ways. Find the value of $\lim_{n \rightarrow \infty} [n/(n+1)]$. (b) Repeat for $a_n = (-1)^n/n$. (c) Repeat for $a_n = (-1)^n n/(n+1)$.

Solution (a) We obtain the terms a_1 through a_6 by substituting $n = 1, 2, 3, \dots, 6$ into the formula for a_n , giving $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$. These values are plotted in Fig. 11.4.2. As n gets larger, the fraction $n/(n+1)$ gets larger and larger but never exceeds 1; we may guess that the limit is equal to 1.

To verify this guess, we look at the difference $1 - n/(n+1)$. We have

$$1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1},$$

which does indeed become arbitrarily small as n increases, so

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

(b) The terms a_1 through a_6 are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$. They are plotted in Fig. 11.4.3. As n gets larger, the number $(-1)^n/n$ seems to get closer to zero. Therefore we guess that $\lim_{n \rightarrow \infty} [(-1)^n/n] = 0$.

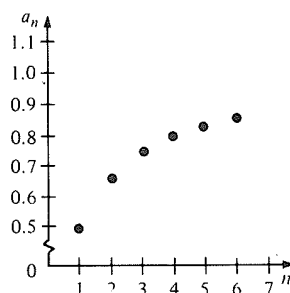
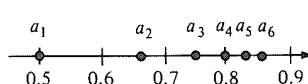


Figure 11.4.2. The sequence $a_n = n/(n+1)$ represented graphically in two different ways.

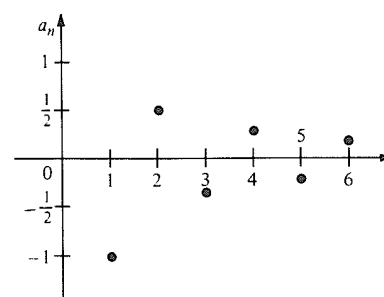
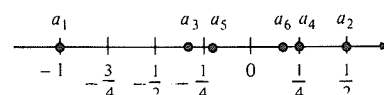


Figure 11.4.3. The sequence $a_n = (-1)^n/n$ plotted in two ways.

(c) We have, for a_1 through a_6 , $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$. They are plotted in Fig. 11.4.4. In this case, the numbers a_n do not approach any particular number. (Some of them are approaching 1, others -1 .) We guess that the sequence does not have a limit. ▲

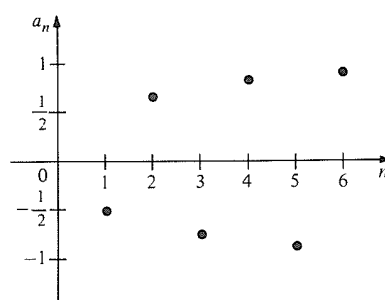
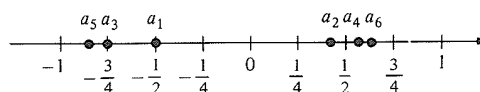


Figure 11.4.4. The sequence $a_n = (-1)^n n / (n+1)$ plotted in two ways.

Just as with the ε - δ definition for limits of functions, there is an ε - N definition for limits of sequences which makes the preceding ideas precise.

Limits of Sequences

The sequence $a_1, a_2, a_3, \dots, a_n, \dots$ approaches l as a limit if a_n gets close to and remains arbitrarily close to l as n becomes large. In this case, we write

$$\lim_{n \rightarrow \infty} a_n = l.$$

In precise terms, $\lim_{n \rightarrow \infty} a_n = l$ if, for every $\varepsilon > 0$, there is an N such that $|a_n - l| < \varepsilon$ for all $n \geq N$.

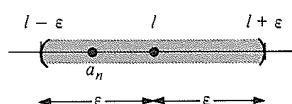


Figure 11.4.5. The relationship between a_n , l , and ε in the definition of the limit of a sequence.

It is useful to think of the number ε in this definition as a *tolerance*, or allowable error. The definition specifies that if l is to be the limit of the sequence a_n , then, given any tolerance, all the terms of the sequence beyond a certain point should be within that tolerance of l . Of course, as the tolerance is made smaller, it will usually be necessary to go farther out in the sequence to bring the terms within tolerance of the limit. (See Fig. 11.4.5.)

The purpose of the ε - N definition is to lay a framework for a precise discussion of limits of sequences and their properties—just as the definitions in Section 11.1.

Let us check the limit of a simple sequence using the ε - N definition.

Example 3 Prove that $\lim_{n \rightarrow \infty} (1/n) = 0$, using the ε - N definition.

Solution To show that the definition is satisfied, we must show that for any $\varepsilon > 0$ there is a number N such that $|1/n - 0| < \varepsilon$ if $n > N$. If we choose $N \geq 1/\varepsilon$, we get, for $n > N$,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

Thus the assertion is proved. ▲

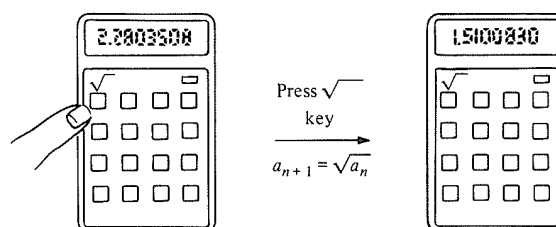
Calculator Discussion

Limits of sequences can sometimes be visualized on a calculator. Consider the sequence obtained by taking successive square roots of a given positive number a :

$$a_0 = a, \quad a_1 = \sqrt{a}, \quad a_2 = \sqrt{\sqrt{a}}, \quad a_3 = \sqrt{\sqrt{\sqrt{a}}},$$

and so forth. (See Fig. 11.4.6.)

Figure 11.4.6. For a recursively defined sequence $a_{n+1} = f(a_n)$, the next member in the sequence is obtained by depressing the “ f ” key. Here $f = \sqrt{}$.



For instance, if we start by entering $a = 5.2$, we get

$$a_0 = 5.2,$$

$$a_1 = \sqrt{5.2} = 2.2803508,$$

$$a_2 = \sqrt{2.2803508} = 1.5100830,$$

$$a_3 = \sqrt{1.5100830} = 1.2288544,$$

and so on. After pressing the $\sqrt{}$ repeatedly you will see the numbers getting closer and closer to 1 until roundoff error causes the number 1 to appear and then stay forever. This sequence has 1 as a limit. (Of course, the calculation does not *prove* this fact, but does suggest it.) Observe that the sequence is defined *recursively*—that is, each member of the sequence is obtained from the previous one by some specific process. The sequence $1, 2, 4, 8, 16, 32, \dots$ is another example; each term is twice the previous one: $a_{n+1} = 2a_n$. ▲

Limits of sequences are closely related to limits of functions. For example, if $f(x)$ is defined for $x \geq 0$, then $a_n = f(n)$ is a sequence. If $\lim_{x \rightarrow \infty} f(x)$ exists, then $\lim_{n \rightarrow \infty} a_n$ exists as well and these limits are equal. This fact can sometimes be used to evaluate some limits. For instance,

$$\lim_{x \rightarrow \infty} \frac{x}{x+1} = \lim_{x \rightarrow \infty} \frac{1}{1+1/x} = \frac{1}{1 + \lim_{x \rightarrow \infty} (1/x)} = \frac{1}{1+0} = 1,$$

and so $\lim_{n \rightarrow \infty} [n/(n+1)] = 1$, confirming our calculations in Example 2(a).

Limits of sequences also obey rules similar to those for functions.⁵ We illustrate:

Example 4 Find (a) $\lim_{n \rightarrow \infty} \left(\frac{n^2 + 1}{3n^2 + n} \right)$
and (b) $\lim_{n \rightarrow \infty} \left(1 - \frac{3}{n} + \frac{n}{n+1} \right)$.

Solution (a) Write

$$\lim_{n \rightarrow \infty} \left(\frac{n^2 + 1}{3n^2 + n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1 + 1/n^2}{3 + 1/n} \right) \quad (\text{dividing numerator and denominator by } n^2)$$

⁵ These are written out formally in Section 12.1.

$$\begin{aligned}
&= \frac{1 + \lim_{n \rightarrow \infty} (1/n^2)}{3 + \lim_{n \rightarrow \infty} (1/n)} \quad (\text{quotient and sum rules}) \\
&= \frac{1 + 0}{3 + 0} = \frac{1}{3}.
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad \lim_{n \rightarrow \infty} \left(1 - \frac{3}{n} + \frac{n}{n+1} \right) &= 1 - 3 \left(\lim_{n \rightarrow \infty} \frac{1}{n} \right) + \lim_{n \rightarrow \infty} \left(\frac{1}{1 + 1/n} \right) \\
&= 1 - 3 \cdot 0 + 1 = 2. \quad \blacktriangle
\end{aligned}$$

The connection with limits of functions allows us to use l'Hôpital's rule to find limits of sequences.

Example 5 (a) Using numerical calculations, guess the value of $\lim_{n \rightarrow \infty} \sqrt[n]{n}$. (b) Use l'Hôpital's rule to verify the result in (a).

Solution (a) Using a calculator we find:

n	$\sqrt[n]{n}$
1	1
5	1.37973
10	1.25893
50	1.08138
100	1.04713
500	1.01251
1000	1.00693
5000	1.00170
10,000	1.00092

Thus it appears that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

(b) To verify this, we use l'Hôpital's rule to show that $\lim_{x \rightarrow \infty} x^{1/x} = 1$. The limit is in ∞^0 form, so we use logarithms:

$$x^{1/x} = e^{(\ln x)/x}.$$

Now $\lim_{x \rightarrow \infty} (\ln x / x)$ is in $\frac{\infty}{\infty}$ form, and l'Hôpital's rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Hence

$$\begin{aligned}
\lim_{x \rightarrow \infty} x^{1/x} &= \exp \left(\lim_{x \rightarrow \infty} \frac{\ln x}{x} \right) \\
&= \exp(0) = 1,
\end{aligned}$$

confirming our numerical calculations. $\blacktriangle$

When we introduced limits of sequences in Example 1, we implicitly used the fact that $\lim_{n \rightarrow \infty} (1/10^n) = 0$. The following general fact is useful.

Limits of Powers

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} \infty, & \text{if } r > 1, \\ 1, & \text{if } r = 1, \\ 0, & \text{if } 0 \leq r < 1. \end{cases}$$

To see this, first consider the case $r > 1$. We write r as $1 + s$ where $s > 0$. If we expand $r^n = (1 + s)^n$, we get $r^n = 1 + ns + (\text{other positive terms})$. Therefore, $r^n \geq 1 + ns$, which goes to ∞ as $n \rightarrow \infty$. Second, if $r = 1$, then $r^n = 1$ for all n , so $\lim_{n \rightarrow \infty} r^n = 1$. Finally, if $0 \leq r < 1$, then excluding the easy case $r = 0$, we let $\rho = 1/r$ so $\rho > 1$, and so $\lim_{n \rightarrow \infty} \rho^n = \infty$. Therefore, $\lim_{n \rightarrow \infty} r^n = \lim_{n \rightarrow \infty} (1/\rho^n) = 0$ (compare the reciprocal test for limits of functions in Section 11.1).

Example 6 Evaluate (a) $\lim_{n \rightarrow \infty} 3^n$, (b) $\lim_{n \rightarrow \infty} e^{-n}$, (c) $\lim_{n \rightarrow \infty} (e + (\frac{2}{3})^n)^4$.

Solution (a) Here $r = 3 > 1$, so $\lim_{n \rightarrow \infty} 3^n = \infty$.
 (b) $e^{-n} = (1/e)^n$, and $1/e < 1$, so $\lim_{n \rightarrow \infty} e^{-n} = 0$.
 (c) $\lim_{n \rightarrow \infty} [e + (\frac{2}{3})^n]^4 = [e + \lim_{n \rightarrow \infty} (\frac{2}{3})^n]^4 = [e + 0]^4 = e^4$. $\blacktriangle$

Another useful test is the comparison test: it says that if $\lim_{n \rightarrow \infty} a_n = 0$ and if $|b_n| \leq |a_n|$, then $\lim_{n \rightarrow \infty} b_n = 0$ as well. This is plausible since b_n is squeezed between $-|a_n|$ and $|a_n|$ which are tending to zero. We ask the reader to supply the proof in Exercise 56.

Comparison Test

If $\lim_{n \rightarrow \infty} a_n = 0$ and $|b_n| \leq |a_n|$ then $\lim_{n \rightarrow \infty} b_n = 0$.

Example 7 Find (a) $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$ and (b) $\lim_{n \rightarrow \infty} \frac{(-1)^n + n}{n}$.

Solution (a) If $a_n = 1/n$ and $b_n = (\sin n)/n$, then $a_n \rightarrow 0$ and $|b_n| \leq |a_n|$, so by the comparison test, $\lim_{n \rightarrow \infty} (\sin n)/n = 0$.
 (b) $|(-1)^n/n| \leq 1/n \rightarrow 0$, so $(-1)^n/n \rightarrow 0$ by the comparison test. Thus

$$\lim_{n \rightarrow \infty} \left(\frac{(-1)^n + n}{n} \right) = \lim_{n \rightarrow \infty} \left(\frac{(-1)^n}{n} \right) + \lim_{n \rightarrow \infty} \frac{n}{n} = 0 + 1 = 1. \quad \blacktriangle$$

Many questions in mathematics and its applications lead to the problem of solving an equation of the form

$$f(x) = 0, \quad (5)$$

where f is some function. The solutions of equation (5) are called the *roots* or *zeros* of f . If f is a polynomial of degree at most 4, one can find the roots of f by substituting the coefficients of f into a general formula (see pp. 17 and 173). On the other hand, if f is a polynomial of degree 5 or greater, or a function involving the trigonometric or exponential functions, there may be no explicit formula for the roots of f , and one may have to search for the solution numerically.

Newton's method uses linear approximations to produce a sequence $x_0, x_1, x_2, \dots$ which converges to a solution of $f(x) = 0$. Let x_0 be a first guess. We seek to correct this guess by an amount Δx so that $f(x_0 + \Delta x) = 0$. Solving this equation for Δx is no easier than solving the original equation (5), so we manufacture an easier problem, replacing f by its first-order approximation at x_0 ; that is, we replace $f(x_0 + \Delta x)$ by $f(x_0) + f'(x_0)\Delta x$. If $f(x_0)$ is not equal to zero, we can solve the equation $f(x_0) + f'(x_0)\Delta x = 0$ to obtain $\Delta x = -f(x_0)/f'(x_0)$, so that our new guess is

$$x_1 = x_0 + \Delta x = x_0 - f(x_0)/f'(x_0).$$

Geometrically, we have found x_1 by following the tangent line to the graph of f at $(x_0, f(x_0))$ until it meets the x axis; the point where it meets is $(x_1, 0)$ (see Fig. 11.4.7).

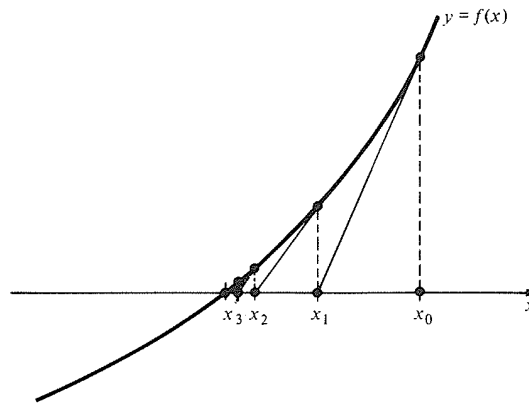


Figure 11.4.7. The geometry of Newton's method.

Now we find a new guess x_2 by repeating the procedure with x_1 in place of x_0 ; that is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, once we have found x_n , we define x_{n+1} by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (6)$$

Let us see how the method works in a case where we know the answer in advance. (This iteration procedure is particularly easy to use on a programmable calculator.)

Example 8 Use Newton's method to find the first few approximations to a solution of the equation $x^2 = 4$, taking $x_0 = 1$.

Solution To put the equation $x^2 = 4$ in the form $f(x) = 0$, we let $f(x) = x^2 - 4$. Then $f'(x) = 2x$, so the iteration rule (6) becomes $x_{n+1} = x_n - (x_n^2 - 4)/2x_n$, which may be simplified to $x_{n+1} = \frac{1}{2}(x_n + 4/x_n)$. Applying this formula repeatedly, with $x_0 = 1$, we get (to the limits of our calculator's accuracy)

$$\begin{aligned} x_1 &= 2.5 \\ x_2 &= 2.05 \\ x_3 &= 2.000609756 \\ x_4 &= 2.000000093 \\ x_5 &= 2 \\ x_6 &= 2 \\ &\vdots \end{aligned}$$

and so on forever. The number 2 is, of course, precisely the positive root of our equation $x^2 = 4$. ▲

Example 9 Use Newton's method to locate a root of $x^5 - x^4 - x + 2 = 0$. Compare what happens with various starting values of x_0 and attempt to explain the phenomenon.

Solution The iteration formula is

$$x_{n+1} = x_n - \frac{x_n^5 - x_n^4 - x_n + 2}{5x_n^4 - 4x_n^3 - 1} = \frac{4x_n^5 - 3x_n^4 - 2}{5x_n^4 - 4x_n^3 - 1}.$$

For the purpose of convenient calculation, we may write this as

$$x_{n+1} = \frac{(4x_n - 3)x_n^4 - 2}{(5x_n - 4)x_n^3 - 1}.$$

Starting at $x_0 = 1$, we find that the denominator is undefined, so we can go no further. (Can you interpret this difficulty geometrically?)

Starting at $x_0 = 2$, we get

$$x_1 = 1.659574468,$$

$$x_2 = 1.372968569,$$

$$x_3 = 1.068606737,$$

$$x_4 = -0.5293374382,$$

$$x_5 = 169.5250382.$$

The iteration process seems to have sent us out on a wild goose chase. To see what has gone wrong, we look at the graph of $f(x) = x^5 - x^4 - x + 2$. (See Fig. 11.4.8.) There is a “bowl” near $x_0 = 2$; Newton's method attempts to take us down to a nonexistent root. (Only after many iterations does one converge to the root—see Exercise 59 and Example 10.)

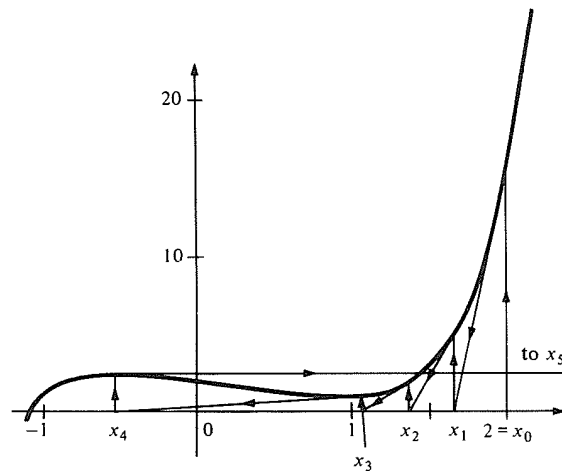


Figure 11.4.8. Newton's method does not always work.

Finally, we start with $x_0 = -2$. The iteration gives

$$x_0 = -2, \quad f(x_0) = -44;$$

$$x_1 = -1.603603604, \quad f(x_1) = -13.61361361;$$

$$x_2 = -1.323252501, \quad f(x_2) = -3.799819057;$$

$$x_3 = -1.162229582, \quad f(x_3) = -0.782974790;$$

$$x_4 = -1.107866357, \quad f(x_4) = -0.067490713;$$

$$x_5 = -1.102228599, \quad f(x_5) = -0.000663267;$$

$$x_6 = -1.102172085, \quad f(x_6) = -0.000000061;$$

$$x_7 = -1.102172080, \quad f(x_7) = -0.000000003.$$

Since the numbers in the $f(x)$ column appear to be converging to zero and those in the x column are converging, we obtain a root to be (approximately) -1.10217208 . Since $f(x)$ is negative at this value (where $f(x) = -0.000000003$) and positive at -1.10217207 (where $f(x) = 0.000000115$), we can conclude, by the intermediate value theorem, that the root is between these two values. $\blacktriangle$

Example 9 illustrates several important features of Newton's method. First of all, it is important to start with an initial guess which is reasonably close to a root—graphing is a help in making such a guess. Second, we notice that once we get near a root, then convergence becomes very rapid—in fact, the number of correct decimal places is approximately *doubled* with each iteration. Finally, we notice that the process for passing from x_n to x_{n+1} is the same for each value of n ; this feature makes Newton's method particularly attractive for use with a programmable calculator or a computer. Human intelligence still comes into play in the choice of the first guess, however.

Newton's Method

To find a root of the equation $f(x) = 0$, where f is a differentiable function such that f' is continuous, start with a guess x_0 which is reasonably close to a root. Then produce the sequence $x_0, x_1, x_2, \dots$ by the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $\lim_{n \rightarrow \infty} x_n = \bar{x}$, then $f(\bar{x}) = 0$.

To justify the last statement in the box above, we suppose that $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Taking limits on both sides of the equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$, we obtain $\bar{x} = \bar{x} - \lim_{n \rightarrow \infty} [f(x_n)/f'(x_n)]$, or $\lim_{n \rightarrow \infty} [f(x_n)/f'(x_n)] = 0$. Now let $a_n = f(x_n)/f'(x_n)$. Then we have $\lim_{n \rightarrow \infty} a_n = 0$, while $f(x_n) = a_n f'(x_n)$. Taking limits as $n \rightarrow \infty$ and using the continuity of f and f' , we find

$$\begin{aligned} \lim_{n \rightarrow \infty} f(x_n) &= \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} f'(x_n), \quad \text{so} \\ f(\bar{x}) &= 0 \cdot f'(\bar{x}) = 0. \end{aligned}$$

Newton's method, applied with care, can also be used to solve equations involving trigonometric or exponential functions.

Example 10 Use Newton's method to find a positive number x such that $\sin x = x/2$.

Solution With $f(x) = \sin x - x/2$, the iteration formula becomes

$$x_{n+1} = x_n - \frac{\sin x_n - x_n/2}{\cos x_n - 1/2} = \frac{2(x_n \cos x_n - \sin x_n)}{2 \cos x_n - 1}.$$

Taking $x_0 = 0$ as our first guess, we get $x_1 = 0$, $x_2 = 0$, and so forth, since zero is already a root of our equation. To find a positive root, we try another guess, say $x_0 = 6$. We get

$x_1 = 13.12652598$	$x_5 = 266.0803351$
$x_2 = 30.50101246$	$x_6 = 143.3754278$
$x_3 = 176.5342378$	$\vdots$
$x_4 = 448.4888306$	$x_9 = -759.1194553$
	$x_{10} = 3,572.554623$

We do not seem to be getting anywhere. To see what might be wrong, we draw a sketch (Fig. 11.4.9). The many places where the graph of $\sin x - x/2$

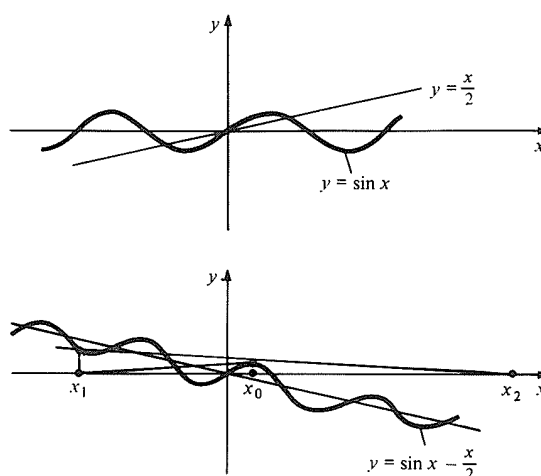


Figure 11.4.9. Newton's method goes awry.

has a horizontal or nearly horizontal tangent causes the Newton sequence to make wild excursions.⁶ We need to make a better first guess; we try $x_0 = 3$. This gives

$$\begin{array}{ll} x_1 = 2.08799541, & x_5 = 1.89549427, \\ x_2 = 1.91222926, & x_6 = 1.89549427, \\ x_3 = 1.89565263, & x_7 = 1.89549427, \\ x_4 = 1.89549428, & x_8 = 1.89549427. \end{array}$$

We conclude that our root is somewhere near 1.89549427. Substituting this value for x in $\sin x - x/2$ gives 1.0×10^{-11} . There may be further doubt about the last figure, due to internal roundoff errors in the calculator; we are probably safe to announce our result as 1.8954943. ▲

You may find it amusing to try other starting values for x_0 in Example 10. For instance, the values 6.99, 7, and 7.01 seem to lead to totally different results. (This was on a HP 15C hand calculator. Numerical errors may be crucial in a calculation such as this.) Recently, the study of sequences defined by iteration has become important as a model for the long-time behavior of dynamical systems. For instance, sequences defined by simple rules of the form $x_{n+1} = ax_n(1 - x_n)$ display very different behavior according to the value of the constant a . (See the supplement to this section and Exercise 59.)

Supplement to Section 11.4 Newton's Method and Chaos

The sequences generated by Newton's method may exhibit several types of strange behavior if the starting guess is not close to a root:

- (a) the sequence $x_0, x_1, x_2, \dots$ may wander back and forth over the real line for some time before converging to a root;

⁶ Try these calculations and those in Example 9 on your calculator and see if you converge to the root after many iterations. You will undoubtedly get different numbers from ours, probably due to roundoff errors, computer inaccuracies and the extreme sensitivity of the calculations. We got four different sets of numbers with four calculators. (The ones here were found on an HP 15C which also has a SOLVE algorithm which cleverly avoids many difficulties.)

- (b) slightly different choices of x_0 or the use of different calculators may lead to very different sequences;
- (c) the sequence $x_0, x_1, x_2, \dots$ may eventually cycle between two or more values, none of which is a root of the equation we are trying to solve;
- (d) the sequence $x_0, x_1, x_2, \dots$ may wander "forever" without ever settling into a regular pattern.

Recent research in pure and applied mathematics has shown that the type of erratic behavior just described is the rule rather than the exception for many mathematical operations and the physical processes which they model (see Exercise 59 for a simple example). Indeed, "chaotic" behavior is observed in fluid flow, chemical reactions, and biological systems, and is responsible for the inherent unpredictability of the weather.

Some references on this work on "chaos", aimed at the nonexpert reader, are:

- M. J. Feigenbaum, "Universal behavior in nonlinear systems," *Los Alamos Science* **1** (Summer 1980), 4–27.
- D. R. Hofstadter, "Metamagical theams," *Scientific American* **245** (November 1981), 22–43.
- L. P. Kadanoff, "Roads to chaos," *Physics Today* **36** (December 1983), 46–53.
- D. Ruelle, "Strange attractors," *Math. Intelligencer* **2** (1980), 126–137.
- D. G. Saari and J. B. Urenko, "Newton's method, circle maps, and chaotic motion," *American Mathematical Monthly* **91** (1984), 3–17; see also **92** (1985) 157–158.

Exercises for Section 11.4

- If $a_n = 1/10 + 1/100 + \dots + 1/10^n$, how large must n be for $\frac{1}{5} - a_n$ to be less than 10^{-6} ?
 - If $a_n = 7/10 + 7/100 + \dots + 7/10^n$, how large must n be for $\frac{7}{5} - a_n$ to be less than 10^{-8} ?
- Find the limits of the sequences in Exercises 3 and 4.
- $a_n = 1 + 1/2 + 1/4 + \dots + 1/2^n$.
 - $a_n = \sin(n\pi/2)$.
- Write down the first six terms of the sequences in Exercises 5–10.
- $k_n = n^2 - 2\sqrt{n}$; $n = 0, 1, 2, \dots$
 - $a_n = (-1)^{n+1}[(n-1)/n!]$; $n = 0, 1, 2, \dots$
($0! = 1$, $n! = n(n-1) \dots 3 \cdot 2 \cdot 1$.)
 - $b_n = nb_{n-1}/(1+n)$; $b_0 = \frac{1}{2}$.
 - $c_{n+1} = -c_n/[2n(4n+1)]$; $c_1 = 2$.
 - $a_{n+1} = [1/(n+1)]\sum_{i=0}^n a_i$; $a_0 = \frac{1}{2}$.
 - $k_n = \sqrt{3n^2 + 2n}$; $n = 1, 2, 3, \dots$

Establish the limits in Exercises 11–14 using the ϵ - N definition.

- $\lim_{n \rightarrow \infty} \frac{3}{n} = 0$
- $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1$
- $\lim_{n \rightarrow \infty} \frac{3}{2n+1} = 0$
- $\lim_{n \rightarrow \infty} \frac{2}{2n+5} = 0$

Evaluate the limits in Exercises 15–24.

- $\lim_{n \rightarrow \infty} \frac{3n}{n+1}$
- $\lim_{n \rightarrow \infty} \frac{2n}{8n-1}$

- $\lim_{n \rightarrow \infty} \frac{n-3n^2}{n^2+1}$
- $\lim_{n \rightarrow \infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$
- $\lim_{n \rightarrow \infty} \left[\frac{3n^2-2n+1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)} \right]^2$
- $\lim_{n \rightarrow \infty} \frac{2+1/n}{(n^2-2)/(n^2+1)}$
- $\lim_{n \rightarrow \infty} \frac{(\sin n)^2}{n+2}$
- $\lim_{n \rightarrow \infty} \frac{(1+n)\cos(n+1)}{n^2+1}$
- $\lim_{n \rightarrow \infty} \frac{(-1)^n \cdot 2}{n+1}$
- $\lim_{n \rightarrow \infty} \frac{\cos(\pi\sqrt{n})}{n^2}$

■ Using numerical calculations, guess the limit as $n \rightarrow \infty$ of the sequences in Exercises 25–28. Verify your answers using l'Hôpital's rule.

- $\sqrt[n]{n/2}$
- $\sqrt[n]{n/3}$
- $\sqrt[n]{n(n+1)/4}$
- $\sqrt[n]{n(n+3)/7}$

Find the limits in Exercises 29–34.

- $\lim_{n \rightarrow \infty} \frac{1}{8^n}$
- $\lim_{n \rightarrow \infty} \pi^n$
- $\lim_{n \rightarrow \infty} \frac{n+(3/4)^n}{n^2+2}$
- Find $\lim_{n \rightarrow \infty} (\pi + (\frac{2}{3})^n)^3$
- $\lim_{n \rightarrow \infty} \left[\frac{3b + (1/2)^{2n}}{n^2-1} \right]^3$; b constant
- $\lim_{n \rightarrow \infty} \left(\frac{2a + e^{-2n}}{n-1} \right)^2$; a constant

35. (a) Use Newton's method to find a solution of $x^3 - 8x^2 + 2x + 1 = 0$. (b) Use division and the quadratic formula to find the other two roots.⁷
36. Use Newton's method to find all real roots of $x^3 - x + \frac{1}{10}$.
37. Use Newton's method to locate a root of $f(x) = x^5 + x^2 - 3$ with starting values $x_0 = 0$, $x_1 = 2$.
38. Use Newton's method to locate a zero for $f(x) = x^4 - 2x^3 - 1$. Use $x_0 = 2$, 3 , and -1 as starting values and compare the results.
39. Use Newton's method to locate a root of $\tan x = x$ in $[\pi/2, 3\pi/2]$.
40. Use Newton's method to find the following numbers: (a) $\sqrt{2}$; (b) $\sqrt[3]{2}$.
41. The equation $\tan x = \alpha x$ appears in heat conduction problems to determine values $\lambda_1, \lambda_2, \lambda_3, \dots$ that appear in the expression for the temperature distribution. The numbers $\lambda_1, \lambda_2, \dots$ are the positive solutions of $\tan x = \alpha x$, listed in increasing order. Find the numbers $\lambda_1, \lambda_2, \lambda_3$ for $\alpha = 2, 3, 5$, by Newton's method. Display your answers in a table.
42. (a) Use Newton's method to solve the equation $x^2 - 2 = 0$ to 8 decimal places of accuracy, using the initial guess $x_0 = 2$.
 (b) Find a constant C such that $|x_n - \sqrt{2}| \leq C|x_{n-1} - \sqrt{2}|^2$ for $n = 1, 2, 3$, and 4 . (See Review Exercise 101 for the theory of the rapid convergence of Newton's method.)
43. Experiment with Newton's method for evaluation of the root $1/e$ of the equation $e^{-ex} = 1/e$.
44. Enter the display value 1.0000000 on your calculator and repeatedly press the "sin" key using the "radian mode". This process generates display numbers $a_1 = 1.0000000$, $a_2 = 0.84147$, $a_3 = 0.74562$, $\dots$.
 (a) Write a formula for a_n , using function notation.
 (b) Conjecture the value of $\lim_{n \rightarrow \infty} a_n$. Explain with a graph.
45. Display the number 2 on your calculator. Repeatedly press the " x^2 " key. You should get the numbers 2, 4, 16, 256, 65536, $\dots$. Express the display value a_n after n repetitions by a formula.
46. Let $f(x) = 1 + 1/x$. Equipped with a calculator with a reciprocal function, complete the following:
 (a) Write out $f(f(f(f(f(f(2)))))$) as a division problem, and calculate the value. We abbreviate this as $f^{(6)}(2)$, meaning to display the value 2, press the " $1/x$ " key and add 1, successively six times.
 (b) Experiment to determine $\lim_{n \rightarrow \infty} [1/f^{(n)}(2)]$ to five decimal places.
47. Suppose that $\lim_{n \rightarrow \infty} a_n = a$ and that $a > 0$. Prove that there is a positive integer N such that $a_n > 0$ for all $n > N$.
48. Let $a_n = 1$ if n is even and -1 if n is odd. Does $\lim_{n \rightarrow \infty} a_n$ exist?
49. If a radioactive substance has a half-life of T , so that half of it decays after time T , write a sequence a_n showing the fraction remaining after time nT . What is $\lim_{n \rightarrow \infty} a_n$?
50. Evaluate:
 (a) $\lim_{n \rightarrow \infty} \left[\frac{(-1)^n}{3} + (-1)^{n+1} \left(\frac{1}{3} + \frac{2}{n} + \frac{1}{n^2} \right) \right]$;
 (b) $\lim_{n \rightarrow \infty} \left\{ \frac{(-1)^n}{2} + n \left[\frac{1 + (-1)^{n+1}(3n+1)}{6n^2 - 5n + 2} \right] \right\}$.
- Find the limit or prove that the limit does not exist in Exercises 51–54.
51. $\lim_{n \rightarrow \infty} \left(\frac{1}{2} + (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right)$.
52. $\lim_{n \rightarrow \infty} \left(\frac{(-1)^n \sin(n\pi/2)}{n} \right) \left(\frac{n^2 + 1}{n + 1} \right)$.
53. $\lim_{n \rightarrow \infty} \left(\frac{3n}{4n+1} + \frac{(-1)^n \sin n}{n+1} \right)$.
54. $\lim_{n \rightarrow \infty} \frac{(1+n)\cos n}{n}$.
- ★55. (a) Give an A - N definition of what $\lim_{n \rightarrow \infty} a_n = \infty$ means. (b) Prove, using your definition in part (a), that $\lim_{n \rightarrow \infty} [(1+n^2)/(1+8n)] = \infty$.
- ★56. If $a_n \rightarrow 0$ and $|b_n| \leq |a_n|$, show that $b_n \rightarrow 0$.
- ★57. Suppose that a_n , b_n , and c_n , $n = 1, 2, 3, \dots$, are sequences of numbers such that for each n , we have $a_n < b_n < c_n$.
 (a) If $\lim_{n \rightarrow \infty} a_n = L$ and if $\lim_{n \rightarrow \infty} b_n$ exists, show that $\lim_{n \rightarrow \infty} b_n = L$. [Hint: Suppose not!]
 (b) If $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$, prove that $\lim_{n \rightarrow \infty} b_n = L$.
- ★58. A rubber ball is released from a height h . Each time it strikes the floor, it rebounds with two-thirds of its previous velocity.
 (a) How far does the ball rise on each bounce? (Use the fact that the height y of the ball at time t from the beginning of each bounce is of the form $y = vt - \frac{1}{2}gt^2$ during the bounce. The constant g is the acceleration of gravity.)
 (b) How long does each bounce take?
 (c) Show that the ball stops bouncing after a finite time has passed.
 (d) How far has the ball travelled when it stops bouncing?
 (e) How would the results differ if this experiment were done on the moon?

⁷ For a computer, this method is preferable to using the formula for the roots of a cubic!

★59. (Research Problems)

- (a) Experiment with the recursion relation $x_{n+1} = ax_n(1 - x_n)$ for various values of the parameter a where $0 < a \leq 4$ and x_0 is in $[0, 1]$. How does the behavior of the sequences change when a varies?
- (b) Study the bizarre behavior of Newton's method in Example 9 for various starting values x_0 . Can you see a pattern? Does x_n always converge?
- (c) Study the bizarre behavior of Newton's method in Example 10.

11.5 Numerical Integration

Integrals can be approximated by sequences which can be computed numerically.

The fundamental theorem of calculus does not solve all our integration problems. The antiderivative of a given integrand may not be easy or even possible to find. The integrand might be given, not by a formula, but by a table of values; for example, we can imagine being given power readings from an energy cell and asked to find the energy stored. In either case, it is necessary to use a method of numerical integration to find an approximate value for the integral.

In using a numerical method, it is important to estimate errors so that the final answer can be said, with confidence, to be correct to so many significant figures. The possible errors include errors in the method, roundoff errors, and roundoff errors in arithmetic operations. The task of keeping careful track of possible errors is a complicated and fascinating one, of which we can give only some simple examples.⁸

The simplest method of numerical integration is based upon the fact that the integral is a limit of Riemann sums (see Section 4.3). Suppose we are given $f(x)$ on $[a, b]$, and divide $[a, b]$ into subintervals $a = x_0 < x_1 < \cdots < x_n = b$. Then $\int_a^b f(x) dx$ is approximated by $\sum_{i=1}^n f(c_i) \Delta x_i$, where c_i lies in $[x_{i-1}, x_i]$. Usually, the points x_i are taken to be equally spaced, so $\Delta x_i = (b - a)/n$ and $x_i = a + i(b - a)/n$. Choosing $c_i = x_i$ or x_{i+1} gives the method in the following box.

Riemann Sums

To calculate an approximation to $\int_a^b f(x) dx$, let $x_i = a + i(b - a)/n$ and form the sum

$$\frac{b-a}{n} [f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad (1a)$$

or

$$\frac{b-a}{n} [f(x_1) + f(x_2) + \cdots + f(x_n)]. \quad (1b)$$

⁸ For a further discussion of error analysis in numerical integration, see, for example, P. J. Davis, *Interpolation and Approximation*, Wiley, New York (1963).

Example 1 Let $f(x) = \cos x$. Evaluate $\int_0^{\pi/2} \cos x \, dx$ by the method of Riemann sums, taking 10 equally spaced points: $x_0 = 0$, $x_1 = \pi/20$, $x_2 = 2\pi/20$, $\dots$, $x_{10} = 10\pi/20 = \pi/2$, and $c_i = x_i$. Compare the answer with the actual value.

Solution Formula (1a) gives

$$\begin{aligned} \int_0^{\pi/2} \cos x \, dx &\approx \frac{\pi}{20} \left(1 + \cos \frac{\pi}{20} + \cos \frac{2\pi}{20} + \dots + \cos \frac{9\pi}{20} \right) \\ &= \frac{\pi}{20} (1 + 0.98769 + 0.95106 + \dots + 0.15643) \\ &= \frac{\pi}{20} (6.85310) = 1.07648. \end{aligned}$$

The actual value is $\sin(\pi/2) - \sin(0) = 1$, so our estimate is about 7.6% off. $\blacktriangle$

Unfortunately, this method is inefficient, because many points x_i are needed to get an accurate estimate of the integral. For this reason we will seek alternatives to the method of Riemann sums.

To get a better method, we estimate the area in each interval $[x_{i-1}, x_i]$ more accurately by replacing the rectangular approximation by a trapezoidal one. (See Fig. 11.5.1.) We join the points $(x_i, f(x_i))$ by straight line segments to obtain a set of approximating trapezoids. The area of the trapezoid between x_{i-1} and x_i is

$$A_i = \frac{1}{2} [f(x_{i-1}) + f(x_i)] \Delta x_i$$

since the area of a trapezoid is its average height times its width.

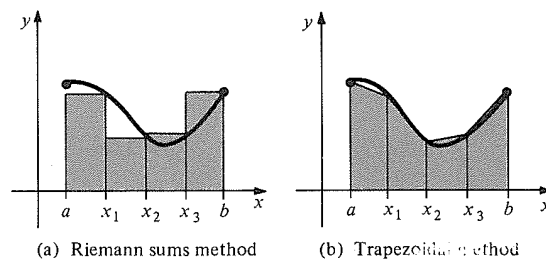


Figure 11.5.1. Comparing two methods of numerical integration.

The approximation to $\int_a^b f(x) \, dx$ given by the trapezoidal rule is $\sum_{i=1}^n \frac{1}{2} [f(x_{i-1}) + f(x_i)] \Delta x_i$. This becomes simpler if the points x_i are equally spaced. Then $\Delta x_i = (b - a)/n$, $x_i = a + i(b - a)/n$, and the sum is

$$\left(\frac{b-a}{n} \right) \sum_{i=1}^n \frac{1}{2} [f(x_{i-1}) + f(x_i)]$$

which can be rewritten as

$$\frac{b-a}{2n} [f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)]$$

since every term occurs twice except those from the endpoints. Although we used areas to obtain this formula, we may apply it even if $f(x)$ takes negative values.

Trapezoidal Rule

To calculate an approximation to $\int_a^b f(x) dx$, let $x_i = a + i(b-a)/n$ and form the sum

$$\frac{b-a}{2n} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]. \quad (2)$$

Formula (2) turns out to be much more accurate than the method of Riemann sums, even though it is just the average of the Riemann sums (1a) and (1b). Using results of Section 12.5, one can show that the error in the *method* (apart from other roundoff or cumulative errors) is $\leq [(b-a)/12]M_2(\Delta x)^2$, where M_2 is the maximum of $|f''(x)|$ on $[a, b]$. Of course, if we are given only numerical data, we have no way of estimating M_2 , but if a formula for f is given, M_2 can be determined. Note, however, that the error depends on $(\Delta x)^2$, so if we divide $[a, b]$ into k times as many divisions, the error goes down by a factor of $1/k^2$. The error in the Riemann sums method, on the other hand, is $\leq (b-a)M_1(\Delta x)$, where M_1 is the maximum of $|f'(x)|$ on $[a, b]$. Here Δx occurs only to the first power. Thus even if we do not know how large M_1 and M_2 are, if n is taken large enough, the error in the trapezoidal rule will eventually be much smaller than that in the Riemann sums method.

Example 2 Repeat Example 1 by using the trapezoidal rule. Compare the answer with the true value.

Solution Now formula (2) becomes

$$\begin{aligned} & \frac{\pi/2}{2 \cdot 10} \left(\cos 0 + 2 \cos \frac{\pi}{20} + \cdots + 2 \cos \frac{9\pi}{20} + \cos \frac{\pi}{2} \right) \\ & \approx \frac{\pi}{40} [1 + 2(0.9877 + 0.9511 + \cdots + 0.1564) + 0] \approx 0.9979. \end{aligned}$$

The answer is correct to within about 0.2%, much better than the accuracy in Example 1. $\blacktriangle$

Example 3 Use the trapezoidal rule with $n = 10$ to estimate numerically the area of the surface obtained by revolving the graph of $y = x/(1+x^2)$ about the x axis, $0 \leq x \leq 1$.

Solution The area is given by formula (2) on p. 483:

$$\begin{aligned} A &= 2\pi \int_0^1 \left(\frac{x}{1+x^2} \right) \sqrt{1 + \left[\frac{d}{dx} \left(\frac{x}{1+x^2} \right) \right]^2} dx \\ &= 2\pi \int_0^1 \frac{x \sqrt{(1+x^2)^4 + (1-x^2)^2}}{(1+x^2)^3} dx. \end{aligned}$$

There is little hope of carrying out this integration, so a numerical approach seems appropriate. We use the trapezoidal rule with the following values:

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y_i = f(x_i)$	0	0.13797	0.25713	0.34668	0.40650	0.44369	0.46684	0.48204	0.49216	0.49807	0.50000

where $f(x) = x\sqrt{(1+x^2)^4 + (1-x^2)^2} / (1+x^2)^3$. Inserting these data in the formula

$$\int_a^b f(x) dx \approx \left(\frac{b-a}{2n}\right) [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

with $x_i = a + [i(b-a)/n]$, $a = 0$ and $b = 1$, gives

$$\int_0^1 \frac{x\sqrt{(1+x^2)^4 + (1-x^2)^2}}{(1+x^2)^3} dx \approx 0.37811,$$

so the area is $A \approx (2\pi)(0.37811) = 2.3757$. Of course, we cannot be sure how many decimal places in this result are correct without an error analysis (see Exercise 17).⁹ ▲

There is a yet more powerful method of numerical integration called Simpson's rule,¹⁰ which is based on approximating the graph by parabolas rather than straight lines. To determine a parabola we need to specify three points through which it passes; we will choose the adjacent points

$$(x_{i-1}, f(x_{i-1})), \quad (x_i, f(x_i)), \quad (x_{i+1}, f(x_{i+1})).$$

It is easily proved (see Exercise 11) that the integral from x_{i-1} to x_{i+1} of the quadratic function whose graph passes through these three points is

$$A_i = \frac{\Delta x}{3} [f(x_{i-1}) + 4f(x_i) + f(x_{i+1})],$$

where $\Delta x = x_i - x_{i-1} = x_{i+1} - x_i$ (equally spaced points). See Fig. 11.5.2.

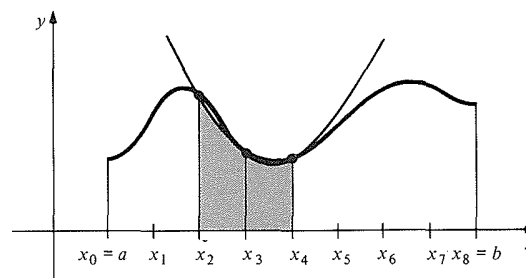


Figure 11.5.2. Illustrating Simpson's rule.

If we do this for every set of three adjacent points, starting at the left endpoint a —that is, for $\{x_0, x_1, x_2\}$, then $\{x_2, x_3, x_4\}$, then $\{x_4, x_5, x_6\}$, and so on—we will get an approximate formula for the area. In order for the points to fill the interval exactly, n should be even, say $n = 2m$.

As in the trapezoidal rule, the contributions from endpoints a and b are counted only once, as are those from the center points of triples $\{x_{i-1}, x_i, x_{i+1}\}$ (that is, x_i for i odd), while the others are counted twice. Thus we are led to Simpson's rule, stated in the box on the next page.

This method is very accurate; the error in using formula (3) does not exceed $[(b-a)/180]M_4(\Delta x)^4$, where M_4 is the maximum of the fourth derivative of $f(x)$ on $[a, b]$. As Δx is taken smaller and smaller, this error decreases much faster than in the other two methods. It is remarkable that juggling the

⁹ The HP 15C has a clever integration program that is careful about errors. It gives 2.3832 for this integral in a few minutes.

¹⁰ It was discussed by Thomas Simpson in his book, *Mathematical Dissertations on Physical and Analytical Subjects* (1743).

coefficients to give formula (3) in place of formula (1) or formula (2) can increase the accuracy so much.

Simpson's Rule

To calculate an approximation to $\int_a^b f(x) dx$, let $n = 2m$ be even and $x_i = a + i(b - a)/n$. Form the sum

$$\frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]. \quad (3)$$

Example 4 Repeat Example 1 using Simpson's rule. Compare the answer with the true value.

Solution Using a calculator, we can evaluate formula (3) by

$$\begin{aligned} \frac{\pi/2}{3 \cdot 10} & \left(\cos 0 + 4 \cos \frac{\pi}{20} + 2 \cos \frac{2\pi}{20} + 4 \cos \frac{3\pi}{20} + 2 \cos \frac{4\pi}{20} \right. \\ & \quad \left. + 4 \cos \frac{5\pi}{20} + 2 \cos \frac{6\pi}{20} + 4 \cos \frac{7\pi}{20} + 2 \cos \frac{8\pi}{20} + 4 \cos \frac{9\pi}{20} + \cos \frac{\pi}{2} \right) \\ & \approx \frac{\pi}{60} (1 + 3.9507534 + \cdots + 0) = \frac{\pi}{60} \cdot 19.098658 \\ & \approx 1.0000034. \end{aligned}$$

The error is less than four parts in a million. ▲

Example 5 Suppose that you are given the following table of data:

$f(0) = 0.846$	$f(0.4) = 1.121$	$f(0.8) = 2.321$
$f(0.1) = 0.928$	$f(0.5) = 1.221$	$f(0.9) = 3.101$
$f(0.2) = 0.882$	$f(0.6) = 1.661$	$f(1.0) = 3.010$
$f(0.3) = 0.953$	$f(0.7) = 2.101$	

Evaluate $\int_0^1 f(x) dx$ by Simpson's rule.

Solution By formula (3),

$$\begin{aligned} \int_0^1 f(x) dx & \approx \frac{1}{30} [f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) \\ & \quad + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1.0)]. \end{aligned}$$

Inserting the given values of f and evaluating on a calculator, we get

$$\int_0^1 f(x) dx = \frac{1}{30} (49.042) = 1.635.$$

This should be quite accurate unless the fourth derivative of f is very large. ▲

Example 6 How small must we take Δx in the trapezoidal rule to evaluate $\int_2^4 e^{-x^2} dx$ to within 10^{-6} ? For Simpson's rule?

Solution Let $f(x) = e^{-x^2}$, $a = 2$, and $b = 4$. The error in the trapezoidal rule is no more than $[(b-a)/12]M_2(\Delta x)^2$, where M_2 is the maximum of $|f''(x)|$ on $[a, b]$. We

find

$$f'(x) = -2xe^{-x^2}, \quad \text{and} \quad f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}.$$

Now $f'''(x) = (12x - 8x^3)e^{-x^2} = 4x(3 - 2x^2)e^{-x^2} < 0$ on $[2, 4]$, so $f''(x)$ is decreasing. Also, $f''(x) > 0$ on $[2, 4]$, so $|f''(x)| = f''(x) \leq f''(2) = 14e^{-4} = M_2$, so the error is at most

$$\frac{b-a}{12} M_2(\Delta x)^2 = \frac{1}{6} \cdot 14e^{-4}(\Delta x)^2.$$

To make this less than 10^{-6} , we should choose Δx so that

$$\frac{1}{6} \cdot 14e^{-4}(\Delta x)^2 < 10^{-6},$$

$$(\Delta x)^2 < e^4 10^{-6} \cdot \frac{3}{7} = 0.0000234,$$

$$\Delta x < 0.0048.$$

That is, we should take at least $n = (b-a)/\Delta x = 416$ divisions.

For Simpson's rule, the error is at most $[(b-a)/180]M_4(\Delta x)^4$. Here

$$f''''(x) = 4(4x^4 - 12x^2 + 3)e^{-x^2}.$$

On $[2, 4]$, we find that $4x^4 - 12x^2 + 3$ is increasing and e^{-x^2} is decreasing, so

$$\begin{aligned} |f''''(x)| &\leq 4(4 \cdot 4^4 - 12 \cdot 4^2 + 3)e^{-4} \\ &= 61.17 = M_4. \end{aligned}$$

Thus $[(b-a)/180]M_4(\Delta x)^4 = \frac{1}{90} \cdot 61.17(\Delta x)^4 = 0.68(\Delta x)^4$. Hence if we are to have error less than 10^{-6} , it suffices to have

$$0.68(\Delta x)^4 \leq 10^{-6},$$

$$\Delta x \leq 0.035.$$

Thus we should take at least $n = (b-a)/\Delta x = 57$ divisions. ▲

Exercises for Section 11.5

Use the indicated numerical method(s) to approximate the integrals in Exercises 1–4.

1. $\int_{-1}^1 (x^2 + 1) dx$. Use Riemann sums with $n = 10$ (that is, divide $[-1, 1]$ into 10 subintervals of equal length). Compare with the actual value.
2. $\int_0^{\pi/2} (x + \sin x) dx$. Use Riemann sums and the trapezoidal rule with $n = 8$. Compare these two approximate values with the actual value.
3. $\int_0^1 [(\sin \pi x/2)/(x^2 + 2x - 1)] dx$. Use the trapezoidal rule and Simpson's rule with $n = 12$.
4. $\int_0^2 (1/\sqrt{x^3 + 1}) dx$. Use the trapezoidal rule and Simpson's rule with $n = 20$.
5. Use Simpson's rule with $n = 10$ to find an approximate value for $\int_0^1 (x/\sqrt{x^3 + 2}) dx$.
6. Estimate the value of $\int_0^3 e^{\sqrt{x}} dx$, using Simpson's rule with $n = 4$. Check your answer using $x = u^2$, $dx = 2u du$.
7. Suppose you are given the following table of

data:

$f(0) = 1.384$	$f(0.4) = 0.915$	$f(0.8) = 0.935$
$f(0.1) = 1.179$	$f(0.5) = 0.768$	$f(0.9) = 1.262$
$f(0.2) = 0.973$	$f(0.6) = 0.511$	$f(1.0) = 1.425$
$f(0.3) = 1.000$	$f(0.7) = 0.693$	

Numerically evaluate $\int_0^1 (x + f(x)) dx$ by the trapezoidal rule.

8. Numerically evaluate $\int_0^1 2f(x) dx$ by Simpson's rule, where $f(x)$ is the function in Exercise 7.
9. Suppose that you are given the following table of data:

$f(0.0) = 2.037$	$f(1.3) = 0.819$
$f(0.2) = 1.980$	$f(1.4) = 1.026$
$f(0.4) = 1.843$	$f(1.5) = 0.799$
$f(0.6) = 1.372$	$f(1.6) = 0.662$
$f(0.8) = 1.196$	$f(1.7) = 0.538$
$f(1.0) = 0.977$	$f(1.8) = 0.555$
$f(1.2) = 0.685$	

- Numerically evaluate $\int_0^{1.8} f(x) dx$ by using Simpson's rule. [Hint: Watch out for the spacing of the points.]
10. Numerically evaluate $\int_0^{1.8} f(x) dx$ by using the trapezoidal rule, where $f(x)$ is the function in Exercise 9.
11. Evaluate $\int_a^b (px^2 + qx + r) dx$. Verify that Simpson's rule with $n = 2$ gives the exact answer. What happens if you use the trapezoidal rule? Discuss.
12. Evaluate $\int_a^b (px^3 + qx^2 + rx + s) dx$ by Simpson's rule with $n = 2$ and compare the result with the exact integral.
13. How large must n be taken in the trapezoidal rule to guarantee an accuracy of 10^{-5} in the evaluation of the integral in Exercise 2? Answer the same question for Simpson's rule.
14. *Gaussian quadrature* is an approximation method based on interpolation. The formula for integration on the interval $[-1, 1]$ is $\int_{-1}^1 f(x) dx = f(1/\sqrt{3}) + f(-1/\sqrt{3}) + R$, where the remainder R satisfies $|R| \leq M/135$, M being the largest value of $f^{(4)}(x)$ on $-1 \leq x \leq 1$.
- (a) The remainder R is zero for cubic polynomials. Check it for $x^3, x^3 - 1, x^3 + x + 1$.
- (b) Find $\int_{-1}^1 [x^2/(1+x^4)] dx$ to two places.
- (c) What is R for $\int_{-1}^1 x^6 dx$? Why is it so large?
15. A tank 15 meters by 60 meters is filled to a depth of 3.2 meters above the bottom. The time T it takes to empty half the tank through an orifice 0.5 meters wide by 0.2 meters high placed 0.1 meters from the bottom is given by

$$\frac{(2.2)T}{10^4} = \frac{1}{\sqrt{19.6}} \int_{130}^{190} \frac{dx}{(x+20)^{3/2} - x^{3/2}}.$$

Compute T from Simpson's rule with $n = 6$.

- ★16. A metropolitan sports and special events complex is circular in shape with an irregular roof that appears from a distance to be almost hemispherical (Fig. 11.5.3).

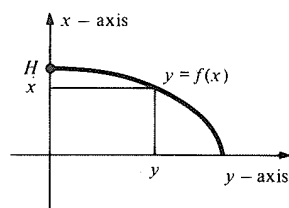


Figure 11.5.3. The profile of the roof of a sports complex.

A summer storm severely damaged the roof, requiring a roof replacement to go out for bid. Responding contractors were supplied with plans of the complex from which to determine an estimate. Estimators had to find the roof profile $y = f(x)$, $0 \leq x \leq H$, which gener-

ates the roof by revolution about the x axis (x and y in feet, x vertical, y horizontal).

- (a) Find the square footage of the roof via a surface area formula. This number determines the amount of roofing material required.
- (b) To check against construction errors, a tape measure is tossed over the roof and the measurement recorded. Give a formula for this measurement using the arc length formula.
- (c) Suppose the curve f is not given explicitly in the plans, but instead $f(0), f(4), f(8), f(12), \dots, f(H)$ are given (complex center-to-ceiling distances every 4 feet). Discuss how to use this information to numerically evaluate the integrals in (a), (b) above, using Example 3 as a guide.
- (d) Find an expression which approximates the surface area of the roof by assuming it is a *conoid* produced by a piecewise linear function constructed from the numbers $f(0), f(4), f(8), \dots, f(H)$.
- ★17. How many digits in the approximate value $A \approx 2.3757$ in Example 3 can be justified by an error analysis?
- ★18. (*Another numerical integration method*)
- (a) Let $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be n points in the plane such that all the x_i 's are different. Show that the polynomial of degree no more than $n - 1$ whose graph passes through the given points is

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x),$$

where $L_i(x) = A_i(x)/A'(x_i)$,

$$A(x) = (x - x_1)(x - x_2) \dots (x - x_n),$$

$$A_i(x) = A(x)/(x - x_i),$$

$$i = 1, 2, \dots, n.$$

(P is called the *Lagrange interpolation polynomial*.)

- (b) Suppose that you are given the following data for an unknown function $f(x)$:

$$f(0) = 0.01, \quad f(0.3) = 1.18,$$

$$f(0.1) = 0.12, \quad f(0.4) = 0.91.$$

$$f(0.2) = 0.82,$$

Estimate the value of $f(0.16)$ by using the Lagrange interpolation formula.

- (c) Estimate $\int_0^{0.4} f(x) dx$ (1) by using the trapezoidal rule, (2) by using Simpson's rule, and (3) by integrating the Lagrange interpolation polynomial.
- (d) Estimate $\int_0^{7/2} \cos x dx$ by using a Lagrange interpolation polynomial with $n = 4$. Compare your result with those obtained by the trapezoidal and Simpson's rules in Examples 2 and 4.

Review Exercises for Chapter 11

Verify the limits in Exercises 1–4 using the ε - δ definition.

1. $\lim_{x \rightarrow 1} (x^2 + x - 1) = 1$
2. $\lim_{x \rightarrow 0} (x^3 + 3x + 2) = 2$
3. $\lim_{x \rightarrow 2} (x^2 - 8x + 8) = -4$
4. $\lim_{x \rightarrow 5} (x^2 - 25) = 0$

Calculate the limits in Exercises 5–16.

5. $\lim_{x \rightarrow 0} \tan\left(\frac{x+1}{x-1}\right)$
6. $\lim_{x \rightarrow 1} \cos\left[\left(\frac{x+1}{x+2}\right) \frac{3\pi}{2}\right]$
7. $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)$
8. $\lim_{x \rightarrow \infty} \left(\frac{x^2+2}{3x^2+2x+1}\right)$
9. $\lim_{x \rightarrow \infty} (\sqrt{2x^2+1} - \sqrt{2}x)$
10. $\lim_{x \rightarrow \infty} (\sqrt{4x^2+1} - 2x)$
11. $\lim_{x \rightarrow 1^-} \frac{1}{\sqrt{1-x}}$
12. $\lim_{x \rightarrow 2^+} \frac{\sin\sqrt{x-2}}{\sqrt{x-2}}$
13. $\lim_{x \rightarrow 0} [x \sin(3/x^2)]$
14. $\lim_{x \rightarrow 0} \sin(\sqrt{x^2+2} - x)$
15. $\lim_{x \rightarrow 0} \frac{x^4+8x}{3x^4+2}$
16. $\lim_{x \rightarrow 0} \frac{x+3}{3x+8}$

17. Find the horizontal asymptotes of the graph $y = \tan^{-1}(3x+2)$. Sketch.

18. Find the vertical asymptotes for the graph of $y = 1/(x^2 - 3x - 10)^2$. Sketch.

Find the horizontal and vertical asymptotes of the functions in Exercises 19 and 20, and sketch.

19. $f(x) = \frac{x-1}{x^2+1}$
20. $f(x) = \frac{2x+3}{3x+5}$

Find the limits if they exist, using l'Hôpital's rule, in Exercises 21–44.

21. $\lim_{x \rightarrow \infty} \frac{x^3+8x+9}{4x^3-9x^2+10}$
22. $\lim_{x \rightarrow \infty} \frac{x}{x+2}$
23. $\lim_{x \rightarrow 0} \frac{1-\cos x}{3^x-2^x}$
24. $\lim_{x \rightarrow 1} \frac{x^x-1}{x-1}$
25. $\lim_{x \rightarrow 0} \frac{(\sqrt{x^2+9}-3)}{\sin x}$
26. $\lim_{x \rightarrow 0} \frac{\sqrt[3]{x^3+27}-3}{x}$
27. $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$
28. $\lim_{x \rightarrow 0} \frac{\tan^2 x}{x^2}$
29. $\lim_{x \rightarrow 2} \frac{\sin(x-2)-x+2}{(x-2)^3}$
30. $\lim_{x \rightarrow 0} \frac{24 \cos x - 24 + 12x^2 - x^4}{x^5}$
31. $\lim_{x \rightarrow 0} \frac{\tan(x+3) - \tan 3}{x}$
32. $\lim_{x \rightarrow \pi^2} \frac{\cos\sqrt{x}+1}{x-\pi^2}$
33. $\lim_{x \rightarrow 0} x \cot x$
34. $\lim_{x \rightarrow 0^+} \frac{\cot x}{\ln x}$

35. $\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$
36. $\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} - \frac{1}{2}\right)$
37. $\lim_{x \rightarrow \infty} x^2 e^{-x}$
38. $\lim_{x \rightarrow 0^+} x^3 (\ln x)^2$
39. $\lim_{x \rightarrow 0^+} x^{\sin x}$
40. $\lim_{x \rightarrow \infty} (\sin e^{-x})^{1/\sqrt{x}}$
41. $\lim_{x \rightarrow 0^+} (1 + \sin 2x)^{1/x}$
42. $\lim_{x \rightarrow 0^+} (\cos 2x)^{1/x}$
43. $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$
44. $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x^2+2x-8}$

Decide which improper integrals in Exercises 45–54 are convergent. Evaluate when possible.

45. $\int_1^\infty \frac{1}{x^2} dx$
46. $\int_{-\infty}^\infty \frac{\sin x}{x^2+3} dx$
47. $\int_2^\infty \frac{dx}{\ln x}$. [Hint: Prove $\ln x \leq x$ for $x \geq 2$.]
48. $\int_1^\infty \frac{dx}{\sqrt{x^2+8x+12}}$
49. $\int_1^2 \frac{1}{\sqrt{x-1}} dx$
50. $\int_{-1}^0 \frac{x+1}{\sqrt{1-x^2}} dx$
51. $\int_0^1 \frac{dx}{(1-x)^{2/5}}$
52. $\int_1^\infty x^2 e^{-x} dx$
53. $\int_0^1 x \ln x dx$
54. $\int_0^\infty (x+2)e^{-(x^2+4x)} dx$

Evaluate the limits in Exercises 55 and 56.

55. $\lim_{x \rightarrow \infty} \int_0^x \frac{dt}{t^2+t+1}$
56. $\lim_{x \rightarrow 0^+} \int_x^1 \frac{dt}{\sqrt{t}}$
57. The region under the curve $y = xe^{-x}$ on $[0, \infty)$ is revolved about the x axis. Find the volume of the resulting solid.
58. The curve $y = \sin x/x^2$ on $[1, \infty)$ is revolved around the x axis. Determine whether the resulting surface has finite area.

Evaluate the limits of the sequences in Exercises 59–72.

59. $\lim_{n \rightarrow \infty} \left(8 + \left(\frac{2}{3}\right)^n\right)^5$
60. $\lim_{n \rightarrow \infty} \sqrt[3]{\frac{8n^3-2n+1}{n^3+1}}$
61. $\lim_{n \rightarrow \infty} \left(1 + \frac{8}{n}\right)^n$
62. $\lim_{n \rightarrow \infty} \left(\frac{n-3}{n}\right)^{-2n}$
63. $\lim_{n \rightarrow \infty} (n^2+3n+1)e^{-n}$
64. $\lim_{n \rightarrow \infty} \frac{2^n}{n^2}$
65. $\lim_{n \rightarrow \infty} \frac{n}{n+2}$
66. $\lim_{n \rightarrow \infty} \frac{n^2+2n}{3n^2+1}$
67. $\lim_{n \rightarrow \infty} \tan\left[\frac{3n}{n+8}\right]$
68. $\lim_{n \rightarrow \infty} [\ln(n^2+1) - \ln(3n^2+5)]$

69. $\lim_{n \rightarrow \infty} \frac{\sin(\pi n/2)}{n^{-3}}$

70. $\lim_{n \rightarrow \infty} \frac{n \cos 4n\pi}{2n+1}$

71. $\lim_{n \rightarrow \infty} \frac{8-2n}{5n}$

72. $\lim_{n \rightarrow \infty} \left(1 - \frac{2+n}{3n+1}\right)$

Using l'Hôpital's rule if necessary, evaluate the limits of the sequences in Exercises 73–76.

73. $\lim_{n \rightarrow \infty} \sqrt[n]{3n}$

74. $\lim_{m \rightarrow \infty} m \log_{10}(2^{-m})$

75. $\lim_{n \rightarrow \infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right]$

76. $\lim_{j \rightarrow \infty} \frac{2e^{3j}}{e^{3j} + j^5}$

Use Newton's method for Exercises 77–80.

▮77. Locate the roots of $x^3 - 3x^2 + 8 = 0$.

▮78. Find the cube root of 21.

▮79. Solve the equation $e^x = 2 + x$.

▮80. Find two numbers, each of whose square is ten times its natural logarithm.

▮81. Evaluate $\int_2^3 (x^2 dx / \sqrt{x^2 + 1})$ by the trapezoidal rule with $n = 10$.

▮82. Evaluate the integral in Exercise 81 by Simpson's rule with $n = 10$.

▮83. Use Simpson's rule with $n = 10$ to calculate the volume obtained by revolving the curve $y = f(x)$ on $[1, 3]$ about the x axis, given the data:

$f(1) = 2.03$	$f(2.2) = 3.16$
$f(1.2) = 2.08$	$f(2.4) = 3.01$
$f(1.4) = 2.16$	$f(2.6) = 2.87$
$f(1.6) = 2.34$	$f(2.8) = 2.15$
$f(1.8) = 2.82$	$f(3) = 1.96$
$f(2) = 3.01$	

▮84. (a) Evaluate $(2/\sqrt{\pi}) \int_0^1 e^{-t^2} dt$ by using Simpson's rule with 10 subdivisions.

(b) Given an upper bound for the error in part (a). (See Example 6 of Section 11.5.)

(c) What does Simpson's rule with 10 subdivisions give for $(2/\sqrt{\pi}) \int_0^{10} e^{-t^2} dt$?

(d) The function $(2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$ is denoted $\text{erf}(x)$ and is called the error function. Its values are tabulated. (For example: *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, June 1964, pp. 310–311.) Compare your results with the tabulated results. *Note:* $\lim_{x \rightarrow \infty} \text{erf}(x) = 1$, and $\text{erf}(10)$ is so close to 1 that it probably won't be listed in the tables. Explain your result in part (c).

85. Let $f(x) = \cos x$ for $x \geq 0$ and $f(x) = 1$ for $x < 0$. Decide whether or not f is continuous or differentiable or both.

86. Let $f(x) = x^{1/\sin(x-1)}$. How should $f(1)$ be defined in order to make f continuous?

87. Find a function on $[0, 1]$ which is integrable (as an improper integral) but whose square is not.

88. Show that $\int_0^\infty [(\sin x)/(1+x)] dx$ is convergent. [*Hint:* Integrate by parts.]

89. (a) Show that

$$f''(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

if f'' is continuous at x_0 . [*Hint:* Use l'Hôpital's rule.]

★(b) Find a similar formula for $f'''(x_0)$.

90. Show that

$$f''(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + 2\Delta x) + f(x_0) - 2f(x_0 + \Delta x)}{(\Delta x)^2}$$

if f'' is continuous at x_0 .

91. Use Riemann sums to evaluate

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\ln n - \ln i) / n.$$

92. Let

$$S_n = \sum_{i=1}^{2n} \left[2 - \cos\left(\frac{i}{n}\pi\right) \right] \frac{1}{n}.$$

Prove that $\lim_{n \rightarrow \infty} S_n = 4$ using Riemann sums.

93. Let

$$S_n = \sum_{i=1}^n \left(\frac{i}{n} + \frac{i^2}{n^2} \right) \frac{1}{n}.$$

Prove that $S_n \rightarrow \frac{5}{6}$ as $n \rightarrow \infty$ by using Riemann sums.

94. Expressing the following sums as Riemann sums, show that:

$$(a) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\sqrt{\frac{i}{n}} - \left(\frac{i}{n}\right)^{3/2} \right] \frac{1}{n} = \frac{4}{15};$$

$$(b) \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3n}{(2n+i)^2} = \frac{1}{2}.$$

95. P dollars is deposited in an account each day for a year. The account earns interest at an annual rate r (e.g., $r = 0.05$ means 5%) compounded continuously. Use Riemann sums to show that the amount in the account at the end of the year is approximately

$$(365P/r)(e^r - 1).$$

★96. Evaluate:

$$\lim_{x \rightarrow \pi^2} \left[\frac{\sin \sqrt{x}}{(\sqrt{x} - \pi)(\sqrt{x} + \pi)} + \tan \sqrt{x} \right].$$

★97. Limits can sometimes be evaluated by geometric techniques. An important instance occurs when the curve $y = f(x)$ is trapped between the two intersecting lines through (a, L) with slopes m and $-m$, $0 < |x - a| \leq h$. Then $\lim_{x \rightarrow a} f(x) = L$, because points approaching $y = f(x)$ from the left or right are forced into a vertex, and therefore to the point (a, L) .

- (a) The equations of the two lines are $y = L + m(x - a)$, $y = L - m(x - a)$. Draw these on a figure and insert a representative graph for f which stays between the lines.
- (b) Show that the algebraic condition that f stay between the two straight lines is

$$\left| \frac{f(x) - L}{x - a} \right| \leq m, \quad 0 < |x - a| \leq h.$$

This is called a *Lipschitz condition*.

- (c) Argue that a Lipschitz condition implies $\lim_{x \rightarrow a} f(x) = L$, by appeal to the definition of limit.

★98. Another geometric technique for evaluation of limits is obtained by requiring that $y = f(x)$ be trapped on $0 \leq |x - a| \leq h$ between two power curves

$$y = L + m(x - a)^\alpha, \quad y = L - m(x - a)^\alpha,$$

where $\alpha > 0$, $m > 0$. The resulting algebraic condition is called a *Hölder condition*:

$$\frac{|f(x) - L|}{|x - a|^\alpha} \leq m, \quad 0 < |x - a| \leq h.$$

- (a) Verify that the described geometry leads to the Hölder condition.
- (b) Argue geometrically that, in the presence of a Hölder condition, $\lim_{x \rightarrow a} f(x) = L$.
- (c) Prove the contention in (b) by appeal to the definition of limit.
- ★99. Prove the chain rule for differentiable functions, $(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$, as follows:

- (a) Let $y = g(x)$ and $z = f(y)$, and write $\Delta y = g'(x_0)\Delta x + \rho(x)$. Show that

$$\lim_{\Delta x \rightarrow 0} \frac{\rho(x)}{\Delta x} = 0$$

Also write $\Delta z = f'(y_0)\Delta y + \sigma(y)$, where $y_0 = g(x_0)$ and show that

$$\lim_{\Delta y \rightarrow 0} \frac{\sigma(y)}{\Delta y} = 0.$$

- (b) Show that
- $$\Delta z = f'(y_0)g'(x_0)\Delta x + f'(y_0)\rho(x) + \sigma(g(x)).$$
- (c) Note that $\sigma(g(x)) = 0$ if $\Delta y = 0$. Thus show that

$$\frac{\sigma(g(x))}{\Delta x} = \begin{cases} \frac{\sigma(g(x))}{\Delta y} \frac{\Delta y}{\Delta x} & \text{if } \Delta y \neq 0 \\ 0 & \text{if } \Delta y = 0 \end{cases} \rightarrow 0$$

as $\Delta x \rightarrow 0$.

- (d) Use parts (a), (b), and (c) above to show that $\lim_{\Delta x \rightarrow 0} [\Delta z / \Delta x] = f'(y_0)g'(x_0)$. (This proof avoids the problem of division by zero mentioned on p. 113.)

- ★100. An alternative to Newton's method for finding solutions of the equation $f(x) = 0$ is the iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

sometimes known as *Picard's method*. Notice that this method requires evaluating f' only at the initial guess x_0 and so requires less computation at each step.

- (a) Show that, if the sequence $x_0, x_1, x_2, \dots$ converges, then $\lim_{n \rightarrow \infty} x_n$ is a solution of $f(x) = 0$.
- (b) Compare Picard's method and Newton's method on the problem $x^5 = x + 1$, using the initial guess $x_0 = 1$ in each case and iterating until the solution is found to six decimal places of accuracy.
- (c) Suppose that $f(q) = 0$ and in addition that $0 < \frac{1}{2}f'(x_0) < f'(x) < \frac{3}{2}f'(x_0)$ for all x in the interval $I = (q - a, q + a)$. Prove that if x_0 is any initial guess in I , then $|x_{n+1} - q| < \frac{1}{2}|x_n - q|$, and so $\lim_{n \rightarrow \infty} x_n = q$. [Hint: $x_{n+1} = P(x_n)$, where $P(x) = x - f(x)/f'(x_0)$. Differentiate $P(x)$ and apply the mean value theorem.] (A similar analysis for Newton's method is presented in the following Review Exercise.)

- ★101. Newton's method for solving $f(x) = 0$ can be described by saying that $x_{n+1} = N(x_n)$, where the *Newton iteration function* N is defined by $N(x) = x - f(x)/f'(x)$ for all x such that $f'(x) \neq 0$.

- (a) Show that $N(x) = x$ if and only if $f(x) = 0$.
- (b) Show that $N'(x) = f(x)f''(x)/[f'(x)]^2$.
- (c) Suppose that $\bar{x}$ is a root of f , that $[a, b]$ is an interval containing $\bar{x}$, and that there are numbers p, q and M such that

$$0 < p \leq f'(x) \leq q \quad \text{and} \quad |f''(x)| \leq M$$

for all x in $[a, b]$. Show that there is a constant C such that

$$|N(x) - \bar{x}| \leq C|x - \bar{x}|^2$$

for all x in $[a, b]$. Express C in terms of p, q and M .

This establishes the "quadratic convergence" of $x_0, x_1, x_2, \dots$ to $\bar{x}$ as soon as some x_i is in $[a, b]$. [Hint: Apply the mean value theorem to N to conclude $N(x) - \bar{x} = N'(\xi)(x - \bar{x})$ for some ξ between x and $\bar{x}$. Use the mean value theorem again to show that $|N'(\xi)| \leq D|\xi|$ for a constant $D > 0$.]

- (d) How many iterations are needed to solve $x^2 - 2 = 0$ to within 20 decimal places, assuming an initial guess in the interval $[1.4, 1.5]$?

Infinite Series

Infinite sums can be used to represent numbers and functions.

The decimal expansion $\frac{1}{3} = 0.3333 \dots$ is a representation of $\frac{1}{3}$ as an infinite sum $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots$. In this chapter, we will see how to represent numbers as infinite sums and to represent functions of x by infinite sums whose terms are monomials in x . For example, we will see that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

and

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots$$

Later in the chapter we shall use our knowledge of infinite series to study complex numbers and some differential equations. There are other important uses of series that are encountered in later courses. One of these is the topic of Fourier series; this enables one, for example, to decompose a complex sound into an infinite series of pure tones.

12.1 The Sum of an Infinite Series

The sum of infinitely many numbers may be finite.

An *infinite series* is a sequence of numbers whose terms are to be added up. If the resulting sum is finite, the series is said to be *convergent*. In this section, we define convergence in terms of limits, give the simplest examples, and present some basic tests. Along the way we discuss some further properties of the limits of sequences, but the reader should also review the basic facts about sequences from Section 11.4.

Our first example of the limit of a sequence was an expression for the number $\frac{1}{3}$:

$$\frac{1}{3} = \lim_{n \rightarrow \infty} \left(\frac{3}{10} + \frac{3}{100} + \dots + \frac{3}{10^n} \right).$$

This expression suggests that we may consider $\frac{1}{3}$ as the sum

$$\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} + \cdots$$

of infinitely many terms. Of course, not every sum of infinitely many terms gives rise to a number (consider $1 + 1 + 1 + \cdots$), so we must be precise about what we mean by adding together infinitely many numbers. Following the idea used in the theory of improper integrals (in Section 11.3), we will define the sum of an infinite series by taking finite sums and then passing to the limit as the sum includes more and more terms.

Convergence of Series

Let $a_1, a_2, \dots$ be a sequence of numbers. The number $S_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$ is called the *n*th partial sum of the a_i 's. If the sequence $S_1, S_2, \dots$ of partial sums approaches a limit S as $n \rightarrow \infty$, we say that the series $a_1 + a_2 + \cdots = \sum_{i=1}^{\infty} a_i$ converges, and we write

$$\sum_{i=1}^{\infty} a_i = S;$$

that is,

$$\sum_{i=1}^{\infty} a_i \text{ is defined as } \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

and is called the *sum* of the series.

If the series $\sum_{i=1}^{\infty} a_i$ does not converge, we say that it *diverges*. In this case, the series has no sum.

In summary:

a series $\sum_{i=1}^{\infty} a_i$ converges if $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$ exists (and is finite);

a series $\sum_{i=1}^{\infty} a_i$ diverges if $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$ does not exist (or is infinite).

Example 1 Write down the first four partial sums for each of the following series:

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$;

(b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$;

(c) $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$;

(d) $\sum_{i=0}^{\infty} \frac{3}{2^{i+1}}$.

Solution (a) $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, $S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$,

and $S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$.

(b) $S_1 = 1$, $S_2 = 1 - \frac{1}{2} = \frac{1}{2}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$,

and $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$.

(c) $S_1 = 1$, $S_2 = 1 + \frac{1}{5} = \frac{6}{5}$, $S_3 = 1 + \frac{1}{5} + \frac{1}{5^2} = \frac{31}{25}$,

and $S_4 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = \frac{156}{125}$.

$$(d) \sum_{i=0}^{\infty} \frac{3}{2^{i+1}} = \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \frac{3}{2^4} + \cdots, \text{ so } S_1 = \frac{3}{2}, S_2 = \frac{3}{2} + \frac{3}{4} = \frac{9}{4},$$

$$S_3 = 3\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{21}{8}, \text{ and } S_4 = 3\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) = \frac{45}{16}. \blacktriangle$$

Do not confuse a *sequence* with a *series*. A sequence is simply an infinite list of numbers (separated by commas): $a_1, a_2, a_3, \dots$. A series is an infinite list of numbers (separated by plus signs) which are meant to be added together: $a_1 + a_2 + a_3 + \cdots$. Of course, the terms in an infinite series may themselves be considered as a sequence, but the most important sequence associated with the series $a_1 + a_2 + \cdots$ is its sequence of partial sums: $S_1, S_2, S_3, \dots$ —that is, the sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$$

We may illustrate the difference between the a_i 's and the S_n 's pictorially. Think of $a_1, a_2, a_3, \dots$ as describing a sequence of “moves” on the real number line, starting at 0. Then $S_n = a_1 + \cdots + a_n$ is the position reached after the n th move. (See Fig. 12.1.1.) Note that the term a_i can be recovered as the difference $S_i - S_{i-1}$.

To study the limits of partial sums, we will need to use some general properties of limits of sequences. The definition of convergence of a sequence was given in Section 11.4. The basic properties we need are proved and used in a manner similar to those for limits of functions (Section 11.1) and are summarized in the following display.

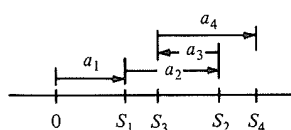


Figure 12.1.1. The term a_i of a series represents the “move” from the partial sum S_{i-1} to S_i . S_n is the cumulative result of the first n moves.

Properties of Limits of Sequences

Suppose that the sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are convergent, and that c is a constant. Then:

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$.
2. $\lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n$.
3. $\lim_{n \rightarrow \infty} (a_n b_n) = (\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n)$.
4. If $\lim_{n \rightarrow \infty} b_n \neq 0$ and $b_n \neq 0$ for all n , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

5. If f is continuous at $\lim_{n \rightarrow \infty} a_n$, then

$$\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right).$$

6. $\lim_{n \rightarrow \infty} c = c$.
7. $\lim_{n \rightarrow \infty} (1/n) = 0$.
8. If $\lim_{x \rightarrow \infty} f(x) = l$, then $\lim_{n \rightarrow \infty} f(n) = l$.
9. If $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$, and if $|r| > 1$ or $r = -1$, $\lim_{n \rightarrow \infty} r^n$ does not exist.

Here are a couple of examples of how the limit properties are used. We will see many more examples as we work with series.

Example 2 Find (a) $\lim_{n \rightarrow \infty} \frac{3+n}{2n+1}$ and (b) $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi n}{2n+1}\right)$.

Solution (a) $\lim_{n \rightarrow \infty} \frac{3+n}{2n+1} = \lim_{n \rightarrow \infty} \frac{3/n+1}{2+1/n} = \frac{3\lim_{n \rightarrow \infty} 1/n + \lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} 1/n} = \frac{3 \cdot 0 + 1}{2 + 0} = \frac{1}{2}.$

This solution used properties 1, 2, and 4 above, together with the facts that $\lim_{n \rightarrow \infty} 1/n = 0$ (property 7) and $\lim_{n \rightarrow \infty} c = c$ (property 6).

(b) Since $\sin x$ is a continuous function, we can use property 5 to get

$$\begin{aligned}\lim_{n \rightarrow \infty} \sin\left(\frac{\pi n}{2n+1}\right) &= \sin\left[\lim_{n \rightarrow \infty} \left(\frac{\pi n}{2n+1}\right)\right] \\ &= \sin\left[\lim_{n \rightarrow \infty} \left(\frac{\pi}{2 + 1/n}\right)\right] \\ &= \sin\left(\frac{\pi}{2}\right) = 1. \quad \blacktriangle\end{aligned}$$

We return now to infinite series. A simple but basic example is the geometric series

$$a + ar + ar^2 + \cdots$$

in which the ratio between each two successive terms is the same. To write a geometric series in summation notation, it is convenient to allow the index i to start at zero, so that $a_0 = a$, $a_1 = ar$, $a_2 = ar^2$, and so on. The general term is then $a_i = ar^i$, and the series is compactly expressed as $\sum_{i=0}^{\infty} ar^i$. In our notation $\sum_{i=1}^{\infty} a_i$ for a general series, the index i will start at 1, but in special examples we may start it wherever we wish. Also, we may replace the index i by any other letter; $\sum_{i=1}^{\infty} a_i = \sum_{j=1}^{\infty} a_j = \sum_{n=1}^{\infty} a_n$, and so forth.

To find the sum of a geometric series, we must first evaluate the partial sums $S_n = \sum_{i=0}^n ar^i$. We write

$$S_n = a + ar + ar^2 + \cdots + ar^n,$$

$$rS_n = ar + ar^2 + \cdots + ar^n + ar^{n+1}.$$

Subtracting the second equation from the first and solving for S_n , we find

$$S_n = \frac{a(1 - r^{n+1})}{1 - r} \quad (\text{if } r \neq 1). \quad (1)$$

The sum of the entire series is the limit

$$\begin{aligned}\sum_{i=0}^{\infty} ar^i &= \lim_{n \rightarrow \infty} S_n \\ &= \lim_{n \rightarrow \infty} \frac{a(1 - r^{n+1})}{1 - r} = \frac{a}{1 - r} \lim_{n \rightarrow \infty} (1 - r^{n+1}) \\ &= \frac{a}{1 - r} \left(1 - \lim_{n \rightarrow \infty} r^{n+1}\right).\end{aligned}$$

(We used limit properties 1, 2, and 6.) If $|r| < 1$, then $\lim_{n \rightarrow \infty} r^{n+1} = 0$ (property 9), so in this case, $\sum_{i=0}^{\infty} ar^i$ is convergent, and its sum is $a/(1 - r)$. If $|r| > 1$ or $r = -1$, $\lim_{n \rightarrow \infty} r^{n+1}$ does not exist (property 9), so if $a \neq 0$, the series diverges. Finally, if $r = 1$, then $S_n = a + ar + \cdots + ar^n = a(n + 1)$, so if $a \neq 0$, the series diverges.

Geometric Series

If $|r| < 1$ and a is any number, then $a + ar + ar^2 + \cdots = \sum_{i=0}^{\infty} ar^i$ converges and the sum is $a/(1 - r)$.

If $|r| \geq 1$ and $a \neq 0$, then $\sum_{i=0}^{\infty} ar^i$ diverges.

Example 3 Sum the series: (a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$, (b) $\sum_{n=0}^{\infty} \frac{1}{6^{n/2}}$, and (c) $\sum_{i=1}^{\infty} \frac{1}{5^i}$.

Solution (a) This is a geometric series with $r = \frac{1}{3}$ and $a = 1$. (Note that a is the first term and r is the ratio of any term to the preceding one.) Thus

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{1 - 1/3} = \frac{3}{2}.$$

(b) $\sum_{n=0}^{\infty} [1/(6^{n/2})] = 1 + (1/\sqrt{6}) + (1/\sqrt{6})^2 + \cdots = a/(1-r)$, where $a = 1$ and $r = 1/\sqrt{6}$, so the sum is $1/(1 - 1/\sqrt{6}) = (6 + \sqrt{6})/5$. (Note that the index here is n instead of i .)

(c) $\sum_{i=1}^{\infty} 1/5^i = 1/5 + 1/5^2 + \cdots = (1/5)/(1 - 1/5) = 1/4$. (We may also think of this as the series $\sum_{i=0}^{\infty} 1/5^i$ with the first term removed. The sum is thus $1/[1 - (1/5)] - 1 = 1/4$.) ▲

The following example shows how a geometric series may arise in a physical problem.

Example 4 A bouncing ball loses half of its energy on each bounce. The height reached on each bounce is proportional to the energy. Suppose that the ball is dropped vertically from a height of one meter. How far does it travel? (Fig. 12.1.2.)

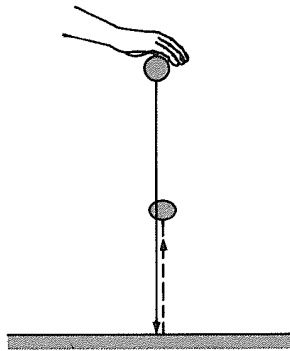


Figure 12.1.2. Find the total distance travelled by the bouncing ball.

Solution Each bounce is $1/2$ as high as the previous one. After the ball falls from a height of 1 meter, it rises to $1/2$ meter on the first bounce, $(1/2)^2 = 1/4$ meter on the second, and so forth. The total distance travelled, in meters, is $1 + 2(1/2) + 2(1/2)^2 + 2(1/2)^3 + \cdots$, which is

$$1 + 2 \sum_{i=1}^{\infty} (1/2)^i = 1 + 2 \left(\frac{1/2}{1 - 1/2} \right) = 3 \text{ meters. } \blacktriangle$$

Two useful general rules for summing series are presented in the box on the following page. To prove the validity of these rules, one simply notes that the identities

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \quad \text{and} \quad \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

are satisfied by the partial sums. Taking limits as $n \rightarrow \infty$ and applying the sum and constant multiple rules for limits of sequences results in the rules in the box.

Algebraic Rules For Series

Sum rule

If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ converge, then $\sum_{i=1}^{\infty} (a_i + b_i)$ converges and

$$\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i.$$

Constant multiple rule

If $\sum_{i=1}^{\infty} a_i$ converges and c is any real number, then $\sum_{i=1}^{\infty} ca_i$ converges and

$$\sum_{i=1}^{\infty} ca_i = c \sum_{i=1}^{\infty} a_i.$$

Example 5 Sum the series $\sum_{i=0}^{\infty} \frac{3^i - 2^i}{6^i}$.

Solution We may write the i th term as

$$\frac{3^i}{6^i} - \frac{2^i}{6^i} = \left(\frac{1}{2}\right)^i - \left(\frac{1}{3}\right)^i = \left(\frac{1}{2}\right)^i + (-1)\left(\frac{1}{3}\right)^i.$$

Since the series $\sum_{i=0}^{\infty} (1/2)^i$ and $\sum_{i=0}^{\infty} (1/3)^i$ are convergent, with sums 2 and $\frac{3}{2}$ respectively, the algebraic rules imply that

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{3^i - 2^i}{6^i} &= \sum_{i=0}^{\infty} \left[\left(\frac{1}{2}\right)^i + (-1)\left(\frac{1}{3}\right)^i \right] \\ &= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i + (-1) \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = 2 - \frac{3}{2} = \frac{1}{2}. \blacktriangle \end{aligned}$$

Example 6 Show that the series $1\frac{1}{2} + 3\frac{3}{4} + 7\frac{7}{8} + 15\frac{15}{16} + \cdots$ diverges. [Hint: Write it as the difference of a divergent and a convergent series.]

Solution The series is $\sum_{i=0}^{\infty} [2^i - (\frac{1}{2})^i]$. If it were convergent, we could add to it the convergent series $\sum_{i=0}^{\infty} (\frac{1}{2})^i$, and the result would have to converge by the sum rule; but the resulting series is $\sum_{i=0}^{\infty} [2^i - (\frac{1}{2})^i + (\frac{1}{2})^i] = \sum_{i=0}^{\infty} 2^i$, which diverges because $2 > 1$, so the original series must itself be divergent. $\blacktriangle$

The sum rule implies that *we may change (or remove—that is, change to zero) finitely many terms of a series without affecting its convergence*. In fact, changing finitely many terms of the series $\sum_{i=1}^{\infty} a_i$ is equivalent to adding to it a series whose terms are all zero beyond a certain point. Such a finite series is always convergent, so adding it to the convergent series produces a convergent result. Of course, the sum of the new series is *not* the same as that of the old one, but rather is the sum of the finite number of added terms plus the sum of the original series.

Example 7 Show that

$$1 + 2 + 3 + 4 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

is convergent and find its sum.

Solution The series $1/4 + 1/4^2 + 1/4^3 + 1/4^4 + \cdots$ is a geometric series with sum $(1/4)/(1 - 1/4) = 1/3$; thus the given series is convergent with sum $1 + 2 +$

$3 + 4 + 1/3 = 10 \frac{1}{3}$. To use the sum rule as stated, one can write

$$\begin{aligned} & 1 + 2 + 3 + 4 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots \\ &= (1 + 2 + 3 + 4 + 0 + 0 + 0 + \cdots) \\ &+ \left(0 + 0 + 0 + 0 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right). \blacktriangle \end{aligned}$$

We can obtain a simple necessary condition for convergence by recalling that $a_i = S_i - S_{i-1}$. If $\lim_{i \rightarrow \infty} S_i$ exists, then $\lim_{i \rightarrow \infty} S_{i-1}$ has the same value. Hence, using properties 1 and 2 of limits of sequences, we find $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} S_i - \lim_{i \rightarrow \infty} S_{i-1} = 0$. In other words, if the series $\sum_{i=1}^{\infty} a_i$ converges, then, the “move” from one partial sum to the next must approach zero (see Fig. 12.1.1).

The i th Term Test

If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i \rightarrow \infty} a_i = 0$.

If $\lim_{i \rightarrow \infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.

If $\lim_{i \rightarrow \infty} a_i = 0$, the test is inconclusive: the series could converge or diverge, and further analysis is necessary.

The i th-term test can be used to show that a series diverges, such as the one in Example 6, but it cannot be used to establish convergence.

Example 8 Test for convergence: (a) $\sum_{i=1}^{\infty} \frac{i}{1+i}$; (b) $\sum_{i=1}^{\infty} (-1)^i \frac{i}{\sqrt{1+i}}$; (c) $\sum_{i=1}^{\infty} \left(\frac{1}{i}\right)$.

Solution (a) Here $a_i = \frac{i}{1+i} = \frac{1}{1/i+1} \rightarrow 1$ as $i \rightarrow \infty$. Since a_i does not tend to zero, the series must diverge.

(b) Here $|a_i| = i/\sqrt{1+i} = \sqrt{i}/\sqrt{1/i+1} \rightarrow \infty$ as $i \rightarrow \infty$. Thus a_i does not tend to zero, so the series diverges.

(c) Here $a_i = 1/i$, which tends to zero as $i \rightarrow \infty$, so our test is inconclusive. $\blacktriangle$

As an example of the “further analysis” necessary when $\lim_{i \rightarrow \infty} a_i = 0$, we consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

from part (c) of Example 8, called the *harmonic series*. We show that the series diverges by noticing a pattern:

$$\begin{array}{rcl} 1 & & \\ \frac{1}{2} & & \\ \frac{1}{3} + \frac{1}{4} & > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} & > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\ \frac{1}{9} + \cdots + \frac{1}{16} & > \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2} \\ \frac{1}{17} + \cdots + \frac{1}{32} & > \frac{1}{32} + \cdots + \frac{1}{32} = \frac{1}{2} \\ \vdots & & \vdots \end{array}$$

and so on. Thus the partial sum S_4 is greater than $1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$, $S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$ and, in general $S_{2^n} > 1 + n/2$, which becomes arbitrarily large as n becomes large. Therefore, the *harmonic series diverges*.

Example 9 Show that the series (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ and (b) $\sum_{i=1}^{\infty} 1/(1+i)$ diverge.

Solution (a) This series is $\sum_{i=1}^{\infty} (1/2i)$. If it converged, so would twice the series $\sum_{i=1}^{\infty} 2 \cdot (1/2i)$, by the constant multiple rule; but $\sum_{i=1}^{\infty} (2 \cdot 1/2i) = \sum_{i=1}^{\infty} 1/i$, which we have shown to diverge.

(b) This series is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is the harmonic series with the first term missing; therefore this series diverges too. $\blacktriangle$

Supplement to Section 12.1: Zeno's Paradox

Zeno's paradox concerns a race between Achilles and a tortoise. The tortoise begins with a head start of 10 meters, and Achilles ought to overtake it. After a certain elapsed time from the start, Achilles reaches the point A where the tortoise started, but the tortoise has moved ahead to point B (Fig. 12.1.3).

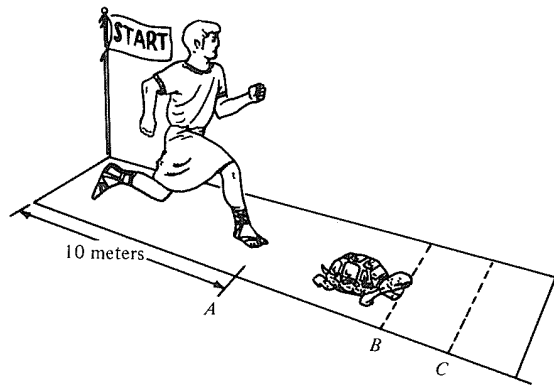


Figure 12.1.3. Will the runner overtake the tortoise?

After a certain further interval of time, Achilles reaches point B , but the tortoise has moved ahead to a point C , and so on forever. Zeno concludes from this argument that Achilles can never pass the tortoise. Where is the fallacy?

The resolution of the paradox is that although the number of time intervals being considered is infinite, the sum of their lengths is finite, so Achilles can overtake the tortoise in a finite time. The word *forever* in the sense of infinitely many terms is confused with “forever” in the sense of the time in the problem, resulting in the apparent paradox.

Exercises for Section 12.1

Write down the first four partial sums for the series in Exercises 1–4.

1. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$
2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
3. $\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i$
4. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Sum the series in Exercises 5–8.

5. $1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots$
6. $2 + \frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + \cdots$
7. $\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^i$
8. $\sum_{n=1}^{\infty} \left(\frac{13}{15}\right)^n$

9. You wish to draw \$10,000 out of a Swiss bank account at age 65, and thereafter you want to draw $\frac{3}{4}$ as much each year as the preceding one. Assuming that the account earns no interest, how much money must you start with to be prepared for an arbitrarily large life span?
10. A decaying radioactive source emits $\frac{9}{10}$ as much radiation each year as the previous one. Assuming that 2000 roentgens are given off in the first year, what is the total emission over all time?

Sum the series (if they converge) in Exercises 11–20.

11. $\sum_{j=1}^{\infty} \frac{1}{13^j}$ 12. $\sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k$
 13. $\sum_{i=0}^{\infty} \frac{2^{3i+4}}{3^{2i+5}}$ 14. $\sum_{l=0}^{\infty} \frac{4^{4l+2}}{5^{3l+80}}$
 15. $\sum_{j=-3}^{\infty} \left(\frac{1}{3}\right)^j$ 16. $\sum_{i=4}^{\infty} 5\left(\frac{1}{3}\right)^{i+1/2}$
 17. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$ 18. $\sum_{k=1}^{\infty} \frac{3^{2k} + 1}{27^k}$
 19. $\sum_{n=5}^{\infty} \frac{2^{n+1}}{3^{n-2}}$
 20. $\sum_{i=1}^{\infty} \left[\left(\frac{1}{2}\right)^i + \left(\frac{1}{3}\right)^{2i} + \left(\frac{1}{4}\right)^{3i+1} \right]$

21. Show that $\sum_{i=1}^{\infty} (1 + 1/2^i)$ diverges.
 22. Show that $\sum_{i=0}^{\infty} (3^i + 1/3^i)$ diverges.
 23. Sum $2 + 4 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$.
 24. Sum $1 + 1/2 + 1/3 + 1/3^2 + 1/3^3 + \dots$.

Test the series in Exercises 25–30 for convergence.

25. $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i+1}}$
 26. $\sum_{i=1}^{\infty} \frac{\sqrt{i} + 1}{\sqrt{i} + 8}$
 27. $\sum_{i=1}^{\infty} \frac{3}{5+5i}$
 28. $\sum_{i=1}^{\infty} \frac{6}{7+7i}$
 29. $1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_2 + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_4 + \underbrace{\frac{1}{16} + \dots}_{8} + \dots$
 30. $1 + \underbrace{\frac{1}{4} + \frac{1}{4}}_2 + \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_4 + \underbrace{\frac{1}{64} + \frac{1}{64} + \dots}_{8} + \dots$

31. Show that the series $\sum_{j=1}^{\infty} (1 - 2^{-j})/j$ diverges.
 32. Show that the series $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \dots$ diverges.
 33. Give an example to show that $\sum_{i=1}^{\infty} (a_i + b_i)$ may converge while both $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ diverge.
 34. Comment on the formula $1 + 2 + 4 + 8 + \dots = 1/(1-2) = -1$.
 35. A *telescoping series*, like a geometric series, can be summed. A series $\sum_{n=1}^{\infty} a_n$ is telescoping if its

n th term a_n can be expressed as $a_n = b_{n+1} - b_n$ for some sequence b_n .

- (a) Verify that $a_1 + a_2 + a_3 + \dots + a_n = b_{n+1} - b_1$; therefore the series converges exactly when $\lim_{n \rightarrow \infty} b_{n+1}$ exists, and $\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} b_{n+1} - b_1$.
 (b) Use partial fraction methods to write $a_n = 1/[n(n+1)]$ as $b_{n+1} - b_n$ for some sequence b_n . Then evaluate the sum of the series $\sum_{n=1}^{\infty} 1/[n(n+1)]$.
36. An experiment is performed, during which time successive excursions of a deflected plate are recorded. Initially, the plate has amplitude b_0 . The plate then deflects downward to form a “dish” of depth b_1 , then a “dome” of height b_2 , and so on. (See Fig. 12.1.4.) The a ’s and b ’s are related by $a_1 = b_0 - b_1, a_2 = b_1 - b_2, a_3 = b_2 - b_3, \dots$. The value a_n measures the amplitude “lost” at the n th oscillation (due to friction, say).

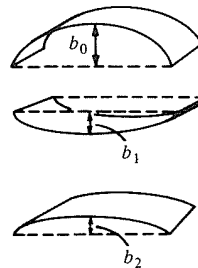


Figure 12.1.4. The deflecting plate in Exercise 36.

- (a) Find $\sum_{n=1}^{\infty} a_n$. Explain why $b_0 - \sum_{n=1}^{\infty} a_n$ is the “average height” of the oscillating plate after a large number of oscillations.
 (b) Suppose the “dishes” and “domes” decay to zero, that is, $\lim_{n \rightarrow \infty} b_{n+1} = 0$. Show that $\sum_{n=1}^{\infty} a_n = b_0$, and explain why this is physically obvious.
37. The joining of the transcontinental railroads occurred as follows. The East and West crews were setting track 12 miles apart, the East crew working at 5 miles per hour, the West crew working at 7 miles per hour. The official with the Golden Spike travelled feverishly by carriage back and forth between the crews until the rails joined. His speed was 20 miles per hour, and he started from the East.
- (a) Assume the carriage reversed direction with no waiting time at each encounter with an East or West crew. Let t_k be the carriage transit time for trip k . Verify that $t_{2n+2} = r^{n+1} \cdot (12/13)$, and $t_{2n+1} = r^n \cdot (12/27)$, where $r = (13/27) \cdot (15/25)$, $n = 0, 1, 2, 3, \dots$.
 (b) Since the crews met in one hour, the total time for the carriage travel was one hour, i.e., $\lim_{n \rightarrow \infty} (t_1 + t_2 + t_3 + t_4 + \dots + t_n) = 1$. Verify this formula using a geometric series.

12.2 The Comparison Test and Alternating Series

A series with positive terms converges if its terms approach zero quickly enough.

Most series, unlike the geometric series, cannot be summed explicitly. If we can prove that a given series converges, we can approximate its sum to any desired accuracy by adding up enough terms.

One way to tell whether a series converges or diverges is to compare it with a series which we already know to converge or diverge. As a fringe benefit of such a “comparison test,” we sometimes get an estimate of the difference between the n th partial sum and the exact sum. Thus if we want to find the sum with a given accuracy, we know how many terms to take.

The comparison test for series is similar to that for integrals (Section 11.3). The test is simplest to understand for series with non-negative terms. Suppose that we are given series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ such that $0 \leq a_i \leq b_i$ for all i :

if $\sum_{i=1}^{\infty} b_i$ converges, then so does $\sum_{i=1}^{\infty} a_i$.

The reason for this is easy to see on an intuitive level. The partial sums $S_n = \sum_{i=1}^n a_i$ are moving to the right on the real number line since $a_i \geq 0$. They must either march off to ∞ or approach a limit. (The proof of this sentence requires a careful study of the real numbers, but we will take it for granted here. Consult the Supplement to this section and the theoretical references listed in the Preface.) However, $\sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i \leq \sum_{i=1}^{\infty} b_i$, since $a_i \leq b_i$ and the partial sums $\sum_{i=1}^n b_i$ are marching to the right toward their limit. Hence all the S_n 's are bounded by the fixed number $\sum_{i=1}^{\infty} b_i$, and so they cannot go to ∞ .

Example 1 Show that $\sum_{i=1}^{\infty} \frac{3}{2^i + 4}$ converges.

Solution We know that $\sum_{i=1}^{\infty} (3/2^i)$ is convergent since it is a geometric series with $a = 3$ and $r = \frac{1}{2} < 1$; but

$$0 < \frac{3}{2^i + 4} < \frac{3}{2^i},$$

so the given series converges by the comparison test. $\blacktriangle$

For series $\sum_{i=1}^{\infty} a_i$ with terms that can be either positive or negative, we replace the condition $0 \leq a_i \leq b_i$ by $|a_i| \leq b_i$. Then if $\sum_{i=1}^{\infty} b_i$ converges, so must $\sum_{i=1}^{\infty} |a_i|$, by the test above. The following fact is true for any series:

if $\sum_{i=1}^{\infty} |a_i|$ converges, so does $\sum_{i=1}^{\infty} a_i$, and $\left| \sum_{i=1}^{\infty} a_i \right| \leq \sum_{i=1}^{\infty} |a_i|$.

A careful proof of this fact is given at the end of this section; for now we simply observe that the convergence of $\sum_{i=1}^{\infty} |a_i|$ implies that the absolute values $|a_i|$ approach zero quickly, and the possibility of varying signs in the a_i 's can only help in convergence. Therefore, if $0 \leq |a_i| \leq b_i$ and $\sum b_i$ converges, then $\sum |a_i|$ converges, and therefore so does $\sum a_i$. (We sometimes drop the “ $i = 1$ ” and “ ∞ ” from $\sum$ if there is no danger of confusion.) This leads to the following test.

Comparison Test

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series such that $|a_i| \leq b_i$. If $\sum_{i=1}^{\infty} b_i$ is convergent, then so is $\sum_{i=1}^{\infty} a_i$.

Example 2 Show that $\sum_{i=1}^{\infty} \frac{(-1)^i}{i3^{i+1}}$ converges.

Solution We can compare the series with $\sum_{i=1}^{\infty} 1/3^i$. Let $a_i = (-1)^i/(i3^{i+1})$ and $b_i = 1/3^i$. Since $i3^{i+1} = (3i) \cdot 3^i > 3^i$, we have

$$|a_i| = \frac{1}{i3^{i+1}} < \frac{1}{3^i} = b_i.$$

Therefore, since $\sum_{i=1}^{\infty} b_i$ converges (it is a geometric series), so does $\sum_{i=1}^{\infty} a_i$. ▲

If the terms of two series $\sum a_i$ and $\sum b_i$ “resemble” one another, we may expect that one of the series converges if the other does. This is the case when the ratio a_i/b_i approaches a limit, as can be deduced from the comparison test. For instance, suppose that $\lim_{i \rightarrow \infty} (|a_i|/b_i) = M < \infty$, with all $b_i > 0$. Then for large enough i , we have $|a_i|/b_i < M + 1$, or $|a_i| < (M + 1)b_i$. Now if $\sum b_i$ converges, so does $\sum (M + 1)b_i$, by the constant multiple rule for series, and hence $\sum a_i$ converges by the comparison test.¹

Example 3 Test for convergence: $\sum_{i=1}^{\infty} \frac{1}{2^i - i}$.

Solution We cannot compare directly with $\sum_{i=1}^{\infty} 1/2^i$, since $1/(2^i - i)$ is *greater* than $1/2^i$. Instead, we look at the ratios a_i/b_i with $a_i = 1/(2^i - i)$ and $b_i = 1/2^i$. We have

$$\lim_{i \rightarrow \infty} \frac{a_i}{b_i} = \lim_{i \rightarrow \infty} \frac{1}{1 - i/2^i} = \frac{1}{1 - 0} = 1$$

($\lim_{i \rightarrow \infty} (i/2^i) = 0$ by l'Hôpital's rule). Since $\sum_{i=1}^{\infty} (1/2^i)$ converges, so does $\sum_{i=1}^{\infty} [1/(2^i - i)]$. ▲

The following tests can both be similarly justified using the original comparison test.

Ratio Comparison Tests

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series, with $b_i > 0$ for all i .

If (1) $|a_i| \leq b_i$ for all i , or if $\lim_{i \rightarrow \infty} (|a_i|/b_i) < \infty$ and
(2) $\sum_{i=1}^{\infty} b_i$ is convergent, then $\sum_{i=1}^{\infty} a_i$ is convergent.

If (1) $a_i \geq b_i$ for all i , or if $\lim_{i \rightarrow \infty} (a_i/b_i) > 0$ and
(2) $\sum_{i=1}^{\infty} b_i$ is divergent, then $\sum_{i=1}^{\infty} a_i$ is divergent.

To choose b_i in applying the ratio comparison test, you should look for the “dominant terms” in the expression for a_i .

¹ Strictly speaking, to apply the comparison test we should have $|a_i| < (M + 1)b_i$ for all i , not just sufficiently large i ; but, as we saw earlier, the convergence or divergence of a series $\sum a_i$ is not affected by the values of its “early” terms, but only the behavior of a_i for large i . Of course, the *sum* of the series depends on all the terms.

Example 4 Show that $\sum_{i=1}^{\infty} \frac{2}{4+i}$ diverges.

Solution As $i \rightarrow \infty$, the dominant term in the denominator $4+i$ is i , that is, if i is very large (like 10^6), 4 is very small by comparison. Hence we are led to let $a_i = 2/(4+i)$, $b_i = 1/i$. Then

$$\lim_{i \rightarrow \infty} \frac{a_i}{b_i} = \lim_{i \rightarrow \infty} \frac{2/(4+i)}{1/i} = \lim_{i \rightarrow \infty} \frac{2i}{4+i} = \lim_{i \rightarrow \infty} \frac{2}{(4/i)+1} = \frac{2}{0+1} = 2.$$

Since $2 > 0$, and $\sum_{i=1}^{\infty} 1/i$ is divergent, it follows that $\sum_{i=1}^{\infty} [2/(4+i)]$ is divergent as well. $\blacktriangle$

The next example illustrates how one may estimate the difference between a partial sum and the full series. We sometimes refer to this difference as a *tail* of the series; it is equal to the sum of all the terms not included in the partial sum.

Example 5 Find the partial sum $\sum_{i=1}^3 \frac{(-1)^i}{i3^{i+1}}$ (see Example 2) and estimate the difference between this partial sum and the sum of the entire series.

Solution The sum of the first three terms is

$$-\frac{1}{3^2 \cdot 1} + \frac{1}{3^3 \cdot 2} - \frac{1}{3^4 \cdot 3} = -\frac{1}{9} + \frac{1}{54} - \frac{1}{243} = -\frac{47}{486} \approx -0.0967.$$

The difference between the full sum of a series and the n th partial sum is given by $\sum_{i=1}^{\infty} a_i - \sum_{i=1}^n a_i = \sum_{i=n+1}^{\infty} a_i$. To estimate this tail in our example, we write

$$\begin{aligned} \left| \sum_{i=1}^{\infty} \frac{(-1)^i}{3^{i+1}i} - (-0.0967) \right| &= \left| \sum_{i=4}^{\infty} \frac{(-1)^i}{3^{i+1}i} \right| \\ &\leq \sum_{i=4}^{\infty} \frac{1}{3^{i+1}i} \quad \left(\text{since } \left| \sum a_i \right| \leq \sum |a_i| \right) \\ &\leq \sum_{i=4}^{\infty} \frac{1}{3^{i+1}} \quad (\text{since } i > 1) \\ &= \frac{1}{3^5} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) \\ &= \frac{1}{3^5} \left(\frac{1}{1-1/3} \right) = \frac{1}{3^5} \cdot \frac{3}{2} = \frac{1}{162} \approx 0.0062. \end{aligned}$$

Thus the error is no more than 0.0062. We may therefore conclude that $\sum_{i=1}^{\infty} [(-1)^i/(3^{i+1}i)]$ lies in the interval $[-0.0967 - 0.0062, -0.0967 + 0.0062] = [-0.103, -0.090]$. $\blacktriangle$

The second kind of series which we will treat in this section is called an *alternating series*. To illustrate, recall that we saw in Section 12.1 that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent even though $\lim_{i \rightarrow \infty} (1/i) = 0$. If we put a minus sign in front of every other term to obtain the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

we might hope that the alternating positive and negative terms “neutralize” one another and cause the series to converge. The alternating series test will indeed guarantee convergence. First we need the following definition.

Alternating Series

A series $\sum_{i=1}^{\infty} a_i$ is called *alternating* if the terms a_i are alternately positive and negative and if the absolute values $|a_i|$ are decreasing to zero; that is, if:

1. $a_1 > 0, a_2 < 0, a_3 > 0, a_4 < 0$, and so on (or $a_1 < 0, a_2 > 0, \dots$);
2. $|a_1| \geq |a_2| \geq |a_3| \geq \dots$;
3. $\lim_{i \rightarrow \infty} |a_i| = 0$.

Conditions 1, 2, and 3 are often easy to verify.

Example 6 Is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ alternating?

Solution The terms alternate in sign, $+ - + - \cdots$, so condition 1 holds. Since the i th term $a_i = (-1)^{i+1}(1/i)$ has absolute value $1/i$, and $1/i > 1/(i+1)$, the terms are decreasing in absolute value, so condition 2 holds. Finally, since $\lim_{i \rightarrow \infty} |a_i| = \lim_{i \rightarrow \infty} (1/i) = 0$, condition 3 holds. Thus the series is alternating. ▲

Later in this section, we will prove that *every alternating series converges*. The proof is based on the idea that the partial sums $S_n = \sum_{i=1}^n a_i$ oscillate back and forth and get closer and closer together, so that they must close in on a limiting value S . This argument also shows that the sum S lies between any two successive partial sums, so that the tail corresponding to the partial sum S_n is less than $|a_{n+1}|$, the size of the first omitted term. (See Fig. 12.2.1.)

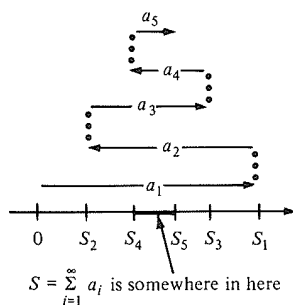


Figure 12.2.1. An alternating series converges, no matter how slowly the terms approach zero. The sum lies between each two successive partial sums.

Alternating Series Test

1. If $\sum_{i=1}^{\infty} a_i$ is a series such that the a_i alternate in sign, are decreasing in absolute value, and tend to zero, then it converges.
2. The error made in approximating the sum by $S_n = \sum_{i=1}^n a_i$ is not greater than $|a_{n+1}|$.

Example 7 Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges, and find its sum with an error of no more than 0.04.

Solution By Example 6, the series is alternating; therefore, by the alternating series test, it converges. To make the error at most $0.04 = \frac{1}{25}$, we must add up all the terms through $\frac{1}{24}$. Using a calculator, we find

$$S_{24} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{24} \approx 0.6727.$$

(Since the sum lies between S_{24} and $S_{25} \approx 0.7127$, an even better estimate is the midpoint $\frac{1}{2}(S_{24} + S_{25}) = 0.6927$, which can differ from the sum by at most 0.02.) ▲

Example 8 Test for convergence:

(a) $\sum_{i=1}^{\infty} \frac{(-1)^i}{(1+i)^2};$

(b) $\frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \frac{2}{6} - \frac{1}{6} + \cdots$

Solution (a) The terms alternate in sign since $(-1)^i = 1$ if i is even and $(-1)^i = -1$ if i is odd. The absolute values, $1/(1+i)^2$, are decreasing and converge to zero. Thus the series is alternating, so it converges.

(b) The terms alternate in sign and tend to zero, but the absolute values are not monotonically decreasing. Thus the series is not an alternating one and the alternating series test does not apply. If we group the terms by twos, we find that the series becomes

$$\left(\frac{2}{2} - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{2}{5} - \frac{1}{5}\right) + \cdots = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which diverges. (Notice that the n th partial sum of the “grouped” series is the $2n$ th partial sum of the original series.) ▲

We noted early in this section that a series $\sum_{i=1}^{\infty} a_i$ always converges if its terms go to zero quickly enough so that the series $\sum_{i=1}^{\infty} |a_i|$ of absolute values is convergent. Such a series $\sum_{i=1}^{\infty} a_i$ is said to be *absolutely convergent*. On the other hand, a series like $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, is convergent only due to the alternating signs of its terms; the series of absolute values, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, is divergent (it is the harmonic series). When $\sum_{i=1}^{\infty} a_i$ converges but $\sum_{i=1}^{\infty} |a_i|$ diverges, the series $\sum_{i=1}^{\infty} a_i$ is said to be *conditionally convergent*.

Example 9 Discuss the convergence of the series $\sum_{i=1}^{\infty} \frac{(-1)^i \sqrt{i}}{i+4}$.

Solution Let $a_i = (-1)^i \sqrt{i}/(i+4)$. We notice that for i large, $|a_i|$ appears to behave like $b_i = 1/\sqrt{i}$. The series $\sum_{i=1}^{\infty} b_i$ diverges by comparison with the harmonic series. To make the comparison between $|a_i|$ and b_i precise, look at the ratios: $\lim_{i \rightarrow \infty} (|a_i|/b_i) = \lim_{i \rightarrow \infty} [i/(i+4)] = 1$, so $\sum_{i=1}^{\infty} |a_i|$ diverges as well; hence our series is not *absolutely convergent*.

The series does look like it could be alternating: the terms alternate in sign and $\lim_{i \rightarrow \infty} a_i = 0$. To see whether the absolute values $|a_i|$ form a decreasing sequence, it is convenient to look at the function $f(x) = \sqrt{x}/(x+4)$. The derivative is

$$f'(x) = \frac{(1/2\sqrt{x})(x+4) - \sqrt{x} \cdot 1}{(x+4)^2} = \frac{2/\sqrt{x} - \sqrt{x}/2}{(x+4)^2} = \frac{4-x}{2\sqrt{x}(x+4)^2},$$

which is negative for $x > 4$, so $f(x)$ is decreasing for $x > 4$. Since $|a_i| = f(i)$, we have $|a_4| > |a_5| > |a_6| > \cdots$ which implies that our series $\sum a_i$, with its first three terms omitted, is alternating. It follows that the series is convergent; since it is not absolutely convergent, it is conditionally convergent. ▲

Absolute and Conditional Convergence

A series $\sum_{i=1}^{\infty} a_i$ is called *absolutely convergent* if $\sum_{i=1}^{\infty} |a_i|$ is convergent. Every absolutely convergent series converges.

A series may converge without being absolutely convergent; such a series is called *conditionally convergent*.

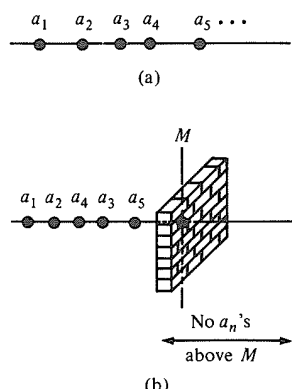


Figure 12.2.2. (a) An increasing sequence; (b) a sequence bounded above by M .

Supplement to Section 12.2: A Discussion of the Proofs of the Comparison and Alternating Series Tests

The key convergence property we need involves increasing sequences. It is similar to the existence of $\lim_{x \rightarrow \infty} f(x)$ if f is increasing and bounded above, which we used in Section 11.3 to establish the comparison test for integrals.

A sequence $a_1, a_2, \dots$ of real numbers is called *increasing* in case $a_1 \leq a_2 \leq \dots$. The sequence is said to be *bounded above* if there is a number M such that $a_n \leq M$ for all n . (See Fig. 12.2.2.)

For example, let $a_n = n/(n+1)$. Let us show that a_n is increasing and is bounded above by M if M is any number ≥ 1 . To prove that it is increasing, we must show that $a_n \leq a_{n+1}$ —that is, that

$$\frac{n}{n+1} \leq \frac{n+1}{(n+1)+1} \quad \text{or} \quad n(n+2) \leq (n+1)^2$$

or

$$n^2 + 2n \leq n^2 + 2n + 1 \quad \text{or} \quad 0 \leq 1.$$

Reversing the steps gives a proof that $a_n \leq a_{n+1}$; i.e., the sequence is increasing. Since $n < n+1$, we have $a_n = n/(n+1) < 1$, so $a_n < M$ if $M \geq 1$.

We will accept without proof the following property of the real numbers (see the references listed in the Preface).

Increasing Sequence Property

If a_n is an increasing sequence which is bounded above, then a_n converges to some number a as $n \rightarrow \infty$. (Similarly, a decreasing sequence bounded below converges.)

The increasing sequence property expresses a simple idea: if the sequence is increasing, the numbers a_n increase, but they can never exceed M . What else could they do but converge? Of course, the limit a satisfies $a_n \leq a$ for all n .

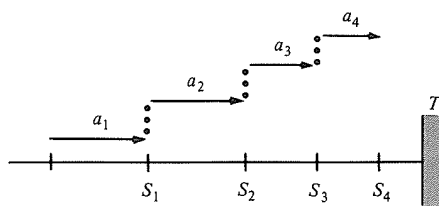
For example, consider

$$a_1 = 0.3, \quad a_2 = 0.33, \quad a_3 = 0.333$$

and so forth. These a_n 's are increasing (in fact, strictly increasing) and are bounded above by 0.4, so we know that they must converge. In fact, the increasing sequence property shows that any infinite decimal expansion converges and so represents a real number.

To prove the comparison test for series with positive terms, we apply the increasing sequence property to the sequence of partial sums. If $\sum_{i=1}^{\infty} a_i$ is a series with $a_i \geq 0$ for each i , then since the partial sums S_n satisfy $S_n - S_{n-1} = a_n \geq 0$, they must be an increasing sequence (see Fig. 12.2.3). If the partial sums are bounded above, the sequence must have a limit, and so the series must converge.

Figure 12.2.3. The partial sums of the series $\sum_{i=1}^{\infty} a_i$ are increasing and bounded above by T .



Now we may simply repeat the argument presented earlier in this section. If $0 \leq a_i \leq b_i$ for all i , and $T_n = \sum_{i=1}^n b_i$, then $S_n \leq T_n$. If the partial sums T_n approach a limit T , then they are bounded above by T , and so $S_n \leq T$ for all n . Thus $\lim_{n \rightarrow \infty} S_n$ exists and is less than or equal to T , i.e., $\sum_{i=1}^{\infty} a_i \leq \sum_{i=1}^{\infty} b_i$.

To complete the proof of the general comparison test, we must show that whenever $\sum_{i=1}^{\infty} |a_i|$ converges, so does $\sum_{i=1}^{\infty} a_i$; in other words, *every absolutely convergent series converges*. Suppose, then, that $\sum |a_i|$ converges.

We define two new series, $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$, by the formulas

$$b_i = \begin{cases} |a_i| & \text{if } a_i \geq 0 \\ 0 & \text{if } a_i < 0 \end{cases} = \begin{cases} a_i & \text{if } a_i \geq 0 \\ 0 & \text{if } a_i < 0 \end{cases},$$

$$c_i = \begin{cases} |a_i| & \text{if } a_i \leq 0 \\ 0 & \text{if } a_i > 0 \end{cases} = \begin{cases} -a_i & \text{if } a_i \leq 0 \\ 0 & \text{if } a_i > 0 \end{cases},$$

These are the “positive and negative parts” of the series $\sum_{i=1}^{\infty} a_i$. It is easy to check that $a_i = b_i - c_i$. The series $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$ are both convergent; in fact, since $b_i \leq |a_i|$, we have $\sum_{i=1}^n b_i \leq \sum_{i=1}^n |a_i| \leq \sum_{i=1}^{\infty} |a_i|$, which is finite since we assumed the series $\sum_{i=1}^{\infty} |a_i|$ to be absolutely convergent. Since $b_i \geq 0$ for all i , $\sum_{i=1}^{\infty} b_i$ is convergent. The same argument proves that $\sum_{i=1}^{\infty} c_i$ is convergent. The sum and constant multiple rules now apply to give the convergence of $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i - \sum_{i=1}^{\infty} c_i$.

Finally, we note that, by the triangle inequality,

$$\left| \sum_{i=1}^n a_i \right| \leq \sum_{i=1}^n |a_i| \leq \sum_{i=1}^{\infty} |a_i|.$$

Since this is true for all n , and

$$\left| \sum_{i=1}^{\infty} a_i \right| = \left| \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \right| = \lim_{n \rightarrow \infty} \left| \sum_{i=1}^n a_i \right|$$

(the absolute value function is continuous), it follows that $|\sum_{i=1}^{\infty} a_i| \leq \sum_{i=1}^{\infty} |a_i|$. (Here we again used the fact that if $b_n \leq M$ for all n and b_n converges to b , then $b \leq M$).

We conclude this section with a proof that every alternating series converges.

Let $\sum_{i=1}^{\infty} a_i$ be an alternating series. If we let $b_i = (-1)^{i+1} a_i$, then all the b_i are positive, and our series is $b_1 - b_2 + b_3 - b_4 + b_5 - \dots$. In addition, we have $b_1 > b_2 > b_3 > \dots$, and $\lim_{i \rightarrow \infty} b_i = 0$. Each even partial sum S_{2n} can be grouped as $(b_1 - b_2) + (b_3 - b_4) + \dots + (b_{n-1} - b_n)$, which is a series of positive terms, so we have $S_2 \leq S_4 \leq S_6 \leq \dots$. On the other hand, the odd partial sums S_{2n+1} can be grouped as $b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - (b_{2n} - b_{2n+1})$, which is a sum of negative terms (except for the first), so we have $S_1 \geq S_3 \geq S_5 \geq \dots$. Next, we note that $S_{2n+1} = S_{2n} + b_{2n+1} \geq S_{2n}$. Thus the even partial sums S_{2n} form an increasing sequence which is bounded above by any member of the decreasing sequence of odd partial sums. (See Fig. 12.2.1.) By the increasing sequence property, the sequence S_{2n} approaches a limit, S_{even} . Similarly, the decreasing sequence S_{2n+1} approaches a limit, S_{odd} .

Thus we have $S_2 \leq S_4 \leq S_6 \leq \dots \leq S_{2n} \leq \dots \leq S_{\text{even}} \leq S_{\text{odd}} \leq \dots \leq S_{2n+1} \leq \dots \leq S_3 \leq S_1$. Now $S_{2n+1} - S_{2n}$ is a_{2n+1} , which approaches zero as $n \rightarrow \infty$; the difference $S_{\text{odd}} - S_{\text{even}}$ is less than $S_{2n+1} - S_{2n}$, so it must be zero; i.e., $S_{\text{odd}} = S_{\text{even}}$. Call this common value S . Thus $|S_{2n} - S| \leq |S_{2n} - S_{2n+1}|$

$= b_{2n+1} = |a_{2n+1}|$ and $|S_{2n+1} - S| \leq |S_{2n+1} - S_{2n+2}| = b_{2n+2} = |a_{2n+2}|$, so each difference $|S_n - S|$ is less than $|a_{n+1}|$. Since $a_{n+1} \rightarrow 0$, we must have $S_n \rightarrow S$ as $n \rightarrow \infty$.

This argument also shows that each tail of an alternating series is no greater than the first term omitted from the partial sum.

Exercises for Section 12.2

Show that the series in Exercises 1–8 converge, using the comparison test for series with positive terms.

1. $\sum_{i=1}^{\infty} \frac{8}{3^i + 2}$
2. $\sum_{i=1}^{\infty} \frac{9}{4^i + 6}$
3. $\sum_{i=1}^{\infty} \frac{1}{3^i - 1}$
4. $\sum_{i=1}^{\infty} \frac{2}{4^i - 3}$
5. $\sum_{i=1}^{\infty} \frac{(-1)^i}{3^i + 2}$
6. $\sum_{i=1}^{\infty} \frac{(-1)^i}{4^i + 1}$
7. $\sum_{i=1}^{\infty} \frac{\sin i}{2^i - 1}$
8. $\sum_{i=1}^{\infty} \frac{\cos(\pi i)}{3^i - 1}$

Show that the series in Exercises 9–12 diverge, by using the comparison test.

9. $\sum_{i=1}^{\infty} \frac{3}{2+i}$
10. $\sum_{i=1}^{\infty} \frac{8}{5+7i}$
11. $\sum_{i=1}^{\infty} \frac{8}{6i-1}$
12. $\sum_{i=1}^{\infty} \frac{3}{2i-1}$

Test the series in Exercises 13–34 for convergence.

13. $\sum_{n=1}^{\infty} \frac{3}{4^n + 2}$
14. $\sum_{n=1}^{\infty} \left(\frac{-4}{2^n + 3} \right)^n$
15. $\sum_{i=1}^{\infty} \frac{1}{2^i + 3^i}$
16. $\sum_{i=1}^{\infty} \frac{(1/2)^i}{i+6}$
17. $\sum_{i=1}^{\infty} \frac{1}{3i+1/i}$
18. $\sum_{i=1}^{\infty} \frac{2}{2i+1}$
19. $\sum_{n=1}^{\infty} \frac{4^n + 5^n}{2^n 3^n}$
20. $\sum_{n=1}^{\infty} \frac{\sqrt{3+n}}{4^n}$
21. $\sum_{i=1}^{\infty} \frac{1 + (-1)^i}{8i + 2^{i+1}}$
22. $\sum_{i=1}^{\infty} \frac{(-2)^i}{3^i + 1}$
23. $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i+2}}$
24. $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i+1}}$
25. $\sum_{i=1}^{\infty} \frac{3i}{2^i}$
26. $\sum_{j=1}^{\infty} \frac{2^j}{j}$
27. $\sum_{i=1}^{\infty} \left(\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} \right)$
28. $\sum_{i=1}^{\infty} \frac{3}{1+3^i}$
29. $\sum_{j=1}^{\infty} \frac{\sin j}{2^j}$
30. $\sum_{n=1}^{\infty} e^{-n}$
31. $\sum_{i=2}^{\infty} \frac{1}{\ln i}$
32. $\sum_{i=1}^{\infty} \left(\frac{i}{i+2} \right)^i$

$$33. \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{33} + \frac{1}{65} + \cdots + \frac{1}{2^n + 1} + \cdots$$

$$34. 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots + \frac{1}{2^n - 1} + \cdots$$

Find the sum of the series in Exercises 35–38 with an error of no more than 0.01.

35. $\sum_{j=1}^{\infty} \frac{1}{j4^j}$
36. $\sum_{k=0}^{\infty} \frac{k}{2^k}$ [Hint: Compare with $\sum_{k=0}^{\infty} \left(\frac{2}{3} \right)^k$.]
37. $\sum_{n=1}^{\infty} \frac{2^n - 1}{5^n + 1}$
38. $\sum_{p=1}^{\infty} \frac{(-1)^p}{2^p + p}$

Test the series in Exercises 39–50 for convergence and absolute convergence.

39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
40. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
41. $\sum_{k=1}^{\infty} \frac{k}{k+1}$
42. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$
43. $\sum_{i=1}^{\infty} \frac{\cos \pi i}{2^i}$
44. $\sum_{n=1}^{\infty} \frac{(-1)^n}{8n+2}$
45. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$
46. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
47. $\sum_{i=1}^{\infty} (-1)^i \frac{i}{i^2 + 1}$
48. $\sum_{i=1}^{\infty} a_i$, where $a_i = 1/(2^i)$ if i is even and $a_i = 1/i$ if i is odd.
49. $\sum_{n=1}^{\infty} (-1)^n \ln[(n+1)/n]$. [Hint: First prove that $\ln(1+a) \geq a/2$ for small $a > 0$.]
50. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln[(n+3)/n]$. (See the hint in 49.)

Estimate the sum of the series in Exercises 51–54 with an error of no more than that specified.

51. $\sum_{i=1}^{\infty} \frac{(-1)^i}{3^i + 1}$; 0.01
52. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^3 + 1}$; 0.005
53. $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{2n} + \frac{1}{5^n} \right)$; 0.02
54. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$; 0.02

55. Test for convergence: $\frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{1}{8} - \frac{1}{8} + \dots$.

56. Does the series $\frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \dots$ converge?

Exercises 57 and 58 deal with an application of the increasing sequence test to inductively defined sequences. For example, let a_n be defined as follows:

$$\begin{aligned} a_0 &= 0, & a_1 &= \sqrt{3}, \\ a_2 &= \sqrt{3 + a_1} = \sqrt{3 + \sqrt{3}}, \\ a_3 &= \sqrt{3 + a_2} = \sqrt{3 + \sqrt{3 + \sqrt{3}}} \dots, \end{aligned}$$

and, in general, $a_n = \sqrt{3 + a_{n-1}}$. If we attempt to write out a_n "explicitly," we quickly find ourselves in a notational nightmare. However, numerical computation suggests that the sequence may be convergent:

$$\begin{array}{lll} a_1 = 1.73205 & a_2 = 2.17533 & a_3 = 2.27493 \\ a_4 = 2.29672 & a_5 = 2.30146 & a_6 = 2.30249 \\ a_7 = 2.30271 & a_8 = 2.30276 & a_9 = 2.30277 \\ a_{10} = 2.30278 & a_{11} = 2.30278 & a_{12} = 2.30278 \dots \end{array}$$

The sequence appears to be converging to a number $l \approx 2.30278 \dots$, but the numerical evidence only suggests that the sequence converges. The increasing sequence test enables us to prove this.

57. Let the sequence a_n be defined inductively by the rules $a_0 = 0$, $a_n = \sqrt{4 + a_{n-1}}$.

(a) Write out a_1 , a_2 , and a_3 in terms of square roots.

■ (b) Calculate a_1 through a_{12} and guess the value of $\lim_{n \rightarrow \infty} a_n$ to four significant figures.

★58. (a) Prove by induction on n that for the sequence in Exercise 57, we have $a_n > a_{n-1}$ and $a_n < 5$.

(b) Conclude that the limit $l = \lim_{n \rightarrow \infty} a_n$ exists.

(c) Show that l must satisfy the equation $l = \sqrt{4 + l}$.

■ (d) Solve the equation in (c) for l and evaluate l to four significant figures. Compare the result with Exercise 57(b).

Show that the sequences in Exercises 59–62 are increasing (or decreasing) and bounded above (or below).

★59. $a_n = \frac{2n}{n+3}$

★60. $a_n = \frac{n}{n^2+1}$

★61. $a_n = \frac{1}{2n} - \frac{1}{n+1}$ ★62. $b_n = n \sin\left(\frac{1}{n}\right)$

★63. Let $B > 0$ and $a_0 = 1$; $a_{n+1} = \frac{1}{2}(a_n + B/a_n)$. Show that $a_n \rightarrow \sqrt{B}$.

★64. Let $a_{n+1} = 3 - (1/a_n)$; $a_0 = 1$. Prove that the sequence is increasing and bounded above. What is $\lim_{n \rightarrow \infty} a_n$?

★65. Let $a_{n+1} = \frac{1}{2}a_n + \sqrt{a_n}$; $a_0 = 1$. Prove that a_n is increasing and bounded above. What is $\lim_{n \rightarrow \infty} a_n$?

★66. Let $a_{n+1} = \frac{1}{2}(1 + a_n)$, and $a_0 = 1$. Show that $\lim_{n \rightarrow \infty} a_n = 1$.

★67. Give an alternative proof that $\lim_{n \rightarrow \infty} r^n = 0$ if $0 < r < 1$ as follows. Show that r^n decreases and is bounded below by zero. If the limit is l , show that $rl = l$ and conclude that $l = 0$. Why does the limit exist?

★68. Suppose that $a_0 = 1$, $a_{n+1} = 1 + 1/(1 + a_n)$. Show that a_n converges and find the limit.

★69. The celebrated example due to Karl Weierstrass of a *nowhere differentiable continuous function* $f(x)$ in $-\infty < x < \infty$ is given by

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x),$$

where $\phi(x+2) = \phi(x)$, and $\phi(x)$ on $0 \leq x \leq 2$ is the "triangle" through $(0, 0)$, $(1, 1)$, $(2, 0)$. By construction, $0 \leq \phi(4^n x) \leq 1$. Verify by means of the comparison test that the series converges for any value of x . [See *Counterexamples in Analysis* by B. R. Gelbaum and J. M. H. Olmsted, Holden-Day, San Francisco (1964), p. 38 for the proof that f is nowhere differentiable.]

★70. Prove that $a_n = (1 + 1/n)^n$ is increasing and bounded above as follows:

(a) If $0 \leq a \leq b$, prove that

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n.$$

That is, prove $b^n[(n+1)a - nb] < a^{n+1}$.

(b) Let $a = 1 + [1/(n+1)]$ and $b = 1 + (1/n)$ and deduce that a_n is increasing.

(c) Let $a = 1$ and $b = 1 + (1/2n)$ and deduce that $(1 + 1/2n)^{2n} < 4$.

(d) Use parts (b) and (c) to show that $a_n < 4$. Conclude that a_n converges to some number (the number is e —see Section 6.3).

12.3 The Integral and Ratio Tests

The integral test establishes a connection between infinite series and improper integrals.

The sum of any infinite series may be thought of as an improper integral. Namely, given a series $\sum_{i=1}^{\infty} a_i$, we define a step function $g(x)$ on $[1, \infty)$ by the formulas:

$$\begin{aligned} g(x) &= a_1 & (1 \leq x < 2) \\ g(x) &= a_2 & (2 \leq x < 3) \\ &\vdots \\ g(x) &= a_i & (i \leq x < i+1) \\ &\vdots \end{aligned}$$

Since $\int_i^{i+1} g(x) dx = a_i$, the partial sum $\sum_{i=1}^n a_i$ is equal to $\int_1^{n+1} g(x) dx$, and the sum $\sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$ exists if and only if the integral $\int_1^{\infty} g(x) dx = \lim_{b \rightarrow \infty} \int_1^b g(x) dx$ does.

By itself, this relation between series and integrals is not very useful. However, suppose now, as is often the case, that the formula which defines the term a_i as a function of i makes sense when i is a real number, not just an integer. In other words, suppose that there is a function $f(x)$, defined for all x satisfying $1 \leq x < \infty$, such that $f(i) = a_i$ when $i = 1, 2, 3, \dots$. Suppose further that f satisfies these conditions:

1. $f(x) > 0$ for all x in $[1, \infty)$;
2. $f(x)$ is decreasing on $[1, \infty)$.

For example, if $a_i = 1/i$, the harmonic series, we may take $f(x) = 1/x$.

We may now compare $f(x)$ with the step function $g(x)$. When x satisfies $i \leq x < i+1$, we have

$$0 \leq f(x) \leq f(i) = a_i = g(x).$$

Hence $0 \leq f(x) \leq g(x)$. (See Fig. 12.3.1.)

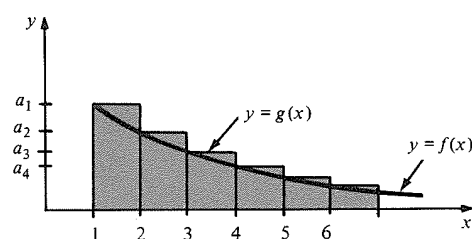


Figure 12.3.1. The area under the graph of f is less than the shaded area, so $0 \leq \int_1^{n+1} f(x) dx \leq \sum_{i=1}^n a_i$.

It follows that, for any n ,

$$0 \leq \int_1^{n+1} f(x) dx \leq \int_1^{n+1} g(x) dx = \sum_{i=1}^n a_i. \quad (1)$$

We conclude that if the series $\sum_{i=1}^{\infty} a_i$ converges, then the integrals $\int_1^{n+1} f(x) dx$ are bounded above by the sum $\sum_{i=1}^{\infty} a_i$, so that the improper integral $\int_1^{\infty} f(x) dx$ converges (see Section 11.3).

In other words, if the integral $\int_1^{\infty} f(x) dx$ diverges, then so does the series $\sum_{i=1}^{\infty} a_i$.

Example 1 Show that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \ln(n+1)$$

and so obtain a new proof that the harmonic series diverges.

Solution We take our function $f(x)$ to be $1/x$. Then, from formula (1) above, we get

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{i=1}^n \frac{1}{i} \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Since $\lim_{n \rightarrow \infty} \ln(n+1) = \infty$, the integral $\int_1^{\infty} (1/x) dx$ diverges; hence the series $\sum_{i=1}^{\infty} (1/i)$ diverges, too. ▲

We would like to turn around the preceding argument to show that if $\int_1^{\infty} f(x) dx$ converges, then $\sum_{i=1}^{\infty} a_i$ converges as well. To do so, we draw the rectangles with height a_i to the *left* of $x = i$ rather than the right; see Fig. 12.3.2. This procedure defines a step function $h(x)$ on $[1, \infty)$ defined by

$$h(x) = a_{i+1} \quad (i \leq x < i+1).$$

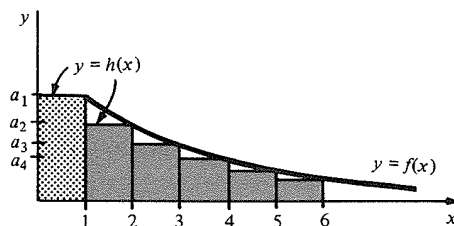
Now we have $\int_i^{i+1} h(x) dx = a_{i+1}$, so $\sum_{i=2}^n a_i = \int_1^n h(x) dx$. If x satisfies $i \leq x < i+1$, we have

$$f(x) \geq f(i+1) = a_{i+1} = h(x) \geq 0.$$

Hence $f(x) \geq h(x) \geq 0$. (See Fig. 12.3.2.) Thus

$$\int_1^n f(x) dx \geq \int_1^n h(x) dx = \sum_{i=2}^n a_i \geq 0. \quad (2)$$

Figure 12.3.2. The area under the graph of f is greater than the shaded area, so $0 \leq \sum_{i=2}^n a_i \leq \int_1^n f(x) dx$.



If the integral $\int_1^{\infty} f(x) dx$ converges, then the partial sums $\sum_{i=1}^n a_i = a_1 + \sum_{i=2}^n a_i$ are bounded above by $a_1 + \int_1^{\infty} f(x) dx$, and therefore the series $\sum_{i=1}^{\infty} a_i$ is convergent (see the Supplement to Section 12.2).

Integral Test

To test the convergence of a series $\sum_{i=1}^{\infty} a_i$ of positive decreasing terms, find a positive, decreasing function $f(x)$ on $[1, \infty)$ such that $f(i) = a_i$.

If $\int_1^{\infty} f(x) dx$ converges, so does $\sum_{i=1}^{\infty} a_i$.

If $\int_1^{\infty} f(x) dx$ diverges, so does $\sum_{i=1}^{\infty} a_i$.

Example 2 Show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges.

Solution This series is $\sum_{i=1}^{\infty} (1/i^2)$. We let $f(x) = 1/x^2$; then

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b}\right) = 1.$$

The indefinite integral converges, so the series does, too. ▲

Example 3 Show that $\sum_{m=2}^{\infty} \frac{1}{m\sqrt{\ln m}}$ diverges, but $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$ converges.

Solution Note that the series start at $m = 2$ rather than $m = 1$. We consider the integral

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^p} dx &= \lim_{b \rightarrow \infty} \int_2^b (\ln x)^{-p} \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \left. \frac{(\ln x)^{-p+1}}{-p+1} \right|_2^b \\ &= \frac{1}{-p+1} \lim_{b \rightarrow \infty} [(\ln b)^{-p+1} - (\ln 2)^{-p+1}]. \end{aligned}$$

The limit is finite if $p = 2$ and infinite if $p = \frac{1}{2}$, so the integral converges if $p = 2$ and diverges if $p = \frac{1}{2}$. It follows that $\sum_{m=2}^{\infty} [1/(m\sqrt{\ln m})]$ diverges and $\sum_{m=2}^{\infty} [1/m(\ln m)^2]$ converges. $\blacktriangle$

Examples 1 and 2 are special cases of a result called the p -series test, which arises from the integral test with $f(x) = 1/x^p$. We recall that $\int_1^{\infty} x^n dx$ converges if $n < -1$ and diverges if $n \geq -1$ (see Example 2, Section 11.3). Thus we arrive at the test in the following box.

p -Series

If $p \leq 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^p}$ diverges.
If $p > 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges.

The p -series are often useful in conjunction with the comparison test.

Example 4 Test for convergence:

$$(a) \sum_{i=1}^{\infty} \frac{1}{1+i^2}; \quad (b) \sum_{j=1}^{\infty} \frac{j^2+2j}{j^4-3j^2+10}; \quad (c) \sum_{n=1}^{\infty} \frac{3n+\sqrt{n}}{2n^{3/2}+2}.$$

Solution (a) We compare the given series with the convergent p series $\sum_{i=1}^{\infty} 1/i^2$. Let $a_i = 1/(1+i^2)$ and $b_i = 1/i^2$. Then $0 < a_i < b_i$ and $\sum_{i=1}^{\infty} b_i$ converges, so $\sum_{i=1}^{\infty} a_i$ does, too.

(b) Let $a_j = (j^2+2j)/(j^4-3j^2+10)$ and $b_j = j^2/j^4 = 1/j^2$. Then

$$\lim_{j \rightarrow \infty} \left| \frac{a_j}{b_j} \right| = \lim_{j \rightarrow \infty} \left| \frac{1+2/j}{1-3/j^2+10/j^4} \right| = 1.$$

Since $\sum_{j=1}^{\infty} b_j$ converges, so does $\sum_{j=1}^{\infty} a_j$, by the ratio comparison test.

(c) Take $a_n = (3n+\sqrt{n})/(2n^{3/2}+2)$ and $b_n = n/(n^{3/2}) = 1/\sqrt{n}$. Then

$$\lim_{n \rightarrow \infty} \frac{3 + (1/\sqrt{n})}{2 + (2/n^{3/2})} = \frac{3}{2}.$$

Since $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} a_n$. $\blacktriangle$

What is the error in approximating a p -series by a partial sum? Let us show that $\sum_{n=1}^N (1/n^p)$ approximates $\sum_{n=1}^{\infty} (1/n^p)$ with error which does not exceed $1/[(p-1)N^{p-1}]$.

Indeed, just as in the proof of formula (2), we have

$$\sum_{n=N+1}^{\infty} \frac{1}{n^p} \leq \int_N^{\infty} \frac{1}{x^p} dx.$$

The left-hand side is the error:

$$\sum_{n=N+1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} - \sum_{n=1}^N \frac{1}{n^p} \leq \int_N^{\infty} \frac{1}{x^p} dx = \frac{1}{(p-1)N^{p-1}}.$$

$$\text{Thus, error} \leq \frac{1}{(p-1)N^{p-1}}. \quad (3)$$

Example 5 It is known that $\sum_{n=1}^{\infty} (1/n^2) = \pi^2/6$. Use this equation² to calculate $\pi^2/6$ with error less than 0.05.

Solution By equation (3), the error in stopping at N terms is at most $1/N$. To have error $< 0.05 = \frac{1}{20}$, we must take 20 terms (note that 100 terms are needed to get two decimal places!). We find:

$$1 = 1,$$

$$1 + \frac{1}{4} = 1.25,$$

$$1 + \frac{1}{4} + \frac{1}{9} = 1.36,$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = 1.42,$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = 1.46,$$

and so forth, obtaining 1.49, 1.51, 1.53, 1.54, 1.55, 1.56, ... Finally, $1 + \frac{1}{4} + \cdots + \frac{1}{400} = 1.596 \dots$ (Notice the "slowness" of the convergence.) We may compare this with the exact value $\pi^2/6 = 1.6449 \dots$ ▲

The idea used in the preceding example can be used to estimate the tail of a series whenever convergence is proven by the integral test. (See Exercise 11.)

Another important test for convergence is called the *ratio test*. This test provides a general way to compare a series with a geometric series, but it formulates the hypotheses in a way which is particularly convenient, since no explicit comparison is needed. Here is the test.

Ratio Test

Let $\sum_{i=1}^{\infty} a_i$ be a series. Suppose that $\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right|$ exists.

1. If $\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right| < 1$, then the series converges (absolutely).
2. If $\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right| > 1$, then the series diverges.
3. If $\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right| = 1$, the test is inconclusive.

² For a proof using only elementary calculus, see Y. Matsuoka, "An Elementary Proof of the Formula $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$," *American Mathematical Monthly* 68(1961): 485–487 (reprinted in T. M. Apostol (ed.), *Selected Papers on Calculus*, Math. Assn. of America (1969), p. 372). The formula may also be proved using Fourier series; see for instance J. Marsden, *Elementary Classical Analysis*, Freeman (1974), Ch. 10.

Do not confuse this test, in which ratios of successive terms in the *same* series are considered, with the ratio comparison test in Section 12.2, where we took the ratios of terms in two *different* series.

**Proof of
the ratio
test**

By definition of the limit, $|a_i/a_{i-1}|$ will be close to its limit l for i large. To prove part 1, let $l < 1$ and let $r = (l + 1)/2$ be the midpoint between l and 1, so that $l < r < 1$. Thus there is an N such that

$$\left| \frac{a_i}{a_{i-1}} \right| < r \quad \text{if } i > N.$$

We will show this implies that the given series converges.

We have $|a_{N+1}/a_N| < r$ so $|a_{N+1}| < |a_N|r$, $|a_{N+2}/a_{N+1}| < r$; hence $|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$ and, in general, $|a_{N+j}| < |a_N|r^j$; but $\sum_{j=1}^{\infty} |a_N|r^j = |a_N|\sum_{j=1}^{\infty} r^j$ is a convergent geometric series since $r < 1$. Hence, by the comparison test, $\sum_{j=1}^{\infty} |a_{N+j}|$ converges. Since we have omitted only $|a_1|, |a_2|, \dots, |a_N|$, the series $\sum_{j=1}^{\infty} |a_j|$ converges as well and part 1 is proved.

For part 2 we find, as in part 1, that $|a_{N+j}| > |a_N|r^j$, where $r = (l + 1)/2$ is now greater than 1. As $j \rightarrow \infty$, $r^j \rightarrow \infty$, so $|a_{N+j}| \rightarrow \infty$. Thus the series cannot converge, since its terms do not converge to zero.

To prove part 3, we consider the p -series with $a_i = i^p$. The ratio is $|a_i/a_{i-1}| = [i/(i-1)]^p$, and $\lim_{i \rightarrow \infty} [i/(i-1)]^p = [\lim_{i \rightarrow \infty} (i/(i-1))]^p = 1^p = 1$ for all $p > 0$; but the p -series is convergent if $p > 1$ and divergent if $p \leq 1$, so the ratio test does not give any useful information for these series. ■

Example 6 Test for convergence: $2 + \frac{2^2}{2^8} + \frac{2^3}{3^8} + \frac{2^4}{4^8} + \dots = 2 + \frac{1}{64} + \frac{8}{6561} + \frac{1}{4096} + \dots$.

Solution We have $a_i = 2^i/i^8$. The ratio a_i/a_{i-1} is

$$\frac{2^i}{i^8} \cdot \frac{(i-1)^8}{2^{i-1}} = 2 \cdot \left(\frac{i-1}{i} \right)^8,$$

so

$$\lim_{i \rightarrow \infty} \frac{a_i}{a_{i-1}} = 2 \left[\lim_{i \rightarrow \infty} \left(\frac{i-1}{i} \right) \right]^8 = 2 \cdot 1^8 = 2$$

which is greater than 1, and so the series diverges. ▲

Example 7 Test for convergence:

(a) $\sum_{n=1}^{\infty} \frac{1}{n!}$, where $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$

(b) $\sum_{j=1}^{\infty} \frac{b^j}{j!}$, b any constant

Solution (a) Here $a_n = 1/n!$, so

$$\frac{a_n}{a_{n-1}} = \frac{1/n(n-1) \cdots 3 \cdot 2 \cdot 1}{1/(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1} = \frac{1}{n}.$$

Thus $|a_n/a_{n-1}| = 1/n \rightarrow 0 < 1$, so we have convergence.

(b) Here $a_j = b^j/j!$, so

$$\frac{a_j}{a_{j-1}} = \frac{b^j/j!}{b^{j-1}/(j-1)!} = \frac{b}{j}.$$

Thus $|a_j/a_{j-1}| = b/j \rightarrow 0$, so we have convergence. In this example, note that the numerator b^j and the denominator $j!$ tend to infinity, but the denominator does so much faster. In fact, since the series converges, $b^j/j! \rightarrow 0$ as $j \rightarrow \infty$. ▲

Let us show that if $|a_n/a_{n-1}| < r < 1$ for $n > N$, then the error made in approximating $\sum_{n=1}^{\infty} a_n$ by $\sum_{n=1}^N a_n$ is no greater than $|a_N|r/(1-r)$. In short,

$$\text{error} \leq \frac{|a_N|r}{1-r}. \quad (4)$$

Indeed, $\sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n$. As in the proof of the ratio test, $|a_{N+1}| < |a_N|r$, and, in general, $|a_{N+j}| < |a_N|r^j$, so $\sum_{j=1}^{\infty} |a_{N+j}| \leq |a_N|r/(1-r)$ by the formula for the sum of a geometric series and the comparison test. Hence the error is no greater than $|a_N|r/(1-r)$.

Example 8 What is the error made in approximating $\sum_{n=1}^{\infty} \frac{1}{n!}$ by $\sum_{n=1}^4 \frac{1}{n!}$?

Solution Here $|a_n/a_{n-1}| = 1/n$, which is $< \frac{1}{5}$ if $n > 4 = N$. By inequality (4), the error is no more than $a_4/5(1 - 1/5) = 1/4 \cdot 4! = 1/96 < 0.0105$. The error becomes small very quickly if N is increased. ▲

Our final test is similar in spirit to the ratio test, in that it is also proved by comparison with a geometric series.

Root Test

Let $\sum_{i=1}^{\infty} a_i$ be a given series, and suppose that $\lim_{i \rightarrow \infty} |a_i|^{1/i}$ exists.

1. If $\lim_{i \rightarrow \infty} |a_i|^{1/i} < 1$, then $\sum_{i=1}^{\infty} a_i$ converges absolutely.
2. If $\lim_{i \rightarrow \infty} |a_i|^{1/i} > 1$, then $\sum_{i=1}^{\infty} a_i$ diverges.
3. If $\lim_{i \rightarrow \infty} |a_i|^{1/i} = 1$, the test is inconclusive.

To prove 1, let $l = \lim_{n \rightarrow \infty} (|a_n|^{1/n})$ and let $r = (1+l)/2$ be the midpoint of 1 and l , so $l < r < 1$. From the definition of the limit, there is an N such that $|a_n|^{1/n} < r < 1$ if $n > N$. Hence $|a_n| < r^n$ if $n > N$. Thus, by direct comparison of $\sum_{n=N+1}^{\infty} |a_n|$ with the geometric series $\sum_{n=N+1}^{\infty} r^n$, which converges since $r < 1$, $\sum_{n=N+1}^{\infty} |a_n|$ converges. Since we have neglected only finitely many terms, the given series converges.

Cases 2 and 3 are left as exercises (see Exercises 37 and 38).

Example 9 Test for convergence: (a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$ and (b) $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$.

Solution (a) Here $a_n = 1/n^n$, so $|a_n|^{1/n} = 1/n$. Thus $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 0 < 1$. Thus, by the root test (with i replaced by n), the series converges (absolutely). [This example can also be done by the comparison test: $1/n^n < 1/n^2$ for $n \geq 2$.]
 (b) Here $a_n = 3^n/n^2$, so $|a_n|^{1/n} = 3/n^{2/n}$; but $\lim_{n \rightarrow \infty} n^{2/n} = 1$, since $\ln(n^{2/n}) = 2(\ln n)/n \rightarrow 0$ as $n \rightarrow \infty$ (by l'Hôpital's rule). Thus $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 3 > 1$, so the series diverges. ▲

The tests we have covered enable us to deal with a wide variety of series. Of course, if the series is geometric, it may be summed. Otherwise, either the ratio test, the root test, comparison with a p -series, the integral test, or the alternating series test will usually work.

Example 10 Test for convergence: (a) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ and (b) $\sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n}$.

Solution (a) We use the ratio test. Here, $a_n = n^n/n!$, so

$$\begin{aligned} \left| \frac{a_n}{a_{n-1}} \right| &= \frac{n^n}{(n-1)^{n-1}} \cdot \frac{(n-1)!}{n!} = \frac{n \cdot n^{n-1}}{(n-1)^{n-1}} \cdot \frac{1}{n} = \left(\frac{n}{n-1} \right)^{n-1} \\ &= \frac{1}{(1 - 1/n)^{n-1}} = \frac{1 - 1/n}{(1 - 1/n)^n}. \end{aligned}$$

The numerator approaches 1 while the denominator approaches e^{-1} (see Section 6.4), so $\lim_{n \rightarrow \infty} |a_n/a_{n-1}| = e > 1$, and the series diverges.

(b) We expect the series to behave like $\sum_{n=1}^{\infty} (1/n^2)$, so we use the ratio comparison test, with $a_i = 1/(i^2 - \ln i)$ and $b_i = 1/i^2$. The ratio between the terms in the two series is

$$\frac{a_i}{b_i} = \frac{i^2}{i^2 - \ln i} = \frac{1}{1 - \ln i/i^2}.$$

Since $\lim_{i \rightarrow \infty} [(\ln i)/i^2] = 0$ (by l'Hôpital's rule), $\lim_{i \rightarrow \infty} a_i/b_i = 1$. The p -series $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} (1/i^2)$ converges, so the series

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{1}{i^2 - \ln i} \text{ converges, too. } \blacktriangle$$

Exercises for Section 12.3

Use the integral test to determine the convergence or divergence of the series in Exercises 1–4.

1. $\sum_{i=1}^{\infty} \frac{i}{i^2 + 1}$
2. $\sum_{i=1}^{\infty} \frac{1}{i^2 + 4}$
3. $\sum_{i=2}^{\infty} \frac{1}{i(\ln i)^{3/2}}$
4. $\sum_{i=2}^{\infty} \frac{1}{i(\ln i)^{2/3}}$

Use the p -series test and a comparison test to test the series in Exercises 5–8 for convergence or divergence.

5. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$
6. $\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$
7. $\sum_{n=1}^{\infty} \frac{n}{n^3 + 4}$
8. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$

Estimate the sums in Exercises 9 and 10 to within 0.05.

9. $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$
10. $\sum_{n=1}^{\infty} \frac{\sin n}{n^4}$

11. Let $f(x)$ be a positive decreasing function on $[1, \infty)$ such that $\int_1^{\infty} f(x) dx$ converges. Show that

$$\left| \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^N f(n) \right| < \int_N^{\infty} f(x) dx.$$

12. Estimate $\sum_{n=1}^{\infty} [(1 + n^2)/(1 + n^8)]$ to within 0.02. (Use the comparison test and the integral test.)

Use the ratio test to determine the convergence or divergence of the series in Exercises 13–16.

13. $\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{3^n}$
14. $\sum_{n=1}^{\infty} \frac{3^n}{2\sqrt{n}}$
15. $\sum_{i=1}^{\infty} \frac{i^3 \cdot 3^i}{i!}$
16. $\sum_{n=1}^{\infty} \frac{2n^2 + n!}{n^5 + (3n)!}$

Estimate the sums in Exercises 17 and 18 to within 0.05.

17. $\sum_{n=0}^{\infty} \frac{\pi^{2n+1}}{(2n+1)!}$
18. $\sum_{n=0}^{\infty} \frac{(\pi/2)^{2n+1}}{(2n+1)!}$

19. Estimate $\sum_{n=1}^{\infty} (1/n!)$: (a) To within 0.05. (b) To within 0.005. (c) How many terms would you need to calculate to get an accuracy of five decimal places?

20. (a) Show that $\sum_{n=1}^{\infty} \frac{\sin(\pi n/2)}{n!}$ converges.

(b) Estimate the sum to within 0.01.

Use the root test to determine the convergence or divergence of the series in Exercises 21–24.

21. $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$
22. $\sum_{n=1}^{\infty} \frac{n^n}{2^n}$
23. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$
24. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Test for convergence in Exercises 25–36.

25. $\sum_{i=1}^{\infty} \frac{1}{i^4}$
26. $\sum_{j=3}^{\infty} \frac{j^2 + \cos j}{j^4 + \sin j}$
27. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$
28. $\sum_{m=1}^{\infty} \frac{(-1)^m(m+1)}{m^2+1}$
29. $\sum_{k=2}^{\infty} \frac{\cos k\pi}{\ln k}$
30. $\sum_{j=1}^{\infty} (-1)^j \sin\left(\frac{\pi}{4j}\right)$
31. $\sum_{j=1}^{\infty} \frac{(j+1)^{100}}{j!}$
32. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + n + n^{3/2}}{\sqrt{n} + n + n^{5/2} + n^3}$

$$33. \sum_{r=0}^{\infty} \frac{2^r}{2^r + 3^r} \quad 34. \sum_{s=1}^{\infty} \frac{s - \ln s}{s^2 + \ln s}$$

$$35. \sum_{t=2}^{\infty} \frac{(-1)^t}{(\ln t)^{1/2}} \quad 36. \sum_{t=1}^{\infty} \frac{(-1)^t}{t^{1/4}}$$

In Exercises 37 and 38, complete the proof of the root test by showing the following.

- ★37. If $\lim_{n \rightarrow \infty} |a_n|^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
 ★38. If $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 1$, the test is inconclusive. (You may use the fact that $\lim_{n \rightarrow \infty} n^{1/n} = 1$.)
 ★39. For which values of p does $\sum_{i=1}^{\infty} [\sin(1/i)]^p$ converge?
 ★40. For which values of p does $\sum_{n=2}^{\infty} [1/n(\ln n)^p]$ converge?
 ★41. For which p does $\sum_{n=2}^{\infty} (1/n^p \ln n)$ converge?
 ★42. For which values of p and q is the series $\sum_{n=2}^{\infty} 1/[n^p(\ln n)^q]$ convergent?
 ★43. (a) Let $f(x)$ be positive and decreasing on $[1, \infty)$, and suppose that $f(i) = a_i$ for $i = 1, 2, 3, \dots$. Show that

$$S - \frac{1}{2} f(n) \leq \sum_{i=1}^{\infty} a_i \leq S + \frac{1}{2} f(n),$$

where

$$S = \sum_{i=1}^n f(i) + \frac{1}{2} \int_n^{n+1} f(x) dx + \int_{n+1}^{\infty} f(x) dx.$$

[Hint: Look at the proof of the integral test; show that $\int_{n+1}^{\infty} f(x) dx \leq \sum_{i=n+1}^{\infty} a_i \leq \int_n^{\infty} f(x) dx$.]

- ▮ (b) Estimate $\sum_{n=1}^{\infty} 1/n^4$ to within 0.0001. How many terms did you use? How much work do you save by using the method of part (a) instead of the formula: error $< 1/(p-1)N^{p-1}$?
 ★44. Using Fourier analysis, it is possible to show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

(a) Show directly that the series on the right is convergent, by means of the integral test.

(b) Determine how many terms are needed to compute $\pi^4/96$ accurate to 20 digits.

- ★45. A bar of length L is loaded by a weight W at its midpoint. At $t = 0$ the load is removed. The deflection $y(t)$ at the midpoint, measured from the straight profile $y = 0$, is given by

$$y(t) = \frac{2WL^2}{\pi^4 EI} \left[\cos r + \frac{\cos(9r)}{3^4} + \frac{\cos(25r)}{5^4} + \dots \right],$$

where $r = \left(\frac{\pi^2}{L^2} \sqrt{\frac{EIg}{\gamma\Omega}} \right) t$. The numbers $E, I, g,$

γ, Ω, L are positive constants.

(a) Show by substitution that the bracketed terms are the first three terms of the infinite series

$$\sum_{n=0}^{\infty} \frac{\cos[(2n+1)^2 r]}{(2n+1)^4}.$$

(b) Make accurate graphs of the first three partial sums

$$S_1(r) = \cos(r),$$

$$S_2(r) = \cos(r) + \frac{1}{3^4} \cos(9r),$$

$$S_3(r) = \cos(r) + \frac{1}{3^4} \cos(9r) + \frac{1}{5^4} \cos(25r).$$

Up to a magnification factor, these graphs approximate the motion of the midpoint of the bar.

(c) Using the integral test and the comparison test, show that the series converges.

12.4 Power Series

Many functions can be expressed as “polynomials with infinitely many terms.”

A series of the form $\sum_{i=0}^{\infty} a_i(x - x_0)^i$, where the a_i 's and x_0 are constants and x is a variable, is called a *power series* (since we are summing the powers of $(x - x_0)$). In this section, we show how a power series may be considered as a function of x , defined on a certain interval. In the next section, we begin with an arbitrary function and show how to find the power series which represents it (if there is such a series).

We first consider power series in which $x_0 = 0$; that is, those of the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{i=0}^{\infty} a_i x^i,$$

where the a_i are given constants. The domain of f can be taken to consist of those x for which the series converges.

If there is an integer N such that $a_i = 0$ for all $i > N$, then the power series is equal to a finite sum, $\sum_{i=0}^N a_i x^i$, which is just a polynomial of degree N . In general, we may think of a power series as a polynomial of “infinite degree”; we will see that as long as they converge, power series may be manipulated (added, subtracted, multiplied, divided, differentiated) just like ordinary polynomials.

The simplest power series, after a polynomial, is the geometric series

$$f(x) = 1 + x + x^2 + \cdots,$$

which converges when $|x| < 1$; the sum is the function $1/(1-x)$. Thus we have written $1/(1-x)$ as a power series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{if } |x| < 1.$$

Convergence of general power series may often be determined by a test similar to the ratio test.

Ratio Test for Power Series

Let $\sum_{i=0}^{\infty} a_i x^i$ be a power series. Assume that

$$\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right| = l$$

exists. Let $R = 1/l$; if $l = 0$, let $R = \infty$, and if $l = \infty$, let $R = 0$. Then:

1. If $|x| < R$, the power series converges absolutely.
2. If $|x| > R$, the power series diverges.
3. If $x = \pm R$, the power series could converge or diverge.

To prove part 1, we use the ratio test for series of numbers; the ratio of successive terms for $\sum_{i=0}^{\infty} a_i x^i$ is

$$\left| \frac{a_i x^i}{a_{i-1} x^{i-1}} \right| = \left| \frac{a_i}{a_{i-1}} \right| |x|.$$

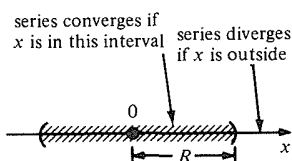


Figure 12.4.1. R is the radius of convergence of $\sum_{i=0}^{\infty} a_i x^i$.

By hypothesis, this converges to $l \cdot |x| < l \cdot R = 1$. Hence, by the ratio test, the series converges absolutely when $|x| < R$. The proof of part 2 is similar, and the examples below will show that at $x = \pm R$, either convergence or divergence can occur.

The number R in this test is called the *radius of convergence* of the series (see Fig. 12.4.1). One can show that a number R (possibly infinity) with the three properties in the preceding box exists for any power series, even if $\lim_{i \rightarrow \infty} |a_i/a_{i-1}|$ does not exist.

Example 1 For which x does $\sum_{i=0}^{\infty} \frac{i}{i+1} x^i$ converge?

Solution Here $a_i = i/(i+1)$. Then

$$\begin{aligned} \frac{a_i}{a_{i-1}} &= \frac{i/(i+1)}{(i-1)/i} = \frac{i^2}{(i+1)(i-1)} \\ &= \frac{1}{(1+1/i)(1-1/i)} \rightarrow 1 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Hence $l = 1$. Thus the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, then $\lim_{i \rightarrow \infty} [i/(i+1)]x^i = 1$, so the series diverges at $x = 1$ since the terms do not go to zero. If $x = -1$, $\lim_{i \rightarrow \infty} [i/(i+1)]x^i = \lim_{i \rightarrow \infty} [i/(i+1)] = 1$, so again the series diverges. $\blacktriangle$

Example 2 Determine the radius of convergence of $\sum_{k=0}^{\infty} \frac{k^5}{(k+1)!} x^k$.

Solution To use the ratio test, we look at

$$l = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k-1}} \right|.$$

Here $a_k = k^5/(k+1)!$, so

$$\begin{aligned} l &= \lim_{k \rightarrow \infty} \left| \frac{k^5}{(k+1)!} \cdot \frac{k!}{(k-1)^5} \right| \\ &= \lim_{k \rightarrow \infty} \left(\frac{k}{k-1} \right)^5 \cdot \lim_{k \rightarrow \infty} \frac{1}{k+1} = 1 \cdot 0 = 0. \end{aligned}$$

Thus $l = 0$, so $R = \infty$ and the radius of convergence is infinite (that is, the series converges for all x). $\blacktriangle$

Example 3 For which x do the following series converge? (a) $\sum_{i=1}^{\infty} \frac{x^i}{i}$ (b) $\sum_{i=1}^{\infty} \frac{x^i}{i^2}$
(c) $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ (By convention, we define $0! = 1$.)

Solution (a) We have $a_i = 1/i$, so

$$l = \lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i-1}} \right| = \lim_{i \rightarrow \infty} \left(\frac{i-1}{i} \right) = 1;$$

the series therefore converges for $|x| < 1$ and diverges for $|x| > 1$. When $x = 1$, $\sum_{i=1}^{\infty} x^i/i$ is the divergent harmonic series; for $x = -1$, the series is alternating, so it converges.

(b) We have $a_i = 1/i^2$, so

$$l = \lim_{i \rightarrow \infty} \frac{(i-1)^2}{i^2} = 1$$

and the radius of convergence is again 1. This time, when $x = 1$, we get the p -series $\sum_{i=1}^{\infty} (1/i^2)$, which converges since $p = 2 > 1$. The series for $x = -1$, $\sum_{i=1}^{\infty} [(-1)^i/i^2]$, converges absolutely, so is also convergent.

(c) Here $a_i = 1/i!$, so $|a_i/a_{i-1}| = (i-1)!/i! = 1/i \rightarrow 0$ as $i \rightarrow \infty$. Thus $l = 0$, so the series converges for all x . $\blacktriangle$

Series of the form $\sum_{i=0}^{\infty} a_i (x - x_0)^i$ are also called power series; their theory is essentially the same as for the case $x_0 = 0$ already studied, because $\sum_{i=0}^{\infty} a_i (x - x_0)^i$ may be written as $\sum_{i=0}^{\infty} a_i w^i$, where $w = x - x_0$.

Example 4 For which x does the series $\sum_{n=0}^{\infty} \frac{4^n}{\sqrt{2n+5}} (x+5)^n$ converge?

Solution This series is of the form $\sum_{i=0}^{\infty} a_i (x - x_0)^i$, with $a_i = 4^i / \sqrt{2i+5}$ and $x_0 = -5$. We have

$$l = \lim_{i \rightarrow \infty} \frac{a_i}{a_{i-1}} = \lim_{i \rightarrow \infty} \frac{4^i}{\sqrt{2i+5}} \cdot \frac{\sqrt{2(i-1)+5}}{4^{i-1}} = \lim_{i \rightarrow \infty} 4 \sqrt{\frac{2i+3}{2i+5}} = 4,$$

so the radius of convergence is $\frac{1}{4}$. Thus the series converges for $|x+5| < \frac{1}{4}$ and diverges for $|x+5| > \frac{1}{4}$. When $x = -5\frac{1}{4}$, the series becomes $\sum_{i=0}^{\infty} [(-1)^i / \sqrt{2i+5}]$, which converges because it is alternating. When $x = -4\frac{3}{4}$, the series is $\sum_{i=0}^{\infty} [1 / \sqrt{2i+5}]$, which diverges by the ratio comparison test with $\sum_{i=1}^{\infty} (1/\sqrt{2i})$ (or by the integral test). Thus our power series converges when $-5\frac{1}{4} \leq x < -4\frac{3}{4}$. $\blacktriangle$

In place of the ratio test, one can sometimes use the root test in the same way.

Root Test for Power Series

Let $\sum_{i=0}^{\infty} a_i x^i$ be a given power series. Assume that $\lim_{i \rightarrow \infty} |a_i|^{1/i} = \rho$ exists. Then the radius of convergence is $R = 1/\rho$.

Indeed, if $|x| < R$, $\lim_{i \rightarrow \infty} |a_i x^i|^{1/i} = \lim_{i \rightarrow \infty} |a_i|^{1/i} |x| = \rho |x| < \rho R = 1$, so the power series converges by the root test.

Example 5 Find the radius of convergence of the series $\sum_{i=1}^{\infty} \frac{x^i}{(2 + 1/i)^i}$.

Solution $\rho = \lim_{i \rightarrow \infty} |a_i|^{1/i} = \lim_{i \rightarrow \infty} (1/(2 + 1/i))^i = \lim_{i \rightarrow \infty} \{1/[2 + (1/i)]\} = \frac{1}{2}$, so the radius of convergence is $R = 2$. $\blacktriangle$

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, defined where the series converges. By analogy with ordinary polynomials, we might guess that

$$f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$$

and that

$$\int f(x) dx = \sum_{i=0}^{\infty} \frac{a_i x^{i+1}}{i+1} + C.$$

In fact, this is true. The proof is contained in (the moderately difficult) Exercises 41–45 at the end of the section.

Example 6 If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, show that $f'(x) = f(x)$. Conclude that $f(x) = e^x$.

Solution By Example 3(c), the series for $f(x)$ converges for all x . Then $f'(x) = \sum_{i=1}^{\infty} (i x^{i-1} / i!) = \sum_{i=1}^{\infty} [x^{i-1} / (i-1)!] = \sum_{i=0}^{\infty} (x^i / i!) = f(x)$. By the

uniqueness of the solution of the differential equation $f'(x) = f(x)$ (see Section 8.2), $f(x)$ must be ce^x for some c . Since $f(0) = 1$, c must be 1, and so $f(x) = e^x$. $\blacktriangle$

Differentiation and Integration of Power Series

To differentiate or integrate a power series within its radius of convergence R , differentiate or integrate it term by term: if $|x - x_0| < R$,

$$\frac{d}{dx} \sum_{i=0}^{\infty} a_i (x - x_0)^i = \sum_{i=1}^{\infty} i a_i (x - x_0)^{i-1},$$

$$\text{and } \int \left[\sum_{i=0}^{\infty} a_i (x - x_0)^i \right] dx = \sum_{i=0}^{\infty} \frac{a_i}{i+1} (x - x_0)^{i+1} + C.$$

(The resulting series converge if $|x - x_0| < R$.)

Example 7 Let $f(x) = \sum_{i=0}^{\infty} \frac{i}{i+1} x^i$. Find a series expression for $f'(x)$. Where is it valid?

Solution By Example 1, $f(x)$ converges for $|x| < 1$. Thus $f'(x)$ also converges if $|x| < 1$, and we may differentiate term by term:

$$f'(x) = \sum_{i=0}^{\infty} \frac{i^2}{i+1} x^{i-1}, \quad |x| < 1 \text{ (this series diverges at } x = \pm 1 \text{).}$$

(Notice that $f'(x)$ is again a power series, so it too can be differentiated. Since this can be repeated, we conclude that f can be differentiated as many times as we please. We say that f is *infinitely differentiable*.) $\blacktriangle$

Example 8 Write down power series for $x/(1+x^2)$ and $\ln(1+x^2)$. Where do they converge?

Solution First, we expand $1/(1+x^2)$ as a geometric series using the general formula $1/(1-r) = 1 + r + r^2 + \cdots$, with r replaced by $-x^2$, obtaining $1 - x^2 + x^4 - \cdots$. Multiplying by x gives $x/(1+x^2) = x - x^3 + x^5 - \cdots$, which converges for $|x| < 1$. (It diverges for $x = \pm 1$.)

Now we observe that $(d/dx)\ln(1+x^2) = 2x/(1+x^2)$, so

$$\begin{aligned} \ln(1+x^2) &= 2 \int \frac{x}{1+x^2} dx = 2 \int (x - x^3 + x^5 - \cdots) dx \\ &= 2 \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots \right) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \end{aligned}$$

(The integration constant was dropped because $\ln(1+0^2) = 0$.) This series converges for $|x| < 1$, and also for $x = \pm 1$, because there it is alternating. $\blacktriangle$

The operations of addition and multiplication by a constant may be performed term by term on power series, just as on polynomials. This may be proved using the limit theorems. The operations of multiplication and division proceed by the same methods one uses for polynomials, but are more subtle to justify. We state the results in the following box.

Algebraic Operations on Power Series

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, with radius of convergence R .

Let $g(x) = \sum_{i=0}^{\infty} b_i x^i$, with radius of convergence S .

If T is the smaller of R and S , then

$$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \quad \text{for } |x| < T;$$

$$cf(x) = \sum_{i=0}^{\infty} (ca_i) x^i \quad \text{for } |x| < R;$$

$$f(x)g(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i \quad \text{for } |x| < T.$$

If $b_0 \neq 0$, then $f(x)/g(x) = \sum_{i=0}^{\infty} c_i x^i$ for x near zero, where the c_i 's may be determined by long division. The determination of the radius of convergence of f/g requires further analysis.

Example 9 Write down power series of the form $\sum_{i=0}^{\infty} a_i x^i$ for $2/(3-x)$, $5/(4-x)$, and $(23-7x)/[(3-x)(4-x)]$. What are their radii of convergence?

Solution We may write

$$\frac{2}{3-x} = \frac{2}{3} \left(\frac{1}{1-x/3} \right) = \frac{2}{3} \sum_{i=0}^{\infty} \left(\frac{x}{3} \right)^i = \sum_{i=0}^{\infty} \frac{2}{3^{i+1}} x^i.$$

The ratio of successive coefficients is $(1/3^{i+1})/(1/3^i) = 1/3$, so the radius of convergence is 3.

Similarly,

$$\frac{5}{4-x} = \sum_{i=0}^{\infty} \frac{5}{4^{i+1}} x^i$$

with radius of convergence 4. Finally, we may use partial fractions (Section 10.2) to write $(23-7x)/[(3-x)(4-x)] = 2/(3-x) + 5/(4-x)$, so we have

$$\frac{23-7x}{(3-x)(4-x)} = \sum_{i=0}^{\infty} \left(\frac{2}{3^{i+1}} + \frac{5}{4^{i+1}} \right) x^i.$$

By the preceding box, the radius of convergence of this series is at least 3. In fact, a limit computation shows that the ratio of successive coefficients approaches $\frac{1}{3}$, so the radius of convergence is exactly 3. $\blacktriangle$

In practice, we do not use the formula for $f(x)g(x)$ in the box above, but merely multiply the series for f and g term by term; in the product, we collect the terms involving each power of x .

Example 10 Write down the terms through x^4 in the series for $e^x/(1-x)$.

Solution We have $e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + \cdots$ (from Example 6) and $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$. We multiply terms in the first series by terms in the second series, in all possible ways.

	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\frac{x^4}{24}$	$\dots$
1	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\frac{x^4}{24}$	$\dots$
x	x	x^2	$\frac{x^3}{2}$	$\frac{x^4}{6}$	$\dots$	
x^2	x^2	x^3	$\frac{x^4}{2}$	$\dots$		
x^3	x^3	x^4	$\dots$			
x^4	x^4	$\dots$				

(Since we want the product series only through x^4 , we may neglect the terms in higher powers of x .) Reading along diagonals from lower left to upper right, we collect the powers of x to get

$$\begin{aligned}
 \frac{e^x}{1-x} &= 1 + (x+x) + \left(x^2 + x^2 + \frac{x^2}{2}\right) + \left(x^3 + x^3 + \frac{x^3}{2} + \frac{x^3}{6}\right) \\
 &\quad + \left(x^4 + x^4 + \frac{x^4}{2} + \frac{x^4}{6} + \frac{x^4}{24}\right) + \dots \\
 &= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \dots \blacktriangle
 \end{aligned}$$

Exercises for Section 12.4

For which x do the series in Exercises 1–10 converge?

- $\sum_{i=0}^{\infty} \frac{2}{i+1} x^i$
- $\sum_{i=0}^{\infty} (2i+1)x^i$
- $\sum_{n=1}^{\infty} \frac{3}{n^2} x^n$
- $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n(n+1)} x^n$
- $\sum_{i=1}^{\infty} \frac{5i+1}{i} (x-1)^i$
- $\sum_{r=0}^{\infty} \frac{r!}{3^{2r}} (x+2)^r$
- $\sum_{n=2}^{\infty} \frac{1}{n! \sin(\pi/n)} x^n$
- $\sum_{i=14}^{\infty} \frac{i(i+3)}{i^3-4i+7} x^i$
- $\sum_{n=1}^{\infty} \frac{x^n}{2^n+4^n}$
- $\sum_{s=1}^{\infty} \left(\frac{2^s+1}{8s^7}\right)^{3/2} x^s$

Find the radius of convergence of the series in Exercises 11–14.

- $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- $1 + \frac{x}{2} + \frac{2!}{4!} x^2 + \frac{3!}{6!} x^3 + \dots$
- $\frac{5x}{2} + \frac{10x^2}{4} + \frac{15x^3}{8} + \frac{20x^4}{16} + \dots$
- $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots$

Find the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n x^n$ in Exercises 15–18 for the given choices of a_n . Discuss convergence at $\pm R$.

- $a_n = 1/(n+1)^n$
- $a_n = (-1)^n/(n+1)$
- $a_n = (n^2+n^3)/(1+n)^5$
- $a_n = n$

Use the root test to determine the radius of convergence of the series in Exercises 19–22.

- $\sum_{n=1}^{\infty} \frac{x^n}{(3+1/n)^n}$
- $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^n}$
- $\sum_{n=1}^{\infty} (-1)^n n^n x^n$
- $\sum_{n=1}^{\infty} \frac{2x^n}{1+5^n}$

23. Let $f(x) = x - x^3/3! + x^5/5! - \dots$. Show that f is defined and is differentiable for all x . Show that $f''(x) + f(x) = 0$. Use the uniqueness of solutions of this equation (Section 8.1) to show that $f(x) = \sin x$.

24. By differentiating the result of Exercise 23, find a series representation for $\cos x$.

25. Let $f(x) = \sum_{i=1}^{\infty} (i+1)x^i$.

- Find the radius of convergence of this series.
- Find the series for $\int_0^x f(t) dt$.
- Use the result of part (b) to sum the series $f(x)$.
- Sum the series $\frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \dots$.

26. (a) Write a power series representing the integral of $1/(1-x)$ for $|x| < 1$. (b) Write a power series for $\ln x = \int (dx/x)$ in powers of $1-x$. Where is it valid?

Write power series representations for the functions in Exercises 27–30.

- e^{-x^2} . (Use Example 6.)
- $(d/dx)e^{-x^2}$

29. $\tan^{-1}x$ and its derivative. [Hint: Do the derivative first.]
 30. The second derivative of $1/(1-x)$.
 31. Find the series for $1/[(1-x)(2-x)]$ by writing

$$\frac{1}{(1-x)(2-x)} = \frac{A}{1-x} + \frac{B}{2-x}$$

and adding the resulting geometric series.

32. Find the series for $x/(x^2 - 4x + 3)$. (See Exercise 31).
 33. Using the result of Exercise 23, write the terms through x^6 in a power series expansion of $\sin^2 x$.
 34. Find the terms through x^6 in the series for $\sin^2 x/x$.
 35. Find series $f(x)$ and $g(x)$ such that the series $f(x) + g(x)$ is not identically zero but has a larger radius of convergence than either $f(x)$ or $g(x)$.
 36. Find series $f(x)$ and $g(x)$, each of them having radius of convergence 2, such that $f(x) + g(x)$ has radius of convergence 3.
 37. (a) By dividing the series for $\sin x$ by that for $\cos x$, find the terms through x^5 in the series for $\tan x$.
 (b) Find the terms through x^4 in the series for $\sec^2 x = (d/dx) \tan x$.
 (c) Using the result of part (b), find the terms through x^4 in the series for $1/\sec^2 x$.
 38. Find the terms through x^5 in the series for

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

39. Find a power series which converges just when $-1 < x \leq 1$.
 40. Why can't $x^{1/3}$ be represented in the form of a series $\sum_{i=0}^{\infty} a_i x^i$, convergent near $x = 0$?

Exercises 41–45 contain the proof of the results on the differentiation and integration of power series. For simplicity, we consider only the case $x_0 = 0$. Refer to the following theorem.

Theorem Suppose that $\sum_{i=0}^{\infty} a_i x^i$ converges for some particular value of x , say $x = x_0$. Then:

1. There is an integer N such that $\sqrt[i]{|a_i|} < 1/|x_0|$ for all $i \geq N$.
2. If $|y| < |x_0|$, then $\sum_{i=0}^{\infty} a_i y^i$ converges absolutely.

Proof For part 1, suppose that $\sqrt[i]{|a_i|} \geq 1/|x_0|$ for arbitrarily large values of i . Then for these values of i we have $|a_i| \geq 1/|x_0|^i$, and $|a_i x_0^i| \geq 1$; but then we could not have $a_i x_0^i \rightarrow 0$, as is required for convergence.

For part 2, let $r = |y|/|x_0|$, so that $|r| < 1$. By part 1, $|a_i y^i| = |a_i| |x_0|^i r^i < r^i$ for all $i \geq N$. By the comparison test, the series $\sum_{i=N}^{\infty} a_i y^i$ converges absolutely; it follows that the entire series converges absolutely as well. ■

- ★41. Prove that the series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$, and $h(x) = \sum_{i=0}^{\infty} [a_i/(i+1)] x^{i+1}$ all have the same radius of convergence. [Hint: Use the theorem above and the definition of radius of convergence on p. 587.]
 ★42. Prove that if $0 < R_1 < R$, where R is the radius of convergence of $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then given any $\epsilon > 0$, there is a positive number M such that, for every number N greater than M , the difference $|f(x) - \sum_{i=0}^N a_i x^i|$ is less than ϵ for all x in the interval $[-R_1, R_1]$. [Hint: Compare $\sum_{i=N+1}^{\infty} a_i x^i$ with a geometric series, using the theorem above.]
 ★43. Prove that if $|x_0| < R$, where R is the radius of convergence of $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then f is continuous at x_0 . [Hint: Use Exercise 42, together with the fact that the polynomial $\sum_{i=0}^N a_i x^i$ is continuous. Given $\epsilon > 0$, write $f(x) - f(x_0)$ as a sum of terms, each of which is less than $\epsilon/3$, by choosing N large enough and $|x - x_0|$ less than some δ .]
 ★44. Prove that if $|x| < R$, where R is the radius of convergence of $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then the integral $\int_0^x f(t) dt$ (which exists by Exercise 43) is equal to $\sum_{i=0}^{\infty} [a_i/(i+1)] x^{i+1}$. [Hint: Use the result of Exercise 42 to show that the difference $|\int_0^x f(t) dt - \sum_{i=0}^N [a_i x^{i+1}/(i+1)]|$ is less than any positive number ϵ .]
 ★45. Prove that if $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$ have radius of convergence R , then $f'(x) = g(x)$ on $(-R, R)$. [Hint: Apply the result of Exercise 44 to $\int_0^x g(t) dt$; then use the alternative version of the fundamental theorem of calculus.]

12.5 Taylor's Formula

The power series which represents a function is determined by the derivatives of the function at a single point.

Up until now, we have used various makeshift methods to find power series expansions for specific functions. In this section, we shall see how to do this systematically. The idea is to assume the existence of a power series and to identify the coefficients one by one.

If $f(x) = \sum_{i=0}^{\infty} a_i(x - x_0)^i$ is convergent for $x - x_0$ small enough, we can find the coefficient a_0 simply by setting $x = x_0$: $f(x_0) = \sum_{i=0}^{\infty} a_i(x_0 - x_0)^i = a_0$. Differentiating and then substituting $x = x_0$, we can find a_1 . Writing out the series explicitly will clarify the procedure:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots, \text{ so } f(x_0) = a_0;$$

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \cdots,$$

$$\text{so } f'(x_0) = a_1.$$

Similarly, by taking more and more derivatives before we substitute, we find

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x - x_0)$$

$$+ 4 \cdot 3a_4(x - x_0)^2 + \cdots$$

$$\text{so } f''(x_0) = 2a_2;$$

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x - x_0) + \cdots$$

$$\text{so } f'''(x_0) = 3 \cdot 2a_3;$$

$$f''''(x) = 4 \cdot 3 \cdot 2a_4 + \cdots$$

$$\text{so } f''''(x_0) = 4 \cdot 3 \cdot 2a_4;$$

etc.

Solving for the a_i 's, we have $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = f''(x_0)/2$, $a_3 = f'''(x_0)/2 \cdot 3$, and, in general, $a_i = f^{(i)}(x_0)/i!$. Here $f^{(i)}$ denotes the i th derivative of f , and we recall that $i! = i \cdot (i - 1) \cdots 3 \cdot 2 \cdot 1$, read " i factorial." (We use the conventions that $f^{(0)} = f$ and $0! = 1$.)

This argument shows that if a function $f(x)$ can be written as a power series in $(x - x_0)$, then this series *must* be

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i.$$

For any f , this series is called the *Taylor series* of f about the point $x = x_0$. (This formula is responsible for the factorials which appear in so many important power series.)

The point x_0 is often chosen to be zero, in which case the series becomes

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

and is called the *Maclaurin*³ series of f .

³ Brook Taylor (1685–1731) and Colin Maclaurin (1698–1746) participated in the development of calculus following Newton and Leibniz. According to the *Guinness Book of World Records*, Maclaurin has the distinction of being the youngest full professor of all time at age 19 in 1717. He was recommended by Newton. Another mathematician-physicist, Lord Kelvin, holds the record for the youngest and fastest graduation from college—between October 1834 and November 1834, at age 10.

Taylor and Maclaurin Series

If f is infinitely differentiable on some interval containing x_0 , the series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

is called the *Taylor series* of f at x_0 .

When $x_0 = 0$, the series has the simpler form

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

and is called the *Maclaurin series* of f .

Example 1 Write down the Maclaurin series for $\sin x$.

Solution We have

$$f(x) = \sin x, \quad f(0) = 0;$$

$$f'(x) = \cos x, \quad f'(0) = 1;$$

$$f''(x) = -\sin x, \quad f''(0) = 0;$$

$$f^{(3)}(x) = -\cos x, \quad f^{(3)}(0) = -1;$$

$$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0;$$

and the pattern repeats from here on. Hence the Maclaurin series is

$$\frac{f(0)}{0!} x^0 + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \blacktriangle$$

Example 2 Find the terms through cubic order in the Taylor series for $1/(1 + x^2)$ at $x_0 = 1$.

Solution *Method 1.* We differentiate $f(x)$ three times:

$$f(x) = \frac{1}{1 + x^2}, \quad f(1) = \frac{1}{2}, \quad a_0 = f(1) = \frac{1}{2};$$

$$f'(x) = \frac{-2x}{(1 + x^2)^2}, \quad f'(1) = -\frac{1}{2}, \quad a_1 = f'(1) = -\frac{1}{2};$$

$$f''(x) = \frac{6x^2 - 2}{(1 + x^2)^3}, \quad f''(1) = \frac{1}{2}, \quad a_2 = \frac{f''(1)}{2!} = \frac{1}{4};$$

$$f'''(x) = \frac{-24x^3 + 24x}{(1 + x^2)^4}, \quad f'''(1) = 0, \quad a_3 = \frac{f'''(1)}{3!} = 0;$$

so the Taylor series begins

$$\frac{1}{1 + x^2} = \frac{1}{2} - \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 + 0 \cdot (x - 1)^3 + \cdots$$

Method 2. Write

$$\begin{aligned} \frac{1}{1 + x^2} &= \frac{1}{1 + [(x - 1) + 1]^2} = \frac{1}{2 + 2(x - 1) + (x - 1)^2} \\ &= \frac{1}{2} \left[\frac{1}{1 + (x - 1) + \frac{1}{2}(x - 1)^2} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[1 - \left((x-1) + \frac{1}{2}(x-1)^2 \right) \right. \\
&\quad \left. + \left((x-1) + \frac{(x-1)^2}{2} \right)^2 - \left((x-1) + \frac{(x-1)^2}{2} \right)^3 + \cdots \right] \quad (\text{geometric series}) \\
&= \frac{1}{2} \left[1 - (x-1) - \frac{1}{2}(x-1)^2 + (x-1)^2 + (x-1)^3 - (x-1)^3 + \cdots \right] \\
&= \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 + 0 \cdot (x-1)^3 + \cdots. \quad \blacktriangle
\end{aligned}$$

Notice that we can write the Taylor series for any function which can be differentiated infinitely often, but we do not yet know whether the series converges to the given function. To understand when this convergence takes place, we proceed as follows. Using the fundamental theorem of calculus, write

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt. \quad (1)$$

We now use integration by parts with $u = f'(t)$ and $v = x - t$. The result is

$$\begin{aligned}
\int_{x_0}^x f'(t) dt &= - \int_{x_0}^x u dv = - \left(uv \Big|_{x_0}^x - \int_{x_0}^x v du \right) \\
&= f'(x_0)(x - x_0) + \int_{x_0}^x (x - t) f''(t) dt.
\end{aligned}$$

Thus we have proved the identity

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - t) f''(t) dt. \quad (2)$$

Note that the first two terms on the right-hand side of formula (2) equal the first two terms in the Taylor series of f . If we integrate by parts again with

$$u = f''(t) \quad \text{and} \quad v = \frac{(x - t)^2}{2},$$

we get

$$\begin{aligned}
\int_{x_0}^x (x - t) f''(t) dt &= - \int_{x_0}^x u dv = -uv \Big|_{x_0}^x + \int_{x_0}^x v du \\
&= \frac{f''(x_0)}{2} (x - x_0)^2 + \int_{x_0}^x \frac{(x - t)^2}{2} f'''(t) dt;
\end{aligned}$$

so, substituting into formula (2),

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} (x - x_0)^2 + \int_{x_0}^x \frac{(x - t)^2}{2} f'''(t) dt. \quad (3)$$

Repeating the procedure n times, we obtain the formula

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} (x - x_0)^2 + \cdots \\
&\quad + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \int_{x_0}^x \frac{(x - t)^n}{n!} f^{(n+1)}(t) dt
\end{aligned} \quad (4)$$

which is called *Taylor's formula with remainder in integral form*. The expression

$$R_n(x) = \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \quad (5)$$

is called the *remainder*, and formula (4) may be written in the form

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i + R_n(x). \quad (6)$$

By the second mean value theorem of integral calculus (Review Exercise 40, Chapter 9), we can write

$$R_n(x) = f^{(n+1)}(c) \left[\int_{x_0}^x \frac{(x-t)^n}{n!} dt \right] = f^{(n+1)}(c) \frac{(x-x_0)^{n+1}}{(n+1)!} \quad (7)$$

for some point c between x_0 and x . Substituting formula (7) into formula (6), we have

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i + \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}. \quad (8)$$

Formula (8), which is called *Taylor's formula with remainder in derivative form*, reduces to the usual mean value theorem when we take $n=0$; that is,

$$f(x) = f(x_0) + f'(c)(x-x_0)$$

for some c between x_0 and x .

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$, then formula (6) tells us that the Taylor series of f will converge to f .

The following box summarizes our discussion of Taylor series.

Convergence of Taylor Series

1. If $f(x) = \sum_{i=0}^{\infty} a_i (x-x_0)^i$ is a convergent power series on an open interval I centered at x_0 , then f is infinitely differentiable and $a_i = f^{(i)}(x_0)/i!$, so

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i.$$

2. If f is infinitely differentiable on an open interval I centered at x_0 , and if $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for x in I , where $R_n(x)$ is defined by formula (5), then the Taylor series of f converges on I and equals f :

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i.$$

- Example 3**
- (a) Expand the function $f(x) = 1/(1+x^2)$ in a Maclaurin series.
 - (b) Use part (a) to find $f''''(0)$ and $f''''''(0)$ without calculating derivatives of f directly.
 - (c) Integrate the series in part (a) to prove that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| < 1.$$

★(d) Justify the formula of Euler:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Solution (a) We expand $1/(1+x^2)$ as a geometric series:

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \\ &= 1 - x^2 + x^4 - x^6 + \cdots \end{aligned}$$

which is valid if $|-x^2| < 1$; that is, if $|x| < 1$. By the box above this is the Maclaurin series of $f(x) = 1/(1+x^2)$.

(b) We find that $f''''''(0)/5!$ is the coefficient of x^5 . Hence, as this coefficient is zero, $f''''''(0) = 0$. Likewise, $f''''''(0)/6!$ is the coefficient of x^6 ; thus $f''''''(0) = -6!$. This is *much* easier than calculating the sixth derivative of $f(x)$.

(c) Integrating from zero to x (justified in Section 12.4) gives

$$\int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots;$$

but we know that the integral of $1/(1+t^2)$ is $\tan^{-1}t$, so

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| < 1.$$

(d) If we set $x = 1$ and use $\tan^{-1}1 = \pi/4$, we get Euler's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots;$$

but this is not quite justified, since the series for $\tan^{-1}x$ is valid only for $|x| < 1$. (It is plausible, though, since $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, being an alternating series, converges.) To justify Euler's formula, we may use the finite form of the geometric series expansion:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + (-1)^{n+1} \frac{t^{2n+2}}{1+t^2}.$$

Integrating from 0 to 1, we have

$$\frac{\pi}{4} = \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2} dt.$$

We will be finished if we can show that the last term goes to zero as $n \rightarrow \infty$. We have

$$0 \leq \int_0^1 \frac{t^{2n+2}}{1+t^2} dt \leq \int_0^1 t^{2n+2} dt = \frac{1}{2n+3}.$$

Since $\lim_{n \rightarrow \infty} [1/(2n+3)] = 0$, the limit of

$$(-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2} dt$$

is zero as well (by the comparison test on p. 543). ▲

There is a simple test which guarantees that the remainder of a Taylor series tends to zero.

Taylor Series Test

To prove that a function $f(x)$ equals its Taylor series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i \quad \text{on } I,$$

it is sufficient to show:

1. f is infinitely differentiable on I ;
2. the derivatives of f grow no faster than a constant C times the powers of a constant M ; that is, for x in I ,

$$|f^{(n)}(x)| \leq CM^n, \quad n = 0, 1, 2, 3, \dots$$

To justify this, we must show that $R_n(x) \rightarrow 0$. By formula (7),

$$|R_n(x)| = \left| f^{(n+1)}(c) \frac{(x - x_0)^{n+1}}{(n+1)!} \right| \leq \frac{CM^{n+1}|x - x_0|^{n+1}}{(n+1)!}.$$

For any number b , however, $b^n/n! \rightarrow 0$, since $\sum_{i=0}^{\infty} (b^i/i!)$ converges by Example 7, Section 12.3. Choosing $b = M|x - x_0|$, we can conclude that $R_n(x) \rightarrow 0$, so the Taylor series converges to f .

Example 4 Prove that:

$$(a) \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{for all } x.$$

$$(b) \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for } x \text{ in } (-\infty, \infty).$$

$$(c) \quad 1 = \frac{\pi}{2} - \frac{\pi^3}{2^3 \cdot 3!} + \frac{\pi^5}{2^5 \cdot 5!} - \frac{\pi^7}{2^7 \cdot 7!} + \cdots$$

Solution (a) Let $f(x) = e^x$. Since $f^{(n)}(x) = e^x$, f is infinitely differentiable. Since all the derivatives at $x_0 = 0$ are 1, the Maclaurin series of e^x is $\sum_{n=0}^{\infty} (x^n/n!)$. To establish equality, it suffices to show $|f^{(n)}(x)| \leq CM^n$ on any finite interval I ; but $f^{(n)}(x) = e^x$, independent of n , so in fact we can choose $M = 1$ and C the maximum of e^x on I .

(b) Since $f'(x) = \cos x$, $f''(x) = -\sin x$, $\dots$, we see that f is infinitely differentiable. Notice that $f^{(n)}(x)$ is $\pm \cos x$ or $\pm \sin x$, so $|f^{(n)}(x)| \leq 1$. Thus we can choose $C = 1$, $M = 1$. Hence $\sin x$ equals its Maclaurin series, which was shown in Example 1 to be $x - x^3/3! + x^5/5! - \dots$.

(c) Let $x = \pi/2$ in part (b). $\blacktriangle$

Some discussion of the limitations of Taylor series is in order. Consider, for example, the function $f(x) = 1/(1+x^2)$, whose Maclaurin series is $1 - x^2 + x^4 - x^6 + \dots$. Even though the function f is infinitely differentiable on the whole real line, its Maclaurin series converges only for $|x| < 1$. If we wish to represent $f(x)$ for x near 1 by a series, we may use a Taylor series with $x_0 = 1$ (see Example 2).

Another instructive example is the function $g(x) = e^{-1/x^2}$, where $g(0) = 0$. This function is infinitely differentiable, but all of its derivatives at $x = 0$ are equal to zero (see Review Exercise 123). Thus the Maclaurin series of g is $\sum_{i=0}^{\infty} 0 \cdot x^i$, which converges (it is zero) for all x , but not to the function g . There also exist infinitely differentiable functions with Taylor series having radius of convergence zero.⁴ In each of these examples, the hypothesis that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ fails, so the assertion in the box above is not contradicted. It simply does not apply. (Functions which satisfy $R_n(x) \rightarrow 0$, and so equal their Taylor series for x close to x_0 , are important objects of study; these functions are called *analytic*).

The following box contains the most basic series expansions. They are worth memorizing.

Some Important Taylor and Maclaurin Series

$$\text{Geometric:} \quad \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i, \quad R = 1.$$

$$\begin{aligned} \text{Binomial:} \quad (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots \\ &= \sum_{i=0}^{\infty} \frac{\alpha(\alpha-1) \cdots (\alpha-i+1)}{i!} x^i, \quad R = 1. \end{aligned}$$

$$\text{Sine:} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}, \quad R = \infty.$$

$$\text{Cosine:} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}, \quad R = \infty.$$

$$\text{Exponential:} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \quad R = \infty.$$

$$\begin{aligned} \text{Logarithm:} \quad \ln x &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \\ &= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(x-1)^i}{i}, \quad R = 1. \end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \quad R = 1.$$

The only formula in the box which has not yet been justified is the binomial series. It may be proved by evaluating the derivatives of $f(x) = (1+x)^\alpha$ at $x = 0$ and verifying convergence by the *method* of the test in the box entitled Taylor series test. (See Review Exercise 124.) If $\alpha = n$ is a positive integer, the series terminates and we get the binomial formula

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

is the number of ways of choosing k objects from a collection of n objects.

⁴ See B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Holden-Day, San Francisco (1964), p. 68.

Example 5 Expand $\sqrt{1+x^2}$ about $x_0 = 0$.

Solution The binomial series, with $\alpha = \frac{1}{2}$ and x^2 in place of x , gives

$$\begin{aligned}(1+x^2)^{1/2} &= 1 + \frac{1}{2}x^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}x^4 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \cdots \\ &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots, \text{ valid for } |x| < 1. \blacktriangle\end{aligned}$$

Taylor's formula with remainder,

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i + R_n(x)$$

can be used to obtain approximations to $f(x)$; we can estimate the accuracy of these approximations using the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

(for some c between x and x_0) and estimating $f^{(n+1)}$ on the interval between x and x_0 . The partial sum of the Taylor series,

$$\sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

is a polynomial of degree n in x called the n th *Taylor (or Maclaurin if $x_0 = 0$) polynomial for f at x_0* , or the n th-order approximation to f at x_0 . The first Taylor polynomial,

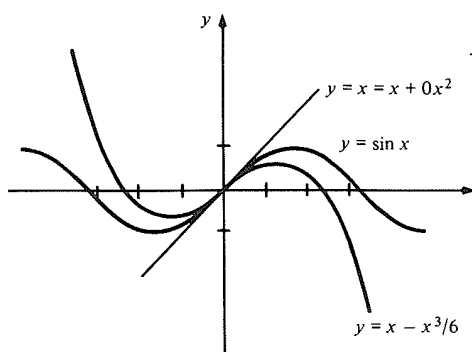
$$f(x_0) + f'(x_0)(x - x_0)$$

is just the *linear approximation* to $f(x)$ at x_0 ; the formula for the remainder $R_1(x) = [f''(c)/2](x - x_0)^2$ shows that we can estimate the error in the first-order approximation in terms of the size of the second derivative f'' on the interval between x and x_0 .

A useful consequence of Taylor's theorem is that for many functions we can improve upon the linear approximation by using Taylor polynomials of higher order.

Example 6 Sketch the graph of $\sin x$ along with the graphs of its Maclaurin polynomials of degree 1, 2, and 3. Evaluate the polynomials at $x = 0.02, 0.2$, and 2 , and compare with the exact value of $\sin x$.

Solution The Maclaurin polynomials of order 1, 2, and 3 are x , $x + 0x^2$, and $x - x^3/6$. They are sketched in Fig. 12.5.1. Evaluating at $x = 0.02, 0.2$, and 2 gives the results shown in the table below.



x	$x - x^3/6$	$\sin x$
0.02	0.0199986667	0.0199986667
0.2	0.1986666	0.1986693
2	0.666666	0.909
20	-1313	0.912

Figure 12.5.1. The first- and third-order approximations to $\sin x$.

The Maclaurin polynomials through degree 71 for $\sin x$ are shown in Fig. 12.5.2.⁵ Notice that as n increases, the interval on which the n th Taylor polynomial is a good approximation to $\sin x$ becomes larger and larger; if we go beyond this interval, however, the polynomials of higher degree “blow up” more quickly than the lower ones.

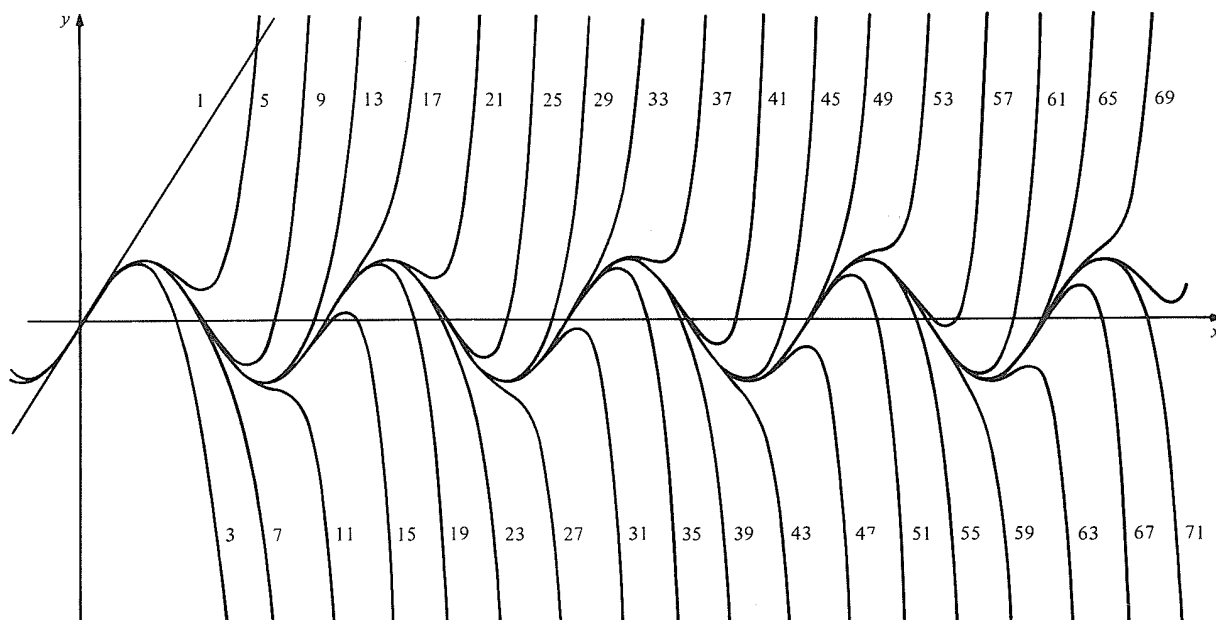


Figure 12.5.2. The Maclaurin polynomials for $\sin x$ through order 71. (The graphs to the left of the y axis are obtained by rotating the figure through 180° .)

The following example shows how errors may be estimated.

Example 7 Write down the Taylor polynomials of degrees 1 and 2 for $\sqrt[3]{x}$ at $x_0 = 27$. Use these polynomials to approximate $\sqrt[3]{28}$, and estimate the error in the second-order approximation by using the formula for $R_2(x)$.

Solution Let $f(x) = x^{1/3}$, $x_0 = 27$, $x = 28$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, and $f'''(x) = \frac{10}{27}x^{-8/3}$. Thus $f(27) = 3$, $f'(27) = \frac{1}{27}$, and $f''(27) = -\frac{2}{3^7}$, so the Taylor polynomials of degree 1 and 2 are, respectively,

$$3 + \frac{1}{27}(x - 27) \quad \text{and} \quad 3 + \frac{1}{27}(x - 27) - \frac{1}{3^7}(x - 27)^2.$$

Evaluating these at $x = 28$ gives $3.0370 \dots$ and $3.0365798 \dots$ for the first- and second-order approximations. The error in the second-order approximation is at most $1/3!$ times the largest value of $(10/27)x^{-8/3}$ on $[27, 28]$, which is

$$\frac{1}{6} \frac{10}{27} \frac{1}{3^8} = \frac{5}{3^{12}} < 0.00001. \quad (\text{Actually, } \sqrt[3]{28} = 3.0365889 \dots) \blacktriangle$$

⁵ We thank H. Ferguson for providing us with this computer-generated figure.

Example 8 By integrating a series for e^{-x^2} , calculate $\int_0^1 e^{-x^2} dx$ to within 0.001.

Solution Substituting $-x^2$ for x in the series for e^x gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

Integrating term by term gives

$$\int_0^x e^{-t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots,$$

and so

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \cdots$$

This is an alternating series, so the error is no greater than the first omitted term. To have accuracy 0.001, we should include $\frac{1}{216}$. Thus, within 0.001,

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.747.$$

This method has an advantage over the methods in Section 11.5: to increase accuracy, we need only add on another term. Rules like Simpson's, on the other hand, require us to start over. (See Review Exercise 84 for Chapter 11.) Of course, if we have numerical data, or a function with an unknown or complicated series, using Simpson's rule may be necessary. ▲

Example 9 Calculate $\sin(\pi/4 + 0.06)$ to within 0.0001 by using the Taylor series about $x_0 = \pi/4$. How many terms would have been necessary if you had used the Maclaurin series?

Solution With $f(x) = \sin x$, and $x_0 = \pi/4$, we have

$$f(x) = \sin x, \quad f(x_0) = \frac{1}{\sqrt{2}};$$

$$f'(x) = \cos x, \quad f'(x_0) = \frac{1}{\sqrt{2}};$$

$$f''(x) = -\sin x, \quad f''(x_0) = -\frac{1}{\sqrt{2}};$$

$$f'''(x) = -\cos x, \quad f'''(x_0) = -\frac{1}{\sqrt{2}};$$

$$f^{(4)}(x) = \sin x, \quad f^{(4)}(x_0) = \frac{1}{\sqrt{2}};$$

and so on. We have

$$R_n(x) = \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n+1)!}$$

for c between $\pi/4$ and $\pi/4 + 0.06$. Since $f^{(n+1)}(c)$ has absolute value less than 1, we have $|R_n(x)| \leq (0.06)^{n+1}/(n+1)!$. To make $|R_n(x)|$ less than 0.0001, it suffices to choose $n = 2$. The second-order approximation to $\sin x$ is

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}} \left(x - \frac{\pi}{4}\right)^2.$$

Evaluating at $x = \pi/4 + 0.06$ gives 0.7483.

If we had used the Maclaurin polynomial of degree n , the error estimate would have been $|R_n(x)| \leq (\pi/4 + 0.06)^{n+1}/(n+1)!$. To make $|R_n(x)|$ less than 0.0001 would have required $n = 6$. $\blacktriangle$

Finally, we show how Taylor series can be used to evaluate limits in indeterminate form. The method illustrated below is sometimes more efficient than l'Hôpital's rule when that rule must be applied several times.

Example 10 Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$ using a Maclaurin series.

Solution Since $\sin x = x - x^3/3! + x^5/5! - \dots$, $\sin x - x = -x^3/3! + x^5/5! - \dots$, and so $(\sin x - x)/x^3 = -1/6 + x^2/5! - \dots$. Since this power series converges, it is continuous at $x = 0$, and so

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}. \quad \blacktriangle$$

Example 11 Use Taylor series to evaluate

$$(a) \quad \lim_{x \rightarrow 0} \frac{\sin x - x}{\tan x - x}$$

(compare Example 4, Section 11.2) and

$$(b) \quad \lim_{x \rightarrow 1} \frac{\ln x}{e^x - e}.$$

Solution (a)
$$\begin{aligned} \frac{\sin x - x}{\tan x - x} &= \frac{(\sin x)(\cos x) - x \cos x}{\sin x - x \cos x} \\ &= \frac{(x - x^3/6 + \dots)(1 - x^2/2 + \dots) - x(1 - x^2/2 + \dots)}{(x - x^3/6 + \dots) - x(1 - x^2/2 + \dots)} \\ &= \frac{x - x^3/2 - x^3/6 + \dots - x + x^3/2 + \dots}{x - x^3/6 + \dots - x + x^3/2 + \dots} \\ &= \frac{-x^3/6 + \dots}{(1/3)x^3 + \dots} = \frac{-1/6 + \dots}{1/3 + \dots} \quad (\text{dividing by } x^3). \end{aligned}$$

Since the terms denoted “ $+\dots$ ” tend to zero as $x \rightarrow 0$, we get

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{\tan x - x} = -\frac{1/6}{1/3} = -\frac{1}{2}.$$

(b)
$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\ln x}{e^x - e} &= \lim_{x \rightarrow 1} \frac{\ln x}{e(e^{x-1} - 1)} \\ &= \frac{1}{e} \lim_{x \rightarrow 1} \frac{(x-1) - (1/2)(x-1)^2 + \dots}{1 + (x-1) + (1/2)(x-1)^2 + \dots - 1} \\ &= \frac{1}{e} \lim_{x \rightarrow 1} \frac{1 - (1/2)(x-1) + \dots}{1 + (1/2)(x-1) + \dots} \\ &= \frac{1}{e} \cdot \frac{1}{1} = \frac{1}{e}. \quad \blacktriangle \end{aligned}$$

For the last example, l'Hôpital's rule would have been a little easier to use.

Exercises for Section 12.5

Write down the Maclaurin series for the functions in Exercises 1–4.

1. $\sin 3x$
2. $\cos 4x$
3. $\cos x + e^{-2x}$
4. $\sin 2x - e^{-4x}$

Find the terms through x^3 in the Taylor series at $x_0 = 1$ for the functions in Exercises 5–8.

5. $1/(1+x^2+x^4)$
6. $1/\sqrt{2-x^2}$
7. e^x
8. $\tan(\pi x/4)$

9. (a) Expand $f(x) = 1/(1+x^2+x^4)$ in a Maclaurin series through the terms in x^6 , using a geometric series. (b) Use (a) to calculate $f^{(6)}(0)$.

10. Expand $g(x) = e^{x^2}$ in a Maclaurin series as far as necessary to calculate $g^{(8)}(0)$ and $g^{(9)}(0)$.

Establish the equalities in Exercises 11–14 for a suitable domain in x .

11. $\ln(1+x) = x - x^2/2 + x^3/3 - \dots$
12. $e^{1+x} = e + ex + \frac{ex^2}{2!} + \frac{ex^3}{3!} + \dots$
13. $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \dots$
14. $\sin x = \frac{1}{\sqrt{2}} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{(x - \pi/4)^2}{2!} - \frac{(x - \pi/4)^3}{3!} + \dots \right]$

15. (a) Write out the Maclaurin series for the function $1/\sqrt{1+x^2}$. (Use the binomial series.)

(b) What is $(d^{20}/dx^{20})(1/\sqrt{1+x^2})|_{x=0}$?

16. (a) Using the binomial series, write out the Maclaurin series for $g(x) = \sqrt{1+x} + \sqrt{1-x}$. (b) Find $g^{(20)}(0)$ and $g^{(2001)}(0)$.

17. Sketch the graphs of the Maclaurin polynomials through degree 4 for $\cos x$.

18. Sketch the graphs of the Maclaurin polynomials through degree 4 for $\tan x$.

19. Calculate $\ln(1.1)$ to within 0.001 by using a power series.

20. Calculate $e^{\ln 2 + 0.02}$ to within 0.0001 using a Taylor series about $x_0 = \ln 2$. How many terms would have been necessary if you had used the Maclaurin series?

21. Use the power series for $\ln(1+x)$ to calculate $\ln 2\frac{1}{2}$, correct to within 0.1. [Hint: $2\frac{1}{2} = \frac{3}{2} \cdot \frac{5}{3}$.]

22. Continue the work of Example 7 by finding the third-, fourth-, fifth- (and so on) order approximations to $\sqrt[3]{28}$. Stop when the round-off errors on your calculator become greater than the remainder of the series.

23. Using the Maclaurin expansion for $1/(1+x)$, approximate $\int_0^{1/2} [dx/(1+x)]$ to within 0.01.

24. Use a binomial expansion to approximate $\int_0^{1/4} \sqrt{1+x^3} dx$ to within 0.01.

25. (a) Use the second-order approximation at x_0 to derive the approximation

$$\int_{x_0-R}^{x_0+R} f(x) dx \approx 2Rf(x_0) + \frac{2f''(x_0)}{3!} R^3.$$

Find an estimate for the error.

- (b) Using the formula given in part (a), find an approximate value for $\int_{-1/2}^{1/2} (dx/\sqrt{1+x^2})$. Compare the answer with that obtained from Simpson's rule with $n = 4$.

26. (a) Can we use the binomial expansion of $\sqrt{1+x}$ to obtain a convergent series for $\sqrt{2}$. Why or why not?

- (b) Writing $2 = \frac{2}{1} \cdot \frac{8}{8}$, we have $\sqrt{2} = \frac{2}{1} \sqrt{8/9}$. Use this equation, together with the binomial expansion, to obtain an approximation to $\sqrt{2}$ correct to two decimal places.

- (c) Use the method of part (b) to obtain an approximation to $\sqrt{3}$ correct to two decimal places.

Evaluate the limits in Exercises 27–30 using Maclaurin series.

$$27. \lim_{x \rightarrow 0} \frac{\sin 2x - 2x}{x^3}$$

$$28. \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

$$29. \lim_{x \rightarrow 0} \left(\frac{1}{x \sin x} - \frac{1}{x^2} \right) \text{ (use a common denominator).}$$

$$30. \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(x - \pi)^2}$$

Expand each of the functions in Exercises 31–36 as a Maclaurin series and determine for what x it is valid.

$$31. \frac{1}{1-x}$$

$$32. \frac{1}{1+x}$$

$$33. \frac{1}{1-x} - \frac{1}{1+x}$$

$$34. \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right)$$

$$35. \frac{1}{1-x^2}$$

$$36. \frac{1}{1-x^2} - \frac{1}{1+x}$$

37. Find the Maclaurin series for $f(x) = (1+x^2)^2$ in two ways:

- (a) by multiplying out the polynomial;
- (b) by taking successive derivatives and evaluating them at $x = 0$ (without multiplying out).

38. Write down the Taylor series for $\ln x$ at $x_0 = 2$.

39. Find a power series expansion for $\int_1^x \ln t dt$. Compare this with the expansion for $x \ln x$. What is your conclusion?

40. Using the Taylor series for $\sin x$ and $\cos x$, find the terms through x^6 in the series for $(\sin x)^2 + (\cos x)^2$.

- Find Maclaurin expansions through the term in x^5 for each of the functions in Exercises 45–48.

49. Find the Taylor polynomial of degree 4 for $\ln x$ at: (a) $x_0 = 1$; (b) $x_0 = e$; (c) $x_0 = 2$.

50. (a) Find a power series expansion for a function $f(x)$ such that $f(0) = 0$ and $f'(x) - f(x) = x$. (Write $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ and solve for the a_i 's one after another.) (b) Find a formula for the function whose series you found in part (a).

Find the first four nonvanishing terms in the power series expansion for the functions in Exercises 51–54.

51. $\ln(1 + e^x)$ 52. e^{x^2+x}
53. $\sin(e^x)$ 54. $e^x \cos x$

- miles per hour per hour. Use the formula $x(t) = x(0) + x'(0)t + \int_0^t (t-s)x''(s)ds$ to estimate the auto's distance from the town 15 minutes later.

- ★57. (a) Let

$$f(x) = \begin{cases} (\sin x)/x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Find $f'(0)$, $f''(0)$, and $f'''(0)$.

- (b) Find the Maclaurin expansion for $(\sin x)/x$.

- ★58. Using Taylor's formula, prove the following inequalities:

- (a) $e^x - 1 \geq x$ for $x \geq 0$.
 (b) $6x - x^3 + x^5/20 \geq 6 \sin x \geq 6x - x^3$ for $x \geq 0$.

- (c) $x^2 - x^4/12 \leq 2 - 2 \cos x \leq x^2$ for $x \geq 0$.

- ★59. Prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

- ★60. (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1}x$. Where do they converge?

- (b) Find the terms through x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ in the series for $\sin^{-1}x$; that is, if $\sin^{-1}x = a_0 + a_1x + a_2x^2 + \cdots$, then

$$\begin{aligned} \sin^{-1}(\sin x) &= a_0 + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) \\ &\quad + a_2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)^2 \\ &\quad + \cdots \end{aligned}$$

- (c) Use the substitution method of part (b) to *obtain* the first five terms of the series for $\sin^{-1}x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 through a_5 .
- (d) Find the terms through x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. (Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g .)

Review Exercises for Chapter 12

In Exercises 1–8, test the given series for convergence. If it can be summed using a geometric series, do so.

1. $\sum_{i=1}^{\infty} \frac{1}{(12)^i}$
2. $\sum_{i=1}^{\infty} \frac{1}{100(i+1)}$
3. $\sum_{i=1}^{\infty} \frac{3^{i+1}}{5^{i-1}}$
4. $\sum_{i=1}^{\infty} \frac{8}{9^i}$
5. $1 + 2 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$
6. $100 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \dots$
7. $\sum_{i=1}^{\infty} \frac{9}{10 + 11i}$
8. $\sum_{i=1}^{\infty} \frac{6}{7 + 8i}$

In Exercises 9–24, test the given series for convergence.

9. $\sum_{n=1}^{\infty} 5^{-n}$
10. $\sum_{n=1}^{\infty} \frac{4^n}{(2n+1)!}$
11. $\sum_{k=1}^{\infty} \frac{k}{3^k}$
12. $\sum_{n=1}^{\infty} \frac{2n}{n+3}$
13. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$
14. $\sum_{n=1}^{\infty} \frac{2n}{n^2+3}$
15. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$
16. $\sum_{j=0}^{\infty} \frac{(-1)^j j}{j^2+8}$
17. $\sum_{n=1}^{\infty} \frac{2n^2}{n!}$
18. $\sum_{i=1}^{\infty} \frac{i}{i^3+8}$
19. $\sum_{n=1}^{\infty} n e^{-n^2}$
20. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 - \sin^2 99n}$
21. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$
22. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}} \right)$
23. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$
24. $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$

■ Sum the series in Exercises 25–32 to within 0.05.

25. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \dots$
26. $\frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \frac{5}{32} - \dots$
27. $\sum_{i=1}^{\infty} \frac{(-1)^i}{2^i + 3}$
28. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{6n^2 - 1}$
29. $\sum_{n=1}^{\infty} \frac{1-n}{3^n}$
30. $\sum_{n=1}^{\infty} \frac{n^2}{n!}$
31. $\sum_{n=1}^{\infty} \frac{\cos n}{n^4 + 1}$
32. $\sum_{n=1}^{\infty} \frac{\cos n}{4^n + 1}$

Tell whether each of the statements in Exercise 33–46 is true or false. Justify your answer.

33. If $a_n \rightarrow 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
34. Every geometric series $\sum_{i=1}^{\infty} r^i$ converges.
35. Convergence or divergence of any series may be determined by the ratio test.
36. $\sum_{i=1}^{\infty} 1/2^i = 1$.
37. $e^{2x} = 1 + 2x + x^2 + x^3/3 + \dots$

38. If a series converges, it must also converge absolutely.

39. The error made in approximating a convergent series by a partial sum is no greater than the first term omitted.

40. $\cos x = \sum_{k=0}^{\infty} (-1)^k x^{2k} / (2k)!$.

41. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=0}^{\infty} b_k$ are both convergent, then $\sum_{j=1}^{\infty} a_j + \sum_{k=0}^{\infty} b_k = b_0 + \sum_{i=1}^{\infty} (a_i + b_i)$.

42. $\sum_{i=1}^{\infty} (-1)^i [3/(i+2)]$ converges conditionally.

43. The convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} (a_n + a_{n+1})$.

44. The convergence of $\sum_{n=1}^{\infty} (a_n + a_{n+1})$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.

45. The convergence of $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$ implies the convergence of $\sum_{n=1}^{\infty} |a_n|$.

46. The convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n^2$.

47. If $0 \leq a_n \leq ar^n$, $r < 1$, show that the error in approximating $\sum_{i=1}^{\infty} a_i$ by $\sum_{i=1}^n a_i$ is less than or equal to $ar^{n+1}/(1-r)$.

■ 48. Determine how many terms are needed to compute the sum of $1 + r + r^2 + \dots$ with error less than 0.01 when (a) $r = 0.5$ and (b) $r = 0.09$.

Find the sums of the series in Exercises 49–52.

49. $\sum_{n=1}^{\infty} 1/9^n$
50. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ [Hint: Use partial fractions.]
51. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ [Hint: Write the numerator as $n+1-1$.]
52. $\sum_{n=1}^{\infty} \frac{1+n}{2^n}$ [Hint: Differentiate a certain power series.]

Find the radius of convergence of the series in Exercises 53–58.

53. $1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{4!} + \dots$
54. $1 + 3x + 5x^2 + 7x^3 + \dots$
55. $\sum_{n=0}^{\infty} \frac{x^n}{(3n)!}$
56. $\sum_{n=0}^{\infty} \frac{(x-1/2)^n}{(n+1)!}$
57. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n$
58. $\sum_{n=0}^{\infty} \frac{n^n}{n!} x^n$

Find the Maclaurin series for the functions in Exercises 59–66.

59. $f(x) = \cos 3x + e^{2x}$
60. $g(x) = \frac{1}{1-x^3}$
61. $f(x) = \ln(1+x^4)$
62. $g(x) = \frac{1}{\sqrt{1-x^4}}$
63. $f(x) = \frac{d}{dx} (\sin x - x)$
64. $g(k) = \frac{d^2}{dk^2} (\cos k^2)$
65. $f(x) = \int_0^x \frac{(e^t - 1)}{t} dt$
66. $g(y) = \int_0^y \sin t^2 dt$

Find the Taylor expansion of each function in Exercises 67–70 about the indicated point, and find the radius of convergence.

67. e^x about $x = 2$ 68. $1/x$ about $x = 1$
 69. $x^{3/2}$ about $x = 1$ 70. $\cos(\pi x)$ about $x = 1$

Find the limits in Exercises 71–74 using series methods.

71. $\lim_{x \rightarrow 0} \frac{1 - \cos \pi x}{x^2}$.
 72. $\lim_{x \rightarrow 0} \left[6 \frac{\sin \pi x}{x^5} - \frac{6\pi - \pi^3 x^2}{x^4} \right]$.
 73. $\lim_{x \rightarrow 0} \frac{(1+x)^{3/2} - (1-x)^{3/2}}{x^2}$.
 74. $\lim_{x \rightarrow \pi} \left[1 + \cos x - \frac{(x - \pi)^2}{2} + \frac{(x - \pi)^4}{24} \right]$.

In Exercises 75–78 find the real part, the imaginary part, the complex conjugate, and the absolute value of the given complex number.

75. $3 + 7i$ 76. $2 - 10i$
 77. $\sqrt{2} - i$ 78. $(2 + i)/(2 - i)$

In Exercises 79–82, plot the given complex numbers, indicating r and θ on your diagram, and write them in polar form $z = re^{i\theta}$.

79. $1 - i$ 80. $\frac{1+i}{1-i}$
 81. $ie^{\pi i/2}$ 82. $(1+i)e^{i\pi/4}$

83. Solve for z : $z^2 - 2z + \pi i = 0$.

84. Solve for z : $z^8 = \sqrt{5} + 3i$.

Find the general solution of the differential equations in Exercises 85–96.

85. $y'' + 4y = 0$ 86. $y'' - 4y = 0$
 87. $y'' + 6y' + 5y = 0$ 88. $y'' - 6y' - 2y = 0$
 89. $y'' + 3y' - 10y = e^x + \cos x$
 90. $y'' - 2y' - 3y = x^2 + \sin x$
 91. $y'' - 6y' + 9y = \cos\left(\frac{x}{2}\right)$
 92. $y'' - 10y + 25 = \cos(2x)$
 93. $y'' + 4y = \frac{x}{\sqrt{x^2 + 1}}$. (Express your answer in terms of integrals.)
 94. $y'' - 3y' - 3y = \frac{\sin x}{\sqrt{\cos^2 x + 1}}$. (Express your answer in terms of integrals.)

95. $y''' + 2y'' + 2y' = 0$

96. $y''' - 3y'' + 3y' - y = e^x$

In Exercises 97–100, identify the equation as a spring equation and describe the limiting behavior as $t \rightarrow \infty$.

97. $x'' + 9x + x' = \cos 2t$.
 98. $x'' + 9x + 0.001x' = \sin(50t)$.
 99. $x'' + 25x + 6x' = \cos(\pi t)$.
 100. $x'' + 25x + 0.001x' = \cos(60\pi t)$.

In Exercises 101–104, find the first few terms of the general solution as a power series in x .

101. $y'' + 2xy = 0$
 102. $y'' - (4 \sin x)y = 0$
 103. $y'' - 2x^2y' + 2y = 0$
 104. $y'' + y' + xy = 0$

Find the first few terms in an appropriate series for at least one solution of the equations in Exercises 105–108.

105. $5x^2y'' + y' + y = 0$.
 106. $xy'' + y' - 4y = 0$.
 107. Bessel's equation with $k = 1$.
 108. Legendre's equation with $\lambda = 3$.

109. The current I in the electric circuit shown in Figure 12.R.1 satisfies

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = \frac{dE}{dt},$$

where E is the applied voltage and L, R, C are constants.

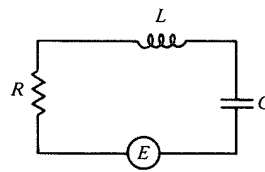


Figure 12.R.1. An electric circuit.

- (a) Find the values of m, k, γ that make this equation a damped spring equation.
 (b) Find $I(t)$ if $I(0) = 0$, $I'(0) = 0$ and $L = 5$, $C = 0.1$, $R = 100$, and $E = 2 \cos(60\pi t)$.

110. Verify formula (16) in Section 12.7.

111. Verify that $\sum_{n=0}^{\infty} x^{2n}(1+x^2)^{-n}$ is a convergent geometric series for $x \neq 0$ with sum $1+x^2$. It also converges to 0 when $x = 0$. (This shows that the sum of an infinite series of continuous terms need not be continuous.)

112. A beam of length L feet supported at its ends carries a concentrated load of P lbs at its center. The maximum deflection D of the beam from equilibrium is

$$D = \frac{2L^3P}{EI\pi^4} \sum_{n=1}^{\infty} \frac{|\sin(n\pi/2)|}{n^4}.$$

- (a) Use the formula $\sum_{n=1}^{\infty} (1/n^4) = \pi^4/90$ to show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \left(\frac{1}{2^4}\right) \left(\frac{\pi^4}{90}\right).$$

[Hint: Factor out 2^{-4} .]

- (b) Show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \left(\frac{15}{16}\right) \left(\frac{\pi^4}{90}\right);$$

hence $D = (1/48)(L^3P/EI)$. [Hint: A series is the sum of its even and odd terms.]

- (c) Use the first two nonzero terms in the series for D to obtain a simpler formula for D . Show that this result differs at most by 0.23% from the theoretical value.

113. The deflection $y(x, t)$ of a string from its straight profile at time t , measured vertically at location x along the string, $0 \leq x \leq L$, is

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).$$

where A_n , L and c are constants.

- (a) Explain what this equation means in terms of limits of partial sums for x, t fixed.
 (b) Initially (at $t = 0$), the deflection of the string is

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

Find the deflection value as an infinite series at the midpoint $x = L/2$.

114. In the study of saturation of a two-phase motor servo, an engineer starts with a transfer function equation $V(s)/E(s) = K/(1 + s\tau)$, then goes to the first-order approximation $V(s)/E(s) = K(1 - s\tau)$, from which he obtains an approximate equation for the saturation dividing line.

- (a) Show that $1/(1 + s\tau) = \sum_{n=0}^{\infty} (-s\tau)^n$, by appeal to the theory of geometric series. Which values of $s\tau$ are allowed?
 (b) Discuss the replacement of $1/(1 + s\tau)$ by $1 - s\tau$; include an error estimate in terms of the value of $s\tau$.

115. Find the area bounded by the curves $xy = \sin x$, $x = 1$, $x = 2$, $y = 0$. Make use of the Taylor expansion of $\sin x$.

116. A wire of length L inches and weight w lbs/inch, clamped at its lower end at a small angle $\tan^{-1}P_0$ to the vertical, deflects $y(x)$ inches due to bending. The displacement $y(L)$ at the upper end is given by

$$y(L) = \frac{2P_0}{3L^{1/3}} \frac{\int_0^{L^{3/2}} u(az) dz}{u(aL^{3/2})},$$

where $a = \frac{2}{3}\sqrt{W/EI}$, and

$$u(az) = \frac{az^{-1/3}}{2} \sum_{k=0}^{\infty} (-1)^k \frac{a^{2k} z^{2k}}{2^{2k} k! \Gamma(k + 2/3)}.$$

The values of the *gamma function* Γ may be found in a mathematical table or on some calculators as $\Gamma(x) = (x-1)!$ [$\Gamma(\frac{2}{3}) = 1.3541$, $\Gamma(\frac{4}{3}) = 0.9027$, $\Gamma(\frac{5}{3}) = 1.5046$, $\Gamma(\frac{4}{3}) = 4.0122$]. The function u is the *Bessel function* of order $-\frac{1}{3}$.

- (a) Find the smallest positive root of $u(az) = 0$ by using the first four terms of the series.
 (b) Evaluate $y(L)$ approximately by using the first four terms of the series.
 117. (a) Use a power series for $\sqrt{1+x}$ to calculate $\sqrt{5}/4$ correct to 0.01. (b) Use the result of part (a) to calculate $\sqrt{5}$. How accurate is your answer?

118. In each of the following, evaluate the indicated derivative:

- (a) $f^{(12)}(0)$, where $f(x) = x/(1+x^2)$;
 (b) $f^{(10)}$, where $f(x) = x^6 e^{x+1}$.

119. Let

$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \cdots.$$

Determine the numbers B_1 , B_2 , and B_3 . (The B_i are known as the *Bernoulli numbers*.)

120. Show in the following two ways that $\sum_{n=1}^{\infty} na^n = a/(1-a)^2$ for $|a| < 1$.

- (a) Consider

$$S_n = a + 2a^2 + 3a^3 + \cdots + na^n,$$

$$aS_n = a^2 + 2a^3 + \cdots + (n-1)a^n + na^{n+1}$$

and subtract.

- (b) Differentiate $\sum_{n=0}^{\infty} a^n = 1/(1-a)$ with respect to a , and then subtract your answer from $\sum_{n=0}^{\infty} na^n = 1/(1-a)$.

121. In highway engineering, a *transitional spiral* is defined to be a curve whose curvature varies directly as the arc length. Assume this curve starts at $(0,0)$ as the continuation of a road coincident with the negative x axis. Then the parametric equations of the spiral are

$$x = k \int_0^{\phi} \frac{\cos \theta}{\sqrt{\theta}} d\theta, \quad y = k \int_0^{\phi} \frac{\sin \theta}{\sqrt{\theta}} d\theta.$$

- (a) By means of infinite series methods, find the ratio x/y for $\phi = \pi/4$.
 (b) Try to graph the transitional spiral for $k = 1$, using accurate graphs of $(\cos \theta)/\sqrt{\theta}$, $(\sin \theta)/\sqrt{\theta}$ and the area interpretation of the integral.

- ★122. The free vibrations of an elastic circular membrane can be described by infinite series, the terms of which involve trigonometric functions and *Bessel functions*. The series

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i! (n+i)!} (x/2)^{n+2i}$$

is called the *Bessel Function* $J_n(x)$; n is an integer ≥ 0 .

- (a) Establish convergence by the ratio test.
 (b) The frequencies of oscillation of the circular membrane are essentially solutions of the equation $J_n(x) = 0$, $x > 0$. Examine the equation $J_0(x) = 0$, and see if you can explain why $J_0(2.404) = 0$ is possible.
 (c) Check that J_n satisfies Bessel's equation (Example 6, Section 12.8).
 ★123. Show that g defined by $g(x) = e^{-1/x^2}$ if $x \neq 0$ and $g(0) = 0$ is infinitely differentiable and $g^{(i)}(0) = 0$ for all i . [Hint: Use the definition of the derivative and the following lemma provable by l'Hôpital's rule: if $P(x)$ is any polynomial, then $\lim_{x \rightarrow 0} P(x)g(x) = 0$.]

- ★124. Let $f(x) = (1+x)^\alpha$, where α is a real number. Show by an induction argument that $f^{(i)}(x) = \alpha(\alpha-1)\cdots(\alpha-i+1)(1+x)^{\alpha-i}$, and hence show that $(1+x)^\alpha$ is analytic for $|x| < 1$.
- ★125. True or false: The convergence of $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ implies absolute convergence of $\sum_{n=1}^{\infty} a_n b_n$.
- ★126. (a) Show that if the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is R , then the radius of convergence of $\sum_{n=1}^{\infty} a_n x^{2n}$ is $\sqrt{R}$.
 (b) Find the radius of convergence of the series $\sum_{n=0}^{\infty} (\pi/4)^n x^{2n}$.
- ★127. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = f(x)/(1-x)$.
 (a) By multiplying the power series for $f(x)$ and $1/(1-x)$, show that $g(x) = \sum_{i=0}^{\infty} b_i x^i$, where $b_i = a_0 + \cdots + a_i$ is the i th partial sum of the series $\sum_{i=0}^{\infty} a_i$.
 (b) Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{i \rightarrow \infty} b_i$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?
- (c) Let $e^x/(1-x) = \sum_{i=0}^{\infty} b_i x^i$. What is $\lim_{i \rightarrow \infty} b_i$?
- ★128. (a) Find the second-order approximation at $T = 0$ to the day-length function S (see the supplement to Chapter 5) for latitude 38° and your own latitude.
 (b) How many minutes earlier (compared with $T = 0$) does the sun set when $T = 1, 2, 10, 30$?
 (c) Compare the results in part (b) with those obtained from the exact formula and with listings in your local newspaper.
 (d) For how many days before and after June 21 is the second-order approximation correct to within 1 minute? Within 5 minutes?
- ★129. Prove that e is irrational, as follows: if $e = a/b$ for some integers a and b , let $k > b$ and let $\alpha = k!(e - 2 - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{k!})$. Show that α is an integer and that $\alpha < 1/k$ to derive a contradiction.

MODULE 3

Text Book 3- Advanced Engineering Mathematics
(6/e) : Dennis G Zill Jones & Bartlett Learning,
LLC(2018)ISBN: 978-1-284-10590-2

Sections 7.6 - 7.7
8.2 – 8.5

43. $(\frac{1}{2}, \frac{3}{4}, -\frac{1}{2})$; $6\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$
 44. $(-1, 1, 0)$; $-\mathbf{i} + \mathbf{j} - \mathbf{k}$

In Problems 45–50, find, if possible, an equation of a plane that contains the given points.

45. $(3, 5, 2)$, $(2, 3, 1)$, $(-1, -1, 4)$
 46. $(0, 1, 0)$, $(0, 1, 1)$, $(1, 3, -1)$
 47. $(0, 0, 0)$, $(1, 1, 1)$, $(3, 2, -1)$
 48. $(0, 0, 3)$, $(0, -1, 0)$, $(0, 0, 6)$
 49. $(1, 2, -1)$, $(4, 3, 1)$, $(7, 4, 3)$
 50. $(2, 1, 2)$, $(4, 1, 0)$, $(5, 0, -5)$

In Problems 51–60, find an equation of the plane that satisfies the given conditions.

51. Contains $(2, 3, -5)$ and is parallel to $x + y - 4z = 1$
 52. Contains the origin and is parallel to $5x - y + z = 6$
 53. Contains $(3, 6, 12)$ and is parallel to the xy -plane
 54. Contains $(-7, -5, 18)$ and is perpendicular to the y -axis
 55. Contains the lines $x = 1 + 3t$, $y = 1 - t$, $z = 2 + t$; $x = 4 + 4s$, $y = 2s$, $z = 3 + s$
 56. Contains the lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6}$;
 $\mathbf{r} = \langle 1, -1, 5 \rangle + t\langle 1, 1, -3 \rangle$
 57. Contains the parallel lines $x = 1 + t$, $y = 1 + 2t$, $z = 3 + t$; $x = 3 + s$, $y = 2s$, $z = -2 + s$
 58. Contains the point $(4, 0, -6)$ and the line $x = 3t$, $y = 2t$, $z = -2t$
 59. Contains $(2, 4, 8)$ and is perpendicular to the line $x = 10 - 3t$, $y = 5 + t$, $z = 6 - \frac{1}{2}t$
 60. Contains $(1, 1, 1)$ and is perpendicular to the line through $(2, 6, -3)$ and $(1, 0, -2)$
 61. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. $\mathcal{P}_1$ and $\mathcal{P}_2$ are orthogonal if $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal and parallel if $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. Determine which of the following planes are orthogonal and which are parallel.
 (a) $2x - y + 3z = 1$ (b) $x + 2y + 2z = 9$
 (c) $x + y - \frac{3}{2}z = 2$ (d) $-5x + 2y + 4z = 0$
 (e) $-8x - 8y + 12z = 1$ (f) $-2x + y - 3z = 5$
 62. Find parametric equations for the line that contains $(-4, 1, 7)$ and is perpendicular to the plane $-7x + 2y + 3z = 1$.

63. Determine which of the following planes are perpendicular to the line $x = 4 - 6t$, $y = 1 + 9t$, $z = 2 + 3t$.
 (a) $4x + y + 2z = 1$ (b) $2x - 3y + z = 4$
 (c) $10x - 15y - 5z = 2$ (d) $-4x + 6y + 2z = 9$
 64. Determine which of the following planes are parallel to the line $(1 - x)/2 = (y + 2)/4 = z - 5$.
 (a) $x - y + 3z = 1$ (b) $6x - 3y = 1$
 (c) $x - 2y + 5z = 0$ (d) $-2x + y - 2z = 7$

In Problems 65–68, find parametric equations for the line of intersection of the given planes.

65. $5x - 4y - 9z = 8$ 66. $x + 2y - z = 2$
 $x + 4y + 3z = 4$ $3x - y + 2z = 1$
 67. $4x - 2y - z = 1$ 68. $2x - 5y + z = 0$
 $x + y + 2z = 1$ $y = 0$

In Problems 69–72, find the point of intersection of the given plane and line.

69. $2x - 3y + 2z = -7$; $x = 1 + 2t$, $y = 2 - t$, $z = -3t$
 70. $x + y + 4z = 12$; $x = 3 - 2t$, $y = 1 + 6t$, $z = 2 - \frac{1}{2}t$
 71. $x + y - z = 8$; $x = 1$, $y = 2$, $z = 1 + t$
 72. $x - 3y + 2z = 0$; $x = 4 + t$, $y = 2 + t$, $z = 1 + 5t$

In Problems 73 and 74, find parametric equations for the line through the indicated point that is parallel to the given planes.

73. $x + y - 4z = 2$
 $2x - y + z = 10$; $(5, 6, -12)$
 74. $2x + z = 0$
 $-x + 3y + z = 1$; $(-3, 5, -1)$

In Problems 75 and 76, find an equation of the plane that contains the given line and is orthogonal to the indicated plane.

75. $x = 4 + 3t$, $y = -t$, $z = 1 + 5t$; $x + y + z = 7$
 76. $\frac{2-x}{3} = \frac{y+2}{5} = \frac{z-8}{2}$; $2x - 4y - z + 16 = 0$

In Problems 77–82, graph the given equation.

77. $5x + 2y + z = 10$ 78. $3x + 2z = 9$
 79. $-y - 3z + 6 = 0$ 80. $3x + 4y - 2z - 12 = 0$
 81. $-x + 2y + z = 4$ 82. $x - y - 1 = 0$

7.6 Vector Spaces

INTRODUCTION In the preceding sections we were dealing with points and vectors in 2- and 3-space. Mathematicians in the nineteenth century, notably the English mathematicians **Arthur Cayley** (1821–1895) and **James Joseph Sylvester** (1814–1897) and the Irish mathematician **William Rowan Hamilton** (1805–1865), realized that the concepts of *point* and *vector* could be generalized. A realization developed that vectors could be described, or defined, by analytic rather than geometric properties. This was a truly significant breakthrough in the history of mathematics. There is no need to stop with three dimensions; ordered quadruples $\langle a_1, a_2, a_3, a_4 \rangle$, quintuples $\langle a_1, a_2, a_3, a_4, a_5 \rangle$, and n -tuples $\langle a_1, a_2, \dots, a_n \rangle$ of real numbers can be thought of as vectors just as well as ordered pairs $\langle a_1, a_2 \rangle$ and ordered triples $\langle a_1, a_2, a_3 \rangle$; the only difference being that we lose our ability to visualize directed line segments or arrows in 4-dimensional, 5-dimensional, or n -dimensional space.

n -Space In formal terms, a **vector in n -space** is any ordered n -tuple $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$ of real numbers called the components of $\mathbf{a}$. The set of all vectors in n -space is denoted by R^n . The concepts of vector addition, scalar multiplication, equality, and so on, listed in Definition 7.2.1 carry over to R^n in a natural way. For example, if $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \dots, b_n \rangle$, then addition and scalar multiplication in n -space are defined by

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, \dots, a_n + b_n \rangle \quad \text{and} \quad k\mathbf{a} = \langle ka_1, ka_2, \dots, ka_n \rangle. \quad (1)$$

The **zero vector** in R^n is $\mathbf{0} = \langle 0, 0, \dots, 0 \rangle$. The notion of **length** or **magnitude** of a vector $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$ in n -space is just an extension of that concept in 2- and 3-space:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}.$$

The length of a vector is also called its **norm**. A **unit vector** is one whose norm is 1. For a nonzero vector $\mathbf{a}$, the process of constructing a unit vector $\mathbf{u}$ by multiplying $\mathbf{a}$ by the reciprocal of its norm, that is, $\mathbf{u} = \frac{1}{\|\mathbf{a}\|} \mathbf{a}$, is referred to as **normalizing $\mathbf{a}$** . For example, if $\mathbf{a} = \langle 3, 1, 2, -1 \rangle$,

then $\|\mathbf{a}\| = \sqrt{3^2 + 1^2 + 2^2 + (-1)^2} = \sqrt{15}$ and a unit vector is

$$\mathbf{u} = \frac{1}{\sqrt{15}} \mathbf{a} = \left\langle \frac{3}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}} \right\rangle.$$

The **standard inner product**, also known as the **Euclidean inner product** or **dot product**, of two n -vectors $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \dots, b_n \rangle$ is the real number defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, \dots, a_n \rangle \cdot \langle b_1, b_2, \dots, b_n \rangle = a_1b_1 + a_2b_2 + \dots + a_nb_n. \quad (2)$$

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n are said to be **orthogonal** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. For example, $\mathbf{a} = \langle 3, 4, 1, -6 \rangle$ and $\mathbf{b} = \langle 1, \frac{1}{2}, 1, 1 \rangle$ are orthogonal in R^4 since $\mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + 4 \cdot \frac{1}{2} + 1 \cdot 1 + (-6) \cdot 1 = 0$.

Vector Space We can even go beyond the notion of a vector as an ordered n -tuple in R^n . A vector can be defined as anything we want it to be: an ordered n -tuple, a number, an array of numbers, or even a function. But we are particularly interested in vectors that are elements in a special kind of set called a **vector space**. Fundamental to the notion of vector space are two kinds of objects, **vectors** and **scalars**, and two algebraic operations analogous to those given in (1). For a set of vectors we want to be able to add two vectors in this set and get another vector in the same set, and we want to multiply a vector by a scalar and obtain a vector in the same set. To determine whether a set of objects is a vector space depends on whether the set possesses these two algebraic operations along with certain other properties. These properties, the *axioms* of a vector space, are given next.

Definition 7.6.1 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following 10 properties are satisfied.

Axioms for Vector Addition:

- (i) If $\mathbf{x}$ and $\mathbf{y}$ are in V , then $\mathbf{x} + \mathbf{y}$ is in V .
- (ii) For all $\mathbf{x}, \mathbf{y}$ in V , $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. ← commutative law
- (iii) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V , $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. ← associative law
- (iv) There is a unique vector $\mathbf{0}$ in V such that
 $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$. ← zero vector
- (v) For each $\mathbf{x}$ in V , there exists a vector $-\mathbf{x}$ such that
 $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$. ← negative of a vector

Axioms for Scalar Multiplication:

- (vi) If k is any scalar and $\mathbf{x}$ is in V , then $k\mathbf{x}$ is in V .
- (vii) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ ← distributive law
- (viii) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ ← distributive law
- (ix) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
- (x) $1\mathbf{x} = \mathbf{x}$

In this brief introduction to abstract vectors we shall take the scalars in Definition 7.6.1 to be real numbers. In this case V is referred to as a **real vector space**, although we shall not belabor this term. When the scalars are allowed to be complex numbers we obtain a **complex vector space**. Since properties (i)–(viii) on page 352 are the prototypes for the axioms in Definition 7.6.1, it is clear that R^2 is a vector space. Moreover, since vectors in R^3 and R^n have these same properties, we conclude that R^3 and R^n are also vector spaces. Axioms (i) and (vi) are called the **closure axioms** and we say that a vector space V is closed under vector addition and scalar multiplication. Note, too, that concepts such as *length* and *inner product* are not part of the axiomatic structure of a vector space.

EXAMPLE 1 Checking the Closure Axioms

Determine whether the sets (a) $V = \{1\}$ and (b) $V = \{0\}$ under ordinary addition and multiplication by real numbers are vector spaces.

SOLUTION (a) For this system consisting of one element, many of the axioms given in Definition 7.6.1 are violated. In particular, axioms (i) and (vi) of closure are not satisfied. Neither the sum $1 + 1 = 2$ nor the scalar multiple $k \cdot 1 = k$, for $k \neq 1$, is in V . Hence V is not a vector space.

(b) In this case the closure axioms are satisfied since $0 + 0 = 0$ and $k \cdot 0 = 0$ for any real number k . The commutative and associative axioms are satisfied since $0 + 0 = 0 + 0$ and $0 + (0 + 0) = (0 + 0) + 0$. In this manner it is easy to verify that the remaining axioms are also satisfied. Hence V is a vector space. $\equiv$

The vector space $V = \{0\}$ is often called the **trivial** or **zero vector space**.

If this is your first experience with the notion of an abstract vector, then you are cautioned to not take the names *vector addition* and *scalar multiplication* too literally. These operations are *defined* and as such you must accept them at face value even though these operations may not bear any resemblance to the usual understanding of ordinary addition and multiplication in, say, R , R^2 , R^3 , or R^n . For example, the addition of two vectors $\mathbf{x}$ and $\mathbf{y}$ could be $\mathbf{x} - \mathbf{y}$. With this forewarning, consider the next example.

EXAMPLE 2 An Example of a Vector Space

Consider the set V of positive real numbers. If x and y denote positive real numbers, then we write vectors in V as $\mathbf{x} = x$ and $\mathbf{y} = y$. Now, *addition* of vectors is defined by

$$\mathbf{x} + \mathbf{y} = xy$$

and *scalar multiplication* is defined by

$$k\mathbf{x} = x^k.$$

Determine whether V is a vector space.

SOLUTION We shall go through all ten axioms in Definition 7.6.1.

- (i) For $\mathbf{x} = x > 0$ and $\mathbf{y} = y > 0$, $\mathbf{x} + \mathbf{y} = xy > 0$. Thus, the sum $\mathbf{x} + \mathbf{y}$ is in V ; V is closed under addition.
- (ii) Since multiplication of positive real numbers is commutative, we have for all $\mathbf{x} = x$ and $\mathbf{y} = y$ in V , $\mathbf{x} + \mathbf{y} = xy = yx = \mathbf{y} + \mathbf{x}$. Thus, addition is commutative.
- (iii) For all $\mathbf{x} = x$, $\mathbf{y} = y$, $\mathbf{z} = z$ in V ,

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = x(yz) = (xy)z = (\mathbf{x} + \mathbf{y}) + \mathbf{z}.$$

Thus, addition is associative.

- (iv) Since $\mathbf{1} + \mathbf{x} = 1x = x = \mathbf{x}$ and $\mathbf{x} + \mathbf{1} = x1 = x = \mathbf{x}$, the zero vector $\mathbf{0}$ is $\mathbf{1} = 1$.

- (v) If we define $-\mathbf{x} = \frac{1}{x}$, then

$$\mathbf{x} + (-\mathbf{x}) = x \frac{1}{x} = 1 = \mathbf{1} = \mathbf{0} \text{ and } (-\mathbf{x}) + \mathbf{x} = \frac{1}{x} x = 1 = \mathbf{1} = \mathbf{0}.$$

Therefore, the negative of a vector is its reciprocal.

(vi) If k is any scalar and $\mathbf{x} = x > 0$ is any vector, then $k\mathbf{x} = x^k > 0$. Hence V is closed under scalar multiplication.

(vii) If k is any scalar, then

$$k(\mathbf{x} + \mathbf{y}) = (xy)^k = x^k y^k = k\mathbf{x} + k\mathbf{y}.$$

(viii) For scalars k_1 and k_2 ,

$$(k_1 + k_2)\mathbf{x} = x^{(k_1 + k_2)} = x^{k_1} x^{k_2} = k_1\mathbf{x} + k_2\mathbf{x}.$$

(ix) For scalars k_1 and k_2 ,

$$k_1(k_2\mathbf{x}) = (x^{k_2})^{k_1} = x^{k_1 k_2} = (k_1 k_2)\mathbf{x}.$$

(x) $1\mathbf{x} = x^1 = x = \mathbf{x}$.

Since all the axioms of Definition 7.6.1 are satisfied, we conclude that V is a vector space. $\equiv$

Here are some important vector spaces—we have mentioned some of these previously. The operations of vector addition and scalar multiplication are the usual operations associated with the set.

- The set R of real numbers
- The set R^2 of ordered pairs
- The set R^3 of ordered triples
- The set R^n of ordered n -tuples
- The set P_n of polynomials of degree less than or equal to n
- The set P of all polynomials
- The set of real-valued functions f defined on the entire real line
- The set $C[a, b]$ of real-valued functions f continuous on the closed interval $[a, b]$
- The set $C(-\infty, \infty)$ of real-valued functions f continuous on the entire real line
- The set $C^n[a, b]$ of all real-valued functions f for which $f, f', f'', \dots, f^{(n)}$ exist and are continuous on the closed interval $[a, b]$

Subspace It may happen that a subset of vectors W of a vector space V is itself a vector space.

Definition 7.6.2 Subspace

If a subset W of a vector space V is itself a vector space under the operations of vector addition and scalar multiplication defined on V , then W is called a **subspace** of V .

Every vector space V has at least two subspaces: V itself and the zero subspace $\{\mathbf{0}\}$; $\{\mathbf{0}\}$ is a subspace since the zero vector must be an element in every vector space.

To show that a subset W of a vector space V is a subspace, it is not necessary to demonstrate that all ten axioms of Definition 7.6.1 are satisfied. Since all the vectors in W are also in V , these vectors must satisfy axioms such as (ii) and (iii). In other words, W inherits most of the properties of a vector space from V . As the next theorem indicates, we need only check the two closure axioms to demonstrate that a subset W is a subspace of V .

Theorem 7.6.1 Criteria for a Subspace

A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under vector addition and scalar multiplication defined on V :

- (i) If $\mathbf{x}$ and $\mathbf{y}$ are in W , then $\mathbf{x} + \mathbf{y}$ is in W .
- (ii) If $\mathbf{x}$ is in W and k is any scalar, then $k\mathbf{x}$ is in W .

EXAMPLE 3 A Subspace

Suppose $\mathbf{f}$ and $\mathbf{g}$ are continuous real-valued functions defined on the entire real line. Then we know from calculus that $\mathbf{f} + \mathbf{g}$ and $k\mathbf{f}$, for any real number k , are continuous and real-valued functions. From this we can conclude that $C(-\infty, \infty)$ is a subspace of the vector space of real-valued functions defined on the entire real line. $\equiv$

EXAMPLE 4 A Subspace

The set P_n of polynomials of degree less than or equal to n is a subspace of $C(-\infty, \infty)$, the set of real-valued functions continuous on the entire real line. $\equiv$

It is always a good idea to have concrete visualizations of vector spaces and subspaces. The subspaces of the vector space R^3 of three-dimensional vectors can be easily visualized by thinking of a vector as a point (a_1, a_2, a_3) . Of course, $\{0\}$ and R^3 itself are subspaces; other subspaces are all lines passing through the origin, and all planes passing through the origin. The lines and planes must pass through the origin since the zero vector $\mathbf{0} = (0, 0, 0)$ must be an element in each subspace.

Similar to Definition 3.1.1 we can define linearly independent vectors.

Definition 7.6.3 Linear Independence

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is said to be **linearly independent** if the only constants satisfying the equation

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0} \quad (3)$$

are $k_1 = k_2 = \cdots = k_n = 0$. If the set of vectors is not linearly independent, then it is said to be **linearly dependent**.

In R^3 , the vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are linearly independent since the equation $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$ is the same as

$$k_1\langle 1, 0, 0 \rangle + k_2\langle 0, 1, 0 \rangle + k_3\langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle \quad \text{or} \quad \langle k_1, k_2, k_3 \rangle = \langle 0, 0, 0 \rangle.$$

By equality of vectors, (iii) of Definition 7.2.1, we conclude that $k_1 = 0$, $k_2 = 0$, and $k_3 = 0$. In Definition 7.6.3, linear dependence means that there are constants $k_1, k_2, \dots, k_n$ *not all zero* such that $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0}$. For example, in R^3 the vectors $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 5, 2, 7 \rangle$ are linearly dependent since (3) is satisfied when $k_1 = 3$, $k_2 = 1$, and $k_3 = -1$:

$$3\langle 1, 1, 1 \rangle + \langle 2, -1, 4 \rangle - \langle 5, 2, 7 \rangle = \langle 0, 0, 0 \rangle \quad \text{or} \quad 3\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}.$$

We observe that two vectors are linearly independent if neither is a constant multiple of the other.

Basis Any vector in R^3 can be written as a linear combination of the linearly independent vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. In Section 7.2, we said that these vectors form a **basis** for the system of three-dimensional vectors.

Definition 7.6.4 Basis for a Vector Space

Consider a set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V . If the set B is linearly independent and if every vector in V can be expressed as a linear combination of these vectors, then B is said to be a **basis** for V .

Standard Bases Although we cannot prove it in this course, every vector space has a basis. The vector space P_n of all polynomials of degree less than or equal to n has the basis $\{1, x, x^2, \dots, x^n\}$ since any vector (polynomial) $p(x)$ of degree n or less can be written as the linear combination $p(x) = c_nx^n + \cdots + c_2x^2 + c_1x + c_0$. A vector space may have many bases. We mentioned previously the set of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis for R^3 . But it can be proved that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \langle 1, 0, 0 \rangle, \quad \mathbf{u}_2 = \langle 1, 1, 0 \rangle, \quad \mathbf{u}_3 = \langle 1, 1, 1 \rangle$$

is a linearly independent set (see Problem 23 in Exercises 7.6) and, furthermore, every vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be expressed as a linear combination $\mathbf{a} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Hence, the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is another basis for R^3 . Indeed, any set of *three* linearly independent vectors is a basis for that space. However, as mentioned in Section 7.2, the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is referred to as the **standard basis** for R^3 . The standard basis for the space P_n is $\{1, x, x^2, \dots, x^n\}$. For the

vector space R^n , the **standard basis** consists of the n vectors

$$\mathbf{e}_1 = \langle 1, 0, 0, \dots, 0 \rangle, \mathbf{e}_2 = \langle 0, 1, 0, \dots, 0 \rangle, \dots, \mathbf{e}_n = \langle 0, 0, 0, \dots, 1 \rangle. \quad (4)$$

If B is a basis for a vector space V , then for every vector $\mathbf{v}$ in V there exist scalars $c_i, i = 1, 2, \dots, n$ such that

$$\mathbf{v} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \dots + c_n \mathbf{x}_n. \quad (5)$$

Read the last sentence several times.

The scalars $c_i, i = 1, 2, \dots, n$, in the linear combination (5) are called the **coordinates of $\mathbf{v}$ relative to the basis B** . In R^n , the n -tuple notation $\langle a_1, a_2, \dots, a_n \rangle$ for a vector $\mathbf{a}$ means that real numbers $a_1, a_2, \dots, a_n$ are the coordinates of $\mathbf{a}$ relative to the standard basis with $\mathbf{e}_i$'s in the precise order given in (4).

Dimension If a vector space V has a basis B consisting of n vectors, then it can be proved that every basis for that space must contain n vectors. This leads to the next definition.

Definition 7.6.5 Dimension of a Vector Space

The number of vectors in a basis B for a vector space V is said to be the **dimension** of the space.

EXAMPLE 5 Dimensions of Some Vector Spaces

- (a) In agreement with our intuition, the dimensions of the vector spaces R, R^2, R^3 , and R^n are, in turn, 1, 2, 3, and n .
- (b) Since there are $n + 1$ vectors in the standard basis $B = \{1, x, x^2, \dots, x^n\}$, the dimension of the vector space P_n of polynomials of degree less than or equal to n is $n + 1$.
- (c) The zero vector space $\{\mathbf{0}\}$ is given special consideration. This space contains only $\mathbf{0}$ and since $\{\mathbf{0}\}$ is a linearly dependent set, it is not a basis. In this case it is customary to take the empty set as the basis and to *define* the dimension of $\{\mathbf{0}\}$ as zero. $\equiv$

If the basis of a vector space V contains a finite number of vectors, then we say that the vector space is **finite dimensional**; otherwise it is **infinite dimensional**. The function space $C^n(I)$ of n times continuously differentiable functions on an interval I is an example of an infinite-dimensional vector space.

Linear Differential Equations Consider the homogeneous linear n th-order differential equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (6)$$

on an interval I on which the coefficients are continuous and $a_n(x) \neq 0$ for every x in the interval. A solution y_1 of (6) is necessarily a vector in the vector space $C^n(I)$. In addition, we know from the theory examined in Section 3.1 that if y_1 and y_2 are solutions of (6), then the sum $y_1 + y_2$ and any constant multiple ky_1 are also solutions. Since the solution set is closed under addition and scalar multiplication, it follows from Theorem 7.6.1 that the solution set of (6) is a subspace of $C^n(I)$. Hence the solution set of (6) deserves to be called the **solution space** of the differential equation. We also know that if $\{y_1, y_2, \dots, y_n\}$ is a linearly independent set of solutions of (6), then its general solution of the differential equation is the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x).$$

Recall that any solution of the equation can be found from this general solution by specialization of the constants $c_1, c_2, \dots, c_n$. Therefore, the linearly independent set of solutions $\{y_1, y_2, \dots, y_n\}$ is a basis for the solution space. The dimension of this solution space is n .

EXAMPLE 6 Dimension of a Solution Space

The general solution of the homogeneous linear second-order differential equation $y'' + 25y = 0$ is $y = c_1 \cos 5x + c_2 \sin 5x$. A basis for the solution space consists of the linearly independent vectors $\{\cos 5x, \sin 5x\}$. The solution space is two-dimensional. $\equiv$

The set of solutions of a nonhomogeneous linear differential equation is not a vector space. Several axioms of a vector space are not satisfied; most notably the set of solutions does not contain a zero vector. In other words, $y = 0$ is not a solution of a nonhomogeneous linear differential equation.

Span If S denotes any set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V , then the set of *all* linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ in S ,

$$\{k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \dots + k_n\mathbf{x}_n\},$$

where the k_i , $i = 1, 2, \dots, n$ are scalars, is called the **span** of the vectors and written $\text{Span}(S)$ or $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$. It is left as an exercise to show that $\text{Span}(S)$ is a subspace of the vector space V . See Problem 33 in Exercises 7.6. $\text{Span}(S)$ is said to be a subspace *spanned* by the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. If $V = \text{Span}(S)$, then we say that S is a **spanning set** for the vector space V , or that S **spans** V . For example, each of the three sets

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}, \quad \{\mathbf{i}, \mathbf{i} + \mathbf{j}, \mathbf{i} + \mathbf{j} + \mathbf{k}\}, \quad \text{and} \quad \{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i} + \mathbf{j}, \mathbf{i} + \mathbf{j} + \mathbf{k}\}$$

are spanning sets for the vector space R^3 . But note that the first two sets are linearly independent, whereas the third set is dependent. With these new concepts we can rephrase Definitions 7.6.4 and 7.6.5 in the following manner:

*A set S of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V is a **basis** for V if S is linearly independent and is a spanning set for V . The number of vectors in this spanning set S is the **dimension** of the space V .*

REMARKS

- (i) Suppose V is an arbitrary real vector space. If there is an inner product defined on V it need not look at all like the standard or Euclidean inner product defined on R^n . In Chapter 12 we will work with an inner product that is a definite integral. We shall denote an inner product that is *not* the Euclidean inner product by the symbol $(\mathbf{u}, \mathbf{v})$. See Problems 30, 31, and 38(b) in Exercises 7.6.
- (ii) A vector space V on which an inner product has been defined is called an **inner product space**. A vector space V can have more than one inner product defined on it. For example, a non-Euclidean inner product defined on R^2 is $(\mathbf{u}, \mathbf{v}) = u_1v_1 + 4u_2v_2$, where $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. See Problems 37 and 38(a) in Exercises 7.6.
- (iii) A lot of our work in the later chapters in this text takes place in an **infinite-dimensional vector space**. As such, we need to extend the definition of linear independence of a finite set of vectors $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ given in Definition 7.6.3 to an infinite set:

*An infinite set of vectors $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots\}$ is said to be **linearly independent** if every finite subset of the set S is linearly independent. If the set S is not linearly independent, then it is **linearly dependent**.*

We note that if S contains a linearly dependent subset, then the entire set S is linearly dependent.

The vector space P of all polynomials has the standard basis $B = \{1, x, x^2, \dots\}$. The infinite set B is linearly independent. P is another example of an infinite-dimensional vector space.

7.6

Exercises

Answers to selected odd-numbered problems begin on page ANS-15.

In Problems 1–10, determine whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless stated to the contrary, assume that vector addition and scalar multiplication are the ordinary operations defined on that set.

1. The set of vectors $\langle a_1, a_2 \rangle$, where $a_1 \geq 0$, $a_2 \geq 0$
2. The set of vectors $\langle a_1, a_2 \rangle$, where $a_2 = 3a_1 + 1$
3. The set of vectors $\langle a_1, a_2 \rangle$, scalar multiplication defined by $k\langle a_1, a_2 \rangle = \langle ka_1, 0 \rangle$

4. The set of vectors $\langle a_1, a_2 \rangle$, where $a_1 + a_2 = 0$
5. The set of vectors $\langle a_1, a_2, 0 \rangle$
6. The set of vectors $\langle a_1, a_2 \rangle$, addition and scalar multiplication defined by

$$\begin{aligned} \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle &= \langle a_1 + b_1 + 1, a_2 + b_2 + 1 \rangle \\ k\langle a_1, a_2 \rangle &= \langle ka_1 + k - 1, ka_2 + k - 1 \rangle \end{aligned}$$

7. The set of real numbers, addition defined by $\mathbf{x} + \mathbf{y} = x - y$

8. The set of complex numbers $a + bi$, where $i^2 = -1$, addition and scalar multiplication defined by

$$(a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$$

$$k(a + bi) = ka + kbi, \quad k \text{ a real number}$$

9. The set of arrays of real numbers $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, addition and scalar multiplication defined by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

10. The set of all polynomials of degree 2

In Problems 11–16, determine whether the given set is a subspace of the vector space $C(-\infty, \infty)$.

11. All functions f such that $f(1) = 0$
12. All functions f such that $f(0) = 1$
13. All nonnegative functions f
14. All functions f such that $f(-x) = f(x)$
15. All differentiable functions f
16. All functions f of the form $f(x) = c_1e^x + c_2xe^x$

In Problems 17–20, determine whether the given set is a subspace of the indicated vector space.

17. Polynomials of the form $p(x) = c_3x^3 + c_1x$; P_3
18. Polynomials p that are divisible by $x - 2$; P_2
19. All unit vectors; R^3
20. Functions f such that $\int_a^b f(x) dx = 0$; $C[a, b]$
21. In 3-space, a line through the origin can be written as $S = \{(x, y, z) | x = at, y = bt, z = ct, a, b, c \text{ real numbers}\}$. With addition and scalar multiplication the same as for vectors $\langle x, y, z \rangle$, show that S is a subspace of R^3 .
22. In 3-space, a plane through the origin can be written as $S = \{(x, y, z) | ax + by + cz = 0, a, b, c \text{ real numbers}\}$. Show that S is a subspace of R^3 .
23. The vectors $\mathbf{u}_1 = \langle 1, 0, 0 \rangle$, $\mathbf{u}_2 = \langle 1, 1, 0 \rangle$, and $\mathbf{u}_3 = \langle 1, 1, 1 \rangle$ form a basis for the vector space R^3 .
 - (a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent.
 - (b) Express the vector $\mathbf{a} = \langle 3, -4, 8 \rangle$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.
24. The vectors $p_1(x) = x + 1$, $p_2(x) = x - 1$ form a basis for the vector space P_1 .
 - (a) Show that $p_1(x)$ and $p_2(x)$ are linearly independent.
 - (b) Express the vector $p(x) = 5x + 2$ as a linear combination of $p_1(x)$ and $p_2(x)$.

In Problems 25–28, determine whether the given vectors are linearly independent or linearly dependent.

25. $\langle 4, -8 \rangle, \langle -6, 12 \rangle$ in R^2
26. $\langle 1, 1 \rangle, \langle 0, 1 \rangle, \langle 2, 5 \rangle$ in R^2
27. $1, (x + 1), (x + 1)^2$ in P_2

28. $1, (x + 1), (x + 1)^2, x^2$ in P_2

29. Explain why $f(x) = \frac{x}{x^2 + 4x + 3}$ is a vector in $C[0, 3]$ but not a vector in $C[-3, 0]$.

30. A vector space V on which a dot or inner product has been defined is called an **inner product space**. An inner product for the vector space $C[a, b]$ is given by

$$(f, g) = \int_a^b f(x)g(x) dx.$$

In $C[0, 2\pi]$ compute $(x, \sin x)$.

31. The **norm** of a vector in an inner product space is defined in terms of the inner product. For the inner product given in Problem 30, the norm of a vector is given by $\|f\| = \sqrt{(f, f)}$. In $C[0, 2\pi]$ compute $\|x\|$ and $\|\sin x\|$.
32. Find a basis for the solution space of

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 10\frac{d^2y}{dx^2} = 0.$$

33. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be any set of vectors in a vector space V . Show that $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ is a subspace of V .

Discussion Problems

34. Discuss: Is R^2 a subspace of R^3 ? Are R^2 and R^3 subspaces of R^4 ?
35. In Problem 9, you should have proved that the set M_{22} of 2×2 arrays of real numbers

$$M_{22} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\},$$

or **matrices**, is a vector space with vector addition and scalar multiplication defined in that problem. Find a basis for M_{22} . What is the dimension of M_{22} ?

36. Consider a finite orthogonal set of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in R^n . Discuss: Is this set linearly independent or linearly dependent?
37. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V , then the axioms of an inner product $(\mathbf{u}, \mathbf{v})$ are

- (i) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
- (ii) $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, k a scalar
- (iii) $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$
- (iv) $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w})$.

Show that $(\mathbf{u}, \mathbf{v}) = u_1v_1 + 4u_2v_2$, where $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, is an inner product on R^2 .

38. (a) Find a pair of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^2 that are not orthogonal with respect to the standard or Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$, but are orthogonal with respect to the inner product $(\mathbf{u}, \mathbf{v})$ in Problem 37.
- (b) Find a pair of nonzero functions f and g in $C[0, 2\pi]$ that are orthogonal with respect to the inner product (f, g) given in Problem 30.

7.7 Gram–Schmidt Orthogonalization Process

INTRODUCTION In Section 7.6 we saw that a vector space V can have many different bases. Recall, the defining characteristics of any basis $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ of a vector space V is that

- the set B is linearly independent, and
- the set B spans the space.

In this context the word *span* means that every vector in the space can be expressed as a linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. For example, every vector $\mathbf{u}$ in R^n can be written as a linear combination of the vectors in the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 = \langle 1, 0, 0, \dots, 0 \rangle, \quad \mathbf{e}_2 = \langle 0, 1, 0, \dots, 0 \rangle, \quad \dots, \quad \mathbf{e}_n = \langle 0, 0, 0, \dots, 1 \rangle.$$

This standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is also an example of an **orthonormal basis**; that is, the $\mathbf{e}_i$, $i = 1, 2, \dots, n$ are mutually orthogonal and are unit vectors; that is,

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, i \neq j \quad \text{and} \quad \|\mathbf{e}_i\| = 1, i = 1, 2, \dots, n.$$

In this section we focus on orthonormal bases for R^n and examine a procedure whereby we can transform or convert any basis B of R^n into an orthonormal basis.

EXAMPLE 1 Orthonormal Basis for R^3

The set of three vectors

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \quad (1)$$

is linearly independent and spans the space R^3 . Hence $B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for R^3 . Using the standard inner product or dot product defined on R^3 , observe

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = 0, \mathbf{w}_1 \cdot \mathbf{w}_3 = 0, \mathbf{w}_2 \cdot \mathbf{w}_3 = 0, \quad \text{and} \quad \|\mathbf{w}_1\| = 1, \|\mathbf{w}_2\| = 1, \|\mathbf{w}_3\| = 1.$$

Hence B is an orthonormal basis. ≡

A basis B for R^n need not be orthogonal nor do the basis vectors need to be unit vectors. In fact, *any* linearly independent set of n vectors can serve as a basis for the n -dimensional vector space R^n . For example, it is a straightforward task to show that the vectors

$$\mathbf{u}_1 = \langle 1, 0, 0 \rangle, \quad \mathbf{u}_2 = \langle 1, 1, 0 \rangle, \quad \mathbf{u}_3 = \langle 1, 1, 1 \rangle$$

in R^3 are linearly independent and hence $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R^3 . Note that B is not an orthogonal basis.

Generally, an orthonormal basis for a vector space V turns out to be the most convenient basis for V . One of the advantages that an orthonormal basis has over any other basis for R^n is the comparative ease with which we can obtain the coordinates of a vector $\mathbf{u}$ relative to that basis.

Theorem 7.7.1 Coordinates Relative to an Orthonormal Basis

Suppose $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is an orthonormal basis for R^n . If $\mathbf{u}$ is any vector in R^n , then

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n.$$

PROOF: The vector $\mathbf{u}$ is in R^n and so it is an element of the set $\text{Span}(B)$. In other words, there exist real scalars k_i , $i = 1, 2, \dots, n$ such that $\mathbf{u}$ can be expressed as the linear combination

$$\mathbf{u} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_n\mathbf{w}_n.$$

The scalars k_i are the coordinates of $\mathbf{u}$ relative to the basis B . These coordinates can be found by taking the dot product of $\mathbf{u}$ with each of the basis vectors:

$$\mathbf{u} \cdot \mathbf{w}_i = (k_1 \mathbf{w}_1 + k_2 \mathbf{w}_2 + \cdots + k_n \mathbf{w}_n) \cdot \mathbf{w}_i = k_1(\mathbf{w}_1 \cdot \mathbf{w}_i) + k_2(\mathbf{w}_2 \cdot \mathbf{w}_i) + \cdots + k_n(\mathbf{w}_n \cdot \mathbf{w}_i). \quad (2)$$

Since B is orthonormal, $\mathbf{w}_i$ is orthogonal to all vectors in B with the exception of $\mathbf{w}_i$ itself. That is, $\mathbf{w}_i \cdot \mathbf{w}_j = 0$, $i \neq j$ and $\mathbf{w}_i \cdot \mathbf{w}_i = \|\mathbf{w}_i\|^2 = 1$. Hence from (2), we obtain $k_i = (\mathbf{u} \cdot \mathbf{w}_i)$ for $i = 1, 2, \dots, n$. $\equiv$

EXAMPLE 2 Coordinates of a Vector in R^3

Find the coordinates of the vector $\mathbf{u} = \langle 3, -2, 9 \rangle$ relative to the orthonormal basis B for R^3 given in (1) of Example 1. Write $\mathbf{u}$ in terms of the basis B .

SOLUTION From Theorem 7.7.1, the coordinates of $\mathbf{u}$ relative to the basis B in (1) of Example 1 are simply

$$\mathbf{u} \cdot \mathbf{w}_1 = \frac{10}{\sqrt{3}}, \quad \mathbf{u} \cdot \mathbf{w}_2 = \frac{1}{\sqrt{6}}, \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w}_3 = -\frac{11}{\sqrt{2}}.$$

Hence we can write

$$\mathbf{u} = \frac{10}{\sqrt{3}} \mathbf{w}_1 + \frac{1}{\sqrt{6}} \mathbf{w}_2 - \frac{11}{\sqrt{2}} \mathbf{w}_3. \quad \equiv$$

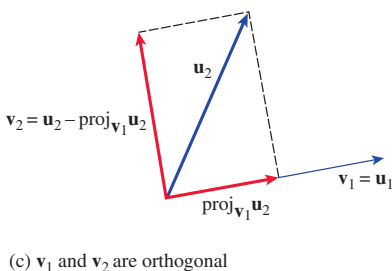
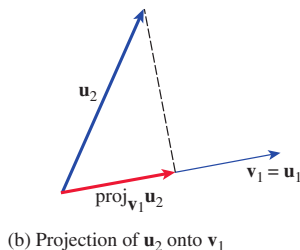
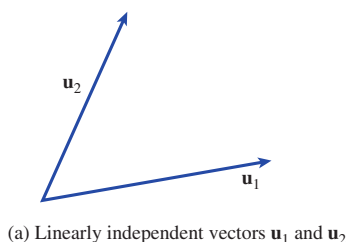


FIGURE 7.7.1 The orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are defined in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$

Gram-Schmidt Orthogonalization Process The procedure known as the **Gram-Schmidt orthogonalization process** is a straightforward algorithm for generating an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ from any given basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ for R^n . We then produce an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ by normalizing the vectors in the orthogonal basis B' . The key idea in the orthogonalization process is vector projection, and so we suggest that you review that concept in Section 7.3. Also, for the sake of gaining some geometric insight into the process, we shall begin in R^2 and R^3 .

Constructing an Orthogonal Basis for R^2 The Gram-Schmidt orthogonalization process for R^n is a sequence of steps; at each step we construct a vector $\mathbf{v}_i$ that is orthogonal to the vector in the preceding step. The transformation of a basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 into an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ consists of two steps. See **FIGURE 7.7.1(a)**. The first step is simple, we merely choose one of the vectors in B , say, $\mathbf{u}_1$, and rename it $\mathbf{v}_1$. Next, as shown in Figure 7.7.1(b), we project the remaining vector $\mathbf{u}_2$ in B onto the vector $\mathbf{v}_1$ and define a second vector to be $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$. Recall from (12) of Section 7.3 that $\text{proj}_{\mathbf{v}_1} \mathbf{u}_2 = \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$. As seen in Figure 7.7.1(c), the vectors

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \end{aligned} \quad (3)$$

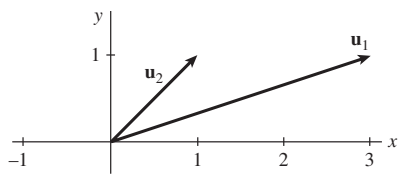
are orthogonal. If you are not convinced of this, we suggest you verify the orthogonality of $\mathbf{v}_1$ and $\mathbf{v}_2$ by demonstrating that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

EXAMPLE 3 Gram-Schmidt Process in R^2

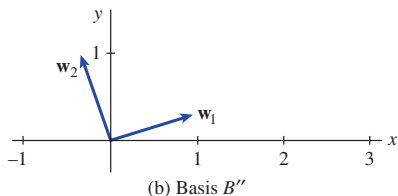
The set $B = \{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = \langle 3, 1 \rangle$, $\mathbf{u}_2 = \langle 1, 1 \rangle$, is a basis for R^2 . Transform B into an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$.

SOLUTION We choose $\mathbf{v}_1$ as $\mathbf{u}_1$: $\mathbf{v}_1 = \langle 3, 1 \rangle$. Then from the second equation in (3), with $\mathbf{u}_2 \cdot \mathbf{v}_1 = 4$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 = 10$, we obtain

$$\mathbf{v}_2 = \langle 1, 1 \rangle - \frac{4}{10} \langle 3, 1 \rangle = \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle.$$



(a) Basis B



(b) Basis B''

FIGURE 7.7.2 The two bases in Example 3

The set $B' = \{\mathbf{v}_1, \mathbf{v}_2\} = \{\langle 3, 1 \rangle, \langle -\frac{1}{5}, \frac{3}{5} \rangle\}$ is an orthogonal basis for R^2 . We finish by normalizing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \quad \text{and} \quad \mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle.$$

The basis B is shown in FIGURE 7.7.2(a), and the new orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is shown in blue in Figure 7.7.2(b). ≡

In Example 3 we are free to choose either vector in $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ as the vector $\mathbf{v}_1$. However, by choosing $\mathbf{v}_1 = \mathbf{u}_2 = \langle 1, 1 \rangle$, we obtain a different orthonormal basis, namely, $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$, where $\mathbf{w}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ and $\mathbf{w}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$. See Problems 5–8 in Exercises 7.7.

|| **Constructing an Orthogonal Basis for R^3** Now suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R^3 . Then the set $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \\ \mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \end{aligned} \quad (4)$$

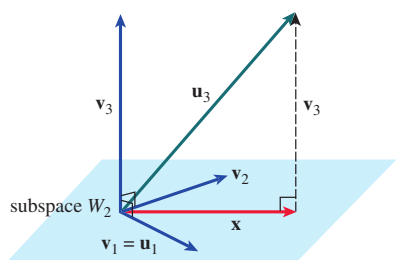


FIGURE 7.7.3 The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (in blue) obtained by the Gram–Schmidt process

is an orthogonal basis for R^3 . Again, if you do not see this, then compute $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_3$, and $\mathbf{v}_2 \cdot \mathbf{v}_3$.

Since the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in the list (4) are by construction orthogonal, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent. See Problem 36 in Exercises 7.6. Thus, $W_2 = \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ is necessarily a two-dimensional subspace of R^3 . Now the vector $\mathbf{x} = \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2$ is a vector in W_2 because it is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vector $\mathbf{x}$ is called the **orthogonal projection** of $\mathbf{u}_3$ onto the subspace W_2 and is usually denoted by $\mathbf{x} = \text{proj}_{W_2} \mathbf{u}_3$. In FIGURE 7.7.3, $\mathbf{x}$ is the red vector. Notice, too, that $\mathbf{x}$ is the sum of two projections. Using (12) of Section 7.3, we can write

$$\mathbf{x} = \text{proj}_{W_2} \mathbf{u}_3 = \overbrace{\left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1}^{\text{proj}_{\mathbf{v}_1} \mathbf{u}_3} + \overbrace{\left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2}^{\text{proj}_{\mathbf{v}_2} \mathbf{u}_3}. \quad (5)$$

The difference $\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{x}$ is orthogonal to $\mathbf{x}$. Indeed, $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$ and to every vector in W_2 . This is precisely the same idea in (3). In that context, $\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{x}$, where $\mathbf{x}$ was the projection of $\mathbf{u}_2$ onto the one-dimensional subspace $W_1 = \text{Span}(\mathbf{v}_1)$ of R^2 . Analogous to (5), we have

$$\mathbf{x} = \text{proj}_{W_1} \mathbf{u}_2 = \overbrace{\left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1}^{\text{proj}_{\mathbf{v}_1} \mathbf{u}_2}. \quad (6)$$

EXAMPLE 4 Gram–Schmidt Process in R^3

The set $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \langle 1, 1, 1 \rangle, \mathbf{u}_2 = \langle 1, 2, 2 \rangle, \mathbf{u}_3 = \langle 1, 1, 0 \rangle$$

is a basis for R^3 . Transform B into an orthonormal basis B'' .

SOLUTION We choose $\mathbf{v}_1$ as $\mathbf{u}_1$: $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$. Then from the second equation in (4), with $\mathbf{u}_2 \cdot \mathbf{v}_1 = 5$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$, we obtain

$$\mathbf{v}_2 = \langle 1, 2, 2 \rangle - \frac{5}{3} \langle 1, 1, 1 \rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle.$$

Now with $\mathbf{u}_3 \cdot \mathbf{v}_1 = 2$, $\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$, $\mathbf{u}_3 \cdot \mathbf{v}_2 = -\frac{1}{3}$, and $\mathbf{v}_2 \cdot \mathbf{v}_2 = \frac{2}{3}$, the third equation in (4) yields

$$\begin{aligned} \mathbf{v}_3 &= \langle 1, 1, 0 \rangle - \frac{2}{3} \langle 1, 1, 1 \rangle + \frac{1}{2} \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \\ &= \langle 1, 1, 0 \rangle + \left\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3} \right\rangle + \left\langle -\frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right\rangle \\ &= \left\langle 0, \frac{1}{2}, -\frac{1}{2} \right\rangle. \end{aligned}$$

The set $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\langle 1, 1, 1 \rangle, \langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \rangle, \langle 0, \frac{1}{2}, -\frac{1}{2} \rangle\}$ is an orthogonal basis for R^3 . As in Example 3, we finish the job by normalizing each vector in B' . Using $\|\mathbf{v}_1\| = \sqrt{3}$, $\|\mathbf{v}_2\| = \sqrt{6}/3$, $\|\mathbf{v}_3\| = \sqrt{2}/2$, and $\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|} \mathbf{v}_i$, $i = 1, 2, 3$, we find that an orthonormal basis for R^3 is $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \quad \mathbf{w}_2 = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle, \quad \mathbf{w}_3 = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle.$$

The set B'' is recognized as the orthonormal basis for R^3 examined in Example 1. ≡

We conclude this section with a theorem that summarizes the most general case of the Gram–Schmidt process for R^n . The orthogonalization process can be used on any linearly independent set S , and so we can use it to find orthonormal bases for subspaces of R^n .

Theorem 7.7.2 Gram–Schmidt Orthogonalization Process

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$, $m \leq n$, be a basis for a subspace W_m of R^n . Then $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$, where

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \\ \mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \\ &\vdots \\ \mathbf{v}_m &= \mathbf{u}_m - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 - \dots - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}} \right) \mathbf{v}_{m-1}, \end{aligned} \tag{7}$$

is an orthogonal basis for W_m . An orthonormal basis for W_m is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\} = \left\{ \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_m\|} \mathbf{v}_m \right\}.$$

REMARKS

Although we have focused on R^n in the foregoing discussion, the orthogonalization process summarized in (7) of Theorem 7.7.2 holds in all vector spaces V on which an inner product $(\mathbf{u}, \mathbf{v})$ is defined. In this case, we replace the symbol R^n in (7) with the words “an inner product space V ” and each dot product symbol $\mathbf{u} \cdot \mathbf{v}$ with $(\mathbf{u}, \mathbf{v})$. See Problems 17 and 18 in Exercises 7.7.

7.7

Exercises

Answers to selected odd-numbered problems begin on page ANS-16.

In Problems 1 and 2, verify that the basis B for the given vector space is orthonormal. Use Theorem 7.7.1 to find the coordinates of the vector $\mathbf{u}$ relative to the basis B . Then write $\mathbf{u}$ as a linear combination of the basis vectors.

1. $B = \left\{ \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle, \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \right\}$, R^2 ; $\mathbf{u} = \langle 4, 2 \rangle$
2. $B = \left\{ \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle, \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle, \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle \right\}$, R^3 ; $\mathbf{u} = \langle 5, -1, 6 \rangle$

In Problems 3 and 4, verify that the basis B for the given vector space is orthogonal. Use Theorem 7.7.1 as an aid in finding the coordinates of the vector $\mathbf{u}$ relative to the basis B . Then write $\mathbf{u}$ as a linear combination of the basis vectors.

3. $B = \{ \langle 1, 0, 1 \rangle, \langle 0, 1, 0 \rangle, \langle -1, 0, 1 \rangle \}$, R^3 ; $\mathbf{u} = \langle 10, 7, -13 \rangle$
4. $B = \{ \langle 2, 1, -2, 0 \rangle, \langle 1, 2, 2, 1 \rangle, \langle 3, -4, 1, 3 \rangle, \langle 5, -2, 4, -9 \rangle \}$, R^4 ; $\mathbf{u} = \langle 1, 2, 4, 3 \rangle$

In Problems 5–8, use the Gram–Schmidt orthogonalization process (3) to transform the given basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 into an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then form an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$.

- (a) First construct B'' using $\mathbf{v}_1, \mathbf{u}_1$.
- (b) Then construct B'' using $\mathbf{v}_1, \mathbf{u}_2$.
- (c) Sketch B and each basis B'' .
5. $B = \{ \langle -3, 2 \rangle, \langle -1, -1 \rangle \}$
6. $B = \{ \langle -3, 4 \rangle, \langle -1, 0 \rangle \}$
7. $B = \{ \langle 1, 1 \rangle, \langle 1, 0 \rangle \}$
8. $B = \{ \langle 5, 7 \rangle, \langle 1, -2 \rangle \}$

In Problems 9–12, use the Gram–Schmidt orthogonalization process (4) to transform the given basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for R^3 into an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then form an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

9. $B = \{ \langle 1, 1, 0 \rangle, \langle 1, 2, 2 \rangle, \langle 2, 2, 1 \rangle \}$
10. $B = \{ \langle -3, 1, 1 \rangle, \langle 1, 1, 0 \rangle, \langle -1, 4, 1 \rangle \}$
11. $B = \{ \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle, \langle -1, 1, -\frac{1}{2} \rangle, \langle -1, \frac{1}{2}, 1 \rangle \}$
12. $B = \{ \langle 1, 1, 1 \rangle, \langle 9, -1, 1 \rangle, \langle -1, 4, -2 \rangle \}$

In Problems 13 and 14, the given vectors span a subspace W of R^3 . Use the Gram–Schmidt orthogonalization process to construct an orthonormal basis for the subspace.

13. $\mathbf{u}_1 = \langle 1, 5, 2 \rangle, \mathbf{u}_2 = \langle -2, 1, 1 \rangle$
14. $\mathbf{u}_1 = \langle 1, 2, 3 \rangle, \mathbf{u}_2 = \langle 3, 4, 1 \rangle$

In Problems 15 and 16, the given vectors span a subspace W of R^4 . Use the Gram–Schmidt orthogonalization process to construct an orthonormal basis for the subspace.

15. $\mathbf{u}_1 = \langle 1, -1, 1, -1 \rangle, \mathbf{u}_2 = \langle 1, 3, 0, 1 \rangle$
16. $\mathbf{u}_1 = \langle 4, 0, 2, -1 \rangle, \mathbf{u}_2 = \langle 2, 1, -1, 1 \rangle, \mathbf{u}_3 = \langle 1, 1, -1, 0 \rangle$

In Problems 17 and 18, an inner product defined on the vector space P_2 of all polynomials of degree less than or equal to 2, is given by

$$(p, q) = \int_{-1}^1 p(x)q(x) dx.$$

Use the Gram–Schmidt orthogonalization process to transform the given basis B for P_2 into an orthogonal basis B' .

17. $B = \{1, x, x^2\}$
18. $B = \{x^2 - x, x^2 + 1, 1 - x^2\}$

For the inner product (p, q) defined on P_2 in Problems 17 and 18, the norm $\|p(x)\|$ of a polynomial p is defined by

$$\|p(x)\|^2 = (p, p) = \int_{-1}^1 p^2(x) dx.$$

Use this norm in Problems 19 and 20.

19. Construct an orthonormal basis B'' from B' obtained in Problem 17.
20. Construct an orthonormal basis B'' from B' obtained in Problem 18.

In Problems 21 and 22, let $p(x) = 9x^2 - 6x + 5$ be a vector in P_2 . Use Theorem 7.7.1 and the indicated orthonormal basis B'' to find the coordinates $p(x)$ relative to B'' . Then write $p(x)$ as a linear combination of the basis vectors.

21. B'' in Problem 19
22. B'' in Problem 20

Discussion Problem

23. The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \langle 1, 1, 3 \rangle, \mathbf{u}_2 = \langle 1, 4, 1 \rangle, \text{ and } \mathbf{u}_3 = \langle 1, 10, -3 \rangle,$$

is linearly dependent in R^3 since $\mathbf{u}_3 = -2\mathbf{u}_1 + 3\mathbf{u}_2$. Discuss what you would expect when the Gram–Schmidt process in (4) is applied to these vectors. Then carry out the orthogonalization process.

Answer Problems 1–30 without referring back to the text. Fill in the blank or answer true/false.

1. The vectors $\langle -4, -6, 10 \rangle$ and $\langle -10, -15, 25 \rangle$ are parallel. _____
 2. In 3-space, any three distinct points determine a plane. _____
 3. The line $x = 1 + 5t$, $y = 1 - 2t$, $z = 4 + t$ and the plane $2x + 3y - 4z = 1$ are perpendicular. _____
 4. Nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. _____
 5. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse, $\mathbf{a} \cdot \mathbf{b} < 0$. _____
 6. If $\mathbf{a}$ is a unit vector, then $\mathbf{a} \cdot \mathbf{a} = 1$. _____
 7. The cross product of two vectors is not commutative. _____
 8. The terminal point of the vector $\mathbf{a} - \mathbf{b}$ is at the terminal point of $\mathbf{a}$. _____
 9. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ _____
 10. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are nonzero coplanar vectors, then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$. _____
 11. The sum of $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ is _____.
 12. If $\mathbf{a} \cdot \mathbf{b} = 0$, the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are _____.
 13. $(-\mathbf{k}) \times (5\mathbf{j}) =$ _____
 14. $\mathbf{i} \cdot (\mathbf{i} \times \mathbf{j}) =$ _____
 15. $\| -12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \| =$ _____
 16. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 5 \\ 0 & 4 & -1 \end{vmatrix} =$ _____
 17. A vector that is normal to the plane $-6x + y - 7z + 10 = 0$ is _____.
 18. The plane $x + 3y - z = 5$ contains the point $(1, -2, \quad)$.
 19. The point of intersection of the line $x - 1 = (y + 2)/3 = (z + 1)/2$ and the plane $x + 2y - z = 13$ is _____.
 20. A unit vector that has the opposite direction of $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ is _____.
 21. If $\overrightarrow{P_1P_2} = \langle 3, 5, -4 \rangle$ and P_1 has coordinates $(2, 1, 7)$, then the coordinates of P_2 are _____.
 22. The midpoint of the line segment between $P_1(4, 3, 10)$ and $P_2(6, -2, -5)$ has coordinates _____.
 23. If $\|\mathbf{a}\| = 7.2$, $\|\mathbf{b}\| = 10$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 135° , then $\mathbf{a} \cdot \mathbf{b} =$ _____.
 24. If $\mathbf{a} = \langle 3, 1, 0 \rangle$, $\mathbf{b} = \langle -1, 2, 1 \rangle$, and $\mathbf{c} = \langle 0, -2, 2 \rangle$, then $\mathbf{a} \cdot (2\mathbf{b} + 4\mathbf{c}) =$ _____.
 25. The x -, y -, and z -intercepts of the plane $2x - 3y + 4z = 24$ are, respectively, _____.
 26. The angle θ between the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{k}$ is _____.
 27. The area of a triangle with two sides given by $\mathbf{a} = \langle 1, 3, -1 \rangle$ and $\mathbf{b} = \langle 2, -1, 2 \rangle$ is _____.
 28. An equation of the plane containing $(3, 6, -2)$ and with normal vector $\mathbf{n} = 3\mathbf{i} + \mathbf{k}$ is _____.
 29. The distance from the plane $y = -5$ to the point $(4, -3, 1)$ is _____.
 30. The vectors $\langle 1, 3, c \rangle$ and $\langle -2, -6, 5 \rangle$ are parallel for $c =$ _____ and orthogonal for $c =$ _____.
 31. Find a unit vector that is perpendicular to both $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
 32. Find the direction cosines and direction angles of the vector $\mathbf{a} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}$.
- In Problems 33–36, let $\mathbf{a} = \langle 1, 2, -2 \rangle$ and $\mathbf{b} = \langle 4, 3, 0 \rangle$. Find the indicated number or vector.
33. $\text{comp}_{\mathbf{b}}\mathbf{a}$
 34. $\text{proj}_{\mathbf{b}}\mathbf{a}$
 35. $\text{proj}_{\mathbf{a}}(\mathbf{a} + \mathbf{b})$
 36. $\text{proj}_{\mathbf{b}}(\mathbf{a} - \mathbf{b})$
 37. Let $\mathbf{r}$ be the position vector of a variable point $P(x, y, z)$ in space and let $\mathbf{a}$ be a constant vector. Determine the surface described by (a) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{r} = 0$ and (b) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{a} = 0$.
 38. Use the dot product to determine whether the points $(4, 2, -2)$, $(2, 4, -3)$, and $(6, 7, -5)$ are vertices of a right triangle.
 39. Find symmetric equations for the line through the point $(7, 3, -5)$ that is parallel to $(x - 3)/4 = (y + 4)/(-2) = (z - 9)/6$.
 40. Find parametric equations for the line through the point $(5, -9, 3)$ that is perpendicular to the plane $8x + 3y - 4z = 13$.
 41. Show that the lines $x = 1 - 2t$, $y = 3t$, $z = 1 + t$ and $x = 1 + 2s$, $y = -4 + s$, $z = -1 + s$ intersect orthogonally.
 42. Find an equation of the plane containing the points $(0, 0, 0)$, $(2, 3, 1)$, $(1, 0, 2)$.
 43. Find an equation of the plane containing the lines $x = t$, $y = 4t$, $z = -2t$, and $x = 1 + t$, $y = 1 + 4t$, $z = 3 - 2t$.
 44. Find an equation of the plane containing $(1, 7, -1)$ that is perpendicular to the line of intersection of $-x + y - 8z = 4$ and $3x - y + 2z = 0$.
 45. A constant force of 10 N in the direction of $\mathbf{a} = \mathbf{i} + \mathbf{j}$ moves a block on a frictionless surface from $P_1(4, 1, 0)$ to $P_2(7, 4, 0)$. Suppose distance is measured in meters. Find the work done.
 46. In Problem 45, find the work done in moving the block between the same points if another constant force of 50 N in the direction of $\mathbf{b} = \mathbf{i}$ acts simultaneously with the original force.
 47. Water rushing from a fire hose exerts a horizontal force $\mathbf{F}_1$ of magnitude 200 lb. See FIGURE 7.R.1. What is the magnitude of the force $\mathbf{F}_3$ that a firefighter must exert to hold the hose at an angle of 45° from the horizontal?

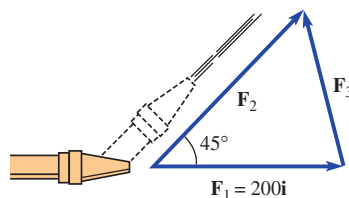


FIGURE 7.R.1 Fire hose in Problem 47

48. A uniform ball of weight 50 lb is supported by two frictionless planes as shown in **FIGURE 7.R.2**. Let the force exerted by the supporting plane $\mathcal{P}_1$ on the ball be $\mathbf{F}_1$ and the force exerted by the plane $\mathcal{P}_2$ be $\mathbf{F}_2$. Since the ball is held in equilibrium, we must have $\mathbf{w} + \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$, where $\mathbf{w} = -50\mathbf{j}$. Find the magnitudes of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. [Hint: Assume the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are normal to the planes $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, and act along lines through the center C of the ball. Place the origin of a two-dimensional coordinate system at C .]

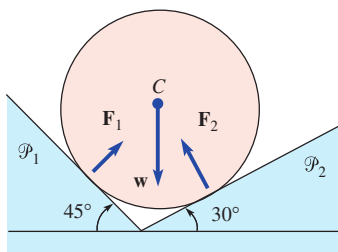


FIGURE 7.R.2 Supported ball in Problem 48

49. Determine whether the set of vectors $\langle a_1, 0, a_3 \rangle$ under addition and scalar multiplication defined by

$$\langle a_1, 0, a_3 \rangle + \langle b_1, 0, b_3 \rangle = \langle a_1 + b_1, 0, a_3 + b_3 \rangle$$

$$k\langle a_1, 0, a_3 \rangle = \langle ka_1, 0, a_3 \rangle$$

is a vector space.

50. Determine whether the vectors $\langle 1, 1, 2 \rangle$, $\langle 0, 2, 3 \rangle$, and $\langle 0, 1, -1 \rangle$ are linearly independent in $\mathbb{R}^3$.
51. Determine whether the set of polynomials in P_n satisfying the condition $d^2p/dx^2 = 0$ is a subspace of P_n . If it is, find a basis for the subspace.
52. Recall that the intersection of two sets W_1 and W_2 is the set of all elements common to both sets, and the union of W_1 and W_2 is the set of elements that are in either W_1 or W_2 . Suppose W_1 and W_2 are subspaces of a vector space V . Prove, or disprove by counterexample, the following propositions:
- (a) $W_1 \cap W_2$ is a subspace of V .
- (b) $W_1 \cup W_2$ is a subspace of V .

P in the spacecraft system after the yaw are related to the coordinates (x, y, z) of P in the fixed coordinate system by the equations

$$\begin{aligned}x_Y &= x \cos \gamma + y \sin \gamma \\y_Y &= -x \sin \gamma + y \cos \gamma \\z_Y &= z\end{aligned}$$

where γ is the angle of rotation.

- (a) Verify that the foregoing system of equations can be written as the matrix equation

$$\begin{pmatrix} x_Y \\ y_Y \\ z_Y \end{pmatrix} = \mathbf{M}_Y \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where
$$\mathbf{M}_Y = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- (b) When the spacecraft performs a pitch, roll, and yaw *in sequence* through the angles α , β , and γ , respectively, the final coordinates of the point P in the spacecraft system (x_S, y_S, z_S) are obtained from the sequence of transformations

$$\begin{aligned}x_P &= x & x_R &= x_P \cos \beta - z_P \sin \beta \\y_P &= y \cos \alpha + z \sin \alpha & y_R &= y_P \\z_P &= -y \sin \alpha + z \cos \alpha; & z_R &= x_P \sin \beta + z_P \cos \beta; \\x_S &= x_R \cos \gamma + y_R \sin \gamma \\y_S &= -x_R \sin \gamma + y_R \cos \gamma \\z_S &= z_R.\end{aligned}$$

Write this sequence of transformations as a matrix equation

$$\begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} = \mathbf{M}_Y \mathbf{M}_R \mathbf{M}_P \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The matrix $\mathbf{M}_Y$ is the same as in part (a). Identify the matrices $\mathbf{M}_R$ and $\mathbf{M}_P$.

- (c) Suppose the coordinates of a point are $(1, 1, 1)$ in the fixed coordinate system. Determine the coordinates of the point in the spacecraft system if the spacecraft performs a pitch, roll, and yaw in sequence through the angles $\alpha = 30^\circ$, $\beta = 45^\circ$, and $\gamma = 60^\circ$.

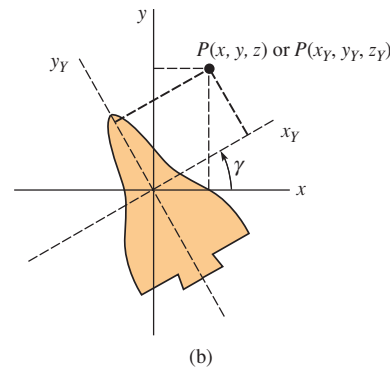
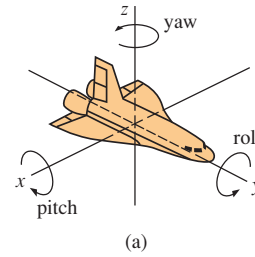


FIGURE 8.1.2 Spacecraft in Problem 51

52. **Project** (a) A matrix $\mathbf{A}$ can be **partitioned** into submatrices. For example the 3×5 and 5×2 matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 & 2 & 4 \\ 1 & 6 & 3 & -1 & 5 \\ 0 & 4 & 6 & -2 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 4 \\ 0 & 7 \\ -4 & 1 \\ -2 & -1 \\ 2 & 5 \end{pmatrix}$$

can be written

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where A_{11} is the upper left-hand block, or submatrix, indicated in blue in $\mathbf{A}$, A_{12} is the upper right-hand block, and so on. Compute the product $\mathbf{AB}$ using the partitioned matrices.

- (b) Investigate how partitioned matrices can be useful when using a computer to perform matrix calculations involving large matrices.

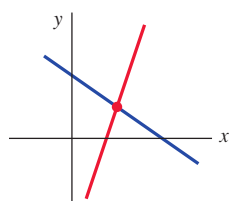
8.2 Systems of Linear Algebraic Equations

INTRODUCTION Recall, any equation of the form $ax + by = c$, where a , b , and c are real numbers is said to be a **linear equation** in the variables x and y . The graph of a linear equation in two variables is a straight line. For real numbers a , b , c , and d , $ax + by + cz = d$ is a linear equation in the variables x , y , and z and is the equation of a plane in 3-space. In general, an equation of the form

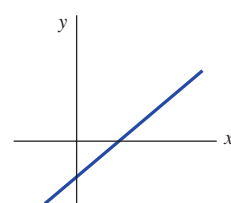
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_n,$$

where $a_1, a_2, \dots, a_n$, and b_n are real numbers, is a linear equation in the n variables $x_1, x_2, \dots, x_n$.

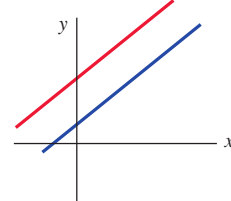
In this section we will study **systems of linear equations**. Systems of linear equations are also called **linear systems**.



(a) Consistent



(b) Consistent



(c) Inconsistent

FIGURE 8.2.1 A linear system of two equations in two variables interpreted as lines in 2-space

General Form A system of m linear equations in n variables, or unknowns, has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \quad (1)$$

The **coefficients** of the variables in the linear system (1) can be abbreviated as a_{ij} , where i denotes the row and j denotes the column in which the coefficient appears. For example, a_{23} is the coefficient of the unknown in the second row and third column (that is, x_3). Thus, $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$. The numbers $b_1, b_2, \dots, b_m$ are called the **constants** of the system. If all the constants are zero, the system (1) is said to be **homogeneous**; otherwise it is **nonhomogeneous**. For example,

this system is homogeneous

$$\begin{aligned} 5x_1 - 9x_2 + x_3 &= 0 \\ x_1 + 3x_2 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \end{aligned}$$

this system is nonhomogeneous

$$\begin{aligned} 2x_1 + 5x_2 + 6x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 9. \end{aligned}$$

Solution A **solution** of a linear system (1) is a set of n numbers $x_1, x_2, \dots, x_n$ that satisfies each equation in the system. For example, $x_1 = 3, x_2 = -1$ is a solution of the system

$$\begin{aligned} 3x_1 + 6x_2 &= 3 \\ x_1 - 4x_2 &= 7. \end{aligned}$$

To see this we replace x_1 by 3 and x_2 by -1 in each equation:

$$3(3) + 6(-1) = 9 - 6 = 3 \quad \text{and} \quad 3 - 4(-1) = 3 + 4 = 7.$$

Solutions of linear systems are also written as an ordered n -tuple $(x_1, x_2, \dots, x_n)$. The solution for the above system is then the ordered pair $(3, -1)$.

A linear system of equations is said to be **consistent** if it has at least one solution, and **inconsistent** if it has no solutions. If a linear system is consistent, it has either

- a unique solution (that is, precisely one solution), or
- infinitely many solutions.

Thus, a system of linear equations cannot have, say, exactly three solutions.

For a linear system with two equations and two variables, the lines in the plane, or 2-space, intersect at one point as in **FIGURE 8.2.1(a)** (unique solution), are identical as in Figure 8.2.1(b) (infinitely many solutions), or are parallel as in Figure 8.2.1(c) (no solutions).

For a linear system with three equations and three variables, each equation in the system represents a plane in 3-space. **FIGURE 8.2.2** shows some of the many ways that this kind of linear system can be interpreted.

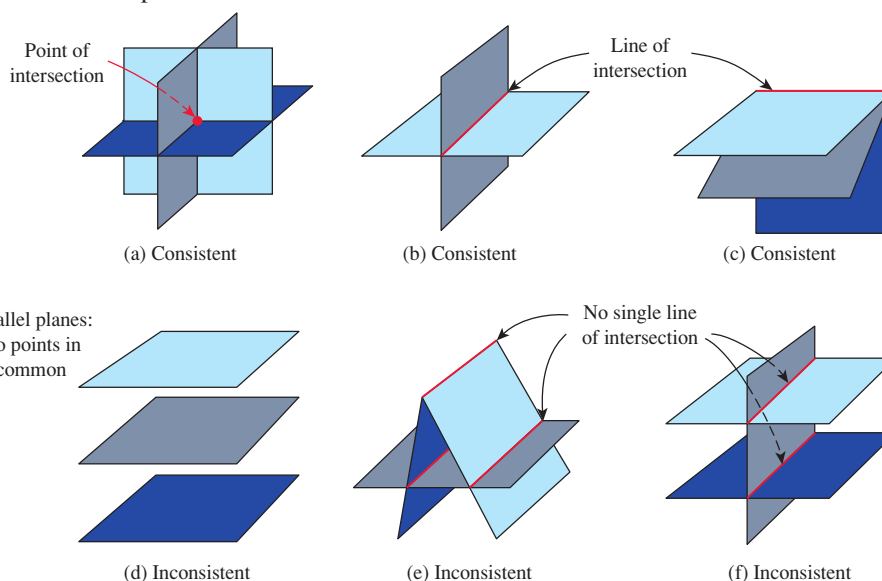


FIGURE 8.2.2 A linear system of three equations in three variables interpreted as planes in 3-space

EXAMPLE 1 Verification of a Solution

Verify that $x_1 = 14 + 7t$, $x_2 = 9 + 6t$, $x_3 = t$, where t is any real number, is a solution of the system

$$2x_1 - 3x_2 + 4x_3 = 1$$

$$x_1 - x_2 - x_3 = 5.$$

SOLUTION Replacing x_1 , x_2 , and x_3 in turn by $14 + 7t$, $9 + 6t$, and t , we have

$$2(14 + 7t) - 3(9 + 6t) + 4t = 1$$

$$14 + 7t - (9 + 6t) - t = 5.$$

For each real number t we obtain a different solution of the system; in other words, the system has an infinite number of solutions. For instance, $t = 0$, $t = 4$, and $t = -2$ give the three solutions

$$x_1 = 14, \quad x_2 = 9, \quad x_3 = 0,$$

$$x_1 = 42, \quad x_2 = 33, \quad x_3 = 4,$$

and

$$x_1 = 0, \quad x_2 = -3, \quad x_3 = -2,$$

respectively. Geometrically, each equation in the system represents a plane in R^3 . In this case, the planes intersect in a line as shown in Figure 8.2.2(b). Parametric equations of the line are $x_1 = 14 + 7t$, $x_2 = 9 + 6t$, $x_3 = t$. The solution can also be written as the ordered triple (x_1, x_2, x_3) or $(14 + 7t, 9 + 6t, t)$. $\equiv$

Solving Systems We can transform a system of linear equations into an equivalent system (that is, one having the same solutions) using the following **elementary operations**:

- (i) Multiply an equation by a nonzero constant.
- (ii) Interchange the positions of equations in the system.
- (iii) Add a nonzero multiple of one equation to any other equation.

As the next example will show, these elementary operations enable us to systematically eliminate variables from the equations of the system.

EXAMPLE 2 Solving a Linear System

Solve

$$2x_1 + 6x_2 + x_3 = 7$$

$$x_1 + 2x_2 - x_3 = -1$$

$$5x_1 + 7x_2 - 4x_3 = 9.$$

SOLUTION We begin by interchanging the first and second rows:

$$x_1 + 2x_2 - x_3 = -1$$

$$2x_1 + 6x_2 + x_3 = 7$$

$$5x_1 + 7x_2 - 4x_3 = 9.$$

Our goal now is to eliminate x_1 from the second and third equations. If we add to the second equation -2 times the first equation, we obtain the equivalent system

$$x_1 + 2x_2 - x_3 = -1$$

$$2x_2 + 3x_3 = 9$$

$$5x_1 + 7x_2 - 4x_3 = 9.$$

By adding to the third equation -5 times the first equation, we get a new equivalent system:

$$\begin{aligned}x_1 + 2x_2 - x_3 &= -1 \\2x_2 + 3x_3 &= 9 \\-3x_2 + x_3 &= 14.\end{aligned}$$

We are now going to use the second equation to eliminate the variable x_2 from the first and third equations. To simplify matters, let us multiply the second equation by $\frac{1}{2}$:

$$\begin{aligned}x_1 + 2x_2 - x_3 &= -1 \\x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\-3x_2 + x_3 &= 14.\end{aligned}$$

Adding to the first equation -2 times the second equation yields

$$\begin{aligned}x_1 - 4x_3 &= -10 \\x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\-3x_2 + x_3 &= 14.\end{aligned}$$

Next, by adding 3 times the second equation to the third equation we get

$$\begin{aligned}x_1 - 4x_3 &= -10 \\x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\\frac{11}{2}x_3 &= \frac{55}{2}.\end{aligned}$$

We shall use the last equation to eliminate the variable x_3 from the first and second equations. To this end, we multiply the third equation by $\frac{2}{11}$:

$$\begin{aligned}x_1 - 4x_3 &= -10 \\x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\x_3 &= 5.\end{aligned}$$

At this point we could use **back-substitution**; that is, substitute the value $x_3 = 5$ back into the remaining equations to determine x_1 and x_2 . However, by continuing with our systematic elimination, we add to the second equation $-\frac{3}{2}$ times the third equation:

$$\begin{aligned}x_1 - 4x_3 &= -10 \\x_2 &= -3 \\x_3 &= 5.\end{aligned}$$

Finally, by adding to the first equation 4 times the third equation, we obtain

$$\begin{aligned}x_1 &= 10 \\x_2 &= -3 \\x_3 &= 5.\end{aligned}$$

It is now apparent that $x_1 = 10, x_2 = -3, x_3 = 5$ is the solution of the original system. The answer written as the ordered triple $(10, -3, 5)$ means that the planes represented by the three equations in the system intersect at a point as in Figure 8.2.2(a). $\equiv$

Augmented Matrix Reflecting on the solution of the linear system in Example 2 should convince you that the solution of the system does not depend on what symbols are used as variables. Thus, the systems

$$\begin{array}{rcl} 2x + 6y + z & = & 7 \\ x + 2y - z & = & -1 \\ 5x + 7y - 4z & = & 9 \end{array} \quad \text{and} \quad \begin{array}{rcl} 2u + 6v + w & = & 7 \\ u + 2v - w & = & -1 \\ 5u + 7v - 4w & = & 9 \end{array}$$

have the same solution as the system in Example 2. In other words, in the solution of a linear system, the symbols used to denote the variables are immaterial; it is the coefficients of the variables and the constants that determine the solution of the system. In fact, we can solve a system of form (1) by dropping the variables entirely and performing operations on the rows of the array of coefficients and constants:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right). \quad (2)$$

This array is called the **augmented matrix** of the system or simply the **matrix of the system** (1).

EXAMPLE 3 Augmented Matrices

(a) The augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 2 \\ 4 & 7 & -1 & 8 \end{array} \right)$ represents the linear system

$$\begin{array}{rcl} x_1 - 3x_2 + 5x_3 & = & 2 \\ 4x_1 + 7x_2 - x_3 & = & 8. \end{array}$$

(b) The linear system

$$\begin{array}{rcl} x_1 - 5x_3 & = & -1 \\ 2x_1 + 8x_2 & = & 7 \\ x_2 + 9x_3 & = & 1 \end{array} \quad \text{is the same as} \quad \begin{array}{rcl} x_1 + 0x_2 - 5x_3 & = & -1 \\ 2x_1 + 8x_2 + 0x_3 & = & 7 \\ 0x_1 + x_2 + 9x_3 & = & 1. \end{array}$$

Thus the matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 2 & 8 & 0 & 7 \\ 0 & 1 & 9 & 1 \end{array} \right). \quad \equiv$$

Elementary Row Operations Since the rows of an augmented matrix represent the equations in a linear system, the three elementary operations on a linear system listed previously are equivalent to the following **elementary row operations** on a matrix:

- (i) Multiply a row by a nonzero constant.
- (ii) Interchange any two rows.
- (iii) Add a nonzero multiple of one row to any other row.

Of course, when we add a multiple of one row to another, we add the corresponding entries in the rows. We say that two matrices are **row equivalent** if one can be obtained from the other through a sequence of elementary row operations. The procedure of carrying out elementary row operations on a matrix to obtain a row-equivalent matrix is called **row reduction**.

Elimination Methods To solve a system such as (1) using an augmented matrix, we shall use either **Gaussian elimination** or the **Gauss–Jordan elimination method**. In the former method, we row-reduce the augmented matrix of the system until we arrive at a row-equivalent augmented matrix in **row-echelon form**:

- (i) The first nonzero entry in a nonzero row is a 1.
- (ii) In consecutive nonzero rows, the first entry 1 in the lower row appears to the right of the 1 in the higher row.
- (iii) Rows consisting of all zeros are at the bottom of the matrix.

In the Gauss–Jordan method, the row operations are continued until we obtain an augmented matrix that is in **reduced row-echelon form**. A reduced row-echelon matrix has the same three properties listed previously, but in addition:

- (iv) A column containing a first entry 1 has zeros everywhere else.

EXAMPLE 4 Echelon Forms

(a) The augmented matrices

$$\left(\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

are in row-echelon form. The reader should verify that the three criteria for this form are satisfied.

(b) The augmented matrices

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

are in reduced row-echelon form. Note that the remaining entries in the columns that contain a leading entry 1 are all zeros. ≡

It should be noted that in Gaussian elimination, we stop when we have obtained *an* augmented matrix in row-echelon form. In other words, by using different sequences of row operations, we may arrive at different row-echelon forms. This method then requires the use of back-substitution. In Gauss–Jordan elimination, we stop when we have obtained *the* augmented matrix in reduced row-echelon form. Any sequence of row operations will lead to the same augmented matrix in reduced row-echelon form. This method does not require back-substitution; the solution of the system will be apparent by inspection of the final matrix. In terms of the equations of the original system, our goal in both methods is simply to make the coefficient of x_1 in the first equation* equal to one and then use multiples of that equation to eliminate x_1 from other equations. The process is repeated for the other variables.

To keep track of the row operations used on an augmented matrix, we shall utilize the following notation:

Symbol	Meaning
$R_i \leftrightarrow R_j$	Interchange rows i and j
cR_i	Multiply the i th row by the nonzero constant c
$cR_i + R_j$	Multiply the i th row by c and add to the j th row

Note: Row operations can lead to different row-echelon forms.

*We can always interchange equations so that the first equation contains the variable x_1 .

EXAMPLE 5 Elimination Methods and Augmented Matrices

Solve the linear system in Example 2 using (a) Gaussian elimination and (b) Gauss–Jordan elimination.

SOLUTION (a) Using row operations on the augmented matrix of the system, we obtain:

$$\begin{aligned}
 & \begin{pmatrix} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{pmatrix} \\
 & \xrightarrow{\substack{-2R_1 + R_2 \\ -5R_1 + R_3}} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & -3 & 1 & 14 \end{pmatrix} \\
 & \xrightarrow{3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{11}{2} & \frac{55}{2} \end{pmatrix} \xrightarrow{\frac{2}{11}R_3} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{pmatrix}.
 \end{aligned}$$

The last matrix is in row-echelon form and represents the system

$$x_1 + 2x_2 - x_3 = -1$$

$$x_2 + \frac{3}{2}x_3 = \frac{9}{2}$$

$$x_3 = 5.$$

Substituting $x_3 = 5$ into the second equation gives $x_2 = -3$. Substituting both these values back into the first equation finally yields $x_1 = 10$.

(b) We start with the last matrix above. Since the first entries in the second and third rows are ones, we must, in turn, make the remaining entries in the second and third columns zeros:

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{pmatrix} \xrightarrow{-2R_2 + R_1} \begin{pmatrix} 1 & 0 & -4 & -10 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{pmatrix} \xrightarrow{\substack{-4R_3 + R_1 \\ -\frac{3}{2}R_3 + R_2}} \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

The last matrix is in reduced row-echelon form. Bearing in mind what the matrix means in terms of equations, we see that the solution of the system is $x_1 = 10$, $x_2 = -3$, $x_3 = 5$. $\equiv$

EXAMPLE 6 Gauss–Jordan Elimination

Use Gauss–Jordan elimination to solve

$$x_1 + 3x_2 - 2x_3 = -7$$

$$4x_1 + x_2 + 3x_3 = 5$$

$$2x_1 - 5x_2 + 7x_3 = 19.$$

SOLUTION Row operations give

$$\begin{aligned}
 & \begin{pmatrix} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{pmatrix} \xrightarrow{\substack{-4R_1 + R_2 \\ -2R_1 + R_3}} \begin{pmatrix} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{pmatrix} \\
 & \xrightarrow{-\frac{1}{11}R_2} \begin{pmatrix} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \end{pmatrix} \xrightarrow{\substack{-3R_2 + R_1 \\ -R_2 + R_3}} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

In this case, the last matrix in reduced row-echelon form implies that the original system of three equations in three variables is really equivalent to two equations in the variables.

Since only x_3 is common to both equations (the nonzero rows), we can assign its values arbitrarily. If we let $x_3 = t$, where t represents any real number, then we see that the system has infinitely many solutions: $x_1 = 2 - t$, $x_2 = -3 + t$, $x_3 = t$. Geometrically, these equations are the parametric equations for the line of intersection of the planes $x_1 + 0x_2 + x_3 = 2$ and $0x_1 + x_2 - x_3 = -3$. $\equiv$

EXAMPLE 7 Inconsistent System

Solve

$$\begin{aligned}x_1 + x_2 &= 1 \\4x_1 - x_2 &= -6 \\2x_1 - 3x_2 &= 8.\end{aligned}$$

SOLUTION In the process of applying Gauss–Jordan elimination to the matrix of the system, we stop at

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array} \right) \xRightarrow{\text{row operations}} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array} \right).$$

The third row of the last matrix means $0x_1 + 0x_2 = 16$ (or $0 = 16$). Since no numbers x_1 and x_2 can satisfy this equation, we conclude that the system has **no solution**. $\equiv$

Worth remembering. $\blacktriangleright$

Inconsistent systems of m linear equations in n variables will always yield the situation illustrated in Example 7; that is, there will be a row in the reduced row-echelon form of the augmented matrix in which the first n entries are zero and the $(n + 1)$ st entry is nonzero.

Networks The currents in the branches of an electrical network can be determined by using Kirchhoff's point and loop rules:

Point rule: The algebraic sum of the currents toward any branch point is 0.

Loop rule: The algebraic sum of the potential differences in any loop is 0.

When a loop is traversed in a chosen direction (clockwise or counterclockwise), an emf is taken to be *positive* when it is traversed from $-$ to $+$ and *negative* when traversed from $+$ to $-$. An iR product is taken to be *positive* if the chosen direction through the resistor is opposite that of the assumed current, and *negative* if the chosen direction is in the same direction as the assumed current.

In **FIGURE 8.2.3**, the branch points of the network are labeled A and B , the loops are labeled L_1 and L_2 , and the chosen direction in each loop is clockwise. Now, applying the foregoing rules to the network yields the nonhomogeneous system of linear equations

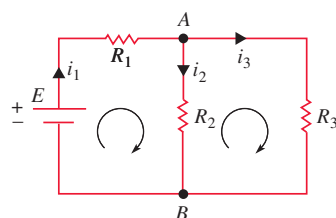


FIGURE 8.2.3 Electrical network

$$\begin{aligned}i_1 - i_2 - i_3 &= 0 & i_1 - i_2 - i_3 &= 0 \\E - i_1R_1 - i_2R_2 &= 0 & \text{or} & i_1R_1 + i_2R_2 &= E \\i_2R_2 - i_3R_3 &= 0 & i_2R_2 - i_3R_3 &= 0.\end{aligned} \quad (3)$$

EXAMPLE 8 Currents in a Network

Use Gauss–Jordan elimination to solve the system (3) when $R_1 = 10$ ohms, $R_2 = 20$ ohms, $R_3 = 10$ ohms, and $E = 12$ volts.

SOLUTION The system to be solved is

$$\begin{aligned}i_1 - i_2 - i_3 &= 0 \\10i_1 + 20i_2 &= 12 \\20i_2 - 10i_3 &= 0.\end{aligned}$$

In this case, Gauss–Jordan elimination yields

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 10 & 20 & 0 & 12 \\ 0 & 20 & -10 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{18}{25} \\ 0 & 1 & 0 & \frac{6}{25} \\ 0 & 0 & 1 & \frac{12}{25} \end{array} \right).$$

Hence, we see that the currents in the three branches are $i_1 = \frac{18}{25} = 0.72$ ampere, $i_2 = \frac{6}{25} = 0.24$ ampere, and $i_3 = \frac{12}{25} = 0.48$ ampere. $\equiv$

Homogeneous Systems All the systems in the preceding examples are nonhomogeneous systems. As we have seen, a nonhomogeneous system can be consistent or inconsistent. By contrast, a homogeneous system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \quad (4)$$

is *always consistent*, since $x_1 = 0, x_2 = 0, \dots, x_n = 0$ will satisfy each equation in the system. The solution consisting of all zeros is called the **trivial solution**. But naturally we are interested in whether a system of form (4) has any solutions for which *some* of the $x_i, i = 1, 2, \dots, n$, are not zero. Such a solution is called a **nontrivial solution**. A homogeneous system either possesses only the trivial solution or possesses the trivial solution along with infinitely many nontrivial solutions. The next theorem, presented without proof, will give us a sufficient condition for the existence of nontrivial solutions.

Theorem 8.2.1 Existence of Nontrivial Solutions

A homogeneous system of form (4) possesses nontrivial solutions if the number m of equations is less than the number n of variables ($m < n$).

EXAMPLE 9 Solving a Homogeneous System

Solve
$$\begin{aligned} 2x_1 - 4x_2 + 3x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0. \end{aligned}$$

SOLUTION Since the number of equations is less than the number of variables, we know from Theorem 8.2.1 that the given system has nontrivial solutions. Using Gauss–Jordan elimination, we find

$$\begin{aligned} \left(\begin{array}{ccc|c} 2 & -4 & 3 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) &\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -4 & 3 & 0 \end{array} \right) \xrightarrow{-2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -6 & 7 & 0 \end{array} \right) \\ &\xrightarrow{-\frac{1}{6}R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -\frac{7}{6} & 0 \end{array} \right) \xrightarrow{-R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{6} & 0 \\ 0 & 1 & -\frac{7}{6} & 0 \end{array} \right). \end{aligned}$$

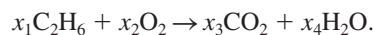
As in Example 6, if $x_3 = t$, then the solution of the system is $x_1 = \frac{5}{6}t, x_2 = \frac{7}{6}t, x_3 = t$. Note we obtain the trivial solution $x_1 = 0, x_2 = 0, x_3 = 0$ for this system by choosing $t = 0$. For $t \neq 0$ we get nontrivial solutions. For example, the solutions corresponding to $t = 6, t = -12$, and $t = 3$ are, in turn, $x_1 = 5, x_2 = 7, x_3 = 6; x_1 = -10, x_2 = -14, x_3 = -12$; and $x_1 = \frac{5}{2}, x_2 = \frac{7}{2}, x_3 = 3$. $\equiv$

Chemical Equations The next example will give an application of homogeneous systems in chemistry.

EXAMPLE 10 Balancing a Chemical Equation

Balance the chemical equation $\text{C}_2\text{H}_6 + \text{O}_2 \rightarrow \text{CO}_2 + \text{H}_2\text{O}$.

SOLUTION We want to find positive integers x_1, x_2, x_3 , and x_4 so that



Because the number of atoms of each element must be the same on each side of the last equation, we have

$$\text{carbon (C): } 2x_1 = x_3 \qquad 2x_1 + 0x_2 - x_3 + 0x_4 = 0$$

$$\text{hydrogen (H): } 6x_1 = 2x_4 \qquad \text{or} \qquad 6x_1 + 0x_2 + 0x_3 - 2x_4 = 0$$

$$\text{oxygen (O): } 2x_2 = 2x_3 + x_4 \qquad 0x_1 + 2x_2 - 2x_3 - x_4 = 0.$$

Gauss–Jordan elimination gives

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array} \right) \xRightarrow{\text{row operations}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{7}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array} \right)$$

and so $x_1 = \frac{1}{3}t$, $x_2 = \frac{7}{6}t$, $x_3 = \frac{2}{3}t$, $x_4 = t$. In this case t must be a positive integer chosen in such a manner so that x_1, x_2 , and x_3 are positive integers. To accomplish this we pick $t = 6$. This gives $x_1 = 2$, $x_2 = 7$, $x_3 = 4$, $x_4 = 6$. The balanced chemical equation is then



Notation In view of matrix multiplication and equality of matrices defined in Section 8.1, if we define matrices **A**, **X**, and **B** as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

then observe that the linear system (1) can be written as the matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or more compactly as

$$\mathbf{AX} = \mathbf{B},$$

where the matrix **A** is, naturally, called the **coefficient matrix**. The augmented matrix of a system $\mathbf{AX} = \mathbf{B}$ is often denoted by $(\mathbf{A}|\mathbf{B})$. A homogeneous system of equations is written $\mathbf{AX} = \mathbf{0}$. This notation makes it easier to discuss linear systems and to prove theorems for such systems.

Theorem 8.2.2 Two Properties of Homogeneous Systems

Let $\mathbf{AX} = \mathbf{0}$ denote a homogeneous system of linear equations.

- (i) If $\mathbf{X}_1$ is a solution of $\mathbf{AX} = \mathbf{0}$, then so is $c\mathbf{X}_1$ for any constant c .
- (ii) If $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of $\mathbf{AX} = \mathbf{0}$, then so is $\mathbf{X}_1 + \mathbf{X}_2$.

PROOF: (i) Because $\mathbf{X}_1$ is a solution, then $\mathbf{AX}_1 = \mathbf{0}$. Now

$$\mathbf{A}(c\mathbf{X}_1) = c(\mathbf{AX}_1) = c\mathbf{0} = \mathbf{0}.$$

This shows that a constant multiple of a solution of a homogeneous linear system is also a solution.

(ii) Because $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions, then $\mathbf{AX}_1 = \mathbf{0}$ and $\mathbf{AX}_2 = \mathbf{0}$. Now

$$\mathbf{A}(\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{AX}_1 + \mathbf{AX}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

This shows that $\mathbf{X}_1 + \mathbf{X}_2$ is a solution. ≡

By combining parts (i) and (ii) of Theorem 8.2.2 we can say that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of $\mathbf{AX} = \mathbf{0}$, then so is the linear combination

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2,$$

where c_1 and c_2 are constants. Moreover, this **superposition principle** extends to three or more solutions of $\mathbf{AX} = \mathbf{0}$.

EXAMPLE 11 Example 9 Revisited

At the end of Example 9, we obtained three distinct solutions

$$\mathbf{X}_1 = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} -10 \\ -14 \\ -12 \end{pmatrix}, \quad \text{and} \quad \mathbf{X}_3 = \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 3 \end{pmatrix}$$

of the given homogeneous linear system. It follows from the preceding discussion that

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -10 \\ -14 \\ -12 \end{pmatrix} + \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \\ -3 \end{pmatrix}$$

is also a solution of the system. Verify this. ≡

Terminology Suppose a linear system has m equations and n variables. If there are more equations than variables, that is, $m > n$, then the system is said to be **overdetermined**. If the system has fewer equations than variables, that is, $m < n$, then the system is called **underdetermined**. An overdetermined linear system may put too many constraints on the variables and so is usually—not always—inconsistent. The system in Example 7 is overdetermined and is inconsistent. On the other hand an underdetermined system is usually—but not always—consistent. The systems in Examples 9 and 10 are underdetermined and are consistent. It should be noted that it is impossible for a consistent underdetermined system to possess a single or unique solution. To see this, suppose that $m < n$. If Gaussian elimination is used to solve the system, then the row-echelon form that is row equivalent to the matrix of the system will contain r nonzero rows where $r \leq m < n$. Thus we can solve for r of the variables in terms of $n - r > 0$ variables or parameters. If the underdetermined system is consistent, then those remaining $n - r$ variables can be chosen arbitrarily and so the system has an infinite number of solutions.

REMARKS

(i) To solve large systems of linear equations, we obviously need the help of a computer. Since the two methods in this section are so systematic, they can be programmed easily. However, the requirement that each nonzero row start with a one may at times demand division by a very small number. Problems can then occur. Large systems are often solved indirectly—that is, by an approximation technique such as **Gauss–Seidel iteration**. See Section 16.1.

(ii) Since Gauss–Jordan elimination avoids the necessity of back-substitution, it would appear that it is the more efficient of the two methods we have considered. Actually, this is not the case. It has been shown that for large systems, Gauss–Jordan elimination can require about 50% more operations than Gaussian elimination.

(iii) A consistent nonhomogeneous linear system $\mathbf{AX} = \mathbf{B}$, $\mathbf{B} \neq \mathbf{0}$, shares a property with nonhomogeneous linear differential equations. If $\mathbf{X}_h$ is a solution of the associated homogeneous system $\mathbf{AX} = \mathbf{0}$, and $\mathbf{X}_p$ is a particular solution of the nonhomogeneous system $\mathbf{AX} = \mathbf{B}$, then the superposition $\mathbf{X}_h + \mathbf{X}_p$ is also a solution of the nonhomogeneous system. This is readily verified by the distributive law of matrix multiplication:

$$\mathbf{A}(\mathbf{X}_h + \mathbf{X}_p) = \mathbf{AX}_h + \mathbf{AX}_p = \mathbf{0} + \mathbf{B} = \mathbf{B}.$$

(iv) The set S of all solution vectors $\mathbf{X}$ of a homogeneous linear system with m equations and n variables,

$$S = \{\mathbf{X} \mid \mathbf{AX} = \mathbf{0}\}$$

is called the **null-space** of the matrix $\mathbf{A}$. If you have studied Section 7.6, then S is a *subspace* of the vector space R^n . See pages 354 and 355. This fact follows immediately from Theorems 7.6.1 and 8.2.2.

8.2

Exercises

Answers to selected odd-numbered problems begin on page ANS-16.

In Problems 1–20, use either Gaussian elimination or Gauss–Jordan elimination to solve the given system or show that no solution exists.

1. $x_1 - x_2 = 11$
 $4x_1 + 3x_2 = -5$
2. $3x_1 - 2x_2 = 4$
 $x_1 - x_2 = -2$
3. $9x_1 + 3x_2 = -5$
 $2x_1 + x_2 = -1$
4. $10x_1 + 15x_2 = 1$
 $3x_1 + 2x_2 = -1$
5. $x_1 - x_2 - x_3 = -3$
 $2x_1 + 3x_2 + 5x_3 = 7$
 $x_1 - 2x_2 + 3x_3 = -11$
6. $x_1 + 2x_2 - x_3 = 0$
 $2x_1 + x_2 + 2x_3 = 9$
 $x_1 - x_2 + x_3 = 3$
7. $x_1 + x_2 + x_3 = 0$
 $x_1 + x_2 + 3x_3 = 0$
8. $x_1 + 2x_2 - 4x_3 = 9$
 $5x_1 - x_2 + 2x_3 = 1$
9. $x_1 - x_2 - x_3 = 8$
 $x_1 - x_2 + x_3 = 3$
 $-x_1 + x_2 + x_3 = 4$
10. $3x_1 + x_2 = 4$
 $4x_1 + 3x_2 = -3$
 $2x_1 - x_2 = 11$
11. $2x_1 + 2x_2 = 0$
 $-2x_1 + x_2 + x_3 = 0$
 $3x_1 + x_3 = 0$
12. $x_1 - x_2 - 2x_3 = 0$
 $2x_1 + 4x_2 + 5x_3 = 0$
 $6x_1 - 3x_3 = 0$
13. $x_1 + 2x_2 + 2x_3 = 2$
 $x_1 + x_2 + x_3 = 0$
 $x_1 - 3x_2 - x_3 = 0$
14. $x_1 - 2x_2 + x_3 = 2$
 $3x_1 - x_2 + 2x_3 = 5$
 $2x_1 + x_2 + x_3 = 1$
15. $x_1 + x_2 + x_3 = 3$
 $x_1 - x_2 - x_3 = -1$
 $3x_1 + x_2 + x_3 = 5$
16. $x_1 - x_2 - 2x_3 = -1$
 $-3x_1 - 2x_2 + x_3 = -7$
 $2x_1 + 3x_2 + x_3 = 8$
17. $x_1 + x_3 - x_4 = 1$
 $2x_2 + x_3 + x_4 = 3$
 $x_1 - x_2 + x_4 = -1$
 $x_1 + x_2 + x_3 + x_4 = 2$
18. $2x_1 + x_2 + x_3 = 3$
 $3x_1 + x_2 + x_3 + x_4 = 4$
 $x_1 + 2x_2 + 2x_3 + 3x_4 = 3$
 $4x_1 + 5x_2 - 2x_3 + x_4 = 16$
19. $x_2 + x_3 - x_4 = 4$
 $x_1 + 3x_2 + 5x_3 - x_4 = 1$
 $x_1 + 2x_2 + 5x_3 - 4x_4 = -2$
 $x_1 + 4x_2 + 6x_3 - 2x_4 = 6$
20. $x_1 + 2x_2 + x_4 = 0$
 $4x_1 + 9x_2 + x_3 + 12x_4 = 0$
 $3x_1 + 9x_2 + 6x_3 + 21x_4 = 0$
 $x_1 + 3x_2 + x_3 + 9x_4 = 0$

In Problems 21 and 22, use a calculator to solve the given system.

21.
$$\begin{aligned} x_1 + x_2 + x_3 &= 4.280 \\ 0.2x_1 - 0.1x_2 - 0.5x_3 &= -1.978 \\ 4.1x_1 + 0.3x_2 + 0.12x_3 &= 1.686 \end{aligned}$$
22.
$$\begin{aligned} 2.5x_1 + 1.4x_2 + 4.5x_3 &= 2.6170 \\ 1.35x_1 + 0.95x_2 + 1.2x_3 &= 0.7545 \\ 2.7x_1 + 3.05x_2 - 1.44x_3 &= -1.4292 \end{aligned}$$

In Problems 23–28, use the procedures illustrated in Example 10 to balance the given chemical equation.

23. $\text{Na} + \text{H}_2\text{O} \rightarrow \text{NaOH} + \text{H}_2$
 24. $\text{KClO}_3 \rightarrow \text{KCl} + \text{O}_2$
 25. $\text{Fe}_3\text{O}_4 + \text{C} \rightarrow \text{Fe} + \text{CO}$
 26. $\text{C}_5\text{H}_8 + \text{O}_2 \rightarrow \text{CO}_2 + \text{H}_2\text{O}$
 27. $\text{Cu} + \text{HNO}_3 \rightarrow \text{Cu}(\text{NO}_3)_2 + \text{H}_2\text{O} + \text{NO}$
 28. $\text{Ca}_3(\text{PO}_4)_2 + \text{H}_3\text{PO}_4 \rightarrow \text{Ca}(\text{H}_2\text{PO}_4)_2$

In Problems 29 and 30, set up and solve the system of equations for the currents in the branches of the given network.

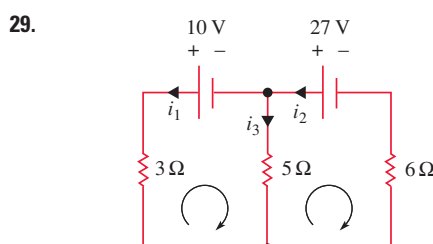


FIGURE 8.2.4 Network in Problem 29

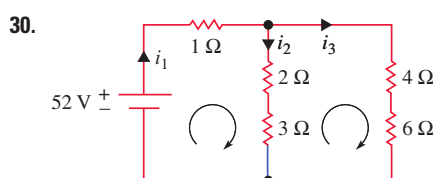


FIGURE 8.2.5 Network in Problem 30

In Problems 31 and 32, write the homogeneous system of linear equations in the form $\mathbf{AX} = \mathbf{0}$. Then verify by matrix multiplication that the given matrix $\mathbf{X}$ is a solution of the system for any real number c_1 .

31.
$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 5x_1 - 2x_2 + 2x_3 = 0 \\ 8x_1 + x_2 + 5x_3 = 0 \end{cases}, \quad \mathbf{X} = c_1 \begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}$$
32.
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ 3x_1 + x_2 + x_3 - x_4 = 0 \end{cases}, \quad \mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

In Problems 33 and 34, write the nonhomogeneous system of linear equations in the form $\mathbf{AX} = \mathbf{B}$. Then verify by matrix multiplication that the given matrix $\mathbf{X}$ is a solution of the system for any real numbers c_1 and c_2 .

33.
$$\begin{cases} 2x_1 - 3x_2 + x_3 = -12 \\ x_1 + x_2 - x_3 = 1 \\ 4x_1 - x_2 - x_3 = -10 \end{cases}, \quad \mathbf{X} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

34.
$$\begin{cases} x_1 + 2x_2 - 3x_3 = 2 \\ 2x_1 + 4x_2 - 6x_3 = 4 \\ 3x_1 + 6x_2 - 9x_3 = 6 \end{cases}, \quad \mathbf{X} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

An **elementary matrix** $\mathbf{E}$ is one obtained by performing a single row operation on the identity matrix $\mathbf{I}$. In Problems 35–39, verify that the given matrix is an elementary matrix.

35.
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
36.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$
37.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$$
38.
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If a matrix $\mathbf{A}$ is premultiplied by an elementary matrix $\mathbf{E}$, the product $\mathbf{EA}$ will be that matrix obtained from $\mathbf{A}$ by performing the elementary row operation symbolized by $\mathbf{E}$. In Problems 39–42, compute the given product for an arbitrary 3×3 matrix $\mathbf{A}$.

39.
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A}$$
40.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \mathbf{A}$$
41.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \mathbf{A}$$
42.
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \mathbf{A}$$

Computer Lab Assignments

In Problems 43–46, use a CAS to solve the given system.

43.
$$\begin{aligned} 1.567x_1 - 3.48x_2 + 5.22x_3 &= 1.045 \\ 3.56x_1 + 4.118x_2 + 1.57x_3 &= -1.625 \end{aligned}$$
44.
$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 0 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 - 6x_2 + 4x_3 &= 0 \\ 4x_1 + 14x_2 - 13x_3 &= 0 \end{aligned}$$
45.
$$\begin{aligned} 1.2x_1 + 3.5x_2 - 4.4x_3 + 3.1x_4 &= 1.8 \\ 0.2x_1 - 6.1x_2 - 2.3x_3 + 5.4x_4 &= -0.6 \\ 3.3x_1 - 3.5x_2 - 2.4x_3 - 0.1x_4 &= 2.5 \\ 5.2x_1 + 8.5x_2 - 4.4x_3 - 2.9x_4 &= 0 \end{aligned}$$
46.
$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 - x_5 &= 5 \\ 6x_1 + 9x_2 - 6x_3 + 17x_4 - x_5 &= 40 \\ 2x_1 + x_2 - 2x_3 + 5x_4 - x_5 &= 12 \\ x_1 + 2x_2 - x_3 + 3x_4 &= 7 \\ x_1 + 2x_2 + x_3 + 3x_4 &= 1 \end{aligned}$$

8.3 Rank of a Matrix

INTRODUCTION In a general $m \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

the rows

$$\mathbf{u}_1 = (a_{11} \ a_{12} \ \cdots \ a_{1n}), \quad \mathbf{u}_2 = (a_{21} \ a_{22} \ \cdots \ a_{2n}), \quad \dots, \quad \mathbf{u}_m = (a_{m1} \ a_{m2} \ \cdots \ a_{mn})$$

and columns

$$\mathbf{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

are called the **row vectors** of $\mathbf{A}$ and the **column vectors** of $\mathbf{A}$, respectively.

Definition As vectors, the set $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is either linearly independent or linearly dependent. We have the following definition.

Definition 8.3.1 Rank of a Matrix

The **rank** of an $m \times n$ matrix $\mathbf{A}$, denoted by $\text{rank}(\mathbf{A})$, is the maximum number of linearly independent row vectors in $\mathbf{A}$.

EXAMPLE 1 Rank of a 3×4 Matrix

Find the rank of the 3×4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}. \quad (1)$$

SOLUTION With $\mathbf{u}_1 = (1 \ 1 \ -1 \ 3)$, $\mathbf{u}_2 = (2 \ -2 \ 6 \ 8)$, and $\mathbf{u}_3 = (3 \ 5 \ -7 \ 8)$, we see that $4\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$. In view of Definition 7.6.3 and the discussion following it, we conclude that the set $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is linearly dependent. On the other hand, since neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ is a constant multiple of the other, the set of row vectors $\mathbf{u}_1, \mathbf{u}_2$ is linearly independent. Hence by Definition 8.3.1, $\text{rank}(\mathbf{A}) = 2$. $\equiv$

See page 352 in Section 7.6. 

Row Space In the terminology of the preceding chapter, the row vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of the matrix (1) are a set of vectors in the *vector space* R^4 . Since $R_{\mathbf{A}} = \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ (the set of all linear combinations of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$) is a subspace of R^4 , we are justified in calling $R_{\mathbf{A}}$ the **row space** of the matrix $\mathbf{A}$. Now the set of vectors $\mathbf{u}_1, \mathbf{u}_2$ is linearly independent and also spans $R_{\mathbf{A}}$; in other words, the set $\mathbf{u}_1, \mathbf{u}_2$ is a *basis* for $R_{\mathbf{A}}$. The dimension (the number of vectors in a basis) of the row space $R_{\mathbf{A}}$ is 2, which is $\text{rank}(\mathbf{A})$.

Rank by Row Reduction Example 1 notwithstanding, it is generally not easy to determine the rank of a matrix by inspection. Although there are several mechanical ways of finding $\text{rank}(\mathbf{A})$, we examine one way that uses the elementary row operations introduced in the preceding section. *Specifically, the rank of $\mathbf{A}$ can be found by row reducing $\mathbf{A}$ to a row-echelon matrix $\mathbf{B}$.* To understand this, first recall that an $m \times n$ matrix $\mathbf{B}$ is row equivalent to an $m \times n$ matrix $\mathbf{A}$ if the rows of $\mathbf{B}$ were obtained from the rows of $\mathbf{A}$ by applying elementary row operations. If we simply interchange two rows in $\mathbf{A}$ to obtain $\mathbf{B}$, then the row space $R_{\mathbf{A}}$ of $\mathbf{A}$ and the row space $R_{\mathbf{B}}$ of $\mathbf{B}$ are equal because the row vectors of $\mathbf{A}$ and $\mathbf{B}$ are the same. When the row vectors of $\mathbf{B}$ are linear combinations of the rows of $\mathbf{A}$, it follows

that the row vectors of $\mathbf{B}$ are in the row space $R_{\mathbf{A}}$, and so $R_{\mathbf{B}}$ is a subset of $R_{\mathbf{A}}$ (written $R_{\mathbf{B}} \subseteq R_{\mathbf{A}}$). Conversely, $\mathbf{A}$ is row equivalent to $\mathbf{B}$, since we can obtain $\mathbf{A}$ by applying row operations to $\mathbf{B}$. Hence the rows of $\mathbf{A}$ are linear combinations of the rows of $\mathbf{B}$, and so it follows that $R_{\mathbf{A}}$ is a subset of $R_{\mathbf{B}}$ ($R_{\mathbf{A}} \subseteq R_{\mathbf{B}}$). From $R_{\mathbf{B}} \subseteq R_{\mathbf{A}}$ and $R_{\mathbf{A}} \subseteq R_{\mathbf{B}}$, we conclude that $R_{\mathbf{A}} = R_{\mathbf{B}}$. Finally, if we row-reduce $\mathbf{A}$ into a row-echelon matrix $\mathbf{B}$, then the nonzero rows of $\mathbf{B}$ are linearly independent. (Why?) The nonzero rows of $\mathbf{B}$ form a basis for the row space $R_{\mathbf{A}}$, and so we have the result that $\text{rank}(\mathbf{A}) = \text{dimension of } R_{\mathbf{A}}$.

We summarize these conclusions in the next theorem.

Theorem 8.3.1 Rank of a Matrix by Row Reduction

If a matrix $\mathbf{A}$ is row equivalent to a row-echelon form $\mathbf{B}$, then

- (i) the row space of $\mathbf{A} =$ the row space of $\mathbf{B}$,
- (ii) the nonzero rows of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$, and
- (iii) $\text{rank}(\mathbf{A}) =$ the number of nonzero rows in $\mathbf{B}$.

EXAMPLE 2 Rank by Row Reduction—Example 1 Revisited

We row-reduce a matrix $\mathbf{A}$ to a row-echelon form $\mathbf{B}$ in exactly the same manner as we row-reduced the augmented matrix of a system of linear equations to an echelon form when we solved the system using Gaussian elimination. Using the matrix (1) in Example 1, elementary row operations give

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \xrightarrow[\begin{smallmatrix} -2R_1 + R_2 \\ -3R_1 + R_3 \end{smallmatrix}]{\Rightarrow} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \xrightarrow[\begin{smallmatrix} \frac{1}{2}R_2 + R_3 \\ -\frac{1}{4}R_2 \end{smallmatrix}]{\Rightarrow} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the last matrix is in row-echelon form, and since the last matrix has two nonzero rows, we conclude from (iii) of Theorem 8.3.1 that $\text{rank}(\mathbf{A}) = 2$. $\equiv$

EXAMPLE 3 Linear Independence/Dependence

Determine whether the set of vectors $\mathbf{u}_1 = \langle 2, 1, 1 \rangle$, $\mathbf{u}_2 = \langle 0, 3, 0 \rangle$, $\mathbf{u}_3 = \langle 3, 1, 2 \rangle$, in R^3 is linearly dependent or linearly independent.

SOLUTION It should be clear from the discussion above that if we form a matrix $\mathbf{A}$ with the given vectors as rows, and if we row-reduce $\mathbf{A}$ to a row-echelon form $\mathbf{B}$ with rank 3, then the set of vectors is linearly independent. If $\text{rank}(\mathbf{A}) < 3$, then the set of vectors is linearly dependent. In this case, it is easy to carry the row reduction all the way to a reduced row-echelon form:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow[\Rightarrow]{\text{row operations}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\text{rank}(\mathbf{A}) = 3$ and the set of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is linearly independent. $\equiv$

As mentioned previously, the vectors in the row-echelon form of a matrix $\mathbf{A}$ can serve as a basis for the row space of the matrix $\mathbf{A}$. In Example 3, we see that a basis for the row space of $\mathbf{A}$ is the standard basis $\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle$ of R^3 .

Rank and Linear Systems The concept of rank can be related back to solvability of linear systems of algebraic equations. Suppose $\mathbf{A}\mathbf{X} = \mathbf{B}$ is a linear system and that $(\mathbf{A}|\mathbf{B})$ denotes the augmented matrix of the system. In Example 7 of Section 8.2, we saw that the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned}$$

was inconsistent. The inconsistency of the system is seen in the fact that, after row reduction of the augmented matrix $(\mathbf{A}|\mathbf{B})$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 4 & -1 & -6 & 8 \\ 2 & -3 & 8 & 8 \end{array}\right) \xRightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 16 & 16 \end{array}\right) \xRightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad (2)$$

the last row of the reduced row-echelon form is nonzero. Of course, this reduction shows that $\text{rank}(\mathbf{A}|\mathbf{B}) = 3$. But note, too, that the result in (2) indicates that $\text{rank}(\mathbf{A}) = 2$ because

$$\left(\begin{array}{cc} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{array}\right) \xRightarrow{\text{row operations}} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right).$$

We have illustrated a special case of the next theorem.

Theorem 8.3.2 Consistency of $\mathbf{AX} = \mathbf{B}$

A linear system of equations $\mathbf{AX} = \mathbf{B}$ is consistent if and only if the rank of the coefficient matrix $\mathbf{A}$ is the same as the rank of the augmented matrix of the system $(\mathbf{A}|\mathbf{B})$.

In Example 6 of Section 8.2, we saw that the system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= -7 \\ 4x_1 + x_2 + 3x_3 &= 5 \\ 2x_1 - 5x_2 + 7x_3 &= 19 \end{aligned} \quad (3)$$

was consistent and had an infinite number of solutions. We solved for two of the variables, x_1 and x_2 , in terms of the remaining variable x_3 , which we relabeled as a parameter t . The number of parameters in a solution of a system is related to the rank of the coefficient matrix $\mathbf{A}$.

Theorem 8.3.3 Number of Parameters in a Solution

Suppose a linear system $\mathbf{AX} = \mathbf{B}$ with m equations and n variables is consistent. If the coefficient matrix $\mathbf{A}$ has rank r , then the solution of the system contains $n - r$ parameters.

For the system (3), we can see from the row reduction

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -7 & -7 \\ 4 & 1 & 3 & 5 & 5 \\ 2 & -5 & 7 & 19 & 19 \end{array}\right) \xRightarrow{\text{row operations}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{B}) = 2$, and so the system is consistent by Theorem 8.3.2. With $n = 3$, we see from Theorem 8.3.3 the number of parameters in the solution is $3 - 2 = 1$.

FIGURE 8.3.1 outlines the connection between the concept of rank of a matrix and the solution of a linear system.

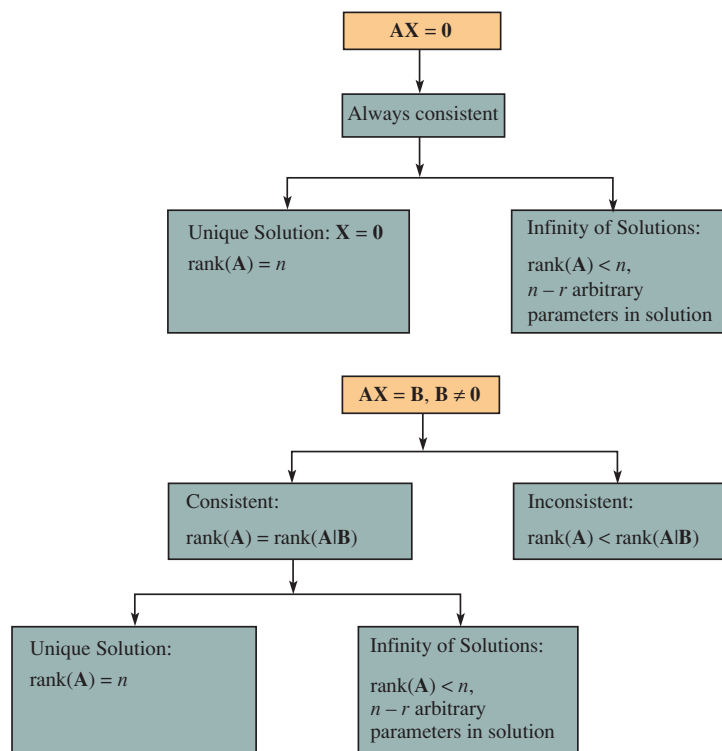


FIGURE 8.3.1 For m linear equations in n variables $\mathbf{AX} = \mathbf{B}$. Two cases: $\mathbf{B} = \mathbf{0}$, $\mathbf{B} \neq \mathbf{0}$. Let $\text{rank}(\mathbf{A}) = r$.

REMARKS

We have not mentioned the connection between the columns of a matrix $\mathbf{A}$ and the rank of $\mathbf{A}$. It turns out that the maximum number of independent columns that a matrix $\mathbf{A}$ can have must equal the maximum number of independent rows. In the terminology of vector spaces, the row space $R_{\mathbf{A}}$ of matrix $\mathbf{A}$ has the same dimension as its column space $C_{\mathbf{A}}$. For example, if we take the transpose of the matrix in (1) and reduce it to a row-echelon form:

$$\mathbf{A}^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ -1 & 6 & -7 \\ 3 & 8 & 8 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we see that the maximum number of rows of $\mathbf{A}^T$ is 2, and so the maximum number of linearly independent columns of $\mathbf{A}$ is 2.

8.3

Exercises

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–10, use (iii) of Theorem 8.3.1 to find the rank of the given matrix.

1. $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 4 \\ -1 & 0 & 3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 3 & -1 & 2 & 0 \\ 6 & 2 & 4 & 5 \end{pmatrix}$

7. $\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$

8. $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{pmatrix}$

$$9. \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{pmatrix}$$

$$10. \begin{bmatrix} 1 & -2 & 1 & 8 & -1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & -1 & 2 & 10 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 8 & -1 & 1 & 2 & 6 \end{bmatrix}$$

In Problems 11–14, determine whether the given set of vectors is linearly dependent or linearly independent.

11. $\mathbf{u}_1 = \langle 1, 2, 3 \rangle$, $\mathbf{u}_2 = \langle 1, 0, 1 \rangle$, $\mathbf{u}_3 = \langle 1, -1, 5 \rangle$
12. $\mathbf{u}_1 = \langle 2, 6, 3 \rangle$, $\mathbf{u}_2 = \langle 1, -1, 4 \rangle$, $\mathbf{u}_3 = \langle 3, 2, 1 \rangle$, $\mathbf{u}_4 = \langle 2, 5, 4 \rangle$
13. $\mathbf{u}_1 = \langle 1, -1, 3, -1 \rangle$, $\mathbf{u}_2 = \langle 1, -1, 4, 2 \rangle$, $\mathbf{u}_3 = \langle 1, -1, 5, 7 \rangle$
14. $\mathbf{u}_1 = \langle 2, 1, 1, 5 \rangle$, $\mathbf{u}_2 = \langle 2, 2, 1, 1 \rangle$, $\mathbf{u}_3 = \langle 3, -1, 6, 1 \rangle$, $\mathbf{u}_4 = \langle 1, 1, 1, -1 \rangle$
15. Suppose the system $\mathbf{A}\mathbf{X} = \mathbf{B}$ is consistent and $\mathbf{A}$ is a 5×8 matrix and $\text{rank}(\mathbf{A}) = 3$. How many parameters does the solution of the system have?
16. Let $\mathbf{A}$ be a nonzero 4×6 matrix.
 - (a) What is the maximum rank that $\mathbf{A}$ can have?
 - (b) If $\text{rank}(\mathbf{A}|\mathbf{B}) = 2$, then for what value(s) of $\text{rank}(\mathbf{A})$ is the system $\mathbf{A}\mathbf{X} = \mathbf{B}$, $\mathbf{B} \neq \mathbf{0}$, inconsistent? Consistent?
 - (c) If $\text{rank}(\mathbf{A}) = 3$, then how many parameters does the solution of the system $\mathbf{A}\mathbf{X} = \mathbf{0}$ have?
17. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the first, second, and third column vectors, respectively, of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 7 \\ 1 & 0 & 2 \\ -1 & 5 & 13 \end{pmatrix}.$$

What can we conclude about $\text{rank}(\mathbf{A})$ from the observation $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$? [Hint: Read the *Remarks* at the end of this section.]

Discussion Problems

18. Suppose the system $\mathbf{A}\mathbf{X} = \mathbf{B}$ is consistent and $\mathbf{A}$ is a 6×3 matrix. Suppose the maximum number of linearly independent rows in $\mathbf{A}$ is 3. Discuss: Is the solution of the system unique?

19. Suppose we wish to determine whether the set of column vectors

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 1 \\ 7 \\ -5 \\ 1 \end{pmatrix}$$

is linearly dependent or linearly independent. By Definition 7.6.3, if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0} \quad (4)$$

only for $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0$, then the set of vectors is linearly independent; otherwise the set is linearly dependent. But (4) is equivalent to the linear system

$$\begin{aligned} 4c_1 + c_2 - c_3 + 2c_4 + c_5 &= 0 \\ 3c_1 + 2c_2 + c_3 + 3c_4 + 7c_5 &= 0 \\ 2c_1 + 2c_2 + c_3 + 4c_4 - 5c_5 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 &= 0. \end{aligned}$$

Without doing any further work, explain why we can now conclude that the set of vectors is linearly dependent.

Computer Lab Assignment

20. A CAS can be used to row-reduce a matrix to a row-echelon form. Use a CAS to determine the ranks of the augmented matrix $(\mathbf{A}|\mathbf{B})$ and the coefficient matrix $\mathbf{A}$ for

$$\begin{aligned} x_1 + 2x_2 - 6x_3 + x_4 + x_5 + x_6 &= 2 \\ 5x_1 + 2x_2 - 2x_3 + 5x_4 + 4x_5 + 2x_6 &= 3 \\ 6x_1 + 2x_2 - 2x_3 + x_4 + x_5 + 3x_6 &= -1 \\ -x_1 + 2x_2 + 3x_3 + x_4 - x_5 + 6x_6 &= 0 \\ 9x_1 + 7x_2 - 2x_3 + x_4 + 4x_5 &= 5. \end{aligned}$$

Is the system consistent or inconsistent? If consistent, solve the system.

8.4 Determinants

INTRODUCTION Suppose $\mathbf{A}$ is an $n \times n$ matrix. Associated with $\mathbf{A}$ is a *number* called the **determinant of $\mathbf{A}$** and is denoted by $\det \mathbf{A}$. Symbolically, we distinguish a matrix $\mathbf{A}$ from the determinant of $\mathbf{A}$ by replacing the parentheses by vertical bars:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

A determinant of an $n \times n$ matrix is said to be a **determinant of order n** . We begin by defining the determinants of 1×1 , 2×2 , and 3×3 matrices.

Definition For a 1×1 matrix $\mathbf{A} = (a)$, we have $\det \mathbf{A} = |a| = a$. For example, if $\mathbf{A} = (-5)$, then $\det \mathbf{A} = |-5| = -5$. In this case the vertical bars $||$ around a number do *not* mean the absolute value of the number.

Definition 8.4.1 Determinant of a 2×2 Matrix

The determinant of $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the number

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \quad (1)$$

As a mnemonic, a determinant of order 2 is thought to be the product of the main diagonal entries of $\mathbf{A}$ minus the product of the other diagonal entries:

$$\begin{array}{ccc} \text{multiply} & \text{multiply} & \text{subtract} \\ & & \text{products} \\ & \searrow & \downarrow \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & = & a_{11}a_{22} - a_{12}a_{21}. \end{array} \quad (2)$$

For example, if $\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 5 & 9 \end{pmatrix}$, then $\det \mathbf{A} = \begin{vmatrix} 6 & -3 \\ 5 & 9 \end{vmatrix} = 6(9) - (-3)(5) = 69$.

Definition 8.4.2 Determinant of a 3×3 Matrix

The determinant of $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the number

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \quad (3)$$

The expression in (3) can be written in a more tractable form. By factoring, we have

$$\det \mathbf{A} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

But in view of (1), each term in parentheses is recognized as the determinant of a 2×2 matrix:

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \left(- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \right) + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \quad (4)$$

Observe that each determinant in (4) is a determinant of a *submatrix* of the matrix $\mathbf{A}$ and corresponds to its coefficient in the following manner: a_{11} is the coefficient of the determinant of the submatrix obtained by deleting the *first row and first column* of $\mathbf{A}$; a_{12} is the coefficient of the negative of the determinant of the submatrix obtained by deleting the *first row and second column* of $\mathbf{A}$; and finally, a_{13} is the coefficient of the determinant of the submatrix obtained by deleting the *first row and third column* of $\mathbf{A}$. In other words, the coefficients in (4) are simply the entries of the first row of $\mathbf{A}$. We say that $\det \mathbf{A}$ has been **expanded by cofactors along the first row**, with the cofactors of a_{11} , a_{12} , and a_{13} being the determinants

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Thus (4) is

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}. \quad (5)$$

In general, the **cofactor of a_{ij}** is the determinant

$$C_{ij} = (-1)^{i+j}M_{ij}, \quad (6)$$

where M_{ij} is the determinant of the submatrix obtained by deleting the i th row and the j th column of $\mathbf{A}$. The determinant M_{ij} is called a **minor determinant**. A cofactor is a *signed* minor determinant; that is, $C_{ij} = M_{ij}$ when $i + j$ is even and $C_{ij} = -M_{ij}$ when $i + j$ is odd.

A 3×3 matrix has nine cofactors:

$$\begin{array}{lll} C_{11} = M_{11} & C_{12} = -M_{12} & C_{13} = M_{13} \\ C_{21} = -M_{21} & C_{22} = M_{22} & C_{23} = -M_{23} \\ C_{31} = M_{31} & C_{32} = -M_{32} & C_{33} = M_{33}. \end{array}$$

Inspection of the above array shows that the sign factor $+1$ or -1 associated with a cofactor can be obtained from the checkerboard pattern:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \quad (7)$$

3×3 matrix

Now observe that (3) can be rearranged and factored again as

$$\begin{aligned} \det \mathbf{A} &= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) \\ &= a_{12} \left(- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \right) + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \left(- \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \right) \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}, \end{aligned} \quad (8)$$

which is the cofactor expansion of $\det \mathbf{A}$ along the second column. It is left as an exercise to show from (3) that $\det \mathbf{A}$ can also be expanded by cofactors along the third row:

$$\det \mathbf{A} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}. \quad (9)$$

We are, of course, suggesting in (5), (8), and (9) the following general result:

The determinant of a 3×3 matrix can be evaluated by expanding $\det \mathbf{A}$ by cofactors along any row or along any column.

EXAMPLE 1 Cofactor Expansion Along the First Row

Evaluate the determinant of $\mathbf{A} = \begin{pmatrix} 2 & 4 & 7 \\ 6 & 0 & 3 \\ 1 & 5 & 3 \end{pmatrix}$.

SOLUTION Using cofactor expansion along the first row gives

$$\det \mathbf{A} = \begin{vmatrix} 2 & 4 & 7 \\ 6 & 0 & 3 \\ 1 & 5 & 3 \end{vmatrix} = 2C_{11} + 4C_{12} + 7C_{13}.$$

Now, the cofactors of the entries in the first row of $\mathbf{A}$ are

$$C_{11} = (-1)^{1+1} \begin{vmatrix} \cancel{2} & \cancel{4} & \cancel{7} \\ 6 & 0 & 3 \\ 1 & 5 & 3 \end{vmatrix} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 5 & 3 \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} \cancel{2} & \cancel{4} & \cancel{7} \\ 6 & 0 & 3 \\ 1 & \cancel{5} & \cancel{3} \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} 6 & 3 \\ 1 & 3 \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} \cancel{2} & \cancel{4} & \cancel{7} \\ 6 & 0 & 3 \\ 1 & 5 & \cancel{3} \end{vmatrix} = (-1)^{1+3} \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix},$$

where the dashed lines indicate the row and column that are deleted. Thus,

$$\begin{aligned} \det \mathbf{A} &= 2(-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 5 & 3 \end{vmatrix} + 4(-1)^{1+2} \begin{vmatrix} 6 & 3 \\ 1 & 3 \end{vmatrix} + 7(-1)^{1+3} \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} \\ &= 2[0(3) - 3(5)] - 4[6(3) - 3(1)] + 7[6(5) - 0(1)] = 120. \end{aligned} \quad \equiv$$

If a matrix has a row (or a column) containing many zero entries, then wisdom dictates that we evaluate the determinant of the matrix using cofactor expansion along that row (or column). Thus, in Example 1, had we expanded the determinant of $\mathbf{A}$ using cofactors along, say, the second row, then

$$\begin{aligned} \det \mathbf{A} &= 6C_{21} + 0C_{22} + 3C_{23} = 6C_{21} + 3C_{23} \\ &= 6(-1)^{2+1} \begin{vmatrix} 4 & 7 \\ 5 & 3 \end{vmatrix} + 3(-1)^{2+3} \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} \\ &= -6(-23) - 3(6) = 120. \end{aligned}$$

EXAMPLE 2 Cofactor Expansion Along the Third Column

Evaluate the determinant of $\mathbf{A} = \begin{pmatrix} 6 & 5 & 0 \\ -1 & 8 & -7 \\ -2 & 4 & 0 \end{pmatrix}$.

SOLUTION Since there are two zeros in the third column, we expand by cofactors of that column:

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 6 & 5 & 0 \\ -1 & 8 & -7 \\ -2 & 4 & 0 \end{vmatrix} = 0C_{13} + (-7)C_{23} + 0C_{33} \\ &= (-7)(-1)^{2+3} \begin{vmatrix} \cancel{6} & \cancel{5} & \cancel{0} \\ -1 & \cancel{8} & \cancel{7} \\ -2 & 4 & \cancel{0} \end{vmatrix} = (-7)(-1)^{2+3} \begin{vmatrix} 6 & 5 \\ -2 & 4 \end{vmatrix} \\ &= 7[6(4) - 5(-2)] = 238. \end{aligned} \quad \equiv$$

Carrying the above ideas one step further, we can evaluate the determinant of a 4×4 matrix by multiplying the entries in a row (or column) by their corresponding cofactors and adding the

products. In this case, each cofactor is a signed minor determinant of an appropriate 3×3 submatrix. The following theorem, which we shall give without proof, states that the determinant of any $n \times n$ matrix $\mathbf{A}$ can be evaluated by means of cofactors.

Theorem 8.4.1 Cofactor Expansion of a Determinant

Let $\mathbf{A} = (a_{ij})_{n \times n}$ be an $n \times n$ matrix. For each $1 \leq i \leq n$, the cofactor expansion of $\det \mathbf{A}$ along the i th row is

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

For each $1 \leq j \leq n$, the cofactor expansion of $\det \mathbf{A}$ along the j th column is

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

The sign factor pattern for the cofactors illustrated in (7) extends to matrices of order greater than 3:

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array} \quad \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

4×4 matrix $n \times n$ matrix

EXAMPLE 3 Cofactor Expansion Along the Fourth Row

Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 & 4 \\ -1 & 0 & 2 & 3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{pmatrix}.$$

SOLUTION Since the matrix has two zero entries in its fourth row, we choose to expand $\det \mathbf{A}$ by cofactors along that row:

$$\det \mathbf{A} = \begin{vmatrix} 5 & 1 & 2 & 4 \\ -1 & 0 & 2 & 3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{vmatrix} = (1)C_{41} + 0C_{42} + 0C_{43} + (-4)C_{44}, \quad (10)$$

$$\text{where } C_{41} = (-1)^{4+1} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \\ 1 & 6 & 1 \end{vmatrix} \quad \text{and} \quad C_{44} = (-1)^{4+4} \begin{vmatrix} 5 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 6 \end{vmatrix}.$$

We then expand both these determinants by cofactors along the second row:

$$\begin{aligned} C_{41} &= (-1) \begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \\ 1 & 6 & 1 \end{vmatrix} = - \left(0(-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix} + 2(-1)^{2+2} \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 3(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} \right) \\ &= 18 \end{aligned}$$

$$C_{44} = \begin{vmatrix} 5 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 6 \end{vmatrix} = (-1)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} + 0(-1)^{2+2} \begin{vmatrix} 5 & 2 \\ 1 & 6 \end{vmatrix} + 2(-1)^{2+3} \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ = -4.$$

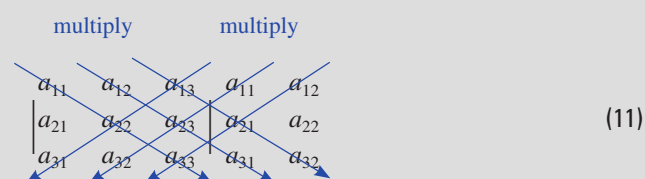
Therefore (10) becomes

$$\det \mathbf{A} = \begin{vmatrix} 5 & 1 & 2 & 4 \\ -1 & 0 & 2 & 3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{vmatrix} = (1)(18) + (-4)(-4) = 34.$$

You should verify this result by expanding $\det \mathbf{A}$ by cofactors along the second column. ≡

REMARKS

In previous mathematics courses you may have seen the following memory device, analogous to (2), for evaluating a determinant of order 3:



- (i) Add the products of the entries on the arrows that go from left to right.
- (ii) Subtract from the number in (i) the sum of the products of the entries on the arrows that go from right to left.

Note: Method illustrated in (11) does not work for determinants of order $n > 3$.

A word of caution is in order here. The memory device given in (11), though easily adapted to matrices larger than 3×3 , does not give the correct results. There are no mnemonic devices for evaluating the determinants of order 4 or greater.

8.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–4, suppose

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \\ -2 & 3 & 5 \end{pmatrix}.$$

Evaluate the indicated minor determinant or cofactor.

1. M_{12} 2. M_{32} 3. C_{13} 4. C_{22}

In Problems 5–8, suppose

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 4 & 0 \\ 1 & 2 & -2 & 3 \\ 5 & 1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Evaluate the indicated minor determinant or cofactor.

5. M_{33} 6. M_{41} 7. C_{34} 8. C_{23}

In Problems 9–14, evaluate the determinant of the given matrix.

9. (-7) 10. (2)

11. $\begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}$ 12. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{pmatrix}$

13. $\begin{pmatrix} 1 - \lambda & 3 \\ 2 & 2 - \lambda \end{pmatrix}$ 14. $\begin{pmatrix} -3 - \lambda & -4 \\ -2 & 5 - \lambda \end{pmatrix}$

In Problems 15–28, evaluate the determinant of the given matrix by cofactor expansion.

15. $\begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{pmatrix}$ 16. $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$17. \begin{pmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{pmatrix}$$

$$19. \begin{pmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$21. \begin{pmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{pmatrix}$$

$$23. \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{pmatrix}$$

$$18. \begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & 9 \end{pmatrix}$$

$$20. \begin{pmatrix} \frac{1}{4} & 6 & 0 \\ \frac{1}{3} & 8 & 0 \\ \frac{1}{2} & 9 & 0 \end{pmatrix}$$

$$22. \begin{pmatrix} 3 & 5 & 1 \\ -1 & 2 & 5 \\ 7 & -4 & 10 \end{pmatrix}$$

$$24. \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 2+x & 3+y & 4+z \end{pmatrix}$$

$$25. \begin{pmatrix} 1 & 1 & -3 & 0 \\ 1 & 5 & 3 & 2 \\ 1 & -2 & 1 & 0 \\ 4 & 8 & 0 & 0 \end{pmatrix}$$

$$26. \begin{pmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{pmatrix}$$

$$27. \begin{vmatrix} 3 & 2 & 0 & 1 & -1 \\ 0 & 1 & 4 & 2 & 3 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$28. \begin{vmatrix} 2 & 2 & 0 & 0 & -2 \\ 1 & 1 & 6 & 0 & 5 \\ 1 & 0 & 2 & -1 & -1 \\ 2 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$

In Problems 29 and 30, find the values of λ that satisfy the given equation.

$$29. \begin{vmatrix} -3-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = 0 \quad 30. \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 3 & 3 & -\lambda \end{vmatrix} = 0$$

8.5 Properties of Determinants

INTRODUCTION In this section we are going to consider some of the many properties of determinants. Our goal in the discussion is to use these properties to develop a means of evaluating a determinant that is an alternative to cofactor expansion.

Properties The first property states that the determinant of an $n \times n$ matrix and its transpose are the same.

Theorem 8.5.1 Determinant of a Transpose

If $\mathbf{A}^T$ is the transpose of the $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{A}^T = \det \mathbf{A}$.

For example, for the matrix $\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 3 & -4 \end{pmatrix}$, we have $\mathbf{A}^T = \begin{pmatrix} 5 & 3 \\ 7 & -4 \end{pmatrix}$. Observe that

$$\det \mathbf{A} = \begin{vmatrix} 5 & 7 \\ 3 & -4 \end{vmatrix} = -41 \quad \text{and} \quad \det \mathbf{A}^T = \begin{vmatrix} 5 & 3 \\ 7 & -4 \end{vmatrix} = -41.$$

Since transposing a matrix interchanges its rows and columns, the significance of Theorem 8.5.1 is that statements concerning determinants and the rows of a matrix also hold when the word *row* is replaced by the word *column*.

Theorem 8.5.2 Two Identical Rows

If any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$ are the same, then $\det \mathbf{A} = 0$.

EXAMPLE 1 Matrix with Two Identical Rows

Since the second and third columns in the matrix $\mathbf{A} = \begin{pmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{pmatrix}$ are the same, it follows from Theorem 8.5.2 that

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{vmatrix} = 0.$$

You should verify this by expanding the determinant by cofactors.

Theorem 8.5.3 Zero Row or Column

If all the entries in a row (column) of an $n \times n$ matrix $\mathbf{A}$ are zero, then $\det \mathbf{A} = 0$.

PROOF: Suppose the i th row of $\mathbf{A}$ consists of all zeros. Hence all the products in the expansion of $\det \mathbf{A}$ by cofactors along the i th row are zero and consequently $\det \mathbf{A} = 0$. $\equiv$

For example, it follows immediately from Theorem 8.5.3 that

$$\begin{array}{c} \text{zero row} \rightarrow \\ \text{zero column} \downarrow \end{array} \begin{array}{c} \left| \begin{array}{cc} 0 & 0 \\ 7 & -6 \end{array} \right| = 0 \quad \text{and} \quad \left| \begin{array}{ccc} 4 & 6 & 0 \\ 1 & 5 & 0 \\ 8 & -1 & 0 \end{array} \right| = 0. \end{array}$$

Theorem 8.5.4 Interchanging Rows

If $\mathbf{B}$ is the matrix obtained by interchanging any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{B} = -\det \mathbf{A}$.

For example, if $\mathbf{B}$ is the matrix obtained by interchanging the first and third rows of

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 9 \\ 6 & 0 & 7 \\ 2 & 1 & 3 \end{pmatrix}, \text{ then from Theorem 8.5.4 we have}$$

$$\det \mathbf{B} = \begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & 7 \\ 4 & -1 & 9 \end{vmatrix} = -\begin{vmatrix} 4 & -1 & 9 \\ 6 & 0 & 7 \\ 2 & 1 & 3 \end{vmatrix} = -\det \mathbf{A}.$$

You should verify this by computing both determinants.

Theorem 8.5.5 Constant Multiple of a Row

If $\mathbf{B}$ is the matrix obtained from an $n \times n$ matrix $\mathbf{A}$ by multiplying a row (column) by a nonzero real number k , then $\det \mathbf{B} = k \det \mathbf{A}$.

PROOF: Suppose the entries in the i th row of $\mathbf{A}$ are multiplied by the number k . Call the resulting matrix $\mathbf{B}$. Expanding $\det \mathbf{B}$ by cofactors along the i th row then gives

$$\begin{aligned} \det \mathbf{B} &= ka_{i1}C_{i1} + ka_{i2}C_{i2} + \cdots + ka_{in}C_{in} \\ &= k(a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}) = k \det \mathbf{A}. \end{aligned}$$

expansion of $\det \mathbf{A}$ by cofactors along the i th row $\equiv$

EXAMPLE 2 Theorems 8.5.5 and 8.5.2

$$\begin{array}{c} \text{from} \\ \text{first column} \\ \downarrow \end{array} \begin{array}{c} \text{from} \\ \text{second column} \\ \downarrow \end{array} \begin{array}{c} \text{from} \\ \text{second row} \\ \downarrow \end{array} \quad \text{(a)} \quad \begin{vmatrix} 5 & 8 \\ 20 & 16 \end{vmatrix} = 5 \begin{vmatrix} 1 & 8 \\ 4 & 16 \end{vmatrix} = 5 \cdot 8 \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = 5 \cdot 8 \cdot 2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 80(1 - 2) = -80$$

$$\begin{array}{ccc}
 & \text{from second column} & \text{from Theorem 8.5.2} \\
 & \downarrow & \downarrow \\
 \text{(b)} \quad \begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ 7 & 4 & -2 \end{vmatrix} & = (-2) \begin{vmatrix} 4 & -1 & -1 \\ 5 & 1 & 1 \\ 7 & -2 & -2 \end{vmatrix} & = (-2) \cdot 0 = 0
 \end{array}
 \quad \equiv$$

Theorem 8.5.6 Determinant of a Matrix Product

If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, then $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$.

In other words, the determinant of a product of two $n \times n$ matrices is the same as the product of the determinants.

EXAMPLE 3 Determinant of a Matrix Product

Suppose $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ -3 & 5 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} -12 & 22 \\ 6 & -9 \end{pmatrix}$. Now $\det \mathbf{AB} = -24$, $\det \mathbf{A} = -8$, $\det \mathbf{B} = 3$, and so we see that

$$\det \mathbf{A} \cdot \det \mathbf{B} = (-8)(3) = -24 = \det \mathbf{AB}. \quad \equiv$$

Theorem 8.5.7 Determinant Is Unchanged

Suppose $\mathbf{B}$ is the matrix obtained from an $n \times n$ matrix $\mathbf{A}$ by multiplying the entries in a row (column) by a nonzero real number k and adding the result to the corresponding entries in another row (column). Then $\det \mathbf{B} = \det \mathbf{A}$.

EXAMPLE 4 A Multiple of a Row Added to Another

Suppose $\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix}$ and suppose the matrix $\mathbf{B}$ is defined as that matrix obtained from $\mathbf{A}$ by the elementary row operation

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix} \xrightarrow{-3R_1 + R_3} \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ -11 & -4 & -2 \end{pmatrix} = \mathbf{B}.$$

Expanding by cofactors along, say, the second column, we find $\det \mathbf{A} = 45$ and $\det \mathbf{B} = 45$. You should verify this result. $\equiv$

Theorem 8.5.8 Determinant of a Triangular Matrix

Suppose $\mathbf{A}$ is an $n \times n$ triangular matrix (upper or lower). Then

$$\det \mathbf{A} = a_{11}a_{22} \cdots a_{nn},$$

where $a_{11}, a_{22}, \dots, a_{nn}$ are the entries on the main diagonal of $\mathbf{A}$.

PROOF: We prove the result for a 3×3 lower triangular matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Expanding $\det \mathbf{A}$ by cofactors along the first row gives

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - 0 \cdot a_{32}) = a_{11}a_{22}a_{33}.$$



EXAMPLE 5 Determinant of a Triangular Matrix

(a) The determinant of the lower triangular matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 5 & 9 & -4 & 0 \\ 7 & 2 & 4 & -2 \end{pmatrix}$$

is
$$\det \mathbf{A} = \begin{vmatrix} 3 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 5 & 9 & -4 & 0 \\ 7 & 2 & 4 & -2 \end{vmatrix} = 3 \cdot 6 \cdot (-4) \cdot (-2) = 144.$$

(b) The determinant of the diagonal matrix $\mathbf{A} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{vmatrix} = (-3) \cdot 6 \cdot 4 = -72.$$



Row Reduction Evaluating the determinant of an $n \times n$ matrix by the method of cofactor expansion requires a Herculean effort when the order of the matrix is large. To expand the determinant of, say, a 5×5 matrix with nonzero entries requires evaluating five cofactors that are determinants of 4×4 submatrices; each of these in turn requires four additional cofactors that are determinants of 3×3 submatrices, and so on. There is a more practical (and programmable) method for evaluating the determinant of a matrix. This method is based on **reducing** the matrix to a triangular form by row operations and the fact that determinants of triangular matrices are easy to evaluate (see Theorem 8.5.8).

EXAMPLE 6 Reducing a Determinant to Triangular Form

Evaluate the determinant of $\mathbf{A} = \begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$.

SOLUTION

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 1 & 2 & 4 \end{vmatrix}$$

← 2 is a common factor in third row: Theorem 8.5.5

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ -4 & -3 & 2 \\ 6 & 2 & 7 \end{vmatrix}$$

← interchange first and third rows: Theorem 8.5.4

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 6 & 2 & 7 \end{vmatrix} \quad \leftarrow 4 \text{ times first row added to second row: Theorem 8.5.7}$$

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & -10 & -17 \end{vmatrix} \quad \leftarrow -6 \text{ times first row added to third row: Theorem 8.5.7}$$

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & 0 & 19 \end{vmatrix} \quad \leftarrow 2 \text{ times second row added to third row: Theorem 8.5.7}$$

$$= (-2)(1)(5)(19) = -190 \quad \leftarrow \text{Theorem 8.5.8}$$

≡

Our final theorem concerns cofactors. We saw in Section 8.4 that a determinant $\det \mathbf{A}$ of an $n \times n$ matrix $\mathbf{A}$ could be evaluated by cofactor expansion along any row (column). This means that the n entries a_{ij} of a row (column) are multiplied by the corresponding cofactors C_{ij} and the n products are added. If, however, the entries a_{ij} of a row (a_{ij} of a column) of $\mathbf{A}$ are multiplied by the corresponding cofactors C_{kj} of a different row (C_{ik} of a different column), the sum of the n products is zero.

Theorem 8.5.9 A Property of Cofactors

Suppose $\mathbf{A}$ is an $n \times n$ matrix. If $a_{i1}, a_{i2}, \dots, a_{in}$ are the entries in the i th row and $C_{k1}, C_{k2}, \dots, C_{kn}$ are the cofactors of the entries in the k th row, then

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \dots + a_{in}C_{kn} = 0 \quad \text{for } i \neq k.$$

If $a_{1j}, a_{2j}, \dots, a_{nj}$ are the entries in the j th column and $C_{1k}, C_{2k}, \dots, C_{nk}$ are the cofactors of the entries in the k th column, then

$$a_{1j}C_{1k} + a_{2j}C_{2k} + \dots + a_{nj}C_{nk} = 0 \quad \text{for } j \neq k.$$

PROOF: We shall prove the result for rows. Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by letting the entries in the i th row of $\mathbf{A}$ be the same as the entries in the k th row—that is, $a_{i1} = a_{k1}, a_{i2} = a_{k2}, \dots, a_{in} = a_{kn}$. Since $\mathbf{B}$ has two rows that are the same, it follows from Theorem 8.5.2 that $\det \mathbf{B} = 0$. Cofactor expansion along the k th row then gives the desired result:

$$\begin{aligned} 0 = \det \mathbf{B} &= a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn} \\ &= a_{i1}C_{k1} + a_{i2}C_{k2} + \dots + a_{in}C_{kn}. \end{aligned}$$

≡

EXAMPLE 7 Cofactors of Third Row/Entries of First Row

Consider the matrix $\mathbf{A} = \begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$. Suppose we multiply the entries of the first row by the cofactors of the third row and add the results; that is,

$$\begin{aligned} a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} &= 6 \begin{vmatrix} 2 & 7 \\ -3 & 2 \end{vmatrix} + 2 \left(- \begin{vmatrix} 6 & 7 \\ -4 & 2 \end{vmatrix} \right) + 7 \begin{vmatrix} 6 & 2 \\ -4 & -3 \end{vmatrix} \\ &= 6(25) + 2(-40) + 7(-10) = 0. \end{aligned}$$

≡

8.5

Exercises

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–10, state the appropriate theorem(s) in this section that justifies the given equality. Do not expand the determinants by cofactors.

$$1. \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$2. \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$$

$$3. \begin{vmatrix} -5 & 6 \\ 2 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -6 & -8 \end{vmatrix}$$

$$4. \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$5. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 18 \\ 5 & 9 & -12 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 5 & 9 & -4 \end{vmatrix}$$

$$6. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 18 \\ 5 & 9 & -12 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 18 \\ 5 & 9 & -12 \\ 1 & 2 & 3 \end{vmatrix}$$

$$7. \begin{vmatrix} 0 & 5 & 0 & 6 \\ 2 & 1 & 0 & 8 \\ 0 & 2 & 0 & -9 \\ 0 & 6 & 0 & 4 \end{vmatrix} = 0$$

$$8. \begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 3 \\ 5 & -8 & -4 \end{vmatrix} = 0$$

$$9. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

$$10. \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix}$$

In Problems 11–16, evaluate the determinant of the given matrix using the result

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5.$$

$$11. \mathbf{A} = \begin{pmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{pmatrix} \quad 12. \mathbf{B} = \begin{pmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{pmatrix}$$

$$13. \mathbf{C} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ b_1 & b_2 & b_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{pmatrix}$$

$$14. \mathbf{D} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$15. \begin{vmatrix} a_1 - 2b_1 + 3c_1 & a_2 - 2b_2 + 3c_2 & a_3 - 2b_3 + 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$16. \begin{vmatrix} 4a_1 - 2a_3 & a_2 & a_3 \\ 4b_1 - 2b_3 & b_2 & b_3 \\ 2c_1 - c_3 & \frac{1}{2}c_2 & \frac{1}{2}c_3 \end{vmatrix}$$

In Problems 17–20, evaluate the determinant of the given matrix without expanding by cofactors.

$$17. \mathbf{A} = \begin{pmatrix} 6 & 1 & 8 & 10 \\ 0 & \frac{2}{3} & 7 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -5 \end{pmatrix} \quad 18. \mathbf{B} = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$19. \mathbf{C} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad 20. \mathbf{D} = \begin{pmatrix} 0 & 7 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

In Problems 21 and 22, verify that $\det \mathbf{A} = \det \mathbf{A}^T$ for the given matrix $\mathbf{A}$.

$$21. \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & -1 \\ 1 & 2 & -1 \end{pmatrix} \quad 22. \mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 7 & 2 & -1 \end{pmatrix}$$

23. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 8 \\ 0 & -1 & 0 \end{pmatrix}.$$

Verify that $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$.

24. Suppose $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{I}$. Then show that $\det \mathbf{A} = \pm 1$.

25. Suppose $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{A}$. Then show that either $\det \mathbf{A} = 0$ or $\det \mathbf{A} = 1$.

26. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then prove or disprove that $\det \mathbf{AB} = \det \mathbf{BA}$.

27. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{pmatrix}.$$

Without expanding, evaluate $\det \mathbf{A}$.

28. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{pmatrix}.$$

Without expanding, show that $\det \mathbf{A} = 0$.

In Problems 29–36, use the procedure illustrated in Example 6 to evaluate the determinant of the given matrix.

$$29. \begin{pmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{pmatrix} \quad 30. \begin{pmatrix} 2 & 4 & 5 \\ 4 & 2 & 0 \\ 8 & 7 & -2 \end{pmatrix}$$

$$31. \begin{pmatrix} -1 & 2 & 3 \\ 4 & -5 & -2 \\ 9 & -9 & 6 \end{pmatrix} \quad 32. \begin{pmatrix} -2 & 2 & -6 \\ 5 & 0 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

$$33. \begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{pmatrix} \quad 34. \begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 5 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 3 & 2 \end{pmatrix}$$

$$35. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{pmatrix} \quad 36. \begin{pmatrix} 2 & 9 & 1 & 8 \\ 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

37. By proceeding as in Example 6, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

38. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$. [Hint: See Problem 37.]

In Problems 39 and 40, verify Theorem 8.5.9 by evaluating $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ and $a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32}$ for the given matrix.

$$39. \mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 4 & -2 & 1 \end{pmatrix} \quad 40. \mathbf{A} = \begin{pmatrix} 3 & 0 & 5 \\ -2 & 3 & -1 \\ 2 & 2 & -3 \end{pmatrix}$$

41. Let $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ -1 & -5 \end{pmatrix}$. Verify that $\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}$.

42. Suppose $\mathbf{A}$ is a 5×5 matrix for which $\det \mathbf{A} = -7$. What is the value of $\det(2\mathbf{A})$?

43. An $n \times n$ matrix $\mathbf{A}$ is said to be **skew-symmetric** if $\mathbf{A}^T = -\mathbf{A}$. If $\mathbf{A}$ is a 5×5 skew-symmetric matrix, show that $\det \mathbf{A} = 0$.

44. It takes about $n!$ multiplications to evaluate the determinant of an $n \times n$ matrix using expansion by cofactors, whereas it takes about $n^3/3$ arithmetic operations using the row-reduction method. Compare the number of operations for both methods using a 25×25 matrix.

8.6 Inverse of a Matrix

INTRODUCTION The concept of the determinant of an $n \times n$, or square, matrix will play an important role in this and the following section.

8.6.1 Finding the Inverse

In the real number system, if a is a nonzero number, then there exists a number b such that $ab = ba = 1$. The number b is called the multiplicative inverse of the number a and is denoted by a^{-1} . For a square matrix $\mathbf{A}$ it is also important to know whether we can find another square matrix $\mathbf{B}$ of the same order such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. We have the following definition.

Definition 8.6.1 Inverse of a Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. If there exists an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \quad (1)$$

where $\mathbf{I}$ is the $n \times n$ identity, then the matrix $\mathbf{A}$ is said to be **nonsingular** or **invertible**. The matrix $\mathbf{B}$ is said to be the **inverse** of $\mathbf{A}$.

For example, the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is nonsingular or invertible since the matrix $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is its inverse. To verify this, observe that

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

and

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Unlike the real number system, where every nonzero number a has a multiplicative inverse, not every nonzero $n \times n$ matrix $\mathbf{A}$ has an inverse.

In Problems 1–20, fill in the blanks or answer true/false.

1. A matrix $\mathbf{A} = (a_{ij})_{4 \times 3}$ such that $a_{ij} = i + j$ is given by _____.
2. If $\mathbf{A}$ is a 4×7 matrix and $\mathbf{B}$ is a 7×3 matrix, then the size of $\mathbf{AB}$ is _____.
3. If $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = (3 \ 4)$, then $\mathbf{AB} =$ _____ and $\mathbf{BA} =$ _____.
4. If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $\mathbf{A}^{-1} =$ _____.
5. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ nonsingular matrices, then $\mathbf{A} + \mathbf{B}$ is necessarily nonsingular. _____
6. If $\mathbf{A}$ is a nonsingular matrix for which $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$. _____
7. If $\mathbf{A}$ is a 3×3 matrix such that $\det \mathbf{A} = 5$, then $\det(\frac{1}{2}\mathbf{A}) =$ _____ and $\det(-\mathbf{A}^T) =$ _____.
8. If $\det \mathbf{A} = 6$ and $\det \mathbf{B} = 2$, then $\det \mathbf{AB}^{-1} =$ _____.
9. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices whose corresponding entries in the third column are the same, then $\det(\mathbf{A} - \mathbf{B}) =$ _____.
10. Suppose $\mathbf{A}$ is a 3×3 matrix such that $\det \mathbf{A} = 2$. If $\mathbf{B} = 10\mathbf{A}$ and $\mathbf{C} = -\mathbf{B}^{-1}$, then $\det \mathbf{C} =$ _____.
11. Let $\mathbf{A}$ be an $n \times n$ matrix. The eigenvalues of $\mathbf{A}$ are the nonzero solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. _____
12. A nonzero scalar multiple of an eigenvector is also an eigenvector corresponding to the same eigenvalue. _____
13. An $n \times 1$ column vector $\mathbf{K}$ with all zero entries is never an eigenvector of an $n \times n$ matrix $\mathbf{A}$. _____
14. Let $\mathbf{A}$ be an $n \times n$ matrix with real entries. If λ is a complex eigenvalue, then $\bar{\lambda}$ is also an eigenvalue of $\mathbf{A}$. _____
15. An $n \times n$ matrix $\mathbf{A}$ always possesses n linearly independent eigenvectors. _____
16. The augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ is in reduced row-echelon form. _____
17. If a 3×3 matrix $\mathbf{A}$ is diagonalizable, then it possesses three linearly independent eigenvectors. _____
18. The only matrices that are orthogonally diagonalizable are symmetric matrices. _____
19. The symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is orthogonal. _____
20. The eigenvalues of a symmetric matrix with real entries are always real numbers. _____
21. An $n \times n$ matrix $\mathbf{B}$ is symmetric if $\mathbf{B}^T = \mathbf{B}$, and an $n \times n$ matrix $\mathbf{C}$ is skew-symmetric if $\mathbf{C}^T = -\mathbf{C}$. By noting the identity $2\mathbf{A} = \mathbf{A} + \mathbf{A}^T + \mathbf{A} - \mathbf{A}^T$, show that any $n \times n$ matrix $\mathbf{A}$ can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
22. Show that there exists no 2×2 matrix with real entries such that $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

23. An $n \times n$ matrix $\mathbf{A}$ is said to be **nilpotent** if, for some positive integer m , $\mathbf{A}^m = \mathbf{0}$. Find a 2×2 nilpotent matrix $\mathbf{A} \neq \mathbf{0}$.

24. (a) Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to **anticommute** if $\mathbf{AB} = -\mathbf{BA}$. Show that each of the **Pauli spin matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $i^2 = -1$, anticommutes with the others. Pauli spin matrices are used in quantum mechanics.

(b) The matrix $\mathbf{C} = \mathbf{AB} - \mathbf{BA}$ is said to be the **commutator** of the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$. Find the commutators of σ_x and σ_y , σ_y and σ_z , and σ_z and σ_x .

In Problems 25 and 26, solve the given system of equations by Gauss–Jordan elimination.

$$25. \begin{pmatrix} 5 & -1 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{pmatrix} \mathbf{X} = \begin{pmatrix} -9 \\ 27 \\ 9 \end{pmatrix} \quad 26. \begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 - 2x_2 + 3x_3 = 2 \\ 2x_1 - 3x_3 = 3 \end{cases}$$

$$27. \text{ Without expanding, show that } \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ bc & ac & ab \end{vmatrix} = 0.$$

$$28. \text{ Show that } \begin{vmatrix} y & x^2 & x & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 4 & 2 & 1 \\ 5 & 9 & 3 & 1 \end{vmatrix} = 0 \text{ is the equation of a parabola}$$

passing through the three points $(1, 2)$, $(2, 3)$, and $(3, 5)$.

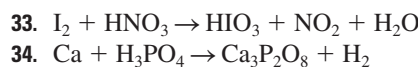
In Problems 29 and 30, evaluate the determinant of the given matrix by inspection.

$$29. \begin{vmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{vmatrix} \quad 30. \begin{pmatrix} -3 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 1 & 3 & 9 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$$

In Problems 31 and 32, without solving, state whether the given homogeneous system has only the trivial solution or has infinitely many solutions.

$$31. \begin{cases} x_1 - x_2 + x_3 = 0 \\ 5x_1 + x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \quad 32. \begin{cases} x_1 - x_2 - x_3 = 0 \\ 5x_1 + x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

In Problems 33 and 34, use Gauss–Jordan elimination to balance the given chemical equation.



In Problems 35 and 36, solve the given system of equations by Cramer's rule.

$$\begin{array}{ll} 35. & x_1 + 2x_2 - 3x_3 = -2 \\ & 2x_1 - 4x_2 + 3x_3 = 0 \\ & 4x_2 + 6x_3 = 5 \end{array} \quad \begin{array}{ll} 36. & x_1 + \quad \quad x_3 = 4 \\ & 2x_1 + 3x_2 + 4x_3 = 5 \\ & x_1 + 4x_2 + 5x_3 = 0 \end{array}$$

37. Use Cramer's rule to solve the system

$$\begin{aligned} X &= x \cos \theta + y \sin \theta \\ Y &= -x \sin \theta + y \cos \theta \end{aligned}$$

for x and y .

38. (a) Set up the system of equations for the currents in the branches of the network given in **FIGURE 8.R.1**.

(b) Use Cramer's rule to show that

$$i_1 = E \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right).$$

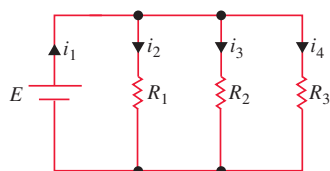


FIGURE 8.R.1 Network in Problem 38

39. Solve the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 6 \\ x_1 - 2x_2 &= -3 \\ -2x_1 + \quad \quad x_3 &= 9 \end{aligned}$$

by writing it as a matrix equation and finding the inverse of the coefficient matrix.

40. Use the inverse of the matrix $\mathbf{A}$ to solve the system $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\text{and the vector } \mathbf{B} \text{ is given by (a) } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ (b) } \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

In Problems 41–46, find the eigenvalues and corresponding eigenvectors of the given matrix.

41. $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

42. $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$

43. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

44. $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

45. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

46. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

47. Supply a first column so that the matrix is orthogonal:

$$\begin{vmatrix} \boxed{} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \boxed{} & 0 & \frac{1}{\sqrt{3}} \\ \boxed{} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix}.$$

48. Consider the symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$.

(a) Find matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ that orthogonally diagonalize the matrix $\mathbf{A}$.

(b) Find the diagonal matrix $\mathbf{D}$ by actually carrying out the multiplication $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

49. Identify the conic section $x^2 + 3xy + y^2 = 1$.

50. Consider the following population data:

Year	1890	1900	1910	1920	1930
Population (in millions)	63	76	92	106	123

The actual population in 1940 was 132 million. Compare this amount with the population predicted from the least squares line for the given data.

In Problems 51 and 52, use the matrix $\mathbf{A} = \begin{pmatrix} 10 & 1 \\ 9 & 1 \end{pmatrix}$ to encode the given message. Use the correspondence (1) of Section 8.14.

51. SATELLITE LAUNCHED ON FRI

52. SEC AGNT ARRVS TUES AM

In Problems 53 and 54, use the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ to

decode the given message. Use the correspondence (1) in Section 8.14.

53. $\mathbf{B} = \begin{pmatrix} 19 & 0 & 15 & 14 & 0 & 20 \\ 35 & 10 & 27 & 53 & 1 & 54 \\ 5 & 15 & -3 & 48 & 2 & 39 \end{pmatrix}$

54. $\mathbf{B} = \begin{pmatrix} 5 & 2 & 21 \\ 27 & 17 & 40 \\ 21 & 13 & -2 \end{pmatrix}$

55. Decode the following messages using the parity check code.

(a) (1 1 0 0 1 1) (b) (0 1 1 0 1 1 0)

56. Encode the word (1 0 0 1) using the Hamming (7, 4) code.

In Problems 57 and 58, solve the given system of equations using LU-factorization.

57. The system in Problem 26

58. The system in Problem 36

59. Find the least squares line for the data (1, -2), (2, 0), (3, 5), (4, -1).

60. Find the least squares parabola for the data given in Problem 59.

MODULE 4

Text Book 3- Advanced Engineering Mathematics
(6/e) : Dennis G Zill Jones & Bartlett Learning,
LLC(2018)ISBN: 978-1-284-10590-2

Sections 8.6, 8.8, 8.9, 8.10, 8.12, 8.13

$$33. \begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{pmatrix} \quad 34. \begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 5 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 3 & 2 \end{pmatrix}$$

$$35. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{pmatrix} \quad 36. \begin{pmatrix} 2 & 9 & 1 & 8 \\ 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

37. By proceeding as in Example 6, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

38. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$. [Hint: See Problem 37.]

In Problems 39 and 40, verify Theorem 8.5.9 by evaluating $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ and $a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32}$ for the given matrix.

$$39. \mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 4 & -2 & 1 \end{pmatrix} \quad 40. \mathbf{A} = \begin{pmatrix} 3 & 0 & 5 \\ -2 & 3 & -1 \\ 2 & 2 & -3 \end{pmatrix}$$

41. Let $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ -1 & -5 \end{pmatrix}$. Verify that $\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}$.

42. Suppose $\mathbf{A}$ is a 5×5 matrix for which $\det \mathbf{A} = -7$. What is the value of $\det(2\mathbf{A})$?

43. An $n \times n$ matrix $\mathbf{A}$ is said to be **skew-symmetric** if $\mathbf{A}^T = -\mathbf{A}$. If $\mathbf{A}$ is a 5×5 skew-symmetric matrix, show that $\det \mathbf{A} = 0$.

44. It takes about $n!$ multiplications to evaluate the determinant of an $n \times n$ matrix using expansion by cofactors, whereas it takes about $n^3/3$ arithmetic operations using the row-reduction method. Compare the number of operations for both methods using a 25×25 matrix.

8.6 Inverse of a Matrix

INTRODUCTION The concept of the determinant of an $n \times n$, or square, matrix will play an important role in this and the following section.

8.6.1 Finding the Inverse

In the real number system, if a is a nonzero number, then there exists a number b such that $ab = ba = 1$. The number b is called the multiplicative inverse of the number a and is denoted by a^{-1} . For a square matrix $\mathbf{A}$ it is also important to know whether we can find another square matrix $\mathbf{B}$ of the same order such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. We have the following definition.

Definition 8.6.1 Inverse of a Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. If there exists an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \quad (1)$$

where $\mathbf{I}$ is the $n \times n$ identity, then the matrix $\mathbf{A}$ is said to be **nonsingular** or **invertible**. The matrix $\mathbf{B}$ is said to be the **inverse** of $\mathbf{A}$.

For example, the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is nonsingular or invertible since the matrix $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is its inverse. To verify this, observe that

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

and

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Unlike the real number system, where every nonzero number a has a multiplicative inverse, not every nonzero $n \times n$ matrix $\mathbf{A}$ has an inverse.

EXAMPLE 1 Matrix with no Inverse

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ has no multiplicative inverse. To see this suppose $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{pmatrix}.$$

Inspection of the last matrix shows that it is impossible to obtain the 2×2 identity matrix $\mathbf{I}$, because there is no way of selecting b_{11} , b_{12} , b_{21} , and b_{22} to get 1 as the entry in the second row and second column. We conclude that the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ has no inverse. $\equiv$

An $n \times n$ matrix that has no inverse is called **singular**. If $\mathbf{A}$ is **nonsingular**, its inverse is denoted by $\mathbf{B} = \mathbf{A}^{-1}$.

Important. $\blacktriangleright$

Note that the symbol -1 in the notation $\mathbf{A}^{-1}$ is not an exponent; in other words, $\mathbf{A}^{-1}$ is not a reciprocal. Also, if $\mathbf{A}$ is nonsingular, its inverse is unique.

Properties The following theorem lists some properties of the inverse of a matrix.

Theorem 8.6.1 Properties of the Inverse

Let $\mathbf{A}$ and $\mathbf{B}$ be nonsingular matrices. Then

- (i) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- (ii) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- (iii) $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$

PROOF OF (i): This part of the theorem states that if $\mathbf{A}$ is nonsingular, then its inverse $\mathbf{A}^{-1}$ is also nonsingular and its inverse is $\mathbf{A}$. To prove that $\mathbf{A}^{-1}$ is nonsingular, we have to show that a matrix $\mathbf{B}$ can be found such that $\mathbf{A}^{-1}\mathbf{B} = \mathbf{BA}^{-1} = \mathbf{I}$. But since $\mathbf{A}$ is assumed to be nonsingular, we know from (1) that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and, equivalently, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$. The last matrix equation indicates that the required matrix, the inverse of $\mathbf{A}^{-1}$, is $\mathbf{B} = \mathbf{A}$. Consequently, $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$. $\equiv$

Theorem 8.6.1(ii) extends to any finite number of nonsingular matrices:

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_1^{-1};$$

that is, the inverse of a product of nonsingular matrices is the product of the inverses in reverse order.

In the discussion that follows we are going to consider two different ways of finding $\mathbf{A}^{-1}$ for a nonsingular matrix $\mathbf{A}$. The first method utilizes determinants, whereas the second employs the elementary row operations introduced in Section 8.2.

Adjoint Method Recall from (6) of Section 8.4 that the cofactor C_{ij} of the entry a_{ij} of an $n \times n$ matrix $\mathbf{A}$ is $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij} ; that is, the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of $\mathbf{A}$.

Definition 8.6.2 Adjoint Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. The matrix that is the transpose of the matrix of cofactors corresponding to the entries of $\mathbf{A}$:

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

is called the **adjoint** of $\mathbf{A}$ and is denoted by $\text{adj } \mathbf{A}$.

The next theorem will give a compact formula for the inverse of a nonsingular matrix in terms of the adjoint of the matrix. However, because of the determinants involved, this method becomes unwieldy for $n \geq 4$.

Theorem 8.6.2 Finding the Inverse

Let $\mathbf{A}$ be an $n \times n$ matrix. If $\det \mathbf{A} \neq 0$, then

$$\mathbf{A}^{-1} = \left(\frac{1}{\det \mathbf{A}} \right) \text{adj } \mathbf{A}. \quad (2)$$

PROOF: For brevity we prove the case when $n = 3$. Observe that

$$\begin{aligned} \mathbf{A}(\text{adj } \mathbf{A}) &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \\ &= \begin{pmatrix} \det \mathbf{A} & 0 & 0 \\ 0 & \det \mathbf{A} & 0 \\ 0 & 0 & \det \mathbf{A} \end{pmatrix} \end{aligned} \quad (3)$$

since $\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$, for $i = 1, 2, 3$, are the expansions of $\det \mathbf{A}$ by cofactors along the first, second, and third rows, and

$$\begin{aligned} a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} &= 0 & a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} &= 0 \\ a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= 0 & a_{21}C_{31} + a_{22}C_{32} + a_{23}C_{33} &= 0 \\ a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13} &= 0 & a_{31}C_{21} + a_{32}C_{22} + a_{33}C_{23} &= 0 \end{aligned}$$

in view of Theorem 8.5.9. Thus, (3) is the same as

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\det \mathbf{A}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\det \mathbf{A}) \mathbf{I}$$

or $\mathbf{A}((1/\det \mathbf{A}) \text{adj } \mathbf{A}) = \mathbf{I}$. Similarly, it can be shown in exactly the same manner that $((1/\det \mathbf{A}) \text{adj } \mathbf{A})\mathbf{A} = \mathbf{I}$. Hence, by definition, $\mathbf{A}^{-1} = (1/\det \mathbf{A}) \text{adj } \mathbf{A}$. $\equiv$

For future reference we observe in the case of a 2×2 nonsingular matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that the cofactors are $C_{11} = a_{22}$, $C_{12} = -a_{21}$, $C_{21} = -a_{12}$, and $C_{22} = a_{11}$. In this case,

$$\text{adj } \mathbf{A} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^T = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

It follows from (2) that

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \quad (4)$$

For a 3×3 nonsingular matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

and so on. After the adjoint of $\mathbf{A}$ is formed, (2) gives

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}. \quad (5)$$

EXAMPLE 2 Inverse of a Matrix

Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$.

SOLUTION Since $\det \mathbf{A} = 10 - 8 = 2$, it follows from (4) that

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 10 & -4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & \frac{1}{2} \end{pmatrix}.$$

$$\text{Check: } \mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 5 - 4 & -2 + 2 \\ 10 - 10 & -4 + 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 5 & -2 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 5 - 4 & 20 - 20 \\ -1 + 1 & -4 + 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \equiv$$

EXAMPLE 3 Inverse of a Matrix

Find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$.

SOLUTION Since $\det \mathbf{A} = 12$, we can find $\mathbf{A}^{-1}$ from (5). The cofactors corresponding to the entries in $\mathbf{A}$ are

$$C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad C_{12} = -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5 \quad C_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2 \quad C_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \quad C_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \quad C_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6.$$

From (5) we then obtain

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & -\frac{1}{6} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The reader is urged to verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ≡

We are now in a position to prove a necessary and sufficient condition for an $n \times n$ matrix $\mathbf{A}$ to have an inverse.

Theorem 8.6.3 Nonsingular Matrices and $\det \mathbf{A}$

An $n \times n$ matrix $\mathbf{A}$ is nonsingular if and only if $\det \mathbf{A} \neq 0$.

PROOF: We shall first prove the sufficiency. Assume $\det \mathbf{A} \neq 0$. Then $\mathbf{A}$ is nonsingular, since $\mathbf{A}^{-1}$ can be found from Theorem 8.6.2.

To prove the necessity, we must assume that $\mathbf{A}$ is nonsingular and prove that $\det \mathbf{A} \neq 0$. Now from Theorem 8.5.6, $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ implies

$$(\det \mathbf{A})(\det \mathbf{A}^{-1}) = (\det \mathbf{A}^{-1})(\det \mathbf{A}) = \det \mathbf{I}.$$

But since $\det \mathbf{I} = 1$ (why?), the product $(\det \mathbf{A})(\det \mathbf{A}^{-1}) = 1 \neq 0$ shows that we must have $\det \mathbf{A} \neq 0$. ≡

For emphasis we restate Theorem 8.6.3 in an alternative manner:

$$\text{An } n \times n \text{ matrix } \mathbf{A} \text{ is singular if and only if } \det \mathbf{A} = 0. \quad (6)$$

EXAMPLE 4 Using (6)

The 2×2 matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ has no inverse; that is, $\mathbf{A}$ is singular, because $\det \mathbf{A} = 6 - 6 = 0$. ≡

Because of the number of determinants that must be evaluated, the foregoing method for calculating the inverse of a matrix is tedious when the order of the matrix is large. In the case of 3×3 or larger matrices, the next method is a particularly efficient means for finding $\mathbf{A}^{-1}$.

Row Operations Method Although it would take us beyond the scope of this book to prove it, we shall nonetheless use the following results:

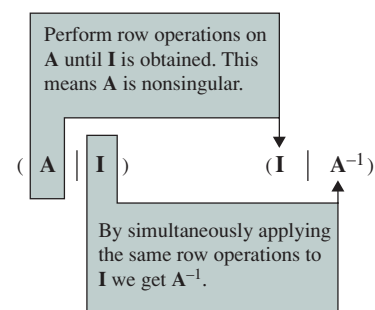
Theorem 8.6.4 Finding the Inverse

If an $n \times n$ matrix $\mathbf{A}$ can be transformed into the $n \times n$ identity $\mathbf{I}$ by a sequence of elementary row operations, then $\mathbf{A}$ is nonsingular. The same sequence of operations that transforms $\mathbf{A}$ into the identity $\mathbf{I}$ will also transform $\mathbf{I}$ into $\mathbf{A}^{-1}$.

It is convenient to carry out these row operations on $\mathbf{A}$ and $\mathbf{I}$ simultaneously by means of an $n \times 2n$ matrix obtained by augmenting $\mathbf{A}$ with the identity $\mathbf{I}$ as shown here:

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array} \right).$$

The procedure for finding $\mathbf{A}^{-1}$ is outlined in the following diagram:



EXAMPLE 5**Inverse by Elementary Row Operations**

Find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$.

SOLUTION We shall use the same notation as we did in Section 8.2 when we reduced an augmented matrix to reduced row-echelon form:

$$\begin{aligned}
 \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array} \right) &\xRightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array} \right) \\
 &\xRightarrow{\substack{2R_1 + R_2 \\ 5R_1 + R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 5 & \frac{17}{2} & \frac{5}{2} & 0 & 1 \end{array} \right) \\
 &\xRightarrow{\substack{\frac{1}{3}R_2 \\ \frac{1}{5}R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{17}{10} & \frac{1}{2} & 0 & \frac{1}{5} \end{array} \right) \\
 &\xRightarrow{-R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{30} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{5} \end{array} \right) \\
 &\xRightarrow{30R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array} \right) \\
 &\xRightarrow{\substack{-\frac{1}{2}R_3 + R_1 \\ -\frac{5}{3}R_3 + R_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -3 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array} \right).
 \end{aligned}$$

Since $\mathbf{I}$ appears to the left of the vertical line, we conclude that the matrix to the right of the line is

$$\mathbf{A}^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}. \quad \equiv$$

If row reduction of $(\mathbf{A}|\mathbf{I})$ leads to the situation

$$(\mathbf{A}|\mathbf{I}) \xRightarrow{\text{row operations}} (\mathbf{B}|\mathbf{C}),$$

where the matrix $\mathbf{B}$ contains a row of zeros, then necessarily $\mathbf{A}$ is singular. Since further reduction of $\mathbf{B}$ always yields another matrix with a row of zeros, we can never transform $\mathbf{A}$ into $\mathbf{I}$.

EXAMPLE 6**A Singular Matrix**

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$ has no inverse, since

$$\begin{aligned}
 \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 6 & 0 & -3 & 0 & 0 & 1 \end{array} \right) &\xRightarrow{-2R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 6 & 0 & -3 & 0 & 0 & 1 \end{array} \right) \\
 &\xRightarrow{-6R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 6 & 9 & -6 & 0 & 1 \end{array} \right)
 \end{aligned}$$

$$\xRightarrow{-R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array} \right).$$

Since the matrix to the left of the vertical bar has a row of zeros, we can stop at this point and conclude that $\mathbf{A}$ is singular. ≡

8.6.2 Using the Inverse to Solve Systems

A system of m linear equations in n variables $x_1, x_2, \dots, x_n$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad (7)$$

can be written compactly as a matrix equation $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Special Case Let us suppose now that $m = n$ in (7) so that the coefficient matrix $\mathbf{A}$ is $n \times n$. In particular, if $\mathbf{A}$ is nonsingular, then the system $\mathbf{AX} = \mathbf{B}$ can be solved by multiplying both of the equations by $\mathbf{A}^{-1}$. From $\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$, we get $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{IX} = \mathbf{X}$, we have

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}. \quad (8)$$

EXAMPLE 7 Using (8) to Solve a System

Use the inverse of the coefficient matrix to solve the system

$$\begin{aligned} 2x_1 - 9x_2 &= 15 \\ 3x_1 + 6x_2 &= 16. \end{aligned}$$

SOLUTION The given system can be written as

$$\begin{pmatrix} 2 & -9 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 16 \end{pmatrix}.$$

Since $\begin{vmatrix} 2 & -9 \\ 3 & 6 \end{vmatrix} = 39 \neq 0$, the coefficient matrix is nonsingular. Consequently, from (4) we get

$$\begin{pmatrix} 2 & -9 \\ 3 & 6 \end{pmatrix}^{-1} = \frac{1}{39} \begin{pmatrix} 6 & 9 \\ -3 & 2 \end{pmatrix}.$$

Using (8) it follows that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{39} \begin{pmatrix} 6 & 9 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 15 \\ 16 \end{pmatrix} = \frac{1}{39} \begin{pmatrix} 234 \\ -13 \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{1}{3} \end{pmatrix},$$

and so $x_1 = 6$ and $x_2 = -\frac{1}{3}$. ≡

EXAMPLE 8 Using (8) to Solve a System

Use the inverse of the coefficient matrix to solve the system

$$\begin{aligned} 2x_1 + \quad \quad \quad x_3 &= 2 \\ -2x_1 + 3x_2 + 4x_3 &= 4 \\ -5x_1 + 5x_2 + 6x_3 &= -1 \end{aligned}$$

SOLUTION We found the inverse of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$$

in Example 4. Thus, (8) gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ 62 \\ -36 \end{pmatrix}.$$

Consequently, $x_1 = 19$, $x_2 = 62$, and $x_3 = -36$. ≡

Uniqueness When $\det \mathbf{A} \neq 0$ the solution of the system $\mathbf{AX} = \mathbf{B}$ is unique. Suppose not—that is, suppose $\det \mathbf{A} \neq 0$ and $\mathbf{X}_1$ and $\mathbf{X}_2$ are two *different* solution vectors. Then $\mathbf{AX}_1 = \mathbf{B}$ and $\mathbf{AX}_2 = \mathbf{B}$ imply $\mathbf{AX}_1 = \mathbf{AX}_2$. Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and so $\mathbf{A}^{-1}(\mathbf{AX}_1) = \mathbf{A}^{-1}(\mathbf{AX}_2)$ and $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_1 = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_2$. This gives $\mathbf{IX}_1 = \mathbf{IX}_2$ or $\mathbf{X}_1 = \mathbf{X}_2$, which contradicts our assumption that $\mathbf{X}_1$ and $\mathbf{X}_2$ were different solution vectors.

Homogeneous Systems A homogeneous system of equations can be written $\mathbf{AX} = \mathbf{0}$. Recall that a homogeneous system always possesses the trivial solution $\mathbf{X} = \mathbf{0}$ and possibly an infinite number of solutions. In the next theorem we shall see that a homogeneous system of n equations in n variables possesses *only* the trivial solution when $\mathbf{A}$ is nonsingular.

Theorem 8.6.5 Trivial Solution Only

A homogeneous system of n linear equations in n variables $\mathbf{AX} = \mathbf{0}$ has only the trivial solution if and only if $\mathbf{A}$ is nonsingular.

PROOF: We prove the sufficiency part of the theorem. Suppose $\mathbf{A}$ is nonsingular. Then by (8), we have the unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$. ≡

The next theorem will answer the question: When does a homogeneous system of n linear equations in n variables possess a nontrivial solution? Bear in mind that if a homogeneous system has one nontrivial solution, it must have an infinite number of solutions.

Theorem 8.6.6 Existence of Nontrivial Solutions

A homogeneous system of n linear equations in n variables $\mathbf{AX} = \mathbf{0}$ has a nontrivial solution if and only if $\mathbf{A}$ is singular.

In view of Theorem 8.6.6, we can conclude that a homogeneous system of n linear equations in n variables $\mathbf{AX} = \mathbf{0}$ possesses

- only the trivial solution if and only if $\det \mathbf{A} \neq 0$, and
- a nontrivial solution if and only if $\det \mathbf{A} = 0$.

The last result will be put to use in Section 8.8.

REMARKS

(i) As a practical means of solving n linear equations in n variables, the use of an inverse matrix offers few advantages over the method of Section 8.2. However, in applications we sometimes need to solve a system $\mathbf{AX} = \mathbf{B}$ several times; that is, we need to examine the solutions of the system corresponding to the same coefficient matrix $\mathbf{A}$ but different input vectors $\mathbf{B}$. In this case, the single calculation of $\mathbf{A}^{-1}$ enables us to obtain these solutions quickly through the matrix multiplication $\mathbf{A}^{-1}\mathbf{B}$.

(ii) In Definition 8.6.1 we saw that if $\mathbf{A}$ is an $n \times n$ matrix and if there exists another $n \times n$ matrix $\mathbf{B}$ that commutes with $\mathbf{A}$ such that

$$\mathbf{AB} = \mathbf{I} \quad \text{and} \quad \mathbf{BA} = \mathbf{I}, \quad (9)$$

then $\mathbf{B}$ is the inverse of $\mathbf{A}$. Although matrix multiplication is in general not commutative, the condition in (9) can be relaxed somewhat in this sense: If we find an $n \times n$ matrix $\mathbf{B}$ for which $\mathbf{AB} = \mathbf{I}$, then it can be proved that $\mathbf{BA} = \mathbf{I}$ as well, and so $\mathbf{B}$ is the inverse of $\mathbf{A}$. As a consequence of this result, in the subsequent sections of this chapter if we wish to prove that a certain matrix $\mathbf{B}$ is the inverse of a given matrix $\mathbf{A}$, it will suffice to show only that $\mathbf{AB} = \mathbf{I}$. We need not demonstrate that $\mathbf{B}$ commutes with $\mathbf{A}$ to give $\mathbf{I}$.

8.6

Exercises

Answers to selected odd-numbered problems begin on page ANS-17.

8.6.1 Finding the Inverse

In Problems 1 and 2, verify that the matrix $\mathbf{B}$ is the inverse of the matrix $\mathbf{A}$.

$$\begin{aligned} 1. \quad \mathbf{A} &= \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \\ 2. \quad \mathbf{A} &= \begin{pmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{pmatrix} \end{aligned}$$

In Problems 3–14, use Theorem 8.6.3 to determine whether the given matrix is singular or nonsingular. If it is nonsingular, use Theorem 8.6.2 to find the inverse.

$$\begin{aligned} 3. \quad & \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} & 4. \quad & \begin{pmatrix} \frac{1}{3} & -1 \\ 4 & 3 \end{pmatrix} \\ 5. \quad & \begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix} & 6. \quad & \begin{pmatrix} -2\pi & -\pi \\ -\pi & \pi \end{pmatrix} \\ 7. \quad & \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 4 \\ 1 & -1 & 1 \end{pmatrix} & 8. \quad & \begin{pmatrix} 2 & 3 & 0 \\ 0 & 11 & 14 \\ -1 & 4 & 7 \end{pmatrix} \\ 9. \quad & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{pmatrix} & 10. \quad & \begin{pmatrix} 2 & -1 & 5 \\ 3 & 0 & -2 \\ 1 & 4 & 0 \end{pmatrix} \\ 11. \quad & \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix} & 12. \quad & \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{pmatrix} \\ 13. \quad & \begin{pmatrix} 0 & -1 & 1 & 4 \\ 3 & 2 & -2 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} & 14. \quad & \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

In Problems 15–26, use Theorem 8.6.4 to find the inverse of the given matrix or show that no inverse exists.

$$\begin{aligned} 15. \quad & \begin{pmatrix} 6 & -2 \\ 0 & 4 \end{pmatrix} & 16. \quad & \begin{pmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ 17. \quad & \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} & 18. \quad & \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 19. \quad & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} & 20. \quad & \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix} \\ 21. \quad & \begin{pmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{pmatrix} & 22. \quad & \begin{pmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ 8 & 10 & -6 \end{pmatrix} \\ 23. \quad & \begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & 24. \quad & \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{pmatrix} \\ 25. \quad & \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix} & 26. \quad & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

In Problems 27 and 28, use the given matrices to find $(\mathbf{AB})^{-1}$.

$$\begin{aligned} 27. \quad \mathbf{A}^{-1} &= \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix} \\ 28. \quad \mathbf{A}^{-1} &= \begin{pmatrix} 1 & 3 & -15 \\ 0 & -1 & 5 \\ -1 & -2 & 11 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix} \\ 29. \quad & \text{If } \mathbf{A}^{-1} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \text{ what is } \mathbf{A}? \end{aligned}$$

30. If $\mathbf{A}$ is nonsingular, then $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. Verify this for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$.

31. Find a value of x such that the matrix $\mathbf{A} = \begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix}$ is its own inverse.

32. Find the inverse of $\mathbf{A} = \begin{pmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{pmatrix}$.

33. A nonsingular matrix $\mathbf{A}$ is said to be **orthogonal** if $\mathbf{A}^{-1} = \mathbf{A}^T$.

(a) Verify that the matrix in Problem 32 is orthogonal.

(b) Verify that $\mathbf{A} = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$ is an orthogonal matrix.

34. Show that if $\mathbf{A}$ is an orthogonal matrix (see Problem 33), then $\det \mathbf{A} = \pm 1$.

35. Suppose $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Then show that $\mathbf{AB}$ is nonsingular.

36. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and that either $\mathbf{A}$ or $\mathbf{B}$ is singular. Then show that $\mathbf{AB}$ is singular.
37. Suppose $\mathbf{A}$ is a nonsingular matrix. Then show that $\det \mathbf{A}^{-1} = 1/\det \mathbf{A}$.
38. Suppose $\mathbf{A}^2 = \mathbf{A}$. Then show that either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A}$ is singular.
39. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $\mathbf{A}$ is nonsingular, and $\mathbf{AB} = \mathbf{0}$. Then show that $\mathbf{B} = \mathbf{0}$.
40. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $\mathbf{A}$ is nonsingular, and $\mathbf{AB} = \mathbf{AC}$. Then show that $\mathbf{B} = \mathbf{C}$.
41. Suppose $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Is $\mathbf{A} + \mathbf{B}$ necessarily nonsingular?
42. Suppose $\mathbf{A}$ is a nonsingular matrix. Then show that $\mathbf{A}^T$ is nonsingular.
43. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ nonzero matrices and $\mathbf{AB} = \mathbf{0}$. Then show that both $\mathbf{A}$ and $\mathbf{B}$ are singular.
44. Consider the 3×3 diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Determine conditions such that $\mathbf{A}$ is nonsingular. If $\mathbf{A}$ is nonsingular, find $\mathbf{A}^{-1}$. Generalize your results to an $n \times n$ diagonal matrix.

8.6.2 Using the Inverse to Solve Systems

In Problems 45–52, use an inverse matrix to solve the given system of equations.

45. $x_1 + x_2 = 4$
 $2x_1 - x_2 = 14$
46. $x_1 - x_2 = 2$
 $2x_1 + 4x_2 = -5$
47. $4x_1 - 6x_2 = 6$
 $2x_1 + x_2 = 1$
48. $x_1 + 2x_2 = 4$
 $3x_1 + 4x_2 = -3$
49. $x_1 + x_3 = -4$
 $x_1 + x_2 + x_3 = 0$
 $5x_1 - x_2 = 6$
50. $x_1 - x_2 + x_3 = 1$
 $2x_1 + x_2 + 2x_3 = 2$
 $3x_1 + 2x_2 - x_3 = -3$
51. $x_1 + 2x_2 + 2x_3 = 1$
 $x_1 - 2x_2 + 2x_3 = -3$
 $3x_1 - x_2 + 5x_3 = 7$
52. $x_1 - x_3 = 2$
 $x_2 + x_3 = 1$
 $-x_1 + x_2 + 2x_3 + x_4 = -5$
 $x_3 - x_4 = 3$

In Problems 53 and 54, write the system in the form $\mathbf{AX} = \mathbf{B}$. Use $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ to solve the system for each matrix $\mathbf{B}$.

53. $7x_1 - 2x_2 = b_1$
 $3x_1 - 2x_2 = b_2$
 $\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 50 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -20 \end{pmatrix}$
54. $x_1 + 2x_2 + 5x_3 = b_1$
 $2x_1 + 3x_2 + 8x_3 = b_2$
 $-x_1 + x_2 + 2x_3 = b_3$
 $\mathbf{B} = \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}$

In Problems 55–58, without solving, determine whether the given homogeneous system of equations has only the trivial solution or a nontrivial solution.

55. $x_1 + 2x_2 - x_3 = 0$
 $4x_1 - x_2 + x_3 = 0$
 $5x_1 + x_2 - 2x_3 = 0$
56. $x_1 + x_2 + x_3 = 0$
 $x_1 - 2x_2 + x_3 = 0$
 $-2x_1 + x_2 - 2x_3 = 0$

57. $x_1 + x_2 - x_3 + x_4 = 0$
 $5x_2 + 2x_4 = 0$
 $x_1 + x_3 - x_4 = 0$
 $3x_1 + 2x_2 - x_3 + x_4 = 0$
58. $x_1 + x_2 - x_3 + x_4 = 0$
 $x_1 + x_2 + x_3 - x_4 = 0$
 $2x_2 + x_3 + x_4 = 0$
 $x_2 - x_3 - x_4 = 0$
59. The system of equations for the currents i_1 , i_2 , and i_3 in the network shown in **FIGURE 8.6.1** is

$$\begin{aligned} i_1 + i_2 + i_3 &= 0 \\ -R_1 i_1 + R_2 i_2 &= E_2 - E_1 \\ -R_2 i_2 + R_3 i_3 &= E_3 - E_2 \end{aligned}$$

where R_k and E_k , $k = 1, 2, 3$, are constants.

- (a) Express the system as a matrix equation $\mathbf{AX} = \mathbf{B}$.
 (b) Show that the coefficient matrix $\mathbf{A}$ is nonsingular.
 (c) Use $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ to solve for the currents.

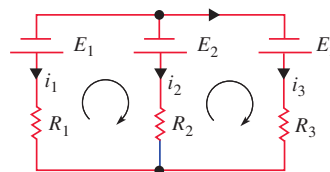


FIGURE 8.6.1 Network in Problem 59

60. Consider the square plate shown in **FIGURE 8.6.2**, with the temperatures as indicated on each side. Under some circumstances it can be shown that the approximate temperatures u_1 , u_2 , u_3 , and u_4 at the points P_1 , P_2 , P_3 , and P_4 , respectively, are given by

$$\begin{aligned} u_1 &= \frac{u_2 + u_4 + 100 + 100}{4} \\ u_2 &= \frac{200 + u_3 + u_1 + 100}{4} \\ u_3 &= \frac{200 + 100 + u_4 + u_2}{4} \\ u_4 &= \frac{u_3 + 100 + 100 + u_1}{4} \end{aligned}$$

- (a) Show that the above system can be written as the matrix equation

$$\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -200 \\ -300 \\ -300 \\ -200 \end{pmatrix}.$$

- (b) Solve the system in part (a) by finding the inverse of the coefficient matrix.

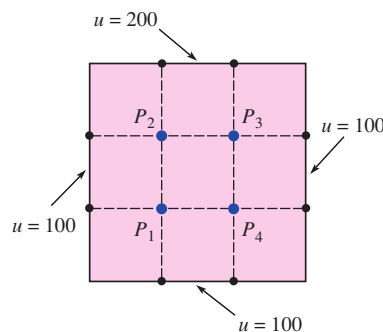


FIGURE 8.6.2 Plate in Problem 60

8.8 The Eigenvalue Problem

INTRODUCTION If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{K}$ is an $n \times 1$ matrix (column vector), then the product $\mathbf{AK}$ is defined and is another $n \times 1$ matrix. It is important in many applications to determine whether there exist nonzero $n \times 1$ matrices $\mathbf{K}$ such that the product vector $\mathbf{AK}$ is a constant multiple λ of $\mathbf{K}$ itself. The problem of solving $\mathbf{AK} = \lambda\mathbf{K}$ for nonzero vectors $\mathbf{K}$ is called the **eigenvalue problem** for the matrix $\mathbf{A}$.

A Definition The foregoing introductory remarks are summarized in the next definition.

Definition 8.8.1 Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix. A number λ is said to be an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero solution vector $\mathbf{K}$ of the linear system

$$\mathbf{AK} = \lambda\mathbf{K}. \quad (1)$$

The solution vector $\mathbf{K}$ is said to be an **eigenvector** corresponding to the eigenvalue λ .

The word *eigenvalue* is a combination of German and English terms adapted from the German word *eigenwert*, which, translated literally, is *proper value*. Eigenvalues and eigenvectors are also called **characteristic values** and **characteristic vectors**, respectively.

Gauss–Jordan elimination introduced in Section 8.2 can be used to find the **eigenvectors** of a square matrix $\mathbf{A}$.

EXAMPLE 1 Verification of an Eigenvector

Verify that $\mathbf{K} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}.$$

SOLUTION By carrying out the multiplication $\mathbf{AK}$ we see

$$\mathbf{AK} = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \overset{\text{eigenvalue}}{\downarrow} (-2)\mathbf{K}.$$

We see from the preceding line and Definition 8.8.1 that $\lambda = -2$ is an eigenvalue of $\mathbf{A}$. $\equiv$

Using properties of matrix algebra, we can write (1) in the alternative form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}, \quad (2)$$

where $\mathbf{I}$ is the multiplicative identity. If we let

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix},$$

then (2) is the same as

$$\begin{aligned} (a_{11} - \lambda)k_1 + a_{12}k_2 + \cdots + a_{1n}k_n &= 0 \\ a_{21}k_1 + (a_{22} - \lambda)k_2 + \cdots + a_{2n}k_n &= 0 \\ \vdots & \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + (a_{nn} - \lambda)k_n &= 0. \end{aligned} \quad (3)$$

Although an obvious solution of (3) is $k_1 = 0, k_2 = 0, \dots, k_n = 0$, we are seeking only nontrivial solutions. Now we know from Theorem 8.6.5 that a homogeneous system of n linear equations in n variables has a nontrivial solution if and only if the determinant of the coefficient matrix is equal to zero. Thus to find a nonzero solution $\mathbf{K}$ for (2), we must have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \quad (4)$$

Inspection of (4) shows that expansion of $\det(\mathbf{A} - \lambda\mathbf{I})$ by cofactors results in an n th-degree polynomial in λ . The equation (4) is called the **characteristic equation** of $\mathbf{A}$. Thus, *the eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation*. To find an eigenvector corresponding to an eigenvalue λ , we simply solve the system of equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$ by applying Gauss–Jordan elimination to the augmented matrix $(\mathbf{A} - \lambda\mathbf{I} | \mathbf{0})$.

EXAMPLE 2 Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}. \quad (5)$$

SOLUTION To expand the determinant in the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0,$$

we use the cofactors of the second row. It follows that the characteristic equation is

$$-\lambda^3 - \lambda^2 + 12\lambda = 0 \quad \text{or} \quad \lambda(\lambda + 4)(\lambda - 3) = 0.$$

Hence the eigenvalues are $\lambda_1 = 0, \lambda_2 = -4, \lambda_3 = 3$. To find the eigenvectors, we must now reduce $(\mathbf{A} - \lambda\mathbf{I} | \mathbf{0})$ three times corresponding to the three distinct eigenvalues.

For $\lambda_1 = 0$, we have

$$\begin{aligned} (\mathbf{A} - 0\mathbf{I} | \mathbf{0}) &= \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array} \right) \xrightarrow[\substack{-6R_1+R_2 \\ R_1+R_3}]{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ &\xrightarrow[\Rightarrow]{-\frac{1}{13}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[\Rightarrow]{-2R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{13} & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Thus we see that $k_1 = -\frac{1}{13}k_3$ and $k_2 = -\frac{6}{13}k_3$. Choosing $k_3 = -13$ gives the eigenvector

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}.$$

For $\lambda_2 = -4$,

$$\begin{aligned}
 (\mathbf{A} + 4\mathbf{I}|\mathbf{0}) &= \left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array} \right) \xrightarrow[\substack{-R_3 \\ R_1 \leftrightarrow R_3}]{\substack{-R_3 \\ R_1 \leftrightarrow R_3}} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array} \right) \\
 &\xrightarrow[\substack{-6R_1+R_2 \\ -5R_1+R_3}]{\substack{-6R_1+R_2 \\ -5R_1+R_3}} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & -8 & 16 & 0 \end{array} \right) \xrightarrow[\substack{-\frac{1}{9}R_2 \\ -\frac{1}{8}R_3}]{\substack{-\frac{1}{9}R_2 \\ -\frac{1}{8}R_3}} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right) \\
 &\xrightarrow[\substack{-2R_2+R_1 \\ -R_2+R_3}]{\substack{-2R_2+R_1 \\ -R_2+R_3}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)
 \end{aligned}$$

implies $k_1 = -k_3$ and $k_2 = 2k_3$. Choosing $k_3 = 1$ then yields a second eigenvector

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Finally, for $\lambda_3 = 3$, Gauss–Jordan elimination gives

$$(\mathbf{A} - 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array} \right) \xrightarrow[\text{row operations}]{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

and so $k_1 = -k_3$ and $k_2 = -\frac{3}{2}k_3$. The choice of $k_3 = -2$ leads to a third eigenvector,

$$\mathbf{K}_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$$

≡

See Figure 8.3.1 on page 392 and reread Theorem 8.2.2(i).

It should be obvious from Example 2 that eigenvectors are not uniquely determined. For example, in obtaining, say, $\mathbf{K}_2$, had we not chosen a specific value for k_3 then a solution of the homogeneous system $(\mathbf{A} + 4\mathbf{I})\mathbf{K} = \mathbf{0}$ can be written

$$\mathbf{K}_2 = \begin{pmatrix} -k_3 \\ 2k_3 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The point of showing $\mathbf{K}_2$ in this form is:

A nonzero constant multiple of an eigenvector is another eigenvector.

For example, $\lambda = 7$ and $\mathbf{K} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$ are, in turn, an eigenvalue and corresponding eigenvector of the matrix $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -4 & 1 \end{pmatrix}$. In practice, it may be more convenient to work with an eigenvector with integer entries:

$$\mathbf{K}_1 = 6\mathbf{K} = 6 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Observe that

$$\mathbf{AK} = \begin{pmatrix} 5 & -3 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ \frac{7}{3} \end{pmatrix} = 7 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \lambda \mathbf{K}$$

and

$$\mathbf{AK}_1 = \begin{pmatrix} 5 & -3 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -21 \\ 14 \end{pmatrix} = 7 \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \lambda \mathbf{K}_1.$$

When an $n \times n$ matrix $\mathbf{A}$ possesses n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, it can be proved that a set of n linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n$ can be found. However, when the characteristic equation has repeated roots, it may not be possible to find n linearly independent eigenvectors for $\mathbf{A}$.

EXAMPLE 3 Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$.

SOLUTION From the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & 4 \\ -1 & 7 - \lambda \end{vmatrix} = (\lambda - 5)^2 = 0,$$

we see $\lambda_1 = \lambda_2 = 5$ is an eigenvalue of multiplicity 2. In the case of a 2×2 matrix, there is no need to use Gauss–Jordan elimination. To find the eigenvector(s) corresponding to $\lambda_1 = 5$, we resort to the system $(\mathbf{A} - 5\mathbf{I})\mathbf{0}$ in its equivalent form

$$\begin{aligned} -2k_1 + 4k_2 &= 0 \\ -k_1 + 2k_2 &= 0. \end{aligned}$$

It is apparent from this system that $k_1 = 2k_2$. Thus, if we choose $k_2 = 1$, we find the single eigenvector $\mathbf{K}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. ≡

EXAMPLE 4 Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$.

SOLUTION The characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 9 - \lambda & 1 & 1 \\ 1 & 9 - \lambda & 1 \\ 1 & 1 & 9 - \lambda \end{vmatrix} = -(\lambda - 11)(\lambda - 8)^2 = 0$$

shows that $\lambda_1 = 11$ and that $\lambda_2 = \lambda_3 = 8$ is an eigenvalue of multiplicity 2.

For $\lambda_1 = 11$, Gauss–Jordan elimination gives

$$(\mathbf{A} - 11\mathbf{I})\mathbf{0} = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) \xRightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Hence $k_1 = k_3$ and $k_2 = k_3$. If $k_3 = 1$, then

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now for $\lambda_2 = 8$ we have

$$(\mathbf{A} - 8\mathbf{I} | \mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

In the equation $k_1 + k_2 + k_3 = 0$ we are free to select two of the variables arbitrarily. Choosing, on the one hand, $k_2 = 1, k_3 = 0$, and on the other, $k_2 = 0, k_3 = 1$, we obtain two linearly independent eigenvectors:

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

corresponding to a single eigenvalue. ≡

Complex Eigenvalues A matrix $\mathbf{A}$ may have **complex eigenvalues**.

Theorem 8.8.1 Complex Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be a square matrix with real entries. If $\lambda = \alpha + i\beta, \beta \neq 0$, is a complex eigenvalue of $\mathbf{A}$, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$. If $\mathbf{K}$ is an eigenvector corresponding to λ , then its conjugate $\bar{\mathbf{K}}$ is an eigenvector corresponding to $\bar{\lambda}$.

PROOF: Since $\mathbf{A}$ is a matrix with real entries, the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is a polynomial equation with real coefficients. From algebra we know that complex roots of such equations appear in conjugate pairs. In other words, if $\lambda = \alpha + i\beta$ is a root, then $\bar{\lambda} = \alpha - i\beta$ is also a root. Now let $\mathbf{K}$ be an eigenvector of $\mathbf{A}$ corresponding to λ . By definition, $\mathbf{A}\mathbf{K} = \lambda\mathbf{K}$. Taking complex conjugates of the latter equation gives

$$\overline{\mathbf{A}\mathbf{K}} = \overline{\lambda\mathbf{K}} \quad \text{or} \quad \mathbf{A}\bar{\mathbf{K}} = \bar{\lambda}\bar{\mathbf{K}},$$

since $\mathbf{A}$ is a real matrix. The last equation indicates that $\bar{\mathbf{K}}$ is an eigenvector corresponding to $\bar{\lambda}$. ≡

EXAMPLE 5 Complex Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$.

SOLUTION The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

From the quadratic formula, we find $\lambda_1 = 5 + 2i$ and $\lambda_2 = \bar{\lambda}_1 = 5 - 2i$.

Now for $\lambda_1 = 5 + 2i$, we must solve

$$\begin{aligned} (1 - 2i)k_1 - k_2 &= 0 \\ 5k_1 - (1 + 2i)k_2 &= 0. \end{aligned}$$

Since $k_2 = (1 - 2i)k_1$,* it follows, after choosing $k_1 = 1$, that one eigenvector is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}.$$

From Theorem 8.8.1, we see that an eigenvector corresponding to $\lambda_2 = \bar{\lambda}_1 = 5 - 2i$ is

$$\mathbf{K}_2 = \bar{\mathbf{K}}_1 = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}. \quad \equiv$$

|| Eigenvalues and Singular Matrices If the number 0 is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then by Definition 8.8.1 the homogeneous linear system

$$\begin{array}{c} \lambda = 0 \\ \downarrow \\ \mathbf{A}\mathbf{K} = 0\mathbf{K} = \mathbf{0} \end{array}$$

of n equations in n variables must possess a *nontrivial* solution vector $\mathbf{K}$. But this fact tells us something important about the matrix $\mathbf{A}$. By Theorem 8.6.6 a homogeneous system of n equations in n variables possesses a nontrivial solution if and only if the coefficient matrix $\mathbf{A}$ is *singular*. We summarize this result in the following theorem.

Theorem 8.8.2 Zero Eigenvalue and a Singular Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the number $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

Put a different way:

A matrix $\mathbf{A}$ is nonsingular if and only if the number 0 is not an eigenvalue of $\mathbf{A}$. (6)

Reexamination of Example 2 shows that $\lambda_1 = 0$ is an eigenvalue of the 3×3 matrix (5). So we can conclude from Theorem 8.8.2 that the matrix (5) is singular, that is, does not possess an inverse. On the other hand, we conclude from (6) that the matrices in Examples 3 and 4 are nonsingular because none of the eigenvalues of the matrices are 0.

The eigenvalues of an $n \times n$ matrix $\mathbf{A}$ are related to $\det \mathbf{A}$. Because the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ is an n th degree polynomial equation, it must, counting multiplicities and complex numbers, have n roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$. By the Factor Theorem of algebra, the characteristic polynomial $\det(\mathbf{A} - \lambda \mathbf{I})$ can then be written

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda). \quad (7)$$

By setting $\lambda = 0$ in (7) we see that

$$\det \mathbf{A} = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n, \quad (8)$$

that is:

The determinant of $\mathbf{A}$ is the product of its eigenvalues.

The result in (8) provides an alternative proof of Theorem 8.8.2: If, say, $\lambda_1 = 0$ then $\det \mathbf{A} = 0$. Conversely, if $\det \mathbf{A} = 0$, then $\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n = 0$ implies that at least one of the eigenvalues of $\mathbf{A}$ is 0.

*Note that the second equation is simply $1 + 2i$ times the first.

EXAMPLE 6 Examples 4 and 5 Revisited

(a) In Example 4 we saw that the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$$

are $\lambda_1 = 11$, $\lambda_2 = \lambda_3 = 8$. With no further work we see that $\det \mathbf{A} = 11 \cdot 8 \cdot 8 = 704$. Because $\det \mathbf{A} \neq 0$ the matrix $\mathbf{A}$ is nonsingular.

(b) In Example 5 it is easily seen that the determinant of the 2×2 matrix $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$ is

$$\det \mathbf{A} = \begin{vmatrix} 6 & -1 \\ 5 & 4 \end{vmatrix} = 29.$$

Also, using the two complex eigenvalues $\lambda_1 = 5 + 2i$, $\lambda_2 = 5 - 2i$ we see that $\det \mathbf{A} = \lambda_1 \lambda_2 = (5 + 2i)(5 - 2i) = 5^2 + 2^2 = 29$. $\equiv$

Eigenvalues and Eigenvectors of $\mathbf{A}^{-1}$ If an $n \times n$ matrix $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1}$ exists and we now know that none of the eigenvalues of $\mathbf{A}$ are 0. Suppose that λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{K}$. By multiplying both sides of the equation $\mathbf{A}\mathbf{K} = \lambda\mathbf{K}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}(\mathbf{A}\mathbf{K}) = \lambda\mathbf{A}^{-1}\mathbf{K}$$

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{K} = \lambda\mathbf{A}^{-1}\mathbf{K}$$

$$\mathbf{I}\mathbf{K} = \lambda\mathbf{A}^{-1}\mathbf{K}$$

$$\text{or} \quad \mathbf{A}^{-1}\mathbf{K} = \frac{1}{\lambda}\mathbf{K}. \quad (9)$$

Comparing the result in (9) with (1) leads us to the following conclusion.

Theorem 8.8.3 Eigenvalues and Eigenvectors of $\mathbf{A}^{-1}$

Let $\mathbf{A}$ be a nonsingular matrix. If λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{K}$, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$ with the same corresponding eigenvector $\mathbf{K}$.

EXAMPLE 7 Eigenvalues of an Inverse

The matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$ has distinct eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since 0 is not an eigenvalue of $\mathbf{A}$ it is nonsingular. So in view of Theorem 8.8.3 we can also say that the reciprocals

$$\lambda_3 = \frac{1}{\lambda_1} = \frac{1}{4} \quad \text{and} \quad \lambda_4 = \frac{1}{\lambda_2} = \frac{1}{3}$$

are, respectively, eigenvalues of $\mathbf{A}^{-1}$ with corresponding eigenvectors $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This is easily verified using $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$:

$$\mathbf{A}^{-1}\mathbf{K}_1 = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\lambda_1} \mathbf{K}_1 = \lambda_3 \mathbf{K}_1$$

$$\mathbf{A}^{-1}\mathbf{K}_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\lambda_2} \mathbf{K}_2 = \lambda_4 \mathbf{K}_2. \quad \equiv$$

Our last theorem follows immediately from the fact that the determinant of an upper triangular, lower triangular, and a diagonal matrix is the product of the main diagonal entries. See Theorem 8.5.8.

Theorem 8.8.4 Triangular and Diagonal Matrices

The eigenvalues of an upper triangular, lower triangular, and diagonal matrix are the main diagonal entries.

Notice in Example 7 that the matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$ is a lower triangular matrix and its eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ are the entries on the main diagonal.

8.8 Exercises

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–6, determine which of the indicated column vectors are eigenvectors of the given matrix $\mathbf{A}$. Give the corresponding eigenvalue.

1. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$,

$\mathbf{K}_3 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 - \sqrt{2} \end{pmatrix}$,

$\mathbf{K}_2 = \begin{pmatrix} 2 + \sqrt{2} \\ 2 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$

3. $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$,

$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$

4. $\mathbf{A} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$\mathbf{K}_2 = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix}$

5. $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,

$\mathbf{K}_2 = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

6. $\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$,

$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

In Problems 7–22, find the eigenvalues and eigenvectors of the given matrix. Using Theorem 8.8.2 or (6), state whether the matrix is singular or nonsingular.

7. $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

9. $\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{pmatrix}$

11. $\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$

12. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

13. $\begin{pmatrix} 4 & 8 \\ 0 & -5 \end{pmatrix}$

14. $\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$

15. $\begin{pmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{pmatrix}$

16. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 0 & 4 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

19. $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

20. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{pmatrix}$

22. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In Problems 23–26, find the eigenvalues and eigenvectors of the given nonsingular matrix $\mathbf{A}$. Then without finding $\mathbf{A}^{-1}$, find its eigenvalues and corresponding eigenvectors.

23. $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

24. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 7 & -1 \end{pmatrix}$

$$25. \mathbf{A} = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{pmatrix} \quad 26. \mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

Discussion Problems

27. Review the definitions of upper triangular, lower triangular, and diagonal matrices on pages 372–373. Explain why the eigenvalues for those matrices are the main diagonal entries.
28. True or false: If λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular. Justify your answer.
29. Suppose λ is an eigenvalue with corresponding eigenvector $\mathbf{K}$ of an $n \times n$ matrix $\mathbf{A}$.
- If $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$, then show that $\mathbf{A}^2\mathbf{K} = \lambda^2\mathbf{K}$. Explain the meaning of the last equation.
 - Verify the result obtained in part (a) for the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$.
 - Generalize the result in part (a).
30. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. The matrix $\mathbf{B}$ is said to be **similar** to the matrix $\mathbf{A}$ if there exists a nonsingular matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. If $\mathbf{B}$ is similar to $\mathbf{A}$, then show that $\mathbf{A}$ is similar to $\mathbf{B}$.
31. Suppose $\mathbf{A}$ and $\mathbf{B}$ are similar matrices. See Problem 30. Show that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues. [Hint: Review Theorem 8.5.6 and Problem 37 in Exercises 8.6.]

Computer Lab Assignment

32. An $n \times n$ matrix $\mathbf{A}$ is said to be a **stochastic matrix** if all its entries are nonnegative and the sum of the entries in each row (or the sum of the entries in each column) add up to 1. Stochastic matrices are important in probability theory.
- Verify that

$$\mathbf{A} = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}, \quad 0 \leq p \leq 1, 0 \leq q \leq 1,$$

$$\text{and} \quad \mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

are stochastic matrices.

- Use a CAS or linear algebra software to find the eigenvalues and eigenvectors of the 3×3 matrix $\mathbf{A}$ in part (a). Make up at least six more stochastic matrices of various sizes, 2×2 , 3×3 , 4×4 , and 5×5 . Find the eigenvalues and eigenvectors of each matrix. If you discern a pattern, form a conjecture and then try to prove it.
- For the 3×3 matrix $\mathbf{A}$ in part (a), use the software to find $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{A}^4$, $\dots$. Repeat for the matrices that you constructed in part (b). If you discern a pattern, form a conjecture and then try to prove it.

8.9 Powers of Matrices

INTRODUCTION It is sometimes important to be able to quickly compute a power $\mathbf{A}^m$, m a positive integer, of an $n \times n$ matrix $\mathbf{A}$:

$$\mathbf{A}^m = \underbrace{\mathbf{A}\mathbf{A}\mathbf{A} \cdots \mathbf{A}}_{m \text{ factors}}$$

Of course, computation of $\mathbf{A}^m$ could be done with the appropriate software or by writing a short computer program, but even then, you should be aware that it is inefficient to simply use brute force to carry out repeated multiplications: $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2$, $\mathbf{A}^4 = \mathbf{A}\mathbf{A}\mathbf{A}\mathbf{A} = \mathbf{A}(\mathbf{A}^3) = \mathbf{A}^2\mathbf{A}^2$, and so on.

Computation of $\mathbf{A}^m$ We are going to sketch an alternative method for computing $\mathbf{A}^m$ by means of the following theorem known as the Cayley–Hamilton theorem.

Theorem 8.9.1 Cayley–Hamilton Theorem

An $n \times n$ matrix $\mathbf{A}$ satisfies its own characteristic equation.

If $(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$ is the characteristic equation of $\mathbf{A}$, then Theorem 8.9.1 states that by replacing λ by $\mathbf{A}$ we have

$$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}. \quad (1)$$

Matrices of Order 2 The characteristic equation of the 2×2 matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ is $\lambda^2 - \lambda - 2 = 0$, and the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. Theorem 8.9.1 implies $\mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = \mathbf{0}$, or, solving for the highest power of $\mathbf{A}$,

$$\mathbf{A}^2 = 2\mathbf{I} + \mathbf{A}. \quad (2)$$

Now if we multiply (2) by $\mathbf{A}$, we get $\mathbf{A}^3 = 2\mathbf{A} + \mathbf{A}^2$, and if we use (2) again to eliminate $\mathbf{A}^2$ on the right side of this new equation, then

$$\mathbf{A}^3 = 2\mathbf{A} + \mathbf{A}^2 = 2\mathbf{A} + (2\mathbf{I} + \mathbf{A}) = 2\mathbf{I} + 3\mathbf{A}.$$

Continuing in this manner—in other words, multiplying the last result by $\mathbf{A}$ and using (2) to eliminate $\mathbf{A}^2$ —we obtain in succession powers of $\mathbf{A}$ expressed solely in terms of the identity matrix $\mathbf{I}$ and $\mathbf{A}$:

$$\begin{aligned} \mathbf{A}^4 &= 6\mathbf{I} + 5\mathbf{A} \\ \mathbf{A}^5 &= 10\mathbf{I} + 11\mathbf{A} \\ \mathbf{A}^6 &= 22\mathbf{I} + 21\mathbf{A} \end{aligned} \quad (3)$$

and so on (verify). Thus, for example,

$$\mathbf{A}^6 = 22 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 21 \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -20 & 84 \\ -21 & 85 \end{pmatrix}. \quad (4)$$

Now we can determine the c_k without actually carrying out the repeated multiplications and resubstitutions as we did in (3). First, note that since the characteristic equation of the matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ can be written $\lambda^2 = 2 + \lambda$, results analogous to (3) must also hold for the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$; that is, $\lambda^3 = 2 + 3\lambda$, $\lambda^4 = 6 + 5\lambda$, $\lambda^5 = 10 + 11\lambda$, $\lambda^6 = 22 + 21\lambda, \dots$. It follows then that the equations

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} \quad \text{and} \quad \lambda^m = c_0 + c_1\lambda \quad (5)$$

hold for the same pair of constants c_0 and c_1 . We can determine the constants c_0 and c_1 by simply setting $\lambda = -1$ and $\lambda = 2$ in the last equation in (5) and solving the resulting system of two equations in two unknowns. The solution of the system

$$\begin{aligned} (-1)^m &= c_0 + c_1(-1) \\ 2^m &= c_0 + c_1(2) \end{aligned}$$

is $c_0 = \frac{1}{3}[2^m + 2(-1)^m]$, $c_1 = \frac{1}{3}[2^m - (-1)^m]$. Now by substituting these coefficients in the first equation in (5), adding the two matrices and simplifying each entry, we obtain

$$\mathbf{A}^m = \begin{pmatrix} \frac{1}{3}[-2^m + 4(-1)^m] & \frac{4}{3}[2^m - (-1)^m] \\ -\frac{1}{3}[2^m - (-1)^m] & \frac{1}{3}[2^{m+2} - (-1)^m] \end{pmatrix}. \quad (6)$$

You should verify the result in (4) by setting $m = 6$ in (6). Note that (5) and (6) are valid for $m \geq 0$ since $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{A}^1 = \mathbf{A}$.

Matrices of Order n If the matrix $\mathbf{A}$ were 3×3 , then the characteristic equation (1) is a cubic polynomial equation, and the analogue of (2) would enable us to express $\mathbf{A}^3$ in terms of

$\mathbf{I}$, $\mathbf{A}$, and $\mathbf{A}^2$. We could proceed as just illustrated to write any power $\mathbf{A}^m$ in terms of $\mathbf{I}$, $\mathbf{A}$, and $\mathbf{A}^2$. In general, for an $n \times n$ matrix $\mathbf{A}$, we can write

$$\mathbf{A}^m = c_0 \mathbf{I} + c_1 \mathbf{A} + c_2 \mathbf{A}^2 + \cdots + c_{n-1} \mathbf{A}^{n-1},$$

and where each of the coefficients c_k , $k = 0, 1, \dots, n-1$, depends on the value of m .

EXAMPLE 1 $\mathbf{A}^m$ for a 3×3 Matrix

Compute $\mathbf{A}^m$ for $\mathbf{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

SOLUTION The characteristic equation of $\mathbf{A}$ is $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$ or $\lambda^3 = -2 + \lambda + 2\lambda^2$, and the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$. From the preceding discussion we know that the same coefficients hold in both of the following equations:

$$\mathbf{A}^m = c_0 \mathbf{I} + c_1 \mathbf{A} + c_2 \mathbf{A}^2 \quad \text{and} \quad \lambda^m = c_0 + c_1 \lambda + c_2 \lambda^2. \quad (7)$$

Setting, in turn, $\lambda = -1$, $\lambda = 1$, $\lambda = 2$ in the last equation generates three equations in three unknowns:

$$\begin{aligned} (-1)^m &= c_0 - c_1 + c_2 \\ 1 &= c_0 + c_1 + c_2 \\ 2^m &= c_0 + 2c_1 + 4c_2. \end{aligned} \quad (8)$$

Solving (8) gives

$$\begin{aligned} c_0 &= \frac{1}{3}[3 + (-1)^m - 2^m], \\ c_1 &= 2[1 - (-1)^m], \\ c_2 &= \frac{1}{6}[-3 + (-1)^m + 2^{m+1}]. \end{aligned}$$

After computing $\mathbf{A}^2$, we substitute these coefficients into the first equation of (7) and simplify the entries of the resulting matrix. The result is

$$\mathbf{A}^m = \begin{pmatrix} \frac{1}{6}[9 - 2^{m+1} - (-1)^m] & \frac{1}{3}[2^m - (-1)^m] & \frac{1}{6}[-9 + 2^{m+1} + 7(-1)^m] \\ 1 - 2^m & 2^m & 2^m - 1 \\ \frac{1}{6}[3 - 2^{m+1} - (-1)^m] & \frac{1}{3}[2^m - (-1)^m] & \frac{1}{6}[-3 + 2^{m+1} + 7(-1)^m] \end{pmatrix}.$$

For example, with $m = 10$,

$$\mathbf{A}^{10} = \begin{pmatrix} -340 & 341 & 341 \\ -1023 & 1024 & 1023 \\ -341 & 341 & 342 \end{pmatrix}. \quad \equiv$$

Finding the Inverse Suppose $\mathbf{A}$ is a nonsingular matrix. The fact that $\mathbf{A}$ satisfies its own characteristic equation can be used to compute $\mathbf{A}^{-1}$ as a linear combination of powers of $\mathbf{A}$.

For example, we have just seen that the nonsingular matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ satisfies $\mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = \mathbf{0}$.

Solving for the identity matrix gives $\mathbf{I} = \frac{1}{2}\mathbf{A}^2 - \frac{1}{2}\mathbf{A}$. By multiplying the last result by $\mathbf{A}^{-1}$, we find $\mathbf{A}^{-1} = \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{I}$. In other words,

$$\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}. \quad (9)$$

REMARKS

There are some obvious problems with the method just illustrated for finding $\mathbf{A}^m$. If, for example, the matrix in Example 1 had an eigenvalue of multiplicity two, then instead of three equations and three unknowns as in (8) we would have only two equations in three unknowns. How do we then find *unique* coefficients c_0, c_1 , and c_2 ? See Problems 11–14 in Exercises 8.9. Also, for larger matrices, even with distinct eigenvalues, solving a large system of equations for $c_0, c_1, c_2, \dots, c_{n-1}$ is far too tedious to do by hand.

8.9

Exercises

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1 and 2, verify that the given matrix satisfies its own characteristic equation.

$$1. \mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} \quad 2. \mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

In Problems 3–10, use the method of this section to compute $\mathbf{A}^m$. Use this result to compute the indicated power of the matrix $\mathbf{A}$.

$$3. \mathbf{A} = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}; \quad m = 3 \quad 4. \mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}; \quad m = 4$$

$$5. \mathbf{A} = \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix}; \quad m = 5 \quad 6. \mathbf{A} = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}; \quad m = 6$$

$$7. \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}; \quad m = 10$$

$$8. \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad m = 6$$

$$9. \mathbf{A} = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}; \quad m = 10$$

$$10. \mathbf{A} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & -2 \end{pmatrix}; \quad m = 8$$

In Problems 11 and 12, show that the given matrix has an eigenvalue λ_1 of multiplicity two. As a consequence, the equations $\lambda^m = c_0 + c_1\lambda$ (Problem 11) and $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ (Problem 12) do not yield enough independent equations to form a system for determining the coefficients c_i . Use the derivative (with respect to λ) of each of these equations evaluated at λ_1 as the extra needed equation to form a system. Compute $\mathbf{A}^m$ and use this result to compute the indicated power of the matrix $\mathbf{A}$.

$$11. \mathbf{A} = \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix}; \quad m = 6$$

$$12. \mathbf{A} = \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & -6 & 0 \end{pmatrix}; \quad m = 5$$

13. Show that $\lambda = 0$ is an eigenvalue of each matrix. In this case, the coefficient c_0 in the characteristic equation (1) is 0. Compute $\mathbf{A}^m$ in each case. In parts (a) and (b), explain why we do not have to solve any system for the coefficients c_i in determining $\mathbf{A}^m$.

$$(a) \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \quad (b) \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$(c) \mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 3 \end{pmatrix}$$

14. In his work *Liber Abbaci*, published in 1202, Leonardo Fibonacci of Pisa, Italy, speculated on the reproduction of rabbits:

How many pairs of rabbits will be produced in a year beginning with a single pair, if every month each pair bears a new pair which become productive from the second month on?

The answer to his question is contained in a sequence known as a Fibonacci sequence.

After Each Month												
Start $n =$	0	1	2	3	4	5	6	7	8	9	10	11 12
Adult pairs	1	1	2	3	5	8	13	21	...			
Baby pairs	0	1	1	2	3	5	8	13	...			
Total pairs	1	2	3	5	8	13	21	34	...			

Each of the three rows describing rabbit pairs is a Fibonacci sequence and can be defined recursively by a **second-order difference equation** $x_n = x_{n-2} + x_{n-1}$, $n = 2, 3, \dots$, where x_0 and x_1 depend on the row. For example, for the first row designating adult pairs of rabbits, $x_0 = 1$, $x_1 = 1$.

- (a) If we let $y_{n-1} = x_{n-2}$, then $y_n = x_{n-1}$, and the difference equation can be written as a system of first-order difference equations

$$\begin{aligned} x_n &= x_{n-1} + y_{n-1} \\ y_n &= x_{n-1} \end{aligned}$$

Write this system in the matrix form $\mathbf{X}_n = \mathbf{A}\mathbf{X}_{n-1}$, $n = 2, 3, \dots$.

(b) Show that

$$\mathbf{A}^m = \begin{pmatrix} \frac{\lambda_2 \lambda_1^m - \lambda_1 \lambda_2^m + \lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1} & \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1} \\ \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1} & \frac{\lambda_2 \lambda_1^m - \lambda_1 \lambda_2^m}{\lambda_2 - \lambda_1} \end{pmatrix}$$

or

$$\mathbf{A}^m = \frac{1}{2^{m+1}\sqrt{5}} \begin{pmatrix} (1 + \sqrt{5})^{m+1} - (1 - \sqrt{5})^{m+1} & 2(1 + \sqrt{5})^m - 2(1 - \sqrt{5})^m \\ 2(1 + \sqrt{5})^m - 2(1 - \sqrt{5})^m & (1 + \sqrt{5})(1 - \sqrt{5})^m - (1 - \sqrt{5})(1 + \sqrt{5})^m \end{pmatrix},$$

where $\lambda_1 = \frac{1}{2}(1 - \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 + \sqrt{5})$ are the distinct eigenvalues of $\mathbf{A}$.

(c) Use the result in part (a) to show $\mathbf{X}_n = \mathbf{A}^{n-1}\mathbf{X}_1$. Use the last result and the result in part (b) to find the number of adult pairs, baby pairs, and total pairs of rabbits after the twelfth month.

In Problems 15 and 16, use the procedure illustrated in (9) to find $\mathbf{A}^{-1}$.

15. $\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$

16. $\mathbf{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

17. A nonzero $n \times n$ matrix $\mathbf{A}$ is said to be **nilpotent of index m** if m is the smallest positive integer for which $\mathbf{A}^m = \mathbf{0}$. Which of the following matrices are nilpotent? If nilpotent, what is its index?

(a) $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$

18. (a) Explain why any nilpotent matrix $\mathbf{A}$ is singular. [Hint: Review Section 8.5.] (b) Show that all the eigenvalues of a nilpotent matrix $\mathbf{A}$ are 0. [Hint: Use (1) of Section 8.8.]

8.10 Orthogonal Matrices

INTRODUCTION In this section we are going to use some elementary properties of **complex numbers**. Suppose $z = a + ib$ denotes a complex number, where a and b are real and the symbol i is defined by $i^2 = -1$. If $\bar{z} = a - ib$ is the **conjugate** of z , then the equality $z = \bar{z}$ or $a + ib = a - ib$ implies that $b = 0$. In other words, if $z = \bar{z}$, then z is a real number. In addition, it is easily verified that the product of a complex number z and its conjugate $\bar{z}$ is a real number: $z\bar{z} = a^2 + b^2$. The **magnitude** of z is defined to be the real number $|z| = \sqrt{a^2 + b^2}$. The magnitude of z can be expressed in terms of the product $z\bar{z}$: $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$, or $|z|^2 = z\bar{z}$. A detailed discussion of complex numbers can be found in Section 17.1.

There are many types of special matrices, but two types occur again and again in applications: symmetric matrices (page 373) and orthogonal matrices (page 413). In this section we are going to examine both these matrices in further detail.

Symmetric Matrices We begin by recalling, in formal terms, the definition of a symmetric matrix.

Definition 8.10.1 Symmetric Matrix

An $n \times n$ matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A} = \mathbf{A}^T$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

The proof of the next theorem depends on the properties of complex numbers discussed in the review at the start of this section.

Theorem 8.10.1 Real Eigenvalues

Let $\mathbf{A}$ be a symmetric matrix with real entries. Then the eigenvalues of $\mathbf{A}$ are real.

PROOF: If $\mathbf{K}$ is an eigenvector corresponding to an eigenvalue λ of $\mathbf{A}$, then $\mathbf{AK} = \lambda\mathbf{K}$. The conjugate of the last equation is

$$\overline{\mathbf{AK}} = \overline{\lambda\mathbf{K}}. \quad (1)$$

Since the entries of $\mathbf{A}$ are real, we have $\mathbf{A} = \overline{\mathbf{A}}$, and so (1) is

$$\mathbf{AK} = \overline{\lambda}\overline{\mathbf{K}}. \quad (2)$$

We now take the transpose of (2), use the fact that $\mathbf{A}$ is symmetric, and multiply the resulting equation on the right by $\mathbf{K}$:

$$\overline{\mathbf{K}}^T \mathbf{AK} = \overline{\lambda} \overline{\mathbf{K}}^T \mathbf{K}. \quad (3)$$

But if we multiply $\mathbf{AK} = \lambda\mathbf{K}$ on the left by $\overline{\mathbf{K}}^T$, we obtain

$$\overline{\mathbf{K}}^T \mathbf{AK} = \lambda \overline{\mathbf{K}}^T \mathbf{K}. \quad (4)$$

Subtracting (4) from (3) then gives

$$0 = (\overline{\lambda} - \lambda) \overline{\mathbf{K}}^T \mathbf{K}. \quad (5)$$

Now $\overline{\mathbf{K}}^T$ is a $1 \times n$ matrix and $\mathbf{K}$ is an $n \times 1$ matrix, so the product $\overline{\mathbf{K}}^T \mathbf{K}$ is the 1×1 matrix $\overline{\mathbf{K}}^T \mathbf{K} = (|k_1|^2 + |k_2|^2 + \cdots + |k_n|^2)$. Since by definition, $\mathbf{K} \neq \mathbf{0}$, the last expression is a positive quantity. Therefore we conclude from (5) that $\overline{\lambda} - \lambda = 0$ or $\overline{\lambda} = \lambda$. This implies that λ is a real number. $\equiv$

Inner Product In R^n the **inner product** or **dot** of two vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \quad (6)$$

Now if $\mathbf{X}$ and $\mathbf{Y}$ are $n \times 1$ column vectors, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, then the matrix analogue of (6) is

$$\mathbf{X} \cdot \mathbf{Y} = \mathbf{X}^T \mathbf{Y} = (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n).^* \quad (7)$$

Of course, for the column vectors given, $\mathbf{Y}^T \mathbf{X} = \mathbf{X}^T \mathbf{Y}$. The **norm** of a column vector $\mathbf{X}$ is given by

$$\|\mathbf{X}\| = \sqrt{\mathbf{X} \cdot \mathbf{X}} = \sqrt{\mathbf{X}^T \mathbf{X}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Theorem 8.10.2 Orthogonal Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Then eigenvectors corresponding to distinct (different) eigenvalues are orthogonal.

*Since a 1×1 matrix is simply a scalar, we will hereafter drop the parentheses and write $\mathbf{X}^T \mathbf{Y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

PROOF: Let λ_1 and λ_2 be two distinct eigenvalues of $\mathbf{A}$ corresponding to the eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively. We wish to show that $\mathbf{K}_1 \cdot \mathbf{K}_2 = \mathbf{K}_1^T \mathbf{K}_2 = 0$.

Now by definition we must have

$$\mathbf{A}\mathbf{K}_1 = \lambda_1\mathbf{K}_1 \quad \text{and} \quad \mathbf{A}\mathbf{K}_2 = \lambda_2\mathbf{K}_2. \quad (8)$$

We form the transpose of the first of these equations, use $\mathbf{A}^T = \mathbf{A}$, and then multiply the result on the right by $\mathbf{K}_2$:

$$\mathbf{K}_1^T \mathbf{A} \mathbf{K}_2 = \lambda_1 \mathbf{K}_1^T \mathbf{K}_2. \quad (9)$$

The second equation in (8) is multiplied on the left by $\mathbf{K}_1^T$:

$$\mathbf{K}_1^T \mathbf{A} \mathbf{K}_2 = \lambda_2 \mathbf{K}_1^T \mathbf{K}_2. \quad (10)$$

Subtracting (10) from (9) yields

$$0 = \lambda_1 \mathbf{K}_1^T \mathbf{K}_2 - \lambda_2 \mathbf{K}_1^T \mathbf{K}_2 \quad \text{or} \quad 0 = (\lambda_1 - \lambda_2) \mathbf{K}_1^T \mathbf{K}_2.$$

Since $\lambda_1 \neq \lambda_2$, it follows that $\mathbf{K}_1^T \mathbf{K}_2 = 0$. ≡

EXAMPLE 1 Orthogonal Eigenvectors

The eigenvalues of the symmetric matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -2$. In turn, the corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Since all the eigenvalues are distinct, it follows from Theorem 8.10.2 that the eigenvectors are orthogonal, that is,

$$\mathbf{K}_1^T \mathbf{K}_2 = (1 \ 0 \ 1) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 = 0$$

$$\mathbf{K}_1^T \mathbf{K}_3 = (1 \ 0 \ 1) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 \cdot 1 + 0 \cdot 2 + 1 \cdot (-1) = 0$$

$$\mathbf{K}_2^T \mathbf{K}_3 = (-1 \ 1 \ 1) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = (-1) \cdot 1 + 1 \cdot 2 + 1 \cdot (-1) = 0. \quad \equiv$$

We saw in Example 3 of Section 8.8 that it may not be possible to find n linearly independent eigenvectors for an $n \times n$ matrix $\mathbf{A}$ when some of the eigenvalues are repeated. But a symmetric matrix is an exception. It can be proved that a set of n linearly independent eigenvectors can always be found for an $n \times n$ symmetric matrix $\mathbf{A}$ even when there is some repetition of the eigenvalues. (See Example 4 of Section 8.8.)

See (2) of Section 7.6 for the definition of the inner product in R^n .

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ in R^n is called **orthonormal** if every pair of distinct vectors is orthogonal and each vector in the set is a unit vector. In terms of the inner product of vectors, the set is orthonormal if

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0, \quad i \neq j, \quad i, j = 1, 2, \dots, n \quad \text{and} \quad \mathbf{x}_i \cdot \mathbf{x}_i = 1, \quad i = 1, 2, \dots, n.$$

The last condition simply states that $\|\mathbf{x}_i\| = \sqrt{\mathbf{x}_i \cdot \mathbf{x}_i} = 1, i = 1, 2, \dots, n$.

Orthogonal Matrix The concept of an orthonormal set of vectors plays an important role in the consideration of the next type of matrix.

Definition 8.10.2 Orthogonal Matrix

An $n \times n$ nonsingular matrix $\mathbf{A}$ is **orthogonal** if $\mathbf{A}^{-1} = \mathbf{A}^T$.

In other words, $\mathbf{A}$ is orthogonal if $\mathbf{A}^T \mathbf{A} = \mathbf{I}$.

EXAMPLE 2 Orthogonal Matrices

(a) The $n \times n$ identity matrix $\mathbf{I}$ is an orthogonal matrix. For example, in the case of the 3×3 identity

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it is readily seen that $\mathbf{I}^T = \mathbf{I}$ and $\mathbf{I}^T \mathbf{I} = \mathbf{I} \mathbf{I} = \mathbf{I}$.

(b) The matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{4} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

is orthogonal. To see this, we need only verify that $\mathbf{A}^T \mathbf{A} = \mathbf{I}$:

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \equiv$$

Theorem 8.10.3 Criterion for an Orthogonal Matrix

An $n \times n$ matrix $\mathbf{A}$ is orthogonal if and only if its columns $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ form an orthonormal set.

PARTIAL PROOF: Let us suppose that $\mathbf{A}$ is an $n \times n$ orthogonal matrix with columns $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$. Hence, the rows of $\mathbf{A}^T$ are $\mathbf{X}_1^T, \mathbf{X}_2^T, \dots, \mathbf{X}_n^T$. But since $\mathbf{A}$ is orthogonal, $\mathbf{A}^T \mathbf{A} = \mathbf{I}$; that is,

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \mathbf{X}_1^T \mathbf{X}_1 & \mathbf{X}_1^T \mathbf{X}_2 & \cdots & \mathbf{X}_1^T \mathbf{X}_n \\ \mathbf{X}_2^T \mathbf{X}_1 & \mathbf{X}_2^T \mathbf{X}_2 & \cdots & \mathbf{X}_2^T \mathbf{X}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_n^T \mathbf{X}_1 & \mathbf{X}_n^T \mathbf{X}_2 & \cdots & \mathbf{X}_n^T \mathbf{X}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It follows from the definition of equality of matrices that

$$\mathbf{X}_i^T \mathbf{X}_j = 0, \quad i \neq j, \quad i, j = 1, 2, \dots, n \quad \text{and} \quad \mathbf{X}_i^T \mathbf{X}_i = 1, \quad i = 1, 2, \dots, n.$$

This means that the columns of the orthogonal matrix form an orthonormal set of n vectors. $\equiv$

If we write the columns of the matrix in part (b) of Example 2 as

$$\mathbf{X}_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \mathbf{X}_3 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix},$$

then the vectors are orthogonal:

$$\mathbf{X}_1^T \mathbf{X}_2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} = 0$$

$$\mathbf{X}_1^T \mathbf{X}_3 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$$\mathbf{X}_2^T \mathbf{X}_3 = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = -\frac{4}{9} + \frac{2}{9} + \frac{2}{9} = 0$$

and are unit vectors:

$$\mathbf{X}_1^T \mathbf{X}_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

$$\mathbf{X}_2^T \mathbf{X}_2 = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$\mathbf{X}_3^T \mathbf{X}_3 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1.$$

Constructing an Orthogonal Matrix If an $n \times n$ real symmetric matrix $\mathbf{A}$ possesses n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, it follows from Theorem 8.10.2 that the corresponding eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n$ are mutually orthogonal. By multiplying each vector by the reciprocal of its norm, we obtain a set of mutually orthogonal unit vectors—that is, an orthonormal set. We can then construct an orthogonal matrix by forming an $n \times n$ matrix $\mathbf{P}$ whose columns are these normalized eigenvectors of $\mathbf{A}$.

EXAMPLE 3 Constructing an Orthogonal Matrix

In Example 1, we verified that the eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

of the given symmetric matrix $\mathbf{A}$ are orthogonal. Now the norms of the eigenvectors are

$$\|\mathbf{K}_1\| = \sqrt{\mathbf{K}_1^T \mathbf{K}_1} = \sqrt{2}, \quad \|\mathbf{K}_2\| = \sqrt{\mathbf{K}_2^T \mathbf{K}_2} = \sqrt{3}, \quad \|\mathbf{K}_3\| = \sqrt{\mathbf{K}_3^T \mathbf{K}_3} = \sqrt{6}.$$

Thus, an orthonormal set of vectors is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

Using these vectors as columns, we obtain the orthogonal matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

You should verify that $\mathbf{P}^T = \mathbf{P}^{-1}$.

≡

We will use the technique of constructing an orthogonal matrix from the eigenvectors of a symmetric matrix in the next section.

Do not misinterpret Theorem 8.10.2. We can always find n linearly independent eigenvectors for an $n \times n$ real symmetric matrix $\mathbf{A}$. However, the theorem does not state that *all* the eigenvectors are mutually orthogonal. The set of eigenvectors corresponding to distinct eigenvalues are orthogonal, but the different eigenvectors corresponding to a repeated eigenvalue may not be orthogonal. Consider the symmetric matrix in the next example.

EXAMPLE 4 Using the Gram–Schmidt Process

For the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$$

we find that the eigenvalues are $\lambda_1 = \lambda_2 = -9$, and $\lambda_3 = 9$. Proceeding as in Section 8.8, for $\lambda_1 = \lambda_2 = -9$, we find

$$(\mathbf{A} + 9\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 16 & 4 & -4 & 0 \\ 4 & 1 & -1 & 0 \\ -4 & -1 & 1 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

From the last matrix we see that $k_1 = -\frac{1}{4}k_2 + \frac{1}{4}k_3$. The choices $k_2 = 1, k_3 = 1$ followed by $k_2 = -4, k_3 = 0$ yield, in turn, the distinct eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}.$$

Now for $\lambda_1 = 9$,

$$(\mathbf{A} - 9\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & 4 & -4 & 0 \\ 4 & -17 & -1 & 0 \\ -4 & -1 & -17 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

indicates that $\mathbf{K}_3 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$ is a third eigenvector.

Observe, in accordance with Theorem 8.10.2, the vector $\mathbf{K}_3$ is orthogonal to $\mathbf{K}_1$ and to $\mathbf{K}_2$, but $\mathbf{K}_1$ and $\mathbf{K}_2$, eigenvectors corresponding to the repeated eigenvalue $\lambda_1 = -9$, are not orthogonal since $\mathbf{K}_1 \cdot \mathbf{K}_2 = -4 \neq 0$.

We use the Gram–Schmidt orthogonalization process (see pages 359–360) to transform the set $\{\mathbf{K}_1, \mathbf{K}_2\}$ into an orthogonal set. Let $\mathbf{V}_1 = \mathbf{K}_1$ and then

$$\mathbf{V}_2 = \mathbf{K}_2 - \left(\frac{\mathbf{K}_2 \cdot \mathbf{V}_1}{\mathbf{V}_1 \cdot \mathbf{V}_1} \right) \mathbf{V}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

The set $\{\mathbf{V}_1, \mathbf{V}_2\}$ is an orthogonal set of vectors (verify). Moreover, the set $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{K}_3\}$ is an orthogonal set of eigenvectors. Using the norms $\|\mathbf{V}_1\| = \sqrt{2}$, $\|\mathbf{V}_2\| = 3$, and $\|\mathbf{K}_3\| = 3\sqrt{2}$, we obtain an orthonormal set of vectors

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{pmatrix},$$

and so the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & -\frac{1}{3\sqrt{2}} \end{pmatrix}$$

is orthogonal. ≡

REMARKS

For a real $n \times n$ symmetric matrix with repeated eigenvalues it is always possible to *find*, rather than *construct*, a set of n mutually orthogonal eigenvectors. In other words, the Gram–Schmidt process does not have to be used. See Problem 23 in Exercises 8.10.

8.10

Exercises

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–4, **(a)** verify that the indicated column vectors are eigenvectors of the given symmetric matrix, **(b)** identify the corresponding eigenvalues, and **(c)** verify that the column vectors are orthogonal.

1. $\begin{pmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 15 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$
2. $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}; \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3. $\begin{pmatrix} 5 & 13 & 0 \\ 13 & 5 & 0 \\ 0 & 0 & -8 \end{pmatrix}; \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{pmatrix}$

$$4. \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}; \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

In Problems 5–10, determine whether the given matrix is orthogonal.

$$5. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$6. \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$7. \begin{pmatrix} 0 & 0 & 1 \\ -\frac{12}{13} & \frac{5}{13} & 0 \\ \frac{5}{13} & \frac{12}{13} & 0 \end{pmatrix}$$

$$8. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$9. \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$10. \begin{pmatrix} 0 & \frac{8}{17} & 0 & -\frac{15}{17} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{15}{17} & 0 & \frac{8}{17} \end{pmatrix}$$

In Problems 11–18, proceed as in Example 3 to construct an orthogonal matrix from the eigenvectors of the given symmetric matrix. (The answers are not unique.)

$$11. \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix}$$

$$12. \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}$$

$$13. \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

$$14. \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$15. \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$16. \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$17. \begin{pmatrix} -8 & 5 & 4 \\ 5 & 3 & 1 \\ 4 & 1 & 0 \end{pmatrix}$$

$$18. \begin{pmatrix} 2 & 8 & -2 \\ 8 & -4 & 10 \\ -2 & 10 & -7 \end{pmatrix}$$

In Problems 19 and 20, use Theorem 8.10.3 to find values of a and b so that the given matrix is orthogonal.

$$19. \begin{pmatrix} \frac{3}{5} & a \\ \frac{4}{5} & b \end{pmatrix}$$

$$20. \begin{pmatrix} 1/\sqrt{5} & b \\ a & 1/\sqrt{5} \end{pmatrix}$$

In Problems 21 and 22, (a) verify that the indicated column vectors are eigenvectors of the given symmetric matrix and (b) identify the corresponding eigenvalues. (c) Proceed as in

Example 4 and use the Gram–Schmidt process to construct an orthogonal matrix $\mathbf{P}$ from the eigenvectors.

$$21. \mathbf{A} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}; \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$22. \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}; \quad \mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

23. In Example 4, use the equation $k_1 = -\frac{1}{4}k_2 + \frac{1}{4}k_3$ and choose two different sets of values for k_2 and k_3 so that the corresponding eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$ are orthogonal.

24. Construct an orthogonal matrix from the eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

25. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ orthogonal matrices. Then show that $\mathbf{AB}$ is orthogonal.

26. Suppose $\mathbf{A}$ is an orthogonal matrix. Is $\mathbf{A}^2$ orthogonal?

27. Suppose $\mathbf{A}$ is an orthogonal matrix. Then show that $\mathbf{A}^{-1}$ is orthogonal.

28. Suppose $\mathbf{A}$ is an orthogonal matrix. Then show that $\det \mathbf{A} = \pm 1$.

29. Suppose $\mathbf{A}$ is an orthogonal matrix such that $\mathbf{A}^2 = \mathbf{I}$. Then show that $\mathbf{A}^T = \mathbf{A}$.

30. Show that the rotation matrix

$$\mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

is orthogonal.

8.11 Approximation of Eigenvalues

INTRODUCTION Recall, to find the eigenvalues for a matrix $\mathbf{A}$ we must find the roots of the polynomial equation $p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = 0$. If $\mathbf{A}$ is a large matrix, the computations involved in obtaining this characteristic equation can be overwhelming. Moreover, even if we can find the exact characteristic equation it is likely that we would have to use a numerical procedure to approximate its roots. There are alternative numerical procedures for approximating eigenvalues and the corresponding eigenvectors. The procedure that we shall consider in this section deals with matrices that possess a **dominant eigenvalue**.

Definition A dominant eigenvalue of a square matrix $\mathbf{A}$ is one whose absolute value is greater than the absolute value of each of the remaining eigenvalues. We formalize the last sentence in the next definition.

In Problems 7–10, use the method of deflation to find the eigenvalues of the given matrix.

7. $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$

9. $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

10. $\begin{pmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 15 \end{pmatrix}$

In Problems 11 and 12, use the inverse power method to find the eigenvalue of least magnitude for the given matrix.

11. $\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$

12. $\begin{pmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{pmatrix}$

13. In Example 4 of Section 3.9 we saw that the deflection curve of a thin column under an applied load P was defined by the boundary-value problem

$$EI \frac{d^2 y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

In this problem we show how to apply matrix techniques to compute the smallest critical load.

Let the interval $[0, L]$ be divided into n subintervals of length $h = L/n$, and let $x_i = ih$, $i = 0, 1, \dots, n$. For small values of h it follows from (6) of Section 6.5 that

$$\frac{d^2 y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},$$

where $y_i = y(x_i)$.

(a) Show that the differential equation can be replaced by the difference equation

$$EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i = 0, \quad i = 1, 2, \dots, n-1.$$

(b) Show that for $n = 4$ the difference equation in part (a) yields the system of linear equations

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{PL^2}{16EI} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Note that this system has the form of the eigenvalue problem $\mathbf{A}\mathbf{Y} = \lambda\mathbf{Y}$, where $\lambda = PL^2/16EI$.

(c) Find $\mathbf{A}^{-1}$.

(d) Use the inverse power method to find, to two decimal places, the eigenvalue of $\mathbf{A}$ of least magnitude.

(e) Use the result of part (d) to compute the approximate smallest critical load. Compare your answer with that given in Section 3.9.

14. Suppose the column in Problem 13 is tapered so that the moment of inertia of a cross-section I varies linearly from $I(0) = I_0 = 0.002$ to $I(L) = I_L = 0.001$.

(a) Use the difference equation in part (a) of Problem 13 with $n = 4$ to set up a system of equations analogous to that given in part (b).

(b) Proceed as in Problem 13 to find an approximation to the smallest critical load.

Computer Lab Assignment

15. In Section 8.9 we saw how to compute a power $\mathbf{A}^m$ for an $n \times n$ matrix $\mathbf{A}$. Consult the documentation for the CAS you have on hand for the command to compute the power $\mathbf{A}^m$. (In *Mathematica* the command is **MatrixPower** $[\mathbf{A}, m]$.) The matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

possesses a dominant eigenvalue.

(a) Use a CAS to compute $\mathbf{A}^{10}$.

(b) Now use (2), $\mathbf{X}_m = \mathbf{A}^m \mathbf{X}_0$, with $m = 10$ and

$$\mathbf{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ to compute } \mathbf{X}_{10}. \text{ In the same manner compute } \mathbf{X}_{12}.$$

Then proceed as in (9) to find the approximate dominant eigenvector $\mathbf{K}$.

(c) If $\mathbf{K}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{A}\mathbf{K} = \lambda\mathbf{K}$. Use this definition and the result in part (b) to find the dominant eigenvalue.

8.12 Diagonalization

INTRODUCTION In Chapter 10 we shall see that eigenvalues, eigenvectors, orthogonal matrices, and the topic of this present section, diagonalization, are important tools in the solution of systems of linear first-order differential equations. The basic question that we shall consider in this section is

For an $n \times n$ matrix $\mathbf{A}$, can we find an $n \times n$ nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ is a diagonal matrix?

A Special Notation We begin by introducing a shorthand notation for the product of two $n \times n$ matrices. This notation will be useful in proving the principal theorem of this section. To illustrate, suppose $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices. Then

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}. \quad (1)$$

column 1 column 2

If we write the columns of the matrix $\mathbf{B}$ as vectors $\mathbf{X}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$, then column 1 and column 2 in the product (1) can be expressed as the products $\mathbf{A}\mathbf{X}_1$ and $\mathbf{A}\mathbf{X}_2$. That is,

$$\mathbf{AB} = (\underbrace{\mathbf{A}\mathbf{X}_1}_{\text{column 1}} \quad \underbrace{\mathbf{A}\mathbf{X}_2}_{\text{column 2}}).$$

In general, for two $n \times n$ matrices

$$\mathbf{AB} = \mathbf{A}(\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_n) = (\mathbf{A}\mathbf{X}_1 \mathbf{A}\mathbf{X}_2 \dots \mathbf{A}\mathbf{X}_n), \quad (2)$$

where $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$, are the columns of $\mathbf{B}$.

Diagonalizable Matrix If an $n \times n$ nonsingular matrix $\mathbf{P}$ can be found so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ is a diagonal matrix, then we say that the $n \times n$ matrix $\mathbf{A}$ can be **diagonalized**, or is **diagonalizable**, and that $\mathbf{P}$ **diagonalizes** $\mathbf{A}$.

To discover how to diagonalize a matrix, let us assume for the sake of discussion that $\mathbf{A}$ is a 3×3 diagonalizable matrix. Then there exists a 3×3 nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ or $\mathbf{AP} = \mathbf{PD}$, where $\mathbf{D}$ is a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}.$$

If $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ denote the columns of $\mathbf{P}$, then it follows from (2) that the equation $\mathbf{AP} = \mathbf{PD}$ is the same as

$$(\mathbf{AP}_1 \mathbf{AP}_2 \mathbf{AP}_3) = (d_{11}\mathbf{P}_1 \ d_{22}\mathbf{P}_2 \ d_{33}\mathbf{P}_3)$$

$$\text{or} \quad \mathbf{AP}_1 = d_{11}\mathbf{P}_1, \quad \mathbf{AP}_2 = d_{22}\mathbf{P}_2, \quad \mathbf{AP}_3 = d_{33}\mathbf{P}_3.$$

But by Definition 8.8.1 we see that d_{11}, d_{22} , and d_{33} are eigenvalues of $\mathbf{A}$ associated with the eigenvectors $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$. These eigenvectors are linearly independent, since $\mathbf{P}$ was assumed to be nonsingular.

We have just discovered, in a particular case, that if $\mathbf{A}$ is diagonalizable, then the columns of the diagonalizing matrix $\mathbf{P}$ consist of linearly independent eigenvectors of $\mathbf{A}$. Since we wish to diagonalize a matrix, we are really concerned with the validity of the converse of the last sentence. In other words, if we can find n linearly independent eigenvectors of an $n \times n$ matrix $\mathbf{A}$ and form an $n \times n$ matrix $\mathbf{P}$ whose columns consist of these eigenvectors, then does $\mathbf{P}$ diagonalize $\mathbf{A}$? The answer is yes and will be proved in the next theorem.

Theorem 8.12.1 Sufficient Condition for Diagonalizability

If an $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n$, then $\mathbf{A}$ is diagonalizable.

PROOF: We shall prove the theorem in the case when $\mathbf{A}$ is a 3×3 matrix. Let $\mathbf{K}_1, \mathbf{K}_2$, and $\mathbf{K}_3$ be linearly independent eigenvectors corresponding to eigenvalues λ_1, λ_2 , and λ_3 ; that is,

$$\mathbf{AK}_1 = \lambda_1\mathbf{K}_1, \quad \mathbf{AK}_2 = \lambda_2\mathbf{K}_2, \quad \text{and} \quad \mathbf{AK}_3 = \lambda_3\mathbf{K}_3. \quad (3)$$

Next form the 3×3 matrix $\mathbf{P}$ with column vectors $\mathbf{K}_1, \mathbf{K}_2$, and $\mathbf{K}_3$: $\mathbf{P} = (\mathbf{K}_1 \ \mathbf{K}_2 \ \mathbf{K}_3)$. $\mathbf{P}$ is nonsingular since, by hypothesis, the eigenvectors are linearly independent. Now using (2) and (3)

we can write the product $\mathbf{AP}$ as

$$\begin{aligned}\mathbf{AP} &= (\mathbf{AK}_1 \mathbf{AK}_2 \mathbf{AK}_3) = (\lambda_1 \mathbf{K}_1 \lambda_2 \mathbf{K}_2 \lambda_3 \mathbf{K}_3) \\ &= (\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \mathbf{PD}.\end{aligned}$$

Multiplying the last equation on the left by $\mathbf{P}^{-1}$ then gives $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. $\equiv$

Note carefully in the proof of Theorem 8.12.1 that the entries in the diagonalized matrix are the eigenvalues of $\mathbf{A}$ and the order in which these numbers appear on the diagonal of $\mathbf{D}$ corresponds to the order in which the eigenvectors are used as columns in the matrix $\mathbf{P}$.

In view of the motivational discussion preceding Theorem 8.12.1, we can state the general result:

Theorem 8.12.2 Criterion for Diagonalizability

An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors.

We saw in Section 8.8 that an $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors whenever it possesses n distinct eigenvalues.

Theorem 8.12.3 Sufficient Condition for Diagonalizability

If an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, it is diagonalizable.

EXAMPLE 1 Diagonalizing a Matrix

Diagonalize $\mathbf{A} = \begin{pmatrix} -5 & 9 \\ -6 & 10 \end{pmatrix}$ if possible.

SOLUTION First we find the eigenvalues of $\mathbf{A}$. The characteristic equation is $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -5 - \lambda & 9 \\ -6 & 10 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. Since the eigenvalues are distinct, we know from Theorem 8.12.3 that $\mathbf{A}$ is diagonalizable.

Next the eigenvectors of $\mathbf{A}$ corresponding to $\lambda_1 = 1$ and $\lambda_2 = 4$ are, respectively,

$$\mathbf{K}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using these vectors as columns, we find that the nonsingular matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ is

$$\mathbf{P} = (\mathbf{K}_1 \mathbf{K}_2) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}.$$

Now
$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix},$$

and so carrying out the multiplication gives

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -5 & 9 \\ -6 & 10 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \mathbf{D}. \quad \equiv$$

In Example 1, had we reversed the columns in $\mathbf{P}$, that is, $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, then the diagonal matrix would have been $\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

EXAMPLE 2 Diagonalizing a Matrix

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$. We saw in Example 2 of Section 8.8 that the eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 0, \quad \lambda_2 = -4, \quad \lambda_3 = 3, \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$$

Since the eigenvalues are distinct, $\mathbf{A}$ is diagonalizable. We form the matrix

$$\mathbf{P} = (\mathbf{K}_1 \ \mathbf{K}_2 \ \mathbf{K}_3) = \begin{pmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{pmatrix}.$$

Matching the eigenvalues with the order in which the eigenvectors appear in $\mathbf{P}$, we know that the diagonal matrix will be

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Now from either of the methods of Section 8.6 we find

$$\mathbf{P}^{-1} = \begin{pmatrix} -\frac{1}{12} & 0 & -\frac{1}{12} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \\ \frac{8}{21} & \frac{1}{7} & \frac{2}{21} \end{pmatrix},$$

$$\begin{aligned} \text{and so} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} -\frac{1}{12} & 0 & -\frac{1}{12} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \\ \frac{8}{21} & \frac{1}{7} & \frac{2}{21} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \mathbf{D}. \end{aligned}$$

≡

A matrix with repeated eigenvalues could be diagonalizable. ►

The condition that an $n \times n$ matrix $\mathbf{A}$ have n distinct eigenvalues is sufficient—that is, a guarantee—that $\mathbf{A}$ is diagonalizable. The condition that there be n distinct eigenvalues is not a necessary condition for the diagonalization of $\mathbf{A}$. In other words, if the matrix $\mathbf{A}$ does not have n distinct eigenvalues, then it may or may not be diagonalizable.

EXAMPLE 3 A Matrix That Is Not Diagonalizable

In Example 3 of Section 8.8 we saw that the matrix $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$ has a repeated eigenvalue $\lambda_1 = \lambda_2 = 5$. Correspondingly we were able to find only a *single* eigenvector $\mathbf{K}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We conclude from Theorem 8.12.2 that $\mathbf{A}$ is not diagonalizable.

≡

EXAMPLE 4 Repeated Eigenvalues Yet Diagonalizable

The eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 1$.

For $\lambda_1 = -1$ we find $\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. For the repeated eigenvalue $\lambda_2 = \lambda_3 = 1$, Gauss–Jordan elimination gives

$$(\mathbf{A} - \mathbf{I}|\mathbf{0}) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the last matrix we see that $k_1 - k_2 = 0$. Since k_3 is not determined from the last matrix, we can choose its value arbitrarily. The choice $k_2 = 1$ gives $k_1 = 1$. If we then pick $k_3 = 0$, we get the eigenvector

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The alternative choice $k_2 = 0$ gives $k_1 = 0$. If $k_3 = 1$, we get another eigenvector corresponding to $\lambda_2 = \lambda_3 = 1$:

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the eigenvectors $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ are linearly independent, a matrix that diagonalizes $\mathbf{A}$ is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Matching the eigenvalues with the eigenvectors in $\mathbf{P}$, we have $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \equiv$$

Symmetric Matrices An $n \times n$ symmetric matrix $\mathbf{A}$ with real entries can always be diagonalized. This is a consequence of the fact that we can always find n linearly independent eigenvectors for such a matrix. Moreover, since we can find n mutually orthogonal eigenvectors, we can use an orthogonal matrix $\mathbf{P}$ to diagonalize $\mathbf{A}$. A symmetric matrix is said to be **orthogonally diagonalizable**.

Theorem 8.12.4 Criterion for Orthogonal Diagonalizability

An $n \times n$ matrix $\mathbf{A}$ can be orthogonally diagonalized if and only if $\mathbf{A}$ is symmetric.

PARTIAL PROOF: We shall prove the necessity part (that is, the “only if” part) of the theorem. Assume an $n \times n$ matrix $\mathbf{A}$ is orthogonally diagonalizable. Then there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ or $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Since $\mathbf{P}$ is orthogonal, $\mathbf{P}^{-1} = \mathbf{P}^T$ and consequently $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$. But from (i) and (iii) of Theorem 8.1.2 and the fact that a diagonal matrix is symmetric, we have

$$\mathbf{A}^T = (\mathbf{P}\mathbf{D}\mathbf{P}^T)^T = (\mathbf{P}^T)^T\mathbf{D}^T\mathbf{P}^T = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{A}.$$

Thus, $\mathbf{A}$ is symmetric. ≡

EXAMPLE 5 Diagonalizing a Symmetric Matrix

Consider the symmetric matrix $\mathbf{A} = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$. We saw in Example 4 of Section 8.8 that the eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 11, \quad \lambda_2 = \lambda_3 = 8, \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

See the Remarks on page 451.

The eigenvectors $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ are linearly independent, but note that they are not mutually orthogonal since $\mathbf{K}_2$ and $\mathbf{K}_3$, the eigenvectors corresponding to the repeated eigenvalue $\lambda_2 = \lambda_3 = 8$, are not orthogonal. For $\lambda_2 = \lambda_3 = 8$, we found the eigenvectors from Gauss–Jordan elimination

$$(\mathbf{A} - 8\mathbf{I}|\mathbf{0}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $k_1 + k_2 + k_3 = 0$. Since two of the variables are arbitrary, we selected $k_2 = 1, k_3 = 0$ to obtain $\mathbf{K}_2$, and $k_2 = 0, k_3 = 1$ to obtain $\mathbf{K}_3$. Now if instead we choose $k_2 = 1, k_3 = 1$ and then $k_2 = 1, k_3 = -1$, we obtain, respectively, two entirely different but orthogonal eigenvectors:

$$\mathbf{K}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Thus, a new set of mutually orthogonal eigenvectors is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Multiplying these vectors, in turn, by the reciprocals of the norms $\|\mathbf{K}_1\| = \sqrt{3}$, $\|\mathbf{K}_2\| = \sqrt{6}$, and $\|\mathbf{K}_3\| = \sqrt{2}$, we obtain an orthonormal set

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

We then use these vectors as columns to construct an orthogonal matrix that diagonalizes $\mathbf{A}$:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$ corresponding to the order in which the eigenvectors appear in $\mathbf{P}$ is then

$$\mathbf{D} = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

This is verified from

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \mathbf{D}. \end{aligned} \quad \equiv$$

Quadratic Forms An algebraic expression of the form

$$ax^2 + bxy + cy^2 \quad (4)$$

is said to be a **quadratic form**. If we let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, then (4) can be written as the matrix product

$$\mathbf{X}^T\mathbf{A}\mathbf{X} = (x \ y) \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (5)$$

Observe that the matrix $\mathbf{A} = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$ is symmetric.

In calculus you may have seen that an appropriate rotation of axes enables us to eliminate the xy -term in an equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

As the next example will illustrate, we can eliminate the xy -term by means of an orthogonal matrix and diagonalization rather than by using trigonometry.

EXAMPLE 6 Identifying a Conic Section

Identify the conic section whose equation is $2x^2 + 4xy - y^2 = 1$.

SOLUTION From (5) we can write the given equation as

$$(x \ y) \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \quad \text{or} \quad \mathbf{X}^T\mathbf{A}\mathbf{X} = 1, \quad (6)$$

where $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now the eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are found to be

$$\lambda_1 = -2, \quad \lambda_2 = 3, \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Observe that $\mathbf{K}_1$ and $\mathbf{K}_2$ are orthogonal. Moreover, $\|\mathbf{K}_1\| = \|\mathbf{K}_2\| = \sqrt{5}$, and so the vectors

$$\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

are orthonormal. Hence, the matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

is orthogonal. If we define the change of variables $\mathbf{X} = \mathbf{P}\mathbf{X}'$ where $\mathbf{X}' = \begin{pmatrix} X \\ Y \end{pmatrix}$, then the quadratic form $2x^2 + 4xy - y^2$ can be written

$$\mathbf{X}^T \mathbf{A} \mathbf{X} = (\mathbf{X}')^T \mathbf{P}^T \mathbf{A} \mathbf{P} \mathbf{X}' = (\mathbf{X}')^T (\mathbf{P}^T \mathbf{A} \mathbf{P}) \mathbf{X}'.$$

Since $\mathbf{P}$ orthogonally diagonalizes the symmetric matrix $\mathbf{A}$, the last equation is the same as

$$\mathbf{X}^T \mathbf{A} \mathbf{X} = (\mathbf{X}')^T \mathbf{D} \mathbf{X}'. \quad (7)$$

Using (7), we see that (6) becomes

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 1 \quad \text{or} \quad -2X^2 + 3Y^2 = 1.$$

This last equation is recognized as the standard form of a hyperbola. The xy -coordinates of the eigenvectors are $(1, -2)$ and $(2, 1)$. Using the substitution $\mathbf{X} = \mathbf{P}\mathbf{X}'$ in the form $\mathbf{X}' = \mathbf{P}^{-1}\mathbf{X} = \mathbf{P}^T\mathbf{X}$, we find that the XY -coordinates of these two points are $(\sqrt{5}, 0)$ and $(0, \sqrt{5})$, respectively. From this we conclude that the X -axis and Y -axis are as shown in **FIGURE 8.12.1**. The eigenvectors, shown in red in the figure, lie along the new axes. The X - and Y -axes are called the **principal axes** of the conic. ≡

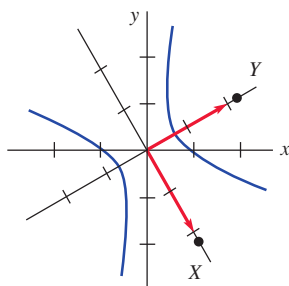


FIGURE 8.12.1 X - and Y -axes in Example 6

REMARKS

The matrix $\mathbf{A}$ in Example 5 is symmetric and as such eigenvectors corresponding to distinct eigenvalues are orthogonal. In the third line of the example, note that $\mathbf{K}_1$, an eigenvector for $\lambda_1 = 11$, is orthogonal to both $\mathbf{K}_2$ and $\mathbf{K}_3$. The eigenvectors $\mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{K}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ corresponding to $\lambda_2 = \lambda_3 = 8$ are not orthogonal. As an alternative to searching for orthogonal eigenvectors for this repeated eigenvalue by performing Gauss–Jordan elimination a second time, we could simply apply the Gram–Schmidt orthogonalization process and transform the set $\{\mathbf{K}_2, \mathbf{K}_3\}$ into an orthogonal set. See Section 7.7 and Example 4 in Section 8.10.

8.12

Exercises

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–20, determine whether the given matrix $\mathbf{A}$ is diagonalizable. If so, find the matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

1. $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

2. $\begin{pmatrix} -4 & -5 \\ 8 & 10 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} -9 & 13 \\ -2 & 6 \end{pmatrix}$

6. $\begin{pmatrix} -5 & -3 \\ 5 & 11 \end{pmatrix}$

7. $\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}$

8. $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$

9. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & -2 \\ -5 & 3 & 8 \end{pmatrix}$

13. $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 0 & -9 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$17. \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 18. \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

$$19. \begin{pmatrix} -8 & -10 & 7 & -9 \\ 0 & 2 & 0 & 0 \\ -9 & -9 & 8 & -9 \\ 1 & 1 & -1 & 2 \end{pmatrix}$$

$$20. \begin{pmatrix} 4 & 2 & -1 & 4 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

In Problems 21–30, the given matrix $\mathbf{A}$ is symmetric. Find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^T \mathbf{A} \mathbf{P}$.

$$21. \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$22. \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$$

$$23. \begin{pmatrix} 5 & \sqrt{10} \\ \sqrt{10} & 8 \end{pmatrix}$$

$$24. \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

$$25. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$26. \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$27. \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}$$

$$28. \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$29. \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

$$30. \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

In Problems 31–34, use the procedure illustrated in Example 6 to identify the given conic section. Graph.

$$31. 5x^2 - 2xy + 5y^2 = 24$$

$$32. 13x^2 - 10xy + 13y^2 = 288$$

$$33. -3x^2 + 8xy + 3y^2 = 20$$

$$34. 16x^2 + 24xy + 9y^2 - 3x + 4y = 0$$

35. Find a 2×2 matrix $\mathbf{A}$ that has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

36. Find a 3×3 symmetric matrix that has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$ and corresponding eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

37. If $\mathbf{A}$ is an $n \times n$ diagonalizable matrix, then $\mathbf{D} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$, where $\mathbf{D}$ is a diagonal matrix. Show that if m is a positive integer, then $\mathbf{A}^m = \mathbf{P} \mathbf{D}^m \mathbf{P}^{-1}$.

38. The m th power of a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$\text{is } \mathbf{D}^m = \begin{pmatrix} a_{11}^m & 0 & \cdots & 0 \\ 0 & a_{22}^m & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^m \end{pmatrix}.$$

Use this result to compute

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}^4.$$

In Problems 39 and 40, use the results of Problems 37 and 38 to find the indicated power of the given matrix.

$$39. \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \mathbf{A}^5 \quad 40. \mathbf{A} = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}, \mathbf{A}^{10}$$

41. Suppose $\mathbf{A}$ is a nonsingular diagonalizable matrix. Then show that $\mathbf{A}^{-1}$ is diagonalizable.

42. Suppose $\mathbf{A}$ is a diagonalizable matrix. Is the matrix $\mathbf{P}$ unique?

8.13 LU-Factorization

INTRODUCTION Just as positive integers and polynomials can be factored, so too a matrix can sometimes be factored into other matrices. For example, in the last section we saw that if an $n \times n$ matrix $\mathbf{A}$ is diagonalizable, then there exists a nonsingular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \mathbf{D}$. When the last equation is written as $\mathbf{A} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1}$ we say that $\mathbf{A}$ has been **factored** or **decomposed** into three matrices. There are many ways of factoring an $n \times n$ matrix $\mathbf{A}$, but in this section we are interested in a special type of factorization that involves triangular matrices.

The notion of a triangular matrix was introduced in Section 8.1. See page 372.

LU-Factorization Recall, a triangular matrix is a square matrix that is either a **lower triangular matrix** whose entries above the main diagonal are all zero or an **upper triangular matrix** whose entries below the main diagonal are all zero:

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}. \quad (1)$$

lower triangular matrix upper triangular matrix

Lower and upper triangular matrices are very important in a variety of computational applications, but in this section we will confine our attention to their use in solving linear systems $\mathbf{AX} = \mathbf{B}$. We will denote a lower triangle matrix by the letter $\mathbf{L}$ and an upper triangular matrix by $\mathbf{U}$. This leads to the following definition.

Definition 8.13.1 LU-Factorization

Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{A}$ can be written as a product $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ and $\mathbf{U}$ are lower triangular and upper triangular matrices, respectively, then we say that $\mathbf{A} = \mathbf{LU}$ is an **LU-factorization** of $\mathbf{A}$.

An LU-factorization of a matrix $\mathbf{A}$ is also called an **LU-decomposition** of the matrix $\mathbf{A}$.

EXAMPLE 1 An LU-Factorization

Suppose $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$. Then for the lower triangular matrix $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and the upper triangular matrix $\mathbf{U} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ it is easily verified by matrix multiplication that an LU-factorization of $\mathbf{A}$ is

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \mathbf{A}. \quad (2) \equiv$$

Finding an LU-Factorization Of course, the obvious question is: How do we find an LU-factorization for a given matrix $\mathbf{A}$? For the sake of discussion let's assume that $\mathbf{A}$ is a 3×3 matrix, although the procedures discussed in this section are applicable to square matrices of any order. We want to find two 3×3 matrices $\mathbf{L}$ and $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$ or

$$\overbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}^{\mathbf{A}} = \overbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}^{\mathbf{L}} \overbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}^{\mathbf{U}}. \quad (3)$$

By carrying out the indicated multiplication on the right-hand side of (3) and equating its entries to the corresponding known entries a_{ij} on the left-hand side we obtain a system of 9 equations in 12 variables, the variables being the l_{ij} and u_{ij} . Recall, such a system is *under-determined* and, if consistent, would have many solutions. This leads us immediately to the

See page 386 at the end of Section 8.2. ►

conclusion that the matrices $\mathbf{L}$ and $\mathbf{U}$ in an LU-factorization are not uniquely determined. In other words:

- An $n \times n$ matrix $\mathbf{A}$ can have several different LU-factorizations.

The factorization depends on the method used.

Doolittle's Method In order to determine unique matrices $\mathbf{L}$ and $\mathbf{U}$ we need to impose three conditions on the variables l_{ij} and u_{ij} in (3). The method that we examine in the examples that follow is called **Doolittle's method** after the American mathematician **Myrick H. Doolittle** (1830–1913). In this method the main diagonal entries l_{ii} of $\mathbf{L}$ in (3) are chosen to be all 1's. The next example illustrates a method for obtaining the remaining nine variables.

EXAMPLE 2 Example 1 Revisited

Use Doolittle's method to find an LU-factorization of

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}.$$

SOLUTION The matrix $\mathbf{A}$ is the one considered in Example 1. We are going to show how the factorization given in (2) can be found. Using (3) with $l_{ii} = 1$, $i = 1, 2, 3$ in the lower triangular matrix $\mathbf{L}$ we write

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}}^{\mathbf{L}} \overbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}^{\mathbf{U}} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}. \quad (4)$$

Then by the definition of equality of matrices we must have

$$\begin{aligned} 1 &= u_{11}, & 1 &= u_{12}, & 1 &= u_{13}, \\ 3 &= l_{21}u_{11}, & 1 &= l_{21}u_{12} + u_{22}, & 2 &= l_{21}u_{13} + u_{23}, \\ 1 &= l_{31}u_{11}, & -1 &= l_{31}u_{12} + l_{32}u_{22}, & 1 &= l_{31}u_{13} + l_{32}u_{23} + u_{33}. \end{aligned} \quad (5)$$

At first glance (5) may look a bit formidable to solve, but it is not. Indeed, (5) is not so much a system of equations as it is a sequence of nine linear equations in *one* variable. The trick in solving (5) is to read each equation beginning with the first row moving left to right and use previously found values to determine the one variable in each equation. The variable determined by each equation is indicated in red in (5). Using the values $u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$ from the first row in the three equations in the second row yield, in turn, $l_{21} = 3$, $u_{22} = -2$, $u_{23} = -1$. Note that we also had to use $l_{21} = 3$ to find u_{22} , and so on. The three equations in the third row then give $l_{31} = 1$, $l_{32} = 1$, $u_{33} = 1$. Substituting these values in the appropriate triangular matrices in (4) gives LU-factorization,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}^{\mathbf{L}} \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}}^{\mathbf{U}}. \quad \equiv$$

An Algorithm Doolittle's method is very popular in that it is easily programmed in computer algebra systems such as *Mathematica* or *MATLAB*. In lieu of the brute force technique illustrated in Example 2, we can carry out Doolittle's method using one of the elementary row operations that we used in Gaussian elimination to create an upper triangular matrix $\mathbf{U}$ out of the matrix $\mathbf{A}$. In particular, we use the notion of *row addition*, that is, adding a nonzero multiple of one row of a matrix $\mathbf{A}$ to another row. Recall from Section 8.2, the notation $cR_i + R_j$ denotes the multiplication of the entries in the i th row of a matrix $\mathbf{A}$ by a nonzero constant c and adding them

to the corresponding entries in the j th row. We state the following two-step algorithm without a formal proof. We assume that $\mathbf{A}$ has an LU-factorization.

- (i) Create an upper triangular matrix $\mathbf{U}$ using row addition to *zero out* the entries below the main diagonal of $\mathbf{A}$. For a 3×3 matrix, this looks like

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xRightarrow{\text{row addition operations}} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{23} & a'_{23} \\ 0 & 0 & a''_{33} \end{pmatrix} = \mathbf{U}. \quad (6)$$

- (ii) Keep a record of the *negatives* of the multipliers of the rows used in step (i) by entering these numbers in an appropriate-sized identity matrix $\mathbf{I}$. If we perform the addition $cR_i + R_j$, then enter $-c$ in the j th row and i th column of $\mathbf{I}$. For example, if we perform $5R_1 + R_2$ on $\mathbf{A}$ in (6) to zero out the entry in the second row and first column, we then replace the 0 in the second row and first column of the 3×3 identity $\mathbf{I}$ by the number -5 :

$$\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When finished with the reduction of $\mathbf{A}$ to $\mathbf{U}$, the matrix created by recording the negatives of all the multipliers of the rows we used is a lower triangular matrix $\mathbf{L}$ because these numbers are placed in $\mathbf{I}$ in exactly the same position as the 0's in the matrix $\mathbf{U}$ (below the diagonal).

EXAMPLE 3 Adding Multiples of Rows to Rows

Find an LU-factorization of

$$\mathbf{A} = \begin{pmatrix} 2 & -8 \\ 3 & 0 \end{pmatrix}.$$

SOLUTION We need only perform one row-addition operation on $\mathbf{A}$ to obtain the desired upper triangular matrix $\mathbf{U}$. We simultaneously create $\mathbf{L}$ using the 2×2 identity $\mathbf{I}$:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 2 & -8 \\ 3 & 0 \end{pmatrix} \xRightarrow{-\frac{3}{2}R_1 + R_2} \begin{pmatrix} 2 & -8 \\ 0 & 12 \end{pmatrix} \\ \mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xRightarrow{\text{record } \frac{3}{2}} \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}. \end{aligned} \quad (7)$$

From (7) we see that

$$\mathbf{U} = \begin{pmatrix} 2 & -8 \\ 0 & 12 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}.$$

An LU-factorization of $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & -8 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & -8 \\ 0 & 12 \end{pmatrix}. \quad \equiv$$

EXAMPLE 4 Adding Multiples of Rows to Rows

Find an LU-factorization of

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -4 \\ 2 & -5 & 10 \\ 3 & 1 & 6 \end{pmatrix}.$$

SOLUTION In this case the reduction of $\mathbf{A}$ to $\mathbf{U}$ requires three row-addition operations. We record the negatives of the multiples of the rows in the 3×3 identity $\mathbf{I}$:

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -1 & 2 & -4 \\ 2 & -5 & 10 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow[\text{record } -3]{\substack{2R_1+R_2 \\ 3R_1+R_3}} \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 7 & -6 \end{pmatrix} \xrightarrow{7R_2+R_3} \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix} \\ \mathbf{I} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{record } -3]{\substack{\text{record } -2 \\ \text{record } -3}} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \xrightarrow{\text{record } -7} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -7 & 1 \end{pmatrix}. \quad (8)\end{aligned}$$

Inspection of (8) yields

$$\mathbf{U} = \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -7 & 1 \end{pmatrix}.$$

Thus an LU-factorization of $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -4 \\ 2 & -5 & 10 \\ 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -7 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix}. \quad \equiv$$

Solving Linear Systems Suppose $\mathbf{AX} = \mathbf{B}$ is a linear system of n equations in n variables. If an $n \times n$ coefficient matrix $\mathbf{A}$ in the system has an LU-factorization $\mathbf{A} = \mathbf{LU}$ then $\mathbf{AX} = \mathbf{B}$ is the same as $(\mathbf{LU})\mathbf{X} = \mathbf{B}$, or $\mathbf{L}(\mathbf{UX}) = \mathbf{B}$. This last system can be solved efficiently in two steps:

- (i) First, let $\mathbf{Y} = \mathbf{UX}$ so that $\mathbf{L}(\mathbf{UX}) = \mathbf{B}$ becomes $\mathbf{LY} = \mathbf{B}$. Solve for $\mathbf{Y}$ by forward-substitution.
- (ii) Then solve $\mathbf{UX} = \mathbf{Y}$ for $\mathbf{X}$ using back-substitution.

The next example illustrates the method.

EXAMPLE 5 Example 1 Revisited

Solve the system

$$\begin{aligned}x_1 + x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 &= 1 \\ x_1 - x_2 + x_3 &= 4.\end{aligned}$$

SOLUTION In terms of the matrix notation $\mathbf{AX} = \mathbf{B}$, the linear system is the same as

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad (9)$$

where the coefficient matrix $\mathbf{A}$ in (9) is the matrix in Example 1. Using the LU-factorization of $\mathbf{A}$ given in (2) of Example 1, (9) can be written

$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}^{\mathbf{L}} \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}}^{\mathbf{UX}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}}^{\mathbf{B}}. \quad (10)$$

If we let $\mathbf{UX} = \mathbf{Y}$ in (10), where $\mathbf{Y}$ is the column matrix

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad (11)$$

then (10) is the same as $\mathbf{LY} = \mathbf{B}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} y_1 & = & 0 \\ 3y_1 + y_2 & = & 1 \\ y_1 + y_2 + y_3 & = & 4. \end{array} \quad (12)$$

We now use forward-substitution in the second form of the system in (12). That is, we substitute $y_1 = 0$ obtained from the first equation into the second equation of (12). This immediately gives $y_2 = 1$. Then substituting both these values into the third equation of (12) yields $y_3 = 3$. Since the column matrix $\mathbf{Y}$ is now determined, we now solve $\mathbf{UX} = \mathbf{Y}$, that is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} x_1 + x_2 + x_3 & = & 0 \\ -2x_2 - x_3 & = & 1 \\ x_3 & = & 3. \end{array} \quad (13)$$

We solve the second form of the system in (13) by back-substitution, that is, we substitute $x_3 = 3$ from the third equation into the second equation to obtain $x_2 = -2$. Substituting these two values into the first equation of (13) yields $x_1 = -1$. Hence the solution of the original system is $x_1 = -1, x_2 = -2, x_3 = 3$. $\equiv$

See Theorem 8.5.6 on page 401.

Relationship to Determinants If an $n \times n$ matrix $\mathbf{A}$ has an LU-factorization $\mathbf{A} = \mathbf{LU}$, then $\det \mathbf{A}$ is easy to evaluate using the determinant property $\det \mathbf{A} = \det \mathbf{L} \cdot \det \mathbf{U}$.

EXAMPLE 6 Determinant of a Factored Matrix

Evaluate the determinant of the matrix $\mathbf{A}$ in Example 4.

SOLUTION From Example 4 an LU-factorization $\mathbf{A} = \mathbf{LU}$ is

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -4 \\ 2 & -5 & 10 \\ 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -7 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix}$$

so that

$$\det \mathbf{A} = \det \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -7 & 1 \end{pmatrix} \det \begin{pmatrix} -1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix}. \quad (14)$$

Now here is the beauty of this technique. Going back to Theorem 8.5.8 we know that the determinant of a triangular matrix is the product of its main diagonal entries. So with no additional work we have immediately that (14) is

$$\det \mathbf{A} = \overbrace{(1 \cdot 1 \cdot 1)}^{\text{diagonal entries of } \mathbf{L}} \overbrace{((-1) \cdot (-1) \cdot 8)}^{\text{diagonal entries of } \mathbf{U}} = 8. \quad \equiv$$

REMARKS

(i) Not every $n \times n$ matrix $\mathbf{A}$ has an LU-factorization. For example, the 2×2 matrix $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ has no LU-factorization. Try the method in Example 2 and see what happens.

In general, if interchanges of rows are required in the method illustrated in Examples 3 and 4 in order to obtain $\mathbf{U}$, then we say that an LU-factorization does not exist.

(ii) When solving a linear system $\mathbf{A}\mathbf{X} = \mathbf{B}$ by Gaussian elimination, the steps of that method must be repeated if the coefficient matrix $\mathbf{A}$ is the same but the column vector $\mathbf{B}$ is changed. The method for solving systems of linear equations illustrated in Example 5 is especially useful if a system $\mathbf{A}\mathbf{X} = \mathbf{B}$ has to be solved for different vectors $\mathbf{B}_i$, $i = 1, 2, \dots, n$, since the matrices $\mathbf{L}$ and $\mathbf{U}$ remain the same in each system and so can be retained in a computer program. See Problems 31–34 in Exercises 8.13.

(iii) If a matrix $\mathbf{A}$ already has a 0 below its main diagonal such as

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 1 & 3 \\ 0 & 4 & 6 \end{pmatrix}$$

then we do not have to zero out the entry in the third row and first column by a row addition. So in the matrix in which we are recording the negatives of the multipliers we simply retain the 0 in the third row and first column.

(iv) Had we chosen the main diagonal entries u_{ii} of $\mathbf{U}$ in (3) to be all 1's, then an LU-factorization of a 3×3 matrix $\mathbf{A}$ looks like

$$\mathbf{A} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{U}} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}. \quad (15)$$

This is called **Crout's method** because it was devised by the American mathematician **Preston D. Crout** (1907–1984). Like the Doolittle method, Crout's method also gives a unique LU-factorization of $\mathbf{A}$. See Problems 41 and 42 in Exercises 8.13.

(v) **Cholesky's method** discovered by the French military officer and mathematician **Andre-Louis Cholesky** (1875–1918) is a way of finding an LU-factorization of a symmetric matrix, that is, a matrix $\mathbf{A}$ for which $\mathbf{A} = \mathbf{A}^T$. The form of the factorization for a 3×3 symmetric matrix is $\mathbf{A} = \mathbf{LU} = \mathbf{LL}^T$:

$$\mathbf{A} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}}_{\mathbf{U} = \mathbf{L}^T} = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}. \quad (16)$$

See Problems 43 and 44 in Exercises 8.13.

8.13

Exercises

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–10, use the procedure illustrated in Example 2 to find the LU-factorization of the given matrix.

$$\begin{array}{llll} 1. \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} & 2. \begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix} & 3. \begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} & 4. \begin{pmatrix} 5 & -4 \\ 15 & 2 \end{pmatrix} \\ 5. \begin{pmatrix} 4 & -2 & 1 \\ -4 & 1 & 2 \\ 12 & 1 & 3 \end{pmatrix} & 6. \begin{pmatrix} -3 & 2 & 1 \\ 9 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix} & 7. \begin{pmatrix} 1 & 2 & 7 \\ 2 & 5 & 6 \\ 7 & 6 & 4 \end{pmatrix} & 8. \begin{pmatrix} -4 & 2 & -10 \\ 4 & 1 & 4 \\ 6 & 6 & -8 \end{pmatrix} \\ 9. \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 6 & 1 \end{pmatrix} & 10. \begin{pmatrix} 1 & 0 & 1 \\ 1 & 9 & 1 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

In Problems 11–20, use the procedure illustrated in Examples 3 and 4 to find the LU-factorization of the given matrix.

$$11. \begin{pmatrix} 3 & 9 \\ 1 & 11 \end{pmatrix}$$

$$12. \begin{pmatrix} -2 & 10 \\ 1 & -4 \end{pmatrix}$$

$$13. \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix}$$

$$14. \begin{pmatrix} 0.2 & 0.3 \\ 0.8 & 3.5 \end{pmatrix}$$

$$15. \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$16. \begin{pmatrix} -1 & 2 & -4 \\ 2 & -5 & 10 \\ 3 & 1 & 6 \end{pmatrix}$$

$$17. \begin{pmatrix} 4 & 2 & 12 \\ 2 & 26 & -4 \\ 12 & -4 & 56 \end{pmatrix}$$

$$18. \begin{pmatrix} 16 & 4 & 20 \\ 4 & 5 & 3 \\ 20 & 3 & 29 \end{pmatrix}$$

$$19. \begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 3 & -2 & 1 \\ 1 & 2 & 4 & 3 \\ 5 & 1 & -1 & 1 \end{pmatrix}$$

$$20. \begin{pmatrix} 1 & -1 & 3 & 2 \\ 3 & -5 & 19 & 3 \\ 2 & -2 & 3 & 10 \\ -1 & 5 & -5 & -2 \end{pmatrix}$$

In Problems 21–30, proceed as in Example 5 and use the corresponding LU-factorization from Problems 1–10 to solve the given linear system of equations.

$$21. \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad 22. \begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

$$23. \begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} \quad 24. \begin{pmatrix} 5 & -4 \\ 15 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$25. \begin{pmatrix} 4 & -2 & 1 \\ -4 & 1 & 2 \\ 12 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 28 \end{pmatrix}$$

$$26. \begin{pmatrix} -3 & 2 & 1 \\ 9 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ -24 \end{pmatrix}$$

$$27. \begin{pmatrix} 1 & 2 & 7 \\ 2 & 5 & 6 \\ 7 & 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 109 \\ 109 \\ 218 \end{pmatrix}$$

$$28. \begin{pmatrix} -4 & 2 & -10 \\ 4 & 1 & 4 \\ 6 & 6 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -20 \\ 14 \\ -18 \end{pmatrix}$$

$$29. \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ -42 \\ 7 \end{pmatrix}$$

$$30. \begin{pmatrix} 1 & 0 & 1 \\ 1 & 9 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 18 \\ 27 \\ -12 \end{pmatrix}$$

In Problems 31–34, use the given LU-factorization

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

to solve the linear system $\mathbf{AX} = \mathbf{B}_i$ for the given column matrix $\mathbf{B}_i, i = 1, 2, 3, 4$.

$$31. \mathbf{B}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$32. \mathbf{B}_2 = \begin{pmatrix} -4 \\ 7 \\ 10 \end{pmatrix}$$

$$33. \mathbf{B}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}$$

$$34. \mathbf{B}_4 = \begin{pmatrix} 30 \\ -42 \\ -18 \end{pmatrix}$$

In Problems 35–40, proceed as in Example 6 and use an LU-factorization to evaluate the determinant $\det \mathbf{A}$ of the indicated matrix $\mathbf{A}$.

35. $\mathbf{A}$ is the matrix in Problem 15

36. $\mathbf{A}$ is the matrix in Problem 16

37. $\mathbf{A}$ is the matrix in Problem 17

38. $\mathbf{A}$ is the matrix in Problem 18

39. $\mathbf{A}$ is the matrix in Problem 19

40. $\mathbf{A}$ is the matrix in Problem 20

In Problems 41 and 42, use Crout's method discussed in (iv) of the *Remarks* to obtain an LU-factorization of the indicated matrix $\mathbf{A}$.

41. $\mathbf{A}$ is the matrix in Problem 15

42. $\mathbf{A}$ is the matrix in Problem 16

In Problems 43 and 44, use Cholesky's method discussed in (v) of the *Remarks* to obtain an LU-factorization of the indicated symmetric matrix $\mathbf{A}$.

43. $\mathbf{A}$ is the matrix in Problem 17

44. $\mathbf{A}$ is the matrix in Problem 18

8.14 Cryptography

INTRODUCTION The word *cryptography* is a combination of two Greek words: *crypto*, meaning “hidden” or “secret,” and *grapho*, which means “writing.” Cryptography then is the study of making “secret writings” or **codes**.

In this section we will consider a system of encoding and decoding messages that requires both the sender of the message and the receiver of the message to know:

- a specified rule of correspondence between a set of symbols (such as letters of the alphabet and punctuation marks from which messages are composed) and a set of integers; and
- a specified *nonsingular* matrix $\mathbf{A}$.

In Problems 1–20, fill in the blanks or answer true/false.

- A matrix $\mathbf{A} = (a_{ij})_{4 \times 3}$ such that $a_{ij} = i + j$ is given by _____.
- If $\mathbf{A}$ is a 4×7 matrix and $\mathbf{B}$ is a 7×3 matrix, then the size of $\mathbf{AB}$ is _____.
- If $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = (3 \ 4)$, then $\mathbf{AB} = \underline{\hspace{2cm}}$ and $\mathbf{BA} = \underline{\hspace{2cm}}$.
- If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $\mathbf{A}^{-1} = \underline{\hspace{2cm}}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ nonsingular matrices, then $\mathbf{A} + \mathbf{B}$ is necessarily nonsingular. _____
- If $\mathbf{A}$ is a nonsingular matrix for which $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$. _____
- If $\mathbf{A}$ is a 3×3 matrix such that $\det \mathbf{A} = 5$, then $\det(\frac{1}{2}\mathbf{A}) = \underline{\hspace{2cm}}$ and $\det(-\mathbf{A}^T) = \underline{\hspace{2cm}}$.
- If $\det \mathbf{A} = 6$ and $\det \mathbf{B} = 2$, then $\det \mathbf{AB}^{-1} = \underline{\hspace{2cm}}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices whose corresponding entries in the third column are the same, then $\det(\mathbf{A} - \mathbf{B}) = \underline{\hspace{2cm}}$.
- Suppose $\mathbf{A}$ is a 3×3 matrix such that $\det \mathbf{A} = 2$. If $\mathbf{B} = 10\mathbf{A}$ and $\mathbf{C} = -\mathbf{B}^{-1}$, then $\det \mathbf{C} = \underline{\hspace{2cm}}$.
- Let $\mathbf{A}$ be an $n \times n$ matrix. The eigenvalues of $\mathbf{A}$ are the nonzero solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. _____
- A nonzero scalar multiple of an eigenvector is also an eigenvector corresponding to the same eigenvalue. _____
- An $n \times 1$ column vector $\mathbf{K}$ with all zero entries is never an eigenvector of an $n \times n$ matrix $\mathbf{A}$. _____
- Let $\mathbf{A}$ be an $n \times n$ matrix with real entries. If λ is a complex eigenvalue, then $\bar{\lambda}$ is also an eigenvalue of $\mathbf{A}$. _____
- An $n \times n$ matrix $\mathbf{A}$ always possesses n linearly independent eigenvectors. _____
- The augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ is in reduced row-echelon form. _____
- If a 3×3 matrix $\mathbf{A}$ is diagonalizable, then it possesses three linearly independent eigenvectors. _____
- The only matrices that are orthogonally diagonalizable are symmetric matrices. _____
- The symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is orthogonal. _____
- The eigenvalues of a symmetric matrix with real entries are always real numbers. _____
- An $n \times n$ matrix $\mathbf{B}$ is symmetric if $\mathbf{B}^T = \mathbf{B}$, and an $n \times n$ matrix $\mathbf{C}$ is skew-symmetric if $\mathbf{C}^T = -\mathbf{C}$. By noting the identity $2\mathbf{A} = \mathbf{A} + \mathbf{A}^T + \mathbf{A} - \mathbf{A}^T$, show that any $n \times n$ matrix $\mathbf{A}$ can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
- Show that there exists no 2×2 matrix with real entries such that $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

23. An $n \times n$ matrix $\mathbf{A}$ is said to be **nilpotent** if, for some positive integer m , $\mathbf{A}^m = \mathbf{0}$. Find a 2×2 nilpotent matrix $\mathbf{A} \neq \mathbf{0}$.

24. (a) Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to **anticommute** if $\mathbf{AB} = -\mathbf{BA}$. Show that each of the **Pauli spin matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $i^2 = -1$, anticommutes with the others. Pauli spin matrices are used in quantum mechanics.

(b) The matrix $\mathbf{C} = \mathbf{AB} - \mathbf{BA}$ is said to be the **commutator** of the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$. Find the commutators of σ_x and σ_y , σ_y and σ_z , and σ_z and σ_x .

In Problems 25 and 26, solve the given system of equations by Gauss–Jordan elimination.

$$25. \begin{pmatrix} 5 & -1 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{pmatrix} \mathbf{X} = \begin{pmatrix} -9 \\ 27 \\ 9 \end{pmatrix} \quad 26. \begin{array}{rcl} x_1 + x_2 + x_3 & = & 6 \\ x_1 - 2x_2 + 3x_3 & = & 2 \\ 2x_1 - & & 3x_3 = 3 \end{array}$$

$$27. \text{ Without expanding, show that } \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ bc & ac & ab \end{vmatrix} = 0.$$

$$28. \text{ Show that } \begin{vmatrix} y & x^2 & x & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 4 & 2 & 1 \\ 5 & 9 & 3 & 1 \end{vmatrix} = 0 \text{ is the equation of a parabola}$$

passing through the three points $(1, 2)$, $(2, 3)$, and $(3, 5)$.

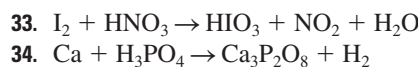
In Problems 29 and 30, evaluate the determinant of the given matrix by inspection.

$$29. \begin{vmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{vmatrix} \quad 30. \begin{pmatrix} -3 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 1 & 3 & 9 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$$

In Problems 31 and 32, without solving, state whether the given homogeneous system has only the trivial solution or has infinitely many solutions.

$$31. \begin{array}{rcl} x_1 - x_2 + x_3 & = & 0 \\ 5x_1 + x_2 - x_3 & = & 0 \\ x_1 + 2x_2 + x_3 & = & 0 \end{array} \quad 32. \begin{array}{rcl} x_1 - x_2 - x_3 & = & 0 \\ 5x_1 + x_2 - x_3 & = & 0 \\ x_1 + 2x_2 + x_3 & = & 0 \end{array}$$

In Problems 33 and 34, use Gauss–Jordan elimination to balance the given chemical equation.



In Problems 35 and 36, solve the given system of equations by Cramer's rule.

$$\begin{array}{ll} 35. \quad x_1 + 2x_2 - 3x_3 = -2 & 36. \quad x_1 + \quad \quad \quad x_3 = 4 \\ \quad \quad 2x_1 - 4x_2 + 3x_3 = 0 & \quad \quad 2x_1 + 3x_2 + 4x_3 = 5 \\ \quad \quad 4x_2 + 6x_3 = 5 & \quad \quad x_1 + 4x_2 + 5x_3 = 0 \end{array}$$

37. Use Cramer's rule to solve the system

$$\begin{aligned} X &= x \cos \theta + y \sin \theta \\ Y &= -x \sin \theta + y \cos \theta \end{aligned}$$

for x and y .

38. (a) Set up the system of equations for the currents in the branches of the network given in **FIGURE 8.R.1**.

(b) Use Cramer's rule to show that

$$i_1 = E \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right).$$

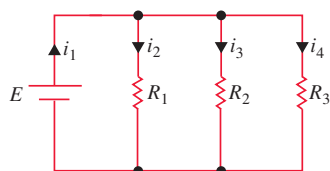


FIGURE 8.R.1 Network in Problem 38

39. Solve the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 6 \\ x_1 - 2x_2 &= -3 \\ -2x_1 + \quad \quad x_3 &= 9 \end{aligned}$$

by writing it as a matrix equation and finding the inverse of the coefficient matrix.

40. Use the inverse of the matrix $\mathbf{A}$ to solve the system $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

and the vector $\mathbf{B}$ is given by (a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

In Problems 41–46, find the eigenvalues and corresponding eigenvectors of the given matrix.

41. $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

42. $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$

43. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

44. $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

45. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

46. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

47. Supply a first column so that the matrix is orthogonal:

$$\begin{vmatrix} \boxed{} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \boxed{} & 0 & \frac{1}{\sqrt{3}} \\ \boxed{} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix}.$$

48. Consider the symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$.

(a) Find matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ that orthogonally diagonalize the matrix $\mathbf{A}$.

(b) Find the diagonal matrix $\mathbf{D}$ by actually carrying out the multiplication $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

49. Identify the conic section $x^2 + 3xy + y^2 = 1$.

50. Consider the following population data:

Year	1890	1900	1910	1920	1930
Population (in millions)	63	76	92	106	123

The actual population in 1940 was 132 million. Compare this amount with the population predicted from the least squares line for the given data.

In Problems 51 and 52, use the matrix $\mathbf{A} = \begin{pmatrix} 10 & 1 \\ 9 & 1 \end{pmatrix}$ to encode the given message. Use the correspondence (1) of Section 8.14.

51. SATELLITE LAUNCHED ON FRI

52. SEC AGNT ARRVS TUES AM

In Problems 53 and 54, use the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ to

decode the given message. Use the correspondence (1) in Section 8.14.

53. $\mathbf{B} = \begin{pmatrix} 19 & 0 & 15 & 14 & 0 & 20 \\ 35 & 10 & 27 & 53 & 1 & 54 \\ 5 & 15 & -3 & 48 & 2 & 39 \end{pmatrix}$

54. $\mathbf{B} = \begin{pmatrix} 5 & 2 & 21 \\ 27 & 17 & 40 \\ 21 & 13 & -2 \end{pmatrix}$

55. Decode the following messages using the parity check code.

(a) (1 1 0 0 1 1) (b) (0 1 1 0 1 1 1 0)

56. Encode the word (1 0 0 1) using the Hamming (7, 4) code.

In Problems 57 and 58, solve the given system of equations using LU-factorization.

57. The system in Problem 26

58. The system in Problem 36

59. Find the least squares line for the data (1, -2), (2, 0), (3, 5), (4, -1).

60. Find the least squares parabola for the data given in Problem 59.