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Advances in Mathematical Economics

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Some various convergence results for multivalued martingales

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Abstract. We prove various convergence results for multivalued martingales, sub-supermartingales and mils with respect to the Mosco topology and the linear topology both in Bochner integration and Pettis integration. We also state some existence theorems of Pettis conditional expectation for multivalued Pettis-integrable multifunctions.

Key words: martingale, submartingale, supermartingale, mil, conditional expectation, Mosco convergence, linear topology, Pettis

1. Introduction

The purpose of this paper is to present various convergence results for martingales, submartingales, supermartingales and mils with respect to the Mosco topology and the linear topology both in Bochner integration and Pettis integration. The paper is organized as follows. In § 2, we set our notation and definitions, and summarize needed results. In § 3, we state some convergence

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theorems for convex weakly compact valued submartingales, mils and unbounded closed convex supermartingales in Bochner integration. In § 4, we present various convergence theorems for convex weakly compact valued Pettis-integrable multifunctions. Several existence theorems of conditional expectation for convex weakly compact valued Pettis-integrable multifunctions and an integral representation theorem for a convex weakly compact valued multifunction defined on L^∞ or more generally on a Köthe space are also provided. In § 5, we provide some versions of Levy's theorem for Pettis-integrable multifunctions. Specific applications to the convergence of multivalued Pettis-integrable martingales are given at the end of § 5.

To our knowledge, not many results in this area are known. We refer to [17–21,25,27–30,32–35,37,40–42] for related results on martingales and mils. The results presented here are motivated by some applications in Mathematical Economics and the Law of Large Numbers (see, e.g. [8,9,12,15,26]).

2. Preliminaries and background

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)_{n\in\mathbf{N}}$ an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\geq 1}\mathcal{F}_n$. Let E be a separable Banach space, E^* the topological dual of E, $D^* = (x_j^*)_{j\in\mathbf{N}}$ a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$, $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) the closed unit ball of E (resp. E^*). Let $cc(E)$ (resp. $cwk(E)$) (resp. $ck(E)$) be the set of nonempty closed convex (resp. weakly compact convex) (resp. compact convex) subsets of E. Given $C \in cc(E)$, the distance function and the support function associated with C are defined respectively by

$$d(x, C) = \inf\{\|x - y\|,\ y \in C\} \quad (x \in E)$$

$$\delta^*(x^*, C) = \sup\{< x^*, y >,\ y \in C\} \quad (x^* \in E^*).$$

A $cc(E)$-valued mapping $C : \Omega \to cc(E)$ is $\mathcal{F}$-measurable, if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E)$, where $\mathcal{B}(E)$ is the Borel tribe of E. For any $C \in cc(E)$, we set

$$|C| = \sup\{\|x\| : x \in C\}.$$

For any $A, B \in cwk(E)$, the Hausdorff distance between A and B is denoted by

$$\mathcal{H}(A, B) = \sup_{x^*\in\overline{B}_{E^*}} |\delta^*(x^*, A) - \delta^*(x^*, B)|.$$

For each $n \in \mathbf{N}\cup\{\infty\}$, we denote by $\mathcal{L}^1_{cwk(E)}(\mathcal{F}_n)$ (with $\mathcal{F}_\infty = \mathcal{F}$) the space of all $\mathcal{F}_n$-measurable $cwk(E)$-valued multifunctions $X : \Omega \to cwk(E)$

such that $\omega \to |X(\omega)|$ is integrable. A sequence $(X_n)_{n\in\mathbf{N}}$ of $cc(E)$-valued multifunctions (mappings for short) is adapted if each X_n is $\mathcal{F}_n$-measurable. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is bounded (resp. uniformly integrable) if the sequence $(|X_n|)_{n\in\mathbf{N}}$ is bounded (resp. uniformly integrable) in $L^1_{\mathbf{R}}(\mathcal{F})$. A $\mathcal{F}$-measurable closed convex valued multifunction $X : \Omega \rightrightarrows E$ is *integrable* if it admits an integrable selection, equivalently if $d(0, X)$ is integrable. A closed convex set C in E is *ball-weakly compact*, if its intersection with any closed ball in E is weakly compact. A $cc(E)$-valued sequence $(X_n)_{n\in\mathbf{N}}$ Mosco-converges [36] to a closed convex set X_∞ if

$$X_\infty = s\text{-}li X_n = w\text{-}ls X_n ,$$

where

$$s\text{-}li\ X_n = \{x \in E : ||x_n - x|| \to 0;\ x_n \in X_n\}$$

and

$$w\text{-}ls\ X_n = \{x \in E : x = w\text{-}\lim_{j\to\infty} x_j,\ x_j \in X_{n_j}\}$$

and s (resp. w) is the strong (resp. weak) topology in E. If $(X_n)_{n\in\mathbf{N}}$ Mosco-converges to X_∞ in $cc(E)$, we write

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty.$$

The linear topology τ_L [3] on $cc(E)$ is the upper bounded of the two following topologies:

(a) The topology of pointwise convergence of support functions on E^*
(b) The topology of pointwise convergence of distance functions on E

Proposition 2.1 *Let $(C_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be a sequence in $cc(E)$, then $C_\infty = \tau_L - \lim_{n\to\infty} C_n$*

$$\Longleftrightarrow \begin{cases} (i)\ \ \delta^*(x^*, C_\infty) = \lim_{n\to\infty} \delta^*(x^*, C_n) \quad \forall x^* \in E^*. \\ (ii)\ d(x, C_\infty) = \lim_{n\to\infty} d(x, C_n) \quad \forall x \in E. \end{cases}$$

Beer [3] showed that the topology τ_L is stronger than the Mosco topology.

If the strong dual E^*_b of E is separable and X is an element of $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, for each $n \in \mathbf{N}$, it is known that the conditional expectation of X with respect to $\mathcal{F}_n$, $E^{\mathcal{F}_n}X$, is the unique element (for = a.s.) of $\mathcal{L}^1_{cwk(E)}(\mathcal{F}_n)$ such that

$$\int_\Omega \delta^*(u(\omega), E^{\mathcal{F}_n}X(\omega))dP(\omega) = \int_\Omega \delta^*(u(\omega), X(\omega))dP(\omega)$$

for all $u \in L^\infty_{E^*}(\mathcal{F}_n)$. See [43, Remark 4 of Theorem 3]. For more information for the conditional expectation of multifunctions, we refer to [27, 43].

Definition 2.1 *Assume that* E_b^* *is separable. A* $cwk(E)$*-valued adapted sequence* $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ *in* $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ *is a mil if for every* $\varepsilon > 0$*, there exists* p *such that for* $n \geq p$*, we have*

$$P(\sup_{n\geq q\geq p} \mathcal{H}(X_q, E^{\mathcal{F}_q} X_n) > \varepsilon) < \varepsilon \ ,$$

where $\mathcal{H}$ *stands for the Hausdorff distance on* $cwk(E)$*.*

From the definition, it is easy to check that, for each $x^* \in \overline{B}_{E^*}$, the sequence $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ is a real-valued mil.

3. Convergence results for $cwk(E)$-valued martingales and mils

Before going further, we present first two simple convergence results which will be applied to the convergence of $cwk(E)$-valued martingales and mils with respect to the Mosco topology and the linear topology.

Proposition 3.1 *Let* E *be a Banach space. Let* $(A_n)_{n\in\mathbf{N}}$ *be a sequence in* $cwk(E)$*. Assume that there exist* $A_\infty \in cwk(E)$ *and a sequence* $(B_n)_{n\in\mathbf{N}}$ *in* $cwk(E)$ *which satisfy:*

(i) $M\text{-}\lim_{n\to\infty} A_n = A_\infty$.
(ii) $\lim_{n\to\infty} \mathcal{H}(A_n, B_n) = 0$.

Then

$$M\text{-}\lim_{n\to\infty} A_n = A_\infty = M\text{-}\lim_{n\to\infty} B_n.$$

Proof. Given x in $M\text{-}\lim_{n\to\infty} A_n$. There exist $x_n \in A_n$ such that $x = \lim_{n\to\infty} x_n$. We have the estimate

$$d(x, B_n) \leq ||x - x_n|| + d(x_n, B_n) \leq ||x - x_n|| + \mathcal{H}(A_n, B_n).$$

By (ii) we deduce that $\lim_{n\to\infty} d(x, B_n) = 0$. Hence $M\text{-}\lim_{n\to\infty} A_n \subset s\text{-}li\ B_n$. Now let $x \in w\text{-}ls\ B_n$. There exist a sequence $(x_k)_{k\in\mathbf{N}}$ weakly converging to x with $x_k \in B_{n_k}$ for all $k \in \mathbf{N}$. Let us pick $y_k \in A_{n_k}$ such that

$$||x_k - y_k|| \leq \mathcal{H}(B_{n_k}, A_{n_k}) + \frac{1}{k}.$$

As $y_k = x_k - (x_k - y_k)$, (y_k) weakly converges to x. Consequently

$$x \in w\text{-}ls\ A_n \subset M\text{-}\lim_{n\to\infty} A_n.$$

Hence $w\text{-}ls\ B_n \subset M\text{-}\lim_{n\to\infty} A_n$ and so $M\text{-}\lim_{n\to\infty} B_n = M\text{-}\lim_{n\to\infty} A_n$. □

Proposition 3.2 *Let E be a separable Banach space. Let $D_1^* = \{e_j^*, j \in \mathbf{N}\}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to Mackey topology $\tau(E^*, E)$. Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $cwk(E)$ and $A_\infty := s\text{-}li\ A_n \in cwk(E)$. Assume that*

$$\lim_{n\to\infty} \delta^*(e_j^*, A_n) = \delta^*(e_j^*, A_\infty) \quad \forall j \in \mathbf{N}.$$

Then the following holds

$$\lim_{n\to\infty} d(x, A_n) = d(x, A_\infty) \quad \forall x \in E.$$

Proof. For each $n \in \mathbf{N}$ and for each $x \in E$ we have

$$d(x, A_n) = \sup_{j\in\mathbf{N}}[< e_j^*, x > -\delta^*(e_j^*, A_n)]. \tag{3.2.1}$$

For each $j \in \mathbf{N}$, the sequence $(< e_j^*, x > -\delta^*(e_j^*, A_n))_{n\in\mathbf{N}}$ converges to $< e_j^*, x > -\delta^*(e_j^*, A_\infty)$. By (3.2.1), for each $j \in \mathbf{N}$ and for each $n \in \mathbf{N}$, we have

$$< e_j^*, x > -\delta^*(e_j^*, A_n) \le d(x, A_n). \tag{3.2.2}$$

By passing to the limit when $n \to \infty$ taking account of the scalar convergence of $(A_n)_{n\ge1}$ we get

$$< e_j^*, x > -\delta^*(e_j^*, A_\infty) \le \liminf_{n\to\infty} d(x, A_n) \tag{3.2.3}$$

for each $j \in \mathbf{N}$. Hence, by taking the supremum on $j \in \mathbf{N}$ in (3.2.3) we get

$$d(x, A_\infty) \le \liminf_{n\to\infty} d(x, A_n). \tag{3.2.4}$$

On the other hand we have the estimate

$$\limsup_{n\to\infty} d(x, A_n) \le d(x, s\text{-}li\ A_n) = d(x, A_\infty) \quad \forall x \in E. \tag{3.2.5}$$

Indeed, let $y \in A_\infty := s\text{-}li\ A_n$, pick $y_n \in A_n$ such that $||y_n - y|| \to 0$. We have $d(x, A_n) \le ||x - y_n||$, so that

$$\limsup_{n\to\infty} d(x, A_n) \le \limsup_{n\to\infty} ||x - y_n|| = ||x - y||. \tag{3.2.6}$$

Whence (3.2.5) follows by taking the infimum on $y \in A_\infty$ in (3.2.6). By combining (3.2.4) and (3.2.5) we conclude that

$$\lim_{n\to\infty} d(x, A_n) = d(x, A_\infty).$$

□

Remark. If A_n and A_∞ are convex compact, we may replace in Proposition 3.2 the set D_1^* by $H_1^* = (h_j^*)_{j\geq 1}$ where $(h_j^*)_{j\in\mathbf{N}}$ is a dense sequence in the closed unit ball $\overline{B}_{E^*}$, with respect to the topology $\sigma(E^*, E)$ that coincides on $\overline{B}_{E^*}$ with the topology of compact convergence.

Now we are ready to state the almost surely convergence with respect to the Mosco topology and the τ_L topology for $cwk(E)$-valued martingales and mils.

Theorem 3.1 *Assume that E_b^* is separable. Let $(X_n)_{n\in\mathbf{N}}$ be a bounded mil in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that for each $x^* \in \overline{B}_{E^*}$, $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$ $a.s.$ Then the following hold:*

(i) $M\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ *a.s.*
(ii) $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$

Consequently, we have

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*},$$

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s. \quad \forall x \in E.$$

Proof. We will proceed in several steps.
Step 1. **Claim** $M\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ a.s.
We have to prove that

$$w\text{-}ls\ E^{\mathcal{F}_n} X_\infty(\omega) \subset X_\infty(\omega) \subset s\text{-}li\ E^{\mathcal{F}_n} X_\infty(\omega)$$

for a.s. $\omega \in \Omega$. Let us check the inclusion

$$X_\infty(\omega) \subset s\text{-}li\ E^{\mathcal{F}_n} X_\infty(\omega) \quad a.s.$$

Let $f \in \mathcal{S}^1_{X_\infty}$. By Levy's theorem for regular martingale $E^{\mathcal{F}_n} f$, we have that $\lim_{n\to\infty} E^{\mathcal{F}_n} f(\omega) = f(\omega)$ a.s. with respect to the norm topology. Since $E^{\mathcal{F}_n} f(\omega) \in E^{\mathcal{F}_n} X_\infty(\omega)$, it follows that $f(\omega) \in s\text{-}li\ E^{\mathcal{F}_n} X_\infty(\omega)$ a.s. Taking a Castaing representation of X_∞ (see [16, Theorem III-37]) we deduce that $X_\infty(\omega) \subset s\text{-}li\ E^{\mathcal{F}_n} X_\infty(\omega)$ a.s. Now we prove the inclusion $w\text{-}ls\ E^{\mathcal{F}_n} X_\infty(\omega) \subset X_\infty(\omega)$ a.s. Let $D^* = (x_j^*)_{j\in\mathbf{N}}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$. Applying Levy's theorem to the L^1-bounded real-valued martingales $E^{\mathcal{F}_n}\delta^*(x_j^*, X_\infty)$ and taking the well-known property for multivalued conditional expectation [43] give

$$\lim_{n\to\infty} \delta^*(x_j^*, E^{\mathcal{F}_n} X_\infty) = \lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(x_j^*, X_\infty) = \delta^*(x_j^*, X_\infty)$$

a.s. Now let $\omega \in \Omega$ be fixed but arbitrary for which the preceding equality holds and let $x \in w\text{-}ls\, E^{\mathcal{F}_n} X_\infty(\omega)$. There is a sequence $(x_k)_{k\in\mathbf{N}}$ in $E^{\mathcal{F}_{n_k}} X_\infty(\omega)$ such that $(x_k)_{k\in\mathbf{N}}$ weakly converges to x. For each $j \in \mathbf{N}$ we have

$$\langle x_j^*, x\rangle = \lim_{k\to\infty} \langle x_j^*, x_k\rangle \le \limsup_{k\to\infty} \delta^*(x_j^*, E^{\mathcal{F}_{n_k}} X_\infty(\omega)) = \delta^*(x_j^*, X_\infty(\omega)).$$

According to Proposition III-35 in [16], we deduce that $x \in X_\infty(\omega)$.
Step 2. **Claim** $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0$ a.s. If $(X_n)_{n\in\mathbf{N}}$ is an E-valued bounded mil in $L^1_E(\mathcal{F})$, then $||X_n - E^{\mathcal{F}_n} X_\infty||$ goes to 0 a.s. by virtue of a result of Talagrand [41, Theorem 6, p. 1193], because for each $x^* \in \overline{B}_{E^*}$, the real-valued L^1-bounded mil $(\langle x^*, X_n - E^{\mathcal{F}_n} X_\infty\rangle)_{n\in\mathbf{N}}$ converges to 0 a.s. In the multivalued case, the claim (ii) is true by using the Definition 2.1 and a careful adaptation of the techniques of Talagrand developed in [41, Theorem 6, p. 1193], namely $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0$ a.s. (see [6, 7] for details).
Step 3 $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}$.
Let $(f_j^*)_{j\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ with respect to the topology of the dual norm. For each $x^* \in \overline{B}_{E^*}$ let us write

$$\begin{aligned} |\delta^*(x^*, X_n) - \delta^*(x^*, X_\infty)| \le{}& |\delta^*(x^*, X_n) - \delta^*(x^*, E^{\mathcal{F}_n} X_\infty)| \\ &+ |\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)|. \end{aligned}$$

From (ii), it is obvious that the first term $|\delta^*(x^*, X_n) - \delta^*(x^*, E^{\mathcal{F}_n} X_\infty)|$ goes to 0 a.s. for all $x^* \in \overline{B}_{E^*}$ when n goes to ∞ and so is the second term

$$|\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)|.$$

Indeed, by Levy's theorem it is obvious that for all $j \in \mathbf{N}$

$$|\delta^*(f_j^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(f_j^*, X_\infty)| \to 0 \quad a.s.$$

when n goes to ∞. Since

$$\sup_{n\in\mathbf{N}} E^{\mathcal{F}_n} |X_\infty|(\omega) < \infty \quad a.s. \quad \omega \in \Omega$$

using a density argument (see, e.g. [5, Theorem 2.7]), it is easy to check that

$$|\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)| \to 0 \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}$$

when n goes to ∞.
Step 4 By (i)–(ii) and Proposition 3.1, we conclude that

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Applying Proposition 3.2 and the above result show that

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

□

The following result is concerned with submartingales.

Theorem 3.2 *Assume that* E^*_b *is separable. Let* $(X_n)_{n\in\mathbf{N}}$ *be a bounded submartingale in* $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ *and* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ *such that for each* $x^* \in \overline{B}_{E^*}$, $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s.$
Then the following hold:
(i) $M\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ *a.s.*
(ii) $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$
Consequently, we have

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*},$$

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s. \quad \forall x \in E.$$

Proof. *Step 1*. **Claim** $M\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ a.s. The proof is the same as the one of Theorem 3.1 and is omitted.
Step 2 **Claim** $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0$ a.s.
Let $D^*_1 = (e^*_j)_{j\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ with respect Mackey topology $\tau(E^*, E)$. We have

$$\mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = \sup_{j\in\mathbf{N}} [\delta^*(e^*_j, X_n) - \delta^*(e^*_j, E^{\mathcal{F}_n} X_\infty)].$$

As $(\delta^*(e^*_j, X_n) - \delta^*(e^*_j, E^{\mathcal{F}_n} X_\infty))_{n\in\mathbf{N}}$ are real-valued bounded submartingales in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0, from Lemma V.2.9 in [39] we see that

$$\begin{aligned}\mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) &= \sup_{j\in\mathbf{N}} [\delta^*(e^*_j, X_n) - \delta^*(e^*_j, E^{\mathcal{F}_n} X_\infty] \\ &\to \sup_{j\in\mathbf{N}} \lim_{n\to\infty} [\delta^*(e^*_j, X_n) - \delta^*(e^*_j, E^{\mathcal{F}_n} X_\infty] = 0\end{aligned}$$

almost surely.
Step 3 $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}$.
Let $(f^*_j)_{j\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ with respect to the topology of the dual norm. For each $x^* \in \overline{B}_{E^*}$ let us write

$$\begin{aligned}|\delta^*(x^*, X_n) - \delta^*(x^*, X_\infty)| \le{}& |\delta^*(x^*, X_n) - \delta^*(x^*, E^{\mathcal{F}_n} X_\infty)| \\ &+ |\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)|.\end{aligned}$$

From (ii), it is obvious that the first term $|\delta^*(x^*, X_n) - \delta^*(x^*, E^{\mathcal{F}_n} X_\infty)|$ goes to 0 a.s. for all $x^* \in \overline{B}_{E^*}$ when n goes to ∞ and so is the second term

$$|\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)|.$$

Indeed, by Levy's theorem it is obvious that for all $j \in \mathbf{N}$

$$|\delta^*(f_j^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(f_j^*, X_\infty)| \to 0 \quad a.s.$$

when n goes to ∞. Since

$$\sup_{n \in \mathbf{N}} E^{\mathcal{F}_n} |X_\infty|(\omega) < \infty \quad a.s. \quad \omega \in \Omega$$

using a density argument it is easy to check that

$$|\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_\infty)| \to 0 \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}$$

when n goes to ∞.
Step 4 By (i)–(ii) and Proposition 3.1, we conclude that

$$M\text{-}\lim_{n \to \infty} X_n = X_\infty \quad a.s.$$

Hence Proposition 3.2 shows that

$$\lim_{n \to \infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

Remark. It is worthy to mention that Theorems 3.1 and 3.2 provide the convergence with respect to the linear topology of the bounded mil (resp. submartingale) $(X_n)_{n \in \mathbf{N}}$ towards X_∞ in the space $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and the relationship with the $cwk(E)$-valued regular martingale $E^{\mathcal{F}_n} X_\infty$, namely

$$\lim_{n \to \infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$$

In Theorem 3.2 we provide an alternative proof of this property for submartingales via Lemma V.2.9 in [39], while in the case of mils, Talagrand's techniques are needed to obtain this result.

We end this section by providing a new version of Mosco convergence results for *unbounded* supermartingales. For this purpose, in the remainder of this section, the conditional expectation is taken in the sense of Hiai–Umegaki [27]. If $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{F}$, $F : \Omega \rightrightarrows E$ is an integrable $\mathcal{F}$-measurable multifunction, Hiai and Umegaki [27] showed the existence of a $\mathcal{B}$-measurable and integrable multifunction G such that

$$S^1_G(\mathcal{B}) = cl\{E^{\mathcal{B}} f : f \in S^1_F(\mathcal{F})\},$$

where $S^1_F(\mathcal{F})$ and $S^1_G(\mathcal{B})$ is the set of all $\mathcal{F}$-measurable (resp. $\mathcal{B}$-measurable) integrable selections of F and G respectively, the closure being taken in $L^1_E(\Omega, \mathcal{F}, P)$. G is the multivalued conditional expectation of F relative to $\mathcal{B}$. The conditional expectation $G := E^{\mathcal{B}} F$ of Hiai and Umegaki can be defined as the essential supremum of $\{E^{\mathcal{B}} f : f \in S^1_F(\mathcal{F})\}$. For more information on the Hiai–Umegaki conditional expectation, see [27]. A $cc(E)$-valued integrable sequence $(X_n)_{n\in\mathbf{N}}$ is a supermartingale if X_n is $\mathcal{F}_n$-measurable for each $n \in \mathbf{N}$ and $E^{\mathcal{F}_n} X_{n+1} \subset X_n$ for each $n \in \mathbf{N}$. For the convenience of the reader we recall and summarize a tightness condition in the space $\mathcal{L}^1_{cwk(E)}(\Omega, \mathcal{F}, P)$. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is $cwk(E)$*-tight* if, for every $\varepsilon > 0$, there is a $cwk(E)$-valued $\mathcal{F}$-measurable multifunction $\Gamma_\varepsilon : \Omega \to E$ such that

$$\sup_{n\in\mathbf{N}} P(\Omega \setminus \{\omega \in \Omega : X_n(\omega) \subset \Gamma_\varepsilon(\omega)\}) \leq \varepsilon.$$

We refer to [11, 14, 15] for various tightness notions. Let us mention a useful result.

Proposition 3.3 *Suppose that E is a separable Banach space, $(X_n)_{n\in\mathbf{N}}$ is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following condition: There is a $\mathcal{F}$-measurable ball-weakly compact $cc(E)$-valued multifunction $K: \Omega \rightrightarrows E$ such that $X_n(\omega) \subset K(\omega)$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$. Then $(X_n)_{n\in\mathbf{N}}$ is $cwk(E)$-tight.*

Proof. See, e.g. [9, Proposition 3.3(i)]. □

We need a preliminary lemma.

Lemma 3.1 *Let $(X_n)_{n\in\mathbf{N}}$ be a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying: There is a $cwk(E)$-valued $\mathcal{F}$-measurable multifunction $K : \Omega \rightrightarrows E$ such that $X_n(\omega) \subset K(\omega)$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$. Then there is $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following properties:*

(a) $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$ *a.s. for all* $x^* \in \overline{B}_{E^*}$.
(b) $\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty)$ *a.s. for all* $x \in E$.
(c) $E^{\mathcal{F}_m} X_\infty \subset X_m$ *a.s. for all* $m \in \mathbf{N}$.

Proof. (a) Let $D^*_1 = (e^*_j)_{j\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology $\tau(E^*, E)$. As $(X_n)_{n\in\mathbf{N}}$ is a uniformly integrable $cwk(E)$-valued supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, for each $j \in \mathbf{N}$, $(\delta^*(e^*_j, X_n))_{n\in\mathbf{N}}$ is a bounded real-valued supermartingale in $L^1_{\mathbf{R}}(\mathcal{F})$. So it converges a.s. for every $j \in \mathbf{N}$ to a function m_j in L^1. Applying [14, Theorem 6.1] to the uniformly integrable $cwk(E)$-tight sequence $(X_n)_{n\in\mathbf{N}}$ provided a subsequence $(X'_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that for each $v \in L^\infty_{E^*}(\mathcal{F})$ the following hold

$$\lim_{n\to\infty} \int_\Omega \delta^*(v, X'_n)dP = \int_\Omega \delta^*(v, X_\infty)dP.$$

So by identifying the limits we get

$$\lim_{n\to\infty} \delta^*(e^*_j, X_n) = m_j = \delta^*(e^*_j, X_\infty) \quad a.s.$$

Therefore (a) follows by a density argument. By applying Lemma V.2.9 in [39] to the family of real-valued L^1- bounded submartingales

$$(\langle e^*_j, x\rangle - \delta^*(e^*_j, X_n))_{n\in\mathbf{N}}$$

gives (b). Now let $m < n$ and $A \in \mathcal{F}_m$. By the supermartingale property we have

$$\int_A \delta^*(e^*_j, X_n)dP \le \int_A \delta^*(e^*_j, X_m)dP.$$

It follows that

$$\int_A \delta^*(e^*_j, X_\infty)dP \le \int_A \delta^*(e^*_j, X_m)dP.$$

Therefore by taking the conditional expectation of X_∞ we get

$$\int_A \delta^*(e^*_j, E^{\mathcal{F}_m} X_\infty)dP = \int_A \delta^*(e^*_j, X_\infty)dP \le \int_A \delta^*(e^*_j, X_m)dP$$

thereby proving (c) (see [16, Proposition III-35]). □

The following result is an extension of a similar one due Choukairi [18, Theorem 2.14] dealing with reflexive separable Banach space and is a variant of a result due to Hess [25, Theorem 5.12]. Compare with similar results obtained in [18, 32, 33] for bounded martingales and sub- and supermartingales in Banach spaces with RNP property and strongly separable dual using the method of selection martingales (see, e.g. [18, Proposition 2.7, Theorem 2.8]). We stress the fact that here we deal with unbounded supermartingales in separable Banach spaces without RNP property.

Theorem 3.3 *Let $(X_n)_{n\in\mathbf{N}}$ be a closed convex integrable supermartingale ($E^{\mathcal{F}_m} X_n \subset X_m$ for $m < n$) satisfying:*

- *(i) There is a ball-weakly compact $cc(E)$-valued $\mathcal{F}$-measurable multifunction $K : \Omega \rightrightarrows E$ such that $X_n(\omega) \subset K(\omega)$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$.*
- *(ii) $(X_n)_{n\in\mathbf{N}}$ admits a regular martingale selection $(f_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_E(\mathcal{F})$.*

Then one can find a $cc(E)$-valued integrable multifunction X_∞ such that

$$M-\lim_n X_n = X_\infty \quad a.s.$$

$$E^{\mathcal{F}_n} X_\infty \subset X_n \quad a.s. \quad \forall n \in \mathbf{N}.$$

Proof. We will proceed in several steps.
Step 1 Here we will use a careful adaptation of a truncation technique developed in [18, Theorem 2.16]. By our assumption there is $f \in L^1_E(\mathcal{F})$ such that $f_n = E^{\mathcal{F}_n} f$ for all $n \in \mathbf{N}$. For each $k \in \mathbf{N}$, let us consider the multifunction

$$X^k_n = X_n \cap [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E].$$

We are going to check that $(X^k_n)_{n\in\mathbf{N}}$ is a uniformly integrable $cwk(E)$-valued supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. By (i) and Proposition 3.3, $(X^k_n)_{n\in\mathbf{N}}$ is $cwk(E)$-tight. Let $m < n$. As $X^k_n \subset X_n$, by supermartingale property and by monotonicity of conditional expectation one has

$$E^{\mathcal{F}_m} X^k_n \subset E^{\mathcal{F}_m} X_n \subset X_m.$$

Taking the conditional expectation $E^{\mathcal{F}_m}$ in the inclusion

$$X^k_n \subset f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E$$

yields

$$E^{\mathcal{F}_m} X^k_n \subset f_m + E^{\mathcal{F}_m}(|f| + k)\overline{B}_E.$$

Hence we get $E^{\mathcal{F}_m} X^k_n \subset X^k_m$. Note that $|X^k_n| \le h^k_n$ for all $n \in \mathbf{N}$, where $h^k_n := |f_n| + E^{\mathcal{F}_n}(|f| + k)$. Further the uniformly integrable submartingale $(h^k_n)_{n\in\mathbf{N}}$ converges a.s. to a positive integrable function h^k. Hence there exists a positive constant r^k depending on $\omega \in \Omega$ such that $h^k_n \le r^k$ a.s. for all $n \in \mathbf{N}$. So $|X^k_n| \le r^k$ a.s. for all $n \in \mathbf{N}$. Applying Lemma 3.1 (a), (b) (c) to the $cwk(E)$-tight uniformly integrable supermartingale $(X^k_n)_{n\in\mathbf{N}}$ provides $X^k_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and a negligible set N_k such that

$$\text{(3.3.1)} \quad \lim_{n\to\infty} \delta^*(x^*, X^k_n) = \delta^*(x^*, X^k_\infty) \quad \forall x^* \in \overline{B}_{E^*} \quad \forall \omega \in \Omega \setminus N_k.$$

$$\text{(3.3.2)} \quad \lim_{n\to\infty} d(x, X^k_n) = d(x, X^k_\infty) \quad \forall x \in E \quad \forall \omega \in \Omega \setminus N_k.$$

$$\text{(3.3.3)} \quad E^{\mathcal{F}_n} X^k_\infty \subset X^k_n \quad \forall n \in \mathbf{N} \quad \forall \omega \in \Omega \setminus N_k$$

so that by (3.3.1) and (3.3.2)

$$M\text{-}\lim_{n\to\infty} X_n^k = X_\infty^k \quad \forall\omega \in \Omega \setminus N_k. \tag{3.3.4}$$

Step 2 Convergence and conclusion.
By construction, we have $X_n^k \subset X_n^{k+1}$, so that $(X_\infty^k)_{k\in\mathbf{N}}$ is increasing. Let us set $N = \cup_{k=1}^\infty N_k$ and

$$X_\infty(\omega) = \begin{cases} cl[\cup_{k=1}^\infty X_\infty^k(\omega)] & \text{if } \ \omega \in \Omega \setminus N, \\ 0 & \text{if } \ \omega \in N. \end{cases}$$

We need to check that

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Let $x \in w\text{-}lsX_n(\omega)$ with $\omega \notin N$. There is a sequence $(x_j)_{j\in\mathbf{N}}$ weakly converging to x with $x_j \in X_{n_j}(\omega)$. Pick a large enough integer $p \in \mathbf{N}$ such that $||x_j|| \le p$ for all $j \in \mathbf{N}$. By Jensen inequality we have

$$\begin{aligned} ||x_j - f_{n_j}(\omega)|| &= ||x_j - E^{\mathcal{F}_{n_j}}(f)(\omega)|| \\ \le p + E^{\mathcal{F}_{n_j}}|f|(\omega) &= E^{\mathcal{F}_{n_j}}(|f| + p)(\omega) \end{aligned}$$

therefore $x_j \in f_{n_j}(\omega) + E^{\mathcal{F}_{n_j}}(|f| + p)(\omega)\overline{B}_E$, and so $x_j \in X_{n_j}^p(\omega)$. Hence using (3.3.4) and the definition of X_∞ we get

$$x \in w\text{-}lsX_{n_j}^p(\omega) \subset X_\infty^p(\omega) \subset X_\infty(\omega).$$

By Theorem 2.1 in [26] we have

$$E^{\mathcal{F}_n}X_\infty(\omega) = cl[\cup_{k=1}^\infty E^{\mathcal{F}_n}X_\infty^k(\omega)] \quad a.s.$$

Hence by (3.3.3)

$$E^{\mathcal{F}_n}X_\infty(\omega) \subset cl[\cup_{k=1}^\infty X_n^k(\omega)] \subset X_n(\omega) \quad a.s. \tag{3.3.5}$$

Let $f \in S^1_{X_\infty}$. By (3.3.5) we have $E^{\mathcal{F}_n}f(\omega) \in X_n(\omega)$. By Levy's theorem we have $\lim_{n\to\infty} E^{\mathcal{F}_n}f = f$ a.s. Hence $f(\omega) \in s\text{-}liX_n(\omega)$ a.s. Taking a Castaing representation of X_∞ we get $X_\infty(\omega) \in s\text{-}liX_n(\omega)$ a.s. ☐

4. Convergences and conditional expectation in Pettis integration

We present in this section some new convergence results for Pettis-integrable $cwk(E)$-valued multifunctions and also we state the existence of conditional expectations for these multifunctions. A multifunction $X : \Omega \Rightarrow cwk(E)$ is *Pettis-integrable* if X is $\mathcal{F}$-measurable and scalarly integrable (that is $\delta^*(x^*, X(.))$ is integrable for every $x^* \in E^*$) and if the scalarly integrable selections of X are Pettis-integrable. Let us denote by $\mathcal{S}^1_{Pe}(X)(\mathcal{F})$ the set of all $\mathcal{F}$-measurable and Pettis-integrable selections of X. By [16, Theorem V-13], $\mathcal{S}^1_{Pe}(X)(\mathcal{F})$ is convex and compact with respect to the topology of pointwise convergence on $L^\infty \otimes E^*$, namely the topology $\sigma(P^1_E(\mathcal{F}), L^\infty(\mathcal{F}) \otimes E^*)$, where $P^1_E(\mathcal{F})$ is the space of all $\mathcal{F}$-measurable and Pettis-integrable E-valued functions defined on $(\Omega, \mathcal{F}, P)$. The usual Pettis norm $||f||_{Pe}$ of $f \in P^1_E(\mathcal{F})$ is defined by

$$||f||_{Pe} = \sup_{x^* \in \overline{B}_{E^*}} \int_\Omega |\langle x^*, f\rangle| dP.$$

$(P^1_E(\mathcal{F}), ||.||_{Pe})$ endowed with $||.||_{Pe}$ is a normed space. The multivalued Aumann–Pettis integral $Pe\text{-}\int_\Omega XdP$ of a $cwk(E)$-valued Pettis-integrable multifunction X is defined by

$$Pe\text{-}\int_\Omega XdP := \{Pe\text{-}\int_\Omega fdP : f \in \mathcal{S}^1_{Pe}(X)(\mathcal{F})\}$$

where $Pe\text{-}\int_\Omega fdP$ denotes the Pettis integral of the Pettis-integrable mapping $f : \Omega \to E$. So $Pe\text{-}\int_\Omega XdP$ is convex and $\sigma(E, E^*)$-compact, furthermore the Strassen formula [16, Theorem V-14] holds

$$\delta^*(x^*, Pe\text{-}\int_A XdP) = \int_A \delta^*(x^*, X)dP \quad \forall x^* \in E^* \quad \forall A \in \mathcal{F}.$$

We will denote by $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ the set of all Pettis-integrable $cwk(E)$-valued multifunctions. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ is adapted if for each $n \in \mathbf{N}$, $X_n \in \mathcal{P}^1_{cwk(E)}(\mathcal{F}_n)$. Given a sub-$\sigma$-algebra $\mathcal{B}$ and a $cwk(E)$-valued Pettis integrable multifunction $X \in \mathcal{P}^1_{cwk(E)}(\mathcal{F})$, the *Pettis conditional expectation* of X is, by definition, a $\mathcal{B}$-measurable $cwk(E)$-valued Pettis-integrable multifunction denoted by $Pe\text{-}E^{\mathcal{B}}X_n$ which satisfies

$$Pe\text{-}\int_A Pe\text{-}E^{\mathcal{B}}XdP = Pe\text{-}\int_A XdP \quad \forall A \in \mathcal{B}.$$

Definition 4.1 *An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ is a mil if for every $\varepsilon > 0$, there exists p such that for $n \geq p$, we have*

$$P(\sup_{p\leq q\leq n} \mathcal{H}(X_q, Pe\text{-}E^{\mathcal{F}_q}X_n) > \varepsilon) < \varepsilon,$$

where $\mathcal{H}$ stands for the Hausdorff distance on $cwk(E)$ and $Pe\text{-}E^{\mathcal{F}_q}X_n$ is the Pettis conditional expectation of X_n associated with the σ-algebra $\mathcal{F}_q$.

Accordingly, our main task is to prove the existence of Pettis conditional expectation for $\mathcal{F}$-measurable $cwk(E)$-valued Pettis-integrable multifunctions. We will provide first some preliminary results in multivalued Pettis integration which lead us to the existence of conditional expectation in this class of multifunctions. It is worth to address the following related question. Given a $\mathcal{F}$-measurable and scalarly integrable $cwk(E)$-valued (resp. $ck(E)$-valued) mapping $X : \Omega \Rightarrow E$, find sufficient conditions for which X is $cwk(E)$- *Pettis* (resp. $ck(E)$- *Pettis*) in the sense: for each $A \in \mathcal{F}$ there is $K_A \in cwk(E)$ (resp. $ck(E)$) such that $\int_A \delta^*(x^*, X)dP = \delta^*(x^*, K_A)$ for all $x^* \in E^*$. In other words, K_A is the $cwk(E)$ (resp. $ck(E)$) *Pettis integral* of X over A. When $X : \Omega \to E$ is $\mathcal{F}$-measurable and scalarly integrable E-valued mapping, the preceding notions coincide with the classical definition of Pettis-integrability. In the following we will provide the answer to this question.

Proposition 4.1 *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ and respectively in $\mathcal{P}^1_{ck(E)}(\mathcal{F})$ and $X : \Omega \to cwk(E)$ $(resp. X : \Omega \to ck(E))$ be a scalarly $\mathcal{F}$-measurable and scalarly integrable mapping satisfying:*

(i) For each $A \in \mathcal{F}$, $\lim_{n\to\infty} \int_A \delta^*(x^*, X_n) = \int_A \delta^*(x^*, X)$ for all $x^* \in E^*$.

(ii) For each $A \in \mathcal{F}$, $(\int_A X_n dP)_{n\in\mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ $(resp. (ck(E), \mathcal{H}))$, here $\mathcal{H}$ is the Hausdorrf distance on $cwk(E)$ $(resp. ck(E))$.

Then for each $A \in \mathcal{F}$, there is $K_A \in cwk(E)$ $(resp. ck(E))$ such that

$$\int_A \delta^*(x^*, X)dP = \delta^*(x^*, K_A)$$

for all $x^* \in E^*$.

Proof. By (ii) For each $A \in \mathcal{F}$, $(\int_A X_n dP)_{n\in\mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$), hence $(\int_A X_n dP)_{n\geq 1}$ converges in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$) to a convex weakly compact (resp. convex compact) set K_A because each $(\int_A X_n dP)$ is convex weakly compact (resp. convex compact), and the space $(cwk(E), \mathcal{H})$ $(resp. (ck(E), \mathcal{H}))$ is complete. By (i) the result follows. □

Theorem 4.1 *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ $(resp. \mathcal{P}^1_{ck(E)}(\mathcal{F}))$ and $X : \Omega \to cwk(E)$ $(resp. X : \Omega \to ck(E))$ be a $\mathcal{F}$-measurable and scalarly integrable mapping satisfying:*

(i) For each $A \in \mathcal{F}$, $\lim_{n\to\infty} \int_A \delta^*(x^*, X_n)dP = \int_A \delta^*(x^*, X)dP$ for all $x^* \in E^*$.

(ii) There is an increasing sequence of $\mathcal{F}$-measurable sets $(B_k)_{k\in\mathbf{N}}$ with $\lim_{k\to\infty} P(B_k) = 1$ such that, for each $k \in \mathbf{N}$, for each $A \in \mathcal{F}$, the sequence $(\int_{A\cap B_k} X_n dP)_{n\in\mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ $(resp.(ck(E), \mathcal{H}))$.

(iii) $(X_n)_{n\in\mathbf{N}}$ is Pettis uniformly integrable, that is, for every $\varepsilon > 0$, there exist $\eta > 0$ such that every $A \in \mathcal{F}$ with $P(A) < \eta$, one has

$$\sup_{n\geq 1} \sup_{x^*\in\overline{B}_{E^*}} \int_A |\delta^*(x^*, X_n)|dP < \varepsilon.$$

Then X is $cwk(E)$-Pettis $(resp.ck(E)\text{-}Pettis)$, that is, for each $A \in \mathcal{F}$, there exist $K_A \in cwk(E)$ $(resp.K_A \in ck(E))$ such that

$$\int_A \delta^*(x^*, X)dP = \delta^*(x^*, K_A)$$

for all $x^* \in E^*$.

Proof. By virtue of Proposition 4.1, we have to prove that for each $A \in \mathcal{F}$, $(\int_A X_n dP)_{n\in\mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$). We have

$$\begin{aligned}
&|\delta^*(x^*, \int_A X_n dP) - \delta^*(x^*, \int_A X_m dP)| \\
&\quad = |\int_A \delta^*(x^*, X_n)dP - \int_A \delta^*(x^*, X_m)dP| \\
&\quad \leq |\int_{A\cap B_k} \delta^*(x^*, X_n)dP - \int_{A\cap B_k} \delta^*(x^*, X_m)dP| \\
&\qquad +|\int_{A\setminus(A\cap B_k)} \delta^*(x^*, X_n)dP - \int_{A\setminus(A\cap B_k)} \delta^*(x^*, X_m)dP| \\
&\quad \leq |\delta^*(x^*, \int_{A\cap B_k} X_n dP) - \delta^*(x^*, \int_{A\cap B_k} X_m dP)| \\
&\qquad + \int_{A\setminus(A\cap B_k)} |\delta^*(x^*, X_n) - \delta^*(x^*, X_m)|dP
\end{aligned}$$

for each $x^* \in \overline{B}_{E^*}$, $m, n \in \mathbf{N}$, each $A \in \mathcal{F}$ and for each $k \in \mathbf{N}$. Let $\varepsilon > 0$. For k large enough, using (iii) we have $P(A \setminus (A \cap B_k)) < \eta$ so that $\sup_{n\geq 1} \sup_{x^*\in\overline{B}_{E^*}} \int_{A\setminus(A\cap B_k)} |\delta^*(x^*, X_n)|dP < \varepsilon$. Hence the last integral is $< 2\varepsilon$. Now k being chosen, the integral

$$|\delta^*(x^*, \int_{A\cap B_k} X_n dP) - \delta^*(x^*, \int_{A\cap B_k} X_m dP)|$$

is $\leq \varepsilon$ for m, n large enough using (ii). By taking the supremum on $\overline{B}_{E^*}$ in the left member of the inequality

$$|\delta^*(x^*, \int_A X_n dP) - \delta^*(x^*, \int_A X_m dP)| < 3\varepsilon$$

we conclude that

$$\mathcal{H}(\int_A X_m dP, \int_A X_n dP) \leq 3\varepsilon$$

for m, n large enough. Hence $(\int_A X_n dP)_{n \in \mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$). ☐

Theorem 4.1 is a version of Vitali theorem for a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$ satisfying the Pettis uniformly integrable condition (iii). Compare with Theorem 1 in [37] dealing Vitali theorem for single-valued Pettis-integrable functions.

In the following we will provide a simple criteria of Pettis integrability via compactness results in Set-Valued Bochner integration [16].

Theorem 4.2 *Let E be a separable Banach space. Let X be a $cwk(E)$ $(resp. ck(E))$-valued Pettis-integrable mapping. Then there exist a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ $(resp. \mathcal{L}^1_{ck(E)}(\mathcal{F}))$ such that, for every $A \in \mathcal{F}$,*

(i) $(\int_A X_n dP)_{n\geq 1}$ is Cauchy in $(cwk(E), \mathcal{H})$ $(resp. (ck(E), \mathcal{H}))$,
(ii) $\int_A \delta^*(x^*, X) dP = \lim_{n\to\infty} \int_A \delta^*(x^*, X_n) dP = \lim_{n\to\infty} \delta^*(x^*, \int_A X_n dP)$ for all $x^* \in E^*$.

Proof. Let $A_n = [|X| \leq n]$ for each $n \in \mathbf{N}$. Then $A_n \in \mathcal{F}$ and

$$\uparrow \lim_{n\to\infty} P(A_n) = 1$$

further, for each $n \in \mathbf{N}$, the multifunction $X_n := 1_{A_n} X$ is $cwk(E)$-valued (resp. $ck(E)$-valued) $\mathcal{F}$-measurable and integrably bounded, consequently the set of $\mathcal{F}$- measurable selections of $1_{A_n} X$ is convex weakly compact in $L^1_E(\mathcal{F})$ (see, e.g. [13, Theorem 6.2.3]), hence $\int_A 1_{A_n} X dP$ is convex weakly compact (resp. convex compact) in E, see [16, Theorem V-15] or [13, Proposition 6.3.4], and furthermore the Strassen formula holds

$$\delta^*(x^*, \int_A 1_{A_n} X dP) = \int_A \delta^*(x^*, 1_{A_n} X) dP$$

for all $A \in \mathcal{F}$ and for all $x^* \in E^*$. Now we assert that for each $A \in \mathcal{F}$, the sequence $(\int_A 1_{A_n} X dP)_{n\in\mathbf{N}}$ is Cauchy in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$). Let $\varepsilon > 0$. Let us choose N large enough so that $P(A \setminus A \cap A_N) \leq \eta$ implies

$$\sup_{x^* \in \overline{B}_{E^*}} \int_{A \setminus A \cap A_N} |\delta^*(x^*, X)| dP < \varepsilon.$$

Now for each $n \geq m \geq N$ and each $x^* \in \overline{B}_{E^*}$ we have

$$\begin{aligned}
&|\delta^*(x^*, \int_A 1_{A_n} X dP) - \delta^*(x^*, \int_A 1_{A_m} X dP)| \\
&\quad = |\int_A \delta^*(x^*, 1_{A_n} X) dP - \int_A \delta^*(x^*, 1_{A_m} X) dP| \\
&\quad \leq \int_{A \cap A_n \setminus A \cap A_m} |\delta^*(x^*, X)| dP \\
&\quad \leq \sup_{x^* \in \overline{B}_{E^*}} \int_{A \cap A_n \setminus A \cap A_m} |\delta^*(x^*, X)| dP \\
&\quad \leq \sup_{x^* \in \overline{B}_{E^*}} \int_{A \setminus A \cap A_N} |\delta^*(x^*, X)| dP < \varepsilon.
\end{aligned}$$

By taking the supremum over $\overline{B}_{E^*}$ in the above estimation

$$|\delta^*(x^*, \int_A 1_{A_n} X dP) - \delta^*(x^*, \int_A 1_{A_m} X dP)| < \varepsilon$$

we get

$$\mathcal{H}(\int_A 1_{A_n} X dP, \int_A 1_{A_m} X dP) < \varepsilon$$

for $n \geq m \geq N$. Hence we conclude that $(\int_A X_n dP)_{n \in \mathbf{N}}$ is a Cauchy sequence in $(cwk(E), \mathcal{H})$ (resp. $(ck(E), \mathcal{H})$). Consequently $(\int_A X_n dP)_{n \in \mathbf{N}}$ converges to a convex weakly compact (convex compact) set K_A with respect to the Hausdorff distance. In particular, we have

$$\lim_{n \to \infty} \delta^*(x^*, \int_A 1_{A_n} X dP) = \delta^*(x^*, K_A)$$

for all $x^* \in E^*$. Whence

$$\begin{aligned}
\lim_{n \to \infty} \delta^*(x^*, \int_A 1_{A_n} X dP) &= \lim_{n \to \infty} \int_{A \cap A_n} \delta^*(x^*, X) dP \\
&= \int_A \delta^*(x^*, X) dP = \delta^*(x^*, K_A).
\end{aligned}$$

□

In the single valued case, namely $X \in P_E^1$, the above results are reduced to the classical Pettis integral. The following characterisation of Pettis integrability is well-known (see, e.g. [13, 24, 38]). An inspection of the proof of Theorem 4.2 provides an alternative proof.

Corollary 4.1 *Let $X : \Omega \to E$ be a scalarly $\mathcal{F}$-measurable and scalarly integrable mapping. Then the following is equivalent*

(a) X is Pettis-integrable.
(b) For every $\varepsilon > 0$, there exist $\eta > 0$ such that $P(A) < \eta$ implies

$$\sup_{x^* \in \overline{B}_{E^*}} \int_A |\langle x^*, X\rangle| dP < \varepsilon.$$

Proof. The implication $(a) \Rightarrow (b)$ is well-known (see, e.g. [1]). To prove that $(b) \Rightarrow (a)$ one can repeat the arguments of the proof of Theorem 4.2. Let $A_n = [|X| \le n]$ for each $n \in \mathbf{N}$. Then $A_n \in \mathcal{F}$ with $\uparrow \lim_n P(A_n) = 1$. Now we assert that for each $A \in \mathcal{F}$, the sequence $(\int_A 1_{A_n} X dP)_{n\in\mathbf{N}}$ is Cauchy in E. Let $\varepsilon > 0$. Let us choose N large enough so that $P(A \setminus A \cap A_N) \le \eta$ implies

$$\sup_{x^* \in \overline{B}_{E^*}} \int_{A\setminus A\cap A_N} |\langle x^*, X\rangle| dP < \varepsilon.$$

Now for each $n \ge m \ge N$ and for each $x^* \in \overline{B}_{E^*}$ we have

$$\begin{aligned}
&|\langle x^*, \int_A 1_{A_n} X dP\rangle - \langle x^*, \int_A 1_{A_m} X dP\rangle| \\
&\quad = |\int_A \langle x^*, 1_{A_n} X\rangle dP - \int_A \langle x^*, 1_{A_m} X\rangle dP| \\
&\quad \le \int_{A\cap A_n \setminus A\cap A_m} |\langle x^*, X\rangle| dP \\
&\quad \le \sup_{x^* \in \overline{B}_{E^*}} \int_{A\cap A_n \setminus A\cap A_m} |\langle x^*, X\rangle| dP \\
&\quad \le \sup_{x^* \in \overline{B}_{E^*}} \int_{A\setminus A\cap A_N} |\langle x^*, X\rangle| dP < \varepsilon.
\end{aligned}$$

By taking the supremum over $\overline{B}_{E^*}$ in the above estimation yields

$$|| \int_A 1_{A_n} X dP - \int_A 1_{A_m} X dP || < \varepsilon.$$

Hence we conclude that $(\int_A X_n dP)_{n\in\mathbf{N}}$ is a Cauchy sequence in E. Consequently $(\int_A X_n dP)_{n\in\mathbf{N}}$ converges to an element $k_A \in E$ with respect to the norm topology. In particular, we have

$$\lim_{n\to\infty} \langle x^*, \int_A 1_{A_n} X dP\rangle = \langle x^*, k_A\rangle$$

for all $x^* \in E^*$. Whence

$$\begin{aligned}\lim_{n\to\infty} \langle x^*, \int_A 1_{A_n} X dP\rangle &= \lim_{n\to\infty} \int_{A\cap A_n} \langle x^*, X\rangle dP \\ &= \int_A \langle x^*, X\rangle dP = \langle x^*, k_A\rangle.\end{aligned}$$

□

Comments. Theorem 4.2 shows that given a $cwk(E)$-valued (resp. $ck(E)$-valued) Pettis-integrable multifunction X, then for each $A \in \mathcal{F}$ there exists a convex weakly compact (resp. convex compact) K_A such that

$$\int_A \delta^*(x^*, X)dP = \delta^*(x^*, K_A).$$

In other words, K_A is the $cwk(E)$-Pettis (resp. $ck(E)$-Pettis) integral of X over A. At this point, the techniques of Theorem 4.2 allow to rediscover similar results obtained by El Amri–Hess [2, Theorems 5.4–5.5]. Other variants of the preceding results can be found in [2, 13, 24, 31].

The above considerations lead to the existence of conditional expectation of a special class of $cwk(E)$-valued Pettis-integrable multifunctions. Compare with [21, 42] for details and comments on this subject. We begin with single-valued Pettis-integrable functions.

Theorem 4.3 *Assume that E is a separable Banach space. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X be a Pettis-integrable E-valued function such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$-measurable, Pettis-integrable E-valued function, denoted by $Pe\text{-}E^{\mathcal{B}}X$, which enjoys the following property: For every $h \in L^\infty(\mathcal{B})$, one has*

$$Pe\text{-}\int_\Omega hPe\text{-}E^{\mathcal{B}}XdP = Pe\text{-}\int_\Omega hXdP.$$

Now we proceed to the existence of conditional expectation in a class of $cwk(E)$-valued Pettis-integrable multifunctions. Namely

Theorem 4.4 *Assume that E^*_b is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be a $cwk(E)$-valued Pettis-integrable multifunction such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$-measurable, $cwk(E)$-valued Pettis-integrable multifunction, denoted by $Pe\text{-}E^{\mathcal{B}}X$, which enjoys the following property: For every $h \in L^\infty(\mathcal{B})$, one has*

$$Pe\text{-}\int_\Omega hPe\text{-}E^{\mathcal{B}}XdP = Pe\text{-}\int_\Omega hXdP,$$

where $Pe\text{-}\int_\Omega hPe\text{-}E^{\mathcal{B}}XdP$ and $Pe\text{-}\int_\Omega hXdP$ denote the $cwk(E)$-valued Aumann–Pettis integral of $hPe\text{-}E^{\mathcal{B}}X$ and hX respectively.

Proof. Both Theorems 4.3 and 4.4 follow from a more general result involving the integral representation theorem (Theorem 4.5) for a class of mapping $M : L^\infty(\mathcal{B}) \to cwk(E)$. ☐

Let us mention a useful corollary.

Corollary 4.2 *Under the hypotheses and notations of Theorem 4.3 and 4.4, the following hold*

(1) *For every* $h \in L^\infty(\mathcal{B})$ *and for every* $x^* \in E^*$ *and for every* $f \in \mathcal{S}^1_{Pe}(X)$*, one has*

$$\begin{aligned}\int_\Omega \langle h \otimes x^*, Pe\text{-}E^{\mathcal{B}} f\rangle dP &= \int_\Omega \langle h \otimes x^*, f\rangle dP \\ &\le \int_\Omega \delta^*(h \otimes x^*, X)dP = \int_\Omega \delta^*(h \otimes x^*, Pe\text{-}E^{\mathcal{B}}X)dP\end{aligned}$$

and hence $Pe\text{-}E^{\mathcal{B}} f(\omega) \in Pe\text{-}E^{\mathcal{B}}X(\omega)$ *a.s.*

(2) *For every* $h \in L^\infty(\mathcal{B})$ *and for every* $x^* \in E^*$*, one has*

$$\begin{aligned}&\delta^*(h \otimes x^*, Pe\text{-}E^{\mathcal{B}}(\mathcal{S}^1_{Pe}(X))) \\ &\quad = \sup\{\langle h \otimes x^*, Pe\text{-}E^{\mathcal{B}} f\rangle : f \in \mathcal{S}^1_{Pe}(X)\} \\ &\quad = \sup\{\langle h \otimes x^*, f\rangle : f \in \mathcal{S}^1_{Pe}(X)\} \\ &\quad = \delta^*(h \otimes x^*, \mathcal{S}^1_{Pe}(X)) = \int_\Omega \delta^*(h \otimes x^*, X)dP \\ &\quad = \int_\Omega \delta^*(h \otimes x^*, Pe\text{-}E^{\mathcal{B}}X)dP = \delta^*(h \otimes x^*, \mathcal{S}^1_{Pe}(Pe\text{-}E^{\mathcal{B}}X)).\end{aligned}$$

(3) *For all* $B \in \mathcal{B}$*, for all* $x^* \in E^*$*,*

$$\int_B \delta^*(x^*, X(\omega))dP = \int_B \delta^*(x^*, Pe\text{-}E^{\mathcal{B}}X(\omega))dP$$

hence

$$\delta^*(x^*, Pe\text{-}E^{\mathcal{B}}X(\omega)) = E^{\mathcal{B}}\delta^*(x^*, X(\omega)) \quad a.s.$$

(4) $\mathcal{S}^1_{Pe}(X)$ *is convex and closed with respect to the Pettis norm* $||.||_{P^1_E}$*, and furthermore the mapping* $E^{\mathcal{B}} : \mathcal{S}^1_{Pe}(X) \to \mathcal{S}^1_{Pe}(E^{\mathcal{B}}X)$ *is a contraction:*

$$||Pe\text{-}E^{\mathcal{B}} f||_{P^1_E} \le ||f||_{P^1_E} \quad \forall f \in \mathcal{S}^1_{Pe}(X).$$

Proof.

(1) Equality $\int_\Omega \langle h \otimes x^*, Pe\text{-}E^{\mathcal{B}} f\rangle dP = \int_\Omega \langle h \otimes x^*, f\rangle dP$ follows from Theorem 4.3 and equality

$$\int_\Omega \delta^*(h \otimes x^*, X)dP = \int_\Omega \delta^*(h \otimes x^*, Pe\text{-}E^{\mathcal{B}} X)dP$$

follows from Theorem 4.4. In particular, by taking the functions $1_A \otimes x_j^*$ where $A \in \mathcal{B}$ and $(x_j^*)_{j\in\mathbf{N}}$ is a dense sequence in E^* for the Mackey topology, we get

$$\int_A \langle x_j^*, Pe\text{-}E^{\mathcal{B}} f\rangle dP \le \int_A \delta^*(x_j^*, Pe\text{-}E^{\mathcal{B}} X)dP$$

and hence

$$\langle x_j^*, Pe\text{-}E^{\mathcal{B}} f\rangle \le \delta^*(x_j^*, Pe\text{-}E^{\mathcal{B}} X)$$

a.s. for all $j \in \mathbf{N}$. By Proposition III-35 in [16], we get

$$Pe\text{-}E^{\mathcal{B}} f(\omega) \in Pe\text{-}E^{\mathcal{B}} X(\omega) \quad \text{a.s.}$$

(2) Follows from the Strassen formula [16] applied to the Aumann–Pettis mulltivalued integrals of the $cwk(E)$-valued Pettis-integrable X and $Pe\text{-}E^{\mathcal{B}} X$.

(3) Follows from the calculus of support functionals in Theorem 4.4.

(4) The Pettis closedness of $\mathcal{S}^1_{Pe}(X)$ is obvious; a more general result is available, see [4, Lemma 3.1], meanwhile the Pettis contraction property follows easily from the property of $E^{\mathcal{B}}$ stated in Theorem 4.3 and the definition of the Pettis norm.

Now establish an integral representation theorem for class of mapping $M : L^\infty(\mathcal{B}) \to cwk(E)$ (alias multivalued Dunford–Pettis theorem) in the vein of [16, Theorem V-17], recovering both Theorems 4.3 and 4.4.

Theorem 4.5 *Assume that E_b^* is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Let us consider a $cwk(E)$-valued mapping $M : L^\infty(\mathcal{B}) \to cwk(E)$ satisfying the following conditions:*

(i) For each $x^* \in E^*$, the scalar function $h \mapsto \delta^*(x^*, M(h))$ is continuous on bounded subset of $L^\infty(\mathcal{B})$ for the topology of convergence in probability.

(ii) $M(f + g) = M(f) + M(g)$ if $fg \ge 0$ for $f, g \in L^\infty(\mathcal{B})$.

(iii) There is a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\Omega, \mathcal{F}, P)$ and a $\mathcal{B}$-measurable partition $(B_n)_{n\in\mathbf{N}}$ of Ω satisfying

$$M(1_{B_n}h) = \int_{B_n} hX_n dP \quad \forall h \in L^\infty(\mathcal{B}) \quad \forall n \in \mathbf{N}.$$

Then there exists a unique $\mathcal{B}$-measurable, $cwk(E)$-valued Pettis-integrable multifunction Γ satisfying the following property:

$$M(h) = Pe\text{-}\int_\Omega h\Gamma dP, \quad \forall h \in L^\infty(\mathcal{B}).$$

Here $Pe\text{-}\int_\Omega h\Gamma dP$ denotes the $cwk(E)$-valued Aumann–Pettis integral of $h\Gamma$.

Proof. By virtue of (iii) and Theorem 3 (Remark 4) in [43], for each $n \in \mathbf{N}$, there is a unique $cwk(E)$-valued $\mathcal{B}$-measurable and integrably bounded multifunction $\Gamma_n := E^{\mathcal{B}}X_n$ satisfying

$$\int_{B_n} h\Gamma_n dP = \int_{B_n} hX_n dP$$

for every $h \in L^\infty(\mathcal{B})$. Coming back to (iii) we have

$$M(h1_{B_n}) = \int_\Omega h1_{B_n}X_n dP = \int_{B_n} h\Gamma_n dP, \quad \forall h \in L^\infty(\mathcal{B}).$$

Let us define $\Gamma(\omega) = \Gamma_n(\omega)$ if $\omega \in B_n$. Then Γ is $\mathcal{B}$-measurable. Using (i) it is not difficult to check that

$$M(\sum_{n=l}^{m} 1_{B_n}h) = \int_\Omega \sum_{n=l}^{m} 1_{B_n}h\Gamma dP \tag{4.5.1}$$

for every $h \in L^\infty(\mathcal{B})$ and for every $n, l \in \mathbf{N}$. Let us consider an arbitrary $\mathcal{B}$-measurable selection g of Γ. Then we have

$$\int_\Omega \sum_{n=l}^{m} 1_{B_n}hgdP \in M(\sum_{n=l}^{m} 1_{B_n}h) \tag{4.5.2}$$

which implies

$$\begin{aligned}\delta^*(-x^*, M(\sum_{n=l}^{m} 1_{B_n}h)) &\le \int_\Omega \langle x^*, \sum_{n=l}^{m} 1_{B_n}hg\rangle dP \\ &\le \delta^*(x^*, M(\sum_{n=l}^{m} 1_{B_n}h)).\end{aligned}$$

for every $x^* \in E$. By (ii) the multifunction $h \mapsto M(h)$ is scalarly continuous on bounded subsets of $L^\infty(\mathcal{B})$ with respect to the convergence in probability, so that from the above estimate and (4.5.1) the sequence

$$(\langle x^*, \sum_{n=1}^{m} 1_{B_n} g\rangle)_{m\in\mathbf{N}}$$

is a $\sigma(L^1(\mathcal{B}), L^\infty(\mathcal{B}))$ Cauchy sequence. But the pointwise limit of this sequence is $\langle x^*, g\rangle$, therefore by classical property of L^1 space we have

$$\lim_{m\to\infty} \sum_{n=1}^{m} 1_{B_n} \langle x^*, g\rangle = \langle x^*, g\rangle \tag{4.5.3}$$

with respect to the norm topology of L^1. By (4.5.3) we conclude that g is Pettis-integrable with $Pe\text{-}\int_\Omega hgdP \in M(h)$ by passing to the limit when m goes to ∞ in (4.5.2). Now we prove that Γ is Pettis-integrable. As any $\mathcal{B}$-measurable selection g of Γ is Pettis integrable, according to our definition it is enough to check that Γ is scalarly integrable. Let $x^* \in E^*$. By the measurable implicit Theorem III-38 in [16], there is $\mathcal{B}$-measurable selection σ of Γ such that $\langle x^*, \sigma\rangle = \delta^*(x^*, \Gamma)$. We conclude that the $cwk(E)$-valued $\mathcal{B}$-measurable multifunction Γ is Pettis-integrable. Let us denote by $\mathcal{S}^1_{Pe}(\Gamma)(\mathcal{B})$ the set of all $\mathcal{B}$-measurable and Pettis-integrable selections of Γ. Then $\mathcal{S}^1_{Pe}(\Gamma)(\mathcal{B})$ is nonempty convex $\sigma(P^1_{E^*}[E](\mathcal{B}), L^\infty(\mathcal{B})\otimes E^*)$ compact, by applying Theorem V.14 in [16]. We finish the proof by showing that

$$M(h) = Pe\text{-}\int_\Omega h\Gamma dP = \{Pe\text{-}\int_\Omega hgdP : g \in \mathcal{S}^1_{Pe}(\Gamma)(\mathcal{B})\}.$$

By invoking Theorem V-14 in [16] we see that the Aumann–Pettis integral $Pe\text{-}\int_\Omega h\Gamma dP$ is convex weakly compact. In order to prove the desired equality, we proceed as in the proof of Theorem V-17 in [16]. It is clear that

$$Pe\text{-}\int_\Omega h\Gamma dP = \{Pe\text{-}\int_\Omega hgdP : g \in \mathcal{S}^1_{Pe}(\Gamma)(\mathcal{B})\} \subset M(h).$$

Assume that there is $\zeta \in M(h) \setminus Pe\text{-}\int_\Omega hYdP$. By Hahn–Banach theorem, there is $x^* \in E^*$ such that

$$\forall g \in \mathcal{S}^1_{Pe}(\Gamma)(\mathcal{B}), \int_\Omega \langle x^*, g\rangle hdP < \langle x^*, \zeta\rangle \leq \delta^*(x^*, M(h)). \tag{4.5.4}$$

By the measurable implicit theorem [16], there is a $\mathcal{B}$-measurable and Pettis-integrable selection $\widetilde{g}$ of Γ such that

$$\langle x^*, \widetilde{g}\rangle h = \delta^*(x^*, h\Gamma), \quad \forall \omega \in \Omega.$$

This implies by integrating on each B_n

$$\int_{B_n} \langle x^*, \widetilde{g}\rangle h dP = \int_{B_n} \delta^*(x^*, h\Gamma) dP = \delta^*(x^*, M(1_{B_n} h)).$$

By (4.5.2) we have

$$\begin{aligned} M(\sum_{n=1}^{m} 1_{B_n} h) &= \int_\Omega \sum_{n=1}^{m} 1_{B_n} h\Gamma dP \\ &= \sum_{n=1}^{m} \int_\Omega 1_{B_n} h\Gamma dP = \sum_{n=1}^{m} M(1_{B_n} h). \end{aligned}$$

Hence we get

$$\delta^*(x^*, M(\sum_{n=1}^{m} 1_{B_n} h)) = \int_\Omega \sum_{n=1}^{m} \langle x^*, 1_{B_n} \widetilde{g}\rangle h dP.$$

By passing to the limit when m goes to ∞, we get by Lebesgue's theorem

$$\delta^*(x^*, M(h)) = \int_\Omega \langle x^*, \widetilde{g}\rangle h dP.$$

This contradicts the inequality

$$\int_\Omega \langle x^*, \widetilde{g}\rangle h dP < \delta^*(x^*, M(h))$$

in (4.5.4) and completes the proof. □

Remark.

(1) Theorem 4.5 holds true without assuming that E_b^* is separable when the mapping M is single-valued.

(2) Theorem 4.5 recovers Theorems 4.3–4.4. Indeed, it is enough to put

$$M(h) = Pe\text{-}\int hXdP \quad h \in L^\infty(\mathcal{B}).$$

Using the assumption $E^{\mathcal{B}}|X| \in [0, +\infty[$ provides a $\mathcal{B}$-measurable partition $(B_n)_{n\in\mathbf{N}}$ of Ω and a sequence $(X_n)_{n\in\mathbf{N}} := (1_{B_n} X)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\Omega, \mathcal{F}, P)$ such that $\forall h \in L^\infty(\mathcal{B}), \forall n \in \mathbf{N}, M(1_{B_n} h) = \int_{B_n} hX_n dP$.

(3) Variants of Theorems 4.3–4.5 are available when dealing with the space $L^1_{E^*}[E](\Omega,\mathcal{F},P)$ and the space $\mathcal{L}^1_{E^*}[E](\Omega,\mathcal{F},P)$ (see [15,43]). Further it is possible to formulate the integral representation 4.5 on a Kothe space instead of L^∞ in the vein of Theorem V-17 in [16]. However the study of the convergence of Pettis conditional expectation for unbounded $cc(E)$-valued Pettis-integrable multifunctions in the same style as in [9,16,27,43] (dealing with Bochner integration) is an open problem.

(4) We will show in next section that convergence of $cwk(E)$-valued Pettis-integrable martingales with respect to the Mosco topology or linear topology is now available, using the above techniques and the convergence of regular martingales of the form $X_n = Pe\text{-}E^{\mathcal{F}_n}X$, where X is a $cwk(E)$-valued Pettis-integrable multifunction and $Pe\text{-}E^{\mathcal{F}_n}X$ is the Pettis conditional expectation of X. For this purpose, we will provide in § 5 some convergence results for the Pettis conditional expectation of the form $Pe\text{-}E^{\mathcal{F}_n}X$ with $X \in \mathcal{P}^1_{cwk(E)}(\mathcal{F})$ and its applications to the convergence of Pettis-integrable martingales and mils.

5. Convergence of $cwk(E)$-valued Pettis-integrable martingales

We present first some versions of Levy's theorem for a $cwk(E)$-valued Pettis-integrable multifunction. This study is a starting point for further applications in the Mosco convergence of Pettis-integrable multivalued martingales, compare with [21, 37, 42] dealing vector-valued Pettis-integrable martingales.

Proposition 5.1 *Assume that E is a separable Banach space and X is a Pettis-integrable E-valued mapping such that $(\mathcal{C}) : E^{\mathcal{F}_n}|X| \in [0,+\infty[$ for each $n \in \mathbf{N}$. Then we have*

$$\lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n}X = X \quad a.s.$$

Proof. By condition $(\mathcal{C})$ and Theorem 4.3, $Pe\text{-}E^{\mathcal{F}_n}X$ is a $\mathcal{F}_n$-measurable and Pettis-integrable E-valued function satisfying

$$(*) \qquad Pe\text{-}\int_\Omega hPe\text{-}E^{\mathcal{F}_n}XdP = Pe\text{-}\int_\Omega hXdP$$

for every $h \in L^\infty(\mathcal{F}_n)$. Applying $(\mathcal{C})$ for $n = 1$, provides a $\mathcal{F}_1$-measurable partition $(B_k)_{k\in\mathbf{N}}$ of Ω such that $X_k := X1_{B_k} \in L^1_E(\mathcal{F})$ for each $k \in \mathbf{N}$. By virtue of Levy's theorem in $L^1_E(\Omega,\mathcal{F},P)$, we have

$$(**) \qquad \lim_{n\to\infty} E^{\mathcal{F}_n}X_k = X_k \quad a.s.$$

for each $k \in \mathbf{N}$. We claim that

$$\lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n} X = X \quad a.s.$$

As $B_k \in \mathcal{F}_1 \subset \mathcal{F}_n$ for every $k \in \mathbf{N}$ and for every $n \in \mathbf{N}$, it is obvious by using (*) that

$$E^{\mathcal{F}_n} X 1_{B_k} = 1_{B_k} Pe\text{-}E^{\mathcal{F}_n} X.$$

By (*) and (**) we have

$$X = \sum_{k=1}^{\infty} X_k = \sum_{k=1}^{\infty} \lim_{n\to\infty} E^{\mathcal{F}_n} X_k$$

$$\sum_{k=1}^{\infty} 1_{B_k} \lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n} X = \lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n} X$$

a.s., thus proving the claim and completes the proof. □

The following is a nontrivial extension of Proposition 5.1.

Proposition 5.2 *Assume that E is a separable Banach space. Let Y be a real-valued positive $\mathcal{F}$-measurable function such that $E^{\mathcal{F}_n} Y \in [0, +\infty[$ for each $n \in \mathbf{N}$. Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of Pettis-integrable E-valued functions. Assume that the following conditions are satisfied:*

(i) $|X_n| \le Y, \quad \forall n \in \mathbf{N}$.

(ii) $(X_n)_{n\in\mathbf{N}}$ *norm converges a.s. to a Pettis-integrable E-valued function X_∞.*

Then the following holds

$$\lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n} X_n = X_\infty \quad a.s.$$

Proof. Existence of $Pe\text{-}E^{\mathcal{F}_n} X_n$ follows from Theorem 4.3 thank to (i) and condition $E^{\mathcal{F}_n} Y \in [0, +\infty[$ for each $n \in \mathbf{N}$. By (i) and (ii) we have $|X_\infty| \le Y$ a.s. so that $E^{\mathcal{F}_n}|X_\infty| \in [0, +\infty[$ for each $n \in \mathbf{N}$, too. Applying now Proposition 5.1 to X_∞ yields

$$(*) \qquad \lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n} X_\infty = X_\infty \quad a.s.$$

From Lemma 2.1 in [23] (or Theorem 3.7 in [22]) , we have

$$(**) \qquad \lim_{n\to\infty} E^{\mathcal{F}_n} |X_n - X_\infty| = 0 \quad a.s.$$

We have the estimate

$$|Pe\text{-}E^{\mathcal{F}_n}X_n - X_\infty| \le |Pe\text{-}E^{\mathcal{F}_n}X_n - Pe\text{-}E^{\mathcal{F}_n}X_\infty| + |Pe\text{-}E^{\mathcal{F}_n}X_\infty - X_\infty|$$
$$\le E^{\mathcal{F}_n}|X_n - X_\infty| + |Pe\text{-}E^{\mathcal{F}_n}X_\infty - X_\infty|.$$

From the above estimate and (*), (**), the result follows. □

Now we provide a multivalued version of the Levy's theorem for $cwk(E)$-valued Pettis-integrable multifunctions extending Proposition 5.1.

Theorem 5.1 *Assume that E_b^* is separable. Let X be a $cwk(E)$-valued Pettis-integrable multifunction such that $E^{\mathcal{F}_n}|X| \in [0, +\infty[$ for each $n \in \mathbf{N}$. Then we have*

$$M\text{-}\lim_{n\to\infty} Pe\text{-}E^{\mathcal{F}_n}X = X \quad a.s.$$

Proof. Since X is Pettis-integrable, the measurable selections of X are Pettis-integrable. So, in view of [16], there is a sequence $(f_p)_{p\in\mathbf{N}}$ in $P_E^1(\mathcal{F})$ such that

$$(*) \qquad X(\omega) = cl\{f_p(\omega) : p \in \mathbf{N}\}$$

for every $\omega \in \Omega$.

Claim 1 $X \subset s\text{-}li\ Pe\text{-}E^{\mathcal{F}_n}X \quad a.s.$

As $E^{\mathcal{F}_n}|X| \in [0, +\infty[$ for each $n \in \mathbf{N}$, by Theorem 4.4 the conditional expectations $Pe\text{-}E^{\mathcal{F}_n}X$ is $\mathcal{F}_n$-measurable and Pettis-integrable. Now let $f \in S^1_{Pe}(X)(\mathcal{F})$. By Theorem 4.3 and Corollary 4.2(1), $Pe\text{-}E^{\mathcal{F}_n}f$ is $\mathcal{F}_n$-measurable and Pettis-integrable and satisfies $Pe\text{-}E^{\mathcal{F}_n}f(\omega) \in Pe\text{-}E^{\mathcal{F}_n}X(\omega)$ a.s. Furthermore, by Proposition 5.1, $\lim_{n\to} Pe\text{-}E^{\mathcal{F}_n}f = f$ a.s. So we conclude that $f \in s\text{-}li\ Pe\text{-}E^{\mathcal{F}_n}X$ a.s. Since this is true for any $f \in S^1_{Pe}(X)(\mathcal{F})$, by invoking (*), we see that Claim 1 is true.

Claim 2 $w\text{-}ls\ Pe\text{-}E^{\mathcal{F}_n}X \subset X \quad a.s.$

Let $(x_j^*)_{j\in\mathbf{N}}$ be a dense sequence in E^* for the Mackey topology $\tau(E^*, E)$. Then the calculus of support functions in the integral representation formula of Theorem 4.4 [cf. Corollary 4.2(3)] imply

$$\delta^*(x_j^*, Pe\text{-}E^{\mathcal{F}_n}X(\omega)) = E^{\mathcal{F}_n}\delta^*(x_j^*, X(\omega))$$

a.s. for all $j \in \mathbf{N}$ and for all $n \in \mathbf{N}$. By Levy's theorem we have

$$\lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(x_j^*, X(\omega)) = \delta^*(x_j^*, X(\omega))$$

a.s. for all $j \in \mathbf{N}$. Let $\omega \in \Omega$ be such that the preceding relations are satisfied. Let $x \in w\text{-}ls\ Pe\text{-}E^{\mathcal{F}_n}X(\omega)$. Then $x_k \to x$ weakly for some $x_k \in Pe\text{-}E^{\mathcal{F}_{n_k}}(X)(\omega)$ and hence

$$\begin{aligned}\langle x_j^*, x\rangle &= \lim_{k\to\infty}\langle x_j^*, x_k\rangle \le \limsup_{k\to\infty}\delta^*(x_j^*, Pe\text{-}E^{\mathcal{F}_{n_k}}(X)(\omega))\\ &= \limsup_{k\to\infty} E^{\mathcal{F}_{n_k}}\delta^*(x_j^*, X(\omega)) \le \lim_{k\to\infty} E^{\mathcal{F}_{n_k}}\delta^*(x_j^*, X(\omega))\\ &= \delta^*(x_j^*, X(\omega)).\end{aligned}$$

So $x \in X(\omega)$ because X is convex weakly compact valued [16, Prop. III-35] and hence Claim 2 follows. □

We end this paper with some new applications to the τ_L convergence of $cwk(E)$-valued Pettis-integrable martingales illustrating the above techniques. Compare with Theorem 3.1 in [21] dealing with Pettis-integrable vector-valued martingales.

Theorem 5.2 *Assume that E_b^* is separable and E is such that c_0 is not isomorphic to a subspace of E ($c_0 \not\hookrightarrow E$). Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be an adapted sequence of $cwk(E)$-valued Pettis-integrable multifunctions satisfying:*

- *(i)* $E^{\mathcal{F}_q}|X_n| < \infty$ *for each* $n \in \mathbf{N}$ *and each* $1 \le q < n$.
- *(ii)* $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ *is a $cwk(E)$-valued Pettis-integrable martingale, that is,* $Pe\text{-}E^{\mathcal{F}_n}X_{n+1} = X_n$ *for all* $n \in \mathbf{N}$.
- *(iii)* $\sup_{n\in\mathbf{N}}\sup_{x^*\in\overline{B}_{E^*}}\int_\Omega |\delta^*(x^*, X_n)|dP < \infty$.
- *(iv) There is a partition $(A_n)_{n\in\mathbf{N}}$ in $\cup_{n=1}^\infty \mathcal{F}_n$ such that for each $m \in \mathbf{N}$, $(X_n|_{A_m})_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(A_m)$.*
- *(v) $(X_n|_{A_m})_{n\in\mathbf{N}}$ is $cwk(E)$-tight for each $m \in \mathbf{N}$.*

Then there is a $cwk(E)$-valued Pettis-integrable multifunction X_∞ such that

$$\lim_{n\to\infty}\delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*},$$

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s. \quad \forall x \in E.$$

Shortly $(X_n)_{n\in\mathbf{N}}$ τ_L-converges a.s. to X_∞.

Proof. By (i) and Theorem 4.4 the Pettis conditional expectations

$$Pe\text{-}E^{\mathcal{F}_q}X_n \quad (1 \le q < n)$$

exist and belong to $\mathcal{P}^1_{cwk(E)}(\mathcal{F})$. Accordingly the $cwk(E)$-valued Pettis-integrable martingale given in (ii) exists. Now for each $m \in \mathbf{N}$, let $n(m) \in \mathbf{N}$ be such that $A_m \in \mathcal{F}_{n(m)}$. Then $(X_n|_{A_m}, \mathcal{F}_n|_{A_m})_{n\ge n(m)}$ is a $cwk(E)$-valued martingale in $\mathcal{L}^1_{cwk(E)}(A_m)$. It follows that, for each $x^* \in \overline{B}_{E^*}$,

$$(\delta^*(x^*, X_n|_{A_m}), \mathcal{F}_n|_{A_m})_{n\ge n(m)}$$

is a real-valued L^1-bounded martingale, so it converges a.s. to a function $h^m_{x^*}$ in $L^1(A_m)$. As $(X_n|_{A_m}, \mathcal{F}_n|_{A_m})_{n \geq n(m)}$ is bounded and $cwk(E)$-tight in $\mathcal{L}^1_{cwk(E)}(A_m)$ by (iv) and (v), from [14, Theorem 6.1], we can find a subsequence $(X'_n|_{A_m})_{n \geq n(m)}$ and $X^m_\infty \in \mathcal{L}^1_{cwk(E)}(A_m)$ such that $(X'_n|_{A_m})_{n \geq n(m)}$ *weakly biting converges* to $X^m_\infty \in \mathcal{L}^1_{cwk(E)}(A_m)$ so that by identifying the limits we get

$$\lim_{n \to \infty, n \geq n(m)} \delta^*(x^*, X_n) = \delta^*(x^*, X^m_\infty) = h^m_{x^*} \quad a.s. \quad \omega \in A_m.$$

So by virtue of Theorem 3.2, we get

$$\lim_{n \to \infty, n \geq n(m)} \delta^*(x^*, X_n) = \delta^*(x^*, X^m_\infty) \quad a.s. \quad \omega \in A_m \quad \forall x^* \in \overline{B}_{E^*}.$$

$$\lim_{n \to \infty, n \geq n(m)} d(x, X_n) = d(x, X^m_\infty) \quad a.s. \quad \omega \in A_m \quad \forall x \in E.$$

Put $X_\infty = \sum_{m=1}^\infty X^m_\infty 1_{A_m}$. Then obviously $(X_n)_{n \in \mathbf{N}}$ τ_L-converges a.s. to X_∞. It is easy to check that X_∞ is $\mathcal{F}$-measurable. By (iii) it follows that, for every $x^* \in \overline{B}_{E^*}$, $(\delta^*(x^*, X_n))_{n \in \mathbf{N}}$ is a real-valued L^1-bounded martingale, so it converges a.s. to a function in L^1. Hence X_∞ is scalarly integrable, and since $(c_0 \not\hookrightarrow E)$ any $\mathcal{F}$-measurable and scalarly integrable selection of X_∞ is Pettis-integrable. Accordingly X_∞ is Pettis-integrable. ☐

We finish this paper with an application of Theorems 4.3–5.2.

Corollary 5.1 *Assume that E is separable and is such that c_0 is not isomorphic to a subspace of E $(c_0 \not\hookrightarrow E)$. Let $(X_n, \mathcal{F}_n)_{n \in \mathbf{N}}$ be an adapted sequence of E-valued Pettis-integrable functions satisfying:*

(i) $E^{\mathcal{F}_q}|X_n| < \infty$ *for each* $n \in \mathbf{N}$ *and each* $1 \leq q < n$.

(ii) $(X_n, \mathcal{F}_n)_{n \in \mathbf{N}}$ *is an E-valued Pettis-integrable martingale, that is,* $X_n = Pe\text{-}E^{\mathcal{F}_n} X_{n+1}$ *for all* $n \in \mathbf{N}$.

(iii) $\sup_{n \in \mathbf{N}} \sup_{x^* \in \overline{B}_{E^*}} \int_\Omega |\langle x^*, X_n \rangle| dP < \infty$.

(iv) There is a partition $(A_n)_{n \in \mathbf{N}}$ *in* $\cup_{n=1}^\infty \mathcal{F}_n$ *such that for each* $m \in \mathbf{N}$, $(X_n|_{A_m})_{n \in \mathbf{N}}$ *is bounded in* $L^1_E(A_m)$.

(v) $(X_n|_{A_m})_{n \in \mathbf{N}}$ *is $cwk(E)$-tight for each* $m \in \mathbf{N}$.

Then there is an E-valued Pettis-integrable function X_∞ such that $(X_n)_{n \in \mathbf{N}}$ norm converges a.s. to X_∞. If (X_n) is Pettis-uniformly integrable in $P^1_E(\mathcal{F})$, then $\lim_{n \to \infty} ||X_n - X_\infty||_{P^1_E} = 0$.

Proof. Under our assumption any L^1-bounded martingale in $L^1_E(\mathcal{F})$ satisfying the $cwk(E)$-tightness condition norm converges a.s. to a function in

$L^1_E(\mathcal{F})$, so the result follows from the arguments given in the proof of Theorem 5.2 while the convergence in Pettis norm follows from [1, Proposition 2.1]. □

Remarks. (1) If E^*_b is separable and if E have the RNP, then one can substitute the tightness condition (v) in Theorem 5.2 by the following: for each $m \in \mathbf{N}$, for each $A \in A_m \cap \mathcal{F}$, $\cup_{n\in\mathbf{N}} \int_A X_n dP$ is relatively weakly compact in E. When E have the RNP, one may recover a former result due to Egghe [21, Corollary 3.2], namely

Corollary 5.2 *Assume that E is separable and have the RNP. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be an adapted sequence of E-valued Pettis-integrable functions satisfying:*

(i) $E^{\mathcal{F}_q}|X_n| < \infty$ *for each* $n \in \mathbf{N}$ *and each* $1 \leq q < n$.
(ii) $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ *is an E-valued Pettis-integrable martingale, that is,* $X_n = Pe\text{-}E^{\mathcal{F}_n} X_{n+1}$ *for all* $n \in \mathbf{N}$.
(iii) $\sup_{n\in\mathbf{N}} \sup_{x^*\in\overline{B}_{E^*}} \int_\Omega |\langle x^*, X_n\rangle| dP < \infty$.
(iv) There is a partition $(A_n)_{n\in\mathbf{N}}$ *in* $\cup_{n=1}^{\infty} \mathcal{F}_n$ *such that for each* $m \in \mathbf{N}$, $(X_n|_{A_m})_{n\in\mathbf{N}}$ *is bounded in* $L^1_E(A_m)$.

Then there is an E-valued Pettis-integrable function X_∞ such that $(X_n)_{n\in\mathbf{N}}$ norm converges a.s. to X_∞. If (X_n) is Pettis-uniformly integrable in $P^1_E(\mathcal{F})$, then $\lim_{n\to\infty} ||X_n - X_\infty||_{P^1_E} = 0$.

Proof. For shortness we omit the details. □

(2) The results presented in § 5 lead us to address the following question: is it possible to prove a version of Theorem 5.1 (resp. Theorem 5.2) for *unbounded* closed convex valued Pettis-integrable mappings (resp. closed convex valued Pettis-integrable supermartingales).

(3) The weak star Kuratowski convergence for sub- supermartingales and mils in a weak star dual of a separable Banach space can be found in a forthcoming paper [10].

References

1. Amrani, A., Castaing, C., Valadier, M.: Convergence in Pettis norm under extreme point condition. Vietnam J. Math. **26**(4), 323–325 (1998)
2. Amri, K.E., Hess, Ch.: On the Pettis integral of closed valued multifunctions. Set-Valued Anal. **8**, 329–360 (2000)
3. Beer, G.: Support and distance functionals for convex sets. Numer. Funct. Anal. Optim. **10**(1–2), 15–36 (1989)

4. Bourras, A., Castaing, C., Guessous, M.: Olech-types lemma and Visintin-types theorem in Pettis integration and $L^1_{E'}[E]$. Josai. Math. Monogr. **1**, 1–26 (1999)
5. Castaing, C.: Méthode de compacité et de décomposition, application: minimization, convergence des martingales, lemme de Fatou multivoque. Ann. Mat. Pura. Appl. (IV) **CLXIV**, 51–75 (1993)
6. Castaing, C., Ezzaki, F.: Some convergence results for multivalued martingales in the limit. In: Séminaire d'Analyse Convexe, Montpellier, exposé No 1 (1990)
7. Castaing, C., Ezzaki, F.: Mosco convergence for multivalued martingales in the limit. C. R. Acad. Sci. Paris Ser. I **312**, 695–698 (1991)
8. Castaing, C., Ezzaki, F.: SLLN for convex random sets and random lower semicontinuous integrands. Atti. Sem. Mat. Fis. Univ. Modena. **XLV**, 527–553 (1997)
9. Castaing, C., Ezzaki, F., Hess, Ch.: Convergence of conditional expectation for unbounded closed convex random sets. Stud. Math. **124**(2), 133–148 (1997)
10. Castaing, C., Ezzaki, F., Lavie, M., Saadoune, M.: Weak star convergence of martingales in a dual space. Working paper, Département de Mathématiques, Université Montpellier II, September 2009
11. Castaing, C., Hess, Ch., Saadoune, M.: Tightness conditions and integrability of the sequential weak upper limit of a sequence of multifunctions. Adv. Math. Econ. **11**, 11–44 (2008)
12. Castaing, C., Raynaud de Fitte, P.: $\mathcal{U}$-Uniform scalar integrability and strong law of large numbers for Pettis-integrable functions with values in a separable locally convex space. J. Theor. Probab. **13**(1), 93–134 (2000)
13. Castaing, C., Raynaud de Fitte, P., Valadier, M.: Young measures on topological spaces. With applications in control theory and probability theory. Dordrecht: Kluwer 2004
14. Castaing, C., Saadoune, M.: Dunford–Pettis-types theorem and convergences in set-valued integration. J. Nonlinear And Convex Anal. **1**(1), 37–71 (1999)
15. Castaing, C., Saadoune, M.: Convergences in a dual space with applications to Fatou lemma. Adv. Math. Econ. **12**, 23–69 (2009)
16. Castaing, C., Valadier, M.: Convex analysis and measurable multifunctions. Lecture notes in mathematics, vol. 580. Berlin: Springer 1977
17. Chatterj, S.D.: Martingale convergence and the Radon–Nikodym theorem. Math. Scand. **22**, 21–44 (1968)
18. Choukairi, A.: M-convergence et régularité des martingales multivoques. In: Séminaire d'Analyse Convexe, Montpellier, 1987, exposé No 5. J. Multivar. Anal. **33**(1), 49–71 (1990)
19. Choukairi, A.: On almost sure convergence of vector valued pramarts and multivalued pramarts. J. Convex Anal. **3**(2), 245–254 (1994)
20. Davis, W.J., Ghoussoub, N., Johnson, W.B., Kwapien, S., Maurey, B.: Weak convergence of vector-valued martingales. In: Probability in Banach spaces 6 (Sandbjerg 1986). Boston: Birkhauser Boston, 1990 pp. 41–40
21. Egghe, L.: Convergence of adapted sequences of Pettis-integrable functions. Pacific J. Math. **114**(2), 345–366 (1984)
22. Ezzaki, F.: Contributions aux problèmes de convergence des suites adaptées et des ensembles aléatoires. Thèse de Doctorat, Université de Montpellier II (1993)
23. Ezzaki, F.: A general dominated convergence theorem for unbounded random sets. Bull. Polish Acad. Sci. Math. **44**(3), 353–361 (1996)

24. Geitz, R.: Pettis integration. Proc. Am. Math. Soc. **8**, 81–86 (1981)
25. Hess, C.: On multivalued martingales whose values may be unbounded: martingale selectors and Mosco convergence. In: Séminaire Analyse convexe Montpellier 1989, exposé 8. J. Multivar. Anal. **39**(1), 175–201 (1991)
26. Hiai, F.: Convergence of conditional expectations and strong laws of large numbers for multivalued random variables. Trans. Am. Math. Soc. **291**(2), 613–627 (1985)
27. Hiai, F., Umegaki, H.: Integrals, conditional expectations and martingales of multivalued functions. J. Multivar. Anal. **7**, 149–182 (1977)
28. Lavie, M.: On the convergence of multivalued martingales in the limit. Monogr. Semin. Mat. Garcia Galdeano **27**, 393–398 (2003)
29. Luu, D.Q.: Decomposition and limits for martingale-like sequences in Banach spaces. Acta. Math. Vietnam **13**, 73–78 (1988)
30. Luu, D.Q.: Convergence of adapted sequences in Banach spaces without the Radon–Nikodym property. Acta. Math. Vietnam **30**(3), 289–297 (2005)
31. Di Piazza, L., Musial, K.: A decomposition theorem for compact valued Henstock integral. Monatshefte. Math. **148**, 119–126 (2006)
32. Li, S., Ogura, Y.: Convergence of set valued sub- and supermartingales in the Kuratowski–Mosco sense. Ann. Probab. **26**(3), 1384–1402 (1998)
33. Li, S., Ogura, Y.: Convergence of set-valued and fuzzy-valued martingales. Fuzzy Sets Syst. **101**, 453–461 (1999)
34. Marraffa, V.: On almost sure convergence of amarts and martingales without the Radon–Nikodym property. J. Theor. Probab. **1**, 255–261 (1988)
35. Marraffa, V.: Stochastic process of vector valued Pettis and MacShane integrable functions. Folia Math. **12**(1), 25–37 (2005)
36. Mosco, U.: Convergence of convex sets and solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
37. Musial, K.: Martingales of Pettis integrable functions. In: Proceedings of the conference on measure theory, Oberwolfach 1979. Lecture notes in mathematics, vol. 794. Berlin: Springer, 1980 pp. 324–339
38. Musial, K.: Topics in the theory of Pettis integration. In: School of measure theory and real analysis, Italy, May 1992
39. Neveu, J.: Martingales à temps discret. Paris: Masson 1972
40. Rongming, W.: Essential (convex) closure of a family of random sets and its apPications. J. Math. Anal. Appl. **262**, 667–687 (2001)
41. Talagrand, M.: Some structure results for martingales in the limit and pramarts. Ann. Probab. **13**(40), 1192–1203 (1985)
42. Uhl, J.J. Jr.: Martingales of strongly measurable Pettis integrable functions. Trans. Am. Math. Soc. **167**, 369–378 (1972)
43. Valadier, M.: On conditional expectation of random sets. Ann. Mat. Pura. Appl. (IV) **CXXVI**, 81–91 (1980)

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A note on Aumann's core equivalence theorem without monotonicity

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Abstract. In an exchange economy with a continuum of traders, we establish the equivalence theorem on the core and the set of competitive allocations without assuming monotonicity of traders' preferences. Under weak assumptions we provide two alternative core equivalence theorems. The first one is for irreducible economies under Debreu's assumption on quasi-equilibria. The second one is an extension of Aumann's theorem under weaker assumptions than monotonicity.

Key words: core, equivalence, monotonicity, quasi-equilibrium, irreducibility

1. Introduction

In his seminal paper, Aumann [1] established the equivalence between the core and the set of competitive equilibrium allocations in an economy with a continuum of traders. His core equivalence theorem is very general in that he assumed a set of very weak conditions on traders' preferences. In fact, in his

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proof, neither irreflexivity nor transitivity is assumed on traders' preference relations. Our purpose is to revisit his equivalence theorem under somewhat different or general assumptions.

In very general economies, Hildenbrand [4, 5] showed that any core allocation is a quasi-equilibrium with relaxing Aumann's monotonicity assumption of preferences to local non-satiation. In our Theorem 1, we first restate his theorem of Hildenbrand [4, 5].

While Aumann's equivalence theorem is extended in our Theorem 3, it is easier to provide Theorem 3 by focusing on the relationship between competitive equilibria and quasi-equilibria. Since any quasi-equilibrium is a competitive equilibrium if each trader's income is positive, and since any equilibrium allocation belongs to the core, it suffices for establishing the equivalence theorem to show the positivity of each trader's income in quasi-equilibria. Taking this point into consideration, we will consider alternative assumptions by which, in quasi-equilibria, the positivity of each trader's income is ensured.

The simplest condition ensuring the positive income of each trader is the positivity of each trader's initial endowment of commodities. Other several conditions are known to ensure the positivity of each trader's income in quasi-equilibria. One of the most general conditions is the irreducibility assumption that was initiated by McKenzie [7]. A weaker version of the irreducibility assumption was introduced by Debreu [2] to prove an existence theorem of competitive equilibria in finite economies. In our Theorem 2, under Debreu's assumption, we show that any quasi-equilibrium becomes a competitive equilibrium and thus establish the equivalence between the core and the set of equilibrium allocations. The result is consistent with the equivalence theorem of Yamazaki [8] where a primitive condition on initial income distributions is assumed.

In the economies under Debreu's assumption, each trader's income in a quasi-equilibrium is positive and each trader participates in trade. On the other hand, in the economy that Aumann [1] considered, there might exist non-negligible individuals who have no endowments and cannot participate in trade. To extend Aumann's equivalence theorem, we would like to look for some conditions which are weaker than both the monotonicity assumption and Debreu's assumption. To include Aumann's economies, we will assume a somewhat weaker condition than monotonicity, which is called the potential desirability of commodities defined by Hara [3] and introduce a weaker form of Debreu's assumption which reflects the potential desirability of commodities. Under the assumptions we will establish the second equivalence theorem which is an extension of Aumann's theorem. Thus, it should be noted that the monotonicity assumption is dispensable to Aumann's equivalence theorem but it is just required that for each commodity there is a non-null coalition of some traders for whom the commodity is desirable.[1] In addition, since our proof is a simple modification of Aumann's proof, it is meant that the technique of his proof is very general and useful.

In what follows, in § 2 we present a model of exchange economy with a continuum of traders and state a well-known proposition that any equilibrium allocation is a core allocation. In § 3, under a set of weaker assumptions on preference relations, we claim that any core allocation is a quasi-equilibrium allocation. In § 4, the first core equivalence theorem is obtained under Debreu's assumption which is weaker than irreducibility. In addition, under an assumption which is related to the desirability of commodities, the second equivalence theorem is established. § 5 is devoted to concluding remarks, in which we refer to the related results of Hildenbrand [4,5].

2. The model

There are n-types of commodities being traded in the economy. A *commodity bundle* is a point in the non-negative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$ and the *consumption set* of each trader is $\mathbb{R}^n_+$. Let the set of traders be the closed unit interval $T = [0,1]$. The space of traders is an atomless measure space $(T, \mathcal{T}, \lambda)$ where $\mathcal{T}$ is a σ-algebra of Borel subsets of $T = [0,1]$ and λ is the Lebesgue measure with $\lambda(T) = 1$.

Following Aumann [1], an *assignment* is a function $\boldsymbol{f} : T \to \mathbb{R}^n_+$ such that $\boldsymbol{f}(t)$ is a commodity bundle assigned to each trader $t \in T$ and each component of the assignment is Lebesgue integrable over T. Let $\boldsymbol{e} : T \to \mathbb{R}^n_+$ be a fixed assignment in which $\boldsymbol{e}(t)$ denotes an *initial endowment* of each trader $t \in T$. The sum of initial endowments $\boldsymbol{e}$ is defined by $\int_{t \in T} \boldsymbol{e}(t) d\lambda$. For simplicity, we omit the symbols of t and $d\lambda$ in the integral, and so $\int_{t \in T} \boldsymbol{e}(t) d\lambda$ is denoted by $\int_T \boldsymbol{e}$. An *allocation* is an assignment $\boldsymbol{f} : T \to \mathbb{R}^n_+$ with $\int_T \boldsymbol{f} = \int_T \boldsymbol{e}$.

Let $\succ_t$ be the *preference relation* of each trader $t \in T$ defined on the consumption set $\mathbb{R}^n_+$, i.e., $\succ_t \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$ which satisfies the following property.

Assumption 1. For a pair of any two assignments $(\boldsymbol{f}, \boldsymbol{g})$, the set $\{t \mid \boldsymbol{f}(t) \succ_t \boldsymbol{g}(t)\}$ is Lebesgue measurable in T.

Note that in this section the measurability on preference relations is just assumed. This assumption is a purely mathematical assumption with no economic interpretation.

Next we state other notations and definitions. A *coalition* of traders is a Lebesgue measurable subset of T. A coalition S with $\lambda(S) > 0$ can *improved*

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[1] Hildenbrand [6] mentioned that the monotonicity is not essential for Aumann's proof.

upon an allocation $\boldsymbol{f} : T \to \mathbb{R}^n_+$ if there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that $\boldsymbol{g}(t) \succ_t \boldsymbol{f}(t)$ for almost every (a.e.) $t \in S$, and $\int_S \boldsymbol{g} = \int_S \boldsymbol{e}$. The *core* is the set of all allocations that no non-null coalition can improve upon. A *competitive equilibrium* is a pair of a *price vector* $p \in \mathbb{R}^n$ with $p \neq 0$ and an *allocation* $\boldsymbol{f}$ such that for a.e. $t \in T$, $\boldsymbol{f}(t)$ is a *maximal* element with respect to $\succ_t$ in t's *budget set* $\{x \in \mathbb{R}^n_+ \mid p \cdot x \leq p \cdot \boldsymbol{e}(t)\}$. An *equilibrium allocation* is the allocation $\boldsymbol{f}$ of the competitive equilibrium, and an *equilibrium price vector* is the price vector p of the competitive equilibrium. In the standard way it is shown that any equilibrium allocation is in the core. We state the following well-known fact without proof.

Proposition 1. *Under Assumption 1, any equilibrium allocation belongs to the core.*

3. The core and quasi-equilibrium allocations

We define the *quasi-equilibrium* as follows.[2] A *quasi-equilibrium* is a pair of a *price vector* $p^* \in \mathbb{R}^n$ with $p^* \neq 0$ and an *allocation* $\boldsymbol{f}^*$ such that for a.e. $t \in T$, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$ and/or $p^* \cdot \boldsymbol{f}^*(t) = p^* \cdot \boldsymbol{e}(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$. A *quasi-equilibrium allocation* is the allocation $\boldsymbol{f}^*$ of the quasi-equilibrium, and a *quasi-equilibrium price vector* is the price vector p^* of the quasi-equilibrium.

Next we assume that for a.e. $t \in T$ preference relation $\succ_t$ satisfies the two assumptions below.

Assumption 2. For a.e. $t \in T$, for any $x \in \mathbb{R}^n_+$ and any $\varepsilon > 0$, there exists a point $y \in \mathbb{R}^n_+$ with $\|y - x\| \leq \varepsilon$ such that $y \succ_t x$.

Assumption 3. For a.e. $t \in T$, for any $x \in \mathbb{R}^n_+$, the upper contour set $\{y \in \mathbb{R}^n_+ \mid y \succ_t x\}$ is open in $\mathbb{R}^n_+$.

Note that we do not necessarily assume that the preference relation of almost every trader satisfies reflexivity, completeness, transitivity, or convexity, etc., which are usually assumed in the existence theorems of competitive equilibria. In his paper, Aumann [1] assumed the following condition of monotonicity of preferences and proved the core equivalence theorem.

Monotonicity: For a.e. $t \in T$, if $y \geq x$ and $y \neq x$ then $y \succ_t x$.

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[2] The notion of quasi-equilibrium was first defined by Debreu [2].

Monotonicity implies local non-satiation, but the converse does not hold. In this paper, instead of monotonicity, we assume the local non-satiation of preference relations.

In order to obtain the core equivalence theorem without Monotonicity, first we will restate the theorem of Hildenbrand [4,5] that any core allocation is a quasi-equilibrium allocation.

Theorem 1. *Under Assumptions 1, 2, and 3, any core allocation is a quasi-equilibrium allocation.*

Applying one of claims of Hildenbrand [6],[3] we can show Theorem 1. For brevity, we omit the proof of Theorem 1.

By $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{Q}$, we denote respectively the set of equilibrium allocations, the core, and the set of quasi-equilibrium allocations. From Proposition 1 and Theorem 1, it follows that $\mathcal{W} \subset \mathcal{C} \subset \mathcal{Q}$ under Assumptions 1, 2, and 3. Therefore, if $\mathcal{W} \supset \mathcal{Q}$, then $\mathcal{W} = \mathcal{C}$, i.e., the core coincides with the set of equilibrium allocations. One of the assumptions under which $\mathcal{W} \supset \mathcal{Q}$ holds is the following.

Positivity of initial endowments: $\boldsymbol{e}(t) \gg 0$ for a.e. $t \in T$.[4]

In fact, $\boldsymbol{e}(t) \gg 0$ implies that $p \cdot \boldsymbol{e}(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for any $p \in \mathbb{R}^n$ with $p \neq 0$. Therefore, by the definition of quasi-equilibrium, any quasi-equilibrium is a competitive equilibrium. Thus, we have the following as a corollary of Theorem 1.

Corollary 1. *In addition to Assumptions 1, 2, and 3, under the assumption of Positivity of initial endowments, the core coincides with the set of equilibrium allocations, i.e.,* $\mathcal{W} = \mathcal{C}$.

4. Irreducible economies

The assumption that every trader has initially a positive amount of every commodity is too strong. In what follows, we assume that the amount of each commodity is positive in the whole economy.

Assumption 4. $\int_T \boldsymbol{e} \gg 0$.

As we have seen in the previous section, we can prove the equivalence between the core and the set of equilibrium allocations by showing that any quasi-equilibrium is a competitive equilibrium. By the definition of

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[3] This claim is obtained by a simple modification of Lemma 4.1 in Aumann [1].

[4] For x and y in $\mathbb{R}^n$, $x \gg y$ means that $x^i > y^i$ for all coordinate i.

quasi-equilibrium, any quasi-equilibrium is a competitive equilibrium if all traders' incomes in the quasi-equilibrium are positive. More accurately, a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$ is a competitive equilibrium if $p^* \cdot \boldsymbol{e}(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for a.e. $t \in T$. The following is a well-known condition to ensure the positivity of traders' incomes.

Irreducibility: An economy is *irreducible* if for any allocation $\boldsymbol{f} : T \to \mathbb{R}^n_+$ and any measurable partition (S, S') of T with $0 < \lambda(S) < 1$, there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that

$$\int_{S'} (\boldsymbol{e} - \boldsymbol{g}) + \int_S \boldsymbol{f} \in \int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}(t)\}.[5]$$

This primitive condition on economies originated with McKenzie [7]. Irreducibility expresses the property that the initial endowments in any coalition are desirable for every trader in its complementary coalition.

Let us assume the following for any quasi-equilibrium.

Assumption 5. In a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, if $p^* \cdot \boldsymbol{f}^*(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders,[6] then it occurs for almost every trader. (Equivalently, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then it occurs for almost every trader.)

Assumption 5 is used by Debreu [2] in order to guarantee the existence of competitive equilibria in a private ownership economy with finite traders.

It is easy to show that under Assumption 2 of local non-satiation, Irreducibility implies Assumption 5. Indeed, let $(p^*, \boldsymbol{f}^*)$ be a quasi-equilibrium, S be a measurable subset of T and S' be its complement defined by

$$\begin{aligned} S &= \{t \in T \mid p^* \cdot \boldsymbol{e}(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}, \\ \text{and} \quad S' &= \{t \in T \mid p^* \cdot \boldsymbol{e}(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}. \end{aligned}$$

We show that $\lambda(S) > 0$ implies $\lambda(S') = 0$. Assume on the contrary that $\lambda(S') > 0$. Then, by Irreducibility, there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that

$$\int_{S'} (\boldsymbol{e} - \boldsymbol{g}) + \int_S \boldsymbol{f}^* \in \int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}^*(t)\}.$$

By the definition of quasi-equilibrium, for a.e. $t \in S$, $\boldsymbol{f}^*$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$. Therefore,

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[5] The integral $\int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}(t)\}$ denotes a set defined by

$$\left\{ \int_S \boldsymbol{h} \mid \boldsymbol{h} : T \to \mathbb{R}^n_+, \ \boldsymbol{h}(t) \succ_t \boldsymbol{f}(t) \ \text{a.e.}\, t \in T \right\}.$$

[6] "Some traders" means that the set of such traders has a positive measure.

$$p^* \cdot \int_S \boldsymbol{f}^* \leq p^* \cdot \int_S \boldsymbol{e} < p^* \cdot \left(\int_{S'} (\boldsymbol{e} - \boldsymbol{g}) + \int_S \boldsymbol{f}^* \right).$$

Hence, we have $0 < p^* \cdot \int_{S'} (\boldsymbol{e} - \boldsymbol{g})$. On the other hand, by definition of S', for each $t \in S'$, $p^* \cdot \boldsymbol{e}(t) \leq p^* \cdot \boldsymbol{g}(t)$, and $p^* \cdot \int_{S'} (\boldsymbol{e} - \boldsymbol{g}) \leq 0$, a contradiction. Now, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then $\lambda(S) > 0$ because $p^* \cdot \boldsymbol{e}(t) \geq p^* \cdot \boldsymbol{f}^*(t)$. Therefore, $\lambda(S') = 0$. In addition, by local non-satiation we can show that $S' = \{t \in T \mid p^* \cdot \boldsymbol{f}^*(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}$. Hence, $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for every trader. This proves that, under Assumption 2, Irreducibility implies Assumption 5.

By using Assumption 5 instead of Monotonicity, we show that the core coincides with the set of equilibrium allocations which is equivalent to the set of quasi-equilibrium allocations.

Theorem 2. *Under Assumptions 1, 2, 3, 4, and 5, the core coincides with the set of equilibrium allocations, i.e.,* $\mathcal{W} = \mathcal{C}$.

Proof. By Proposition 1, the set of equilibrium allocations is a subset of the core. To prove the converse, let $\boldsymbol{f} : T \to \mathbb{R}^n_+$ be a core allocation. Then, by Theorem 1, there exists a price vector $p \neq 0$ such that $(p, \boldsymbol{f})$ is a quasi-equilibrium.

CASE 1: If vector p has some negative components, then $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = -\infty$. Therefore, $p \cdot \boldsymbol{f}(t) > -\infty = \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for a.e. $t \in T$.

CASE 2: If vector p has no negative component, then $\inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\} = 0$. Also, by Assumption 4, $p \cdot \int_T \boldsymbol{f} = p \cdot \int_T \boldsymbol{e} > 0$.

Therefore, $p \cdot \boldsymbol{f}(t) > 0 = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in T$. Thus, by Assumption 5, $p \cdot \boldsymbol{f}(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for a.e. $t \in T$. Hence, in any case, by the definition of quasi-equilibrium, $\boldsymbol{f}(t)$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p \cdot x \leq p \cdot \boldsymbol{e}(t)\}$ for a.e. $t \in T$, i.e., $\boldsymbol{f}$ is an equilibrium allocation. □

Since Irreducibility implies Assumption 5, we have the following corollary of Theorem 2.

Corollary 2. *In addition to Assumptions 1, 2, 3, and 4, under the assumption of Irreducibility, the core coincides with the set of equilibrium allocations, i.e.,* $\mathcal{W} = \mathcal{C}$.

The following is an example of economies which do not satisfy Irreducibility, but Assumption 5.

Example 1. *Let* $n = 2$ *and* (S^0, S^1) *be a measurable partition of* T *such that* $\lambda(S^0) = \lambda(S^1) = 0.5$. *For each trader* t, *initial endowment* $\boldsymbol{e}(t)$ *and* *utility function* U_t *which corresponds to preference relation* $\succ_t$ *are defined by the following:*

$$\boldsymbol{e}(t) = \begin{cases} (1,3) \text{ for } t \in S^0, \\ (1,0) \text{ for } t \in S^1, \end{cases} \quad \text{and} \quad U_t(x_1,x_2) = \begin{cases} \min\{x_1,x_2\} & \text{for } t \in S^0, \\ x_1 & \text{for } t \in S^1. \end{cases}$$

A pair $(p^*, \boldsymbol{f}^*)$ *of a price vector and an allocation defined by*

$$p^* = (1,0) \quad \text{and} \quad \boldsymbol{f}^*(t) = \begin{cases} (1, 3-y) & \text{for } t \in S^0 \\ (1,y) & \text{for } t \in S^1 \end{cases} \quad (\text{where } 0 \le y \le 2)$$

is a competitive equilibrium as well as a quasi-equilibrium.

In the economy of Example 1, every trader has a positive income in quasi-equilibrium $(p^*, \boldsymbol{f}^*)$ for any y with $0 \le y \le 2$, and therefore Assumption 5 is satisfied. However, Irreducibility is not satisfied. In fact, let $y = 2$ and, in the definition of Irreducibility, put $\boldsymbol{f} = \boldsymbol{f}^*$, $S = S^0$, and $S' = S^1$. Then, to make traders in S^0 better off, both commodities are needed, while traders in S^1 have only commodity 1.

Next, we would like to consider an assumption which is weaker than Monotonicity and to introduce the concept of *potential desirability of commodities* defined by Hara [3]. The intuition of the potential desirability is that for any commodity there exists a group of traders with influential power for whom the commodity is desirable. Note that the potential desirability does not require that any small amount of each commodity is desirable for a group, but just that some constant amount of the commodity are desirable for the group. Let us define the potential desirability and introduce an assumption of potential desirability as follows.

Definition 1. *Commodity* m *is potentially desirable for a coalition* $S \subset T$ *with respect to a constant number* $\alpha > 0$ *if* $x + \alpha \mathbf{1}_m \succ_t x$ *for all* $x \in \mathbb{R}^n_+$ *and for a.e.* $t \in S$.[7]

Assumption 6. For each commodity $m = 1, 2, \dots, n$, there exists a coalition $S^m \subset T$ with $\lambda(S^m) > 0$ and a number $\alpha^m > 0$ such that commodity m is potentially desirable for coalition S^m with respect to α^m.

This assumption says that for each commodity there are some traders for whom the commodity is desirable, but the commodity is not necessarily desirable for almost every traders. It also says that a particular positive amount of the commodity is desirable for the traders.

As a simple economic interpretation of Assumption 6, let us consider a realistic setting in which some durable goods are traded. In the setting, it seems

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[7] By $\mathbf{1}_m$ we denote a vector whose m-th coordinate is 1 and whose other coordinates are 0.

natural to presume that each commodity is desirable only for a part of buyers. The presumption may be supported by the following actual situation: A recycle shop owner sells used household goods, which are not desirable for the buyers who sell them to the owner but for some other buyers. Furthermore, even if it is possible to trade tiny amounts of commodities between sellers and buyers, it seems plausible that trading some large amounts of them often occurs in order to satisfy desires and appetites. This reflects that we usually trade divisible commodities according to a certain unit.

Moreover, we add the following assumption which reflects both properties of Assumptions 5 and 6.

Assumption 7. In a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then it occurs for some traders in S^m for all $m = 1, 2, \ldots, n$.[8]

Clearly, Monotonicity implies Assumption 6, whereas the converse is not true. Note that it is possible for a trader to be included in more than one coalition of $S^1, S^2, \ldots, S^n$. Under Monotonicity, every trader belongs to S^m for all $m = 1, 2, \ldots, n$. Thus, Monotonicity implies Assumption 7.

Now let us consider economies in which Assumption 7 is satisfied while Assumption 5 is not. The following is an example of such economies.

Example 2. *Let* $n = 2$ *and* (S^0, S^1, S^2) *be a measurable partition of* T *such that* $\lambda(S^1) = \lambda(S^2) > 0$*. For each trader* t*, the initial endowment* $\boldsymbol{e}(t)$ *and the preference relation* $\succ_t$ *which is depicted by a utility function* U_t *are defined by the following:*

$$\boldsymbol{e}(t) = \begin{cases} (0,0) \text{ for } t \in S^0, \\ (0,1) \text{ for } t \in S^1, \\ (1,0) \text{ for } t \in S^2, \end{cases} \quad U_t(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{for } t \in S^0, \\ x_1 & \text{for } t \in S^1, \\ x_2 & \text{for } t \in S^2. \end{cases}$$

Here, commodity 1 is desirable for traders in S^1*, commodity 2 is desirable for those in* S^2*, both are desirable for those in* S^0*. A pair* $(p^*, \boldsymbol{f}^*)$ *of a price vector and an allocation defined by*

$$p^* = (1,1) \quad \text{and} \quad \boldsymbol{f}^*(t) = \begin{cases} (0,0) \text{ for } t \in S^0 \\ (1,0) \text{ for } t \in S^1 \\ (0,1) \text{ for } t \in S^2 \end{cases}$$

is a competitive equilibrium as well as a quasi-equilibrium that is unique.

In the above example, Assumption 6 is satisfied. Moreover, Assumption 7 is satisfied for quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, while Assumption 5 isn't if $\lambda(S^0) > 0$.

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[8] S^m is the non-null coalition defined in Assumption 6 for each $m = 1, 2, \ldots, n$.

From now on, in order to obtain a core equivalence theorem for economies which satisfy Assumptions 1, 2, 3, 4, 6 and 7, we show the following lemmas.

Lemma 1. *Under Assumption 6, for any quasi-equilibrium* $(p^*, \boldsymbol{f}^*)$, $p^* \geq 0$ *with* $p^* \neq 0$.

Proof. Suppose that $p^*_m < 0$ for some m. Then, by Assumption 6, for a.e. $t \in S^m$,

$$\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m \succ_t \boldsymbol{f}^*(t) \text{ and } p^* \cdot (\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m) < p^* \cdot \boldsymbol{f}^*(t) \leq p^* \cdot \boldsymbol{e}(t),$$

i.e., $\boldsymbol{f}^*(t)$ is not maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$.

On the other hand, $p^* \cdot \boldsymbol{f}^*(t) > -\infty = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for a.e. $t \in T$. Therefore, by the definition of quasi-equilibrium, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set for a.e. $t \in T$, a contradiction. □

Lemma 2. *Under Assumptions 4, 6, and 7, for any quasi-equilibrium* $(p^*, \boldsymbol{f}^*)$, $p^* \gg 0$.

Proof. Since $p^* \geq 0$ by Lemma 1, $\inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\} = 0$. Also, by Assumption 4, $p^* \cdot \int_T \boldsymbol{f}^* = p^* \cdot \int_T \boldsymbol{e} > 0$. Therefore, $p^* \cdot \boldsymbol{f}^*(t) > 0 = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in T$. Thus, by Assumption 7, for each $m = 1, 2, \ldots, n$, $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in S^m$. Therefore, by the definition of quasi-equilibrium, for each $m = 1, 2, \ldots, n$, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set for some $t \in S^m$.

Now, suppose $p^*_m = 0$ for some m. Then, by Assumption 6, for a.e. $t \in S^m$,

$$\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m \succ_t \boldsymbol{f}^*(t) \text{ and } p^* \cdot (\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m) = p^* \cdot \boldsymbol{f}^*(t) \leq p^* \cdot \boldsymbol{e}(t),$$

i.e., $\boldsymbol{f}^*(t)$ is not maximal with respect to $\succ_t$ in t's budget set, a contradiction. □

By using Lemmas 1 and 2, we can get the following theorem.

Theorem 3. *Under Assumptions 1, 2, 3, 4, 6, and 7, the core coincides with the set of equilibrium allocation, i.e.,* $\mathcal{W} = \mathcal{C}$. *Moreover, every equilibrium price vector is strictly positive.*

Proof. By Theorem 1, any core allocation is a quasi-equilibrium allocation. By Lemma 2, the price vector associated with any quasi-equilibrium is strictly positive. Let $\boldsymbol{f} : T \to \mathbb{R}^n_+$ be a core allocation. Then, there exists $p \gg 0$ such that $(p, \boldsymbol{f})$ is a quasi-equilibrium. Note that $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = 0$.

Case 1: For $t \in T$ with $p \cdot \boldsymbol{e}(t) > 0$, by the definition of quasi-equilibrium, $\boldsymbol{f}(t)$ is maximal with respect to $\succ_t$ in t's budget set, since $p \cdot \boldsymbol{e}(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$.

Case 2: For $t \in T$ with $p \cdot \boldsymbol{e}(t) = 0$, clearly $\boldsymbol{e}(t) = \boldsymbol{f}(t) = 0$. Suppose that $\boldsymbol{f}(t)$ is not maximal with respect to $\succ_t$ in t's budget set. Then, $0 \succ_t \boldsymbol{f}(t)$, since t's budget set contains only the origin 0 of $\mathbb{R}^n_+$. If the set S of traders t for whom this happens has positive measure, then S can improve upon $\boldsymbol{f}$ via $\boldsymbol{e}$, contradicting that $\boldsymbol{f}$ is a core allocation. Thus, the set S is null and can be ignored.

This proves that $(p, \boldsymbol{f})$ is a competitive equilibrium where $p \gg 0$. □

Note that Assumptions 6 and 7 are weaker than Monotonicity. This fact means that the assumption of Theorem 3 is weaker than that of Aumann's theorem, i.e., Theorem 3 is an extension of Aumann's equivalence theorem.

Since Example 2 satisfies all the assumptions in Theorem 3, we can apply the theorem to the example and conclude that $\boldsymbol{f}^*$ is a unique core allocation. Under the assumptions of Theorem 3, the core might be smaller than the set of quasi-equilibrium allocations. However, the example suggests the following corollary.

Corollary 3. *Let the preference relation of almost every trader be irreflexive. Under Assumptions 1, 2, 3, 4, 6, and 7, the following three sets of allocations: (i) the core, (ii) the set of equilibrium allocations, (iii) the set of quasi-equilibrium allocations are equivalent, i.e.,* $\mathcal{W} = \mathcal{C} = \mathcal{Q}$.

Proof. The argument in the proof of Theorem 3 can be applied not only to any core allocation, but also to any quasi-equilibrium allocation, since the assertion of Case 2 is obviously true under the irreflexivity assumption. □

Example 3. *Everything is the same as Example 2 except for the preference relations of traders in* S^0. *For* $t \in S^0$, *let* $\succ_t$ *be a relation such that* $\mathbf{0} \succ_t x$ *for all* $x \in \mathbb{R}^2_+$. *Namely, traders in* S^0 *prefer nothing but the origin* $\mathbf{0}$ *of* $\mathbb{R}^2_+$. *Note that their preference relations are not irreflexive since* $\mathbf{0} \succ_t \mathbf{0}$.

In Example 3, if $\lambda(S^0) > 0$, the pair $(p^*, \boldsymbol{f}^*)$ defined in Example 2 is a unique quasi-equilibrium, while it is not a competitive equilibrium. Since Assumption 7 is satisfied for $(p^*, \boldsymbol{f}^*)$, Theorem 3 holds. However, since $\boldsymbol{f}^*$ is not a core allocation, in this case both the core and the set of equilibrium allocations are empty. On the other hand, if $\lambda(S^0) = 0$, this is a case of Corollary 3, and $\boldsymbol{f}^*$ is a core allocation as well as an equilibrium allocation.

5. Conclusion

As we have shown, the monotonicity of preferences is not essential for Aumann's equivalence theorem. On the other hand, general equivalence theorems have been established by Hildenbrand [4, 5]. In his paper (1968) he presented a coalition production economy and proved an equivalence theorem in a very general measure-theoretic framework. Especially, he allowed consumption sets to vary with traders and proved a theorem that any core allocation is a quasi-equilibrium. The theorem is more general than Theorem 1 in this paper, since the coalition production economy includes our exchange economy as a special case. However, in asserting that the equivalence theorem holds, he assumed the monotonicity of preference relations. Furthermore, in his book (1974) he proved an equivalence theorem in an atomless exchange economy by assuming some regular conditions on preference relations such as irreflexivity, transitivity, and monotonicity.[9] Thus, his equivalence theorems are not more general than Theorems 2 and 3 in this paper, which hold even for some reducible economies without Monotonicity such as those in Examples 1 and 2.

References

1. Aumann, R.J.: Market with a continuum of traders. Econometrica **32**, 39–50 (1964)
2. Debreu, G.: New concepts and techniques for equilibrium analysis. Int. Econ. Rev. **3**, 257–273 (1962)
3. Hara, C.: An equilibrium existence theorem for atomless economies without the monotonicity assumption. Econ. Bull. **34**, 1–5 (2006)
4. Hildenbrand, W.: The core of an economy with a measure space of economic agents. Rev. Econ. Stud. **35**, 443–452 (1968)
5. Hildenbrand, W.: Core and equilibria of a large economy. Princeton: Princeton University Press 1974
6. Hildenbrand, W.: Core of an economy. In: Arrow K.J., Intriligator M.D. (eds.): Handbook of mathematical economics, chap. 18, vol. 2. Amsterdam: North-Holland, 1982 pp. 829–877
7. McKenzie, L.W.: On the existence general equilibrium for a competitive markets. Econometrica **27**, 54–71 (1959)
8. Yamazaki, A.: On the pseudo-competitive allocations and the core of a large economy. J. Math. Econ. **5**, 217–228 (1978)

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[9] In Problem 9 of Hildenbrand [5, p. 143], he claimed that it is possible to prove an equivalence theorem for an irreducible exchange economy by using an analogous argument to Theorem 1 in Hildenbrand [5, p. 133], but it seems that the monotonicity assumption is indispensable in the argument. However, there is no problem with using the method of proof in Hildenbrand [4].

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On two classical turnpike results for the Robinson–Solow–Srinivasan model*

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Abstract. Turnpike theory, as originally conceived by Samuelson, pertains to optimal programs over a large but finite time horizon with given initial and terminal stocks. In this paper, we present two turnpike results in the context of a model proposed by Robinson, Solow and Srinivasan, and the subject of extensive recent analysis as the RSS model. Our results are classical except that they are phrased in terms of (i) approximately optimal programs, and (ii) golden-rule stocks rather than their parent facet, and they underscore the distinction between the original theory and the asymptotic stability of optimal infinite horizon programs. Our results, and the

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This paper is dedicated to the memory of David Cass: one of the authors learnt (in 1969–1970) the rudiments of capital theory and Pontryagin's principle at his hands, and to his everlasting regret, was not afforded the chance of also working with him on his Yale Ph.D. dissertation.

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arguments used to prove them, go beyond the RSS model to contribute to the general theory.

Key words: approximately optimal program, asymptotic stability, choice of technique, good program, large but finite time horizon, optimal program, turnpike

1. Introduction

It is now well-understood that the turnpike property furnishes a fruitful point of entry to the general theory of inter-temporal resource allocation. In the introduction to his 1986 Handbook chapter, McKenzie [21] presents both the undiscounted and the discounted theory under the aegis of the word *cluster*:

> We will seek to determine the asymptotic behavior of maximal paths, which display a tendency to cluster in the sufficiently distant future from whatever capital stocks they start. Other types of turnpike behavior that have been studied are clustering in early periods for finite optimal paths that start from the same initial stocks, but have different terminal stocks, and clustering in the middle parts of paths that may start and end with different stocks. In models with stationary utility functions, perhaps subject to discounting, the clustering has been seen as convergence to a stationary path along which capital stocks are constant.

In this substantial broadening of the reach of turnpike theory, and the corresponding delineation of the "long-term tendencies of paths of capital accumulation, the structure of the problem is [seen to be] characteristic of all economizing over time whether on the social scale, or the scale of the individual or the firm," and that the "subject is best described as the study of economizing over time."[1]

Ten years earlier, in 1976, the idea was already put forward as the "basic result that optimal paths converge to each other in appropriate circumstances, whatever their shapes may be,"[2] and under the typology of *middle, early* and *late* turnpikes, these shapes were usefully diagrammed; the first three figures in [20] are reproduced here for the reader's convenience as Fig. 1. Indeed, in this clustering phenomenon, not only is the privileged position of a balanced growth path to be de-emphasized,[3] but one can proceed even beyond

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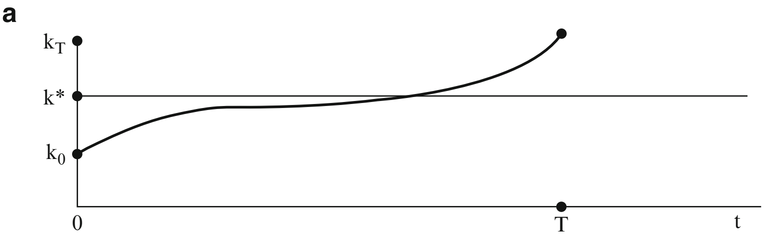
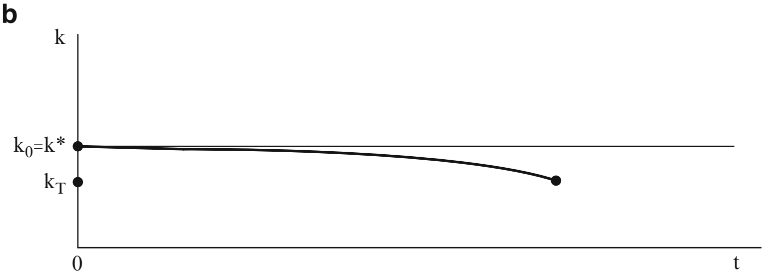
[1] This quote joins the first two sentences of McKenzie's introduction to its concluding sentence; see [21, pp. 1281, 1284]. For the indented quote above, see the fourth paragraph of the introduction to [21, p. 1282].

[2] See [20, p. 844] and also the later Ely Lecture [23, § 5] titled "Turnpikes".

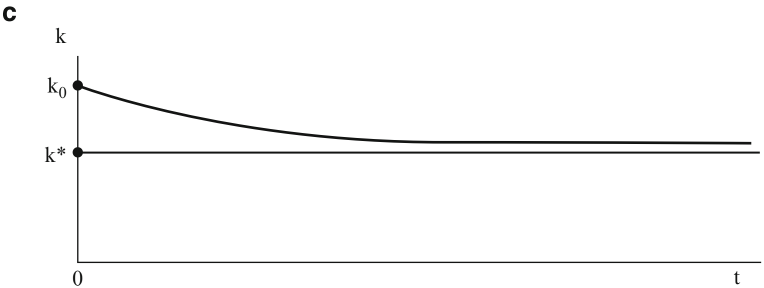
[3] In [23, § 5] McKenzie writes "Almost all the attention to asymptotic convergence has been concentrated on convergence to balanced paths, although it is not clear that



The middle turnpike.



The carly turnpike.



The late turnpike.

Fig. 1 **a** The middle turnpike. **b** The early turnpike. **c** The late turnpike

optimal programs *per se;* to say that the turnpike property is a property of programs which are "fairly good," or "not so bad;" to recognize, in a phrase, that optimality is sufficient for the turnpike property, but certainly not necessary.[4]

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optimal balanced paths will exist. This type of path is virtually impossible to believe in... I made this point in articles published in 1974 and 1976." Also see a repetition of this rather basic point in [24, p. 390].

[4] The phraseology is Tapan Mitra's; the terms *good* and *bad* have precise technical meaning going back to Gale [8], as in the definitions below.

Indeed, "the turnpike property of good programs is the key to developing a completely satisfactory theory of dynamic programming in the undiscounted case," and one can justifiably see the average turnpike property of *good* programs as "the general result of the subject."[5]

In Samuelson's reading of the turnpike literature, and in his contributions to it, the term is limited to what McKenzie categorizes as one of the two "other types of turnpike behavior": the *middle* turnpike and the "arching property of catenary motions" of optimal control trajectories over a finite horizon. In this somewhat narrower but classical conception, the issue of asymptotic stability of optimal programs is made to stand separately on its own, and turnpike theory is restricted to optimal trajectories on large but finite intervals as opposed to those on an infinite horizon.[6] Optimal trajectories on infinite horizon are sighted only for their theoretical interest in that they offer an approximate benchmark, turnpike to be sure, for the optimal solutions to temporally finite problems. The theory is seen to encapsulate this, and only this, particular interest. Thus, in his Nobel Lecture, in a section titled "dynamics and maximizing," Samuelson [33] presents the following "tongue-twister":

> An interesting triple limit is involved: as the horizon becomes *large,* you spend an indefinitely *large* fraction of your time within a *small* distance of the turnpike.

In other words, for any given levels of the initial and terminal capital stocks, z_0 and z_1, and for any two given levels of approximation, ϵ_1 and ϵ_2, one can find a large but finite time horizon $T(z_0, z_1; \epsilon_1, \epsilon_2)$ (henceforth, simply T), such that any optimal program starting from z_0 and guaranteed to furnish z_1 at its termination, and extending over T time periods, spends $(1 - \epsilon_1)$-proportion of the time, in an ϵ_2 proximity to the turnpike. Samuelson uses the term "most efficient" rather than "optimal," and leaves open the particular turnpike (the Ramseyian or the von-Neumann one) he has in mind. He also does not give symbolic prominence to the initial and terminal conditions on capital stocks, and in particular as to whether the large but finite time horizon T is uniformly independent of them. Formal treatments in the context of models, both aggregative and multi-sectoral are available in Part III of [34], and the theme of large but finite programs obtains the same uniform expression in all of these papers. In [32], Samuelson diagrams the basic

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[5] See [13, pp. 343, 347]. For overviews of Brock's 1970 result and the average turnpike property, see [22, § 8, concluding paragraph] and [23].

[6] The phrase "turnpike property of good programs" already serves as a counterpoint. Note that in his recent survey, Mitra uses the phrase "turnpike property" synonymously with asymptotic stability, [27, first paragraph]. Also see [26] and the reference to the "often called turnpike property" in [28, Introduction and Conclusion].

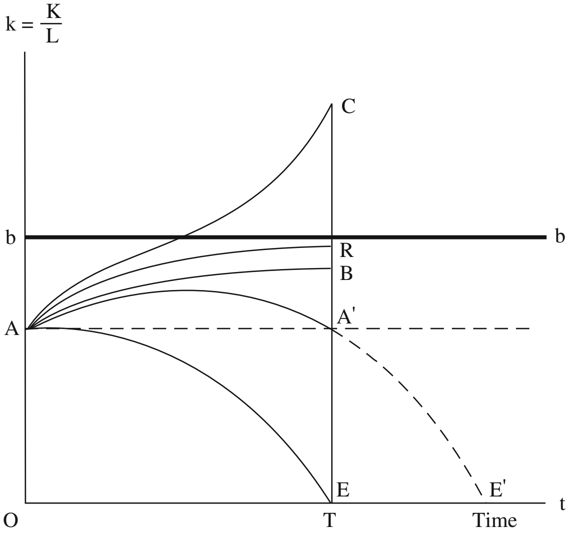

Fig. 2

ideas, and his Fig. 3 is reproduced here for the reader's convenience as Fig. 2 and to be contrasted with Fig. 1. Referring to the trajectory AE, he observes that it "gives the maximum utility for the finite interval $(0, T)$, ending up like a feckless annuitant with zero terminal capital. For it, the world ends neither in a bang nor a whimper, but on an upbeat of consumption."[7]

This difference between two conceptions, Samuelson's and McKenzie's, while neither subtle nor entirely obvious, is not rehearsed here merely for doctrinal considerations pertaining to the history of economic thought;[8] it constitutes an essential backdrop to the two theorems that we present in this paper. Rather than the asymptotic stability of infinite horizon programs,[9] the two turnpike results presented here revert to the classical Samuelsonian

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[7] Such a trajectory will be of particular relevance to what follows in the sequel.

[8] The question has to do with whether the von-Neumann growth model serves as a possible discontinuity in the evolution of the application of techniques of maximization and of the calculus of variations to capital theory; whether "maximization of an objective function had no part in his [von-Neumann's] theory." In his Nobel Lecture, [33], Samuelson motivates his work on the turnpike by recounting this episode, and McKenzie also refers to it in his Ely Lecture, [23]. Whether von-Neumann's position is to be granted half or full credit is another interesting shade of difference between McKenzie and Samuelson.

[9] For the latter, see [11], and under a title referring to a "turnpike of the third kind", [16].

conception, and their essential difficulty, and surely their substantive interest, arises from the fact that they concern large but finite optimal, and approximately optimal, programs. This is the first point that needs to be emphasized in their introduction; the second addresses itself to the fact that they are set in the context of a multi-sectoral model due to Robinson, Solow and Srinivasan, the so-called RSS model.[10] Why this particular model in the context of turnpike theory, however narrowly or broadly conceived? It is our contention, and especially to the reader unaware of recent work on this model, that this interest is warranted on both substantive and technical grounds, and especially if one takes into account the current stage of the development of the theory.[11] We begin with issues of economic substance.

Leaving aside the recent revival of interest in vintage capital theory from the viewpoint of macroeconomics,[12] the question of "choice of technique" in the context of development planning has led the RSS model to enjoy a renewed resurgence.[13] The original, and still challenging, 1968 result due to Stiglitz [36] established, for a continuous time setting, that both undiscounted and discounted optimality, with linear felicities, dictate the following: only *one* type of machine, say σ, out of a finite spectrum of types, is used and produced, and whereas other machine types identified as "better" are used, they are never produced. The result continues to remain a challenge for at least two reasons. First, the characterization of optimal policy when the felicity function is strictly concave has remained open during all this time. Even the intermediate case of a setting with linear felicities, coupled with an exogenously-given minimum consumption constraint, has not yielded any appreciable advance on the analysis of the model, see [37] and [6]. Second, Stiglitz's result still lacks an asymptotic implementation to a setting with discrete time with arbitrarily small but non-zero time intervals.[14] Indeed, if the continuous time setting has proved to be largely intractable, the discrete time setting has proved to be even more so.

The discrete time analysis has yielded results rather different in form and spirit to those of Stiglitz. The entire analysis has been shown to rest on a

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[10] For a discussion of this model, as well as the antecedent references to the papers of these authors, see, for example, [11] and [14].

[11] It is an interesting and enlightening exercise to base one reading of this current stage by coupling [21] and [35].

[12] See [16, § 1] for references to this literature from the viewpoint of the RSS mode.

[13] As [11] made clear, this resurgence is relative to [6, 34, 35]; also see [11] for their references to the earlier vintage of work by Robinson, Solow and Srinivasan.

[14] As readers of Samuelson's writings in financial economics are well aware, this is yet another way of lengthening the time horizon. For a rigorous replication of the Stiglitz analysis in the undiscounted case, but still with continuous time, see [12] and [41].

parametric specification ξ_σ pertaining to the machine-type σ that identifies the marginal rate of transformation between two adjacent time periods, and therefore perhaps by necessity, missed out in the continuous formulation. Even though Stiglitz policies are optimal for all values of $\xi_\sigma \leq 1$, they are not monotonic, and furthermore, in the case where $\xi_\sigma > 1$, they yield a *bad* program, leave alone an optimal one. Thus the characterization of optimal policy, even with linear felicities, remains unresolved for this particular parametric specification. On the other hand, for strictly concave felicities, a decisive example shows that optimality requires that machine types other than σ are also produced and used.[15] Since this example concerns a two-sector (one machine-type) RSS model with a piecewise linear felicity function, a complete characterization of optimal policies for the general model remains as distant a hope as it did for Cass and Stiglitz in the early 1970s. To be sure, substantial progress has been achieved in the special case of one machine type, where, as in the Ramseyian aggregative setting, considerations of the choice of technique are totally absent, but even here, a complete characterization in which the optimal policy function is based securely on the curvature of the felicity function, has yet to be achieved.[16]

Given this state of affairs, it is natural to fall back on turnpike theory, and ask whether Stiglitz policies are relevant not only for the very long run, but also for programs over a large but finite time horizon with given final terminal stocks. This is really to ask whether the golden-rule capital stock based solely on the machine-type σ serves as the turnpike both as regards asymptotic stability of infinite programs, as well as the classical middle-time benchmark for Samuelsonian turnpike theory.[17] Since the question pertaining to the former has already been satisfactorily answered,[18] the latter, presumably more

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[15] See Examples 2 and 3 in [11]. We also note here that Example 1 in this paper shows the existence of a two-period optimal cycle, thereby negating Stiglitz' monotonicity result in the continuous case.

[16] See [15] for the surprising differences between the linear and strictly concave theories. Note, however, that throughout this paper, we limit ourselves to an undiscounted setting, and do not comment on work, by now substantial, on the RSS model with a positive discount factor. It is also for this reason that in the discussion of the Ramseyian turnpike, there is no reference in the sequel to the theorems of Cass and Koopmans. We also draw the reader's attention to work with non-concave felicities as in [42].

[17] To quote McKenzie again, but now in the context of this more delimited turnpike theory, "The real ground for the result is the tendency for optimal paths to bunch together in the middle time, and this tendency is preserved even in models that are time-dependent;" see [21, p. 843]. In [19], McKenzie refers to the middle-turnpike for the von-Neumann model as the "Samuelson Turnpike".

[18] See [11, Theorem 3], and for a uniform asymptotic theorem, [16].

difficult, answer remains open and outstanding. After all, one can argue that the genesis of turnpike theory precisely lies in an attempt to circumvent detailed computations relating to the configuration and composition of the capital stock, types of machines used and produced, in the very long-run, and since this consists solely of the one-machine type σ, the computational economy is substantial indeed in this model. In summary, given that we know the measurable parameters underlying all of the machine types (the input–output coefficients and the labor-input coefficients), and therefore the distance of the initial and terminal capital-stock configurations from the golden-rule stocks based on σ, the hope is that these turnpike theorems will bear out, in some approximate sense, that only one type of machine will be used and constructed even in the middle-run of a large but finite optimal program. This hope is fulfilled, with a little additional room to spare, in the two theorems that we report and prove here.

These substantive considerations lead to issues of method and technique, the second arm of this introduction. Given the specificity of the RSS model – Leontieff technologies for the production of the consumption good, a finite number of machine types with a common depreciation rate, and each type produced by a single linear activity using labor alone – the question arises as to why any analysis is necessary beyond an invocation and application of the relevant theorems of the general theory of intertemporal resource allocation? The point is that even a cursory familiarity with the results of the two-sector (one machine-type) RSS model bears out the fact that they have necessitated new methods and techniques,[19] and the general theory is simply not in a position to answer the specific questions that have been posed. But staying here with turnpike theory in particular, we note McKenzie's identification of three possible methods of proof: (i) the primal approach that does not refer to any supporting prices, (ii) the value-loss approach, and (iii) the functional-analytic approach involving transformations of a Banach space. Primal methods under (i) involve strict convexity of the reduced-form utility function (uniform strict convexity in the non-stationary setting), and thereby preclude the RSS model, where the Leontieff technologies render the reduced-form utility function piecewise linear even when the original felicity function is strictly concave. Functional-analytical methods under (iii) are based on the Euler-Lagrange conditions and its implicit differentiability assumptions, and are again precluded for the RSS model by the Leontieff kink

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[19] See, for example, the synthesis of value-loss methods and a novel theory of undiscounted dynamic programming presented in [13, 15]. And as mentioned in Footnote 14, we avoid any reference to the discounted setting.

in the reduced-form utility functions at the golden-rule stock.[20] Thus, from the technical point of view, our only option here is to proceed with an adaptation of the value-loss methods of Radner [31], Atsumi [3] and McKenzie [17].[21] We postpone to the sequel a detailed discussion of the results and their relationship to previous work, and limit ourselves in this introduction to one basic observation: the two results presented here pertain to approximately optimal, large but finite programs, and they use the golden-rule stocks, the unique von-Neumann ray, as the relevant turnpike benchmark rather than the facet, the non-trivial von-Neumann facet, in which they lie. The fact that modifications of Radner's value-loss method still work so successfully is both a surprise and a testimony to its versatile durability.

The remainder of this paper is structured in two parts for two types of readers: § 2–5 are written for readers primarily interested in growth theory, new or old,[22] and § 6–9 for the expert in turnpike theory who wants to see the detailed execution of the arguments. § 2 is a relatively fast-paced and rigorous introduction to the RSS model and its basic categories and results. § 3 presents the two principal results and motivates them in terms of the ideas laid out in the introduction. § 4 is devoted to the heuristics of the proofs, and in particular, a framed discussion of the basic argumentative footholds that gird them, and that are presented in § 6 and 7. § 8 and 9 use these footholds to complete the proofs of the respective theorems. It is because of this two-part structure that we present some open questions, and remarks of a concluding nature as § 5, right before the symbolic technicalities of § 6–9.

2. The model and antecedent results

We begin with some preliminary notation. Let R (R_+) be the set of real (non-negative) numbers and let R^n be a finite-dimensional Euclidean space with non-negative orthant $R^n_+ = \{x \in R^n : \ x_i \geq 0, \ i = 1, \ldots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x \gg y$, $x > y$,

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[20] Whether methods of non-smooth analysis, and in particular, the subdifferential of the felicity function $w(\cdot)$ at the golden-rule stock $\hat{x}$, can be substituted for the differential, remains a hope, though a distant one.

[21] The one cautionary flag in the context of the earlier work on linear models relates to the fact that rather than the Ramseyian setting of the maximization of aggregate utility, the works relate to the von Neumann objective of the maximization of the utility of terminal capital stocks; see [18] and the references therein.

[22] See [9] and [35] for an overview of "neoclassical growth theory," with the latter also including the Ramsey aggregate capital accumulation model within its ambit; and [23] for the relevance of "turnpike theory" to the "new growth theory".

$x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $||x||$ denote the Euclidean norm of x.

We consider an economy capable of producing a finite number n of alternative types of machines. We begin with a formal description of this technological structure, and follow it by a specification of the planner's preferences.[23]

For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Thus, the production possibilities of the economy are represented by an (labor) input-coefficients vector, $a = (a_1, \dots, a_n) \gg 0$ and an output-coefficients vector, $b = (b_1, \dots, b_n) \gg 0$. We assume that all machines depreciate at a rate $d \in (0, 1)$.[24] Thus the effective labor cost of producing a unit of output on a machine of type i is given by $(1 + da_i)/b_i$: the direct labor cost of producing unit output, and the indirect cost of replacing the depreciation of the machine in this production. Let $c_i = b_i/(1 + da_i)$, $i = 1, \dots, n$. We shall assume the following:[25]

(2.1)

There exists $\sigma \in \{1, \dots, n\}$ such that for all $i \in \{1, \dots, n\} \setminus \{\sigma\}$, $c_\sigma > c_i$.

For each nonnegative integer t, let the amounts of the n types of machines that are available in time-period t be denoted by $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$, and the amounts of the n types of machines used for production of the consumption good, $by(t)$, during period $(t + 1)$ by $y(t) = (y_1(t), \dots, y_n(t)) \geq 0$. Let the total labor force of the economy be stationary and positive. We shall normalize it to be unity. Also $y(t)$ representing the use of available machines for manufacture of the consumption good, will require $ey(t)$ units of labor in period t. Thus, the availability of labor constrains employment in the consumption and investment sectors by $a(x(t + 1) - (1 - d)x(t)) + ey(t) \leq 1$. Note that the flow of consumption and of investment (new machines) are in gestation during the period and available at the end of it.

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[23] For the reader innocent of economic theory, as we shall see below, the model is a rather standard one in discrete optimal control theory in infinite time.

[24] Note that in restricting d to the open unit interval, we rule out the case of circulating capital. There is little doubt that our analysis can be extended to cover this case. See [7] for an analysis of the optimal policy function with circulating capital in a model that includes the two-sector RSS model.

[25] This is a fundamental assumption that guarantees a unique golden-rule stock, and thereby a unique von-Neumann ray. For a detailed discussion in the context of the RSS model, the reader is referred to [11] and [15].

The planner's preferences are formalized by a function $w : [0, \infty) \to R$ be a continuous, strictly increasing, concave and differentiable.

Now that the model is specified, we raise a cautionary flag regarding notation.[26] Throughout this paper, we shall reserve, and scrupulously so, the alphabets a and b as the n-vectors of labor and output coefficients, and the alphabet d for the scalar rate of depreciation. As specified above, these constitute the basic parameters of the model, and have substantive content. The same is true for the capital stocks: $2n$-vectors (x, x'), or their counterparts with the time-period explicitly specified, $(x(t), x(t+1))$. It is this substantive specification that allows us to depart from custom[27] and blur the distinction between inner products and multiplication by numbers: ax, bx and ex are real numbers while dx is a n-vector.

The fundamental concept of the subject is the notion of a *program.*

Definition 2.1. *A sequence $\{x(t), y(t)\}_{t=0}^{\infty}$ is called a program if for each integer $t \geq 0$*

$$(x(t), y(t)) \in R^n_+ \times R^n_+, \; x(t+1) \geq (1-d)x(t), \tag{2.2}$$
$$0 \leq y(t) \leq x(t), \; a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1.$$

Definition 2.2. *Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences*

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$ and for each integer t satisfying $T_1 \leq t < T_2$ relations (2.2) hold.

We now present a basic proposition that allows a restriction of the analysis to a compact subset.[28]

Proposition 2.1. *Let $m_0 > 0$. Then there is $m_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies $x(0) \leq m_0 e$ the inequality $x(t) \leq m_1 e$ holds for all integers $t \in [0, T]$.*

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[26] This paragraph is a response to a referee's insistence that the choice of notation does not (i) contradict minimal aesthetical requirements, and (ii) present a serious obstacle to understanding of the material. We hope that these criteria are now being adhered to as a result of the explanation in this paragraph.

[27] As a referee pointed out, "it is customary in modern literature to use different alphabets for vectors and numbers" and that "inner product is not a group operation." On the other side, changing the notation now when it has been established in more than fifteen published papers on the RSS model would cause confusion in another branch of the literature.

[28] See [11]. In this context, also see [21, Assumption II] for the general model.

Next, we recast the model in the Gale–Mckenzie reduced form,[29] but before doing this, we make some elementary observations on the model from an optimal control perspective. Note that the basic optimization problem is given by

$$\text{maximize} \sum_{t=0}^{\infty} w(by(t))$$

$$\text{subject to } (i)\ x(t+1) - x(t) = z(t+1) - dx(t) \text{ (a difference equation),}$$
$$(ii)\ az(t+1) + ey(t) \leq 1,\ z(t+1) \geq 0 \text{ and } y(t) \geq 0$$
$$\text{(the control constraints),}$$
$$(iii)\ x(t) - y(t) \geq 0 \text{ (a mixed constraint).}$$

Apart from the fact that optimization over an infinite time interval may lead the series defining the objectives to not converge, the problem, even given the mixed constraint (iii), is clear enough: a concave benefit function (actually a composition of a concave function of one variable and a linear function) and with infinitely-many constraints all linear. One way of proceeding is to ensure the existence of a solution and to characterize it by first-order necessary conditions. Indeed, in his characterization of transition dynamics, Stiglitz [36] proceeds from such conditions for the problem posed in continuous time.[30] In this paper too, Lemma 2.1 below revolves around them in regard to stationary programs. However, we are interested in the necessary conditions of optimality for the full problem only to the extent that they throw light on the turnpike property of finite optimal programs, which is to say, on the clustering of the solutions of the finite version of the problem to the solution of the problem itself. To be sure, both Samuelson [32] and Cass [5] use these conditions in a continuous-time setting to develop the turnpike (catenary) property for what has now come to be known as the Ramsey–Koopmans–Cass one-sector model, but the analysis here is focussed on approximately optimal finite programs. In this case, necessary conditions cannot easily get off the ground. Indeed the fecundity of the Gale–McKenzie

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[29] As emphasized in their introduction and abstract, this is the point of departure of Khan–Mitra [11].

[30] It bears emphasis that Stiglitz did not offer sufficient condition for the existence of an optimal program, and that these were furnished only recently by [12,41]. But more to the point, his results applied only to the case of $w(\cdot)$ being linear, and the problem of characterizing transition dynamics with concave functions remains open even in continuous-time models; see [6,37]. In the discrete time setting, the problems has not be fully solved even for the case of one machine ($n = 1$), the two-sector RSS model, leave alone the multi-sectoral one being considered in this paper; see [11,15].

reduced form, and Radner's [31] breakthrough, have to be judged precisely from this perspective.[31]

Towards this end, set

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}.$$

We have a correspondence $\Lambda : \Omega \to R^n_+$ given by

$$\Lambda(x, x') = \{y \in R^n_+ : 0 \leq y \leq x \text{ and } ey \leq 1 - a(x' - (1-d)x)\}, \ (x, x') \in \Omega.$$

For any $(x, x') \in \Omega$ define

$$u(x, x') = \max\{w(by) : \ y \in \Lambda(x, x')\}.$$

We can now work with the specific pair (Ω, u), the reduced form model of the general theory. Figure 3 provides a pictorial representation of the two-sector RSS model, and the reader, especially one unfamiliar with the RSS model, is invited to use it to check the concepts and results presented in this section.[32] For a pictorial representation of the model and its basic concepts, see Fig. 3.

Next, we turn to the basic benchmark, the golden-rule stock $\widehat{x}$: Theorem 2.1 establish its existence and Lemma 2.1, the so-called *value-loss lemma* establishes its characterization using the golden-rule prices $\widehat{p}$.

Definition 2.3. *A golden-rule stock is* $\widehat{x} \in R^n_+$ *such that* $(\widehat{x}, \widehat{x})$ *is a solution to the problem: maximize* $u(x, x')$ *subject to* (i) $x' \geq x$, (ii) $(x, x') \in \Omega$.

Theorem 2.1. *There exists a unique golden-rule stock* $\widehat{x} = (1/(1 + da_\sigma))e(\sigma)$.

It is not difficult to see that $\widehat{x}$ is a solution to the problem $\operatorname{argmax}_{y \in \Lambda(\widehat{x}, \widehat{x})} w(by)$. Set $\widehat{y} = \widehat{x}$, and for $i = 1, \ldots, n$, set

$$\widehat{q}_i = a_i b_i / (1 + da_i), \ \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{2.3}$$

The following is an important auxiliary result that drives, in some sense, the entire analysis to follow. With strict inequality, it goes back to Radner [31].[33]

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[31] To be able to prove the turnpike results reported here from the above suggested optimal control theory point of view has remained an open problem right from the inception of the subject.

[32] See [24, p. 389] for 1964 as the original date of introduction of this reduced form model. For a comprehensive and complete diagrammatic analysis of the two-sector RSS model in the undiscounted case, see [14]. For the general case, Figs. 7 and 8 in [24], and Figs. 7.3–7.5 in [25].

[33] We shall come back to this lemma in the discussion in § 4 below.

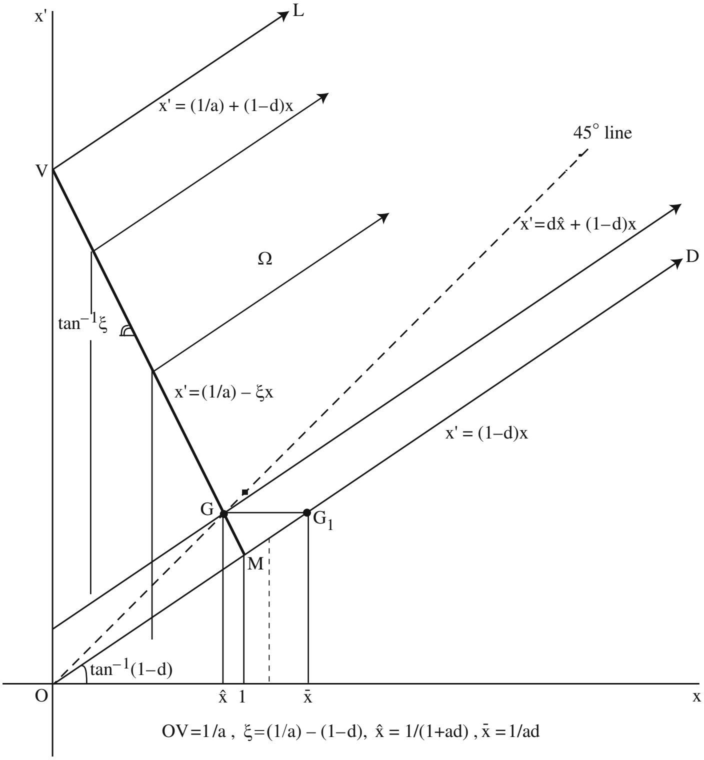

Fig. 3 Basic geometrical benchmarks of the two-sector RSS model

Lemma 2.1. *$w(b\widehat{x}) \geq w(by) + \widehat{p}x' - \widehat{p}x$ for any $(x, x') \in \Omega$ and for any $y \in \Lambda(x, x')$.*

For any $(x, x') \in \Omega$ and any $y \in \Lambda(x, x')$ set

$$\delta(x, y, x') = \widehat{p}(x - x') - (w(by) - w(b\widehat{y})). \tag{2.4}$$

Remark 2.1. By Lemma 2.1, $\delta(x, y, x') \geq 0$ for $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$.

Next, we turn to precise definitions of *good* and *bad* programs, as formulated in [8] and [4], and then, in the next three results, note conditions for their existence and for their convergence to the golden-rule stock.[34]

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[34] Note that the conditions furnished in Theorem 2.2 are only sufficient conditions; it is entirely conceivable that the convergence of good programs to the golden-rule

Definition 2.4. *A program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *is called good if there exists* $M \in R$ *such that*

$$\sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) \geq M \text{ for all } T \geq 0.$$

A program is called bad if $\lim_{T\to\infty} \sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) = -\infty$.

Proposition 2.2. *(i) Any program that is not good is bad. (ii) For any initial stock* $x_0 \in R_+^n$ *there exists good program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *such that* $x(0) = x_0$. *(iii) A program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *is good if and only if* $\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty$.

Let $x_0 \in R_+^n$. Set

$$\Delta(x_0) = \inf\{\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) : \{x(t), y(t)\}_{t=0}^{\infty} \text{ is a program starting from } x_0\}. \tag{2.5}$$

Since there exists a good program from x_0, it follows from Proposition 2.2 that

$$\Delta(x_0) < \infty. \tag{2.6}$$

Proposition 2.3. *Let* $x_0 \in R_+^n$. *Then there exists a program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *from* x_0 *such that* $\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) = \Delta(x_0)$.

Theorem 2.2. *For each good program* $\{x(t), y(t)\}_{t=0}^{\infty}$, $\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x})$ *if at least one of the following assumptions holds: (i) the function* w *is strictly concave, or (ii) and* $\xi_\sigma \neq -1$ *where*

$$\xi_\sigma = 1 - d - (1/a_\sigma). \tag{2.7}$$

Next, we turn to precise definitions of optimal programs, as originally formulated in [3] and [38], and extended by [8] and [4].[35] Theorem 2.3 is the existence theorem; its second part, coupled with Theorem 2.2, makes precise the fact that for "almost all" versions of the RSS model, optimal programs, as opposed to weakly optimal programs, exist.[36]

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stock, (the, so-called, turnpike property of good programs) holds with felicity functions that are only strictly concave only in the neighborhood of the golden-rule stock. See [8] for this assumption; it is highlighted in [24, p. 389].

[35] As is well understood, the issue goes back to Ramsey's original 1928 paper; see [25, p. 256] for a textbook treatment. In [4], what we call *weakly optimal* is termed *weakly maximal.*

[36] As emphasized in the introduction, our primary concern in this paper is with optimal, and approximately optimal, large but finite programs.

Definition 2.5. *A program* $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is called optimal if*

$$\limsup_{T\to\infty} \sum_{t=0}^{T} [w(by(t)) - w(by^*(t))] \le 0$$

for every program $\{x(t), y(t)\}_{t=0}^{\infty}$ *satisfying* $x(0) = x^*(0)$.

Definition 2.6. *A program* $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is called weakly optimal if for each program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *satisfying* $x(0) = x^*(0)$ *the following inequality holds:*

$$\liminf_{T\to\infty} \sum_{t=0}^{T}[w(by(t)) - w(by^*(t))] \le 0.$$

Theorem 2.3. *(i) For each initial stock* $x_0 \in R^n_+$ *there exists a weakly optimal program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *which satisfies* $x(0) = x_0$. *(ii) If, for each good program* $\{x(t), y(t)\}_{t=0}^{\infty}$, $\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x})$, *then for each initial stock* $x_0 \in R^n_+$ *there exists an optimal program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *which satisfies* $x(0) = x_0$.

Finally, we conclude this section by presenting a characterization of the von-Neumann facet, a concept that has played a fundamental role in the (continuing) analysis of the RSS model.[37] We also present a technical proposition that is primarily geared to the multi-sector case; in the two-sector RSS model it is trivially satisfied, as can be verified through Fig. 3.

Theorem 2.4. *The von Neumann facet*

$$\{(x, x') \in \Omega : \text{ there is } y \in \Lambda(x, x') \text{ such that } \delta(x, y, x') = 0\}$$

is a subset of

$$\{(x, x') \in \Omega : x'_i = x_i = 0 \text{ for all } i \ne \sigma,\ x'_\sigma = (1/a_\sigma) + \xi_\sigma x_\sigma\}$$

with equality if the function w *is linear. If the function* w *is strictly concave, then the facet is the singleton* $\{(\widehat{x}, \widehat{x})\}$.

Proposition 2.4. *Let* $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $x, x' \in R^n_+$ *satisfying* $||x - \widehat{x}||, ||x' - \widehat{x}|| \le \delta$ *there are* $\bar{x} \ge x'$, $y \in R^n_+$ *such that*

$$(x, \bar{x}) \in \Omega, \quad y \in \Lambda(x, \bar{x}), \quad ||y - \widehat{x}|| \le \epsilon, \quad ||\bar{x} - \widehat{x}|| \le \epsilon.$$

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[37] See, for example, [24, pp. 387–388], and the reference to the unpublished dissertation of Takahashi for the first dimensionality results regarding the facet. For the use of the concept as a basic engine of analysis in the form of value-loss lines in the two-sector RSS model, see [14].

None of the results presented in this section are new, and we present a detailed bibliography for the interested reader. The proofs of Propositions 2.1, 2.2 and 2.3 are all available in [11] as proofs of Propositions 1, 2, 7, and 8 respectively. The proof of Proposition 2.4 is available in [16] as the proof of Proposition 5.5. Theorem 2.1 and Theorem 2.4 are both taken from [11]. Theorem 2.2 is due to [13] and [15] for the two-sector RSS model, and to [39] for the general case. Theorem 2.3(i) is taken from [11], and Theorem 2.3(ii) from [39].

3. The principal results

With a description of the model and the relevant antecedent results behind us, we can turn to the two principal results of this paper. We develop notation for the aggregate value of finite optimal programs: the first refers to trajectories like AE in Fig. 2 that leave zero terminal stocks, and the second to general finite trajectories with specified terminal stocks. Let $z \in R^n_+$ and $T \geq 1$ be a natural number. Set

(3.1)
$$U(z,T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : (\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = z\}.$$

Clearly $U(z,T)$ is a finite number. Let $x_0, x_1 \in R^n_+$, T_1, T_2 be integers, $0 \leq T_1 < T_2$. Define

(3.2)
$$U(x_0,x_1,T_1,T_2) = \sup\{\sum_{t=T_1}^{T_2-1} w(by(t)) : (\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = x_0,\ x(T_2) \geq x_1\}.$$

It is clear from Fig. 3 that a too high a choice of the terminal stock x_1 leads to a supremum over empty set by precluding the existence of an optimal program. We assume this to be $-\infty$. Clearly $U(x_0,x_1,T_1,T_2) < \infty$. It is also clear that for any $z \in R^n_+$ and any integer $T \geq 1$, $U(z,T) = U(z,0,0,T)$.

We shall also assume the following hypothesis for each of our results.

Standing Hypothesis: Each good program converges to the golden-rule stock $(\widehat{x}, \widehat{x})$.

A preliminary question arises as to what happens when the Standing Hypothesis does not hold? Given Theorem 2.2, this is to ask, in particular, what happens when the felicity function is linear with a parametric configuration $\xi_\sigma \neq -1$ that "almost never"[38] holds? The conclusions of either of the theorems presented below are false in this case, and rather than the turnpike property, only the averages of the stocks converge to the golden-rule stock, the so-called average turnpike property.[39]

With Card(A) denoting the cardinality of a finite set A, we can present our first result.

Theorem A. *Let* M, ϵ *be positive numbers and* $\Gamma \in (0,1)$. *Then there exists a natural number* L *such that for each integer* $T > L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \leq Me$ *and* $az_1 \leq \Gamma d^{-1}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0,\ x(T) \geq z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - M,$$

the following inequality holds:

$$Card\{i \in \{0, \ldots, T-1\} :\ \max\{||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}||\} > \epsilon\} \leq L.$$

Next, we present our second result.

Theorem B. *Let* M, ϵ *be positive numbers and* $\Gamma \in (0,1)$. *Then there exist a natural number* L *and a positive number* γ *such that for each integer* $T > 2L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \leq Me$ *and* $az_1 \leq \Gamma d^{-1}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0,\ x(T) \geq z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - \gamma, \tag{3.3}$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T - L, T], \tag{3.4}$$
$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \leq \epsilon \text{ for all } t = \tau_1, \ldots, \tau_2 - 1 \text{ and } ||x(\tau_2) - \widehat{x}|| \leq \epsilon.$$

Moreover if $||x(0) - \widehat{x}|| \leq \gamma$ *then* $\tau_1 = 0$.

As emphasized in the introduction, both results are classical in that they pertain to the following situation: if an initial configuration of capital stocks is given, and a particular terminal configuration is stipulated, then with enough

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[38] If we endow the parameter space $(a, d) \subset (R_+ \times (0,1)) \subset R^2_+$ with Lebesgue measure, the set $\{(a,d) : (1/a) = 2 - d\}$ has zero measure.

[39] See [11, Theorem 3 and Example 1], and, for the two-sector case, [13–15].

time at his disposal, the planner ought to stay near the golden-rule configuration. Indeed, the larger the time horizon allowed the planner, the less of an error he or she in departing from the golden-rule configuration.

Where the results are not classical is that they take the notion of optimality one step further: rather than the desirability of the terminal stocks as in the case of the Samuelson turnpike for the von-Neumann model, or the optimality of aggregate consumption as in the Ramsey turnpike for the aggregate one-good model,[40] Theorems A and B emphasize approximate optimality. Thus, rather than the Samuelsonian triple limit alluded to in the introduction, an interesting "quarter limit" seems to be involved, and four separate considerations are being quantified: in addition to the length of the time-horizon, the proximity to the turnpike, and the length of time that is spent within this proximity, a "degree of slack" in the attainment of the objective is also brought into play. In terms of the language of the introduction, for any given levels of the initial and terminal capital stocks, z_0 and z_1, and for any two given levels of approximation, ϵ_1 and ϵ_2, one can find a large but finite time horizon $T(z_0, z_1; \epsilon_1, \epsilon_2)$ such that any "fairly good program" starting from z_0 and guaranteed to furnish at least z_1 at its termination, and extending over T time periods, spends $(1 - \epsilon_1)$-proportion of the time, in an ϵ_2 proximity to the turnpike.

It is precisely relative to this consideration of approximate optimality that the two results differ. Theorem A, in allowing the choice of M, ϵ and Γ, allows not only the choice of the interval of given initial, and pre-specified terminal, stocks, and the proximity level to the turnpike, through M, but also the particular "degree of slack" or the "level of goodness" that can be tolerated in the attainment of the objective.[41] The price of this is that the number of periods $(T - L)$ spent close to the turnpike cannot be guaranteed to be consecutive, the Radner-situation [31] that Nikaido [30] referred to as a "hop-skip-jumping turnpike theorem".[42] Theorem B prevents this "anomalous situation" by not allowing the choice of approximate optimality, but by

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[40] The terminology is McKenzie's [23, § II and III respectively,] as already referred to in Footnote 17.

[41] The parameter Γ is necessitated by a particular structural characteristic of the RSS model. The reader can refer to Fig. 3 to check out that in the two-sector RSS model, the maximal sustainable capital stock $(1/ad)$ can never be attained by a finite program starting from an initial stock less than it. Note that in this special case, $n = 1$, and therefore a is a scalar.

[42] Also see Inada [10] and his emphasis on uniformity considerations in turnpike theory. We leave for future work a detailed comparison of our results to those in these papers. It is clear that the RSS model is a particular case of the Leontieff model without joint production and many (but a finite number) techniques for the production of the consumption good.

letting "degree of goodness" number γ be determined as a consequence of ϵ, Γ and M, the latter now solely parametrizing the choice of the interval of given initial stocks. Thus both results go in a direction not emphasized in the literature: how far can the notion of optimality be weakened to yield a *uniform* middle turnpike theorem of consequence, uniform or doubly uniform in the sense that T (in the case of Theorem A), or T and γ (in the case of Theorem B), do not depend on the precise value of the initial and terminal stocks, z_0 and z_1, only that they lie within arbitrarily given ranges? And note that both theorems give the *actual* number of time periods, rather than a proportion of time, that the program spends in proximity to the turnpike.[43]

After this discussion of the theorems, we turn to the literature and focus on finite optimal, rather than approximately optimal, programs. The point to be re-emphasized in this comparison is that Theorems A and B both pertain to the unique golden-rule stock rather than the non-trivial (in the case of linear felicity function) von-Neumann facet in which they may lie. Thus, Theorems 2 and 3 in [17] both concern the facets F and W. Theorem 3 in [25, p. 254] re-presents the basic result in the specific context of what is termed the "multisectoral" Ramsey model. In his discussion of his 1968 results in [22], McKenzie presents the ideas in terms of asymptotic stability of the golden-rule stock (discussed below as the *late* turnpike), and writes:

> The most general assumption is that the von Neumann facet F is stable, in the sense that all paths that remain on the facet forever converge uniformly to a maximal stationary path. The assumption in this general form was proposed by Inada (1964).

We shall return to this statement in the sequel; for the moment, note that it enables us to see Theorems A and B, and especially their corollary presented below, as providing sufficient conditions for the facet in the RSS model to be stable.[44] Note also that this stability question does not necessarily reduce to the stability of the differential equation describing the path that lies on the facet; in the case $\xi_\sigma > 1$, the facet is stable but the root of the differential

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[43] An anonymous referee brought Theorem 5 in Chap. 4 of Arkin–Evstigneev [2] to our attention in the context of approximate optimality. This theorem establishes in a stochastic setting a version of the "middle turnpike theorem (in the spirit of Theorem A) for a family of uniformly good finite programs, and since optimal finite programs are uniformly good, the same is true for a family of approximately optimal programs."

[44] See [24, p. 389] for this point in the context of the general theory.

equation has modulus greater than unity.[45] Theorem 2 in [20] concerns the theorem in its non-stationary setting and relies on "uniform reachability" assumptions to subdue the non-stationarity aspects of the model.[46]

McKenzie notes that the "features of the model that allow the turnpike results to be reached are quite similar for the three cases, so their differences are sometimes a matter of form rather than substance" and that "a turnpike theorem of the first kind [*middle* turnpike] will usually imply a theorem of the second kind [*early* turnpike], but there are other cases as well."[47] This statement is silent on the derivation of the *late* turnpike theorem, and we observe here that *both* Theorem A and B yield a *late* turnpike, asymptotic stability, result.[48]

Corollary 1. *Let M, ϵ be positive numbers. Then there exists a natural number T_0 such that for each optimal program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \leq Me$,*

$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \leq \epsilon \text{ for all integers } t \geq T_0.$$

This corollary follows by taking into account that optimal program converges to the golden-rule stock, $\widehat{x}$, and by restricting an optimal program to a finite interval with a large enough T.

In his Handbook article, McKenzie [21] presents three *late* turnpike theorems without discounting: Theorems 8.1, 8.2 and 9.3, with the only the last one concerned with a stationary model. It is phrased under the sufficient assumption that the facet F is stable.[49] The conclusions of Theorems 8.1 and 8.2 are the same as that of Corollary 1, but without the assumption of strict uniform concavity (Theorem 8.1) or that of strictly uniform value-loss (Theorem 8.2). As mentioned in the introduction, the RSS model precludes

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[45] For a complete characterization of the policy correspondence in the two-sector RSS model, including the situation when the Standing Hypothesis does not necessarily hold, see [13–15].

[46] The form that McKenzie chooses as final version suitable for a text, [25, Theorem 3] referred to above, obviously relies on [18, 19] in its evolution. However, since these latter references are phrased in terms of the maximization of a function of the terminal stock configuration, the Samuelson turnpike, so to speak, we do not discuss them in any detail.

[47] See page 845 in [20] for the first quote, and pages 850, 851 for the second.

[48] This is the principal result of [16, Theorem 3.3], which itself is a generalization of [11, Theorem 3].

[49] It is of interest in the light of Theorem 2.3(ii) above, that McKenzie's theorem 9.3 also furnishes an existence result that asserts the optimality of such a converging path, if it is maximal, (weakly maximal in the terminology used in this paper).

the former assumption. We shall have more to say about the latter, but only in the stationary setting of course, when we get to a discussion of the proofs.[50]

The final consideration in this section relates to what McKenzie calls the "turnpike of the second kind," the *early* turnpike for which he ascribes the priority to Brock, and whose possibility is not even conceived in Fig. 2 taken from Samuelson. It is a testimony to the power of Theorems A and B that they trivially yield the following corollaries.[51] Note that the second rules out non-Nikaido antics at the price of restricting the initial stocks to be *precisely* the golden-rule stocks.

Corollary 2a. *Let* M, ϵ *be positive numbers and* $\Gamma \in (0,1)$. *Then there exists a natural number* L *such that for each integer* $T > L$, *each* $z_0, z_1 \in R^n_+$ *satisfying* $z_0 \leq Me$ *and* $az_1 \leq \Gamma d^{-1}$ *and any two programs* $(\{\bar{x}(t)\}_{t=0}^{T}, \{\bar{y}(t)\}_{t=0}^{T-1})$ *and* $(\{\tilde{x}(t)\}_{t=0}^{T}, \{\tilde{y}(t)\}_{t=0}^{T-1})$ *which satisfy*[52]

$$\bar{x}(0) = \tilde{x}(0) = z_0, \ \bar{x}(T) \geq z_1, \ \tilde{x}(T) \geq z_1,$$
$$\sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - M,$$

the following inequality holds:

$$Card\{i \in \{0, \dots, T-1\} : \ \max\{||\bar{x}(t) - \tilde{x}(t)||, \ ||\bar{y}(t) - \tilde{y}(t)\} > \epsilon\} \leq L.$$

Corollary 2b. *Let* ϵ *be a positive number and* $\Gamma \in (0,1)$. *Then there exist a natural number* L *such that for each integer* $T > 2L$, *each* $z_1 \in R^n_+$ *satisfying* $az_1 \leq \Gamma d^{-1}$ *and any two programs* $(\{\bar{x}(t)\}_{t=0}^{T}, \{\bar{y}(t)\}_{t=0}^{T-1})$ *and* $(\{\tilde{x}(t)\}_{t=0}^{T}, \{\tilde{y}(t)\}_{t=0}^{T-1})$ *which satisfy*[53]

$$\bar{x}(0) = \tilde{x}(0) = \widehat{x}, \ \bar{x}(T) \geq z_1, \ \tilde{x}(T) \geq z_1, \quad \sum_{t=0}^{T-1} w(by(t)) = U(z_0, z_1, 0, T),$$

there is an integer $\tau \in [T-L, T]$, *such that* $||\bar{x}(\tau) - \tilde{x}(\tau)|| \leq \epsilon$ *and*

$$||\bar{x}(t) - \tilde{x}(t)||, \ ||\bar{y}(t) - \tilde{y}(t)|| \leq \epsilon \textit{ for all } t = 1, \dots, \tau.$$

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[50] For a most fruitful exploitation of differentiable methods for this question in the context of the non-stationary aggregative one-good model, see [26, Theorem 3] and [28, Theorem 5.1].

[51] An anonymous referee pointed out that methods of proof of Theorem 1 in [1] could possibly be used to provide a direct and unified proof of the corollaries. We leave this for future work.

[52] In the succeeding line, $y(t)$ stands for $\bar{y}(t)$ and $\tilde{y}(t)$ alternatively.

[53] In the succeeding line, $y(t)$ stands for $\bar{y}(t)$ and $\tilde{y}(t)$ alternatively.

4. The heuristics of the proofs

The proofs of Theorems A and B presented in § 6–9 below are involved and technical, and in this section we preface them with a heuristic introduction that identifies the basic footholds and relates them to arguments that are by now standard for turnpike theory.

Radner's 1961 proof of the turnpike theorem is a basic template and a fundamental point of entry into the structure of classical turnpike theorems, certainly in discrete time. As is well-known, it exploits the fact that maximal balanced growth path in the von-Neumann model leads to a saddlepoint in the space of the stocks and prices, and this minimax aspect is effectively materialized as the value-loss lemma. Thus it bears emphasis that the value-loss lemma is a property of this saddlepoint and has little to do with the optimality or non-optimality of the large but finite horizon program under consideration. As such it makes direct contact with the value loss lemma presented for the maximal balanced growth path and the resulting golden-rule stocks and prices pair $(\hat{x}, \hat{p})$ presented as Lemma 2.1 above. The simplicity of the RSS model and the constructive proof of this lemma should not be obscure the fact that it is a solution to a non-linear (concave) programming problem, and as such, a consequence of supporting the optimal solution by Karush–Kuhn–Tucker multipliers obtained by the separating hyperplane theorem.[54] Whereas there is no presumption that this lemma can be strengthened to inequality in the case of linear felicities (see Theorem 2.4), Radner's assumption of the strict convexity of the technology and that of the unique von-Neumann ray allows him this inequality,[55] and thereby the claim that for any $\epsilon > 0$, there exists a $\delta > 0$ such that production plans ϵ-away from golden-rule stocks lead to greater than δ-value losses.[56] The rest, as is well-knows is a routine arithmetical verification based on his terminal optimality criterion to complete

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[54] Thus Radner's emphasis on Karlin's proof of the von-Neumann theorem is hardly incidental from this substantive point of view.

[55] Note that it is only the assumption of strict convexity that does not hold in the RSS model; the von-Neumann ray is indeed unique by virtue of the fundamental assumption embodied in 2.1. Note also that in this statement, as well as loose treatment of the basic ideas in this section, we are using the golden-rule pair $(\hat{x}, \hat{p})$ of the RSS model as the relevant analogue of the von-Neumann balanced growth path and the associated shadow prices. The assumption of exogenously given labor supply in each period obviously vitiates the constant returns to scale assumption in the von-Neumann model. It is of course the constant returns to scale assumption that necessitates the particular metric that Radner uses.

[56] In addition to [31], see [10]. Also see the reference to Atsumi in [22] and [25, Lemma 4, p. 252]. A preliminary statement for the two-sector RSS model with strictly concave felicities, but one that ignores the complications of initial and terminal

the proof of his theorem. It is thus hardly incidental that one of the basic footholds in the substantiation of Theorems A and B is Lemma 7.3, nothing if not a restatement of the value-loss lemma in the RSS context, and adjusted for the initial and terminal stocks.

It is intuitively clear that the Radner arithmetic, however effective in ensuring convergence to the von-Neumann facet, is of itself largely irrelevant in guaranteeing convergence to the von-Neumann ray. If there are programs other than the stationary maximal program that give zero-value losses, it seems "intuitively obvious" that one cannot use the Gale–Brock or von-Neumann prices to characterize their convergence properties, and so new methods based on other prices need to be found.[57] In short, the value-loss lemma has to be supplemented by the Inada–McKenzie arguments as to the stability of the facet, and it is this stability result that is embodied in Lemma 7.2. Note, however, that the form of the result is not that any sequence of production plans on the facet converges to the unique golden-rule stock; this would be a straightforward consequence of the fact that any sequence with zero aggregate value-loss is good, and therefore its convergence is simply guaranteed by the Standing Hypothesis.[58] The statement says rather that the only way a (doubly infinite) sequence of plans can stay on the facet, or even a part of the facet, is by being the constant sequence of golden-rule stocks! And it is of interest that this statement is used in the proof of Lemma 7.3, the value-loss lemma discussed above.

The role of the stability lemma in the proofs of Theorems A and B can be usefully contrasted with the use, in the early 1960s, that Morishima and McKenzie make of support prices of the optimal program to prove their turnpike theorems.[59] As is well-known and well-understood, Pontryagin's principle is also prefixed on efficiency prices, and its use in the context of the turnpike theorem by Cass and his followers involves the use of the transversality condition to ensure the convergence of the efficiency prices to the (Gale–Brock) golden-rule prices: what McKenzie sees as Yano's *dual turnpike* theorem.[60] Stiglitz's result on asymptotic stability, and his characterization of optimal policies in the RSS model is also totally based on efficiency

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capital stocks, is available in [15, Footnote 4]. For the non-stationary case, see [21, Lemma 8.1].

[57] In any case, as the two-sector RSS case brings out, the turnpike result does not hold in the case $\xi = 1$, and so it can hardly be proved by any method.

[58] See Footnote 37 above and the accompanying text.

[59] See [29] for the reference to Morishima (1961) and to [25] for that to McKenzie (1963). Malinvaud famous 1953 paper is also of obvious relevance in this connection.

[60] See [23, Concluding paragraph of § 4]. It is of interest that both Samuelson and Koopmans avoid the use of Pontryagin's principle and couch all their arguments in the classical phraseology of the calculus of variations.

prices and their convergence.[61] The methodology is hitched to the fact that the program is optimal or efficient in some sense. The point is that Radner, in maximizing a function of the terminal capital stock, is also subscribing to inter-temporal efficiency, and even though even though efficiency prices may be associated with that program, he does *not* use them. The formalities of his proof limit themselves only to a binary comparison: all that is required is that the utility of the final output of the program be more than that of an hypothetical path constructed relative to the maximally balanced program. Thus, the arguments reported here exploit the Radner method to show the dispensability of McKenzie–Morishima (Cass–Stiglitz) efficiency prices even in the presence of a non-trivial von-Neumann facet.

The proof of Lemma 7.2, the stability lemma, is a consequence of Lemma 7.1, a result that we shall term the "visiting lemma". For any arbitrarily small ϵ, and any degree of approximation m_1, it guarantees a natural number τ, such that all τ-period programs that start within the specified range of initial stocks furnish *average* utility approximately higher than that obtained at the golden-rule stocks, the degree of approximation, governed by (m_1/τ), must be ϵ-close to the turnpike at least in *one* period. Thus approximate optimality dictates that the program must *visit* the turnpike at least once. This result is reminiscent of the Nikaido–Inada [10, 30] strengthening of Radner's result [31] that an optimal program spends consecutive periods in close proximity to the turnpike, and we leave it to future work a precise delineation of its relationship to their arguments.[62]

Lemmas 7.1–7.3, the *visiting, stability* and *value-loss* lemmas, coupled with the first four propositions of § 6, are enough to structure the proof of Theorem A. But before turning to a discussion of the rather technical results of § 6, we consider Lemma 7.4. This is an *aggregate value-loss* lemma: given any ϵ, it finds a degree of approximation γ, such that *each* finite γ-approximately optimal program with γ-regulated ranges of initial and terminal capital stocks, has its aggregate value-losses bounded below by ϵ. Note that this is opposite in spirit to the value-loss lemma, Lemma 7.3, which takes a "small" value-loss as a hypothesis, and asserts a conclusion concerning the distance of plans from golden-rule stocks. At any rate, as Remark 7.1 makes clear, the proof of Lemma 7.4 is the only result out of the four in § 7 that does not utilize the Standing Hypothesis that all good programs converge to the golden-rule stocks.

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[61] See [36]; also [6, 37]. For a supplementation of Stiglitz's arguments by existence theorems, see [12, 41].

[62] To our knowledge, a result of this form is not available in the economic literature though it is well-known to students of optimal control and the variational calculus; see [40].

We now turn to the results of § 6 consisting of six technical propositions. The first of these, Proposition 6.1 can be coupled with Proposition 2.1: whereas the latter guarantees the boundedness of the programs depending on where they start, the former specifies the minimum length of the a finite program if it is to attain a given terminal stock. Figure 3 can be used to understand either statement. Propositions 6.2 and 6.3 both concern the levels of approximation if each finite program is to attain at least an average utility approximately greater than that of the golden-rule consumption: the first pertains to finite programs that with zero terminal stocks, and the second to given initial and terminal stocks.[63] Proposition 6.3 follows as a direct consequence of Propositions 6.1 and 6.2, and intuition for the straightforward proofs of the latter can be had by tracing out the trajectories of the relevant programs in Fig. 3. Proposition 6.4 complements the definition of good programs (Definition 2.4 above) by its assertion that the aggregate value-losses of any program starting from a given range of initial stocks are bounded from above. It is a straightforward consequence of Lemma 2.1 asserting the minimax property of the pair $(\hat{x}, \hat{p})$. Finally, Proposition 6.6, based on the elementary Proposition 6.5 complements Propositions 6.2 and 6.3 by referring to approximate aggregate utility rather than the average, but it translates the relevant property from a finite program to its restrictions on any subintervals. It is to be noted that Propositions 6.5 and 6.6 stand alone from the other four in that their proofs are independent of them.

With a preparatory description of the six propositions and the four lemmata at our disposal, all that remains is a narrative of how they are used to structure the proofs of Theorems A and B. We leave this an exercise for the interested reader. The standing hypothesis in this paper has been the asymptotic stability of good programs, and at the very least, such an exercise will make the reader appreciate all that is required to go beyond asymptotic stability of optimal programs to obtain classical turnpike theorems for approximately optimal large but finite programs.

5. Concluding observations

The very specificity of the RSS model has allowed detailed answers not possible in the more abstract formulations of the general theory of intertemporal resource allocation. At the very same time, it has provided a more concrete understanding of its concepts and the framework in which they are set. And so, prior to a presentation of the technical arguments, in this hinge section, we conclude this more informal part of the paper with two open questions, one specific and the other general, chosen out of a whole range of directions

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[63] The distinction between the two kinds of trajectories was referred to in the introduction and illustrated in Fig. 2.

for extension and study. Both seem within reach and would go a long way towards the completion of the Samuelson–McKenzie turnpike research program in the simpler undiscounted case, a logically antecedent prelude to the undiscounted theory.

The specific question concerns the *early* turnpike theorem presented as Corollary 2b. It is not clear how to extend such a result for a range of initial stocks, initially for a neighborhood of the golden-rule stocks, and then for an arbitrarily large range as is considered in the statements of Theorems A and B. An ideal situation would be one where it could be deduced from the two principal results of this paper, but at this stage, it is not even clear as to the form in which such a result would be true.[64]

The general question concerns the extension of the results to the nonstationary case. From the substantive "choice of technique" point of view, such an extension is overdue. Furthermore, in some strong sense, this setting is the very raison d'etre for turnpike theory, a situation where evolving technologies and tastes render exact solutions, especially to infinite horizon optimization problems, as meaningful only if they are to be rough guides for decisions in the medium- to short-run.[65]

6. Six auxiliary propositions

We now present six technical but rather straightforward results. In their statements, wherever necessary, we suppose that the sum over an empty set is zero.

Proposition 6.1. *Let* $\Gamma \in (0,1)$. *Then there exists a natural number* $k(\Gamma)$ *such that for each* $z_0 \in R^n_+$ *and each* $z_1 \in R^n_+$ *satisfying* $az_1 \le \Gamma d^{-1}$ *there is a program* $(\{x(t)\}_{t=0}^{k(\Gamma)}, \{y(t)\}_{t=0}^{k(\Gamma)-1})$ *such that* $x(0) = z_0$, $x(k(\Gamma)) \ge z_1$.

Proof. Since $\sum_{i=0}^{\infty}(1-d)^i = 1/d$ there is a natural number $k(\Gamma)$ such that

$$\sum_{i=0}^{k(\Gamma)-1} (1-d)^i > \Gamma d^{-1}.$$

Assume that $z_0 \in R^n_+$ and $z_1 \in R^n_+$ satisfies $az_1 \le \Gamma d^{-1}$. Put[66]

$$z_2 = \Gamma^{-1} d z_1,$$

$$x(0) = z_0,\ x(t+1) = (1-d)x(t) + z_2 \text{ and } y(t) = 0 \text{ for all integers } t \ge 0.$$

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[64] The fact that asymptotic stability (the *late* turnpike property) implies other turnpike properties is studied, in continuous time and in a general setting, in [40, Chaps. 3–5].

[65] See Footnote 3 and the accompanying text.

[66] The reader is invited to diagram the program in Fig. 3.

It is easy to see that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program and that

$$x(k(\Gamma)) \geq \sum_{i=0}^{k(\Gamma)-1} (1-d)^i z_2 \geq \Gamma d^{-1} z_2 = z_1.$$

Proposition 6.1 is proved. □

In the sequel with each $\Gamma \in (0, 1)$ we associate a natural number $k(\Gamma)$ for which the assertion of Proposition 6.1 holds.

Proposition 6.2. *There is $m > 0$ such that for each $z \in R_+^n$ and each natural number T*

$$U(z, T) \geq Tw(b\widehat{x}) - m.$$

Proof. Put[67]

(6.1) $y(0) = 0,\ y(t+1) = (1-d)y(t) + d\widehat{x}$ for all integers $t \geq 0$.

It is not difficult to see that the sequence $\{y(t)\}_{t=0}^{\infty}$ is nondecreasing and converges to $\widehat{x}$ as $t \to \infty$. Put

(6.2)

$$m = d^{-1}(b\widehat{x} - by(1))|w'(db\widehat{x})| + |w(by(0)) - w(b\widehat{x})| + |w(by(1)) - w(b\widehat{x})|.$$

Let $z \in R_+^n$. Set

(6.3)
$$x(0) = z,$$
$$x(t+1) = (1-d)x(t) + d\widehat{x} \text{ for all integers } t \geq 0.$$

It is not difficult to see that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. It follows from the definition of the sequence $\{y(t)\}_{t=0}^{\infty}$ that for all natural numbers t

(6.4) $$by(t) \geq db\widehat{x}$$

and that for all integers $t \geq 2$

(6.5) $$by(t) - b\widehat{x} = (1-d)^{t-1}(by(1) - b\widehat{x}).$$

Since the function w is concave and strictly increasing it follows from (6.4) and (6.5) that for all integers $t \geq 2$

$$w(b\widehat{x}) - w(by(t)) \leq w'(by(t))(b\widehat{x} - by(t)) \leq w'(db\widehat{x})(b\widehat{x} - by(t))$$
$$\leq w'(db\widehat{x})(b\widehat{x} - by(1))(1-d)^{t-1}.$$

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[67] The reader is invited to diagram the trajectories (6.1) and (6.3) in Fig. 3.

By the relation above, (6.3) and (6.2) for each natural number T,

$$\begin{aligned}U(z,T) &\geq \sum_{t=0}^{T-1} w(by(t)) = \sum\{w(by(t)):\ t\in\{0,1\} \text{ and } t\leq T-1\}\\ &\quad+\sum\{w(by(t)):\ t \text{ is an integer and } 2\leq t<T\}\\ &\geq Tw(b\widehat{x})-|w(by(0))-w(b\widehat{x})|-|w(by(1))-w(b\widehat{x})|\\ &\quad-\sum\{|w(b\widehat{x})-w(by(t))|:\ t \text{ is an integer and } 2\leq t<T\}\\ &\geq Tw(b\widehat{x})-|w(by(0))-w(b\widehat{x})|-|w(by(1))-w(b\widehat{x})|\\ &\quad-w'(db\widehat{x})(b\widehat{x}-by(1))\sum_{t=2}^{T+1}(1-d)^{t-1}\\ &\geq Tw(b\widehat{x})-|w(by(0))-w(b\widehat{x})|-|w(by(1))-w(b\widehat{x})|\\ &\quad-w'(db\widehat{x})(b\widehat{x}-by(1))d^{-1}\\ &= Tw(b\widehat{x})-m.\end{aligned}$$

Proposition 6.2 is proved. □

Proposition 6.3. *Let* $\Gamma\in(0,1)$. *Then there exists* $m>0$ *such that for each* $z_0\in R^n_+$, *each* $z_1\in R^n_+$ *satisfying* $az_1\leq\Gamma d^{-1}$ *and each natural number* $T>k(\Gamma)$,

$$U(z_0,z_1,0,T)\geq Tw(b\widehat{x})-m.$$

Proof. By Proposition 6.2 there is $m_0>0$ such that

(6.6)
$$U(z,T)\geq Tw(b\widehat{x})-m_0 \text{ for each } z\in R^n_+ \text{ and each natural number } T.$$

Put

$$m=m_0+k(\Gamma)(|w(b\widehat{x})|+|w(0)|).$$

Assume that $z_0\in R^n_+$, $z_1\in R^n_+$ satisfies $az_1\leq\Gamma d^{-1}$ and a natural number $T>k(\Gamma)$. By (6.6) there is a program $(\{x(t)\}_{t=0}^{T-k(\Gamma)},\{y(t)\}_{t=0}^{T-k(\Gamma)-1})$ such that

$$\sum_{t=0}^{T-k(\Gamma)-1} w(by(t))\geq(T-k(\Gamma))w(b\widehat{x})-m_0. \tag{6.7}$$

By the choice of $k(\Gamma)$, Proposition 6.1 and the inequality $az_1\leq\Gamma d^{-1}$ there is a program $(\{x(t)\}_{t=T-k(\Gamma)}^{T},\{y(t)\}_{t=T-k(\Gamma)}^{T-1})$ such that $x(T)\geq z_1$. By (6.6) and (6.7),

$$\begin{aligned}
U(z_0, z_1, 0, T) &\geq \sum_{t=0}^{T-1} w(by(t)) \geq \sum_{t=0}^{T-k(\Gamma)-1} w(by(t)) - k(\Gamma))|w(0)| \\
&\geq (T - k(\Gamma))w(b\widehat{x}) - m_0 - k(\Gamma)|w(0)| \\
&\geq Tw(b\widehat{x}) - k(\Gamma)|w(b\widehat{x})| - m_0 - k(\Gamma)|w(0)| \\
&\geq Tw(b\widehat{x}) - m.
\end{aligned}$$

Proposition 6.3 is proved. □

Proposition 6.4. *Let* $m_0 > 0$*. Then there exists* $m_2 > 0$ *such that for each natural number* T *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies* $x(0) \leq m_0 e$ *the following inequality holds:*

$$\sum_{t=0}^{T-1} [w(by(t)) - w(b\widehat{x})] \leq m_2.$$

Proof. By Proposition 2.1 there exists $m_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies $x(0) \leq m_0 e$ we have

$$x(t) \leq m_1 e \text{ for all } t = 0, \dots, T. \tag{6.8}$$

Choose a number

$$m_2 \geq 2||\widehat{p}||m_1 n. \tag{6.9}$$

Assume that T is a natural number and a program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ satisfies $x(0) \leq m_0 e$. Then (6.8) holds.

By (2.4) and Remark 2.1 for each integer $t \in [0, T-1]$

$$w(b\widehat{y}) \geq w(by(t)) + \widehat{p}(x(t+1)) - \widehat{p}x(t).$$

Together with (6.8) and (6.9) this implies that

$$\begin{aligned}
\sum_{t=0}^{T-1} [w(by(t)) - w(b\widehat{x})] &\leq \sum_{t=0}^{T-1} [\widehat{p}x(t) - \widehat{p}x(t+1)] \\
&= \widehat{p}x(0) - \widehat{p}x(T) \leq 2||\widehat{p}||nm_1 \leq m_2.
\end{aligned}$$

Proposition 6.4 is proved. □

It is easy to see that the following auxiliary result holds.

Proposition 6.5. *Assume that* T_1, T_2 *are nonnegative integers,* $T_1 < T_2$*,*

$$(\{x(t)\}_{t=T_1}^{T_2}, \ \{y(t)\}_{t=T_1}^{T_2-1})$$

is a program and $u \in R^n_+$*. Then* $(\{x(t) + (1-d)^{t-T_1}u\}_{t=T_1}^{T_2}, \ \{y(t)\}_{t=T_1}^{T_2-1})$ *is also a program.*

Proposition 6.6. *Let τ be a natural number, $M > 0$, $x_0, x_1 \in R^n_+$ and let*

$$(\{x(t)\}_{t=0}^{\tau},\ \{y(t)\}_{t=0}^{\tau-1})$$

be a program such that

$$x(0) = x_0,\ x(\tau) \geq x_1,\ \sum_{t=0}^{\tau-1} w(by(t)) \geq U(x_0, x_1, 0, \tau) - M. \tag{6.10}$$

Then for each pair of integers S_1, S_2 satisfying

$$0 \leq S_1 < S_2 \leq \tau \tag{6.11}$$

the following inequality holds:

$$\sum_{t=S_1}^{S_2-1} w(by(t)) \geq U(x(S_1), x(S_2), S_1, S_2) - M.$$

Proof. Let us assume the contrary. Then there exists a pair of integers S_1, S_2 satisfying (6.11) such that

$$\sum_{t=S_1}^{S_2-1} w(by(t)) < U(x(S_1), x(S_2), S_1, S_2) - M. \tag{6.12}$$

By (3.2) and (6.12) there exists a program $(\{x'(t)\}_{t=S_1}^{S_2}, \{y'(t)\}_{t=S_1}^{S_2-1})$ such that

$$\begin{gathered} x'(S_1) = x(S_1),\ x'(S_2) \geq x(S_2), \\ \sum_{t=S_1}^{S_2-1} w(by(t)) < \sum_{t=S_1}^{S_2-1} w(by'(t)) - M. \end{gathered} \tag{6.13}$$

Set

$$z = x'(S_2) - x(S_2). \tag{6.14}$$

Define sequences $\{\bar{x}(t)\}_{t=0}^{\tau},\ \{\bar{y}(t)\}_{t=0}^{\tau-1}$ as follows:

$$\begin{gathered} \bar{x}(t) = x(t), \quad t = 0, \dots, S_1, \quad \bar{y}(t) = y(t), \quad t = 0, \dots, S_1 - 1, \\ \bar{x}(t) = x'(t), \quad t = S_1 + 1, \dots, S_2, \quad \bar{y}(t) = y'(t), \quad t = S_1, \dots, S_2 - 1, \\ \bar{x}(t) = x(t) + (1-d)^{t-S_2} z \text{ for all integers } t \text{ satisfying } S_2 < t \leq \tau, \\ \bar{y}(t) = y(t) \text{ for all integers } t \text{ satisfying } S_2 \leq t \leq \tau - 1. \end{gathered} \tag{6.15}$$

It follows from Proposition 6.5, (6.13), (6.14) and (6.15) that $(\{\bar{x}(t)\}_{t=0}^{\tau}, \{\bar{y}(t)\}_{t=0}^{\tau-1})$ is a program. In view of (6.10), (6.13), (6.14) and (6.15)

$$\bar{x}(0) = x(0) = x_0, \quad \bar{x}(\tau) \geq x(\tau).$$

By (6.13) and (6.15)

$$\sum_{t=0}^{\tau-1} w(b\bar{y}(t)) - \sum_{t=0}^{\tau-1} w(by(t)) = \sum_{t=S_1}^{S_2-1} w(by'(t)) - \sum_{t=S_1}^{S_2-1} w(by(t)) > M.$$

This contradicts (6.10). The contradiction we have reached proves Proposition 6.6. □

7. Four substantive lemmata

In this section, we turn to the four lemmata discussed in § 4: the visiting lemma, the stability lemma, the value-loss lemma and the aggregate value-loss lemma. For the validity of the first three lemmata, we suppose that for each program $\{u(t), v(t)\}_{t=0}^{\infty}$

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}). \tag{7.1}$$

Lemma 7.1. *Let* $m_0, m_1 > 0$, $\epsilon > 0$. *Then there is a natural number* τ *such that for each program* $(\{x(t)\}_{t=0}^{\tau}, \{y(t)\}_{t=0}^{\tau-1})$ *satisfying*

$$x(0) \leq m_0 e, \sum_{t=0}^{\tau-1} w(by(t)) \geq \tau w(b\widehat{x}) - m_1$$

there is an integer $t \in [0, \tau]$ *such that*

$$||y(t) - \widehat{x}||, \; ||x(t) - \widehat{x}|| \leq \epsilon.$$

Proof. Let us assume the contrary. Then for each natural number k there exists a program $(\{x_t^{(k)}\}_{t=0}^{k}, \; \{y_t^{(k)}\}_{t=0}^{k-1})$ such that

$$x^{(k)}(0) \leq m_0 e, \sum_{t=0}^{k-1} w(by^{(k)}(t)) \geq kw(b\widehat{x}) - m_1, \tag{7.2}$$

$$\max\{||y_t^{(k)} - \widehat{x}||, \; ||x_t^{(k)} - \widehat{x}||\} > \epsilon, \; t = 0, \ldots, k. \tag{7.3}$$

In view of (7.2) and Proposition 2.1 there is $m_2 > m_0$ such that for each natural number k

$$x^{(k)}(t) \le m_2 e,\ t = 0, \dots, k. \tag{7.4}$$

By Proposition 6.4 there is $m_3 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) \le m_2 e$ the following inequality holds:

$$\sum_{t=0}^{T-1} (w(by(t)) - w(b\widehat{y})) \le m_3. \tag{7.5}$$

Let k be a natural number and let an integer s satisfy $0 < s < k$.

It follows from (7.4) and the choice of m_3 (see (7.5)) that

$$\sum_{t=s}^{k-1} [w(by^{(k)}(t)) - w(b\widehat{y})] \le m_3.$$

Combined with (7.2) this implies that

$$\sum_{t=0}^{s-1} [w(by^{(k)}(t)) - w(b\widehat{y})] = \sum_{t=0}^{k-1} [w(by^{(k)}(t)) - w(b\widehat{y})] - \sum_{t=s}^{k-1} [w(by^{(k)}(t)) - w(b\widehat{y})] \ge -m_1 - m_3.$$

Thus for each pair of natural numbers s, k satisfying $s < k$

$$\sum_{t=0}^{s-1} [w(by^{(k)}(t)) - w(b\widehat{y})] \ge -m_1 - m_3. \tag{7.6}$$

By extracting subsequence and using (7.4) and diagonalization process we obtain that there exist a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and sequences $\{x^*(t)\}_{t=0}^{\infty}, \{y^*(t)\}_{t=0}^{\infty} \subset R^n$ such that

$$x^{(k_j)}(t) \to x^*(t),\ y^{(k_j)}(t) \to y^*(t) \text{ as } j \to \infty \text{ for all integers } j \ge 0. \tag{7.7}$$

It is not difficult to see that $\{x^*(t),\ y^*(t)\}_{t=0}^{\infty}$ is a program. In view of (7.4) and (7.7)

$$x^*(t) \le m_2 e \text{ for all integers } t \ge 0. \tag{7.8}$$

By (7.7) and (7.6) for all natural numbers s

$$\sum_{t=0}^{s-1} (w(by^*(t)) - sw(b\widehat{x})) \ge -m_3 - m_1.$$

This implies that $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is a good program and by (7.1)

$$y^*(t) \to \widehat{x},\ x^*(t) \to \widehat{x} \text{ as } t \to \infty. \tag{7.9}$$

On the other hand it follows from (7.7) and (7.3) that for all integers $t \geq 0$

$$\max\{||y^*(t) - \widehat{x}||,\ ||x^*(t) - \widehat{x}||\} \geq \epsilon.$$

This contradicts (7.9). The contradiction we have reached proves Lemma 7.1. □

Lemma 7.2. *Let* $S_0 > 0$ *and let a sequence* $\{x(t), y(t)\}_{t=-\infty}^{\infty} \subset R_+^{2n}$ *satisfy the following conditions:*

$$\begin{gathered}
||x(t)||,\ ||y(t)|| \leq S_0 \text{ for all integers } t,\\
x(t+1) \geq (1-d)x(t), \quad 0 \leq y(t) \leq x(t) \text{ for all integers } t,\\
a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1 \text{ for all integers } t,\\
\delta(x(t), y(t), x(t+1)) = 0 \text{ for all integers } t.
\end{gathered}$$

Then $x(t) = y(t) = \widehat{x}$ *for all integers* t.

Proof. Assume that the lemma does not hold. Then we may assume without loss of generality that

$$(x(0), y(0)) \neq (\widehat{x}, \widehat{x}).$$

Set

$$\kappa = \min\{1, \max\{||x(0) - \widehat{x}||,\ ||y(0) - \widehat{x}||\}\}. \tag{7.10}$$

By Proposition 2.4 there exists a strictly increasing sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0, \kappa)$ such that for each natural number k the following property holds:

(P1) If $x, x' \in R_+^n$, $||x - \widehat{x}||, ||x' - \widehat{x}|| \leq 4\delta_k$, then there exist $\bar{x} \geq x'$, $y \in R^n$ such that

$$\begin{aligned}
&(x, \bar{x}) \in \Omega, \quad y \in \Lambda(x, \bar{x}), \quad ||y - \widehat{x}|| \leq 8^{-k}\kappa, \quad ||\bar{x} - \widehat{x}|| \leq 8^{-k}\kappa,\\
&\qquad \delta(x, y, \bar{x}) \leq 8^{-k}\kappa.
\end{aligned}$$

By Lemma 7.1 for each natural number k there exist natural numbers $T_k, S_k > 4$ such that

$$(1-d)^{\min\{T_k, S_k\}} < \min\{\delta_k,\ 8^{-k}\kappa\}, \tag{7.11}$$

$$\begin{aligned}
&||x(T_k) - \widehat{x}||,\ ||y(T_k) - \widehat{x}||,\ ||x(-S_k) - \widehat{x}||,\ ||y(-S_k) - \widehat{x}||\\
&\leq \min\{\delta_k,\ 8^{-k}\kappa\}.
\end{aligned} \tag{7.12}$$

Set

(7.13)
$$\tau_0=0, \quad \tau_1=T_1+S_1, \quad \tau_{k+1}=\tau_k+T_{k+1}+S_{k+1}+1 \text{ for any integer } k \ge 1.$$

By induction, we construct a program $\{u(t), v(t)\}_{t=0}^{\infty}$. Set

(7.14)
$$u(t)=x(t-S_1), \quad t=0,\dots,\tau_1, \quad v(t)=y(t-S_1), \quad t=0\dots,\tau_1-1.$$

Now assume that k is a natural number, we defined $u(t)$, $t=0,\dots,\tau_k$, $v(t)$, $t=0,\dots,\tau_k-1$ such that $(\{u(t)\}_{t=0}^{\tau_k}, \{v(t)\}_{t=0}^{\tau_k-1})$ is a program,

$$||u(\tau_k)-\widehat{x}|| \le 2\min\{\delta_k, 8^{-k}\kappa\}, \tag{7.15}$$

$$\sum_{t=0}^{\tau_k-1} \delta(u(t), v(t), u(t+1)) \le (1+4||\widehat{p}||)\sum_{j=0}^{k-1} 8^{-j}, \tag{7.16}$$

and if $k \ge 2$, then for $j=1,\dots,k-1$

$$\max\{||v(\tau_j+S_{j+1}+1)-\widehat{x}||,\ ||u(\tau_j+S_{j+1}+1)-\widehat{x}||\} \ge 2^{-1}\kappa. \tag{7.17}$$

(It is not difficult to see that for $k=1$ all these assumptions hold.)

Define $u(t)$, $t=\tau_k+1,\dots,\tau_{k+1}$, $v(t)$, $t=\tau_k,\dots,\tau_{k+1}-1$. First define $u(\tau_k+1)$, $v(\tau_k)$. By (7.15), (7.12) and (P1) there exist

$$u(\tau_k+1) \ge x(-S_{k+1}),\ v(\tau_k) \in R^n \tag{7.18}$$

such that

$$(u(\tau_k), u(\tau_k+1)) \in \Omega,\ v(\tau_k) \in \Lambda(u(\tau_k), u(\tau_k+1)), \tag{7.19}$$

$$||u(\tau_k+1)-\widehat{x}|| \le 8^{-k}\kappa,\ ||v(\tau_k)-\widehat{x}|| \le 8^{-k}\kappa, \tag{7.20}$$

$$\delta(u(\tau_k), v(\tau_k), u(\tau_k+1)) \le 8^{-k}\kappa. \tag{7.21}$$

For $t=\tau_k+1,\dots,\tau_{k+1}-1$ set

$$v(t)=y(t-\tau_k-S_{k+1}-1) \tag{7.22}$$

and for $t=\tau_k+2,\dots,\tau_{k+1}$ set

$$u(t)=x(t-\tau_k-S_{k+1}-1)+(1-d)^{t-\tau_k-1}[u(\tau_k+1)-x(-S_{k+1})]. \tag{7.23}$$

By Proposition 6.5, (7.22), (7.23), (7.18) and (7.19) $(\{u(t)\}_{t=0}^{\tau_{k+1}}, \{v(t)\}_{t=0}^{\tau_{k+1}-1})$ is a program.

By (7.23), (7.13), (7.12), (7.20), (7.11) and (7.10),

$$\begin{aligned}||u(\tau_{k+1})-\widehat{x}|| &= ||x(T_{k+1})+(1-d)^{T_{k+1}+S_{k+1}}[u(\tau_k+1)\\ &\quad -x(-S_{k+1})]-\widehat{x}||\\ &\le ||x(T_{k+1})-\widehat{x}||+(1-d)^{T_{k+1}+S_{k+1}}(||u(\tau_k+1)\\ &\quad -\widehat{x}||+||\widehat{x}-x(-S_{k+1})||)\\ &\le \min\{\delta_{k+1},\ 8^{-k-1}\kappa\}+(1-d)^{T_{k+1}+S_{k+1}}(8^{-k}\kappa+8^{-k}\kappa)\\ &\le 2\min\{\delta_{k+1},8^{-k-1}\kappa\}\end{aligned}$$

and

$$||u(\tau_{k+1})-\widehat{x}||\le 2\min\{\delta_{k+1},\ 8^{-k-1}\kappa\}. \tag{7.24}$$

By (7.23), (7.22), (7.20), (7.12) and (7.11),

$$v(\tau_k+1+S_{k+1})=y(0),$$

$$\begin{aligned}||u(\tau_k+1+S_{k+1})-x(0)|| &= (1-d)^{S_{k+1}}||u(\tau_k+1)-x(-S_{k+1})||\\ &\le (1-d)^{S_{k+1}}(||u(\tau_k+1)-\widehat{x}||+||\widehat{x}-x(-S_{k+1})||)\\ &\le (1-d)^{S_{k+1}}(3\cdot 8^{-k}\kappa)\le 8^{-1}\kappa. \qquad (7.25)\end{aligned}$$

By (7.25)

$$||v(\tau_k+1+S_{k+1})-\widehat{x}||=||y(0)-\widehat{x}||,$$

$$\begin{aligned}||u(\tau_k+1+S_{k+1})-\widehat{x}|| &\ge ||\widehat{x}-x(0)||-||x(0)-u(\tau_k+1+S_{k+1})||\\ &\ge ||\widehat{x}-x(0)||-8^{-1}\kappa.\end{aligned}$$

Together with (7.10) this implies that

$$\begin{aligned}&\max\{||v(\tau_k+1+S_{k+1})-\widehat{x}||,\ ||u(\tau_k+1+S_{k+1})-\widehat{x}||\}\\ &\quad \ge \max\{||y(0)-\widehat{x}||-8^{-1}\kappa,\ ||\widehat{x}-x(0)||-8^{-1}\kappa\}\ge \kappa/2. \qquad (7.26)\end{aligned}$$

By (7.16) and (7.21)

$$\begin{aligned}&\sum_{t=0}^{\tau_{k+1}-1}\delta(u(t),v(t),u(t+1))\\ &\quad =\sum_{t=0}^{\tau_k-1}\delta(u(t),v(t),u(t+1))+\delta(u(\tau_k),v(\tau_k),u(\tau_k+1))\\ &\qquad +\sum_{t=\tau_k+1}^{\tau_{k+1}-1}\delta(u(t),v(t),u(t+1))\\ &\quad \le (1+4||\widehat{p}||)\sum_{j=0}^{k-1}8^{-j}+8^{-k}\kappa+\sum_{t=\tau_k+1}^{\tau_{k+1}-1}\delta(u(t),v(t),u(t+1)). \qquad (7.27)\end{aligned}$$

By (2.4), (7.22), (7.23), the equality $\delta(x(t), y(t), x(t+1)) = 0$ which holds for all integers t, (7.12), (7.20), (7.18) and (7.24),

$$\sum_{t=\tau_k+1}^{\tau_{k+1}-1} \delta(u(t), v(t), u(t+1)) = \sum_{t=\tau_k+1}^{\tau_{k+1}-1} [w(b\widehat{x}) - w(bv(t)) + \widehat{p}(u(t) - u(t+1))]$$

$$= \sum_{t=\tau_k+1}^{\tau_{k+1}-1} [w(b\widehat{x}) - w(by(t - \tau_k - S_{k+1} - 1))] + \widehat{p}(u(\tau_k + 1) - u(\tau_{k+1}))$$
$$= \sum_{t=-S_{k+1}}^{T_{k+1}-1} [w(b\widehat{x}) - w(by(t)) + \widehat{p}(x(t) - x(t+1))]$$
$$-\widehat{p}(x(-S_{k+1}) - x(T_{k+1})) + \widehat{p}(u(\tau_k + 1) - u(\tau_{k+1}))$$
$$= -\widehat{p}(x(-S_{k+1}) - x(T_{k+1})) + \widehat{p}(u(\tau_k + 1) - u(\tau_{k+1}))$$
$$\leq ||\widehat{p}||2 \cdot 8^{-k-1}\kappa + ||\widehat{p}||(2 \cdot 8^{-k-1}\kappa + 8^{-k-1}\kappa) \leq |\widehat{p}||8^{-k}\kappa.$$

Together with (7.27) this implies that

$$\sum_{t=0}^{\tau_{k+1}-1} \delta(u(t), v(t), u(t+1)) \leq (1 + 4||\widehat{p}||) \sum_{j=0}^{k} 8^{-j}. \tag{7.28}$$

It follows from (7.24), (7.28) and (7.26) that in such a manner we constructed by induction a program $\{(u(t), v(t)\}_{t=0}^{\infty}$ such that for each integer $k \geq 1$, (7.15) and (7.16) hold and for each integer $j \geq 1$ (7.17) holds.

By Proposition 2.2(iii) and (7.16) $\{u(t), v(t)\}_{t=0}^{\infty}$ is a good program. In view of (7.1)

$$u(t) \to \widehat{x},\ v(t) \to \widehat{x} \text{ as } t \to \infty.$$

This contradicts (7.17) which holds for any integer $j \geq 1$. The contradiction we have reached proves Lemma 7.2. □

Lemma 7.3. *Let $\epsilon > 0$. Then there exists $\gamma > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ which satisfies*

$$||x(0) - \widehat{x}||,\ ||x(T) - \widehat{x}|| \leq \gamma,$$
$$\delta(x(t), y(t), x(t+1)) \leq \gamma, \quad t = 0, 1, \ldots, T-1$$

the following inequality holds:

$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \leq \epsilon, \quad t = 0, \ldots, T-1. \tag{7.29}$$

Proof. Let $\{\gamma_k\}_{k=1}^{\infty}$ be a strictly decreasing sequence of positive numbers such that

$$\gamma_1 < 8^{-1}\epsilon, \quad \lim_{k\to\infty} \gamma_k = 0. \tag{7.30}$$

Assume that the lemma does not hold. Then for each number k there exists a natural number T_k, a program $(\{x^{(k)}(t)\}_{t=0}^{T_k}, \{y^{(k)}(t)\}_{t=0}^{T_k-1})$ such that

$$||x^{(k)}(0) - \widehat{x}|| \le \gamma_k, \quad ||x^{(k)}(T_k) - \widehat{x}|| \le \gamma_k, \tag{7.31}$$

$$\delta(x^{(k)}(t), y^{(k)}(t), x^{(k)}(t+1)) \le \gamma_k, \quad t = 0, \dots, T_k - 1 \tag{7.32}$$

and there is a natural number

$$S_k \in [0, T_k - 1] \tag{7.33}$$

such that

$$\max\{||x^{(k)}(S_k) - \widehat{x}||, \ ||y^{(k)}(S_k) - \widehat{x}||\} > \epsilon. \tag{7.34}$$

By (7.31) and Proposition 2.1 there exists $M > 0$ such that for all natural numbers k

$$x^{(k)}(t) \le Me \text{ for all } t = 0, \dots, T_k. \tag{7.35}$$

Extracting a subsequence and re-indexing we may assume that one of the following cases holds:

(1) $\sup\{T_k : k = 1, 2, \dots\} < \infty$.
(2) $\sup\{S_k : k = 1, 2, \dots\} < \infty$, $T_k - S_k \to \infty$ as $k \to \infty$.
(3) $S_k \to \infty$ as $k \to \infty$, $\sup\{T_k - S_k : k = 1, 2, \dots\} < \infty$.
(4) $S_k \to \infty$, $T_k - S_k \to \infty$ as $k \to \infty$.

Assume that the case (1) holds. Extracting a subsequence and re-indexing we may assume without loss of generality that

$$T_k = T_1, \ S_k = S_1 \text{ for all natural numbers } k,$$

for each integer $t \in [0, T_1]$ there exists $x(t) = \lim_{k\to\infty} x^{(k)}(t)$ and that for each $t \in [0, T_1 - 1]$ there is $y(t) = \lim_{k\to\infty} y^{(k)}(t)$. It is not difficult to see that $(\{x(t)\}_{t=0}^{T_1}, \{y(t)\}_{t=0}^{T_1-1})$ is a program.

In view of (7.32), (7.30), (7.31) and (7.34)

$$\begin{gathered} \delta(x(t), y(t), x(t+1)) = 0, \quad t = 0, 1, \dots, T_1 - 1, \\ x(T_1) = x(0) = \widehat{x}, \\ \max\{||x(S_1) - \widehat{x}||, \ ||y(S_1) - \widehat{x}||\} \ge \epsilon. \end{gathered} \tag{7.36}$$

Set

$$x(t) = \widehat{x} \text{ for all integers } t \in (-\infty, 0) \cup (T_1, \infty),$$
$$y(t) = \widehat{x} \text{ for all integers } t \in (-\infty, 0) \cup (T_1, \infty).$$

Clearly $\{x(t), y(t)\}_{t=-\infty}^{\infty}$ satisfies the assumptions of Lemma 7.2 which in its turn implies that $x(t) = y(t) = \widehat{x}$ for all integers t. This contradicts (7.36). The contradiction we have reached proves that the case (1) does not hold.

Assume that the case (2) holds. Extracting a subsequence and re-indexing we may assume without loss of generality that

$$S_k = S_1 \text{ for all natural numbers } k \tag{7.37}$$

and that for each integer $t \geq 0$ there exist

$$x(t) = \lim_{k\to\infty} x^{(k)}(t), \quad y(t) = \lim_{k\to\infty} y^{(k)}(t). \tag{7.38}$$

It is clear that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. By (7.38), (7.35), (7.31), (7.37), (7.34) and (7.32)

$$\begin{gathered} x(t) \leq Me \text{ for all integers } t \geq 0, \\ \delta(x(t), y(t), x(t+1)) = 0 \text{ for all integers } t \geq 0, \\ x(0) = \widehat{x}, \\ \max\{||x(S_1) - \widehat{x}||,\ ||y(S_1) - \widehat{x}||\} \geq \epsilon. \end{gathered} \tag{7.39}$$

Set

$$x(t) = \widehat{x} \text{ for all integers } t < 0, \quad y(t) = \widehat{x} \text{ for all integers } t < 0.$$

Clearly $\{x(t), y(t)\}_{t=-\infty}^{\infty}$ satisfies the assumptions of Lemma 7.2 which in its turn implies that $x(t) = y(t) = \widehat{x}$ for all integers t. This contradicts (7.39). The contradiction we have reached proves that the case (2) does not hold.

Assume that the case (3) holds. Extracting a subsequence and re-indexing we may assume that

$$T_k - S_k = T_1 - S_1 \text{ for all natural numbers } k. \tag{7.40}$$

For each natural number k set

$$\begin{gathered} \tilde{x}^{(k)}(t) = x^{(k)}(t + T_k),\ t = -T_k. \dots, 0, \\ \tilde{y}^{(k)}(t) = y^{(k)}(t + T_k),\ t = -T_k, \dots, -1. \end{gathered} \tag{7.41}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that there exist

$$x(t) = \lim_{k\to\infty} \tilde{x}^{(k)}(t) \text{ for each integer } t \le 0 \text{ and}$$
$$y(t) = \lim_{k\to\infty} \tilde{y}^{(k)}(t) \text{ for each integer } t < 0. \tag{7.42}$$

By (7.42), (7.35), (7.30), (7.31), (7.32), (7.40), (7.41) and (7.34),

$$\begin{gathered} x(t) \le Me \text{ for al integers } t \le 0, \\ \delta(x(t), y(t), x(t+1)) = 0 \text{ for all integers } t < 0, \\ x(0) = \widehat{x}, \\ \max\{||x(S_1 - T_1) - \widehat{x}||,\ ||y(S_1 - T_1) - \widehat{x}||\} \ge \epsilon. \end{gathered} \tag{7.43}$$

Set

$$\begin{gathered} x(t) = \widehat{x} \text{ for all integers } t > 0, \\ y(t) = \widehat{x} \text{ for all integers } t \ge 0. \end{gathered}$$

Clearly $\{x(t), y(t)\}_{t=-\infty}^{\infty}$ satisfies the assumptions of Lemma 7.2 which in its turn implies that $x(t) = y(t) = \widehat{x}$ for all integers t. This contradicts (7.43). The contradiction we have reached proves that the case (3) does not hold.

Assume that the case (4) holds. For each natural number k define sequences $\{\tilde{x}^{(k)}(t)\}_{t=-S_k}^{T_k-S_k}$, $\{\tilde{y}^{(k)}(t)\}_{t=-S_k}^{T_k-S_k-1}$ by

$$\begin{gathered} \tilde{x}^{(k)}(t) = x^{(k)}(t + S_k), \quad t = -S_k, \dots., T_k - S_k, \\ \tilde{y}^{(k)}(t) = y^{(k)}(t + S_k), \quad t = -S_k, \dots, T_k - S_k - 1. \end{gathered} \tag{7.44}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that for each integer t there exists

$$x_*(t) = \lim_{k\to\infty} \tilde{x}^{(k)}(t), \quad y_*(t) = \lim_{k\to\infty} \tilde{y}^{(k)}(t). \tag{7.45}$$

By (7.45), (7.44), (7.35), (7.32), (7.30) and (7.34) for each integer t

$$\begin{gathered} x_*(t) \le Me, \\ \delta(x_*(t), y_*(t), x_*(t+1)) = 0, \\ \max\{||x_*(0) - \widehat{x}||,\ ||y_*(0) - \widehat{x}||\} \ge \epsilon. \end{gathered} \tag{7.46}$$

Clearly $\{x_*(t), y_*(t)\}_{t=-\infty}^{\infty}$ satisfies the assumptions of Lemma 7.2 which in its turn implies that $x_*(t) = y_*(t) = \widehat{x}$ for all integers t. This contradicts

(7.46). The contradiction we have reached proves that the case (4) does not hold.

Therefore in the all four cases we have reached a contradiction which proves that Lemma 7.3 holds. □

Lemma 7.4. *Let* $\epsilon > 0$. *Then there exists* $\gamma > 0$ *such that for each natural number* T *and each program* $(\{x(t)\}_{t=0}^{T}, \ \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$||x(0) - \widehat{x}|| \le \gamma, \ ||x(T) - \widehat{x}|| \le \gamma, \tag{7.47}$$

$$\sum_{t=0}^{T-1} w(by(t)) \ge U(x(0), x(T), 0, T) - \gamma \tag{7.48}$$

the following inequality holds:

$$\sum_{t=0}^{T-1} \delta(x(t), y(t), x(t+1)) \le \epsilon.$$

Proof. Choose a positive number δ_0 such that

$$2(||\widehat{p}|| + 1)\delta_0 < \epsilon/8, \tag{7.49}$$
$$|w(b\widehat{x}) - w(bz)| < \epsilon/8 \text{ for any } z \in R^n_+ \text{ satisfying } ||z - \widehat{x}|| \le \delta_0.$$

By Proposition 2.4 there is $\gamma \in (0, 1)$ such that $\gamma < \delta_0$ and the following property holds:

(P2) If $x, x' \in R^n_+$ satisfy $||x - \widehat{x}||, \ ||x' - \widehat{x}|| \le \gamma$, then there exist $\bar{x} \ge x'$, $y \in R^n$ such that

$$(x, \bar{x}) \in \Omega, \quad y \in \Lambda(x, \bar{x}), \quad ||y - \widehat{x}|| \le \delta_0, \quad ||\bar{x} - \widehat{x}|| \le \delta_0.$$

Let T be a natural number and let a program $(\{x(t)\}_{t=0}^{T}, \ \{y(t)\}_{t=0}^{T-1})$ satisfy (7.47) and (7.48). We construct a program $(\{\tilde{x}(t)\}_{t=0}^{T}, \ \{\tilde{y}(t)\}_{t=0}^{T-1})$. Set

$$\tilde{x}(0) = x(0). \tag{7.50}$$

There are two cases: $T = 1$ and $T \ge 2$.

Consider the case with $T = 1$. By (7.47) and (P2), there exist $\tilde{x}(1)$ and $\tilde{y}(0)$ such that

$$\tilde{x}(1) \ge x(1), \ \tilde{y}(0) \in R^n, \tag{7.51}$$
$$(\tilde{x}(0), \tilde{x}(1)) \in \Omega, \ \tilde{y}(0) \in \Lambda(\tilde{x}(0), \tilde{x}(1)), \ ||\tilde{y}(0) - \widehat{x}|| \le \delta_0, \ ||\tilde{x}(1) - \widehat{x}|| \le \delta_0.$$

By (2.4), (7.48), (7.50), (7.51) and (7.47)

$$\begin{aligned}\delta(x(0), y(0), x(1)) &= w(b\widehat{x}) - w(by(0)) + \widehat{p}(x(0) - x(1))\\ &\le w(b\widehat{x}) - w(b\tilde{y}(0)) + \widehat{p}(x(0) - x(1)) + \gamma\\ &\le w(b\widehat{x}) - w(b\tilde{y}(0)) + \|\widehat{p}\|2\gamma + \gamma < \delta_0(2\|\widehat{p}\| + 1) < \epsilon.\end{aligned}$$

Thus in the case $T = 1$ the lemma holds.

Assume that $T > 1$. By (7.47), (7.50) and (P2) there exist $\tilde{x}(1)$, $\tilde{y}(0)$ such that

$$\begin{gathered}\tilde{x}(1) \ge \widehat{x}, \quad \tilde{y}(0) \in R^n,\\ (\tilde{x}(0), \tilde{x}(1)) \in \Omega, \quad \tilde{y}(0) \in \Lambda(\tilde{x}(0), \tilde{x}(1)), \quad \|\tilde{y}(0) - \widehat{x}\| \le \delta_0,\\ \|\tilde{x}(1) - \widehat{x}\| \le \delta_0.\end{gathered} \tag{7.52}$$

If an integer t satisfies $1 < t \le T - 1$, then we put

$$\tilde{y}(t-1) = \widehat{x}, \quad \tilde{x}(t) = \widehat{x} + (1-d)^{t-1}(\tilde{x}(1) - \widehat{x}). \tag{7.53}$$

It is clear that $(\{\tilde{x}(t)\}_{t=0}^{T-1}, \{\tilde{y}(t)\}_{t=0}^{T-2})$ is a program.

By (7.47) and (P2) there exist $z \ge x(T)$, $\tilde{y}(T-1) \in R^n_+$ such that

$$\begin{gathered}(\widehat{x}, z) \in \Omega, \quad \tilde{y}(T-1) \in \Lambda(\widehat{x}, z),\\ \|\tilde{y}(T-1) - \widehat{x}\| \le \delta_0, \quad \|z - \widehat{x}\| \le \delta_0.\end{gathered} \tag{7.54}$$

Set

$$\tilde{x}(T) = z + (1-d)^{T-1}[\tilde{x}(1) - \widehat{x}]. \tag{7.55}$$

By (7.54), (7.55), (7.52) and (7.53) $(\{\tilde{x}(t)\}_{t=0}^{T}, \ \{\tilde{y}(t)\}_{t=0}^{T-1})$ is a program. By (7.50), (7.52), (7.55) and (7.54)

$$\tilde{x}(0) = x(0), \quad \tilde{x}(T) \ge x(T). \tag{7.56}$$

In view of (7.48) and (7.56)

$$\sum_{t=0}^{T-1} w(by(t)) \ge \sum_{t=0}^{T-1} w(b\tilde{y}(t)) - \gamma. \tag{7.57}$$

By (2.4), (7.57), (7.53), (7.47), (7.54), (7.49) and (7.52)

$$\begin{aligned}\sum_{t=0}^{T-1} \delta(x(t), y(t), x(t+1)) &= \sum_{t=0}^{T-1}[-w(by(t)) + w(b\widehat{x})] + \widehat{p}(x(0) - x(T))\\ &\le \sum_{t=0}^{T-1}[w(b\widehat{x}) - w(b\tilde{y}(t))] + \gamma + \|\widehat{p}\|(\|x(0) - \widehat{x}\| + \|x(T) - \widehat{x}\|)\\ &\le |w(b\widehat{x}) - w(b\tilde{y}(0))| + |w(b\widehat{x}) - w(b\tilde{y}(T-1))| + 2\|\widehat{p}\|\gamma\\ &\le \epsilon/4 + 2\|\widehat{p}\|\delta_0 < \epsilon.\end{aligned}$$

Lemma 7.4 is proved. □

Remark 7.1. As emphasized above, Lemma 7.4 does not rely on the Standing Hypothesis for its validity, and the proof offered above substantiates this.

8. Proofs: proof of Theorem A

In this section, we complete the proof of Theorem A. We shall feel free to draw on all the results of § 6 and 7. Note, however, that we have no need for Lemma 7.4 that bounds the sum of the value-losses of the entire program.

By Proposition 2.1 there is $M_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) \le Me$ the following inequality holds:

$$x(t) \le M_1 e \text{ for all integers } t \in [0, T]. \tag{8.1}$$

By Proposition 6.3 there exists $M_2 > 0$ such that for each $z_0 \in R^n_+$, each $z_1 \in R^n_+$ satisfying $az_1 \le \Gamma d^{-1}$ and each natural number $T > k(\Gamma)$,

$$U(z_0, z_1, 0, T) \ge Tw(b\widehat{x}) - M_2. \tag{8.2}$$

By Lemma 7.3 there is $\epsilon_1 \in (0, \epsilon)$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\quad \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$||x(0) - \widehat{x}|| \le \epsilon_1, \quad ||x(T) - \widehat{x}|| \le \epsilon_1, \tag{8.3}$$

$$\delta(x(t), y(t), x(t+1)) \le \epsilon_1, \quad t = 0, \dots, T-1 \tag{8.4}$$

the following inequality holds:

$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \le \epsilon, \quad t - 0, \dots, T-1. \tag{8.5}$$

By Lemma 7.1 there exists a natural number L_0 such that for each program

$$(\{x(t)\}_{t=0}^{L_0}, \quad \{y(t)\}_{t=0}^{L_0-1})$$

satisfying

$$x(0) \le M_1 e, \tag{8.6}$$

$$\sum_{t=0}^{L_0-1} w(by(t)) \ge L_0 w(b\widehat{y}) - M_2 - M - 4||\widehat{p}||nM_1 \tag{8.7}$$

there is an integer $t \in [0, L_0]$ such that

$$||y(t) - \widehat{x}||,\ ||x(t) - \widehat{x}|| \le \epsilon_1. \tag{8.8}$$

Choose a natural number

$$L > 8L_0 + 4k(\Gamma) + (2L_0 + 1)[2 + \epsilon_1^{-1}(M + M_2 + 2M_1 n||\widehat{p}||)]. \tag{8.9}$$

Assume that an integer $T > 2L$,

$$z_0, \quad z_1 \in R^n_+, \quad z_0 \le Me, \quad az_1 \le \Gamma d^{-1} \tag{8.10}$$

and that a program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ satisfies

$$x(0) = z_0, \quad x(T) \ge z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \ge U(z_0, z_1, 0, T) - M. \tag{8.11}$$

In view of (8.10) and (8.11) the relation (8.1) holds. By (8.10), (8.11), (8.2) and the inequality $T > L > k(\Gamma)$,

$$\sum_{t=0}^{T-1} w(by(t)) \ge U(z_0, z_1, 0, T) - M \ge Tw(b\widehat{x}) - M_2 - M. \tag{8.12}$$

It follows from (2.4), (8.12) and (8.1) that

$$\begin{aligned} \sum_{t=0}^{T-1} \delta(x(t), y(t), x(t+1)) &= \sum_{t=0}^{T-1}[w(b\widehat{x}) - w(by(t))] + \widehat{p}(x(0) - x(T)) \\ &\le M + M_2 + 2M_1 n||\widehat{p}||. \end{aligned} \tag{8.13}$$

It follows from (2.4), (8.1) and (8.13) that for each pair of integers S_1, S_2 satisfying $0 \le S_1 < S_2 \le T$

$$\begin{aligned} |\sum_{t=S_1}^{S_2-1}[w(by(t)) - w(b\widehat{x})]| &\le \sum_{t=S_1}^{S_2-1} \delta(x(t), y(t), x(t+1)) + 2nM_1||\widehat{p}|| \\ &\le M + M_2 + 4nM_1||\widehat{p}||. \end{aligned} \tag{8.14}$$

It follows from the choice of L_0 (see (8.6)–(8.8)), (8.1) and (8.14) that the following property holds:

(P3) For each integer S satisfying $0 \le S \le T - L_0$ there is an integer $t \in [S, S + L_0]$ such that

$$||y(t) - \widehat{x}||,\ ||x(t) - \widehat{x}|| \le \epsilon_1.$$

Set $t_0 = 0$. By (P3), (8.9) and the relation $T > L$ there is an integer $t_1 \in [L_0, 2L_0]$ such that

$$||y(t_1) - \widehat{x}||,\ ||x(t_1) - \widehat{x}|| \le \epsilon_1.$$

Assume that an integer $j \ge 1$ and we defined integers $t_i,\ i = 0, \dots, j$ such that $t_j \le T$,

$$t_{i+1} \in [t_i + L_0, t_i + 2L_0]$$

for $i = 0, \dots, j - 1$ and that

$$||y(t_i) - \widehat{x}||,\ ||x(t_i) - \widehat{x}|| \le \epsilon_1 \tag{8.15}$$

for $i = 1, \dots, j$. If $t_j = T$, then the construction of the sequence is completed. Consider the case $t_j < T$. If $t_j + 2L_0 > T$, then put $t_{j+1} = T$ and complete the construction of the sequence.

If $t_j + 2L_0 \le T$, then by (P3) there is an integer $t_{j+1} \in [t_j + L_0, t_j + 2L_0]$ such that

$$||x(t_{j+1}) - \widehat{x}||,\ ||y(t_{j+1}) - \widehat{x}|| \le \epsilon_1.$$

In such a way we construct by induction a finite sequence of nonnegative integers $\{t_i\}_{i=0}^{k}$ such that $t_0 = 0$, $t_k = T$, $1 \le t_{i+1} - t_i \le 2L_0$ for $i = 0, \dots, k - 1$ and that for $i = 1, \dots, k - 1$ the inequalities (8.15) hold.

Assume that an integer i satisfies $1 \le i \le k - 2$ and that

$$\sum_{j=t_i}^{t_{i+1}-1} \delta(x(t), y(t), x(t+1)) \le \epsilon_1.$$

Together with (8.15) and the choice of ϵ_1 (see (8.3)–(8.5)) this implies that

$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \le \epsilon,\ t = t_i, \dots, t_{i+1}. \tag{8.16}$$

By (8.16)

$$\begin{aligned} &\{t \in \{0, \dots, T - 1\} : \max\{||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}||\} > \epsilon\} \\ &\subset \{0, \dots, t_1\} \cup \{t_{k-1}, \dots, T\} \cup \{\{t_i, \dots, t_{i+1}\} : \\ &i \in \{1, \dots, k - 2\} \text{ and } \sum_{j=t_i}^{t_{i+1}-1} \delta(x(t), y(t), x(t+1)) > \epsilon_1\}. \end{aligned}$$

This implies that

$$\begin{aligned} &\mathrm{Card}\{t \in \{0, \dots, T - 1\} : \max\{||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}||\} > \epsilon\} \\ &\le 2(2L_0 + 1) + (2L_0 + 1)\mathrm{Card}\{i \in \{1, \dots, k - 2\} : \\ &\sum_{j=t_i}^{t_{i+1}-1} \delta(x(t), y(t), x(t+1)) > \epsilon_1\}. \end{aligned} \tag{8.17}$$

By (8.13)

$$\begin{aligned}\mathrm{Card}\{i \in \{1,\dots,k-2\} : \sum_{j=t_i}^{t_{i+1}-1} \delta(x(t),y(t),x(t+1)) > \epsilon_1\} \\ \le \epsilon_1^{-1}\sum_{t=0}^{T-1}\delta(x(t),y(t),x(t+1)) \le \epsilon_1^{-1}(M+M_2+2M_1n\|\widehat{p}\|).\end{aligned}$$

Together with (8.9) and (8.17), this implies that

$$\begin{aligned}&\mathrm{Card}\{t \in \{0,\dots,T-1\} : \max\{\|x(t)-\widehat{x}\|,\ \|y(t)-\widehat{x}\|\} > \epsilon\} \\ &\quad\le (2L_0+1)[2+\epsilon_1^{-1}(M+M_2+2M_1n\|\widehat{p}\|)] < L.\end{aligned}$$

This completes the proof of Theorem A. □

9. Proofs: proof of Theorem B

In this section, we complete the proof of Theorem B.

By Proposition 2.1 there is $M_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) \le Me$ the following inequality holds:

(9.1) $$x(t) \le M_1 e \text{ for all integers } t \in [0,T].$$

By Proposition 6.3 there exists $M_2 > 0$ such that for each $z_0 \in R_+^n$, each $z_1 \in R_+^n$ satisfying $az_1 \le \Gamma d^{-1}$ and each natural number $T > k(\Gamma)$,

(9.2) $$U(z_0,z_1,0,T) \ge Tw(b\widehat{x}) - M_2.$$

By Proposition 6.4 there exists $M_3 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) \le M_1 e$ the following inequality holds:

(9.3) $$\sum_{t=0}^{T-1}[w(by(t)) - w(b\widehat{y})] \le M_3.$$

By Lemma 7.3 there is $\epsilon_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1})$ which satisfies

(9.4) $$\|x(0)-\widehat{x}\| \le \epsilon_1, \quad \|x(T)-\widehat{x}\| \le \epsilon_1,$$

(9.5) $$\delta(x(t),y(t),x(t+1)) \le \epsilon_1, \quad t = 0,\dots,T-1$$

the following inequality holds:

$$||x(t) - \widehat{x}||,\ ||y(t) - \widehat{x}|| \le \epsilon, \quad t = 0, \dots, T - 1. \tag{9.6}$$

By Lemma 7.4 there exists

$$\gamma \in (0, \min\{1, \epsilon, \epsilon_1\}) \tag{9.7}$$

such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$||x(0) - \widehat{x}|| \le \gamma, \quad ||x(T) - \widehat{x}|| \le \gamma, \\ \sum_{t=0}^{T-1} w(by(t)) \ge U(x(0), x(T), 0, T) - \gamma \tag{9.8}$$

the following inequality holds:

$$\sum_{t=0}^{T-1} \delta(x(t), y(t), x(t+1)) \le \epsilon_1. \tag{9.9}$$

By Lemma 7.1 there exists a natural number L_0 such that for each program

$$(\{x(t)\}_{t=0}^{L_0},\ \{y(t)\}_{t=0}^{L_0-1})$$

satisfying

$$x(0) \le M_1 e, \quad \sum_{t=0}^{L_0-1} w(by(t)) \ge L_0 w(b\widehat{y}) - M_2 - M_3 - 1 \tag{9.10}$$

there is an integer $t \in [0, L_0]$ such that

$$||y(t) - \widehat{x}||,\ ||x(t) - \widehat{x}|| \le \gamma. \tag{9.11}$$

Put

$$L = L_0 + k(\Gamma). \tag{9.12}$$

Assume that an integer $T > 2L$,

$$z_0, \quad z_1 \in R_+^n, \quad z_0 \le Me, \quad az_1 \le \Gamma d^{-1} \tag{9.13}$$

and that a program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ satisfies (3.3).

In view of (3.3) and (9.13), the relation (9.1) holds. By (3.3), (9.2), (9.7), (9.12) and (9.13),

$$\sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - \gamma \geq T w(b\widehat{x}) - M_2 - 1. \tag{9.14}$$

It follows from the choice of M_3 (see (9.3)) and (9.1) that

$$\sum_{t=L_0}^{T-1} [w(by(t)) - w(b\widehat{x})] \leq M_3, \qquad \sum_{t=0}^{T-L_0-1} [w(by(t)) - w(b\widehat{x})] \leq M_3. \tag{9.15}$$

By (9.14) and (9.15),

$$\begin{aligned} \sum_{t=0}^{L_0-1} [w(by(t)) - w(b\widehat{x})] &= \sum_{t=0}^{T-1} [w(by(t)) - w(b\widehat{x})] - \sum_{t=L_0}^{T-1} [w(by(t)) - w(b\widehat{x})] \\ &\geq -M_2 - 1 - M_3, \end{aligned} \tag{9.16}$$

$$\begin{aligned} \sum_{t=T-L_0}^{T-1} [w(by(t)) - w(b\widehat{x})] &= \sum_{t=0}^{T-1} [w(by(t)) - w(b\widehat{x})] \\ &- \sum_{t=0}^{T-L_0-1} [w(by(t)) - w(b\widehat{x})] \geq -M_2 - 1 - M_3. \end{aligned} \tag{9.17}$$

It follows from (9.1), (9.16), (9.17), and the choice of L_0, that there exist integers $\tau_1 \in [0, L_0]$, $\tau_2 \in [T - L_0, T]$ such that

$$||x(\tau_i) - \widehat{x}||, \; ||y(\tau_i) - \widehat{x}|| \leq \gamma, \quad i = 1, 2. \tag{9.18}$$

(If $||x(0) - \widehat{x}|| \leq \gamma$, then we put $\tau_1 = 0$). By (3.3) and Proposition 6.6

$$\sum_{t=\tau_1}^{\tau_2-1} w(by(t)) \geq U(x(\tau_1), x(\tau_2), \tau_1, \tau_2) - \gamma. \tag{9.19}$$

In view of (9.18), (9.19), the choice of γ (see (9.8), (9.9))

$$\sum_{t=\tau_1}^{\tau_2-1} \delta(x(t), y(t), x(t+1)) \leq \epsilon_1. \tag{9.20}$$

It follows from (9.20), (9.18), (9.7), the choice of ϵ_1 (see (9.4)–(9.6)) that

$$||x(t)-\widehat{x}||,\ ||y(t)-\widehat{x}|| \leq \epsilon, \quad t=\tau_1,\dots,\tau_2-1.$$

Theorem B is proved. □

References

1. Anoulova, S.V., Evstigneev, I.V., Gundlach, V.M.: Turnpike theorems for positive multivalued stochastic operators. Adv. Math. Econ. **2**, 1–20 (2000)
2. Arkin, V.I., Evstigneev, I.V.: Stochastic models of control and economic dynamics. New York: Academic 1987
3. Atsumi, H.: Neoclassical growth and the efficient program of capital accumulation. Rev. Econ. Stud. **32**, 127–136 (1965)
4. Brock, W.A.: On existence of weakly maximal programmes in a multi-sector economy. Rev. Econ. Stud. **37**, 275–280 (1970)
5. Cass, D.: Optimum growth in an aggregative model of capital accumulation: a turnpike theorem. Econometrica **34**, 833–850 (1966)
6. Cass, D., Stiglitz, J.E.: The implications of alternative saving and expectations hypotheses for choices of technique and patterns of growth. J. Polit. Econ. **77**, 586–627 (1969)
7. Fujio, M.: Undiscounted optimal growth in a Leontief two-sector model with circulating capital: the case of a capital-intensive consumption good. J. Econ. Behav. Organ. **66**, 420–436 (2008)
8. Gale, D.: On optimal development in a multi-sector economy. Rev. Econ. Stud. **34**, 1–18 (1967)
9. Hahn, F.H., Matthews, R.C.O.: The theory of economic growth. Econ. J. **74**, 779–902 (1964)
10. Inada, K.: Some structural characteristics of turnpike theorems. Rev. Econ. Stud. **31**, 43–58 (1964)
11. Khan, M. Ali, Mitra, T.: On choice of technique in the Robinson–Solow–Srinivasan model. Int. J. Econ. Theory **1**, 83–110 (2005)
12. Khan, M. Ali, Mitra, T.: Optimal growth in the two-sector RSS model: a continuous time analysis. In: Proceedings of the seventh Portuguese conference on automatic control, Electronic publication 2006
13. Khan, M. Ali, Mitra, T.: Undiscounted optimal growth in the two-sector Robinson–Solow–Srinivasan model: a synthesis of the value-loss approach and dynamic programming. Econ. Theory **29**, 341–362 (2006)
14. Khan, M. Ali, Mitra, T.: Optimal growth in a two-sector model without discounting: a geometric investigation. Japanese Econ. Rev. **58**, 191–225 (2007)
15. Khan, M. Ali, Mitra, T.: Growth in the Robinson–Solow–Srinivasan model: undiscounted optimal policy with a strictly concave welfare function. J. Math. Econ. **44**, 707–732 (2008)

16. Khan, M. Ali, Zaslavski, A.J.: On a uniform turnpike of the third kind in the Robinson–Solow–Srinivasan model. J. Econ. **92**, 137–166 (2006)
17. McKenzie, L.W.: Accumulation programs of maximum utility and the von Neumann facet. In: Wolfe J.N. (ed.): Value, capital and growth. Edinburgh: Edinburgh University Press, 1968 pp. 353–383
18. McKenzie, L.W.: Capital accumulation optimal in the final state. Zeitschrift für Nationalökonomie Supplement **1**, 107–120 (1971)
19. McKenzie, L.W.: Turnpike theorems with technology and the welfare function variable. In: Loś J., Loś M.W. (eds.): Mathematical models in economics. New York: American Elsevier, 1974 pp. 271–287
20. McKenzie, L.W.: Turnpike theory. Econometrica **44**, 841–866 (1976)
21. McKenzie, L.W.: Optimal economic growth, turnpike theorems and comparative dynamics. In: Arrow K.J., Intrilligator M. (eds.): Handbook of mathematical economics, vol. 3. New York: North-Holland, 1986 pp. 1281–1355
22. McKenzie, L.W.: Turnpike theory. In: Eatwell J., Milgate M., Newman P.K. (eds.): The New Palgrave, vol. 4. London: MacMillan, 1987 pp. 712–720
23. McKenzie, L.W.: Turnpikes. Am. Econ. Rev. Pap. Proc. **88**, 1–14 (1998)
24. McKenzie, L.W.: Equilibrium, trade and capital accumulation. Japanese Econ. Rev. **50**, 369–397 (1999)
25. McKenzie, L.W.: Classical general equilibrium theory. Cambridge: The MIT Press 2002
26. Mitra, T.: On optimal economic growth with variable discount rates: existence and stability results. Int. Econ. Rev. **20**, 133–145 (1979)
27. Mitra, T.: Characterization of the turnpike property of optimal paths in the aggregative model of intertemporal allocation. Int. J. Econ. Theory **1**, 247–275 (2005)
28. Mitra, T., Zilcha, I.: On optimal economic growth with changing technology and tastes: characterization and stability results. Int. Econ. Rev. **22**, 221–238 (1981)
29. Morishima, M.: Theory of economic growth. Oxford: Oxford University Press 1964
30. Nikaido, H.: Persistence of continual growth near the von Neumann ray: a strong version of the Radner turnpike theorem. Econometrica **32**, 151–162 (1964)
31. Radner, R.: Paths of economic growth that are optimal with regard only to final states; a turnpike theorem. Rev. Econ. Stud. **28**, 98–104 (1961)
32. Samuelson, P.A. A catenary turnpike theorem involving consumption and the golden rule. Am. Econ. Rev. **55**, 486–496 (1965)
33. Samuelson, P.A.: Maximum principles in analytical economics. Les Prix Nobel en 1970. Stockholm: The Nobel Foundation, 1971 pp. 273–288
34. Samuelson, P.A.: In: Merton R.C. (ed.): The collected scientific papers of Paul A. Samuelson, vol. III. Cambridge: MIT Press 1972
35. Solow, R.M.: Neoclassical growth theory. In: Taylor J.B., Woodford M. (eds.): Handbook of macroeconomics, vol. 1. New York: Elsevier, 1999 pp. 637–667
36. Stiglitz, J.E.: A note on technical choice under full employment in a socialist Economy. Econ. J. **78**, 603–609 (1968)
37. Stiglitz, J.E.: Recurrence of techniques in a dynamic economy. In: Mirrlees J., Stern N.H. (eds.): Models of economic growth. New York: Wiley 1973

38. von Weizsäcker, C.C.: Existence of optimal programs of accumulation for an infinite horizon. Rev. Econ. Stud. **32**, 85–104 (1965)
39. Zaslavski, A.J.: Optimal programs in the RSS model. Int. J. Econ. Theory **1**, 151–165 (2005)
40. Zaslavski, A.J.: Turnpike properties in the calculus of variations and optimal control. New York: Springer 2005
41. Zaslavski, A.J.: Optimal programs in the continuous time RSS model. In: Proceedings of the seventh Portuguese conference on automatic control, Electronic publication 2006
42. Zaslavski, A.J.: Good programs in the RSS model with a nonconcave utility function. J. Ind. Manage. Optim. **2**, 399–423 (2006)

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A certain limit of iterated conditional tail expectation

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Abstract. We consider the continuous limit of iterated Conditional Tail Expectation for a Brownian filtration and show the existence of the limit for some special random variables.

Key words: Risk measure, Value at risk, CTE

1. Introduction

There is a lot of attempts to measure market risk for multi-period models (e.g. [1–4]). In particular, Hardy and Wirch [5] introduced iterated CTE (Conditional Tail Expectation) and it is a quite natural extension of CTE for one period models. We give a precise definition of CTE later. But roughly speaking $CTE_\alpha(X)$, $\alpha \in (0,1)$, for random variable X which express a loss of a firm in the end of a term, is given by

$$CTE_\alpha(X) = E[X|X < q],$$

where q is the α quantile, i.e. $P(X < q) = \alpha$.

On the other hand, Morimoto and the author [6] introduced a kind of iterated law-invariant coherent risk measures for multiperiod models and also

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discussed their continuous limits in continuous time models. However, the assumptions for Theorem 7 in [6] is too strong to apply it to iterated CTE. In the present paper, we study about the continuous limits of iterated CTE for Brownian filtration models.

Let $\mathcal{L}_1$ denote the set of probability measures ν on $\mathbf{R}$ such that $\int_{\mathbf{R}} |x|\nu(dx) < \infty$. For $\nu \in \mathcal{L}^1$, let us define $Z : [0,1) \times \mathcal{L} \to \mathbf{R}$ by

$$Z(x,\nu) = \inf\{z; \nu((-\infty,z]) > x\}, \qquad x \in [0,1),\ \nu \in \mathcal{L}^1.$$

Then $Z(\cdot,\nu) : [0,1) \to \mathbf{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot,\nu)$ under Lebesgue measure on $[0,1)$ is ν.

For each $\alpha \in (0,1]$, let $\eta_\alpha : \mathcal{L}_1 \to \mathbf{R}$ be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x,\nu)dx, \qquad \nu \in \mathcal{L}_1.$$

Let $(\Omega, \mathcal{F}, P)$ be an atomless standard probability space. Then for any integrable random variable X, we see that

$$\eta_\alpha(\mu_X) = \inf\{E[\rho X]; \rho \text{ is a random variable with } 0 \leqq \rho \leqq 1/\alpha \text{ and } E[\rho] = 1\},$$

where μ_X denotes the probability law of X.

We denote $\eta_\alpha(\mu_X)$ by $\eta_\alpha(X)$ for simplicity. Then η_α is a coherent monetary utility function.

The precise definition of CTE is

$$CTE_\alpha(X) = -\eta_{1-\alpha}(-X),$$

if a random variable X is regarded as a loss of a firm in the end of a term.

Definition 1 *Let* $\alpha \in (0,1]$.

(1) *For any integrable random variable* X *and any sub-*σ*-algebra* $\mathcal{G}$*, we define a* $\mathcal{G}$*-measurable random variable* $\eta_\alpha(X|\mathcal{G})$ *by*

$$\eta_\alpha(X|\mathcal{G}) = \eta_\alpha(P(X \in dx|\mathcal{G})),$$

where $P(X \in dx|\mathcal{G})$ *is a regular conditional probability law of* X *given a sub-*σ*-algebra* $\mathcal{G}$.

(2) *For any integrable random variable* X *and any filtration* $\{\mathcal{F}_k\}_{k=0}^n$*, we define an adapted process* $\{Z_k\}_{k=0}^n$ *inductively by*

$$Z_n = \eta_\alpha(X|\mathcal{F}_n),$$

$$Z_{k-1} = \eta_\alpha(Z_k|\mathcal{F}_{k-1}), \qquad k = n, n-1, \ldots, 1.$$

We denote an $\mathcal{F}_0$*-measurable random variable* Z_0 *by* $\eta_\alpha(X|\{\mathcal{F}_k\}_{k=0}^n)$.

If a random variable X is regarded as a loss of a firm, $-\eta_{1-\alpha}(-X|\{\mathcal{F}_k\}_{k=0}^n)$ is essentially the same as $ICTE_\alpha$ defined by Hardy and Wirch [5].

Now let Φ be the standard normal distribution function, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\frac{y^2}{2})dy \qquad x \in \mathbf{R}.$$

Let $b > 0$ and $\alpha : (0,\infty) \to (0,1)$ be a function. We assume the following.

(A) $\Phi^{-1}(\alpha(h))^2 - \log(1/h) + \log(2\pi) \to -2\log b, \qquad h \downarrow 0.$

Here $\Phi^{-1} : (0,1) \to \mathbf{R}$ is the inverse function of Φ.

Let $\{B_t\}_{t\geqq 0}$ be a N-dimensional Brownian motion starting from the origin and let $\mathcal{F}_t = \sigma\{B_s;\ s \in [0,t]\}$, $t \geqq 0$. Also, let $\mathcal{K}_T$, $T \geqq 0$, be a set of martingales $\{\rho(t)\}_{t\in[0,T]}$ such that there is an $\{\mathcal{F}_t\}_{t\geqq 0}$ adapted measurable process $\varphi : [0,\infty) \times \Omega \to \mathbf{R}^N$, for which

$$P(|\varphi(t)| \leqq b \text{ for any } t \in [0,T]) = 1$$

and

$$\rho(t) = \exp(\int_0^t \varphi(s)dB(s) - \frac{1}{2}\int_0^t |\varphi(s)|^2 ds), \qquad t \in [0,T].$$

Then we have the following.

Theorem 2 *Let* $f : \mathbf{R}^N \to \mathbf{R}$ *be a continuous function such that there are* $\gamma > 0$ *and* $C > 0$ *for which*

$$|f(x)| \leqq C\exp(\gamma|x|), \qquad x \in \mathbf{R}^N.$$

Then under the assumption (A) *we have for any* $T > 0$

$$\lim_{h\downarrow 0} \eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) = \inf\{E[\rho(T)f(B(T))]; \rho \in \mathcal{K}_T\}.$$

Corollary 3 *Suppose that* $N = 1$. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuous non-decreasing function such that there are* $\gamma > 0$ *and* $C > 0$ *for which*

$$|f(x)| \leqq C\exp(\gamma|x|), \qquad x \in \mathbf{R}.$$

Then under the assumption (A) *we have for any* $T > 0$

$$\lim_{h\downarrow 0} \eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1})$$
$$= E[\exp(-bB(T) - \frac{b}{2}T)f(B(T))] = E[f(B(T) - bT)].$$

Let us give an example for Corollary 3. Let $N = 1$, $\sigma > 0$, and $\mu \in \mathbf{R}$. Let $S(t)$ be the solution to the following SDE.

$$dS(t) = S(t)(\sigma dB(t) + \mu dt), \quad t > 0, \qquad S(0) = S_0,$$

where $S_0 > 0$ is non-random. Then we see that

$$S(t) = S_0 \exp(\sigma B(t) + (\mu - \frac{\sigma^2}{2})t).$$

Let $K > 0$. Then we have

$$(K - S_T) \vee 0 = f(B(T)),$$

where

$$f(x) = (K - \exp(\sigma x + (\mu - \frac{\sigma^2}{2})T) \vee 0, \qquad x \in \mathbf{R}.$$

Then $-f$ is a bounded continuous non-decreasing function. Therefore we see that

$$\begin{aligned} &\lim_{h \downarrow 0} -\eta_{\alpha(h)}(-f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) \\ &\quad = E[(K - \exp(\sigma B(T) + (\mu - \sigma b - \frac{\sigma^2}{2})T) \vee 0]. \end{aligned}$$

2. Associated nonlinear PDE

Let $C_b(\mathbf{R}^N)$ denote the set of bounded continuous functions defined in $\mathbf{R}^N$, and $||f||_\infty$ denote $\max_{x \in \mathbf{R}^N} |f(x)|$ for any $f \in C_b(\mathbf{R}^N)$. Let $C_b^n(\mathbf{R}^N)$, $n \geqq 1$, denote the set of $f \in C_b(\mathbf{R}^N)$ for which f is n-times continuously differentiable and its partial derivatives up to order n belong to $C_b(\mathbf{R}^N)$.

Let P_t, $t \geqq 0$, be linear operators in $C_b(\mathbf{R}^N)$ given by

$$(P_t f)(x) = \int_{\mathbf{R}^N} g(t, x - y) f(y) dy, \qquad f \in C_b(\mathbf{R}^N),\ t > 0$$

where

$$g(t, x) = (\frac{1}{2\pi t})^{N/2} \exp(-\frac{|x|^2}{2}), \qquad (t, x) \in (0, \infty) \times \mathbf{R}^N,$$

and P_0 is the identity operator.

Proposition 4 *For any $f \in C_b^3(\mathbf{R}^N)$, there is a continuous function $u : [0, \infty) \times \mathbf{R}^N \to \mathbf{R}$ satisfying the following.*

(1) *$u(t,x)$ is continuously differentiable in x and $\nabla_x u(t,x)$ is continuous in (t,x).*

(2) *For any $T > 0$,*

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^N} (|u(t,x)| + |\nabla_x u(t,x)|) < \infty.$$

(3) *For any $(t,x) \in [0,\infty)\times\mathbf{R}^N$*

$$u(t,x) = (P_t f)(x) - b\int_0^t P_{t-s}(|\nabla_x u(s,\cdot)|)(x)ds.$$

(4) *For any $(t,x) \in [0,\infty)\times\mathbf{R}^N$*

$$\begin{aligned} u(T-t, x+B(t)) = &\int_0^t \nabla_x u(T-s, x+B(s))dB(s) \\ &+b\int_0^t |\nabla_x u(T-s, x+B(s))|ds. \end{aligned}$$

(5) *For any $T > 0$ and $\beta \in (0,1)$,*

$$\sup_{t\in[0,T],x,y\in\mathbf{R}^N,y\neq 0,|y|\leqq 1} |y|^{-\beta}|\nabla_x u(t,x+y) - \nabla_x u(t,x)| < \infty.$$

Also, for any $T > 0$ and $\beta \in (0,1/2)$,

$$\sup_{t\in[0,T],x,y\in\mathbf{R}^N,s\in(0,1]} s^{-\beta}|\nabla_x u(t+s,x) - \nabla_x u(t,x)| < \infty.$$

Remark 5 We will show in the next section that

$$\lim_{h\downarrow 0} \eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) = u(T,0)$$

for any $f \in C_b^3(\mathbf{R}^N)$.

Proof. The assertions (1), (2), (3), (4) have been shown in § 4 in [6].

Note that

$$\frac{\partial^m}{\partial t^m}\nabla_x^n(P_t f))(x) = \int_{\mathbf{R}^N} \nabla_x^n \frac{\partial^m g}{\partial t^m}(t, x-z)f(z)dz$$

for any $f \in C_b(\mathbf{R}^N)$, $n, m \geqq 0$, and

$$\frac{\partial^m}{\partial t^m}\nabla_x^n(P_t f))(x) = \int_{\mathbf{R}^N} g(t,z)(\nabla_x^n(\frac{1}{2}\Delta)^m f)(x-z)dz$$

for any $f \in C_b^3(\mathbf{R}^N)$, $n, m \geqq 0$, with $n + 2m \leqq 3$. Also, we see that

$$\int_{\mathbf{R}^N} |\nabla_x^n \frac{\partial^m g}{\partial t^m}(t, z)| dz = t^{-m-n/2} \int_{\mathbf{R}^N} |\nabla_x^n \frac{\partial^m g}{\partial t^m}(1, z)| dz$$

for any $n, m \geqq 0$. Let

$$C = \sum_{n,m=0}^{2} \int_{\mathbf{R}^N} |\nabla_x^n \frac{\partial^m g}{\partial t^m}(1, z)| dz.$$

Then we see that

$$|\nabla_x (P_t f)(x + y) - \nabla_x (P_t f)(x)| \leqq C|y| \, ||\nabla^2 f||_\infty,$$

and

$$|\nabla_x (P_{t+s} f)(x) - \nabla_x (P_t f)(x)| \leqq CNs ||\nabla^3 f||_\infty,$$

for any $f \in C_b^3(\mathbf{R}^N)$.

Also, we have

$$|\nabla_x (P_t f)(x)| \leqq Ct^{-1/2} ||f||_\infty,$$

$$|\nabla_x (P_t f)(x + y) - \nabla_x (P_t f)(x)| \leqq Ct^{-1} |y| ||f||_\infty,$$

$$|\nabla_x (P_{t+s} f)(x) - \nabla_x (P_t f)(x)| \leqq Cst^{-3/2} ||f||_\infty,$$

for any $f \in C_b(\mathbf{R}^N)$. So we have

$$|\nabla_x (P_t f)(x + y) - \nabla_x (P_t f)(x)| \leqq 2Ct^{-(1+\beta)/2} |y|^\beta ||f||_\infty,$$

and

$$|\nabla_x (P_{t+s} f)(x) - \nabla_x (P_t f)(x)| \leqq 2Cs^\beta t^{-(1+2\beta)/2} ||f||_\infty,$$

for any $f \in C_b(\mathbf{R}^N)$ $t \geqq 0$, $x \in \mathbf{R}^N$ and $\beta \in (0, 1)$.

So we see that

$$\begin{aligned} &|\nabla_x u(t, x + y) - \nabla_x u(t, x)| \\ &\quad \leq |y| ||\nabla^2 f||_\infty + 2bC|y|^\beta \Big(\int_0^t ds (t - s)^{(1+\beta)/2}\Big) \\ &\quad \sup_{(s,z) \in [0,T] \times \mathbf{R}^N} |\nabla_x u(s, z)| \end{aligned}$$

and

$$\begin{aligned} &|\nabla_x u(t + s, x) - \nabla_x u(t, x)| \\ &\quad \leq |\nabla_x (P_{t+s} f)(x) - \nabla_x (P_t f)(x)| \end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t+s} dr|\nabla_x(P_{t+s-r}(|\nabla_x u(s,\cdot)|))(x)| \\
&+\int_0^t dr|\nabla_x(P_{t+s-r}(|\nabla_x u(s,\cdot)|))(x)-\nabla_x(P_{t-r}|\nabla_x u(s,\cdot)|)(x)| \\
&\leqq NCs||\nabla^3 f||_\infty + b(2Cs^\beta(\int_0^t dr(t-r)^{-(1+2\beta)/2}) \\
&+\int_t^{t+s} dr(t+s-r)^{-1/2}) \sup_{(s,z)\in[0,T]\times\mathbf{R}^N} |\nabla_x u(s,z)|.
\end{aligned}
$$

These imply the assertion (5). □

3. Proof of main results

First we notice the following.

Proposition 6 *Let* $\alpha \in (0,1]$ *and* $p \in [1,\infty)$. *Then*

$$|\eta_\alpha(X) - E[X]| \leqq 2(1-\alpha)^{1-1/p}\alpha^{-1}E[|X|^p]^{1/p}$$

for any $X \in L^p(\Omega, \mathcal{F}, P)$. *Moreover, it holds that*

$$|\eta_\alpha(X|\mathcal{G}) - \eta_\alpha(Y|\mathcal{G}) - E[X-Y|\mathcal{G}]| \leqq 2(1-\alpha)^{1-1/p}\alpha^{-1}E[|X-Y|^p|\mathcal{G}]^{1/p}$$

for any $X, Y \in L^p(\Omega, \mathcal{F}, P)$ *and a sub* σ*-algebra* $\mathcal{G}$.

Proof. Let K be the set of random variables ρ satisfying $E[\rho] = 1$ and $0 \leqq \rho \leqq 1/\alpha$. Then for any $\rho \in K$, we have

$$\frac{1-\alpha}{\alpha} \leqq E[(\rho-1)\vee 0] = E[(1-\rho)\vee 0],$$

and so

$$E[|\rho-1|^p] \leqq E[|\rho-1|]\alpha^{-(p-1)} \leqq 2(1-\alpha)\alpha^{-p}.$$

Therefore we see that

$$|\eta_\alpha(X) - E[X]| \leqq \sup\{|E[\rho X] - E[X]|;\ \rho \in K\} \leqq 2(1-\alpha)^{1-1/p}\alpha^{-1}E[|X|^p]^{1/p}.$$

This proves the first assertion. Note that (cf. [6, Propositions 12 and 13])

$$
\begin{aligned}
&|\eta_\alpha(X|\mathcal{G}) - \eta_\alpha(Y|\mathcal{G}) - E[X-Y|\mathcal{G}]| \\
&\leqq \eta_\alpha(X - Y - E[X-Y]|\mathcal{G}]|\mathcal{G}))|. \vee \eta_\alpha(Y - X - E[Y-X]|\mathcal{G}]|\mathcal{G}))|.
\end{aligned}
$$

So we have the second assertion from the first assertion. □

Now let $\alpha : (0, \infty) \to (0.1)$ and $b > 0$ be as in Introduction, and assume (A). Let

$$r(h) = \Phi^{-1}(\alpha(h))^2 - \log(1/h) + \log(2\pi) + 2\log b, \qquad h > 0.$$

Then $r(h) \to 0$, $h \downarrow 0$. Let $q(h) = h^{1/2}\Phi^{-1}(\alpha(h))$, $h \in (0, 1]$. Then we have

$$\alpha(h) = \Phi(h^{-1/2}q(h)) \text{ and } h^{-1/2}q(h) = \{\log(1/h) - \log(2\pi b^2) + r(h)\}^{1/2}.$$

So we see that $h^{-1/2}q(h) \to \infty$, as $h \downarrow 0$.

It is well known (e.g. Williams [7, p. 141]) that

$$(1 - \Phi(x))x(2\pi)^{1/2}\exp(\frac{x^2}{2}) \to 1, \qquad x \to \infty.$$

So we can see that

$$\frac{(\log(1/h))^{1/2}}{h^{1/2}}(1 - \alpha(h)) \to b, \qquad h \downarrow 0. \tag{3.1}$$

Also, we see that

$$\begin{aligned}\frac{1}{h}\eta_{\alpha(h)}(B(h)) &= \frac{1}{h}(\frac{1}{2\pi h})^{1/2}\int_{-\infty}^{-q(h)} x\exp(-\frac{x^2}{2h})dx \\ &= -h^{-1/2}(\frac{1}{2\pi})^{1/2}\exp(-\frac{q(h)^2}{2h}) \to -b, \qquad h \downarrow 0. \quad (3.2)\end{aligned}$$

Proposition 7 *Let* $f \in C_b^3(\mathbf{R}^N)$ *and* $T > 0$. *Let* $u : [0, \infty) \times \mathbf{R}^N \to \mathbf{R}$ *be as in Proposition 4. Then for any* $\beta \in (0, 1)$, *there is a constant* $C > 0$ *such that*

$$\begin{aligned}&|\eta_\alpha(u(T - (t + s), x + B(t + s))|\mathcal{F}_t) - u(T - t, x + B(t))) \\ &-(\eta_\alpha(B(s)) + bs)|\nabla_x u(T - t, x + B(t))|| \\ &\quad \leqq C((1 - \alpha)^{1/2}s^{(1+\beta)/2} + s^{(2+\beta)/2})\end{aligned}$$

for any $\alpha \in [1/2, 1)$, $x \in \mathbf{R}^N$ *and* $t \geqq 0$, $s \in (0, 1]$ *with* $s + t \leqq T$.

Proof. Let

$$C_1 = \sup_{t\in[0,T],x,y\in\mathbf{R}^N,y\neq 0} |y|^{-\beta}|\nabla_x u(t, x + y) - \nabla_x u(t, x)| < \infty$$

and

$$C_2 = \sup_{t\in[0,T],x,y\in\mathbf{R}^N,s\in(0,1]} s^{-\beta/2}|\nabla_x u(t + s, x) - \nabla_x u(t, x)| < \infty.$$

Note that

$$u(T-(t+s), x+B(t+s)) = Y_0 + Y_1 + Y_2,$$

where

$$Y_0 = u(T-t, x+B(t)) + |\nabla_x u(T-t, x+B(t))|)dr \\ +\nabla_x u(T-t, x+B(t))(B(t+s)-B(t)),$$

$$Y_1 = \int_t^{t+s} \nabla_x u(T-r, x+B(r))dB(r) - \nabla_x u(T-t, x+B(t))(B(t+s)-B(t)),$$

and

$$Y_2 = \int_t^{t+s} (|\nabla_x u(T-r, x+B(r))| - |\nabla_x u(T-t, x+B(t))|)dr.$$

Then we see that

$$\eta_\alpha(Y_0|\mathcal{F}_t) = u(T-t, x+B(t)) + |\nabla_x u(T-t, x+B(t))|)dr \\ +|\nabla_x u(T-t, x+B(t))|\eta_\alpha(B(s)).$$

Also, we see that $E[Y_1|\mathcal{F}_t] = 0$ and

$$\begin{aligned} E[Y_1^2|\mathcal{F}_t] &= E[\int_t^{t+s} |\nabla_x u(T-r, x+B(r)) - \nabla_x u(T-t, x+B(t))|^2 dr|\mathcal{F}_t] \\ &\leqq 2(C_1+C_2)\int_t^{t+s} (|r-t|^\beta + E[|B(r)-B(t)|^{2\beta}|\mathcal{F}_t])dr \\ &\leqq 2(C_1+C_2)s^{1+\beta}(1+E[|B(1)|^{2\beta}]). \end{aligned}$$

Then we have

$$E[|Y_2|^2|\mathcal{F}_t] \leqq sE[\int_t^{t+s} |\nabla_x u(T-r, x+B(r)) \\ -\nabla_x u(T-t, x+B(t))|^2 dr|\mathcal{F}_t] \leqq Cs^{2+\beta}.$$

So we see that

$$|E[Y_2|\mathcal{F}_t]| \leqq C^{1/2}s^{(2+\beta)/2}.$$

So by Propositions 4 and 6, we have our assertion. □

Proposition 8 *Let* $\rho(h) = |h^{-1}\eta_{\alpha(h)}(B(h)) + b|$, $h \in (0,1]$. *Let* $f \in C_b^3(\mathbf{R}^N)$ *and* $u : [0,\infty)\times\mathbf{R}^N \to \mathbf{R}$ *be as in Proposition 4. Let* $x \in \mathbf{R}^N$, *and* $Z_n^{(h)}$, $n = 0, 1, \ldots, [T/h]+1$ *be random variables given by*

$$Z^{(h)}_{[T/h]+1} = f(x + B(T)),$$

$$Z^{(h)}_{n-1} = \eta_{\alpha(h)}(Z^{(h)}_n | \mathcal{F}_{(n-1)h}), \qquad n = [T/h] + 1, [T/h], \ldots, 1.$$

Then there are $h_0 \in (0, 1]$*, and* $C' > 0$ *such that*

$$\begin{aligned} &||Z^{(h)}_n - u((T - nh) \vee 0, x + B((nh) \wedge T))||_{L^\infty(\Omega,\mathcal{F},P)} \\ &\quad \leqq C'([T/h] + 1 - n)h(\rho(h) + h^{1/12}) \end{aligned}$$

for any $h \in (0, h_0]$ *and* $n = 0, 1, \ldots, [T/h] + 1$.

In particular, we have

$$\lim_{h\downarrow 0} \eta_{\alpha(h)}(f(x + B_T)|\{\mathcal{F}_{hk}\}^{[T/h]+1}_{k=0}) = u(T, x), \quad x \in \mathbf{R}^N.$$

Proof. Note that there is an $h_0 \in (0, 1]$ such that $1 - \alpha(h) \leqq h^{1/3}$ for any $h \in (0, h_0]$. Let $\beta = 5/6$, and let $C > 0$ be as in Proposition 7. Now let

$$C' = 2C + \sup\{|\nabla_y u(t, y)|;\ t \in [0, T],\ y \in \mathbf{R}^N\}.$$

We prove the first assertion by induction in n. For $n = [T/h] + 1$, the first assertion is obvious. Suppose that the first assertion is valid for n. Then we see that

$$\begin{aligned} &|Z_{n-1} - u(T - (n - 1)h, x + B((n - 1)h \wedge T))| \\ &\quad \leqq ||Z_n - u((T - nh) \vee 0, x + B(nh \wedge T))||_{L^\infty(\Omega,\mathcal{F},P)} \\ &\qquad + |\eta_{\alpha(h)}(u((T - nh) \vee 0, x + B(nh \wedge T))|\mathcal{F}_{(n-1)h}) \\ &\qquad - u(T - (n - 1)h, x + B((n - 1)h \wedge T))| \\ &\quad \leqq C'([T/h] + 1 - n)h(\rho(h) + h^{1/12}) \\ &\qquad + C((1 - \alpha(h))^{1/2} h^{(1+\beta)/2} + h^{(2+\beta)/2}) + C'\rho(h). \end{aligned}$$

This shows that the first assertion is valid for $n - 1$. So the induction is complete.

The latter assertion is an immediate consequence of the first assertion for $n = 0$. □

For any $\gamma \in \mathbf{R}$, let

$$M^{(\gamma)}(t) = \exp(\gamma B(t) - \frac{\gamma^2 t}{2}), \qquad t \geqq 0.$$

Proposition 9 *For any* $\gamma \in \mathbf{R}$*, there is a constant* $c(\gamma) > 0$ *such that*

$$\eta_\alpha(-M^{(\gamma)}(t + s)|\mathcal{F}_t) \geqq -(1 + c(\gamma)(|\eta_\alpha(B(s))| + (1 - \alpha)^{1/2} s))M^{(\gamma)}(t)$$

for any $\alpha \in [1/2, 1)$*, and* $t, s \geqq 0$.

Proof. It is easy to see that

$$\eta_\alpha(-M^{(\gamma)}(t+s)|\mathcal{F}_t) = \eta_\alpha(-M^{(\gamma)}(s))M^{(\gamma)}(t).$$

Note that

$$M^{(\gamma)}(s) = 1 + \gamma B(s) + \gamma \int_0^s (M^{(\gamma)}(r) - 1)dB(r).$$

Then we have by Proposition 6

$$\begin{aligned} |\eta_\alpha(-\int_0^s (M^{(\gamma)}(r) - 1)dB(r))| &\leqq 2\alpha^{-1}(1-\alpha)^{1/2} E[\int_0^s (M^{(\gamma)}(r) - 1)^2 dr]^{1/2} \\ &= 2\alpha^{-1}(1-\alpha)^{1/2}(\int_0^s (\exp(\gamma^2 r) - 1)dr)^{1/2}. \end{aligned}$$

Since

$$\eta_\alpha(-1 - \gamma B(s)) = -(1 + \gamma|\eta_\alpha(B(s))|),$$

we have our assertion. □

Similarly to the proof of Proposition 8, we have the following.

Proposition 10 *For any* $\gamma \in \mathbf{R}$ *and* $T > 0$ *there is a* $C(\gamma, T) > 0$ *such that let*

$$|\eta_{\alpha(h)}(-M^{(\gamma)}(T)|\{\mathcal{F}_{nh}\}_{n=0}^{[T/h]+1})| \leqq C(\gamma, T)$$

for any $h \in (0, 1]$.

Now let us prove Theorem 2.

First, let us assume that $f \in C_b^3(\mathbf{R}^N)$. Then by Proposition 8, we see that

$$\lim_{h\downarrow 0} \eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) = u(T, 0), \quad x \in \mathbf{R}^N.$$

Then by the argument in [6, § 5] we see that Theorem 2 is valid for any $f \in C_b^3(\mathbf{R}^N)$. Now let $f : \mathbf{R}^N \to \mathbf{R}$ is as in Theorem 2. Then we can find a sequence $\{f_n\}_{n=1}^\infty$ in $C_b^3(\mathbf{R}^N)$ such that

$$a_n = \sup_{x\in\mathbf{R}^N} \exp(-2\gamma|x|)|f(x) - f_n(x)| \to 0, \qquad n \to \infty.$$

Then we see by [6, Proposition 15] that

$$\begin{aligned} &|\eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) - \eta_{\alpha(h)}(f_n(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1})| \\ &\quad \leqq -a_n \eta_{\alpha(h)}(-\exp(2\gamma|B_T|)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) \\ &\quad \leqq -a_n(\eta_{\alpha(h)}(-M^{(2\gamma)}(T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) \\ &\quad +\eta_{\alpha(h)}(-M^{(-2\gamma)}(T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1})) \\ &\quad \leqq 2a_n(C(2\gamma, T) + C(-2\gamma, T)), \qquad h \in (0, h_0] \end{aligned}$$

for some $h_0 \in (0,1]$. So we see that

$$\sup_{h\in(0,h_0]} |\eta_{\alpha(h)}(f(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1}) \\ -\eta_{\alpha(h)}(f_n(B_T)|\{\mathcal{F}_{hk}\}_{k=0}^{[T/h]+1})| \to 0, \qquad n \to \infty.$$

For $\rho \in \mathcal{K}_T$, let Q_ρ be a probability measure given by $dQ_\rho = \rho(T)dP$. Then we see that $\tilde{B}(t) = B(t) - \int_0^t \varphi(s)ds$, $t \in [0,T]$ is a Q_ρ-Brownian motion. Therefore we have

$$|E[\rho(T)f(B(T))] - E[\rho(T)f_n(B(T))]| \leqq a_n E^{Q_\rho}[\exp(2\gamma|\tilde{B}(T) \\ + \int_0^T \varphi(s)ds|)] \leqq a_n \exp(2\gamma bT + 2\gamma^2 T).$$

These imply Theorem 2.

Now let us prove Corollary 3. We have

$$E[\rho(T)f(B(T))] = E^{Q_\rho}[f(\tilde{B}_\rho(T) + \int_0^T \varphi(s)ds)],$$

where

$$\tilde{B}_\rho(t) = B(t) - \int_0^{t\wedge T} \varphi(s)ds, \qquad t \geqq 0.$$

Then $\tilde{B}_\rho(t), t \geqq 0$, is a Q_ρ-Brownian motion.

Since $\int_0^T \varphi(s)ds \geqq -bT\ Q_\rho - a.s.$ and we can take $-b$ as $\varphi(t)$, we see that

$$\inf\{E[\rho(T)f(B(T))]; \rho \in \mathcal{K}_T\} = E[f(B(T) - bT)].$$

This completes the proof of Corollary 3.

References

1. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Multiperiod risk and coherent multiperiod risk measurement (2003, preprint)
2. Artzner, P., Delbaen, F., Eber, J.M., Heath, D., Ku, H.: Coherent muliperiod risk measurement (2003, preprint)
3. Artzner, P., Delbaen, F., Eber, J.M., Heath, D., Ku, H.: Coherent multiperiod risk adjusted values and Bellman's principle. Ann. Oper. Res. **152**, 5–22 (2007)
4. Cheridito, P., Delbaen, F., Kupper, M.: Coherent and convex risk measure for bounded cádlág processes. Stoch. Proc. Appl. **112**, 1–22 (2004)

5. Hardy, M.R., Wirch, J.L.: The iterated CTE: a dynamic risk measure. North Am. Actuar. J. **8**, 62–75 (2004)
6. Kusuoka, S., Morimoto, Y.: Homogeneous law invariant multiperiod value measures and their limits. J. Math. Sci. Univ. Tokyo **14**, 117–156 (2007)
7. Williams, D.: Probability with martingales. Cambridge: Cambridge University Press 1991

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Set-valued optimization in welfare economics

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Abstract. This paper mainly concerns applications of advanced techniques of variational analysis and generalized differentiation to nonconvex models of welfare economics with finite-dimensional and infinite-dimensional commodity spaces. We pay special attention to establishing new relationships between necessary conditions in multiobjective/set-valued optimization and appropriate extensions of the second fundamental theorem of welfare economics to nonconvex economies with general preference relations. The variational approach developed in this paper allows us to obtain new necessary conditions for various types of local optimal solutions to constrained multiobjective problems and to derive from them new versions of the second welfare theorem applied to Pareto as well as weak, strict, and strong Pareto optimal allocations of nonconvex economies under certain qualification conditions developed in the paper. We also establish relationships of the latter conditions with some versions of Mas-Colell's uniform properness.

Key words: multiobjective optimization, second welfare theorem, variational analysis, welfare economics

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1. Introduction

It has been well recognized that vector optimization has its roots in economic modeling and general equilibrium theory. Recently, there has been increasing attention to mathematical areas studying optimal (in certain senses) points of sets and set-valued mappings that relate to *set-valued optimization*. The reader is referred to, e.g., books [10, 12, 14, 27], more recent papers [1, 2, 4], and the bibliographies therein to basic concepts and results in set and set-valued optimization with their numerous applications. We generally prefer to use the conventional term *multiobjective optimization* to unify problems of vector optimization (with single-valued objectives) as well as their set and set-valued counterparts, which are mainly considered in this paper.

This paper pursues the following *two-fold goal*: to derive new necessary conditions for general notions of (Pareto-type) optimality in constrained multiobjective problems that are largely motivated by economic applications and then to apply them to nonconvex models of *welfare economics*. Both issues are developed in this paper by employing appropriate tools of advanced *variational analysis* and *generalized differentiation*.

Let us first formulate a conventional *model of welfare economics* studied in what follows. Given a normed *commodity space* E, consider an *economy*

$$\mathcal{E} = (C_1, \ldots, C_n, S_1, \ldots, S_m, W) \tag{1.1}$$

involving $m \in I\!N := \{1, 2, \ldots\}$ firms with *production sets* $S_j \subset E$ $(j = 1, \ldots, m)$, $n \in I\!N$ customers with *consumption sets* $C_i \subset E$ $(i = 1, \ldots, n)$, and a *net demand constraint set* W representing constraints related to the initial inventory of commodities in the economy $\mathcal{E}$. Denote production strategies by $y = (y_1, \ldots, y_m) \in S_1 \times \cdots \times S_m$ and consumption plans by $z = (z_1, \ldots, z_n) \in C_1 \times \cdots \times C_n$ assuming that all the consumption sets C_i are *closed*. Then the pair (y, z) is an *admissible state* of the economy $\mathcal{E}$. Further, associate with each consumer a *preference set* $P_i(z)$ that consists of elements in C_i preferred to z_i by this consumer at the consumption plan z. The corresponding preference mappings $P_i : Z \rightrightarrows E$ are *set-valued* with $Z := E^n$. By definition we have $z_i \notin P_i(z)$ for every $i = 1, \ldots, n$ and naturally suppose that $P_i(z) \neq \emptyset$ at least for some $i \in \{1, \ldots, n\}$. Put for convenience $\mathrm{cl}\, P_i(z) := \{z_i\}$ of $P_i(z) = \emptyset$, where the symbol "cl" stands for the closure of the set in question.

Definition 1.1 (feasible allocations). *An admissible state* (y, z) *of the economy* $\mathcal{E}$ *in* (1.1) *is a* feasible allocation *of* $\mathcal{E}$ *if*

$$w := \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \in W. \tag{1.2}$$

Definition 1.2 (Pareto-type optimal allocations). *Let* $(\bar{y}, \bar{z}) \in E^m \times E^n$ *be a feasible allocation of the economy* $\mathcal{E}$. *We say that:*

(i) The pair $(\bar{y}, \bar{z})$ *is a* local weak Pareto optimal allocation *of* $\mathcal{E}$ *if there is a neighborhood* $\mathcal{O} \subset E^m \times E^n$ *of* $(\bar{y}, \bar{z})$ *such that for every feasible allocation* $(y, z) \in \mathcal{O}$ *we have* $z_i \notin P_i(\bar{z})$ *for some* $i \in \{1, \ldots, n\}$.

(ii) The pair $(\bar{y}, \bar{z})$ *is a* local Pareto optimal allocation *of* $\mathcal{E}$ *if there is a neighborhood* $\mathcal{O}$ *of* $(\bar{y}, \bar{z})$ *such that for every feasible allocation* $(y, z) \in \mathcal{O}$ *either* $z_i \notin \operatorname{cl} P_i(\bar{z})$ *for some* $i \in \{1, \ldots, n\}$, *or* $z_i \notin P_i(\bar{z})$ *for all* $i \in \{1, \ldots, n\}$.

(iii) The pair $(\bar{y}, \bar{z})$ *is a* local strict Pareto optimal allocation *of* $\mathcal{E}$ *if there is a neighborhood* $\mathcal{O}$ *of* $(\bar{y}, \bar{z})$ *such that for every feasible allocation* $(y, z) \in \mathcal{O}$ *with* $z \neq \bar{z}$ *we have* $z_i \notin \operatorname{cl} P_i(\bar{z})$ *for some* $i \in \{1, \ldots, n\}$.

(iv) The pair $(\bar{y}, \bar{z})$ *is a* local strong Pareto optimal allocation *of* $\mathcal{E}$ *if there is a neighborhood* $\mathcal{O}$ *of* $(\bar{y}, \bar{z})$ *such that for every feasible allocation* $(y, z) \in \mathcal{O}$ *with* $(y, z) \neq (\bar{y}, \bar{z})$ *we have* $z_i \notin \operatorname{cl} P_i(\bar{z})$ *for some* $i \in \{1, \ldots, n\}$.

(v) We omit the adjective "local" in (i)–(iv) *or replace it by* "global" *if* $\mathcal{O} = E^m \times E^n$ *above.*

It is clear from the definitions that (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii)$\Longrightarrow$(i) but not vice versa; the same implications hold for the global version in (v). Note that the notions of (both local and global) *weak Pareto* and *Pareto* optimal allocations are conventional in welfare economics. They correspond to similar Pareto-type concepts (weakly efficient and efficient solutions) for standard problems of vector optimization in the case of preferences given by utility functions; see, e.g., [14,20,22] and the references therein. The notion of *strong Pareto* optimal allocations was introduced by Khan [17] for models of welfare economics and then was intensively studied in [21,24,25,27]. We have not been familiar with any counterpart of this notion in general theory of multiobjective optimization before establishing such relationships in this paper. The notion of *strict Pareto* optimal allocations first appeared, to the best of our knowledge, in [27, Remark 8.15] as a modified version of the strong Pareto concept that shared some features of the latter. In this paper we show that local strict Pareto optimal allocations in welfare economics can be reduced in fact to the so-called fully localized minimizers in associated problems of constrained set-valued optimization. Note finally that if the extended preference sets $P_i(\bar{z}) \cup \{\bar{z}\}$ are *locally closed* around $\bar{z}$ for all $i = 1, \ldots, n$, then the notions of Pareto, weak Pareto, and strict Pareto optimal allocations are identical while they are generally *essentially different*.

Let us now formulate a *constrained set-valued optimization problem*, which we associate with the above model of welfare economics. Define a set-valued mapping $F\colon E^{m+1} \rightrightarrows E^n$ by

$$F(y,w) := \Big\{ z \in Z \,\Big|\, w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \Big\} \quad \text{with} \quad Z := E^n$$

and a constraint set $\Omega \subset E^{m+1}$ by

$$\Omega := \Big(\prod_{j=1}^{m} S_j \Big) \times W.$$

The multiobjective problem under consideration is written as

$$\text{minimize} \quad F(y,w) \quad \text{subject to} \quad (y,w) \in \Omega, \tag{1.3}$$

where "minimization" is understood with respect to some *preference* on Z determined by a level-set mapping $L\colon Z \rightrightarrows Z$ in the form

$$L(z) := \prod_{i=1}^{n} P_i(z).$$

We introduce several notions of optimal solutions to problem (1.3) as well as to more general problems of multiobjective optimization and establish their relationships with the Pareto-type optimal allocations in models of welfare economics from Definition 1.2. Employing advanced tools of *variational analysis* and *generalized differentiation* allows us to obtain new *necessary optimality conditions* to general multiobjective problems and then to apply them, through (1.3), to deriving new versions of the extended second welfare theorem in the economy $\mathcal{E}$ for all the above notions of local Pareto-type optimal allocations.

The rest of the paper is organized as follows. § 2 contains some basic definitions and *preliminaries* from variational analysis and generalized differentiation widely used in the sequel. In § 3 we introduce new *fully localized* versions of optimal solutions to general constrained set-valued optimization problems with respect to abstract *preference relations*. In fact, these notions are inspired by the local Pareto-type optimal allocations in welfare economics (1.1) defined above and broaden conventional notions of optimal solutions in multiobjective optimization. The main result of this section provides unified *first-order necessary conditions* for the introduced notions of optimal solutions to general multiobjective problems obtained under the new *asymptotic closedness* property of preference sets at local minimizers. The key ingredient of our device is the *extremal principle* of variational analysis in product spaces and the corresponding calculus rules of generalized differentiation [26, Chap. 3].

§ 4 is devoted to establishing precise relationships between *local Pareto-type* optimal allocations of the welfare economy (1.1) and optimal solutions to the constrained multiobjective problem (1.3). These allow us to derive from the first-order necessary optimality conditions of § 3 the new unified versions of the *extended second welfare theorem* for nonconvex economies under the asymptotic closedness property of the corresponding preference sets imposed at the local optimal allocations of the economy $\mathcal{E}$ from Definition 1.2. In this way *marginal prices* at Pareto-type optimal allocations in the obtained versions of the second welfare theorem are naturally interpreted as *generalized Lagrange multipliers* in necessary optimality conditions for multiobjective problems. Furthermore, in this section we establish efficient conditions for the validity of the asymptotic closedness property and its relationships with several versions of Mas-Colell's *uniform properness* in models of welfare economics.

The final § 5 concerns deriving *global* versions of the second welfare theorem for (global) *strict* and *strong* Pareto optimal allocations of the economy $\mathcal{E}$ under new *net demand qualification conditions*, which are weaker than the asymptotic closedness property of the preference sets imposed in § 4. We unify these results with appropriate versions of the second welfare theorem for *global* Pareto and weak Pareto optimal allocations of the nonconvex economies clarifying the corresponding results from [27, Chap. 8].

The notation of this paper is basically standard; see, e.g., [26, 27]. For a Banach space X, denote its norm by $\|\cdot\|$ and consider the dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ stands for the canonical pairing between X and X^*. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, recall that

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \;\Big|\; \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \\ \text{with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\Big\}$$

signifies the *sequential Painlevé–Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^*. Note finally that the symbol $I\!B$ stands for the closed unit ball of the space in question.

2. Tools of variational analysis and generalized differentiation

This section contains a brief overview of some basic tools of variational analysis and generalized differentiation widely used in formulations and proofs of the major results of the paper. We mainly follow the books [26, 27] referring the reader also to [8, 13, 18, 30, 31] and the bibliographies therein for related and additional material.

Since the major results of the paper using generalized differentiation require the Asplund structure of the Banach spaces in question, we present the definition of our basic generalized differential constructions in the Asplund space framework. Recall that a Banach space X is *Asplund* if each of its separable subspaces has a separable dual. There are many other equivalent descriptions of the original Asplund property, which can be found, e.g., in [26, Chap. 2]. Observe, in particular, that every reflexive Banach space is Asplund.

Let $\Omega \subset X$ be a subset of an Asplund space X locally closed around some point $\bar{x} \in \Omega$. The (basic, limiting, Mordukhovich) *M-normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} \widehat{N}(x;\Omega) \tag{2.4}$$

via the sequential Painlevé–Kuratowski outer limit of the so-called Fréchet/regular normal cone

$$\widehat{N}(x;\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \le 0 \right\}, \quad x \in \Omega, \tag{2.5}$$

where the symbol $u \overset{\Omega}{\to} x$ means that $u \to x$ with $u \in \Omega$, and where $\widehat{N}(x;\Omega) : = \emptyset$ for $x \notin \Omega$. If X is a finite-dimensional Euclidean space and if Ω is *locally closed* around $\bar{x} \in \Omega$ (i.e., there is a neighborhood U of $\bar{x}$ such that the set $\Omega \cap U$ is closed), then the M-normal cone (2.4) admits the equivalent representation

$$N(\bar{x};\Omega) = \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} \big[\mathrm{cone}\big(x - P(x;\Omega)\big)\big]$$

via the Euclidean projector $P(\cdot;\Omega)$ to Ω, where "cone" stands for the conic hull of a set.

It is easy to check by the definitions that both the M-normal cone (2.4) and the Fréchet one (2.5) admit the product property

$$N(\bar{x};\prod_{i=1}^{n}\Omega_i) = \prod_{i=1}^{n} N(\bar{x}_i;\Omega_i), \quad \widehat{N}(\bar{x};\prod_{i=1}^{n}\Omega_i) = \prod_{i=1}^{n} \widehat{N}(\bar{x}_i;\Omega_i). \tag{2.6}$$

In contrast to (2.5), the M-normal cone (2.4) is often *nonconvex*, while it and the associated coderivative and subdifferential constructions enjoy *full calculus* in Asplund spaces that is mainly based on the *variational/extremal principles* of variational analysis.

One of the most important ingredients of variational analysis in infinite-dimensional spaces is the necessity to impose some "normal compactness" properties, which are automatic in finite dimensions and allow us, in particular, to perform limiting procedures of deriving nontrivial calculus rules and optimality conditions. Recall that a set $\Omega \subset X$ in the product space $X = \prod_{i=1}^{n} X_i$ is *partially sequentially normally compact* (PSNC) at $\bar{x} \in \operatorname{cl}\Omega$ with respect to $\{X_i \mid i \in I\}$ as $I \subset \{1,\ldots,n\}$ (or with respect to the indices I) if for any sequences of elements $(x_k, x_k^*) \in X \times X^*$ with $x_k = (x_{1k},\ldots,x_{nk})$ and $x_k^* = (x_{1k}^*,\ldots,x_{nk}^*)$ satisfying

$$x_k \overset{\Omega}{\to} \bar{x} \quad \text{and} \quad x_k^* \in \widehat{N}(x_k;\Omega) \ \text{ for all } \ k \in I\!N \tag{2.7}$$

we have the implication

$$\left[x_{ik}^* \overset{w^*}{\to} 0, i \in I, \text{ and } \|x_{ik}^*\| \to 0, i \in \{1,\ldots,n\}\backslash I\right] \Longrightarrow \|x_{ik}^*\| \to 0, i \in I.$$

The *strong PSNC* property of Ω at $\bar{x}$ with respect to $\{X_i \mid i \in I\}$ with $I \subset \{1,\ldots,n\}$ means that for any sequences (x_k, x_k^*) satisfying (2.7) we have the implication

$$\left[x_{ik}^* \overset{w^*}{\to} 0,\ i \in \{1,\ldots,n\}\right] \quad \Longrightarrow \quad \|x_{ik}^*\| \to 0,\ i \in I.$$

In the extreme case of $I = \{1,\ldots,n\}$, both PSNC and strong PSNC properties do not depend on the product structure and reduce to the *sequentially normally compact* (SNC) property of Ω at $\bar{x}$.

We refer the reader to [26] for efficient conditions ensuring the fulfillment of these properties and their preservation under various operations; the latter *SNC calculus* is based on the extremal principle of variational analysis. Let us mention, in particular, that a set Ω is SNC at $\bar{x} \in \Omega$ if it is compactly epi-Lipschitzian (CEL) around this point. Moreover, both properties agree for a broad class of Banach spaces; see, e.g., [8, 13, 26] for more details and references.

Given a set-valued mapping $F\colon X \rightrightarrows Z$ between Banach spaces, we associate it with the graph

$$\operatorname{gph} F := \left\{(x,z) \in X \times Z \,\middle|\, z \in F(x)\right\},$$

which is a subset of the product space. It is well known that the product of Asplund spaces is also Asplund. The (basic) *coderivative* of F at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a positively homogeneous mapping $D^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ defined via the M-normal cone (2.4) by

$$D^*F(\bar{x},\bar{z})(z^*) := \left\{x^* \in X^* \,\middle|\, (x^*,-z^*) \in N\big((\bar{x},\bar{z});\operatorname{gph} F\big)\right\}. \tag{2.8}$$

When $F = f: X \to Z$ is single-valued at $\bar{z} = f(\bar{x})$, we omit $\bar{z}$ in the coderivative notation (2.8). If $f: X \to Z$ is strictly differentiable at $\bar{x}$ with the derivative $\nabla f(\bar{x}): X \to Z$, in the sense that

$$\lim_{x,u \to \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u \rangle}{\|x - u\|} = 0$$

(which is automatic when it is C^1 around this point), then we have

$$D^* f(\bar{x})(z^*) = \left\{ \nabla f(\bar{x})^* z^* \right\} \text{ for all } z^* \in Z^*,$$

i.e., the coderivative (2.8) reduces to the adjoint derivative operator. In [26, 27] and the references therein, the reader can find equivalent descriptions of the coderivative (2.8) and its numerous properties including comprehensive calculus rules and computations for various classes of mappings in finite and infinite dimensions.

Employing the above SNC/PSNC properties to the graphical set $\operatorname{gph} F \subset X \times Z$, we say that:

- F is SNC at $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ if the set $\operatorname{gph} F$ is SNC at this point.
- F is PSNC at $(\bar{x}, \bar{z})$ if the set $\operatorname{gph} F$ is PSNC with respect to X at $(\bar{x}, \bar{z})$.
- F^{-1} is PSNC at $(\bar{z}, \bar{x})$ if the set $\operatorname{gph} F$ is PSNC with respect to Z at $(\bar{x}, \bar{z})$.

Recall that a mapping $F: X \rightrightarrows Z$ between Banach spaces is PSNC at $(\bar{x}, \bar{z})$ if it exhibits the *Lipschitz-like property* around this point, in the sense that there is a constant $\ell \geq 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$F(x) \cap V \subset F(u) + \ell \|x - u\| I\!B \text{ for all } x, u \in U,$$

where $I\!B$ stands for the closed unit ball of Z. This property, which is also known as the Aubin or pseudo-Lipschitzian property (cf. the discussions in [26, 30]), agrees with the classical local Lipschitzian behavior in the case of single-valued mappings and reduces to the standard (Hausdorff) local Lipschitzian property of set-valued mappings when $V = Z$. Furthermore, it is equivalent to both *metric regularity* and *linear openness* properties of the inverse mapping F^{-1}; see the afore-mentioned books and the references therein.

The main tool of our analysis in this paper is the *extremal principle* for systems of sets that can be treated as a far-going *variational* counterpart of the classical separation theorem in the case of nonconvex sets involving the M-normal cone (2.4); see [26, Chap. 2]. In fact, we need its extended version in product spaces established in [27, Lemma 5.58]. Recall that $\bar{x} \in \Omega_1 \cap \Omega_2$ is a *local extremal point* of the set system $\{\Omega_1, \Omega_2\}$ in X if there exists a neighborhood V of $\bar{x}$ such that for any $\varepsilon > 0$ we can find $a \in \varepsilon I\!B$ with

$$\Omega_1 \cap \big(\Omega_2 + a\big) \cap V = \emptyset. \tag{2.9}$$

The Extremal Principle. *Let* $\bar{x}$ *be a local extremal point of the set system* $\{\Omega_1, \Omega_2\}$*, where both* Ω_1 *and* Ω_2 *are locally closed around* $\bar{x}$ *in the product* $\prod_{i=1}^{n} X_i$ *of Asplund spaces* X_i*,* $i = 1, \ldots, n$*. Take two index sets* $I, J \subset \{1, \ldots, n\}$ *with* $I \cup J = \{1, \ldots, n\}$ *and assume that one of the following PSNC conditions is satisfied for* Ω_1 *and* Ω_2*:*

- Ω_1 *is PSNC at* $\bar{x}$ *with respect to* I *and* Ω_2 *is strongly PSNC at* $\bar{x}$ *with respect to* J*.*
- Ω_1 *is strongly PSNC at* $\bar{x}$ *with respect to* I *and* Ω_2 *is PSNC at* $\bar{x}$ *with respect to* J*.*

Then there is $x^* \in X^*$ *such that*

$$0 \neq x^* \in N(\bar{x}; \Omega_1) \cap \big(-N(\bar{x}; \Omega_2)\big). \tag{2.10}$$

This result is a modified version of [27, Lemma 5.58] in the product space $(\prod_{i \in I} X_i) \times (\prod_{j \in J} X_i)$ with the corresponding index sets I and J therein satisfying the condition $I \cap J = \emptyset$.

3. Solution notions and necessary optimality conditions in constrained multiobjective optimization

In this section we study a general problem of *set-valued optimization with geometric constraints*

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in \Omega, \tag{3.1}$$

where the cost mapping $F\colon X \rightrightarrows Z$ is set-valued between Banach spaces, $\Omega \subset X$ is a subset in X, and "minimization" in (3.1) is understood with respect to some *preference relation* on Z. Usually a general/abstract preference relation on Z is defined as follows; see, e.g., [27, § 5.3.1]. Given a subset $Q \subset Z \times Z$, we say that z_1 is preferred to z_2 and write $z_1 \prec z_2$ if $(z_1, z_2) \in Q$.

Consider the *level-set* mapping $L\colon Z \rightrightarrows Z$ associated with the preference $\prec$ by

$$L(z) := \big\{u \in Z \big| \, u \prec z\big\}. \tag{3.2}$$

To be useful in multiobjective optimization and applications, abstract preferences $\prec$ have to satisfy some requirements. The three properties imposed in

[27, Definition 5.55] to postulate the notion of "closed preference relations" are as follows:

- The preference $\prec$ is *nonreflexive* meaning that $(z,z) \notin Q$ for all $z \in Z$.
- Given some $\bar{z} \in Z$ (a local minimizer in the sequel), the preference $\prec$ is *locally satiated* around $\bar{z}$ meaning that $z \in \operatorname{cl} L(z)$ for all z in a neighborhood of $\bar{z}$.
- The preference $\prec$ is *almost transitive* in the sense that

$$\big[u \in L(z),\ v \in \operatorname{cl} L(u)\big] \Longrightarrow v \in L(z). \tag{3.3}$$

The almost transitivity property is widely used (under different names) in the study of multiobjective optimization problems; see, e.g., [5,27,28,32,33] and the references therein. However, it turns out to be rather restrictive, in contrast to the first two properties of $\prec$ formulated above. In particular, for the so-called "generalized Pareto optimality" defined by

$$z_1 \prec z_2 \Longleftrightarrow z_2 - z_1 \in \Theta \setminus \{0\} \tag{3.4}$$

via an ordering cone $\Theta \subset Z$, the preference $\prec$ is almost transitive if and only if Θ is convex and pointed, i.e., $\Theta \cap (-\Theta) = \{0\}$; see [27, Proposition 5.56]. It happens that we do not have such a property for the lexicographical order on $I\!R^n$ and other natural preference relations important in multiobjective optimization and its applications including those to welfare economics.

In this section we develop another approach to preference relations in multiobjective optimization that is motivated by applications to models of welfare economics and allows us to avoid the almost transitivity restriction (3.3), which does not usually hold for the economies described in § 1. Similarly to the setting of (1.1), this approach operates not in terms of a preference $\prec$ on Z described via a subset $Q \subset Z \times Z$, but directly in terms of a given *preference mapping* $L\colon Z \rightrightarrows Z$ defined as follows: $u \in Z$ is preferred to z if $u \in L(z)$. The latter corresponds to the level-set mapping (3.2) determined by a given preference $\prec$.

In what follows we fully drop the almost transitivity properties from [27, Definition 5.55] and require the most natural local satiation property of the preference mapping L formulated *at* the reference optimal solution instead of its "around" counterpart presented above. Furthermore, in this way we introduce the following notions of optimal solutions to the constrained multiobjective problem (3.1) described directly via the preference set $L(\bar{z})$ at the reference point.

Definition 3.1 (fully localized minimizers for constrained multiobjective problems). *Let* $(\bar{x},\bar{z}) \in \operatorname{gph} F$ *with* $\bar{x} \in \Omega$. *Then we say that:*

(i) $(\bar{x}, \bar{z})$ *is a* fully localized weak minimizer *for* (3.1) *if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that there is no $z \in F(\Omega \cap U) \cap V$ preferred to $\bar{z}$, i.e.,*

$$F(\Omega \cap U) \cap L(\bar{z}) \cap V = \emptyset \quad \textit{with} \quad F(\Omega \cap U) \\ := \bigcup \big\{F(x) \big| \; x \in \Omega \cap U\big\}. \tag{3.5}$$

(ii) $(\bar{x}, \bar{z})$ *is a* fully localized minimizer *for* (3.1) *if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that there is no $z \in F(\Omega \cap U) \cap V$ with $z \neq \bar{z}$ and $z \in \operatorname{cl} L(\bar{z})$, i.e.,*

$$F(\Omega \cap U) \cap \operatorname{cl} L(\bar{z}) \cap V = \big\{\bar{z}\big\}. \tag{3.6}$$

(iii) $(\bar{x}, \bar{z})$ *is a* fully localized strong minimizer *for* (3.1) *if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that there is no $(x, z) \in \operatorname{gph} F \cap (U \times V)$ with $(x, z) \neq (\bar{x}, \bar{z})$ satisfying $x \in \Omega$ and $z \in \operatorname{cl} L(\bar{z})$, i.e.,*

$$\operatorname{gph} F \cap \big(\Omega \times \operatorname{cl} L(\bar{z})\big) \cap (U \times V) = \big\{(\bar{x}, \bar{z})\big\}. \tag{3.7}$$

It is easy to see that (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i) in Definition 3.1. If $\Omega = X$ therein, we speak about the corresponding fully localized minimizers for the mapping F.

The underlying feature of all the notions in Definition 3.1 is that we introduce the *image localization* of minimizers in constructions (3.5)–(3.7). This seems to be new even for minimizers and weak minimizers of single-valued objectives $F = f : X \to Z$ and allows us to study *local* Pareto-type optimal allocations of welfare economies; see § 4. To the best of our knowledge, the concept of (global or local) *strong minimizers* has not been considered earlier in the literature on multiobjective optimization; it is inspired here by Khan's notion of strong Pareto optimal allocations for models of welfare economics; see Definition 1.2(iii) and the corresponding relationships established in § 4.

Let us illustrate some particular features of the optimality notions introduced in Definition 3.1. Observe first that a *fully* localized strong minimizer may not provide a *partially* localized (i.e., with $V = Z$) minimum or weak minimum in (3.5) and (3.6). Indeed, considering a set-valued mapping $F : I\!R \rightrightarrows I\!R$ defined by

$$F(x) := \begin{cases} -x & \text{if } \; x < 0, \\ \{0, 1\} & \text{if } \; x = 0, \\ x + 1 & \text{if } \; x > 0, \end{cases}$$

with respect to the usual order on $I\!R$ generating the level sets $L(z) = (-\infty, z)$, it is not hard to check that condition (3.7) is fulfilled at $(0, 1)$ with

$U = (-1/2, 1/2)$ and $V = (1/2, 3/2)$ while conditions (3.5) and (3.6) do not hold with $V = I\!R$.

Observe further that the localized minimizers and weak minimizers in Definition 3.1 are identical in the case of scalar set-valued optimization with $Z = I\!R$ and $L(z) = (-\infty, z)$, but they may be quite different in the vector case. For example, considering a cost mapping $F\colon I\!R \rightrightarrows I\!R^2$ with $F(x) \equiv I\!R^2 \setminus \operatorname{int} I\!R^2_-$ and the usual weak Pareto preference on $I\!R^2$ with the level sets $L(z) = z - \operatorname{int} I\!R^2_-$, we have that $(0,0) \in I\!R \times I\!R^2$ is a localized weak minimizer for F, but it is not a localized minimizer for this mapping. Note finally that localized strong minimizers reduce to standard *isolate* minimizers for scalar single-valued optimization problems.

To derive necessary conditions for the fully localized minimizers of the set-valued optimization problem defined above, we develop a variational approach based on the *extremal principle* and *generalized differential calculus* [26]. Besides [27, Chap. 5], this approach has been recently implemented in [1, 2, 4] for establishing necessary conditions for various type of minimizers in multiobjective optimization. Employing this approach in the setting under consideration, we intend first to find appropriate assumptions ensuring that minimizers $(\bar{x}, \bar{z})$ from Definition 3.1 reduce to *local extremal points* of the set system $\{\operatorname{gph} F, \Omega \times \operatorname{cl} L(\bar{z})\}$. Let us now introduce a property of sets that allows us to efficiently proceed in this direction.

Definition 3.2 (asymptotic closedness property). *Let $\Xi \subset Z$ be a subset in a Banach space Z, and let $\bar{z} \in \operatorname{cl} \Xi$. We say that Ξ is* asymptotically closed *at $\bar{z}$ if there is a neighborhood V of $\bar{z}$ and for any $\varepsilon > 0$ there is $c \in \varepsilon I\!B$ such that*

$$(\operatorname{cl} \Xi + c) \cap V \subset \Xi \setminus \{\bar{z}\}. \tag{3.8}$$

This property (which is fully independent of the local closedness of a set) holds in many rather general settings and is also satisfied under natural assumptions in models of welfare economics; see § 4. Let us present some sufficient conditions ensuring the fulfillment of the asymptotic closedness of sets while referring the reader to [3] for more results, discussions, and applications:

- Every proper convex subcone $\Xi \subset Z$ with nonempty interior and its nonconvex complement $Z \setminus \Xi$ have the asymptotic closeness property at the origin.
- Every closed, convex, cone $\Xi \subset Z$ with $\Xi \setminus (-\Xi) \neq \emptyset$ has the asymptotic closedness property at the origin.
- The epigraph of an extended-real-valued function $\varphi\colon X \to I\!R \cup \{\infty\}$ has the asymptotic closedness property at $(\bar{x}, \varphi(\bar{x}))$ provided that φ is lower semicontinuous around $\bar{x}$.

Note that the above condition $\Xi \setminus (-\Xi) \neq \emptyset$ means that the cone Ξ is not a linear subspace of Z; this is more general than the pointedness requirement $\Xi \cap (-\Xi) = \{0\}$ meaning that Ξ does not contain a linear subspace. As has been already mentioned, the generalized Pareto optimality (3.4) defined by a closed convex cone Θ does not correspond to a closed preference relation unless the ordering cone Θ is pointed, which is not required by the asymptotic closedness property.

The next result needed in what follows ensures the asymptotic closedness property of product sets via this property of a part of their components.

Proposition 3.3 (asymptotic closedness property of Cartesian products of sets). *Let* $\bar{z} \in \operatorname{cl} \prod_{i=1}^{n} \Xi_i \subset \prod_{i=1}^{n} Z_i$ *in the Banach space setting, let* $I \subset \{1, \ldots, n\}$ *be a nonempty index set, and let* $J := \{1, \ldots, n\} \setminus I$. *Assume that the sets* Ξ_i *are asymptotically closed at* $\bar{z}_i \in \operatorname{cl} \Xi_i$ *for* $i \in I$ *while the other sets* Ξ_j *are locally closed around* $\bar{z}_j$ *for* $j \in J$. *Then the product set*

$$\Xi := \prod_{i=1}^{n} \Xi_i \tag{3.9}$$

enjoys the asymptotic closedness property at $\bar{z}$.

Proof. Without loss of generality, assume that $I = \{1, \ldots, m\}$ with some $0 < m \leq n$. Since for each $i \in I$ the set Ξ_i is asymptotically closed at $\bar{z}_i$, there is a neighborhood U_i of $\bar{z}_i$ such that, whenever $\varepsilon > 0$, we have

$$(\operatorname{cl} \Xi_i + c_i) \cap U_i \subset \Xi_i \setminus \{\bar{z}_i\} \text{ with some } c_i \in \varepsilon I\!B_{Z_i}, \quad i \in I.$$

On the other hand, by the assumed local closedness of Ξ_j around $\bar{z}_j$, for each $j \in J$ we find a neighborhood U_j of $\bar{z}_j$ such that

$$\operatorname{cl} \Xi_j \cap U_j \subset \Xi_j, \quad j \in J.$$

It is obvious that the set $U := \prod_{i \in I} U_i \times \prod_{j \in J} U_j$ is a neighborhood of $\bar{z}$ in the product of Banach spaces $Z := \prod_{i=1}^{n} Z_i$ equipped with the maximum norm. Furthermore, for any number $\varepsilon > 0$ there is $c := (c_1, \ldots, c_m, 0, \ldots, 0) \in \varepsilon I\!B_Z$ satisfying

$$\begin{aligned}(\operatorname{cl} \Xi + c) \cap U &= \Big(\prod_{i \in I} (\operatorname{cl} \Xi_i + c_i) \cap U_i\Big) \times \Big(\prod_{j \in J} (\operatorname{cl} \Xi_j \cap U_j)\Big) \\ &\subset \Big(\prod_{i \in I} \big(\Xi_i \setminus \{\bar{z}_i\}\big)\Big) \times \Big(\prod_{j \in J} \Xi_j\Big) \subset \Big(\prod_{i=1}^{n} \Xi_i\Big) \setminus \{\bar{z}\} = \Xi \setminus \{\bar{z}\},\end{aligned}$$

where the last inclusion holds due to $I \neq \emptyset$. This gives (3.8) and thus justifies the asymptotic closedness property of Ξ at $\bar{z}$. □

Now we are ready to derive first-order necessary optimality conditions for all the types of fully localized minimizers from Definition 3.1 for the constrained multiobjective problem (3.1).

Theorem 3.4 (first-order necessary conditions for localized minimizers in multiobjective optimization). *Let* $F\colon X \rightrightarrows Z$ *be a set-valued mapping between Asplund spaces with the graph* $\operatorname{gph} F$ *locally closed around some point* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$, *let* $\Omega \subset X$ *be locally closed around* $\bar{x}$, *and let* $L\colon Z \rightrightarrows Z$ *be a preference mapping on* Z *locally satiated at* $\bar{z}$. *Impose also one of the following SNC assumptions on the initial data* (F, Ω, L):

(a) $\operatorname{gph} F$ *is SNC at* $(\bar{x}, \bar{z})$.
(b) Ω *is SNC at* $\bar{x}$ *and* $\operatorname{cl} L(\bar{z})$ *is SNC at* $\bar{z}$.
(c) F *is PSNC at* $(\bar{x}, \bar{z})$ *and* $\operatorname{cl} L(\bar{z})$ *is SNC at* $\bar{z}$.
(d) Ω *is SNC at* $\bar{x}$ *and* F^{-1} *is PSNC at* $(\bar{z}, \bar{x})$.

Then there is a pair $(0,0) \neq (x^*, z^*) \in X^* \times Z^*$ *satisfying the necessary optimality conditions*

$$x^* \in D^*F(\bar{x}, \bar{z})(z^*) \cap \big(-N(\bar{x}; \Omega)\big) \ \text{ and } \ z^* \in N\big(\bar{z}; \operatorname{cl} L(\bar{z})\big) \tag{3.10}$$

in each of the following cases:

- $(\bar{x}, \bar{z})$ *is a* fully localized weak minimizer *for* (3.1) *provided that the set* $L(\bar{z})$ *is asymptotically closed at* $\bar{z} \in \operatorname{cl} L(\bar{z})$.
- $(\bar{x}, \bar{z})$ *is a* fully localized minimizer *for* (3.1) *provided that the set* $\operatorname{cl} L(\bar{z})$ *is asymptotically closed at* $\bar{z}$.
- $(\bar{x}, \bar{z})$ *is a* fully localized strong minimizer *for* (3.1) *provided that either* $\operatorname{cl} L(\bar{z})$ *or* Ω *is asymptotically closed at* $\bar{z}$ *or* $\bar{x}$, *respectively.*

Proof. Arguing in a unifying way, take any fully localized minimizer $(\bar{x}, \bar{z})$ for problem (3.1) considered in the theorem and reduce it to a local extremal point of some system of sets in the product space $X \times Z$. Indeed, define the sets $\Omega_1, \Omega_2 \subset X \times Z$ by

$$\Omega_1 := \operatorname{gph} F \quad \text{and} \quad \Omega_2 := \Omega \times \operatorname{cl} L(\bar{z}) \tag{3.11}$$

and observe that they are locally closed around $(\bar{x}, \bar{z})$ in the Asplund space $X \times Z$ under the local closedness assumptions made. Let us check that $(\bar{x}, \bar{z})$ is a *local extremal point* of the system $\{\Omega_1, \Omega_2\}$ in each case of fully localized minimizers under consideration. We first conclude that $(\bar{x}, \bar{z}) \in \Omega_1 \cap \Omega_2$ due to the *local satiation* property of the preference mapping L at $\bar{z}$.

Next let us show that there is a sequence $\{a_k\} \subset X \times Z$ with $a_k \to 0$ as $k \to \infty$ such that

$$\Omega_1 \cap (\Omega_2 + a_k) \cap \mathcal{O} = \emptyset \text{ for all } k \in I\!N, \tag{3.12}$$

where $\mathcal{O}$ is a neighborhood of $(\bar{x}, \bar{z})$ specified later. We choose an appropriate sequence $\{a_k\}$ in (3.12) in the following way for each type of local minimizers considered in the theorem.

• Let $(\bar{x}, \bar{z})$ be a fully localized *weak minimizer* for problem (3.1). Since $L(\bar{z})$ is assumed to be *asymptotically closed* at $\bar{z}$ in this case, there exist a neighborhood $\widetilde{V}$ of $\bar{z}$ and a sequence $\{c_k\} \subset Z$ with $c_k \to 0$ as $k \to \infty$ such that

$$\big(\mathrm{cl}\, L(\bar{z}) + c_k\big) \cap \widetilde{V} \subset L(\bar{z}) \setminus \{\bar{z}\} = L(\bar{z}). \tag{3.13}$$

Put $a_k := (0, c_k) \in X \times Z$ for all $k \in I\!N$ and $\mathcal{O} := U \times (V \cap \widetilde{V})$, where U is a neighborhood of $\bar{x}$ and V is a neighborhood of $\bar{z}$ from definition (3.5) of the localized weak minimality of $(\bar{x}, \bar{z})$. Then we get from (3.13) and the structures of Ω_1, Ω_2 in (3.11) that

$$\begin{aligned} &\Omega_1 \cap (\Omega_2 + a_k) \cap \mathcal{O} \\ &\quad = \mathrm{gph}\, F \cap \big(\Omega \times (\mathrm{cl}\, L(\bar{z}) + c_k) \cap \widetilde{V}\big) \cap (U \times V) \\ &\quad \subset \mathrm{gph}\, F \cap \big(\Omega \times (L(\bar{z}) \setminus \{\bar{z}\})\big) \cap (U \times V) = \emptyset, \end{aligned} \tag{3.14}$$

where the last equality is due to (3.5). This justifies the extremality condition (3.12) in the case of fully localized weak minimizers.

• Let $(\bar{x}, \bar{z})$ is a fully localized *minimizer* for problem (3.1). In this case we use the same arguments as for weak minimizers above replacing now the set $L(\bar{z})$ in (3.13) and in the last line of (3.14) by its closure $\mathrm{cl}\, L(\bar{z})$. This can be done, since the set $\mathrm{cl}\, L(\bar{z})$ is assumed to be *asymptotically closed* at $\bar{z}$. Thus we get relationship (3.12) in the case of fully localized minimizers.

• Let $(\bar{x}, \bar{z})$ is a fully localized *strong minimizer* for problem (3.1). Applying Proposition 3.3 to the product set $\Omega_2 = \Omega \times \mathrm{cl}\, L(\bar{z})$, we get from the imposed assumptions on the sets Ω and $\mathrm{cl}\, L(\bar{z})$ in this case that Ω_2 is *asymptotically closed* at $(\bar{x}, \bar{z})$. Hence there are a neighborhood $\mathcal{O}$ of $(\bar{x}, \bar{z})$ (without loss of generality we assume that $\mathcal{O} \subset U \times V$) and a sequence $\{a_k\} \subset X \times Z$ with $a_k \to 0$ as $k \to \infty$ such that

$$(\Omega_2 + a_k) \cap \mathcal{O} \subset \Omega_2 \setminus \big\{(\bar{x}, \bar{z})\big\}. \tag{3.15}$$

Thus we get from (3.15) that

$$\begin{aligned}
&\Omega_1 \cap (\Omega_2 + a_k) \cap \mathcal{O} \\
&\quad = \Omega_1 \cap \big((\Omega_2 + a_k) \cap \mathcal{O}\big) \cap \mathcal{O} \\
&\quad \subset \Omega_1 \cap \big(\Omega_2 \backslash \{(\bar{x}, \bar{z})\}\big) \cap (U \times V) \\
&\quad = \operatorname{gph} F \cap \big((\Omega \times \operatorname{cl} L(\bar{z})) \backslash \{(\bar{x}, \bar{z})\}\big) \cap (U \times V) = \emptyset,
\end{aligned}$$

where the last equality holds by the strong optimality (3.7). This justifies the extremality condition (3.12) in the strong minimum case and shows that $(\bar{x}, \bar{z})$ is a local extremal point of the set system $\{\Omega_1, \Omega_2\}$ in all the cases under consideration.

To proceed further, we apply to the system $\{\Omega_1, \Omega_2\}$ at $(\bar{x}, \bar{z})$ the *extremal principle* in the product of Asplund spaces $X \times Z$ formulated in § 2. Observe that each of the SNC/PSNC conditions (a)–(d) imposed in the theorem ensure the fulfillment of the PSNC conditions for the sets Ω_1 and Ω_2 required in the extremal principle. Indeed, denoting $X_1 := X$ and $X_2 := Z$ therein, we have the following:

- Ω_1 is strongly PSNC at $(\bar{x}, \bar{z})$ with respect to $I = \{1, 2\}$ if condition (a) holds.
- Ω_2 is strongly PSNC at $(\bar{x}, \bar{z})$ with respect to $J = \{1, 2\}$ if the conditions in (b) hold.
- Ω_1 is PSNC at $(\bar{x}, \bar{z})$ with respect to $I = \{1\}$ and Ω_2 is strongly PSNC at $(\bar{x}, \bar{z})$ with respect to $J = \{2\}$ if the conditions in (c) hold.
- Ω_1 is PSNC at $(\bar{x}, \bar{z})$ with respect to $I = \{2\}$ and Ω_2 is strongly PSNC at $(\bar{x}, \bar{z})$ with respect to $J = \{1\}$ if the conditions in (d) hold.

Thus the relationships in (3.10) are followed from (2.10) by taking into account the definition of coderivative (2.8) and the exact formula for the M-normal cone of product sets in (2.6). That is, there is $(x^*, z^*) \in X^* \times Z^*$ with $(x^*, z^*) \neq 0$ satisfying

$$(-x^*, z^*) \in N\big((\bar{x}, \bar{z}); \Omega_2\big) = N(\bar{x}; \Omega) \times N\big(\bar{z}; \operatorname{cl} L(\bar{z})\big),$$

$$(x^*, -z^*) \in N\big((\bar{x}, \bar{z}); \operatorname{gph} F\big), \quad \text{i.e.,} \quad x^* \in D^* F(\bar{x}, \bar{z})(z^*),$$

which completes the proof of the theorem. □

Remark 3.5 (PSNC conditions in Theorem 3.4). Note first that the PSNC property of F in (c) and this property of F^{-1} in (d) hold *automatically* provided that F is *Lipschitz-like* around $(\bar{x}, \bar{z})$ and it is *metrically regular* around this point, respectively; see [26, Theorems 4.10 and 4.18]. Note further that in the case of the product spaces $X = \prod_{i=1}^{n} X_i$ and $Z = \prod_{j=1}^{m} Z_j$, conditions (a)–(d) of Theorem 3.4 can be replaced by the following:

- Ω_1 is *PSNC* at $(\bar{x}, \bar{z})$ with respect to some $I \subset \{1, \dots, n; 1, \dots, m\}$ and Ω_2 is *strongly PSNC* $(\bar{x}, \bar{z})$ with respect to some $J \subset \{1, \dots, n; 1, \dots, m\}$, where $I \cup J = \{1, \dots, n; 1 \dots, m\}$.

This follows directly from the proof of Theorem 3.4 due to the employed extremal principle. We use this observation below in the proof of Theorem 4.3 for models of welfare economics.

Remark 3.6 (comparison of Theorem 3.4 with known results). To the best of our knowledge, the most advanced first-order necessary optimality conditions for multiobjective problems with respect to abstract preference relations are given in [27, Theorem 5.73]; see also the references and commentaries therein. Besides the fact that our new Theorem 3.4 addresses *set-valued* (vs. single-valued as in [27]) multiobjective optimization problems, it has the following clear advantages over the corresponding pointwise necessary optimality conditions of [27, Theorem 5.73]:

- In contrast to [27, Theorem 5.73] that concerns only Pareto-type minimizers with no image localization, we equally deal in Theorem 3.4 with *fully localized minimizers, weak minimizers*, and *strong minimizers* for problem (3.1).
- We now *drop* the *almost transitivity* requirements imposed on the preferences under consideration satisfying the *asymptotic closedness* property, which is largely motivated by applications to welfare economics.
- The results of [27, Theorem 5.73] establishes necessary optimality conditions for problem (3.1) with single-valued objectives under the SNC assumption (a) of Theorem 3.4 and a version of (b) therein, where the SNC property of the set $\operatorname{cl} L(\bar{z})$ at $\bar{z}$ is replaced by the *more restrictive* ISNC property of the level-set mapping $L\colon Z \rightrightarrows Z$ at $(\bar{z}, \bar{z})$; see [27, Definition 5.71]. There are *no counterparts* of assumptions (c) and (d) in [27, Theorem 5.73].
- Necessary conditions of type (3.10) are obtained in [27, Theorem 5.73] via the *extended normal cone* to the set $\operatorname{cl} L(\bar{z})$ at $\bar{z}$ defined by

  $$N_+\big(\bar{z}; \operatorname{cl} L(\bar{z})\big) := \mathop{\operatorname{Lim\,sup}}_{\substack{(z,u)\to(\bar{z},\bar{z})\\ (z,u)\in\operatorname{gph}(\operatorname{cl} L)}} \widehat{N}\big(u; \operatorname{cl} L(z)\big)$$

  instead of $N\big(\bar{z}; \operatorname{cl} L(\bar{z})\big)$ in Theorem 3.4. We always have the inclusion

  $$N\big(\bar{z}; \operatorname{cl} L(\bar{z})\big) \subset N_+\big(\bar{z}; \operatorname{cl} L(\bar{z})\big),$$

  which is strict in general; see [27, § 5.3] for more results and discussions.

4. Local versions of the second welfare theorem

This section is devoted to applications of necessary optimality conditions in constrained set-valued optimization obtained in § 3 to the general nonconvex model of welfare economics described in § 1. First of all, we reduce the welfare economic model (1.1) to a special set-valued optimization problem of type (3.1) and establish relationships between the notions of *fully localized* minimizers for (3.1) from Definition 3.1 and the *local* Pareto-type optimal allocations of (1.1) given in Definition 1.2. Then we use the necessary optimality conditions for multiobjective problems obtained in Theorem 3.4 to derive new versions of the *second fundamental theorem of welfare economics* for local *weak, strict*, and *strong* Pareto optimal allocations of (1.1) under economic counterparts of the *asymptotic closedness* condition. Finally, we establish relationships between economic counterparts of asymptotic closedness and previously developed properties in this vein (including several nonconvex versions of *Mas-Colell's properness*), which allow us to compare the new results with those known in the literature.

Let us consider the following set-valued optimization problem in form (3.1) with $X = E^{m+1}$, $x = (y, w)$, and $Z := E^n$ constructed upon the initial data of the welfare economy $\mathcal{E}$:

$$\begin{cases} \text{minimize} \;\; F(x) := \left\{ z \in Z \,\middle|\, w = \sum\limits_{i=1}^{n} z_i - \sum\limits_{j=1}^{m} y_j \right\} \\ \text{subject to} \;\; x \in \Omega := \Big(\prod\limits_{j=1}^{m} S_j \Big) \times W \subset X, \end{cases} \tag{4.1}$$

where "minimization" is understood with respect to the *preference/level-set mapping* $L\colon Z \rightrightarrows Z$ defined in the product form

$$L(z) := \prod_{i=1}^{n} P_i(z), \quad z \in Z, \tag{4.2}$$

via the given preference mappings $P_i\colon Z \rightrightarrows E$ of the welfare economy $\mathcal{E}$.

The first result of this section shows that the notions of local weak Pareto, strict Pareto, and strong Pareto optimal allocations of the welfare economy (1.1) from Definition 1.2 are *equivalent* to, respectively, the notions of fully localized weak minimizers, minimizers, and strong minimizers in the sense of Definition 3.1 for the set-valued optimization problem (4.1). This will allow us to derive new versions of the second welfare theorem for (1.1) from the corresponding necessary optimality conditions of Theorem 3.4. Note that we do not provide below an equivalent description of local Pareto optimal allocations of the economy $\mathcal{E}$ via multiobjective optimization, just observing that the version of the second welfare theorem obtained in this section for local weak Pareto optimal allocations of $\mathcal{E}$ surely holds for local Pareto ones. The final § 5 contains versions of the second welfare theorem held for *global* optimal allocations of all the kinds considered in Definition 1.2 under appropriate qualification conditions.

Theorem 4.1 (equivalence between local Pareto-type optimal allocations in welfare economics and fully localized minimizers in set-valued optimization). *Let $(\bar{y}, \bar{z})$ be a feasible allocation of the welfare economy $\mathcal{E}$ in (1.1) with the preference sets $P_i(z)$, and let $\bar{x} := (\bar{y}, \overline{w})$ with $\overline{w} := \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j$. Then we have the following equivalence relationships:*

(i) $(\bar{y}, \bar{z})$ is a local weak Pareto *optimal allocation of $\mathcal{E}$ if and only if $(\bar{x}, \bar{z})$ is a fully localized* weak minimizer *for the multiobjective optimization problem (4.1) with respect to the preference L defined in (4.2).*

(ii) $(\bar{y}, \bar{z})$ is a local strict Pareto *optimal allocation of $\mathcal{E}$ if and only if $(\bar{x}, \bar{z})$ is a fully localized* minimizer *for (4.1) with respect to L.*

(iii) $(\bar{y}, \bar{z})$ is a local strong Pareto *optimal allocation of $\mathcal{E}$ if and only if it is a fully localized* strong minimizer *for (4.1) with respect to L.*

Proof. Let us first justify assertion (i). Assuming that $(\bar{y}, \bar{z})$ is a local *weak Pareto* optimal allocation of the economy $\mathcal{E}$, we get by Definition 1.2(i) and by the structure of L in (4.2) a neighborhood $\mathcal{O} = \mathcal{O}_y \times \mathcal{O}_z$ of $(\bar{y}, \bar{z})$ such that

$$z \notin L(\bar{z}) \quad \text{for all feasible allocations} \quad (y, z) \in \mathcal{O}. \tag{4.3}$$

Choosing $V := \mathcal{O}_z$ and $U := \mathcal{O}_y \times \mathcal{O}_w$, where $\mathcal{O}_w := \{w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \mid (y, z) \in \mathcal{O}\}$ is a neighborhood of $\overline{w}$, we claim that

$$F(\Omega \cap U) \cap L(\bar{z}) \cap V = \emptyset. \tag{4.4}$$

Indeed, the violation of (4.4) means that there is $z \in F(\Omega \cap U) \cap L(\bar{z}) \cap V$. Taking into account in this case the constructions of F and Ω in (4.1), we find $y \in \prod_{j=1}^{m} S_j \cap \mathcal{O}_y$ satisfying the relationships $w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \in W$. This implies that $(y, z) \in \mathcal{O}$ is a feasible allocation of $\mathcal{E}$ with $z \in L(\bar{z})$. The latter clearly contradicts (4.3) and thus justifies (4.4), which means that $(\bar{x}, \bar{z})$ is a fully localized weak minimizer for the set-valued optimization problem (4.1).

Conversely, let $(\bar{x}, \bar{z})$ be a fully localized *weak minimizer* for problem (4.1) with $\bar{x} = (\bar{y}, \overline{w})$. By Definition 3.1(i), find a neighborhood $U = \mathcal{O}_y \times \mathcal{O}_w$ of $(\bar{y}, \overline{w})$ and a neighborhood V of $\bar{z}$ such that (4.4) holds and that the set $\Big\{w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \mid (y,z) \in \mathcal{O}_y \times V\Big\}$ is contained in $\mathcal{O}_w$. For any feasible allocation (y,z) from the neighborhood $\mathcal{O} := \mathcal{O}_y \times V$ of $(\bar{y},\bar{z})$, we get by Definition 1.1 of feasible allocations and the above choice of the neighborhoods that $(\bar{y},\overline{w}) \in \Omega \cap U$, $\bar{z} \in F(\bar{y},\overline{w}) \subset F(\Omega \cap U)$, and

$$z_i \notin P_i(\bar{z}) \quad \text{for some} \quad i \in \{1,\ldots,n\}. \tag{4.5}$$

Indeed, the violation of (4.5) reads that $z_i \in P_i(\bar{z})$ for all $i \in \{1,\ldots,n\}$, which implies by (4.2) that $z \in L(\bar{z})$ and thus $z \in F(\Omega \cap U) \cap L(\bar{z}) \cap V$. The latter surely contradicts (4.4) and hence gives (4.5) justifying therefore that $(\bar{y},\bar{z})$ is a local weak Pareto optimal allocation of the economy $\mathcal{E}$. This completes the proof of assertion (i) of the theorem.

To justify assertion (ii) next, pick a local *strict Pareto* optimal allocation of the economy $\mathcal{E}$ and find by Definition 1.2(iii) a neighborhood $\mathcal{O} = \mathcal{O}_y \times \mathcal{O}_z$ of $(\bar{y},\bar{z})$ such that

$$z \notin \operatorname{cl} L(\bar{z}) \quad \text{for all feasible allocations} \quad (y,z) \in \mathcal{O} \ \text{ with } \ z \neq \bar{z}. \tag{4.6}$$

We claim that the minimum condition (3.6) is satisfied with $U = \mathcal{O}_y \times \mathcal{O}_w$ and $V = \mathcal{O}_z$, where

$$\mathcal{O}_w := \Big\{w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \in E \,\Big|\, (y,z) \in \mathcal{O}\Big\} \tag{4.7}$$

is obviously a neighborhood of $\overline{w}$. Suppose on the contrary that (3.6) is violated and get a pair (x,z) with $z \neq \bar{z}$ that belongs to the left-hand side of (3.6). Taking into account the structures of F and Ω in (4.1), we have $x = (y,w) \in \Omega \cap U$ and $z \in F(x) \cap V$ such that

$$\begin{aligned}(y,z) \in \mathcal{O},\ y \in \prod_{j=1}^{m} S_j \times \mathcal{O}_y,\ w\\ = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \in W,\ z_i \in \operatorname{cl} P_i(\bar{z}) \ \text{ as } \ i \in \{1,\ldots.n\}.\end{aligned} \tag{4.8}$$

Since the consumption sets C_i are closed and since $P_i(\bar{z}) \subset C_i$ for all $i = 1,\ldots,n$, we get that $(y,z) \in \mathcal{O}$ is a feasible allocation of $\mathcal{E}$ with $z_i \in \operatorname{cl} P_i(\bar{z})$ for all $i = 1,\ldots,n$, which surely contradicts (4.6) and thus justifies the claim.

To prove the converse implication in (ii), pick an arbitrary fully localized *minimizer* $(\bar{x},\bar{z})$ for (4.1) with $\bar{x} = (\bar{y},\overline{w})$ and find neighborhoods $U = \mathcal{O}_y \times \mathcal{O}_w$ of $(\bar{y},\overline{w})$ and V of $\bar{z}$ such that (3.6) holds with the preference set $L(\bar{z})$ defined in (4.2). We claim that

$$z_i \notin \operatorname{cl} P_i(\bar{z}) \quad \text{for some} \quad i \in \{1,\ldots,n\} \tag{4.9}$$

whenever $(y,z) \in \mathcal{O}_y \times V$ is a feasible allocation of $\mathcal{E}$ with $z \neq \bar{z}$. Indeed, the violation of (4.9) and the structure of L in (4.2) yield that $z \in \operatorname{cl} L(\bar{z})$. Due to the inclusion

$$\Big\{w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \;\Big|\; (y,z) \in \mathcal{O}_y \times V\Big\} \subset \mathcal{O}_w$$

for sufficiently small neighborhoods $\mathcal{O}_y$ and V, we have $z \in F(\Omega \cap U) \cap V$, which contradicts (3.6) and thus completes the proof of assertion (ii).

Let us finally justify assertion (iii) of the theorem. Pick any local *strong Pareto* optimal allocation $(\bar{y},\bar{z})$ of the economy $\mathcal{E}$ and find by Definition 1.2(iv) a neighborhood $\mathcal{O} = \mathcal{O}_y \times \mathcal{O}_z$ of $(\bar{y},\bar{z})$ such that (4.9) holds for every feasible allocation $(y,z) \in \mathcal{O}$ of $\mathcal{E}$ with $(y,z) \neq (\bar{y},\bar{z})$. Then similarly to the proof of assertion (ii) we claim that the strong minimum condition (3.7) is satisfied with $U = \mathcal{O}_y \times \mathcal{O}_w$ and $V = \mathcal{O}_z$, where $\mathcal{O}_w$ from (4.7) is a neighborhood of $\overline{w}$. Indeed, supposing that (3.7) is violated allows us to find some pair $(x,z) \neq (\bar{x},\bar{z})$ that belongs to the left-hand side of (3.7). Taking into account the structures of F and Ω in (4.1), we have $x = (y,w) \in \Omega \cap U$ and $z \in F(x) \cap V$ satisfying (4.8). It is obvious that $(y,z) \neq (\bar{y},\bar{z})$. Then we get a contradiction with (4.9) and thus justifies the claim arguing as in the proof of (ii).

To finish the proof of assertion (iii), it remains to justify the converse implication therein. Picking an arbitrary fully localized *strong minimizer* $(\bar{x},\bar{z})$ for (4.1) with $\bar{x} = (\bar{y},\overline{w})$, we find a neighborhood $U = \mathcal{O}_y \times \mathcal{O}_w$ of $(\bar{y},\overline{w})$ and a neighborhood V of $\bar{z}$ such that condition (3.7) holds with $L(\bar{z})$ from (4.2). Our goal is to show that $(\bar{y},\bar{z})$ is a local strong Pareto optimal allocation of the economy $\mathcal{E}$, i.e., condition (4.9) is satisfied for any feasible allocation $(y,z) \neq (\bar{y},\bar{z})$ in some neighborhood $\mathcal{O}$ of $(\bar{y},\bar{z})$. Indeed, the violation of the latter means that

$$z_i \in \operatorname{cl} P_i(\bar{z}) \;\text{ for all }\; i = 1,\ldots,n. \tag{4.10}$$

Since (y,z) is a feasible allocation of $\mathcal{E}$ and since $x = (y,w)$ for $w \in W$ from (1.2), we have that $(x,z) \in \operatorname{gph} F$ with F defined in (4.1). Furthermore, it follows from (4.10) and the construction of Ω in (4.1) and of L in (4.2) that

$$(y,z) \in \prod_{j=1}^{m} S_j \times W \;\text{ and }\; z \in \prod_{i=1}^{n} \operatorname{cl} P_i(\bar{z}) = \operatorname{cl} L(\bar{z}). \tag{4.11}$$

Combining (4.11) with $x = (y,w) \in \operatorname{gph} F$ and taking into account the above choice of the neighborhoods, we arrive at the relationships

$$(\bar{x},\bar{z}) \neq (x,z) \in \operatorname{gph} F \cap \big(\Omega \times \operatorname{cl} L(\bar{z})\big) \cap (U \times V),$$

which clearly contradict (3.7) and thus justify (4.9) for all feasible allocations $(y, z) \neq (\bar{y}, \bar{z})$ from the neighborhood $\mathcal{O}$ of $(\bar{y}, \bar{z})$. This completes the proof of the theorem. □

To establish the main result of this section giving new versions of the second welfare theorem for local Pareto-type optimal allocations, we need the following proposition specifying the asymptotic closedness property for product sets involved into economy (1.1).

Proposition 4.2 (asymptotic closedness property for product sets in models of welfare economics). *Consider the economy* $\mathcal{E}$ *from* (1.1) *with the preference sets* $P_i(z)$. *Then:*

(i) The level set $L(\bar{z})$ *defined in* (4.2) *enjoys the asymptotic closedness property at* $\bar{z}$ *provided that all the preference sets* $P_i(\bar{z})$ *as* $i = 1, \ldots, n$ *have this property at* $\bar{z}_i$, *respectively.*

(ii) The closure of the level set $\operatorname{cl} L(\bar{z})$ *enjoys the asymptotic closedness property at* $\bar{z}$ *if there is* $i \in \{1, \ldots, n\}$ *such that* $\operatorname{cl} P_i(\bar{z})$ *has this property at* $\bar{z}_i$.

(iii) Assume that the production sets S_j, $j = 1, \ldots, m$, *and the net demand set* W *are locally closed around the points in question. Then the set*

$$\Omega \times \operatorname{cl} L(\bar{z}) := \prod_{j=1}^{m} S_j \times W \times \prod_{i=1}^{n} \operatorname{cl} P_i(\bar{z}) \tag{4.12}$$

enjoys the asymptotic closedness property at $(\bar{y}, \overline{w}, \bar{z})$ *if there is at least one set from* $\{S_1, \ldots, S_j, W, \operatorname{cl} P_i(\bar{z}), \ldots, \operatorname{cl} P_n(\bar{z})\}$ *having this property at the corresponding points.*

Proof. Follows directly from Proposition 3.3. □

Now we are ready to establish new extended versions of the *second fundamental theorem of welfare economics* for *local* weak Pareto, strict Pareto, and strong Pareto optimal allocations of the economy $\mathcal{E}$ under the *asymptotic closedness* condition imposed at the corresponding optimal allocation on appropriate sets involved in the economy. The following theorem ensures the existence of a *marginal price* supporting the aforementioned local Pareto-type optimal allocations that are formalized via the *M-normal cone* (2.4) computed at the optimal allocation in question. This theorem is derived directly from the corresponding first-order necessary conditions for *fully localized* minimizers in constrained *set-valued optimization* established in Theorem 3.4.

Theorem 4.3 (extended second welfare theorem for local Pareto-type optimal allocations). *Let* $(\bar{y}, \bar{z})$ *be a local optimal allocation of economy* (1.1) *in the senses listed below with respect to the preference sets* $P_i(z)$ *under the local satiation requirement:*

$$\bar{z}_i \in \mathrm{cl}\, P_i(\bar{z}) \ \textit{for all} \ i = 1, \ldots, n. \tag{4.13}$$

Assume that the commodity space E *is Asplund, that the sets* $S_1, \ldots, S_m$*, and* W *are locally closed around the points in question, and that one of the sets*

$$\mathrm{cl}\, P_i(\bar{z}),\ i = 1, \ldots, n, \quad S_j,\ j = 1, \ldots, m, \ \textit{and} \ W \tag{4.14}$$

is SNC at $\bar{z}_i$, $\bar{y}_j$*, and* $\overline{w} = \sum_{i=1}^{n} \bar{z}_i - \sum_{j=1}^{m} \bar{y}_j$*, respectively. Then there exists a nonzero marginal price* $p^* \in E^*$ *satisfying the conditions*

$$\begin{cases} -p^* \in N\big(\bar{x}_i; \mathrm{cl}\, P_i(\bar{z})\big), & i = 1, \ldots, n, \\ p^* \in N(\bar{y}_j; S_j), & j = 1, \ldots, m, \\ p^* \in N(\overline{w}; W) \end{cases} \tag{4.15}$$

in each of the following cases of local optimal allocations of the economy $\mathcal{E}$*:*

- $(\bar{y}, \bar{z})$ *is a* local weak Pareto optimal allocation *of* $\mathcal{E}$ *provided that all the sets* $P_i(\bar{z})$*,* $i = 1, \ldots, n$*, are asymptotically closed at the corresponding points* $\bar{z}_i$*.*
- $(\bar{y}, \bar{z})$ *is a* local strict Pareto optimal allocation *of* $\mathcal{E}$ *provided that there is an index* $i \in \{1, \ldots, n\}$ *such that the set* cl $P_i(\bar{z})$ *is asymptotically closed at* $\bar{z}_i$*.*
- $(\bar{y}, \bar{z})$ *is a* local strong Pareto optimal allocation *of* $\mathcal{E}$ *provided that one of sets in* (4.14) *is asymptotically closed at the corresponding point.*

Proof. Observe first that the local satiation property of the general preference mapping L in Theorem 3.4 reduces for the preference L defined in (4.2) to the local satiation requirement (4.13) on P_i imposed in this theorem. Using further the equivalence relationships of Theorem 4.1 between the local Pareto-type optimal allocations of the economy $\mathcal{E}$ and the fully localized minimizers for the set-valued optimization problem (4.1) with respect to the preference L from (4.1) and then the results of Proposition 4.2 on the asymptotic closedness property of the product sets in the welfare economy, the three statements of the theorem reduce to the following ones:

- $(\bar{x}, \bar{z})$ is a *fully localized weak minimizer* for (4.1) with respect to (4.2) provided that the set $L(\bar{z})$ is asymptotically closed at $\bar{z}$.
- $(\bar{x}, \bar{z})$ is a *fully localized Pareto minimizer* for (4.1) with respect to (4.2) provided that the set cl $L(\bar{z})$ is asymptotically closed at $\bar{z}$.

- $(\bar{x}, \bar{z})$ is a *fully localized strong Pareto minimizer* for (4.1) with respect to (4.2) provided that the set $\Omega \times \operatorname{cl} L(\bar{z})$ from (4.12) is asymptotically closed at $(\bar{x}, \bar{z}) = (\bar{y}, \overline{w}, \bar{z})$.

To deduce the statements above from the necessary optimality conditions of Theorem 3.4 applied to problem (4.1) with preference (4.2), it remains to check the validity of the corresponding PSNC properties in Remark 3.5 imposed on the sets

$$\Omega_1 := \operatorname{gph} F \quad \text{and} \quad \Omega_2 := \prod_{j=1}^{m} S_j \times W \times \prod_{i=1}^{n} \operatorname{cl} P_i(\bar{z}), \tag{4.16}$$

with F is defined in (4.1), under the SNC assumption on one of the sets in (4.14) made in the theorem. To proceed, we rename the sets and the reference points as follows:

$$\prod_{i=1}^{m+n+1} X_i := X \times Z = E^{m+n+1} \text{ with } X_i := E \text{ for } i = 1, \ldots, m+n+1,$$

$$\Theta_1 := S_1, \ldots, \Theta_m := S_m, \ \Theta_{m+1} := W, \ \Theta_{m+2} := \operatorname{cl} P_1(\bar{z}), \ldots, \ \Theta_{m+n+1} := \operatorname{cl} P_n(\bar{z}),$$

$$\bar{x}_1 := -\bar{y}_1, \ldots, \bar{x}_m := -\bar{y}_m, \ \bar{x}_{m+1} := -\overline{w}, \ \bar{x}_{m+2} := \bar{z}_1, \ldots, \ \bar{x}_{m+n+1} := \bar{z}_n.$$

Since one of the sets S_j, W, and $\operatorname{cl} P_i(\bar{z})$ is SNC at $\bar{y}_j$, $\overline{w}$, and $\bar{z}_i$, respectively, there is an index $i_0 \in \{1, \ldots, m+n+1\}$ such that the set Ω_2 from (4.16) is represented as

$$\Omega_2 = \prod_{i=1}^{m+n+1} \Theta_i$$

and is *strongly PSNC* with respect to the index set $I := \{i_0\}$. Consider further a Lipschitz continuous mapping $f \colon \prod_{i=1}^{i_0-1} X_i \times \prod_{i_0+1}^{m+n+1} X_i \to X_{i_0}$ defined by

$$f(x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{m+n+1}) := -\sum_{\substack{i=1 \\ i \neq i_0}}^{m+n+1} x_i. \tag{4.17}$$

It follows from the structures of F and f in (4.1) and (4.17), respectively, that the set Ω_1 in (4.16) admits the representation

$$\Omega_1 = \big\{(-x_1, \ldots, -x_{m+1}, x_{m+2}, \ldots, x_{m+n+1}) \big| \ (x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{m+n+1}, x_{i_0}) \in \operatorname{gph} f\big\}.$$

Since f is Lipschitz continuous, it is PSNC at $(\bar{x}_1, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i_0+1}, \ldots, \bar{x}_{m+n+1}, \bar{x}_{i_0})$ by [26, Proposition 1.68], and hence the graphical set Ω_1 is

PSNC at $(\bar{y}, \overline{w}, \bar{z})$ with respect to the index set $J := \{1, \ldots, m+n+1\} \backslash \{i_0\}$ by definition. Observing that $I \cup J = \{1, \ldots, m+n+1\}$ concludes that the PSNC assumptions of Remark 3.5 are satisfied for the sets Ω_1 and Ω_2 in (4.16), and thus we can apply the necessary optimality conditions of Theorem 3.4 to the set-valued optimization problem (4.1) with respect to preference (4.2). It follows in this way by taking into account the structures of the initial data in (4.1), (4.2), and (4.16) that there are $(0,0) \neq (x^*, z^*) \in X^* \times Z^*$ and $p^* \in E^*$ satisfying the relationships

$$\begin{aligned} &(-x^*, z^*) \in N\big((\bar{x}, \bar{z}); \Omega_2\big) = \prod_{j=1}^{m} N(\bar{y}_j; S_j) \times N(\overline{w}; W) \\ &\quad \times \prod_{i=1}^{n} N\big(\bar{z}_i; \mathrm{cl}\, P_i(\bar{z})\big), \\ &(x^*, -z^*) \in N\big((\bar{x}, \bar{z}); \Omega_1\big) = \prod_{j=1}^{m} \{-p^*\} \times \{-p^*\} \times \prod_{i=1}^{n} \{p^*\}, \end{aligned}$$

where the latter one obviously implies that $p^* \neq 0$. Thus we arrive at all the conditions (4.15) of the extended second welfare theorem and so complete the proof. □

Let us now consider economies with commodity spaces E *ordering* by their closed positive cones $E_+ \subset E$ and discuss some efficient conditions, well-recognized in welfare economics, that imply the asymptotic closedness property of the corresponding sets. The following preliminary result of its independent interest reveals a general setting when the asymptotic closedness property holds in ordered Banach spaces E with *generating* closed positive cones $E_+ - E_+ = E$.

Proposition 4.4 (asymptotic closedness property in ordered Banach spaces). *Let E be an ordered Banach space with the generating closed positive cone E_+, let Ξ be a closed subset of E satisfying the condition*

$$\Xi - E_+ \subset \Xi, \tag{4.18}$$

and let $\bar{z} \in \Xi$ be a boundary point of Ξ. Then the set Ξ is asymptotically closed at $\bar{z}$.

Proof. Since $\bar{z}$ is a boundary point of Ξ, we find a sequence $\{z_k\} \subset E$ with $z_k \to 0$ as $k \to \infty$ and $\bar{z} + z_k \notin \Xi$ for all $k \in I\!N$. The classical Krein–Šmulian theorem in ordered spaces with generating positive cones ensures the existence of a constant $M > 0$ such that for each $e \in E$ there are

$$u, v \in E_+ \ \text{ with } \ e = u - v \ \text{ and } \ \max\big\{\|u\|, \|v\|\big\} \leq M\,\|e\|.$$

Hence we get two sequences $\{u_k\} \subset E$ and $\{v_k\} \subset E$ satisfying

$$z_k = u_k - v_k,\ u_k \to 0, \text{ and } v_k \to 0 \text{ as } k \to \infty.$$

To justify the asymptotic closedness property of Ξ at $\bar{z}$, it suffices to show that condition (4.18) implies that $\bar{z} \notin \Xi - u_k$ for all large $k \in I\!N$. Arguing by contradiction, fix $k \in I\!N$ and suppose that there is some $z \in \Xi$ such that $\bar{z} = z - u_k$. This yields

$$z = \bar{z} + u_k = \bar{z} + z_k + v_k \in \Xi, \text{ i.e., } \bar{z} + z_k = z - v_k \in \Xi - E_+ \subset \Xi,$$

which contradicts the choice of $\{z_k\}$ and thus completes the proof of the proposition. □

According to the conventional terminology in welfare economics, we say that the economy $\mathcal{E}$ with the preference sets $P_i(z)$ exhibits:

- The *implicit free disposal of commodities* if

$$\text{cl}\, W - E_+ \subset \text{cl}\, W. \tag{4.19}$$

- The *free disposal of production* if

$$\text{cl}\, S_j - E_+ \subset \text{cl}\, S_j \text{ for some } j \in \{1, \ldots, m\}. \tag{4.20}$$

- The *desirability condition* if

$$\text{cl}\, P_i(\bar{z}) + E_+ \subset \text{cl}\, P_i(\bar{z}) \text{ for some } i \in \{1, \ldots, n\}. \tag{4.21}$$

The following consequence of Theorem 4.3 employs the result of Proposition 4.4 providing in this way efficient implementations of the established extended versions of the second welfare theorem for the cases of local *strict Pareto* and *strong Pareto* optimal allocations of nonconvex economies with ordered commodity spaces.

Corollary 4.5 (extended second welfare theorem for local strict Pareto and strong Pareto optimal allocations with ordered commodities). *In addition to the general assumptions of Theorem 4.3, suppose that the commodity space E is ordered and generated by its closed positive cone E_+. Then there is a positive marginal price $p^* \in E^*_+ \setminus \{0\}$ satisfying the conditions of the second welfare theorem* (4.15) *in each of the following cases:*

- $(\bar{y}, \bar{z})$ *is a* local strict Pareto optimal allocation *of the economy $\mathcal{E}$ exhibiting the desirability condition* (4.21) *with respect to its preferences.*
- $(\bar{y}, \bar{z})$ *is a* local strong Pareto optimal allocation *of the economy $\mathcal{E}$ exhibiting either the implicit free disposal of commodities* (4.19)*, or the free disposal of production* (4.20)*, or the desirability condition* (4.21).

Proof. Observe first that by [27, Lemma 8.12] the price *positivity* $p^* \in E^*_+$ in ordered commodity spaces follows directly from assertions (4.15) of the extended second welfare theorem under the fulfillment of either one of the underlying conditions (4.19)–(4.21). Employing further Proposition 4.4 and the respectful statements of Theorem 4.3, we arrive at both conclusions claimed in the corollary by showing that:

(i) There is a consumer index $i \in \{1, \ldots, n\}$ such that the corresponding component $\bar{z}_i$ of $(\bar{y}, \bar{z})$ is a *boundary point* of the set cl $P_i(\bar{z})$ provided that $(\bar{y}, \bar{z})$ is a local *strict* Pareto optimal allocation of the economy $\mathcal{E}$ exhibiting the *desirability condition* (4.21).

(ii) Each component $\bar{z}_i$ and $\bar{y}_j$ of $(\bar{y}, \bar{z})$ as well as the combination $\overline{w} = \sum\limits_{i=1}^{n} \bar{z}_i - \sum\limits_{j=1}^{m} \bar{y}_j$ is a *boundary point* of the corresponding set cl $P_i(\bar{z})$, S_j, and W provided that $(\bar{y}, \bar{z})$ is a local *strong* Pareto optimal allocation of the economy $\mathcal{E}$ exhibiting *at least one* of the free disposal/desirability properties (4.19)–(4.21).

Both assertions (i) and (ii) above can be easily derived, arguing by contradiction, from the definitions of local strict Pareto and strong Pareto optimal allocations in § 1. This completes the proof of the corollary. □

Note that, while the strict Pareto result of Corollary 4.5 is new, the strong Pareto counterpart of the corollary was first established in [24, Theorem 5.5] and then in [27, Theorem 8.14] by a direct proof, with no reduction to the asymptotic closedness property of the corresponding set in Theorem 4.3 under one of the free disposal/desirability conditions (4.19)–(4.21).

Let us further establish relationships between the asymptotic closedness introduced in Definition 3.8 and some *properness* properties in economic modeling originated by Mas-Colell [22, 23] and then well recognized and developed in welfare economics. In the *rest of this section* we consider the economy $\mathcal{E}$ as in (1.1) whose commodity space is a *Banach lattice* and suppose for simplicity that $C_1 = \ldots = C_n = E_+$, that $m = 1$ with S standing for the *total production set*, and that $W = \{\overline{w}\}$, which is the classical *markets clear* condition. Recall first the most advanced (to the best of our knowledge) properness properties recently developed and applied in [11].

Definition 4.6 (properness properties). *Given the economy $\mathcal{E}$ as described above whose commodity space is a Banach lattice, we say that:*

(A1) $\mathcal{E}$ satisfies the properness property for preferences *if there are positive numbers δ, λ, and θ such that*

$$\Big(\big(P_i(\bar z)\cap(\bar z_i+\theta I\!B)\big)+\Gamma\Big)\cap E_+\subset P_i(\bar z)\quad \textit{with}$$
$$\Gamma:=\bigcup_{t\in(0,\lambda]} t\Big(\frac{1}{(n+1)}\overline{w}+\delta I\!B\Big). \tag{4.22}$$

(A2) $\mathcal{E}$ satisfies the properness property for production *if the total production set S is closed and there are positive numbers δ, λ, and θ such that*

$$(y-\Gamma)\cap\big\{z\in E\;\big|\;z^+\le y^+\big\}\subset Y \;\; \textit{for all} \;\; y\in S\cap(\bar y+\theta I\!B), \tag{4.23}$$

where Γ is defined as in (A1).

Note that the cone Γ in (4.22) depends on δ,λ, and θ while we do not indicate this dependence in notation for simplicity. When the preference sets are derived from a transitive and complete preference on consumption sets, the properness condition for preferences ($A1$) is implied by the *uniform properness property* introduced by Mas-Colell in [23]; see also [22] for that in finite-dimensional spaces. Following Mas-Colell, we say that the preference relation $\prec$ is *proper* at $z\in E_+$ if there are $\alpha>0$, $\varepsilon>0$, $\overline{w}\in E_+$, and a neighborhood of the origin $\mathcal{O}$ such that

$$u\in L \quad \text{and} \quad \big[z-\alpha\overline{w}+u\prec x \implies u\in\varepsilon\mathcal{O}\big].$$

The preference $\prec$ is *uniformly proper* if it is proper at every $z\in E_+$ while $\overline{w}$ and V can be chosen independent of z. Geometrically the properness at z means that there is an open cone $\Gamma\subset E$ containing positive vectors such that

$$(-\Gamma)\cap\big\{u-z\in E_+\;|\;u\prec z\big\}=\emptyset.$$

Note further that the fulfilment of the properness for preferences (A1) implies the so-called F-properness introduced in [29] as follows: $P_i(\bar z)$ is *F-proper* at $\bar z_i$ if there is some $\overline{w}\in L$ and a neighborhood $\mathcal{O}$ of the origin such that:

- $\bar z_i+\overline{w}\in E_+$.
- If $u\in\mathcal{O}$, then $\bar z_i+\lambda(\overline{w}-u)\in E_+$ yields $\bar z_i+t(\overline{w}-u)\in P_i(\bar z)$ for all $t>0$ sufficiently small, i.e., there is some $\lambda>0$ satisfying

$$(\bar z_i+\Lambda)\cap E_+\subset P_i(\bar z)\quad\text{with}\quad \Lambda:=\bigcup_{t\in(0,\lambda]} t\big(\overline{w}+\mathcal{O}\big).$$

The next proposition establishes relationships between the properness for preference property (A1) and the asymptotic closedness of preference sets in the economic model under consideration.

Proposition 4.7 (relationships between properness and asymptotic closedness properties). *The following assertions hold for the economy $\mathcal{E}$ under consideration:*

(i) The properness property (A1) *of* $P_i(\bar{z})$ *at* $\bar{z}_i$ *implies the asymptotic closedness property of* $P_i(\bar{z})$ *at this point.*
(ii) If (A1) *is satisfied, then the set*

$$\widetilde{P}_i(\bar{z}) := P_i(\bar{z}) \cap (\bar{z}_i + \theta I\!B) + \Gamma$$

is asymptotically closed at $\bar{z}_i$*.*

Proof. Let us first justify (i). Assume that $P_i(\bar{z})$ is proper at $\bar{z}_i$. To derive from (A1) the asymptotic closedness property of $P_i(\bar{z})$ at $\bar{z}_i$, we intend to show that

$$\big(\mathrm{cl}\, P_i(\bar{z}) + c_k\big) \cap V \subset P_i(\bar{z}) \ \text{ for all } \ k \ \text{ sufficiently large,} \tag{4.24}$$

where the set $V \subset Z$ and the sequence $\{c_k\} \subset Z$ are defined by

$$V := \Big(\bar{z}_i + \frac{\theta}{2} I\!B\Big) \ \text{ and } \ c_k := \frac{\lambda}{k}\Big(\frac{1}{n+1}\overline{w} + \delta c\Big) \ \text{ with } \ c \in E_+ \cap I\!B.$$

Proceeding in this way, fix $k \in I\!N$ so large that $c_k \in (\theta/2)I\!B$. Take further any $z \in (\mathrm{cl}\, P_i(\bar{z}) + c_k) \cap V$ and find a sequence $\{z_m\} \subset P_i(\bar{z}) \subset E_+$ satisfying $z_m \to \widetilde{z} \in \mathrm{cl}\, P_i(\bar{z})$ as $m \to \infty$, $z_m + c_k \in V$, and $z = \widetilde{z} + c_k$. Then $z_m \in -c_k + V \subset (\bar{z}_i + \theta I\!B)$. Since $c_k \in \mathrm{int}\, \Gamma$, there is $\gamma > 0$ such that

$$z_m + c_k \in z_m + c_k + \gamma I\!B \subset \big(P_i(\bar{z}) \cap (\bar{z}_i + \theta I\!B) + \Gamma\big) \ \text{ for all } \ m \in I\!N.$$

Passing above to the limit as $m \to \infty$, we get from (4.22) that

$$z = \widetilde{z} + c_k \in \big(P_i(\bar{z}) \cap (\bar{z}_i + \theta I\!B) + \Gamma\big) \cap E_+ \subset P_i(\bar{z}).$$

Taking into account that $z \in (\mathrm{cl}\, P_i(\bar{z}) + c_k) \cap V$ was chosen arbitrarily, the latter implies (4.24), which gives therefore the asymptotic closedness property of the set $P_i(\bar{z})$ at $\bar{z}_i$ claimed in (i). Furthermore, this implies the asymptotic closedness property of the set $\widetilde{P}_i(\bar{z})$ at $\bar{z}_i$ in assertion (ii) and thus completes the proof of the proposition. □

Observe that the converse implication to (i) in Proposition 4.7 does not hold even in finite-dimensional commodity spaces. To illustrate it, take $E = I\!R^3$ and consider the preference set $P(0)$ at the origin given by

$$P(0) := \big\{(a, b, 0) \in I\!R^3 \ \big|\ a, b \ge 0\big\} \setminus \{0\}.$$

Since $P(0) \cup \{0\}$ is a closed and convex cone, the set $P(0)$ clearly has the asymptotic closedness property at 0. It turns out however that

$$I\!R^3_+ \cap \big(P(0) \cap \gamma I\!B + \Gamma\big) \not\subset P(0)$$

for every $\gamma > 0$ with Γ given in (4.22). To see this, take any triple $v = (v_1, v_2, v_3) \in \Gamma \cap I\!R^3_+$ with $v_3 > 0$; the existence of such v is ensured by $\operatorname{int}\Gamma \neq \emptyset$. Then for every $u = (u_1, u_2, 0) \in P(0) \cap \gamma I\!B$ we have $u_3 + v_3 = v_3 > 0$, and thus $u + v \notin P(0)$. This shows that $P(0)$ does not have the properness property at the origin. Observe in general that it happens when $\operatorname{span} P_i(\bar{z}) \subset L$, where $L \neq E$ is a subspace of E.

Let us finally present consequences of Theorem 4.3 for the case of local *weak Pareto* optimal allocations under properness assumptions and compare them with the main result of [11]. To proceed, we need the following proposition allowing us to reduce, under the properness assumptions, local weak Pareto optimal allocations of the economy $\mathcal{E}$ under consideration to those of a modified economy with *nonempty interior* properties of preference and production sets.

Proposition 4.8 (weak Pareto optimal allocations under properness assumptions). *Let $(\bar{y}, \bar{z})$ be a local weak Pareto optimal allocation of the economy $\mathcal{E} = (P_1, \ldots, P_n, S, \overline{w})$ under the properness assumptions* (A1) *and* (A2) *of Definition 4.6 held for all $i = 1, \ldots, n$. Then $(\bar{y}, \bar{z})$ is a local weak Pareto optimal allocation of the modified economy $\widetilde{\mathcal{E}} = (\widetilde{P}_1, \ldots, \widetilde{P}_n, \widetilde{S}, \overline{w})$ with*

$$\widetilde{P}_i(\bar{z}) := P_i(\bar{z}) \cap (\bar{z}_i + \theta I\!B) + \Gamma,\ i = 1, \ldots, n,\ \text{ and } \qquad \widetilde{S} := S \cap (\bar{y} + \theta I\!B) - \Gamma \tag{4.25}$$

Proof. Arguing by contradiction, suppose that $(\bar{y}, \bar{z})$ is *not* a local weak Pareto optimal allocation of $\widetilde{\mathcal{E}}$ and thus find (y, z) satisfying

$$y \in \widetilde{S},\ \ \overline{w} = \sum_{i=1}^{n} z_i - y,\ \text{ and }\ z_i \in \widetilde{P}_i(\bar{z})\ \text{ for all }\ i = 1, \ldots, n.$$

The latter relationships imply that

$$\overline{w} \in \sum_{i=1}^{n} \Big(\big(P_i(\bar{z}) \cap (\bar{z}_i + \theta I\!B)\big) + \Gamma \Big) - \Big(\big(S \cap (\bar{y} + \theta I\!B)\big) - \Gamma \Big),$$

which contradicts [11, Claim 4.1] under the properness assumptions (A1) and (A2) and hence verifies the local weak Pareto optimality of the allocation $(\bar{y}, \bar{z})$ of the economy $\widetilde{\mathcal{E}}$. □

Based on the fact from Proposition 4.7(i) that the properness for preferences implies the asymptotic closedness property, we derive from Theorem 4.3 a version of the extended second welfare theorem for the economy $\mathcal{E}$ under the properness (A1) and SNC assumptions in terms of the initial data

(P_i, S). Moreover, the nonempty interior of the convex cone Γ in (4.22) and the fact that $P_i(\bar{z}) \subset \widetilde{P}_i(\bar{z})$ allow us to obtain from Theorem 4.3 and Propositions 4.7(ii), 4.8 a modified version of the second welfare theorem with *no SNC* requirements.

Corollary 4.9 (extended second welfare theorem for local weak Pareto optimal allocations under properness assumptions). *Let $(\bar{y}, \bar{z})$ be a local weak Pareto optimal allocation of the economy $\mathcal{E} = (P_i, S, \overline{w})$ with an ordered Asplund commodity space E. Then the following assertions hold:*

(i) Assume that the economy $\mathcal{E}$ satisfies the properness for preferences assumption (A1) *for all $i = 1, \ldots, n$ and that one of the sets* $\operatorname{cl} P_i(\bar{z})$, *$i = 1, \ldots, n$, and S is SNC at $\bar{z}_i$ and $\bar{y}$, respectively. Then there is a nonzero marginal price $p^* \in E^* \setminus \{0\}$ such that*

$$-p^* \in N\big(\bar{z}_i; \operatorname{cl} P_i(\bar{z})\big), \quad i = 1, \ldots, n, \quad \textit{and} \quad p^* \in N(\bar{y}; S). \tag{4.26}$$

(ii) Assume that both properness properties (A1) *as $i = 1, \ldots, n$ and* (A2) *from Definition 4.6 are satisfied. Then there is a nonzero marginal price $p^* \in E^* \setminus \{0\}$ such that*

$$-p^* \in N\big(\bar{z}_i; \operatorname{cl} \widetilde{P}_i(\bar{z})\big), \quad i = 1, \ldots, n, \quad \textit{and} \quad p^* \in N(\bar{y}; \widetilde{S}) \tag{4.27}$$

via the modified preference and total production sets $\widetilde{P}_i(\bar{z})$ and $\widetilde{S}$ defined in (4.25)*.*

Proof. To justify assertion (i), we imply Proposition 4.7(i) ensuring, under the properness assumption (A1), the asymptotic closedness property of each preference set $P_i(\bar{z})$ at $\bar{z}_i$ as $i = 1, \ldots, n$ imposed in Theorem 4.3 for weak Pareto optimal allocations. This allows us to get (4.26) with $p^* \neq 0$ under the SNC assumptions made.

To prove assertion (ii), we use Proposition 4.8 justifying that $(\bar{y}, \bar{z})$ is a local weak Pareto optimal allocation to the modified economy $\widetilde{\mathcal{E}}$ with data (4.25). It follows from Proposition 4.7(ii) that each modified preference sets $\widetilde{P}_i(\bar{z})$ is asymptotically closed at $\bar{z}_i$, $i = 1, \ldots, n$. For applying now Theorem 4.3 to the weak Pareto optimal allocation $(\bar{y}, \bar{z})$ of the economy $\widetilde{\mathcal{E}}$, it is sufficient to demonstrate that at least *one* of the sets $\operatorname{cl} \widetilde{P}_i(\bar{z}_i)$. $i = 1, \ldots, n$, and $\widetilde{S}$ is SNC at the corresponding point. Let us show that in fact *all* of them are SNC at the references points. Due to the structure of the sets (4.25) and the construction of the Fréchet normal cone (2.5) in the definition of the SNC property, we consider without loss of generality the set

$$\widetilde{P}(\bar{z}) := P(\bar{z}) \cap (\bar{z} + \theta I\!B) + \Gamma \tag{4.28}$$

with the cone Γ defined in (4.22), and show that set (4.28) is SNC at $\bar{z} \in \operatorname{cl} \widetilde{P}(\bar{z})$. By definition of the SNC property in § 2, take a sequence $\{z_k, z_k^*\} \subset Z \times Z^*$ satisfying

$$z_k \to \bar{z} \text{ and } z_k^* \xrightarrow{w^*} 0 \text{ as } k \to \infty \text{ with } z_k^* \in \widehat{N}\big(z_k; \widetilde{P}(\bar{z})\big), \quad k \in I\!N.$$

Taking into account the *decreasing property* of the Fréchet normal cone with respect to set inclusions (see [26, p. 5]) and the *convexity* of the cone Γ, we have

$$z_k^* \in \widehat{N}(\widetilde{z}_k; \Gamma) \subset \widehat{N}(0; \Gamma), \quad k \in I\!N,$$

with some sequence $\{\widetilde{z}_k\} \subset Z$, which implies by the nonempty interior of Γ that $\|z_k^*\| \to 0$ as $k \to \infty$; see [26, Proposition 1.25 and Theorem 1.26]. This justifies the SNC property of set (4.28) at $\bar{z}$ (and of all the sets (4.25) at the corresponding points) and thus completes the proof. □

Observe that inclusions (4.26) follow from a more general result obtained in [11, Theorem 3.4] for local weak Pareto optimal allocations under imposing both properness properties of Definition 4.6. Inclusions (4.27) seem to be new and generally independent of (4.26). In fact we can deduce (4.26) from (4.27) provided that the sets $\operatorname{cl} P_i(\bar{z})$, $i = 1, \ldots, n$ and S are order semicontinuous at $\bar{z}_i$ and $\bar{y}$, respectively, in the sense of [2, Definition 4.1] specified for a constant set-valued mapping; see [2] for efficient conditions ensuring this property. Recall that a subset $\Xi \subset Z$ of a Banach space with an ordering convex cone $\Theta \subset Z$ is *order semicontinuous* at $\bar{z} \in \Xi$ if

$$\begin{aligned}&\text{for all } \{z_k\} \subset \Xi + \Theta \quad \text{with} \quad z_k \to \bar{z} \text{ there is } \{\widetilde{z}_k\} \subset \Xi\\ &\text{such that} \quad \widetilde{z}_k \to \bar{z}.\end{aligned}$$

It follows from [2, Proposition 4.3] that

$$N(\bar{z}; \Xi + \Theta) \subset N(\bar{z}; \Omega) \tag{4.29}$$

provided that Ξ is order semicontinuous at $\bar{z}$. This ensures that assertion (ii) implies (i) in Corollary 4.9 under the order semicontinuity of the underlying preference and production sets. It may be different when the latter property is violated. Considering, e.g., the set

$$\Xi := \big\{(x, y) \in I\!R^2 \big| \ (x+1)^2 + (y+1)^2 = 1 \text{ and either } x \ge 0 \text{ or } y \ge 0\big\}$$

and the ordering cone $\Theta = I\!R^2_+$, it is easy to check that

$$N(0; \Xi) = \big\{(x, y \in I\!R^2 \big| \ x - y = 0\big\} \quad \text{while}$$

$$N(0; \Xi + \Theta) = N(0; I\!R^2 \setminus I\!R^2_<) = \big\{(x, y) \in I\!R^2_\le \big| \ xy = 0\big\}.$$

5. Global versions of the second welfare theorem

In this section we consider the general nonconvex welfare economy formulated in § 1 and focus on deriving extended versions of the second welfare theorem formalized via the M-normal cone for *global* Pareto-type optimal allocations of all the four kinds described in Definition 1.2.

Since any global optimal allocation is a local one of the same type, the results of § 4 obtained for local weak Pareto, strict Pareto, and strong Pareto optimal allocations surely hold for their global counterparts. In this section we show, by using somewhat different arguments, that the global nature of all the optimal allocations under consideration in Definition 1.2 allows us to derive extended versions of the second welfare theorem held under *more general qualification* conditions in comparison with those employed above.

Note that the differences between necessary optimality conditions for local and global solutions have been well recognized in several areas of optimization and equilibrium theories. In this vein we mention the classical calculus of variations and optimal control and also models of bilevel programming/Stackelberg games recently illuminated in [9]. To the best of our knowledge, the discrepancy between local and global versions of the second welfare theorem and similar results of economic modeling has not been revealed in microeconomic theory, where the vast majority of publications concern global optimal allocations.

In what follows we derive global versions of the extended second welfare theorem from § 4 under the so-called *net demand qualification conditions* known for the cases of Pareto optimal and weak Pareto optimal allocations and introduced here for the cases of strict Pareto and strong Pareto ones. All these versions of net demand qualifications are (properly) implied by the asymptotic closedness property of the corresponding sets from Theorem 4.3.

Definition 5.1 (net demand qualification conditions). *Let* $(\bar{y}, \bar{z})$ *be a feasible allocation of the economy* $\mathcal{E}$ *in* (1.1) *with the preference sets* $P_i(z)$*, and let* $\overline{w} := \sum_{i=1}^{n} \bar{z}_i - \sum_{j=1}^{m} \bar{y}_j \in W$*. Rename the sets* $P_i(\bar{z})$*,* S_j*, and* W *as:*

$$\Xi_i := P_i(\bar{z}),\ i = 1, \ldots, n, \quad \Xi_{n+j} := -S_j,$$
$$j = 1, \ldots, m,\ \text{and}\ \Xi_{m+n+1} := -W$$

with $\bar{x}_i := \bar{z}_i$*,* $\bar{x}_{n+j} := -\bar{y}_{n+j}$*, and* $\bar{x}_{m+n+1} := -\overline{w}$*. Given* $\varepsilon > 0$*, consider the set*

$$\Delta_\varepsilon := \sum_{i=1}^{n+m+1} \operatorname{cl} \Xi_i \cap (\bar{z}_i + \varepsilon I\!B) \tag{5.1}$$

and define the following qualification conditions at the allocation $(\bar{y}, \bar{z})$*:*

(i) *The* net demand weak qualification (NDWQ) condition *holds at* $(\bar{y}, \bar{z})$ *if there are* $\varepsilon > 0$ *and a sequence* $\{e_k\} \subset E$ *with* $e_k \to 0$ *as* $k \to \infty$ *such that*

$$\Delta_\varepsilon + e_k \subset \sum_{i=1}^{n} \Xi_i + \sum_{i=n+1}^{n+m+1} \text{cl}\, \Xi_i \quad \textit{for all} \\ k \in I\!N \ \ \textit{sufficiently large}. \tag{5.2}$$

(ii) *The* net demand qualification (NDQ) condition *holds at* $(\bar{y}, \bar{z})$ *if there are* $\varepsilon > 0$*, a sequence* $\{e_k\} \subset E$ *with* $e_k \to 0$ *as* $k \to \infty$*, and an index* $i_0 \in \{1, \ldots, n\}$ *such that*

$$\Delta_\varepsilon + e_k \subset \Xi_{i_0} + \sum_{\substack{i=1 \\ i \neq i_0}}^{n+m+1} \text{cl}\, \Xi_i \quad \textit{for all} \\ k \in I\!N \ \ \textit{sufficiently large}. \tag{5.3}$$

(iii) *The* net demand strict qualification (NDSQ) condition *holds at* $(\bar{y}, \bar{z})$ *if there are* $\varepsilon > 0$*, a sequence* $\{e_k\} \subset E$ *with* $e_k \to 0$ *as* $k \to \infty$*, and an index* $i_0 \in \{1, \ldots, n\}$ *such that*

$$\Delta_\varepsilon + e_k \subset \text{cl}\, \Xi_{i_0} \setminus \{\bar{x}_{i_0}\} + \sum_{\substack{i=1 \\ i \neq i_0}}^{n+m+1} \text{cl}\, \Xi_i \quad \textit{for all} \\ k \in I\!N \ \ \textit{sufficiently large}. \tag{5.4}$$

(iv) *The* net demand strong qualification (NDSGQ) condition *holds at* $(\bar{y}, \bar{z})$ *if there are* $\varepsilon > 0$*,* $\{e_k\} \subset E$ *with* $e_k \to 0$ *as* $k \to \infty$*, and* $i_0 \in \{1, \ldots, m+n+1\}$ *such that*

$$\Delta_\varepsilon + e_k \subset \text{cl}\, \Xi_{i_0} \setminus \{\bar{x}_{i_0}\} + \sum_{\substack{i=1 \\ i \neq i_0}}^{m+n+1} \text{cl}\, \Xi_i \\ \textit{for all} \ \ k \in I\!N \ \ \textit{sufficiently large}. \tag{5.5}$$

Note that both NDQ and NDWQ conditions appeared in [24], while the previous version of the NDQ condition for nonconvex economies with $W = \{\overline{w}\}$ (markets clear) and closed production sets was formulated in [15] (and in [16] submitted in 1998) under the name of "asymptotically included condition." We refer the reader to [6–8, 11, 13, 15–19, 21, 24, 27, 33] for particular cases, modifications, and more discussions on qualification conditions ensuring various versions of the second welfare theorem for Pareto and weak Pareto optimal allocations. The NDSQ and NDSGQ conditions used in what follows to establish global versions of the second welfare theorem for strict and strong Pareto optimal allocations appear here for the first time.

It is easy to check (see, e.g., [27, Proposition 8.4]) that the NDQ condition holds if there is some index $i_0 \in \{1, \ldots, n\}$ such that the $P_{i_0}(\bar{z})$ is asymptotically closed at $\bar{z}_{i_0}$ while the NDWQ condition is fulfilled when all the sets $P_i(\bar{z})$ are asymptotically closed at $\bar{z}_i$ for $i = 1, \ldots, n$, respectively. We can similarly conclude that the NDSQ condition is satisfied if there is some index $i_0 \in \{1, \ldots, n\}$ such that the set $\operatorname{cl} P_{i_0}(\bar{z})$ is asymptotically closed at $\bar{z}_{i_0}$, and that the NDSGQ condition holds if at least one set from

$$\big\{\operatorname{cl} P_1(\bar{z}), \ldots, \operatorname{cl} P_n(\bar{z}), S_1, \ldots, S_m, W\big\}$$

is asymptotically closed at $\bar{z}_i$, $\bar{y}_j$, and $\overline{w}$, respectively. Roughly speaking, the asymptotic closedness property always implies the corresponding net demand qualification condition. However, the converse does not hold. Indeed, consider the markets clear economy $\mathcal{E}$ with $E = I\!R^2$, $n = m = 1$, and $W = \{0\}$, and

$$C = I\!R^2_+,\ S = \Xi := \big\{(a, b) \in I\!R^2_+ \big|\ ab = 0\big\},\ P(z) \equiv \Xi \setminus \{(0, 0)\},\ \bar{z} = (0, 0).$$

It is obvious that $P(0)$ does not have the asymptotic closedness property at the origin, while the NDQ condition (5.3) is satisfied at $(\bar{y}, \bar{z}) = (0, 0) \in I\!R^4$, since

$$\Big(\frac{1}{k}, \frac{1}{k}\Big) + \Xi - \Xi \subset \Xi \backslash \{(0, 0)\} - \Xi.$$

The next theorem establishes *global* versions of the second welfare theorem for all the four types of Pareto optimal allocations from Definition 1.2 valid under the corresponding net demand qualification conditions of Definition 5.1.

Theorem 5.2 (extended second welfare theorem for global Pareto-type optimal allocations). *Let $(\bar{y}, \bar{z})$ be a global optimal allocation of economy (1.1) in the senses listed below with respect to the preference sets $P_i(z)$ under the local satiation requirement (4.13). Assume that the commodity space E is Asplund, that the sets $S_1, \ldots, S_m$, and W are locally closed around the points in question, and that one of the sets (4.14) is SNC at $\bar{z}_i$, $\bar{y}_j$, and $\overline{w} = \sum_{i=1}^{n} \bar{z}_i - \sum_{j=1}^{m} \bar{y}_j$, respectively. Then there exists a nonzero marginal price $p^* \in E^*$ satisfying all the relationships (4.15) of the extended second welfare theorem in each of the following cases of global optimal allocations of the economy $\mathcal{E}$:*

- $(\bar{y}, \bar{z})$ *is a* global weak Pareto optimal allocation *provided that the net demand weak qualification condition* (5.2) *is satisfied.*
- $(\bar{y}, \bar{z})$ *is a* global Pareto optimal allocation *provided that the net demand qualification condition* (5.3) *is satisfied.*
- $(\bar{y}, \bar{z})$ *is a* global strict Pareto optimal allocation *provided that the net demand strict qualification condition* (5.4) *is satisfied.*
- $(\bar{y}, \bar{z})$ *is a* global strong Pareto optimal allocation *provided that the net demand strong qualification condition* (5.5) *is satisfied.*

Proof. We follow the pattern based on applying the extremal principle and developed in the proofs of [27, Theorems 8.5 and 8.8]. Note that the latter results were justified in fact for global Pareto and weak Pareto optimal allocations while the claims were made for their local counterparts; see Remark 5.3 for more discussions. The arguments below verify the proof for global Pareto and weak Pareto optimal allocations and justify the new results for strict and strong Pareto ones. To proceed, consider two subsets of the Asplund spaces E^{m+n+1} defined by

$$\begin{cases} \Omega_1 := \Pi_{j=1}^m S_j \times \Pi_{i=1}^n \operatorname{cl} P_i(\bar{z}) \times W, \\ \Omega_2 := \left\{ (y, z, w) \in E^{m+n+1} \,\middle|\, \sum\limits_{i=1}^n z_i - \sum\limits_{j=1}^m y_j - w = 0 \right\}, \end{cases} \tag{5.6}$$

which are locally closed around the point $(\bar{y}, \bar{z}, \overline{w})$, and show that this point is *locally extremal* for the system $\{\Omega_1, \Omega_2\}$ under the fulfillment of the NDWQ/NDQ/NDSQ/NDSGQ conditions held for the corresponding weak Pareto/Pareto/strict Pareto/strong Pareto optimal allocation of the economy $\mathcal{E}$. Indeed, we always have $(\bar{y}, \bar{z}, \overline{w}) \in \Omega_1 \cap \Omega_2$, and it remains to check that there exist a sequence $\{a_k\} \subset E^{m+n+1}$ with $a_k \to 0$ as $k \to \infty$ and a neighborhood U of $(\bar{y}, \bar{z}, \overline{w})$ such that

$$\Omega_1 \cap (\Omega_2 - a_k) \cap U = \emptyset \quad \text{for all large} \quad k \in I\!N \tag{5.7}$$

when the corresponding qualification condition (5.2)–(5.5) is satisfied. Arguing in a unified way, take $\{e_k\} \subset E$ from the corresponding qualification condition of Definition 5.1 and form the sequence $a_k := (0, \ldots, 0, e_k) \in E^{m+n+1}$. Take further

$$U := \prod_{j=1}^m (\bar{y}_j + \varepsilon I\!B) \times \prod_{i=1}^n (\bar{z}_i + \varepsilon I\!B) \times (\overline{w} + \varepsilon I\!B)$$

and show that the extremality relationship (5.7) holds under this choice. Assuming the contrary, find a sequence of triples $(y_k, z_k, w_k) \in \Omega_1$ with $(y_k, z_k, w_k) + a_k \in \Omega_2$ and get by the structure of $\{\Omega_1, \Omega_2\}$ in (5.6) and the above choice of $\{a_k\}$ and U that

$$\begin{cases} y_{jk} \in S_j \cap (\bar{y}_j + \varepsilon I\!B), & j = 1, \ldots, m, \\ z_{ik} \in \mathrm{cl}\, P_i(\bar{z}) \cap (\bar{z}_i + \varepsilon I\!B), & i = 1, \ldots, n, \\ w_k \in W \cap (\overline{w} + \varepsilon I\!B), & \text{and} \\ 0 = \displaystyle\sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j - w_k + e_k \in \Delta_\varepsilon + e_k \end{cases} \tag{5.8}$$

for the set Δ_ε defined in (5.1). We check now that the relationships in (5.8) lead to a contradiction with the global Pareto-type optimality under consideration provided the fulfillment of the corresponding net demand qualification condition.

- Assuming that $(\bar{y}, \bar{z})$ is a *global weak Pareto* optimal allocation and that the net demand weak qualification condition (5.2) is satisfied, we get

$$0 \in \sum_{i=1}^{n} P_i(\bar{z}) - \sum_{j=1}^{m} S_j - W,$$

i.e., there are $z_i \in P_i(\bar{z})$ for $i = 1, \ldots, n$, $y_j \in S_j$ for $j = 1, \ldots, m$, and $w \in W$ such that

$$w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j. \tag{5.9}$$

By (5.9) with $w \in W$ we have that the allocation (y, z) is feasible for $\mathcal{E}$ while the inclusions $z_i \in P_i(\bar{z})$ as $i = 1, \ldots, n$ imply that $(\bar{y}, \bar{z})$ is not a global weak Pareto optimal allocation of $\mathcal{E}$ by Definition 3.1(i) with $\mathcal{O} = E^{m+n}$. Contradiction.

- Assuming that $(\bar{y}, \bar{z})$ is a *global Pareto* optimal allocation and that the net demand qualification condition (5.3) holds, we get

$$0 \in P_{i_0}(\bar{z}) + \sum_{\substack{i=1 \\ i \neq i_0}}^{n} \mathrm{cl}\, P_i(\bar{z}) - \sum_{j=1}^{m} S_j - W,$$

i.e., there are $z_{i_0} \in P_{i_0}(\bar{z})$, $z_i \in \mathrm{cl}\, P_i(\bar{z})$ for $i = 1, \ldots, n$, $y_j \in S_j$ for $j = 1, \ldots, m$, and $w \in W$ satisfying (5.9). These relationships clearly contradict the global Pareto optimality of the allocation $(\bar{y}, \bar{z})$ by Definition 3.1(ii) with $\mathcal{O} = E^{m+n}$.

- Assuming that $(\bar{y}, \bar{z})$ is a *global strict Pareto* optimal allocation and that the net demand strict qualification condition (5.4) holds, we get

$$0 \in \mathrm{cl}\, P_{i_0}(\bar{z}) \setminus \{\bar{z}_{i_0}\} + \sum_{\substack{i=1\\ i\neq i_0}}^{n} \mathrm{cl}\, P_i(\bar{z}) - \sum_{j=1}^{m} S_j - W,$$

i.e., there are $z_i \in \mathrm{cl}\, P_i(\bar{z})$ as $i = 1,\dots,n$ with $z \neq \bar{z}$, $y_j \in S_j$ as $j = 1,\dots,m$, and $w \in W$ satisfying (5.9). Thus the allocation (y,z) is feasible for the economy $\mathcal{E}$ with $z_i \in \mathrm{cl}\, P_i(\bar{z})$ for all $i = 1,\dots,n$. This implies that $(\bar{y},\bar{z})$ is not a global strict Pareto optimal allocation of $\mathcal{E}$ by Definition 3.1(iii) with $\mathcal{O} = E^{m+n}$. Contradiction.

• Assuming finally that $(\bar{y},\bar{z})$ is a *global strong Pareto* optimal allocation and that the net demand strong qualification condition (5.5) holds, we get the relationships

$$\begin{cases} \text{either} & 0 \in \mathrm{cl}\, P_{i_0}(\bar{z}) \setminus \{\bar{z}_{i_0}\} + \sum_{\substack{i=1\\ i\neq i_0}}^{n} \mathrm{cl}\, P_i(\bar{z}) - \sum_{j=1}^{m} S_j - W, \\ \text{or} & 0 \in \sum_{i=0}^{n} \mathrm{cl}\, P_i(\bar{z}) - S_{j_0} \setminus \{\bar{y}_{j_0}\} - \sum_{\substack{j=1\\ j\neq j_0}}^{m} S_j - W, \\ \text{or} & 0 \in \sum_{i=0}^{n} \mathrm{cl}\, P_i(\bar{z}) - \sum_{j=1}^{m} S_j - W\setminus\{\overline{w}\}. \end{cases}$$

Each of the latter conditions allows us to find $z_i \in \mathrm{cl}\, P_i(\bar{z})$ as $i = 1,\dots,n$, $y_j \in S_j$ as $j = 1,\dots,m$, and $w \in W$ satisfying (5.9) such that $(y,z) \neq (\bar{y},\bar{z})$. Thus we clearly arrive at a contradiction with the global strong Pareto optimality of the feasible allocation $(\bar{y},\bar{z})$ of the economy $\mathcal{E}$ by Definition 3.1(iv) with $\mathcal{O} = E^{m+n}$.

The rest of the proof for all the four kinds of Pareto-type optimal allocations are similar to the proofs of [27, Theorems 8.5 and 8.8] by employing the extremal principle and the SNC property imposed on one of the sets (4.14). □

Remark 5.3 (global vs. local Pareto optimal allocations). Let us demonstrate that the net demand qualification conditions from Definition 5.1 are appropriate to deal with *global*, not *local* Pareto-type optimal allocations. Consider the economy $\mathcal{E}$ with $E = I\!R^2$, $n = m = 1$, $C = I\!R^2_+$, $W = \{0\}$, and

$$S := \big\{(a,-a+2) \in I\!R^2 \big|\ a \le 0\big\} \cup \big\{(a,b) \in I\!R^2_+ \big|\ a^2 + (b-1)^2 = 1\big\} \cup \big\{(a,-a) \big|\ a \ge 0\big\}.$$

The customer uses the preference generated by a nonconvex cone Θ as follows:

$$P(z) := z + \Theta \setminus \{0\}, \quad \text{where } \Theta := \big\{z = (a,b) \in I\!R^2 \big|\ ab = 0\big\}.$$

Since $P(z) \cup \{z\}$ is a closed set, there is no difference between weak Pareto/Pareto/strict Pareto optimal allocations as well as between NDWQ/NDQ/NDSQ properties from Definition 5.1. It is easy to check that $(\bar{y}, \bar{z}) = (0,0) \in I\!R^2 \times I\!R^2$ is a *local* (not global) Pareto optimal allocation of the economy $\mathcal{E}$ with the ball neighborhood $\mathcal{O} := \operatorname{int} I\!B \times \operatorname{int} I\!B$. Furthermore, the net demand qualification condition is satisfied, since $P(0) - S$ contains the unit ball of $I\!R^2$, and thus the underlying inclusion

$$\Delta_1 + e_k \in P(0) - S$$

holds with $\varepsilon > 0$ sufficiently small (in fact $\varepsilon \leq 1$) and $k \in I\!N$ sufficiently large. For this example the inclusion $0 \in \Delta_1 + e_k$ implies that $0 \in P(0) - S$. The latter gives $z = (0,2) \in P(0)$ and $y = (0,2) \in S$. Observe that there is no contradiction with local optimality of the reference Pareto optimal allocation $(\bar{y}, \bar{z}) = (0,0)$, since the feasible allocation (y, z) found above does not belong to the aforementioned neighborhood $\mathcal{O}$ of $(\bar{y}, \bar{z})$.

Remark 5.4 (global versions of the second welfare theorem under improved qualification conditions). Quite recently [13], Ioffe has established a global version of the second welfare theorem, with equilibrium prices formalized via an abstract normal cone/subdifferential satisfying natural requirements, under an improved net demand qualification condition of type (5.3) for *global Pareto* optimal allocations. The improvement of (5.3) consists of replacing the set $\Xi_{i_0} = P_{i_0}(\bar{z})$ with some index $i_0 \in \{1, \ldots, n\}$ on the right-hand side of (5.3) by the *union* of these sets over all $i_0 \in \{1, \ldots, n\}$. Observe here that such an improvement can be also obtained by using the approach of Theorem 5.2 and, furthermore, in this way we can establish improved versions of Theorem 5.2 for global *strict Pareto* and *strong Pareto* optimal allocations under similarly improved qualification conditions (5.4) and (5.5).

References

1. Bao, T.Q., Mordukhovich, B.S.: Existence of minimizers and necessary conditions in set-valued optimization with equilibrium constraints. Appl. Math. **26**, 452–562 (2007)
2. Bao, T.Q., Mordukhovich, B.S.: Necessary conditions for super minimizers in constrained multiobjective optimization. J. Global. Optim. **43**, 533–552 (2009)
3. Bao, T.Q., Mordukhovich, B.S.: Asymptotic closedness of sets and necessary conditions for minimal points with respect to general preferences (2009, preprint)
4. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers in multiobjective optimization: existence and optimality conditions. Math. Program. **122**, 301–347 (2010)

5. Bellaassali, S., Jourani, A.: Lagrange multipliers for multiobjective programs with a general preference. Set-Valued Anal. **16**, 229–243 (2008)
6. Bonnisseau, J.-M., Cornet, B.: Valuation equilibrium and Pareto optimum in nonconvex economies. J. Math. Econ. **17**, 293–308 (1988)
7. Bonnisseau, J.-M., Lachiri, O.: (2006) About the second theorem of welfare economics with stock markets. Pacific J. Optim. **2**, 469–485
8. Borwein, J.M., Zhu, Q.J.: Techniques of variational analysis. CMS books in mathematics, vol. 20. New York: Springer 2005
9. Dempe, S., Dutta, J.: Is bilevel programming a special case of a mathematical program with complementarity constraints (2009, preprint)
10. Dempe, S., Kalashnikov, V. (eds.): Optimization with multivalued mappings: theory, applications and algorithms. Springer optimization and its applications. Berlin: Springer 2006
11. Florenzano, M., Gourdel, P., Jofré, A.: Supporting weakly Pareto optimal allocations in infinite-dimensional nonconvex economies. J. Econ. Theory **29**, 549–564 (2006)
12. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational methods in partially ordered spaces. CMS books in mathematics, vol. 17. New York: Springer 2003
13. Ioffe, A.D.: Variational analysis and mathematical economics, I: subdifferential calculus and the second theorem of welfare economics. Adv. Math. Econ. **12**, 71–95 (2009)
14. Jahn, J.: Vector optimization: theory, applications and extensions. Berlin: Springer 2004
15. Jofré, A.: A second welfare theorem in nonconvex economies. In: Théra M. (ed.): Constructive, experimental and nonlinear analysis. CMS conference proceedings, vol. 27, 2000 pp. 175–184
16. Jofré, A., Rivera Cayupi, J.: A nonconvex separation property and some applications. Math. Program **108**, 37–51 (2006)
17. Khan, M.A.: Ioffe's normal cone and the foundation of welfare economics: the infinite-dimensional theory. J. Math. Anal. Appl. **161**, 284–298 (1991)
18. Khan, M.A.: The Mordukhovich normal cone and the foundations of welfare economics. J. Public Econ. Theory **1**, 309–338 (1999)
19. Khan, M.A., Vohra, R.: An extension of the second welfare theorem to economies with non-convexities and public goods. Q. J. Econ. **102**, 223–245 (1987)
20. Luc, D.T.: Theory of vector optimization. Berlin: Springer 1989
21. Malcolm, G.G., Mordukhovich, B.S.: Pareto optimality in nonconvex economies with infinite-dimensional commodity spaces. J. Global Optim. **20**, 323–346 (2001)
22. Mas-Colell, A.: Pareto optima and equilibria: the finite-dimensional case. In: Aliprantis C., Burkinshaw O., Rothman N. (eds.): Advances in equilibrium theory. Lecture notes in economics and mathematical systems, vol. 244. Berlin: Springer, 1985 pp. 25–42
23. Mas-Colell, A.: The price equilibrium existence problem in topological vector lattices. Econometrica **54**, 1039–1053 (1986)
24. Mordukhovich, B.S.: Abstract extremal principle with applications to welfare economics. J. Math. Anal. Appl. **161**, 187–212 (2000)

25. Mordukhovich, B.S.: Nonlinear prices in nonconvex economics with classical Pareto and strong Pareto allocations. Positivity **9**, 541–568 (2005)
26. Mordukhovich, B.S.: Variational analysis and generalized differentiation, I: Basic theory. Grundlehren series (Fundamental principles of mathematical sciences), vol. 330. Berlin: Springer 2006
27. Mordukhovich, B.S.: Variational analysis and generalized differentiation, II: Applications. Grundlehren series (Fundamental principles of mathematical sciences), vol. 331. Berlin: Springer 2006
28. Mordukhovich, B.S., Treiman, J.S., Zhu, Q.J.: An extended extremal principle with applications to multiobjective optimization. SIAM J. Optim. **14**, 359–379 (2003)
29. Podczeck, K.: Equilibria in vector lattices without ordered preferences or uniform properness. J. Math. Econ. **25**, 465–485 (1996)
30. Rockafellar, R.T., Wets, R.J.-B.: Variational analysis. Grundlehren series (Fundamental principles of mathematical sciences), vol. 317. Berlin: Springer 1998
31. Schirotzek, W.: Nonsmooth analysis. Universitext. Berlin: Springer 2007
32. Zhu, Q.J.: Hamiltonian necessary conditions for a multiobjective optimal control problem. SIAM J. Control. Optim. **39**, 97–112 (2000)
33. Zhu, Q.J.: Nonconvex separation theorem for multifunctions, subdifferential calculus and applications. Set-Valued Anal. **12**, 275–290 (2004)

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Convexity of the lower partition range of a concave vector measure

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Abstract. This paper investigates a class of nonadditive measures on σ-algebras, named *concave measures*, to establish a Lyapunov-type convexity theorem. To this end, we introduce convex combinations of measurable sets in terms of a nonatomic vector measure and demonstrate the convexity of the lower partition range of a concave vector measure. The main result is applied to a fair division problem along the lines of L.E. Dubins and E.H. Spanier (Amer Math Monthly 68:1–17, 1961).

Key words: Nonatomic finite measure; Concave measure; Convexity; Lower partition range; Fair division

1. Introduction

The problem of fairly partitioning limited divisible resources (often called the cake) among a group of people or institutions (often called the players) has a long and rich history. From the early mathematically oriented works,

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the papers by Steinhaus [11, 12] and the paper by Dubins and Spanier [2] are amongst the most influential. These papers, and a large portion of the subsequent mathematical literature, are concerned with situations where the cake is a nonempty set Ω, the system of feasible parts of the cake is an algebra or a σ-algebra $\mathscr{F}$ of subsets of Ω, and the players' preferences are represented by finitely additive or countably additive measures over $\mathscr{F}$. Often it is the case that all measures representing the players' preferences are nonatomic probability measures.

Various notions of fairness have been introduced and studied within this framework. One such notion, the α-fairness, is defined as follows. Suppose that there are n players, the value of the empty set is 0, and the value of the whole cake is 1 for all players. Let α be an n-tuple $(\alpha_1, \dots, \alpha_n)$ of nonnegative real numbers whose sum is 1. If $(A_1, \dots, A_n)$ is an ordered partition of Ω into n subsets, then we say that $(A_1, \dots, A_n)$ is *α-fair* when, for each player i, the value of his or her measure at A_i is at least α_i.

Obviously, some of the mentioned assumptions, especially those of additivity and nonatomicity, limit possible applications. For example, some important preference relations have no numerical representations. Moreover, even if numerical representations exist, the requirement of additivity is too restrictive for some applications in social sciences. For example, in economics it implies utility functions that are marginally constant. To investigate preference relations that are representable by nonadditive set functions, Sagara and Vlach [8] introduced the convexity of preference relations on a σ-algebra $\mathscr{F}$ and demonstrated that convex preference relations were representable by quasiconcave functions on $\mathscr{F}$.

While the hypothesis of the measures being nonatomic is difficult to relax, because without it fair divisions do not always exist, under some relaxations of the additivity assumption, the characterization and demonstration of existence of solutions seem possible with certain continuity conditions. Recent attempts to relax the additivity assumption can be found in Dall'Aglio and Maccheroni [1], Maccheroni and Marinacci [5], Sagara [6, 7], and Sagara and Vlach [10].

In this short note we introduce a certain notion of concavity of functions on σ-algebras and prove the convexity of their lower partition ranges (Theorem 3.1). Consequently, we obtain a new type of generalization of the Dubins and Spanier [2] result for the existence of α-fair divisions (Corollary 4.1).

2. Convex combinations of measurable sets

Let $\mathscr{F}$ be a σ-algebra of subsets of a nonempty set Ω. A measure μ on $\mathscr{F}$ is *nonatomic* if, for every $A \in \mathscr{F}$ with $\mu(A) > 0$, there exists some $E \in \mathscr{F}$ such that $0 < \mu(E) < \mu(A)$.

Let $\mu_1, \ldots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathscr{F})$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ be an $\mathbb{R}^n$-valued vector measure. For an arbitrarily given $A \in \mathscr{F}$ and $t \in [0,1]$, we define the family $\mathscr{K}_t^{\vec{\mu}}(A)$ of measurable subsets of A by:

$$\mathscr{K}_t^{\vec{\mu}}(A) = \{E \in \mathscr{F} \mid E \subset A \text{ and } \vec{\mu}(E) = t\vec{\mu}(A)\}.$$

By Lyapunov's convexity theorem, $\mathscr{K}_t^{\vec{\mu}}(A)$ is nonempty for every $A \in \mathscr{F}$ and $t \in [0,1]$. Furthermore, for an arbitrarily given $A, B \in \mathscr{F}$ and $t \in [0,1]$, we denote by $\mathscr{K}_t^{\vec{\mu}}(A, B)$ the family of sets $C \in \mathscr{F}$ such that C is the union of some disjoint sets $E \in \mathscr{K}_t^{\vec{\mu}}(A)$ and $F \in \mathscr{K}_{1-t}^{\vec{\mu}}(B)$. For a nonatomic scalar measure μ, we use $\mathscr{K}_t^{\mu}(A, B)$. It is evident that $C \in \mathscr{K}_t^{\vec{\mu}}(A, B)$ if and only if $C \in \mathscr{K}_t^{\mu_i}(A, B)$ for each $i = 1, \ldots, n$, and hence $\mathscr{K}_t^{\vec{\mu}}(A, B) = \bigcap_{i=1}^n \mathscr{K}_t^{\mu_i}(A, B)$ for every $A, B \in \mathscr{F}$ and $t \in [0,1]$.

The following result can be obtained as an easy consequence of the proof of Lemma 4 in Halmos [4].

Theorem 2.1. *$\mathscr{K}_t^{\vec{\mu}}(A, B)$ is nonempty for every $A, B \in \mathscr{F}$ and $t \in [0,1]$.*

Proof. Let $A, B \in \mathscr{F}$ and $t \in [0,1]$ and let $E_1 \in \mathscr{K}_t^{\vec{\mu}}(A \cap B)$, $E_2 \in \mathscr{K}_t^{\vec{\mu}}(A \setminus B)$, $F_1 = (A \cap B) \setminus E_1$ and $F_2 \in \mathscr{K}_{1-t}^{\vec{\mu}}(B \setminus A)$. To prove the nonemptiness of $\mathscr{K}_t^{\vec{\mu}}(A, B)$, we show that $E_1 \cup E_2 \cup F_1 \cup F_2$ belongs to $\mathscr{K}_t^{\vec{\mu}}(A, B)$. It follows immediately from the construction that the sets $E_1 \cup E_2$ and $F_1 \cup F_2$ are disjoint. Thus, it remains to show that $E_1 \cup E_2 \in \mathscr{K}_t^{\vec{\mu}}(A)$ and $F_1 \cup F_2 \in \mathscr{K}_{1-t}^{\vec{\mu}}(B)$. By construction, we know that E_1, E_2, F_1 and F_2 are pairwise disjoint, and by an easy calculation we obtain:

- $\vec{\mu}(E_1 \cup E_2) = t\vec{\mu}(A \cap B) + t\vec{\mu}(A \setminus B) = t\vec{\mu}(A)$.
- $\vec{\mu}(F_1 \cup F_2) = (1-t)\vec{\mu}(A \cap B) + (1-t)\vec{\mu}(B \setminus A) = (1-t)\vec{\mu}(B)$.

Hence $E_1 \cup E_2 \in \mathscr{K}_t^{\vec{\mu}}(A)$ and $F_1 \cup F_2 \in \mathscr{K}_{1-t}^{\vec{\mu}}(B)$. □

An *m-partition* of Ω is an m-tuple $(A_1, \ldots, A_m)$ of mutually disjoint elements $A_1, \ldots, A_m$ in $\mathscr{F}$ whose union is Ω. We denote by $\mathscr{P}^m$ the set of m-partitions of Ω.

Theorem 2.2. *For every pair of m-partitions $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_m)$ of Ω and every real number t in $[0,1]$, there is an m-partition $(C_1, \ldots, C_m)$ of Ω such that $C_j \in \mathscr{K}_t^{\vec{\mu}}(A_j, B_j)$ for each $j = 1, \ldots, m$.*

Proof. Let E_{ij} be a subset of $A_i \cap B_j$ satisfying $E_{ij} \in \mathscr{K}_t^{\vec{\mu}}(A_i \cap B_j)$, and put $E_i = \bigcup_{j=1}^n E_{ij}$ and $F_j = B_j \setminus \bigcup_{i=1}^n E_{ij}$ for each i, j. By construction, we have $E_i \cap E_j = \emptyset$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. Since $E_{ik} \cap B_j = \emptyset$ for $k \neq j$, we have

$$E_i \cap F_j = \bigcup_{k=1}^{m} E_{ik} \cap \left(B_j \setminus \bigcup_{k=1}^{m} E_{kj} \right) = E_{ij} \cap \left(B_j \setminus \bigcup_{k=1}^{m} E_{kj} \right) \subset E_{ij} \cap \left(B_j \setminus E_{ij} \right) = \emptyset,$$

for each i, j. Since

$$\vec{\mu}(E_i) = \sum_{j=1}^{m} \vec{\mu}(E_{ij}) = t \sum_{j=1}^{m} \vec{\mu}(A_i \cap B_j) = t\vec{\mu}(A_i),$$

we have $E_i \in \mathscr{K}_t^{\vec{\mu}}(A_i)$ for each i. Similarly,

$$\vec{\mu}(F_j) = \vec{\mu}(B_j) - \sum_{i=1}^{m} \vec{\mu}(E_{ij}) = \vec{\mu}(B_j) - t \sum_{i=1}^{m} \vec{\mu}(A_i \cap B_j) = \vec{\mu}(B_j) - t\vec{\mu}(B_j) = (1-t)\vec{\mu}(B_j).$$

Thus, $F_j \in \mathscr{K}_{1-t}^{\vec{\mu}}(B_j)$ for each j. Define $C_j = E_j \cup F_j$ for each j. Then the resulting partition $(C_1, \dots, C_m)$ is such that $C_j \in \mathscr{K}_t^{\vec{\mu}}(A_j, B_j)$ for each j. □

3. Convex structure of concave measures

The notion of concave measures on σ-algebras presented in the following definition bears an obvious resemblance to that of concave functions on real vector spaces. These measures have recently been introduced and studied by Sagara and Vlach [8–10].

Definition 3.1. A set function $\nu : \mathscr{F} \to \mathbb{R}$ is a *concave measure* if $\nu(\emptyset) = 0$ and there exists a nonatomic finite measure μ such that $A, B \in \mathscr{F}$ and $t \in [0,1]$ imply $t\nu(A) + (1-t)\nu(B) \leq \nu(C)$ for every $C \in \mathscr{K}_t^{\mu}(A, B)$.

For set functions $\nu_1, \dots, \nu_n$, define the *lower range* by

$$\underline{\mathscr{R}}(\nu_1, \dots, \nu_n) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists A \in \mathscr{F} : x_i \leq \nu_i(A),\ i = 1, \dots, n\}$$

and the *lower partition range* by

$$\underline{\mathscr{R}}^m(\nu_1, \dots, \nu_n) = \left\{ (x_{ij}) \in \mathbb{R}^{nm} \,\middle|\, \begin{array}{l} \exists (A_1, \dots, A_m) \in \mathscr{P}^m : x_{ij} \leq \nu_i(A_j) \\ i = 1, \dots, n;\ j = 1, \dots, m \end{array} \right\}.$$

The main result of this paper is as follows.

Theorem 3.1. *If $\nu_1, \dots, \nu_n$ are concave measures, then $\underline{\mathscr{R}}(\nu_1, \dots, \nu_n)$ and $\underline{\mathscr{R}}^m(\nu_1, \dots, \nu_n)$ are convex in $\mathbb{R}^n$ and $\mathbb{R}^{nm}$ respectively.*

Proof. Let $x, y \in \underline{\mathscr{R}}(\nu_1, \dots, \nu_n)$ and $t \in [0, 1]$. Then there are A and B in $\mathscr{F}$ such that $x_i \leq \nu_i(A)$ and $y_i \leq \nu_i(B)$ for each i. Since each ν_i is a concave measure, there exists a nonatomic finite measure μ_i associated with ν_i and by Theorem 2.1, $\mathscr{K}_t^{\vec{\mu}}(A, B)$ is nonempty for $\vec{\mu} = (\mu_1, \dots, \mu_n)$. Choose any $C \in \mathscr{K}_t^{\vec{\mu}}(A, B)$. We then have $tx_i + (1-t)y_i \leq t\nu_i(A) + (1-t)\nu_i(B) \leq \nu_i(C)$ for each i in view of $C \in \bigcap_{i=1}^n \mathscr{K}_t^{\mu_i}(A, B)$. Therefore, $tx + (1-t)y \in \underline{\mathscr{R}}(\nu_1, \dots, \nu_n)$.

For every $(x_{ij}), (y_{ij}) \in \underline{\mathscr{R}}^m(\nu_1, \dots, \nu_n)$ and $t \in [0, 1]$, there are $(A_1, \dots, A_m)$ and $(B_1, \dots, B_m)$ in $\mathscr{P}^m$ such that $x_{ij} \leq \nu_i(A_j)$ and $y_{ij} \leq \nu_i(B_j)$ for each i, j. By Theorem 2.2, there exists a $(C_1, \dots, C_m) \in \mathscr{P}^m$ such that $C_j \in \mathscr{K}_t^{\vec{\mu}}(A_j, B_j)$ for each j. We then have $tx_{ij} + (1-t)y_{ij} \leq t\nu_i(A_j) + (1-t)\nu_i(B_j) \leq \nu_i(C_j)$ in view of $C_j \in \bigcap_{i=1}^n \mathscr{K}_t^{\mu_i}(A_j, B_j)$ for each j. Therefore, $t(x_{ij}) + (1-t)(y_{ij}) \in \underline{\mathscr{R}}^m(\nu_1, \dots, \nu_n)$.

Note that if each ν_i is a nonatomic probability measure, Theorem 3.1 can be strengthened to guarantee the compactness of the lower and lower partition ranges. This is a consequence of Lyapunov's convexity theorem (see also Dvoretsky et al. [3]).

4. An application to the fair division problem

There are n players, indexed by $i = 1, \dots, n$. Each player's preference on $\mathscr{F}$ is given by a real-valued function $\nu_i : \mathscr{F} \to \mathbb{R}$, called a *utility function*, in terms of which, the inequality $\nu_i(A) \geq \nu_i(B)$ means that A is preferred over B or equally valued as B by player i.

Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$; that is:

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \alpha_i = 1 \text{ and } \alpha_i \geq 0, i = 1, \dots, n \right\}.$$

The following corollary of Theorem 3.1 generalizes a result of Dubins and Spanier [2].

Corollary 4.1. *If $\nu_1, \dots, \nu_n$ are concave measures with $0 \leq \nu_i \leq 1$ and $\nu_i(\Omega) = 1$ for each $i = 1, \dots, n$, then for every $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ there is an n-partition of Ω such that $\nu_i(A_i) \geq \alpha_i$ for each $i = 1, \dots, n$.*

Proof. If $P_k = (E_1, \dots, E_n)$ is a partition of Ω in which $E_k = \Omega$ and $E_j = \emptyset$ for $j \neq k$, then a partition matrix $M(P_k) = (\nu_i(E_j))$ has values of 1 in the kth column and values of zero elsewhere. Since $M(P_1), \dots, M(P_n)$ belong to $\underline{\mathscr{R}}^n(\nu_1, \dots, \nu_n)$, Theorem 3.1 implies that $\sum_{i=1}^n \alpha_i M(P_i)$ is in $\underline{\mathscr{R}}^n(\nu_1, \dots, \nu_n)$ for every $\alpha \in \Delta^{n-1}$. Therefore, there is a partition

$P = (A_1, \dots, A_n)$ of Ω such that $M(P) \geq \sum_{i=1}^{n} \alpha_i M(P_i)$; that is, $\nu_i(A_j) \geq \alpha_j$ for each i, j. □

The partitions whose existence is guaranteed by the corollary are said to be α-*fair*. Without resorting to any convexity hypothesis, Sagara and Vlach [10] demonstrated the existence of Pareto-optimal α-fair partitions for the class of *average anti-monotone measures*, which is more general than that of concave measures. Dall'Aglio and Maccheroni [1] and Maccheroni and Marinacci [5] obtained the existence of α-fair partitions under nonatomic submodular capacities. For a recent development of fair division theory, see the survey article by Sagara [7].

References

1. Dall'Aglio, M., Maccheroni, F.: Fair division without additivity. Amer. Math. Monthly **112**, 363–365 (2005)
2. Dubins, L.E., Spanier, E.H.: How to cut a cake fairly. Amer. Math. Monthly **68**, 1–17 (1961)
3. Dvoretsky, A., Wald, A., Wolfowitz, J.: Relations among certain ranges of vector measures. Pacific J. Math. **1**, 59–74 (1951)
4. Halmos, P.R.: The range of a vector measure. Bull. Amer. Math. Soc. **54**, 416–421 (1948)
5. Maccheroni, F., Marinacci, M.: How to cut a pizza fairly: fair division with decreasing marginal evaluations. Soc. Choice Welf. **20**, 457–465 (2003)
6. Sagara, N.: A characterization of α-maximin solutions of fair division problems. Math. Social Sci. **55**, 273–280 (2008)
7. Sagara, N.: Fair division problems with nonadditive evaluations: existence of solutions and representation of preference orderings. In: Varela J., Acuña S. (eds.): Handbook of optimization theory: decision analysis and application. Hauppauge: Nova 2010, forthcoming
8. Sagara, N., Vlach, M.: Representation of preference relations on σ-algebras of nonatomic measure spaces: convexity and continuity. Fuzzy Sets and Systems **160**, 624–634 (2009)
9. Sagara, N., Vlach, M.: Convex functions on σ-algebras of nonatomic measure spaces. Pac. J. Optim. **6**, 89–102 (2010)
10. Sagara, N., Vlach, M.: A new class of convex games on σ-algebras and optimal partitioning of measurable spaces. Faculty of Economics, Hosei University, mimeo. http://home.v00.itscom.net/nsagara/ (2009)
11. Steinhaus, H.: The problem of fair division. Econometrica **16**, 101–104 (1948)
12. Steinhaus, H.: Sur la division pragmatique. Econometrica **17**(Suppl), 315–319 (1949)

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Good locally maximal programs for the Robinson–Solow–Srinivasan model

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Abstract. In this paper we obtain results on the good choice of techniques in the long-run in the model proposed by Robinson, Solow and Srinivasan. We study this model with a nonconcave utility function which represents the preferences of the planner and establish some properties of locally maximal programs. We obtain a useful estimation for consumption over good locally maximal programs and show that a limit of good locally optimal programs is also a good locally maximal program.

Key words: Good program, locally maximal program, RSS model

1. Introduction

The study of the existence and structure of optimal programs and good programs in models of economic growth is an important area of the economic theory. In their paper Khan and Mitra [6] studied the problem of optimal economic growth in the Robinson–Solow–Srinivasan model [13] (henceforth, the RSS model). Khan and Mitra [6] established the existence of a golden-rule stock, with support prices, showed that the golden-rule stock is unique and showed the existence of good programs and maximal programs. In Zaslavski [16] we obtained an extension of some of their results and established

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the existence of an overtaking optimal program for any initial stock. Optimal growth in the two-sector RSS model was studied in [7–10]. A uniform turnpike result of the third kind in the Robinson–Solow–Srinivasan model was established in [11]. It should we mentioned that Khan and Mitra [6–10] Khan and Zaslavski [11] and Zaslavski [16] assumed that the preferences of the planner are represented by a concave function w, a usual assumption in mathematical economics. In Zaslavski [17] we studied the RSS model without the assumption that the function w is concave and established the existence of good programs and locally maximal programs. In the present paper we continue to study the locally maximal programs introduced in Zaslavski [17].

We begin with some preliminary notation. Let R (R_+) be the set of real (non-negative) numbers and let R^n be a finite-dimensional Euclidean space with non-negative orthant $R^n_+ = \{x \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x \gg y$, $x > y$, $x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $||x||$ denote the Euclidean norm of x.

We now give a formal description of the RSS model. For its economic interpretation see [6].

Let $a = (a_1, \dots, a_n) \gg 0$, $b = (b_1, \dots, b_n) \gg 0$ and let $d \in (0, 1]$.

A sequence $\{x(t), y(t)\}_{t=0}^\infty$ is called a program if for each integer $t \geq 0$

$$(x(t), y(t)) \in R^n_+ \times R^n_+, \ x(t+1) \geq (1-d)x(t), \tag{1.1}$$

$$0 \leq y(t) \leq x(t), \ a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1.$$

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$ and for each integer t satisfying $T_1 \leq t < T_2$ relations (1.1) hold.

Let $w : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function which represents the preferences of the planner.

For each $x_0 \in R^n_+$ and each integer $T > 0$ set

$$U(x_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : \ (\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$$

$$\text{is a program such that } \ x(0) = x_0\}. \tag{1.2}$$

In the sequel we assume that supremum of empty set is $-\infty$.

Let $x_0, \ \tilde{x}_0 \in R^n_+$ and let T be a natural number. Set

$$U(x_0, \tilde{x}_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : \ (\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$$

$$\text{is a program such that } \ x(0) = x_0, \ x(T) \geq \tilde{x}_0\}. \tag{1.3}$$

Set

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' \geq (1-d)x \text{ and } a(x'-(1-d)x) \leq 1\}. \tag{1.4}$$

We have a correspondence $\Lambda : \Omega \to R^n_+$ given by

$$\Lambda(x, x') = \{y \in R^n_+ : \ 0 \leq y \leq x \text{ and}$$

$$ey \leq 1 - a(x' - (1-d)x)\}, \ (x, x') \in \Omega. \tag{1.5}$$

Let $M_0 > 0$ and let T be a natural number. Set

$$\widehat{U}(M_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : \tag{1.6}$$

$$(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) \leq M_0 e\}.$$

It is clear that $\widehat{U}(M_0, T)$ is finite.

We begin with the following result established in [17, Theorem 1.1] which allows us to define an important constant μ introduced in [17].

Theorem 1.1. *Let* $M_1, M_2 > \max\{(da_i)^{-1} : \ i = 1, \dots, n\}$. *Then there exist finite limits*

$$\lim_{p\to\infty} \widehat{U}(M_i, p)/p, \ i = 1, 2$$

and

$$\lim_{p\to\infty} \widehat{U}(M_1, p)/p = \lim_{p\to\infty} \widehat{U}(M_2, p)/p.$$

Define

$$\mu = \lim_{p\to\infty} \widehat{U}(M, p)/p, \tag{1.7}$$

where $M > \max\{(da_i)^{-1} : \ i = 1, \dots, n\}$. By Theorem 1.1 μ is well defined and does not depend on M. We showed in [17] that $\mu > w(0)$.

The next theorem was also proved in [17].

Theorem 1.2. *Let* $M_0 > \max\{(da_i)^{-1} : i = 1, \dots, n\}$. *Then there exists* $M > 0$ *such that*

$$|\widehat{U}(M_0, p) - p\mu| \le M \text{ for all integers } p \ge 1.$$

Corollary 1.1. *Let* $M_0 > \max\{(da_i)^{-1} : i = 1, \dots, n\}$. *Then there exists* $M > 0$ *such that for each program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *satisfying* $x(0) \le M_0 e$ *and each integer* $T \ge 1$

$$\sum_{t=0}^{T-1} [w(by(t)) - \mu] \le M.$$

Note that Corollary 1.1 easily follows from Theorem 1.2.

The following result was also proved in [17, Proposition 1.3].

Proposition 1.1. *Let* $\{x(t), y(t)\}_{t=0}^{\infty}$ *be a program. Then either the sequence* $\{\sum_{t=0}^{T-1} [w(by(t)) - \mu]\}_{T=1}^{\infty}$ *is bounded or*

$$\lim_{T \to \infty} \sum_{t=0}^{T-1} [w(by(t)) - \mu] = -\infty.$$

We use the following notion of good programs introduced by Gale [5].

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called good if there exists $M \in R$ such that

$$\sum_{t=0}^{T} (w(y(t)) - \mu) \ge M \text{ for all } T \ge 0.$$

A program is called bad if

$$\lim_{T \to \infty} \sum_{t=0}^{T} (w(y(t)) - \mu) = -\infty.$$

The last two results of this section was also proved in Zaslavski [17].

Theorem 1.3 [17, Theorem 1.3]. *Let* $M_0 > \max\{(da_i)^{-1}: i = 1, \dots, n\}$. *Then there exists* $M > 0$ *such that for each* $x_0 \in R^n_+$ *satisfying* $x_0 \le M_0 e$ *there exists a program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *such that* $x(0) = x_0$, *for each integer* $T_1 \ge 0$ *and each integer* $T_2 > T_1$

$$|\sum_{t=T_1}^{T_2 - 1} w(by(t)) - \mu(T_2 - T_1)| \le M$$

and that for each integer $T > 0$

$$\sum_{t=0}^{T-1} w(by(t)) = U(x(0), x(T), T). \tag{1.8}$$

Theorem 1.3 establishes that for any initial state $x_0 \geq 0$ there exists a good program $\{x(t),\ y(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$. In addition this program satisfies (1.8) for each integer $T > 0$. This leads us to the following optimality criterion which was introduced in [17].

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called locally maximal if equality (1.8) holds for all integers $T > 0$.

Note that prototypes of this optimality criterion were introduced in Aubry and Le Daeron [2] and studied in [2, 12, 15] for certain physical models. In these works the main goal was to study the existence and properties of (locally) minimal energy configurations. In [2,15] it was considered minimal energy configurations in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals while in [12] it was considered minimal energy configurations in the theory of thermodynamical equilibrium for materials.

In the literature there are other optimality criterions which are stronger than the notion of the local maximality [1, 3, 14] but in order to use them we need the concavity of the function w [6, 7, 16]. By Theorem 1.3 locally maximal programs exist without this concavity assumption.

Theorem 1.4 [17, Theorem 1.4]. *Let* $\{x(t), y(t)\}_{t=0}^{\infty}$ *be a locally maximal program such that* $\limsup_{t\to\infty} by(t) > 0$. *Then the program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *is good.*

By Theorem 1.4 in general locally maximal programs are good. Clearly, there are good programs which are not locally maximal. For example, consider a program which is good and locally maximal, fix a natural number T and construct a new program changing only consumption at moments $t = 0, \dots, T-1$ and assuming that it is zero. It is clear that this new program is good but not locally optimal.

The discussion above shows the importance of locally maximal programs and we continue to study them in the present paper. The paper contains three main results. In the first result (Theorem 2.1) we establish a continuity property of the function $U(\cdot, \cdot, T)$ for any fixed natural number T. In the second main result (Theorem 2.2) we obtain an estimation of consumption over good locally maximal programs and in the third main result (Theorem 2.3) we show that a limit of a convergent sequence of good locally maximal programs is also a good locally maximal program. Theorems 2.2 and 2.3 have prototypes in [2, 15].

2. Main results

We assume for simplicity that $w(0) = 0$ and begin with the following result which establishes the continuity of the function $U(\cdot, \cdot, T)$. This result will be proved in § 3.

Theorem 2.1. *Let $T > 0$ be an integer, $x_0,\ \tilde{x}_0 \in R^n_+$, $U(x_0, \tilde{x}_0, T) > 0$ and let one of the following conditions holds: $d < 1$; $d = 1$ and $a\tilde{x}_0 < 1$.*

Then the function $(y, z) \to U(y, z, T)$, $y, z \in R^n_+$ is continuous at $(x_0, \tilde{x}_0)$.

The following result will be proved in § 4.

Theorem 2.2. *Let $M_0 > \max\{(da_i)^{-1} : i = 1, \dots, n\}$. Then there exists $M_1 > 0$ such that for each good locally maximal program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \le M_0 e$ and each pair of integers $S_1 \ge 0$ and $S_2 > S_1$,*

$$|\sum_{t=S_1}^{S_2-1} w(by(t)) - \mu(S_2 - S_1)| \le M_1. \tag{2.1}$$

By definition for any good program $\{x(t), y(t)\}_{t=0}^{\infty}$ there is a constant $M_1 > 0$ such that (2.1) holds. In view of Theorem 2.2 the constant M_1 depends only on the constant M_0 and the inequality (2.1) holds for all programs $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \le M_0 e$.

The next theorem is our final result which will be proved in § 5.

Theorem 2.3. *Let $\{x^{(k)}(t), y^{(k)}(t)\}_{t=0}^{\infty}$, $k = 1, 2, \dots$ be good locally maximal programs. Assume that for any integer $t \ge 0$ there exists $x(t) = \lim_{k\to\infty} x^{(k)}(t)$, $y(t) = \lim_{k\to\infty} y_t^{(k)}$. Then $\{x(t), y(t)\}_{t=0}^{\infty}$ is a good locally maximal program.*

It should be mentioned that in Theorem 2.1 the case d = 1 is also considered. In this connection see [4].

The proofs of our results have the following structure. The proof of Theorem 2.1 is based on two lemmas. In the first lemma we prove upper semicontinuity of the function $U(\cdot, \cdot, T)$ while lower semicontinuity of this function is established in the second lemma. The proof of Theorem 2.2 is based on Theorem 1.3, Corollary 1.1 and Lemma 2.1 of [17]. The main ingredients of the proof of Theorem 2.3 are Theorems 2.1 and 2.2.

3. Proof of Theorem 2.1

In the sequel we use the following simple auxiliary result of [17, Lemma 1.1].

Lemma 3.1. *Let a number* $M_0 > \max\{(a_i d)^{-1} : i = 1, \dots, n\}$, $(x, x') \in \Omega$ *and let* $x \le M_0 e$. *Then* $x' \le M_0 e$.

Lemma 3.2. *Let* T *be a natural number,* $x_0, \tilde{x}_0 \in R^n_+$ *satisfy* $U(x_0, \tilde{x}_0, T) > 0$ *and let* $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $y_0, \tilde{y}_0 \in R^n_+$ *satisfying* $||y_0 - x_0||$, $||\tilde{y}_0 - \tilde{x}_0|| \le \delta$ *the following inequality holds:*

$$U(y_0, \tilde{y}_0, T) < U(x_0, \tilde{x}_0, T) + \epsilon.$$

Proof. Let us assume the contrary. Then for each integer $k \ge 1$ there exist $x_0^{(k)}$, $\tilde{x}_0^{(k)} \in R^n_+$ such that

$$||x_0 - x_0^{(k)}|| \le k^{-1}, \ ||\tilde{x}_0 - \tilde{x}_0^{(k)}||] \le k^{-1}, \tag{3.1}$$

$$U(x_0^{(k)}, \tilde{x}_0^{(k)}, T) \ge U(x_0, \tilde{x}_0, T) + \epsilon. \tag{3.2}$$

For each natural number k there exists a program $(\{x^{(k)}(t)\}_{t=0}^{T}, \{y^{(k)}(t)\}_{t=0}^{T-1})$ such that

$$x^{(k)}(0) = x_0^{(k)}, \ x_T^{(k)} \ge \tilde{x}_0^{(k)}, \tag{3.3}$$

$$\sum_{t=0}^{T-1} w(by^{(k)}(t)) \ge U(x_0^{(k)}, \tilde{x}_0^{(k)}, T) - k^{-1}. \tag{3.4}$$

By (3.1), (3.3) and Lemma 3.1 the set

$$\{x^{(k)}(t) : \ t = 0, 1, \dots, \ k = 1, 2, \dots\}$$

is bounded. Extracting a subsequence and re-indexing we may assume without loss of generality that for $t = 0, \dots, T$ there exists

$$x(t) = \lim_{k \to \infty} x^{(k)}(t) \tag{3.5}$$

and for $t = 0, \dots, T - 1$ there exists

$$y(t) = \lim_{k \to \infty} y^{(k)}(t). \tag{3.6}$$

It is clear that $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ is a program. By (3.1), (3.3) and (3.5),

$$x(0) = x_0,\ x(T) \geq \tilde{x}_0^{(0)}. \tag{3.7}$$

In view of (3.2), (3.4), (3.6) and (3.7),

$$U(x_0, \tilde{x}_0, T) \geq \sum_{t=0}^{T-1} w(by(t)) = \lim_{k\to\infty} \sum_{t=0}^{T-1} w(by^{(k)}(t))$$

$$= \limsup_{k\to\infty} U(x_0^{(k)}, \tilde{x}_0^{(k)}, T) \geq U(x_0, \tilde{x}_0, T) + \epsilon.$$

The contradiction we have reached proves the lemma. □

Lemma 3.3. *Let T be a natural number, $x_0,\ \tilde{x}_0 \in R^n_+$ be such that $U(x_0, \tilde{x}_0, T) > 0$ and let $\epsilon > 0$. Assume that one of the following conditions holds: $d < 1$; $d = 1$ and $a\tilde{x}_0 < 1$. Then there exists $\delta > 0$ such that for each $y_0, \tilde{y}_0 \in R^n_+$ satisfying*

$$||y_0 - x_0||,\ ||\tilde{y}_0 - \tilde{x}_0|| \leq \delta \tag{3.8}$$

the following inequality holds:

$$U(y_0, \tilde{y}_0, T) > U(x_0, \tilde{x}_0, T) - \epsilon.$$

Proof. Since $U(x_0, \tilde{x}_0, T) > 0$ it follows from Lemma 3.1 and the continuity of w that there exists a program $(\{\bar{x}(t)\}_{t=0}^{T}, \{\bar{y}(t)\}_{t=0}^{T-1})$ such that

$$\bar{x}(0) = x_0,\ \bar{x}(T) \geq \tilde{x}_0, \tag{3.9}$$

$$\sum_{t=0}^{T-1} w(b\bar{y}(t)) = U(x_0, \tilde{x}_0, T) > 0.$$

By (3.9) there exists an integer $\tau_1 \in [0, T-1]$ such that

$$w(b\bar{y}(\tau_1)) > 0; \tag{3.10}$$

$$\text{if an integer } t \text{ satisfies } 0 \leq t < \tau_1 \text{ then } w(b\bar{y}(t)) = 0.$$

Then

$$\max\{\bar{y}_i(\tau_1) :\ i = 1, \dots, n\} > 0.$$

Fix $i_1 \in \{1, \dots, n\}$ such that

$$\bar{y}_{i_1}(\tau_1) > 0. \tag{3.11}$$

Choose a positive number Δ such that

$$\Delta < 4^{-1}\bar{y}_{i_1}(\tau_1),\ \Delta < (be)^{-1} b\bar{y}(\tau_1)/8, \tag{3.12}$$

$$w(b\bar{y}(\tau_1) - \Delta be) > w(b\bar{y}(\tau_1)) - \epsilon/8.$$

Let us define a constant $\delta > 0$. We consider the two cases separately.

If $d < 1$, then we choose $\delta > 0$ such that for all $i = 1, \dots, n$

$$\delta < (4n)^{-1}(1 + a_i)^{-1}\Delta(1 - d)^T. \tag{3.13}$$

Now consider the case with

$$d = 1 \text{ and } a\tilde{x}_0 < 1.$$

Since w is strictly increasing it follows from (3.9) that

$$\bar{x}(T) = \tilde{x}_0,\ b\bar{y}(T - 1) > 0. \tag{3.14}$$

By (3.14) there is $i_2 \in \{1, \dots, n\}$ such hat

$$\bar{y}_{i_2}(T - 1) > 0. \tag{3.15}$$

In view of (3.15) we choose $\delta > 0$ such that for all $i = 1, \dots, n$

$$\delta < (4n)^{-1}(1 + a_i)^{-1}\Delta, \tag{3.16}$$

$$(ae + 1)\delta < \bar{y}_{i_2}(T - 1)/4,$$

$$w(b\bar{y}(T - 1) - b_{i_2}\delta ae) > w(b\bar{y}(T - 1)) - \epsilon/8.$$

Thus the constant δ is defined in the both cases.

Assume that $y_0, \tilde{y}_0 \in R^n_+$ satisfy (3.8). Set

$$x(0) = y_0. \tag{3.17}$$

For each integer t satisfying $0 \le t < \tau_1$ set

$$x(t + 1) = (1 - d)x(t) + \bar{x}(t + 1) - (1 - d)\bar{x}(t),\ y(t) = 0. \tag{3.18}$$

It is clear that if $\tau_1 > 0$, then $(\{x(t)\}_{t=0}^{\tau_1}, \{y(t)\}_{t=0}^{\tau_1 - 1})$ is a program. By (3.8), (3.9), (3.17) and (3.18),

$$||x(\tau_1) - \bar{x}(\tau_1)|| \le ||x(0) - \bar{x}(0)|| = ||y_0 - x_0|| \le \delta. \tag{3.19}$$

Note that (3.19) holds if $\tau_1 > 0$ and if $\tau_1 = 0$. Set

(3.20)

$$y_i(\tau_1) = \max\{\bar{y}_i(\tau_1) - \delta, 0\},\ i \in \{1, \dots, n\} \setminus \{i_1\},\ y_{i_1}(\tau_1) = \bar{y}_{i_1}(\tau_1) - \Delta,$$

(3.21)

$$x(\tau_1 + 1) = (1 - d)x(\tau_1) + \bar{x}(\tau_1 + 1) - (1 - d)\bar{x}(\tau_1) + n^{-1}(a_1^{-1}, \dots, a_n^{-1})\Delta.$$

In view of (3.13), (3.16), (3.19) and (3.20)

$$0 \le y(\tau_1) \le x(\tau_1). \tag{3.22}$$

Clearly,

$$x(\tau_1 + 1) \ge (1 - d)x(\tau_1). \tag{3.23}$$

By (3.21) and (3.20)

$$\begin{aligned} &a(x(\tau_1 + 1) - (1 - d)x(\tau_1)) + ey(\tau_1) \\ &= a(\bar{x}(\tau_1 + 1) - (1 - d)\bar{x}(\tau_1)) + \Delta + ey(\tau_1) \\ &\le a(\bar{x}(\tau_1 + 1) - (1 - d)\bar{x}(\tau_1)) + \Delta + e\bar{y}(\tau_1) - \Delta \le 1. \end{aligned}$$

Together with (3.22) and (3.23) this implies that $(\{x(t)\}_{t=0}^{\tau_1+1}, \{y(t)\}_{t=0}^{\tau_1})$ is a program. It follows from (3.21), (3.19), (3.16) and (3.13) that for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} x_i(\tau_1 + 1) &\ge \bar{x}_i(\tau_1 + 1) - ||\bar{x}(\tau_1) - \bar{x}(\tau_1)|| + n^{-1}a_i^{-1}\Delta \\ &\ge \bar{x}_i(\tau_1 + 1) - \delta + n^{-1}a_i^{-1}\Delta \\ &\ge \bar{x}_i(\tau_1 + 1) + 2^{-1}n^{-1}a_i^{-1}\Delta \end{aligned}$$

and

$$x(\tau_1 + 1) \ge \bar{x}(\tau_1 + 1) + (2n)^{-1}\Delta(a_1^{-1}, \dots, a_n^{-1}). \tag{3.24}$$

By (3.8), (3.10), (3.12), (3.13), (3.16) and (3.20),

$$\begin{aligned} &\sum_{t=0}^{\tau_1} w(by(t)) - \sum_{t=0}^{\tau_1} w(b\bar{y}(t)) = w(by(\tau_1)) - w(b\bar{y}(\tau_1)) \\ &= w(\sum_{i=1}^{n} b_i y_i(\tau_1)) - w(\sum_{i=1}^{n} b_i \bar{y}_i(\tau_1)) \\ &\ge w(\max\{\sum_{i=1}^{n} b_i(\bar{y}_i(\tau_1) - \Delta), 0\}) - w(\sum_{i=1}^{n} b_i \bar{y}_i(\tau_1)) \end{aligned}$$

$$\ge w(\sum_{i=1}^{n} b_i \bar{y}_i(\tau_1) - \Delta \sum_{i=1}^{n} b_i) - w(\sum_{i=1}^{n} b_i \bar{y}_i(\tau_1)) > -\epsilon/8. \tag{3.25}$$

If $\tau_1 + 1 = T$, then by (3.24), (3.13), (3.16) and (3.9),

$$x(T) = x(\tau_1 + 1) \ge \bar{x}(\tau_1 + 1) + \delta e \ge \tilde{x}(0) + \delta e \ge \tilde{y}(0) \tag{3.26}$$

and by (3.26), (3.17), (3.25) and (3.9)

$$U(y_0, \tilde{y}_0, T) \geq \sum_{t=0}^{\tau_1} w(by(t)) \geq \sum_{t=0}^{\tau_1} w(b\bar{y}(t)) - \epsilon/8 = U(x_0, \tilde{x}_0, T) - \epsilon/8.$$

Thus in the case $\tau_1 + 1 = T$ the assertion of the lemma holds.

Assume that $\tau_1+1 < T$. There are two cases: $d < 1$; $d = 1$ and $a\tilde{x}_0 < 1$.

Consider the case $d < 1$. For each integer t satisfying $\tau_1 + 1 \leq t < T$ set

$$(3.27) \quad x(t+1) = (1-d)x(t) + \bar{x}(t+1) - (1-d)\bar{x}(t), \ y(t) = \bar{y}(t).$$

In view of (3.14) and (3.27) $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ is a program. By (3.24) and (3.27),

$$\begin{aligned} x(T) - \bar{x}(T) &= (1-d)^{T-\tau_1-1}(x(\tau_1+1) - \bar{x}(\tau_1+1)) \\ &\geq (1-d)^{T-\tau_1-1}(2n)^{-1}\Delta(a_1^{-1}, \dots, a_n^{-1}) \\ &\geq (1-d)^{T}(2n)^{-1}\Delta(a_1^{-1}, \dots, a_n^{-1}). \end{aligned}$$

Together with (3.8), (3.9) and (3.13) this implies that

$$(3.28) \quad x(T) \geq \bar{x}(T) + (1-d)^T(2n)^{-1}\Delta(a_1^{-1}, \dots, a_n^{-1}) \geq \tilde{x}_0 + \delta e \geq \tilde{y}_0.$$

By (3.28), (3.17), (3.9), (3.27) and (3.25)

$$\begin{aligned} U(y_0, \tilde{y}_0, T) - U(x_0, \tilde{x}_0, T) &\geq \sum_{t=0}^{T-1} w(by(t)) - \sum_{t=0}^{T-1} w(b\bar{y}(t)) \\ &= \sum_{t=0}^{\tau_1} w(by(t)) - \sum_{t=0}^{\tau-1} w(b\bar{y}(t)) > -\epsilon/8. \end{aligned}$$

Thus the assertion of the lemma holds if $d < 1$.

Now assume that $d = 1$ and $a\tilde{x}_0 < 1$. Then (3.14)-(3.16) hold. Set for each integer t satisfying $\tau_1 + 1 \leq t < T - 1$

$$(3.29) \quad x(t+1) = \bar{x}(t+1), \ y(t) = \bar{y}(t),$$

$$x(T) = \bar{x}(T) + \delta e, \ y_{i_2}(T-1) = \bar{y}_{i_2}(T-1) - \delta a e,$$

$$\bar{y}_i(T-1) = y_i(T-1), \ i \in \{1, \dots, n\} \setminus \{i_2\}.$$

By (3.16) and (3.29) $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ is a program. In view of (3.8), (3.9) and (3.29),

$$x(T) = \bar{x}(T) + \delta e \geq \tilde{x}_0 + \delta e \geq \tilde{y}_0. \tag{3.30}$$

It follows from (3.30), (3.17), (3.9), (3.29), (3.25) and (3.16) that

$$U(y, \tilde{y}_0, T) - U(x_0, \tilde{x}_0, T) \geq \sum_{t=0}^{T-1} w(by(t)) - \sum_{t=0}^{T-1} w(b\bar{y}(t))$$

$$= \sum_{t=0}^{\tau_1} w(by(t)) - \sum_{t=0}^{\tau_1} w(b\bar{y}(t)) + w(by(T-1)) - w(b\bar{y}(T-1))$$

$$\geq -\epsilon/8 + w(b\bar{y}(T-1) - b_{i_2}\delta ae) - w(b\bar{y}(T-1)) \geq -\epsilon/4$$

and the assertion of the lemma holds if $d = 1$ and $a\bar{x}(T) < 1$. Lemma 3.3 is proved. □

Theorem 2.1 now easily follows from Lemmas 3.2 and 3.3.

4. Proof of Theorem 2.2

In the sequel we use the following simple auxiliary result of [17, Lemma 2.1].

Lemma 4.1. *Let* $\delta > 0$ *and let*

$$M_0 > \max\{(a_i d)^{-1} : i = 1, \ldots, n\}.$$

Then there exists a natural number $T_0 \geq 4$ *such that for each integer* $\tau \geq T_0$, *each program* $(\{x(t)\}_{t=0}^{\tau}, \{y(t)\}_{t=0}^{\tau-1})$ *which satisfies*

$$x(0) \leq M_0 e, \ by(\tau - 1) \geq \delta$$

and each $\tilde{x}_0 \in R_+^n$ *which satisfies* $\tilde{x}_0 \leq M_0 e$ *there exists a program*

$$(\{\tilde{x}(t)\}_{t=0}^{\tau}, \{\tilde{y}(t)\}_{t=0}^{\tau-1})$$

such that

$$\tilde{x}(0) = \tilde{x}_0, \ \tilde{x}(\tau) \geq x(\tau).$$

Proof of Theorem 2.2 There is $\delta > 0$ such that

$$w(t) < \mu/8 \text{ for each } t \in [0, \delta]. \tag{4.1}$$

Let $M > 0$ be as guaranteed by Theorem 1.3. By Corollary 1.1 there is $\tilde{M} > 0$ such that the following property holds:

(P1) for each program $\{x(t), y(t)\}_{t=0}^{\infty}$ satisfying $x(0) \le M_0 e$ and each integer $T \ge 1$,

$$\sum_{t=0}^{T-1} [w(by(t)) - \mu] \le \tilde{M}.$$

By Lemma 4.1 there exists a natural number $p \ge 4$ such that the following property holds:

(P2) for each integer $\tau \ge p$, each program $(\{x(t)\}_{t=0}^{\tau}, \{y(t)\}_{t=0}^{\tau-1})$ which satisfies

$$x(0) \le M_0 e,\ by(\tau - 1) \ge \delta$$

and each $\tilde{x}_0 \in R^n_+$ which satisfies $\tilde{x}_0 \le M_0 e$ there exists a program

$$(\{\bar{x}(t)\}_{t=0}^{\tau}, \{\bar{y}(t)\}_{t=0}^{\tau-1})$$

such that

$$\bar{x}(0) = \tilde{x}_0,\ \bar{x}(\tau) \ge x(\tau).$$

Set

$$M_1 = 2\tilde{M} + M + pM. \tag{4.2}$$

Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a good program such that

$$x(0) \le M_0 e \tag{4.3}$$

and that for all integers $T \ge 1$,

$$\sum_{t=0}^{T-1} w(by(t)) = U(x(0), x(T), T). \tag{4.4}$$

By the choice of M and Theorem 1.3 there exists a program $\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{\infty}$ such that

$$\bar{x}(0) = x(0) \tag{4.5}$$

and that for each integer $S_1 \ge 0$ and each integer $S_2 > S_1$,

$$|\sum_{t=S_1}^{S_2-1} w(b\bar{y}(t)) - \mu(S_2 - S_1)| \le M. \tag{4.6}$$

By (4.6) there exists an increasing sequence of natural numbers$\{S_k\}_{k=1}^{\infty}$ such that

$$w(by(T_k-1)) \geq \mu/2,\ k=1,2,\dots \tag{4.7}$$

In view of (4.1) and (4.7),

$$by(T_k-1) > \delta,\ k=1,2,\dots \tag{4.8}$$

We may assume without loss of generality that $T_k > p$ for all integers $k \geq 1$.

Let $k \geq 1$ be an integer and set

$$\tilde{x}(t)=\bar{x}(t),\ t=0,\dots,T_k-p,\ \tilde{y}(t)=\bar{y}(t),\ t=0,\dots,T_k-p-1. \tag{4.9}$$

By (4.3), (4.5), (4.8), (4.9), Lemma 3.1 and property (P2) applied to the program

$$(\{x(t)\}_{t=T_k-p}^{T_k}, \{y(t)\}_{t=T_k-p}^{T_k-1})$$

there exists a program $(\{\tilde{x}(t)\}_{t=T_k-p}^{T_k}, \{\tilde{y}(t)\}_{t=T_k-p}^{T_k-1})$ such that

$$\tilde{x}(T_k) \geq x(T_k). \tag{4.10}$$

By (4.4), (4.9), (4.10), (4.5) and (4.6),

$$\sum_{t=0}^{T_k-1} w(by(t)) = U(x(0),x(T_k),T_k) \geq \sum_{t=0}^{T_k-1} w(b\tilde{y}(t))$$

$$= \sum_{t=0}^{T_k-p-1} w(b\bar{y}(t)) + \sum_{t=T_k-p}^{T_k-1} w(b\tilde{y}(t)) \geq \mu(T_k-p)-M$$

$$\geq \mu(T_k)-M-pM.$$

Thus for each integer $k \geq 1$

$$\sum_{t=0}^{T_k-1} w(by(t)) \geq \mu(T_k)-M-p\mu. \tag{4.11}$$

Assume that integers $S_1 \geq 0$ and $S_2 > S_1$. Choose a natural number k such that $T_k > S_2$. By (P1), Lemma 3.1, (4.3) and (4.11)

$$\sum_{t=S_1}^{S_2-1} w(by(t)) = \sum_{t=0}^{T_k-1} w(by(t))$$

$$-\sum\{t \text{ is an integer and } 0 \leq t \leq S_1-1:\ w(by(t))\} - \sum_{t=S_2}^{T_k-1} w(by(t))$$

(4.12)
$$\geq \mu T_k - M - p\mu - [\mu S_1 + \tilde{M}] - [T_k - S_2]\mu - \tilde{M} = \mu(S_2 - S_1) - M - p\mu - 2\tilde{M}$$

(here we assume that the sum over an empty set is zero). In view of (4.3), (P1) and Lemma 3.1,

$$\sum_{t=S_1}^{S_2-1} w(by(t)) \leq \tilde{M} + (S_2 - S_1)\mu.$$

Together with (4.2) and (4.12) this implies that

$$|\sum_{t=S_1}^{S_2-1} w(by(t)) - \mu(S_2 - S_1)| \leq 2\tilde{M} + M + \mu p = M_1.$$

Theorem 2.2 is proved. □

5. Proof of Theorem 2.3

Clearly, $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. By Theorem 2.2 there is $M_1 > 0$ such that for each integer $k \geq 1$ and each pair integers $S_1 \geq 0$ and $S_2 > S_1$,

$$|\sum_{t=S_1}^{S_2-1} w(by^{(k)}(t)) - \mu(S_2 - S_1)| \leq M_1. \tag{5.1}$$

This implies that for each pair of integers $S_1 \geq 0$ and $S_2 > S_1$,

$$|\sum_{t=S_1}^{S_2-1} (w(by(t)) - \mu(S_2 - S_1)| \leq M_1. \tag{5.2}$$

In view of (5.2) the program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good. By (5.2) there is a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{\infty}$ such that

$$w(by(T_j - 1)) > \mu/2 \text{ for all natural numbers } j. \tag{5.3}$$

Let $j \geq 1$ be an integer. Applying Theorem 2.1 to the program

$$(\{x(t)\}_{t=0}^{T_j}, \{y(t)\}_{t=0}^{T_j-1})$$

we obtain that

$$\begin{aligned} \sum_{t=0}^{T_j-1} w(by(t)) &= \lim_{k\to\infty} \sum_{t=0}^{T_j-1} w(by^{(k)}(t)) \\ &= \lim_{k\to\infty} U(x^{(k)}(0), x^{(k)}(T_j), T_j) = U(x(0), x(T_j), T_j). \end{aligned}$$

This implies that $\{x(t), y(t)\}_{t=0}^{\infty}$ is locally maximal. Theorem 2.3 is proved.

References

1. Atsumi, H.: Neoclassical growth and the efficient program of capital accumulation. Rev. Econ. Stud. **32**, 127–136 (1965)
2. Aubry, S., Le Daeron, P.Y.: The discrete Frenkel–Kontorova model and its extensions. Physica D. **8**, 381–422 (1983)
3. Brock, W.A.: On existence of weakly maximal programmes in a multi-sector economy. Rev. Econ. Stud. **37**, 275–280 (1970)
4. Fujio, M.: Undiscounted optimal growth in a Leontief two-sector model with circulating capital: the case of a capital-intensive consumption good. J. Econ. Behav. Organ. **66**, 420–436 (2008)
5. Gale, D.: On optimal development in a multi-sector economy. Rev. Econ. Stud. **34**, 1–18 (1967)
6. Ali Khan, M., Mitra, T.: On choice of technique in the Robinson–Solow–Srinivasan model. Int. J. Econ. Theory **1**, 83–110 (2005)
7. Ali Khan, M., Mitra, T.: Discounted optimal growth in the two-sector RSS model: a geometric investigation. Adv. Math. Econ. **8**, 349–387 (2006)
8. Ali Khan, M., Mitra, T.: Optimal growth in the two-sector RSS model: a continuous time analysis. In: Proceedings of the seventh Portuguese conference on automatic control, Electronic publication 2006
9. Ali Khan, M., Mitra, T.: Undiscounted optimal growth in the two-sector Robinson–Solow–Srinivasan model: a synthesis of the value-loss approach and dynamic programming. Econ. Theory **29**, 341–362 (2006)
10. Ali Khan, M., Mitra, T.: Optimal growth in a two-sector model without discounting: a geometric investigation. Japanese Econ. Rev. **58**, 191–225 (2007)
11. Ali Khan, M., Zaslavski, A.J.: On a uniform turnpike of the third kind in the Robinson–Solow–Srinivasan model. J. Econ. **92**, 137–166 (2007)
12. Leizarowitz, A., Mizel, V.J.: One dimensional infinite horizon variational problems arising in continuum mechanics. Arch. Ration. Mech. Anal. **106**, 161–194 (1989)
13. Stiglitz, J.E.: A note on technical choice under full employment in a socialist Economy. Econ. J. **78**, 603–609 (1968)
14. von Weizsacker, C.C.: Existence of optimal programs of accumulation for an infinite horizon. Rev. Econ. Stud. **32**, 85–104 (1965)
15. Zaslavski, A.J.: Ground states in Frenkel–Kontorova model. Math. USSR Izvestiya. **29**, 323–354 (1987)
16. Zaslavski, A.J.: Optimal programs in the RRS model. Int. J. Econ. Theory **1**, 151–165 (2005)
17. Zaslavski, A.J.: Good programs in the RSS model with a nonconcave utility function. J. Ind. Manage. Optim. **2**, 399–423 (2006)

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Pythagorean mathematical idealism and the framing of economic and political theory

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Abstract. The Greek culture evolved in the shadow of two dominant and sophisticated economies: the Chaldeans (Babylonians), with their strong arithmetical and administrative culture and the Egyptians, who contributed a geometric orientation. In the shadow of these two traditions, the Pythagoreans and Plato assimilated a mystic perspective of an ideal world of mathematics. Later Greeks developed a system of fair division that absorbed an arithmetic dyad. Aristotle analyzed two-party isolated exchange using the harmonic proportion introduced by the Pythagoreans whose "ideal types" influenced later market perspectives. These traditions informed modern regulated political, legal and economic institutions.

Key words: Pythagoras, Plato, Xenophon, Aristotle, Boethius, rational numbers, irrational numbers, hedonic calculus, harmonic proportion, dyad, Fibonacci series, fair division, bargaining, isolated exchange, theory of friendly numbers, number ladder, the divided line, bounded universe, sacred ratio, Golden Number, divide and choose, justice in exchange, international trade theory, pentagon, dodecahedron

1. Introduction

The development and use of basic arithmetic and geometric concepts can be traced to the early Chaldean (Babylonian) and Egyptian cultures. The later, more primitive Greek culture emerged in a vacuum between these two lines of

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quantitative, numerically rational and sophisticated thought, and it embraced influences from both earlier cultures with awe and reverence.

2. Pythagoras and his secret societies

It might seem strange to start a discussion of ancient mathematical influences on economic ideas with the contributions of Pythagoras (c. 570–480 BCE), since he left no writings and only oblique commentaries on his mathematical theories [16]. Pythagoras' ideas are tantalizingly obscure, because his mathematical and social principles were part of the rituals of the secret societies he founded, and only bits and pieces of his thought were revealed to the public by occasional dissidents.

However, some of Pythagoras' thought can be resurrected from the writings of his near contemporaries, Plato (c. 427–347 BCE) and Aristotle (c. 384–322 BCE), both of whom left a large body of literature that made a major contribution to the European intellectual heritage. Plato's Academy lasted 900 years and perpetuated his extensive writings. Although a few of Aristotle's writings were found in European monastic libraries, they were primarily preserved in the Arabic world and reintroduced to Europe by the Scholastics in the thirteenth century AD.

Born in Samos around 570 BCE, Pythagoras was influenced by both the Chaldean and Egyptian cultures, possibly because he traveled to Mesopotamia and the Nile. He eventually settled in Crotona, a small Greek city in southern Italy and established an authoritarian regime run by the members of his semireligious order. Pythagoras' regime was probably somewhat similar to John Calvin's control of Geneva in the early 1500s, or the modern Taliban administration in Afghanistan, and he remained in power for some 20 years until the populace rose up and drove him and his cohorts from the city.

In daily life, Pythagoras advocated vegetarianism (except for meat at some sacrifices), but he excluded beans for some rather strange and mystical reasons. Little more is known about his personal life, but he left a following of semimonastic groups that were dedicated to the study of the relationship between geometric, mathematic and harmonic proportions as they were related to the political and physical world.

Pythagoras' political thought was nurtured in a vacuum being filled by ideas from the advanced thought of the two major cultures of the time. The Chaldean tradition focused on rational calculation and bookkeeping, and the Egyptians followed a geometric orientation derived from their practice of surveying. As he struggled to digest the complexities of these bodies of mathematical thought, Pythagoras seems to have created a third or intermediate perspective which included mystical correlations between numbers and forms as they applied to physical and political reality. To Pythagoras, the rational relationships between musical intervals and the geometric patterns expressed in crystals seemed to suggest a structure of preordained patterns that could be examined through the study of geometry and arithmetic.[1]

3. Early use of mathematics for rational administration

As early as the fourth millennium BCE, the Egyptians had developed geometry (earth measurement) which they used to design pyramids and re-establish boundary lines and field allotments. The annual flooding of the Nile left a layer of raw mud obscuring boundary lines on large areas of agricultural land. Furthermore, the area of allotments had to be adjusted when the river moved its banks during flood stage. The design theorem by which a right angle was laid out, ($3^2 + 4^2 = 5^2$), was developed in Egypt more than 1,000 years before Pythagoras' time and was later attributed to him.

As opposed to Egyptian geometry, Chaldean arithmetic had reached an extremely high level of development during the same period. The recent analysis of the Erlenmeyer Tablets, a collection 88 of clay tablets found in a large jar, provides insight into the development of both numbers and written symbols in Mesopotamia from about 2200–2100 BCE. These tablets were bookkeeping records for a 3-year period during which the total agricultural land (some 75,000 acres) of the city-state was listed with the anticipated annual yields and recorded shortfalls. The average yields were recorded at about 12.5 bushels per acre, with three-quarters of a bushel retained for seed – a 6% re-investment surplus [15]. There are also records of the milling operation with recorded inputs of grain and "female labor days" with annual yields and shortfalls to be made up in subsequent years. This emphasis on bookkeeping records is consistent with documents from other early economies such as the Cretan society.

By the time of Hamurabi (eighteenth century BCE in Babylon), a sexagesimal system had been developed in which columns and placement were used. This numeric system ran up to 60 for each column, so that 33 in our numerical pattern would be 183 in theirs, since the right-hand 3 (in the "ones" column) would stand for 3, and the left-hand 3 (in the second, "sixties" column – 3×60) would stand for 180. This level of arithmetic skill enabled the Babylonians to calculate compound interest on massive, long-term irrigation projects [14]. The sexagesimal pattern clearly influenced the ancient

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[1] It has been surmised that Pythagoras' exposure to the existence of regular crystalline forms used by his father in his jewelry trade informed the development of Pythagoras' concept that the universe is composed of regular solids.

Greek monetary system in which 60 mine equaled a talent. Aspects of the sexagesimal system also persisted in the scientific culture supported by the great library in Alexandria, Egypt (which was burned in the seventh century AD), and influenced the modern use of seconds, minutes, and degrees.

It appears that Pythagoras was exposed to enough Chaldean arithmetic and Egyptian geometry to lead him to develop a mystic symbolism underpinning both political and natural processes. In trying to understand these concepts as part of a cosmic whole, he created a mystic theoretical abstraction of the quantitative relationships he absorbed from these rational enumerations and geometric forms. The essence of his thought seems to have been the quantitative coherence of all things. This is consistent with the Democritean concept of atoms as the smallest material unit in the universe. Pythagoras believed that the integer – the "one" – was the essence of all things, as was the point, or dot, in geometry. All geometric entities were built of points, lines, planes, and volume, with the point as the universal common denominator. The "one" was the universal element of any number, and all phenomena – both physical and social – could be correlated with precision by whole number relationships. This interpretation viewed all things as rational – capable of being consistently correlated to one another in terms of whole number ratios. To Pythagoras, the Chaldean capacity to numerically document administrative processes would seem to illustrate the internal numerical consistency within comprehensive systems. Pythagorean ideas that are perpetuated in Plato's writings and among the Neoplatonists seem to indicate that the search for a connection between the unit and the universal and the exhaustive quantification of component forms was a very important part of Pythagoras' semireligious mathematics.

Much of the research by Pythagoreans was dedicated to the analysis of musical proportions. For example, a single string under tension tuned to middle C would make a tone an octave higher if a moving bridge reduced the length of the string by half. If a string with a length 12 units (point 12) long was shortened with a bridge to a length 6 (point 6) units long, it would increase the tone by an octave. The intervals at point 8 and point 9 would increase the tone by a musical fourth and fifth. Similar harmonic ratios were found in the sounds made by iron objects of different sizes when struck by a hammer.

Pythagoreans correlated these numerical relationships with the cube, a regular solid in geometry, and one of their symbols of comprehensive order. The cube has 12 edges, 6 faces, 8 corners, and 9 symmetry planes. Thus, this geometric shape has a correlation with musical intervals, suggesting further, more complex relationships which apparently extended to the belief that there was a harmony in planetary motion, creating "the music of the spheres." Pythagoreans further applied these arithmetic, geometric and harmonic proportions to define economic processes and political and social interactions.

4. Plato's dedication to rational ideal forms

During the eighth and seventh centuries BCE, Greek cities embarked on an extensive period of colonization and sent their surplus populations to establish new colonies in different parts of the Mediterranean. One such ideal city-state with a formal street layout and a legal code, Thurii (443 BCE), was established by Pericles in southern Italy. The concept of planning and the development of a basic layout for a city indicates the adoption by the Greeks of the use of geometry and mathematics from the Egyptians and Chaldeans.

The Republic, Plato's description of an ideal state, can be assumed to be his response to the democratic ideals behind the establishment of Thurii. He contended that mathematics and geometry should be used in dealing with mundane processes in military and civic life, and he also endorsed the Pythagorean perspective that mathematics is essential in directing people's attention to higher abstractions (*Republic*, Book VII, 522–534). The character of his expositions is in terms of quantifiable absolutes in which individuals with basic skills need to be organized into mutually reciprocating groups to provide an efficient economic structure. The basis of the origin of the state is predicated on the social contract theory in which two parties mutually calculate the benefits of restraint and complimentary activities. This simplistic commitment to rational self-interest as a fundamental human trait easily characterizes a quantifiable economic process (*Republic*, Book II).

In Plato's dialogue *Protagorus*, the hedonic calculus is used to develop the comparative value of "goods" and "bads" and the role time plays in human choices (e.g., the willingness to accept a current small loss in exchange for a large future benefit, and vice versa). Protagoras' outline of the hedonic calculus was appropriated almost entirely by Jeremy Bentham, who changed the calculus to values associated with material goods in his *Principles of Morals and Legislation*. Although the pattern of his exposition follows the *Protagorus* dialogue almost exactly, Bentham did not credit Protagoras as he embraced the postulate of the absolute level of human rationality. In contrast, Protagoras suggested that human beings are not dependably rational and are frequently overcome by the temptations of short-term pleasure, thereby rejecting the Pythagorean assumption of rationality implicit in the social contract thesis. Protagoras' idea of absolute rationality also surfaces in Adam Smith's *The Theory of Moral Sentiments*, in which Smith reiterates Protagoras' (uncredited) theory that human society could not function until it developed "fellow feeling" (human sympathy) and a sense of justice which allowed mankind to organize with a sense of trust. With his doctrine of human sympathy, Smith was undercutting the rationalistic base of the social contract theory.

Returning to Plato's *Republic*, it's clear that Plato was dedicated to the premise that there was an absolute, unequivocal system available to human beings that could be applied with mathematical precision by a capable administrator. This obviously reflects some influence from Pythagoras' approach to political administration. Plato believed that people's competencies could be evaluated and their role in society established early in life so that a perfectible, integrated economy could be developed. All participants in the society should recognize that they would benefit from the efficient regulation and administration by the most intelligent and competent person in the society. Those who did not appreciate the benefits of such administration did not deserve to be in the society and could be eliminated. To demonstrate the advantages of unfettered administration, Plato described how passengers on a ship would benefit from having a captain exercise his navigational skills to sail the ship to its destination. Karl Popper, in his *The Open Society and Its Enemies*, acerbically commented on this Platonic perspective by suggesting that, although the captain may well be the most capable to pilot the ship to his perceived destination, passengers might expect to have some choice as to where they wanted to go. Plato thought there was no question as to how society should be organized and what its basic objectives should be. Assuming that Plato was drawing on Pythagorean ideas of administrative absolutism (likely revealed to him by a Pythagorean dissident, Philolaus of Tarentum (c. 480–? BCE), we can understand why the citizens of Crotona tired of Pythagoras' belief in his superiority and drove him and his followers from the city.

5. The Chaldean tradition of efficient administration

Plato used the Chaldean tradition to support his notion that rationality could raise administration to the level of an "ideal form." In contrast, Xenophon applied rational administration in a very realistic and practical way. It is clear he was also very familiar with the Chaldean culture as evidenced by his extensive biographical account of the Chaldean ruler, Cyrus the Great (c. 580–529 BCE).

In his book, the *Oeconomicus*, Xenophon developed the principles of effective management of an agricultural estate. His main orientation was that human beings are the primary natural resource. He believed the effective leadership, training, and organization of human workers was the essence of efficient management, and that praise and prizes were more effective than punishment in encouraging optimum performance by workers [9]. Xenophon also saw good order as one of the primary sources of efficiency, and he idealized the layout of the Phoenician galley, a ship in which everything was always carefully put in its place but was ready to hand when it was needed.

In his *Hiero*, he advocated the awarding of prizes for technical innovations and for the highest level of agricultural production. He pointed out that the awarding of prizes is the least expensive way of purchasing desired results.

One of the most telling presentations on rational efficiency is found in Xenophon's *Banquet* (VII.1–5), in which a Syracusan impresario who is providing entertainment for the party, challenges Socrates to validate his reputation as a theorist [11, p. 81]. In response, Socrates makes a theoretical suggestion to the impresario that the increased risk of having his acrobats perform over sharp swords is not justified, since the increased risk far exceeds the increased value of the entertainment. Here we have a clear a formulation of the analysis that marginal costs cannot exceed marginal revenue and still maintain efficiency. This is not a simple cost–benefit analysis but a marginalist one. Joan Robinson's analysis of imperfect competition and Edward Chamberlin's analysis of monopolistic competition apply this administrative doctrine in the context of a market process.

Xenophon also presented a description of efficiency related to specialization (division of labor). He described how, in the kitchen of Cyrus the Great, specialized cooks turned out the highest quality dishes. He then developed the thesis that, in small cities, carpenters do a variety of work, whereas in larger cities they specialize, thereby presenting the idea that specialization is limited by the extent of the market (Xenophon's *Cyropaedia* VII.2.5–6) [11, p. 68 ff]. Xenophon further observes that in large cities one could find shoemaking establishment with specific workers assembling different parts of the shoe, using prepared pieces, with one person responsible for distributing the standard units to each workman. Not only is this a precursor of Eli Whitney's discovery of the principle of replaceable parts, but it also deals with quantitative efficiency through specialization. Adam Smith apparently drew on this passage from Xenophon's *Cyropaedia* for his principle that specialization is limited by the extent of the market. Karl Marx also drew specific attention to this passage about specialized workers in *Cyropaedia*, but he treated it as an example of qualitative efficiency tied to handicraft production.

Another administrative idea that probably has a rich Chaldean background is found in Xenophon's treatment of Cyrus the Great's arrangement between the Chaldeans and the Armenians. After Cyrus conquered the area, he brought representatives of both groups before him and suggested that the Armenians, a pastoral people, had extensive agricultural land of which they were not making good use. The neighboring Chaldeans, an agricultural people, had extensive mountain land that they were not using as pasture. Cyrus observed they could rent these lands to one another, and both would benefit. The surpluses resulting from this mutually beneficial reallocation of resources could contribute to supporting a constabulary which Cyrus would leave to superintend the exchange and help support his rule. This is an administrative approach to a mutually beneficial contract with clear rational components and provides the practical base for the harmonic proportion to be discussed below.

6. Bureaucratic individualism and dispute resolution

Two branches of thought emerged from the simple tradition of rational administration which was expressed arithmetically by the Chaldeans and geometrically by the Egyptians. In one direction, the authoritative system developed customs for coping with disputes between individuals of equal standing that did not affect the authority of the administrative system – primarily problems of fair division or distribution. Such issues were corollaries of the tradition of bureaucratic individualism. The distribution of rations to large numbers of people building irrigation systems or pyramids required a predetermined rational allocation per person with the requirement that it be efficiently portioned out. A presumption of equal rights could not be asserted by the recipient, but uniformity of distribution was required for the efficient management of the operation. The same can be said for the development of coined money, which is commonly believed to have originated where large numbers of troops had to be issued their monthly stipends in precious metal. It was inefficient to weigh out gold dust or bits of silver for hundreds of soldiers lined up to received their pay. Pre-stamped weighed-out coins were the answer. This method of payment blossomed in the late seventh century BCE in the kingdom of Lydia. Although standard balance weights had long been in use, they had not yet been adapted into monetary circulation.

The strength of this tradition of administering and distributing rations must be appreciated as a dominant perspective in formal reflections on the development of social processes. The Egyptians not only developed earth measurement or surveying to a high degree of precision, they also used balance measurement as a significant system for regulating the stability of their economy. Since grain storage was simple and dependable in a dry climate, public granaries were an important institution. Large balances were used to weigh in the grain in bushel or two-bushel quantities. In ancient times, a talent was approximately 60 pounds, the weight of a current bushel of grain. At the end of each year, a formal accounting of the grain stores was held. There was both a record of the amount of grain delivered to the granary and of the amount of grain ordered out; therefore, the amount still on hand had to tally out in order for the steward to be justified. At this annual formal accounting, the steward was essentially on trial, and, if his granary was found short, he would suffer severe penalties. This administrative tradition of an annual public accounting of officials was maintained by the Greeks and persisted into modern times in the Latin American culture as well as in Czarist Russia.

As a result of this long-standing agricultural tallying system, the image of scales or a balance became pervasive in Egyptian culture. The image of the scales also occurs in areas of daily life. For instance, the scales became a symbol of the final judgment of the dead, in which the deceased's life was "weighed" to determine his fate in the afterlife [3]. The image of the level balance square to the plumb line is at the heart of the rational tradition and became a common symbol for ideas of justice and fairness in all aspects of life. However, the balance was primarily a judicial, rather than a commercial, symbol.

7. Plato's assimilation of Pythagorean thought

Pythagoras apparently had a very charismatic personality, and he effectively communicated his enthusiasm for mystical insights associated with mathematics to the Greeks. Although Pythagoras' societies kept their ritual mathematics and beliefs secret, we can find much of his thought reflected in Plato's writings. Similarly, Plato also insisted on keeping his mathematical ideas from being published: "There is no writing of mine about these matters, nor will here ever be one. For this knowledge is not something that can be put into words like other sciences; but after long-continued intercourse between teacher and pupil, in joint pursuit of the subject, suddenly, like light flashing forth when a fire is kindled, it is born in the soul and straightaway nourishes itself" (Plato, *Letter VII*, 341 c–d).

Plato developed a theory of forms and "the ideas" that suggested the pre-existence of blueprints for all natural and social processes. He believed these blueprints or forms were "ideal types" that had an existence of their own and that actual processes strove to conform to them. Therefore, by using rational analysis, one could determine true relationships more dependably than was possible by simply observing imperfect expressions in the physical world. When we fully appreciate the widespread acceptance of Plato's views, we can better understand why the discovery of irrational numbers caused such an intellectual crisis among the Pythagoreans.

The mystic mathematics of the Pythagoreans and the Platonists assumed that everything was derived from the unit, the one, equivalent to the point in geometry or the atom in matter. The unit was the building blocks of all things, and everything in existence, being an accumulation of units, was commensurable. In other words, all quantitative relationships could be expressed in whole number ratios. At some point, as arithmetic was refined, certain numbers seemed to be elusive – primarily the square root of 2. Ideally, the diagonal of a unit square should be definable as a whole number relationship to the sides, just as it could be derived by the Theorem of Pythagoras in which the sum of the squares of two sides is equal to the square of the diagonal or hypotenuse. According to Neugebauer [14], there is a Babylonian tablet dating from about 1700 BCE showing the square root of 2 developed to the equivalent of five decimal places.

It is rumored that Plato bought a book from Philolaus which revealed the mathematical secrets of the Pythagorean movement. Plato insisted that geometry was a basis of education, but the only dialogue that he wrote that specifically addresses his cosmology is *Timaeus*, in which the idea that precise models should be followed is put forward but with some additions. Plato also presents what is clearly a Pythagorean notion that says "the god" constructed models for the formation of the earth, and human beings strive to discern these models (proportions, ratios, and geometric patterns) and use them to understand the blueprints that govern the design of the physical and natural world (*Timaeus* 36–39 [4]). Furthermore, "the god" created the earth out of the regular solids: the tetrahedron, octahedron, icosahedron and the cube, constituting fire, air, water and earth. These geometric forms created a cosmos and were mixed in a proportion that harmonized and completely exhausted all potential space to form a bounded universe (*Timaeus* 54e–55). The dodecahedron, a potential sphere, was a solid of twelve pentagonal faces and became a symbol of the universe for the Pythagoreans. It was also the known technique for making a ball in which twelve equal-sized pentagonal patches of leather were stitched together and stuffed with wool. When stuffed tightly while wet, the leather would shrink and create a ball. Modern soccer balls perpetuate this appearance.

8. Solving the problem of irrational numbers: the dyad of the greater and the smaller

Despite Plato's and Pythagoras' secrecy,[2] we can reconstruct from other writers an idea of the role the discovery of rational numbers played in antiquity. The Pythagorean movement was nearly destroyed by the discovery of the

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[2] Plato's emulation of Pythagoras' secrecy is reflected in the absence of any discussion of irrational numbers in his writings. This secrecy regarding irrational numbers was also perpetuated by Euclid and was apparently also characteristic of Nicomachus, who wrote an extended book on arithmetic at the beginning of the second century AD, much of which is lost. Iamblichus wrote a book on Pythagoreanism dating from the late third or early fourth century AD, but the chapters on arithmetic, apparently drawn from Nicomachus, have also been lost. We have to go to Boethius' writing in the late fifth and early sixth century AD, for a record of Nicomachus' work, but Boethius' commentary does not elaborate the problem of irrational numbers.

existence of irrational numbers. The Pythagoreans' religious base rested on a premise that all things were rational and built from basic units and were therefore commensurable. They believed all geometric relationships could be reduced to a common denominator, and some whole number ratio could be arrived at if one pursued the arithmetic far enough. Throughout the history of mathematics, the idea of rational commensurability has persisted in the search for the solution to the challenging problem of the squaring of the circle, the trisection of the angle, and the duplication of the cube. The dimensions of the circle should provide the information by which, with straight edge and compass, a square with the same area as the circle could be drawn.

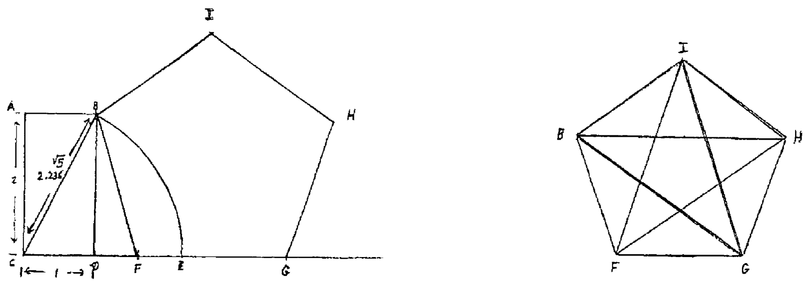
The specific problem that apparently caused the crisis within the Pythagorean tradition was the question of how to reduce the square root of two to a whole number ratio. If one drew a diagonal in a unit square, then that diagonal should be equal to the square root of two. There should be a whole number arithmetic ratio between the diagonal of the square and a side. This could not be geometrically derived by meticulous measurement.

It is known that Eudoxus brought the system of number ladders to Plato's Academy in the first half of the fourth century BCE. The number ladder for arriving at the square root of two is built by a sequence in which two columns were developed with adjacent pairs of numbers starting with 1-1, then 2–3, 5–7, 12–17 . . . . Each of these pairs was generated by adding the two numbers from the previous pair to make a left-hand number, then adding the left-hand number with the left-hand number from the previous pair to make the right hand number.

x	y	x/y	$2x^2 - y^2 = \pm 1$
1	1	1.0	$2 - 1 = +1$
2	3	1.50	$8 - 9 = -1$
5	7	1.40	$50 - 49 = +1$
12	17	1.416	$288 - 289 = -1$
29	41	1.413	$1,682 - 1,681 = +1$

If one develops ratios with the right-hand number in each pair as enumerator and the left-hand number in that pair as the denominator, one will generate a series of ratios that are, alternately, a little more and a little less than the square root of two, but which come closer and closer to it. D'Arcy Thompson [19] presents this material in his essay *Excess and Defects*. An uncanny facet of this number ladder is that if you subtract the square of the right-hand number in a pair from twice the square of the left-hand number in a pair, the result is alternately plus one and minus one. This recalls the Pythagorean notion of the ultimate mystic relevance of unity – the one and the many. To reiterate, if you subtract the square of three from twice the square of two, the result is minus one; if you subtract the square of seven from twice the square of five, the result is plus one, etc. – the dyad of the greater and the smaller.

There is no question that Plato was aware of this approach to rational numbers, and there are many allusions in both Plato's and Aristotle's writings to the dyads of the more and the less, the greater and the smaller, and excess and defect, indicating they had some grasp of how to deal with irrational numbers. Although Plato did not leave any written discussion of irrational numbers, he may have treated it in a lost dialogue on *The Good*. He seems to have clung to the earlier Pythagorean emphasis on geometry, and he must have known of the geometrical basis for developing the pentagon which involves the derivation of what is called "the divided line", to which he makes some reference. "The divided line" involves an irrational number, the square root of five, minus one, divided by two. This number is the "sacred ratio," 0.618 .... From the geometric diagram used to divide a line to establish this ratio, one can lift the angle (108 °) necessary to develop the regular pentagon which is a symbol for man's relation to the cosmos. If you connect the vertices within a pentagon to each other with straight lines, you create a five-pointed star embracing a smaller pentagon at its center (see Fig. 1). If the sides of the pentagon are extended with straight lines, they will, of course, generate a larger five-pointed star whose points can be connected to make a larger pentagon, ad infinitum. Extrapolating from this, a human being can be presented, spread eagle, within the confines of a five-pointed star or pentagon encased within a circle to symbolize man's relationship to the infinitely small (the microcosm) and the infinitely large (the macrocosm). This material was surely part of the Pythagorean cult's secret knowledge of geometry, and Plato was undoubtedly aware of it.



$$\frac{\sqrt{5}-1}{2} = .618\ldots \text{ (Golden Number)}$$

Fig. 1 The derivation, using straight edge and compass, of the Golden Number (the Sacred or Fibonacci ratio) and its use in constructing the pentagon

The number ladder by which the "sacred ratio" could be developed did not reach European literature until Fibonacci, Leonardo of Pisa (c. 1170–1250 AD), brought this material back from Algeria after a long stay in a counting house in the early thirteenth century AD. This indicates the arithmetic of the early Greeks must have survived in Arabic copies. Fibonacci also brought Arabic numbers and bookkeeping techniques to northern Italy which facilitated the keeping of running accounts in neat columns – a format that was nearly impossible using Roman numerals.

The number ladder that generates the Fibonacci series is a simple additive one. Each number is generated by adding the two previous numbers: i.e., 1, 1, 2, 3, 5, 8, 13, 21, 34, ... The sequence of ratios generated by any pair of these numbers – 3/5, 5/8, 8/13 – are alternately a little more and a little less than 0.618..., although they approach it closer and closer as the numbers progress. Also, if one takes any set of three numbers in this series, multiplies the first and third then subtracts from the product the square of the second, one gets alternately, plus one, minus one. Thus:

$$\begin{aligned} 2 \times 5 = 10 - 3^2 = +1 \\ 3 \times 8 = 24 - 5^2 = -1 \\ 5 \times 13 = 65 - 8^2 = +1. \end{aligned}$$

Once again, we have the dyad of the greater and the smaller.

What is more remarkable about the Fibonacci series is that it is replicated in the incremental growth of many botanical patterns. For instance, in the seed face of a sunflower or any of the daisy family one finds two patterns of alignment going in different directions that fall into some pair of numbers in the Fibonacci series: 8 and 13 or 13 and 21, etc. The same is true of the scales of a pine cone. Renaissance artists assumed that the "sacred ratio" of 0.618... was an esthetic verity expressed in the ratio of joints of the fingers, hands, and arm, as well as the ratio of a man's height from the navel to the ground in proportion to his total height. This ratio was also expressed in Renaissance architecture in which the shape of windows, width to height, and the layout of the floor plan of buildings follow this ratio. Known as the "sacred number," 0.618... it came to be called the "Golden Number" or the "Golden Mean" in the mid-nineteenth century.[3]

This tradition of mathematical mysticism, with its seemingly eerie natural physical expressions, apparently has had an underlying influence on the reassurance that mathematical frameworks are the basis or guiding principles behind observed reality. This was clearly Plato's view as expressed in

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[3] An exhaustive analysis of the uncanny number of correlations implicit in this ratio can be found in *Le nombre d'or* by Ghyka [6].

his dialogue, *Timaeus*, and these ideas are the most likely summary by Plato of the Pythagorean views, which he embraced. Like the Pythagoreans, Plato was committed to an authoritarian rational administrative tradition, and just as these beliefs led to the expulsion of the Pythagorean cult from its control over at least one south Italian city, they also led to Plato's isolation from the Athenian democratic tradition.

9. Prometheus and fair division

The legacy of rational distribution in an administrative setting had a very strong influence on Greek culture, and it dominated the mathematical and reflective thought in classical times. The economies of the Greek city-states were based on semisubsistence agriculture significantly augmented by military adventures designed to supplement the community's income with booty [7]. The economy, therefore, had a primary dependence on fair division or distribution among individuals who shared some established relationship, either of equality or relative merit. This system of proportional distribution drew on a very ancient hunter/gatherer tradition that attempted to resolve the problem of the division of spoils from the hunt between individuals who were not part of an extended family structure.

The paradigm for the problem of fair division is found in the myth in which Prometheus divided an ox with Zeus and is recorded in Hesiod's *Theogony* (535–45). In this account, Prometheus slaughtered an ox and spread the skin, flesh side up, upon which he placed the remains in two piles. In one pile he arranged the lean meat and covered it with the stomach and other entrails. He put the bones covered with white fat in the other pile. Zeus was then permitted to choose which pile he preferred. This type of division – "I divided and you choose" – was a well-established institution. Its virtue was that both parties were locked into the division by their own voluntary choice, since the divider could not complain about having made one pile bigger than the other, and the chooser could not complain about having taken the smaller share. A number of subtleties occur behind this division [11, Chap. 5; 12]. In Hesiod's account, Zeus was aware of the apparent deception, but he chose the bones and fat anyway. The reality was, however, that the only way Zeus (in Heaven) could receive his portion was through a burnt offering that could only ascend into the heavens as smoke, and bones and fat were more readily transmitted by burning. Prometheus, who represented the interest of mankind, preferred the lean meat. In that time, it was customary at public sacrifices to distribute the lean meat to be barbecued on skewers. Thus, this division allowed for the preferences of both the gods and mankind [12]. This ceremonial division – burning the bones and fat as offerings to the gods and keeping the meat to be consumed by the people – became institutionalized as a religious tradition and dominated religious sacrifice in the eastern Mediterranean and Near East and is found in the Hebraic tradition as well.

Aesop presents a version of this distribution tradition in some of his fables. One fable recounts a hunt in which four animals, including a lion, made a kill, and one of the animals was assigned the job of dividing the meat into four equal piles. The lion presumed to choose first, and, placing his paw on one pile, announced that, "This is mine because I was the leader of the hunt." Indicating the second pile he announced, "This is mine because I am the King of the Beasts." He gestured to the third pile and announced, "This is mine because this is my share in the hunt," and, pointing the fourth pile, he hesitated and finally said, "Well, I dare anybody to try to take this away from me." This fable is one source of the adage, "taking the lion's share." In another fable, the lion, the ass and the fox were engaged in a joint hunt. When the ass divided the spoils into three equal piles, the lion became infuriated and killed the ass. He then told the fox to divide the proceeds of the hunt into two piles. The fox divided it into one large pile and one very small one. When the lion asked the fox where he had learned to divide so fairly, the fox replied, "I learned it from the ass." These fables indicate that the Greeks were quite conscious that any fair distributive ritual could be distorted by the assertion of raw power. The system of "divide and choose" was also used in ancient times to settle the inheritance to be shared by several heirs, and has been adapted for modern use.

Despite the influence of individual preferences and imbalances of power having an effect on fairly dividing the spoils, the system of fair division was widely used in antiquity and was the basis by which booty was divided after military ventures. When a random assortment of booty had to be distributed between the participants in a raid, an inverted form of auction took place in which the distributor would suggest a particular item as a fair share, and one of the participants would volunteer to accept it. If no one thought that portion was enough, something would be added to it, e.g., a helmet could be added to a copper kettle or a tripod could be added to a slave. Each recipient had to estimate how much was left to be divided and worry about the value being eroded if the earlier distributions were too high. By the same token, the recipient could take a chance that the earlier offers would be a little low and that, by waiting, the later shares might be more valuable. There was also room for individual preference to influence choice. There are current mathematical analyses of this type of distribution [2].

This institution of choice-oriented distributions deals with problems between participants rather than the public administration characteristic of the empires. When a person in authority had to settle a dispute between two individuals in which the administrative authority had no particular interest, the only concern was that some system of amiable settlement could be achieved. Xenophon presents the essence of this problem in his *Cyropaedia*. Cyrus the Great was considered an exemplar of efficient and just rule. While being trained as a young boy for his future role as king, Cyrus was assigned the job of adjudicating disputes among his playmates. Xenophon recounts that a tall boy with a short tunic forcibly exchanged tunics with a short boy with a long tunic. When presented with the complaint by the short boy, Cyrus ruled that since both parties now had better-fitting tunics the exchange was fair. Cyrus' instructors corrected him and told him that an exchange would only be fair if it was voluntary. This notion that volition was a precondition for fairness became a classical Greek tradition. It not only became a principle in sales but was also considered in settling claims for liability for injuries. It was reported that Pericles and Protagoras spent a whole day arguing as to whether a boy was unjustly injured when he darted across a field and was killed by a thrown javelin. A legal tradition held individuals responsible for the consequences of their acts, making the javelin thrower liable for an injustice to the boy and owing damages to his family. But could there be an injustice if the injury was the result of a voluntary, careless act on the part of the boy? We find an echo of this problem in the modern legal tradition of voluntary assumption of risk. This principle of volition has been a dominant consideration in administering justice in disputed exchanges since ancient times.

10. The process of arriving at settlements: the dyad

The egalitarian element in the more primitive Greek society appears as early as Homer's *Iliad*, in which the process of public debate over war policy is recorded. Although the right to speak in the public assembly is clearly limited to the leadership of different contingents, an orderly process is maintained by circulating a staff that gives the holder the right to speak [11]. This process was perpetuated in fifth and fourth century Athenian culture and is illustrated in Aeschylus' *Eumenides* [1]. In this tragedy, Orestes is faced with a dilemma because he fulfilled his obligation to avenge the death of his father by killing his mother, the murderess, and her paramour, but it violated his obligation to honor his mother. He is being hounded by the Furies, which represent the matriarchal tradition, and he goes to the temple of Athene to seek refuge. The Furies charge that, "He murdered his mother by deliberate choice" (*Eumenides*, 425). Athene responds by saying "Here are two sides, and only half the argument" (*Eumenides*, 428). She appoints a jury of twelve and insists that both sides use short arguments in presenting their cases. Greek tragedians were considered the educators of the Athenian populace, and this play sets the standard for trials in criminal cases before the classic Athenian court made up of twelve jurors. This legal process seems to draw on the idea of the dyad that approached justice by prescribing alternating arguments by each side. Although there is no mention of a dyad in *Eumenides*, which dates from the late fifth century, the echo of the Pythagorean search for irrational numbers can be heard in this judicial pattern of short, sequential argument.

This concept of dialectical argument developed by Plato and the institution of formal argument in the law courts can, at the least, be seen as a cultural interaction with the mathematics of a dyad. Which method – the legal argument or the mathematical dyad – influenced the other and to what degree is a matter of speculation. Even in modern parlance the resolution of conflicting legal issues is commonly formulated as a “tension” between two opposing perspectives and is a close parallel to the use of a dyad to close on an irrational number. In antiquity, the science of rhetoric was conceived as the refinement of the efficient presentation of cogent arguments and was designed to facilitate the development of just decisions in the courts and sound conclusions in public debate. Aristotle’s theory of rhetoric emphasizes the development of choices between alternative utilities, and he draws on Protagoras’ hedonic calculus for evaluating subjective choices in both his *Topics* and *Rhetoric*. Aristotle clearly develops the subjective perspective from a given side and elaborates many marginalist ideas in arriving at value judgments [11, pp. 176–180].

The Greeks understood vision as a process in which light was emitted from the eyes and reflected back to the person who was observing a specific phenomenon. Comprehension of what was viewed was considered to be primarily subjective. This concept is found in Plato’s thought, which was concerned with the lack of uniformity human beings experienced in developing a sound grasp of the “forms” and “ideas” – the true blueprints controlling reality. This flexibility in settling on a cardinal reality is at the heart of the concept of an approach to justice or fairness which, by definition, is an interaction between two or more people, each with their own individual perceptions. We must, therefore, abandon any notion of a cardinal justice when we approach Aristotle’s analysis of justice in exchange.

11. Aristotle’s analysis of exchange and the harmonic proportion

When we begin to investigate Aristotle’s theory of exchange, there are two important points to keep in mind. First, Aristotle was thinking in terms of individual exchange growing out of barter arrangements that had grown into monetary exchange. Secondly, he approached the process of exchange from a judicial standpoint, not from an overview of market theory. While it is well established that Aristotle was skeptical about Platonic and Pythagorean ideas of natural mathematic principles, he did utilize mathematics to describe phenomena to which they could be applied. His classic treatment of exchange is in Book V ("Justice") of his *Nicomachean Ethics*. In his discussion of particular justice, he deals with corrective justice, which he illustrates with an arithmetic proportion, and he uses a geometric proportion to exemplify distributive justice in which relative shares are calculated. As has been described above, distribution was a major consideration in the economic life of antiquity, and shares were frequently prorated in terms of proportional contributions or merit.

Aristotle's discussion of exchange involves a reference to a third proportion which is not clearly spelled out in his transcribed texts,[4] and this has given rise to considerable controversy. Many classical scholars have failed to see the economic relevance of proportions to which Aristotle alludes and find his analysis confusing. Aristotle's illustration of exchange involving a diagram with crossing diagonals within a square that appears in many medieval translations is reproduced in the commentaries of many Scholastic authors, but their explanations of the nuances of the basic principles of two-party exchange are unsatisfactory. The Scholastics' difficulty with Aristotle's meaning can be attributed to their approach toward a cardinal line of allocation appropriate to the geometric proportion. Without reviewing the literature in depth here (see [11, Chap. 7]), the issue can be clarified by looking at the ancient references to the harmonic or third proportion to which Aristotle undoubtedly alluded in Book Vv. In Plato's *Gorgias* 508b, Socrates observes that "... where there's no partnership there's no friendship ... wise men claim that partnership, friendship, orderliness, self-control, and justice hold together heaven and earth.... You've failed to notice that proportionate equality has great power among both gods and men, and you suppose that you ought to practice getting the greater share. That's because you neglect geometry."

Another general mathematical illustration of the concept of mutuality viewed from two perspectives is the "theory of friendly numbers," which was based on the calculation of the submultiples of a number. The classic illustration is the pair of friendly numbers, 220 and 284, in which the total of the submultiples of 220 is equal to 284, and the total of the submultiples of 284 is equal to 220. The point is that there are ample illustrations in both

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[4] The basic problem is that Aristotle's text on exchange is rather sketchy and has probably suffered from centuries of re-copying and translation, frequently done by people who did not clearly understand the subject matter.

mathematical and social thought of an interdependence measured from two subjective perspectives rather than in terms of an objective quantitative base. The discussion of the arrangement between the agrarian Chaldeans and the pastoral Armenians imposed by Cyrus the Great, discussed earlier, indicates this was also a facet of administrative thought. In that instance, both parties benefited by an enhancement of their own individual value systems, creating a sum-sum game, and were bound together by the mutual advantage of the relationship. This pattern of thought is articulated in what was referred to as the harmonic proportion, which tradition suggests that Pythagoras learned from the Babylonians. Because the term "harmonic proportion" is often assumed to be associated with musical intervals (6, 8, and 12), its relevance to social interactions is often overlooked.

Although Archytas of Tarentum (428–347 BCE) is noted for developing musical harmonies, he described the harmonic proportion in a way that is not particularly relevant to musical intervals: "by whatever part of itself the first exceeds the second, the second exceeds the third by the same part of the third" [11, p. 200]. In other words, the mean term (8) is designated subjectively from the extreme term (12) as one-third of 12 less than 12 and from the extreme term (6) as one-third of 6 greater than 6. Thus, we have a proportion where the mean is defined from two different subjective bases. This is supported by Boethius' (480–525 AD) comment dating from the early sixth century AD that all the ancients knew three major proportions – the arithmetic, the geometric, and the harmonic – and used them to elucidate social and political relations [13, Book II, § 43]. One of Boethius' illustrations of the harmonic proportion is completely unrelated to musical intervals; he uses ratios to show numeric relationships, i.e., 16 is the harmonic mean between 10 and 40, since it is 60% of 10 greater than 10 and 60% of 40 less than 40. Not only does Boethius devote § 40–51 in the second part of his commentary *De Institutione Arithmetica* to the use of arithmetic, geometric and harmonic proportions, but classical scholars attempting to translate Aristotle's *Nicomachean Ethics* Vv seem to be completely unfamiliar with this work, which is considered to be a commentary on Nicomachus' arithmetic. It is unquestionable that Aristotle was prepared to analyze exchange from the dual perspective of opposing sides in a two-party exchange.

In a bargaining situation, a fair exchange is one that is defined in modern terms as an arrangement between a willing buyer and a willing seller. As long as both parties are satisfied with the settlement, it is considered fair. In the absence of a market reference, there can be a range of mutually acceptable exchanges. For example, if A is willing to pay as much as \$100 for B's cow, and B would be willing to sell the cow for as little as \$75, there are a number of prices between those two sums that would be considered fair. Whether the cow was sold for \$90 or \$80, both parties could conceivably consider themselves fortunate, having made a better bargain than they may have expected. Aristotle's discussion in Book Vv has vague allusions to this nuance, indicating that the parties are drawn together by anticipated exchange but at one point suggests that a judge should serve as a "halver." These unstructured observations fall into place when we consider the Greek law on sales which was obviously paramount in Aristotle's consciousness when he recorded his ideas in Book Vv. The Greeks did not combine contract and sale, and their tradition of volunteerism mandated that no sale was finalized unless it was completely voluntary up to the point of final execution by both parties – payment given and delivery received. The theory of the sales contract which permitted economic planning was developed in Roman law, although there are suggestions that these Aristotelian ideas on fair value in exchange were influential [18].

The judicial setting of which Aristotle was writing involved a well-established system of dispute resolution. Plescia [17], whose work has been generally ignored by classical scholars, has explained how Greek law developed from reliance on public oath to settle disputes to a sophisticated tradition of arbitration. Parties could choose an arbitrator and argue their cases if there was some question as to the fairness of the exchange in dispute. If either or both parties were unhappy with the private arbitrator's decision, they could go to a public arbitrator, and if unsatisfied with his resolution, they could go through the process of a trial before a large jury. The problem with a trial was that if a litigant did not receive at least 25% of the jury's votes, he was subject to penalty for bringing a frivolous claim. Once again, we have the image of the dyad – bargaining back and forth to approach an ideal point – persisting in public policy.

Aristotle's analysis of isolated exchange is clearly paralleled in modern international trade theory where two-party negotiations take place in the absence of an organized structured market. The nineteenth century discussions of comparative advantage are, of course, a specific nuance in this type of analysis. Edgeworth developed these issues in his contract curve and in the Edgeworth Box.

What is important from our modern perspective is that for some 2,000 years most economic discussions were in terms of isolated exchange with publicly structured markets and rules to facilitate information regarding the quantity and quality of goods available. In a handicraft society the variations in quality were always sufficiently significant that each transaction had its own unique facets. Anyone familiar with a third-world market has seen this validated. The revival of Renaissance naturalism in eighteenth-century British literature suggests that the idealization of a free market as an ideal type may be a recent economic phenomenon. The parallelism with the Neo-Pythagorean concept of a dyad approaching a mathematically structured "ideal type" seems to persist in the minds of those who are less conscious of the institutional frameworks and legal restraints that have structured modern markets. Pythagoreanism seems to die hard.

We generally consider Aristotle to have been a skeptic regarding Pythagorean mathematical verities, but as D'arcy Thompson [19] points out, Aristotle's biological thought shows strong Pythagorean and Platonic influences. According to Thompson, Aristotle's definition of "genus" in the classification of fish presents the notion that the evolution of different species of fish suggests an attempt to move toward the expression of the "ideal type" characterized by the genera. This replicates the concept of the dyad approaching the golden mean.

Aristotle's position and his influence in modern thought is probably best characterized by John Donne's [5] observation in his *First Anniversary* in which he characterizes a geometric design as a human creation rather than an "ideal type." This passage may be the origin of the concept of an intellectual grid [10], which has been embraced by the sociology of knowledge.

> Of Meridians and Parallels,
> Man hath weav'd out a net, and this net throwne
> Upon the Heavens, and now they are his owne.

12. In summary

The secrecy that pervaded Pythagorean and Platonic thought in antiquity obscures an extremely deep and rich legacy of arithmetic and geometry. The use of the golden mean or "divided line" in the design of the Great Pyramid of Egypt and in the layouts of ancient temples suggests that this geometric relationship had a very long history. Those who have visited the archaeological site of the palace of Minos on the island of Crete have seen the carefully chipped altar stone, about $2\frac{1}{2}$ feet long and $1\frac{1}{2}$ feet wide and less than a foot thick. When I visited the site, inspired by a reference I had seen, I took out my pocket tape and measured this archeological treasure that dates from around 1400 BCE. The ratio of the thickness of the stone to its width and of the width to the length are about as close to the ratio of $0.618\ldots/1$ as you can get, considering it was an unpolished stone.

The intriguing correlations and complexity found in the geometry of the Fibonacci series, and the fact that it only surfaced in modern European thought when Fibonacci brought the concept from Algeria around 1220 AD, is difficult to comprehend. One explanation may be that the secrecy with which this body of mathematics was held in early Greek times resulted in its being lost or perpetuated in only a very few secret societies. However, this early mathematics apparently survived in Arabic culture through medieval times. We do have a vivid record of the profundity with which this body of mathematics was expressed in Renaissance architecture and art and influenced the imagination of European mathematicians.

The pervasive underlying presence of mathematical and geometrical forms in human culture is difficult to evaluate. The ability to develop mathematical models can lead to the assumption that there are behavior patterns that are implicit in mathematical relationships. If you can create a model that is mathematically consistent within itself, you can then explore the potentials of this process without concern for the effects of the foibles of human nature and its failure to live up to rational standards of behavior. Plato taught us that ideal forms that are mathematically rational and consistent within themselves do exist, and the material and social world struggles to replicate these forms but can only approximate them. The actual world is always imperfect, so we must go to mathematical formulations to discover the true process or ideal type. However, the ability of thought patterns to mold our observations of reality, and the feedback of these observations as they confirm our theoretical preconceptions is an intellectual hazard of which we must be constantly aware.

References

1. Aeschylus (n.d.) Eumenides
2. Brams, S.J., Taylor, A.D.: Fair division: from cake-cutting to dispute resolution. Cambridge, UK: Cambridge University Press 1996
3. Brandon, S.G.F.: The weighing of the soul. In: Kitagawa J.M., Long C.H. (eds.): Myths and symbols. Chicago: The University of Chicago Press 1969
4. Cooper, J.M., et al.: Plato complete works. Indianapolis: Hackett 1997
5. Donne, J.: First anniversary 11.278. Quoted in: Nicholson M.H. (1962) The breaking of the circle, rev. edn. New York: Columbia University Press 1972
6. Ghyka, M.C.: Le nombre d'or. Paris: Editions du la Librairie Gallimard 1931
7. Hanson, V.D.: The other Greeks. New York: The Free Press 1995
8. Krader, L.: The Asiatic mode of production: sources, development and critique in the writings of Karl Marx. Assen: Van Gorcum 1975
9. Lowry, S.T.: The classical Greek theory of natural resource economics. Land. Econ. **41**, 204–208 (1965)
10. Lowry, S.T.: The archaeology of the circulation concept in economic theory. J. Hist. Ideas **35**(3), 429–444 (1974)
11. Lowry, S.T.: The archaeology of economic ideas. Durham: Duke University Press 1987
12. Lowry, S.T.: Distributive economics and the Promethean meat division: myth, folktale and legal precedent. In: Silver M. (ed.): Ancient economy in mythology: east and west. Savage, MD: Roman & Littlefield, 1991 pp. 45–58

13. Masi, M.: Boethian number theory: a translation of the De Institutione Arithmetica. Amsterdam: Editions Rodopi 1983
14. Neugebauer, O.: The exact sciences in antiquity, 2nd edn. New York: Dover 1969
15. Nissen, H.J., Damereow, P., Englund, R.K.: Archaic bookkeeping: early writing and techniques of economic administration in the ancient near east. Chicago: The University of Chicago Press 1994
16. O'Meara, D.J.: Pythagoras revived. Oxford: Oxford University Press 1989
17. Plescia, J.: The oath and perjury in ancient Greece. Gainesville, FL: University Press of Florida 1970
18. Pringsheim, F.: The Greek law of sale. Weimar: Böhlau 1950
19. Thompson, D.W.: "Excess and defects" in Science and the classics. London: Oxford University Press 1940

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