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Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics

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Preface

In 2009 I have organized several “journées spéciales” in LATP, Université de Provence, Marseille (France). These were series of lectures on various aspects of complex Monge–Ampère equations (regularity issues, geometric properties of solutions, etc) both in domains of $\mathbb{C}^n$ and on compact Kähler manifolds (with or without boundary).

The speakers – with the help of a few participants – have produced a set of lecture notes, working hard in making them accessible to non-experts. This volume presents them in a unified way.

It is a pleasure to thank all the participants of the “journées spéciales” for their enthusiasm and for creating a very pleasant atmosphere of work. Special thanks of course to the lecturers

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- Boris KOLEV (CNRS and LATP, France)
- Ahmed ZERIAHI (Institut Mathématiques de Toulouse, France)

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Marseille *Vincent Guedj*

Contents

Chapter 1
Introduction

Vincent Guedj

1.1 Motivation

The complex Monge–Ampère operator is a fully non-linear differential operator of order 2 which generalizes the Laplace operator in several complex variables. In local coordinates in $\mathbb{C}^n$, it applies to smooth functions φ by

$$MA(\varphi) := (dd^c\varphi)^n = c_n \det\left(\frac{\partial^2\varphi}{\partial z_i \partial \overline{z_j}}\right) dV(z),$$

where

$$dV(z) = \frac{i}{2} dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge \frac{i}{2} dz_n \wedge d\overline{z_n}$$

denotes the Lebesgue measure in $\mathbb{C}^n$ and $d = \partial + \overline{\partial}, d^c = \frac{1}{2i\pi}\left(\partial - \overline{\partial}\right)$ are real operators. The normalization factor is chosen so that

$$\int_{\mathbb{C}^n} MA\left(\frac{1}{2}\log[1 + \|z\|^2]\right) = 1.$$

One usually restricts the complex Monge–Ampère operator to the cone of *plurisubharmonic functions* φ, for which the Levi form $dd^c\varphi$ is non negative in the sense that

$$\sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial z_i \partial \overline{z_j}}(z) w_i \overline{w_j} \geq 0$$

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for all $z, w \in \mathbb{C}^n$. Note that this inequality is still meaningful in the sense of distributions (or rather currents), one simply needs φ to be locally integrable and one further asks that φ be upper semi-continuous: plurisubharmonicity is thus equivalent to being subharmonic on each complex line in $\mathbb{C}^n$.

1.1.1 The Local Dirichlet Problem

A basic question is to solve the Dirichlet problem for the complex Monge–Ampère operator, say on a bounded domain $\Omega \subset \mathbb{C}^n$. This amounts to finding a plurisubharmonic function u in Ω which is "smooth up to the boundary", such that

$$MA(u) = \mu \text{ in } \Omega \quad \text{ and } \quad u = \varphi \text{ on } \partial\Omega,$$

where μ is a given positive Radon measure and φ are prescribed boundary values.

This problem turns out to be very difficult, involving the geometry of Ω, the regularity of μ, φ, as well as the positivity properties of μ. When Ω is the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, the *homogeneous Dirichlet problem* (i.e. $\mu = 0$) was solved by Bedford and Taylor in [BT76]: they showed that when $\varphi \in \mathcal{C}^{1,1}(\partial\mathbb{B}^n)$, there is a unique plurisubharmonic solution u which is $\mathcal{C}^{1,1}$-smooth in $\mathbb{B}^n$ and Lipschitz up to the boundary. It turns out that u is not more than $\mathcal{C}^{1,1}$-smooth, even if φ is real-analytic. This is a strong indication that one should extend the domain of definition of the complex Monge–Ampère operator, so that it can be applied to mildly regular functions φ.

An important breakthrough was made by Bedford and Taylor in [BT82], where they developed a very consistent theory of this operator acting on *bounded* plurisubharmonic functions: given a bounded plurisubharmonic function u and u_j a sequence of smooth plurisubharmonic functions decreasing to u, they showed the existence of a positive Radon measure $MA(u)$ such that the smooth measures $(dd^c u_j)^n$ converge, in the weak sense of Radon measures, towards $MA(u)$, the latter being independent of the sequence (u_j).

An important consequence of their theory is that a bounded plurisubharmonic function is *maximal* if and only if it satisfies $MA(u) = 0$. Here a plurisubharmonic function u is called maximal if whenever $v \leq u$ on the boundary ∂D of an open subset D, v a plurisubharmonic function on D, then $v \leq u$ in D. In this sense maximal plurisubharmonic functions are the natural generalization of harmonic ones. If u is harmonic along the leaves of a holomorphic foliation, it is easy to check that u is maximal. The converse holds when u is regular enough, as follows from Frobenius theorem, however it does not (at all!) when the regularity of u is below $\mathcal{C}^{1,1}$, as we shall see in the first part of these notes.

1.1.2 Extremal Kähler Metrics

In Kähler geometry, the Ricci curvature is expressed in terms of the complex Monge–Ampère operator. If X is a compact Kähler manifold of complex dimension n equipped with a Kähler metric ω which writes, in local coordinates,

$$\omega = \sum_{\alpha,\beta} g_{\alpha,\beta} \frac{i}{2} dz_\alpha \wedge d\overline{z}_\beta,$$

then the *Ricci form* of ω is

$$Ric(\omega) = -c \sum_{\alpha,\beta} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log\left(\det g_{ij}\right) \frac{i}{2} dz_\alpha \wedge d\overline{z}_\beta.$$

This is a smooth closed real $(1,1)$-form that represents the first Chern class $c_1(X)$ of X. The *scalar curvature* of ω is the trace of the Ricci curvature form with respect to ω. It is the function $s(\omega)$ defined by

$$Ric(\omega) \wedge \frac{\omega^{n-1}}{(n-1)!} = s(\omega) \frac{\omega^n}{n!}.$$

An important question in Kähler geometry is the problem of the existence and uniqueness of constant scalar curvature Kähler metrics. Calabi proposed in [Cal82] to consider the space

$$\mathcal{H} := \{\omega' \text{ Kähler form cohomologous to } \omega\}$$

of all Kähler metrics in the same cohomology class as ω, and to try and minimize the functional

$$\mathrm{Ca} : \omega' \in \mathcal{H} \mapsto \int_X s(\omega')^2 \, (\omega')^n.$$

It easily follows from the Cauchy–Schwarz inequality that a metric of constant scalar curvature is an absolute minimum of Ca hence it is a critical point. More generally, critical points of Ca are called *extremal metrics*.

The Euler–Lagrange equation of Ca was computed by Calabi: a metric $\omega \in \mathcal{H}$ is a critical point of Ca if and only if $\partial^2 s(\omega)/\partial \overline{z}_\alpha \partial \overline{z}_\beta \equiv 0$ so that the vector field

$$V := \sum_\alpha \left[\sum_\beta g^{\alpha\overline{\beta}} \frac{\partial s(\omega)}{\partial \overline{z}_\beta} \right] \frac{\partial}{\partial z_\alpha}$$

is holomorphic. If X does not admit (non zero) holomorphic vector fields, every extremal metric then has constant scalar curvature, however there are examples (e.g. on ruled surfaces) of extremal metrics with varying scalar curvature (see [Gau10]).

The existence problem for constant scalar curvature metrics is very difficult and will not be discussed at all in this volume (see [PS08] for a recent overview). The uniqueness issue is also quite involved. Donaldson observed in [Don99] that it could be reduced to finding a "smooth" solution to a certain homogeneous Dirichlet problem on a compact Kähler manifold with boundary (to be discussed in the next subsection).

When the cohomology class $\{\omega\}$ is proportional to the first Chern class $c_1(X)$, a metric $\omega' \in \mathcal{H}$ has constant scalar curvature if and only if it is Kähler–Einstein, i.e., its Ricci curvature form $Ric(\omega')$ is proportional to ω'. It follows from the dd^c-lemma that ω' decomposes as $\omega' = \omega + dd^c\varphi$ for some smooth function φ. In this case the problem boils down to solving a complex Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi}\mu$$

for some appropriate smooth volume form μ and $\lambda \in \mathbb{R}$.

The existence of Kähler–Einstein metrics was proved by Aubin [Aub76] and Yau [Yau78] when $\lambda < 0$ (negative curvature case), and by Yau [Yau78] when $\lambda = 0$ (Ricci flat case). In both cases, the uniqueness can be proved by an appropriate use of the *comparison principle*. When $\lambda > 0$ (positive curvature case), Kähler–Einstein metrics do not always exist and this problem is nowadays a very active area of research (see [Aub98, Tianbook, PS06]). The uniqueness issue is also more subtle when the curvature is positive: it was established (up to the action of a holomorphic vector field) by Bando and Mabuchi in [BM85]. An alternate proof was given in [BBGZ09] using the convexity properties of the Aubin–Mabuchi functional along geodesics.

1.1.3 The Space of Kähler Metrics

Let again $\mathcal{H}$ denote the space of all Kähler metrics within a fixed cohomology class. Mabuchi discovered in the 1980s [Mab87] that it can be formally regarded as a non-positively curved infinite dimensional locally symmetric space: one defines a Riemannian structure on $\mathcal{H}$ (the L^2-metric) by setting

$$\langle f, g\rangle_\varphi := \int_X f\,\overline{g}\,(\omega + dd^c\varphi)^n,$$

where $\varphi \in \mathcal{H}$ and f, g are tangent vectors in $T_\varphi(\mathcal{H}) \simeq \mathcal{C}^\infty(X)$. It follows from the dd^c-lemma that any Kähler form $\omega' \in \mathcal{H}$ writes $\omega' = \omega + dd^c\varphi$ for some smooth function φ which is uniquely determined (once properly normalized).

Mabuchi defines a natural Levi–Civita connection such that the "product formula" holds. A geodesic between two points $\varphi_0, \varphi_1 \in \mathcal{H}$ is an extremal for the Energy functional

$$E(\Phi) := \frac{1}{2} \int_0^1 \int_X \dot{\varphi_t}^2 (\omega + dd^c\varphi_t)^n dt$$

where $\Phi = (\varphi_t)_{0 \leq t \leq 1}$ denotes a path in $\mathcal{H}$ joining φ_0 to φ_1.

A direct computation shows that Φ is a geodesic if and only if it satisfies

$$\ddot{\varphi_t} = \frac{1}{2} \|\nabla \dot{\varphi}_t\|^2_{\varphi_t}$$

where the gradient is computed with respect to the metric $\omega + dd^c\varphi_t$. An important motivation for introducing this structure is that several important functionals on $\mathcal{H}$ (the Mabuchi functional, the Aubin–Mabuchi functional, Donaldson's functionals, etc) are convex along geodesics.

Semmes realized in [Sem92] that by complexifying the time parameter, the geodesic equation can be reformulated as a Dirichlet problem for the complex Monge–Ampère operator on $X \times A$, where $A = \{z \in \mathbb{C}/1 \leq |z| \leq e\}$ denotes an annulus in $\mathbb{C}$. Namely setting $z = e^{t+is} \in A$, $\beta(x, z) = \omega(x)$ and $\psi(x, z) = \varphi_t(x)$, the path (φ_t) is a geodesic joining φ_0 to φ_1 if and only if

$$(\beta + dd^c\psi)^{n+1} = 0 \text{ in } X \times A,$$

with $\psi(\cdot, 1) \equiv \varphi_0$ and $\psi(\cdot, e) \equiv \varphi_1$. Here the derivatives are taken with respect to the $(n+1)$-variables $(x, z) \in X \times A$.

All these facts have been rediscovered by Donaldson in [Don99]. This latter paper has been very influential, as Donaldson pointed out how a better understanding of geodesics in $\mathcal{H}$ would have important consequences in complex differential geometry. He noticed in particular that the uniqueness problem for extremal Kähler metrics would follow from the existence of $\mathcal{C}^2$-geodesics.

From there on, the regularity properties of solutions to homogeneous complex Monge–Ampère equations have become of crucial importance and there has been a lot of activity in this direction.

Our motivation in these lectures has been to propose a general overview on these problems, covering a variety of techniques (from complex analysis, probability theory, Kähler geometry, non linear PDE's), so as to prepare the audience before going to more advanced and specialized texts such as [CT08, PS08, Don09].

1.2 Contents

We now describe the precise contents of the volume. It is divided in four parts and eight chapters.

The first part deals with the Dirichlet problem for the homogeneous complex Monge–Ampère equation in domains of $\mathbb{C}^n$. The second part exposes the probabilistic approach to the regularity properties of the solutions. The third part presents a self-contained proof of Yau's solution to the Calabi conjecture. The fourth part studies the properties of geodesics in the space of Kähler metrics.

We now review in some detail the contents of each chapter.

Chapter 2

The second chapter is probably the most elementary of all. It is an expanded set of notes, written by V. Guedj and A. Zeriahi, after the lecture A. Zeriahi delivered in LATP, Marseille (France), in March 2009.

It presents a detailed proof of the following classical and fundamental results: let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi \in \mathcal{C}^0(\partial\Omega, \mathbb{R})$ be boundary values, then there exists a unique solution $u \in PSH(\Omega) \cap \mathcal{C}^0(\Omega)$ to the homogeneous Dirichlet problem

$$(dd^c u)^n = 0 \text{ in } \Omega \quad \text{ and } \quad u_{|\partial\Omega} \equiv \varphi,$$

such that moreover

- $u \in \mathcal{C}^0(\overline{\Omega})$ (Bremermann–Walsh, Bedford–Taylor)
- u is Lipschitz up to the boundary if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ (Bedford–Taylor)
- $u \in \mathcal{C}^{1,1}(\Omega)$ if $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $\Omega = \mathbb{B}^n$ (Bedford–Taylor).

The reader will also find some examples and comments emphasizing that

- u is no more than $\mathcal{C}^{1,1}$-smooth, even if φ is real analytic
- the regularity drops if Ω is weakly pseudoconvex
- higher order regularity up to the boundary is still open (see however Chap. 4 for a presentation of the decisive results by Krylov).

There are also a few comments on the domain of definition of the complex Monge–Ampère operator and on the most important maximum principle in this area, the *comparison principle*.

Chapter 3

The third chapter is an expanded set of notes, written by R. Dujardin and V. Guedj, after the lecture R. Dujardin delivered in LATP, Marseille (France), in March 2009.

A bounded plurisubharmonic function φ in a domain $\Omega \subset \mathbb{C}^2$ is *maximal* if it satisfies $(dd^c\varphi)^2 = 0$. As explained earlier (and in Chap. 2), maximal plurisubharmonic functions are the natural generalizations of harmonic functions. The goal of this lecture is to study the geometric properties that can underly the maximality of such a function.

When $\varphi \in \mathcal{C}^3(\Omega)$, it follows from Frobenius' theorem that $T = dd^c\varphi$ is a *foliated cycle*: the interior of $Supp\,T$ is foliated by complex leaves along which φ is harmonic (the so called Monge–Ampère foliation [BK77]).

When φ is less regular, it is tempting to think that a similar picture holds, replacing the foliated cycle structure by the more supple notion of *laminar current*. However various examples are presented for which the support of $T = dd^c\varphi$ has no analytic structure at all if the regularity of φ is strictly below $\mathcal{C}^{1,1}$.

These constructions, due to R. Dujardin [Duj09], are inspired by classical constructions of polynomial hulls with no analytic structure. These are recalled, together with the connection made by Bremermann [Bre59] between the polynomial hull of some Hartogs domains and the homogeneous Dirichlet problem for the complex Monge–Ampère operator.

Chapter 4

The fourth chapter is an expanded set of notes, written by F. Delarue, after the lecture he delivered in LATP, Marseille (France), in December 2009. It is the longest and probably the most difficult chapter of all.

On the other hand it is also the most original aspect of this volume, as it exposes in a self-contained and pedagogical way the works of Krylov, a probabilistic approach to the regularity theory for the complex Monge–Ampère equation.

The notes focus on the proof of the following result: let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain, $\Phi \in \mathcal{C}^4(\partial\Omega)$ and $f \geq 0$ a non negative density such that $f^{1/n} \in \mathcal{C}^2(\Omega)$. Then the unique plurisubharmonic solution φ of the Dirichlet problem

$$(dd^c\varphi)^n = f dV_{Leb} \text{ in } \Omega \quad \text{and} \quad \varphi \equiv \Phi \text{ on } \partial\Omega$$

is in $\mathcal{C}^{1,1}(\overline{\Omega})$ (i.e. has Lipschitz first order derivatives up to the boundary).

The whole analysis relies on a probabilistic writing of the solution as the value function of a stochastic optimal control problem. Such a representation has been introduced by Gaveau [Gav77] in the late 1970s and used in an exhaustive way by Krylov in a series of papers published in the late 1980s [Kry80, Kry82, Kry87, Kry89b] up to the final paper [Kry89] in which the $\mathcal{C}^{1,1}$-estimate of the solution is eventually established.

Krylov's proof is a real tour de force. The proof here given in Chap. 4 follows most of the original arguments. It is thus also very useful as it exposes in a unified way the main steps of the Krylov works. Before reaching its final point, the reader will benefit from a nice introduction to the

- Kolmogorov representation of the Dirichlet problem with constant coefficients by means of the Brownian motion
- Basic rules of stochastic calculus
- Probabilistic representation of Monge–Ampère due to Gaveau
- Stochastic differential equations

Original Krylov's argument applies to a much larger class of equations, the so called *Hamilton–Jacobi–Bellman equations*. The proof is here restricted to the Monge–Ampère equation, but the connection with Hamilton–Jacobi–Bellman equations is briefly discussed as well.

After Chap. 2, the reader will realize that Krylov's approach yields the only known proof of the $\mathcal{C}^{1,1}$-regularity up to the boundary, even when the domain Ω is the unit ball!

Chapter 5

The fifth chapter are the lecture notes written by Z. Błocki, after a course he delivered at a Winter School in complex analysis and geometry in Toulouse (France), in January 2005.

Let X be a compact Kähler manifold (of complex dimension n) and fix $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler cohomology class. Calabi conjectured in [Cal56] that if η is any smooth closed real $(1,1)$-form representing the first Chern class of X, then one can find a Kähler form ω in α such that $Ric(\omega) = \eta$. In particular if $c_1(X) = 0$, this implies the existence of a Ricci-flat Kähler metric in α. This conjecture was solved by Yau in [Yau78].

This lecture presents a self-contained proof of the Calabi–Yau theorem. Fixing $\omega \in \alpha$ an arbitrary Kähler form, there exists $h \in \mathcal{C}^\infty(X)$ such that $Ric(\omega) = \eta + dd^c h$. One then looks for $\omega' = \omega + dd^c\varphi > 0$ such that $Ric(\omega') = \eta$. This is equivalent to solving the complex Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = e^h \omega^n.$$

The proof, by the continuity method, consists in establishing various a priori estimates. The original estimates by Yau are sketched and detailed proofs of subsequent simplifications are given.

Although we are primarily interested in more degenerate equations (notably homogeneous type complex Monge–Ampère equations), these are naturally studied through approximating them by non degenerate ones (as we shall see in Chap. 7).

Chapter 6

The sixth chapter is an expanded set of notes, written by B. Kolev, after the lecture he delivered in LATP, Marseille (France), in March 2009.

Let X be a compact Kähler manifold, $\alpha \in H^{1,1}(X, \mathbb{R})$ a fixed Kähler class and let $\mathcal{H}$ denote the set of all Kähler metrics in α. The goal of this lecture is to explore, after Mabuchi [Mab87], Semmes [Sem92] and Donaldson [Don99], the Riemannian structure of $\mathcal{H}$.

As already mentioned above, the space $\mathcal{H}$ can be formally regarded as a locally symmetric space of non-positive curvature. The infinite dimensional setting makes it however difficult to establish the existence of geodesics, or to measure lengths and distances.

The first part of the lecture aims at making the reader familiar with a few problems encountered in infinite dimensional Riemannian geometry. It introduces various Riemannian structures on (sub)groups of diffeomorphisms of a compact Riemannian manifold M (on the whole group $\mathrm{Diff}(M)$, or on the subgroup of diffeomorphisms that preserve a volume form, on the subgroup of symplectomorphims, etc). It is for instance explained that the Riemannian L^2-pseudo-distance on the component of identity $\mathrm{Diff}^0(M)$ of $\mathrm{Diff}(M)$ vanishes identically!

The second part of the lecture focuses on the Riemannian structure induced on $\mathcal{H}$ by the Mabuchi L^2-metric. The geodesic equation is derived, as well as its equivalent formulation in terms of a homogeneous complex Monge–Ampère Dirichlet problem on the compact manifold with boundary $X \times A$, where A denotes an annulus in $\mathbb{C}$.

The Aubin–Mabuchi functional (a primitive of the Monge–Ampère operator) is shown to be affine along geodesics, while the Mabuchi energy functional is proved to be convex along geodesics. This property is crucial when trying to prove the uniqueness of constant scalar curvature metrics, as the latter are critical points of the Mabuchi functional. This however requires the existence of smooth geodesics, a question to be treated in the next chapter...

Chapter 7

The seventh chapter is an expanded set of notes, written by S. Boucksom, after the lecture he delivered in LATP, Marseille (France), in March 2009.

The primarily goal of this lecture was to present a detailed proof of the existence of almost $\mathcal{C}^{1,1}$-geodesics in the space $\mathcal{H}$ of Kähler metrics, a result due to X.X. Chen. As explained above, this amounts to solving a homogeneous Dirichlet problem for the complex Monge–Ampère equation in a compact Kähler manifold with boundary.

Although the regularity obtained is not sufficient to directly use Donaldson's observation (for proving the uniqueness of constant scalar curvature metrics in a fixed Kähler class), this result has been very influential; allowing for example, Chen to prove that the space $\mathcal{H}$ endowed with the Mabuchi L^2-metric is indeed a metric space.

The proof goes by first solving non degenerate equations, producing smooth solutions, then establishing uniform a priori estimates on the Laplacian of the latter, and finally letting the equations degenerate to the homogeneous one.

As the method of proof is very similar to the one used to solve the corresponding equations in smoothly bounded domains of $\mathbb{C}^n$ (works of Caffarelli–Kohn–Nirenberg–Spruck and Guan), the author succeeds – by using a recent work of Blocki – in presenting a unified frame, solving non degenerate complex Monge–Ampère equations on a compact Kähler manifold with boundary.

The proofs are self-contained, but for some $\mathcal{C}^2$-estimates that are explained in Blocki's lecture (Chap. 5).

Chapter 8

The eighth chapter is an expanded set of notes, written by R. Berman and J. Keller, after the lecture R. Berman delivered in LATP, Marseille (France), in March 2009.

This final lecture is again devoted to the study of geodesics in the space $\mathcal{H}$ of Kähler metrics in a fixed cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$. One assumes here that $\alpha = c_1(L)$ is the first Chern class of an ample line bundle and considers the spaces $\mathcal{H}_k$ of Bergman metrics of L^k: these are the pull-backs of Fubini–Study metrics by the embeddings of the manifold into the projective space $\mathbb{P}H^0(X, L^k)$. A result of Bouche [Bou90] and Tian [Tian90] asserts that every $\varphi \in \mathcal{H}$ is canonically approximated, as $k \to +\infty$, by a sequence $P_k(\varphi) \in \mathcal{H}_k$ of Bergman metrics.

The spaces $\mathcal{H}_k$ are finite dimensional symmetric spaces naturally equipped with a Riemannian structure. Motivated by the seminal work of Donaldson [Don01], it is natural to ask whether geodesics in the Bergman spaces $\mathcal{H}_k$ converge towards geodesics in the space $\mathcal{H}$. A positive answer was furnished by Phong–Sturm [PS06] and later on refined by Berndtsson [Bern09b]. The goal of this lecture is to survey the proofs of these results.

Part I
The Local Homogeneous Dirichlet Problem

Chapter 2
Dirichlet Problem in Domains of $\mathbb{C}^n$

Vincent Guedj and Ahmed Zeriahi

Abstract This lecture treats the Dirichlet problem for the homogeneous complex Monge–Ampère equation in domains $\Omega \subset \mathbb{C}^n$. The most important result, due to Bedford and Taylor [BT76], yields the optimal interior regularity of the solution when $\Omega = \mathbb{B}$ is the unit ball. We provide a complete proof, following the simplifications of Demailly [Dem93].

2.1 Introduction

The goal of this lecture is to study the Dirichlet problem in bounded domains of $\mathbb{C}^n$ for the complex Monge–Ampère operator. If $\Omega \Subset \mathbb{C}^n$ is such a domain and $\varphi : \Omega \to \mathbb{R}$ are continuous boundary values, the goal is to find a plurisubharmonic function $u : \Omega \to \mathbb{R}$ solution of the following nonlinear PDE with prescribed boundary values,

$$\mathrm{DirMA}(\Omega, \varphi) := \begin{cases} (dd^c u)^n = 0 \text{ in } \Omega \\ u_{|\partial\Omega} = \varphi \end{cases}$$

and to study regularity properties of u in terms of those of φ. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$ are real operators so that

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$$(dd^c u)^n = c \det \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) dV,$$

when $u \in \mathcal{C}^2(\Omega)$, for some normalizing constant $c > 0$ and dV denotes the Lebesgue volume form in $\mathbb{C}^n$.

The complex Monge–Ampère operator $(dd^c u)^n$ still makes sense when u is poorly regular, as we shall explain in Sect. 2.5.1. The property "$u_{|\partial\Omega} = \varphi$" has to be understood as

$$\lim_{\Omega \ni z \to \zeta} u(z) = \varphi(\zeta), \quad \text{for all } \zeta \in \partial\Omega.$$

Whether it holds depends both on the continuity properties of φ and on the geometry of $\partial\Omega$. We shall usually assume Ω is smooth and *strictly pseudoconvex*, a notion recalled in Sect. 2.3.1.

Nota Bene. These notes are written by Vincent Guedj and Ahmed Zeriahi after the lecture delivered by Ahmed Zeriahi in Marseille, March 2009. There is no claim for any originality, all the material presented here being quite classical. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

2.2 The Classical Dirichlet Problem in $\mathbb{C}$

In dimension one, the Monge–Ampère operator coincides with the Laplacian. It is thus much easier to study. We briefly recall here how to solve the Dirichlet problem in this case, first in the unit disk by using the Poisson representation formula – a tool not available in higher dimension-, then in general bounded domains of $\mathbb{C}$ using the method of barriers which can be adapted in higher dimension.

2.2.1 Unit Disk

We study here DirMA$(\mathbb{D}, \varphi)$ where $\mathbb{D} = \{\zeta \in \mathbb{C} \,/\, |\zeta| < 1\}$ is the unit disk. It admits a unique solution u_φ which can be expressed by averaging against the Poisson kernel.

Proposition 2.1 *Assume $\varphi \in \mathcal{C}^0(\partial\mathbb{D})$. Then*

$$u_\varphi(z) := \int_0^1 \frac{1 - |z|^2}{|z - e^{2i\pi\theta}|^2} \varphi(e^{2i\pi\theta}) d\theta$$

is the unique solution to DirMA$(\mathbb{D}, \varphi)$. *It is harmonic (hence real-analytic) in* $\mathbb{D}$ *and continuous up to the boundary.*

Proof. Observe that the Poisson kernel is the real part of a holomorphic function in $\mathbb{D}$,

$$\frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} = \Re\left(\frac{e^{2i\pi\theta}+z}{z-e^{2i\pi\theta}}\right).$$

This shows that u_φ is harmonic in $\mathbb{D}$, as an average of harmonic functions.

We now establish the continuity up to the boundary. Fix $\zeta = e^{2i\pi\theta_0} \in \partial\mathbb{D}$ and $\varepsilon > 0$. Since φ is assumed to be continuous at ζ, we can find $\delta > 0$ such that $|\varphi(e^{2i\pi\theta}) - \varphi(\zeta)| < \varepsilon/2$ whenever $|e^{2i\pi\theta} - \zeta| < \delta$. Observing that

$$\int_0^1 \frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} d\theta \equiv 1,$$

we infer

$$|u_\varphi(z) - \varphi(\zeta)| \leq \varepsilon/2 + 2M \int_{|e^{2i\pi\theta}-\zeta|\geq\delta} \frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} d\theta,$$

where $M = \sup_{S^1} |\varphi|$. Note that $|z - e^{2i\pi\theta}| \geq \delta/2$ if z is close enough to ζ and $|e^{2i\pi\theta} - \zeta| \geq \delta$. The latter integral is therefore bounded from above by $4(1-|z|^2)/\delta^2$ hence converges to zero as z approaches the unit circle. $\square$

It is clear from the proof above that one can control the modulus of continuity of u_φ on $\overline{\mathbb{D}}$ in terms of that of φ. For instance if φ is Hölder continuous, then so is u_φ. Let us denote by

$$Lip_\alpha(K) := \{u : K \to \mathbb{R} \,/\, \exists C > 0, \forall x, y \in K, |u(x) - u(y)| \leq C|x-y|^\alpha\}$$

the set of α-Hölder continuous functions on a Borel set K, $0 < \alpha \leq 1$.

Exercise 2.2

(1) Show that $\varphi \in Lip_\alpha(\partial\mathbb{D}) \Rightarrow u_\varphi \in Lip_\alpha(\overline{\mathbb{D}})$ *when* $0 < \alpha < 1$.
(2) By considering $\varphi(e^{2i\pi\theta}) = |\sin\theta|$, *show that the result does not hold with* $\alpha = 1$.

Beware that the exercise is trickier than it perhaps seems at first glance: following the proof of the previous proposition, you should be able to obtain $u_\varphi \in Lip_\beta(\overline{\mathbb{D}})$ with $\beta = \alpha/(\alpha+2)$. Proving that u_φ is actually α-Hölder is slightly more subtle, give it a try!

The fact that the class Lip_1 does not behave well for the Dirichlet problem is a classical fact in the study of elliptic PDE's. Note that one can similarly show (see [GT83]) that

$$\varphi \in \mathcal{C}^{k,\alpha}(\partial\mathbb{D}) \Rightarrow u_\varphi \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$$

for all $k \in \mathbb{N}$ and $0 < \alpha < 1$. In particular

$$\varphi \in \mathcal{C}^{\infty}(\partial\mathbb{D}) \Rightarrow u_\varphi \in \mathcal{C}^{\infty}(\overline{\mathbb{D}}).$$

We will soon see that this is far from being true in higher dimension.

2.2.2 Perron Envelopes

We now consider DirMA(Ω, φ), the Dirichlet problem corresponding to a bounded domain $\Omega \Subset \mathbb{C}$. Here $\varphi : \partial\Omega \to \mathbb{R}$ is a fixed continuous function on the boundary of Ω.

It follows from the maximum principle for harmonic functions that if a solution exists, it is unique. More generally if u, v are subharmonic functions on Ω such that $\Delta u \leq \Delta v$ in the weak sense of measures on Ω and $(u-v)_* \geq 0$ on $\partial\Omega$ (i.e. $u \geq v$ on $\partial\Omega$), then $u \geq v$ in Ω. Indeed, $v - u$ is subharmonic on Ω and $(v-u)^* \leq 0$ on $\partial\Omega$, so that $v - u \leq 0$ in Ω by the maximum principle for subharmonic functions.

This shows that if u is the solution of the Dirichlet problem DirMA(Ω, φ), then any "subsolution" $v \in SH(\Omega)$ such that $v^* \leq \varphi$ on $\partial\Omega$ satisfies $v \leq u$ on Ω. Therefore

$$u_\varphi := \sup\{v \,/\, v \in SH(\Omega), v^* \leq \varphi \text{ on } \partial\Omega\} \leq u.$$

Observe that u itself is a subsolution so that actually $u = u_\varphi$. In other words, if the Dirichlet problem DirMA(Ω, φ) admits a solution, then it is the "Perron envelope" u_φ defined above [Per23].

One can easily show, by "balayage" (using a max construction together with solutions of the Dirichlet problem in small disks) that u_φ is harmonic in Ω. The problem is therefore reduced to checking whether u_φ has the right boundary values. This depends on the geometry of $\partial\Omega$.

Definition 2.3 *A barrier at the point $\zeta_0 \in \partial\Omega$ is a non positive subharmonic function $b \in SH(\Omega)$ such that $\lim_{\zeta\to\zeta_0} b(\zeta) = 0$ and $b^* < 0$ in $\overline{\Omega}\setminus\{\zeta_0\}$.*

The interest in this notion lies in the following

Lemma 2.4 *If there exists a barrier at a boundary point $\zeta_0 \in \partial\Omega$, then*

$$\lim_{z\to\zeta_0} u_\varphi(z) = \varphi(\zeta_0).$$

Proof. Fix $\varepsilon > 0$. Since φ is continuous we can find $\delta > 0$ such that

$$\varphi(\zeta_0) - \varepsilon \leq \varphi(\zeta) \leq \varphi(\zeta_0) + \varepsilon \text{ for } \zeta \in \partial\Omega \text{ with } |\zeta - \zeta_0| \leq \delta.$$

Since $b^* < 0$ on the compact subset $\partial\Omega\backslash\mathbb{D}(\zeta_0, \delta)$, it follows from upper semi-continuity of b^* that for $A > 1$ large enough, $Ab^* + \varphi(\zeta_0) - \varepsilon \leq \varphi$ on $\partial\Omega$. Thus $Ab + \varphi(\zeta_0) - \varepsilon$ is a subsolution, hence

$$Ab(z) + \varphi(\zeta_0) - \varepsilon \leq u_\varphi(z), \quad \forall z \in \Omega.$$

Letting $z \to \zeta_0$ and then $\varepsilon \to 0$ shows that $\varphi(\zeta_0) \leq \liminf_{z\to\zeta_0} u_\varphi(z)$.

Consider now the Dirichlet problem $\mathrm{DirMA}(\Omega, -\varphi)$. It follows from the maximum principle that $u_\varphi + u_{-\varphi} \leq 0$ in Ω, hence $u_\varphi \leq -u_{-\varphi}$. Previous reasoning thus yields

$$\varphi(\zeta_0) \geq -(-\varphi(\zeta_0)) \geq -\liminf_{z\to\zeta_0} u_{-\varphi}(z) \geq \limsup_{z\to\zeta_0} u_\varphi(z),$$

hence finally $\lim_{z\to\zeta_0} u(z) = \varphi(\zeta_0)$. □

Constructing barriers is thus the final step towards a solution of the Dirichlet problem. It turns out that they always exist when the boundary $\partial\Omega$ is Lipschitz. Note that some hypothesis on $\partial\Omega$ has to be made: the problem $\mathrm{DirMA}(\mathbb{D}^*, \varphi)$ has no solution when $\Omega = \mathbb{D}^*$ is the unit disk minus the origin and φ is zero on the unit circle and 1 at the origin: in this case u_φ is the constant function zero, hence it does not have the right boundary value at the origin.

2.3 Maximal Plurisubharmonic Functions

We now start to consider similar questions in higher dimension. Observe that some further constraints have to be put either on the geometry of $\partial\Omega$ or on the behavior of the boundary values φ: if $f(\mathbb{D}) \subset \partial\Omega$ is a holomorphic disk (image of the unit disk by a non constant holomorphic map) lying within the boundary, then φ has to be subharmonic along $f(\mathbb{D})$ if the Dirichlet problem $\mathrm{DirMA}(\Omega, \varphi)$ ever has a solution. In order to avoid difficulties related to such questions, we restrict ourselves to considering smooth *strictly pseudoconvex* domains Ω.

2.3.1 Strictly Pseudoconvex Domains

Although it makes sense to study the Dirichlet problem for the complex Monge–Ampère operator on a general domain $\Omega \Subset \mathbb{C}^n$, we will restrict ourselves and consider only domains that are *bounded*, with smooth boundary and such that the latter has a certain convexity property:

Definition 2.5 *A bounded domain $\Omega \Subset \mathbb{C}^n$ is* **strictly pseudoconvex** *if there exists a smooth strictly plurisubharmonic function ρ on some open neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{z \in \Omega' \,/\, \rho(z) < 0\}$.*

A classical result asserts that a bounded domain is strictly pseudoconvex if and only if it is locally biholomorphic to a strictly convex domain. Slightly more general are *weakly pseudoconvex domains* and *hyperconvex domains*. The former coincide with *domains of holomorphy* (this is the famous Levi problem), while the latter are still defined as $\{\rho < 0\}$ but for a function that is only weakly (i.e. not necessarily strictly) plurisubharmonic and exhaustive.

There do exist some interesting results concerning the Dirichlet problem on these more general domains, as well as on non pseudoconvex ones (see e.g. [Sad82, Bl00, Guan02]). These are technically more involved and beyond the scope of this lecture.

2.3.2 Perron–Bremermann Envelope

Following the one variable solution to the Dirichlet problem, it is natural to consider

$$u_\varphi := \sup\{v \,/\, v \in \mathcal{B}(\Omega, \varphi)\}$$

where

$$\mathcal{B}(\Omega, \varphi) := \left\{ v \in PSH(\Omega);\; v^*(\zeta) := \limsup_{z \to \zeta} v(z) \le \varphi(\zeta),\; \forall \zeta \in \partial\Omega \right\},$$

is the family of subsolutions for the boundary data φ.

The function u_φ is called the Perron–Bremermann envelope associated to the boundary data φ. Bremermann [Bre59] has shown that the function u_φ is a plurisubharmonic function in Ω with boundary values φ, Walsh [Wa68] further showed that u_φ is continuous in Ω:

Theorem 2.6 *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex subset of $\mathbb{C}^n$. The upper envelope u_φ is a continuous plurisubharmonic function on $\overline{\Omega}$ with boundary values φ i.e.*

$$\lim_{z \to \zeta} u_\varphi(z) = \varphi(\zeta) \;\; \textit{for all } \zeta \in \partial\Omega.$$

Proof. Let ρ be a strictly plurisubharmonic defining function of $\Omega = \{\rho < 0\}$. Observe that the family $\mathcal{B}(\Omega, \varphi)$ is not empty: for $A \gg 1$ large enough, the function $A(\rho - 1)$ is one of its members (recall that φ is continuous hence bounded from below on $\partial\Omega$). Note also that $\mathcal{B}(\Omega,\varphi)$ is locally uniformly bounded from above in Ω: the constant function $\sup\varphi$ dominates all members of $\mathcal{B}(\Omega,\varphi)$.

It follows that the upper semi-continuous regularization $U_\varphi := u_\varphi^*$ is plurisubharmonic in Ω. We are going to prove that U_φ has boundary values φ. This will imply that $U_\varphi \in \mathcal{B}(\Omega,\varphi)$, so that $u_\varphi = U_\varphi$ in Ω.

As in one variable, we plan to construct a plurisubharmonic barrier function at each point $\zeta_0 \in \partial\Omega$. Since ρ is strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$, we can choose $A > 1$ large enough so that the function $b_0 := A\rho(z) - |z-\zeta_0|^2$ is a plurisubharmonic barrier at the point ζ_0 i.e. b_0 is plurisubharmonic in Ω, continuous up to the boundary and such that $b_0(\zeta_0) = 0$ with $b_0 < 0$ in the complement $\overline{\Omega}\setminus\{\zeta_0\}$.

Fix $\varepsilon > 0$ and take $\eta > 0$ such that $\varphi(\zeta_0) - \varepsilon \leq \varphi(\zeta)$ for $\zeta \in \partial\Omega$ and $|\zeta-\zeta_0| \leq \eta$. Choose $C > 1$ big enough so that $Cb_0 + \varphi(\zeta_0) - \varepsilon \leq \varphi$ on $\partial\Omega$. This implies that the function $v(z) := Cb_0 + \varphi(\zeta_0) - \varepsilon$ is plurisubharmonic in a neighborhood of $\overline{\Omega}$ and such that $v \leq \varphi$ on $\partial\Omega$. Thus we have $v \leq u_\varphi$ on Ω, which implies that $\varphi(\zeta_0) - \varepsilon \leq \liminf_{z\to\zeta_0} u_\varphi(z)$. We infer

$$\liminf_{z\to\zeta} U_\varphi(z) \geq \liminf_{z\to\zeta} u_\varphi(z) \geq \varphi(\zeta) \tag{2.1}$$

for all $\zeta \in \partial\Omega$.

In the same way, we can construct a plurisubharmonic subsolution w for the boundary data $-\varphi$ such that $\lim_{z\to\zeta_0} w(z) = -\varphi(\zeta_0)-\varepsilon$. By the maximum principle, for any $v \in \mathcal{B}(\Omega,\varphi)$, we have $v + w \leq 0$ in Ω, hence $u_\varphi + w \leq 0$ on Ω. By upper regularization we infer $U_\varphi + w \leq 0$ in Ω, which implies

$$\limsup_{z\to\zeta_0} U_\varphi(z) \leq -\liminf_{z\to\zeta_0} w(z) = \varphi(\zeta_0) + \varepsilon.$$

Therefore we have proved that

$$\limsup_{z\to\zeta} U_\varphi(z) \leq \varphi(\zeta), \quad \forall\zeta \in \partial\Omega. \tag{2.2}$$

This shows that $U_\varphi \in \mathcal{B}(\Omega,\varphi)$ hence $U_\varphi \leq u_\varphi$ in Ω so that $U_\varphi \equiv u_\varphi$. Inequalities (2.1), (2.2) show that the envelope u_φ has boundary values φ.

It remains to prove that $u = u_\varphi$ is lower semi-continuous in Ω. Fix $\varepsilon > 0$. Since $\partial\Omega$ is compact, we can choose $\eta > 0$ so small that

$$z \in \Omega, \quad \zeta \in \partial\Omega, \quad |z-\zeta| \leq \eta \Longrightarrow |u(z) - \varphi(\zeta)| \leq \varepsilon. \tag{2.3}$$

Fix $a \in \mathbb{C}^n$ with $\|a\| < \eta$ and set $\widetilde{\Omega} := \Omega - a$. Then $u(\zeta + a) \leq \varphi(\zeta) + \varepsilon$ if $\zeta \in \widetilde{\Omega}\cap\partial\Omega$ and $u^*(z+a) \leq \varphi(z+a) + \varepsilon \leq u(z) + 2\varepsilon$ if $z \in \Omega\cap\partial\widetilde{\Omega}$. It follows that the function

$$v(z) := \begin{cases} \sup\{u(z), u(z+a) - 2\varepsilon\} & \text{for } z \in \Omega \cap \widetilde{\Omega} \\ u(z) \quad \text{for } z \in \Omega \setminus \widetilde{\Omega} \end{cases}$$

is plurisubharmonic in Ω and satisfies the condition $v^* \leq \varphi$ on $\partial\Omega$. Therefore $v \leq u_\varphi = u$ in Ω, in particular

$$u(z+a) - 2\varepsilon \leq u(z) \quad \text{for } z \in \Omega \quad \text{and} \quad a \in \mathbb{C}^n, \ \ \|a\| < \eta.$$

This shows that $u = u_\varphi$ is lower semi-continuous in Ω. □

2.3.3 Maximal Plurisubharmonic Functions

Recall that harmonic functions are "above" sub-harmonic ones. This property actually characterizes harmonicity and was illustrated in Sect. 2.2 by the fact that we could recover the harmonic solution to the Dirichlet problem as an upper envelope.

It is therefore natural to consider, among all plurisubharmonic functions, those which are maximal, a notion introduced by Sadullaev [Sad81].

Definition 2.7 *A plurisubharmonic function $u : \Omega \to [-\infty, +\infty[$ is said to be maximal in Ω if for any plurisubharmonic function v defined on a subdomain $D \Subset \Omega$, $v \leq u$ on ∂D implies $v \leq u$ in D.*

Of course a pluriharmonic function is maximal (and smooth, as it is locally the real part of holomorphic function). However, in contrast to the one variable case, maximal plurisubharmonic functions need not be continuous: any (discontinuous) subharmonic function in the unit disk $\mathbb{D}$ gives rise to a maximal plurisubharmonic in $\mathbb{D}^2$ when considered as a function of two complex variables. This is a particular case of the following criterion of maximality.

Lemma 2.8 *Let $u : \Omega \to [-\infty, +\infty[$ be a plurisubharmonic function in Ω. If for any $z_0 \in \Omega$ there is a complex curve $Z \Subset \Omega$ containing z_0 such that $u|Z$ is harmonic on $Z \cap \Omega$, then u is maximal in Ω.*

We leave the easy proof as an exercise. One may wonder whether maximality can always be explained by the existence of "harmonic disks". This is indeed true if the function is regular enough (by Theorem 2.20 below and Frobenius theorem), however there are less regular maximal psh functions with no harmonic disk: this is the topic of the lecture by Dujardin [DG09].

As one can guess, the Perron–Bremermann envelope is maximal:

Proposition 2.9 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $\varphi \in \mathcal{C}^0(\partial\Omega)$ a continuous function on $\partial\Omega$. Then u_φ is the unique maximal plurisubharmonic function on Ω with boundary values φ.*

Proof. We first show that u_φ is maximal on Ω. Let v be a plurisubharmonic function in some subdomain $D \Subset \Omega$ such that $v \le u_\varphi$ on ∂D. Then the function

$$w := \begin{cases} \sup\{u_\varphi, v\} & \text{in } D \\ u_\varphi & \text{in } \Omega \setminus D \end{cases}$$

is plurisubharmonic in Ω and satisfies $w^* \le \varphi$ on $\partial\Omega$. Therefore $w \le u_\varphi$ in Ω hence $v \le w \le u_\varphi$ in D, which proves our claim.

We now prove uniqueness. Let v a maximal plurisubharmonic function in Ω such that $\lim_{z\to\zeta} v(z) = \varphi(\zeta)$ for any $\zeta \in \partial\Omega$. It follows that $v \le u_\varphi$ in Ω while for any fixed $\varepsilon > 0$, the set $\{v + \varepsilon < u_\varphi\} \Subset \Omega$ is relatively compact in Ω. Let $D \Subset \Omega$ be any domain such that $\{v + \varepsilon < u_\varphi\} \Subset D$. Then $v + \varepsilon$ is a maximal plurisubharmonic function satisfying $v + \varepsilon \ge u_\varphi$ on ∂D. Therefore $v + \varepsilon \ge u_\varphi$ in D. Letting ε decrease to zero and D increase to Ω we infer $u_\varphi \le v$ in Ω. $\square$

In dimension one the upper envelope u_φ is harmonic on Ω, hence it is smooth and satisfies the partial differential equation $\Delta u_\varphi = 0$ on Ω. It is natural to wonder whether a similar result holds in higher dimension as well. We study the regularity question in the next section. The PDE characterization is postponed to the last section.

2.4 Regularity of Perron–Bremermann Envelopes

In this section we study the propagation of regularity from φ to u_φ. We start by explaining the fundamental result of Bedford and Taylor [BT76], following a simplified proof due to Demailly [Dem93]. We then list various results, open questions and examples that illustrate some of the difficulties encountered with DirMA(Ω, φ) when $n \ge 2$.

2.4.1 Unit Ball

Our goal here is to prove the following result due to Bedford and Taylor [BT76].

Theorem 2.10 *Let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$.*
If $\varphi \in \mathcal{C}^{1,1}(\partial\mathbb{B}, \mathbb{R})$ then u_φ is a $\mathcal{C}^{1,1}$-function in $\mathbb{B}$.

Recall that a function $f : M \to \mathbb{R}$ defined on a smooth real submanifold is $\mathcal{C}^{1,1}$ if f is differentiable and df is a locally Lipschitz 1-form on M. Observe that a $\mathcal{C}^{1,1}$-function has locally bounded second order derivatives almost everywhere.

Proof. We will show in Proposition 2.12 below that $u = u_\varphi$ is Lipschitz continuous up to the boundary. We focus here on the second order estimates. By Lemma 2.11 below, it suffices to prove that for any $z \in \Omega$ and $h \in \mathbb{C}^n$ with $|h| \ll 1$ we have

$$u(z+h) + u(z-h) - 2u(z) \leq C_2 \|h\|^2.$$

The idea is to study the boundary behavior of the plurisubharmonic function $z \longmapsto \frac{1}{2}(u(z+h)+u(z-h))$ in order to compare it with the function u in Ω. This does not make sense since the translations do not preserve the boundary. We are instead going to move point z by automorphisms of the unit ball: the group of holomorphic automorphisms of the latter acts transitively on it and this is the main reason why we prove this result for the unit ball rather than for a general strictly pseudoconvex domain (which has generically few such automorphisms).

Fix a point $a \in \mathbb{B} \setminus \{0\}$ and consider the mapping

$$F_a(z) := \frac{P_a(z) - a + (1 - \|a\|^2)^{1/2}(z - P_a(z))}{1 - \langle z, a \rangle}$$

where $P_a(z) := \|a\|^{-2}\langle z, a\rangle a$ is the orthogonal projection on the complex line $\mathbb{C}\cdot a$. Here $\langle z, a\rangle = \sum_{i=1}^n z_i \overline{a_i}$ denotes the hermitian scalar product of z and a. We let the reader check that F_a is an holomorphic automorphism of the unit ball $\mathbb{B}$ which sends a to the origin. The interested reader will find further information on these automorphisms in [Ru80].

An elementary computation yields

$$F_a(z) = \frac{z - a + O(|a|^2)}{1 - \langle z, a\rangle} = z - a + \langle z, a\rangle z + O(\|a\|^2) = z - h + O(\|a\|^2),$$

where $h := a - \langle z, a \rangle z$ and $O(\|a\|^2)$ is uniform with respect to $z \in \overline{\mathbb{B}}$.

Consider the function $v(z) := u \circ F_a(z) + u \circ F_{-a}(z)$. It is plurisubharmonic in $\mathbb{B}$ and has boundary values equal to

$$g(z) := \varphi(F_a(z)) + \varphi(F_{-a}(z))$$

since F_a preserves $\partial\mathbb{B}$. We can extend φ as a $\mathcal{C}^{1,1}$-smooth function so that $\varphi(F_{\pm a}(z)) \leq \varphi(z \mp h) + C_1\|a\|^2$ and

$$\varphi(z+h) + \varphi(z-h) - 2\varphi(z) \leq A\|h\|^2$$

whenever $z \in \mathbb{B}$ and $\|h\| \leq \delta$. Altogether this yields

$$g(z) \leq \varphi(z+h) + \varphi(z-h) + 2C_1\|a\|^2 \leq 2\varphi(z) + A\|h\|^2 + 2C_1\|a\|^2$$

for $z \in \partial\mathbb{B}$. We infer $v(z) \leq 2u(z) + A\|h\|^2 + 2C_1\|a\|^2$ when $z \in \mathbb{B}$.

Observe that the mapping $a \mapsto h = h(a,z)$ is a local diffeomorphism in a neighborhood of the origin as long as $\|z\| < 1$. An easy computation shows that the inverse map $h \mapsto a$ has norm $\leq (1 - \|z\|^2)^{-1}$.

Fix a compact set $K \subset \mathbb{B}$. Then there is exists $\delta > 0$ small enough and a constant $C_2 = C_2(K) > 0$ such that for $z \in K$ and $|h| \leq \delta$ we have $|a| \leq C_2|h|$.

It follows that for any $z \in K$ and $|h| \leq \delta$,

$$u(z+h) + u(z-h) - 2u(z) \leq C_3\|h\|^2,$$

where $C_3 > 0$ is a uniform constant depending on K, which proves the required estimates. □

Let us stress that we haven't proved that u_φ is $\mathcal{C}^{1,1}$ up to the boundary of the unit ball. This would require further regularity of the boundary values (see Sect. 2.4.3). In other words the constant C_2 in the proof above depends on $\mathrm{dist}(\mathrm{z}, \partial\mathbb{B})$.

It remains to prove the following criterion.

Lemma 2.11 *Let u be a plurisubharmonic function in a domain $\Omega \Subset \mathbb{C}^n$. Assume that there exists constants $A, \delta > 0$ such that*

$$u(z+h) + u(z-h) - 2u(z) \leq A\|h\|^2, \quad \forall 0 < \|h\| < \delta$$

and for all $z \in \Omega, \mathrm{dist}(\mathrm{z}, \partial\Omega) > \delta$. Then u is $\mathcal{C}^{1,1}$-smooth and its second derivatives, which exist almost everywhere, satisfy $\|D^2 u\|_{L^\infty(\Omega)} \leq A$.

Moreover the Monge–Ampère measure $(dd^c u)^n$ is absolutely continuous w.r.t. the Lebesgue measure dV in Ω, with

$$(dd^c u)^n = c_n \det\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{z_j} \partial \bar{\mathrm{z}}_\mathrm{k}}\right) \beta_\mathrm{n},$$

where β_n is the standard volume form on $\mathbb{C}^n$.

Proof. Let $u_\varepsilon := u \star \rho_\varepsilon$ denote the standard regularization of u defined in $\Omega_\varepsilon := \{z \in \Omega \,/\, \mathrm{dist}(\mathrm{z}, \partial\Omega) > \varepsilon\}$ for $0 < \varepsilon \ll 1$. Fix $\delta > 0$ small enough and $0 < \varepsilon < \delta/2$. Then for $\|h\| < \delta/2$, we have

$$u_\varepsilon(z+h) + u_\varepsilon(z-h) - 2u_\varepsilon(z) \leq A\|h\|^2.$$

It follows from Taylor's formula that if $z \in \Omega_\delta$,

$$\frac{d^2}{dt^2} u_\varepsilon(z+th)|_{t=0} = \lim_{t \to 0^+} \frac{u_\varepsilon(z+th) + u_\varepsilon(z-th) - 2u_\varepsilon(z)}{t^2},$$

therefore $D^2 u_\varepsilon(z) \cdot h^2 \leq A\|h\|^2$ for all $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. Now for $z \in \Omega_\varepsilon$,

$$D^2 u_\varepsilon(z) \cdot h^2 = \sum_{j,k=1}^n \left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial z_k} h_j h_k + 2 \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k + \frac{\partial^2 u_\varepsilon}{\partial \bar{z}_j \partial \bar{z}_k} \bar{h}_j \bar{h}_k \right).$$

Recall that u_ε is plurisubharmonic in Ω_ε, hence

$$D^2 u_\varepsilon(z) \cdot h^2 + D^2 u_\varepsilon(z) \cdot [ih]^2 = \sum_{j,k=1}^n 4 \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k \geq 0.$$

The above upper-bound therefore also yields a lower-bound,

$$D^2 u_\varepsilon(z) \cdot h^2 \geq -D^2 u_\varepsilon(z) \cdot [ih]^2 \geq -A\|h\|^2,$$

for any $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. This implies that $\|D^2 u_\varepsilon(z)\|_{L^\infty(\Omega_\varepsilon)} \leq A$.

We have thus shown that Du_ε is uniformly Lipschitz in Ω_ε. We infer that Du is Lipschitz in Ω and $Du_\varepsilon \to Du$ uniformly on compact subsets of Ω. Since the dual of L^1 is L^∞, it follows from the Alaoglu–Banach theorem that, up to extracting a subsequence, there exists a bounded function V such that $D^2 u_\varepsilon \to V$ weakly in L^∞. Now $D^2 u_\varepsilon \to D^2 u$ in the sense of distributions hence $V = D^2 u$. Therefore u is $\mathcal{C}^{1,1}$-smooth in Ω, its second order derivatives exist almost everywhere with $\|D^2 u(z)\|_{L^\infty} \leq A$.

Recall that if $f \in L^n_{loc}(\Omega)$ then $f_\varepsilon := f \star \rho_\varepsilon \to f$ in $L^n_{loc}(\Omega)$. In particular

$$\frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} \longrightarrow \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \text{ in } L^n_{loc}(\Omega) \supset L^\infty(\Omega).$$

Using generalized Hölder's inequality, we infer $\det\left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k}\right) \to \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$ in $L^1_{loc}(\Omega)$.

Recall that $(dd^c u)^n$ is well defined in the weak sense of Bedford–Taylor [BT82] as the weak limit of the smooth forms $(dd^c u_\varepsilon)^n$, since (u_ε) decreases to u as ε decreases to 0. Since convergence in L^1_{loc} implies weak convergence, we infer that

$$(dd^c u)^n = c_n det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\beta_n. \qquad \square$$

2.4.2 Strictly Pseudoconvex Domains

We now consider the more general case of a smoothly bounded strictly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$.

We show here that the upper envelope u_φ is Lipschitz up to the boundary as soon as the boundary value φ is $\mathcal{C}^{1,1}$.

Proposition 2.12 *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex set. If $\varphi \in \mathcal{C}^{1,1}(\partial\Omega, \mathbb{R})$ then the envelope u_φ is Lipschitz continuous on $\overline{\Omega}$.*

Proof. Let ρ be a smooth defining function of Ω which is strictly psh in a neighborhood Ω' of $\overline{\Omega}$.

We can find a $\mathcal{C}^{1,1}$-extension F of φ with compact support in $\mathbb{C}^n$ such that $\|F\|_{\mathcal{C}^{1,1}(\mathbb{C}^n)} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$. Replacing F by $F + A\rho$, with $A \gg 1$, we can further assume that F is plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$.

Applying the same process to the boundary data $-\varphi$ we choose a $\mathcal{C}^{1,1}$ psh function G in Ω' such that $G = -\varphi$ on $\partial\Omega$. Observe that F is a subsolution while the function $-G$ is a supersolution, hence $F \leq u \leq -G$ in Ω.

Since $F \leq u$ in Ω, the envelope u can be extended as a psh function in Ω' by setting $u = F$ in $\Omega' \setminus \Omega$. Fix $\delta > 0$ so small that $z + h \in \Omega'$ whenever $z \in \Omega$ and $\|h\| < \delta$. Fix $h \in \mathbb{C}^n$ such that $\|h\| < \delta$. Recall that F and G are Lipschitz, thus

$$|F(z+h) - F(z)| \leq C_1\|h\| \quad \text{and} \quad |G(z+h) - G(z)| \leq C_1\|h\|$$

for any $z \in \overline{\Omega}$.

Observe that the function $v(z) := u(z+h) - C_1\|h\|$ is well defined and psh in the open set Ω. If $z \in \partial\Omega$ and $z \in \Omega_h$ (i.e. $z + h \in \Omega$), then

$$v(z) = u(z+h) - C_1\|h\| \leq -G(z+h) - C_1\|h\| \leq -G(z) = \varphi(z) = u(z).$$

This shows that the function w defined by

$$w(z) := \begin{cases} \max\{v(z), u(z)\} & \text{if } z \in \Omega \cap \Omega_h \\ u(z) & \text{if } z \in \Omega \setminus \Omega_h \end{cases}$$

is plurisubharmonic in Ω. Since $w \leq \varphi$ on $\partial\Omega$ we get $w \leq u$ in Ω, hence $v \leq u$ in Ω. We have thus shown that

$$u(z+h) - u(z) \leq C_1 \|h\|$$

whenever $z \in \Omega \cap \Omega_h$, $\|h\| \leq \delta$ and $z \in \Omega_h$. Replacing h by $-h$ shows that $|u(z+h) - u(z)| \leq C_1 \|h\|$, which proves that u is Lipschitz on $\overline{\Omega}$. □

The $\mathcal{C}^{1,1}$-regularity in Ω of u_φ has been established by Krylov [Kry89] by a probabilistic approach based on controlled diffusion process, as advocated by Gaveau [Gav77]. We refer the reader to the notes by Delarue [Del09] (see Chap. 4) for an introduction to this point of view.

2.4.3 Further Results and Counterexamples

2.4.3.1 No More than $\mathcal{C}^{1,1}$

It is tempting to think that the envelope u_φ is $\mathcal{C}^\infty$-smooth when so is φ, as it is the case in dimension one. This fails when $n \geq 2$. The following example of Gamelin and Sibony shows that the envelope u_φ is not better than $\mathcal{C}^{1,1}$ even if φ is real analytic.

Example 2.13 *Let $\mathbb{B} \Subset \mathbb{C}^2$ be the open unit ball. For $(z,w) \in \partial\mathbb{B}$, set*

$$\varphi(z,w) := (|z|^2 - 1/2)^2 = (|w|^2 - 1/2)^2.$$

Observe that φ is real-analytic on $\partial\mathbb{B}$. We claim that

$$u_\varphi(z,w) = \max\{\psi(z), \psi(w)\}, \quad (z,w) \in \mathbb{B},$$

where

$$\psi(z) := \left(\max\{0, |z|^2 - 1/2\}\right)^2, \quad z \in \mathbb{C}.$$

Indeed denote by u the right hand side of the above formula. It has the right boundary values so we simply have to check that it is maximal. Now observe that if $(z,w) \in \mathbb{B}$ then either $|z|^2 < 1/2$ or $|w|^2 < 1/2$. In each case u depends only on one variable hence it is maximal. Therefore $u = u_\varphi$ and the reader will easily check that it is not $\mathcal{C}^2$-smooth.

It is perhaps worth mentioning that in the non degenerate case, the unique solution of the Dirichlet problem $MA(u) = dV$ (=volume form) with smooth boundary values φ is smooth, as was established by Caffarelli et al. [CKNS85]. The reader will find a detailed proof of this result in Boucksom's lecture [Bou09] (Chap. 7).

On the other hand when the domain is merely weakly pseudoconvex, the regularity theory breaks down dramatically as shown in [Co97, Li04].

2.4.3.2 Regularity Up to the Boundary

Looking carefully at the proof of Theorem 2.10, the reader will convince himself that the $\mathcal{C}^{1,1}$-norm of u_φ does not blow up faster than $1/\mathrm{dist}(\cdot, \partial\Omega)^2$ as one approaches the boundary.

It is expected that u_φ is $\mathcal{C}^{1,1}$-smooth up to the boundary when $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$. This has been established by Caffarelli et al. [CNS86] for the *real* homogeneous Monge–Ampère equation. The only known approach to the complex case is due to Krylov (see Delarue's lecture [Del09, Chap. 3]).

The following example (adaptation of an example in [CNS86]) shows that there is a necessary loss in the regularity up to the boundary:

Example 2.14 *Consider $u(z,w) = (1 + \Re(w))^{2\alpha}$, where $0 < \alpha < 1$. This is a plurisubharmonic function in the unit ball $\mathbb{B} \Subset \mathbb{C}^2$ which is smooth and maximal, continuous up to the boundary $\overline{\mathbb{B}}$, hence it coincides with u_φ for the boundary values*

$$\varphi(z,w) = (1 + \Re(w))^{2\alpha} \in Lip_{4\alpha}(\partial\mathbb{B})$$

The only problematic point is of course $(0, -1) \in \partial\mathbb{B}$.

Observe that $u = u_\varphi$ is only in $Lip_{2\alpha}(\overline{\mathbb{B}})$: this can be seen by a radial approach to the boundary point $(0, -1)$, while the tangential (boundary) approach allows to gain a factor 2.

2.4.3.3 Hölder Regularity

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. We have given above the proof due to Bedford and Taylor [BT76] that u_φ is Lipschitz on $\overline{\Omega}$ whenever φ is $\mathcal{C}^{1,1}$-smooth. In the same vein, these authors have shown that $u_\varphi \in \mathrm{Lip}_\beta(\overline{\Omega})$ is Hölder continuous on $\overline{\Omega}$ with exponent

$$\beta = \begin{cases} \frac{1+\alpha}{2} & \text{if } \varphi \in \mathcal{C}^{1,\alpha}(\partial\Omega), \quad 0 \le \alpha \le 1 \\[2ex] \frac{\alpha}{2} & \text{if } \varphi \in \mathrm{Lip}_\alpha(\partial\Omega), \quad 0 \le \alpha \le 1 \end{cases}$$

When Ω is merely weakly pseudoconvex, a similar result holds with a weaker exponent β when Ω is of "finite type" [Co97]. It has been moreover proved by Coman that this propagation of Hölder regularity characterizes finite type domains (see also [Li04]).

2.5 Dirichlet Problem in Domains of $\mathbb{C}^n$

In this section we apply Bedford–Taylor's result to show that the Perron–Bremermann envelope u_φ solves the Dirichlet problem DirMA(Ω, φ). Since u_φ is not very regular, this requires to first extend the definition of the complex Monge–Ampère operator.

2.5.1 Domain of Definition of MA

Let $\varphi \in PSH(\Omega)$ be a plurisubharmonic function. When φ is smooth, the Monge–Ampère measure $MA(\varphi)$ is absolutely continuous with respect to the Euclidean Lebesgue measure dV,

$$MA(\varphi) = (dd^c\varphi)^n := c \det\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}\right) dV,$$

for some normalizing constant $c > 0$. We would like to extend the definition of this operator and apply it to non smooth functions φ.

It is known that one can not define the Monge–Ampère measure $MA(\varphi)$ for any such function: Kiselman gives in [Kis83] an elementary example of a function $\varphi \in PSH(\mathbb{B})$ which is smooth but along some hyperplane H, hence $MA(\varphi)$ is well defined in $\mathbb{B} \backslash H$ but it has locally infinite mass near H.

Following Bedford and Taylor [BT82], we say that φ belongs to the domain of definition of the complex Monge–Ampère operator in Ω ($\varphi \in DomMA(\Omega)$) if for every $x \in \Omega$ and for every sequence φ_j of smooth and psh functions decreasing to φ in a neighborhood V_x of x, the sequence of positive measures $MA(\varphi_j)$ converges, in the weak sense of Radon measures, to a measure μ_φ independent of the sequence (φ_j). One then sets $MA(\varphi) := \mu_\varphi$.

Although this definition may seem cumbersome, this is precisely the way one usually computes derivatives in the sense of distributions. It is moreover motivated by the following result established by Bedford and Taylor in [BT82].

Theorem 2.15 *$PSH \cap L^\infty_{loc}(\Omega) \Subset DomMA(\Omega)$.*

Thus the complex Monge–Ampère operator is well defined for psh functions that are locally bounded, which is what we basically needs here since u_φ is continuous. It follows straightforwardly from the definition that the operator *MA* is continuous along decreasing sequences.

More involved is the continuity along increasing sequences which was also established by Bedford and Taylor in [BT82]. Note however that MA is discontinuous along non monotonic sequences. We propose one example as an exercise for the reader.

Exercise 2.16 *Set* $\varphi_j(z,w) = \frac{1}{2j}\log[1+|z^j+w^j|^2]$.

1) Verify that the functions φ_j *are smooth, psh, with* $MA(\varphi_j)=0$ *in* $\mathbb{C}^2$.
2) Show that (φ_j) *converges in* $L^1_{loc}(\mathbb{C}^2)$ *towards*

$$\varphi(z,w) = \log\max(1,|z|,|w|) \in PSH \cap L^\infty_{loc}(\mathbb{C}^2)$$

and verify that $MA(\varphi)$ *is the Lebesgue measure on the real torus* $S^1\times S^1$.

When $n=2$, it was observed by Bedford and Taylor in [BT78] that one can define $MA(\varphi)$ as soon as $\nabla\varphi \in L^2_{loc}(\Omega)$. In this case $d\varphi\wedge d^c\varphi$ is well defined hence so is the current $\varphi dd^c\varphi$ (by integration by parts) and one can thus set

$$MA(\varphi) := dd^c(\varphi dd^c\varphi)$$

where the derivatives are taken in the sense of distributions (currents). It turns out in this case that if φ_j are smooth, psh, and decrease to φ, then φ_j converge to φ in the Sobolev norm $W^{1,2}_{loc}$. It was recently shown by Blocki [Bl04] that one can not make sense of $MA(\varphi)$ when $n=2$ and $\nabla\varphi\notin L^2_{loc}(\Omega)$.

2.5.2 The Comparison Principle

The comparison principle is one of the most effective tools in pluripotential theory. It is a non linear version of the classical maximum principle. The central result, again due to Bedford and Taylor [BT87] is the following:

Theorem 2.17 *Let* u,v *be locally bounded psh functions in a domain* $\Omega\Subset\mathbb{C}^n$. *Then*

$$\mathbf{1}_{\{u>v\}}(dd^c\max\{u,v\})^n = \mathbf{1}_{\{u>v\}}(dd^c u)^n,$$

in the sense of Borel measures in Ω.

Proof. Set $D := \{u>v\}$. Observe that if u is continuous then the set D is an open subset of Ω and $\max\{u,v\}=u$ in D. Therefore we have

$$(dd^c\max\{u,v\})^n = (dd^c u)^n,$$

weakly in the open set D, as desired.

The general case proceeds by approximation: one can approximate u from above by a decreasing sequence of psh continuous functions (by local convolutions) and it suffices to establish fine convergence results in order to pass to the limit. These convergence results are of course the hard technical part of the argument and will not be reproduced here. Let us simply mention that the key properties for these to hold is that u is *quasicontinuous*, i.e. it coincides with a continuous function on a set of arbitrary large size with respect to the Monge–Ampère measures involved. □

We derive from this identity two corollaries which are often called "maximum principle" in the literature.

Corollary 2.18 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain and let u, v be locally bounded psh functions such that $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \geq 0$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Proof. Since $\{u - \varepsilon < v\} \nearrow \{u < v\}$ as $\varepsilon \searrow 0$, we can assume that $\liminf_{z\to\partial\Omega}(u(z) - v(z)) > \varepsilon > 0$. We can thus fix a compact subset $K \Subset \Omega$ such that $u(z) - v(z) > \varepsilon$ on $\Omega \setminus K$. Therefore $\max\{u, v\} = u$ on $\Omega \setminus K$.

We infer the following "mass conservation property",

$$\int_\Omega (dd^c \max\{u, v\})^n = \int_\Omega (dd^c u)^n.$$

Indeed set $w := \max\{u, v\}$ and observe that $(dd^c w)^n - (dd^c u)^n = dd^c S$ weakly in the sense of currents on Ω, where $S := w(dd^c w)^{n-1} - u(dd^c u)^{n-1}$. Since $w = u$ on $\Omega \setminus K$, it follows that $S = 0$ in the open set $\Omega \setminus K$ thus the support of the current $dd^c S$ is contained in K. Taking a smooth test function χ on Ω such that $\chi \equiv 1$ in a neighborhood of K, we conclude that $\int_\Omega dd^c S = \int_\Omega \chi dd^c S = \int_\Omega S \wedge dd^c \chi = 0$, since $dd^c \chi = 0$ on the support of current S.

The mass conservation property together with Theorem 2.17 yields

$$\begin{aligned}
\int_{\{u<v\}} (dd^c v)^n &= \int_{\{u<v\}} (dd^c \max\{u, v\})^n \\
&= \int_\Omega (dd^c \max\{u, v\})^n - \int_{\{u\geq v\}} dd^c \max\{u, v\})^n \\
&\leq \int_\Omega (dd^c u)^n - \int_{\{u>v\}} (dd^c \max\{u, v\})^n \\
&= \int_\Omega (dd^c u)^n - \int_{\{u>v\}} (dd^c u)^n = \int_{\{u\leq v\}} (dd^c u)^n.
\end{aligned}$$

We have thus shown that $\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u\leq v\}} (dd^c u)^n$. Replacing u by $u - \varepsilon$ and letting ε decrease to zero yields the desired inequality. □

Corollary 2.19 *Let u, v be locally bounded psh functions in a bounded domain $\Omega \Subset \mathbb{C}^n$ such that* $\liminf_{z\to\partial\Omega}(u(z)-v(z)) \geq 0$. *Then*

$$(dd^c u)^n \leq (dd^c v)^n \Longrightarrow v \leq u \text{ in } \Omega.$$

Proof. Define for $\varepsilon > 0$, $v_\varepsilon := v + \varepsilon\rho$, where $\rho(z) := \|z\|^2 - R^2$ is chosen so that $\rho < 0$ in Ω. Observe that $\{u < v_\varepsilon\} \Subset \{u < v\} \Subset \Omega$. It follows therefore from the previous corollary that

$$\int_{\{u<v_\varepsilon\}} (dd^c v_\varepsilon)^n \leq \int_{\{u<v_\varepsilon\}} (dd^c u)^n.$$

Since $(dd^c v_\varepsilon)^n \geq (dd^c v)^n + \varepsilon^n (dd^c\rho)^n > (dd^c u)^n$, we infer $\int_{\{u<v_\varepsilon\}} (dd^c\rho)^n = 0$. This means that the sets $\{u < v_\varepsilon\}$ all have Lebesgue measure zero, $\varepsilon > 0$. Since $\{u < v\} = \bigcup_{j\geq 1}\{u < v_{1/j}\}$, it follows that the set $\{u < v\}$ also has Lebesgue measure 0 so that $v \leq u$ in Ω by the submean value inequality. □

2.5.3 Characterization of Maximal Plurisubharmonic Functions

Theorem 2.20 *A function $u \in PSH \cap L^\infty_{loc}(\Omega)$ is maximal if and only if $MA(u) = 0$. In particular the Perron–Bremermann envelope u_φ satisfies $MA(u_\varphi) = 0$ hence it is the unique solution to the Dirichlet problem* DirMA(Ω, φ).

Proof. If $(dd^c u)^n = 0$ on Ω, it follows from the comparison principle that u is a maximal plurisubharmonic function on Ω.

Conversely assume that u is maximal on Ω and let $\mathbb{B} \Subset \Omega$ be an Euclidean ball. Let φ be the restriction of u to the boundary $\partial\mathbb{B}$. Since u is maximal, it coincides with the Perron-Bremmerman envelope $u = u_\varphi$ with respect to the domain $\mathbb{B}$.

Let (φ_j) be a decreasing sequence of $\mathcal{C}^2$-smooth functions on $\partial\mathbb{B}$ which converges to φ on the boundary $\partial\mathbb{B}$. We let the reader check that $u_j := u_{\varphi_j}$ decreases to $u = u_\varphi$. By Bedford–Taylor's result, u_j is $\mathcal{C}^{1,1}(\mathbb{B})$, hence it satisfies $(dd^c u_j)^n = 0$ on $\mathbb{B}$ by Lemma 2.21 below. Since the Monge–Ampère operator is continuous along decreasing sequences we infer $(dd^c u)^n = 0$ in $\mathbb{B}$. Since $\mathbb{B}$ was arbitrary this yields $(dd^c u)^n = 0$ in all of Ω. □

It remains to check that regular maximal functions have zero Monge–Ampère measure.

Lemma 2.21 *Let $u : \Omega \to \mathbb{R}$ be a maximal plurisubharmonic function. If u is $\mathcal{C}^{1,1}$-smooth then $MA(u) = 0$.*

Proof. It follows from Lemma 2.11 that u admits second derivatives at almost every point and that its Monge–Ampère measure $MA(u)$ is absolutely continuous with respect to Lebesgue measure, with density defined almost everywhere by $\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$. We are going to show that the latter is zero whenever defined.

The second order Taylor expansion of u at z_0 gives,

$$u(z_0 + h) = \Re P(h) + \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0) h_j \bar{h}_k + o(\|h\|^2),$$

where

$$P(h) := u(z_0) + 2 \sum_j \frac{\partial u}{\partial z_j}(z_0) h_j + \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial z_k}(z_0) h_j h_k.$$

Assume that $\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0)\right) > 0$. Then there exists $c > 0$ and $r > 0$ small enough such that for $\|h\| = r$, we have $u(z_0 + h) = \Re P(h) + c\|h\|^2 > \Re P(h)$. Therefore the function $v(z) := \Re P(z_0 + z)$ is a plurisubharmonic function such that $v(z_0) = u(z_0)$ and $v(z) < u(z)$ on the boundary of the ball $\mathbb{B}(z_0, r)$, which contradicts the fact that u is maximal on Ω. □

Remark 2.22 *One can similarly show that a psh function φ which belongs to the domain of definition of the complex Monge–Ampère operator is maximal if and only if $MA(\varphi) = 0$ [Bl04].*

Chapter 3
Geometric Properties of Maximal psh Functions

Romain Dujardin and Vincent Guedj

Abstract We review the geometric properties of maximal plurisubharmonic functions u and of the associated closed positive currents $dd^c u$, in two complex dimensions. When u is regular enough (at least of class $\mathcal{C}^3$), $dd^c u$ is "foliated" by Riemann surfaces along which u is harmonic. On the other hand when u is less than $\mathcal{C}^{1,1}$ this picture breaks down completely, as recent examples show.

3.1 Introduction

The goal of this lecture is to study the geometric properties of *maximal plurisubharmonic functions*. A plurisubharmonic function u defined in a domain $\Omega \subset \mathbb{C}^n$ is maximal if for all plurisubharmonic function v defined in $U \subset\subset \Omega$, the following holds:

$$v \leq u \text{ on } \partial U \Longrightarrow v \leq u \text{ in } U.$$

In dimension $n = 1$, a maximal psh function u is harmonic. In higher dimension $n \geq 2$, maximality is characterized by the non linear PDE $(dd^c u)^n = 0$, as explained in Zeriahi's lecture (Chap. 2).

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We seek here for a geometric understanding of the maximality property of a given psh function u. If one can fill out a small neighborhood of any point $x \in \Omega$ by holomorphic disks along which u is harmonic, then u is certainly maximal. When u is regular enough, this situation actually always happen, and follows from the Frobenius Integrability Theorem as we explain in *Sect. 3.2*. In lower regularity, this is not anymore the case as shown by a striking construction due to the first author (see Sect. 3.5). It seems that the critical regularity should be $\mathcal{C}^{1,1}$, as we will try to explain. In order to simplify the exposition, we restrict ourselves throughout this note to the two dimensional situation $n = 2$.

Nota Bene. These notes are written by Romain Dujardin and Vincent Guedj and grew up from a lecture delivered by Romain Dujardin in Marseille in March 2009. Most results are standard, except for the last section which briefly explains a recent result of the first author [Duj09]. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

3.2 Monge–Ampère Foliations

3.2.1 Preliminaries on Currents

For this paragraph the reader is referred to Demailly's book [Dembook]. Recall that a *current* of bidimension (p, q) in a domain $\Omega \subset \mathbb{C}^n$ is a continuous linear form on the space of smooth differential forms with compact support (test forms) of bidegree (p, q). It can be canonically identified with a differential form of bidegree $(n-p, n-q)$ with distribution coefficients. Since we are primarily interested in this case, *from now on we make the assumption that $p = q = 1$ and $n = 2$*, that is, we work with currents of bidegree $(1, 1)$ in a domain of $\mathbb{C}^2$.

Such a current T is *closed* if $\langle T, d\eta \rangle = 0$ for every test form of degree 1. The current T is *positive* if $\langle T, \theta \rangle \geq 0$ for every positive test form of bidegree $(1, 1)$. A test form is positive if it belongs to the closed convex set generated by forms of the type $\chi\omega$ where χ is a positive test function and ω is a Kähler form. Thus, by continuity, checking the positivity of a current T amounts to verify that $\langle T, \chi\omega \rangle \geq 0$, for all $\chi \geq 0$ and $\omega > 0$. A smooth current $T = \sum T_{pq} i dz_p \wedge d\overline{z}_q$ is positive if and only if (T_{pq}) is a non negative hermitian matrix at every point.

Any positive closed current of bidegree $(1, 1)$ is locally given as $T = dd^c u$, where u is a plurisubharmonic function, $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$ and the derivatives are taken in a weak sense.

Here are two fundamental examples of such currents:

Example 3.1 *If V is a (closed) complex curve in a domain $\Omega \subset \mathbb{C}^2$, let us denote by $[V]$ the current of integration along V, defined by*

$$\langle [V], \theta \rangle := \int_V \theta.$$

It is immediate to check that this is a well defined positive closed $(1,1)$-current in Ω.

It is a classical result due to Lelong that when V is merely an analytic subset of (complex) dimension one, it is still possible to consider $[V]$ by integrating along the regular points $\mathrm{Reg(V)}$ *of V. This requires to show that the current of integration along* $\mathrm{Reg(V)}$ *has locally finite mass near* $\mathrm{Sing(V)}$ *and that the resulting current (extended by zero through* $\mathrm{Sing(V)}$*) is still closed.*

An important result of Siu asserts that these currents are extremal points of the convex cone of all positive closed $(1,1)$-currents. More precisely, any positive closed $(1,1)$-current supported on an irreducible complex curve V is a (positive) multiple of $[V]$.

Example 3.2 *Let ω be a Kähler form in Ω, e.g. $\omega = dd^c\rho$ where ρ is a smooth strictly plurisubharmonic function in Ω. Then ω defines a positive closed current of bidegree $(1,1)$ by setting*

$$\langle \omega, \theta \rangle := \int_\Omega \theta \wedge \omega.$$

Both families are dense in the cone of positive closed currents of bidegree $(1,1)$: one can regularize psh functions (using standard convolutions), add $\varepsilon||z||^2$ and hence approximate any positive closed current by Kähler forms. Likewise, any plurisubharmonic function u is the limit in L^1_{loc} of rational multiples of $\log|f_j|$, f_j holomorphic functions, as follows for instance from Hörmander's L^2-estimates. Thus, the current $T = dd^c u$ is the weak limit of (rational multiples of) the currents of integration along the analytic sets $\{f_j = 0\}$.

One important aspect of $(1,1)$-positive closed currents is that it is often possible to wedge them. Our primary interest here is on *self-intersections*. If u is a psh function s.t. $\nabla u \in L^2_{\mathrm{loc}}$, then $(dd^c u)^2$ is a well-defined positive measure. Indeed $du \wedge d^c u$ is well defined by assumption, hence so is $udd^c u$ (integrate by parts). One thus defines $(dd^c u)^2 := dd^c(u\, dd^c u)$. Blocki [Bl04] has shown that the Monge–Ampère measure $(dd^c u)^2$ cannot be reasonably defined when $\nabla u \notin L^2_{loc}$.

Abusing terminology, we say that a $(1,1)$-positive closed current $T = dd^c u$ in $\mathbb{C}^2$, with $\nabla u \in L^2_{\mathrm{loc}}$, is *maximal* when $T \wedge T = (dd^c u)^2 = 0$. Of course this notion does not depend on the choice of the potential u.

3.2.2 Foliated Cycles

Recall that a $\mathcal{C}^k$ *foliation* of a domain $\Omega \subset \mathbb{C}^2$ by complex leaves is given by a covering (Ω_α) by *coherent foliated charts* (or *flow boxes*), that is, each Ω_α is provided with a $\mathcal{C}^k$-diffeomorphism

$$\phi_\alpha : \mathbb{D}_z \times \mathbb{D}_w \longrightarrow \Omega_\alpha$$

which is holomorphic in the z coordinate. Coherence here means that the transition maps between charts preserve the "plaques" $\{w = C^{st}\}$. By definition, a connected immersed submanifold L is a *leaf* if for each α, $L \cap \Omega_\alpha$ is a union of plaques. A *transversal* to the foliation is a piece of submanifold which is transverse to the leaves.

All the problems we consider in these notes are local, so most often we restrict to a single foliated chart. We denote by $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha\in\mathbb{D}}$ the corresponding family of leaves. Abusing slightly, the notation $\mathcal{L}$ will also refer to the foliation itself.

In this section we assume that $k \geq 1$ (while in the next section we focus on the case $k = 0$), in which case an alternative definition is that $\mathcal{L}$ is defined as the integral curves of the kernel of the differential form $\phi_*(dw \wedge d\bar{w})$.

Given a probability measure μ on $\mathbb{D}_w$, let us consider the current

$$T_\mu := \int [\mathcal{L}_\alpha] d\mu(\alpha).$$

This is a geometric current of bidegree $(1,1)$ which is *positive* and *closed*: it acts on a smooth form θ of bidegree $(1,1)$ by integrating along each leaf and averaging against μ,

$$\langle T, \theta \rangle = \int \left(\int_{\mathcal{L}_\alpha} \theta \right) d\mu(\alpha),$$

the result being nonnegative if θ is positive and zero if θ is exact.

We want to define a *foliated cycle* on a foliation as a positive closed current which is "locally of the above form". Passing from local to global here requires a little bit of care. An *invariant transverse measure* is the data of a locally finite measure on each transversal, which is invariant under holonomy (that is, transport along the leaves). We see that in a given coordinate chart $\mathbb{D}_z \times \mathbb{D}_w$, it is determined by its value μ on $\mathbb{D}_w$, and gives rise to a well-defined current T_μ as above.

Definition 3.3 *A positive closed current of bidegree* $(1,1)$ *in a domain* $\Omega \subset \mathbb{C}^2$ *is called a foliated cycle if there exists a foliation of* Ω *by complex curves and an invariant transverse measure whose* T *is the associated current.*

This concept was introduced by Sullivan [Su76] and has proved to be of fundamental importance in the theory of foliations.

Observe that foliated cycles interpolate between smooth differential forms and currents of integration: if $\mathcal{L}$ is the foliation by horizontal lines $\{w = cst\}$ and μ is the Dirac mass at the origin, then $T_\mu = [w = 0]$ is the current of integration along the complex line $(w = 0)$, while if μ is the Lebesgue measure then $T_\mu = c idw \wedge d\overline{w}$ is smooth.

It is an easy fact that foliated cycles are maximal currents:

Proposition 3.4 *If T is a $\mathcal{C}^1$-smooth foliated cycle, then $T \wedge T = 0$.*

Proof. Working in a foliated chart, we can assume that $\mathcal{L}_\alpha = \{w = \alpha\}$. The current of integration along $\mathcal{L}_\alpha$ can be formally written as $[\mathcal{L}_\alpha] = \delta_\alpha idw \wedge d\overline{w}$ so that $T_\mu = \chi idw \wedge d\overline{w}$ for some $\mathcal{C}^1$-smooth function χ. Elementary calculus on differential forms then shows that $T \wedge T = 0$. □

It turns out that the converse is also true, as follows from the Frobenius Integrability Theorem.

Theorem 3.5 *Let T be a $\mathcal{C}^1$-smooth positive closed differential form of bidegree $(1,1)$, satisfying the equation $T \wedge T = 0$. Then there exists a foliation by complex curves on the interior of $\operatorname{supp}(T)$ and T is a foliation cycle associated to this foliation.*

The foliation by complex curves induced by T in the interior of its support is called a *Monge–Ampère foliation*. The reader is referred to [BK77] for a thorough discussion on the connections between Monge–Ampère equations and foliations.

Proof. Since T is a positive differential form of bidegree $(1,1)$, it can be decomposed as

$$T = \sum_{p,q=1}^{2} T_{pq} \frac{i}{2} dz_p \wedge d\overline{z}_q,$$

where (T_{pq}) is a nonnegative hermitian matrix at every point. Now

$$T \wedge T = c \det(T_{pq})(idz_1 \wedge d\overline{z}_1) \wedge (idz_2 \wedge d\overline{z}_2) = 0$$

hence (T_{pq}) has complex rank ≤ 1 in general and exactly 1 at interior points of the support of T. This shows that $\ker T$ defines a $\mathcal{C}^1$ distribution of complex lines on the interior of $\operatorname{supp}(T)$. Notice that, having continuous coefficients, T gives zero mass to the boundary of its support.

Let Ω be an open subset of the support of T. We would like to show that the complex lines defined by $\ker T$ are tangent to a $\mathcal{C}^1$-foliation in Ω. By the Frobenius theorem, we need to check that the distribution $\ker T$ is *involutive*, i.e. for every pair of smooth vector fields X, Y belonging to $\ker T$, then

$[X, Y] \in \ker T$. Let Z be any vector field. By standard calculus on differential forms (see e.g. [GHL, p.44]), and using the fact that T is closed we get that

$$\begin{aligned} 0 &= dT(X, Y, Z) \\ &= X(T(Y, Z)) - Y(T(X, Z)) + Z(T(X, Y)) - T([X, Y], Z) \\ &\quad -T(Y, [X, Z]) + T(X, [Y, Z]), \end{aligned}$$

whence $T([X, Y], Z) = 0$, so $[X, Y] \in \ker T$, which was the desired result. The leaves are complex curves because their tangent space is complex at every point.

Let us denote these curves by $\mathcal{L}_\alpha$ and show that T is an average of currents of integration along the $\mathcal{L}_\alpha$. We provide a proof which is not the most simple, but carries over to a wider context (see e.g. [Duj06]).

Working in a single flow box again, the space of leaves is compact for the topology of currents. By the Choquet Integral Representation Theorem, it is enough to prove that T belongs to the closed convex cone generated by the $\mathcal{L}_\alpha$'s.

Assume this is not the case. It then follows from the Hahn–Banach theorem that there exists a test form θ such that $\langle T, \theta\rangle > 0$ while $\langle [\mathcal{L}_\alpha], \theta\rangle \leq -1$ for all α.

Fix a transversal τ to the foliation and let $\{\chi_i\}$ be a partition of unity subordinate to an open covering of τ by open sets of diameter $\leq 1/2$. We extend the functions χ_i to Ω by making them constant along each leaf. Since $T = \sum \chi_i T$, there exists i_0 such that $\langle \chi_{i_0} T, \theta\rangle > 0$. Set $T_1 := \chi_{i_0} T / ||\chi_{i_0} T||$.

Observe that if χ is any function which is constant along the leaves, then χT is closed. Indeed there are C^1 real coordinates[1] (x_1, x_2, x_3, x_4) in which the leaves are defined by the equations $\{x_3 = a_3,\ x_4 = a_4\}$, and T is a multiple of $dx_3 \wedge dx_4$. It is then clear that if χ depends only on (x_3, x_4), $d\chi \wedge T = 0$. In particular T_1 is closed.

Now we repeat the above procedure, that is we consider a covering of $\operatorname{supp}(\chi_{i_0})$ by open sets of diameter $\leq 1/4$ and an associated partition of unity, and build a current $T_2 = \chi_{1,i_1} T_1 / ||\chi_{1,i_1} T_1||$ of mass 1 such that

$$\langle T_2, \theta\rangle > 0 \text{ and } \langle [\mathcal{L}_\alpha], \theta\rangle \leq -1 \text{ for all } \alpha.$$

We thus inductively obtain a sequence of positive closed currents T_n of bidegree $(1, 1)$ and mass 1 whose support is contained in an arbitrarily small neighborhood of a leaf $\mathcal{L}_{\alpha_n}$, and satisfying $\langle T_n, \theta\rangle > 0$. Any cluster point σ of (T_n) is supported on a leaf $\mathcal{L}_{\alpha_0}$ hence coincides with the current of integration along $\mathcal{L}_{\alpha_0}$. By definition of θ, $\langle \sigma, \theta\rangle \leq -1$, but on the other hand, being a limit of (T_n), $\langle \sigma, \theta\rangle \geq 0$ and this contradiction finishes the proof. □

[1] Caution is in order here because positivity makes no sense in these coordinates.

3.2.3 Geometric Maximality

Our main interest in these series of lectures is the understanding of various Dirichlet problems for complex Monge–Ampère operators. Recall from [GZ09] that a continuous psh function u is maximal in a domain $\Omega \subset \mathbb{C}^2$ if and only if it satisfies $(dd^c u)^2 = 0$. In dimension 1, maximal functions are harmonic hence very regular. This is not necessarily the case in higher dimension, however the expectation is that they should be somehow harmonic along a foliation by complex curves. We can reformulate Theorem 3.5 in this spirit:

Theorem 3.6 *Let u be a maximal plurisubharmonic function in some domain $\Omega \subset \mathbb{C}^2$. If $u \in \mathcal{C}^3(\Omega)$ then $T = dd^c u$ is a foliated cycle in the interior of* Supp T *and u is harmonic along the leaves of the Monge–Ampère foliation.*

A natural question is therefore to understand what happens when u is less regular. This will be the subject of the remaining sections. Notice that this type of question was already raised at the very beginnings of pluripotential theory (see e.g. [B88]).

3.3 Laminations and Laminar Currents

It is natural to expect that a result like Theorem 3.6 should hold under weaker regularity assumptions on u. This leads to the concepts of uniformly laminar and laminar currents. Although it is not clear how to extend the Frobenius Theorem to weaker regularity, it is still interesting to explore the properties of these objects, which have recently played an important role in the context of complex dynamics.

3.3.1 Laminations and Holomorphic Motions

An *embedded lamination* by complex curves is given by a covering of a closed subset $X \subset \Omega$ by a coherent system of flow boxes

$$\phi_\alpha : \mathbb{D}_z \times K \longrightarrow \Omega_\alpha$$

which are holomorphic along the leaves (i.e. in the z coordinate) and only continuous in the transverse direction (the w coordinate). Here K denotes a compact subset of $\mathbb{D}_w$.

It follows from a celebrated result of Mañé–Sad–Sullivan [MSS83] that the assumption that K is closed is superfluous, and that ϕ_α is automatically Hölder continuous.

Proposition 3.7 *Fix $K \subset \mathbb{D}$ an arbitrary subset and let $\{\mathcal{L}_\alpha\}_{\alpha\in K}$ be a bounded family of disjoint graphs over K, $\mathcal{L}_\alpha := \{(z,w) \in \mathbb{D}^2 \,/\, w = f_\alpha(z)\}$, where f_α is a holomorphic function in $\mathbb{D}$ with $f_\alpha(0) = \alpha$. Then*

(1) $\{\mathcal{L}_\alpha\}_{\alpha\in K}$ extends uniquely to a family of disjoint graphs parameterized by $\overline{K}$;
(2) the collection $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha\in\overline{K}}$ form a lamination, in the sense that the holonomy map $\{0\}\times\mathbb{D} \to \{z\}\times\mathbb{D}$ is automatically continuous.

Laminations by graphs over the unit disk are often called *holomorphic motions*

Proof. Assume without loss of generality that $|f_\alpha| < 1$. Then for $\alpha \neq \beta$, the function $h = h_{\alpha,\beta} := -\log(|f_\alpha(z) - f_\beta(z)|/2)$ is harmonic and positive in the unit disk $\mathbb{D}$. It therefore follows from the Harnack inequality that

$$\frac{1-|z|}{1+|z|}h(0) \le h(z), \text{ for all } z \in \mathbb{D}.$$

Since $f_\alpha(0) = \alpha$, we infer that

$$|f_\alpha(z) - f_\beta(z)| \le 2\left(\frac{|\alpha-\beta|}{2}\right)^{\frac{1-|z|}{1+|z|}}. \tag{3.1}$$

This shows that the holonomy map $\{0\}\times\mathbb{D} \to \{z\}\times\mathbb{D}$ is locally uniformly Hölder continuous.

Since the family (f_α) is bounded, it thus uniquely extends to $\overline{K}$. For $\alpha \in \overline{K}$ we simply set $f_\alpha = \lim f_{\alpha_n}$, where α_n is any sequence converging to α (by the previous Hölder estimate, the limiting map f_α is independent of the choice both of the cluster point of f_{α_n} and of the sequence (α_n)). Note finally that by Rouché's Theorem, the extended family is still a family of disjoint graphs. □

Remark 3.8 *Concluding of long series of previous works, Slodkowski [Slod91] proved that any holomorphic motion of a compact subset $K \subset \mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$. By [MSS83] the holonomy is then a quasiconformal mapping $\mathbb{C} \to \mathbb{C}$. Conversely any such quasiconformal map can be realized as the holonomy map (at time $t = 1$) of a lamination by disjoint graphs. To find such a holomorphic motion, one simply takes the Beltrami coefficient of this quasiconformal map and multiplies it by the complex parameter t. The interested reader will find more information on the topic in [AM01].*

3.3.2 Uniformly Laminar Currents

Uniformly laminar currents are the natural generalization of foliated cycles to laminations:

Definition 3.9 *A positive closed* $(1,1)$*-current* T *in a domain* $\Omega \subset \mathbb{C}^2$ *is called uniformly laminar if there exists an embedded lamination* $\mathcal{L}$ *in* Ω *such that in any flow box* $\{\mathcal{L}_\alpha\}_{\alpha\in K}$*,* T *has the form*

$$T = \int_{\alpha\in K} [\mathcal{L}_\alpha]\, d\mu(\alpha),$$

for some positive measure μ *on* K*.*

We say that T is *subordinate* to the lamination $\mathcal{L}$. Globally speaking, a uniformly laminar current induces an invariant transverse measure on the lamination to which it is subordinate. The following result of Demailly [Dem82] provides an interesting class of uniformly laminar currents.

Proposition 3.10 *Let* T *be a positive closed current supported on a* C^1 *Levi-flat hypersurface* Σ *in* $\Omega \subset \mathbb{C}^2$*. Then* T *is uniformly laminar.*

Recall that in $\mathbb{C}^2$ a Levi flat hypersurface is a real hypersurface which is foliated by holomorphic curves. If a Levi-flat hypersurface carries a positive closed current, then its underlying foliation possesses an invariant transverse measure.

Proof. We treat the case when the hypersurface Σ is real-analytic. For the general case where Σ is merely C^1 we refer the reader to the "second theorem of support" in [Dem93].

A real-analytic Levi flat hypersurface in $\mathbb{C}^2$ is locally of the form $\Re(z) = 0$ in some holomorphic system of coordinates (z, w). We work locally so Σ writes as $\Re(z) = 0$. Put $z = x + iy$. Then since $x = 0$ on $\operatorname{supp}(T)$ and T has measure coefficients, we infer that $xT = 0$. Since T is ∂ and $\overline{\partial}$ closed, we infer that $\partial(xT) = dz \wedge T = 0$ and $\overline{\partial}(xT) = d\overline{z} \wedge T = 0$. Now if χ is a test function which is constant along the leaves, that is, which depends only on z along $\operatorname{supp}(T)$, we infer that χT is closed. We conclude as in Theorem 3.5. □

We now prove that the potentials of uniformly laminar currents are maximal.

Exercise 3.11 *Let* $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha\in K} \subset \Omega$ *be a lamination by disjoint graphs,* $\mathcal{L}_\alpha = \{(z, w) \in \mathbb{D} \times \mathbb{C} \,/\, w = f_\alpha(z)\}$*. Fix a probability measure* μ *on* $K \subset \mathbb{C}$ *and consider* $T = T_\mu$ *the corresponding uniformly laminar current. Check that* $T = dd^c u$*, where*

$$u(z, w) = \int_{\alpha\in K} \log|w - f_\alpha(z)| d\mu(\alpha).$$

Proposition 3.12 *If* $T = dd^c u$ *is a uniformly laminar current such that* $\nabla u \in L^2_{loc}$ *then*

$$T \wedge T = (dd^c u)^2 = 0.$$

Proof. We work in a flow box. With notation as in Exercise 3.11, fix $\beta \in K$. Note that μ has no atom since $\nabla u \in L^2_{loc}$ thus

$$u(z,w) = \int_{\alpha \in K \setminus \{\beta\}} \log |w - f_\alpha(z)| d\mu(\alpha)$$

is harmonic on $\mathcal{L}_\beta$ as an average of harmonic functions, or identically $-\infty$. Since u is locally integrable with respect to $dd^c u$, $u_{|\mathcal{L}_\beta}$ is harmonic on μ-almost every leaf. We infer that $T \wedge [\mathcal{L}_\beta] = 0$, whence

$$T \wedge T = \int_{\beta \in K} T \wedge [\mathcal{L}_\beta]\, d\mu(\beta) = 0.$$

□

Laminar currents which are made up of graphs over the unit disk have the following important compactness property (this precise version is taken from [BS98]).

Proposition 3.13 *Let T_n be a sequence of uniformly laminar currents in $\mathbb{D}^2$, respectively subordinate to a sequence of laminations $\mathcal{L}_n$ by graphs over the unit disk. Assume that T_n converges to T. Then $(\mathcal{L}_n)$ converges to a limit lamination $\mathcal{L}$ and T is subordinate to $\mathcal{L}$.*

What we mean by convergence for the sequence of laminations $\mathcal{L}$ is the following. Fix the transversal $\{0\} \times \mathbb{D}$, and denote by $\mathcal{L}_{n,\alpha}$ the leaf of $\mathcal{L}_n$ through $(0,\alpha)$. Let $K_n = \mathcal{L}_n \cap (\{0\} \times \mathbb{D})$. We say that $\mathcal{L}_n$ converges to $\mathcal{L}$ if for every $\alpha \in \limsup K_n$ there exists a unique graph $\mathcal{L}_\alpha$ through $(0,\alpha)$ such that if $\alpha_n \in K_n$ is any sequence converging to α, $\mathcal{L}_{n,\alpha}$ converges to $\mathcal{L}_\alpha$.

Proof. Of course a family of uniformly bounded graphs over the unit disk is compact relative to the compact-open topology. Write

$$T_n = \int [\mathcal{L}_{n,\alpha}] d\mu_n(\alpha),$$

with μ_n a positive measure supported in $\{0\} \times \mathbb{D}$. Likewise, denote by $\mu_n(z)$ the measure induced by holonomy on $\{z\} \times \mathbb{D}$. It can also be expressed as $\mu_n = T_n \wedge [\{z\} \times \mathbb{D}]$. Since the currents T_n have locally uniformly bounded mass, so do the μ_n. Restricting μ_n to $\{0\} \times D(0, 1-\varepsilon)$ if necessary, we can always assume that its mass is uniformly bounded, say by 1.

Since $T_n \to T$ it is a basic consequence of Slicing Theory that for Lebesgue a.e. z, $\mu_n(z)$ converges to some $\mu(z)$. Now for every test function $\varphi(w)$, by the Hölder continuity property (3.1) the family $\int \varphi d\mu_n(z)$ is equicontinuous in z so we get that $\mu_n(z)$ converges for all z, in particular at $z = 0$.

It remains to prove that the laminations $\mathcal{L}_n$ converge in the previous sense to some lamination $\mathcal{L}$. It will then be clear that the T_n converge to $T = \int [\mathcal{L}_\alpha] d\mu(\alpha)$.

Let $\mathcal{L}$ be the set of graphs over $\mathbb{D}$ which are cluster values of $\mathcal{L}_{n_j,\alpha_j}$ for some subsequence (n_j). We need to show that the graphs of $\mathcal{L}$ do not intersect. Then by Proposition 3.7, they will form a lamination. Notice that this is more subtle than just applying the Hurwitz Theorem because the leaves of different $\mathcal{L}_n$ can intersect. We use the convergence of currents instead.

It suffices to show the following fact : "if a sequence $\mathcal{L}_{n_j,\alpha_j}$ satisfies $\alpha_j \to \alpha \in \operatorname{supp}(\mu)$ then the sequence converges". Suppose this is not the case: then there exists two subsequences $\mathcal{L}_{n^i_j,\alpha^i_j} \to \mathcal{L}^i_\alpha$, $i = 1, 2$, and $\mathcal{L}^1_\alpha \neq \mathcal{L}^2_\alpha$. If α is not an atom of μ, we can assume that $\mathcal{L}^1_\alpha$ and $\mathcal{L}^2_\alpha$ are transverse at $(0, \alpha)$: if not we can move the $\mathcal{L}_{n^1_j,\alpha^1_j}$ slightly so that they will converge to a $\mathcal{L}^1_\beta$ close to $\mathcal{L}^1_\alpha$ and disjoint from it, by the Hurwitz Theorem. Then $\mathcal{L}^1_\beta$ and $\mathcal{L}^2_\alpha$ intersect transversely (see [BLS93, Lemma 6.4]).

Now all the graphs near $\mathcal{L}_{n^i_j,\alpha^i_j}$ have slope close to that of the limiting graph $\mathcal{L}^i_\alpha$. Since $\alpha \in \operatorname{supp}(\mu)$ and $\mathcal{L}^1_\alpha$ and $\mathcal{L}^2_\alpha$ are transverse at $(0, \alpha)$, this contradicts the convergence of currents.

The case where μ has an atom at α is similar and we leave it to the reader. □

3.3.3 Laminar Currents

Laminar currents are a generalization of uniformly laminar currents, suitable for dynamical applications, which were introduced by Bedford et al. [BLS93].

Definition 3.14 *A positive closed $(1, 1)$-current T in a domain $\Omega \subset \mathbb{C}^2$ is called* **laminar** *if for every $\varepsilon > 0$ there exists a locally uniformly laminar current T_ε in a subdomain $\Omega_\varepsilon \subset \Omega$ such that $0 \le T_\varepsilon \le T$ and $||T - T_\varepsilon|| \le \varepsilon$.*

It may not seem obvious at first glance why this definition should be so different from Definition 3.9. The following example is very illustrative.

Example 3.15 *Let $T = dd^c \log \max(|z|, |w|, 1)$ in $\mathbb{C}^2$. The structure of this current was studied thoroughly in [Dem82] where it was proved to be the first example of extremal positive closed current not supported on an irreducible subvariety. This was a negative answer to a conjecture of Lelong's [L72]. We claim that T is laminar but not uniformly laminar.*

We first show that T is laminar. The support of T can be decomposed as

$$\begin{aligned} \operatorname{supp}(T) &= \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \mathbb{T} \\ &= \{|z| < 1,\ |w| = 1\} \cup \{|w| < 1,\ |z| = 1\} \\ &\quad \cup \{|z| = |w| > 1\} \cup \{|z| = |w| = 1\} \end{aligned}$$

Each of the Σ_i is Levi-flat, so we infer that T is uniformly laminar outside the unit torus $\mathbb{T} = \{|z| = |w| = 1\}$. Furthermore, Demailly proves that T gives *zero mass to* $\mathbb{T}$ *(this is a variation on Proposition 3.10). So we conclude that* T *is laminar, even in a very strong form, since it is uniformly laminar outside a fixed closed subset. Of course it is clear that* T *is not locally uniformly laminar near any point of* $\mathbb{T}$*.*

On the other hand it is easy to prove that $T \wedge T$ *is, up to normalization, the Lebesgue measure on* $\mathbb{T}$*, in particular it is not zero. Hence this example shows that the potentials of laminar currents are not maximal in general.*

Example 3.16 *Complex dynamics is a source of interesting examples of maximal psh functions. For instance, the invariant currents of polynomial automorphisms of* $\mathbb{C}^2$ *are laminar currents with continuous potentials and* $T \wedge T = 0$*, but in general they are not* uniformly *laminar (examples are provided e.g. by mappings with indifferent periodic points). This shows that it is too much to expect for a continuous maximal psh function* u *that* $dd^c u$ *is uniformly laminar. On the other hand it is unclear at this point whether* $dd^c u$ *should be expected to be laminar in general. We give an answer to this problem in Sect. 3.5 below.*

3.3.4 $\mathcal{C}^2$ Maximal psh Functions

In view of the results of Sect. 3.2.2 it is natural to ask:

Question 3.17 *Assume* $u \in PSH \cap \mathcal{C}^2(\Omega)$ *is maximal. Is* $T = dd^c u$ *uniformly laminar?*

The assumption that u is C^2 is natural for $dd^c u$ then determines a *continuous* field of complex lines in the tangent bundle, which we can hope could be integrated into a lamination. A positive answer to this question has been given by Kruzhilin [Kr84] when u has rotational symmetry in z, i.e. $u(z, w) = u(|z|, w)$, using some properties of solutions to real Monge–Ampère equations. The general case remains open.

3.4 Polynomial Hulls

The notion of polynomial hull is a central concept in analysis in several complex variables. We briefly recall its definition and the connection made by Bremermann with the Dirichlet problem for the complex Monge–Ampère operator. We then indicate a construction due to Stolzenberg [St63] and Wermer [We82] of a polynomial hull without complex structure (i.e. containing no holomorphic disk). A variation on this construction will be used in Sect. 3.5 to exhibit maximal currents without complex structure.

3.4.1 The Bremermann Construction

Definition 3.18 *Let $K \subset \mathbb{C}^n$ be a compact subset. The polynomial hull $\hat{K}$ of K is the set*

$$\hat{K} := \left\{ z \in \mathbb{C}^n \,/\, |P(z)| \leq \sup_K |P| \text{ for all polynomials } P \text{ on } \mathbb{C}^n \right\}.$$

In dimension $n = 1$, the hull $\hat{K}$ is easy to understand by using the maximum principle: $\hat{K}$ is the union of K and the bounded connected components of $\mathbb{C} \setminus K$. This is a much more subtle notion in higher dimension which is not invariant by biholomorphic change of coordinates. The reader will easily convince himself that the following tori have very different polynomial hulls,

$$K_1 := \{(e^{i\theta}, 0) \in \mathbb{C}^2 \,/\, \theta \in \mathbb{R}\} \text{ and } K_2 := \{(e^{i\theta}, e^{-i\theta}) \in \mathbb{C}^2 \,/\, \theta \in \mathbb{R}\}.$$

Indeed K_2 is polynomially convex (that is $\hat{K_2} = K_2$) while $\hat{K_1} = \overline{\mathbb{D}} \times \{0\}$.

These examples are very peculiar. It is in general very difficult (if not impossible) to determine the polynomial hull of a given compact set.

As any plurisubharmonic function in $\mathbb{C}^n$ can be (well) approximated by rational multiples of $\log |P|$, P polynomial, an alternative definition of the polynomial hull is

$$\hat{K} := \{z \in \mathbb{C}^n \,/\, \varphi(z) \leq \sup_K \varphi \text{ for all } \varphi \in PSH(\mathbb{C}^n)\}. \tag{3.2}$$

Bremermann made in [Bre59] an interesting connection between the construction of certain polynomial hulls and the Dirichlet problem for the complex Monge–Ampère operator. Let $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain and Φ be a smooth function on $\partial\Omega$. We let

$$u(z) := \sup\{v(z) \,/\, v \in PSH(\Omega) \text{ with } \limsup v \leq \Phi \text{ on } \partial\Omega\}$$

denote the Perron–Bremermann envelope (which Bremermann introduced for this purpose). This is a maximal plurisubharmonic function in Ω which is continuous up to the boundary, with boundary values $u_{|\partial\Omega} = \Phi$. We refer the reader to [GZ09] for an up-to-date discussion of such envelopes. Consider

$$K := \{(z, w) \in \partial\Omega \times \mathbb{C} \,/\, |w| \leq \exp(-\Phi(z))\}.$$

Proposition 3.19 *The polynomial hull of K is*

$$\hat{K} = \{(z, w) \in \overline{\Omega} \times \mathbb{C} \,/\, |w| \leq \exp(-u(z))\}.$$

Proof. Set $F := \{(z,w) \in \overline{\Omega} \times \mathbb{C} \,/\, |w| \leq \exp(-u(z))\}$. Note that ρ admits a plurisubharmonic extension to $\mathbb{C}^n$ (see e.g. the proof of Lemma 3.22 below) so that it easily follows from (3.2) that $\hat{K} \subset \overline{\Omega} \times \mathbb{C}$. We can also extend u as a psh function in $\mathbb{C}^n$ and use the function $\psi(z,w) := u(z) + \log|w| \in PSH(\mathbb{C}^{n+1})$ to check that $\hat{K} \subset F$.

We now prove the reverse inclusion. By Lemma 3.20 below, it suffices to consider psh functions of the type $\varphi(z,w) = \log|w| + v(z)$, $v \in PSH(\mathbb{C}^n)$ to compute $\hat{K}$. Fix $(z_0,w_0) \in \overline{\Omega} \times \mathbb{C} \setminus \hat{K}$ and $v \in PSH(\mathbb{C}^n)$ such that

$$\log|w_0| + v(z_0) > 0 = \sup_{(z,w)\in K} [\log|w| + v(z)] \,.$$

We need to show that $(z_0,w_0) \notin F$, i.e. $|w_0| > \exp(-u(z_0))$. Observe that $v \leq \Phi$ on $\partial\Omega$, as follows from the condition $\sup_{(z,w)\in K} [\log|w| + v(z)] = 0$. Therefore

$$|w_0| > \exp(-v(z_0)) \geq \exp(-u(z_0)),$$

as desired. □

Lemma 3.20 *Let $K \subset \mathbb{C}^n \times \mathbb{C}$ be a compact subset that is invariant by rotation in the last coordinate, i.e. $(z, e^{i\theta}w) \in K$ whenever $(z,w) \in K$ and $\theta \in \mathbb{R}$. Then*

$$\hat{K} = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C} \,/\, |A(z,w)| \leq \sup_K |A|, \ \textit{for all polynomials } A(z,w) = P(z)w^j \right\}.$$

Proof. Let $\check{K}$ denote the hull on the right hand side, i.e. the polynomial hull restricted to special polynomials of the form $A(z,w) = P(z)w^j$. By definition $\hat{K} \subset \check{K}$.

Assume $(z_0,w_0) \in \check{K}$. Fix $0 < t < 1$. We are going to show that $(z_0, tw_0) \in \hat{K}$. Since $\hat{K}$ is closed, we thus infer that $(z_0,w_0) \in \hat{K}$ by letting t increase to 1. It is an exercise to show that for all $c \geq 1$,

$$\hat{K} = \hat{K}_c := \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C} \,/\, |P(z,w)| \leq c \sup_K |P|, \ \text{for all polynomials } P \right\}.$$

It is therefore sufficient to show that $(z_0, tw_0) \in \hat{K}_c$ where $c = 1/(1-t)$.

Let $P(z,w) = \sum_j P_j(z) w^j$ be the decomposition of a polynomial $P(z,w)$. Note that $\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta}w) e^{-ij\theta}\, d\theta = P_j(z) w^j$, thus the invariance property of K yields

$$\sup_K |P_j(z) w^j| \leq \sup_K |P|.$$

Since $(z_0,w_0) \in \check{K}$ we infer

$$|P(z_0, tw_0)| \le \sum_j t^j |P_j(z_0) w_0^j| \le \frac{1}{1-t} \sup_K |P|.$$

Therefore $(z_0, tw_0) \in \hat{K}$, hence $(z_0, w_0) \in \hat{K}$. □

It is a particular feature of this construction that $\hat{K} \setminus K$ is filled in with holomorphic disks. This is not the case in general, as we now explain.

3.4.2 Stolzenberg and Wermer Examples

Let $K \subset \mathbb{C}^n$ be a compact subset. It is an immediate consequence of the maximum principle that if a complex submanifold V of the unit ball $\mathbb{B}$ has boundary in K, $\partial V \subset K \cap \partial \mathbb{B}$, then $V \subset \hat{K}$. This suggests that $\hat{K} \setminus K$ may be filled in with complex subvarieties whose boundary lies in K.

This is actually far from being true in general. Stolzenberg has produced in [St63] an example of a compact set $K \subset \partial \mathbb{B}$ such that $\hat{K} \setminus K$ is non empty and does not contain any germ of holomorphic disk.

It is a general principle that the hulls of subsets of $\partial \mathbb{D} \times \mathbb{D}$ are better behaved than those of general compact sets. Thus, one may guess that for compact subsets $K \subset \partial \mathbb{D} \times \mathbb{D}$, these complex subvarieties exist. This is equally wrong and Wermer has produced [We82] an example of a compact set $K \subset \partial \mathbb{D} \times \mathbb{D}$ with non trivial polynomial hull and such that $\hat{K} \setminus K$ does not contain any germ of holomorphic disk. More generally, we define a *Wermer example* as a closed horizontal subset of the unit bidisk, which contains no holomorphic disk, and which is the polynomial hull of $\overline{X} \cap (\partial \mathbb{D} \times \mathbb{D})$. Recall that a subset of $\mathbb{D}^2$ is said to be horizontal if it is contained in $\mathbb{D} \times D(0, 1-\varepsilon)$ for some $\varepsilon > 0$.

We now present a construction of Wermer examples, following a slight modification of the original Wermer construction due to Duval and Sibony [DS95].

Let (a_n) be a dense sequence in $\mathbb{D}$, and (r_n) a sequence of positive real numbers, decreasing to zero. We construct (P_n) and (δ_n) by induction as follows. Set $P_0 = w$ and $\delta_0 = 1/2$ so that X_0 is the cylinder $\mathbb{D} \times \{|w| < 1/2\}$.

Define P_1 by $P_1(z, w) = w^2 - \varepsilon_1(z - a_1)$. Choose ε_1 small enough so that $\{P_1 = 0\} \subset X_0$. If δ_1 is small enough, $X_1 := \{|P_1| < \delta_1\}$ is contained in X_0 and neither contain any continuous section (relative to the first coordinate) over $D(a_1, r_1)$, nor any vertical disk of size r_1.

Repeat the same process by induction, by setting

$$P_{n+1}(z, w) = P_n^2(z, w) - \varepsilon_{n+1}(z - a_{n+1})^2,$$

and choosing the constants δ_{n+1} and ε_{n+1} so small that

- $X_{n+1} \subset X_n$ where we define X_n as $X_n = \{|P_n| < \delta_n\}$.
- X_{n+1} neither contains any graph over $D(a_{n+1}, r_{n+1})$, nor any vertical disk of size r_{n+1}.

It is now clear that $X = \bigcap X_n$ contains no germ of holomorphic disk. Furthermore it is easily shown that $X = \hat{K} \setminus K$, where $K = \overline{X} \cap (\partial\mathbb{D} \times \mathbb{D})$. Thus X is a Wermer example.

It is obvious that X supports positive closed currents: consider indeed any cluster value T of the sequence $\frac{1}{2^n}[P_n = 0]$. This mere observation provides a second proof of the fact that there exist extremal closed positive currents which are not supported on analytic varieties (cf. Example 3.15): this is the case for the currents appearing in the Choquet decomposition of this current T.

There is no reason for these currents to have well-defined self intersection – especially if the δ_n are too small, in which case X tends to become pluripolar. In Sect. 3.5.2 below we address this problem and find examples of Wermer examples supporting regular maximal currents T.

3.5 Maximality with No Holomorphic Disk

3.5.1 The Sibony Construction

Using Stolzenberg-like examples, Sibony exhibited examples of positive closed $(1,1)$ currents T in the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ with $\mathcal{C}^{1,1}$ potential, and which are not uniformly laminar (see [BF79, B88]). Let us explain this construction.

Let $X \subset \partial\mathbb{B}$ be a Stolzenberg example, i.e. a compact subset of $\partial\mathbb{B}$ such that the polynomial hull $\hat{X}$ is non trivial yet does not contain any holomorphic disk $\mathbb{D}$. Fix $\Phi \in \mathcal{C}^\infty(\partial\mathbb{B})$ a non-negative function such that $X = \{\Phi = 0\}$. By Bedford–Taylor's result [BT76], there exists a unique $u \in PSH(\mathbb{B}) \cap Lip(\overline{\mathbb{B}}) \cap \mathcal{C}^{1,1}(\mathbb{B})$ such that

$$(dd^c u)^2 = 0 \text{ in } \mathbb{B} \qquad \text{and} \qquad u_{|\partial\mathbb{B}} = \Phi.$$

This is actually the Perron–Bremermann envelope,

$$u(z) := \sup\{v(z) \,/\, v \in PSH(\Omega) \text{ with } \limsup v \le \Phi \text{ on } \partial\Omega\}.$$

The main step is the following.

Proposition 3.21 *The current $T = dd^c u$ is not uniformly laminar.*

Proof. Let us assume for the moment that $\hat{X} = \{u = 0\}$. Let $p \in \hat{X} \cap \mathbb{B}$, and let us show that T is not uniformly laminar near p. Otherwise there would exist a holomorphic disk Δ through p such that $u_{|\Delta}$ is harmonic. Observe that

$u \geq 0$ in $\mathbb{B}$ since $\Phi \geq 0$. Therefore since u vanishes at p, it has to be identically zero on $\mathbb{D}$ by the maximum principle so that $\mathbb{D} \subset \hat{X}$, a contradiction. □

It remains to establish the following result:

Lemma 3.22 $\hat{X} = \{u = 0\}$.

Proof. Observe that u admits a plurisubharmonic extension to $\mathbb{C}^2$. Indeed consider first Φ_1 a smooth extension of Φ with compact support in $\mathbb{C}^2$ and add a large multiple of $\log[1 + ||z||^2] - \log 2$ to obtain an extension Φ_2 of Φ which is moreover plurisubharmonic. By its upper envelope nature, the function u dominates Φ_2 in $\mathbb{B}$ and coincides with it on $\partial\mathbb{B}$, thus we can extend u by setting $u(z) = \Phi_2(z)$ in $\mathbb{C}^2 \setminus \mathbb{B}$.

It readily follows from the definition of $\hat{X}$ via psh functions that $\hat{X} \subset \{u = 0\}$.

For the reverse inclusion, note that $u > 0$ in $\mathbb{C}^2 \setminus (\mathbb{B} \cup X)$ so that $\hat{X} \subset \mathbb{B} \cup X$. What remains to be proved is thus that $u(p) > 0$ whenever $p \in \mathbb{B} \setminus \hat{X}$. Let $p \in \mathbb{B} \setminus \hat{X}$. There exists a psh function on $\mathbb{C}^2$ such that $v < 0$ on X and $v(p) > 0$. Fix δ such that $v < 0$ on $\Phi \leq \delta$, and M such that $v \leq M$ on $\partial\mathbb{B}$. If $0 < \varepsilon < \delta/M$, we infer that $\varepsilon v \leq \Phi$ on $\partial\mathbb{B}$. From the definition of u as an upper envelope, we deduce that $u \geq \varepsilon v$ on $\mathbb{B}$. In particular $u(p) > 0$. □

This interesting example shows that the answer to Question 3.17 is "no" in the $\mathcal{C}^{1,1}$ setting. Regularity $\mathcal{C}^{1,1}$ is important in pluripotential theory for it is the regularity of solutions to the homogeneous Monge–Ampère equation with smooth boundary data. On the other hand it can be shown (see [Duj09, Prop. 4.1]) that for these examples, the Stolzenberg example $\hat{X}$ has zero trace measure, that is, we are proving non-laminarity on a *negligible* set for T. This raises the following natural question:

Question 3.23 *Is the above current T laminar?*

We don't know the answer to this question. What we explain in the next paragraph is that, at the expense of a small loss of regularity, we can ensure non-laminarity *everywhere* on the support of T.

3.5.2 No Holomorphic Disk at All

We finish these notes by explaining the main ideas of the proof of the following recent result of the first author [Duj09]:

Theorem 3.24 *There exists a plurisubharmonic function u in the unit polydisk $\mathbb{D}^2$ such that*

1. *u is of class $\mathcal{C}^{1,\alpha}$ for all $0 < \alpha < 1$*
2. *u is maximal (i.e. $(dd^c u)^2 = 0$)*
3. *the support of $dd^c u$ does not contain any holomorphic disk.*

The basic idea of the proof is to reconsider the Wermer construction of [DS95], as presented in Sect. 3.4.2 and make explicit estimates of the quantities δ_n, ε_n in terms of r_n. Then we take a cluster value T of the sequence $T_n := \frac{1}{2^n}[P_n = 0]$ (which will actually be convergent), and arrange the parameters so that T has continuous potential u. It turns out that such T are always maximal. An unfortunate fact is that in this construction the potential is never more regular than merely continuous (and even never Hölder continuous!). So to achieve $\mathcal{C}^{1,\alpha}$ regularity we will need to find a way of "thickening" the Wermer example.

Let us be more specific. We work in $\mathbb{D}^2$. Let (a_n) be a dense sequence in $\mathbb{D}$ and (r_n) a sequence of real numbers decreasing to zero. We define by induction a sequence of polynomials by the formula $P_{n+1} = P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$ and a sequence of horizontal subsets $X_n = \{|P_n| < \delta_n\}$ so that

(i) $X_{n+1} \subset X_n$
(ii) X_n does not contain any holomorphic graph over $\mathbb{D}(a_n, r_n)$.

This imposes some explicit conditions on the parameters, respectively

(i) $\delta_{n+1} + \varepsilon_{n+1} < \delta_n^2$
(ii) $\delta_n < \varepsilon_n r_n$.

An essentially optimal choice for this is (our goal is to get estimates from below for δ_n):

$$\varepsilon_{n+1} = \frac{\delta_n^2}{2} \text{ and } \delta_{n+1} = \frac{\delta_n^2 r_{n+1}}{2}$$

($r_n \leq 1/10$, say). Note that, declaring that $\delta_0 = 1/2$ as above, (δ_n) depends only on (r_n).

Now introduce the potential $u_n = \frac{1}{2^n}\max(\log|P_n|, \log\delta_n)$ of T_n. Using the definition of P_{n+1} in terms of P_n it may be shown that $|u_{n+1} - u_n| = O\big(\frac{1}{2^n}|\log r_n|\big)$. In particular if the series $\sum \frac{1}{2^n}|\log r_n|$ converges, we infer that u_n converges uniformly to some continuous function u. In particular the sequence T_n converges to a current T with continuous potential, supported on X.

It is an easy fact that $(dd^c u_n)^2 = 0$ so by uniform convergence we infer that $(dd^c u)^2 = 0$.

With our definition of P_n, we are actually only certain that X does not contain any holomorphic graph, but we cannot rule out the possibility of vertical disks. For this, we slightly modify the inductive step by introducing an oblique projection: we assume that (a_{2n}) and (a_{2n+1}) are dense and define $P_{n+1} = P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$ if n is even, and $P_n^2 - \varepsilon_{n+1}(z + \frac{w}{100} - a_{n+1})$ if n is odd.

At this point we have constructed a maximal current with continuous potential, whose support does not contain any holomorphic disk. We want to understand more precisely the regularity of its potential.

For this we analyze the vertical slice measures of T. Keeping the same construction, we enlarge the bidisk and look at the curves $[P_n = 0]$ and the sets $X_n = \{|P_n| < \delta_n\}$ in $3\mathbb{D} \times \mathbb{C}$. Over the annulus $3\mathbb{D} \setminus 2\mathbb{D}$ the curves $\{P_n = s\}$, $|s| < \delta_n$ have no ramifications (relative to the vertical projection), so the intersection of each of these curves with a vertical fiber consists of exactly 2^n points. Thus in those vertical fibers,[2] X_n is the union of 2^n disjoint topological disks, and each component of X_n contains two components of X_{n+1}. These disks are actually essentially "round" (in the sense of distortion theory of conformal mappings), and we can estimate their size. Up to exponential terms, the size is of the order of magnitude of $\prod_1^n r_k$ (which is superexponentially small). To say it differently, each component of X_{n+1} has relative size $\approx r_{n+1}$ times the size of the component of X_n in which it sits. In particular the fibers of X have zero Hausdorff dimension.

On the other hand, the slice measure of T (which is the same as the Laplacian of u restricted to the vertical fiber) is the "balanced" measure on the Cantor set X, so that each component of X_n has mass 2^{-n}. This prevents u from being Hölder continuous. Indeed, the Laplacian of a Hölder continuous plane subharmonic function gives mass $O(r^\alpha)$ to a disk of radius r (α is the Hölder exponent).

Another way to say this is that a measure with Hölder continuous potential cannot charge sets of Hausdorff dimension 0.

Therefore, to upgrade the regularity of our examples, we need to modify the construction in order to "thicken" the set X. To understand this modification, let us first imagine a model situation. Suppose that we have a process which at time n replaces a disk D of radius r in the plane by two smaller disks $D_i \subset D$, of *relative* size r_n and with mutual distance $r/2$. Start with the unit disk, and apply this process repeatedly. Then at time n we have 2^n disjoint disks of size $\prod_1^n r_k$. Taking their union we get a nested sequence of subsets, and the result is a Cantor set of Hausdorff dimension zero.

To increase dimension we do as follows. We now consider a process in two steps. Given a disk D of radius r, we first replace it by $\approx N^2$ evenly spaced small disks of radius r/N, and then we apply the previous doubling process to each of the small disks. Then if at each step, N is sufficiently large with respect to r_n, the limiting Cantor set will have dimension 2.

To implement this strategy for our Wermer examples, before the ramification process $P_n \mapsto P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$, we replace $\{|P_n| < \delta_n\}$ by a large number of smaller subsets $\left\{|P_n - s| < \frac{\delta_n}{N}\right\}$, filling out most of $\{|P_n| < \delta_n\}$ (s ranges over a finite set $\mathcal{S}_n$). Then we apply the ramification process to each $P_n - s$ and get a family of polynomials $P_{n+1,s}$. Let then $X_n := \bigcup_s \{|P_{n+1,s}| < \delta_{n+1}\}$ for well chosen δ_{n+1} and $X = \bigcap X_n$. It can be shown that the vertical slices of X_n over $3\mathbb{D} \setminus 2\mathbb{D}$ indeed "look like" the

[2]We use the same notation for X_n (resp. X) and its vertical fibers.

model situation described above, in the sense of plane conformal geometry. In particular we can ensure Hausdorff dimension 2 for the vertical fibers of X.

Let further u_n be defined by

$$u_n = \frac{1}{2^n \# \mathcal{S}_n} \sum_{s \in \mathcal{S}_n} \log \max \left(|P_{n+1,s}| , \delta_{n+1} \right).$$

As before, (u_n) converges to a maximal continuous psh function u, such that $\operatorname{supp}(dd^c u)$ is contained in X, thus contains no holomorphic disk.

By standard estimates in potential theory, if at each step N is chosen to be sufficiently large, u will be of class $\mathcal{C}^{1,\alpha}$ for all $\alpha < 1$ in $(3\mathbb{D} \setminus 2\mathbb{D}) \times \mathbb{C}$. Then by regularity theory for solutions of Monge–Ampère equations, this regularity propagates to $3\mathbb{D} \times \mathbb{C}$. The reader is referred to [Duj09] for details.

Part II
Stochastic Analysis for the Monge–Ampère Equation

Chapter 4
Probabilistic Approach to Regularity

François Delarue

Abstract We here gather in a single note several original probabilistic works devoted to the analysis of the $\mathcal{C}^{1,1}$ regularity of the solution to the possibly degenerate complex Monge–Ampère equation. The whole analysis relies on a probabilistic writing of the solution as the value function of a stochastic optimal control problem. Such a representation has been introduced by Gaveau [Gaveau, J. Functional Analysis 25 (1977), no. 4, 391–411] in the late 1970s and used in an exhaustive way by Krylov in a series of papers published in the late 1980s up to the final paper [Krylov, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 66–96; translation in Math. USSR-Izv. 34 (1990), no. 1, 65–95] in which the $\mathcal{C}^{1,1}$-estimate is eventually established. All the arguments we here use follow from these seminal works.

Nota Bene. This is an expanded version of the notes prepared by François Delarue for a series of lectures he delivered in LATP, Marseille, in December 2009. As the audience consisted of non specialists, he has tried to make these lecture notes accessible with only few prerequisites.

4.1 Introduction

4.1.1 Background

This chapter is devoted to the stochastic analysis of the possibly degenerate Monge–Ampère equation and specifically to the probabilistic proof of the $\mathcal{C}^{1,1}$-estimate of the solutions under some suitable assumption.

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For a complete review of the stakes of such a result, we refer the reader to Chaps. 1 and 2 by V. Guedj and A. Zeriahi: we here focus on the probabilistic counterpart only and keep silent about the geometric motivations that are hidden behind.

The idea of understanding the complex Monge–Ampère equation from a probabilistic point of view goes back to the earlier paper by Gaveau [Gav77] in the late 1970s. Therein, the solution is shown to write as the value function of a stochastic optimal control problem, i.e. as the minimal value of some averaged cost computed along the trajectories of different diffusion processes evolving inside the underlying domain.

In some sense, this representation formula is a *compact* (or *closed*) representation formula that appears as a generalization of the Kolmogorov formula for the heat equation: the solution of the heat equation may be expressed as some averaged value computed along the trajectories of the Brownian motion. Brownian motion might be understood as follows: at any given time and at any given position, the diffusive particle at hand moves at random, independently of the past and in an isotropic way. Actually, Kolmogorov formula extends to linear (say to simplify purely) second-order partial differential equations with a variable diffusion coefficient: the solution is then understood as some averaged value computed along the solution of a differential equation of stochastic type driven by the coefficient of the PDE at hand. This appears as a stochastic method of characteristics: at any given time and at any given position, the diffusive particle associated with the stochastic differential equation moves at random, independently of the past, but in a non-isotropic way; the most likely directions are given by the main eigenvectors of the diffusion matrix at the current point. In the case of Monge–Ampère, the story might read as follows: at any given time and at any given position, the particle at hand moves at random, independently of the past, and the diffusion coefficient is chosen among all the possible diffusion coefficient of determinant 1 according to some local optimization criterion or, equivalently, to some local cost.

4.1.2 Purpose of the Note

In his paper, Gaveau managed to derive some Hölder continuity property of the solution to Monge–Ampère from the probabilistic formulation, but the exhaustive use of the formula for the analysis of the regularity of the derivatives goes back to Krylov. The reference paper on the subject is [Kry89]: the solution is shown to be $\mathcal{C}^{1,1}$ on the whole domain (i.e. up to the boundary) under some suitable assumption that may include the degenerate case. Basically, it applies to a much more general framework than the Monge–Ampère one: it applies to a general class of Hamilton–Jacobi–Bellman equations, i.e. to a general class of equations summarizing the dynamics of the value function of some stochastic optimization problem.

Actually, the paper [Kry89] is not *self-contained*. It must be seen as the conclusion of a series of papers initiated in the 1980s: see, among others, [Kry87,Kry89b,Kry90] and, finally, [Kry89]. This note is an attempt to gather in a single manuscript most of the ingredients of the whole proof, at least in the specific case of Monge–Ampère: from the basic rules of stochastic calculus to the detailed computations of the final estimate of the first- and second-order derivatives.

However, the proof we here provide is a bit different from the original one and may appear as less straightforward. In some sense, the objective is here both mathematical and... pedagogical: the idea is both to provide an *almost* complete and self-contained proof of the $\mathcal{C}^{1,1}$ estimate and to explain to the reader the way we are following to reach it.

4.1.3 A Short Review of the Strategy

The arguments used by Krylov have been developed since the 1970s. Some of them may be found in the seminal work by Malliavin [Mal76, Mal78], even if used differently. In short, Malliavin initiated a program to prove by means of stochastic arguments only the *Sum of Squares Theorem* by Hörmander: *Sum of Squares Theorem* provides some sufficient condition on the Lie algebra generated by the vector fields of a possibly degenerate diffusion matrix to let the corresponding operator be hypoelliptic. The program consists in an exhaustive analysis of the stochastic flow generated by the associated differential equation of stochastic type. (For the purely Laplace operator, the flow is trivial since the current diffusion process reduces to a Brownian motion plus a starting point.) A part of the problem is then to investigate the regularity of the flow.

In the current framework, the main idea of Krylov consists in reducing the analysis of the $\mathcal{C}^{1,1}$ regularity of the solution to Monge–Ampère to a long-run analysis of the derivatives of the flow of the diffusion processes behind. Roughly speaking, the point is to control the first- and second-order derivatives of the flow both in time and in the optimization parameter. At first sight, it turns out to be really challenging. By the way, it is in some sense: stated under this form, the objective may not be reachable. Here is the key-point of the proof: the required long-run estimate of the derivatives of order one and two of the flow may be relaxed according to the underlying second-order differential structure. As an example, the analysis may benefit from some uniform ellipticity (or non-degeneracy) property: when applied to a non-degenerate linear second-order partial differential equation instead of the Monge–Ampère equation, the original required long-run estimate of the derivatives of the flow can be relaxed to a much more less restrictive version (and in fact can almost be cancelled) thanks to the non-degeneracy assumption itself. (The argument is explained in the note.) In the case of

Monge–Ampère, the equation may degenerate, but the analysis may benefit from the description of the boundary: if the domain is strictly pseudo-convex, the original required long-run estimate of the derivatives of the flow can be relaxed as well (but cannot be cancelled); that is, strict pseudo-convexity plays the role of a *weak non-degeneracy assumption*. Finally, the analysis may also benefit from the Hamilton–Jacobi–Bellman formulation, i.e. from the writing of the Monge–Ampère equation as an equation deriving from a stochastic optimization problem: the structure is indeed kept invariant under some transformations of the optimization parameters. As explained below, this may also help to reduce the long-run constraint on the derivatives of the flow.

As mentioned, the way the required long-run constraint on the derivatives of the flow is relaxed is detailed in the note. At least, we may here specify the keyword only: *perturbation*. Indeed, the strategy is common to the Malliavin point of view and consists of a well-chosen perturbation of the original probabilistic representation. This is a general *meta*-principle in stochastic analysis: from a probabilistic point of view, regularity properties are understood through the reaction of the stochastic system under consideration to an external perturbation.

4.1.4 Main Result

In the end, the result we here prove is the following:

Theorem 4.1.1 *Let (**A**) stand for the assumption:*

- *$\mathcal{D}$ is a bounded domain of $\mathbb{C}^d$, $d \geq 1$, described by some $\mathcal{C}^4$ function ψ in the neighborhood of $\bar{\mathcal{D}}$, i.e.*

$$\mathcal{D} := \big\{ z \in \mathbb{C}^d : \psi(z) > 0 \big\}.$$

- *The function ψ is assumed to be plurisuperharmonic in the neighborhood of $\bar{\mathcal{D}}$, i.e.*

$$\forall a \in \mathcal{H}_d^+ : \mathrm{Trace}(a) = 1, \quad \forall z \in \bar{\mathcal{D}},\ \mathrm{Trace}\big(aD^2_{z,\bar{z}}\psi(z)\big) < 0,$$

where $\mathcal{H}_d^+$ stands for the set of non-negative Hermitian matrices of size $d \times d$.

- *The function ψ is non-singular in the neighborhood of the boundary of $\mathcal{D}$, i.e.*

$$\exists \delta > 0,\ \forall z \in \partial\mathcal{D}, \quad |D_z\psi(z)| \geq \delta.$$

- *f and g are two functions of class $\mathcal{C}^2$ and $\mathcal{C}^4$ on $\bar{\mathcal{D}}$ with values in $\mathbb{R}_+$ and $\mathbb{R}$ respectively.*

*Then, under Assumption (**A**), there exists a function u from $\bar{\mathcal{D}}$ to $\mathbb{R}$, of class $\mathcal{C}^{1,1}$ on the whole $\bar{\mathcal{D}}$ (i.e. with Lipschitz first-order derivatives on the closure of the domain $\mathcal{D}$), plurisubharmonic, i.e.*

$$\forall a \in \mathcal{H}_d^+ : \mathrm{Trace}(a) = 1, \quad \text{a.e. } z \in \mathcal{D}, \ \mathrm{Trace}\big(a D^2_{z,\bar{z}} u(z)\big) \geq 0,$$

and

$$\det{}^{1/d}\big(D^2_{z,\bar{z}} u(z)\big) = \frac{f(z)}{d} \text{ a.e. } z \in \mathcal{D}, \quad u(z) = g(z), \ z \in \partial\mathcal{D}, \tag{4.1}$$

i.e. u satisfies the Monge–Ampère equation on $\mathcal{D}$ with f^d (up to some normalizing constant) as source term and g as boundary condition. (Compare with Chap. 1, Sect. 1.1, by V. Guedj.)

Pay attention that Theorem 4.1.1 does not recover Theorem 2.10 in Chap. 2 by V. Guedj and A. Zeriahi (that holds for the ball only) since the boundary condition therein is $\mathcal{C}^{1,1}$ only.

4.1.5 Organization of the Note

The note is organized as follows. In Sect. 4.2, we explain the basic optimization principle on which the whole proof relies. In Sects. 4.3 and 4.4, we introduce the Kolmogorov representation of the Dirichlet problem with constant coefficients by means of the Brownian motion. We then give a short overview of the basic rules of stochastic calculus. In Sect. 4.5, we introduce the probabilistic representation of Monge–Ampère, as originally considered by Gaveau. The program for the analysis of the representation is explained in Sect. 4.6. Section 4.7 is a short presentation of the differentiability properties of the flow of a stochastic differential equation. In Sect. 4.8, we give a first sketch of the proof of the $\mathcal{C}^1$-regularity. As explained therein, it fails for the second-order derivatives. The right argument is given in Sect. 4.9.

4.1.6 Useful Notation

Below, the gradient of a function is understood as a row vector and for any pair of vectors (x, y) (of the same dimension d) with real or complex coordinates, the notation $\langle x, y\rangle$ stands for $\sum_{i=1}^d x_i y_i$.

4.2 Hamilton–Jacobi–Bellman Formulation

We here introduce the Hamilton–Jacobi–Bellman formulation of the Monge–Ampère equation.

4.2.1 Optimization Problem

Generally speaking, Hamilton–Jacobi–Bellman equations describe the dynamics – in space only for a stationary problem and in time as well for an evolution equation – of the value function of an optimal (possibly stochastic) control problem.

In the specific case of Monge–Ampère, the Hamilton–Jacobi–Bellman formulation follows from a simple Lemma taken from the original article by Gaveau [Gav77]:

Lemma 4.2.1 *Given a non-negative Hermitian matrix H of size $d \times d$, the determinant of H is the solution of the minimization problem:*

$$\det{}^{1/d}(H) = \frac{1}{d} \inf\bigl\{\mathrm{Trace}[aH]\ ;\ a \in \mathcal{H}_d^+,\ \det(a) = 1\bigr\}.$$

Proof. Up to a diagonalization, we may assume H to be diagonal. Denoting by $(\lambda_1, \ldots, \lambda_d)$ its (non-negative real) eigenvalues, we obtain for some $a \in \mathcal{H}_d^+$

$$\mathrm{Trace}[aH] = \sum_{i=1}^{d} a_{i,i}\lambda_i.$$

Noting that the elements $(a_{i,i})_{1 \le i \le d}$ are non-negative, the standard inequality between the arithmetic and geometric means yields

$$\frac{1}{d}\mathrm{Trace}[aH] \ge \left(\prod_{i=1}^{d} a_{i,i}\lambda_i\right)^{1/d} = \det{}^{1/d}(H)\left(\prod_{i=1}^{d} a_{i,i}\right)^{1/d}.$$

Finally, Hadamard inequality says that $\mathrm{Trace}[aH] \ge d \det^{1/d}(H)$, that is

$$\inf\{\mathrm{Trace}[aH]; a \in \mathcal{H}_d^+, \det(a) = 1\} \ge d \det{}^{1/d}(H).$$

To prove the equality between both quantities, we choose $a_{i,i} = \lambda_i^{-1} \det^{1/d}(H)$ (and $a_{i,j}$ equal to zero for i and j different) when H is non-degenerate (so that the *infimum* then reads as a *minimum*). In the degenerate case, it is sufficient to choose $a_{i,i} = \varepsilon$ when $\lambda_i > 0$ and $a_{i,i} = N$ when $\lambda_i = 0$, with ε small and N large to be chosen so that the determinant be equal to 1 (again, $a_{i,j}$ is set equal to 0 for i and j different). $\square$

Lemma 4.2.1 suggests us to write, at least formally, Monge–Ampère equation (4.1) under the form:

$$\sup_{a \in \mathcal{H}_d^+,\ \det(a)=1} \left[-\mathrm{Trace}[aD^2_{z,\bar{z}} u](z) \right] + f(z) = 0, \quad z \in \mathcal{D}. \tag{4.2}$$

(With the same boundary condition.) This formulation makes the family of diffusion operators $(\mathrm{Trace}[aD^2_{z,\bar{z}} \cdot])_{a \in \mathcal{H}_d^+,\ \det(a)=1}$ appear.

Roughly speaking, an equation driven by an *infimum* (or a *supremum*) taken over a family of second-order operators is called a second-order *Hamilton–Jacobi–Bellman* equation.

4.2.2 First-Order Case

We first explain how minimization (or maximization) may affect a family of first-order partial differential equations. In such a case, the resulting equation is called a first-order *Hamilton–Jacobi–Bellman* equation. Consider to this end a very simple one-dimensional evolution problem:

$$D_t u(t,x) - \sup_{a \in \mathbb{R},\ |a|=1} \left[a D_x u \right](t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{4.3}$$

with a given regular boundary condition $u(0,\cdot) = u_0(\cdot)$. This is a non-linear equation with

$$D_t u(t,x) - \left| D_x u \right|(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R},$$

as explicit form.

The purpose is here to understand how the method of characteristics may write for such an equation. When the *parameter* or *control* a is frozen, the equation

$$D_t u(t,x) - a D_x u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{4.4}$$

is a simple transport equation with $-a$ as constant velocity, whose solution is explicitly known:

$$u(t,x) = u_0(x + at), \quad (t,x) \in [0,+\infty) \times \mathbb{R}.$$

Said differently, the initial shape u_0 is translated at velocity $-a$: as an example, the value of u at time t and a point $-at$ is $u_0(0)$. Said differently, the mapping $t \geq 0 \mapsto u(t, x - at)$ is constant.

Here, the linear mapping $t \geq 0 \mapsto x + at$ is called a *backward characteristic* of the transport equation (4.4).

Go now back to the general case. We understand that the *supremum* in (4.3) favours the velocity fields of same sign as the local spatial variation of the solution. Said differently, the possible characteristics must now be sought among paths driven by positive or negative speed according to the values of the gradient of the solution of the PDE. We thus consider paths of the form

$$x_t = x_0 + \int_0^t a_s ds, \quad t \geq 0, \tag{4.5}$$

where $(a_t)_{t \geq 0}$ is a (measurable) function with values in $\{-1, 1\}$ and x_0 is an arbitrary initial condition. The whole point is then to understand the behavior of the solution to the PDE along all these trajectories. To do so, wa may differentiate, at least formally, u along some $(x_t)_{t \geq 0}$ as in (4.5). For a given time $T > 0$ and some $t \in [0, T]$, we write

$$\begin{aligned} \frac{d}{dt}\big[u(T-t, x_t)\big] &= -D_t u(T-t, x_t) + a_t D_x u(T-t, x_t) \\ &= -|D_x u|(T-t, x_t) + a_t D_x u(T-t, x_t) \leq 0, \end{aligned}$$

by taking into account the equality $|a_t| = 1$. Therefore,

$$u(T, x_0) \geq u_0\bigg(x_0 + \int_0^T a_s ds\bigg),$$

that is[1]

$$u(T, x_0) \geq \sup_{(a_t)_{0 \leq t \leq T} : |a_t| = 1} \bigg[u_0\bigg(x_0 + \int_0^T a_s ds\bigg)\bigg]. \tag{4.6}$$

Now, the formal choice $(a_t = \text{sign}[D_x u(T-t, x_t)])_{t \geq 0}$ says that equality might hold. We thus derive as a (possible) *closed* representation formula of u:

$$u(T, x_0) = \sup_{(a_t)_{0 \leq t \leq T} : |a_t| = 1} \big[u_0(x_T^a)\big], \tag{4.7}$$

with

$$x_t^a = x_0 + \int_0^t a_s ds, \quad t \geq 0.$$

[1] Obviously, the right-hand side in (4.6) has the simpler form $\sup_{|a| \leq T}[u_0(x_0 + a)]$, but it is useless for our specific purpose.

The argument is here formal only. However, it suggests some possible *closed* representation for the solution of (4.3) as the value function of a deterministic control problem: the so-called *control parameter* is of the form $(a_t)_{t\geq 0}$ with $|a_t| = 1$, $t \geq 0$, and the resulting controlled path is of the form $(x_t^a)_{t\leq 0}$. We stress out that the *supremum* in (4.3) is kept preserved in the representation formula (4.7). This follows from a maximum principle argument: by the maximum principle, the solution to (4.3) is above the solution to any linear transport PDE with the same initial condition u_0 and with a (possibly time-dependent) velocity field of norm 1. (See (4.6).)

We also emphasize that the theory of viscosity solutions provides a rigorous framework to the formal argument we have here given. (See for example Chap. 2, Lemma 2.1, in the monograph by Barles [Ba94].)

4.2.3 Second-Order Equations

Go now back to the Hamilton–Jacobi–Bellman formulation (4.2). In comparison with the previous subsection, we may distinguish two main differences. On the hand, (4.2) has a source term. On the other hand, the underlying operator is of second-order. (The reader may also notice that the equation is also stationary and that it is set on a bounded domain of the space only. We will come back to these two points later.)

Plugging a source term (say f in the right-hand side) in the Hamilton–Jacobi formulation (4.3) would not really modify the analysis we just performed. In a such a case, the right form of (4.7) would be

$$u(T, x_0) = \sup_{(a_t)_{0\leq t\leq T}:|a_t|=1} \left[u_0(x_T^a) + \int_0^T f(x_t^a)dt \right]. \tag{4.8}$$

(That is, the source term would be integrated along the controlled trajectories.)

Replacing the first-order operator by a second-order one is actually much more difficult to understand. To do so, the first point consists in going back to the frozen problem without any optimization, i.e. to the case when the diffusion coefficient in (4.2) is given by some fixed $a \in \mathcal{H}_d^+$, and then in seeking for the right characteristics in that framework.

Under this form, the problem is not well-posed. The whole point is the following: for a second-order operator, there are no *true characteristics*; the only possible way to obtain a closed formula for the solution consists in introducing an additional parameter, i.e. some randomness, and then in considering *random characteristics*. This follows from some scale factors: there is no way to balance, in a single differentiation, first-order terms in time and in space and second-order terms in space. More precisely, to balance first-order terms in time and second-order terms in space, the point is to introduce some characteristics with unbounded variation and, in fact, characteristics that are not absolutely continuous w.r.t. the Lebesgue measure. Randomness may be useless for the construction of such trajectories: as we will see below, randomness permits to get rid of some parasitic terms of order one by a simple integration w.r.t. to the underlying probability measure.

The typical case is the purely Laplace one. When a matches the identity matrix I_d, the operator $\mathrm{Trace}[D^2_{z,\bar{z}}\cdot]$ admits the complex Brownian motion of dimension d as *random characteristic*. Actually, $\mathrm{Trace}[D^2_{z,\bar{z}}\cdot]$ may be expanded in real coordinates as

$$\mathrm{Trace}[D^2_{z,\bar{z}}\cdot] = \frac{1}{4}\big[\Delta_{x,x} + \Delta_{y,y}\big],$$

so that it is equivalent to consider the real Brownian motion of dimension $2d$ as *random characteristic*: Brownian motion is the right *stochastic process* associated with the heat equation.

4.3 Brownian Motion

We first explain what Brownian motion is in the simplest case when the dimension is 1.

4.3.1 Gaussian Density

The connection between Brownian motion and heat equation is well-understood through the so-called marginal laws, that is the laws of the positions of a Brownian motion at a given time. Recall indeed that the time–space heat equation in dimension 1

$$D_t u(t,x) - \frac{1}{2} D^2_{x,x} u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{4.9}$$

with an initial condition of the form $u(0,\cdot) = u_0(\cdot)$ admits as solution (say if u_0 is bounded and continuous)

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(x-y) \exp\left(-\frac{|y|^2}{2t}\right) dy, \quad (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{4.10}$$

Said differently, the solution may be expressed as the convolution of the initial condition by the Gaussian density of zero mean and of variance t, i.e. the function

$$y \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|y|^2}{2t}\right) dy.$$

The density is here said to be of zero mean and of variance t since

$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y \exp\left(-\frac{|y|^2}{2t}\right) dy = 0$$
$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^2 \exp\left(-\frac{|y|^2}{2t}\right) dy = t.$$

(The second result follows from a simple change of variable .)

Convolution by a Gaussian kernel may be expressed in a simple probabilistic way. Indeed, if $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete[2] probability space and $(B_t)_{t\geq 0}$ a family of random variables (i.e. of measurable functions from $(\Omega, \mathcal{F})$ to $\mathbb{R}$ endowed with its Borel sets) such that, for any $t > 0$, B_t has a Gaussian density of zero mean and variance t, i.e. (below, $\mathbb{E}$ stands for the expectation)

$$\forall f \in \mathcal{C}_b(\mathbb{R}), \quad \mathbb{E}\big[f(B_t)\big] = \int_\Omega f(X_t(\omega))d\mathbb{P}(\omega)$$
$$= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y) \exp\left(-\frac{|y|^2}{2t}\right) dy,$$

and $\mathbb{P}\{B_0 = 0\} = 1$, then

$$u(t, x) = \mathbb{E}\big[u_0(x + B_t)\big], \quad t \geq 0. \tag{4.11}$$

4.3.2 Dynamics

The connection we just gave between heat equation and Gaussian variables is actually too much "static" to be fully relevant. Nothing is said about the joint behavior of the variables $(B_t)_{t\geq 0}$ ones with others.

To understand the dynamics, we use a discretization *artifact*. Assume indeed that we are applying a finite difference numerical scheme to solve heat equation (4.9). Specifically, for a small time step Δt and a small spatial step Δx, assume that we are seeking for a family of reals $(u_{n,k})_{n\in\mathbb{N},k\in\mathbb{Z}}$ approximating the "true" values $(u(n\Delta t, k\Delta x))_{k\in\mathbb{Z}}$. A common scheme consists in defining $(u_{n,k})_{n\in\mathbb{N},k\in\mathbb{Z}}$ through the iterative procedure

[2] The completeness is used in the sequel.

$$\frac{u_{n+1,k} - u_{n,k}}{\Delta t} = \frac{1}{2} \frac{u_{n,k+1} + u_{n,k-1} - 2u_{n,k}}{\Delta x^2}, \quad n \in \mathbb{N}, \ k \in \mathbb{Z}, \tag{4.12}$$

with $u_{n,k} = u_0(k\Delta x)$ as initial condition. Obviously, in the above equation, the left-hand side is understood as an approximation of the time-derivative of u and the right-hand side of its second-order spatial derivative.

We can write (4.12) as

$$u_{n+1,k} = \big(1 - \frac{\Delta t}{\Delta x^2}\big) u_{n,k} + \frac{\Delta t}{\Delta x^2} \frac{u_{n,k+1} + u_{n,k-1}}{2}.$$

Choosing $\Delta t = \Delta x^2$, we obtain the simpler formula

$$u_{n+1,k} = \frac{u_{n,k+1} + u_{n,k-1}}{2}, \quad n \in \mathbb{N}, \ k \in \mathbb{Z} \tag{4.13}$$

Replace now the approximating values $(u_{n,k})_{k \in \mathbb{Z}, n \geq 0}$ in (4.13) by the true quantities and write

$$\begin{aligned} u\big((n+1)\Delta t, k\Delta x\big) &\approx \frac{u\big(n\Delta t, (k+1)\Delta x\big) + u\big(n\Delta t, (k-1)\Delta x\big)}{2} \\ &= \mathbb{E}\big[u(n\Delta t, k\Delta x + \Delta x\, \varepsilon)\big], \end{aligned}$$

where ε is a random variable taking the values 1 and -1 with probability $1/2$. Notice that it is possible to repeat the argument by approximating $u(n\Delta t, \cdot)$ with a new expectation (computed w.r.t. a new random variable, independent of ε). Therefore,

$$u\big((n+1)\Delta t, k\Delta x\big) \approx \mathbb{E}\big[u\big((n-1)\Delta t, k\Delta x + \Delta x(\varepsilon_1 + \varepsilon_2)\big)\big],$$

where ε_1 and ε_2 are two independent random variables taking the values 1 and -1 with probability $1/2$. Iterating the procedure N times, we deduce that

$$u\big(N\Delta t, k\Delta x\big) \approx \mathbb{E}\big[u\big(0, k\Delta x + \Delta x(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_N)\big)\big]. \tag{4.14}$$

Clearly, the symbol $\approx$ is not really meaningful because of the numerous approximations we just performed. However, choosing to simplify $k = 0$ and $N\Delta t = 1$, so that $\Delta x = N^{-1/2}$ since $\Delta t = \Delta x^2$, we understand that the random variable in the right-hand side in (4.14) has the form

$$N^{-1/2}\big[\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_N\big].$$

Central Limit Theorem says that it converges, in the weak sense, towards the Gaussian law of zero mean and variance 1. (Here, weak convergence means weak convergence of probability measures.) In particular, passing to the limit in (4.14), we recover (4.11).

Actually, this non-rigorous argument says that the right structure for $(B_t)_{t\geq 0}$ in (4.11) is of independent increment type. Indeed, we understand that, on disjoint intervals, the underlying variables $(\varepsilon_n)_{n\geq 1}$ are asked to be independent. Moreover, the structure is stationary: randomness between times 0 and $t-s$ is the same in law as the randomness plugged into the system between times s and t. This says that the right choice for $(B_t)_{t\geq 0}$ is

Definition 4.3.1 *A family of random variables* $(B_t)_{t\geq 0}$ *is a Brownian motion starting from* 0 *if*

1. $\mathbb{P}\{B_0 = 0\} = 1$,
2. *For any* $n \geq 1$, *for any* $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, *the increments* B_{t_1}, $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent,*
3. *For any* $0 < s < t$, *the increment* $B_t - B_s$ *has a Gaussian law of zero mean and variance* $t - s$.
4. *With probability 1, the paths* $t \geq 0 \mapsto B_t(\omega)$ *are continuous.*

The last condition is the most technical one: roughly speaking, it says that the differential structure associated with Brownian motion is local. Add also that, by definition, a Brownian motion starting from x is nothing else but x plus a Brownian motion starting from 0.

4.3.3 Differential Rules

To understand if Brownian motion is the right characteristic for heat equation, the point is to compute the infinitesimal variation of $(u(T-t, B_t))_{0\leq t\leq T}$, for a given $T > 0$, where u is given by (4.9). We here expand by Taylor's formula

$$
\begin{aligned}
&u(T-(t+h), B_{t+h} - B_t + B_t) \\
&\quad = u(T-t, B_t) - D_t u(t, B_t) h + D_x u(t, B_t)(B_{t+h} - B_t) \\
&\qquad + \frac{1}{2} D^2_{x,x} u(t, B_t)(B_{t+h} - B_t)^2 + \frac{1}{2} D^2_{t,t} u(t, B_t) h^2 \\
&\qquad - D^2_{t,x} u(t, B_t)(B_{t+h} - B_t) h + \ldots
\end{aligned}
$$

Expansion is given at least of order two: we aim to recover heat equation. (Moreover, it makes sense since u is regular away from the boundary.)

Actually, it is enough to stop the expansion at order two: by definition of a Brownian motion, $\mathbb{E}[(B_{t+h} - B_t)^2] = h$; using a simple Gaussian argument, this result may be generalized as $\mathbb{E}[(B_{t+h} - B_t)^{2p}] = C_p h^p$ for any integer p, the constant C_p being universal. In particular, the only term of order 1 in h among the derivatives of order two is the term in spatial derivatives. The others are of order $h^{3/2}$ and h^2. Therefore, we write

$$\begin{aligned} &u(T-(t+h), B_{t+h}-B_t+B_t) \\ &\quad = u(T-t, B_t) - D_t u(t,B_t)h + D_x u(t,B_t)(B_{t+h}-B_t) \\ &\qquad + \frac{1}{2} D^2_{x,x} u(t,B_t)(B_{t+h}-B_t)^2 + \dots \end{aligned} \tag{4.15}$$

Here, we wish to replace $(B_{t+h}-B_t)^2$ by h. Using a Gaussian argument again,

$$\mathbb{E}\big[\big((B_{t+h}-B_t)^2-h\big)^2\big] = 2h^2.$$

Clearly, this does not show that the term $(B_{t+h}-B_t)^2-h$ is less than h. However, on the long run, the sum of the terms of this type, i.e.

$$\sum_{i=0}^{n-1} \big[(B_{t_{i+1}}-B_{t_i})^2-h\big]^2 \tag{4.16}$$

for a subdivision $0<t_1<t_2<\dots<t_n$ of stepsize h is a sum of independent random variables of variance $2h^2$. In the independent case, the variance is additive: the variance of the sum is equal to $2nh^2$. Noting that nh is macroscopic, we understand that the action of this term is negligible from a macroscopic point of view.

The reader can check that the argument still holds when the quantity $D^2_{x,x}u(t,B_t)$ is added to sum as in (4.15).

Finally, we write

$$\begin{aligned} &u(T-(t+h), B_{t+h}-B_t+B_t) \\ &\quad = u(T-t, B_t) - D_t u(t,B_t)h + D_x u(t,B_t)(B_{t+h}-B_t) \\ &\qquad + \frac{1}{2} D^2_{x,x} u(t,B_t)h + o(h) \\ &\quad = u(T-t,B_t) + D_x u(t,B_t)(B_{t+h}-B_t) + o(h), \end{aligned}$$

the second line being obtained by using the PDE. From an infinitesimal point of view (i.e. when getting rid of the negligible terms), we write

$$d\big[u(T-t,B_t)\big] = D_x u(t,B_t)dB_t, \quad 0 \le t \le T, \tag{4.17}$$

We emphasize that the result is not zero! Said differently, the variation of $(u(T-t,B_t))_{0\le t\le T}$ is not zero, as for equations of order one. Actually, understanding $D_x u(t,B_t)dB_t$ as $D_x u(t,B_t)(B_{t+h}-B_t)$, we deduce from the independence of $D_x u(t,B_t)$ and $B_{t+h}-B_t$ that the expectation of the increment is zero. Therefore, $(u(T-t,B_t))_{0\le t\le T}$ is constant... in expectation.

4.3.4 Differential Rules

In the end, everything works as if we had written

$$d\big[u(T-t,B_t)\big] = -D_t u(t,B_t)dt + \frac{1}{2}D^2_{x,x}u(t,B_t)dB_t^2 + D_x u(t,B_t)dB_t,$$

and set $dB_t^2 = dt$. We will use this rule below.

Theorem 4.3.2 (Itô's formula) *Let $(B_t)_{t\geq 0}$ a real Brownian motion and f a function of class $\mathcal{C}^{1,2}([0,+\infty),\mathbb{R})$. Then, the infinitesimal variation of $(f(t,B_t))_{0\leq t\leq T}$ writes*

$$d\big[f(t,B_t)\big] = \left[D_t f(t,B_t) + \frac{1}{2}D^2_{x,x}f(t,B_t)\right]dt + D_x f(t,B_t)dB_t.$$

Said differently, Itô's formula is a Taylor formula with convention $dB_t^2 = dt$.

4.4 Stochastic Integral

We here explain the basic steps of the construction of the stochastic integral. Specifically, the problem is to give a meaning, from a macroscopic point of view, to the term

$$D_x u(t,B_t)dB_t, \tag{4.18}$$

in the statement of Theorem 4.3.2.

4.4.1 Heuristics

Under a macroscopic form, the term in (4.18) reads as a stochastic integral

$$\int_0^T D_x u(t,B_t)dB_t.$$

This integral is not defined in the Lebesgue sense: Brownian motion paths are not of bounded variation. However, it may be understood in a specific way, as the limit (in a certain sense) of some Riemann sums. Indeed, the integral is understood as the L^2 limit of the sum

$$\sum_{i=0}^{n-1} D_x u(t_i,B_{t_i})\big(B_{t_{i+1}} - B_{t_i}\big),$$

where $0 = t_0 < t_1 < \cdots < t_n$ is a subdivision of $[0,T]$ of (say uniform) stepsize, equal to T/n.

Define now the process (i.e. a family of random variables depending on time)

$$\alpha_t^n = \sum_{i=0}^{n-1} D_x u(t_i, B_{t_i}) \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

As a definition of the stochastic integral of such a simple process, we then set

$$\int_0^T \alpha_t^n dB_t := \sum_{i=0}^{n-1} D_x u(t_i, B_{t_i}) \big(B_{t_{i+1}} - B_{t_i} \big).$$

As we already said, this term is of zero expectation. The variance is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T \alpha_t^n dB_t \Big)^2 \Big] \\
&\quad = \sum_{i=0}^{n-1} \mathbb{E}\big[\big| D_x u(t_i, B_{t_i}) \big|^2 |B_{t_{i+1}} - B_{t_i}|^2 \big] \\
&\qquad + 2 \sum_{0 \le i < j \le n-1} \mathbb{E}\big[D_x u(t_i, B_{t_i}) D_x u(t_j, B_{t_j}) \big(B_{t_{i+1}} - B_{t_i} \big) \big(B_{t_{j+1}} - B_{t_j} \big) \big].
\end{aligned}$$

In the first sum, we may take advantage of the independence of $B_{t_{i+1}} - B_{t_i}$ and B_{t_i} to split the expectations. Similarly, in the second sum, the expectation of $B_{t_{j+1}} - B_{t_j}$ may be isolated: it is equal to 0. Therefore,

$$\mathbb{E}\Big[\Big(\int_0^T \alpha_t^n dB_t \Big)^2 \Big] = h \sum_{i=0}^{n-1} \mathbb{E}\big[\big| D_x u(t_i, B_{t_i}) \big|^2 \big] = \mathbb{E} \int_0^T (\alpha_t^n)^2 dt.$$

Said differentily, we just built an isometry between $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $L^2([0,T] \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, dt \otimes \mathbb{P})$. It is well-seen that the sequence $(\alpha_t^n)_{0 \le t \le T}$ converges (at least pointwise) towards $(D_x u(t, B_t))_{0 \le t \le T}$. It may be assumed to be bounded if the initial condition u_0 in (4.9) is Lipschitz. Therefore, it has a limit in $L^2([0,T] \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, dt \otimes \mathbb{P})$ and, thus, is Cauchy. As a consequence, the sequence

$$\Big(\int_0^T \alpha_t^n dB_t \Big)_{0 \le t \le T}$$

is Cauchy in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ as well. It is convergent: by definition, the limit is the stochastic integral

$$\int_0^T D_x u(t, B_t) dB_t.$$

4.4.2 Construction

[The reader may skip this part.] Actually, the procedure may be generalized to integrate more general stochastic processes. To do so, we first specify some elements of the theory of stochastic processes (keep in mind that $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a complete probability space):

Definition 4.4.1 *We call a filtration any non-decreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub σ-fields of $\mathcal{F}$.*

In practice, a filtration stands for the available information by observation of the events occurred between the initial and present times. In what follows, filtrations are assumed to be right-continuous, i.e. $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ and complete, i.e. containing sets of zero measure. This is necessary to state some fundamental results for stochastic processes.

Definition 4.4.2 *A process $(X_t)_{t\geq 0}$ is said to be adapted w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ if, for any $t \geq 0$, X_t is $\mathcal{F}_t$-measurable. (That is, the value of X_t is known at time t.)*

Definition 4.4.3 *A Brownian motion $(B_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion if it is adapted w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and if, for any $(t, h) \in \mathbb{R}_+^2$, the increment $B_{t+h} - B_t$ is independent of $\mathcal{F}_t$. For instance, a Brownian motion $(B_t)_{t\geq 0}$ is always a Brownian motion w.r.t. its natural filtration*

$$\mathcal{F}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}, \quad t \geq 0. \tag{4.19}$$

Here, $\sigma(B_s, s \leq t)$ stands for the smallest filtration for which the variables $(B_s)_{0\leq s\leq t}$ are measurable and $\mathcal{N}$ for the collection of sets of zero-measure.

We are now in position to generalize the definition of the stochastic integral:

Definition 4.4.4 *A simple process w.r.t. to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a process of the form*

$$H_t = \sum_{i=0}^{n-1} H^i \mathbf{1}_{(t_i, t_{i+1}]}(t),$$

where H^i is a square-integrable $\mathcal{F}_{t_i}$-measurable random variable and $0 < t_1 < t_2 < \cdots < t_n$. Then, the stochastic integral is

$$\int_0^{+\infty} H_t dB_t = \sum_{i=0}^{n-1} H^i \big(B_{t_{i+1}} - B_{t_i}\big). \tag{4.20}$$

Using, as above, the independence of H^i and of $B_{t_{i+1}} - B_{t_i}$, we can show that

$$\mathbb{E}\left[\left(\int_0^{+\infty} H_t dB_t\right)^2\right] = \mathbb{E}\int_0^{+\infty} H_t^2 dt.$$

As announced above, the integral defines an isometry. By density, we can extend the definition of the integral to the class of so-called *progressively-measurable processes*:

Definition 4.4.5 *A process $(H_t)_{t\geq 0}$ is said to be progressively-measurable w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ if, at any time $T \geq 0$, the joint mapping*

$$(t, \omega) \in [0, T] \times \Omega \mapsto X_t(\omega)$$

is measurable for the product σ-field $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$.

Given a progressively-measurable process such that

$$\mathbb{E}\int_0^{+\infty} H_t^2 dt < +\infty,$$

there exists a sequence $(H_t^n)_{t\geq 0}$ of simple processes converging in $L^2([0, +\infty) \times \Omega, \mathcal{B}([0, +\infty)) \otimes \mathcal{F}, dt \otimes \mathbb{P})$ towards $(H_t)_{t\geq 0}$ so that

$$\int_0^{+\infty} H_s dB_s$$

exists as a limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of a Cauchy sequence. It satisfies Itô's isometry, i.e.

$$\mathbb{E}\left[\left(\int_0^{+\infty} H_s dB_s\right)^2\right] = \mathbb{E}\int_0^{+\infty} H_s^2 ds.$$

The notion of progressive-measurability is necessary: as the isometry property shows, the process is seen as joint function of time and randomness. As example, it may be proven that any (left- or right-)continuous adapted process is progressively-measurable.

4.4.3 Variation of the Integration Bound

To make the connection between Definition 4.4.5 and

$$\int_0^T D_x u(t, B_t) dB_t,$$

we understand the above stochastic integral as

$$\int_0^{+\infty} \mathbf{1}_{(0,T]}(t) D_x u(t, B_t) dB_t.$$

Below, we use the first writing only. Going back to (4.17), we finally write (replacing $(B_t)_{t\geq 0}$ by $(x+B_t)_{t\geq 0}$), for all $t \geq 0$,

$$u(T-t, x+B_t) = u(T,x) + \int_0^t D_x u(T-s, x+B_s) dB_s. \tag{4.21}$$

This writing is a bit awkward because of the time reversal. To obtain a straightforward probabilistic formulation, it turns out to be easier to set (4.9) in a backward sense itself, i.e. with a terminal boundary condition. Actually, in the specific case of Monge–Ampère, this has no real influence since the equation is stationary.

However, we understand from (4.21) how it may be useful to see the stochastic integral as a process, indexed by the upper integration bound. Actually, it is not so easy to do: the integral being defined as an element of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, it is defined up to an event of zero measure only. To let the upper integration bound vary, it is necessary to choose a suitable version at each time:

Proposition 4.4.6 *Given a progressively-measurable stochastic process $(H_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that*

$$\forall t \geq 0, \quad \mathbb{E}\int_0^t H_s^2 ds < +\infty,$$

it is possible to choose for any $t \geq 0$ a version of the stochastic integral

$$\int_0^t H_s dB_s = \int_0^{+\infty} \mathbf{1}_{]0,t]}(s) H_s dB_s,$$

such that the process

$$\left(\int_0^t H_s dB_s\right)_{t\geq 0}$$

be of continuous paths. (That is, is continuous ω by ω.)

Notice that the continuity property is well-understood in (4.21) since the left-hand side therein is continuous.

4.4.4 Martingale Property

There is another remarkable property of the stochastic integral: it is of zero expectation. Said differently, taking the expectation in (4.21) when $t = T$, we obtain

$$u(T, x) = \mathbb{E}\big[u_0(x + B_T)\big].$$

This is nothing but the representation announced in (4.11): this representation is referred as *Feynman-Kac formula.*

Actually, the centering property for the stochastic integral may be seen as a consequence of a more general property: the stochastic integral is a martingale. The martingale property is a projective property based upon the notion of conditional expectation:

Definition 4.4.7 *An adapted process $(M_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a martingale if it is integrable at any time and*

$$\forall 0 \leq s \leq t, \quad \mathbb{E}\big[M_t|\mathcal{F}_s\big] = M_s.$$

In particular, a martingale has a constant expectation.

Go now back to Definition 4.4.4. Considering (4.20), we notice, with the same notations, that

$$\int_0^{t_j} H_r dB_r = \sum_{i=0}^{j-1} H^i (B_{t_{i+1}} - B_{t_i}),$$

for $0 \leq j \leq n$. By conditioning w.r.t. $\mathcal{F}_{t_{j-1}}$, we obtain

$$\mathbb{E}\Big[\int_0^{t_j} H_r dB_r | \mathcal{F}_{t_{j-1}}\Big] = \sum_{i=0}^{j-2} H^i (B_{t_{i+1}} - B_{t_i}) + \mathbb{E}\big[H^{j-1}(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}\big],$$

since the $j-1$ first terms are measurable w.r.t. the σ-field $\mathcal{F}_{t_{j-1}}$. Examinate now the remaining part: we know that H^{j-1} is measurable w.r.t. $\mathcal{F}_{t_{j-1}}$ and that the increment $(B_{t_j} - B_{t_{j-1}})$ is independent of $\mathcal{F}_{t_{j-1}}$. Therefore, the product of both is orthogonal to $L^2(\Omega, \mathcal{F}_{t_{j-1}}, \mathbb{P})$: the conditional expectation is zero. Finally,

$$\mathbb{E}\Big[\int_0^{t_j} H_r dB_r | \mathcal{F}_{t_{j-1}}\Big] = \int_0^{t_{j-1}} H_r dB_r.$$

The argument is actually true for any conditioning by $\mathcal{F}_{t_\ell}$, $0 \leq \ell \leq j-1$. Moreover, noting that any pair (s, t), $0 \leq s \leq t$, may be understood as a subset of the subdivision $\{t_0, \ldots, t_n\}$, we obtain that

$$\mathbb{E}\left[\int_0^t H_r dB_r | \mathcal{F}_s\right] = \int_0^s H_r dB_r,$$

for any s and t. By a density argument, we deduce

Proposition 4.4.8 *Given a progressively-measurable process $(H_t)_{t \geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t \geq 0}$ and satisfying*

$$\forall t \geq 0, \quad \mathbb{E}\left[\int_0^t H_s^2 ds\right] < +\infty,$$

the stochastic integral

$$\left(\int_0^t H_s dB_s\right)_{t \geq 0}$$

is a martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$.

4.4.5 Stopping Times

The reader may wonder about the connection between a process of zero mean and a martingale. Actually, a martingale is a process whose expectation is zero when stopped at any suitable random times, called *stopping times*.

Here is the definition (together with an example):

Definition 4.4.9 *Given a filtration $(\mathcal{F}_t)_{t \geq 0}$, a random variable τ with non-negative (but possibly infinite) values is called a stopping-time if*

$$\forall t \geq 0, \quad \{\tau \leq t\} \in \mathcal{F}_t.$$

As an example, a continuous and adapted process $(X_t)_{t \geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t \geq 0}$ and a closed subset $F \subset \mathbb{R}$, the variable

$$\tau := \inf\{t \geq 0 : X_t \in F\},$$

is a stopping time (the infimum being set as $+\infty$ is the set is empty).

Stopping times are really useful because of the following Doob Theorem:

Theorem 4.4.10 *Given a martingale $(M_t)_{t \geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t \geq 0}$ and a stopping time τ, $(M_{t \wedge \tau})_{t \geq 0}$ is also a martingale (w.r.t. the same filtration). (Here $t \wedge \tau = \min(t, \tau)$.)*

In particular, if τ is bounded by some T, then $\mathbb{E}[M_\tau] = \mathbb{E}[M_{T \wedge \tau}] = \mathbb{E}[M_0]$.

In the above statement, $t \wedge \tau$, for some deterministic time t, is a stopping time again. Indeed, we let the reader check that the minimum of two stopping times is a stopping time as well.

Below, we will also make use of the following version of Doob's theorem:

Theorem 4.4.11 *For a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a stopping time τ (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$), we call σ-field of events occurred before time τ, the σ-field*

$$\mathcal{F}_\tau := \big\{ A \in \mathcal{F} : \forall t \geq 0,\ A \cap \{\tau \leq t\} \in \mathcal{F}_t \big\}.$$

Then, for a martingale $(M_t)_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and for another stopping time $\sigma \geq \tau$,

$$\forall t \geq 0, \quad \mathbf{1}_{\{\tau \leq t\}} \mathbb{E}\big[M_{\sigma\wedge t} | \mathcal{F}_\tau\big] = \mathbf{1}_{\{\tau \leq t\}} M_{\sigma \wedge t}.$$

(Again, it is an easy exercise to check that $\{\tau \leq t\}$ is in $\mathcal{F}_\tau$. Indeed, $\mathcal{F}_\tau$ must be understood as the collection of events for which it may be decided if they have occurred or not at time τ.)

4.5 Probabilistic Writing of Monge–Ampère

We now go back to Sect. 4.2. In order to give a probabilistic representation of (4.2), we first investigate the probabilistic writing of the solution to the Dirichlet problem

$$\mathrm{Trace}\big[aD^2_{z,\bar{z}}u\big](z) = f(z), \quad z \in \mathcal{D}, \tag{4.22}$$

with the boundary condition $u(z) = g(z)$, $z \in \partial\mathcal{D}$, the non-negative Hermitian matrix a being given.

4.5.1 Real Dirichlet Problem

It may be simpler to start with the real case:

$$\mathrm{Trace}\big[aD^2_{x,x}u\big](x) + f(x) = 0, \quad x \in \mathcal{D}\ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

the matrix a being *real*, symmetric and non-negative. Obviously, in this writing, the coefficients f and g together with the domain $\mathcal{D}$ are supposed to be of real structure.

In the case when a is equal to the identity matrix, the process associated with the differential operator $\mathrm{Trace}[D^2_{x,x}\cdot]$ is (up to a multiplicative constant) the d-dimensional Brownian motion, as defined by

Definition 4.5.1 *A process $(B^1_t,\dots,B^d_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is called a d-dimensional Brownian motion if each process $(B^i_t)_{t\geq 0}$, $1\leq i\leq d$, is a Brownian motion and if all of them are independent, i.e. for any time-indices $0<t_1<\dots<t_n$, $n\geq 1$, the vectors $(B^1_{t_1},\dots,B^1_{t_n})$, ..., $(B^d_{t_1},\dots,B^d_{t_n})$ are independent.*

Generally speaking, the stochastic integration theory works in dimension d as in dimension 1. Specifically, the point is to consider a common reference filtration: the natural choice consists in replacing B_s in (4.19) by $(B^1_s,\dots,B^d_s)$. It is also necessary to extend the differential rules given in the statement of Theorem 4.3.2 to the multi-dimensional case.

Theorem 4.5.2 *Itô's formula (or stochastic Taylor formula) in Theorem 4.3.2 extends to the multi-dimensional setting. For a d-dimensional Brownian motion $(B_t=(B^1_t,\dots,B^d_t))_{t\geq 0}$ and a function $f\in\mathcal{C}([0,+\infty)\times\mathbb{R}^d,\mathbb{R})$, the infinitesimal variation of $(f(t,B_t))_{t\geq 0}$ expands as*

$$d\big[f(t,B_t)\big] = \left[D_t f(t,B_t)+\frac{1}{2}\sum_{i=1}^d D^2_{x_i,x_i}f(t,B_t)\right]dt+\sum_{i=1}^d D_{x_i}f(t,B_t)dB^i_t,\quad t\geq 0.$$

Sketch of the Proof. We just provide the main idea. Generally speaking, the proof relies on the d-dimensional Taylor formula. The only problem is to understand how behave the infinitesimal products $dB^i_t dB^j_t$, $1\leq i,j\leq d$.

Obviously, $dB^i_t dB^i_t=dt$ for any $1\leq i\leq d$. When $i\neq j$, $dB^i_t dB^j_t$ is set as 0. This definition may be understood by discretizing the underlying dynamics with a microscopic stepsize. Indeed, if $0=t_0<t_1<\dots<t_n$ is a time-grid of stepsize h, we may compute

$$\mathbb{E}\left[\left(\sum_{k=0}^{n-1}(B^i_{t_{k+1}}-B^i_{t_k})(B^j_{t_{k+1}}-B^j_{t_k})\right)^2\right],$$

as in (4.16).

The idea is then the same as in (4.16). Variables are clearly independent and of zero expectation so that the expectation of the square of the sum is equal to the sum of the variances. Now, since $\mathbb{E}[(B^i_{t_{k+1}}-B^i_{t_k})^2(B^j_{t_{k+1}}-B^j_{t_k})^2]=h^2$, the sum is equal to nh^2. It is thus microscopic at the macroscopic level according to the same argument as in (4.16). Macroscopic contributions of the crossed terms are therefore zero. □

We now provide an example of application. (In what follows, we will write B_t for $(B_t^1, \dots, B_t^d)$, so that B_t stands for a vector.)

When $a = (1/2)I_d$ and f and g are regular enough (say f is Hölder continuous and g has Hölder continuous second-order derivatives), it is well-known that the real Dirichlet problem

$$\frac{1}{2}\Delta u(x) + f(x) = 0, \quad x \in \mathcal{D} \; ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

has a unique classical solution, with bounded derivatives. For $x \in \mathcal{D}$, we write the infinitesimal dynamics of $(u(x + B_t))_{t \geq 0}$. We obtain

$$\begin{aligned} du(x + B_t) &= \sum_{i=1}^d D_{x_i} u(x + B_t) dB_t^i + \frac{1}{2} \sum_{i=1}^d D^2_{x_i, x_i} u(x + B_t) dt \\ &= \sum_{i=1}^d D_{x_i} u(x + B_t) dB_t^i - f(x + B_t) dt. \end{aligned} \tag{4.23}$$

On the macroscopic scale, we obtain (with $B_0 = 0$)

$$u(x + B_t) = u(x) - \int_0^t f(x + B_s) ds + \sum_{i=1}^d \int_0^t D_{x_i} u(x + B_t) dB_t^i.$$

This writing is actually unsatisfactory: it holds when $x + B_t$ belongs to $\mathcal{D}$ only; otherwise, it is meaningless. To make things rigorous, we introduce the stopping time:

$$\tau^x := \inf\{t \geq 0 : x + B_t \in \mathcal{D}^{\complement}\}.$$

We are then able to write

$$\begin{aligned} &u(x + B_t) \\ &\quad = u(x) - \int_0^t f(x + B_s) ds + \sum_{i=1}^d \int_0^t D_{x_i} u(x + B_t) dB_t^i, \quad 0 \leq t \leq \tau^x. \end{aligned}$$

We emphasize that the martingale term is well-defined since the gradient is bounded. (Actually, for what follows, it would be sufficient that the gradient be continuous inside $\mathcal{D}$ and thus bounded on every compact subset of $\mathcal{D}$.) Taking the expectation at time $t \wedge \tau^x$ and applying Doob's Theorem de Doob 4.4.10, we obtain

$$\mathbb{E}\big[u(x + B_{t \wedge \tau^x})\big] = u(x) - \mathbb{E} \int_0^{t \wedge \tau^x} f(x + B_s) ds. \tag{4.24}$$

We then intend to let t tend to the infinity. This is possible if $\mathbb{E}[\tau^x] < +\infty$.

Theorem 4.5.3 *For any $x \in \mathcal{D}$, define τ^x as $\tau^x := \inf\{t \geq 0 : x + B_t \in \mathcal{D}^{\complement}\}$. Then, for any $x \in \mathcal{D}$, $\mathbb{E}[\tau^x] < +\infty$.*

In particular, if f is Hölder continuous on $\mathcal{D}$ and g has Hölder continuous second-order derivatives in the neighborhood of $\bar{\mathcal{D}}$, then the solution u to the Dirichlet problem

$$\frac{1}{2}\Delta u(x) + f(x) = 0, \quad x \in \mathcal{D} \ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

admits the following Feynman-Kac representation

$$u(x) = \mathbb{E}\left[g(x + B_{\tau^x}) + \int_0^{\tau^x} f(x + B_s)ds\right].$$

Proof. It is sufficient to prove $\mathbb{E}[\tau^x] < +\infty$. Feynman–Kac formula then follows by letting t to $+\infty$ in (4.24).

To prove $\mathbb{E}[\tau^x] < +\infty$, we use the non-degeneracy property of the identity matrix in one arbitrarily chosen direction of the space. Compute indeed

$$\begin{aligned} d|x + B_t|^2 = d\left[\sum_{i=1}^d |x_i + B_t^i|^2\right] &= \sum_{i=1}^d \big[2(x_i + B_t^i)dB_t^i + (dB_t^i)^2\big] \\ &= 2\sum_{i=1}^d (x_i + B_t^i)dB_t^i + d\,dt. \end{aligned}$$

Take expectation at time $t \wedge \tau^x$. Since $\mathcal{D}$ is bounded, we obtain

$$\sup_{t \geq 0} \mathbb{E}\big[t \wedge \tau^x\big] < +\infty.$$

By monotonous convergence Theorem, we complete the proof. □

When the identity matrix is replaced by a non-zero symmetric matrix a, Brownian motion is replaced by the process

$$X_t := x + \int_0^t \sigma dB_s, \quad t \geq 0, \tag{4.25}$$

where σ is a square-root of a, i.e. $\sigma\sigma^* = a$. This writing must be understood as

$$X_t^i = x_i + \sum_{j=1}^d \int_0^t \sigma_{i,j} dB_s^j, \quad t \geq 0.$$

Following (4.23), we then obtain

$$du(X_t) = \sum_{i=1}^d D_{x_i} u(X_t) dX_t^i + \frac{1}{2}\sum_{i,j=1}^d D^2_{x_i,x_j} u(X_t) dX_t^i dX_t^j, \quad t \geq 0. \tag{4.26}$$

Here, $dX_t^i = \sum_{j=1}^d \sigma_{i,j} dB_t^j$ and the differential rules have the form

$$dX_t^i dX_t^j = \sum_{k,\ell=1}^d \sigma_{i,k}\sigma_{j,\ell} dB_t^k dB_t^\ell = \sum_{k=1}^d \sigma_{i,k}\sigma_{j,k} dt = (\sigma\sigma^*)_{i,j} dt.$$

If $\det(a) \neq 0$, we then obtain an analogous representation to the one obtained for the Laplace operator.

Theorem 4.5.4 *Consider a positive symmetrix matrix a with σ as square-root, i.e. $a = \sigma\sigma^*$. For any $x \in \mathcal{D}$, consider $(X_t^x)_{t\geq 0}$ as in (4.25) and set $\tau^x := \inf\{t \geq 0 : X_t \in \mathcal{D}^\complement\}$. Then, $\mathbb{E}[\tau^x] < +\infty$.*

Moreover, if f is Hölder continuous on $\mathcal{D}$ and g has Hölder continuous second-order derivatives in the neighborhood of $\bar{\mathcal{D}}$, then the solution u to the Dirichlet problem

$$\frac{1}{2}\mathrm{Trace}\big[aD^2_{x,x}u\big](x) + f(x) = 0, \quad x \in \mathcal{D}\ ; \quad u(x) = g(x), \quad x \in \partial\mathcal{D},$$

admits the Feynman-Kac representation

$$u(x) = \mathbb{E}\bigg[g(X^x_{\tau^x}) + \int_0^{\tau^x} f(X_s^x) ds\bigg].$$

Sketch of the Proof. The boundedness of the expectation of the hitting time is proved as in Theorem 4.5.3. By Itô's formula (4.26), we complete the proof. □

4.5.2 Complex Brownian Motion

Consider now the complex Dirichlet problem. With the same notation as above (but understood in the complex sense), we are seeking for a representation of the solution u to

$$\mathrm{Trace}\big[aD_{z,\bar{z}}u\big](z) + f(z) = 0, \quad z \in \mathcal{D}\ ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D}.$$

Here, the matrix a is a non-negative Hermitian matrix.

The solution u may be represented as above. We are going to reproduce the same computations, but with respect to the complex Brownian motion:

Definition 4.5.5 *A complex Brownian motion of dimension d is a d-dimensional process $(B_t = (B_t^1, \ldots, B_t^d))_{t\geq 0}$ with values in $\mathbb{C}^d$ given by*

$$B_t^j = \frac{W_t^{j,1} + \sqrt{-1}\, W_t^{j,2}}{\sqrt{2}}, \quad t \geq 0,\ 1 \leq j \leq d,$$

where the processes $(W_t^{j,1}, W_t^{j,2})_{1 \le j \le d}$ *are independent real Brownian motions.*

We emphasize that the coefficient $\sqrt{2}$ is here to normalize the expectation of the square modulus of B_t, i.e. $\mathbb{E}[|B_t|^2] = t$, $t \ge 0$.

Differential rules are given by

Proposition 4.5.6 *Let* $(B_t = (B_t^1, \dots, B_t^d))_{t \ge 0}$ *be a complex Brownian motion of dimension* d*. Then, Itô's formula in Theorem 4.5.2 holds with* f *function of the complex variable of dimension* d *and with the differential rules*

$$dB_t^i dB_t^j = 0,\ dB_t^i d\bar{B}_t^j = \mathbf{1}_{\{i=j\}} dt, \quad 1 \le i, j \le d.$$

Sketch of the Proof. For $1 \le i \le d$,

$$dB_t^i dB_t^i = \frac{(dW_t^{i,1})^2 - (dW_t^{i,2})^2 + 2\sqrt{-1}\, dW_t^{i,1} dW_t^{i,2}}{2} = 0.$$

Similarly, $d\bar{B}_t^i d\bar{B}_t^i = 0$ and

$$dB_t^i d\bar{B}_t^i = \frac{(dW_t^{i,1})^2 + (dW_t^{i,2})^2 + 2\sqrt{-1}\, dW_t^{i,1} dW_t^{i,2}}{2} = dt.$$

Finally, for $1 \le i < j \le d$,

$$dB_t^i dB_t^j = dB_t^i d\bar{B}_t^j = 0.$$

This completes the proof. □

Give now several examples.

Example (a). If $d = 1$ and $(Z_t^1)_{t \ge 0}$ and $(Z_t^2)_{t \ge 0}$ admit

$$\begin{aligned} dZ_t^1 &= \sigma_t^1 dB_t + b_t^1 dt \\ dZ_t^2 &= \sigma_t^2 dB_t + b_t^2 dt, \quad t \ge 0, \end{aligned}$$

as dynamics, we obtain

$$\begin{aligned} d(Z_t^1 Z_t^2) &= Z_t^1 dZ_t^2 + Z_t^2 dZ_t^1 + dZ_t^1 dZ_t^2 \\ &= (Z_t^1 \sigma_t^2 + Z_t^2 \sigma_t^1) dB_t + (Z_t^1 b_t^2 + Z_t^2 b_t^1) dt + \sigma_t^1 \sigma_t^2 dB_t dB_t, \quad t \ge 0. \end{aligned}$$

(Pay attention that the absolutely continuous parts $b_t^1 dt$ and $b_t^2 dt$ play no role in the product $dZ_t^1 dZ_t^2$: all the terms they induce are least of order $dt^{3/2}$.) Now, $dB_t dB_t = 0$ in the above equation.

However,

$$
\begin{aligned}
&d(Z_t^1 \bar{Z}_t^2)\\
&\quad = Z_t^1 d\bar{Z}_t^2 + \bar{Z}_t^2 dZ_t^1 + dZ_1 d\bar{Z}_t^2\\
&\quad = (Z_t^1 \bar{\sigma}_t^2 d\bar{B}_t + \bar{Z}_t{}^2 \sigma_t^1 dB_t) + (Z_t^1 \bar{b}_t^2 + \bar{Z}_t^2 b_t^1)dt + \sigma_t^1 \bar{\sigma}_t^2 dB_t d\bar{B}_t, \quad t \geq 0.
\end{aligned}
$$

Here, $dB_t \cdot d\bar{B}_t = dt$.

In particular, if

$$
Z_t = \sum_{j=1}^n \sigma_j dB_t^j, \quad t \geq 0,
$$

where $((B_t^j)_{t\geq 0})_j$ are independent complex Brownian motion (i.e. $(B_t = (B_t^1, \dots, B_t^d))_{t\geq 0}$ is a complex Brownian motion of dimension d), then

$$
\begin{aligned}
d|Z_t|^2 &= Z_t d\bar{Z}_t + \bar{Z}_t dZ_t + dZ_t d\bar{Z}_t\\
&= Z_t \sum_{j=1}^n \bar{\sigma}_j d\bar{B}_t^j + \bar{Z}_t \sum_{j=1}^n \sigma_j dB_t^j + \sum_{j=1}^n \sigma_j \bar{\sigma}_j dt, \quad t \geq 0.
\end{aligned}
$$

For example, if $\sigma_j = (\sigma\xi)_j$ for a matrix σ, then the last term is equal to $|\sigma\xi|^2$, i.e. to $\langle \bar{\xi}, a\xi \rangle$ where $a = \bar{\sigma}^* \sigma$. This is also equal to $\langle a^* \bar{\xi}, \xi \rangle$.

Example (b). Assume that $d = 1$ and consider an holomorphic function f on $\mathbb{C}$. Then,

$$
df(B_t) = f_z'(B_t)dB_t + \frac{1}{2} f_{z,z}''(B_t)dB_t dB_t = f_z'(B_t)dB_t, \quad t \geq 0.
$$

In particular, if $\tau_R := \inf\{t \geq 0 : |B_t| \geq R\}$, $R > 0$, then $(f(B_{t\wedge\tau_R}))_{t\geq 0}$ is a martingale. (Here, the stopping time is necessary to guarantee that the martingale is integrable: such an argument is called "a localization argument".) We will say that $(f(B_t))_{t\geq 0}$ is a local martingale.

Example (c). Assume now that $d \geq 1$. Consider a function u with real values of class $\mathcal{C}^2$ on the domain $\mathcal{D}$ and compute $du(X_t)$, $t \geq 0$, where

$$
X_t = z + \int_0^t \sigma dB_s, \quad t \geq 0,
$$

with σ complex matrix of size $d \times d$. We obtain, for any $t \geq 0$,

$$
\begin{aligned}
du(X_t) & \\
&= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{z_i,z_j} u(X_t)(dX_t)^i (dX_t)^j + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{\bar{z}_i,\bar{z}_j} u(X_t)(dX_t)^i (dX_t)^j \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{z_i,\bar{z}_j} u(X_t)(dX_t)^i (d\bar{X}_t)^j + \frac{1}{2} \sum_{i,j=1}^{d} D^2_{\bar{z}_i,z_j} u(X_t)(d\bar{X}_t)^i (dX_t)^j.
\end{aligned}
$$

It is well-seen that $(dX_t)^i(dX_t)^j = 0$ and $(d\bar{X}_t)^i(d\bar{X}_t)^j = 0$, $1 \le i,j \le d$. Moreover, $(dX_t)^i(d\bar{X}_t)^j = \sum_{\ell=1}^{d} \sigma_{i,k}\bar{\sigma}_{k,j} dt = (\sigma\bar{\sigma}^*)_{i,j} dt$. Therefore,

$$
\begin{aligned}
du(X_t) &= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i \\
&\qquad + \frac{1}{2} \mathrm{Trace}\big[a D^2_{z,\bar{z}} u(X_t)\big] dt + \frac{1}{2} \mathrm{Trace}\big[\bar{a} D^2_{\bar{z},z} u(X_t)\big] dt, \quad t \ge 0.
\end{aligned}
$$

Finally, since a and $D^2_{z,\bar{z}} u$ are Hermitian, we deduce

$$
\begin{aligned}
du(X_t) & \\
&= \sum_{i=1}^{d} D_{z_i} u(X_t) dX_t^i + \sum_{i=1}^{d} D_{\bar{z}_i} u(X_t) d\bar{X}_t^i + \mathrm{Trace}\big[a D^2_{z,\bar{z}} u(X_t)\big] dt, \quad t \ge 0.
\end{aligned}
$$

Obviously, this is true for $t \le \tau^z := \inf\{t \ge 0 : X_t \notin \mathcal{D}\}$ only. We then deduce the analog of Theorem 4.5.3:

Theorem 4.5.7 *Let a be a positive Hermitian complex matrix of size $d \times d$ and σ be an Hermitian square-root of a, i.e. $a = \sigma\bar{\sigma}^*$. For a given $z \in \mathcal{D}$ ($\mathcal{D}$ being here assumed to be of the complex variable of dimension d), set*

$$
X_t^z = z + \int_0^t \sigma dB_s, \quad t \ge 0,
$$

together with $\tau^z := \inf\{t \ge 0 : X_t \notin \mathcal{D}\}$. Then, $\mathbb{E}[\tau^z] < +\infty$.

Moreover, for given real-valued functions f and g of the complex variable of dimension d, satisfying the same assumption as in Theorem 4.5.3, the solution u to the complex Dirichlet problem

$$
\mathrm{Trace}\big[a D^2_{z,\bar{z}} u(z)\big] + f(z) = 0, \quad z \in \mathcal{D}\ ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D},
$$

admits the Feynman-Kac representation

$$u(z) = \mathbb{E}\Big[g(X^z_{\tau^z}) + \int_0^{\tau^z} f(X^z_s) ds\Big].$$

4.5.3 *Formulation* "à la Gaveau"

We are now in position to give a probabilistic representation of the solution of the Monge–Ampère equation. In light of (4.2) and (4.8), a natural candidate to solve the Monge–Ampère equation is

$$\forall z \in \bar{\mathcal{D}}, \quad u(z) = \inf \mathbb{E}\Big[g(X^{\sigma;z}_{\tau^{\sigma,z}}) - \int_0^{\tau^{\sigma,z}} f(X^{\sigma,z}_t) dt\Big], \tag{4.27}$$

the *infimum* being here taken over all progressively-measurable processes $(\sigma_t)_{t \geq 0}$ with values in the set of complex matrices of size d and of determinant of modulus 1, i.e. $\det(\sigma_t \bar{\sigma}^*_t) = 1$ for all $t \geq 0$, with

$$X^{\sigma,z}_t = z + \int_0^t \sigma_s dB_s, \quad t \geq 0 \; ; \quad \tau^{\sigma,z} := \inf\{t \geq 0 : X^{\sigma,z}_t \in \mathcal{D}^{\complement}\}. \tag{4.28}$$

We emphasize that this is an *infimum* and not a *supremum* despite the *supremum* in (4.2). The reason may be understood as follows.

Proposition 4.5.8 *Let σ be a (non-zero) complex matrix of size $d \times d$ and u be a $\mathcal{C}(\bar{\mathcal{D}}) \cap \mathcal{C}^2(\mathcal{D})$ function satisfying*

$$-\mathrm{Trace}\big[a D^2_{z,\bar{z}} u(z)\big] + f(z) \leq 0, \quad z \in \mathcal{D} \; ; \quad u(z) = g(z), \quad z \in \partial\mathcal{D}, \tag{4.29}$$

*where $a = \sigma\sigma^*$ and f and g are functions from $\mathcal{D}$ into $\mathbb{R}$ as in Theorem 4.5.7 (or as in Assumption (**A**)).*

For a given $z \in \mathcal{D}$, define $(X^z_t)_{t \geq 0}$ and τ^z as in Theorem 4.5.7. Then,

$$u(z) \leq \mathbb{E}\Big[g(X^z_{\tau^z}) - \int_0^{\tau^z} f(X^z_s) ds\Big].$$

Sketch of the Proof. The proof is similar to the proof of Theorem 4.5.7 and relies on a simple application of Itô's formula. □

Pay attention that u is here assumed to be smooth. In particular, the reader may object that the solution to the Monge–Ampère equation is not assumed to be of class $\mathcal{C}^2$, so that Proposition 4.5.8 does not apply to it. Actually, Proposition 4.5.8 must be understood as some heuristics towards the probabilistic formulation of Monge–Ampère.

In PDE theory, a function u satisfying (4.29) is called a *subsolution* to the Dirichlet problem driven by a, f and g. From a probabilistic point of view, it says that the process $(u(X_t^z))_{t\geq 0}$ is a sub-martingale when $f \geq 0$, i.e. the infinitesimal variation of $(u(X_t^z))_{t\geq 0}$ is greater than the infinitesimal variation of a martingale.

Proposition 4.5.8 may be seen a variation of the maximum principle: there exists a comparison principle between the solutions of the Dirichlet problems driven by the same matrix a. Going back to the formulation (4.2) of Monge–Ampère, we then understand that the solution to Monge–Ampère is expected to be less than the solution to any Dirichlet problem driven by the same f and g as in Monge–Ampère and by any non-negative Hermitian matrix of determinant 1.

We derive the following representation principle, which may be seen as a probabilistic variation of the Perron-Bremermann method discussed in Chap. 2 by V. Guedj and A. Zeriahi (see Sect. 2.2 therein).[3]

Definition 4.5.9 *Let f and g be as in Assumption ($\mathbf{A}$) and $(B_t)_{t\geq 0}$ be a complex Brownian motion of dimension d. We call* Gaveau representation *or* Gaveau candidate for the Monge–Ampère equation *the function u given by*

$$\forall z \in \bar{\mathcal{D}}, \quad u(z) = \inf \mathbb{E}\left[g\left(X^{\sigma,z}_{\tau^{\sigma,z}}\right) - \int_0^{\tau^{\sigma,z}} f(X_s^{\sigma,z}) ds \right],$$

the infimum *being taken over the set of progressively-measurable processes $(\sigma_t)_{t\geq 0}$ with values in $\mathbb{C}^{d\times d}$ such that $\det(\sigma_t \bar{\sigma}_t^*) = 1$, $t \geq 0$, the process $(X_t^{\sigma,z})_{t\geq 0}$ being given by*

$$X_t^{\sigma,z} = z + \int_0^t \sigma_s dB_s, \quad t \geq 0,$$

and the stopping time $\tau^{\sigma,z}$ by $\tau^{\sigma,z} = \inf\{t \geq 0 : X_t^{\sigma,z} \notin \mathcal{D}\}$.

As the reader may guess, Definition 4.5.9 goes back to the earlier paper by Gaveau [Gav77]. In fact, it is different from the one used by Krylov in his works and thus different from the one we use below. The reason why Krylov introduced a different representation in his own analysis may be explained as follows: in Definition 4.5.9, the control σ is poorly controlled! Said differently, the condition on the determinant of $\sigma\bar{\sigma}^*$ is really weak since the norm of the matrix $\sigma\bar{\sigma}^*$ may be as large as possible.

[3]We here say "variation" of the Perron-Bremermann method since the optimization below is not performed over a set of plurisubharmonic functions as in the Perron-Bremermann method. Plurisubharmonicity is here hidden in the very large choice for the stochastic process $(\sigma_t)_{t\geq 0}$: this is the reason why we say "probabilistic variation".

Nevertheless, we emphasize that the connection between the candidate u in Definition 4.5.9 and the Monge–Ampère equation is rigorously established in the original paper by Gaveau. We refer the reader to it for the complete argument.

4.5.4 Krylov Point of View

Krylov's strategy is a bit different. The starting point consists in writing the original Monge–Ampère formulation

$$\det{}^{1/d}\big[D^2_{z,\bar{z}}u(z)\big] = \frac{1}{d}f(z), \quad z \in \mathcal{D}, \tag{4.30}$$

under the form

$$\sup\big\{-\mathrm{Trace}\big(aD^2_{z,\bar{z}}u(z)\big) + \det{}^{1/d}(a)f(z)\,;\, a = \bar{a}^* \geq 0,\ \mathrm{Trace}(a) = 1\big\} = 0, \tag{4.31}$$

$z \in \mathcal{D}$. Obviously, the first problem is to prove that any $\mathcal{C}^{1,1}$ solution u to (4.31) satisfies (4.30) as well.

Assume therefore that there exists a $\mathcal{C}^{1,1}$ function u from $\mathcal{D}$ to $\mathbb{R}$ solving (4.31) almost everywhere in $\mathcal{D}$. Since u is $\mathcal{C}^{1,1}$, $D^2_{z,\bar{z}}u(z)$ exists for almost every $z \in \mathcal{D}$. By (4.31) and by the sign condition $f \geq 0$, for almost every $z \in \mathcal{D}$, $\mathrm{Trace}(aD^2_{z,\bar{z}}u(z)) \geq 0$ for any non-negative Hermitian matrix a, so that u is plurisubharmonic. Choose now some $z \in \mathcal{D}$ at which $D^2_{z,\bar{z}}u(z)$ exists. If $D^2_{z,\bar{z}}u(z)$ is equal to zero, we can find a positive Hermitian matrix a (with a non-zero determinant) with 1 as trace such that $\mathrm{Trace}(aD^2_{z,\bar{z}}u(z)) = 0$. In particular, (4.31) says that $f(z) \leq 0$ so that $f(z) = 0$ since f is non-negative: (4.30) holds at point z. If the determinant in non-zero at z, the complex Hessian $D^2_{z,\bar{z}}u(z)$ is non-degenerate. In particular it is positive. Therefore, for any sequence $(a_n)_{n\geq 1}$ of non-degenerate matrices approximating the *supremum* in (4.31), the determinant of a_n, $n \geq 1$, is away from zero, uniformly in n. (If the determinant has some vanishing subsequence, we can find a non-zero non-negative Hermitian matrix a such that $\mathrm{Trace}(aD^2_{z,\bar{z}}u(z)) = 0$: by Lemma 4.2.1, $D^2_{z,\bar{z}}u(z)$ is of zero determinant.) Therefore, by compactness, there exists a matrix a with 1 as determinant such that

$$-\mathrm{Trace}\big(aD^2_{z,\bar{z}}u(z)\big) + f(z) = 0.$$

By Lemma 4.2.1, we understand that $\det^{1/d}(D^2_{z,\bar{z}}u(z)) \leq f(z)/d$. Now, choosing the matrix a in (4.31) as $a = (D^2_{z,\bar{z}}u(z))^{-1}/\mathrm{Trace}[(D^2_{z,\bar{z}}u(z))^{-1}]$, we obtain

$$-d + \det^{-1/d}(D^2_{z,\bar{z}}u(z))f(z) \le 0,$$

i.e. $f(z)/d \le \det^{1/d}(D^2_{z,\bar{z}}u(z))$, so that equality holds.

The value function associated with the optimal control problem (4.31) admits the following (formal) probabilistic representation

$$\forall z \in \mathcal{D}, \quad u(z) = \inf \mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) - \int_0^{\tau^{\sigma,z}} \det^{1/d}(\sigma_t\bar{\sigma}^*_t)f(X^{\sigma,z}_t)dt\Big],$$

the *infimum* being here taken over the progressively-measurable processes $(\sigma_t)_{t\ge 0}$ with values in the set of complex matrices of size d such that $\mathrm{Trace}(\sigma_t\bar{\sigma}^*_t) = 1$ for any $t \ge 0$, with

$$X^{\sigma,z}_t = z + \int_0^t \sigma_s dB_s, \quad t \ge 0\ ; \quad \tau^{\sigma,z} := \inf\{t \ge 0 : X^{\sigma,z}_t \in \mathcal{D}^{\complement}\}.$$

In what follows, we will investigate $-u$ instead of u itself. Changing g into $-g$ in the original Monge–Ampère equation, we set

Definition 4.5.10 *Let f and g be as in Assumption (**A**) and $(B_t)_{t\ge 0}$ be a complex Brownian motion of dimension d. We call* Krylov formulation *of the Monge–Ampère equation driven by the source term f and the boundary condition $-g$ (and not g) the function $-v$, where*

$$v(z) = \sup_{\sigma} v^{\sigma}(z), \quad z \in \bar{\mathcal{D}}, \tag{4.32}$$

the supremum being here taken over the set of progressively-measurable processes $(\sigma_t)_{t\ge 0}$ with values in $\mathbb{C}^{d\times d}$ such $\mathrm{Trace}(\sigma_t\bar{\sigma}^*_t) = 1$, *$t \ge 0$, and v^{σ} being given by*

$$v^{\sigma}(z) = \mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det^{1/d}(a_t)f(X^{\sigma,z}_t)dt\Big], \quad a_t = \sigma_t\bar{\sigma}^*_t, \tag{4.33}$$

the process $(X^{\sigma,z}_t)_{t\ge 0}$ by

$$X^{\sigma,z}_t = z + \int_0^t \sigma_s dB_s, \quad t \ge 0,$$

and the stopping time $\tau^{\sigma,z}$ by $\tau^{\sigma,z} = \inf\{t \ge 0 : X^{\sigma,z}_t \notin \mathcal{D}\}$.

If v is $\mathcal{C}^{1,1}$ on $\mathcal{D}$ and $-v$ satisfies (4.31) almost everywhere, i.e.

$$\sup\big\{\mathrm{Trace}\big(aD^2_{z,\bar{z}}v(z)\big) + \det^{1/d}(a)f(z)\,;\, a = \bar{a}^* \ge 0,\ \mathrm{Trace}(a) = 1\big\} = 0, \tag{4.34}$$

a.e. $z \in \mathcal{D}$, then $-v$ is plurisubharmonic and satisfies the Monge–Ampère equation (4.30). If $-v$ is continuous up to the boundary $\partial\mathcal{D}$, it admits $-g$ as boundary condition.

The reader may worry about the boundary condition. First, why is it satisfied? Second, may we expect the solution to be continuous up to the boundary $\partial\mathcal{D}$? The answer to the first question is quite obvious: when $z \in \partial\mathcal{D}$, the stopping time $\tau^{\sigma,z}$ is zero, so that $X^{\sigma,z}_{\tau^{\sigma,z}} = z$. Concerning the second question, we will see below that the answer is clearly positive under Assumption (**A**).

4.5.5 Dynamic Programming Principle

The Definition 4.5.10 is not completely satisfactory. The right question is now: may we claim that $-v$ given by (4.32) is a solution to Monge–Ampère without making any reference to the Hamilton–Jacobi–Bellman equation (4.31)?

We will see below that the answer is almost positive. We say almost because, to say so, we need some regularity property on v, as in Definition 4.5.10.

Proposition 4.5.11 *Under the notation of Definition 4.5.10, assume that the family $(v^\sigma)_\sigma$ is equicontinuous on every compact subset of $\mathcal{D}$ and that v is $\mathcal{C}^{1,1}$ on $\mathcal{D}$. Then, $-v$ satisfies (4.31) almost everywhere and thus satisfies the Monge–Ampère equation (4.30).*

Proof. The proof relies on a variation of the so-called "Dynamic Programming Principle" (or Bellman Principle). The main point is to split the cost (4.33) of reaching the boundary of $\mathcal{D}$ when starting from a given point z into two parts: the cost of reaching the boundary of a subdomain from z and the cost of reaching $\partial\mathcal{D}$ when starting from the boundary of the subdomain.

We thus fix a given point $z \in \mathcal{D}$ at which v is twice differentiable in the sense of Taylor, i.e. admits a Taylor expansion of order two at z. (Have in mind that v is almost-everywhere twice differentiable in the sense of Taylor since belongs to $\mathcal{C}^{1,1}(\mathcal{D})$.) Fix also a positive real ε such that the closed (complex) ball $\bar{B}(z,\varepsilon)$ of center z and radius ε is included in $\mathcal{D}$. For any $(\sigma_t)_{t\geq 0}$ as in Definition 4.5.10, define ρ^σ as the first exit time from the open ball $B(z,\varepsilon)$ by the process $X^{z,\sigma}$, i.e. $\rho^\sigma := \inf\{t \geq 0 : |X^{z,\sigma}_t - z| \geq \varepsilon\}$. Then, the Dynamic Programming Principle writes

Lemma 4.5.12 *Under the notation of Definition 4.5.10, assume that the family $(v^\sigma)_\sigma$ is equicontinuous on every compact subset of $\mathcal{D}$. Then, the Dynamic Programming Principle holds in the following way*

$$v(z) = \sup_\sigma \mathbb{E}\Big[v(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t)dt\Big], \quad a_t = \sigma_t\bar{\sigma}^*_t, \qquad (4.35)$$

the supremum *being here taken w.r.t. the processes* $(\sigma_t)_{t\geq 0}$ *as in Definition 4.5.10.*

Proof of the Lower Bound in Lemma 4.5.12. By (4.33),

$$v^\sigma(z) = \mathbb{E}\bigg\{\mathbb{E}\bigg[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_{\rho^\sigma}^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t)dt | \mathcal{F}_{\rho^\sigma}\bigg] + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t)dt\bigg\}. \tag{4.36}$$

A part of the trick for the Dynamic Programming Principle is the following: the conditional expectation above is less than $v(X_{\rho^\sigma})$. Indeed, for $t \geq \rho^\sigma$,

$$X^{\sigma,z}_t = X^{\sigma,z}_{\rho^\sigma} + \int_{\rho^\sigma}^t \sigma_s dB_s,$$

so that the conditional expectation may be understood as an integration with respect to the trajectories of $(X^{\sigma,z}_t)_{t\geq \rho^\sigma}$ with $X^{\sigma,z}_{\rho^\sigma}$ as starting point. (In particular, the interval $[\rho^\sigma, \tau^{\sigma,z}]$ on which $(\det^{1/d}(a_t) f(X^{\sigma,z}_t))_{t\geq 0}$ is integrated in the conditional expectation represents the time passed from ρ^σ up to the exit time from $\mathcal{D}$.) Therefore,

$$v^\sigma(z) \leq \mathbb{E}\bigg[v(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t)dt\bigg]. \tag{4.37}$$

Taking the supremum w.r.t. σ, we complete the proof of the lower bound.

Proof of the Subsolution Property in Monge–Ampère. We now deduce the subsolution property from the lower bound in the Dynamic Programming Principle. Since v is twice Taylor differentiable at z, we can write

$$v(X^{\sigma,z}_{\rho^\sigma}) = v(z) + 2\mathrm{Re}\big[D_z v(z)\big(X^{\sigma,z}_{\rho^\sigma} - z\big)\big] + \frac{1}{2}\big[H^0[v(z)](X^{\sigma,z}_{\rho^\sigma} - z)\big] + o_\varepsilon(1)\varepsilon^2, \tag{4.38}$$

the notation $o_\varepsilon(1)$ standing for the Landau notation (i.e. $o_\varepsilon(1)$ tends to 0 with ε) and being independent of the control σ and the underlying randomness ω. Above $H^0[v(z)](\nu)$, for $\nu \in \mathbb{C}^d$, stands for $H^0[v(z)](\nu) = \sum_{i,j=1}^d (D^2_{z_i,z_j} v(z)\nu_i\nu_j + D^2_{z_i,\bar{z}_j} v(z)\nu_i\bar{\nu}_j + D^2_{\bar{z}_i,z_j} v(z)\bar{\nu}_i\nu_j + D^2_{\bar{z}_i,\bar{z}_j} v(z)\bar{\nu}_i\bar{\nu}_j)$. By Itô's formula, it is plain to see that

$$\mathbb{E}[H^0[v(z)](X^{\sigma,z}_{\rho^\sigma} - z)] = 2\mathbb{E}\bigg[\int_0^{\rho^\sigma} \mathrm{Trace}\big(a_t D^2_{z,\bar{z}} v(z)\big)dt\bigg].$$

It is also clear that $\mathrm{Re}[D_z v(z)(X^{\sigma,z}_{\rho^\sigma} - z)]$ in (4.38) has zero expectation.

Add now $\int_0^{\rho^\sigma} \det^{1/d}(a_t) f(X_t^{\sigma,z})dt$ to both sides in (4.38) and take the expectation. Then,

$$\mathbb{E}\bigg[v(X_{\rho^\sigma}^{\sigma,z}) + \int_0^{\rho^\sigma} \det^{1/d}(a_t) f(X_t^{\sigma,z})dt\bigg]$$
$$= v(z) + \mathbb{E}\bigg[\int_0^{\rho^\sigma} \big[\mathrm{Trace}\big(a_t D^2_{z,\bar{z}} v(z)\big) + \det^{1/d}(a_t) f(X_t^{\sigma,z})\big]dt\bigg] + o_\varepsilon(1)\varepsilon^2.$$

Therefore, applying (4.37) and using the continuity of f,

$$v^\sigma(z) \le v(z) + \sup_{a=\bar{a}^*\ge 0, \mathrm{Trace}(a)=1} \big[\mathrm{Trace}\big(aD^2_{z,\bar{z}} v(z)\big) + \det^{1/d}(a) f(z)\big]\mathbb{E}[\rho^\sigma]$$
$$+ o_\varepsilon(1)\big(\mathbb{E}[\rho^\sigma] + \varepsilon^2\big).$$

By Ito's formula, $\varepsilon^2 = \mathbb{E}[|X^\sigma_{\rho^\sigma} - z|^2] = \mathbb{E}[\rho^\sigma]$. Taking the supremum over σ, dividing by ε^2 and letting ε tend to 0, we deduce that

$$\sup_{a=\bar{a}^*\ge 0, \mathrm{Trace}(a)=1} \big[\mathrm{Trace}\big(aD^2_{z,\bar{z}} v(z)\big) + \det^{1/d}(a) f(z)\big] \ge 0.$$

Proof of the Upper Bound in Lemma 4.5.12. To prove the supersolution property, we first prove the upper bound in Lemma 4.5.12. By assumption, we know that the functions $(v^\sigma)_\sigma$ are equicontinuous. Therefore, for a given $\delta > 0$, we can find N points $y_1, \dots, y_N$ on the surface of the ball $B(z,\varepsilon)$ such that, for any $(\sigma_t)_{t\ge 0}$ as above and any $y \in \partial B(z,\varepsilon)$, there exists an index $i(y)$ (say the smallest one) such that $|v^\sigma(y) - v^\sigma(y_{i(y)})| \le \delta$. (Taking the *supremum*, the same holds for v, i.e. $|v(y) - v(y_{i(y)})| \le \delta$.) Moreover, by definition of the *supremum*, for any index $i \in \{1,\dots,N\}$, we can find a δ-optimal control σ^i such that $v^{\sigma^i}(y_i) + \delta \ge v(y_i) \ge v^{\sigma^i}(y_i)$.

Consider now a control $(\sigma_t)_{t\ge 0}$ of the same type as above. It must be understood as a progressively-measurable functional of the Brownian paths $(B_t)_{t\ge 0}$ and of the (possibly random) initial condition X_0, i.e. something as $(\sigma_t)_{t\ge 0} = (\sigma_t((B_s)_{0\le s\le t}, X_0))_{t\ge 0}$. In particular, we emphasize that the value of ρ^σ depends on the values of $(\sigma_t)_{0\le t<\rho^\sigma}$ only. Moreover, we can modify the values of $(\sigma_t)_{t\ge \rho^\sigma}$ without changing ρ^σ itself. For instance, we can choose σ_t, for $t \ge \rho^\sigma$, as $\sigma_t = \sigma'_{t-\rho^\sigma}((B_{r+\rho^\sigma} - B_{\rho^\sigma})_{0\le r\le t-\rho^\sigma}, X^{\sigma,z}_{\rho^\sigma})$ for a new process $(\sigma'_t)_{t\ge 0}$, i.e. we can choose σ_t, for $t \ge \rho^\sigma$, as the new process σ', but shifted in time, the time shift being given by ρ^σ.

For such a choice of $(\sigma_t)_{t\ge 0}$, we are able to compute the conditional expectation in (4.36) explicitly. Indeed, for $(\sigma_t)_{t\ge 0}$ as described above,

$$\mathbb{E}\bigg[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det^{1/d}(a_t) f(X_t^{\sigma,z})dt | \mathcal{F}_{\rho^\sigma}\bigg]$$

$$= \mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_{\rho^\sigma}^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt | \mathcal{F}_{\rho^\sigma} \Big]$$
$$+ \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt. \tag{4.39}$$

Write now $X^{\sigma,z}_t = X^{\sigma,z}_{\rho^\sigma} + \int_{\rho^\sigma}^t \sigma_s dB_s$. Written in a non-rigorous way, this has the form:

$$X^{\sigma,z}_t = X^{\sigma,z}_{\rho^\sigma} + \int_{\rho^\sigma}^t \sigma'_{s-\rho^\sigma}\big((B_{r+\rho^\sigma} - B_{\rho^\sigma})_{0\le r\le s}, X^{\sigma,z}_{\rho^\sigma}\big) d\big(B_s - B_{\rho^\sigma}\big).$$

When computing the conditional expectation in the last line of (4.39), everything works as an integration with respect to the trajectories of $(B_t - B_{\rho^\sigma})_{t\ge\rho^\sigma}$: this is a Brownian motion, independent of the past before ρ^σ. Everything thus restarts afresh from $X^{\sigma,z}_{\rho^\sigma}$. Therefore, because of the specific form of σ after ρ^σ (this is the crucial point), the conditional expectation reduces to compute $v^{\sigma'}$ at point $X^{\sigma,z}_{\rho^\sigma}$, so that

$$\mathbb{E}\Big[g(X^{\sigma,z}_{\tau^{\sigma,z}}) + \int_0^{\tau^{\sigma,z}} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt | \mathcal{F}_{\rho^\sigma} \Big]$$
$$= v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt.$$

Taking the expectation, we deduce a kind of martingale property:

$$v^\sigma(z) = \mathbb{E}\Big[v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt \Big]. \tag{4.40}$$

Here is the choice of σ'. Rigoroulsy, we choose σ'_t as $\sigma^{i(X_0)}_t$ where X_0 stands for the (possibly random) initial condition of the process X. Clearly, this means that $\sigma_t = \sigma^{i(X^\sigma_{\rho^{\sigma,z}})}_{t-\rho^\sigma}$, $t > \rho^\sigma$. For this choice of $(\sigma_t)_{t\ge 0}$, we have from (4.40)

$$\begin{aligned} v(z) &\ge v^\sigma(z) \\ &= \mathbb{E}\Big[v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt \Big] \\ &\ge \mathbb{E}\Big[v(X^{\sigma,z}_{\rho^\sigma}) + \int_0^{\rho^\sigma} \det{}^{1/d}(a_t) f(X^{\sigma,z}_t) dt \Big] - \mathbb{E}\big[|v(X^{\sigma,z}_{\rho^\sigma}) - v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma})| \big]. \end{aligned} \tag{4.41}$$

Now, by the choice of the points $(y_i)_{1\le i\le N}$, we know that $|v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma}) - v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^\sigma})})| \le \delta$ and $|v(X^{\sigma,z}_{\rho^\sigma}) - v(y_{i(X^{\sigma,z}_{\rho^\sigma})})| \le \delta$. Moreover, by definition,

$v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^\sigma})}) = v^{\sigma^j}(y_j)$ with $j = i(X^{\sigma,z}_{\rho^\sigma})$ so that $|v^{\sigma'}(y_{i(X^{\sigma,z}_{\rho^\sigma})}) - v(y_{i(X^{\sigma,z}_{\rho^\sigma})})| \leq \delta$. Therefore

$$\mathbb{E}\big[|v(X^{\sigma,z}_{\rho^\sigma}) - v^{\sigma'}(X^{\sigma,z}_{\rho^\sigma})|\big] \leq 3\delta. \tag{4.42}$$

Plugging (4.42) into (4.41) and letting δ tend to 0, we obtain the upper bound in Lemma 4.5.12 and thus the equality, i.e. the complete Bellman Principle.

Proof of the Supersolution Property. To deduce the supersolution property in Monge–Ampère, we perform a suitable choice for $(\sigma_t)_{0 \leq t \leq \rho^\sigma}$ up to time ρ^σ. We choose it to be constant between 0 and ρ^σ, the constant value being denoted by σ for more simplicity. Expanding $v(X^{\sigma,z}_{\rho^{\sigma,z}})$ in (4.41) as in (4.38) and letting δ and then ε tend to 0, we obtain

$$\mathrm{Trace}\big(aD^2_{z,\bar{z}}v(z)\big) + \det{}^{1/d}(a)f(z) \leq 0, \quad \text{with } a = \sigma\bar{\sigma}^*.$$

This completes the proof of Proposition 4.5.11. □

4.5.6 Plurisubharmonicity by Bellman Principle

We finally emphasize that the Bellman Principle is nothing but a probabilistic version of the plurisubharmonicity property:

Proposition 4.5.13 *Assume that, for any $z \in \mathcal{D}$, any $\varepsilon > 0$ such that $\bar{B}(z,\varepsilon) \subset \mathcal{D}$ and any $\mathbb{C}^{d\times d}$-valued control $(\sigma_t)_{t\geq 0}$ such that $\mathrm{Trace}(\sigma_t\bar{\sigma}_t^*) = 1$, $t \geq 0$, the process $(X^{\sigma,z}_t)_{t\geq 0}$ given by Definition 4.5.10 satisfies the Bellman Principle stated in Lemma 4.5.12 where ρ^σ stands therein for the stopping time $\rho^\sigma = \inf\{t \geq 0 : |X^{\sigma,z}_t - z| \geq \varepsilon\}$. Assume also that v is continuous on $\mathcal{D}$. Then, v is plurisuperharmonic on $\mathcal{D}$.*

In particular, v is plurisuperharmonic if the family $(v^\sigma)_\sigma$ in Definition 4.5.10 is equicontinuous on every compact subset of $\mathcal{D}$.

Proof. Given $z \in \mathcal{D}$ and $\varepsilon > 0$ such that $\bar{B}(z,\varepsilon) \subset \mathcal{D}$, it is enough to prove that, for any $\nu \in \mathbb{C}^d$, $|\nu| = 1$,

$$v(z) \geq \frac{1}{2\pi}\int_0^{2\pi} v\big(z + \varepsilon e^{i\theta}\nu\big)d\theta. \tag{4.43}$$

In (4.35), we choose σ as the (deterministic) projection matrix on ν, i.e. $\sigma = \nu\bar{\nu}^*$, ν being understood as a column vector. Since f is non-negative, we deduce

$$v(z) \geq \mathbb{E}\big[v\big(X^{\sigma,z}_{\rho^\sigma}\big)\big], \tag{4.44}$$

with

$$X^{\sigma,z}_{\rho^\sigma} = z + \nu\bar{\nu}^* B_{\rho^\sigma}. \tag{4.45}$$

We now emphasize that $(\bar{\nu}^* B_t)_{t\geq 0}$ is a complex Brownian motion of dimension 1. Indeed, independence of the increments is well-seen and continuity of the trajectories is obviously true as well. It remains to see that $(\mathrm{Re}(\bar{\nu}^* B_t))_{t\geq 0}$ and $(\mathrm{Im}(\bar{\nu}^* B_t))_{t\geq 0}$ are independent non-standard[4] Brownian motions with increments of variance $\Delta/2$ over intervals of length Δ.

Clearly, $\mathrm{Re}(\bar{\nu}^*(B_t - B_s))$, for $0 \leq s \leq t$, is equal to $[\bar{\nu}^*(B_t - B_s) + \nu^*(\bar{B}_t - \bar{B}_s)]/2$. By standard computations, the expectation of the square is equal to $(t-s)/2$, as announced. Similar computations hold for $\mathrm{Im}(\bar{\nu}^*(B_t - B_s))$.

To prove independence, it is sufficient to prove that $\mathrm{Re}(\bar{\nu}^*(B_t - B_s))$ and $\mathrm{Im}(\bar{\nu}^*(B_t - B_s))$ are orthogonal in $L^2(\Omega, \mathbb{P})$ for any $0 \leq s \leq t$.[5] This is easily checked.

Finally, (4.45) yields

$$\varepsilon = |X^{\sigma,z}_{\rho^\sigma} - z| = |\nu\bar{\nu}^* B_{\rho^\sigma}| = |\bar{\nu}^* B_{\rho^\sigma}|,$$

so that ρ^σ stands for the first time when $(\bar{\nu}^* B_t)_{t\geq 0}$ hits the circle of radius ε. By isotropy, the distribution of the hitting point, i.e. $\bar{\nu}^* B_{\rho^\sigma}$, is uniform on the circle. We deduce (4.43) from (4.44). □

4.6 Program for the Probabilistic Analysis

Krylov's program now consists in establishing

Theorem 4.6.1 *Let Assumption (**A**) be in force. Then, the value function v in Definition 4.5.10 belongs to $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$. Moreover, the assumption of Proposition 4.5.11 is satisfied so that $-v$ solves almost everywhere the Monge–Ampère equation with f as source term and $-g$ as boundary condition.*

As the reader may notice, there are two parts in the statement of Theorem 4.6.1. The first part must be understood as the main result: it provides the $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$ property for the solution to Monge–Ampère under Assumption (**A**). The second part makes the connection between Krylov's formulation and the original Monge–Ampère equation: the only additional point to prove is the

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[4]Non-standard means that the variance of the increments is not normalized.

[5]This argument is false for general processes. It is here true because processes under consideration are of Gaussian type with independent increments. We refer the reader to any lecture on Gaussian vectors and processes.

equicontinuity property for the family $(v^\sigma)_\sigma$ on every compact subset of $\mathcal{D}$. Actually, we prove more right below: we prove that equicontinuity holds on the whole $\bar{\mathcal{D}}$ so that v is continuous up to the boundary and satisfies g as boundary condition.

4.6.1 Equicontinuity of $(v^\sigma)_\sigma$

We here prove the very first step of our program:

Proposition 4.6.2 *Under Assumption (**A**) and the notation of Definition 4.5.10, the functions $(v^\sigma)_\sigma$ are equicontinuous on $\bar{\mathcal{D}}$.*

Proof. We here follow the proof by Gaveau [Gav77]. Below, the control $(\sigma_t)_{t\geq 0}$ is fixed as in Definition 4.5.10. For given $z, z' \in \mathcal{D}$,

$$\begin{aligned}
&|v^\sigma(z) - v^\sigma(z')| \\
&\quad \leq \mathbb{E}\big[\big|g(X^{\sigma,z}_{\tau^{\sigma,z}}) - g(X^{\sigma,z'}_{\tau^{\sigma,z'}})\big|\big] + \mathbb{E}\int_0^{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}} \big|f(X^{\sigma,z}_s) - f(X^{\sigma,z'}_s)\big|ds \\
&\qquad + \mathbb{E}\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \big|f(X^{\sigma,z}_s)\big|ds + \mathbb{E}\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z'}} \big|f(X^{\sigma,z'}_s)\big|ds.
\end{aligned}$$

(Keep in mind that $\det(a_t) \leq \mathrm{Trace}(a_t) = 1$.) By Assumption (**A**), we can find a constant C, depending on (**A**) only (and whose value may vary from line to line), such that

$$\begin{aligned}
|v^\sigma(z) - v^\sigma(z')| &\leq C\mathbb{E}\big[\big|X^{\sigma,z}_{\tau^{\sigma,z}} - X^{\sigma,z'}_{\tau^{\sigma,z'}}\big|\big] + C\mathbb{E}\int_0^{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}} \big|X^{\sigma,z}_s - X^{\sigma,z'}_s\big|ds \\
&\quad + C\mathbb{E}\big[|\tau^{\sigma,z'} - \tau^{\sigma,z}|\big] \\
&= T_1 + T_2 + T_3. \qquad (4.46)
\end{aligned}$$

Above, $a \vee b$ stands for $\max(a,b)$ and $a \wedge b$ for $\min(a,b)$.

To deal with T_2 in (4.46), we emphasize that $X^{\sigma,z}_s - X^{\sigma,z'}_s = z - z'$, $0 \leq s \leq \tau^{\sigma,z} \wedge \tau^{\sigma,z'}$, so that

$$T_2 \leq C|z - z'|\,\mathbb{E}\big[\tau^{\sigma,z}\big].$$

To treat T_1, we notice that

$$\mathbb{E}\big[\big|X^{\sigma,z}_{\tau^{\sigma,z}} - X^{\sigma,z'}_{\tau^{\sigma,z'}}\big|\big] \leq |z - z'| + \mathbb{E}\bigg[\bigg|\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \sigma_s dB_s\bigg|\bigg]$$

$$\le |z - z'| + \mathbb{E}\left[\left|\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \sigma_s dB_s\right|^2\right]^{1/2}$$

$$= |z - z'| + \mathbb{E}\left[\int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}\vee\tau^{\sigma,z'}} \mathrm{Trace}(\sigma_s\bar{\sigma}_s^*)ds\right]^{1/2}$$

$$= |z - z'| + \mathbb{E}\big[|\tau^{\sigma,z} - \tau^{\sigma,z'}|\big]^{1/2}.$$

To complete the proof, it is thus sufficient to prove

Lemma 4.6.3 *There exists a constant C, depending on (**A**) only, such that for any $z, z' \in \mathcal{D}$, $\mathbb{E}[\tau^{\sigma,z}] \le C$ and $\mathbb{E}[|\tau^{\sigma,z} - \tau^{\sigma',z}|] \le C|z - z'|$.*

Proof (Lemma 4.6.3). Given two different points z and z' in $\mathcal{D}$, we know that $X_t^{\sigma,z} - X_t^{\sigma,z'} = z - z'$ for any $t \le \tau^{\sigma,z} \wedge \tau^{\sigma,z'}$.

Moreover, on the event $\{\tau^{\sigma,z} \ge \tau^{\sigma,z'}\}$,

$$X_{\tau^{\sigma,z'}}^{\sigma,z'} = X_{\tau^{\sigma,z'}}^{\sigma,z'} - X_{\tau^{\sigma,z'}}^{\sigma,z} + X_{\tau^{\sigma,z'}}^{\sigma,z} = z - z' + X_{\tau^{\sigma,z'}}^{\sigma,z}, \tag{4.47}$$

so that $\mathrm{dist}(X_{\tau^{\sigma,z'}}^{\sigma,z}, \partial\mathcal{D}) \le |z - z'|$ when $\tau^{\sigma,z} \ge \tau^{\sigma',z}$.

As a consequence, $\mathrm{dist}(X_{\tau^{\sigma,z'}\wedge\tau^{\sigma,z}}^{\sigma,z}, \partial\mathcal{D}) \le |z - z'|$ on the whole probability space.

Apply now Itô's formula to $(\psi(X_t^{\sigma,z}))_{t\ge 0}$. We obtain

$$\psi\big(X_{\tau^{\sigma,z}}^{\sigma,z}\big) = \psi\big(X_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\sigma,z}\big) + \int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \mathrm{Trace}\big(a_s D_{z,\bar{z}}^2\psi(X_s^{\sigma,z})\big)ds$$
$$+ \int_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\tau^{\sigma,z}} \big(D_z\psi(X_s^{\sigma,z})\sigma_s dB_s + D_{\bar{z}}\psi(X_s^{\sigma,z})\bar{\sigma}_s d\bar{B}_s\big).$$

We emphasize that the LHS is zero. Taking the expectation, we deduce from the plurisuperharmonicity property that

$$\mathbb{E}\big[\psi\big(X_{\tau^{\sigma,z}\wedge\tau^{\sigma,z'}}^{\sigma,z}\big)\big] \ge C\mathbb{E}\big[\tau^{\sigma,z} - \tau^{\sigma,z} \wedge \tau^{\sigma,z'}\big],$$

for some constant $C > 0$ depending on (**A**) only.

By (4.47), we deduce (C possibly varying from line to line) that

$$\mathbb{E}\big[\big(\tau^{\sigma,z} - \tau^{\sigma,z'}\big)^+\big] = \mathbb{E}\big[\tau^{\sigma,z} - \tau^{\sigma,z} \wedge \tau^{\sigma,z'}\big] \le C|z - z'|.$$

By symmetry,

$$\mathbb{E}\big[\big|\tau^{\sigma,z} - \tau^{\sigma,z'}\big|\big] \le C|z - z'|.$$

This completes the proof. □

4.6.2 Semi-Convexity Argument

The main idea to prove the regularity is to reduce the analysis to a convexity problem:

Proposition 4.6.4 *Assume that the function v is Lipschitz continuous and semi-convex in the whole $\bar{\mathcal{D}}$, i.e. there exists a constant N such that the function $z \in \bar{\mathcal{D}} \mapsto v(z) + N|z|^2$ is convex in any ball included in $\bar{\mathcal{D}}$. Then v belongs to $\mathcal{C}^{1,1}(\bar{\mathcal{D}})$.*

Proof. Proposition 4.6.4 follows from Lemma 2.11 in Chap. 2 by V. Guedj and A. Zeriahi. Indeed, by Proposition 4.5.13 and Proposition 4.6.2, $-v$ is plurisubharmonic. Moreover, the semi-convexity property provides the required estimate in Lemma 2.11. □

Remark 4.6.5 *Below, we will also apply Proposition 4.6.4 on compact subsets of $\mathcal{D}$ (instead of the whole $\bar{\mathcal{D}}$). Obviously, the result then remains true.*

4.6.3 Getting Rid of the Supremum

A very natural idea, to investigate v, is to get rid of, as most as possible, of the *supremum*. In some sense, this is not so difficult since both Lipschitz continuity and (semi-)convexity are stable by *supremum*:

Proposition 4.6.6 *Let $(w^\beta)_\beta$ be a family of (bounded) functions of the real variable, indexed by some parameter β, for which we can find two functions r_1 and r_2, of the real variable as well, satisfying for any β,*

$$|w^\beta(s) - w^\beta(0)| \leq r_1(s), \quad s \in \mathbb{R},$$

and

$$s \mapsto w^\beta(s) + r_2(s)$$

is convex. Then, the function $s \mapsto \sup_\beta w^\beta(s)$ satisfies the same properties.

The proof is straightforward. The key point is to think of $w^\beta(s)$ as $v^\sigma(\gamma(s))$ for some path $s \in \mathbb{R} \mapsto \gamma(s)$ with values in the domain $\mathcal{D}$, v^σ being given by Definition 4.5.10. The functions $s \in \mathbb{R} \mapsto r_1(s)$ and $s \in \mathbb{R} \mapsto r_2(s)$ may be understood as $s \in \mathbb{R} \mapsto Ns$ et $s \in \mathbb{R} \mapsto Ns^2$, for some constant N. In such a case, the first inequality in Proposition 4.6.6 is understood as a Lipschitz property and the second one as a semi-convexity property.

4.6.4 Differentiation Under the Symbol $\mathbb{E}$

As we just said, the strategy consists in applying Proposition 4.6.6 to each function v^σ in Definition 4.5.10 along a path γ with values in $\mathcal{D}$: this is the way we are able to transfer regularity from the family $(v^\sigma)_\sigma$ to its supremum, i.e. to the function v.

Therefore, the whole problem is now to estimate v^σ uniformly in σ: specifically, we are to estimate the Lipschitz constant and to bound from below the second-order derivatives.

The most natural idea to do so is to differentiate under the symbol $\mathbb{E}$ with respect to the initial condition z in the definition of v^σ, see (4.32), σ being fixed. Remember indeed that the coefficients f and g are differentiable. Remember also that, for each σ, the value $X_t^{\sigma,z}$ of the controlled process at time t is easily differentiable with respect to z, whatever the randomness may be.

Unfortunately, the picture is not so simple. The big deal is the following: the stopping times $\tau^{\sigma,z}$ are not differentiable w.r.t. z.

4.6.5 Modification of the Representation

To be able to differentiate under the symbol $\mathbb{E}$, it is necessary to get rid of the boundary. This means the following: we are to get rid of the boundary condition and to force the representation process to stay in $\mathcal{D}$ forever.

To get rid of the boundary condition, it is sufficient to consider $v^\sigma - g$. Indeed, stochastic differentiation rules given in Sect. 4.5 show that $v^\sigma - g$ may be written as

$$\big(v^\sigma - g\big)(z) = \mathbb{E}\int_0^{\tau^{\sigma,z}} \big[\det{}^{1/d}(a_t) f(X_t^{\sigma,z}) + \mathrm{Trace}\big(a_t D^2_{z,\bar z} g(X_t^{\sigma,z})\big)\big]\,dt,$$

with $a_t = \sigma_t \bar\sigma_t^*$, $t \geq 0$. Obviously, the function g being assumed to be $\mathcal{C}^4$ with bounded derivatives, this operation doesn't modify the regularity property of the second member. However, it may modify its sign.

To recover the right sign, we may use the plurisuperharmonicity condition. Indeed, since

$$\sup_a \sup_{z\in\mathcal{D}} \mathrm{Trace}\big(a D^2_{z,\bar z}\psi(z)\big) < 0,$$

(with a as above), we can add $N_0\psi$ to $v^\sigma - g$, for N_0 as large as necessary.

We emphasize that this transform cannot be understood as a modification of the original second member f of the Monge–Ampère equation. Indeed, the coefficients we here remove depend on σ in a more general way than $\det^{1/d}(a_t) f$ does so that the expectation we have to investigate has the form

$$\widetilde{v}^{\sigma}(z) := \mathbb{E}\int_0^{\tau^{\sigma,z}} F(\det(a_t), a_t, X_t^{\sigma,z})dt, \tag{4.48}$$

which is much more general than the original one in Definition 4.5.10. We also notice that the general coefficient F is $\mathcal{C}^2$ with respect to the second and third parameters. (Above, $a_t = \sigma_t \bar{\sigma}_t^*$, $t \geq 0$.)

It now remains to get rid of the boundary itself! The idea is to slow down the process $(X_t)_{t \geq 0}$ (forget for the moment the superscripts z and σ to simplify the notations) in the neighborhood of the boundary by means of the function ψ. Consider indeed a stochastic process $(Z_t)_{t \geq 0}$ with the following dynamics:

$$dZ_t = \psi^{1/2}(Z_t)\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t)dt, \quad t \geq 0, \tag{4.49}$$

and with $Z_0 = z$ as initial condition. Since the dynamics depend on $(Z_t)_{t \geq 0}$ itself, the process $(Z_t)_{t \geq 0}$ is said to satisfy a Stochastic Differential Equation (SDE for short): we give in the next section a short overview of conditions ensuring existence and uniqueness of solutions. Roughly speaking, we will see that the basic conditions are the same as in the theory of Ordinary Differential Equations: (4.49) is solvable in infinite horizon under global Lipschitz conditions; if the coefficients are locally Lipschitz only on a bounded open subset $\mathcal{U}$, then existence and uniqueness hold up to the first exit time of $\mathcal{U}$. The point is then to discuss whether $(Z_t)_{t \geq 0}$ may reach the boundary of the domain $\mathcal{D}$ or not.

Proposition 4.6.7 *Given an initial condition $z \in \mathcal{D}$ and a control $(\sigma_t)_{t \geq 0}$ with values in the set of complex matrices of size $d \times d$ such that* $\mathrm{Trace}(\sigma_t \bar{\sigma}_t^*) = 1$, *$t \geq 0$, the SDE*

$$dZ_t^{\sigma,z} = \psi^{1/2}(Z_t^{\sigma,z})\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^{\sigma,z})dt, \quad t \geq 0, \tag{4.50}$$

with the initial condition $Z_0^{\sigma,z} = z$ admits a unique solution. It stays inside $\mathcal{D}$ forever.

Said differently, the stopping time $\tau_\infty^{\sigma,z} := \inf\{t \geq 0 : Z_t^{\sigma,z} \notin \mathcal{D}\}$ (with $\tau_\infty^{\sigma,z} = +\infty$ if the underlying set is empty) is almost-surely infinite.

Proof. The proof relies on a so-called localization argument. For the sake of simplicity, we remove below the superscript (σ, z) in $Z^{\sigma,z}$ and in $\tau_\infty^{\sigma,z}$.

Assume for the moment that (4.50) is indeed solvable. On the interval $[0, \tau_\infty)$, we then compute

$$\begin{aligned} d\psi^{-1}(Z_t) &= -\psi^{-3/2}(Z_t)D_z\psi(Z_t)\sigma_t dB_t - \psi^{-3/2}(Z_t)D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t \\ &\quad - \psi^{-1}(Z_t)\mathrm{Trace}\big[a_t D_{z,\bar{z}}\psi(Z_t)\big]dt, \quad 0 \leq t < \tau_\infty, \end{aligned}$$

with $a_t = \sigma_t \bar{\sigma}_t^*$, $t \geq 0$. Here, the dt term must be understood as

$$\begin{aligned} & -2\psi^{-2}(Z_t) D_z \psi(Z_t) a_t D_{\bar{z}}^* \psi(Z_t) + \psi(Z_t) \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2 \big(\psi^{-1}\big)(Z_t)\big] \\ & \quad = -\psi^{-1}(Z_t) \mathrm{Trace}\big[a_t D_{z,\bar{z}} \psi(Z_t)\big]. \end{aligned}$$

Therefore,

$$\begin{aligned} & d\bigg[\psi^{-1}(Z_t) \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2 \psi(Z_s)\big]\, ds\bigg)\bigg] \\ & \quad = \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2 \psi(Z_s)\big]\, ds\bigg) \\ & \qquad \times \big[-\psi^{-3/2}(Z_t) D_z \psi(Z_t) \sigma_t dB_t - \psi^{-3/2}(Z_t) D_{\bar{z}} \psi(Z_t) d\bar{B}_t\big], \quad 0 \leq t < \tau_\infty. \end{aligned} \tag{4.51}$$

We obtain a (local) martingale.

Indeed, setting $\tau_n := \inf\{t \geq 0 : \psi(Z_t) \leq 1/n\}$, the stochastic integral may be defined rigorously between 0 and τ_n.[6] Therefore, for any $t \geq 0$,

$$\mathbb{E}\bigg[\psi^{-1}(Z_{t\wedge\tau_n}) \exp\bigg(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[a_s D_{z,\bar{z}}^2 \psi(Z_s)\big]\, ds\bigg)\bigg] = \psi^{-1}(z). \tag{4.52}$$

Noting that $\psi^{-1}(Z_{t\wedge\tau_n}) = n$ if $\tau_n \leq t$, we deduce that, for some constant $C > 0$ independent of n and t,

$$n \exp(-Ct) \mathbb{P}\big\{\tau_n \leq t\big\} \leq \psi^{-1}(z). \tag{4.53}$$

Thus,

$$\forall n \geq 1,\ t \geq 0, \quad n \exp(-Ct) \mathbb{P}\big\{\tau_\infty \leq t\big\} \leq \psi^{-1}(z),$$

since $\tau_\infty \geq \tau_n$. Dividing by n and letting it tend to $+\infty$, we obtain

$$\forall t \geq 0, \quad \mathbb{P}\big\{\tau_\infty \leq t\big\} = 0.$$

In particular, $\tau_\infty = +\infty$ almost-surely.

It now remains to prove that both existence and uniqueness hold. Actually, we can solve the truncated version of (4.50)

$$dZ_t^n = \big(\varphi_n \psi^{1/2}\big)(Z_t^n) \sigma_t dB_t + \varphi_n(Z_t^n) a_t D_{\bar{z}}^* \psi(Z_t^n) dt, \quad t \geq 0, \tag{4.54}$$

[6] This is the reason why the proof consists of a "localizing" argument.

where φ_n is some smooth cut-off function with values in $[0,1]$ matching 1 on the set $\{\psi \geq 1/n\}$ and 0 on the set $\{\psi \leq 1/(2n)\}$, $n \geq 1$. It is clear that (4.54) is uniquely solvable. (See Sect. 4.7.1.) Up to the stopping time $\rho_n := \inf\{t \geq 0 : \psi(Z_t^n) \leq 1/n\}$, it satisfies (4.50) as well. In particular, (4.53) holds with ρ_n instead of τ_n, so that $\rho_n \to +\infty$ almost-surely (as $n \to +\infty$). Moreover, by uniqueness of the solution of a Cauchy-Lipschitz SDE, for $m \geq n$, $(Z_t^n)_{t\geq 0}$ and $(Z_t^m)_{t\geq 0}$ are equal up to time $\min(\rho_n, \rho_m) = \rho_n$.

We then set $Z_t = \lim_{n\to+\infty} Z_t^n$. For $t \leq \rho_n$, $n \geq 0$, $Z_t = Z_t^n$ so that $(Z_t)_{0\leq t\leq \rho_n}$ satisfies (4.50) up to time ρ_n. Letting n tend to $+\infty$, we deduce that $(Z_t)_{t\geq 0}$ satisfies (4.50) over the whole $\mathbb{R}_+$.

Uniqueness follows from the same argument. Any other solution $(Z'_t)_{t\geq 0}$ (with the same initial condition) matches $(Z_t)_{t\geq 0}$ up to the first time it exits from $\{\psi \geq 1/n\}$. Letting n tend to $+\infty$, we deduce that there exists a unique solution. □

Obviously, changing $(X_t^{\sigma,z})_{t\geq 0}$ into $(Z_t^{\sigma,z})_{t\geq 0}$ breaks down the representation of v^σ given in Definition 4.5.10 (and in (4.48)). The point is thus to provide a representation of v (or of $-v$, i.e. of the candidate to solve Monge–Ampère) in terms of the family $((Z_t^{\sigma,z})_{t\geq 0})_\sigma$.

To do so, we first investigate the representation of $\widetilde{v}^\sigma$ when $(\sigma_t)_{t\geq 0}$ is deterministic and constant, i.e. $\sigma_t = \sigma$ deterministic, with $\det(\sigma) \neq 0$.

In the deterministic and constant case, we know that $\widetilde{v}^\sigma$ given in (4.48) satisfies the PDE

$$-\mathrm{Trace}\big[aD^2_{z,\bar z}\widetilde{v}^\sigma(z)\big] = F(\det(a), a, z), \quad z \in \mathcal{D},$$

with zero as boundary condition. (Have in mind that F is here given by adding the $\mathrm{Trace}[aD^2_{z,\bar z}(g - N_0\psi)(z)]$ to the original source term $\det^{1/d}(a)f(z)$.)

By Theorem 4.5.7, we know that $\widetilde{v}^\sigma$ is $\mathcal{C}^2$ inside $\mathcal{D}$ and continuous up to the boundary. In particular, we can apply Itô's formula to $(\psi^{-1}(Z_t^{\sigma,z})\widetilde{v}^\sigma(Z_t^{\sigma,z}))_{t\geq 0}$:

Lemma 4.6.8 *Under the notation of Proposition 4.6.7, for any (possibly random) control $(\sigma_t)_{t\geq 0}$ (with values in the set of complex matrices of size $d \times d$ such that $\mathrm{Trace}(\sigma_t\bar\sigma_t^*) = 1$, $t \geq 0$) and for any function G in $\mathcal{C}^2(\mathcal{D})$ with real values,*

$$\begin{aligned} d\Big[G(Z_t^{\sigma,z})\exp\Big(\int_0^t \mathrm{Trace}\big[a_sD^2_{z,\bar z}\psi(Z_s^{\sigma,z})\big]\,ds\Big)\Big] \\ = \exp\Big(\int_0^t \mathrm{Trace}\big[a_sD^2_{z,\bar z}\psi(Z_s^{\sigma,z})\big]\,ds\Big)\big[D_zG(Z_t^{\sigma,z})\sigma_t dB_t + D_{\bar z}G(Z_t^{\sigma,z})\bar\sigma_t d\bar B_t\big] \\ + \exp\Big(\int_0^t \mathrm{Trace}\big[a_sD^2_{z,\bar z}\psi(Z_s^{\sigma,z})\big]\,ds\Big)\mathrm{Trace}\big[a_tD^2_{z,\bar z}(\psi G)(Z_t^{\sigma,z})\big]\,dt, \quad t \geq 0, \end{aligned}$$

with $a_t = \sigma_t\bar\sigma_t^*$, $t \geq 0$.

In particular, if σ is constant and non-degenerate, we obtain by choosing $G = \psi^{-1}\widetilde{v}^{\sigma}$

$$\psi^{-1}(z)\widetilde{v}^{\sigma}(z) \\ = \mathbb{E}\int_0^{+\infty} \exp\Big(\int_0^t \mathrm{Trace}\big[aD^2_{z,\bar{z}}\psi(Z^{\sigma,z}_s)\big]\,ds\Big) F(\det(a), a, Z^{\sigma,z}_t)dt, \quad z \in \mathcal{D}. \tag{4.55}$$

Proof. For simplicity, we get rid of the superscript (σ, z) in $(Z^{\sigma,z}_t)_{t\geq 0}$. The first part of the proof is similar to the proof of (4.51). For the second part, it is necessary to localize the dynamics of $(Z_t)_{t\geq 0}$ up to the stopping time $\tau^n = \inf\{t \geq 0 : \psi(Z_t) \leq 1/n\}$ as in (4.52). For $\psi(z) \geq 1/n$, we obtain

$$\psi^{-1}(z)\widetilde{v}^{\sigma}(z) = \mathbb{E}\Big[\exp\Big(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[aD^2_{z,\bar{z}}\psi(Z_s)\big]\,ds\Big)\psi^{-1}(Z_{t\wedge\tau_n})\widetilde{v}^{\sigma}(Z_{t\wedge\tau_n})\Big] \\ + \mathbb{E}\int_0^{t\wedge\tau_n} \exp\Big(\int_0^s \mathrm{Trace}\big[aD^2_{z,\bar{z}}\psi(Z_r)\big]\,dr\Big)F(\det(a), a, Z_s)ds.$$

We emphasize that the plurisuperharmonicity condition here plays a crucial role: it says that the second integral is exponentially convergent. In particular, the second term in the RHS clearly converges towards the announced quantity as n and t tend to the infinity. The first term in the RHS may be a bit more difficult to handle. By (4.52), we can bound

$$\mathbb{E}\Big[\exp\Big(\int_0^{t\wedge\tau_n} \mathrm{Trace}\big[aD^2_{z,\bar{z}}\psi(Z_s)\big]\,ds\Big)\psi^{-1}(Z_{t\wedge\tau_n})\widetilde{v}^{\sigma}(Z_{t\wedge\tau_n}); \tau_n \leq t\Big] \tag{4.56}$$

by $\psi^{-1}(z)\sup\{\widetilde{v}^{\sigma}(z'),\ \psi(z') \leq 1/n\}$: this quantity tends to 0 as n tends to the infinity by continuity of $\widetilde{v}^{\sigma}$ up to the boundary. On the complementary, i.e. on $\{\tau_n > t\}$, we use the plurisuperharmonicity condition to bound (4.56) by $C\exp(-Ct)n$, for a constant C independent of n and t. Letting t tend first to the infinity, and then n, we complete the proof. □

We shall now explain what happens when the control $(\sigma_t)_{t\geq 0}$ in (4.48) and (4.50) is random and evolves with time. Formally, when σ is non-constant, (4.55) breaks down: the term $\psi^{1/2}$ in (4.50) is understood as a change of time speed[7] and the process $(Z^{\sigma,z}_t)_{t\geq 0}$ appears as a slower version of the original $(X^{\sigma,z}_t)_{t\geq 0}$, so that the process $(\sigma_t)_{t\geq 0}$ inside (4.55) cannot be the same as the original one in (4.48).

[7]For the reader who knows a bit of stochastic analysis, the drift term in (4.50) follows from a Girsanov transform.

The main idea is the following: (4.55) cannot be a general formula for $\widetilde{v}^{\sigma}$, but, taking the supremum w.r.t. σ, we recover a representation formula for $\sup_{\sigma} \widetilde{v}^{\sigma}$. The idea is not so surprising. Indeed, going back to the proof of the Dynamic Programming Principle, see Lemma 4.5.12, we understand that the global supremum in (4.32) may be localized, i.e. the values of $(\sigma_t)_{t\geq 0}$ may be locally frozen. Since the representation of $\widetilde{v}^{\sigma}$ in (4.55) holds for a constant control, we may expect the supremum w.r.t. to (general) σ to satisfy a similar representation formula.

This result turns out to be true: representation (4.55) holds for the value function of the optimization problem. We thus claim

Proposition 4.6.9 *Given a control $(\sigma_t)_{t\geq 0}$ with values in the set of $d \times d$ complex matrices such that* $\mathrm{Trace}(\sigma_t \bar{\sigma}_t^*) = 1$, $t \geq 0$, *consider the function v^{σ} as in Definition 4.5.10 and modify it into $\widetilde{v}^{\sigma} = v^{\sigma} - g + N_0 \psi$ as in (4.48) for some large enough N_0, so that*

$$\begin{aligned}
&\big(\widetilde{v}^{\sigma} - g + N_0\psi\big)(z) \\
&\quad = \mathbb{E}\int_0^{\tau^{\sigma,z}} \Big[\det{}^{1/d}(a_t) f(X_t^{\sigma,z}) + \mathrm{Trace}\big(a_t D^2_{z,\bar{z}} g(X_t^{\sigma,z})\big) \\
&\qquad\qquad - N_0 \mathrm{Trace}\big(a_t D^2_{z,\bar{z}} \psi(X_t^{\sigma,z})\big)\Big] dt, \\
&\quad := \mathbb{E}\int_0^{\tau^{\sigma,z}} F(\det(a_t), a_t, X_t^{\sigma,z}) dt, \quad z \in \mathcal{D},
\end{aligned}$$

with F non-negative.

For a given initial condition $z \in \mathcal{D}$, consider also the SDE

$$dZ_t^{\sigma,z} = \psi^{1/2}(Z_t^{\sigma,z})\sigma_t dB_t + a_t D_{\bar{z}}\psi^*(Z_t^{\sigma,z})dt, \quad t \geq 0, \tag{4.57}$$

with the initial condition $Z_0^{z,\sigma} = z \in \mathcal{D}$.

Then, the value function $\sup_{\sigma}[v^{\sigma} - g + N_0\psi]$ at point z may be expressed as

$$v(z) - g(z) + N_0\psi(z) = \sup_{\sigma}\big[\big(v^{\sigma} - g + N_0\psi\big)(z)\big] = \psi(z)\sup_{\sigma}\big[V^{\sigma}(z)\big],$$

where

$$\begin{aligned}
&V^{\sigma}(z) \\
&\quad = \sup_{\sigma}\mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}\big[a_s D^2_{z,\bar{z}}\psi(Z_s^{\sigma,z})\big] ds\bigg) F(\det(a_t), a_t, Z_t^{\sigma,z}) dt\bigg],
\end{aligned}$$

$z \in \mathcal{D}$. Below, we set $V(z) := \sup_{\sigma} V^{\sigma}(z)$.

4.7 Derivative Quantity

By Proposition 4.6.9, we can now forget the boundary constraints. In comparison with the formulation of the complex Monge–Ampère equation given in Sect. 4.5, the new representation formula is set in infinite time: we may think of differentiating with respect to the initial condition without taking care of the exit phenomenon.

Unfortunately, there is a price to pay for the new writing. The dynamics of the controlled paths involved in the new representation formula are much less simple to handle with than the original ones. Even without any specific knowledge in stochastic differential equations, it is well-guessed that the derivative of Z in (4.50), if exists, is the solution of a new stochastic differential equation, obtained by differentiation: the whole problem is now to investigate the differentiated equation on the long-run.

4.7.1 A Word on SDEs

We said very few about stochastic differential equations. We here specify some elementary facts. (To simplify, things are here stated for real valued processes, but all of them are extendable to the complex case in a standard way.)

A stochastic differential equation may be set in real or complex coordinates. It has the general form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0. \tag{4.58}$$

Here, the coefficient b is called the *drift* of the equation. It may depend on time, on the solution at current time and on the randomness as well. The same is true for the *diffusion coefficient* σ. Obviously, B here stands for a Brownian motion (with real or complex values according to the framework). We also indicate that the dimension of X may be different from the dimension of B. This is not the case in Proposition 4.6.9 since the matrix σ is of size $d \times d$. When necessary, we will specify by d the dimension of X and by d_B the dimension of B, so that σ is a matrix of size $d \times d_B$.

Here are the standard solvability conditions. The standard framework for the regularity in space is the Lipschitz one, as we said above: coefficients are assumed to be Lipschitz in space, uniformly in randomness and in time in compact subsets, i.e. $\forall T > 0$, $\exists K_T \geq 0$, $\forall \omega \in \Omega$, $\forall t \in [0, T]$, $\forall x, x'$,

$$|\sigma(t, x) - \sigma(t, x')| + |b(t, x) - b(t, x')| \leq K_T |x - x'|. \tag{4.59}$$

To be sure that the underlying integrals are well-defined, some measurability property is necessary: for any x, the processes $(b(t, x))_{t \geq 0}$ and $(\sigma(t, x))_{t \geq 0}$ are progressively-measurable.

Finally, to control the growth of the coefficients, we ask

$$\forall T \geq 0, \quad \mathbb{E} \int_0^T \big[|b(s,0)|^2 + |\sigma(s,0)|^2\big] ds < +\infty. \tag{4.60}$$

Under these three conditions, existence and uniqueness of a solution to (4.58) with a given initial condition in L^2 hold, on the whole $[0, +\infty)$. The solution has continuous paths that are adapted to the filtration generated by B. Moreover, the supremum of the solution is in L^2, locally in time:

$$\forall T \geq 0, \quad \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_T|^2\right] < +\infty. \tag{4.61}$$

In the case when the initial condition is in L^p, for some $p > 2$, and (4.60) holds in L^p as well, for the same p, then (4.61) also holds in L^p.

Actually, global Lipschitz conditions may be relaxed. Under local Lipschitz conditions in space, the solution exists on a random interval and may blow up at some random time. As easily-guessed, the blow-up time is a stopping time. It corresponds to the limit of the stopping times (*first time when the modulus of the solution is larger than* $m)_m$.

Below, we will compare the solutions to stochastic differential equations driven by different coefficients. The following result will be referred to as *a stability property*:

Proposition 4.7.1 *Consider two sets of coefficients* (b, σ) *and* (b', σ') *satisfying* (4.59) *and* (4.60) *and denote by* $(X_t)_{t \geq 0}$ *and* $(X'_t)_{t \geq 0}$ *the associated solutions for some initial conditions* X_0 *and* X'_0 *in* L^2*. Then, for any* $T > 0$*, there exists a constant* $C_T \geq 0$*, only depending on* T *and* K_T*, such that, for any event* $A \in \mathcal{F}_0$*,*

$$\mathbb{E}\left[\mathbf{1}_A \sup_{0 \leq t \leq T} |X_t - X'_t|^2\right] \leq C_T \left\{ \mathbb{E}\big[\mathbf{1}_A |X_0 - X'_0|^2\big] + \mathbb{E}\left[\mathbf{1}_A \int_0^T \big(|b - b'|^2(t, X_t) + |\sigma - \sigma'|^2(t, X_t)\big) dt\right]\right\}.$$

A similar version holds in L^p*, for* $p > 2$*, if the initial conditions are in* L^p *and* (4.60) *holds in* L^p *both for* (b, σ) *and* (b', σ')*.*

(The indicator $\mathbf{1}_A$ here permits to localize the stability property w.r.t. the values of the initial conditions.)

In what follows, the generic equation we consider is of real structure, the complex case being a particular case of the real one by doubling the dimension. The equation is also assumed to be set on the whole space. ((4.50) is indeed set on the whole space provided ψ be extended to the whole $\mathbb{C}^d$, but the solution stays inside $\mathcal{D}$ forever.)

4.7.2 Differentiation of the Flow Generated by a SDE

Clearly, we have in mind to differentiate under the symbol $\mathbb{E}$ in the representation formula of Proposition 4.6.9. To do so, we here give some preliminary results about the differentiability of the flow generated by a stochastic differential equation.

Specifically, the following result guarantees the differentiability of the paths $(X_t^x)_{t\geq 0}$ with respect to the starting point x, the coordinates of x being possibly real or complex.

Theorem 4.7.2 *Assume that, for every $t \geq 0$ and (almost) every $\omega \in \Omega$, the coefficients $b(t,\cdot) : x \in \mathbb{R}^d \mapsto b(t,x)$ and $\sigma(t,\cdot) : x \in \mathbb{R}^d \mapsto \sigma(t,x)$ are of class $\mathcal{C}^3$, with bounded derivatives, uniformly in ω and in t in compact sets. Then, $\mathbb{P}$-almost surely, for all $t \geq 0$, the mapping $x \in \mathbb{R}^d \mapsto X_t^x$ is twice differentiable with respect to x.*

In particular, for any family of initial conditions $(X_0^s)_{s\in\mathbb{R}}$ such that, $\mathbb{P}$-a.s., $s \in \mathbb{R} \mapsto X_0^s$ is $\mathcal{C}^3$, with bounded derivatives, uniformly in ω, the mappings $(s \mapsto X_t^s := X_t^{X_0^s})_{t\geq 0}$ are, $\mathbb{P}$ almost-surely, differentiable with respect to s for all $t \geq 0$. Moreover, $(D_s[X_t^s])_{t\geq 0}$ and $(D^2_{s,s}[X_t^s])_{t\geq 0}$ satisfy linear stochastic differential equations (with random coefficients):

$$\xi_t^s = \gamma'(s) + \int_0^t D_x b(r, X_r^s)\xi_r^s dr + \int_0^t \sum_{j=1}^{d_B} D_x \sigma_{\cdot,j}(r, X_r^s)\xi_r^s dW_r^j, \tag{4.62}$$

and

$$\begin{aligned} \eta_t^s = \gamma''(s) &+ \int_0^t \big[D_x b(r, X_r^s)\eta_r^s + D^2_{x,x} b(r, X_r^s)\xi_r^s \otimes \xi_r^s\big] dr \\ &+ \int_0^t \sum_{j=1}^{d_B} \big(D_x\sigma_{\cdot,j}(r, X_r^s)\eta_r^s + D^2_{x,x}\sigma_{\cdot,j}(r, X_r^s)\xi_r^s \otimes \xi_r^s\big) dW_r^j, \end{aligned} \tag{4.63}$$

that is $D_s[X_t^s] = \xi_t^s$ and $D^2_{s,s}[X_t^s] = \eta_t^s$, $t \geq 0$, $s \in \mathbb{R}$.

Proof. We refer the reader to the monograph by Protter [Pro90, Chap. V, Sect. 7, Thm. 39] for the proof. □

Below, the differentiability property in Theorem 4.7.2 is referred to as *pathwise twice differentiability*, that is the paths of the process are twice differentiable, randomness by randomness. In some sense, *pathwise differentiability* is too much demanding for our purpose. Indeed, as we recalled above, the point below is to differentiate under the symbol $\mathbb{E}$ only, so that weaker notions of differentiability turn out to be sufficient:

Definition 4.7.3 *Under the notations of Theorem 4.7.2, the process $(X_t^s)_{t\geq 0}$ is said to be twice differentiable in probability w.r.t. s if (4.62) and (4.63) are uniquely solvable and, for any $T > 0$ and any $s \in \mathbb{R}$,*

$$\forall \nu > 0, \quad \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \left| \delta_\varepsilon X_t^s - \xi_t^s \right| \geq \nu \right\} = 0,$$
$$\lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \left| \delta_\varepsilon \xi_t^s - \eta_t^s \right| \geq \nu \right\} = 0, \tag{4.64}$$

with the generic notation $\delta_\varepsilon F_t^s = \varepsilon^{-1}(F_t^{s+\varepsilon} - F_t^s)$ *for some functional* F *depending on* t, s *and possibly* ω.

The process $(X_t^s)_{t \geq 0}$ *is said to be twice differentiable in the mean w.r.t.* s *if* (4.62) *and* (4.63) *are uniquely solvable and, for any* $T > 0$ *and any* $s \in \mathbb{R}$,

$$\forall p \geq 1, \quad \lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \delta_\varepsilon X_t^s - \xi_t^s \right|^p \right] = 0,$$
$$\lim_{\varepsilon \to 0, \varepsilon \neq 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \delta_\varepsilon \xi_t^s - \eta_t^s \right|^p \right] = 0. \tag{4.65}$$

It turns out that *differentiability in the mean* holds under weaker assumptions than *pathwise differentiability*:

Theorem 4.7.4 *Assume that, for every* $t \geq 0$ *and (almost) every* $\omega \in \Omega$, *the coefficients* $b(t, \cdot) : x \in \mathbb{R}^d \mapsto b(t, x)$ *and* $\sigma(t, \cdot) : x \in \mathbb{R}^d \mapsto \sigma(t, x)$ *are of class* $\mathcal{C}^2$, *with bounded derivatives, uniformly in* t. *Consider a family of initial conditions* $(X_0^s)_{s \in \mathbb{R}}$ *that is twice differentiable in probability, i.e. such that, for any* $s \in \mathbb{R}$,

$$\xi_0^s = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \delta_\varepsilon X_0^s \text{ and } \eta_0^s = \lim_{\varepsilon \to 0, \varepsilon \neq 0} \delta_\varepsilon \xi_0^s \tag{4.66}$$

exist in probability, i.e. as in (4.64). *Then, the process* $(X_t^s)_{t \geq 0}$ *is twice differentiable in probability w.r.t.* s.

If the random variables $(X_0^s)_{s \in \mathbb{R}}$ *have finite* p*-moments of any order* $p \geq 1$ *and are differentiable in the mean, i.e.* (4.66) *holds as in* (4.65), *then the process* $(X_t^s)_{t \geq 0}$ *is twice differentiable in the mean w.r.t.* s.

The proof is a consequence of the stability property for SDEs. (See Proposition 4.7.1.)

We now say a word about the connection between the different kinds of differentiability. As easily guessed by the reader, *pathwise differentiability* is stronger than *differentiability in probability*. (This is a straightforward consequence of Lebesgue dominated convergence Theorem. This is also well-understood by comparing the assumptions of Theorems 4.7.2 and 4.7.4.) By Markov inequality, it is also clear that *differentiability in the mean* implies *differentiability in probability*.

The converse is true provided some uniform integrability conditions. Consider for example a family of initial conditions $(X_0^s)_{s \in \mathbb{R}}$, with finite p-moments of any order $p \geq 1$, such that the mapping $s \in \mathbb{R} \mapsto X_0^s$ is $\mathcal{C}^3$

almost-surely, with derivatives in any L^p, $p \geq 1$, uniformly in s in compact sets, and assume that, for some stopping τ, $(X_t^s)_{0 \leq t \leq \tau}$ is *twice differentiable in probability*, uniformly in $t \in [0, \tau]$. (That is T in (4.65) is replaced by τ.) If $\sup_{0 \leq t \leq \tau} |\xi_t^s|$ and $\sup_{0 \leq t \leq \tau} |\eta_t^s|$ are in any L^p, $p \geq 1$, uniformly in s in compact sets, then *twice differentiability in the mean* holds uniformly on $[0, \tau]$. As announced, the proof relies on a classical argument in probability theory: convergence in probability implies convergence in any L^p, $p \geq 1$, provided uniform integrability in any L^p, $p \geq 1$. Specifically, the point is to prove that, for any $s \in \mathbb{R}$ and $p \geq 1$, $\sup_{0 \leq t \leq \tau} |\delta_\varepsilon X_t^s|$ and $\sup_{0 \leq t \leq \tau} |\delta_\varepsilon \zeta_t^s|$ are in L^p, uniformly in ε in a neighborhood of 0 (ε being different from zero). This may be seen as a consequence of the bounds:

$$\mathbb{E}\left[\sup_{0 \leq t \leq \tau} |\delta_\varepsilon X_t^s|^p\right] \leq \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \mathbb{E}\left[\sup_{0 \leq t \leq \tau} |\zeta_t^r|^p\right] dr,$$
$$\mathbb{E}\left[\sup_{0 \leq t \leq \tau} |\delta_\varepsilon \zeta_t^s|^p\right] \leq \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \mathbb{E}\left[\sup_{0 \leq t \leq \tau} |\eta_t^r|^p\right] dr, \qquad (4.67)$$

for $\varepsilon > 0$. (Within the framework of Theorem 4.7.4 and with a similar inequality for $\varepsilon < 0$.) The above inequalities are a straightforward consequence of the first-order Taylor formula when the family $((X_t^s)_{t \geq 0})_{s \in \mathbb{R}}$ is twice differentiable in the pathwise sense, that is when the coefficients b and σ in Theorem 4.7.2 are smooth. When they are $\mathcal{C}^2$ only, we can approximate them by a sequence of mollified coefficients: by the stability property for SDEs, the derivatives of the solutions to the mollified equations converge towards the derivatives of the true equation; passing to the limit in (4.67), we obtain the expected bounds.

Unless specified, we will work below under the $\mathcal{C}^2$ framework of Theorem 4.7.4.

4.7.3 Derivative Quantity

In the whole subsection, we choose $X_0^s = \gamma(s)$, γ here standing for a $\mathcal{C}^2$ deterministic curve from $\mathbb{R}$ to $\mathbb{R}^d$, with bounded derivatives. As a consequence of Theorem 4.7.4, we claim:

Corollary 4.7.5 *Keep the assumption and notation of Theorem 4.7.4. Given $T > 0$ and a bounded progressively-measurable random function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^2$ with respect to the spatial parameter and with bounded derivatives, uniformly in time t and in randomness, the real-valued function of the real variable*

$$s \in [-1, 1] \mapsto w_T(s) = \mathbb{E} \int_0^T f(r, X_r^s) dr$$

admits as first and second-order derivatives:

$$w'_T(s) = \mathbb{E}\int_0^T D_x f(r, X_r^s)\xi_r^s dr$$

$$w''_T(s) = \mathbb{E}\int_0^T \big(D_x f(r, X_r^s)\eta_r^s + D^2_{x,x} f(r, X_r^s)\xi_r^s \otimes \xi_r^s\big) dr.$$

Corollary 4.7.5 permits to bound w'_T and w''_T. Indeed, since the equations satisfied by $(\xi_t^s)_{t\geq 0}$ and $(\eta_t^s)_{t\geq 0}$ are linear (with random coefficients), standard stability techniques, based on Gronwall's Lemma, would show that:

$$\forall p \geq 0, \quad \forall T > 0, \quad \sup_{0\leq t\leq T} \mathbb{E}\big[|\xi_t^s|^p + |\eta_t^s|^p\big] \leq C(p,T), \tag{4.68}$$

$C(p,T)$ depending on p, T and the bounds for the derivatives of the coefficients.

Unfortunately, Corollary 4.7.5 doesn't apply to Proposition 4.6.9 since T is infinite in Proposition 4.6.9. Therefore, we must discuss the long-run behavior of $(|\xi_t^s|)_{t\geq 0}$ and $(|\eta_t^s|)_{t\geq 0}$ carefully and, specifically, investigate the long-run integrability against the exponential weight generated by the plurisuperharmonic function ψ, exactly as in the representation formula of Proposition 4.6.9.

In this framework, we emphasize the following facts. First, in light of Corollary 4.7.5, it is sufficient to analyze the long-run behavior of the second-order moments of $(|\xi_t^s|)_{t\geq 0}$ and the first-order moments of $(|\eta_t^s|)_{t\geq 0}$. Moreover, the linear structure of $(\eta_t^s)_{t\geq 0}$ being close to the one of $(\xi_t^s)_{t\geq 0}$ (the nonlinear terms in the dynamics of $(|\eta_t^s|)_{t\geq 0}$ being controlled by $(|\xi_t^s|^2)_{t\geq 0}$), it is more or less sufficient to investigate the long-run behavior of $(|\xi_t^s|^2)_{t\geq 0}$.

Therefore, we now compute the form of $d|\xi_t^s|^2$. Using Itô's formula, we obtain

$$\begin{aligned} d|\xi_t^s|^2 &= 2\sum_{i,j=1}^{d_B} (\xi_t^s)^i D_{x_j} b^i(t, X_t^s)(\xi_t^s)^j dt \\ &\quad + \sum_{i=1}^{d}\sum_{j=1}^{n}\left(\sum_{k=1}^{d} D_{x_k}\sigma_{i,j}(t, X_t^s)(\xi_t^s)^k\right)^2 dt + dm_t, \end{aligned} \tag{4.69}$$

dm_t standing for a martingale term, which has no role when computing the expectation. In comparison with Krylov's original proof, we emphasize that Krylov makes use of the following shorten notation:

$$D_\xi b_t^i := \sum_{j=1}^{d} D_{x_j} b^i(t, X_t^s)(\xi_t^s)^j, \quad D_\xi \sigma_t^{i,j} := \sum_{k=1}^{d} D_{x_k}\sigma_{i,j}(t, X_t^s)(\xi_t^s)^k,$$

so that the dynamics of $|\xi_t^s|^2$ have the form:

$$d|\xi_t^s|^2 = \left[2\langle \xi_t^s, D_\xi b_t\rangle + |D_\xi \sigma_t|^2\right] dt + dm_t. \tag{4.70}$$

A typical condition to obtain a long-run control for $(|\xi_t^s|^2)_{t\geq 0}$ is

$$2\langle \xi_t^s, D_\xi b_t\rangle + |D_\xi \sigma_t|^2 \leq 0, \quad t \geq 0. \tag{4.71}$$

Indeed, (4.71) implies that $(\mathbb{E}[|\xi_t^s|^2])_{t\geq 0}$ is bounded.

Actually, the reader must understand that the choice we here make is very restrictive: instead of investigating the dynamics of $(|\xi_t^s|^2)_{t\geq 0}$, we could also investigate the dynamics of $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$ for some smooth function A from $\mathbb{R}^d$ into the set of positive symmetric matrices of dimension d. Indeed, if the spectrum of A is in a compact subset of $(0, +\infty)$, it is equivalent to obtain a long-run control for $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$ and a long-run control for $(|\xi_t^s|^2)_{t\geq 0}$.

By choosing A possibly different from the identity, we are able to plug some freedom into (4.70) and thus to relax the condition (4.71).

In what follows, we will call:

Definition 4.7.6 *Under the notation and assumption of Theorem 4.7.4 and for a smooth function A from $\mathbb{R}^d$ into the set of positive symmetric matrices of size d, we call* derivative quantity *the quadratic process $(\langle A(X_t^s)\xi_t^s, \xi_t^s\rangle)_{t\geq 0}$, denoted by $(\Gamma_t^s)_{t\geq 0}$, and we call* dynamics of the derivative quantity *its absolutely continuous part, denoted by $(\partial\Gamma_t^s)_{t\geq 0}$.*

Specifically, we call dynamics of derivative quantity *(at point $\gamma(s)$) the process (also denoted by $(\partial\Gamma_t(X_t^s, \xi_t^s))_{t\geq 0}$) given by*

$$\begin{aligned}\partial\Gamma_t^s = {} & 2\langle \xi_t^s, A(X_t^s) D_x b(t, X_t^s)\xi_t^s\rangle \\ & + \langle D_x\sigma(t, X_t^s)\xi_t^s, A(X_t^s) D_x\sigma(t, X_t^s)\xi_t^s\rangle \\ & + 2\mathrm{Trace}\left[(D_x\sigma^*(t, X_t^s)\xi_t^s)(D_x A(X_t^s)\xi_t^s)\sigma(t, X_t^s)\right] \\ & + \langle \xi_t^s, (L_t A)(X_t^s)\xi_t^s\rangle, \quad t \geq 0,\end{aligned}$$

where

$$L_t = \sum_{i=1}^d b_i(t, \cdot) D_{x_i} + (1/2) \sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(t, \cdot) D^2_{x_i, x_j}$$

$$\begin{aligned}&\langle D_x\sigma(t, X_t^s)\xi_t^s, A(X_t^s) D_x\sigma(t, X_t^s)\xi_t^s\rangle \\ &\quad = \sum_{j=1}^{d_B} \langle D_x\sigma_{\cdot,j}(t, X_t^s)\xi_t^s, A(X_t^s) D_x\sigma_{\cdot,j}(t, X_t^s)\xi_t^s\rangle\end{aligned} \tag{4.72}$$

$$
\begin{aligned}
&\mathrm{Trace}\big[(D_x\sigma^*(t,X_t^s)\xi_t^s)(D_xA(X_t^s)\xi_t^s)\sigma(t,X_t^s)\big]\\
&\quad=\sum_{i,k=1}^{d}\sum_{j=1}^{d_B}(D_x\sigma_{i,j}(t,X_t^s)\xi_t^s)\big((D_xA_{\cdot,k}(\xi_t^s)_k)\sigma(t,X_t^s)\big)_{i,j}.
\end{aligned}
$$

Following (4.70), *it satisfies*

$$
d\Gamma_t^s = d\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle = \partial\Gamma_t^s dt + dm_t, \quad t\geq 0. \tag{4.73}
$$

(In the complex case, A is an Hermitian functional and Γ_t^s has the form $\langle \xi_t^s, A(X_t^s)\bar{\xi}_t^s\rangle$.)

We claim

Proposition 4.7.7 *Together with the notations given above, we are also given a real $\delta>0$ and an $[\delta,+\infty)$-valued (progressively-measurable) random function c both depending on the randomness $\omega\in\Omega$ and on $(t,x)\in[0,+\infty)\times\mathbb{R}^d$ such that, for every $t\geq 0$ and for (almost) every $\omega\in\Omega$, $c(t,\cdot): x\in\mathbb{R}^d\mapsto c(t,x)\in[\delta,+\infty)$ is of class $\mathcal{C}^2$, with bounded derivatives, uniformly in t and in ω.*

Given an open subset $\mathcal{U}\subset\mathbb{R}^d$ such that $\gamma(s)\in\mathcal{U}$ for some $s\in[-1,1]$, assume that $\partial\Gamma_t^s=\partial\Gamma_t(X_t^s,\xi_t^s)\leq (c(t,X_t^s)-\delta)\Gamma_t^s$ up to the exit time from $\mathcal{U}$, i.e. for $t\leq\tau_U:=\inf\{t\geq 0: X_t^s\notin\mathcal{U}\}$, then, for any $t\geq 0$,

$$
\mathbb{E}\Big[\exp\Big(-\int_0^{t\wedge\tau_{\mathcal{U}}}(c(r,X_r^s)-\delta)dr\Big)\Gamma_{t\wedge\tau_{\mathcal{U}}}\Big]\leq\langle\gamma'(s),A(\gamma(s))\gamma'(s)\rangle. \tag{4.74}
$$

Assume for example that $U=\mathbb{R}^d$. Then, with the notation and assumption of Corollary 4.7.5, there exists a constant C depending on δ and the L^∞ norms (on $\mathcal{U}$) of A^{-1}, D_xc, f and D_xf only such that, for any $T>0$, the function

$$
s\in[-1,1]\mapsto w_T(s)=\mathbb{E}\Big[\int_0^T\exp\Big(-\int_0^t c(r,X_r^s)dr\Big)f(t,X_t^s)dt\Big], \tag{4.75}
$$

satisfy $|w_T'(s)|\leq C|\gamma'(s)|$. In particular, the Lipschitz constant of w_T is independent of T.

Proof. The proof is almost straightforward. By (4.73),

$$
\begin{aligned}
&d\Big[\exp\Big(-\int_0^t\big(c(r,X_r^s)-\delta\big)dr\Big)\Gamma_t^s\Big]\\
&d\Big[\exp\Big(-\int_0^t\big(c(r,X_r^s)-\delta\big)dr\Big)\langle\xi_t^s,A(X_t^s)\xi_t^s\rangle\Big]\\
&\quad=\exp\Big(-\int_0^t\big(c(r,X_r^s)-\delta\big)dr\Big)\big[\big(\partial\Gamma_t^s-(c(t,X_t^s)-\delta)\Gamma_t^s\big)dt+dm_t\big].
\end{aligned}
$$

Taking the expectation, we get rid of the martingale term. Having, in mind the sign condition on $\partial\Gamma_t^s - (c(t,X_t^s)-\delta)\Gamma_t^s$, we directly deduce (4.74).

To prove the Lipschitz estimate, we first emphasize that, for any $s \in [-1,1]$,

$$\begin{aligned}
|w_T'(s)| &= \bigg| \mathbb{E} \int_0^T \exp\Big(- \int_0^t c(r, X_r^s) dr \Big) \\
&\qquad \times \Big[D_x f(t, X_t^s) \xi_t^s - f(t, X_t^s) \int_0^t D_x c(r, X_r^s) \xi_r^s dr \Big] \bigg| \\
&\le C \mathbb{E} \bigg[\int_0^T \exp\Big(- \int_0^t c(r, X_r^s) dr \Big) \Big[|\xi_t^s| + \int_0^t |\xi_r^s| dr \Big] dt \bigg], \qquad (4.76)
\end{aligned}$$

for some constant C depending on $\|f\|_\infty$, $\|D_x f\|_\infty$ and $\|D_x c\|_\infty$ only.

The result then follows from Lemma 4.7.8 below. □

Lemma 4.7.8 *Consider a non-negative process $(c_t)_{t\ge 0}$ together with an $\mathbb{R}^d$-valued process $(\xi_t)_{t\ge 0}$ such that $c_t \ge \delta$, $t \ge 0$, and*

$$\mathbb{E}\bigg[\exp\Big(- \int_0^t c_r dr \Big) |\xi_t|^2 \bigg] \le C \exp(-\delta t), \quad t \ge 0,$$

for some $C \ge 0$ and $\delta > 0$, then

$$\mathbb{E}\bigg[\int_0^{+\infty} \exp\Big(- \int_0^t c_r dr \Big) \Big(|\xi_t| + \int_0^t |\xi_r| dr \Big) dt \bigg] \le C',$$

for some C' depending on C and δ only.

Proof. From Cauchy-Schwarz inequality and from the bound $c \ge \delta$, we obtain the L^1 version:

$$\begin{aligned}
\mathbb{E}\bigg[\exp\Big(- \int_0^t c_r dr \Big) |\xi_t^s| \bigg] &\le \mathbb{E}\bigg[\exp\Big(-2 \int_0^t c_r dr \Big) |\xi_t^s|^2 \bigg]^{1/2} \\
&\le \exp\Big(-\frac{\delta}{2} t \Big) \mathbb{E}\bigg[\exp\Big(- \int_0^t c_r dr \Big) |\xi_t^s|^2 \bigg]^{1/2} \\
&\le C^{1/2} \exp(-\delta t), \quad t \ge 0. \qquad (4.77)
\end{aligned}$$

In particular, since c is always larger than δ, Inequality (4.77) yields

$$\mathbb{E}\bigg[\int_0^{+\infty} \exp\Big(- \int_0^t c_r dr \Big) \Big(|\xi_t| + \int_0^t |\xi_r| dr \Big) dt \bigg]$$

$$
\begin{aligned}
&\le \mathbb{E}\bigg[\int_0^{+\infty}\bigg(\exp\Big(-\int_0^t c_r dr\Big)|\xi_t| \\
&\qquad\qquad + \exp(-\delta t)\int_0^t \exp(\delta r)\exp\Big(-\int_0^r c_u du\Big)|\xi_r| dr\bigg) dt\bigg] \\
&\le C^{1/2}\int_0^{+\infty}\exp(-\delta t)\big(1+t\big)dt. \qquad (4.78)
\end{aligned}
$$

This completes the proof. □

We now perform a similar analysis, but for the second-order derivative $(\langle \eta_t^s, A(X_t^s)\eta_t^s\rangle)_{t\ge 0}$ (see Theorems 4.7.2 and 4.7.4) and then for $w_T''(s)$.

Proposition 4.7.9 *Assume that the assumption of Proposition 4.7.7 are in force and that σ is bounded. For any $s \in [-1,1]$, denote by $(\Delta_t^s)_{t\ge 0}$ (or by $(\Gamma_t(X_t^s,\eta_t^s))_{t\ge 0}$) the process $(\langle \eta_t^s, A(X_t^s)\eta_t^s\rangle)_{t\ge 0}$ and by $(\partial\Delta_t^s)_{t\ge 0}$ the process*

$$
\begin{aligned}
\partial\Delta_t^s &= 2\langle \eta_t^s, A(X_t^s)D_x b(t,X_t^s)\eta_t^s\rangle \\
&\quad + \langle D_x\sigma(t,X_t^s)\eta_t^s, A(X_t^s)D_x\sigma(t,X_t^s)\eta_t^s\rangle \\
&\quad + 2\mathrm{Trace}\big[(D_x\sigma^*(t,X_t^s)\eta_t^s)(D_x A(X_t^s)\eta_t^s)\sigma(t,X_t^s)\big] \\
&\quad + \langle \eta_t^s, (L_t A)(X_t^s)\eta_t^s\rangle, \quad t\ge 0.
\end{aligned}
$$

(Be careful that $(\partial\Delta_t^s)_{t\ge 0}$ is not the absolutely continuous part of $(\Delta_t^s)_{t\ge 0}$. It is obtained by replacing $(\xi_t^s)_{t\ge 0}$ by $(\eta_t^s)_{t\ge 0}$ in the definition of $(\partial\Gamma_t^s)_{t\ge 0}$.)

Given an open subset $\mathcal{U}\subset\mathbb{R}^d$ such that $\gamma(s)\in\mathcal{U}$ for some $s\in[-1,1]$, assume that, for all $t\le\tau_U := \inf\{t\ge 0 \ : \ \Gamma_t^s \notin \mathcal{U}\}$, $\partial\Delta_t \le (c(t,X_t^s)-\delta)\Delta_t$. (Pay attention that this is exactly the same inequality as the one in Proposition 4.7.7, but with $(\xi_t^s)_{t\ge 0}$ replaced by $(\eta_t^s)_{t\ge 0}$. Clearly, if the one in Proposition 4.7.7 is true, the current one is expected to be true as well.) Then, there exists a constant C, depending on δ and the L^∞ norms (on $\mathcal{U}$) of A, A^{-1}, D_xA, c, $D^2_{x,x}b$, σ, $D_x\sigma$ and $D^2_{x,x}\sigma$ only, such that, for any $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\Big(-\int_0^{t\wedge\tau_{\mathcal{U}}}(c(r,X_r^s)-\delta/2)dr\Big)\big([\Gamma^s_{t\wedge\tau_{\mathcal{U}}}]^2+\Delta^s_{t\wedge\tau_{\mathcal{U}}}\big)^{1/2}\bigg] \\
&\quad \le \big(\langle\gamma'(s),A(\gamma(s))\gamma'(s)\rangle^2 + \langle\gamma''(s),A(\gamma(s))\gamma''(s)\rangle\big)^{1/2} \\
&\qquad + C\langle\gamma'(s),A(\gamma(s))\gamma'(s)\rangle. \qquad (4.79)
\end{aligned}
$$

For example if $\mathcal{U}=\mathbb{R}^d$, the function w_T in (4.75) satisfies

$$
|w_T''(s)| \le \frac{C}{(1\wedge\delta)^3}\big(|\gamma'(s)|^2+|\gamma''(s)|\big), \quad s\in[-1,1].
$$

for a possible modified value of the constant C, depending on the L^∞ norms (on $\mathcal{U}$) of $D_x c$, $D^2_{x,x}c$, f, $D_x f$ and $D^2_{x,x}f$ as well. (In particular, it is independent of T and s.)

Proof. For simplicity, we make use of Krylov's notations, i.e. we set: $D_\eta b^s_t := D_x b(t, X^s_t)\eta^s_t$, $D^2_{\xi,\xi} b^s_t := D^2_{x,x} b(t, X^s_t)\xi^s_t \otimes \xi^s_t$, $D_\eta(\sigma^s_t)_{\cdot,j} := D_x\sigma_{\cdot,j}(t, X^s_t)\eta^s_t$ and finally $D^2_{\xi,\xi}(\sigma^s_t)_{\cdot,j} := D^2_{x,x}\sigma_{\cdot,j}(t, X^s_t)\xi^s_t \otimes \xi^s_t$. With these notations, η in Theorem 4.7.2 has the form:

$$d\eta^s_t = D_\eta b^s_t dt + D^2_{\xi,\xi} b^s_t dt + \sum_{j=1}^{d_B} D_\eta(\sigma^s_t)_{\cdot,j} dW^j_t + \sum_{j=1}^{d_B} D^2_{\xi,\xi}(\sigma^s_t)_{\cdot,j} dW^j_t,$$

$t \geq 0$. Considering the quadratic form driven by A, we obtain (with the notation $A^s_t = A(X^s_t)$)

$$\begin{aligned} & d\langle \eta^s_t, A^s_t \eta^s_t\rangle \\ & \quad = 2\langle \eta^s_t, A^s_t D_\eta b^s_t\rangle dt + 2\langle \eta^s_t, A^s_t D^2_{\xi,\xi} b^s_t\rangle dt \\ & \qquad + 2\mathrm{Trace}\big[(D_\eta \sigma^*(t, X^s_t) + D^2_{\xi,\xi}\sigma^*(t, X^s_t))(D_x A(X^s_t)\eta^s_t)\sigma(t, X^s_t)\big] dt \\ & \qquad + \langle \big(D_\eta \sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big), A^s_t \big(D_\eta\sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big)\rangle dt + \langle \eta^s_t, L_t A^s_t \eta^s_t\rangle dt + dm_t, \end{aligned}$$

$t \geq 0$, $(m_t)_{t\geq 0}$ standing for a generic martingale term that is (more or less) useless in what follows. (See (4.72) for the definition of $\langle \big(D_\eta \sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big), A^s_t \big(D_\eta\sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big)\rangle$.) Following the proof of (4.73),

$$d\big[\langle \xi^s_t, A^s_t \xi^s_t\rangle^2\big] = 2\langle \xi^s_t, A^s_t\xi^s_t\rangle \partial\Gamma^s_t dt + \big|2A^s_t D_\xi \sigma^s_t \xi^s_t + \langle \xi^s_t, D_\sigma A(X^s_t)\xi^s_t\rangle\big|^2 dt + dm_t.$$

(Here again, the generic notation $(m_t)_{t\geq 0}$ stands for a martingale. Moreover, the term $|2A^s_t D_\xi \sigma^s_t\xi^s_t + \langle \xi^s_t, D_\sigma A(X^s_t)\xi^s_t\rangle|^2$ stands for $\sum_{j=1}^{d_B} |2\langle A^s_t D_\xi (\sigma^s_t)_{\cdot,j}, \xi^s_t\rangle + \sum_{k=1}^{d}\langle \xi^s_t, D_{x_k}A(X^s_t)\xi^s_t\rangle \sigma_{k,j}(t, X^s_t)|^2$.) Therefore,

$$\begin{aligned} & d\big(\langle \xi^s_t, A^s_t\xi^s_t\rangle^2 + \langle \eta^s_t, A^s_t\eta^s_t\rangle\big) \\ & \quad = 2\langle \eta^s_t, A^s_t D_\eta b^s_t\rangle dt + 2\langle \eta^s_t, A^s_t D^2_{\xi,\xi} b^s_t\rangle dt + \langle \eta^s_t, L_t A^s_t\eta^s_t\rangle dt \\ & \qquad + 2\mathrm{Trace}\big[(D_\eta \sigma^*(t, X^s_t) + D^2_{\xi,\xi}\sigma^*(t, X^s_t))(D_x A(X^s_t)\eta^s_t)\sigma(t, X^s_t)\big] dt \\ & \qquad + \langle \big(D_\eta \sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big), A^s_t \big(D_\eta\sigma^s_t + D^2_{\xi,\xi}\sigma^s_t\big)\rangle dt + 2\langle \xi^s_t, A^s_t\xi^s_t\rangle\partial\Gamma^s_t dt \\ & \qquad + \big|2A^s_t D_\xi\sigma^s_t\xi^s_t + \langle \xi^s_t, D_\sigma A(X^s_t)\xi^s_t\rangle\big|^2 dt + dm_t. \end{aligned}$$

Apply now the function $x \in \mathbb{R} \mapsto (a + x)^{1/2}$, for some small $a > 0$. It is a concave function, so that the second-order term deriving from Itô's formula is non-increasing. In particular, we write (in a little bit crude way)

$$
\begin{aligned}
&d\big(a + \langle \xi_t^s, A_t^s \xi_t\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{1/2} \\
&\quad \leq \frac{1}{2}\big(a + \langle \xi_t^s, A_t^s \xi_t\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{-1/2}\big[2\langle \eta_t^s, A_t^s D_\eta b_t^s\rangle + 2\langle \eta_t^s, A_t^s D_{\xi,\xi}^2 b_t^s\rangle \\
&\qquad + 2\text{Trace}\big[(D_\eta \sigma^*(t, X_t^s) + D_{\xi,\xi}^2 \sigma^*(t, X_t^s))(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big]dt \\
&\qquad + \langle \eta_t^s, L_t A_t^s \eta_t^s\rangle + \langle \big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big), A_t^s\big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big)\rangle \\
&\qquad + 2\langle \xi_t^s, A_t^s \xi_t^s\rangle \partial \Gamma_t^s dt + \big|2A_t^s D_\xi \sigma_t^s \xi_t^s + \langle \xi_t^s, D_\sigma A(X_t^s)\xi_t^s\rangle\big|^2\big] + dm_t.
\end{aligned}
\tag{4.80}
$$

We now claim that

$$
\begin{aligned}
&2\langle \eta_t^s, A_t^s D_\eta b_t^s\rangle + \langle \eta_t^s, L_t A_t^s \eta_t^s\rangle + \langle \big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big), A_t^s\big(D_\eta \sigma_t^s + D_{\xi,\xi}^2 \sigma_t^s\big)\rangle \\
&\qquad + 2\text{Trace}\big[(D_\eta \sigma^*(t, X_t^s) + D_{\xi,\xi}^2 \sigma^*(t, X_t^s))(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big] \\
&\quad = \partial \Delta_t^s + 2\langle D_\eta \sigma_t^s, A_t^s D_{\xi,\xi}^2 \sigma_t^s\rangle + \langle D_{\xi,\xi}^2 \sigma_t^s, A_t^s D_{\xi,\xi}^2 \sigma_t^s\rangle \\
&\qquad + 2\text{Trace}\big[D_{\xi,\xi}^2 \sigma^*(t, X_t^s)(D_x A(X_t^s)\eta_t^s)\sigma(t, X_t^s)\big] \\
&\quad = \partial \Delta_t^s + O\big((a + |\xi_t^s|^4 + |\eta_t^s|^2)^{1/2}|\xi_t^s|^2\big),
\end{aligned}
$$

the notation $O(\dots)$ standing for the Landau notation. Here, we emphasize that the underlying constant in $O(\cdots)$ depends on the L^∞ norms (on $\mathcal{U}$) of A, $D_x A$, σ, $D_x \sigma$ and $D_{x,x}^2 \sigma$ only and, in particular, is independent of t and ω. Actually, all the remaining terms in (4.80) except the martingale term can be bounded by $O((a + |\xi_t^s|^4 + |\eta_t^s|^2)^{1/2}|\xi_t^s|^2)$ as well, the underlying constant in $O(\cdots)$ possibly depending on the L^∞ norms (on $\mathcal{U}$) of A^{-1}, c and $D_{x,x}^2 b$ also. Therefore, we can find some constant $C > 0$, depending on the L^∞ norms (on $\mathcal{U}$) of A, A^{-1}, $D_x A$, c, $D_{x,x}^2 b$, σ, $D_x \sigma$ and $D_{x,x}^2 \sigma$ only, such that

$$
\begin{aligned}
&d\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{1/2} \\
&\quad \leq \frac{1}{2}\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{-1/2}\partial \Delta_t^s dt + C|\xi_t^s|^2 dt + dm_t.
\end{aligned}
$$

Finally, following the proof of Proposition 4.7.7,

$$
\begin{aligned}
&d\Big[\exp\Big(-\int_0^t (c(r, X_r^s) - \delta)dr\Big)\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{1/2}\Big] \\
&\quad \leq \frac{1}{2}\exp\Big(-\int_0^t (c(r, X_r^s) - \delta)dr\Big)\big\{\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)^{-1/2} \\
&\qquad \times \big[\partial \Delta_t^s - 2(c(t, X_t^s) - \delta)\big(a + \langle \xi_t^s, A_t^s \xi_t^s\rangle^2 + \langle \eta_t^s, A_t^s \eta_t^s\rangle\big)\big] \\
&\qquad\quad + C|\xi_t^s|^2 dt + dm_t\big\}.
\end{aligned}
$$

By assumption, $\partial\Delta_t \le (c(t,X_t^s)-\delta)\langle\eta_t^s, A_t^s\eta_t^s\rangle \le 2(c(t,X_t^s)-\delta)\langle\eta_t^s, A_t^s\eta_t^s\rangle$ since c is greater than δ, so that

$$d\left[\exp\left(-\int_0^t (c(r,X_r^s)-\delta)dr\right)\left(a+\langle\xi_t^s, A_t^s\xi_t^s\rangle^2+\langle\eta_t^s, A_t^s\eta_t^s\rangle\right)^{1/2}\right]$$
$$\le \exp\left(-\int_0^t (c(r,X_r^s)-\delta)dr\right)\left\{C|\xi_t^s|^2dt+dm_t\right\}.$$

Integrating from 0 to $t\wedge\tau_{\mathcal{U}}$, taking the expectation and letting a tend to 0,

$$\mathbb{E}\left[\exp\left(-\int_0^{t\wedge\tau_{\mathcal{U}}}\big(c(r,X_r^s)-\delta\big)dr\right)\left(\langle\xi_{t\wedge\tau_{\mathcal{U}}}^s, A_{t\wedge\tau_{\mathcal{U}}}^s\xi_{t\wedge\tau_{\mathcal{U}}}^s\rangle^2\right.\right.$$
$$\left.\left.+\langle\eta_{t\wedge\tau_{\mathcal{U}}}^s, A_{t\wedge\tau_{\mathcal{U}}}^s\eta_{t\wedge\tau_{\mathcal{U}}}^s\rangle\right)^{1/2}\right]$$
$$\le \left(\langle\gamma'(s), A(\gamma(s))\gamma'(s)\rangle^2+\langle\gamma''(s), A(\gamma(s))\gamma''(s)\rangle\right)^{1/2}$$
$$+C\mathbb{E}\int_0^{t\wedge\tau_{\mathcal{U}}}\left[\exp\left(-\int_0^r (c(u,X_u^s)-\delta)du\right)|\xi_r^s|^2\right]dr.$$

Obviously, the above inequality applies with $\delta/2$ instead of δ. Then, from Proposition 4.7.7, the last term in the RHS has the form

$$\mathbb{E}\int_0^{t\wedge\tau_{\mathcal{U}}}\left[\exp\left(-\int_0^r (c(u,X_u^s)-\delta/2)du\right)|\xi_r^s|^2\right]dr$$
$$\le \int_0^{+\infty}\left[\exp\big(-(\delta/2)r\big)\mathbb{E}\left[\exp\left(-\int_0^{r\wedge\tau_{\mathcal{U}}}(c(u,X_u^s)-\delta)du\right)|\xi_{r\wedge\tau_{\mathcal{U}}}^s|^2\right]dr\right]$$
$$\le C\langle\gamma'(s), A(\gamma(s)), \gamma'(s)\rangle\int_0^{+\infty}\exp\big(-(\delta/2)r\big)dr,$$

for a possibly new value of C, possibly depending on δ as well. This completes the proof of (4.79).

We now investigate w_T''. Following the proof of (4.76), we claim

$$|w_T''(s)| \le C\mathbb{E}\left[\int_0^T \exp\left(-\int_0^t c(r,X_r^s)dr\right)\left[|\eta_t^s|+\int_0^t|\eta_r^s|dr\right.\right.$$
$$\left.\left.+|\xi_t^s|^2+\int_0^t|\xi_r^s|^2dr+|\xi_t^s|\int_0^t|\xi_r^s|dr+\left(\int_0^t|\xi_r^s|dr\right)^2\right]\right]. \tag{4.81}$$

We now apply (4.78) and (4.79). For some possibly new value of the constant C, also depending on the L^∞ norms (on $\mathcal{U}$) of c, D_xc, $D_{x,x}^2c$, f, D_xf and $D_{x,x}^2f$,

$$\mathbb{E}\Big[\int_0^T \exp\Big(-\int_0^t c(r,X_r^s)dr\Big)\Big[|\eta_t^s| + \int_0^t |\eta_r^s|dr\Big]\Big] \leq C\big(|\gamma'(s)|^2 + |\gamma''(s)|\big). \tag{4.82}$$

This shows how to deal with the terms in η^s in (4.81). The terms in ξ^s can be handled as follows. Note from Young's inequality and Cauchy-Schwarz inequality that

$$\begin{aligned} &|\xi_t^s|^2 + \int_0^t |\xi_r^s|^2 dr + |\xi_t^s| \int_0^t |\xi_r^s| dr + \Big(\int_0^t |\xi_r^s| dr\Big)^2 \\ &\leq C\Big(|\xi_t^s|^2 + (1+t)\int_0^t |\xi_r^s|^2\Big), \quad t \geq 0. \end{aligned} \tag{4.83}$$

Following (4.78), we complete the proof. □

4.7.4 Conclusion

Before we carry on the analysis of the Monge–Ampère equation, we mention the following points:

1. We let the reader adapt the statements of Propositions 4.7.7 and 4.7.9 to the complex case, then considering A as an Hermitian functional.
2. As well guessed from Proposition 4.6.9, the (random) function c in the statements of Propositions 4.7.7 and 4.7.9 must be understood as $\mathrm{Trace}(a_t D^2_{z,\bar{z}}\psi(z))$ in the specific framework of Monge–Ampère.
3. We also emphasize how the rule obtained by Krylov has a very simple form. The whole problem is now to compare two quadratic (or Hermitian in the complex case) forms: $\xi \in \mathbb{R}^d \mapsto \partial\Gamma_t(x,\xi)$ and $\xi \in \mathbb{R}^d \mapsto (c(t,x)-\delta)|\xi|^2$, with $t \geq 0$ and $x \in \mathbb{R}^d$ (or x in a domain of $\mathbb{R}^d$ or $\mathbb{C}^d$: for instance $\mathcal{D}$ in the Monge–Ampère case). If comparison holds, then both the first and second-order derivatives of w_T in the statement of Proposition 4.7.7 can be controlled uniformly in T. In the Hamilton–Jacobi–Bellman framework, the comparison rule between $\partial\Gamma_t(x,\xi)$ and $(c(t,x)-\delta)|\xi|^2$ must hold for any value of the underlying parameter (denoted by σ in the specific case of Monge–Ampère, see Proposition 4.6.9). Obviously, establishing such a comparison rule might be really challenging in practice: it is indeed in the Monge–Ampère case!
4. Below, we sometimes call the process $(\partial\Gamma_t^s)_{t\geq 0}$ in Definition 4.7.6 *derivative quantity* itself whereas the *derivative quantity* stands for the process $(\langle \xi_t^s, A(X_t^s)\xi_t^s\rangle)_{t\geq 0}$. We feel that it is not confusive for the reader.

4.8 Almost Proof of the $\mathcal{C}^1$ Regularity

In this section, we explain how to derive the $\mathcal{C}^1$ property of the solution to Monge–Ampère equation from the program developed in the previous Sect. 4.7.4. Unfortunately, we are not able to provide a completely rigorous proof at this stage of the notes: some "holes" are indeed left open in the proof. Specifically, some quantities under consideration are not rigorously shown to be differentiable. The plan is thus the following: we here explain how things work without paying too much attention to the differentiability arguments and we postpone to the final Sect. 4.9 the complete argument. We will deal with the second-order estimates in Sect. 4.9 as well.

For all these reasons, the following statement is called a "Meta-Theorem":

Meta-Theorem 4.8.1 *Assume that Assumption (**A**) is in force and keep the notation of Proposition 4.6.9. Then, up to the proof of some differentiability properties, it may be shown that, for any compact subset $\mathcal{K} \subset \mathcal{D}$, there exists a constant C, depending on (**A**) and $\mathcal{K}$ only, such that, for every smooth curve $\gamma : [-1, 1] \to \mathcal{D}$, the function $s \mapsto V(\gamma(s))$ is Lipschitz with $C\|\gamma'\|_\infty$ as Lipschitz constant.*

Obviously, the whole idea is to apply Points (2) and (3) in Conclusion 4.7.4 to the solution of the rescaled SDE (4.50), i.e.

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^s)dt, \quad t \geq 0. \tag{4.84}$$

with $Z_0^s = \gamma(s)$, where $\gamma : s \in [-1, 1] \mapsto \gamma(s) \in \mathcal{D}$ is a curve as in the statement of Theorem 4.8.1. (Note that the compact set $\mathcal{K}$ is not specified at this stage of the proof.) Here, $(\sigma_t)_{t\geq 0}$ denotes a generic control process (i.e. a progressively-measurable process with values in $\mathbb{C}^{d\times d}$ such that $\text{Trace}(\sigma\bar{\sigma}_t^*) = 1$.)

The reader may then easily understand what "Meta" means: because of the exponent $1/2$, the function $\psi^{1/2}$ is singular at the boundary so that Theorems 4.7.2 and 4.7.4 do not apply to (4.84). In particular, it may be a bit tricky to establish the differentiability of $(Z_t^s)_{t\geq 0}$ w.r.t. s. As announced above, we forget this difficulty in the whole section and assume that (4.84) is differentiable in the mean w.r.t. s. Setting $\zeta_t^s = dZ_t^s/ds$, $t \geq 0$, we write (at least formally)

$$\begin{aligned} d\zeta_t^s &= \psi^{-1/2}(Z_t^s)\text{Re}\big[D_z\psi(Z_t^s)\zeta_t^s\big]\sigma_t dB_t \\ &\quad + \big[a_t D_{\bar{z},z}^2\psi(Z_t^s)\zeta_t^s + a_t D_{\bar{z},\bar{z}}^2\psi(Z_t^s)\bar{\zeta}_t^s\big]dt. \end{aligned} \tag{4.85}$$

Applying Itô's formula, we could compute the dynamics of $(|\zeta_t^s|^2)_{t\geq 0}$ as in (4.73) and thus express the form of the associated derivative quantity. We won't do it here: the strategy fails when applied in a straightforward way. Said differently, there are very little chances to be able to bound the *derivative quantity* as in the statements of Propositions 4.7.7 and 4.7.9.

4.8.1 Procedure to Estimate the **Derivative Quantity** *in the General Case*

The major idea of Krylov consists in *perturbing* as most as possible the probabilistic ingredients of the Monge–Ampère equation to improve the long-run control of the *derivative quantity*. Here, the word "perturbing" doesn't mean that we are seeking for another new representation: the general structure given by Proposition 4.6.9 is the right one. The whole problem is to *perturb* it in a convenient way to obtain the desired long-run estimate.

There are three general ways to perturb the system:

1. Since the problem is stationary, time speed may be changed,
2. Using stochastic processes theory, the underlying probability measure may be perturbed itself,
3. Finally, additional "ghost" control parameters may be plugged into the control representation and used as perturbation parameters.

We here try to explain the main ideas of this *perturbation* procedure. In the next subsections, we will show how to apply them to the Monge–Ampère equation explicitly. Unfortunately, to do so, the method given in Proposition 4.6.6 must be revisited first.

Having in mind the general notation used in Proposition 4.6.6, the revisited strategy may be explained as follows. Consider indeed a generic family:

$$w^\beta(s) = \mathbb{E}\int_0^{+\infty} F(\beta_r, X_r^{s,\beta})dr, \tag{4.86}$$

where

$$dX_t^{s,\beta} = \sigma(\beta_t, X_t^{s,\beta})dB_t + b(\beta_t, X_t^{s,\beta})dt, \quad t \geq 0\ ; \quad X_0^{s,\beta} = \gamma(s),$$

just as in Propositions 4.6.6 and 4.6.9. Assume also that, for a given s, we are able to find a family $(\hat{w}^\beta(s+\varepsilon))_\varepsilon$, indexed by a small parameter ε, such that, for any β,

$$\hat{w}^\beta(s+\varepsilon) \leq W(s+\varepsilon) := \sup_\beta w^\beta(s+\varepsilon) \quad \text{and} \quad \hat{w}^\beta(s) = w^\beta(s). \tag{4.87}$$

If the Lipschitz assumption of Proposition 4.6.6 is satisfied for the family $\hat{w}^\beta(s+\varepsilon)$, i.e.

$$\left|\hat{w}^{\beta}(s+\varepsilon) - \hat{w}^{\beta}(s)\right| \leq r_1(\varepsilon), \tag{4.88}$$

(say) for $s, s+\varepsilon \in (-1,1)$ and some function r_1, then

$$W(s+\varepsilon) - w^{\beta}(s) \geq -r_1(\varepsilon),$$

by the inequality in (4.87), so that

$$W(s+\varepsilon) - W(s) \geq -r_1(\varepsilon), \tag{4.89}$$

by using the equality in (4.87) and by taking the infimum with respect to β. Obviously, if the argument holds for any s in $(-1,1)$, s and $s+\varepsilon$ may be exchanged to bound the increment from above.

Similarly, if the convexity assumption of Proposition 4.6.6 is satisfied for the family $\hat{w}^{\beta}(s+\varepsilon)$, i.e.

$$\varepsilon \mapsto \hat{w}^{\beta}(s+\varepsilon) + r_2(s+\varepsilon) \tag{4.90}$$

is convex (say) for $s, s+\varepsilon, s-\varepsilon \in (-1,1)$ and some function r_2, then, for all β,

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \varepsilon^{-2}\big(&W(s+\varepsilon) + r_2(s+\varepsilon) + W(s-\varepsilon) + r_2(s-\varepsilon) \\
&- 2W(s) - 2r_2(s)\big) \\
\geq \liminf_{\varepsilon\to 0} \varepsilon^{-2}\big(&\hat{w}^{\beta}(s+\varepsilon) + r_2(s+\varepsilon) + \hat{w}^{\beta}(s-\varepsilon) + r_2(s-\varepsilon) \\
&- 2W(s) - 2r_2(s)\big).
\end{aligned}$$

Choosing β of the form β^{ε} so that

$$w^{\beta^{\varepsilon}}(s) \geq W(s) - \varepsilon^3,$$

we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \varepsilon^{-2}\big(&W(s+\varepsilon) + r_2(s+\varepsilon) + W(s-\varepsilon) + r_2(s-\varepsilon) \\
&- 2W(s) - 2r_2(s)\big) \\
\geq \liminf_{\varepsilon\to 0} \varepsilon^{-2}\big(&\hat{w}^{\beta^{\varepsilon}}(s+\varepsilon) + r_2(s+\varepsilon) + \hat{w}^{\beta^{\varepsilon}}(s-\varepsilon) + r_2(s-\varepsilon) \\
&- 2\hat{w}^{\beta^{\varepsilon}}(s) - 2r_2(s)\big).
\end{aligned} \tag{4.91}$$

Now, by convexity, the right-hand side is non-negative. (Pay attention, we say so without passing to the limit.) If such a strategy holds for all s in $(-1,1)$, we deduce that $W + r_2$ is convex.

4.8.2 Enlarging the Set of Controls

We now explain how the family $(\hat{w}^\beta)_{\beta>0}$ can be constructed in the framework of Monge–Ampère.

The starting point is the following: in the specific case of Hamilton–Jacobi–Bellman equations, the set of controls may exhibit some invariance properties; if so, it is conceivable to perturb the system along some transformation that let the system invariant. For instance, for the Monge–Ampère equation, the generic matricial control $(\sigma_t)_{t\geq 0}$ can be replaced by $(\exp(p_t)\sigma_t)_{t\geq 0}$ for some process $(p_t)_{t\geq 0}$ with values in the set of anti-Hermitian matrices: obviously, the trace of $\exp(p_t)a_t\exp(\bar{p}_t^*) = \exp(p_t)a_t\exp(-p_t)$ is still equal to 1.

The auxiliary control parameter $(p_t)_{t\geq 0}$ appears as a "ghost" parameter along which the system may be perturbed. To explain how things work, we go back to (4.84):

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^*\psi(Z_t^s)dt, \quad t \geq 0, \tag{4.92}$$

which is the generic controlled equation used to represent the Monge–Ampère equation as the value function of some optimization problem with an infinite horizon.

As said in introduction of Sect. 4.8, we may consider a curve $(\gamma(s))_{s\in[-1,1]}$ with values in $\mathcal{D}$. For a fixed value of s, we define $(\hat{Z}_t^s)_{t\geq 0}$ as above: it is the solution of (4.92) (or equivalently of (4.84)) with $\hat{Z}_0^s = \gamma(s)$ as initial solution, so that $\hat{Z}_t^s = Z_t^s$ for any $t \geq 0$. Now, for ε in the neighborhood of 0 (but different from 0), we define $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ as the solution of

$$\begin{aligned} d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\exp\bigl(P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s)\bigr)\sigma_t dB_t \\ &\quad + \exp\bigl(P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s)\bigr)a_t \exp\bigl(\bar{P}^*(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s)\bigr)D_{\bar{z}}^*\psi(\hat{Z}_t^{s+\varepsilon})dt, \end{aligned} \tag{4.93}$$

$t \geq 0$, with $\hat{Z}_0^{s+\varepsilon} = \gamma(s+\varepsilon)$ as initial condition. Here $P(z,z')$ is some function of the parameters z in $\mathcal{D}$ and z' in $\mathbb{C}^d$ with values in the set of anti-Hermitian matrices. It is assumed to be regular in z', with bounded derivatives, uniformly in z so that existence and uniqueness hold for (4.93). (See the proof of Proposition 4.6.7.) It is also assumed to satisfy $P(z,0) = 0$ so that $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ matches $(Z_t^s)_{t\geq 0}$ in (4.92) when $\varepsilon = 0$.

The typical choice we perform below for $P(z,z')$ is (at least for z close to the boundary so that $D_z\psi(z)$ is non-zero)

$$\begin{aligned} P(z,z') = \rho\bigl(&|D_z\psi(z)|^{-2}\bigl[D_{\bar{z},z}^2\psi(z)z'D_z\psi(z) + D_{\bar{z},\bar{z}}^2\psi(z)\bar{z}'D_z\psi(z) \\ &- D_{\bar{z}}^*\psi(z)(D_{z,\bar{z}}^2\psi(z)\bar{z}')^* - D_{\bar{z}}^*\psi(z)(D_{z,z}^2\psi(z)z')^*\bigr]\bigr), \end{aligned} \tag{4.94}$$

where ρ is some smooth function from $\mathbb{C}^{d\times d}$ into itself, with compact support, matching the identity on the neighborhood of 0 and preserving the anti-Hermitian structure.[8] (Have in mind that $D_z\psi(z)$ above is seen as a row vector and z' as a column vector.) We let the reader check that $P(z,z')$ is anti-Hermitian.

For ε as above, we set $p_t^{s+\varepsilon} = P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s) = P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$, so that (4.93) has the form

$$d\hat{Z}_t^{s+\varepsilon} = \psi^{1/2}(\hat{Z}_t^{s+\varepsilon}) \exp(p_t^{s+\varepsilon})\sigma_t dB_t + \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}) D_{\bar{z}}^* \psi(\hat{Z}_t^{s+\varepsilon}) dt,$$

$t \geq 0$. Now, we can follow Proposition 4.6.9 and consider

$$\begin{aligned} &\hat{V}^\sigma(s+\varepsilon) \\ &\quad = \mathbb{E}\int_0^{+\infty} \Big[\exp\Big(\int_0^t \mathrm{Trace}[\exp(p_r^{s+\varepsilon}) a_r \exp(-p_r^{s+\varepsilon}) D^2_{z,\bar{z}}\psi(\hat{Z}_r^{s+\varepsilon})] dr\Big) \\ &\qquad\qquad \times F\big(\det(a_t), \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}), \hat{Z}_t^{s+\varepsilon}\big)\Big] dt. \end{aligned} \tag{4.95}$$

(Pay attention that the determinant of a_t is the same as the determinant of the perturbed matrix $\exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon})$.) Clearly, we have $\hat{V}^\sigma(s) = V^\sigma(\gamma(s))$ (see the notation of Proposition 4.6.9). Moreover, $\hat{V}^\sigma(s+\varepsilon) \leq \sup_\sigma(V^\sigma(\gamma(s+\varepsilon)))$. (The control $(\exp(p_t^{s+\varepsilon})\sigma_t)_{t\geq 0}$ is a particular control of the same type as $(\sigma_t)_{t\geq 0}$.)

Differentiating (4.95) with respect to ε, we expect[9] a generic expression of the form

$$\begin{aligned} &\frac{d}{d\varepsilon}\big[\hat{V}^\sigma(s+\varepsilon)\big]_{|\varepsilon=0} \\ &\quad = \mathbb{E}\int_0^{+\infty} \Big[\exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)] dr\Big) \\ &\qquad \times \Big\{\Lambda_t^{1,s}\pi_t^s + \bar{\Lambda}_t^{1,s}\bar{\pi}_t^s + \Lambda_t^{2,s}\hat{\zeta}_t^s + \bar{\Lambda}_t^{2,s}\bar{\hat{\zeta}}_t^s \\ &\qquad\quad + \int_0^t \big(\Lambda_r^{3,s}\pi_r^s + \bar{\Lambda}_r^{3,s}\bar{\pi}_r^s + \Lambda_r^{4,s}\hat{\zeta}_r^s + \bar{\Lambda}_r^{4,s}\bar{\hat{\zeta}}_r^s\big) dr\Big\} dt\Big]. \end{aligned} \tag{4.96}$$

---

[8]Think of

$$\rho : (z_{i,j})_{1\leq i,j\leq d} \in \mathbb{C}^{d\times d} \mapsto \rho_1\Big(\sum_{i,j=1}^d |z_{i,j}|^2\Big)(z_{i,j})_{1\leq i,j\leq d},$$

where ρ_1 stands for a smooth function from $\mathbb{R}$ to $\mathbb{R}$ with a compact support matching 1 in the neighborhood of zero.

[9]We here say "expect" only since the differentiation argument under the integral symbol is not justified at this stage of the proof.

Here, $\Lambda_r^{i,s}$, $\bar{\Lambda}_r^{i,s}$, $i = 1, 2$, stand for the derivatives of the coefficients appearing in (4.95) and

$$\hat{\zeta}_t^s = \frac{d}{d\varepsilon}\big[\hat{Z}_t^{s+\varepsilon}\big]_{|\varepsilon=0} \quad \text{and} \quad \pi_t^s = \frac{d}{d\varepsilon}\big[p_t^{s+\varepsilon}\big]_{|\varepsilon=0}.$$

Since $p_t^{s+\varepsilon} = P(\hat{Z}_t^s, \hat{Z}_t^{s+\varepsilon} - \hat{Z}_t^s)$, the term π_t^s writes as $D_{z'}P(\hat{Z}_t^s, 0)\hat{\zeta}_t^s + D_{\bar{z}'}P(\hat{Z}_t^s, 0)\bar{\hat{\zeta}}_t^s$ so that (4.96) reduces to

$$\begin{aligned}
&\frac{d}{d\varepsilon}\big[\hat{V}^\sigma(s+\varepsilon)\big]_{|\varepsilon=0} \\
&\quad = \mathbb{E}\int_0^{+\infty}\Bigg[\exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\bigg) \\
&\qquad \times \bigg\{\hat{\Lambda}_t^{1,s}\hat{\zeta}_t^s + \bar{\hat{\Lambda}}_t^{1,s}\bar{\hat{\zeta}}_t^s + \int_0^t \big(\hat{\Lambda}_r^{2,s}\hat{\zeta}_r^s + \bar{\hat{\Lambda}}_r^{2,s}\bar{\hat{\zeta}}_r^s\big)dr\bigg\}\Bigg]dt,
\end{aligned} \tag{4.97}$$

for two new coefficients $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$.

Before we carry on the analysis, we emphasize that the rigorous proof of (4.97) is far from being easy: it relies on a differentiation argument under the integral symbol that may be very difficult to justify because of the long-run integration. To overcome this problem, a possible strategy is to multiply F by some smooth cut-off function $\phi(\cdot/S)$, S standing for a large positive real and ϕ for a function matching 1 on some $[0, 1]$ and vanishing on $[2, +\infty)$. In that case, the differentiation is expected to make sense: for example, it makes sense in the framework of Definition 4.7.3 because of the supremum over t in $[0, T]$ in the differentiability property. Obviously, the infinite horizon framework can be recovered by letting S tend to $+\infty$ at the end of the analysis, provided the bound we have for the RHS in (4.97) is uniform in the cut-off procedure.[10]

The basic argument to bound the RHS in (4.97) is the following. By the very assumption on the coefficients and for the typical choice of P we have in mind, the terms $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ are bounded in the neighborhood of the boundary only, i.e. for $\hat{Z}_t^s = Z_t^s$ close to $\partial\mathcal{D}$. (Indeed, have in mind that $D_z\psi$ is non-zero in the neighborhood of $\partial\mathcal{D}$.) Just for the moment, assume that they are bounded on the whole time interval $[0, +\infty)$. Then, to bound the right-hand side above, it is sufficient to prove an equivalent of (4.74), i.e.

$$\mathbb{E}\bigg[\exp\bigg(-\int_0^t c_r dr\bigg)|\hat{\zeta}_t^s|^2\bigg] \le \exp(-\delta t)|\hat{\zeta}_0^s|^2 = \exp(-\delta t)|\gamma'(s)|^2 \tag{4.98}$$

for all $t \ge 0$, with $-c_r = \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]$.

[10] We will detail this argument in Sect. 4.9 rigorously.

In some sense, we are reduced to the original problem of long-run estimate for the derivative of the diffusion process, but for a new derivative $\hat{\zeta}^s$, namely for the solution of the SDE

$$
\begin{aligned}
d\hat{\zeta}^s_t &= \big[D_z[\psi^{1/2}](Z^s_t) + \psi^{1/2}(Z^s_t) D_{z'} P(Z^s_t, 0)\big]\hat{\zeta}^s_t \sigma_t dB_t \\
&\quad + \big[D_{\bar{z}}[\psi^{1/2}](Z^s_t) + \psi^{1/2}(Z^s_t) D_{\bar{z}'} P(Z^s_t, 0)\big]\bar{\hat{\zeta}}^s_t \sigma_t dB_t \\
&\quad + \big\{\big(D_{z'} P(Z^s_t, 0)\hat{\zeta}^s_t + D_{\bar{z}'} P(Z^s_t, 0)\bar{\hat{\zeta}}^s_t\big) a_t \\
&\qquad - a_t \big(D_{z'} P(Z^s_t, 0)\hat{\zeta}^s_t + D_{\bar{z}'} P(Z^s_t, 0)\bar{\hat{\zeta}}^s_t\big)\big\} D^*_{\bar{z}}\psi(Z^s_t) dt \\
&\quad + a_t \big[\big(D^2_{\bar{z},z}\psi(Z^s_t)\big)^* \hat{\zeta}^s_t + \big(D^2_{\bar{z},\bar{z}}\psi(Z^s_t)\big)^* \bar{\hat{\zeta}}^s_t\big] dt, \quad t \geq 0, \qquad (4.99)
\end{aligned}
$$

with the initial condition $\hat{\zeta}^s_0 = \gamma'(s)$. The whole point is then to check that the typical choice (4.94) for $P(z, z')$ permits to derive the long-run estimate (4.98). Unfortunately, we will see below that it permits to obtain (4.98) for Z^s_t close to $\partial\mathcal{D}$ only. (Actually, this is well-guessed: remember that, for the typical choice we have in mind for $P(z, z')$, we cannot bound $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ away from the boundary. Indeed, $P(z, z')$ may explode for z away from the boundary.)

The strategy we follow below consists in localizing the perturbation argument. If the starting point $\gamma(s)$ of Z^s is close enough to the boundary, the perturbation argument applies up to the stopping time $\mathfrak{t} := \inf\{t \geq 0 : \psi(Z^s_t) \geq \epsilon\}$, ϵ standing for some small positive parameter[11]; if the starting point $\gamma(s)$ of Z^s is far away from the boundary, we can apply the perturbation argument when $(\psi(Z^s_t))_{t\geq 0}$ becomes small enough, i.e. when $(Z^s_t)_{t\geq 0}$ enters into the neighborhood of $\partial\mathcal{D}$. Specifically, if $\mathfrak{s}$ is some (finite) stopping time at which $\psi(Z^s_{\mathfrak{s}}) < \epsilon$, we can apply the perturbation argument up to the stopping time $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \geq \epsilon\}$:

Proposition 4.8.2 *Let $S > 0$ be a positive real, ϕ be a smooth function from $\mathbb{R}_+$ to $[0, 1]$ matching 1 on $[0, 1]$ and 0 outside $[0, 2]$, $\epsilon > 0$ be a small enough real such that $|D_z\psi(z)| > 0$ for $\psi(z) \leq \epsilon$ and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z^s_{\mathfrak{s}}) < \epsilon$. For $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \geq \epsilon\}$, consider some process $(\hat{Z}^{s+\varepsilon}_t)_{0\leq t\leq \mathfrak{t}}$ for which $([d/d\varepsilon](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ and $([d^2/d\varepsilon^2](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ exist and for which the perturbed SDE (4.93) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and define*

$$
\begin{aligned}
&\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon) \\
&\quad = \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \bigg[\exp\bigg(\int_0^t \mathrm{Trace}[\exp(p^{s+\varepsilon}_r) a_r \exp(-p^{s+\varepsilon}_r) D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)] dr\bigg) \\
&\qquad\qquad \times F\big(\det(a_t), \exp(p^{s+\varepsilon}_t) a_t \exp(-p^{s+\varepsilon}_t), \hat{Z}^{s+\varepsilon}_t\big)\phi\bigg(\frac{t}{S}\bigg)\bigg] dt, \quad (4.100)
\end{aligned}
$$

[11] Pay attention that ϵ and ε stand for two different parameters.

as the cut-off localized version of (4.95)*, with* $p_t^{s+\varepsilon} = P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$*,* $\mathfrak{s} \leq t \leq \mathfrak{t}$*,* P *being given by* (4.94)*.*

If the differentiation operator w.r.t. ε *and the expectation and integral symbols in the RHS of* (4.100) *can be exchanged, then there exists a constant* $C > 0$*, depending on Assumption* (**A**) *and on* ϵ *only (in particular, it is independent of* S *and* $(\sigma_t)_{t\geq 0}$*), such that*

$$\left| \frac{d}{d\varepsilon} \big[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \big]_{|\varepsilon=0} \right| \\ \leq C\mathbb{E}\bigg[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr \bigg) \bigg[|\hat{\zeta}_t^s| + \int_0^t |\hat{\zeta}_r^s| dr \bigg] dt \bigg],$$

where $\hat{\zeta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

Say a word about the concrete meaning of Proposition 4.8.2: from time 0 to time $\mathfrak{s}$, the process $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{s}}$ is chosen arbitrarily provided it be twice differentiable (in the mean) w.r.t. ε. Below, we explicitly say how it is chosen: roughly speaking, it is built from another (local) perturbation argument. We also emphasize, that the value function $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ has no straightforward connection with the original V: again, we will see below how to gather all the local value functions into a single one, directly connected to Monge–Ampère.

Obviously, we can iterate the argument to bound the second-order derivatives:

Proposition 4.8.3 *Keep the assumption and notation of Proposition 4.8.2 and assume that the second-order differentiation operator w.r.t.* ε *and the expectation and integral symbols in the RHS of* (4.100) *can be exchanged, then there exists a constant* $C > 0$*, depending on Assumption* (**A**) *and on* ϵ *only, such that*

$$\left| \frac{d^2}{d\varepsilon^2} \big[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \big] \right| \\ \leq C\mathbb{E}\bigg[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr \bigg) \\ \times \bigg[|\hat{\eta}_t^s| + |\hat{\zeta}_t^s|^2 + \int_0^t |\hat{\eta}_r^s| dr + \int_0^t |\hat{\zeta}_r^s|^2 dr + \bigg(\int_0^t |\hat{\zeta}_r^s| dr \bigg)^2 \bigg] dt \bigg],$$

where $\hat{\eta}_t^s = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

4.8.3 Time Change

Here is another example of perturbation. The starting point is the following. In the Hamilton–Jacobi–Bellman formulation (4.34) of Monge–Ampère, the normalizing condition for the trace of the matrix a is purely arbitrary. Indeed, the equation remains unchanged when multiplied by any positive constant, so that the trace may be asked to match any other positive real value.

Intuitively, this means that, in (4.85), the normalizing condition for the trace of $(a_t)_{t\geq 0}$ might be useless, or said differently, that we might consider a rescaled version of $(a_t)_{t\geq 0}$ instead of $(a_t)_{t\geq 0}$ itself.

Now, have in mind that we are here seeking for a perturbed writing of (4.85) when initialized at $\gamma(s+\varepsilon)$ for ε in the neighborhood of zero. We are thus thinking of rescaling $(a_t)_{t\geq 0}$ by some positive scale function $(|\tau_t^\varepsilon|^2)_{t\geq 0}$ depending on the perturbation variable ε. Here, $(\tau_t^\varepsilon)_{t\geq 0}$ stands for an arbitrary progressively-measurable real-valued process that is differentiable with respect to the parameter ε. Specifically, we consider the perturbed SDE

$$d\hat{Z}_t^{s+\varepsilon} = \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\tau_t^\varepsilon \sigma_t dB_t + |\tau_t^\varepsilon|^2 a_t D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon})dt, \quad t \geq 0. \tag{4.101}$$

with $\hat{Z}_0^{s+\varepsilon}$ as initial condition. (Solvability is proven as in Proposition 4.6.7.)

Exactly as in the previous subsection, the perturbation we here choose vanishes at $\varepsilon = 0$, i.e. τ_t^ε is chosen as $T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$ for a smooth function $T : (z, z') \in \mathcal{D} \times \mathbb{C}^d \to \mathbb{R}$ such that $T(z, 0) = 1$. In other words, $\hat{Z}^s$ and Z^s stand for the same process. In particular, when differentiating $T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)$ with respect to 0, we obtain $2\mathrm{Re}[D_{z'}T(Z_t^s, 0)\hat{\zeta}_t^s]$ where $\hat{\zeta}_t^s$ stands for the derivative of $Z_t^{s+\varepsilon}$ with respect to ε at $\varepsilon = 0$, i.e.

$$\hat{\zeta}_t^s := \frac{d}{d\varepsilon}\big[\hat{Z}_t^{s+\varepsilon}\big]_{|\varepsilon=0}.$$

The typical choice we have in mind for $T(z, z')$ is

$$T(z, z') = 1 + \rho\big(\psi^{-1}(z)\mathrm{Re}\big[D_z\psi(z)z'\big]\big), \tag{4.102}$$

where ρ is some smooth function with values in $[-1/2, 1/2]$, such that $\rho(0) = 0$ and $\rho'(0) = 1$, so that

$$\mathrm{Re}\big[D_{z'}T(z, 0)\zeta\big] = \psi^{-1}(z)\mathrm{Re}\big[D_z\psi(z)\zeta\big], \quad \zeta \in \mathbb{C}^n,$$

and

$$\frac{d}{d\varepsilon}\big[T^2\big(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s\big)\big]_{|\varepsilon=0} = 2\psi^{-1}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\hat{\zeta}_t^s\big]. \tag{4.103}$$

The resulting dynamics for $(\hat{\zeta}_t^s)_{t\geq 0}$ is computed below.

The problem is to understand first how this perturbed process is connected with the representation of the solution of Monge–Ampère. Here is the whole point: the process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ appears as a time-change solution of a SDE of the same type as (4.85). Said in a non-rigorous way, we may think of $(\hat{Z}_t^{s+\varepsilon})$ as $(Z^{s+\varepsilon}_{\mathfrak{T}_t^\varepsilon})_{t\geq 0}$ where $\dot{\mathfrak{T}}_t^\varepsilon = |\tau_t^\varepsilon|^2$, $t \geq 0$, and

$$dZ_t^{s+\varepsilon} = \psi^{1/2}(Z_t^{s+\varepsilon}) \frac{\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}}{|\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}|} \sigma_{(\mathfrak{T}^\varepsilon)_t^{-1}} dB_t + a_{(\mathfrak{T}^\varepsilon)_t^{-1}} D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon}) dt, \quad t \geq 0. \tag{4.104}$$

(Here, $(\mathfrak{T}^\varepsilon)^{-1}$ stands for the converse of $\mathfrak{T}^\varepsilon$. We will explain right below why we keep the same notation for this $Z^{s+\varepsilon}$ as in the original (4.84).) We won't provide a rigorous proof for this time-change formula,[12] but the idea is very intuitive: roughly speaking, the action of the time-change on the dB_t term must be understood as a multiplication by $[\dot{\mathfrak{T}}_t^{s+\varepsilon}]^{1/2}$ since dB_t is understood itself as $[dt]^{1/2}$; obviously, the action of the time-change on the dt terms is the same as in an ODE.

Actually, (4.104) is false. The reader might guess that, one way or another, the time-change affects the dynamics of the Brownian motion $(B_t)_{t\geq 0}$. The right version is

$$dZ_t^{s+\varepsilon} = \psi^{1/2}(Z_t^{s+\varepsilon}) \frac{\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}}{|\tau^\varepsilon_{(\mathfrak{T}^\varepsilon)_t^{-1}}|} \sigma_{(\mathfrak{T}^\varepsilon)_t^{-1}} d\hat{B}_t^\varepsilon + a_{(\mathfrak{T}^\varepsilon)_t^{-1}} D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon}) dt, \quad t \geq 0, \tag{4.105}$$

where

$$\hat{B}_t^\varepsilon = \int_0^{(\mathfrak{T}^\varepsilon)_t^{-1}} |\tau_r^\varepsilon| dB_r, \quad t \geq 0.$$

Here, $(\hat{B}_t^\varepsilon)_{t\geq 0}$ is a Brownian motion again[13] w.r.t. to the time-rescaled filtration $(\mathcal{F}_{(\mathfrak{T}_t^\varepsilon)^{-1}})_{t\geq 0}$.

---

[12] We refer the reader to the original paper by Krylov [Kry90] for the complete argument.

[13] Clearly, $(\hat{B}_t^\varepsilon)_{t\geq 0}$ is a martingale with values in $\mathbb{C}^d$. Actually, for any coordinates $1 \leq j, k \leq d$,

$$d[(\hat{B}_t^\varepsilon)^j (\hat{B}_t^\varepsilon)^k] = 0, \quad d[(\hat{B}_t^\varepsilon)^j \overline{(\hat{B}_t^\varepsilon)^k}] = \delta_{j,k} dt, \tag{4.106}$$

where $\delta_{j,k}$ stands for the Kronecker symbol. There is a famous theorem in stochastic calculus, due to Paul Lévy, that says that any continuous martingale starting from 0 and satisfying (4.106) is a complex Brownian motion of dimension d. Actually, this may be explained as follows: (4.106), together with the martingale property, provide the local infinitesimal dynamics of $\hat{B}^\varepsilon$; this makes the connection between W and the Laplace operator in $\mathbb{R}^{2d}$ through Itô's formula. In some sense, there is one and only one stochastic process associated with the Laplace operator in $\mathbb{R}^{2d}$: the $2d$-dimensional real Brownian

Now, the time-rescaled term $((\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})^{-1}_t}/|\tau^{\varepsilon}_{(\mathfrak{T}^{\varepsilon})^{-1}_t}|)\sigma_{(\mathfrak{T}^{\varepsilon})^{-1}_t})_{t\geq 0}$ may be seen as a new control process with $(a_{(\mathfrak{T}^{\varepsilon})^{-1}_t})_{t\geq 0}$ as Hermitian square, so that we are reduced to the original formulation of Monge–Ampère, but w.r.t. to a different Brownian set-up (the set-up is the pair given by the Brownian motion and the underlying filtration). It may be well-understood that the representation of the Monge–Ampère equation is kept preserved by modification of the underlying Brownian set-up,[14] so that

$$V\big(\gamma(s+\varepsilon)\big) \geq \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}[a_{(\mathfrak{T}^{\varepsilon})^{-1}_r} D^2_{z,\bar{z}}\psi(Z^{s+\varepsilon}_r)]dr\bigg) \times F(\det(a_{(\mathfrak{T}^{\varepsilon})^{-1}_t}), a_{(\mathfrak{T}^{\varepsilon})^{-1}_t}, Z^{s+\varepsilon}_t)dt\bigg].$$

(Use Proposition 4.6.9.) Changing time-speed in the integrals above, we deduce that $V(\gamma(s+\varepsilon)) \geq \hat{V}^{\sigma}(s+\varepsilon)$ where

$$\begin{aligned}\hat{V}^{\sigma}(s+\varepsilon) &:= \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \dot{\mathfrak{T}}^{\varepsilon}_r \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\bigg) \\ &\qquad\qquad \times F(\det(a_t), a_t, \hat{Z}^{s+\varepsilon}_t)\dot{\mathfrak{T}}^{\varepsilon}_t dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t |\tau^{\varepsilon}_r|^2 \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\bigg) \\ &\qquad\qquad \times F(\det(a_t), a_t, \hat{Z}^{s+\varepsilon}_t)|\tau^{\varepsilon}_t|^2 dt\bigg]. \end{aligned} \tag{4.107}$$

Of course, when $\varepsilon = 0$, $\hat{V}^{\sigma}(s) = V^{\sigma}(s)$ so that $\sup_{\sigma}[\hat{V}^{\sigma}(s)] = V(\gamma(s))$.

The reader may notice that everything works as if $(a_t)_{t\geq 0}$ had been multiplied by the scaling factor $(|\tau^{\varepsilon}_t|^2)_{t\geq 0}$ as discussed at the very beginning of the paragraph: remember indeed that F is homogeneous with respect to a.

It now remains to understand what happens when differentiating (4.107) w.r.t. ε. We let the reader check that the resulting formula for $[d/d\varepsilon](\hat{v}^{\sigma}(s+\varepsilon))$ is similar to (4.97). Specifically, the terms $\hat{\Lambda}^{1,s}$ and $\hat{\Lambda}^{2,s}$ therein are bounded in the current framework if $D_{z'}T(z,0)$ is bounded. With the typical choice (4.102) we have in mind, it is bounded away from the boundary, i.e. for $\psi(z)$ away from 0. Actually, the main technical problem is the same as in (4.96): the

motion or, equivalently, the d-dimensional complex Brownian motion. (For further details, we refer the reader to [Pro90, Thm II. 40].)

[14] Actually, the proof is not so easy: the problem is to understand how the modification of the Brownian paths and of the underlying filtration affects the representation. We refer the reader to the monograph by Krylov [Kry80], Remark III.3.10 for a complete discussion.

point is to justify the differentiation. To do, we use the same trick as in the previous subsection by considering some cut-off version of F. We thus deduce the analogs of Propositions 4.8.2 and 4.8.3:

Proposition 4.8.4 *Let S be a positive real, ϕ be a smooth function matching one on $[0,1]$ and vanishing outside $[0,2]$, ϵ be a positive real and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z^s_{\mathfrak{s}}) > \epsilon$. For $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \leq \epsilon\}$, consider some process $(\hat{Z}^{s+\varepsilon}_t)_{0\leq t\leq \mathfrak{t}}$ for which $([d/d\varepsilon](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ and $([d^2/d\varepsilon^2](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ exist and for which the perturbed SDE (4.101) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and define*

$$\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon) = \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \biggl[\exp\biggl(\int_0^t |\tau^\varepsilon_r|^2 \text{Trace}[a_r D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\biggr) \times F\bigl(\det(a_t), a_t, \hat{Z}^{s+\varepsilon}_t\bigr)\phi\biggl(\frac{\mathfrak{T}^\varepsilon_t}{S}\biggr)|\tau^\varepsilon_t|^2\biggr]dt,$$

as the localized version of (4.107), with $\tau^\varepsilon_t = T(Z^s_t, \hat{Z}^{s+\varepsilon}_t - Z^s_t)$, $\mathfrak{s} \leq t \leq \mathfrak{t}$, T being given by (4.102), and $\dot{\mathfrak{T}}^\varepsilon_t = |\tau^\varepsilon_t|^2$ (with $\mathfrak{T}^\varepsilon_0 = 1$).

If the differentiation operators of order 1 and 2 w.r.t. ε and the expectation and integral symbols in the definition of $\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S$ can be exchanged, there exists a constant $C > 0$, depending on Assumption ($\mathbf{A}$) and on ϵ only, such that

$$\biggl|\frac{d}{d\varepsilon}\bigl[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon)\bigr]\biggr| \leq C\mathbb{E}\biggl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\biggl(\int_0^t \text{Trace}[a_r D^2_{z,\bar{z}}(Z^s_t)]dr\biggr) \times \biggl[|\hat{\zeta}^s_t| + \int_0^t |\hat{\zeta}^s_r|dr\biggr]dt\biggr]$$

$$\biggl|\frac{d^2}{d\varepsilon^2}\bigl[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S(s+\varepsilon)\bigr]\biggr| \leq C\mathbb{E}\biggl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\biggl(\int_0^t \text{Trace}[a_r D^2_{z,\bar{z}}(Z^s_t)]dr\biggr) \times \biggl[|\hat{\eta}^s_t| + |\hat{\zeta}^s_t|^2 + \int_0^t |\hat{\eta}^s_r|dr + \int_0^t |\hat{\zeta}^s_r|^2 dr + \biggl(\int_0^t |\hat{\zeta}^s_r|dr\biggr)^2\biggr]dt\biggr],$$

where $\hat{\zeta}^s_t = [d/d\varepsilon](Z^{s+\varepsilon}_t)_{|\varepsilon=0}$ and $\hat{\eta}^s_t = [d^2/d\varepsilon^2](Z^{s+\varepsilon}_t)_{|\varepsilon=0}$.

The reader may wonder about the specific choice for the cut-off. First, the time-change is plugged as an argument of the cut-off function: when performing the change of variable, we recover $(\phi(t/S))_{t\geq 0}$ as cut-off. Second, we emphasize that the cut-off permits to get rid of times t at which $\mathfrak{T}^\varepsilon_t \geq 2S$. By assumption, we know that $|\tau^\varepsilon|^2$ is always greater than $1/4$ so that $\mathfrak{T}^\varepsilon_t$ is always greater than $t/4$, $t \geq 0$. In particular, the cut-off vanishes at times t at which $t/4 \geq 2S$. In other words, the definition of $\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}_S$

is understood as a finite horizon value function: this permits to justify the differentiation argument w.r.t. ε provided $(\hat{Z}_t^{s+\varepsilon})_{0\le t\le \mathfrak{t}}$ satisfies the assumption of Corollary 4.7.5. (Have in mind that Corollary 4.7.5 holds in finite horizon.) Unfortunately, because of the singularity of the coefficient $\psi^{1/2}$ in (4.84) in the neighborhood of $\partial\mathcal{D}$, it is not so easy to prove that $(\hat{Z}_t^{s+\varepsilon})_{0\le t\le \mathfrak{t}}$ satisfies the assumption of Corollary 4.7.5. At this stage of the proof, this point is left open: this is the "meta"-part of Meta-Theorem 4.8.1.

4.8.4 Perturbation of the Measure: Girsanov Theorem

The last perturbation method we here discuss consists in modifying the measure of the underlying probability space. This a typical probabilistic way to estimate the solution of a partial differential equation of second-order: we may refer the reader to the lectures by Krylov in Pisa [Kry04] for a detailed overview; we also mention the personal work [Del03] and the references therein.

We here explain first how the probability measure may be changed to establish some smoothness property for the solution of a second-order partial differential equation. Generally speaking, the modification of the reference measure is a common argument in stochastic analysis, which turns out to be really efficient to quantify the sensitivity of a system with respect to the input noise. More or less, this is the starting point of the Malliavin Calculus, used to prove by probabilistic tools the so-called "Sum of squares" Theorem due to Hörmander. (See the monograph [Mal97].)

In the specific case of heat equation, the problem may be understood as follows. Indeed, as already explained in (4.9) and (4.10), the solution of the one-dimensional heat equation

$$D_t u(t,x) - \frac{1}{2} D_{x,x}^2 u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R},$$

with an initial condition of the form $u(0,\cdot) = u_0(\cdot)$ (say, with u_0 continuous and bounded) is given by

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \exp\left(- \frac{|x-y|^2}{2t} \right) dy, \quad x \in \mathbb{R},$$

Clearly, at fixed $t > 0$, and for any $\varepsilon \in \mathbb{R}$, the Gaussian measures

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t} \right) dy \quad \text{and} \quad \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x+\varepsilon-y|^2}{2t} \right) dy$$

are equivalent, so that $u(t,x+\varepsilon)$ can be written as

$$\begin{aligned} &u(t,x+\varepsilon) \\ &\quad = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \exp\left(-\frac{|x+\varepsilon-y|^2-|x-y|^2}{2t}\right) \exp\left(-\frac{|x-y|^2}{2t}\right) dy \\ &\quad = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(x+y) \exp\left(-\frac{|\varepsilon-y|^2-|y|^2}{2t}\right) \exp\left(-\frac{|y|^2}{2t}\right) dy. \end{aligned}$$

Thinking of the Gaussian density as the density of the (marginal) law of the position of some Brownian B at time t, we may write as well:

$$\begin{aligned} u(t,x+\varepsilon) &= \mathbb{E}\left[u_0(x+B_t) \exp\left(-\frac{|\varepsilon-B_t|^2-|B_t|^2}{2t}\right)\right] \\ &= \mathbb{E}\left[u_0(x+B_t) \exp\left(\varepsilon\frac{B_t}{t}-\frac{\varepsilon^2}{2t}\right)\right]. \end{aligned}$$

Now, the term $M^\varepsilon = \exp(\varepsilon B_t/t - \varepsilon^2/(2t))$ appears as a density on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the Brownian motion is defined. Said differently, the representation of $u(t,x+\varepsilon)$ consists in integrating $u_0(x+B_t)$, as for $u(t,x)$, but under the measure $M^\varepsilon \cdot \mathbb{P}$. In particular, the smoothness of $u(t,\cdot)$ with respect to the spatial parameter is directly given by the smoothness of the density M^ε with respect to the parameter ε.

This example is very simple because the change of measure is of finite dimension. Nevertheless, there exists an infinite dimensional counterpart, known as Girsanov Theorem.[15]

To understand how things work, go back to the statement of Theorem 4.7.2 and consider a curve γ of the form $\gamma(s) = x_0 + (T-s)\nu$, where T is some positive real, and x_0 and ν some vectors in $\mathbb{R}^d$. (Recall that, for more simplicity, the framework of Theorem 4.7.2 is real and not complex.) The whole idea now consists in considering $(X_t^{\gamma(t)})_{0 \le t \le T}$: it both depends on time t through the time index of X and through the initial condition $\gamma(t)$. (Keep in mind that $X_0^{\gamma(t)} = \gamma(t)$.) It can be proven (see e.g. the monograph by Kunita [Kun90]) that

$$dX_t^{\gamma(t)} = b(t, X_t^{\gamma(t)})dt + \sigma(t, X_t^{\gamma(t)})dB_t + \xi_t^{\gamma(t)}dt,$$

where $\xi_t^{\gamma(t)}$ is the value of $\xi_t^s = D_s[X_t^{\gamma(s)}]$ at $s=t$. (That is, $\xi_t^{\gamma(t)} = D_x X_t^{\gamma(t)} \gamma'(t)$. See the statement of Theorem 4.7.2.)

[15] We won't give the explicit form of Girsanov Theorem here. It would require an additional effort which seems useless. We refer to the monograph by Protter [Pro90].

The big deal is the following. If σ is invertible and σ^{-1} is bounded, uniformly in time and space, we write

$$dX_t^{\gamma(t)} = b(t, X_t^{\gamma(t)})dt + \sigma(t, X_t^{\gamma(t)})\big(dB_t + \sigma^{-1}(t, X_t^{\gamma(t)})\xi_t^{\gamma(t)}dt\big).$$

What Girsanov Theorem says is: we can find a new measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on the σ-algebra generated by $(B_t)_{0 \le t \le T}$, such that the process in parentheses be a Brownian motion, i.e.

$$\left(B_t + \int_0^t \sigma^{-1}(r, X_r^{\gamma(r)})\xi_r^{\gamma(r)}dr\right)_{0 \le t \le T},$$

is a Brownian motion under $\mathbb{Q}$.[16] As a consequence, under the new probability measure $\mathbb{Q}$, the process $(X_t^{\gamma(t)})_{0 \le t \le T}$ behaves as the initial process $(X_t^{\gamma(0)})_{0 \le t \le T}$ under $\mathbb{P}$. In particular, if u stands for the solution of the Cauchy problem

$$D_t u(t,x) + \langle b(t,x), D_x u(t,x)\rangle + \frac{1}{2}\mathrm{Trace}\big[a(t,x)D_{x,x}^2 u(t,x)\big] = 0,$$

with the boundary condition $u(T,x) = u_T(x)$. (Note that the problem is set in a backward way for notational simplicity only), the initial condition $u(0, \gamma(0))$ can be written on the same model as (4.24) as $\mathbb{E}_{\mathbb{P}}[u_T(X_T^{\gamma(0)})]$ and therefore as $\mathbb{E}_{\mathbb{Q}}[u_T(X_T^{\gamma(T)})]$. (Here, the indices $\mathbb{P}$ and $\mathbb{Q}$ denote the probability used to perform the integration.) In particular,

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{Q}}\big[u(T, X_T^{\gamma(T)})\big].$$

Now, the trick is: $\gamma(T) = x_0$ so that

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{Q}}\big[u(T, X_T^{x_0})\big].$$

Finally, it remains to give the form of $\mathbb{Q}$. It is given by Girsanov Theorem as

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \rho_T^{\nu}\\ &:= \exp\left(-\int_0^T \langle \sigma^{-1}(r, X_r^{\gamma(r)})\xi_t^{\gamma(t)}, dB_t\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}(t, X_t^{\gamma(t)})\xi_t^{\gamma(t)}|^2 dt\right).\end{aligned}$$

[16] The reader who knows Girsanov Theorem already may notice that the exponential martingale property should be checked to apply the theorem. Obviously, it should be: actually, the whole argument relies on a localization procedure that is a little bit involved. For simplicity, we do not discuss it here.

Finally,

$$u(0, x_0 + T\nu) = \mathbb{E}_{\mathbb{P}}\big[u(T, X_T^{x_0})\rho_T^{\nu}\big].$$

In other words, the regularity of u with respect to the spatial parameter follows from the regularity of ρ_T^{ν}, independently of the regularity of the boundary condition: this is the typical probabilistic argument to understand the regularizing effect of non-degenerate diffusion operators. Of course, the price to pay is the same as in analysis: the underlying diffusion matrix has to be non-degenerate.

Obviously, this is not the case in the Monge–Ampère problem. However, we will use Girsanov Theorem as a perturbation tool.

The idea is the following: go back to (4.84) and consider at $s + \varepsilon$ the perturbed dynamics

$$\begin{aligned} d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})\sigma_t\big[dB_t + G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)dt\big] \\ &\quad + a_t D_{\bar{z}}^* \psi(\hat{Z}_t^{s+\varepsilon})dt, \quad t \geq 0. \end{aligned} \tag{4.108}$$

Here, the function G satisfies $G(z, 0) = 0$ so that $(\hat{Z}_t^s)_{t\geq 0}$ and $(Z_t^s)_{t\geq 0}$ are equal as required in the perturbation method. When G (seen as a function of two arguments) is a smooth function with a compact support, the unique solvability of (4.108) may be proven as in Proposition 4.6.7: the sketch is given in footnote below.[17] (The reader can skip it.) To make the connection with the original dynamics, we are then seeking for a new measure $\mathbb{P}^{\varepsilon}$ under which the process

[17] The argument is almost the same as in Proposition 4.6.7 but the right martingale to consider in (4.51) is

$$\begin{aligned} m_t = \psi^{-1}(\hat{Z}_t^{s+\varepsilon}) \times \exp\bigg(&\int_0^t \mathrm{Trace}[a_r D_{z,\bar{z}}^2 \psi(Z_r^{s+\varepsilon})]dr \\ &- \int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] - \int_0^t |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\bigg), \end{aligned}$$

$t \geq 0$. Indeed, by Itô's formula, we can prove that it is a local martingale.

Then, denoting by $\tau_n = \inf\{t \geq 0 : \psi^{-1}(Z_t^{s+\varepsilon}) \leq 1/n\}$,

$$\begin{aligned} n^{1/2}\mathbb{P}\{\tau_n \leq t\} &\leq \mathbb{E}\big[\psi^{-1/2}(\hat{Z}_t^{s+\varepsilon})\big] \\ &\leq \mathbb{E}\bigg[\psi^{-1}(\hat{Z}_t^{s+\varepsilon}) \exp\bigg(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\bigg)\bigg]^{1/2} \\ &\quad \times \mathbb{E}\bigg[\exp\bigg(\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\bigg)\bigg]^{1/2} \\ &\leq C\exp(Ct)\mathbb{E}[m_t] = C\exp(Ct)\psi^{-1}(z). \end{aligned}$$

$$\left(\hat{B}_t^\varepsilon := B_t + \int_0^t G(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s) dr \right)_{t \geq 0}$$

is a Brownian motion. (So that, under $\mathbb{P}^\varepsilon$, the process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ has the right dynamics.)

What Girsanov Theorem[18] says is the following: if G is bounded, there exists a measure $\mathbb{P}^\varepsilon$ given by

$$\begin{aligned} \mathbb{P}^\varepsilon(A) = \mathbb{E}\bigg[\exp\bigg(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] \\ - \int_0^t |G|^2 (Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s) dr\bigg) \mathbf{1}_A\bigg], \quad A \in \mathcal{F}_t, \quad t \geq 0, \end{aligned} \tag{4.109}$$

under which $(\hat{B}_t^\varepsilon)_{t\geq 0}$ is a complex Brownian motion of dimension d. (In particular, $\mathbb{P}^\varepsilon$ admits a density with respect to $\mathbb{P}$ (and is even equivalent to $\mathbb{P}$) when restricted to the σ-subalgebra $\mathcal{F}_t$, $t \geq 0$.)

We now go back to (4.108): we understand that $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ has the same dynamics as $(Z_t^{s+\varepsilon})_{t\geq 0}$ in (4.84) but with $(B_t)_{t\geq 0}$ replaced by $(\hat{B}_t^\varepsilon)_{t\geq 0}$. Since $(\hat{B}_t^\varepsilon)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}^\varepsilon$, we expect $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ to have the same dynamics (i.e. the same distribution) under $\mathbb{P}^\varepsilon$ as $(Z_t^{s+\varepsilon})_{t\geq 0}$ under $\mathbb{P}$. Under local Cauchy-Lipschitz like type assumption on the coefficients of (4.84), this is true: this is the so-called Yamada and Watanabe Theorem, see e.g. Stroock and Varadhan [StroV79].

Consider now the perturbed value function

$$\begin{aligned} &\hat{V}^\sigma(s+\varepsilon) \\ &= \int_0^{+\infty} \mathbb{E}\bigg[\exp\bigg(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big] \\ &\qquad - \int_0^t |G|^2 (Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s) dr\bigg) \\ &\qquad \times \exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}} \psi(\hat{Z}_r^{s+\varepsilon})] dr\bigg) F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big)\bigg] dt. \end{aligned} \tag{4.110}$$

The last line follows from the bound

$$\mathbb{E}\bigg[\exp\bigg(\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\big]\bigg)\bigg]^{1/2} \leq \exp(C\|G\|_\infty t).$$

See Rogers and Williams [RW87].

[18] Pay attention that Girsanov Theorem is here given for the complex Brownian motion.

(Note that the integral and the expectation have been exchanged in comparison with the original formulation in Proposition 4.6.9. This new writing permits to apply Girsanov Theorem easily. Nevertheless, by boundedness of F and superharmonicity of ψ, Fubini's Theorem applies and the integrals may be exchanged.) We may write it as

$$\hat{V}^{\sigma}(s+\varepsilon) = \int_0^{+\infty} \mathbb{E}_{\mathbb{P}^{\varepsilon}} \Big[\exp\Big(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}} \psi(\hat{Z}_r^{s+\varepsilon})\big] dr \Big) \times F\big(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\big) \Big] dt,$$

where $\mathbb{E}_{\mathbb{P}^{\varepsilon}}$ denotes the expectation under $\mathbb{P}^{\varepsilon}$. We then replace $\hat{Z}^{s+\varepsilon}$ by $Z^{s+\varepsilon}$ by saying that the dynamics of the first one under $\mathbb{P}^{\varepsilon}$ are the same as the dynamics of the second one under $\mathbb{P}$. We deduce that the supremum $\sup_{\sigma} \hat{v}^{\sigma}(s+\varepsilon)$ is equal to $V(\gamma(s+\varepsilon))$.[19]

It now remains to specify the choice for G. Actually, we can choose it such that

$$\frac{d}{d\varepsilon}\big[\bar{G}(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0} = \Xi(Z_t^s)\hat{\zeta}_t^s, \tag{4.111}$$

where $\Xi(z)$ is a complex matrix of size $d \times d$ and $\hat{\zeta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$. (Choose for example $G(z,z') = \Xi(z)\rho(z')$, the function ρ being bounded and satisfying $\rho(0)=0$, $D_{z'}\rho(0) = I_d$ and $D_{\bar{z}'}\rho(0) = 0$.[20]) Below, the matrix $\Xi(z)$ we use is bounded in z on every compact subset of $\mathcal{D}$ only. (In particular, $\Xi(z)$ may explode as z tends to $\partial\mathcal{D}$.)

To complete the argument, it remains to explain what happens when differentiating (4.110) w.r.t. ε. (Again, we assume that we can do so: this is a part of the "meta" in Meta-Theorem 4.8.1.) The story is a bit different from what we explained above for the two other perturbations. Indeed, when differentiating (4.110), we obtain a new term to bound which is

$$\mathbb{E}\int_0^{+\infty} \Big| \int_0^t \langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle \exp\Big(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}\psi(Z_r^s)\big] dr\Big)\Big| dt.$$

Here is what we can say:

Lemma 4.8.5 *Consider a process* $(\varsigma_t)_{t\geq 0}$ *with values in* $\mathbb{C}^d$, *solution to a SDE of the form*

$$d\varsigma_t = \big(\beta_t \varsigma_t + \beta'_t \bar{\varsigma}_t\big)dt + (\alpha_t \varsigma_t + \alpha'_t \bar{\varsigma}_t) dB_t,$$

[19] Here, the story is the same as for time-change. To have a completely rigorous argument, we should check first that the representation of Monge–Ampère remains the same when the underlying Brownian motion is modified. Again, we refer to Remark III.3.10 in the monograph [Kry80] for a complete discussion.

[20] A typical example is $\rho(z') = (\rho_0(z'_i))_{1\leq i\leq d}$ with $\rho_0(z'_i) = z'_i \exp(-|z'_i|^2)$, $z'_i \in \mathbb{C}$.

the coefficients $(\beta_t)_{t\geq 0}$, $(\beta'_t)_{t\geq 0}$ *and* $(\alpha_t)_{t\geq 0}$, $(\alpha'_t)_{t\geq 0}$ *being* $\mathbb{C}^d\otimes\mathbb{C}^d$ *and* $\mathbb{C}^{d\times d}\otimes\mathbb{C}^d$*-valued respectively (i.e.* $\beta_t\varsigma_t$ *and* $\beta'_t\bar{\varsigma}_t$ *are in* $\mathbb{C}^d$ *and* $\alpha_t\varsigma_t$ *and* $\alpha'_t\bar{\varsigma}_t$ *are in* $\mathbb{C}^{d\times d}$*) and being possibly random as well. Set*

$$m_t = \int_0^t \langle \Xi_r\varsigma_r, dB_r\rangle, \quad t \geq 0,$$

for another bounded $\mathbb{C}^{d\times d}$*-valued process* $(\Xi_t)_{t\geq 0}$*. Assume finally that* $(\Xi_t)_{t\geq 0}$ *vanishes when the process* $(\psi(Z_t^s))_{t\geq 0}$ *is less than some* $\epsilon_{00} > 0$*. Then, for a non-positive process* $(c_t)_{t\geq 0}$*,*

$$\mathbb{E}\biggl[|m_t|\exp\biggl(\int_0^t c_r dr\biggr)\biggr] \leq C\mathbb{E}\biggl[\int_0^t |\varsigma_r|\bigl(1+r^{-1/2}\bigr)\exp\biggl(\int_0^r c_u du\biggr)dr\biggr],$$

the constant C *only depending on the bound of* Ξ *and on the bounds of* α, α', β *and* β' *at times* t *for which* $\psi(Z_t^s) > \epsilon_{00}/2$*.*

Proof. We follow the proof of (4.80). We consider a smooth cut-off function φ with values in $[0,1]$ matching 1 on $[\epsilon_{00},+\infty)$ and vanishing on $(-\infty,\epsilon_{00}/2]$. Applying Itô's formula, we write

$$\begin{aligned} d\bigl[\varphi(\psi(Z_t^s))\bigr] &= \varphi'(\psi(Z_t^s))d_t^{(1)}dt + \varphi''(\psi(Z_t^s))\bigl|d_t^{(2)}\bigr|^2 dt \\ &\quad + \varphi'(\psi(Z_t^s))\langle d_t^{(2)}, dB_t\rangle + \varphi'(\psi(Z_t^s))\langle \bar{d}_t^{(2)}, d\bar{B}_t\rangle, \end{aligned}$$

$t \geq 0$, where $(d_t^{(1)})_{t\geq 0}$ and $(d_t^{(2)})_{t\geq 0}$ stand for the coefficients of the Itô expansion of $(\psi(Z_t^s))_{t\geq 0}$, i.e.

$$d\bigl[\psi(Z_t^s)\bigr] = d_t^{(1)}dt + \langle d_t^{(2)}, dB_t\rangle + \langle \bar{d}_t^{(2)}, d\bar{B}_t\rangle, \quad t \geq 0.$$

Note also that

$$\begin{aligned} d\bigl[|\varsigma_t|^2\bigr] &= \bigl(2\mathrm{Re}\bigl[\langle\bar{\varsigma}_t, \beta_t\varsigma_t + \beta'_t\bar{\varsigma}_t\rangle\bigr] + |\alpha_t\varsigma_t + \alpha'_t\bar{\varsigma}_t|^2\bigr)dt \\ &\quad + 2\mathrm{Re}\bigl[\langle(\alpha_t\varsigma_t + \alpha'_t\bar{\varsigma}_t)^*\bar{\varsigma}_t, dB_t\rangle\bigr], \quad t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &d\bigl(|m_t|^2 + t\varphi(\psi(Z_t^s))|\varsigma_t|^2\bigr) \\ &\quad = \bigl[|\Xi_t\varsigma_t|^2 + \varphi(\psi(Z_t^s))|\varsigma_t|^2 \\ &\qquad + 2t\varphi(\psi(Z_t^s))\mathrm{Re}\bigl[\langle\bar{\varsigma}_t, \beta_t\varsigma_t + \beta'_t\bar{\varsigma}_t\rangle\bigr] + t\varphi(\psi(Z_t^s))\bigl|\alpha_t\varsigma_t + \alpha'_t\bar{\varsigma}_t\bigr|^2 \\ &\qquad + t|\varsigma_t|^2\varphi'(\psi(Z_t^s))d_t^{(1)} + t|\varsigma_t|^2\varphi''(\psi(Z_t^s))\bigl|d_t^{(2)}\bigr|^2 \\ &\qquad + 2t\varphi'(\psi(Z_t^s))\mathrm{Re}\bigl[\langle(\alpha_t\varsigma_t + \alpha'_t\bar{\varsigma}_t)^*\bar{\varsigma}_t, \bar{d}_t^{(2)}\rangle\bigr]dt + dn_t, \quad t \geq 0, \end{aligned}$$

where $(n_t)_{t\geq 0}$ stands for a new martingale term whose value may vary from line to line. Then, for any small $a>0$, by concavity of the function $x\in\mathbb{R}_+\mapsto(a+x)^{1/2}$ and by the bound $|\Xi_t\varsigma_t|^2\leq\varepsilon_{00}^{-1/2}|\Xi_t\mathbf{1}_{\{\psi(Z_t^s)\geq\varepsilon_{00}\}}|^2\varphi^{1/2}(\psi(Z_t^s))|\varsigma_t|^2$,

$$\begin{aligned}
&d\big(a+|m_t|^2+t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{1/2}\\
&\quad\leq\frac{1}{2}\big(a+|m_t|^2+t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{-1/2}\big\{|\Xi_t\varsigma_t|^2+\varphi(\psi(Z_t^s))|\varsigma_t|^2\\
&\qquad+2t\varphi(\psi(Z_t^s))\mathrm{Re}\big[\langle\bar{\varsigma}_t,\beta_t\varsigma_t+\beta_t'\bar{\varsigma}_t\rangle\big]+t\varphi(\psi(Z_t^s))\big|\alpha_t\varsigma_t+\alpha_t'\bar{\varsigma}_t\big|^2\\
&\qquad+t|\varsigma_t|^2\varphi'(\psi(Z_t^s))d_t^{(1)}+t|\varsigma_t|^2\varphi''(\psi(Z_t^s))\big|d_t^{(2)}\big|^2\\
&\qquad+2t\varphi'(\psi(Z_t^s))\mathrm{Re}\big[\langle(\alpha_t\varsigma_t+\alpha_t'\bar{\varsigma}_t)^*\bar{\varsigma}_t,\bar{d}_t^{(2)}\rangle\big]\big\}dt+dn_t,\\
&\quad\leq C\big(1+t^{-1/2}\big)|\varsigma_t|dt+dn_t,
\end{aligned}\tag{4.112}$$

the constant C here depending on the bound of $(\Xi_t)_{t\geq 0}$, the bounds of the processes $(\alpha_t\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$, $(\alpha_t'\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$, $(\beta_t\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$ and $(\beta_t'\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$ and the supremum norm of $\varphi'/\varphi^{1/2}$ and $\varphi''/\varphi^{1/2}$. (Note that $(d_t^{(1)})_{t\geq 0}$ and $(d_t^{(2)})_{t\geq 0}$ are bounded by known constants.) In particular, C is independent of a.

Now, we can choose φ such that $\varphi'/\varphi^{1/2}$ and $\varphi''/\varphi^{1/2}$ be bounded. For example, think of $\varphi(x)=\exp[-\epsilon_{00}^2/(x^2-(\epsilon_{00}/2)^2)]$ for $x\in(\epsilon_{00}/2,\epsilon_{00}/\sqrt{2})$, $\varphi(x)=0$ for $x\leq\epsilon_{00}/2$, $\varphi(x)=1$ for $x\geq\epsilon_{00}$ and $\varphi(x)\in[\exp(-4),1]$ for $x\in(\epsilon_{00}/\sqrt{2},\epsilon_{00})$. As a consequence, we can assume that the constant C in (4.112) only depends on the bounds of $(\Xi_t)_{t\geq 0}$, $(\alpha_t\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$ and $(\beta_t\mathbf{1}_{\{\psi(Z_t^s)>\epsilon_{00}/2\}})_{t\geq 0}$.

Finally, using the non-positivity of $(c_t)_{t\geq 0}$, we deduce

$$\begin{aligned}
&d\Big[\big(a+|m_t|^2+t\varphi(\psi(Z_t^s))|\varsigma_t|^2\big)^{1/2}\exp\Big(\int_0^t c_r dr\Big)\Big]\\
&\qquad\leq C\big(1+t^{-1/2}\big)|\varsigma_t|\exp\Big(\int_0^t c_r dr\Big)dt+dn_t,\quad t\geq 0.
\end{aligned}$$

Taking the expectation and letting a tend to 0, we complete the proof. □

Obviously, we wish to apply Lemma 4.8.5 with

$$\varsigma_t=\hat{\zeta}_t^s,\ \ \Xi_t=\Xi(Z_t^s),\ \ c_t=\mathrm{Trace}\big[a_t D_{z,\bar{z}}^2\psi(Z_t^s)\big],$$

provided we have a bound for the term $\Xi(Z_t^s)$ in (4.111) and for ϵ_{00} to be fixed later on. (Basically, we cannot choose $\epsilon_{00}=0$ since the coefficients driving the SDE satisfied by $(\hat{\zeta}_t^s)_{t\geq 0}$ are expected to be singular in the neighborhood of the boundary. See (4.84).)

As explained above, for the choice of Ξ we use below, the term $\Xi(Z_t^s)$ is bounded for Z_t^s away from the boundary of the domain only. Following Propositions 4.8.2 and 4.8.4, we are to localize the perturbation argument. Specifically,

Definition 4.8.6 *For some real $S > 0$, some smooth cut-off function $\phi : \mathbb{R}_+ \to [0,1]$ matching 1 on $[0,1]$ and 0 outside $[2,+\infty)$, some given positive real $\epsilon > 0$ and some (finite) stopping time $\mathfrak{s}$ at which $\psi(Z_{\mathfrak{s}}^s) > \epsilon$, we call* localized perturbation argument of Girsanov type *from time $\mathfrak{s}$ to time $\mathfrak{t} := \inf\{t > \mathfrak{s} : \psi(Z_t^s) \leq \epsilon\}$ ($\mathfrak{t}$ being possibly infinite) the perturbation of the Brownian motion $(B_t)_{t\geq 0}$ on the interval $[\mathfrak{s},\mathfrak{t}]$ only. In such a case, the change of measure in (4.109) takes the form*

$$\mathbb{P}^{\varepsilon}(A) = \mathbb{E}\biggl[\exp\biggl(-\int_{\mathfrak{s}}^{t\wedge\mathfrak{t}} 2\mathrm{Re}\bigl[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\bigr] \\ - \int_{\mathfrak{s}}^{t\wedge\mathfrak{t}} |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\biggr)\mathbf{1}_A\biggr], \quad A \in \mathcal{F}_t,\ t \geq 0,$$

and the perturbed value function (with cut-off) in (4.110) writes

$$\begin{aligned} &\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \\ &= \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\biggl[\exp\biggl(-\int_{\mathfrak{s}}^{t} 2\mathrm{Re}\bigl[\langle \bar{G}(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s), dB_r\rangle\bigr] \\ &\qquad - \int_{\mathfrak{s}}^{t} |G|^2(Z_r^s, \hat{Z}_r^{s+\varepsilon} - Z_r^s)dr\biggr) \\ &\qquad \times \exp\biggl(\int_0^t \mathrm{Trace}[a_r D_{z,\bar{z}}^2\psi(\hat{Z}_r^{s+\varepsilon})]dr\biggr) F\bigl(\det(a_t), a_t, \hat{Z}_t^{s+\varepsilon}\bigr)\phi\biggl(\frac{t}{T}\biggr)\biggr]dt, \end{aligned} \tag{4.113}$$

for some (progressively-measurable) extension of $(\hat{Z}_t^{s+\varepsilon})_{0\leq t\leq\mathfrak{s}}$ to the time indices less than $\mathfrak{s}$ for which $([d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ and $([d^2/d\varepsilon^2] (\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq\mathfrak{t}}$ exist. In such a case, by Lemma 4.8.5,

$$\begin{aligned} &\mathbb{E}\biggl[\biggl|\int_{\mathfrak{s}}^{t\wedge\mathfrak{t}} \langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle\biggr| \exp\biggl(\int_0^{t\wedge\mathfrak{t}} \mathrm{Trace}[a_r D_{z,\bar{z}}^2\psi(Z_r^s)]dr\biggr)\biggr] \\ &\quad \leq C\mathbb{E}\biggl[\int_0^{t\wedge\mathfrak{t}} (1+r^{-1/2})|\hat{\zeta}_r^s| \exp\biggl(\int_0^r \mathrm{Trace}[a_u D_{z,\bar{z}}^2\psi(Z_u^s)]du\biggr)dr\biggr], \end{aligned}$$

for some constant $C > 0$, only depending on $(\mathbf{A})$ and on the bounds of $(\Xi(Z_t^s))_{\mathfrak{s}\le t\le \mathfrak{t}}$ and of the coefficients appearing in the Itô writing of $(\zeta_t^s)_{0\le t\le \mathfrak{t}}$ at times $0 \le t \le \mathfrak{t}$ for which $\psi(Z_t^s) \ge \epsilon/2$. (Pay attention that we here start from time 0 to benefit from a as initial condition in (4.112).)

We then deduce the analog of Proposition 4.8.2.

Proposition 4.8.7 *Keep the assumptions of Definition 4.8.6 and assume that the function Ξ is bounded on the set $\{\psi \ge \epsilon\}$ If the differentiation operator w.r.t. ε and the expectation and integral symbols in the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ can be exchanged, then there exists a constant $C > 0$, only depending on Assumption $(\mathbf{A})$ and on the bounds of $(\Xi(\zeta_t^s))_{\mathfrak{s}\le t\le \mathfrak{t}}$ and of the coefficients appearing in the Itô writing of $(\zeta_t^s)_{0\le t\le \mathfrak{t}}$ at times $0 \le t \le \mathfrak{t}$ for which $\psi(Z_t^s) \ge \epsilon/2$, such that*

$$\left| \frac{d}{d\varepsilon}\big[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\big]\right| \le C\mathbb{E}\left[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\left(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)]dr\right)\left[|\hat{\zeta}_t^s| + \int_0^t (1+r^{-1/2})|\hat{\zeta}_r^s|dr\right]dt\right],$$

where $\hat{\zeta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

Actually, the same strategy applies when differentiating twice in (4.113). It is then necessary to bound

$$\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\left[\left|\int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r\rangle\right|^2 \exp\left(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(Z_r^s)\big]\,dr\right)\right]dt, \quad (4.114)$$

and

$$\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\left|\int_{\mathfrak{s}}^{t}\langle \Xi(Z_r^s)\hat{\eta}_r^s, dB_r\rangle \exp\left(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(Z_r^s)\big]\,dr\right)\right|dt, \quad (4.115)$$

with $\hat{\eta}_t^s = [d/d\varepsilon](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$, and

$$\mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}}\left|\int_{\mathfrak{s}}^{t}\langle \big(D_z\Xi(Z_r^s)\hat{\zeta}_r^s + D_{\bar{z}}\Xi(Z_r^s)\bar{\hat{\zeta}}_r^s\big)\hat{\zeta}_r^s, dB_r\rangle \times \exp\left(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(Z_r^s)\big]\,dr\right)\right|dt. \quad (4.116)$$

For (4.115) and (4.116), the proof is the same as the one of Lemma 4.8.5. With the same notations as the ones used therein, the point is to consider (for $a > 0$)

$$d\Big[\big(a + |m_t|^2 + t\varphi(\psi(Z_t^s))[|\hat{\zeta}_t^s|^4 + |\hat{\eta}_t^s|^2]\big)^{1/2}\Big], \quad \mathfrak{s} \le t \le \mathfrak{t},$$

with

$$m_t = \int_{\mathfrak{s}}^{t} \langle \Xi(Z_r^s)\hat{\eta}_r^s, dB_r \rangle, \quad \mathfrak{s} \le t \le \mathfrak{t},$$

or

$$m_t = \int_0^t \langle \big(D_z \Xi(Z_r^s)\hat{\zeta}_r^s + D_{\bar{z}} \Xi(Z_r^s)\overline{\hat{\zeta}}_r \big)\hat{\zeta}_r^s, dB_r \rangle, \quad \mathfrak{s} \le t \le \mathfrak{t}.$$

For (4.114), it is sufficient to expand

$$\left[\left| \int_{\mathfrak{s}}^{t} \langle \Xi(Z_r^s)\hat{\zeta}_r^s, dB_r \rangle \right|^2 \exp\left(\int_0^t \mathrm{Trace}\big[a_r D_{z,\bar{z}}^2 \psi(Z_r^s) \big] dr \right) \right]_{\mathfrak{s} \le t \le \mathfrak{t}}$$

by Itô's formula to get an analog of Lemma 4.8.5.

We then deduce

Proposition 4.8.8 *Keep the assumption Proposition 4.8.7. If the differentiation operator of order 2 w.r.t.* ε *and the expectation and integral symbols in the definition of* $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$ *can be exchanged, then there exists a constant* $C > 0$*, only depending on Assumption (**A**) and on the bounds of* $(\Xi(\zeta_t^s))_{\mathfrak{s} \le t \le \mathfrak{t}}$ *and of the coefficients appearing in the Itô writing of* $(\zeta_t^s)_{0 \le t \le \mathfrak{t}}$ *and* $(\eta_t^s)_{0 \le t \le \mathfrak{t}}$ *at times* $0 \le t \le \mathfrak{t}$ *for which* $\psi(Z_t^s) \ge \epsilon/2$*, such that*

$$\begin{aligned} \left| \frac{d^2}{d\varepsilon^2} \big[\hat{V}^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \big] \right| \le C\mathbb{E}\Bigg[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\left(\int_0^t \mathrm{Trace}[a_r D_{z,\bar{z}}^2 (Z_t^s)] dr \right) \\ \times \Bigg[|\hat{\eta}_t^s| + |\hat{\zeta}_t^s|^2 + \int_0^t (1 + r^{-1/2}) |\hat{\eta}_r^s| dr \\ + \int_0^t (1 + r^{-1/2}) |\hat{\zeta}_r^s|^2 dr + \left(\int_0^t |\hat{\zeta}_r^s| dr \right)^2 \Bigg] dt \Bigg], \end{aligned}$$

where $\hat{\eta}_t^s = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon})_{|\varepsilon=0}$.

4.8.5 Explicit Computations at the Boundary

We are now in position to expand the computations. We start with the so-called "enlargement of the set of controls" method. Following the localization procedure described in the statement of Proposition 4.8.2, the time indices t we consider below are always assumed to belong to the interval $[\mathfrak{s}, \mathfrak{t}]$, the choice of the parameter ϵ in Proposition 4.8.2 being clearly specified at the end of the discussion. Recall that for $t \in [\mathfrak{s}, \mathfrak{t}]$, $\psi(Z_t^s)$ is less than ϵ. Recall also from (4.93) that the perturbation reads

$$d\hat{Z}_t^{s+\varepsilon} = \psi^{1/2}(\hat{Z}_t^{s+\varepsilon}) \exp\bigl(P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\bigr)\sigma_t dB_t$$
$$+ \exp\bigl(P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) a_t \exp\bigl(\bar{P}^*(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\bigr) D_{\bar{z}}^* \psi(\hat{Z}_t^{s+\varepsilon}) dt, \tag{4.117}$$

where $t \in [\mathfrak{s}, \mathfrak{t}]$, and (see (4.94))

$$\frac{d}{d\varepsilon}\bigl[P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\bigr] = |D_z\psi_t|^{-2}\bigl[D^2_{\bar{z},z}\psi_t\zeta_t D_z\psi_t + D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t D_z\psi_t$$
$$- D_{\bar{z}}^*\psi_t(D^2_{z,\bar{z}}\psi_t\bar{\zeta}_t)^* - D_{\bar{z}}^*\psi_t(D^2_{z,z}\psi_t\zeta_t)^*\bigr],$$
$$:= Q_t\zeta_t, \quad t \in [\mathfrak{s}, \mathfrak{t}], \tag{4.118}$$

ζ_t being given by $\zeta_t = [d/d\varepsilon][Z_t^{s+\varepsilon}]$, $t \in [\mathfrak{s}, \mathfrak{t}]$.

We emphasize that (4.118) makes sense for ϵ small enough: since $\psi(Z_t^s) \leq \epsilon$ for $t \in [\mathfrak{s}, \mathfrak{t}]$, $|D_z\psi_t(Z_t^s)| \neq 0$ for ϵ small enough and $t \in [\mathfrak{s}, \mathfrak{t}]$.

We also make use of the following abbreviated notation: we get rid of the symbol hat "^" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s}\leq t\leq \mathfrak{t}}$ (compare with the statement of Proposition 4.8.2); we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_t D^2_{z,\bar{z}}\psi(Z_t^s)]$, $\mathfrak{s} \leq t \leq \mathfrak{t}$.

We then write the derivative $(\zeta_t)_{\mathfrak{s}\leq t\leq\mathfrak{t}}$ as the solution of[21]

$$d\zeta_t = \bigl\{\psi_t^{-1/2}\mathrm{Re}\bigl[D_z\psi_t\zeta_t\bigr] + \psi_t^{1/2}Q_t\zeta_t\bigr\}\sigma_t dB_t$$
$$+ \bigl[a_t D^2_{\bar{z},z}\psi_t\zeta_t + a_t D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\bigr]dt + \bigl[Q_t\zeta_t a_t D_{\bar{z}}^*\psi_t - a_t Q_t\zeta_t D_{\bar{z}}^*\psi_t\bigr]dt.$$

Above, the vector $(\sum_{j,k=1}^d (a_t)_{i,j} D^2_{\bar{z}_j,z_k}\psi(Z_t^s)(\zeta_t)_k)_{1\leq i\leq d}$ is represented by the product $a_t D_{\bar{z},z}\psi_t\zeta_t$.

From (4.118), we have (pay attention that $D_z\psi_t a_t D_{\bar{z}}^*\psi_t$ and $[(D^2_{z,\bar{z}}\psi_t\bar{\zeta}_t)^* + (D^2_{z,z}\psi_t\zeta_t)^*]D_{\bar{z}}^*\psi_t$ below stand for scalar quantities as products of row and column vectors)

$$a_t D^2_{\bar{z},z}\psi_t\zeta_t + a_t D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t + Q_t\zeta_t a_t D_{\bar{z}}^*\psi_t - a_t Q_t\zeta_t D_{\bar{z}}^*\psi_t$$
$$= |D_z\psi_t|^{-2} D_z\psi_t a_t D_{\bar{z}}^*\psi_t\bigl(D^2_{\bar{z},z}\psi_t\zeta_t + D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\bigr)$$
$$- |D_z\psi_t|^{-2} D_{\bar{z}}^*\psi_t\bigl[(D^2_{z,\bar{z}}\psi_t\bar{\zeta}_t)^* + (D^2_{z,z}\psi_t\zeta_t)^*\bigr] a_t D_{\bar{z}}^*\psi_t$$
$$+ |D_z\psi_t|^{-2}\bigl[(D^2_{z,\bar{z}}\psi_t\bar{\zeta}_t)^* + (D^2_{z,z}\psi_t\zeta_t)^*\bigr] D_{\bar{z}}^*\psi_t a_t D_{\bar{z}}^*\psi_t$$
$$:= |D_z\psi_t|^{-2} D_z\psi_t a_t D_{\bar{z}}^*\psi_t\bigl(D^2_{\bar{z},z}\psi_t\zeta_t + D^2_{\bar{z},\bar{z}}\psi_t\bar{\zeta}_t\bigr) + H_t a_t D_{\bar{z}}^*\psi_t, \tag{4.119}$$

[21] Again, the differentiation is purely formal since no differentiability property has been established yet. This is the so-called "meta" part of Meta-Theorem 4.8.1.

$(H_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$ here standing for the auxiliary process

$$H_t = |D_z\psi_t|^{-2}\big\{-D_{\bar z}^*\psi_t\big[(D^2_{z,\bar z}\psi_t\bar\zeta_t)^* + (D^2_{z,z}\psi_t\zeta_t)^*\big] \\ + \big[(D^2_{z,\bar z}\psi_t\bar\zeta_t)^* + (D^2_{z,z}\psi_t\zeta_t)^*\big]D_{\bar z}^*\psi_t\big\}, \tag{4.120}$$

with values in $\mathbb{C}^{d\times d}$.

We deduce that

$$d\zeta_t = \big\{\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big] + \psi_t^{1/2}Q_t\zeta_t\big\}\sigma_t dB_t \\ + |D_z\psi_t|^{-2}D_z\psi_t a_t D_{\bar z}^*\psi_t\big(D^2_{\bar z,z}\psi_t\zeta_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t\big)dt + H_t a_t D_{\bar z}^*\psi_t dt.$$

Taking the square norm, we obtain

$$\begin{aligned} d|\zeta_t|^2 &= 2|D_z\psi_t|^{-2}D_z\psi_t a_t D_{\bar z}^*\psi_t \mathrm{Re}\big[\langle\bar\zeta_t, \big(D^2_{\bar z,z}\psi_t\zeta_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t\big)\big]dt \\ &\quad + 2\mathrm{Re}\big[\langle\bar\zeta_t, H_t a_t D_{\bar z}^*\psi_t\rangle\big]dt \\ &\quad + \mathrm{Trace}\big[\big(\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]I_d + \psi_t^{1/2}Q_t\zeta_t\big) \\ &\qquad\quad \times a_t\big(\psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]I_d - \psi_t^{1/2}\big(Q_t\zeta_t\big)^*\big)\big]dt \\ &\quad + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\bar\zeta_t, \sigma_t dB_t\rangle + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\zeta_t, \bar\sigma_t d\bar B_t\rangle \\ &\quad + \psi_t^{1/2}\big[\langle\bar\zeta_t, Q_t\zeta_t\sigma_t dB_t\rangle + \langle\zeta_t, \overline{Q_t\zeta_t}\bar\sigma_t d\bar B_t\rangle\big], \quad \mathfrak{s}\le t\le\mathfrak{t}. \end{aligned} \tag{4.121}$$

In what follows, we modify the choice of ψ according to the following observation: for any constant $c > 0$, $c\psi$ is again a plurisuperharmonic function describing the domain. To make things clear, we denote by ψ^0 some *reference plurisuperharmonic function* such that, for any Hermitian matrix a of trace 1 and for any $z \in \mathcal{D}$, $\mathrm{Trace}[aD^2_{z,\bar z}\psi^0(z)] \le -1$. Then, we understand ψ as $N\psi^0$ for some free parameter $N \ge 1$ that will be fixed later on.

As a first application, we can simplify the form of $d|\zeta_t|^2$, or at least we can bound it. As already said, for $\epsilon > 0$ small, $\psi_t^0 \le N\psi_t^0 \le \epsilon$, $t \in [\mathfrak{s},\mathfrak{t}]$, so that $|D_z\psi_t^0| \ge \kappa$ for some given constant $\kappa > 0$, $\mathfrak{s}\le t\le\mathfrak{t}$. For example, we notice that $|Q_t\zeta_t|$ in (4.118) and $|H_t|$ in (4.120) by can be bounded by $C|\zeta_t|$, i.e.

$$|Q_t\zeta_t|, |H_t| \le C|\zeta_t|, \quad \mathfrak{s}\le t\le\mathfrak{t}, \tag{4.122}$$

for some constant C depending on κ, $\|D\psi^0\|_\infty$ and $\|D^2\psi^0\|_\infty$, but independent of N. Therefore, denoting by $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ a generic bounded process, bounded by some constant C at any time in $[\mathfrak{s},\mathfrak{t}]$, and setting $\mathcal{E}_t^0 := D_z\psi_t^0 a_t D_{\bar z}^*\psi_t^0$, we write

$$
\begin{aligned}
d|\zeta_t|^2 &= \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big]dt + \mathrm{Re}\big[D_z\psi_t\zeta_t\big]|\zeta_t|r_t dt + \psi_t|\zeta_t|^2 r_t dt \\
&\quad + N|\zeta_t|^2\big((\mathcal{E}_t^0)^{1/2} + \mathcal{E}_t^0\big)r_t dt \\
&\quad + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\bar{\zeta}_t, \sigma_t dB_t\rangle + \psi_t^{-1/2}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\langle\zeta_t, \bar{\sigma}_t d\bar{B}_t\rangle \\
&\quad + \psi_t^{1/2}\big[\langle\bar{\zeta}_t, Q_t\zeta_t\sigma_t dB_t\rangle + \langle\zeta_t, \overline{Q_t\zeta_t}\bar{\sigma}_t d\bar{B}_t\rangle\big], \quad \mathfrak{s} \le t \le \mathfrak{t}, \qquad (4.123)
\end{aligned}
$$

the constant C in the bound of $(r_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ depending on $(\mathbf{A})$ only (and not on N). In particular, C may depend on κ. (Above, the writing $((\mathcal{E}_t^0)^{1/2}+\mathcal{E}_t^0)r_t$ is an abuse of notation. It stands for $(\mathcal{E}_t^0)^{1/2}r_t + \mathcal{E}_t^0 r_t$ for possibly different values of r. We will use this simplification quite often below.) One way or another, we understand that the terms $(\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t])_{t\ge0}$ and $(\mathcal{E}_t^0)_{t\ge0}$ are to be controlled to control the *derivative quantity* according to the program announced in Sect. 4.7.

The strategy we here develop (and inspired by the one of Krylov) consists in considering a modified version of the *derivative quantity*. Below, we consider

$$
\bar{\Gamma}_t = \exp(-K\psi_t)|\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big], \quad \mathfrak{s} \le t \le \mathfrak{t}, \qquad (4.124)
$$

for some constant $K > 0$ to be chosen later on.

To compute $(d\bar{\Gamma}_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$, we use the following writing for $(d\psi_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$

$$
\begin{aligned}
d\psi_t &= \psi_t^{1/2}\big[D_z\psi_t\sigma_t dB_t + D_{\bar{z}}\psi_t\bar{\sigma}_t d\bar{B}_t\big] \\
&\quad + 2D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt + \psi_t L\psi_t dt, \quad \mathfrak{s} \le t \le \mathfrak{t}. \qquad (4.125)
\end{aligned}
$$

(Apply Itô's formula to $(\psi(Z_t^s))_{\mathfrak{s}\le t\le\mathfrak{t}}$ and have in mind that $P(Z^s, \hat{Z}^s - Z^s) = 0$ when $\hat{Z}^s$ in (4.117) is Z^s itself.) We first write

$$
\begin{aligned}
&d\exp(-K\psi_t) \\
&= -2K\exp(-K\psi_t)\psi_t^{1/2}\mathrm{Re}\big[D_z\psi(Z_t^s)\sigma dB_t\big] \\
&\quad + [K^2\psi_t - 2K]\exp(-K\psi_t)\langle D_z\psi_t, a_t D_{\bar{z}}\psi_t\rangle dt \\
&\quad - K\exp(-K\psi_t)\psi_t L\psi_t dt \\
&= -2K\exp(-K\psi_t)\psi_t^{1/2}\mathrm{Re}\big[D_z\psi(Z_t^s)\sigma dB_t\big] \\
&\quad + N^2[K^2\psi_t - 2K]\exp(-K\psi_t)\mathcal{E}_t^0 dt - NK\exp(-K\psi_t)\psi_t L\psi_t^0 dt. \qquad (4.126)
\end{aligned}
$$

Using (4.123),

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\
&\quad = \exp(-K\psi_t)\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t] + \mathrm{Re}[D_z\psi_t\zeta_t]|\zeta_t|r_t \\
&\qquad + \psi_t|\zeta_t|^2 r_t + N|\zeta_t|^2\big((\mathcal{E}_t^0)^{1/2} + \mathcal{E}_t^0\big)r_t\big]\,dt \\
&\qquad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\psi_t - 2K]\mathcal{E}_t^0 - NK\psi_t L\psi_t^0\big]\,dt \\
&\qquad + NK\exp(-K\psi_t)\big[\mathrm{Re}[D_z\psi_t\zeta_t]|\zeta_t| + \psi_t|\zeta_t|^2\big]r_t + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t},
\end{aligned}
$$

where $(m_t)_{t\ge 0}$ stands for a generic martingale term. We are now in position to compute $d\bar{\Gamma}_t$ at any time $t \in [\mathfrak{s},\mathfrak{t}]$. Have in mind that, for such t's, ψ_t is less than ϵ and $(r_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$ is a generic process satisfying $|r_t| \le C$, for some C depending on $(\mathbf{A})$ only. Think in particular of the useful bound: $|\mathrm{Re}[D_z\psi_t\zeta_t]| \le \epsilon^{1/2}\psi_t^{-1/2}|\mathrm{Re}[D_z\psi_t\zeta_t]|$, $t \in [\mathfrak{s},\mathfrak{t}]$. Then, applying Young's inequality to the term $N(\mathcal{E}_t^0)^{1/2}$, the above equation has the form

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\
&\quad \le \exp(-K\psi_t)\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t] + C(1+\epsilon^{1/2}+\epsilon)|\xi_t|^2 \\
&\qquad\qquad + C(N+N^2)|\zeta_t|^2\mathcal{E}_t^0\big]\,dt \\
&\qquad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\epsilon - 2K]\mathcal{E}_t^0 + CNK\epsilon\big]\,dt \\
&\qquad + NK\exp(-K\psi_t)\big[C\epsilon^{1/2}|\xi_t|^2 + C\epsilon|\xi_t|^2\big] + dm_t,
\end{aligned} \tag{4.127}
$$

where $|\xi_t|^2 = |\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]$. To complete the analysis (in the neighborhood of the boundary), we must compute $d[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]]$, $\mathfrak{s} \le t \le \mathfrak{t}$. To do so, we start with (4.125) at $s+\varepsilon$ (so that a_t is understood as $\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)$). Taking the square root, we write

$$
\begin{aligned}
&d\psi^{1/2}(Z_t^{s+\varepsilon}) \\
&\quad = \frac{1}{2}\big[D_z\psi(Z_t^{s+\varepsilon})\exp(p_t^\varepsilon)\sigma_t dB_t + D_{\bar z}\psi(Z_t^{s+\varepsilon})\exp(\bar{p}_t^\varepsilon)\bar{\sigma}_t d\bar{B}_t\big] \\
&\qquad + \frac{3}{4}\psi^{-1/2}(Z_t^{s+\varepsilon})D_z\psi(Z_t^{s+\varepsilon})\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)D_{\bar z}^*\psi(Z_t^{s+\varepsilon})dt \\
&\qquad + \frac{1}{2}\psi^{1/2}(Z_t^{s+\varepsilon})\mathrm{Trace}\big[\exp(p_t^\varepsilon)a_t\exp(-p_t^\varepsilon)D_{z,\bar z}^2\psi(Z_t^{s+\varepsilon})\big]\,dt.
\end{aligned}
$$

We now differentiate with respect to ε at $\varepsilon = 0$. We obtain (with the notation $\mathcal{E}_t = D_z\psi_t a_t D_{\bar z}^*\psi_t = N^2\mathcal{E}_t^0$)

$$
\begin{aligned}
&\frac{1}{2}d\big[\psi_t^{-1/2}\mathrm{Re}[D_z\psi(Z_t)\zeta_t]\big] \\
&\quad = \mathrm{Re}\big[\big((D^2_{z,z}\psi_t\zeta_t)^* + (D^2_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t\big)\sigma_t dB_t\big] \\
&\qquad - \frac{3}{4}\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
&\qquad + \frac{3}{4}\psi_t^{-1/2}\big[\big((D^2_{z,z}\psi_t\zeta_t)^* + (D^2_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t\big)a_t D_{\bar z^*}\psi_t\big]dt \\
&\qquad + \frac{3}{4}\psi_t^{-1/2}\big[D_z\psi_t a_t\big(D^2_{\bar z,z}\psi_t\zeta_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t - Q_t\zeta_t D^*_{\bar z}\psi_t\big)\big]dt \\
&\qquad + \frac{1}{2}\psi_t^{1/2}\mathrm{Trace}\big[\big(Q_t\zeta_t a_t - a_t Q_t\zeta_t\big)D^2_{z,\bar z}\psi_t + a_t D^3_{z,\bar z,z}\psi_t\zeta_t + a_t D^3_{z,\bar z,\bar z}\psi_t\bar\zeta_t\big]dt \\
&\qquad + \frac{1}{2}\mathrm{Re}[D_z\psi_t\zeta_t]\psi_t^{-1/2}L\psi_t dt. \qquad (4.128)
\end{aligned}
$$

Plugging the definition of $(Q_t\zeta_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$ (see (4.118)), we deduce

$$
\begin{aligned}
&(D^2_{z,z}\psi_t\zeta_t)^* + (D^2_{z,\bar z}\psi_t\bar\zeta_t)^* + D_z\psi_t Q_t\zeta_t \\
&\quad = |D_z\psi_t|^{-2}\big(D_z\psi_t D^2_{\bar z,z}\psi_t\zeta_t + D_z\psi_t D^2_{\bar z,\bar z}\psi_t\bar\zeta_t\big)D_z\psi_t \\
&\quad = r_t|\zeta_t|D_z\psi_t. \qquad (4.129)
\end{aligned}
$$

It is important to notice that the process $(r_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$ in (4.129) is scalar as the product of row and column vectors. (It is also bounded independently of N.) We deduce

$$
\begin{aligned}
&d\big[\psi_t^{-1/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
&\quad = 2\mathrm{Re}\big[r_t|\zeta_t|D_z\psi_t\sigma_t dB_t\big] - \frac{3}{2}\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
&\qquad + \psi_t^{-1/2}r_t\mathcal{E}_t|\zeta_t|dt + N\psi_t^{1/2}r_t|\zeta_t|dt + \mathrm{Re}[D_z\psi_t\zeta_t]\psi_t^{-1/2}L\psi_t dt.
\end{aligned}
$$

Taking the square, we finally claim (use the following trick to pass from the equality to the inequality: $\psi_t^{-1}r_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t \le \psi_t^{-2}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t + r_t^2|\zeta_t|^2\mathcal{E}_t$, $Nr_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t] \le N\psi_t r_t^2|\zeta_t|^2 + N\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]$ and $L\psi_t \le -N$)

$$
\begin{aligned}
&d\big[\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] \\
&\quad = dm_t + r_t|\zeta_t|^2\mathcal{E}_t dt - 3\psi_t^{-2}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t dt + \psi_t^{-1}r_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t dt \\
&\qquad + Nr_t|\zeta_t|\mathrm{Re}[D_z\psi_t\zeta_t]dt + 2\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt \\
&\quad \le dm_t + C(1+\mathcal{E}_t)|\zeta_t|^2 dt + CN\psi_t|\zeta_t|^2 dt - N\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt, \qquad (4.130)
\end{aligned}
$$

for a possibly new value of C.

Making the sum with (4.127) and assuming $\epsilon < 1$ and $N \geq 1$, we deduce

$$\begin{aligned} d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt \\ + |\xi_t|^2\big(C' + C'N\epsilon^{1/2} + C'NK\epsilon^{1/2}\big)dt \\ + |\zeta_t|^2\exp(-K\psi_t)N^2\big[K^2\epsilon - 2K + C'\exp(K\psi_t)\big]\mathcal{E}_t^0 dt + dm_t, \end{aligned}$$

the constant C' depending on C only. (In particular, C' is independent of K, N, ϵ, s and t.)

Choose now $K = \epsilon^{-1/4}$. We obtain

$$\begin{aligned} d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + 2(C' + 2C'N\epsilon^{1/4})|\xi_t|^2 dt \\ + |\zeta_t|^2\exp(-K\psi_t)N^2\big[\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{3/4})\big]\mathcal{E}_t^0 dt + dm_t. \end{aligned}$$

Choose ϵ small enough such that

$$\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{3/4}) < 0. \tag{4.131}$$

Then,

$$d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + 2\big(C' + 2C'N\epsilon^{1/4}\big)|\xi_t|^2 dt + dm_t,$$

for $\mathfrak{s} \leq t \leq \mathfrak{t}$. Finally for $N = \epsilon^{-1/4}$ and $\exp(\epsilon^{3/4}) \leq 2$, we obtain:

$$d\bar{\Gamma}_t \leq 6C'|\xi_t|^2 dt + dm_t \leq 6C'\exp(\epsilon^{3/4})\bar{\Gamma}_t + dm_t \leq 12C'\bar{\Gamma}_t + dm_t, \tag{4.132}$$

for $\mathfrak{s} \leq t \leq \mathfrak{t}$.

Exactly as in the statement of Proposition 4.7.7 (see in particular (4.74)), the right quantity to consider is

$$\exp\bigg(\int_0^t L\psi_r dr\bigg)\bar{\Gamma}_t = \exp\bigg(\int_0^t NL\psi_r^0 dr\bigg)\bar{\Gamma}_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}.$$

Again, for $\mathfrak{s} \leq t \leq \mathfrak{t}$,

$$\begin{aligned} d\bigg[\exp\bigg(\int_0^t NL\psi_r^0 dr\bigg)\bar{\Gamma}_t\bigg] \\ \leq \exp\bigg(\int_0^t NL\psi_r^0 dr\bigg)\bigg[NL\psi_t^0\bar{\Gamma}_t + 12C'\bar{\Gamma}_t\bigg]dt + dm_t \\ \leq \exp\bigg(\int_0^t NL\psi_r^0 dr\bigg)\bigg[-N\bar{\Gamma}_t + 12C'\bar{\Gamma}_t\bigg]dt + dm_t. \end{aligned}$$

Having in mind that $N = \epsilon^{-1/4}$, we deduce that, for $\epsilon^{-1/4} \geq 12C'$ (obviously, this is compatible with (4.131)),

$$d\left[\exp\left(\int_0^t NL\psi_r^0 dr\right)\bar{\Gamma}_t\right] \leq dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{4.133}$$

Actually, it is plain to see that, for ϵ small enough, the same holds with $NL\psi_s^0$ replaced by $(N-1)L\psi_s^0$, i.e.

$$d\left[\exp\left(\int_0^t (N-1)L\psi_s^0 ds\right)\bar{\Gamma}_t\right] \leq dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{4.134}$$

We deduce

Proposition 4.8.9 *There exists a positive real ϵ_1 such that for $0 < \epsilon < \epsilon_1$, for $N = K = \epsilon^{-1/4}$, for $\psi = N\psi^0$, where ψ^0 is the* reference plurisuperharmonic function *describing $\mathcal{D}$ such that* $\mathrm{Trace}[aD^2_{z,\bar{z}}\psi^0(z)] \leq -1$*, $z \in \mathcal{D}$, for a stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) < \epsilon$, the* derivative quantity *obtained by perturbing the control parameter as in* (4.93) *and* (4.118)

$$\bar{\Gamma}_t^{(1)} = \exp\bigl(-K\psi(Z_t^s)\bigr)|\zeta_t|^2 + \psi^{-1}(Z_t^s)\mathrm{Re}^2\bigl[D_z\psi(Z_t^s)\zeta_t\bigr], \quad t \geq \mathfrak{s},$$

satisfies up to time $\mathfrak{t} = \inf\{t \geq \mathfrak{s} : \psi(Z_t^s) \geq \epsilon\}$ (provided that $(\hat{Z}_t^{s+\varepsilon})_{0 \leq t \leq \mathfrak{t}}$ is well differentiable w.r.t. ε)

$$\mathbb{E}\left[\exp\left(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right)\bar{\Gamma}^{(1)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\right]$$
$$\leq \exp\left(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right)\bar{\Gamma}^{(1)}_{\mathfrak{s}}, \quad t \geq \mathfrak{s},$$

with $\delta = 1/N = \epsilon^{1/4}$.

4.8.6 Away from the Boundary

We now investigate the *derivative quantity* away from the boundary. The idea consists in perturbing the system in two different ways as the same time, or said differently, in applying two perturbations. In the subsections above, this possibility has not been discussed, but we feel it quite simple to understand: it is even plain to see that provided that the corresponding versions of Propositions 4.8.2, 4.8.4 or 4.8.7 be true for each perturbation under consideration, the common action of both perturbations on the perturbed value function is of the same type, i.e. the statements of Propositions 4.8.2, 4.8.4 or 4.8.7 (according to the framework) remain true under the common action.

Away from the boundary, the idea is to perturb both the underlying time speed, as explained in Sect. 4.8.3, and the probability measure, as explained in Sect. 4.8.4.

Following the localization procedure described in the statement of Propositions 4.8.4 and 4.8.7, the time indices t we consider in this subsection are always assumed to belong to the interval $[\mathfrak{s}, \mathfrak{t}]$, where $\mathfrak{s}$ is some stopping time at which $\psi(Z_t^s) > \epsilon$[22] and $\mathfrak{t} = \inf\{t > \mathfrak{s} : \psi(Z_t^s) \leq \epsilon\}$. (As above, the choice of the parameter ϵ is clearly specified at the end of the discussion.) In particular for $t \in [\mathfrak{s}, \mathfrak{t}]$, $\psi(Z_t^s)$ is greater than ϵ.

We also make use of the same abbreviated notation as above: we get rid of the symbol hat "^" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s} \leq t \leq \mathfrak{t}}$; we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)]$, $\mathfrak{s} \leq t \leq \mathfrak{t}$. Finally, we emphasize that ψ is here arbitrary: the connection with the form $\psi = N\psi^0$ used in Proposition 4.8.9 is explained later on.

The time-change we here use is given by a variation of (4.103), namely

$$\frac{d}{d\varepsilon}\big[T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0} = -\psi^{-1}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t\big], \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{4.135}$$

Moreover, the measure perturbation we choose in (4.111) is

$$\frac{d}{d\varepsilon}\big[G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\big]_{|\varepsilon=0} = -\Lambda\bar{\sigma}_t^*\zeta_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \tag{4.136}$$

for some constant Λ to be chosen. (In other words, $\Xi(Z_t^s) = -\Lambda\bar{\sigma}_t^*$ in (4.111).)

We emphasize that both perturbations (4.135) and (4.136) are linear functionals of ζ, with a bounded linear coefficient. (Again, $\psi^{-1}(Z_t^s)$ is bounded by ϵ^{-1} away for $t \in [\mathfrak{s}, \mathfrak{t}]$.)

The dynamics of $(\hat{Z}_t^{s+\varepsilon})_{\mathfrak{s} \leq t \leq \mathfrak{t}}$ then read (compare with (4.101) and (4.108))

$$\begin{aligned} d\hat{Z}_t^{s+\varepsilon} &= \psi^{1/2}(\hat{Z}_t^{s+\varepsilon})T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)\sigma_t\big[dB_t + G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)dt\big] \\ &\quad + T^2(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s)a_t D_{\bar{z}}^*\psi(Z_t^{s+\varepsilon})dt, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \end{aligned}$$

Differentiating (at least formally), we obtain

$$\begin{aligned} d\zeta_t &= -\Lambda a_t\zeta_t dt + a_t\big(D_{\bar{z},z}^2\psi_t\big)^*\zeta_t dt + a_t\big(D_{\bar{z},\bar{z}}^2\psi_t\bar{\zeta}_t\big)^* dt \\ &\quad - 2\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]a_t D_{\bar{z}}^*\psi_t dt. \end{aligned}$$

(Pay attention that the dB_t terms cancel.)

[22] Pay attention that the values of ϵ may be different from the ones given by Proposition 4.8.9.

Then,

$$d|\zeta_t|^2 = -2\Lambda\langle\bar{\zeta}_t, a_t\zeta_t\rangle dt + 2\mathrm{Re}\big[\langle\bar{\zeta}_t, a_t\big(D^2_{\bar{z},z}\psi_t\big)^*\zeta_t\rangle + \langle\bar{\zeta}_t, a_t\big(D^2_{\bar{z},\bar{z}}\psi_t\big)^*\bar{\zeta}_t\rangle\big]dt$$
$$- 4\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]\mathrm{Re}[D_z\psi_t a_t\zeta_t]dt.$$

Have in mind that $\psi_t \geq \epsilon$ for $t \in [\mathfrak{s}, \mathfrak{t}]$. Then, by Young's inequality, we can find some constant $C(\epsilon, \psi)$ depending on ϵ and ψ only,[23] such that

$$d|\zeta_t|^2 \leq \big[C(\epsilon,\psi) - 2\Lambda\big]\langle\bar{\zeta}_t, a_t\zeta_t\rangle dt + \epsilon^2|\zeta_t|^2 dt, \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{4.137}$$

Consider now some real R such that $R^2 \geq 2\sup_{z\in\mathcal{D}}[|z|^2]$. Then, by Lemma 4.6.8,

$$d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}\big]\exp\bigg(\int_0^t L\psi_r dr\bigg)\bigg] = -\exp\bigg(\int_0^t L\psi_r dr\bigg)dt + dm_t,$$

where $(m_t)_{t\geq 0}$ stands for a generic martingale term whose value may vary from line to line. In particular, for a small real $\delta > 0$,

$$d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}\big]\exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)\bigg]$$
$$= \big[-\delta(R^2 - |Z_t|^2)\psi_t^{-1}L\psi_t - 1\big]\exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)dt + dm_t, \quad t \geq 0. \tag{4.138}$$

Finally, from (4.137) and (4.138),

$$d\bigg[\big[(R^2 - |Z_t|^2)\psi_t^{-1}|\zeta_t|^2\big]\exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)\bigg]$$
$$\leq \big[\big(\epsilon^2 - \delta L\psi_t\big)(R^2 - |Z_t|^2)\psi_t^{-1} - 1\big]|\zeta_t|^2\exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)dt$$
$$+ \big(C(\epsilon,\psi) - 2\Lambda\big)\big[(R^2 - |Z_t|^2)\psi_t^{-1}\big]\langle\bar{\zeta}_t, a_t\zeta_t\rangle\exp\bigg(\int_0^t (1-\delta)L\psi_r dr\bigg)dt$$
$$+ dm_t, \tag{4.139}$$

[23] We here specify the dependence on ψ since ψ may vary in the statement of Proposition 4.8.9.

for $\mathfrak{s} \le t \le \mathfrak{t}$. Choose ϵ small enough such that $\epsilon R^2 \le 1/2$ and then δ small enough such that

$$\delta^{-1} \ge 2R^2\epsilon^{-1} \sup\bigl\{-\mathrm{Trace}(aD^2_{z,\bar{z}}\psi(z)),\ z \in \mathcal{D},\ a \in \mathcal{H}_d : \mathrm{Trace}(a) = 1\bigr\}, \tag{4.140}$$

so that

$$\delta R^2\epsilon^{-1} \sup\bigl\{-\mathrm{Trace}(aD^2_{z,\bar{z}}\psi(z)),\ z \in \mathcal{D},\ a \in \mathcal{H}_d : \mathrm{Trace}(a) = 1\bigr\} \le \frac{1}{2}.$$

Then, for any $\mathfrak{s} \le t \le \mathfrak{t}$,

$$\bigl(\epsilon^2 - \delta L\psi_t\bigr)(R^2 - |Z_t|^2)\psi_t^{-1} - 1 \le \bigl(\epsilon^2 - \delta L\psi_t\bigr)R^2\epsilon^{-1} - 1 \le 0,$$

so that the first term in the RHS in (4.139) is non-positive. Choose finally $\Lambda = C(\epsilon, \psi)/2$ to cancel the second term in the RHS in (4.139). Then,

$$d\biggl[\bigl[(R^2 - |Z_t|^2)\psi_t^{-1}|\zeta_t|^2\bigr]\exp\biggl(\int_0^t (1-\delta)L\psi_r dr\biggr)\biggr] \le dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}.$$

Finally,

Proposition 4.8.10 *Let ψ be a plurisuperharmonic function describing the domain $\mathcal{D}$ as in (**A**). Then, there exists a positive real $\epsilon_3 > 0$ such that for any $0 < \epsilon < \epsilon_3$, we can find a constant $C(\epsilon, \psi)$, depending on ϵ and ψ only, such that, for any stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) > \epsilon$, for $\Lambda = C(\epsilon, \psi)/2$ in (4.136) and $R^2 \ge 2\sup_{z\in\mathcal{D}}[|z|^2]$, the* derivative quantity *obtained by perturbing the time speed as in (4.135) and the measure as in (4.136)*

$$\bar{\Gamma}^{(3)}_t = (R^2 - |Z^s_t|^2)\psi^{-1}(Z^s_t)|\zeta_t|^2, \quad t \ge \mathfrak{s},$$

satisfies up to time $\mathfrak{t} = \inf\{t \ge \mathfrak{s} : \psi(Z^s_t) \le \epsilon\}$ (provided that $(\hat{Z}^{s+\varepsilon}_t)_{0\le t\le \mathfrak{t}}$ is well differentiable w.r.t. ε)

$$\begin{aligned}\mathbb{E}\biggl[\exp\biggl(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\biggr)\bar{\Gamma}^{(3)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\biggr]\\ \le \exp\biggl(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\biggr)\bar{\Gamma}^{(3)}_{\mathfrak{s}}, \quad t \ge \mathfrak{s},\end{aligned}$$

with δ as in (4.140).

4.8.7 Interpolation Between the Interior and the Boundary

It now remains to gather the estimates at and away the boundary. To do, we introduce an interpolated version of the *derivative quantity*.

The idea is the same as in the previous subsection: we couple at the same time several perturbations. Specifically, we here make use of the three possible types of perturbations discussed in Sects. 4.8.1, 4.8.2 and 4.8.4: the control perturbation is given by (4.94) and (4.118), i.e.

$$\begin{aligned}
&\frac{d}{d\varepsilon}\big[P(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big]\\
&\quad= |D_z\psi_t|^{-2}\big[D^2_{\bar z,z}\psi_t\zeta_t D_z\psi_t + D^2_{\bar z,\bar z}\psi_t\bar\zeta_t D_z\psi_t\\
&\qquad\qquad - D^*_{\bar z}\psi_t(D^2_{z,\bar z}\psi_t\bar\zeta_t)^* - D^*_{\bar z}\psi_t(D^2_{z,z}\psi_t\zeta_t)^*\big],\\
&\quad:= Q_t\zeta_t,
\end{aligned}\tag{4.141}$$

the time-change perturbation is given by a variation of (4.103), namely

$$\frac{d}{d\varepsilon}\big[T(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big]_{|\varepsilon=0} = (\lambda-1)\psi^{-1}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t\big],\tag{4.142}$$

for some real $\lambda\in(0,1)$ to be chosen later on, and the measure perturbation is given as a variation of (4.111):

$$\begin{aligned}
&\frac{d}{d\varepsilon}\big[G(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big]_{|\varepsilon=0}\\
&\quad= (-2\lambda+\lambda^2+2)\psi^{-3/2}(Z_t^s)\mathrm{Re}\big[D_z\psi(Z_t^s)\zeta_t\big]\bar\sigma_t^* D_{\bar z}\psi(Z_t^s).
\end{aligned}\tag{4.143}$$

(We here say a variation of (4.111) since the perturbation now involves $(\bar\zeta_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$ as well. Obviously, this doesn't change the global strategy.) The dynamics of $(\hat Z_t^{s+\varepsilon})_{\mathfrak{s}\le t\le\mathfrak{t}}$ then read (compare with (4.93), (4.101) and (4.108))

$$\begin{aligned}
d\hat Z_t^{s+\varepsilon} &= \psi^{1/2}(\hat Z_t^{s+\varepsilon})T(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\exp\big(P(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big)\\
&\quad\times\sigma_t\big[dB_t+G(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)dt\big]\\
&\quad+T^2(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\exp\big(P(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big)\\
&\quad\times a_t\exp\big(-P(Z_t^s,\hat Z_t^{s+\varepsilon}-Z_t^s)\big)D^*_{\bar z}\psi(Z_t^{s+\varepsilon})dt,
\end{aligned}$$

for $\mathfrak{s}\le t\le\mathfrak{t}$.

Following the localization procedure described in the statement of Propositions 4.8.2, 4.8.4 and 4.8.7, the time indices t we consider in this subsection are always assumed to belong to the interval $[\mathfrak{s},\mathfrak{t}]$, where $\mathfrak{s}$ is some stopping time at which $\epsilon' < \psi(Z_t^s) < \epsilon$, for an additional positive real ϵ'[24] and $\mathfrak{t} = \inf\{t > \mathfrak{s} : \psi(Z_t^s) \notin]\epsilon', \epsilon[\}$. In particular for $t \in [\mathfrak{s}, \mathfrak{t}]$, $\psi(Z_t^s)$ belongs to $[\epsilon', \epsilon]$.

We also make use of the same abbreviated notation as above: we get rid of the symbol hat "^" and of the superscript s for more simplicity in $(\hat{\zeta}_t^s)_{\mathfrak{s} \le t \le \mathfrak{t}}$; we also write ψ_t for $\psi(Z_t^s)$ and $L\psi_t$ for $\mathrm{Trace}[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)]$, $\mathfrak{s} \le t \le \mathfrak{t}$.

Then, we can differentiate the dynamics of $(\hat{Z}_t^{s+\varepsilon})_{t \ge 0}$ according to the rules prescribed above. Following (4.119), we obtain

$$
\begin{aligned}
d\zeta_t &= \big[\lambda \psi_t^{-1/2} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] + \psi_t^{1/2} Q_t \zeta_t\big] \sigma_t dB_t + \psi_t^{1/2} \Xi_t a_t D_{\bar{z}}^* \psi_t dt \\
&\quad + \big(N^{-1} \mathcal{E}_t + \mathcal{E}_t^{1/2}\big) |\zeta_t| r_t dt + 2(\lambda - 1) \psi_t^{-1} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] a_t D_{\bar{z}}^* \psi_t dt, \quad t \ge 0,
\end{aligned}
$$

where $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ stands for a generic process, bounded by some constant C depending on $(\mathbf{A})$ only. (Here and only here $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ has values in $\mathbb{C}^d$. Below, it has values in $\mathbb{C}$.) Above, $\mathcal{E}_t := D_z \psi_t a_t D_{\bar{z}}^* \psi_t$ and N denotes a real greater than 1 such that $\psi = N \psi^0$ where ψ^0 is some reference choice of the plurisuperharmonic function describing $\mathcal{D}$, such that $\mathrm{Trace}[a D_{z,\bar{z}}^2 \psi^0(z)] \le -1$ for any $z \in \mathcal{D}$ and any positive Hermitian matrix a of trace 1.

Now, $N^{-1} \mathcal{E}_t$ is bounded by $C \mathcal{E}_t^{1/2}$, $\mathfrak{s} \le t \le \mathfrak{t}$, up to a modification of C. (Pay attention that C is independent of N.) Therefore, using the boundedness of $|Q_t \zeta_t| / |\zeta_t|$ (see (4.118)), $\mathfrak{s} \le t \le \mathfrak{t}$,

$$
\begin{aligned}
& d|\zeta_t|^2 \\
&= \lambda \psi_t^{-1/2} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] \big(\langle \bar{\zeta}_t, \sigma_t dB_t \rangle + \langle \zeta_t, \bar{\sigma}_t d\bar{B}_t \rangle\big) \\
&\quad + \psi_t^{1/2} \big(\langle \bar{\zeta}_t, Q_t \zeta_t \sigma_t dB_t \rangle + \langle \zeta_t, \overline{Q_t \zeta_t} \bar{\sigma}_t d\bar{B}_t \rangle\big) \\
&\quad + 4(\lambda - 1) \psi_t^{-1} \mathrm{Re}\big[D_z \psi_t \zeta_t\big] \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt + 2 \psi_t^{1/2} \Xi_t \mathrm{Re}\big[D_z \psi_t a_t \zeta_t\big] dt \\
&\quad + \big[\lambda^2 \psi_t^{-1} \mathrm{Re}^2 [D_z \psi_t \zeta_t] + \lambda N |\zeta_t|^2 r_t + \psi_t |\zeta_t|^2 r_t + \mathcal{E}_t^{1/2} |\zeta_t|^2 r_t\big] dt.
\end{aligned} \tag{4.144}
$$

Now, from (4.125),

$$
\begin{aligned}
d\psi_t^{-\lambda} &= -\lambda \psi_t^{-\lambda - 1/2} \big[D_z \psi_t \sigma_t dB_t + D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t\big] \\
&\quad - \lambda (1 - \lambda) \psi_t^{-(1+\lambda)} D_z \psi_t a_t D_{\bar{z}}^* \psi_t dt - \lambda \psi_t^{-\lambda} L \psi_t dt.
\end{aligned} \tag{4.145}
$$

By (4.126), for $K \ge 1$ to be chosen later on,

$$
\begin{aligned}
& d\big[\exp(-K \psi_t)\big] \\
&= -K \exp(-K \psi_t) \psi_t^{1/2} \big[D_z \psi_t \sigma_t dB_t + D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t\big] \\
&\quad + [K^2 \psi_t - 2K] \exp(-K \psi_t) D_z \psi_t a_t D_{\bar{z}}^* \psi_t dt - K \exp(-K \psi_t) \psi_t L \psi_t dt,
\end{aligned}
$$

[24] The values of both ϵ and ϵ' will be specified later on.

so that

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)\psi_t^{-\lambda}\big] \\
&\quad = -\big[\lambda\psi_t^{-1/2} + K\psi_t^{1/2}\big]\exp(-K\psi_t)\psi_t^{-\lambda}\big[D_z\psi_t\sigma_t dB_t + D_{\bar z}\psi_t\bar\sigma_t d\bar B_t\big] \\
&\qquad + \big[K^2\psi_t + 2\lambda K - 2K - \lambda(1-\lambda)\psi_t^{-1}\big]\exp(-K\psi_t)\psi_t^{-\lambda}D_z\psi_t a_t D_{\bar z}^*\psi_t dt \\
&\qquad - \big[\lambda + K\psi_t\big]\exp(-K\psi_t)\psi_t^{-\lambda}L\psi_t dt.
\end{aligned}
$$

Then, by (4.144) and the above equality,

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2\big] \\
&\quad = (4\lambda - 2\lambda^2 - 4 - 2\lambda K\psi_t)\exp(-K\psi_t)\psi_t^{-(1+\lambda)}\mathrm{Re}\big[D_z\psi_t\zeta_t\big]\mathrm{Re}\big[D_z\psi_t a_t\zeta_t\big]dt \\
&\qquad + 2\Xi_t\exp(-K\psi_t)\psi_t^{1/2-\lambda}\mathrm{Re}\big[D_z\psi_t a_t\zeta_t\big]dt \\
&\qquad + \exp(-K\psi_t)\psi_t^{-\lambda}\big[\lambda^2\psi_t^{-1}\mathrm{Re}^2[D_z\psi_t\zeta_t] + \lambda N|\zeta_t|^2 r_t \\
&\qquad\qquad\qquad + NK\psi_t|\zeta_t|^2 r_t + \mathcal{E}_t^{1/2}|\zeta_t|^2 r_t\big]dt \\
&\qquad + \big[K^2\psi_t + 2\lambda K - 2K - \lambda(1-\lambda)\psi_t^{-1}\big]\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt \\
&\qquad - \big[\lambda + K\psi_t\big]\exp(-K\psi_t)\psi_t^{-\lambda}L\psi_t|\zeta_t|^2 dt + dm_t,
\end{aligned} \tag{4.146}
$$

where $(m_t)_{\mathfrak{s}\le t\le\mathfrak{s}}$ stands for a generic martingale term.

By the specific choice we made for $(\Xi_t)_{\mathfrak{s}\le t\le\mathfrak{t}}$, see (4.143), and by Young's inequality,

$$
\begin{aligned}
&d\big[\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2\big] \\
&\quad \le \lambda^2\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt \\
&\qquad + C\big(\lambda K N^2 + \lambda N + K^{-1} + NK\psi_t\big)\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\
&\qquad + \big[K^2\psi_t - (1-2\lambda)K\big]\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt \\
&\qquad - \big[\lambda + K\psi_t\big]\exp(-K\psi_t)\psi_t^{-\lambda}L\psi_t|\zeta_t|^2 dt + dm_t.
\end{aligned} \tag{4.147}
$$

Replacing $-\lambda$ by $(1-\lambda)/2$ in (4.145), we obtain in a similar way

$$
\begin{aligned}
&d\psi_t^{(1-\lambda)/2} \\
&\quad = \frac{1-\lambda}{2}\psi_t^{-\lambda/2}\big[D_z\psi_t\sigma_t dB_t + D_{\bar z}\psi_t\bar\sigma_t d\bar B_t\big] \\
&\qquad + \frac{(1-\lambda)(3-\lambda)}{4}\psi_t^{-(1+\lambda)/2}D_z\psi_t a_t D_{\bar z}^*\psi_t dt + \frac{1-\lambda}{2}\psi_t^{(1-\lambda)/2}L\psi_t dt.
\end{aligned} \tag{4.148}
$$

Below, we make use of (4.148) but at point $s+\varepsilon$ instead of ε. We obtain

$$
\begin{aligned}
& d\big[\psi^{(1-\lambda)/2}(\hat{Z}_t^{s+\varepsilon})\big] \\
&\quad = \frac{1-\lambda}{2}\psi^{-\lambda/2}(\hat{Z}_t^{s+\varepsilon})T(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s) \\
&\qquad \times \big[D_z\psi(\hat{Z}_t^{s+\varepsilon})\exp\big(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)\sigma_t\big(dB_t+G(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)dt\big) \\
&\qquad + D_{\bar{z}}\psi(\hat{Z}_t^{s+\varepsilon})\exp\big(\bar{P}(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)\bar{\sigma}_t\big(d\bar{B}_t+\bar{G}(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)dt\big)\big] \\
&\qquad + \frac{(1-\lambda)(3-\lambda)}{4}\psi^{-(1+\lambda)/2}(\hat{Z}_t^{s+\varepsilon})T^2(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)D_z\psi(\hat{Z}_t^{s+\varepsilon}) \\
&\qquad \times \exp\big(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)a_t\exp\big(-P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)D_{\bar{z}}^*\psi(\hat{Z}_t^{s+\varepsilon})dt \\
&\qquad + \frac{1-\lambda}{2}\psi^{(1-\lambda)/2}(\hat{Z}_t^{s+\varepsilon})T^2(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s) \\
&\qquad \times \mathrm{Trace}\big[\exp\big(P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big) \\
&\qquad \times a_t\exp\big(-P(Z_t^s,\hat{Z}_t^{s+\varepsilon}-Z_t^s)\big)D_{z,\bar{z}}^2\psi(\hat{Z}_t^{s+\varepsilon})\big]dt.
\end{aligned}
\tag{4.149}
$$

We now differentiate according to the rules prescribed above (see in particular (4.141)–(4.143)). Using (4.129), we obtain

$$
\begin{aligned}
& (1-\lambda)d\big[\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
&\quad = \frac{1-\lambda}{2}\psi_t^{-\lambda/2}\big[-\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]+r_t|\zeta_t|\big]\big[D_z\psi_t\sigma_t dB_t+D_{\bar{z}}\psi_t\bar{\sigma}_t d\bar{B}_t\big] \\
&\qquad + (1-\lambda)\psi_t^{-\lambda/2}\Xi_t D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt \\
&\qquad + \frac{(1-\lambda)(3-\lambda)}{4}\big[-1-\lambda-2+2\lambda\big]\psi_t^{-(3+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt \\
&\qquad + \frac{(1-\lambda)(3-\lambda)}{4}\psi_t^{-(1+\lambda)/2}r_t|\zeta_t|D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt \\
&\qquad + \frac{1-\lambda}{2}\big[1-\lambda-2+2\lambda\big]\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]L\psi_t dt \\
&\qquad + (1-\lambda)\psi_t^{(1-\lambda)/2}\big[\mathrm{Re}\big(D_z L\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D_{z,\bar{z}}^2\psi_t]\big)\big]dt.
\end{aligned}
$$

In a shorter way,

$$
\begin{aligned}
& d\big[\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]\big] \\
&\quad = \frac{1}{2}\psi_t^{-\lambda/2}\big[-\psi_t^{-1}\mathrm{Re}[D_z\psi_t\zeta_t]+r_t|\zeta_t|\big]\big[D_z\psi_t\sigma_t dB_t+D_{\bar{z}}\psi_t\bar{\sigma}_t d\bar{B}_t\big] \\
&\qquad + \frac{3-\lambda}{4}(\lambda-3)\psi_t^{-(3+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]D_z\psi_t a_t D_{\bar{z}}^*\psi_t dt
\end{aligned}
$$

$$+\frac{3-\lambda}{4}\psi_t^{-(1+\lambda)/2}r_t|\zeta_t|D_z\psi_t a_t D_{\bar z}^*\psi_t dt + \psi_t^{-\lambda/2}\Xi_t D_z\psi_t a_t D_{\bar z}^*\psi_t dt$$
$$+\frac{1}{2}(\lambda-1)\psi_t^{-(1+\lambda)/2}\mathrm{Re}[D_z\psi_t\zeta_t]L\psi_t dt$$
$$+\psi_t^{(1-\lambda)/2}\big[\mathrm{Re}\big(D_z L\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D^2_{z,\bar z}\psi_t]\big)\big]dt.$$

Finally, taking the square, we obtain

$$\begin{aligned}
&d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big]\\
&\quad=\bigg\{\frac{1}{2}\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]+\psi_t^{-\lambda}r_t|\zeta_t|^2+\psi_t^{-(1+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]r_t|\zeta_t|\\
&\qquad+\frac{(3-\lambda)}{2}(\lambda-3)\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]+2\psi_t^{-(1/2+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\Xi_t\bigg\}\\
&\qquad\times D_z\psi_t a_t D_{\bar z}^*\psi_t dt\\
&\qquad+(\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt\\
&\qquad+2\psi_t^{-\lambda}\mathrm{Re}[D_z\psi_t\zeta_t]\big[\mathrm{Re}\big(D_zL\psi_t\zeta_t\big)+\mathrm{Re}\big(\mathrm{Trace}[Q_t\zeta_t a_t D^2_{z,\bar z}\psi_t]\big)\big]dt+dm_t.
\end{aligned}$$

In abbreviated notations, we deduce

$$\begin{aligned}
&d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big]\\
&\quad=\frac{1+(3-\lambda)(\lambda-3)}{2}\psi_t^{-(2+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\mathcal{E}_t dt\\
&\qquad+2\psi_t^{-(1/2+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\Xi_t\mathcal{E}_t dt\\
&\qquad+\psi_t^{-(1+\lambda)}\mathrm{Re}[D_z\psi_t\zeta_t]\mathcal{E}_t|\zeta_t|r_t dt\\
&\qquad+(\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt+N^2\psi_t^{-\lambda}|\zeta_t|^2 r_t dt+dm_t.
\end{aligned}$$

Recall now from (4.143) that $\Xi_t=(-2\lambda+\lambda^2+2)\psi_t^{-3/2}\mathrm{Re}[D_z\psi_t\zeta_t]$. Then, applying Young's inequality to the second term in the above RHS,

$$\begin{aligned}
&d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big]\\
&\quad\le\Big(-\frac{1}{2}\lambda+\frac{3}{2}\lambda^2\Big)\psi_t^{-(2+\lambda)}\mathcal{E}_t\mathrm{Re}^2[D_z\psi_t\zeta_t]dt\\
&\qquad+C(\lambda^{-1}\mathcal{E}_t+N^2)\psi_t^{-\lambda}|\zeta_t|^2dt+(\lambda-1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt+dm_t.
\end{aligned}\tag{4.150}$$

Choose now $\epsilon \le 1$ and $\lambda \le \epsilon$ small enough such that $-\lambda/2 + 3\lambda^2/2 < 0$ and $N = K = \epsilon^{-1/4}$. Then, (4.147) writes for $\psi_t \le \epsilon$

$$\begin{aligned} d\big[\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2\big] \\ &\le \lambda^2 \psi_t^{-(1+\lambda)} \mathrm{Re}^2[D_z\psi_t\zeta_t]dt + C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + \big[3\epsilon^{1/2} - \epsilon^{-1/4}\big]\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt + dm_t. \end{aligned} \tag{4.151}$$

In the same way, (4.150) has the form

$$\begin{aligned} d\big[\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]\big] &\le C(\lambda^{-1}\mathcal{E}_t + \epsilon^{-1/2})\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + (\lambda - 1)\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]L\psi_t dt + dm_t. \end{aligned} \tag{4.152}$$

Consider now the *modified derivative quantity*

$$\bar{\Gamma}_t = \exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 + 2\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t].$$

From (4.151) and (4.152), we obtain

$$\begin{aligned} d\bar{\Gamma}_t &\le C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + \big(C\epsilon^{1/4} - \epsilon^{-1/4}\big)\exp(-K\psi_t)\psi_t^{-\lambda}\mathcal{E}_t|\zeta_t|^2 dt \\ &\quad + \big[2\lambda(\lambda-1)\epsilon^{1/4}L\psi_t + \lambda^2\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t. \end{aligned}$$

For $C\epsilon^{1/4} - \epsilon^{-1/4} < 0$, we deduce

$$\begin{aligned} d\bar{\Gamma}_t &\le C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + \big[\lambda^2(2L\psi_t^0 + 1) - 2\lambda\epsilon^{1/4}L\psi_t\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t. \end{aligned}$$

Since $L\psi_t^0 \le -1$, we finally deduce

$$\begin{aligned} d\bar{\Gamma}_t &\le C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + \big[\lambda^2 L\psi_t^0 - 2\lambda\epsilon^{1/4}L\psi_t\big]\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t \\ &\le C\epsilon^{1/4}\exp(-K\psi_t)\psi_t^{-\lambda}|\zeta_t|^2 dt \\ &\quad + 2\big[(\lambda/2 - 1)L\psi_t\big]\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi_t\zeta_t]dt + dm_t. \end{aligned}$$

Following (4.133) and (4.134), we deduce that

$$d\left[\exp\left(\int_0^t (1-\lambda/2)L\psi_r dr\right)\bar{\Gamma}_t\right] \le dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}, \tag{4.153}$$

for ϵ small enough and $\lambda \le \epsilon$.

We deduce

Proposition 4.8.11 *Let ψ be a plurisuperharmonic function describing the domain $\mathcal{D}$ as in (**A**). Then, there exists a positive real $\epsilon_2 > 0$ such that for any $0 < \epsilon' < \epsilon < \epsilon_2$ and $0 < \lambda < \epsilon$, for $N = K = \epsilon^{-1/4}$, $\psi = N\psi^0$ (with ψ^0 as in the statement of Proposition 4.8.9) and any stopping time $\mathfrak{s}$ at which $\psi(Z^s_{\mathfrak{s}}) \in [\epsilon', \epsilon]$, the* derivative quantity *obtained by perturbing the control parameter as in (4.141), the time speed as in (4.142) and the measure as in (4.143):*

$$\bar{\Gamma}^{(2)}_t = \exp(-K\psi_t)\psi^{-\lambda}(Z^s_t)|\zeta_t|^2 + 2\lambda\epsilon^{1/4}\psi_t^{-(1+\lambda)}\mathrm{Re}^2[D_z\psi(Z^s_t)\zeta_t], \quad t \ge \mathfrak{s},$$

satisfies up to time $\mathfrak{t} = \inf\{t \ge \mathfrak{s} : \psi(Z^s_t) \notin]\epsilon', \epsilon[\}$ (provided that $(\hat{Z}^{s+\varepsilon}_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$ is well differentiable w.r.t. ε)

$$\mathbb{E}\left[\exp\left(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\right)\bar{\Gamma}^{(2)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\right]$$
$$\le \exp\left(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\right)\bar{\Gamma}^{(2)}_{\mathfrak{s}}, \quad t \ge \mathfrak{s},$$

with $\delta = \lambda/2$.

4.8.8 Global Derivative Quantity

The reader might understand the problem we are facing right now: above, we have defined three different *derivative quantities* according to the position of the underlying representation process in the domain $\mathcal{D}$. Surely, we must gather into a single one the three different parts to control the dynamics on the whole space.

Actually, the strategy is not so complicated. In what follows, we are given $0 < \epsilon < \min(\epsilon_1, \epsilon_2, \epsilon_3)$ in the statements of Propositions 4.8.9–4.8.11 and we choose $\psi = \epsilon^{-1/4}\psi^0$ in each statement and $\lambda = \epsilon^2$ in the statement of Proposition 4.8.11. Then, the three different *derivative quantities* have the forms

$$\bar{\Gamma}^{(1)}_t = \exp(-\epsilon^{-1/4}\psi_t)|\zeta_t|^2 + \psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big],$$

$$\bar{\Gamma}_t^{(2)} = \exp(-\epsilon^{-1/4}\psi_t)\psi_t^{-\epsilon^2}|\zeta_t|^2 + 2\epsilon^{9/4}\psi_t^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big],$$
$$\bar{\Gamma}_t^{(3)} = (R^2 - |Z_t|^2)\psi_t^{-1}|\zeta_t|^2. \tag{4.154}$$

At this stage of the proof, the definitions of $\bar{\Gamma}^{(1)}$, $\bar{\Gamma}^{(2)}$ and $\bar{\Gamma}^{(3)}$ are purely formal since the perturbed process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ has not been defined in a global way yet. Obviously, $(Z_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$ and $(D_z\psi_t)_{t\geq 0}$ will be understood as $(Z_t^s)_{t\geq 0}$, solution of (4.92), $(\psi(Z_t^s))_{t\geq 0}$, $([d/d\varepsilon][\hat{Z}_t^{s+\varepsilon}])_{t\geq 0}$ and $(D_z\psi(Z_t^s))_{t\geq 0}$.

For the moment, we claim

Proposition 4.8.12 *Let $(Z_t)_{t\geq 0}$ be a process with values in $\mathcal{D}$ and $(\zeta_t)_{t\geq 0}$ be another process with values in $\mathbb{C}^d$. Setting $\psi_t = \psi(Z_t)$ and $D_z\psi_t = D_z\psi(Z_t)$, $t \geq 0$, consider $(\bar{\Gamma}_t^{(1)})_{t\geq 0}$, $(\bar{\Gamma}_t^{(2)})_{t\geq 0}$ and $(\bar{\Gamma}_t^{(3)})_{t\geq 0}$ as in (4.154).*

Then, there exists a real $0 < \epsilon_0 < \min(\epsilon_1, \epsilon_2, \epsilon_3)$, depending on Assumption ($\mathbf{A}$) only, such that for $\epsilon < \epsilon_0$, we can find three reals $\epsilon_4 < \epsilon/4$ and $\mu_2, \mu_3 > 0$, depending on ϵ and ($\mathbf{A}$) only, such that

$$\psi_t = \epsilon \Rightarrow \mu_2\bar{\Gamma}_t^{(2)} \geq \mu_3\bar{\Gamma}_t^{(3)}$$
$$\psi_t = \epsilon/2 \Rightarrow \bar{\Gamma}_t^{(1)} \geq \mu_2\bar{\Gamma}_t^{(2)} \left(+(1-2\epsilon^{9/4})\psi_t^{-1}\mathrm{Re}^2\big[D_z\psi_t\zeta_t\big]\right)$$
$$\psi_t = \epsilon/4 \Rightarrow \mu_3\bar{\Gamma}_t^{(3)} \geq \mu_2\bar{\Gamma}_t^{(2)}$$
$$\psi_t = \epsilon_4 \Rightarrow \mu_2\bar{\Gamma}_t^{(2)} \geq \bar{\Gamma}_t^{(1)} \left(+ \left[\left(\frac{\epsilon}{2\epsilon_4}\right)^{\epsilon^2} - 1\right]|\zeta_t|^2\right).$$

Above, additional terms in parentheses are positive for ϵ_0 small enough. They are useless in the whole Sect. 4.8. They will be useful in Sect. 4.9.

Proposition 4.8.12 may be understood through Fig. 4.1 below. Each drawn curve stands for a possible graph of one of the three *derivative quantities* in Proposition 4.8.12. The boundary points of each curve (except the ones in 0 and ϵ) are bounded from below by the current point of another curve.

Proof. When $\psi_t = \epsilon/2$, it is clear that

$$\left(\frac{\epsilon}{2}\right)^{\epsilon^2} \bar{\Gamma}_t^{(2)} \leq \bar{\Gamma}_t^{(1)},$$

provided $2\epsilon^{9/4} \leq 1$ (which is obviously true for ϵ small enough).

If $2(\epsilon/2)^{\epsilon^2}\epsilon^{9/4}\psi_t^{-\epsilon^2} = 1$ (i.e. $\psi_t = \epsilon_4$, with ϵ_4 much more smaller than $\epsilon/2$), then

$$\left(\frac{\epsilon}{2}\right)^{\epsilon^2} \bar{\Gamma}_t^{(2)} \geq \bar{\Gamma}_t^{(1)}.$$

We thus choose $\mu_2 = (\epsilon/2)^{\epsilon^2}$.

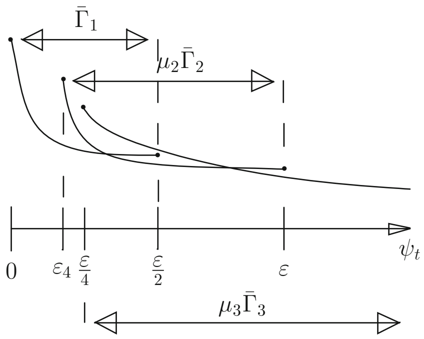

Fig. 4.1 Representation of the *derivative quantities*

When $\psi_t = \epsilon$,

$$\epsilon^{1-\epsilon^2} R^{-2} \exp(-\epsilon^{3/4}) \bar\Gamma_t^{(3)} \le \bar\Gamma_t^{(2)}. \tag{4.155}$$

When $\psi_t = \vartheta\epsilon$,

$$\begin{aligned}\bar\Gamma_t^{(2)} &\le (\vartheta\epsilon)^{1-\epsilon^2} \psi_t^{-1} |\zeta_t|^2 + 2\epsilon^{9/4-\epsilon^2} \vartheta^{-\epsilon^2} \|D_z\psi\|_\infty^2 \psi_t^{-1} |\zeta_t|^2 \\ &\le (\vartheta\epsilon)^{1-\epsilon^2} \psi_t^{-1} |\zeta_t|^2 + 2\epsilon^{7/4-\epsilon^2} \vartheta^{-\epsilon^2} \|D_z\psi^0\|_\infty^2 \psi_t^{-1} |\zeta_t|^2,\end{aligned}$$

since $\psi = \epsilon^{-1/4}\psi^0$.

Since $R^2 \ge 2\sup_{z\in\mathcal{D}}[|z|^2]$, we have $R^2 - \sup_{z\in\mathcal{D}}[|z|^2] \ge R^2/2$ so that $\bar\Gamma_t^{(3)} \ge (R^2/2)\psi_t^{-1}|\zeta_t|^2$. We deduce

$$\bar\Gamma_t^{(2)} \le 2R^{-2}\big[(\vartheta\epsilon)^{1-\epsilon^2} + 2\epsilon^{7/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\bar\Gamma_t^{(3)}.$$

Finally,

$$\begin{aligned}\bar\Gamma_t^{(2)} &\le 2R^{-2}\big[(\vartheta\epsilon)^{1-\epsilon^2} + 2\epsilon^{7/4-\epsilon^2}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\bar\Gamma_t^{(3)} \\ &\le 2\exp(\epsilon^{3/4})\big[\vartheta^{1-\epsilon^2} + 2\epsilon^{3/4}\vartheta^{-\epsilon^2}\|D_z\psi^0\|_\infty^2\big]\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar\Gamma_t^{(3)}.\end{aligned}$$

Choose $\vartheta = 1/4$. Then,

$$\bar\Gamma_t^{(2)} \le 2\exp(\epsilon^{3/4})\big[4^{-1+\epsilon^2} + 2\cdot 4^{\epsilon^2}\epsilon^{3/4}\|D_z\psi^0\|_\infty^2\big]\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar\Gamma_t^{(3)}.$$

Then, for ϵ small enough,

$$\bar\Gamma_t^{(2)} \le \epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})\bar\Gamma_t^{(3)}.$$

We finally choose $\mu_3 = [\epsilon^{1-\epsilon^2}R^{-2}\exp(-\epsilon^{3/4})]\mu_2$, so that $\mu_2\bar\Gamma_t^{(2)} \le \mu_3\bar\Gamma_t^{(3)}$ when $\psi_t = \epsilon/4$. By (4.155), $\mu_3\bar\Gamma_t^{(3)} \le \mu_2\bar\Gamma_t^{(2)}$ when $\psi_t = \epsilon$. □

Proposition 4.8.13 *Let $\epsilon \in (0, \epsilon_0)$ and ϵ_4 be as in Proposition 4.8.12, define the following sets:*

$$U_0 = \{z \in \mathcal{D} : \psi(z) \le \epsilon_4\}$$
$$U_1 = \{z \in \mathcal{D} : \epsilon_4 \le \psi(z) \le \epsilon/2\}$$
$$U_2 = \{z \in \mathcal{D} : \epsilon/4 \le \psi(z) \le \epsilon\}$$
$$U_3 = \{z \in \mathcal{D} : \psi(z) \ge \epsilon\}.$$

Let γ be a smooth path from $[-1, 1]$ into U_3, s be some fixed point in $(-1, 1)$ and $(Z_t^s)_{t\ge 0}$ be the solution of (4.84) with $\gamma(s)$ as initial condition.

Define as well $(\tau_n)_{n\ge 1}$ as the sequence of exit times of the process $(\psi(Z_t^s))_{t\ge 0}$ from the sets $[\epsilon/4, +\infty)$, $[\epsilon_4, \epsilon]$ and $[0, \epsilon/2]$, i.e.

$$\tau_1 := \inf\{t \ge 0 : \psi_t = \psi(Z_t^s) \le \epsilon/4\},$$
$$\tau_2 := \inf\{t > \tau_1 : \psi_t \notin]\epsilon_4, \epsilon[\},$$
$$\tau_3 := \inf\{t > \tau_2 : \psi_t \notin [0, \epsilon/2[\} \quad \text{if } \psi_{\tau_2} = \epsilon_4,$$
$$\tau_3 := \inf\{t > \tau_2 : \psi_t \le \epsilon/4\} \quad \text{if } \psi_{\tau_2} = \epsilon,$$
$$\ldots$$

(If $\tau_n = +\infty$, then $\tau_{n+1} = +\infty$ as well, $n \ge 1$.)

For initial conditions of the form $\gamma(s + \varepsilon)$, consider the perturbed version $(\hat{Z}_t^{s+\varepsilon})_{0\le t\le\tau_1}$ as in Proposition 4.8.10 ($\epsilon/4$ playing the role of ϵ) up to time τ_1. If $\tau_1 < +\infty$, extend the perturbed process as $(\hat{Z}_t^{s+\varepsilon})_{\tau_1\le t\le\tau_2}$ according to the perturbation of Proposition 4.8.11 ($\epsilon/2$ playing the role of ϵ, ϵ' being equal to ϵ_4 and λ to ϵ^2) up to time τ_2. And so on... according to Fig. 4.2 below.

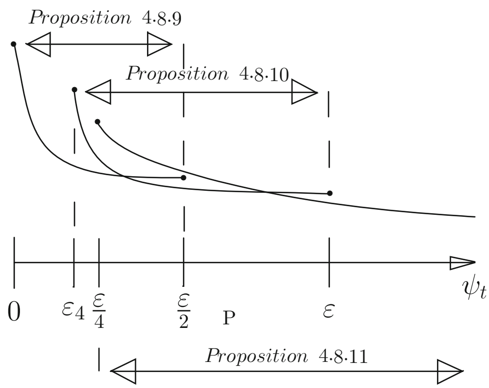

Fig. 4.2 Choice of the perturbations

Assume that the whole process $(\hat{Z}_t^{s+\varepsilon})_{t\geq 0}$ *is differentiable in the mean w.r.t.* ε *and that the derivative process* $(\zeta_t = (d/d\varepsilon)[\hat{Z}_t^{s+\varepsilon}]_{|\varepsilon=0})_{t\geq 0}$ *satisfies the SDE obtained by differentiation of the coefficients of the perturbations as in Theorem 4.7.2. Then, from time* 0 *to time* τ_1*, consider as* derivative quantity *the process* $(\mu_3\bar{\Gamma}_t^{(3)})_{0\leq t\leq \tau_1}$ *defined in Proposition 4.8.10. From time* τ_1 *(if finite) to time* τ_2*, consider as* derivative quantity *the process* $(\mu_2\bar{\Gamma}_t^{(2)})_{\tau_1<t\leq \tau_2}$ *defined in Proposition 4.8.11. And so on... according to Fig. 4.1. Denote by* $(\bar{\Gamma}_t)_{t\geq 0}$ *the resulting global* derivative quantity. *(So that the process is left-continuous.)*

Then, we can find $\alpha \in (0,1)$*, depending on (***A***) and* ϵ *only, such that*

$$\mathbb{E}\left[\bar{\Gamma}_t \exp\left(\int_0^t \alpha L\psi(Z_r^s)dr\right)\right] \leq \bar{\Gamma}_0, \quad t\geq 0.$$

Moreover, there exists a constant $C \geq 0$*, depending on (***A***) and* ϵ *only, such that*

$$\mathbb{E}\left[|\zeta_t|^2 \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq C\bar{\Gamma}_0, \quad t\geq 0. \tag{4.156}$$

Proof. By Proposition 4.8.10, we can find some exponent $\alpha < 1$ (depending on (**A**) and ϵ only) such that

$$d\left[\bar{\Gamma}_t^{(3)} \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq dm_t, \quad 0\leq t\leq \tau_1, \tag{4.157}$$

$(m_t)_{t\geq 0}$ standing for a generic martingale term (whose value may vary from line to line).

Consider the case when $\tau_1 < +\infty$. By Proposition 4.8.11, we can modify α so that

$$d\left[\bar{\Gamma}_t^{(2)} \exp\left(\int_0^t \alpha L\psi_r dr\right)\right] \leq dm_t, \quad \tau_1\leq t\leq \tau_2 \tag{4.158}$$

We then gather both *derivative quantities* $(\mu_3\bar{\Gamma}_t^{(3)})_{0\leq t\leq\tau_1}$ and $(\mu_2\bar{\Gamma}_t^{(2)})_{\tau_1\leq t\leq\tau_2}$ into a single one, denoted by $(\bar{\Gamma}_t)_{0\leq t\leq\tau_2}$. Obviously, it may be discontinuous at time τ_1: by convention, we assume it to be left-continuous so that $\bar{\Gamma}_{\tau_1} = \mu_3\bar{\Gamma}_{\tau_1}^{(3)}$. Then, we can rewrite (4.157) and (4.158) as

$$\mathbb{E}\left[\mu_3\bar{\Gamma}_{\tau_1}^{(3)} \exp\left(\int_0^{\tau_1} \alpha L\psi_r dr\right)\mathbf{1}_{\{\tau_1<+\infty\}}\right] \leq \mu_3\bar{\Gamma}_0^{(3)} = \bar{\Gamma}_0$$

$$\mathbb{E}\Big[\mu_2 \bar{\Gamma}^{(2)}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) | \mathcal{F}_{\tau_1}\Big] \mathbf{1}_{\{\tau_1 < +\infty\}}$$
$$\leq \mu_2 \bar{\Gamma}^{(2)}_{t\wedge\tau_1} \exp\Big(\int_0^{t\wedge\tau_1} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 < +\infty\}}. \quad (4.159)$$

(The second inequality above is obviously true if $t \leq \tau_1$: in that case, everything is known at time $t \wedge \tau_2$ and the conditional expectation is useless. Otherwise, i.e. if $t > \tau_1$, the second inequality is a consequence of (4.158). Add also that $\{\tau_1 < +\infty\} \in \mathcal{F}_{\tau_1}$: at time τ_1, τ_1 is known to be finite or not.)

We now apply Proposition 4.8.12. If $\tau_1 < +\infty$ and $t > \tau_1$, we know that $\psi_{t\wedge\tau_1} = \psi_{\tau_1} = \epsilon/4$ so that $\mu_2 \bar{\Gamma}^{(2)}_{\tau_1} \leq \mu_3 \bar{\Gamma}^{(3)}_{\tau_1}$. Then, for $t > \tau_1$ (and $\tau_1 < +\infty$), (4.159) yields

$$\mathbb{E}\Big[\mu_2 \bar{\Gamma}^{(2)}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) | \mathcal{F}_{\tau_1}\Big] \leq \mu_3 \bar{\Gamma}^{(3)}_{\tau_1} \exp\Big(\int_0^{\tau_1} \alpha L \psi_r dr\Big),$$

i.e.

$$\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) | \mathcal{F}_{\tau_1}\Big] \leq \bar{\Gamma}_{\tau_1} \exp\Big(\int_0^{\tau_1} \alpha L \psi_r dr\Big).$$

Finally, for any $t \geq 0$,

$$\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big)\Big]$$
$$= \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 < t\}}\Big]$$
$$+ \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 \geq t\}}\Big]$$
$$= \mathbb{E}\Big[\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 < t\}}\Big] | \mathcal{F}_{\tau_1}\Big]$$
$$+ \mathbb{E}\Big[\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 \geq t\}}\Big] | \mathcal{F}_{\tau_1}\Big].$$

Therefore,

$$\mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_2} \exp\Big(\int_0^{t\wedge\tau_2} \alpha L \psi_r dr\Big)\Big]$$
$$\leq \mathbb{E}\Big[\bar{\Gamma}_{\tau_1} \exp\Big(\int_0^{\tau_1} \alpha L \psi_r dr\Big) \mathbf{1}_{\{\tau_1 < t\}}\Big]$$

$$+ \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_1} \exp\Big(\int_0^{t\wedge\tau_1} \alpha L\psi_r dr\Big) \mathbf{1}_{\{\tau_1 \geq t\}}\Big]$$

$$= \mathbb{E}\Big[\bar{\Gamma}_{t\wedge\tau_1} \exp\Big(\int_0^{t\wedge\tau_1} \alpha L\psi_r dr\Big)\Big] \leq \bar{\Gamma}_0.$$

In other words, we are able to gather the two inequalities in (4.159) into a single one over the whole interval $[0, \tau_2)$. By induction, we can process further: if $\tau_2 < +\infty$ and $\psi_{\tau_2} = \epsilon_4$, we make use of Proposition 4.8.9 up to $\tau_3 = \inf\{t > \tau_2 : \psi_t \geq \epsilon/2\}$; if $\tau_2 < +\infty$ and $\psi_{\tau_2} = \epsilon$, we make use of Proposition 4.8.10 up to $\tau_3 = \inf\{t > \tau_2 : \psi_t \leq \epsilon/4\}$; we then extend $\bar{\Gamma}_t$ to $[0, \tau_3)$ by using Proposition 4.8.12 (at time τ_2, $\mu_2\bar{\Gamma}^{(2)}_{\tau_2}$ is greater than the two other *derivative quantities*); and so on... We then extend the *derivative quantity* to the whole $[0, +\infty)$ in such a way that

$$\mathbb{E}\Big[\bar{\Gamma}_t \exp\Big(\int_0^t \alpha L\psi_r dr\Big)\Big] \leq \bar{\Gamma}_0.$$

Of course, the value of $\bar{\Gamma}_t$ is given by one of the three original *derivative quantities* $\bar{\Gamma}^{(1)}_t$, $\mu_2\bar{\Gamma}^{(2)}_t$ and $\mu_3\bar{\Gamma}^{(3)}_t$ according to the position of Z^s_t in $\mathcal{D}$. (See Fig. 4.1.) What is important is that, in any case, $\bar{\Gamma}_t \geq c|\zeta_t|^2$, for some positive c depending on (**A**) and ϵ only. Equation (4.156) follows. □

4.8.9 Conclusion

It now remains to gather all the localized value functions into a single one:

Proposition 4.8.14 *Keep the assumption and notation of Proposition 4.8.13. (In particular, s stands below for some fixed real in* $(-1, 1)$*.) Given* $S > 0$ *and* ε *with* $s + \varepsilon \in (-1, 1)$*, define the globally perturbed analog of* V *in Proposition 4.6.9*

$$\hat{V}^\sigma_S(s+\varepsilon) = \mathbb{E}\int_0^{+\infty} \Big[\exp\Big(-\int_0^t 2\mathrm{Re}\big[\langle \bar{G}(Z^s_r, \hat{Z}^{s+\varepsilon}_r - Z^s_r), dB_r\rangle\big] - \int_0^t |G|^2(Z^s_r, \hat{Z}^{s+\varepsilon}_r - Z^s_r)dr\Big)$$
$$\times \exp\Big(\int_0^t |\tau^\varepsilon_r|^2 \mathrm{Trace}[\exp(p^\varepsilon_r)a_r \exp(-p^\varepsilon_r) D^2_{z,\bar{z}}\psi(\hat{Z}^{s+\varepsilon}_r)]dr\Big)$$
$$\times F\big(\det(a_t), \exp(p^\varepsilon_t)a_t \exp(-p^\varepsilon_t), \hat{Z}^{s+\varepsilon}_t\big)\phi\Big(\frac{\mathfrak{T}^\varepsilon_t}{S}\Big)\Big]|\tau^\varepsilon_t|^2 dt, \tag{4.160}$$

where the quantities $(p_t^\varepsilon = P(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, $(\tau_t^\varepsilon = T(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ *and* $(G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ *stand for the different possible perturbations used in Proposition 4.8.13. Precisely,* p^ε *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 4.8.2 applies,* τ^ε *is set equal to* 1 *outside the intervals on which the perturbation of Proposition 4.8.4 applies and* $(G(Z_t^s, \hat{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 4.8.7 applies. Moreover,* $\dot{\mathfrak{T}}_t^\varepsilon = |\tau_t^\varepsilon|^2$, $t \geq 0$.

Then, at point s, $\sup_\sigma \hat{V}_S^\sigma(s) = V_S(\gamma(s))$ *exactly, where* $V_S(\gamma(s))$ *stands for the finite-horizon version of* $V(\gamma(s))$ *in Proposition 4.6.9, i.e.*

$$V_S(z) = \sup_\sigma V_S^\sigma(z), \quad z \in \mathcal{D},$$

where

$$V_S^\sigma(z) = \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(Z_r^{\sigma,z})\big]dr\bigg) \times F(\det(a_t), a_t, Z_t^{\sigma,z})\phi\bigg(\frac{t}{S}\bigg)dt\bigg].$$

Moreover, for any control $(\sigma_t)_{t\geq 0}$, $\hat{V}_S^\sigma(s+\varepsilon) \leq V_S(\gamma(s+\varepsilon))$.

Sketch of the Proof. The equality $\sup_\sigma \hat{V}_S^\sigma(s) = V_S(\gamma(s))$ is easily seen.

The proof of the inequality $\sup_\sigma \hat{V}_S^\sigma(s+\varepsilon) \leq V_S(\gamma(s+\varepsilon))$ is a bit more challenging. We won't perform it in a complete way. We refer the reader to the original articles by Krylov [Kry89b, Kry90]: the argument is explained therein in a very detailed way. However the idea is quite clear and consists in coupling the arguments given in Sects. 4.8.2–4.8.4: modification of the control, of the time speed and of the measure. □

Here is the final step:

Proposition 4.8.15 *Keep the assumption and notation of Propositions 4.8.13 and 4.8.14. Assume in addition that, for any* $S > 0$ *and* $s \in [-1, 1]$,

$$\limsup_{\varepsilon\to 0} \sup_\sigma \sup_{|\varepsilon'|<|\varepsilon|} \bigg[\bigg|\frac{\partial}{\partial\varepsilon'}\big[\hat{V}_S^\sigma(s+\varepsilon')\big]\bigg|\bigg] = \bigg|\frac{\partial}{\partial\varepsilon'}\big[\hat{V}_S^\sigma(s+\varepsilon')\big]_{|\varepsilon'=0}\bigg|. \quad (4.161)$$

Assume also that, for every compact interval $I \subset (-1, 1)$, *for* ε *small enough, the quantity* $\sup_\sigma \sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\hat{V}_S^\sigma(s+\varepsilon')]|]$ *is uniformly bounded w.r.t.* $s \in I$. *(Pay attention that the definition of the function* $\hat{V}_S^\sigma$ *depends on* s *itself.)*

Then, there exists a constant $C > 0$, *depending on* (**A**) *and the distance from* $\gamma([-1, 1])$ *to* $\partial\mathcal{D}$ *only, such that, for any* $S > 0$, *the function* $s \in (-1, 1) \mapsto V_S(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ *is non-decreasing and the function* $s \in (-1, 1) \mapsto V_S(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ *is non-increasing.*

Proof. Without loss of generality, we can assume ϵ to be small enough so that $\gamma([-1,1]) \subset U_3$, with U_3 as in Proposition 4.8.13. Following the proofs of Propositions 4.8.2, 4.8.4 and 4.8.7, we then claim that (C being as in the statement)

$$\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right| \le C\mathbb{E}\left[\int_0^{+\infty}\exp\left(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\right)\right.$$
$$\left.\times\left[|\zeta_t| + \int_0^t (1+r^{-1/2})|\zeta_r|dr\right]dt\right]. \quad (4.162)$$

Recall that $\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)] \le -N = \epsilon^{-1/4}$. By (4.156), we deduce

$$\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right|$$
$$\le C\mathbb{E}\left[\int_0^{+\infty}\exp\big(-N(1-\alpha/2)t\big)\left[|\hat{\zeta}_t|\exp\left(\int_0^t(\alpha/2)L\psi_r dr\right)\right.\right.$$
$$\left.\left.+\int_0^t(1+r^{-1/2})|\hat{\zeta}_r|\exp\left(\int_0^r(\alpha/2)L\psi_u du\right)dr\right]dt\right]$$
$$= C\int_0^{+\infty}\exp\big(-N(1-\alpha/2)t\big)\left\{\mathbb{E}\left[|\hat{\zeta}_t|\exp\left(\int_0^t(\alpha/2)L\psi_r dr\right)\right]\right.$$
$$\left.+\int_0^t(1+r^{-1/2})\mathbb{E}\left[|\hat{\zeta}_r|\exp\left(\int_0^r(\alpha/2)L\psi_u du\right)\right]dr\right\}dt$$
$$\le C\bar{\Gamma}_0^{1/2}\int_0^{+\infty}\exp\big(-N(1-\alpha/2)t\big)(1+t)dt \le C\bar{\Gamma}_0^{1/2},$$

the last line following from Cauchy-Schwarz inequality.

Since $\bar{\Gamma}_0 = \bar{\Gamma}_0^{(3)}$, we deduce that

$$\left|\frac{d}{d\varepsilon}\left[\hat{V}_S^\sigma(s+\varepsilon)\right]_{|\varepsilon=0}\right| \le CR|\gamma(s)|^{-1/2}|\gamma'(s)|. \quad (4.163)$$

Unfortunately, the above estimate is a bit weaker than (4.88) and is not sufficient to recover

$$\liminf_{\varepsilon\to 0,\varepsilon\neq 0}\frac{V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s))}{|\varepsilon|} \ge -CR|\gamma(s)|^{-1/2}|\gamma'(s)|, \quad (4.164)$$

as in (4.89).

To recover (4.89), we take benefit of (4.161). Indeed, by the mean value Theorem, we can generalize (4.163) and write (for a possibly new value of the constant C)

$$
\begin{aligned}
V_S(\gamma(s+\varepsilon)) - V_S(\gamma(s)) &\geq \inf_{\sigma} \big[\hat{V}_S^{\sigma}(s+\varepsilon) - \hat{V}_S^{\sigma}(s) \big] \\
&\geq -C|\varepsilon| \sup_{|\varepsilon'|<|\varepsilon|} \sup_{\sigma} \bigg[\bigg| \frac{d}{d\varepsilon'} \big[\hat{V}_S(s+\varepsilon') \big] \bigg| \bigg]. \qquad (4.165)
\end{aligned}
$$

By (4.161) and (4.163), we deduce (4.164). Modifying the constant C in (4.164) (have in mind that C may depend on ϵ but is independent of S and s), we deduce that

$$
\begin{aligned}
\liminf_{\varepsilon\to 0, \varepsilon>0} \varepsilon^{-1} \bigg[& V_S(\gamma(s+\varepsilon)) + CR \int_0^{s+\varepsilon} |\gamma'(r)| dr \\
& - V_S(\gamma(s)) - CR \int_0^{s} |\gamma'(r)| dr \bigg] \geq 0. \qquad (4.166)
\end{aligned}
$$

Actually, (4.165) says a little bit more. Since $\sup_{|\varepsilon'|<|\varepsilon|} \sup_{\sigma} [|[d/d\varepsilon'](\hat{V}_S(s+\varepsilon'))|]$ is bounded in s in compact subsets of $(-1,1)$ (at least for $|\varepsilon|$ small enough), we deduce that the function $V_S \circ \gamma$ is Lipschitz continuous and thus continuous. (Pay attention that the Lipschitz constant may depend on S at this stage of the proof.) Indeed, the LHS in (4.165) being bounded from below uniformly in s, the points s and $s+\varepsilon$ may be exchanged, so that the bound holds from above as well.

We then deduce from (4.166) that the function $s \in (-1,1) \mapsto V_S(\gamma(s)) + C\int_0^s |\gamma'(r)| dr$ is non-decreasing.

Letting S tend to $+\infty$, we deduce that the function $s \in (-1,1) \mapsto V(\gamma(s)) + C\int_0^s |\gamma'(r)| dr$ is non-decreasing. Similarly (i.e. by changing ε into $-\varepsilon$), we can prove that the function $s \in (-1,1) \mapsto V(\gamma(s)) - C\int_0^s |\gamma'(r)| dr$ is non-increasing.

To complete the proof of Meta-Theorem 4.8.1, it remains to choose γ. For some point z such that $\psi(z) > \epsilon$, we can set $\gamma(s) = z + s\nu$, $s \in [-1,1]$, for some $\nu \in \mathbb{C}^d$ such that the complex closed ball of center z and of radius $|\nu|$ be included in U_3. (See the definition of U_3 in the statement of Proposition 4.8.13.) Then, $V(\gamma(1)) - V(\gamma(0)) + C|\nu| \geq 0$ and $V(\gamma(1)) - V(\gamma(0)) - C|\nu| \leq 0$, i.e. $|V(z+\nu) - V(z)| \leq C|\nu|$, the constant C here depending on ϵ. Going back to the connection between V and the solution to Monge–Ampère in Proposition 4.6.9, we understand that the solution to Monge–Ampère is Lipschitz continuous in every compact subset of $\mathcal{D}$. □

Unfortunately, the argument fails for the second-order derivatives. The reason is quite simple. Indeed, we wish to apply Proposition 4.7.9. Replacing $(\zeta_t)_{t\geq 0}$ by $(\eta_t = [d^2/d\varepsilon^2](\hat{Z}_t^{s+\varepsilon}))_{t\geq 0}$ in the definition of $\bar{\Gamma}_t^{(1)}$, $\bar{\Gamma}_t^{(2)}$ and $\bar{\Gamma}_t^{(3)}$ in (4.154), the problem is to prove that the resulting global second-order *derivative quantity*, denoted by $(\bar{\Gamma}_t(\eta_t))_{t\geq 0}$, satisfies (compare with (4.156))

$$\mathbb{E}\Big[\bar{\Gamma}_t^{1/2}(\eta_t)\exp\Big(\int_0^t \alpha L\psi(Z_r^s)dr\Big)\Big] \leq C\bar{\Gamma}_0^{1/2}, \quad t \geq 0.$$

In some sense, this matches (4.79) in Proposition 4.7.9.

The problem is not to prove $\partial\bar{\Gamma}_t(\eta_t) \leq \alpha' L\psi(Z_r^s)\bar{\Gamma}_t(\eta_t)$, $t \geq 0$. (The notation $(\partial\bar{\Gamma}_t(\eta_t))_{t\geq 0}$ has the same meaning as in Proposition 4.7.9.) Basically, if the inequality is satisfied for ζ_t, it is satisfied for η_t as well: it is sufficient to replace ζ_t by η_t therein. The problem is somewhere else: in Proposition 4.7.9, the derivative quantity is assumed to be driven by a quadratic form equivalent to the Hermitian (Euclidean in the real case) one. Obviously, this is not the case when using $(\bar{\Gamma}_t(\eta_t))_{t\geq 0}$ since $(\bar{\Gamma}_t^{(1)})_{t\geq 0}$ in (4.154), which is the *derivative quantity* we used in the neighborhood of the boundary, has some singular coefficient inside: $(\psi_t^{-1})_{t\geq 0}$.

4.9 Proof of the $\mathcal{C}^{1,1}$-Regularity up to the Boundary

We now complete the proof of Theorem 4.6.1.

In comparison with Sect. 4.8, Krylov's program consists in introducing an alternative representation of the solution of the Monge–Ampère equation in the neighborhood of the boundary and to associate a new *derivative quantity* with it, free of any singularities, so that Proposition 4.7.9 may apply.

4.9.1 Representation Process on a Zero Surface

The trick consists in introducing a parameterized version of (4.57) in the statement of Proposition 4.6.9. In what follows, we thus consider the system (with values in $\mathbb{C}^d \times \mathbb{C}^2$)

$$\begin{aligned} dZ_t &= \sum_{i=1,2} Y_t^i \sigma_t dB_t^i + a_t D_{\bar{z}}\psi^*(Z_t)dt, \\ dY_t^i &= D_{\bar{z}}\psi(Z_t)\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2}Y_t^i \mathrm{Trace}\big[a_t D^2_{z,\bar{z}}\psi(Z_t)\big]dt, \quad t \geq 0, \quad i = 1, 2, \end{aligned} \tag{4.167}$$

where B^1 and B^2 denote two independent complex Brownian motion of dimension d. At that point of the proof, we don't know whether the process $(Z_t)_{t\geq 0}$ stays inside $\mathcal{D}$ or not: since ψ is $\mathcal{C}^4$ in the neighborhood of $\bar{\mathcal{D}}$, we can extend it to the whole $\mathbb{C}^d$ into a $\mathcal{C}^4$ bounded function with bounded derivatives. For such an extension and for a given initial condition (Z_0, Y_0), the above system has locally Lipschitz coefficients and is therefore uniquely solvable on some interval $[0, \tau)$, τ here standing for a stopping time.

In what follows, we set $\Phi(z, y) = \psi(z) - |y|^2$ for $z \in \mathbb{C}^d$ (ψ being extended to the whole space) and $y \in \mathbb{C}^2$. We prove below that, for $Z_0 \in \mathcal{D}$, the solution $(Z_t, Y_t)_{0 \le t < \tau}$ lives in a level set of the function Φ so that it can be extended to the whole $[0, +\infty)$, i.e. $\tau = +\infty$. (Indeed, the level set property says that $(Y_t)_{0 \le t < \tau}$ is bounded by a universal constant.) To do so, we compute for $0 \le t < \tau$:

$$d\psi(Z_t) = \sum_{i=1,2} Y_t^i D_z \psi(Z_t) \sigma_t dB_t^i + \sum_{i=1,2} \bar{Y}_t^i D_{\bar{z}} \psi(Z_t) \bar{\sigma}_t d\bar{B}_t^i \\ + 2 D_z \psi(Z_t) a_t D_{\bar{z}} \psi^*(Z_t) dt + |Y_t|^2 \mathrm{Trace}(a_t D^2_{z,\bar{z}} \psi(Z_t)) dt. \tag{4.168}$$

Above, $|Y_t|^2 = |Y_t^1|^2 + |Y_t^2|^2$. Now, we write for $i \in \{1, 2\}$ and $0 \le t < \tau$:

$$d|Y_t^i|^2 = Y_t^i D_z \psi(Z_t) \sigma_t dB_t^i + \bar{Y}_t^i D_{\bar{z}} \psi(Z_t) \bar{\sigma}_t d\bar{B}_t^i \\ + |Y_t^i|^2 \mathrm{Trace}\big[a_t D^2_{z,\bar{z}} \psi(Z_t)\big] dt + D_z \psi(Z_t) a_t D_{\bar{z}} \psi^*(Z_t) dt. \tag{4.169}$$

As a consequence, we obtain that

$$d\big(\psi(Z_t) - |Y_t|^2\big) = 0, \quad 0 \le t < \tau, \tag{4.170}$$

so that the process $(\psi(Z_t) - |Y_t|^2)_{0 \le t < \tau}$ lives on a level set of the function Φ. Therefore, $(Y_t)_{0 \le t < \tau}$ is bounded by some universal constant, so that (4.167) appears as a Lipschitz system.

It now remains to understand how the dynamics of (Z, Y) are connected with the original ones of Z in (4.57). To this end, we set

$$W_t = \sum_{i=1,2} \int_0^t \left(\frac{Y_s^i}{|Y_s|} \mathbf{1}_{\{|Y_s| > 0\}} + \frac{1}{\sqrt{2}} \mathbf{1}_{\{|Y_s| = 0\}} \right) dB_s^i, \quad t \ge 0. \tag{4.171}$$

Clearly, $(W_t)_{t \ge 0}$ is a martingale with values in $\mathbb{C}^d$. Actually, for any coordinates $1 \le j, k \le d$,

$$d[W_t^j W_t^k] = 0, \quad d[W_t^j \bar{W}_t^k] = \delta_{j,k} dt, \tag{4.172}$$

where $\delta_{j,k}$ stands for the Kronecker symbol. Following Footnote 13, $(W_t)_{t \ge 0}$ is a complex Brownian motion of dimension d. Moreover, (4.171) implies

$$|Y_t| dW_t = \sum_{i=1,2} Y_t^i dB_t^i, \quad t \ge 0. \tag{4.173}$$

Choose now $Z_0 \in \mathcal{D}$ and $Y_0 \in \mathbb{C}^2$ such that $\psi(Z_0) = |Y_0|^2$. By (4.170), $\psi(Z_t) = |Y_t|^2$ for any $t \geq 0$ so that (4.173) has the form

$$\psi^{1/2}(Z_t)dW_t = \sum_{i=1,2} Y_t^i dB_t^i, \quad t \geq 0.$$

In particular, $(Z_t)_{t\geq 0}$ satisfies

$$dZ_t = \psi^{1/2}(Z_t)\sigma_t dW_t + a_t D_{\bar{z}}\psi^*(Z_t)dt, \quad t \geq 0, \tag{4.174}$$

i.e. (4.57). Clearly, (4.174) says that Proposition 4.6.7 applies to $(Z_t)_{t\geq 0}$, that is $(Z_t)_{t\geq 0}$ does not leave $\mathcal{D}$, and that we can use the parameterized version (4.167) of (4.57) in Proposition 4.6.9. (See Footnote 14 as well.) When doing so, the representation formula holds at some point $z \in \mathcal{D}$: it is the initial condition of Z. However, we stress out that the right initial condition of (4.167) is the complete initial condition of the pair (Z, Y): given the starting point of Z, the starting point of Y is chosen in such a way that (Z_0, Y_0) is a zero of Φ.

Here is a possible choice:

Proposition 4.9.1 *Let $\gamma = (\gamma_0, \gamma_1)$ be a smooth path from $[-1, 1]$ into $\mathcal{D} \times \mathbb{C}^2$ such that, for any $s \in [-1, 1]$, $\Phi(\gamma(s)) = 0$, where $\Phi(z, y) = \psi(z) - |y|^2$, $z \in \mathcal{D}$, $y \in \mathbb{C}^2$. Then, for any $s \in [-1, 1]$, the solution $(Z_t^s, Y_t^s)_{t\geq 0}$ to*

$$dZ_t^s = \sum_{i=1,2} (Y_t^s)^i \sigma_t dB_t^i + a_t D_{\bar{z}}\psi^*(Z_t^s)dt,$$

$$d(Y_t^s)^i = D_{\bar{z}}\psi(Z_t^s)\bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2}(Y_t^s)^i \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2\psi(Z_t^s)\big]dt, \quad t \geq 0, \ i = 1, 2,$$

with $(Z_0^s, Y_0^s) = \gamma(s)$ as initial condition, stays in the zero surface of Φ. (Above, $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ stand for two independent complex Brownian motions of dimension d.)

Moreover, the value function V in Proposition 4.6.9 may be represented at point $\gamma(s)$ as the supremum of $V^\sigma(\gamma(s))$ obtained by plugging the above choice for $(Z_t^s)_{t\geq 0}$ into the definition of Proposition 4.6.9.

A possible choice for γ is $\gamma_0(s) = z + s\nu$, $z \in \mathcal{D}$ and $\nu \in \mathbb{C}^d \setminus \{0\}$ (such that $B(z, |\nu|) \subset \mathcal{D}$) and $\gamma_1 = (\gamma_{1,1}, \gamma_{1,2})$ solution of the ODE

$$\dot{\gamma}_{1,1}(s) = \bar{\gamma}_{1,1}^{-1}(s)D_z\psi(\gamma_0(s))\nu, \quad \dot{\gamma}_{1,2}(s) = 0, \quad s \in [-1, 1], \tag{4.175}$$

with $|\gamma_{1,1}(0)|^2 = \psi(z)$ and $\gamma_{1,2}(0) = 0$.

Proof. The first part of the statement has been already proven. Turn now to the ODE (4.175). It is solvable on a short time interval around zero as soon as $\gamma_1(0)$ is non zero. Actually, a simple computation shows that, in the neighborhood of 0,

$$\frac{d\big[|\gamma_{1,1}(s)|^2 - \psi(\gamma_0(s))\big]}{ds} = 2\text{Re}\big[D_z\psi(\gamma_0(s))\nu\big] - 2\text{Re}\big[D_z\psi(\gamma_0(s))\nu\big] = 0,$$

so that $|\gamma_{1,1}(s)|^2 = \psi(\gamma_0(s))$ for s in the neighborhood of 0. As $\psi(\gamma_0(s))$ doesn't vanish for $s \in [-1, 1]$, γ_1 may be defined on the whole $[-1, 1]$ (at least). □

Below, the objective is to compute the derivatives of the pair $(Z_t^s, Y_t^s)_{t\geq 0}$ and to consider a suitable derivative quantity for it. Specifically, we emphasize that the situation is different from the original one in Proposition 4.6.9: here, the coefficients of the SDE of the pair $(Z_t^s, Y_t^s)_{t\geq 0}$ are smooth up to the boundary. (Because of the exponent $1/2$ in ψ, they are not in the original Proposition 4.6.9.)

4.9.2 Example: Estimate on a Ball

To explain how things work, we first focus on the specific case when the domain is a ball, say the ball of center 0 and radius R. In such a case, we may choose $\psi(z) = R^2 - |z|^2$ so that (4.57) has the form

$$dZ_t = \big[R^2 - |Z_t|^2\big]^{1/2}\sigma_t dB_t - a_t Z_t dt, \tag{4.176}$$

with $Z_0 = z \in B(0, R) = \{z' \in \mathbb{C}^d : |z'|^2 < R^2\}$.

We then apply Proposition 4.9.1 with $\Phi(z, y) = \psi(z) - |y|^2 = R^2 - |z|^2 - |y|^2$, $z \in B(0, R)$ and $y \in \mathbb{C}^2$. The parameterized version (4.167) of (4.176) has the form:

$$\begin{aligned} dZ_t &= \sum_{i=1,2} Y_t^i \sigma_t dB_t^i - a_t Z_t dt \\ dY_t^i &= -\langle Z_t, \bar{\sigma}_t d\bar{B}_t^i\rangle - \frac{1}{2} Y_t^i dt, \quad i = 1, 2, \end{aligned} \tag{4.177}$$

where $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are two independent Brownian motions with values in $\mathbb{C}^d$.

We are now in position to complete the analysis on a ball. To do so, we compute the derivatives of the pair (Z, Y): specifically, we initialize the pair at some $\gamma(s)$, s in the neighborhood of zero and for some curve γ on a level set of Φ. (Choose for example γ as in (4.175).) The resulting pair (Z, Y) is denoted by (Z^s, Y^s) as above. The derivative process is denoted by (ζ_t^s, ϱ_t^s). It is understood as ξ^s with the notations of Theorem 4.7.2. Equation (4.177) being linear, Theorem 4.7.2 applies and we obtain:

$$d\zeta_t^s = \sum_{i=1,2} (\varrho_t^s)^i \sigma_t dB_t^i - a_t \zeta_t dt$$

$$d(\varrho_t^s)^i = -\langle \zeta_t, \bar{\sigma}_t d\bar{B}_t^i \rangle - \frac{1}{2}(\varrho_t^s)^i dt, \quad i = 1, 2.$$

Have in mind that $d(|Z_t|^2 + |Y_t|^2) = d(-R^2 + |Z_t|^2 + |Y_t|^2) = d[-\psi(Z_t) + |Y_t|^2] = 0$. Similarly, the pair $(\zeta_t^s, \varrho_t^s)_{t \geq 0}$ satisfies

$$d\big(|\zeta_t^s|^2 + |\varrho_t^s|^2\big) = 0.$$

In comparison with Definition 4.7.6, this means that the *derivative quantity* is zero, i.e.

$$d\Gamma_t^s = 0, \quad t \geq 0,$$

with $\Gamma_t^s = |\xi_t^s|^2 = |\zeta_t^s|^2 + |\varrho_t^s|^2$. In particular,

$$\exp\bigg(-\int_0^t c_s ds\bigg)|\xi_t^s|^2 = \exp(-t)|\xi_0^s|^2,$$

where $c_t = -\mathrm{Trace}[a_t D^2_{z,\bar{z}}\psi(Z_t)] = 1$.

We then recover the conclusion of Proposition 4.8.13 but the constant C in (4.156) we now obtain is independent of the distance from γ to the boundary $\partial\mathcal{D}$. Moreover, the matrix A in Proposition 4.7.9 is simply the identity matrix so that a similar bound is expected for the square-root of the second-order *derivative quantity*. This makes the whole difference with Sect. 4.8.

4.9.3 Perturbed Version

Obviously, the case of the ball is very specific. In the general case, we go back to the perturbation strategy developed in Sect. 4.8 but for the pair (Z, Y) solution of (4.167).

Specifically, we consider a $\mathcal{C}^2$ curve $\gamma : s \in [-1, 1] \mapsto \gamma(s)$ such that $\Phi(\gamma(s)) = 0$ for any $s \in [-1, 1]$. For a given (fixed) $s \in (-1, 1)$ and for ε in the neighborhood of 0, we denote by $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{t \geq 0}$ the solution of[25]

$$\begin{aligned} dZ_t^{s+\varepsilon} &= \sum_{i=1,2} (Y_t^{s+\varepsilon})^i \exp(p_t^\varepsilon)\sigma_t dB_t^i + \exp(p_t^\varepsilon) a_t \exp(-p_t^\varepsilon) D^*_{\bar{z}}\psi(Z_t^{s+\varepsilon})dt, \\ d(Y_t^{s+\varepsilon})^i &= D_{\bar{z}}\psi(Z_t^{s+\varepsilon}) \exp(\bar{p}_t^\varepsilon)\bar{\sigma}_t d\bar{B}_t^i \\ &\quad + \frac{1}{2}(Y_t^{s+\varepsilon})^i \mathrm{Trace}\big[\exp(p_t^\varepsilon) a_t \exp(-p_t^\varepsilon) D^2_{z,\bar{z}}\psi(Z_t^{s+\varepsilon})\big] dt, \\ &\qquad t \geq 0, \quad i = 1, 2, \end{aligned} \tag{4.178}$$

with the initial condition $(Z_0^{s+\varepsilon}, Y_0^{s+\varepsilon}) = \gamma(s + \varepsilon)$

[25] For more simplicity, we forget the symbol "^" used in Sect. 4.8.2.

Here, the process $(p_t^{\varepsilon})_{t\geq 0}$ denotes a ghost parameter with values into the set of anti-Hermitian matrices, exactly as in (4.93). Specifically, $p_t^{s+\varepsilon} = P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)$ as in (4.93) with P as in (4.94). As in Sect. 4.9.1, ψ is here extended to the whole $\mathbb{C}^d$ into a $\mathcal{C}^4$ function with bounded derivatives, so that the above system has Lipschitz coefficients on the whole space and is therefore uniquely solvable for any given initial condition (Z_0, Y_0).

Following the proof of Proposition 4.9.1, we can compute $d(\psi(Z_t^{s+\varepsilon}) - |Y_t^{s+\varepsilon}|^2)$ for any $t \geq 0$ and prove that it is zero, so that the process $(\psi(Z_t^{s+\varepsilon}) - |Y_t^{s+\varepsilon}|^2)_{t\geq 0}$ lives on the zero set of the function $\Phi : (z,y) \in \mathcal{D} \times \mathbb{C}^2 \mapsto \psi(z) - |y|^2$. (In particular, $(Z_t^{s+\varepsilon})_{t\geq 0}$ does not leave $\mathcal{D}$.)

Here is the analog of Propositions 4.8.2 and 4.8.3

Proposition 4.9.2 *Let $S > 0$ be a positive real, ϕ be a smooth function from $\mathbb{R}_+$ to $[0,1]$ matching 1 on $[0,1]$ and 0 outside $[0,2]$, $\epsilon > 0$ be a small enough real such that $|D_z\psi(z)| > 0$ for $\psi(z) \leq \epsilon$ and $\mathfrak{s}$ be some (finite) stopping time such that $\psi(Z_{\mathfrak{s}}^s) < \epsilon$. For $\mathfrak{t} := \inf\{t \geq \mathfrak{s} : \psi(Z_t^s) \geq \epsilon\}$, consider some process $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{0\leq t\leq \mathfrak{t}}$ for which $([d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ and $([d^2/d\varepsilon^2](Z_t^{s+\varepsilon})_{|\varepsilon=0})_{0\leq t\leq \mathfrak{t}}$ exist and for which the perturbed SDE (4.178) holds from $\mathfrak{s}$ to $\mathfrak{t}$ and define*

$$\begin{aligned}
&\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon) \\
&\quad = \mathbb{E}\int_{\mathfrak{s}}^{\mathfrak{t}} \biggl[\exp\biggl(\int_0^t \mathrm{Trace}[\exp(p_r^{s+\varepsilon}) a_r \exp(-p_r^{s+\varepsilon}) D^2_{z,\bar{z}}\psi(Z_r^{s+\varepsilon})] dr\biggr) \\
&\qquad\qquad \times F\bigl(\det(a_t), \exp(p_t^{s+\varepsilon}) a_t \exp(-p_t^{s+\varepsilon}), Z_t^{s+\varepsilon}\bigr)\phi\biggl(\frac{t}{S}\biggr)\biggr] dt,
\end{aligned}$$

with $p_t^{s+\varepsilon} = P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)$, $\mathfrak{s} \leq t \leq \mathfrak{t}$, P being given by (4.94).

*Assume that the differentiation operator w.r.t. ε and the expectation and integration symbols can be exchanged in the definition of $\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}$. Then, we can find a constant $C > 0$, depending on Assumption (**A**) and on ϵ only (in particular, it is independent of C), such that*

$$\begin{aligned}
&\biggl|\frac{d}{d\varepsilon}\bigl[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\bigr]\biggr| \\
&\qquad \leq C\mathbb{E}\biggl[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\biggl(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z_t^s)] dr\biggr)\biggl[|\zeta_t^s| + \int_0^t |\zeta_r^s| dr\biggr] dt\biggr],
\end{aligned}$$

where $\zeta_t^s = [d/d\varepsilon](Z_t^{s+\varepsilon})_{|\varepsilon=0}$.

Similarly,

$$\biggl|\frac{d^2}{d\varepsilon^2}\bigl[\hat{V}_S^{\sigma,\mathfrak{s},\mathfrak{t}}(s+\varepsilon)\bigr]\biggr|$$

$$\leq C\mathbb{E}\Big[\int_{\mathfrak{s}}^{\mathfrak{t}} \exp\Big(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}(Z^s_t)]dr\Big)$$
$$\times \Big[|\eta^s_t| + |\zeta^s_t|^2 + \int_0^t |\eta^s_r|dr + \int_0^t |\zeta^s_r|^2 dr + \Big(\int_0^t |\zeta^s_r|dr\Big)^2\Big]dt\Big],$$

where $\eta^s_t = [d^2/d\varepsilon^2](\hat{Z}^{s+\varepsilon}_t)_{|\varepsilon=0}$.

4.9.4 Derivative Quantity

We now prove the analog of Proposition 4.8.9:

Proposition 4.9.3 *Keep the assumption and notation of Proposition 4.9.2. Then, there exists a positive real* ϵ'_1 *such that for* $0 < \epsilon < \epsilon'_1$, *for* $N = K = \epsilon^{-1/4}$, *for* $\psi = N\psi^0$, *where* ψ^0 *is the* reference plurisuperharmonic function *describing* $\mathcal{D}$ *such that* $\mathrm{Trace}[aD^2_{z,\bar{z}}\psi^0(z)] \leq -1$, $z \in \mathcal{D}$, *for a stopping time* $\mathfrak{s}$ *at which* $\psi(Z^s_{\mathfrak{s}}) < \epsilon$, *the* derivative quantity *obtained by perturbing the control parameter as in* (4.178)

$$\bar{\Gamma}^{(1)}_t = \exp\big(-K\psi(Z^s_t)\big)|\zeta_t|^2 + |\rho_t|^2, \quad t \geq \mathfrak{s},$$

with $\zeta_t = [d/d\varepsilon](Z^{s+\varepsilon}_t)_{|\varepsilon=0}$ *and* $\rho^s_t = [d/d\varepsilon](Y^{s+\varepsilon}_t)_{|\varepsilon=0}$, *satisfies up to time* $\mathfrak{t} = \inf\{t \geq \mathfrak{s} : \psi(Z^s_t) \geq \epsilon\}$

$$\mathbb{E}\Big[\exp\Big(\int_0^{t\wedge\mathfrak{t}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\Big)\bar{\Gamma}^{(1)}_{t\wedge\mathfrak{t}}|\mathcal{F}_{\mathfrak{s}}\Big]$$
$$\leq \exp\Big(\int_0^{\mathfrak{s}} (1-\delta)\mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z^s_r)]dr\Big)\bar{\Gamma}^{(1)}_{\mathfrak{s}}, \quad t \geq \mathfrak{s},$$

with $\delta = 1/N = \epsilon^{1/4}$.

Proof. The proof is similar to the one of Proposition 4.8.9. The derivatives of $(Z^{s+\varepsilon}_t, Y^{s+\varepsilon}_t)_{t\geq 0}$ with respect to ε at $\varepsilon = 0$ are denoted by

$$\zeta_t = \frac{d}{d\varepsilon}[Z^{s+\varepsilon}_t]_{|\varepsilon=0}, \ \varrho_t = \frac{d}{d\varepsilon}[Y^{s+\varepsilon}_t]_{|\varepsilon=0}, \quad t \geq 0.$$

As $(Y^{s+\varepsilon}_t)$ is $\mathbb{C}^2$-valued, so is $(\varrho_t)_{t\geq 0}$. Below, we denote by $(\varrho^1_t)_{t\geq 0}$ and $(\varrho^2_t)_{t\geq 0}$ the two coordinates of $(\varrho_t)_{t\geq 0}$. We also use the following notations:

$$\psi_t = \psi(Z^s_t), \quad (L\psi)_t = \mathrm{Trace}\big(a_t D^2_{z,\bar{z}}\psi(Z^s_t)\big),$$
$$Q_t\zeta_t = \frac{d}{d\varepsilon}\big[P(Z^s_t, Z^{s+\varepsilon}_t - Z^s_t)\big]_{|\varepsilon=0}, \quad t \geq 0.$$

Moreover, I_d stands for the identity matrix of size d. By Theorem 4.7.4, the pair $(\zeta_t, \varrho_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ satisfies the equation[26]:

$$
\begin{aligned}
d\zeta_t &= \sum_{i=1,2} \big[\varrho_t^i I_d + Y_t^i Q_t \zeta_t\big] \sigma_t dB_t^i + \big[a_t D^2_{\bar{z},z} \psi_t \zeta_t + a_t D^2_{\bar{z},\bar{z}} \psi_t \bar{\zeta}_t\big] dt \\
&\qquad + \big[Q_t \zeta_t a_t D^*_{\bar{z}} \psi_t - a_t Q_t \zeta_t D^*_{\bar{z}} \psi_t\big] dt \\
d\varrho_t^i &= \big[\big(D^2_{\bar{z},z} \psi_t \zeta_t\big)^* + \big(D^2_{\bar{z},\bar{z}} \psi_t \bar{\zeta}_t\big)^* - D_{\bar{z}} \psi_t (Q_t \zeta_t)^*\big] \bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2} \varrho_t^i L\psi_t dt \\
&\quad + \frac{1}{2} Y_t^i \big[D_z (L\psi)_t \zeta_t + D_{\bar{z}} (L\psi)_t \bar{\zeta}_t\big] dt \\
&\quad + \frac{1}{2} Y_t^i \big[\mathrm{Trace}(Q_t \zeta_t a_t D^2_{z,\bar{z}} \psi_t) - \mathrm{Trace}(a_t Q_t \zeta_t D^2_{z,\bar{z}} \psi_t)\big] dt, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathfrak{s} \le t \le \mathfrak{t}, \quad i = 1, 2.
\end{aligned}
$$

Using the anti-Hermitian property of $Q_t \zeta_t$, we have:

$$
\begin{aligned}
&\overline{\mathrm{Trace}(Q_t \zeta_t a_t D^2_{z,\bar{z}} \psi_t)} \\
&\quad = -\mathrm{Trace}((Q_t \zeta_t)^* a_t^* (D^2_{z,\bar{z}} \psi_t)^*) \\
&\quad = -\mathrm{Trace}(D^2_{z,\bar{z}} \psi_t a_t Q_t \zeta_t) = -\mathrm{Trace}(a_t Q_t \zeta_t D^2_{z,\bar{z}} \psi_t), \quad \mathfrak{s} \le t \le \mathfrak{t}.
\end{aligned}
$$

Taking the complex conjugate in (4.129), we deduce

$$
\begin{aligned}
d\varrho_t^i &= r_t |\zeta_t| D_{\bar{z}} \psi_t \bar{\sigma}_t d\bar{B}_t^i + \frac{1}{2} \varrho_t^i L\psi_t dt \\
&\quad + Y_t^i \mathrm{Re}\big[D_z (L\psi)_t \zeta_t\big] dt \\
&\quad + Y_t^i \mathrm{Re}\big[\mathrm{Trace}(Q_t \zeta_t a_t D^2_{z,\bar{z}} \psi_t)\big] dt, \quad \mathfrak{s} \le t \le \mathfrak{t}, \quad i = 1, 2,
\end{aligned}
$$

where $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ stands for a generic process scalar process bounded in terms of (**A**) only. (The values of $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ may vary from line to line.)

We are now in position to compute the norm of the derivative process $((\zeta_t, \varrho_t))_{\mathfrak{s} \le t \le \mathfrak{t}}$.

$$
\begin{aligned}
d|\zeta_t|^2 &= 2\mathrm{Re}\langle \bar{\zeta}_t, a_t D^2_{\bar{z},z} \psi_t \zeta_t + a_t D^2_{\bar{z},\bar{z}} \psi_t \bar{\zeta}_t \rangle dt \\
&\quad + 2\mathrm{Re}\langle \bar{\zeta}_t, Q_t \zeta_t a_t D^*_{\bar{z}} \psi_t - a_t Q_t \zeta_t D^*_{\bar{z}} \psi_t \rangle dt
\end{aligned}
$$

---

[26] The reader may understand that Theorem 4.7.4 provides both the form of the equation for the pair $(\zeta_t, \varrho_t)_{\mathfrak{s} \le t \le \mathfrak{s}}$ and the differentiability property of the process $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{\mathfrak{s} \le t \le \mathfrak{t}}$ w.r.t. ε. Indeed, (4.167) satisfies the assumption of Theorem 4.7.4: there is no singular term inside contrary to (4.84). (Since the component Y is bounded, the coefficients may be considered as $\mathcal{C}^2$ coefficients with bounded derivatives.)

$$+ \sum_{i=1,2} \text{Trace}\big[\big(\varrho^i I_d + (Y_t^s)^i Q_t \zeta_t\big) a_t \big(\bar{\varrho}^i I_d - (\bar{Y}_t^s)^i Q_t \zeta_t\big)\big] dt$$

$$+ dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{4.179}$$

Similarly,

$$d|\varrho_t|^2 = |\varrho_t|^2 L\psi_t dt$$
$$+ 2\text{Re}\big(\langle \varrho_t, \bar{Y}_t^s \rangle\big) \big[\text{Re}(D_z(L\psi)_t \zeta_t) + \text{Re}\big(\text{Trace}(Q_t \zeta_t a_t D^2_{z,\bar{z}} \psi_t)\big)\big] dt$$
$$+ r_t D_z \psi_t a_t D^*_{\bar{z}} \psi_t |\zeta_t|^2 dt + dm_t, \quad t \ge 0. \tag{4.180}$$

In what follows, we follow Sect. 4.8 and modify the choice of ψ according to the observation we made therein: for any constant $c > 0$, $c\psi$ is again a plurisuperharmonic function describing the domain and we denote by ψ^0 some choice of the plurisuperharmonic function such that, for any Hermitian matrix a of trace 1 and for any $z \in \mathcal{D}$, $\text{Trace}[a D^2_{z,\bar{z}} \psi^0(z)] \le -1$. Then, we understand ψ as $N\psi^0$ for some free parameter N that will be fixed later on.

As a first application, we can simplify the form of $d|\varrho_t|^2$, or at least we can bound it, for $\mathfrak{s} \le t \le \mathfrak{t}$. To this end, have in mind that $|\psi_t| \le \epsilon$ for $\mathfrak{s} \le t \le \mathfrak{t}$ so that $|D_z \psi_t^0| \ge \kappa$ for some given constant $\kappa > 0$ (for $\mathfrak{s} \le t \le \mathfrak{t}$ and for ϵ small enough). Therefore, from (4.180), we claim

$$d|\varrho_t|^2 = N|\varrho_t|^2 L\psi_t^0 dt + N|\varrho_t||\zeta_t||Y_t^s| r_t dt + N^2 |\zeta_t|^2 \mathcal{E}_t^0 r_t dt + dm_t, \quad \mathfrak{s} \le t \le \mathfrak{t}, \tag{4.181}$$

where $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ is a generic notation for a process, bounded by some constant C depending on $(\mathbf{A})$ and κ only. (The values of $(r_t)_{\mathfrak{s} \le t \le \mathfrak{t}}$ may vary from line to line.) Above, $(\psi_t^0)_{\mathfrak{s} \le t \le \mathfrak{t}}$ is understood as $(\psi^0(Z_t^s))_{\mathfrak{s} \le t \le \mathfrak{t}}$ and $(\mathcal{E}_t^0)_{\mathfrak{s} \le t \le \mathfrak{t}}$ stands for $(\mathcal{E}_t^0 := \langle D_z^* \psi_t^0, a_t D_{\bar{z}}^* \psi_t^0 \rangle))_{\mathfrak{s} \le t \le \mathfrak{t}}$.

By (4.119),

$$d|\zeta_t|^2 = |\varrho_t|^2 dt + |Y_t^s||\varrho_t||\zeta_t| r_t dt + |Y_t^s|^2 |\zeta_t|^2 r_t dt$$
$$+ N|\zeta_t|^2 \mathcal{E}_t^0 r_t dt + N|\zeta_t|^2 (\mathcal{E}_t^0)^{1/2} r_t dt$$
$$+ 2 \sum_{i=1,2} \text{Re}\big[\langle \bar{\zeta}_t, \big(\varrho_t^i I_d + (Y_t^s)^i Q_t \zeta_t\big) \sigma_t dB_t^i \rangle\big], \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{4.182}$$

We now consider the derivative quantity

$$\bar{\Gamma}_t = \exp(-K\psi_t)|\zeta_t|^2 + |\varrho_t|^2, \quad \mathfrak{s} \le t \le \mathfrak{t}. \tag{4.183}$$

for some constant $K > 0$ to be chosen later on.

To compute $(d\bar{\Gamma}_t)_{\mathfrak{s}\le t\le \mathfrak{t}}$, we first note that

$$d\psi_t = 2\sum_{i=1,2}(Y_t^s)^i \mathrm{Re}\big[D_z\psi_t\sigma_t dB_t^i\big] + 2\langle D_z\psi_t, a_t D_{\bar{z}}^*\psi_t\rangle dt + |Y_t^s|^2 L\psi_t dt,$$

so that

$$\begin{aligned} d\big[\exp(-K\psi_t)\big] &= -2K\exp(-K\psi_t)\sum_{i=1,2}\mathrm{Re}\big[(Y_t^s)^i D_z\psi(Z_t^s)\sigma dB_t^i\big] \\ &\quad + [K^2|Y_t^s|^2 - 2K]\exp(-K\psi_t)\langle D_z\psi_t, a_t D_{\bar{z}}\psi_t\rangle dt \\ &\quad - K\exp(-K\psi_t)|Y_t^s|^2 L\psi_t dt \\ &= -2K\exp(-K\psi_t)\sum_{i=1,2}\mathrm{Re}\big[(Y_t^s)^i D_z\psi(Z_t^s)\sigma dB_t^i\big] \\ &\quad + N^2[K^2|Y_t^s|^2 - 2K]\exp(-K\psi_t)\mathcal{E}_t^0 dt \\ &\quad - NK\exp(-K\psi_t)|Y_t^s|^2 L\psi_t^0 dt, \quad \mathfrak{s}\le t\le \mathfrak{t}. \end{aligned} \tag{4.184}$$

Therefore, from (4.184) and (4.182),

$$\begin{aligned} & d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\ &\quad = \exp(-K\psi_t)\big[|\varrho_t|^2 + |Y_t^s||\varrho_t||\zeta_t|r_t + |Y_t^s|^2|\zeta_t|^2 r_t \\ &\qquad + N|\zeta_t|^2\mathcal{E}_t^0 r_t + N|\zeta_t|^2(\mathcal{E}_t^0)^{1/2} r_t\big]dt \\ &\qquad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2|Y_t^s|^2 - 2K]\mathcal{E}_t^0 - NK|Y_t^s|^2 L\psi_t^0\big]dt \\ &\qquad + NK\exp(-K\psi_t)\big[|Y_t^s||\zeta_t||\varrho_t|r_t + |Y_t^s|^2|\zeta_t|^2 r_t\big] + dm_t, \quad \mathfrak{s}\le t\le \mathfrak{t}. \end{aligned}$$

We are now in position to compute $d\bar{\Gamma}_t$ for $\mathfrak{s}\le t\le \mathfrak{t}$. To this end, have in mind that $L\psi_t^0 \le -1$ and that $|Y_t^s|^2 = \psi_t \le \epsilon$, $\mathfrak{s}\le t\le \mathfrak{t}$. Then, applying Young's inequality to the term $N(\mathcal{E}_t^0)^{1/2}$, the above equation has the form

$$\begin{aligned} & d\big[\exp(-K\psi_t)|\zeta_t|^2\big] \\ &\quad \le \exp(-K\psi_t)\big[|\varrho_t|^2 + C(1+\epsilon^{1/2}+\epsilon)|\xi_t|^2 + C(N+N^2)|\zeta_t|^2\mathcal{E}_t^0\big]dt \\ &\qquad + |\zeta_t|^2\exp(-K\psi_t)\big[N^2[K^2\epsilon - 2K]\mathcal{E}_t^0 + CNK\epsilon\big]dt \\ &\qquad + NK\exp(-K\psi_t)\big[C\epsilon^{1/2}|\xi_t|^2 + C\epsilon|\xi_t|^2\big] + dm_t, \quad \mathfrak{s}\le t\le \mathfrak{t}, \end{aligned} \tag{4.185}$$

where $|\xi_t|^2 = |\zeta_t|^2 + |\varrho_t|^2$. (Actually, $(\xi_t)_{t\ge 0}$ must be understood as the derivative process $(\zeta_t, \varrho_t)_{t\ge 0}$.) Similarly, from (4.181),

$$d|\varrho_t|^2 \leq -N|\varrho_t|^2 dt + CN\epsilon^{1/2}|\xi_t|^2 dt + CN^2|\zeta_t|^2 \mathcal{E}_t^0 dt + dm_t \quad \mathfrak{s} \leq t \leq \mathfrak{t}. \tag{4.186}$$

Therefore, assuming $\epsilon < 1$ and $N \geq 1$, we deduce from (4.185) and (4.186)

$$\begin{aligned} d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt \\ &+ |\xi_t|^2\big(C' + C'N\epsilon^{1/2} + C'NK\epsilon^{1/2}\big)dt \\ &+ |\zeta_t|^2 \exp(-K\psi_t)N^2\big[K^2\epsilon - 2K + C'\exp(K\psi_t)\big]\mathcal{E}_t^0 dt \\ &+ dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}, \end{aligned}$$

the constant C' depending on C only. (In particular, C' is independent of K, N, ϵ, s and t.)

Choose now $K = \epsilon^{-1/4}$. We obtain

$$\begin{aligned} d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt + 2|\xi_t|^2\big(C' + C'N\epsilon^{1/4}\big)dt \\ + |\zeta_t|^2 \exp(-K\psi_t)N^2\big[\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{1/4})\big]\mathcal{E}_t^0 dt + dm_t. \end{aligned}$$

Choose ϵ small enough such that $\epsilon^{1/2} - 2\epsilon^{-1/4} + C'\exp(\epsilon^{1/4}) < 0$. Then,

$$d\bar{\Gamma}_t \leq \exp(-K\psi_t)(1-N)|\varrho_t|^2 dt + 2|\xi_t|^2\big(C' + C'N\epsilon^{1/4}\big)dt + dm_t, \quad \mathfrak{s} \leq t \leq \mathfrak{t}.$$

Finally for $N = \epsilon^{-1/4}$, we obtain:

$$d\bar{\Gamma}_t \leq 4C'|\xi_t|^2 dt + dm_t \leq 4C'\exp(\epsilon^{1/4})\bar{\Gamma}_t + dm_t \leq 12C'\bar{\Gamma}_t + dm_t. \tag{4.187}$$

The end of the proof is similar to the one of Proposition 4.8.9. □

4.9.5 Global Derivative Quantity

Proposition 4.9.4 *Let $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ be two independent complex Brownian motions of dimension d, the pair being independent of $(B_t)_{t\geq 0}$. Moreover, let ϵ and ϵ_4 be as in Proposition 4.8.12, ϵ being less than ϵ_1' in Proposition 4.9.3 as well, γ_0 be a path from $[-1,1]$ into $\mathcal{D}$ and s be a point in $(-1,1)$ such that $\psi(\gamma_0(s)) > \epsilon$.*

For a given progressively-measurable (w.r.t. the filtration generated by the triple of processes $(B_t, B_t^1, B_t^2)_{t\geq 0}$) control $(\sigma_t)_{t\geq 0}$ with values in the set of complex matrices of size $d\times d$ such that $\mathrm{Trace}(\sigma_t\bar{\sigma}_t^*) = 1$, $t \geq 1$, define $(Z_t^s)_{t\geq 0}$ as follows. Set $\mathfrak{r}_0 = 0$. Up to time $\mathfrak{r}_1 = \{t \geq 0 : \psi_t = \psi(Z_t^s) \leq \epsilon_4\}$, define $(Z_t^s)_{0\leq t\leq \mathfrak{r}_1}$ as the solution of the SDE (4.84) with $\gamma_0(s)$ as initial condition. At time $\mathfrak{r}_1$, set $Y_{\mathfrak{r}_1}^s = (\psi^{1/2}(Z_{\mathfrak{r}_1}^s), 0) \in \mathbb{C}^2$ and then define $(Z_t^s, Y_t^s)_{\mathfrak{r}_1\leq t\leq \mathfrak{r}_2}$

(with values into $\mathcal{D}\times\mathbb{C}^2$*) up to time* $\mathfrak{r}_2 = \{t \geq \mathfrak{r}_1 : \psi_t = \psi(Z_t^s) \geq \epsilon/2\}$ *as the solution of* (4.167)*. At time* $\mathfrak{r}_2$*, define* $(Z_t^s)_{\mathfrak{r}_2 \leq t \leq \mathfrak{r}_3}$ *up to time* $\mathfrak{r}_3 = \{t \geq \mathfrak{r}_1 : \psi_t = \psi(Z_t^s) \leq \epsilon_4\}$ *as the solution of the SDE* (4.84) *and so on..., that is*

$$dZ_t^s = \psi^{1/2}(Z_t^s)\sigma_t dB_t + a_t D_{\bar{z}}^* \psi(Z_t^s)dt, \quad t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}],\ k \geq 0, \tag{4.188}$$

with $Z_0^s = \gamma(s)$ *as initial condition (above,* $\mathfrak{r}_0 = 0$*), and*

$$\begin{aligned} dZ_t^s &= \sum_{i=1,2} (Y_t^s)^i \sigma_t dB_t^i + a_t D_{\bar{z}}^* \psi(Z_t^s)dt \\ d\big(Y_t^s\big)^i &= D_{\bar{z}}\psi(Z_t^s)\bar{\sigma}_t d\bar{B}_t^i \\ &\quad + \frac{1}{2}\big(Y_t^s\big)^i \mathrm{Trace}\big[a_t D_{z,\bar{z}}^2 \psi(Z_t^s)\big]dt, \quad t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}],\ k \geq 0,\ i = 1, 2, \end{aligned} \tag{4.189}$$

with $Y_{\mathfrak{r}_{2k+1}} = (\psi^{1/2}(Z^s_{\mathfrak{r}_{2k+1}}), 0)$ *as initial condition.*

Define also $(\tau_n)_{n\geq 1}$ *as the sequence of exit times of the process* $(\psi(Z_t^s))_{t\geq 0}$ *from the sets* $[\epsilon/4, +\infty)$*,* $[\epsilon_4, \epsilon]$ *and* $[0, \epsilon/2]$*. When the process* $(\psi(Z_t^s))_{t\geq 0}$ *belongs to* $[\epsilon/4, +\infty)$ *consider the perturbation given by Proposition 4.8.10; when* $(\psi(Z_t^s))_{t\geq 0}$ *belongs to* $[\epsilon_4, \epsilon]$ *consider the perturbation given by Proposition 4.8.11: the perturbation is then given by a process of the form* $(Z_t^{s+\varepsilon})_{\mathfrak{r}_{2k} \leq t \leq \mathfrak{r}_{2k+1}}$*, with* $k \geq 0$*. When* $(\psi(Z_t^s))_{t\geq 0}$ *belongs to* $[0, \epsilon/2]$ *consider the perturbation given by Proposition 4.9.3: the perturbation is then given by a pair of the form* $(Z_t^{s+\varepsilon}, Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}}$*,* $k \geq 0$*, with* $Y^{s+\varepsilon}_{\mathfrak{r}_{2k+1}} = (\psi^{1/2}(Z^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}), 0)$ *as initial condition. Specifically,*

$$\begin{aligned} dZ_t^{s+\varepsilon} &= T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\psi^{1/2}(Z_t^{s+\varepsilon}) \exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times \sigma_t\big(dB_t + G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)dt\big) \\ &\quad + |T|^2(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s) \exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times a_t \exp\big(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon})dt, \quad \mathfrak{r}_{2k} \leq t \leq \mathfrak{r}_{2k+1}, \end{aligned}$$

with $Z_0^{s+\varepsilon} = \gamma(s+\varepsilon)$ *as initial condition, and*

$$\begin{aligned} dZ_t^{s+\varepsilon} &= \sum_{i=1}^{2} (Y_t^{s+\varepsilon})^i dB_t^i \\ &\quad + \exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) a_t \exp\big(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) D_{\bar{z}}^* \psi(Z_t^{s+\varepsilon})dt \\ d\big(Y_t^{s+\varepsilon}\big)^i &= D_{\bar{z}}\psi(Z_t^{s+\varepsilon}) \exp\big(\bar{P}(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big)\bar{\sigma}_t d\bar{B}_t^i \end{aligned}$$

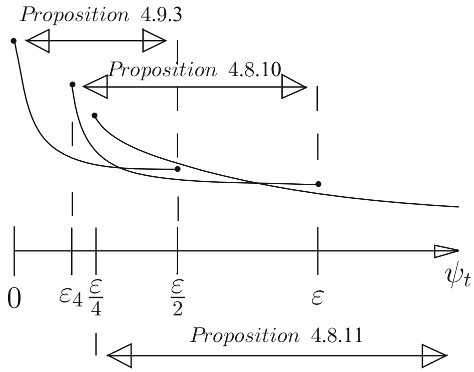

Fig. 4.3 Choice of the perturbations with the new representation

$$+\frac{1}{2}\big(Y_t^{s+\varepsilon}\big)^i \mathrm{Trace}\big[\exp\big(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big)$$
$$\times a_t \exp\big(-P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s)\big) D^2_{z,\bar{z}}\psi(Z_t^{s+\varepsilon})\big]\,dt,$$
$$\mathfrak{r}_{2k+1} \le t \le \mathfrak{r}_{2k+2}, \quad i = 1, 2,$$

with $Y^{s+\varepsilon}_{\mathfrak{r}_{2k+1}} = (\psi^{1/2}(Z^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}), 0)$ *as initial condition.*

Above, $(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$, $(T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$, *and* $(G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$, *stand for the different possible perturbations used in Propositions 4.8.10, 4.8.11 and 4.9.3. Precisely,* $(P(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 4.8.2 applies,* $(T(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$ *is set equal to* 1 *outside the intervals on which the perturbation of Proposition 4.8.4 applies and* $(G(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s))_{t\ge 0}$ *is set equal to* 0 *outside the intervals on which the perturbation of Proposition 4.8.7 applies. As a summary, Fig. 4.3 below is the analog of Fig. 4.2.*

Then, the family of processes $(Z_t^{s+\varepsilon})_{t\ge 0}$, ε *in the neighborhood of* 0, *is twice differentiable in probability w.r.t.* ε *at* $\varepsilon = 0$, *with time continuous derivatives. Similarly, for each* $k \ge 0$, *the family of processes* $(Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le \mathfrak{r}_{2k+2}}$, ε *in the neighborhood of* 0, *is twice differentiable in probability w.r.t.* ε *at* $\varepsilon = 0$, *with continuous derivatives. Moreover, the dynamics of the derivatives are obtained by differentiating (w.r.t.* ε*) the dynamics of* $(Z_t^{s+\varepsilon})_{t\ge 0}$ *and* $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le \mathfrak{r}_{2k+2}})_{k\ge 0}$ *formally at* $\varepsilon = 0$, *as done in the* meta-part *of Sect. 4.8.*

Define then the derivative quantity $(\bar{\Gamma}_t)_{t\ge 0}$ *as* $\mu_2\bar{\Gamma}_t^{(2)}$, $\mu_3\bar{\Gamma}_t^{(3)}$ *in Proposition 4.8.12 and* $\bar{\Gamma}_t^{(1)}$ *in Proposition 4.9.3. (In particular,* $(\bar{\Gamma}_t)_{t\ge 0}$ *is left-continuous.) Then, we can find* $\alpha \in (0,1)$, *depending on (***A***) and* ϵ *only, such that*

$$\mathbb{E}\Big[\bar{\Gamma}_t \exp\Big(\int_0^t \alpha L\psi_r dr\Big)\Big] \le \bar{\Gamma}_0, \quad t \ge 0.$$

Proof. Differentiability properties will be established below. (See Proposition 4.9.6 below.) In comparison with Sect. 4.8.8, the only difference is here to show that

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar\Gamma_t \le \bar\Gamma_{\mathfrak{r}_{2k+1}},\ \lim_{t\to\mathfrak{r}_{2k}+}\bar\Gamma_t \le \bar\Gamma_{\mathfrak{r}_{2k}}\quad k\ge 0.$$

When $t\to\mathfrak{r}_{2k}+$, $\bar\Gamma_t$ is given by $\mu_2\bar\Gamma_t^{(2)}$, so that, by Proposition 4.8.12 (recall that $\psi_{\mathfrak{r}_{2k}}=\epsilon/2$),

$$\begin{aligned}\lim_{t\to\mathfrak{r}_{2k}+}\bar\Gamma_t &= \mu_2\bar\Gamma^{(2)}_{\mathfrak{r}_{2k}}\\ &= \mu_2\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\zeta_{\mathfrak{r}_{2k}}|^2+2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}]\\ &\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\zeta_{\mathfrak{r}_{2k}}|^2+\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big].\end{aligned}\tag{4.190}$$

Now, have in mind that $|Y_{\mathfrak{r}_{2k}}^{s+\varepsilon}|^2=\psi(Z_{\mathfrak{r}_{2k}}^{s+\varepsilon})$ so that, by differentiation,

$$\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]=\mathrm{Re}\big[Y^1_{\mathfrak{r}_{2k}}(\bar\varrho_{\mathfrak{r}_{2k}})^1\big]+\mathrm{Re}\big[Y^2_{\mathfrak{r}_{2k}}(\bar\varrho_{\mathfrak{r}_{2k}})^2\big].\tag{4.191}$$

Therefore,

$$\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]\big|\le|Y^1_{\mathfrak{r}_{2k}}||\varrho^1_{\mathfrak{r}_{2k}}|+|Y^2_{\mathfrak{r}_{2k}}||\varrho^2_{\mathfrak{r}_{2k}}|\le|Y_{\mathfrak{r}_{2k}}||\rho_{\mathfrak{r}_{2k}}|.\tag{4.192}$$

Since $|Y_{\mathfrak{r}_{2k}}|=\psi^{1/2}_{\mathfrak{r}_{2k}}$,

$$\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\zeta_{\mathfrak{r}_{2k}}\big]\big|\le|\varrho_{\mathfrak{r}_{2k}}|^2.$$

From (4.190), we deduce

$$\lim_{t\to\mathfrak{r}_{2k}+}\bar\Gamma_t\le\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\zeta_{\mathfrak{r}_{2k}}|^2+\psi_{\mathfrak{r}_{2k}}^{-1}|\varrho_{\mathfrak{r}_{2k}}|^2=\bar\Gamma_{\mathfrak{r}_{2k}}.$$

It now remains to prove the bound at time $\mathfrak{r}_{2k+1}$. When $t\to\mathfrak{r}_{2k+1}+$, $\bar\Gamma_t$ is given by $\bar\Gamma_t^{(1)}$, i.e.

$$\bar\Gamma_t=\exp(-\epsilon^{-1/4}\psi_t)|\zeta_t|^2+|\varrho_t|^2.$$

Therefore,

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar\Gamma_t=\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\zeta_{\mathfrak{r}_{2k+1}}|^2+|\varrho_{\mathfrak{r}_{2k+1}}|^2.\tag{4.193}$$

Have in mind that, at time $t=\mathfrak{r}_{2k+1}$, $Y^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}=(\psi^{1/2}(Z^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}),0)$, so that, by differentiation,

$$\varrho_{\mathfrak{r}_{2k+1}} = \big(\psi_{\mathfrak{r}_{2k+1}}^{-1/2}\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\zeta_{\mathfrak{r}_{2k+1}}\big],0\big). \tag{4.194}$$

We deduce that

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Gamma}_t = \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\zeta_{\mathfrak{r}_{2k+1}}|^2 + \psi_{\mathfrak{r}_{2k+1}}^{-1}\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\zeta_{\mathfrak{r}_{2k+1}}\big]\big|^2. \tag{4.195}$$

Applying Proposition 4.8.12 (recall that $\psi_{\mathfrak{r}_{2k+1}} = \epsilon_4$), we obtain

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Gamma}_t \le \mu_2\bar{\Gamma}^{(2)}_{\mathfrak{r}_{2k+1}} = \bar{\Gamma}_{\mathfrak{r}_{2k+1}}.$$

This completes the proof. □

We deduce

Corollary 4.9.5 *Keep the notation of Proposition 4.9.4 and define the second-order derivatives of* $(Z_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1}}$, $k\ge 0$, *by setting* $\eta_t^s = [d^2/d\varepsilon^2][Z_t^{s+\varepsilon}]_{|\varepsilon=0}$, *for* $\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1}$, $k\ge 0$, *and define the second-order derivatives of* $(Z_t^{s+\varepsilon},Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2}}$, $k\ge 0$, *by setting* $(\eta_t^s,\pi_t^s) = [d^2/d\varepsilon^2][(Z_t^{s+\varepsilon},Y_t^{s+\varepsilon})]_{|\varepsilon=0}$, *for* $\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1}$, $k\ge 1$.

Define the analogs of $\bar{\Gamma}_t^{(1)}$, $\mu_2\bar{\Gamma}_t^{(2)}$ *and* $\mu_3\bar{\Gamma}_t^{(3)}$, $t\ge 0$, *i.e.*

$$\begin{aligned}
\bar{\Delta}_t^{(1)} &= \exp\big(-\epsilon^{-1/4}\psi(Z_t^s)\big)|\eta_t|^2 + |\pi_t|^2,\\
\bar{\Delta}_t^{(2)} &= \exp(-\epsilon^{-1/4}\psi_t)\psi_t^{-\epsilon^2}|\eta_t|^2 + 2\epsilon^{9/4}\psi_t^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_t\eta_t\big],\\
\bar{\Delta}_t^{(3)} &= (R^2-|Z_t|^2)\psi_t^{-1}|\eta_t|^2,
\end{aligned}$$

for some ϵ *as in the statement of Proposition 4.8.12. Define the global second-order derivative quantity* $(\bar{\Delta}_t)_{t\ge 0}$ *as the analog of* $(\bar{\Gamma}_t)_{t\ge 0}$. *(In particular, mention that* $(\bar{\Delta}_t)_{t\ge 0}$ *is left-continuous.)*

Then, we can find $\alpha\in(0,1)$ *and* $C>0$, *depending on (**A**) and* ϵ *only, such that*

$$\mathbb{E}\bigg[\big(\bar{\Delta}_t^{1/2}+\bar{\Gamma}_t\big)\exp\bigg(\int_0^t \alpha L\psi_r dr\bigg)\bigg] \le \bar{\Delta}_0^{1/2} + C\bar{\Gamma}_0, \quad t\ge 0.$$

Proof. Following the proof of Proposition 4.7.9, we can prove that on each $[\tau_n,\tau_{n+1})$, $n\ge 0$, with $\tau_0=0$ and $(\tau_n)_{n\ge 1}$ as in Proposition 4.9.4, and for any $a>0$,

$$d\bigg[\exp\bigg(\int_0^t \alpha L\psi_r dr\bigg)\big(a+\bar{\Delta}_t+\bar{\Gamma}_t^2\big)^{1/2}\bigg] \le C\bar{\Gamma}_t\exp\bigg(\int_0^t\alpha L\psi_r dr\bigg)dt. \tag{4.196}$$

The proof of (4.196) relies on two points. First, what is called $(\partial\bar{\Gamma}_t(X_t^s,(\eta_t^s,\pi_t^s)))_{t\geq 0}$ in the statement of Proposition 4.7.9 (or equivalently $(\partial\bar{\Delta}_t)_{t\geq 0}$ with the current notation) satisfies the same bound as $(\partial\bar{\Gamma}_t)_{t\geq 0}$. Precisely, $(\partial\bar{\Gamma}_t)_{t\geq 0}$ corresponds to the dt term obtained by differentiating the form $(\bar{\Gamma}_t)_{t\geq 0}$ and then by replacing $(\zeta_t^s,\varrho_t^s)_{t\geq 0}$ therein by $(\eta_t^s,\pi_t^s)_{t\geq 0}$. In the current case, we know that $\partial\bar{\Gamma}_t \leq \alpha L\psi_t\bar{\Gamma}_t$ for any $t\in(\tau_n,\tau_{n+1})$ and for any possible values of the pair $(\zeta_t^s,\varrho_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$. Replacing $(\zeta_t^s,\varrho_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$ by $(\eta_t^s,\pi_t^s)_{\tau_n\leq t\leq\tau_{n+1}}$, we deduce that $\partial\bar{\Delta}_t \leq \alpha L\psi_t\bar{\Delta}_t$ for any $t\in(\tau_n,\tau_{n+1})$. Second, the proof of (4.196) relies on the equivalence of the quadratic form driving $(\bar{\Gamma}_t)_{t\geq 0}$ and $(\bar{\Delta}_t)_{t\geq 0}$ and the current Hermitian form: of (complex) dimension d for $t\in(\mathfrak{r}_{2k},\mathfrak{r}_{2k+1}]$, $k\geq 0$, and of (complex) dimension $d+2$ for $t\in(\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]$. This equivalence makes the difference between Sects. 4.8 and 4.9.

As a consequence of (4.196), we only need to check the boundary conditions to recover the statement, i.e. we only need to prove that $\lim_{t\to\tau_n+}\bar{\Delta}_t\leq\bar{\Delta}_{\tau_n}$.

If τ_n is different from some $\mathfrak{r}_k$, the result follows from Proposition 4.8.12.

If τ_n is equal to some $\mathfrak{r}_{2k}$, we follow (4.190). (Keep in mind that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k}+$ and by $\bar{\Delta}_t^{(1)}$ as $t\to\mathfrak{r}_{2k}-$.) The point is to bound $\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}]$ in terms of $|\pi_{\mathfrak{r}_{2k}}|^2$. We have the analog of (4.191), but with quadratic first-order terms in addition, i.e.

$$\mathrm{Re}\big[Y_{\mathfrak{r}_{2k}}^1(\bar{\pi}_{\mathfrak{r}_{2k}})^1\big]+\mathrm{Re}\big[Y_{\mathfrak{r}_{2k}}^2(\bar{\pi}_{\mathfrak{r}_{2k}})^2\big]=\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]+O\big(|\zeta_{\mathfrak{r}_{2k}}|^2+|\varrho_{\mathfrak{r}_{2k}}|^2\big). \tag{4.197}$$

(Here, the constants in the Landau notation $O(\dots)$ only depend on (**A**).) As in (4.192), we deduce that

$$\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\leq|\pi_{\mathfrak{r}_{2k}}|^2+O\Big(\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]\big|\big(|\zeta_{\mathfrak{r}_{2k}}|^2+|\varrho_{\mathfrak{r}_{2k}}|^2\big)\Big)$$
$$+O\big(|\zeta_{\mathfrak{r}_{2k}}|^4+|\varrho_{\mathfrak{r}_{2k}}|^4\big). \tag{4.198}$$

(Here, the Landau term $O(\dots)$ may depend on ϵ as well. Indeed, $\psi_{\mathfrak{r}_{2k}}=\epsilon/2$.) As a consequence, for any small $a>0$, we can write

$$\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]$$
$$\leq|\pi_{\mathfrak{r}_{2k}}|^2+a\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]+(1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k}}|^4+|\varrho_{\mathfrak{r}_{2k}}|^4\big).$$

By Proposition 4.8.12, we then deduce that (recall that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k}+$)

$$\lim_{t\to\mathfrak{r}_{2k}+}\bar{\Delta}_t$$
$$=\mu_2\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\eta_{\mathfrak{r}_{2k}}|^2+2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big]$$

$$\begin{aligned}
&\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\eta_{\mathfrak{r}_{2k}}|^2 + \psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big] \\
&\quad - \big(1-2\epsilon^{9/4}\big)\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big] \\
&\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})|\eta_{\mathfrak{r}_{2k}}|^2 + |\pi_{\mathfrak{r}_{2k}}|^2 + \big(a-1+2\epsilon^{9/4}\big)\psi_{\mathfrak{r}_{2k}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big] \\
&\quad + (1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k}}|^4 + |\varrho_{\mathfrak{r}_{2k}}|^4\big).
\end{aligned}$$

Choosing a small enough (in terms of ϵ), we deduce that

$$\lim_{t\to\mathfrak{r}_{2k}+}\bar{\Delta}_t \le \bar{\Delta}_{\mathfrak{r}_{2k}} + C\big(|\zeta_{\mathfrak{r}_{2k}}|^4 + |\varrho_{\mathfrak{r}_{2k}}|^4\big). \tag{4.199}$$

We apply the same strategy when $t\to\mathfrak{r}_{2k+1}+$. (Keep in mind that $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(1)}$ as $t\to\mathfrak{r}_{2k+1}+$ and by $\bar{\Delta}_t^{(2)}$ as $t\to\mathfrak{r}_{2k+1}-$.) Following (4.193), we claim

$$\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Delta}_t = \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\eta_{\mathfrak{r}_{2k+1}}|^2 + |\pi_{\mathfrak{r}_{2k+1}}|^2.$$

Now, as in (4.194),

$$|\pi_{\mathfrak{r}_{2k+1}}| = \psi_{\mathfrak{r}_{2k+1}}^{-1/2}\big|\mathrm{Re}\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big]\big| + O(|\zeta_{\mathfrak{r}_{2k+1}}|^2).$$

(Here as well, $O(\cdots)$ may depend on ϵ and ϵ_4. Indeed, $\psi_{\mathfrak{r}_{2k+1}} = \epsilon_4$.)

In particular, for any small $a>0$,

$$|\pi_{\mathfrak{r}_{2k+1}}|^2 \le (1+a)\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] + (1+a^{-1})O(|\zeta_{\mathfrak{r}_{2k+1}}|^4).$$

Following (4.195) and using Proposition 4.8.12, we deduce (as $t\to\mathfrak{r}_{2k+1}+$, $\bar{\Delta}_t$ is given by $\bar{\Delta}_t^{(1)}$)

$$\begin{aligned}
&\lim_{t\to\mathfrak{r}_{2k+1}+}\bar{\Delta}_t \\
&\quad\le \exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k+1}})|\eta_{\mathfrak{r}_{2k+1}}|^2 + \psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] \\
&\qquad + a\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] + (1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k+1}}|^4\big). \\
&\quad\le \mu_2\exp(-\epsilon^{-1/4}\psi_{\mathfrak{r}_{2k}})\psi_{\mathfrak{r}_{2k}}^{-\epsilon^2}|\eta_{\mathfrak{r}_{2k}}|^2 + 2\mu_2\epsilon^{9/4}\psi_{\mathfrak{r}_{2k}}^{-(1+\epsilon^2)}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k}}\eta_{\mathfrak{r}_{2k}}\big] \\
&\qquad + a\psi_{\mathfrak{r}_{2k+1}}^{-1}\mathrm{Re}^2\big[D_z\psi_{\mathfrak{r}_{2k+1}}\eta_{\mathfrak{r}_{2k+1}}\big] - \left[\left(\frac{\epsilon}{2\epsilon_4}\right)^{\epsilon^2} - 1\right]|\eta_{\mathfrak{r}_{2k+1}}|^2 \\
&\qquad + (1+a^{-1})O\big(|\zeta_{\mathfrak{r}_{2k+1}}|^4\big).
\end{aligned}$$

Choosing a small enough in terms of ϵ and ϵ_4, we deduce the analog of (4.199), i.e.

$$\lim_{t \to \mathfrak{r}_{2k+1}+} \bar{\Delta}_t \le \bar{\Delta}_{\mathfrak{r}_{2k+1}} + C|\zeta_{\mathfrak{r}_{2k+1}}|^4. \tag{4.200}$$

From (4.195) and (4.200), we deduce that, at least, for any $n \ge 0$,

$$\lim_{t \to \tau_n+} \bar{\Delta}_t \le \bar{\Delta}_{\tau_n} + C\bar{\Gamma}^2_{\tau_n},$$

the constant C here depending on (**A**), ϵ and ϵ_4, that is

$$\lim_{t \to \tau_n+} \left(\bar{\Delta}_t + \bar{\Gamma}^2_t\right) \le \bar{\Delta}_{\tau_n} + \bar{\Gamma}^2_{\tau_n} + C\bar{\Gamma}^2_{\tau_n}. \tag{4.201}$$

(Equation (4.201) must be seen as a version of (4.79).)

Inequality (4.201) is not very helpful. To get rid of the term $C\bar{\Gamma}^2_{\tau_n}$, we shall add a correction to the term $(\bar{\Delta}_t + \bar{\Gamma}^2_t)_{t\ge 0}$.

Choose indeed a non-negative smooth function θ with compact support included in $(0, +\infty)$ such that $\theta(\epsilon_4) = 1$ and $\theta(\epsilon/2) = 3$ and consider the processes

$$\begin{aligned}
\bar{\Phi}^{(1)}_t &= \bar{\Delta}^{(1)}_t + (1 + \theta(\psi_t)C)\left(\bar{\Gamma}^{(1)}_t\right)^2,\\
\bar{\Phi}^{(2)}_t &= \bar{\Delta}^{(2)}_t + (1 + 2C)\left(\bar{\Gamma}^{(2)}_t\right)^2,\\
\bar{\Phi}^{(3)}_t &= \bar{\Delta}^{(3)}_t + (1 + 2C)\left(\bar{\Gamma}^{(3)}_t\right)^2, \quad t \ge 0,
\end{aligned}$$

and define the global process $(\bar{\Phi}_t)_{t\ge 0}$ by gathering the three processes above according to the position of $(\psi_t)_{t\ge 0}$ as done to define $(\bar{\Gamma}_t)_{t\ge 0}$ and $(\bar{\Delta}_t)_{t\ge 0}$.

It is well seen that (4.196) still holds for Φ, i.e.

$$d\left[\exp\left(\int_0^t \alpha L\psi_r dr\right)\left(1 + \bar{\Phi}_t\right)^{1/2}\right] \le C\bar{\Gamma}_t \exp\left(\int_0^t \alpha L\psi_r dr\right)dt. \tag{4.202}$$

It thus remains to check the boundary conditions. When t tends to $\mathfrak{r}_{2k}+$, $\bar{\Phi}_t$ is given by $\bar{\Phi}^{(2)}_t$ and $\psi_t \to \epsilon/2$. Therefore, by (4.201)

$$\lim_{t \to \mathfrak{r}_{2k}+} \bar{\Phi}_t = \lim_{t \to \mathfrak{r}_{2k}+} \bar{\Phi}^{(2)}_t \le \bar{\Delta}_{\mathfrak{r}_{2k}} + (1 + 3C)\bar{\Gamma}^2_{\mathfrak{r}_{2k}} = \bar{\Phi}^{(1)}_{\mathfrak{r}_{2k+1}} = \bar{\Phi}_{\mathfrak{r}_{2k}}.$$

Similarly, when t tends to $\mathfrak{r}_{2k+1}+$, $\bar{\Phi}_t$ is given by $\bar{\Phi}^{(1)}_t$ and $\psi_t \to \epsilon_4$. Therefore, by (4.201)

$$\lim_{t \to \mathfrak{r}_{2k+1}+} \bar{\Phi}_t = \lim_{t \to \mathfrak{r}_{2k+1}+} \bar{\Phi}^{(1)}_t \le \bar{\Delta}_{\mathfrak{r}_{2k+1}} + (1 + 2C)\bar{\Gamma}^2_{\mathfrak{r}_{2k+1}} = \bar{\Phi}^{(2)}_{\mathfrak{r}_{2k+1}} = \bar{\Phi}_{\mathfrak{r}_{2k+1}}.$$

This completes the proof. □

4.9.6 Proof of the Differentiability Properties

Proposition 4.9.6 *Choose $0 < \check{\epsilon} < \epsilon_4 < \epsilon < \min(\epsilon_0, \epsilon_1')$, with ϵ_0 as in Proposition 4.8.12 and ϵ_1' as in Proposition 4.9.3, and consider a cut-off function φ_1 from $\mathbb{C}^d$ into $[0,1]$ matching 1 on the subset $\{z \in \mathcal{D} : \psi(z) \geq \check{\epsilon}\}$ and vanishing on the subset $\{z \in \mathcal{D} : \psi(z) \leq \check{\epsilon}/2\}$. Consider another cut-off function φ_2 from $\mathbb{C}$ to $\mathbb{C}$, matching 1 on $\{y \in \mathbb{C} : |y| \leq r_0\}$, $r_0 = \sup_{z \in \mathcal{D}} \psi^{1/2}(z)$, and vanishing outside $\{y \in \mathbb{C} : |y| \leq 2r_0\}$.*

For any $k \geq 0$, define on $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}]$, $\check{Z}^{\varepsilon}$ as the solution of

$$\begin{aligned} d\check{Z}_t^{s+\varepsilon} &= T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)(\varphi_1 \psi^{1/2})(\check{Z}_t^{s+\varepsilon}) \exp\big(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times \sigma_t \big(dB_t + G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)dt\big) \\ &\quad + |T|^2(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s) \exp\big(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times a_t \exp\big(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big)\big(\varphi_1 D_{\bar{z}}^* \psi\big)(\check{Z}_t^{s+\varepsilon})dt, \quad \mathfrak{r}_{2k} \leq t \leq \mathfrak{r}_{2k+1}, \end{aligned} \tag{4.203}$$

with $\check{Z}_0^{s+\varepsilon} = \gamma(s+\varepsilon)$ as initial condition. Above, $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, $(T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, and $(G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$, stand for the different possible perturbations used in Proposition 4.9.4. Precisely, $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 0 outside the intervals on which the perturbation of Proposition 4.9.3 applies, $(T(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 1 outside the intervals on which the perturbation of Proposition 4.8.4 applies and $(G(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is set equal to 0 outside the intervals on which the perturbation of Proposition 4.8.7 applies.

On $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}]$, define $\check{Z}^{s+\varepsilon}$ as the first coordinate of the pair $(\check{Z}_t^{s+\varepsilon}, \check{Y}_t^{\varepsilon})$ solution of

$$\begin{aligned} d\check{Z}_t^{s+\varepsilon} &= \sum_{i=1}^{2} \varphi_2\big[(\check{Y}_t^{s+\varepsilon})^i\big] dB_t^i \\ &\quad + \exp\big(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) a_t \exp\big(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) D_{\bar{z}}^* \psi(\check{Z}_t^{s+\varepsilon})dt \\ d\big(\check{Y}_t^{s+\varepsilon}\big)^i &= D_{\bar{z}}\psi(\check{Z}_t^{s+\varepsilon}) \exp\big(\bar{P}(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big)\bar{\sigma}_t d\bar{B}_t^i \\ &\quad + \frac{1}{2}\varphi_2\big[\big(\check{Y}_t^{s+\varepsilon}\big)^i\big] \mathrm{Trace}\big[\exp\big(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) \\ &\quad \times a_t \exp\big(-P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s)\big) D_{z,\bar{z}}^2 \psi(\check{Z}_t^{s+\varepsilon})\big] dt, \\ &\qquad \mathfrak{r}_{2k+1} \leq t \leq \mathfrak{r}_{2k+2}, \quad i = 1, 2, \end{aligned} \tag{4.204}$$

with $\check{Y}_t^{s+\varepsilon} = ((\varphi_1 \psi^{1/2})(\check{Z}_t^{s+\varepsilon}), 0)$ as initial condition. (Above, ψ is understood as any smooth extension with compact support of the original ψ to the whole space $\mathbb{C}^d$. The perturbation $(P(Z_t^s, \check{Z}_t^{s+\varepsilon} - Z_t^s))_{t\geq 0}$ is the same as in (4.203).)

Then, the process $(\check{Z}_t^{s+\varepsilon})_{t\ge 0}$ *is twice differentiable in the mean w.r.t.* ε*, with time continuous first and second order derivatives, and, the process* $(\sum_{k\ge 0}\check{Y}_t^{s+\varepsilon}\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t))_{t\ge 0}$ *is also twice differentiable w.r.t.* ε*, with time continuous first and second order derivatives on every* $[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]$*,* $k\ge 0$*.*

Moreover, for any $S>0$ *and any integer* $p\ge 1$*,*

$$\sup_{0<|\varepsilon'|<|\varepsilon|}\ \sup_{\sigma}\mathbb{E}\Big[\sup_{0\le t\le S}\big(|\check{\zeta}_t^{s+\varepsilon'}|^p+|\check{\eta}_t^{s+\varepsilon'}|^p\big)\Big]<+\infty, \tag{4.205}$$

and

$$\sup_{0<|\varepsilon'|<|\varepsilon|}\ \sup_{\sigma}\mathbb{E}\Big[\sup_{k\ge 0}\ \sup_{\mathfrak{r}_{2k+1}\le t\le \mathfrak{r}_{2k+2},t\le S}\big(|\check{\varrho}_t^{s+\varepsilon'}|^p+|\check{\pi}_t^{s+\varepsilon'}|^p\big)\Big]<+\infty, \tag{4.206}$$

and

$$\lim_{\varepsilon\to 0}\sup_{\sigma}\mathbb{E}\Big[\sup_{0\le t\le S}\big(\big|\check{\zeta}_t^{s+\varepsilon}-\check{\zeta}_t^{s}\big|^p+\big|\check{\eta}_t^{s+\varepsilon}-\check{\eta}_t^{s}|^p\big)\Big]=0, \tag{4.207}$$

where $\check{\zeta}_t^{s+\varepsilon}=[d/d\varepsilon][\check{Z}_t^{s+\varepsilon}]$*,* $\check{\varrho}_t^{s+\varepsilon}=[d/d\varepsilon][\check{Y}_t^{s+\varepsilon}]\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t)$*, and* $\check{\eta}_t^{s+\varepsilon}=[d^2/d\varepsilon^2][\check{Z}_t^{s+\varepsilon}]$*,* $\check{\pi}_t^{s+\varepsilon}=[d^2/d\varepsilon^2][\check{Y}_t^{s+\varepsilon}]\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t)$*,* $t\ge 0$*,* $k\ge 0$*.*

Proof. We first establish differentiability in probability. By Theorem 4.7.4, twice differentiability in probability holds on $[0,\mathfrak{r}_1]$, i.e. $(\check{\zeta}_t^{s+\varepsilon})_{0\le t\le \mathfrak{r}_1}$ and $(\check{\eta}_t^{s+\varepsilon})_{0\le t\le \mathfrak{r}_1}$ exist for any ε in the neighborhood of 0, and, for any $S>0$,

$$\lim_{\varepsilon'\to 0,\varepsilon'\ne 0}\ \sup_{0\le t\le S\wedge\mathfrak{r}_1}\big\{\big|\delta_{\varepsilon'}\check{Z}_t^{s+\varepsilon}-\check{\zeta}_t^{s}\big|+\big|\delta_{\varepsilon'}\check{\zeta}_t^{s+\varepsilon}-\check{\eta}_t^{s+\varepsilon}\big|\big\}=0,$$

in $\mathbb{P}$-probability, i.e. in the sense of (4.64).

In particular, in $\mathbb{P}$-probability,

$$\lim_{\varepsilon'\to 0,\varepsilon'\ne 0}\big\{\big|\delta_{\varepsilon'}\check{Z}_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}-\check{\zeta}_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}\big|+\big|\delta_{\varepsilon'}\check{\zeta}_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}-\check{\eta}_{S\wedge\mathfrak{r}_1}^{s+\varepsilon}\big|\big\}=0,$$

so that we can apply Theorem 4.7.4 again, but on the time interval $[\mathfrak{r}_1,\mathfrak{r}_2]\cap[0,S]$, or equivalently on $[\mathfrak{r}_1,\mathfrak{r}_2\wedge S]$ and on the event $\{\mathfrak{r}_1\le S\}$. Indeed, the dynamics of $(\check{Z}^{s+\varepsilon},\check{Y}^{s+\varepsilon})$ on $[\mathfrak{r}_1,\mathfrak{r}_2]\cap[0,S]$ are given by (4.204): (4.204) satisfies Theorem 4.7.4. We deduce that $(\check{\zeta}_t^{s+\varepsilon},\check{\rho}_t^{s+\varepsilon})_{\mathfrak{r}_1\le t\le\mathfrak{r}_2,t\le S}$ and $(\check{\eta}_t^{s+\varepsilon},\check{\pi}_t^{s+\varepsilon})_{\mathfrak{r}_1\le t\le\mathfrak{r}_2,t\le S}$ exist and

$$\lim_{\varepsilon'\to 0,\varepsilon'\ne 0}\ \sup_{\mathfrak{r}_1\le t\le\mathfrak{r}_2,t\le S}\big\{\big|(\delta_{\varepsilon'}\check{Z}_t^{s+\varepsilon},\delta_{\varepsilon'}\check{Y}_t^{s+\varepsilon})-(\check{\zeta}_t^{s+\varepsilon},\check{\rho}_t^{s+\varepsilon})\big| \\ +\big|(\delta_{\varepsilon'}\check{\zeta}_t^{s+\varepsilon},\delta_{\varepsilon'}\check{\rho}_t^{s+\varepsilon})-(\check{\eta}_t^{s+\varepsilon},\check{\pi}_t^{s+\varepsilon})\big|\big\}=0,$$

in $\mathbb{P}$-probability. Then, the procedure can be applied again but on $[\mathfrak{r}_2,\mathfrak{r}_3]\cap[0,S]$, and so on by induction. This proves that twice differentiability in probability holds for the pair process $(\check{Z}^{s+\varepsilon}_{t\wedge\mathfrak{r}_n}, \sum_{k\geq 0} \check{Y}^{s+\varepsilon}_{t\wedge\mathfrak{r}_n}\mathbf{1}_{[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}]}(t\wedge\mathfrak{r}_n))_{0\leq t\leq S}$, $n\geq 0$. Since $\mathfrak{r}_n\to+\infty$ a.s., twice differentiability in probability follows on the whole $[0,S]$, for any $S>0$. (We emphasize that $\mathfrak{r}_n\to+\infty$ a.s. since the process $(\psi(Z^s_t))_{t\geq 0}$ is a.s. continuous: it cannot switch from ϵ_4 to $\epsilon/2$ an infinite number of times on a compact set.) Twice differentiability in the mean will follow from (4.205), (4.206) and (4.67).

To prove (4.205), we emphasize that, for any $k\geq 0$, we can find a constant C, independent of ε, γ, k and σ, such that, on each $[\mathfrak{r}_{2k},\mathfrak{r}_{2k+1})$,[27]

$$d\big[\exp(-Ct)\big|\check{\zeta}^{s+\varepsilon}_t\big|^{2p}\big] \leq dm_t, \quad \mathfrak{r}_{2k}\leq t<\mathfrak{r}_{2k+1}, \tag{4.208}$$

$(m_t)_{\mathfrak{r}_{2k}\leq t<\mathfrak{r}_{2k+1}}$ standing for a generic martingale term. (The proof is the same as the proof of Corollary 4.7.5.)

Similarly, up to a modification of the constant C, on each $[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2})$, $k\geq 0$,

$$d\big[\exp(-Ct)\big(\big|\check{\zeta}^{s+\varepsilon}_t\big|^{2p}+\big|\check{\varrho}^{s+\varepsilon}_t\big|^{2p}\big]\big)\big] \leq dm_t, \quad \mathfrak{r}_{2k+1}\leq t<\mathfrak{r}_{2k+2}. \tag{4.209}$$

To gather (4.208) and (4.209), it is sufficient to check what happens at boundary times $\mathfrak{r}_n$, $n\geq 0$. The relationship $\check{Y}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}} = ((\varphi_1\psi^{1/2})(\check{Z}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}),0)$ yields

$$\big|\check{\varrho}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}\big| = \big|\mathrm{Re}\big[D_z\big(\varphi_1\psi^{1/2}\big)(\check{Z}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}})\check{\zeta}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}\big]\big| \leq C'\big|\check{\zeta}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}\big|,$$

for some constant C' (independent of ε, γ, k and σ).

Below, we consider a non-negative smooth function θ with values in $[0,1]$, matching 1 in ϵ_4 and 0 in $\epsilon/2$. Then, for any $k\geq 0$,

$$\begin{aligned}
&\lim_{t\to\mathfrak{r}_{2k+1}-}\big[\big(1+C'\theta\big(\psi(\check{Z}^s_t)\big)\big)\big|\check{\zeta}^{s+\varepsilon}_t\big|^{2p}\big] \geq \big|\check{\zeta}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}\big|^{2p}+\big|\check{\varrho}^{s+\varepsilon}_{\mathfrak{r}_{2k+1}}\big|^{2p},\\
&\lim_{t\to\mathfrak{r}_{2k+2}-}\big[\big|\check{\zeta}^{s+\varepsilon}_t\big|^{2p}+\big|\check{\varrho}^{s+\varepsilon}_t\big|^{2p}\big] \geq \big|\check{\zeta}^{s+\varepsilon}_{\mathfrak{r}_{2k+2}}\big|^{2p} = \big(1+C'\theta\big(\psi(\check{Z}^s_{\mathfrak{r}_{2k+2}})\big)\big)\big|\check{\zeta}^{s+\varepsilon}_{\mathfrak{r}_{2k+2}}\big|^{2p}.
\end{aligned} \tag{4.210}$$

Indeed, $\psi(\check{Z}^s_{\mathfrak{r}_{2k+1}})=\epsilon_4$ and $\psi(\check{Z}^s_{\mathfrak{r}_{2k+2}})=\epsilon/2$, $k\geq 0$. (Obviously, $(\check{Z}^{s+\varepsilon}_t)_{t\geq 0}$ is continuous in time.) Now, it remains to see that

$$d\big[\exp(-Ct)\big(1+C'\theta\big(\psi(\check{Z}^s_t)\big)\big)\big|\check{\zeta}^{s+\varepsilon}_t\big|^{2p}\big] \leq dm_t, \quad \mathfrak{r}_{2k}\leq t<\mathfrak{r}_{2k+1}, \quad k\geq 0,$$

for a possibly new value of C. (This follows from Itô's formula.)

[27] Here, we feel simpler to use right-continuous versions of the processes at hand. Actually, this has an interest for $(\check{\varrho}^{s+\varepsilon}_t)_{t\geq 0}$ only since $(\check{\zeta}^{s+\varepsilon}_t)_{t\geq 0}$ is continuous.

Set finally

$$M_t^p := \begin{cases} \exp(-Ct)\big(1 + C'\theta\big(\psi(\check{Z}_t^s)\big)\big)\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p}, \ \mathfrak{r}_{2k} \le t < \mathfrak{r}_{2k+1}, \\ \exp(-Ct)\big(\big|\check{\zeta}_t^{s+\varepsilon}\big|^{2p} + \big|\check{\varrho}_t^{s+\varepsilon}\big|^{2p}\big), \ \mathfrak{r}_{2k+1} \le t < \mathfrak{r}_{2k+2}, \end{cases}, \quad k \ge 0.$$

Then, for any $n \ge 0$, $t \mapsto \mathbb{E}[M_{t\wedge \mathfrak{r}_n}]$ is non-increasing. (Use the martingale property and (4.210)). This proves the part related to the first-order derivatives in (4.205) and (4.206), but with the supremum outside the expectation. To get the supremum inside the expectation, we can use so-called Doob's inequality. It says that, for any square integrable progressively-measurable process $(H_t)_{0\le t\le S}$ with values in $\mathbb{C}^d$,

$$\mathbb{E}\left[\sup_{0\le t\le S}\left|\int_0^t \langle H_s, dB_s\rangle\right|^2\right] \le c\mathbb{E}\int_0^S |H_t|^2 dt,$$

for some universal $c > 0$. We then choose $(m_t)_{0\le t\le S}$ for $(\int_0^t \langle H_s, dB_s\rangle)_{0\le t\le S}$. We notice that the corresponding process $(H_t)_{0\le t\le S}$ is always bounded by $C'|\check{\zeta}^{s+\varepsilon}|^{2p}$ for $t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0, S]$, $k \ge 0$, and by $C'(|\check{\zeta}^{s+\varepsilon}|^{2p} + |\check{\rho}^{s+\varepsilon}|^{2p})$ for $t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0, S]$, $k \ge 0$, for some constant C' independent of ε, γ, k and σ. Using the bounds for $(\mathbb{E}[M_t^{2p}])_{0\le t\le S}$, (4.205) and (4.206) follow. A similar argument holds for the second-order derivatives (handling the boundary condition by considering $(|\check{\zeta}_t^{s+\varepsilon}|^{4p})_{t\ge 0}$ as in the proof of Corollary 4.9.5).

We finally turn to (4.207). It relies on the stability property of SDEs. (See Proposition 4.7.1.) Basically, Proposition 4.7.1 applies on any interval $[\mathfrak{r}_n, \mathfrak{r}_{n+1}]$. By induction, we obtain

$$\forall n \ge 1, \quad \lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le S}\big(\big|\check{\zeta}_t^{s+\varepsilon} - \check{\zeta}_t^s\big|^p + \big|\check{\eta}_t^{s+\varepsilon} - \check{\eta}_t^s|^p\big); S \le \mathfrak{r}_n\right] = 0. \quad (4.211)$$

To get the same estimate but on the whole space, we first notice that

$$\lim_{n\to+\infty}\sup_{\sigma}\mathbb{P}\{S \le \mathfrak{r}_n\} = 1. \quad (4.212)$$

Equation (4.212) follows from a tightness argument. Since the coefficients of $(Z_t^s)_{t\ge 0}$ are bounded, uniformly in σ, the paths of $(Z_t^s)_{0\le t\le S}$ are continuous, uniformly in σ, with large probability: specifically, given a small positive real ν, we can find a compact subset $\mathcal{K} \subset \mathcal{C}([0, S], \mathbb{C}^d)$, such that, for any σ, $(Z_t^s)_{0\le t\le S}$ belongs to $\mathcal{K}$ with probability greater than $1-\nu$. To prove (4.212), it then remains to see that $\mathfrak{r}_{2n}/n$ is greater than the smallest amount of time $(Z_t^s)_{t\ge 0}$ needs to switch from ϵ_4 to $\epsilon/2$: clearly, on $[0, S]$, this smallest amount of time is controlled from below in terms of the modulus of continuity of $(Z_t^s)_{0\le t\le S}$ only. In particular, when $(Z_t^s)_{0\le t\le S}$ belongs to $\mathcal{K}$, S must be less than $\mathfrak{r}_{2n}$ for n larger than some n_0, n_0 depending on $\mathcal{K}$ and S only.

In particular,

$$\lim_{n\to+\infty}\sup_{\sigma}\mathbb{P}\{S>\mathfrak{r}_n\}=0.$$

By (4.205), (4.206) and Cauchy-Schwarz inequality,

$$\lim_{n\to+\infty}\sup_{\sigma}\mathbb{E}\Big[\sup_{0\le t\le S}\big(|\check{\varrho}_t^{s+\varepsilon'}|^p+|\check{\pi}_t^{s+\varepsilon'}|^p\big);S>\mathfrak{r}_n\Big]=0, \tag{4.213}$$

uniformly in ε' in a neighborhood of 0.

By (4.211) and (4.213), we complete the proof of (4.207). □

We are now in position to justify the *meta-statements*:

Corollary 4.9.7 *Keep the assumption and notation of Propositions 4.9.4 and 4.9.6. Then, for any $S>0$ and for $\check{\epsilon}$ as in Proposition 4.9.6, there exist a decreasing sequence of positive reals $(\varepsilon_n)_{n\ge1}$, a countable family of increasing events $(\Omega_n)_{n\ge1}$ (i.e. $\Omega_n\subset\Omega_{n+1}$, $n\ge1$), such that $\mathbb{P}(\Omega_n)\to1$ as $n\to+\infty$, and continuous processes $((\zeta_t^{s+\varepsilon})_{0\le t\le S},((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_0}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S},((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_0}$ such that, for any $n\ge1$, $((Z_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ is twice differentiable in probability on the event Ω_n, with $((\zeta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ as first and second order derivatives, that is, with the notations of Theorem 4.7.4,*

$$\forall\varepsilon\in(-\varepsilon_n,\varepsilon_n),\ \forall\nu>0,\ \lim_{\varepsilon'\to0,\varepsilon'\ne0}\mathbb{P}\Big\{\sup_{0\le t\le S}\big|\delta_{\varepsilon'}Z_t^{s+\varepsilon}-\zeta_t^{s+\varepsilon}\big|>\nu,\Omega_n\Big\}=0,$$

$$\lim_{\varepsilon'\to0,\varepsilon'\ne0}\mathbb{P}\Big\{\sup_{0\le t\le S}\big|\delta_{\varepsilon'}\zeta_t^{s+\varepsilon}-\eta_t^{s+\varepsilon}\big|>\nu,\Omega_n\Big\}\ =0,$$

and, for every $k\ge0$ and $n\ge1$, the family $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ is twice differentiable in probability on Ω_n, with $((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\le t\le\mathfrak{r}_{2k+1},t\le S})_{|\varepsilon|<\varepsilon_n}$ as first and second order derivatives.

Moreover, on each Ω_n, the dynamics of the processes $((\zeta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ and $((\eta_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ are obtained by differentiating w.r.t. ε the dynamics of $((Z_t^{s+\varepsilon})_{0\le t\le S})_{|\varepsilon|<\varepsilon_n}$ formally, as done in the meta-part *of Sect. 4.8. The same holds for the processes $((\rho_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_n}$ and $((\pi_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},t\le S})_{k\ge0})_{|\varepsilon|<\varepsilon_n}$.*

Finally, a.s.,

$$\zeta_t^s=\frac{d}{d\varepsilon}\big[\check{Z}_t^{s+\varepsilon}\big]_{|\varepsilon=0},\qquad \eta_t^s=\frac{d^2}{d\varepsilon^2}\big[\check{Z}_t^{s+\varepsilon}\big],\quad t\ge0,$$

$$\rho_t^s=\frac{d}{d\varepsilon}\big[\check{Y}_t^{s+\varepsilon}\big]_{|\varepsilon=0},\qquad \pi_t^s=\frac{d^2}{d\varepsilon^2}\big[\check{Y}_t^{s+\varepsilon}\big],\quad \mathfrak{r}_{2k+1}\le t\le\mathfrak{r}_{2k+2},\quad k\ge0. \tag{4.214}$$

Before we make the proof, we emphasize the following: the reader may worry about the properties of differentiability of the processes $(Z_t^{s+\varepsilon})_{t\geq 0}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ at $\varepsilon = 0$. Indeed, we here discussed the notion of differentiability in probability only whereas we used the notion of differentiability in the mean in the meta-statements of Sect. 4.8. The reason is the following: all the differentiations we perform below under the symbol $\mathbb{E}$ hold on the families $(\check{Z}_t^{s+\varepsilon})_{t\geq 0}$ and $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ only, so that differentiability in the mean of $(Z_t^{s+\varepsilon})_{t\geq 0}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ is useless. By Proposition 4.9.6, the families $(\check{Z}_t^{s+\varepsilon})_{t\geq 0}$ and $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}})_{k\geq 0}$ are known to be differentiable in the mean.

Proof. For an arbitrary $\check{\epsilon}$ as in the statement of Proposition 4.9.6 we know that $(Z_t^s)_{t\geq 0}$ and $(\check{Z}_t^s)_{t\geq 0}$ coincide. (Cut-off functions match 1 because of the stopping times.) Similarly, $((Y_t^s)_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}})_{k\geq 0}$ and $((\check{Y}_t^s)_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}})_{k\geq 0}$ coincide.

By Theorem 4.7.2, we know that the mappings $((t,\varepsilon) \in \mathbb{R}_+ \times [-\varepsilon_0, \varepsilon_0] \mapsto \check{Z}_t^{s+\varepsilon}$ are once-continuously differentiable for every $\check{\epsilon}$ as in Proposition 4.9.6. (Here ε_0 stands for a small enough positive real such that $[s-\varepsilon, s+\varepsilon] \subset [-1,1]$). In particular, they are continuous, so that $\sup_{|\varepsilon'|<\varepsilon} \sup_{0\leq t\leq S} |\check{Z}_t^{s+\varepsilon'} - \check{Z}_t^s|$ tends to 0 a.s. as ε tends to 0. Therefore, we can find ε_n small enough such that the event

$$\mathcal{N}_n := \left\{ \inf_{|\varepsilon'|<\varepsilon_n} \inf_{k\geq 0} \inf_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}, t\leq S} \psi(\check{Z}_t^{s+\varepsilon'}) \leq \check{\epsilon} \right\},$$

has probability less than $1/n$.

Set $\Omega_n = (\mathcal{N}_n)^{\complement}$ so that $\mathbb{P}(\Omega_n) \geq 1 - 1/n$. On Ω_n, $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ coincide with $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ and $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}, t\leq S})_{k\geq 0}$ coincide with the process $((\check{Y}_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}, t\leq S})_{k\geq 0}$ for any $\varepsilon \in (-\varepsilon_n, \varepsilon_n)$. (Indeed, on each $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0,S]$, $k \geq 0$, the process $(\psi(Z_t^{s+\varepsilon}))_{\mathfrak{r}_{2k}\leq t\leq \mathfrak{r}_{2k+1}, t\leq S}$ is above $\check{\epsilon}$ so that $\varphi_1(\check{Z}_t^{s+\varepsilon})$ in (4.203) and in the initial condition of (4.204) matches 1. As a consequence, on each $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0,S]$, $k \geq 0$, $|\check{Y}_t^{s+\varepsilon}|^2 = \psi(\check{Z}_t^{s+\varepsilon})$.) Twice differentiability in probability of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ on Ω_n easily follows.

We now check that, on each Ω_n, $n \geq 1$, the dynamics of the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ w.r.t. $\varepsilon \in (-\varepsilon_n, \varepsilon_n)$ are obtained by differentiating the dynamics of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ formally. This is well-seen since the dynamics of the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ are obtained by differentiating the dynamics of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ formally and since the cut-off functions φ_1 and φ_2 in the dynamics of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ match 1 on Ω_n.

In particular, on each Ω_n, $n \geq 1$, the derivatives of $(Z_t^{s+\varepsilon})_{0\leq t\leq S}$ at $\varepsilon = 0$ and the derivatives of $(\check{Z}_t^{s+\varepsilon})_{0\leq t\leq S}$ at $\varepsilon = 0$ coincide. Taking the union over $n \geq 1$, this shows that equality holds almost-surely.

A similar argument holds for $((Y_t^{s+\varepsilon})_{\mathfrak{r}_{2k+1}\leq t\leq \mathfrak{r}_{2k+2}, t\leq S})_{k\geq 0}$. □

4.9.7 Differentiability Under the Symbol $\mathbb{E}$

We now claim

Proposition 4.9.8 *With the choice made for* $(Z_t^s)_{t\geq 0}$ *and* $(Z_t^{s+\varepsilon})_{t\geq 0}$ *in Proposition 4.9.4, for a smooth path* γ *from* $[-1,1]$ *into* $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$ *and for a given* $s \in [-1,1]$*, define* $\hat{V}_S^g$*,* V_S^g *and* V *as in Proposition 4.8.14. Then, the conclusion of Proposition 4.8.14 is still true.*

Sketch of the Proof. The proof follows the argument used to establish Proposition 4.9.1. (See (4.170)–(4.173).)

Consider $(Z_t^s)_{t\geq 0}$ and define the process

$$W_t = \sum_{n\geq 0}\left(\int_0^t \mathbf{1}_{\{\mathfrak{r}_{2n}\leq r<\mathfrak{r}_{2n+1}\}} dB_r\right) + \sum_{i=1,2}\sum_{n\geq 0}\left(\int_0^t \mathbf{1}_{\{\mathfrak{r}_{2n+1}\leq r<\mathfrak{r}_{2n+2}\}}\left(\frac{Y_r^i}{|Y_r|}\mathbf{1}_{\{|Y_r|>0\}} + \frac{1}{\sqrt{2}}\mathbf{1}_{\{|Y_r|=0\}}\right)dB_r^i\right), \quad t \geq 0.$$

Then, $(W_t)_{t\geq 0}$ is a complex Brownian motion of dimension d. Moreover,

$$dZ_t^s = \psi^{1/2}(Z_t^s)dW_t + a_t D_{\bar{z}}^*\psi(Z_t^s)dt, \quad t \geq 0.$$

Therefore, for $(Z_t^s)_{t\geq 0}$, everything works as in Proposition 4.8.14 but with $(B_t)_{t\geq 0}$ replaced by $(W_t)_{t\geq 0}$.

A similar argument holds for $(Z_t^{s+\varepsilon})_{t\geq 0}$ w.r.t. some $(W_t^\varepsilon)_{t\geq 0}$ (obtained in a similar way). To do so, we emphasize that $(\langle \bar{G}(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s), dB_t\rangle)_{t\geq 0}$ in (4.160) is equal to $(\langle \bar{G}(Z_t^s, Z_t^{s+\varepsilon} - Z_t^s), dW_t^\varepsilon\rangle)_{t\geq 0}$ since G is set equal to 0 on $[\mathfrak{r}_{2n+1}, \mathfrak{r}_{2n+2}]$, $n \geq 0$. □

We now deduce

Proposition 4.9.9 *Keep the assumption and notation of Proposition 4.9.8 and consider in particular a smooth path* γ *from* $[-1,1]$ *into* $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$*. Then, there exists a constant* $C > 0$*, depending on* $(\mathbf{A})$ *only, such that, for any* $S > 0$*, the function* $s \in (-1,1) \mapsto V_S(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ *is non-decreasing, the function* $s \in (-1,1) \mapsto V_S(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ *is non-increasing and the function* $s \in (-1,1) \mapsto V_S(\gamma(s)) + C\int_0^s [(s-r)(|\gamma'(r)|^2 + |\gamma''(r)|)]dr$ *is convex.*

Proof. It is sufficient to find some constant C, depending on $(\mathbf{A})$ only, such that for any $s \in (-1,1)$,

$$\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))}{|\varepsilon|}\geq -C|\gamma'(s)|,$$
$$\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon))+V_S(\gamma(s-\varepsilon))-2V_S(\gamma(s))}{\varepsilon^2}\geq -C\big(|\gamma'(s)|^2+|\gamma''(s)|\big), \tag{4.215}$$

and to prove that $V_S\circ\gamma$ is continuous. To do so, we first claim:

Lemma 4.9.10 *Choose $\epsilon=\min(\epsilon_0,\epsilon_1')/2$, with ϵ_0 as in Proposition 4.8.12 and ϵ_1' as in Proposition 4.9.3.*

Define

$$\check{p}_t^{\varepsilon}=P(Z_r^s,\check{Z}_r^{s+\varepsilon}-Z_r^s),\ \check{\tau}_t^{\varepsilon}=T(Z_r^s,\check{Z}_r^{s+\varepsilon}-Z_r^s),$$
$$\check{\Xi}_t^{\varepsilon}=G(Z_r^s,\check{Z}_r^{s+\varepsilon}-Z_r^s),\quad t\geq 0.$$

For a given smooth cut-off function ρ with values in $[0,1]$ matching the identity on $[1/2,3/2]$ and vanishing outside a compact subset, set as well

$$\begin{aligned}&\check{V}_S^{\sigma}(s+\varepsilon)\\&=\mathbb{E}\int_0^{+\infty}\Big[\rho\Big(\exp\Big(-\int_0^t 2\mathrm{Re}\big[\langle\bar{\check{\Xi}}_r^{\varepsilon},dB_r\rangle\big]-\int_0^t|\check{\Xi}_r^{\varepsilon}|^2dr\Big)\Big)\\&\qquad\times\exp\Big(\int_0^t|\check{\tau}_r^{\varepsilon}|^2\mathrm{Trace}[\exp(\check{p}_r^{\varepsilon})a_r\exp(-\check{p}_r^{\varepsilon})D^2_{z,\bar z}\psi(\check{Z}_r^{s+\varepsilon})]dr\Big)\\&\qquad\times F\big(\det(a_t),\exp(\check{p}_t^{\varepsilon})a_t\exp(-\check{p}_t^{\varepsilon}),\check{Z}_t^{s+\varepsilon}\big)\phi\big(\frac{\check{\mathfrak{T}}_t^{\varepsilon}}{S}\big)\Big]|\check{\tau}_t^{\varepsilon}|^2dt,\end{aligned}\tag{4.216}$$

with $[d/dt](\check{\mathfrak{T}}_t^{\varepsilon})=(\check{\tau}_t^{\varepsilon})^2$, $t\geq 0$.

*Then, $\sup_\sigma[\check{V}_S^{\sigma}(s)]=V_S(\gamma(s))$ and, for ε in the neighborhood of 0, $\sup_\sigma[\check{V}_S^{\sigma}(s+\varepsilon)]\leq V_S(\gamma(s+\varepsilon))+C\varepsilon^3$, for a constant C depending on (**A**) and S only.*

Moreover, we can find a constant C such that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\Big|\frac{d}{d\varepsilon'}\big[\check{V}_S^{\sigma}\big(\gamma(s+\varepsilon')\big)\big]\Big|\\&\quad\leq\mathbb{E}\Big[\int_0^{+\infty}\exp\Big(\int_0^t\mathrm{Trace}[a_rD^2_{z,\bar z}\psi(Z_r^s)]dr\Big)\\&\qquad\times\Big(|\bar{\Gamma}_t|+\int_0^t(1+r^{-1/2})|\bar{\Gamma}_r|dr\Big)dt\Big],\end{aligned}\tag{4.217}$$

and,

$$\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\left|\frac{d^2}{d\varepsilon'^2}\left[\check{V}_S^{\sigma}\big(\gamma(s+\varepsilon')\big)\right]\right|$$
$$\leq \mathbb{E}\bigg[\int_0^{+\infty}\exp\bigg(\int_0^t \mathrm{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\bigg)$$
$$\times\bigg(|\bar{\Gamma}_t|^2+|\bar{\Delta}_t|+\int_0^t(1+r^{-1/2})\big(|\bar{\Gamma}_r|^2+|\bar{\Delta}_r|\big)dr\bigg)dt\bigg]. \quad (4.218)$$

Finally, for every compact interval $I\subset(-1,1)$ and for ε small enough, the quantity $\sup_\sigma\sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\check{V}_S^{\sigma}(\gamma(s+\varepsilon'))]|]$ is uniformly bounded w.r.t. $s\in I$. (Pay attention that the definition of $\check{V}_S^{\sigma}$ depends on s itself.)

End of the Proof of Proposition 4.9.9. Before we prove Lemma 4.9.10, we complete the proof of Proposition 4.9.9. Clearly, by Lemma 4.9.10

$$\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))}{|\varepsilon|}\geq\lim_{\varepsilon\to 0}\frac{1}{|\varepsilon|}\left[\inf_{\sigma}\big(\check{V}^{\sigma}(s+\varepsilon)-\check{V}^{\sigma}(s)\big)\right]$$
$$\geq-\lim_{\varepsilon\to 0}\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\left|\frac{d}{d\varepsilon'}\left[\check{V}^{\sigma}(s+\varepsilon')\right]\right|.$$

By Lemma 4.9.10, we deduce that

$$\lim_{\varepsilon\to 0}\frac{V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))}{|\varepsilon|}$$
$$\geq-\sup_{\sigma}\mathbb{E}\bigg[\int_0^{+\infty}\exp\bigg(\int_0^t\mathrm{Trace}\big[a_r D^2_{z,\bar{z}}\psi(Z_r^{\sigma})\big]dr\bigg)$$
$$\times\bigg(|\bar{\Gamma}_t|+\int_0^t(1+r^{-1/2})|\bar{\Gamma}_r|dr\bigg)dt\bigg].$$

By Proposition 4.9.4, we deduce that there exists a constant C, depending on (**A**) only, such that the first inequality in (4.215) holds. The same strategy holds to prove the second inequality in (4.215).

It remains to prove that $V_S\circ\gamma$ is continuous. Basically,

$$V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))\geq\sup_{\sigma}\big[\check{V}_S^{\sigma}(s+\varepsilon)\big]-\sup_{\sigma}\big[\check{V}_S^{\sigma}(s)\big]-C|\varepsilon|^3$$
$$\geq-|\varepsilon|\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\left[\left|\frac{\partial\check{V}_S^{\sigma}}{\partial\varepsilon'}(s+\varepsilon')\right|\right]-C|\varepsilon|^3.$$

Therefore, for any compact interval $I\subset(-1,1)$, for ε small enough, we can find some constant C' such that

$$V_S(\gamma(s+\varepsilon))-V_S(\gamma(s))\geq -C'|\varepsilon|,$$

when s and $s+\varepsilon$ are in I. Exchanging the roles of $s+\varepsilon$ and s, this proves that $V_S \circ \gamma$ is continuous. □

We now prove Lemma 4.9.10.

Proof of Lemma 4.9.10. The equality $\sup_\sigma[\check{V}_S^\sigma(s)] = V_S(\gamma(s))$ is easily taken since $\check{V}_S^\sigma(s) = \hat{V}_S^\sigma(s)$, with $\hat{V}_S^\sigma$ as in Proposition 4.9.8.

We now establish the inequality $\sup_\sigma[\check{V}_S^\sigma(s+\varepsilon)] \le V_S(\gamma(s+\varepsilon)) + C\varepsilon^3$. It is well-seen that all the terms under the integral symbol in (4.216) are bounded by some constant C depending on $(\mathbf{A})$ and S only.

Therefore, for some $\epsilon' > 0$ to be chosen later,

$$\begin{aligned}
&\check{V}_S^\sigma(s+\varepsilon) \\
&\quad = \mathbb{E}\bigg\{\int_0^{+\infty}\bigg[\rho\bigg(\exp\bigg(-\int_0^t 2\mathrm{Re}\big[\langle\check{\Xi}_r^\varepsilon, dB_r\rangle\big] - \int_0^t |\check{\Xi}_r^\varepsilon|^2 dr\bigg)\bigg) \\
&\qquad\qquad \times \exp\bigg(\int_0^t |\check{\tau}_r^\varepsilon|^2 \mathrm{Trace}[\exp(\check{p}_r^\varepsilon) a_r \exp(-\check{p}_r^\varepsilon) D^2_{z,\bar{z}}\psi(\check{Z}_r^{s+\varepsilon})]dr\bigg) \\
&\qquad\qquad \times F\big(\det(a_t), \exp(\check{p}_t^\varepsilon) a_t \exp(-\check{p}_t^\varepsilon), \check{Z}_t^{s+\varepsilon}\big)\phi\bigg(\frac{\check{\mathfrak{T}}_t^\varepsilon}{S}\bigg)\bigg]|\check{\tau}_t^\varepsilon|^2 dt; \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon} - Z_t^s| \le \epsilon'\bigg\} \\
&\qquad\qquad + O\big(\mathbb{P}\{\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon} - Z_t^s| \ge \epsilon'\}\big).
\end{aligned} \tag{4.219}$$

(Here, the Landau term $O(\cdots)$ is uniform w.r.t. ε.)

As long as the process $(|\check{Z}_t^{s+\varepsilon} - Z_t^s|)_{t\ge 0}$ stays below ϵ', the process $(|\psi(\check{Z}_t^{s+\varepsilon}) - \psi(Z_t^s)|)_{t\ge 0}$ stays below some $C\epsilon'$, C depending on ψ only. In particular, we can choose ϵ' small enough such that $C\epsilon' < \check{\epsilon}/2$. (See Proposition 4.9.6 for the definition of $\check{\epsilon}$.)

On each $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}]$, $k \ge 0$, as in Proposition 4.9.4, the process $(\psi(Z_t^s))_{\mathfrak{r}_{2k}\le t\le \mathfrak{r}_{2k+1}}$ is above $\epsilon_4 > 2\check{\epsilon}$. Therefore, on each $[\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0,S]$, $k \ge 0$, the condition $\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon} - Z_t^s| \le \epsilon'$ implies (recall that $a \wedge b$ stands for $\min(a,b)$)

$$\psi(\check{Z}_t^{s+\varepsilon}) > \check{\epsilon}, \quad t \in [\mathfrak{r}_{2k}, \mathfrak{r}_{2k+1}] \cap [0,S],$$

so that $\varphi_1(\check{Z}_t^{s+\varepsilon})$ in (4.203) and in the initial condition of (4.204) matches 1. As a consequence, on each $[\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0,S]$, $k \ge 0$, the condition $\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon} - Z_t^s| \le \epsilon'$ implies

$$|\check{Y}_t^{s+\varepsilon}|^2 = \psi(\check{Z}_t^{s+\varepsilon}), \quad t \in [\mathfrak{r}_{2k+1}, \mathfrak{r}_{2k+2}] \cap [0,S].$$

Finally, under the condition $\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon}-Z_t^s|\le \epsilon'$, processes $(\check{Z}_t^{s+\varepsilon})_{0\le t\le S}$ and $(Z_t^{s+\varepsilon})_{0\le t\le S}$ have the same dynamics on the whole $[0,S]$.

As a consequence, the first term in (4.219) is less than $\hat{V}_S^{\sigma}(s+\varepsilon)$. (Use $F\ge 0$ to say so.) It thus remains to bound the second term.

The idea consists in using Markov inequality. For any $p\ge 1$, it says that

$$\mathbb{P}\Big\{\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon}-Z_t^s|\ge \check{\epsilon}/2\Big\}\le 2^p\check{\epsilon}^{-p}\mathbb{E}\Big[\sup_{0\le t\le S}|\check{Z}_t^{s+\varepsilon}-Z_t^s|^p\Big]. \tag{4.220}$$

Using the stability property for SDEs, see Proposition 4.7.1, we know that

$$\begin{aligned}\mathbb{E}&\Big[\sup_{0\le t\le S}\big|\check{Z}_t^{s+\varepsilon}-Z_t^s\big|^p\Big]\\ &\le C\varepsilon^p+C\mathbb{E}\int_0^S\big(\big|\check{Z}_r^{s+\varepsilon}-Z_r^s\big|^p+\big|\check{Y}_r^{s+\varepsilon}-Y_r^s\big|^p\big)dr\\ &\le C\varepsilon^p\bigg(1+\int_0^S\sup_{|\varepsilon'|\le\varepsilon}\mathbb{E}\big[\big|\check{\zeta}_r^{s+\varepsilon'}\big|^p+\big|\check{\varrho}_r^{s+\varepsilon'}\big|^p\big]dr\le C\varepsilon^p\bigg).\end{aligned} \tag{4.221}$$

Plugging the above bound in (4.220) and then in (4.219), we complete the proof of the bound $\sup_\sigma[\check{V}_S^{\sigma}(s+\varepsilon)]\le V_S(\gamma(s+\varepsilon))+C\varepsilon^3$.

The proof of the inequalities (4.217) is now straightforward: it follows from (4.165), (4.205), (4.207) and (4.214):

$$\begin{aligned}\lim_{\varepsilon\to 0}&\sup_{|\varepsilon'|<|\varepsilon|}\sup_{\sigma}\left|\frac{d}{d\varepsilon}\big[\check{V}^{\sigma}\big(\gamma(s+\varepsilon)\big)\big]\right|\\ &\le\sup_{\sigma}\mathbb{E}\bigg[\int_0^{+\infty}\exp\bigg(\int_0^t\mathrm{Trace}[a_rD^2_{z,\bar{z}}\psi(Z_r^s)]dr\bigg)\\ &\quad\times\bigg(|\zeta_t^s|+\int_0^t|\zeta_r^s|dr+\bigg|\int_0^t\mathrm{Re}\big[\langle D_{z'}G(Z_r^s,0)\zeta_r^s,dB_r\rangle\big]\bigg|\bigg)dt\bigg].\end{aligned} \tag{4.222}$$

Following the proof of Proposition 4.8.8 (and specifically using a variant of Lemma 4.8.5[28]), we obtain

[28] In Sect. 4.8, the process $(\varsigma_t)_{t\ge0}$ in the statement of Lemma 4.8.5 is understood as $(\zeta_t^s)_{t\ge0}$. Here, ς_t, $t\ge0$, is to be understood as ζ_t^s or (ζ_t^s,ϱ_t^s) according to the cases: $t\in[\mathfrak{r}_{2k},\mathfrak{r}_{2k+1}[$ or $t\in[\mathfrak{r}_{2k+1},\mathfrak{r}_{2k+2}[$, $k\ge0$. For this reason, it may be simpler to plug $(\bar{\Gamma}_t)_{t\ge0}$ itself into $(\varsigma_t)_{t\ge0}$.

However, since it is discontinuous, $(\bar{\Gamma}_t)_{t\ge0}$ does not satisfy the assumption of Lemma 4.8.5. Actually, it is sufficient to apply Itô's formula to $((a+m_t+\bar{\Gamma}_t)^{1/2})_{t\ge0}$ on each $(\mathfrak{r}_{2k},\mathfrak{r}_{2k+1})$, a standing for a small positive real, and then to check the boundary conditions. In particular, it is useless to localize the proof as done in the proof of Lemma 4.8.5 since there is no singularity anymore in the dynamics of the derivative processes.

$$\frac{d}{d\varepsilon}\big[V_S\big(\gamma(s+\varepsilon)\big)\big] \le \mathbb{E}\bigg[\int_0^{+\infty} \exp\bigg(\int_0^t \text{Trace}[a_r D^2_{z,\bar{z}}\psi(Z_r^s)]dr\bigg) \times \bigg(|\bar{\Gamma}_t| + \int_0^t \big(1+r^{-1/2}\big)|\bar{\Gamma}_r|dr\bigg)dt\bigg]. \quad (4.223)$$

The same argument holds for the second-order derivatives.

Finally, for every compact interval $I \subset (-1,1)$ and for ε small enough, the quantity $\sup_\sigma \sup_{|\varepsilon'|<|\varepsilon|}[|(\partial/\partial\varepsilon')[\check{V}_S^\sigma(\gamma(s+\varepsilon'))]|]$ is shown to be uniformly bounded w.r.t. $s \in I$ by a similar argument and by (4.205). □

4.9.8 Final Step

We now complete the proof of Theorem 4.6.1.

Passing to the limit in $S \to +\infty$ in Proposition 4.9.9, we deduce that V in Proposition 4.6.9 satisfies the same property as V_S, i.e. for any smooth curve γ from $[-1,1]$ into $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$, the function $s \in [-1,1] \mapsto V(\gamma(s)) + C\int_0^s |\gamma'(r)|dr$ is non-decreasing, the function $s \in [-1,1] \mapsto V(\gamma(s)) - C\int_0^s |\gamma'(r)|dr$ is non-increasing and the function $s \in [-1,1] \mapsto V(\gamma(s)) + C\int_0^s [(s-r)(|\gamma''(r)| + |\gamma'(r)|^2)]dr$ is convex.

Choosing γ as a straight path of the form $s \in [-1,1] \mapsto z + \nu s$, for $\psi(z) > \epsilon_4$ and $\nu \in \mathbb{C}^d$, with $|\nu|$ small enough, we deduce that V is Lipschitz and semi-convex away from the boundary, i.e. on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$. In particular, $v - g + N_0\psi$ in Proposition 4.6.9 is Lipschitz and semi-convex on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$ as well. By Proposition 4.6.4 and Remark 4.6.5, v is $\mathcal{C}^{1,1}$ on $\{z \in \mathcal{D} : \psi(z) > \epsilon_4\}$. Since ϵ_4 may be chosen as small as desired, we deduce that v is $\mathcal{C}^{1,1}$ in $\mathcal{D}$.

We emphasize that the Lipschitz and semi-convexity constants are bounded in terms of (**A**) only on every compact subset. The problem is then to bound the Lipschitz and semi-convexity constants up to the boundary.

To do so, we consider a path γ_0 from $[-1,1]$ into $\{z \in \mathcal{D} : \psi(z) < \epsilon/2\}$, for the same ϵ as in Propositions 4.9.4 and 4.9.6. Then, we can define $(Z_t^s)_{0 \le t \le \mathfrak{r}_1}$ as in (4.189) first, i.e. as the first coordinate of the pair $(Z_t^s, Y_t^s)_{0 \le t \le \mathfrak{r}_1}$, $\mathfrak{r}_1$ now standing for $\inf\{t \ge 0 : \psi(Z_t^s) > \epsilon/2\}$. and switch to (4.188) from $\mathfrak{r}_1$ to $\mathfrak{r}_2$, with $\mathfrak{r}_2 = \inf\{t \ge \mathfrak{r}_1 : \psi(Z_t^s) < \epsilon_4\}$, and so on... Here, Z_0^s is chosen as $\gamma_0(s)$ and Y_0^s is chosen in such a way that $|Y_0^s|^2 = \psi(Z_0^s) = \psi(\gamma(s))$. Obviously, we can apply the same procedure for the perturbed process and first consider $(\check{Z}_t^{s+\varepsilon}, \check{Y}_t^{s+\varepsilon})_{0 \le t \le \mathfrak{r}_1}$ as in (4.204).

The whole question then lies in the choice of the initial condition $(\check{Z}_0^{s+\varepsilon}, \check{Y}_0^{s+\varepsilon})$. Surely, we choose $\check{Z}_0^{s+\varepsilon}$ as $\gamma_0(s+\varepsilon)$ and $\check{Y}_0^{s+\varepsilon}$ such that $|\check{Y}_0^{s+\varepsilon}|^2 = \psi(\check{Z}_0^{s+\varepsilon})$. Assume therefore that $\check{Y}_0^{s+\varepsilon} = \gamma_1(s+\varepsilon)$ for some smooth path γ_1 defined on $[-1,1]$ such that $\psi(\gamma_0(s)) = |\gamma_1(s)|^2$, $s \in [-1,1]$. Then, Proposition 4.9.9 remains true with $\gamma = (\gamma_0, \gamma_1)$, the proof being exactly the same. In particular, the constant C therein depends on (**A**) only (and is independent of the distance of γ_0 to the boundary). Since V is now known to be $\mathcal{C}^{1,1}$ in $\mathcal{D}$ (see Remark 4.6.5), this may be read as

$$\left|\frac{d[V(\gamma_0(s))]}{ds}\right| \le C|\gamma'(s)| \quad s \in [-1,1]$$
$$\left|\frac{d^2[V(\gamma_0(s))]}{ds^2}\right| \le C\big(|\gamma'(s)|^2 + |\gamma''(s)|\big) \quad \text{a.e. } s \in [-1,1]. \tag{4.224}$$

To obtain the Lipschitz property up to the boundary, we fix some z with $\psi(z) < \epsilon/2$ and we choose γ as in Proposition 4.9.1, i.e. $\gamma = (\gamma_0, \gamma_1)$ with $\gamma_0(s) = z + s\nu$, $s \in [-1,1]$, for $\nu \in \mathbb{C}^d$ with a small enough norm, and $\gamma_1 = (\gamma_{1,1}, 0)$, with

$$(\gamma_{1,1})'(s) = (\bar{\gamma}_{1,1})^{-1}(s) D_z\psi(\gamma_0(s))\nu \quad |\gamma_{1,1}(0)|^2 = \psi(z), \quad s \in [-1,1].$$

Keep in mind that $|\gamma_{1,1}(s)|^2 = \psi(\gamma_0(s))$ for $s \in [-1,1]$.

Now, compute for a differentiable function $w(s)$:

$$\left|\frac{d[w(s)\psi(\gamma_0(s))]}{ds}\right| = \left|\psi(\gamma_0(s))\frac{dw}{ds}(s) + 2w(s)\mathrm{Re}\big[D_z\psi(\gamma_0(s))\nu\big]\right|.$$

Choose now $w = V \circ \gamma_0$ and deduce from (4.224) that

$$\left|\frac{d[V(\gamma_0(s))\psi(\gamma_0(s))]}{ds}\right|$$
$$\le C\psi(\gamma_0(s))\big[|\nu| + |\bar{\gamma}_{1,1}^{-1}(s)||D_z\psi(\gamma_0(s))\nu|\big] + C\|V\|_\infty|\nu|.$$

Modifying the constant C if necessary, we deduce that $[\psi V](\gamma_0(s))$ is Lipschitz continuous of constant $C|\nu|$. We emphasize that the constant C is independent of the distance from z to the boundary since $|\psi(\gamma_0(s))\bar{\gamma}_{1,1}^{-1}| = \psi^{1/2}(\gamma_0(s))$ is bounded. This procedure directly applies to Proposition 4.6.9: we deduce that $v - g + N_0\psi$ is Lipschitz continuous up to the boundary. This is the first part in Theorem 4.8.1.

It now remains to investigate the second-order derivatives. To obtain an estimate that holds up to the boundary, we consider another parameterized curve. Let $(\gamma_0^a, \gamma_{1,1}^a)$ and $(\gamma_0^b, \gamma_{1,1}^b)$ be two pairs with values in $\mathcal{D} \times \mathbb{R}$ such that

$$\dot{\gamma}_0^i(s) = \gamma_{1,1}^i(s)\nu, \quad \dot{\gamma}_{1,1}^i(s) = \mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big], \quad i = a, b. \tag{4.225}$$

(Pay attention that $\gamma_{1,1}^i$ is real-valued.) The initial boundary condition has the form: $\gamma_0^i(0) = z$ (with $\psi(z) < \epsilon/2$) and $\gamma_{1,1}^i(0) = y_0^i \in \mathbb{R}$, with y_0^i to be chosen later on. Clearly, for each $i = a, b$, the system is (at least) solvable on a small interval around 0. Now,

$$\begin{aligned}
&\frac{d}{ds}\big[\psi(\gamma_0^i(s)) - |\gamma_{1,1}^i(s)|^2\big] \\
&\quad = 2\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\dot{\gamma}_0^i(s)\big] - 2\gamma_{1,1}^i(s)\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big] \\
&\quad = 0.
\end{aligned} \tag{4.226}$$

Now, for $w^i = V \circ \gamma_0^i$ and for s in the interval of definition of $(\gamma_0^i, \gamma_{1,1}^i)$,

$$\begin{aligned}
&\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big] \\
&\quad = 2\frac{d}{ds}\big\{\gamma_{1,1}^i(s)\mathrm{Re}\big[D_z V(\gamma_0^i(s))\nu\big]\big\} \\
&\quad = 2\mathrm{Re}\big[D_z\psi(\gamma_0^i(s))\nu\big]\mathrm{Re}\big[D_z V(\gamma_0^i(s))\nu\big] + |\gamma_{1,1}^i(s)|^2\big[D^2V(\gamma_0^i(s))\big](\nu),
\end{aligned}$$

where $[D^2V(\gamma_0^i(s))](\nu)$ stands for the action of the second-order derivatives of V at point $\gamma_0^i(s)$ on the vector ν.[29] Choosing $s = 0$ and making the sum over $i = a, b$, we obtain:

$$\begin{aligned}
\sum_{i=a,b}\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big]_{|s=0} &= 4\mathrm{Re}\big[D_z\psi(z)\nu\big]\mathrm{Re}\big[D_z V(z)\nu\big] \\
&\quad + (|y_0^a|^2 + |y_0^b|^2)\big[D^2V(z)\big](\nu).
\end{aligned}$$

The whole trick now consists in choosing $|y_0^a|^2 = |y_0^b|^2 = \psi(z)/2$ so that

$$\begin{aligned}
\big[D^2\big(\psi V\big)(z)\big](\nu) &= \big[D^2\psi(z)\big](\nu)V(z) \\
&\quad + 4\mathrm{Re}\big[D_z\psi(z)\nu\big]\mathrm{Re}\big[D_z V(z)\nu\big] + \psi(z)\big[D^2V(z)\big](\nu) \\
&= \big[D^2\psi(z)\big](\nu)V(z) + \sum_{i=a,b}\frac{d^2}{ds^2}\big[V(\gamma_0^i(s))\big]_{|s=0}.
\end{aligned}$$

To apply (4.224), we need to specify what the second coordinate of each γ_1^i is. We set $\gamma_1^i(s) = (\gamma_{1,1}^i(s), (\psi(z)/2)^{1/2})$ for s in the interval of definition of $(\gamma_0^i, \gamma_{1,1}^i)$. By (4.226), it satisfies $\psi(\gamma_0^i(s)) - |\gamma_1^i(s)|^2 = 0$, so that (γ_0^i, γ_1^i), $i = a, b$, is a zero of the function $\Phi(z, y) = \psi(z) - |y|^2$. (In particular, γ_0^i cannot exit from $\mathcal{D}$ and the solution to (4.225) may be extended to

[29] That is, $D^2[V(z)](\nu) = \sum_{k,\ell=1}^d (D^2_{z_k,z_\ell}V(z)\nu_k\nu_\ell + D^2_{\bar{z}_k,z_\ell}V(z)\bar{\nu}_k\nu_\ell + D^2_{z_k,\bar{z}_\ell}V(z)\nu_k\bar{\nu}_\ell + D^2_{\bar{z}_k,\bar{z}_\ell}V(z)\bar{\nu}_k\bar{\nu}_\ell)$.

the whole $[-1,1]$. Indeed, γ_1^i cannot vanish since $\gamma_{1,2}^i(s) = (\psi(z)/2)^{1/2}$.) We now apply (4.224) (with s in the neighborhood of 0 only). Then, we obtain that $D^2[\psi(z)V(z)](\nu) \geq -C|\nu|^2$, for some constant C, independent from the distance from z to the boundary. Since $\psi V = v - g + N_0\psi$, this proves that the semi-convexity constant of v is uniform up to the boundary. By Proposition 4.6.4, we complete the proof of Theorem 4.8.1.

4.9.9 Conclusion

We here paid some price to gather into a single one the two different representations $((Z_t^s)_{\mathfrak{r}_{2k} \leq t < \mathfrak{r}_{2k+1}})_{k \geq 0}$ and $((Z_t^s)_{\mathfrak{r}_{2k+1} \leq t < \mathfrak{r}_{2k+2}})_{k \geq 0}$ according to the position of the process $(Z_t^s)_{t \geq 0}$ inside the domain $\mathcal{D}$.

A natural way to simplify things consists in considering the parameterized representation (4.167) in the whole space and in forgetting the original (4.84). Actually, this is exactly what Krylov does in the papers mentioned in the references below.

The reason why we here decided to split the representation into two pieces is purely pedagogical even if a bit heavy to detail. Indeed, Sect. 4.8 exactly shows what works and fails when dealing with the first approach. In some sense, this may justify in a more understandable way the reason why the parameterized version is the one used by Krylov. We also emphasize that the computations performed in Sect. 4.8 for the single process $(Z_t^s)_{t \geq 0}$ turn out to be really cumbersome for the pair process $(Z_t^s, Y_t^s)_{t \geq 0}$: this is another reason why we kept both representations in the whole proof.

Part III
Monge–Ampère Equations on Compact Kähler Manifolds

Chapter 5
The Calabi–Yau Theorem

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Abstract This lecture, based on a course given by the author at Toulouse in January 2005, surveys the proof of Yau's celebrated solution to the Calabi conjecture, through the solvability of inhomogeneous complex Monge–Ampère equations on compact Kähler manifolds.

5.1 Introduction

Our main goal is to present a complete proof of the Calabi–Yau theorem [Yau78] (Theorem 5.3 below). In Sect. 5.2 we collect basic notions of the Kähler geometry (proofs can be found for example in [KN69]). We then formulate the Calabi conjecture and reduce it to solving a Monge–Ampère equation. Kähler–Einstein metrics are also briefly discussed. In Sect. 5.3 we prove the uniqueness of solutions and reduce the proof of existence to a priori estimates using the continuity method and Schauder theory. Since historically the uniform estimate has caused the biggest problem, we present two different proofs of this estimate in Sect. 5.4. The first is the classical simplification of the Yau proof due to Kazdan, Aubin and Bourguignon and its main tool is the Moser iteration technique. The second is essentially due to Kolodziej and is more in the spirit of pluripotential theory. In Sect. 5.5 we show the estimate for the mixed second order complex derivatives of solutions which can also be applied in the degenerate case. The $C^{2,\alpha}$ estimate can be proved

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locally using general Evans–Krylov–Trudinger theory coming from (real) fully nonlinear elliptic equations. This is done in Sect. 5.6. Finally, in Sect. 5.7 we study a corresponding Dirichlet problem for weak (continuous) solutions.

We concentrate on the PDE aspects of the subject, whereas the geometric problems are presented only as motivation. In particular, without much more effort we could also solve the Monge–Ampère equation (5.9) below for $\lambda < 0$ and thus prove the existence of the Kähler–Einstein metric on compact complex manifolds with negative first Chern class.

We try to present as complete proofs as possible. We assume that the reader is familiar with main results from the theory of linear elliptic equations of second order with variable coefficients (as covered in [GT83, Part I]) and basic theory of functions and forms of several complex variables. Good general references are [Aub98, Dembook, GT83, KN69], whereas the lecture notes [Siu87] and [Tianbook] (as well as [Aub98]) cover the subject most closely. In Sect. 6 we assume the Bedford–Taylor theory of the complex Monge–Ampère operator in $\mathbb{C}^n$ but in fact all the results of that part are proved by means of certain stability estimates that are equally difficult to show for smooth solutions.

When proving an a priori estimate by $C_1, C_2, \ldots$ we will denote constants which are as in the hypothesis of this estimate and call them *under control*.

5.2 Basic Concepts of Kähler Geometry

In this section we collect the basic notions of Kähler geometry. Let M be a complex manifold of dimension n. By TM denote the (real) tangent bundle of M - it is locally spanned over $\mathbb{R}$ by $\partial/\partial x_j, \partial/\partial y_j$, $j = 1, \ldots, n$. The complex structure on M defines the endomorphism J of TM given by $J(\partial/\partial x_j) = \partial/\partial y_j$, $J(\partial/\partial y_j) = -\partial/\partial x_j$. Every hermitian form on M

$$\omega(X, Y) = \sum_{i,j=1}^{n} g_{i\bar{j}} X_i \overline{Y}_j, \quad X, Y \in TM,$$

we can associate with a real (this means that $\omega = \overline{\omega}$) (1,1)-form

$$2\sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \tag{5.1}$$

(it is easy to check that they are transformed in the same way under a holomorphic change of coordinates). If ω is positive then $\widetilde{\omega} := \operatorname{Re}\omega$ is the Riemannian form on M. Let ∇ be the Levi–Civita connection defined by $\widetilde{\omega}$ - it is the unique torsion-free connection satisfying $\nabla\widetilde{\omega} = 0$, that is

$$(\nabla_X \widetilde{\omega})(Y,Z) = \widetilde{\omega}(\nabla_X Y, Z) + \widetilde{\omega}(Y, \nabla_X Z) - X\widetilde{\omega}(Y,Z) = 0, \quad X,Y,Z \in TM.$$

One can show that for a hermitian manifold (M,ω)

$$d\omega = 0 \;\Leftrightarrow\; \nabla\omega = 0 \;\Leftrightarrow\; \nabla J = 0. \tag{5.2}$$

Hermitian forms ω satisfying equivalent conditions (5.2) are called *Kähler*. This means that the complex structure of M is compatible with the Riemannian structure given by ω. Manifold M is called *Kähler* if there exists a Kähler form on M.

We shall use the operators $\partial, \overline{\partial}$, so that $d = \partial + \overline{\partial}$ and $2\sqrt{-1}\partial\overline{\partial} = dd^c$, where $d^c := \sqrt{-1}(\overline{\partial} - \partial)$.

Proposition 5.1 *Let ω be a closed, real (1,1) form on M. Then locally $\omega = dd^c\eta$ for some smooth η.*

Proof. Locally we can find a real 1-form γ such that $\omega = d\gamma$. We may write $\gamma = \alpha + \beta$, where α is a (1,0)-form and β a (0,1)-form. We have $\overline{\alpha} = \beta$, since γ is real. Moreover,

$$\omega = (\partial + \overline{\partial})(\alpha + \beta) = \partial\alpha + \overline{\partial}\alpha + \partial\beta + \overline{\partial}\beta,$$

and thus $\partial\alpha = 0$, $\overline{\partial}\beta = 0$, since ω is a (1,1)-form. Then locally we can find a complex-valued, smooth function f with $\beta = \overline{\partial} f$ and

$$\omega = \partial\overline{\beta} + \overline{\partial}\beta = dd^c(\operatorname{Im} f).$$

□

The condition $d\omega = 0$ reads

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad i,j,k = 1,\dots,n,$$

and by Proposition 5.1 this means that locally we can write $\omega = dd^c g$ for some smooth, real-valued g. We will use the notation $f_i = \partial f/\partial z_i$, $f_{\bar{j}} = \partial f/\partial \overline{z}_j$, it is then compatible with (5.1). If ω is Kähler then g is strongly plurisubharmonic (shortly psh). From now on we assume that ω is a Kähler form and g is its local potential.

By $T_{\mathbb{C}}M$ denote the complexified tangent bundle of M - it is locally spanned over $\mathbb{C}$ by $\partial_j := \partial/\partial z_j, \partial_{\bar{j}} := \partial/\partial \overline{z}_j$, $j = 1,\dots,n$. Then J, ω and ∇ can be uniquely extended to $T_{\mathbb{C}}M$ in a $\mathbb{C}$-linear way. One can check that

$$J(\partial_j) = \sqrt{-1}\partial_j, \quad J(\partial_{\bar{j}}) = -\sqrt{-1}\partial_{\bar{j}},$$
$$\omega(\partial_i, \partial_j) = \omega(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0, \quad \omega(\partial_i, \partial_{\bar{j}}) = g_{i\bar{j}},$$
$$\nabla_{\partial_i}\partial_{\bar{j}} = \overline{\nabla_{\partial_{\bar{i}}}\partial_j} = 0, \quad \nabla_{\partial_i}\partial_j = \overline{\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}}} = g^{k\bar{l}} g_{i\bar{l}j}\partial_k, \tag{5.3}$$

where $(g^{k\bar{l}})$ is the inverse transposed to $(g_{i\bar{j}})$, that is

$$g^{k\bar{l}} g_{j\bar{l}} = \delta_{jk}. \tag{5.4}$$

We have the following curvature tensors from Riemannian geometry

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ R(X,Y,W,Z) &= \omega(R(X,Y)Z, W), \\ Ric(Y,Z) &= \mathrm{tr}\{X \longmapsto R(X,Y)Z\}. \end{aligned}$$

One can then show that

$$R_{i\bar{j}k\bar{l}} = R(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}}) = -g_{i\bar{j}k\bar{l}} + g^{p\bar{q}} g_{p\bar{j}k} g_{i\bar{l}\bar{q}}$$

$$Ric_{k\bar{l}} = Ric(\partial_k, \partial_{\bar{l}}) = g^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log\det(g_{i\bar{j}}). \tag{5.5}$$

Since this is the moment where the Monge–Ampère operator appears in complex geometry, let us have a look at the last equality. Let D, Q be any linear first order differential operators with constant coefficients. Then

$$Q \log\det(g_{i\bar{j}}) = \frac{a^{i\bar{j}} Q g_{i\bar{j}}}{\det(g_{i\bar{j}})} = g^{i\bar{j}} Q g_{i\bar{j}}, \tag{5.6}$$

where $(a^{i\bar{j}})$ is the (transposed) adjoint matrix of $(g_{i\bar{j}})$. Differentiating (5.4) we get

$$D g^{i\bar{j}} = -g^{i\bar{q}} g^{p\bar{j}} D g_{p\bar{q}},$$

thus

$$DQ \log\det(g_{i\bar{j}}) = g^{i\bar{j}} DQ g_{i\bar{j}} - g^{i\bar{q}} g^{p\bar{j}} D g_{p\bar{q}} Q g_{i\bar{j}}, \tag{5.7}$$

and (5.5) follows.

The (real) Laplace–Beltrami operator of a smooth function u is defined as the trace of $X \longmapsto \nabla_X \nabla u$, where $\widetilde{\omega}(X, \nabla u) = Xu$, $X \in TM$. In the complex case it is convenient to define this operator as the double of the real one – then

$$\Delta u = g^{i\bar{j}} u_{i\bar{j}}$$

and

$$dd^c u \wedge \omega^{n-1} = \frac{1}{n} \Delta u \, \omega^n.$$

The form ω^n will be the volume form for us (in fact, it is $4^n n!$ times the standard volume form) and we will denote $V := \mathrm{vol}(M) = \int_M \omega^n$. Note that the local formulas for the quantities we have considered (the Christoffel symbols (5.3), the curvature tensors, the Laplace–Beltrami operator) are simpler in the Kähler case than in the real Riemannian case. It will also be convenient to use the notation $R_\omega = -dd^c \log\det(g_{i\bar{j}})$ $(= 2Ric_\omega$ by (5.5)).

The formula (5.5) has also the following consequence: if $\widetilde{\omega}$ is another Kähler form on M then $R_\omega - R_{\widetilde{\omega}} = dd^c\eta$, where η is a globally defined function (this easily follows from Proposition 5.1), and thus R_ω, $R_{\widetilde{\omega}}$ are cohomologous (we write $R_\omega \sim R_{\widetilde{\omega}}$). The cohomology class of R_ω is precisely $c_1(M)$, the first Chern class of M, which does not depend on ω but only on the complex structure of M.

The so called *dd^c-lemma* says that in the compact case every d-exact (1,1)-form is dd^c-exact:

Lemma 5.2 *Let α be a real, d-exact (1,1)-form on a compact Kähler manifold M. Then there exists $\eta \in C^\infty(M)$ such that $\alpha = dd^c\eta$.*

Proof. Write $\alpha = d\beta$ and let ω be a Kähler form on M. Let η be the solution of the following Poisson equation

$$dd^c\eta \wedge \omega^{n-1} = \alpha \wedge \omega^{n-1}.$$

(This equation is solvable since $\int_M \alpha \wedge \omega^{n-1} = \int_M d(\beta \wedge \omega^{n-1}) = 0$.) Define $\gamma := \beta - d^c\eta$. We then have $d\gamma \wedge \omega^{n-1} = 0$ and we have to show that $d\gamma = 0$. For this we will use the Hodge theory. Note that

$$\int_M \langle d\gamma, d\gamma\rangle dV = \int_M \langle \gamma, d^* d\gamma\rangle dV,$$

it is therefore enough to show that $d^* d\gamma = 0$. From now on the argument is local: by Proposition 5.1 we may write $d\gamma = dd^c h$ and $dd^c h \wedge \omega^{n-1} = 0$ is equivalent to $d^* dh = 0$. We then have

$$d^* d\gamma = d^* dd^c h = -d^* d^c dh = d^c d^* dh = 0,$$

where we have used the equality

$$d^* d^c + d^c d^* = 0$$

(see e.g. [Dembook]). □

From now on, we always assume that M is a **compact** manifold of dimension $n \geq 2$ and ω a Kähler form with local potential g.

Calabi conjecture. [Cal56] Let $\widetilde{R}$ be a (1,1) form on M cohomologous to R_ω. Then we ask whether there exists another Kähler form $\widetilde{\omega} \sim \omega$ on M such that $\widetilde{R} = R_{\widetilde{\omega}}$. In other words, the problem is if every form representing $c_1(M)$ is the Ricci form of a certain Kähler metric on M coming from one cohomology class.

By the dd^c-lemma we have $R_\omega = \widetilde{R} + dd^c\eta$ for some $\eta \in C^\infty(M)$. We are thus looking for $\varphi \in C^\infty(M)$ such that in local coordinates $(\varphi_{i\bar{j}} + g_{i\bar{j}}) > 0$ and

$$dd^c(\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - \eta) = 0.$$

However, $\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - \eta$ is globally defined, and since it is pluriharmonic on a compact manifold, it must be constant. This means that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{c+\eta}\det(g_{i\bar{j}}),$$

which is equivalent to

$$(\omega + dd^c\varphi)^n = e^{c+\eta}\omega^n.$$

Since $(\omega + dd^c\varphi)^n - \omega^n$ is exact, from the Stokes theorem we infer

$$\int_M (\omega + dd^c\varphi)^n = V,$$

and thus the constant c is uniquely determined. Therefore, solving the Calabi conjecture is equivalent to solving the following Dirichlet problem for the complex Monge–Ampère operator on M.

Theorem 5.3 [Yau78] *Let $f \in C^\infty(M)$, $f > 0$, be such that $\int_M f\omega^n = V$. Then there exists, unique up to a constant, $\varphi \in C^\infty(M)$ such that $\omega + dd^c\varphi > 0$ and*

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{5.8}$$

Kähler–Einstein metrics. A Kähler form $\widetilde{\omega}$ is called *Kähler–Einstein* if $R_{\widetilde{\omega}} = \lambda\widetilde{\omega}$ for some $\lambda \in \mathbb{R}$. Since $\lambda\widetilde{\omega} \in c_1(M)$, it follows that a necessary condition for a complex manifold M to posses a Kähler–Einstein metric is that either $c_1(M) < 0$, $c_1(M) = 0$ or $c_1(M) > 0$, that is there exists an element in $c_1(M)$ which is either negative, zero or positive. In such a case we can always find a Kähler form ω on M with $\lambda\omega \in c_1(M)$, that is $R_\omega = \lambda\omega + dd^c\eta$ for some $\eta \in C^\infty(M)$, since M is compact. We then look for $\varphi \in C^\infty(M)$ such that $\widetilde{\omega} := \omega + dd^c\varphi > 0$ (from the solution of the Calabi conjecture we know that $c_1(M) = \{R_{\widetilde{\omega}} : \widetilde{\omega} \sim \omega\}$, so we only have to look for Kähler forms that are cohomologous to the given ω) and $R_{\widetilde{\omega}} = \lambda\widetilde{\omega}$, which, similarly as before, is equivalent to

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi+\eta+c}\omega^n. \tag{5.9}$$

To find a Kähler–Einstein metric on M we thus have to find an *admissible* (that is $\omega + dd^c\varphi \geq 0$) solution to (5.9) (for some constant c).

If $c_1(M) = 0$ then $\lambda = 0$ and the solvability of (5.9) is guaranteed by Theorem 5.3. If $c_1(M) < 0$ one can solve the equation (5.9) in a similar way as (5.8). In fact, the uniform estimate for (5.9) with $\lambda < 0$ is very simple (see [Aub76], [Yau78, p. 379], and Exercise 5.9 below) and in this case, (5.9) was independently solved by Aubin [Aub76]. The case $c_1(M) > 0$ is the most difficult and it turns out that only the uniform estimate is the problem. There was a big progress in this area in the last 20 years (especially thanks to G. Tian) and, indeed, there are examples of compact manifolds with positive first Chern class not admitting a Kähler–Einstein metric. We refer to [Tianbook] for details and further references.

5.3 Reduction to A Priori Estimates

The uniqueness in Theorem 5.3 is fairly easy.

Proposition 5.4 [Cal55] *If $\varphi, \psi \in C^2(M)$ are such that $\omega + dd^c\varphi > 0$, $\omega + dd^c\psi \geq 0$ and $(\omega + dd^c\varphi)^n = (\omega + dd^c\psi)^n$ then $\varphi - \psi = const.$*

Proof. We have

$$0 = (\omega + dd^c\varphi)^n - (\omega + dd^c\psi)^n = dd^c(\varphi - \psi) \wedge T,$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-1-j}$$

is a positive, closed $(n-1, n-1)$-form. Integrating by parts we get

$$0 = \int_M (\psi - \varphi)((\omega + dd^c\varphi)^n - (\omega + dd^c\psi)^n) = \int_M d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge T$$

and we conclude that $D(\varphi - \psi) = 0$. □

In subsequent sections we will show the following a priori estimate: there exists $\alpha \in (0, 1)$ and $C > 0$, depending only on M and on upper bounds for $||f||_{1,1}$ and $1/\inf_M f$, such that for any admissible solution $\varphi \in C^4(M)$ of (5.8) satisfying the normalization condition $\int_M \varphi\omega^n = 0$ we have

$$||\varphi||_{2,\alpha} \leq C, \tag{5.10}$$

where we use the following notation: in any chart $U \subset M$

$$||\varphi||_{C^{k,\alpha}(U)} := \sum_{0 \leq j \leq k} \sup_U |D^j\varphi| + \sup_{x,y \in U, x \neq y} \frac{|D^k\varphi(x) - D^k\varphi(y)|^\alpha}{|x - y|}$$

and $||\varphi||_{k,\alpha} := \sum_i ||\varphi||_{C^{k,\alpha}(U_i)}$ for a fixed finite atlas $\{U_i\}$ (for any two such atlases the obtained norms will be equivalent). In this convention

$$||f||_{k,1} = \sum_{0\le j\le k+1} \sup_M |D^j f|.$$

The aim of this section is to reduce the proof of Theorem 5.3 to showing the estimate (5.10). It will be achieved using the continuity method (which goes back to Bernstein) and the Schauder theory for linear elliptic equations of second order.

Continuity method. Fix arbitrary integer $k \ge 2$, $\alpha \in (0,1)$ and let f be as in Theorem 5.3. By S we denote the set of $t \in [0,1]$ such that we can find admissible $\varphi_t \in C^{k+2,\alpha}(M)$ solving

$$(\omega + dd^c\varphi_t)^n = (tf + 1 - t)\omega^n$$

and such that $\int_M \varphi_t\omega^n = 0$. It is clear that $0 \in S$ and if we show that $1 \in S$ then we will have a $C^{k+2,\alpha}$ solution of (5.8). It will be achieved if we prove that S is open and closed in $[0,1]$.

The complex Monge–Ampère operator $\mathcal{N}$, determined by

$$(\omega + dd^c\varphi)^n = \mathcal{N}(\varphi)\,\omega^n,$$

in local coordinates given by

$$\mathcal{N}(\varphi) = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})}$$

smoothly maps the set

$$\mathcal{A} = \left\{\varphi \in C^{k+2,\alpha}(M) : \omega + dd^c\varphi > 0, \int_M \varphi\omega^n = 0\right\}$$

to

$$\mathcal{B} = \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f}\omega^n = \int_M \omega^n\right\}.$$

Then $\mathcal{A}$ is an open subset of the Banach space

$$\mathcal{E} = \left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta\omega^n = 0\right\}$$

and $\mathcal{B}$ is a hyperplane of the Banach space $C^{k+2,\alpha}(M)$ with the tangent space

$$\mathcal{F} = \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f}\omega^n = 0\right\}.$$

We want to show that for every $\varphi \in \mathcal{A}$ the differential $D\mathcal{N}(\varphi) : \mathcal{E} \to \mathcal{F}$ is an isomorphism. For $\eta \in \mathcal{E}$, denoting $\widetilde{\omega} = \omega + dd^c\varphi$, we have

$$D\mathcal{N}(\varphi).\eta = \frac{d}{dt}\mathcal{N}(\varphi + t\eta)|_{t=0} = \frac{\det(\widetilde{g}_{i\bar{j}})}{\det(g_{i\bar{j}})}\widetilde{g}^{i\bar{j}}\eta_{i\bar{j}} = \mathcal{N}(\varphi)\widetilde{\Delta}\eta.$$

It immediately follows that $D\mathcal{N}(\varphi)$ is injective. From the real theory on compact Riemannian manifolds it is known that the Laplace–Beltrami operator bijectively maps

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f} = 0\right\}$$

(see e.g. [Aub98, Theorem 4.7]). This, applied to $(M, \widetilde{\omega})$, implies that $D\mathcal{N}(\varphi)$ is indeed surjective, and thus an isomorphism. Therefore $\mathcal{N}$ is locally invertible and in particular $\mathcal{N}(\mathcal{A})$ is open in $\mathcal{B}$, and S is open in $[0, 1]$.

If we knew that the set $\{\varphi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}(M)$ then from its every sequence, by the Arzela–Ascoli theorem, we could choose a subsequence whose all partial derivatives of order $\leq k + 1$ converged uniformly. Thus, to show that S is closed, we need an a priori estimate

$$||\varphi||_{k+2,\alpha} \leq C \tag{5.11}$$

for the solutions of (5.8). We now sketch how to use (locally) the Schauder theory to show that (5.10) implies (5.11).

Schauder theory. We first analyze the complex Monge–Ampère operator

$$F(D^2u) = \det(u_{i\bar{j}})$$

for smooth psh functions u – we see that the above formula defines the real operator of second order. It is elliptic if the $2n \times 2n$ real symmetric matrix $A := (\partial F/\partial u_{pq})$ (here by u_{pq} we denote the elements of the real Hessian D^2u) is positive. Matrix A is determined by

$$\frac{d}{dt}F(D^2u + tB)|_{t=0} = \operatorname{tr}(AB^T).$$

Exercise 5.5 Show that

$$\lambda_{min}(\partial F/\partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{max}(u_{i\bar{j}})}, \quad \lambda_{max}(\partial F/\partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{min}(u_{i\bar{j}})},$$

where $\lambda_{min}A$, resp. $\lambda_{max}A$, denotes the minimal, resp. maximal, eigenvalue of A.

Thus the operator F is elliptic (in the real sense) for smooth strongly psh functions and in our case when (5.10) is satisfied (then Δu is under control and hence so are the complex mixed derivatives $u_{i\bar{j}}$) is even uniformly elliptic, that is

$$|\zeta|^2/C \leq \sum_{p,q=1}^{2n} \partial F/\partial u_{pq}\zeta_p\zeta_q \leq C|\zeta|^2, \quad \zeta \in \mathbb{C}^n = \mathbb{R}^{2n}$$

for some uniform constant C. We can now apply the standard elliptic theory (see [GT83, Lemma 17.16] for details) to the equation

$$F(D^2u) = f.$$

For a fixed unit vector ζ and small $h > 0$ we consider the difference quotient

$$u^h(x) = \frac{u(x+h\zeta) - u(x)}{h}$$

and

$$a_h^{pq} = \int_0^1 \frac{\partial F}{\partial u_{pq}}\big(tD^2u(x+h\zeta) + (1-t)D^2u(x)\big)dt.$$

Then

$$a_h^{pq}(x)u_{pq}^h(x) = \frac{1}{h}\int_0^1 \frac{d}{dt}F\big(tD^2u(x+h\zeta) + (1-t)D^2u(x)\big)dt = f^h(x).$$

From the Schauder theory for linear elliptic equations with variable coefficients we then infer (all corresponding estimates are uniform in h)

$$u \in C^{2,\alpha} \implies a_h^{pq} \in C^{0,\alpha} \overset{\text{Schauder}}{\implies} u^h \in C^{2,\alpha} \implies u \in C^{3,\alpha} \implies \dots$$

Coming back to our equation (5.8) for $k \geq 1$ we thus get

$$\varphi \in C^{2,\alpha},\ f \in C^{k,\alpha} \implies \varphi \in C^{k+2,\alpha}$$

and

$$||\varphi||_{k+2,\alpha} \leq C,$$

where $C > 0$ depends only on M and on upper bounds for $||\varphi||_{2,\alpha}$, $||f||_{k,\alpha}$. Hence, we get (5.11), $\varphi \in C^\infty(M)$, and to prove Theorem 5.3, it is enough to establish the a priori estimate (5.10).

5.4 Uniform Estimate

The main goal of this section will be to prove the uniform estimate. We will use the notation $||\varphi||_p = ||\varphi||_{L^p(M)}$, $1 \le p \le \infty$.

Theorem 5.6 *Assume that $\varphi \in C^2(M)$ is admissible and $(\omega + dd^c\varphi)^n = f\omega^n$. Then*

$$\operatorname*{osc}_M \varphi := \sup_M \varphi - \inf_M \varphi \le C,$$

where $C > 0$ depends only on M and on an upper bound for $||f||_\infty$.

The L^p estimate for $p < \infty$ follows easily for any admissible φ (without any knowledge on f).

Proposition 5.7 *For any admissible $\varphi \in C^2(M)$ with $\max_M \varphi = 0$ one has*

$$||\varphi||_p \le C(M,p), \quad 1 \le p < \infty.$$

Proof. The case $p = 1$ follows easily from the following estimate (applied in finite number of local charts to $u = \varphi + g$): if u is a negative subharmonic function in $B(y, 3R)$ in $\mathbb{R}^m$ then for $x \in B(y,R)$ we have

$$u(x) \le \frac{1}{\operatorname{vol}(B(x,2R))} \int_{B(x,2R)} u \le \frac{1}{\operatorname{vol}(B(y,2R))} \int_{B(y,R)} u$$

and thus

$$||u||_{L^1(B(y,R))} \le \operatorname{vol}(B(y,2R)) \inf_{B(y,R)} (-u).$$

For $p > 1$ we now use the following estimate: if u is a negative psh in $B(y, 2R)$ in $\mathbb{C}^n$ then

$$||u||_{L^p(B(y,R))} \le C(n,p,R)||u||_{L^1(B(y,2R))}.$$

□

We will now present two different proofs of the uniform estimate. The first one (see [Siu87, p. 92] or [Tianbook, p. 49]) is similar to the original proof of Yau, subsequently simplified by Kazdan [Kaz78] for $n = 2$ and by Aubin and Bourguignon for arbitrary n (for the detailed historical account we refer to [Yau78, p. 411] and [Siu87, p. 115]).

First proof of Theorem 5.6. Without loss of generality we may assume that $\int_M \omega^n = 1$ and $\max_M \varphi = -1$, so that $||\varphi||_p \le ||\varphi||_q$ if $p \le q < \infty$. We have

$$(f-1)\omega^n = (\omega + dd^c\varphi)^n - \omega^n = dd^c\varphi \wedge T,$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\varphi)^j \wedge \omega^{n-1-j} \geq \omega^{n-1}.$$

Integrating by parts we get for $p \geq 1$

$$\begin{aligned}\int_M (-\varphi)^p (f-1)\omega^n &= \int_M (-\varphi)^p dd^c\varphi \wedge T = -\int_M d(-\varphi)^p \wedge d^c\varphi \wedge T \\ &= p\int_M (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge T \geq p\int_M (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \\ &= \frac{4p}{(p+1)^2}\int_M d(-\varphi)^{(p+1)/2} \wedge d^c(-\varphi)^{(p+1)/2} \wedge \omega^{n-1}\end{aligned}$$

so that

$$\int_M (-\varphi)^p (f-1)\omega^n = \frac{c_n p}{(p+1)^2} ||D(-\varphi)^{(p+1)/2}||_2^2. \tag{5.12}$$

The Sobolev inequality on compact a Riemannian manifold M with real dimension m states that

$$||v||_{mq/(m-q)} \leq C(M,q)\left(||v||_q + ||Dv||_q\right), \quad v \in W^{1,q}(M),\ q < m. \tag{5.13}$$

(it easily follows from the Sobolev inequality for $u \in W_0^{1,q}(\mathbb{R}^m)$ applied in charts forming a finite covering of M). Using (5.13) with $q = 2$ and (5.12)

$$\begin{aligned}&||(-\varphi)^{(p+1)/2}||_{2n/(n-1)} \\ &\quad \leq C_M\left(||(-\varphi)^{(p+1)/2}||_2 + \frac{p+1}{\sqrt{p}}\left(\int_M (-\varphi)^p (f-1)\omega^n\right)^{1/2}\right).\end{aligned}$$

From this (replacing $p+1$ with p) and since $|\varphi| \leq 1$ we easily get

$$||\varphi||_{np/(n-1)} \leq (Cp)^{1/p} ||\varphi||_p, \quad p \geq 2. \tag{5.14}$$

We will now apply Moser's iteration scheme (see [Mos60] or the proof of [GT83, Theorem 8.15]). Set

$$p_0 := 2, \quad p_k := \frac{np_{k-1}}{n-1}, \quad k = 1, 2, \ldots,$$

so that $p_k = 2(n/(n-1))^k$. Then by (5.14)

$$||\varphi||_\infty = \lim_{k\to\infty} ||\varphi||_{p_k} \le ||\varphi||_2 \prod_{j=0}^{\infty} (Cp_j)^{1/p_j}.$$

Taking the logarithm one can show that

$$\prod_{j=0}^{\infty} (Cp_j)^{1/p_j} = (n/(n-1))^{n(n-1)/2}(2C)^{n/2}$$

and it is enough to use Proposition 5.7 (for $p = 2$).

Exercise 5.8 Slightly modifying the above proof show that the uniform estimate follows if we assume that $||f||_q$ is under control for some $q > n$.

Exercise 5.9 Consider the equation

$$(\omega + dd^c\varphi)^n = F(\cdot, \varphi)\omega^n,$$

where $F \in C^\infty(M \times \mathbb{R})$ is positive. Show that if an admissible solution $\varphi \in C^\infty(M)$ attains maximum at $y \in M$ then $F(y, \varphi(y)) \le 1$. Deduce a uniform estimate for admissible solutions of (5.9) when $\lambda < 0$.

The second proof of the uniform estimate is essentially due to Kolodziej [Kol98] who studied pluripotential theory on compact Kähler manifolds (see also [TZ00]). The Kolodziej argument gave the uniform estimate under weaker conditions than in Theorem 5.6 – it is enough to assume that $||f||_q$ is under control for some $q > 1$. For $q = \infty$ (and even $q > 2$) this argument was simplified in [Bl05] and we will follow that proof.

The main tool in the second proof of Theorem 5.6 will the following L^2 stability for the complex Monge–Ampère equation. It was originally established by Cheng and Yau (see [B88, p. 75]). The Cheng–Yau argument was made precise by Cegrell and Persson [CP92].

Theorem 5.10 *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that $u \in C(\overline{\Omega})$ is psh and C^2 in Ω, $u = 0$ on $\partial\Omega$. Then*

$$||u||_{L^\infty(\Omega)} \le C(n, \operatorname{diam}\Omega)\, ||f||^{1/n}_{L^2(\Omega)},$$

where $f = \det(u_{i\bar{j}})$.

Proof. We use the theory of convex functions and the real Monge–Ampère operator. From the Alexandrov–Bakelman–Pucci principle [GT83, Lemma 9.2] we get

$$||u||_{L^\infty(\Omega)} \le \frac{\operatorname{diam}\Omega}{\lambda_{2n}^{1/2n}} \left(\int_\Gamma \det D^2 u \right)^{1/2n},$$

where $\lambda_{2n} = \pi^n/n!$ is the volume of the unit ball in $\mathbb{C}^n$ and

$$\Gamma := \{x \in \Omega : u(x) + \langle Du(x), y - x\rangle \le u(y)\ \forall\, y \in \Omega\} \subset \{D^2 u \ge 0\}.$$

It will now be sufficient to prove the pointwise estimate

$$D^2 u \ge 0 \implies \det D^2 u \le c_n (\det(u_{i\bar{j}}))^2.$$

We may assume that $(u_{i\bar{j}})$ is diagonal. Then

$$\begin{aligned}\det(u_{i\bar{j}}) &= 4^{-n}(u_{x_1x_1} + u_{y_1y_1})\dots(u_{x_nx_n} + u_{y_ny_n})\\ &\ge 2^{-n}\sqrt{u_{x_1x_1}u_{y_1y_1}\dots u_{x_nx_n}u_{y_ny_n}}\\ &\ge \sqrt{\det D^2u/c_n},\end{aligned}$$

where the last inequality follows because for real nonnegative symmetric matrices (a_{pq}) one easily gets $\det(a_{pq}) \le m!a_{11}\dots a_{mm}$ (because $|a_{pq}| \le \sqrt{a_{pp}a_{qq}}$); from Lemma 5.16 below one can deduce that in fact $\det(a_{pq}) \le a_{11}\dots a_{mm}$. □

From the comparison principle for the complex Monge–Ampère operator one can immediately obtain the estimate

$$||u||_{L^\infty(\Omega)} \le (\operatorname{diam}\Omega)^2\, ||f||^{1/n}_{L^\infty(\Omega)}$$

in Theorem 5.10. It is however not sufficient for our purposes, because it does not show that if $\operatorname{vol}(\Omega)$ is small then so is $||u||_{L^\infty(\Omega)}$.

Exercise 5.11 Using the Moser iteration technique from the first proof of Theorem 5.6 show the L^q stability for $q > n$, that is Theorem 5.10 with $||f||_{L^2(\Omega)}$ replaced with $||f||_{L^q(\Omega)}$.

The uniform estimate will easily follow from the next result.

Proposition 5.12 *Let Ω be a bounded domain in $\mathbb{C}^n$ and u is a negative C^2 psh function in Ω. Assume that $a > 0$ is such that the set $\{u < \inf_\Omega u + a\}$ is nonempty and relatively compact in Ω. Then*

$$||u||_{L^\infty(\Omega)} \le a + (C/a)^{2n}||u||_{L^1(\Omega)}||f||^2_{L^\infty(\Omega)},$$

where $f = \det(u_{i\bar{j}})$ and $C = C(n, \operatorname{diam}\Omega)$ is the constant from Theorem 5.10.

Proof. Set $t := \inf_\Omega u + a$, $v := u - t$ and $\Omega' := \{v < 0\}$. By Theorem 5.10

$$a = ||v||_{L^\infty(\Omega')} \le C\,(\operatorname{vol}(\Omega'))^{1/2n}\, ||f||^{1/n}_{L^\infty(\Omega')}.$$

On the other hand,

$$\operatorname{vol}(\Omega') \leq \frac{||u||_{L^1(\Omega)}}{|t|} = \frac{||u||_{L^1(\Omega)}}{||u||_{L^\infty(\Omega)} - a}$$

and the estimate follows. □

Second proof of Theorem 5.6 Let $y \in M$ be such that $\varphi(y) = \min_M \varphi$. The Taylor expansion of g about y gives

$$\begin{aligned} g(y+h) &= \operatorname{Re} P(h) + \sum_{i,j=1}^{n} g_{i\bar{j}}(y) h_i \overline{h}_j + \frac{1}{3!} D^3 g(\widetilde{y}).h^3 \\ &\geq \operatorname{Re} P(h) + c_1 |h|^2 - c_2 |h|^3, \end{aligned}$$

where

$$P(h) = g(y) + 2\sum_i g_i(y) h_i + \sum_{i,j} g_{ij}(y) h_i h_j$$

is a complex polynomial, $\widetilde{y} \in [y, y+h]$ and $c_1, c_2 > 0$ depend only on M. Modifying g by a pluriharmonic function (and thus not changing ω), we may thus assume that there exists $a, r > 0$ depending only on M such that $g < 0$ in $B(y, 2r)$, g attains minimum in $B(y, 2r)$ at y and $g \geq g(y) + a$ on $B(y, 2r) \setminus B(y, r)$. Proposition 5.12 (for $\Omega = B(y, 2r)$ and $u = g + \varphi$) combined with Proposition 5.7 (for $p = 1$) gives the required estimate.

Slightly improving the proof of Proposition 5.12 (using the Hölder inequality) we see that the second proof of Theorem 5.6 implies that we can replace $||f||_\infty$ with $||f||_q$ for any $q > 2$. Moreover, since Kolodziej [Kol96] showed (with more complicated proof using pluripotential theory) that the (local) L^q stability for the complex Monge–Ampère equation holds for every $q > 1$ (and even for a weaker Orlicz norm), we can do this on M also for every $q > 1$. This was proved in [Kol98], where the local techniques from [Kol96] had to be repeated on M. The above argument allows to easily deduce the global uniform estimate from the local results. Exercises 5.8 and 5.11 show that both proofs of Theorem 5.6, although quite different, are related.

5.5 Second Derivative Estimate

In this section we will show the a priori estimate for the mixed complex derivatives $\varphi_{i\bar{j}}$ which is equivalent to the estimate of $\Delta\varphi$. The main idea is the same as the one in the original Yau proof [Yau78] who used the method of Pogorelov [Pog71] from the real Monge–Ampère equation. We will present an improvement of the Yau estimate that can be applied to the degenerate case (when $f \geq 0$) because it does not quantitatively depend on $\inf_M f$. It uses the idea of Guan [Gua97] (see also [GTW99]) who obtained regularity results for the degenerate real Monge–Ampère equation. It also simplifies some computations from [Yau78].

Theorem 5.13 [Bl03] *Let $\varphi \in C^4(M)$ be such that $\omega + dd^c\varphi > 0$ and $(\omega + dd^c\varphi)^n = f\omega^n$. Then*

$$\sup_M |\Delta\varphi| \leq C,$$

where C depends only on M and on an upper bound for $||f^{1/(n-1)}||_{1,1}$.

Proof. By Theorem 5.6 we may assume that

$$-C_1 \leq \varphi \leq 0. \tag{5.15}$$

Note that for any admissible φ we have $(g_{i\bar{j}} + \varphi_{i\bar{j}}) \geq 0$ and thus

$$\Delta\varphi = g^{i\bar{j}}\varphi_{i\bar{j}} \geq -n.$$

It is therefore enough to estimate $\Delta\varphi$ from above. In local coordinates the function $u = g + \varphi$ is strongly psh. It is easy to see that the expression

$$\eta := \max_{|\zeta|=1} \frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}} = \max_{\zeta\neq 0} \frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}},$$

(where $u_\zeta = \sum_i \zeta_i u_i$, $u_{\bar{\zeta}} = \sum_i \overline{\zeta}_i u_{\bar{i}}$, and $u_{\zeta\bar{\zeta}} = \sum_{i,j} \zeta_i\overline{\zeta}_j u_{i\bar{j}}$, $\zeta \in \mathbb{C}^n$) is independent of holomorphic change of coordinates, and thus η is a continuous, positive, globally defined function on M. Set

$$\alpha := \log\eta - A\varphi,$$

where $A > 0$ under control will be specified later. Since M is compact and α is continuous, we can find $y \in M$, where α attains maximum. After rotation we may assume that the matrix $(u_{i\bar{j}})$ is diagonal and $u_{1\bar{1}} \geq \cdots \geq u_{n\bar{n}}$ at y. Fix $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, such that $\eta = u_{\zeta\bar{\zeta}}/g_{\zeta\bar{\zeta}}$ at y. Then the function

$$\widetilde{\alpha} := \log\frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}} - A\varphi,$$

defined in a neighborhood of y, also has maximum at y. Moreover, $\widetilde{\alpha} \leq \alpha$ and $\widetilde{\alpha}(y) = \alpha(y)$. Since

$$u_{\zeta\bar{\zeta}}(y) \leq u_{1\bar{1}}(y) \leq C_2 u_{\zeta\bar{\zeta}}(y), \tag{5.16}$$

by (5.15) it is clear that to finish the proof it is sufficient to show the estimate

$$u_{1\bar{1}}(y) \leq C_3. \tag{5.17}$$

We will use the following local estimate.

Lemma 5.14 *Let u be a C^4 psh function with $F := \det(u_{i\bar{j}}) > 0$. Then for any direction ζ*

$$u^{i\bar{j}}(\log u_{\zeta\bar{\zeta}})_{i\bar{j}} \geq \frac{(\log F)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}}.$$

Proof. Differentiating (logarithm of) the equation $\det(u_{i\bar{j}}) = F$ twice, similarly as in (5.6), (5.7) we get

$$\begin{aligned} u^{i\bar{j}} u_{i\bar{j}\zeta} &= (\log F)_\zeta, \\ u^{i\bar{j}} u_{i\bar{j}\zeta\bar{\zeta}} &= (\log F)_{\zeta\bar{\zeta}} + u^{i\bar{l}} u^{k\bar{j}} u_{i\bar{j}\zeta} u_{k\bar{l}\bar{\zeta}}. \end{aligned}$$

Using this we obtain

$$\begin{aligned} u_{\zeta\bar{\zeta}}\, u^{i\bar{j}}(\log u_{\zeta\bar{\zeta}})_{i\bar{j}} &= u^{i\bar{j}} u_{i\bar{j}\zeta\bar{\zeta}} - \frac{1}{u_{\zeta\bar{\zeta}}} u^{i\bar{j}} u_{\zeta\bar{\zeta}i} u_{\zeta\bar{\zeta}\bar{j}} \\ &= (\log F)_{\zeta\bar{\zeta}} + u^{i\bar{l}} u^{k\bar{j}} u_{i\bar{j}\zeta} u_{k\bar{l}\bar{\zeta}} - \frac{1}{u_{\zeta\bar{\zeta}}} u^{i\bar{j}} u_{\zeta\bar{\zeta}i} u_{\zeta\bar{\zeta}\bar{j}}. \end{aligned}$$

At a given point we may assume that the matrix $(u_{i\bar{j}})$ is diagonal. Then

$$u^{i\bar{j}} u_{\zeta\bar{\zeta}i} u_{\zeta\bar{\zeta}\bar{j}} = \sum_i \frac{|u_{\zeta\bar{\zeta}i}|^2}{u_{i\bar{i}}}$$

and

$$|u_{\zeta\bar{\zeta}i}|^2 = \left| \sum_j \bar{\zeta}_j u_{i\bar{j}\zeta} \right|^2 \leq \sum_j |\zeta_j|^2 u_{j\bar{j}} \sum_j \frac{|u_{i\bar{j}\zeta}|^2}{u_{j\bar{j}}}$$

by Schwarz inequality. Therefore

$$u^{i\bar{j}} u_{\zeta\bar{\zeta}i} u_{\zeta\bar{\zeta}\bar{j}} \leq u_{\zeta\bar{\zeta}} \sum_{i,j} \frac{|u_{i\bar{j}\zeta}|^2}{u_{i\bar{i}} u_{j\bar{j}}} = u_{\zeta\bar{\zeta}}\, u^{i\bar{l}} u^{k\bar{j}} u_{i\bar{j}\zeta} u_{k\bar{l}\bar{\zeta}}$$

and the lemma follows. □

As noticed by Bo Berndtsson, Lemma 4.2 has a geometric context. If ζ is a holomorphic vector field on a Kähler manifold (with potential u) then one can show that

$$\sqrt{-1}\partial\bar\partial\log|\zeta|^2 \geq -\frac{R(\zeta,\zeta,\cdot,\cdot)}{|\zeta|^2}.$$

Taking the trace and using that $Ric_{i\bar j} = -(\log F)_{i\bar j}$, one obtains the statement of the lemma.

Proof of Theorem 5.13 (continued) Using the fact that $\widetilde\alpha$ has maximum at y, by Lemma 5.14 with $F = f\det(g_{i\bar j})$ we get

$$0 \geq u^{i\bar j}\widetilde\alpha_{i\bar j} \geq \frac{(\log f)_{\zeta\bar\zeta}}{u_{\zeta\bar\zeta}} + \frac{(\log\det(g_{p\bar q}))_{\zeta\bar\zeta}}{u_{\zeta\bar\zeta}} + Au^{i\bar j}g_{i\bar j} - nA.$$

By (5.16) and the elementary inequality (following from differential calculus of functions of one real variable)

$$||\sqrt{h}||_{0,1} \leq C_M(1+||h||_{1,1}), \quad h\in C^2(M),\ h\geq 0,$$

we get, denoting $\widetilde f := f^{1/(n-1)}$,

$$\frac{(\log f)_{\zeta\bar\zeta}}{u_{\zeta\bar\zeta}} = \frac{n-1}{u_{\zeta\bar\zeta}}\left(\frac{\widetilde f_{\zeta\bar\zeta}}{\widetilde f} - \frac{|\widetilde f_\zeta|^2}{\widetilde f^2}\right) \geq -\frac{C_4}{u_{1\bar 1}\widetilde f}.$$

Therefore, using (5.16) again (recall that $(u_{i\bar j})$ is diagonal at y),

$$0 \geq -\frac{C_4}{u_{1\bar 1}\widetilde f} - \frac{C_5}{u_{1\bar 1}} + (-C_6 + A/C_7)\sum_i \frac{1}{u_{i\bar i}} - nA,$$

where $1/C_7 \leq \lambda_{min}(g_{i\bar j}(y))$. We choose A such that $-C_6 + A/C_7 = \max\{1, C_5\}$. The inequality between arithmetic and geometric means gives

$$\sum_{i\geq 2}\frac{1}{u_{i\bar i}} \geq \frac{n-1}{(u_{2\bar 2}\dots u_{n\bar n})^{1/(n-1)}} = (n-1)\frac{u_{1\bar 1}^{1/(n-1)}}{\widetilde f}.$$

We arrive at

$$u_{1\bar 1}^{n/(n-1)} - C_8 u_{1\bar 1} - C_9 \leq 0$$

(at y) from which (5.17) immediately follows. □

In the proof of Theorem 5.13, unlike in [Yau78], we used standard derivatives in local coordinates and not the covariant ones – it makes some calculations simpler.

It is rather unusual in the theory of nonlinear elliptic equations of second order that the second derivative estimate can be obtained directly from the uniform estimate, bypassing the gradient estimate. The gradient estimate follows locally (and hence globally on M) from the estimate for the Laplacian for arbitrary solutions of the Poisson equation (see e.g. [GT83, Theorem 3.9] or use the Green function and differentiate under the sign of integration).

5.6 $C^{2,\alpha}$ Estimate

Aubin [Aub70] and Yau [Yau78] proved a priori estimates for third-order derivatives of φ. The estimate from [Yau78], due to Nirenberg (see [Yau78, Appendix A]), was based on an estimate for the real Monge–Ampère equation of Calabi [Cal58]. In the meantime, a general theory of nonlinear elliptic equations of second order has been developed. It allows to obtain an interior $C^{2,\alpha}$-estimate, once an estimate for the second derivatives is known. It was done by Evans [Ev82, Ev83] (and also independently by Krylov [Kry82]) and his method was subsequently simplified by Trudinger [Trud83]. Although the complex Monge–Ampère operator is uniformly elliptic in the real sense (see Exercise 5.5), we cannot apply the estimate from the real theory directly. The reason is that Sect. 5.5 gives the control for the mixed complex derivatives $\varphi_{i\bar{j}}$ but not for $D^2\varphi$, which is required in the real estimate. We can however almost line by line repeat the real method in our case. It has been done in [Siu87], and also in [Bl00, Theorem 3.1], where an idea from [Sch86] and [WJ85] was used to write the equation in divergence form. We will get the following a priori estimate for the complex Monge–Ampère equation.

Theorem 5.15 *Let u be a C^4 psh function in an open $\Omega \subset \mathbb{C}^n$ such that $f := \det(u_{i\bar{j}}) > 0$. Then for any $\Omega' \Subset \Omega$ there exist $\alpha \in (0,1)$ depending only on n and on upper bounds for $||u||_{C^{0,1}(\Omega)}$, $sup_\Omega \Delta u$, $||f||_{C^{0,1}(\Omega)}$, $1/\inf_\Omega f$, and $C > 0$ depending in addition on a lower bound for $\operatorname{dist}(\Omega', \partial\Omega)$ such that*

$$||u||_{C^{2,\alpha}(\Omega')} \le C.$$

A similar estimate can be proved for more general equations of the complex Hessian of the form

$$F((u_{i\bar{j}}), Du, u, z) = 0.$$

Here F is a smooth function of $\mathcal{G} \times \mathbb{R}^{2n} \times \mathbb{R} \times \Omega$, where $\mathcal{G}$ is an open subset of the set of all $n \times n$ hermitian matrices $\mathcal{H}$. In case of the complex Monge–Ampère operator we take $\mathcal{G} = \mathcal{H}_+ := \{A \in \mathcal{H} : A > 0\}$. The crucial assumption that has to be made on F in order for the Evans–Trudinger method to work is that it is concave with respect to $(u_{i\bar{j}})$. In case of the complex Monge–Ampère equation one has to use the fact that the mapping

$$\mathcal{H}_+ \ni A \longmapsto (\det A)^{1/n} \in \mathbb{R}_+ \tag{5.18}$$

is concave. This can be immediately deduced from the following very useful lemma.

Lemma 5.16 [Gav77]

$$(\det A)^{1/n} = \frac{1}{n} \inf\{\operatorname{tr}(AB) : B \in \mathcal{H}_+,\ \det B = 1\}, \quad A \in \mathcal{H}_+.$$

Proof. For every $B \in \mathcal{H}_+$ there is unique $C \in \mathcal{H}_+$ such that $C^2 = B$. We denote $C = B^{1/2}$. Then $B^{1/2}AB^{1/2} \in \mathcal{H}_+$ and after diagonalizing it, from the inequality between arithmetic and geometric means we get

$$\begin{aligned}(\det A)^{1/n}(\det B)^{1/n} &= (\det(B^{1/2}AB^{1/2}))^{1/n} \\ &\leq \tfrac{1}{n}\operatorname{tr}(B^{1/2}AB^{1/2}) = \tfrac{1}{n}\operatorname{tr}(AB)\end{aligned}$$

and $\leq$ follows. To show $\geq$ we may assume that A is diagonal and then we easily find B for which the infimum is attained. □

Lemma 5.16 also shows that the Monge–Ampère operator is an example of a Bellman operator.

Proof of Theorem 5.15 Fix $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$. Differentiating the logarithm of both sides of the equation

$$\det(u_{i\bar{j}}) = f,$$

similarly as in (5.7) or in the proof of Lemma 5.14, we obtain

$$u^{i\bar{j}} u_{\zeta\bar{\zeta}i\bar{j}} = (\log f)_{\zeta\bar{\zeta}} + u^{i\bar{l}} u^{k\bar{j}} u_{\zeta i\bar{j}} u_{\bar{\zeta}k\bar{l}} \geq (\log f)_{\zeta\bar{\zeta}}. \tag{5.19}$$

The inequality $u^{i\bar{l}} u^{k\bar{j}} u_{\zeta i\bar{j}} u_{\bar{\zeta}k\bar{l}} \geq 0$ is equivalent to the concavity of the mapping

$$\mathcal{H}_+ \ni A \longmapsto \log\det A \in \mathbb{R}$$

which also follows from concavity of (5.18). It will be convenient to write (5.19) in divergence form. Set $a^{i\bar{j}} := f u^{i\bar{j}}$. Then for any fixed i

$$(a^{i\bar{j}})_{\bar{j}} = f(u^{i\bar{j}} u^{k\bar{l}} - u^{i\bar{l}} u^{k\bar{j}}) u_{k\bar{l}\bar{j}} = 0$$

and by (5.19)

$$(a^{i\bar{j}} u_{\zeta\bar{\zeta}i})_{\bar{j}} \geq f_{\zeta\bar{\zeta}} - \frac{|f_\zeta|^2}{f} \geq -C_1 + \sum_j \left(\frac{\partial f^j}{\partial x_j} + \frac{\partial f^{j+n}}{\partial y_j}\right),$$

where $||f^l||_{L^\infty(\Omega)} \leq C_2$, $l = 1, \ldots, 2n$. By the assumptions on u (and Exercise 5.5) the operator $\partial_{\bar{j}}(a^{i\bar{j}}\partial_i)$ is uniformly elliptic (in the real sense) and from the weak Harnack inequality [GT83, Theorem 8.18] we now get

$$r^{-2n}\int_{B_r}\big(\sup_{B_{4r}} u_{\zeta\bar\zeta}-u_{\zeta\bar\zeta}\big)\le C_3\big(\sup_{B_{4r}} u_{\zeta\bar\zeta}-\sup_{B_r} u_{\zeta\bar\zeta}+r\big), \tag{5.20}$$

where $B_{4r}=B(z_0,4r)\subset\Omega$ and $z_0\in\Omega'$.

On the other hand, for $x,y\in\Omega$ by Lemma 5.16 we have

$$a^{i\bar j}(y)\big(u_{i\bar j}(y)-u_{i\bar j}(x)\big)\le nf(y)^{1-1/n}\big(f(y)^{1/n}-f(x)^{1/n}\big)\le C_4|x-y|. \tag{5.21}$$

We are going to combine (5.20) with (5.21). For that we will need to choose an appropriate finite set of directions ζ. The following lemma from linear algebra will be crucial.

Lemma 5.17 *Let $0<\lambda<\Lambda<\infty$ and by $S(\lambda,\Lambda)$ denote the set of hermitian matrices whose eigenvalues are in the interval $[\lambda,\Lambda]$. Then one can find unit vectors $\zeta_1,\dots,\zeta_N\in\mathbb{C}^n$ and $0<\lambda_*<\Lambda_*<\infty$, depending only on n,λ, and Λ, such that every $A\in S(\lambda,\Lambda)$ can be written as*

$$A=\sum_{k=1}^{N}\beta_k\zeta_k\otimes\overline{\zeta}_k,\quad i.e.\quad a_{i\bar j}=\sum_k\beta_k\zeta_{ki}\overline{\zeta}_{kj},$$

where $\beta_k\in[\lambda_*,\Lambda_*]$, $k=1,\dots,N$. The vectors $\zeta_1,\dots,\zeta_N$ can be chosen so that they contain a given orthonormal basis of $\mathbb{C}^n$.

Proof. [Siu87, p. 103] The space $\mathcal{H}$ of all hermitian matrices is of real dimension n^2. Every $A\in\mathcal{H}$ can be written as

$$A=\sum_{k=1}^{n}\lambda_k w_k\otimes\overline{w}_k,$$

where $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ are the eigenvalues of A and $w_1,\dots,w_n\in\mathbb{C}^n$ the corresponding unit eigenvectors. It follows that there exist unit vectors $\zeta_1,\dots,\zeta_{n^3}\in\mathbb{C}^n$ such that the matrices $\zeta_k\otimes\overline{\zeta}_k$, $k=1,\dots,n^3$, span $\mathcal{H}$ over $\mathbb{R}$. For such sets of vectors we consider the sets of matrices

$$U(\zeta_1,\dots,\zeta_{n^3})=\left\{\sum_k\beta_k\zeta_k\otimes\overline{\zeta}_k:0<\beta_k<2\Lambda\right\}.$$

They form an open covering of $S(\lambda/2,\Lambda)$, a compact subset of $\mathcal{H}$. Choosing a finite subcovering we get unit vectors $\zeta_1,\dots,\zeta_N\in\mathbb{C}^n$ such that

$$S(\lambda/2,\Lambda)\subset\left\{\sum_{k=1}^{N}\beta_k\zeta_k\otimes\overline{\zeta}_k:0<\beta_k<2\Lambda\right\}.$$

For $A \in S(\lambda, \Lambda)$ we have

$$A - \frac{\lambda}{2N} \sum_{k=1}^{N} \zeta_k \otimes \overline{\zeta}_k \in S(\lambda/2, \Lambda)$$

and the lemma follows. We see that may take arbitrary $\lambda_* < \lambda/N$ and $\Lambda_* > \Lambda$. □

Proof of Theorem 5.15. (continued) The eigenvalues of $(u_{i\bar{j}})$ are in $[\lambda, \Lambda]$, where $\lambda, \Lambda > 0$ are under control. By Lemma 5.17 we can find unit vectors $\zeta_1, \dots, \zeta_N \in \mathbb{C}^n$ such that for $x, y \in \Omega$

$$a^{i\bar{j}}(y)\big(u_{i\bar{j}}(y) - u_{i\bar{j}}(x)\big) = \sum_{k=1}^{N} \beta_k(y)\big(u_{\zeta_k\bar{\zeta}_k}(y) - u_{\zeta_k\bar{\zeta}_k}(x)\big),$$

where $\beta_k(y) \in [\lambda^*, \Lambda^*]$ and $\lambda^*, \Lambda^* > 0$ are under control. Set

$$M_{k,r} := \sup_{B_r} u_{\zeta_k\bar{\zeta}_k}, \quad m_{k,r} := \inf_{B_r} u_{\zeta_k\bar{\zeta}_k},$$

and

$$\eta(r) := \sum_{k=1}^{N} (M_{k,r} - m_{k,r}).$$

We need to show that $\eta(r) \le Cr^\alpha$. Since $\gamma_1, \dots, \gamma_N$ can be chosen so that they contain the coordinate vectors, it will then follow that $||\Delta u||_{C^\alpha(\Omega')}$ is under control and by the Schauder estimates for the Poisson equation [GT83, Theorem 4.6] also that $||D^2 u||_{C^\alpha(\Omega')}$ is under control. The condition $\eta(r) \le Cr^\alpha$ is equivalent to

$$\eta(r) \le \delta\eta(4r) + r, \quad 0 < r < r_0, \tag{5.22}$$

where $\delta \in (0, 1)$ and $r_0 > 0$ are under control (see [GT83, Lemma 8.23]).

From (5.21) we get

$$\sum_{k=1}^{N} \beta_k(y)\big(u_{\zeta_k\bar{\zeta}_k}(y) - u_{\zeta_k\bar{\zeta}_k}(x)\big) \le C_4|x - y|. \tag{5.23}$$

Summing (5.20) over $l \ne k$, where k is fixed, we obtain

$$r^{-2n} \int_{B_r} \sum_{l \ne k} \big(M_{l,4r} - u_{\zeta_l\bar{\zeta}_l}\big) \le C_3(\eta(4r) - \eta(r) + r). \tag{5.24}$$

By (5.23) for $x \in B_{4r}, y \in B_r$ we have

$$\begin{aligned}\beta_k(y)\big(u_{\zeta_k\bar\zeta_k}(y) - u_{\zeta_k\bar\zeta_k}(x)\big) &\le C_4|x-y| + \sum_{l\neq k}\beta_l(y)\big(u_{\zeta_l\bar\zeta_l}(x) - u_{\zeta_l\bar\zeta_l}(y)\big)\\ &\le C_5 r + \Lambda^*\sum_{l\neq k}(M_{l,4r} - u_{\zeta_l\bar\zeta_l}(y)).\end{aligned}$$

Thus

$$u_{\zeta_k\bar\zeta_k}(y) - m_{k,4r} \le \frac{1}{\lambda^*}\left(C_5 r + \Lambda^*\sum_{l\neq k}\big(M_{l,4r} - u_{\zeta_l\bar\zeta_l}(y)\big)\right)$$

and (5.24) gives

$$r^{-2n}\int_{B_r}\big(u_{\zeta_k\bar\zeta_k} - m_{k,4r}\big) \le C_6(\eta(4r) - \eta(r) + r).$$

This coupled with (5.20) easily implies that

$$\eta(r) \le C_7(\eta(4r) - \eta(r) + r),$$

and (5.22) follows.

5.7 Weak Solutions

The theory of the complex Monge–Ampère operator $(dd^c)^n$ for nonsmooth psh functions has been developed by Bedford and Taylor (see [BT76, BT82] and also general references [Dem93, Dembook, Klimbook, Bl96, Blobook, Ceg88, Kol05]). In particular, one can define $(dd^c u)^n$ as a nonnegative regular Borel measure if u is a locally bounded psh function, and this operator is continuous for monotone sequences (in the weak* topology of measures). We define the class of weakly admissible functions on M in a natural way: $\varphi : M \to \mathbb{R}\cup\{-\infty\}$ is called *admissible* (or *ω-psh*) if locally $g+\varphi$ is psh. Therefore, if φ is locally bounded and admissible then $\mathcal{M}(\varphi) := (\omega + dd^c\varphi)^n$, locally equal to $(dd^c(g+\varphi))^n$, is a measure such that $\int_M \mathcal{M}(\varphi) = V$.

We will show the following version of Theorem 5.3 for weak solutions.

Theorem 5.18 [Kol98, Kol03] *Let $f \in C(M)$, $f \ge 0$, be such that $\int_M f\omega^n = V$. Then there exists a, unique up to a constant, admissible $\varphi \in C(M)$ such that $\mathcal{M}(\varphi) = f\omega^n$.*

The existence part of Theorem 5.18 was shown in [Kol98], also for $f \in L^q(M)$, $q > 1$. As we will see, this part for $f \in C(M)$ can be proved in a simpler way. It will immediately follow from Theorem 5.3 and appropriate stability of smooth solutions (Theorem 5.21 below).

Concerning the uniqueness in Theorem 5.18 it was later shown in [Kol03] (also for more general densities f). One can however consider the uniqueness problem without any assumption on density of the Monge–Ampère measure: does $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ imply that $\varphi - \psi = const$? It was proved in [BT89] for $M = \mathbb{P}^n$ but it is true for arbitrary M and can be shown much simpler than in [BT89]. We have the following most general uniqueness result with the simplest proof.

Theorem 5.19 [Bl03b] *If $\varphi, \psi \in L^\infty(M)$ are admissible and $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ then $\varphi - \psi = const$.*

Proof. Set $\rho := \varphi - \psi$ and $\omega_\varphi := \omega + dd^c\varphi$. We start as in the proof of Proposition 5.4. We will get

$$d\rho \wedge d^c\rho \wedge \omega_\varphi^j \wedge \omega_\psi^{n-1-j} = 0, \quad j = 0, 1, \ldots, n-1, \tag{5.25}$$

and we have to show that $d\rho \wedge d^c\rho \wedge \omega^{n-1} = 0$. To describe the further method we assume that $n = 2$. Using (5.25) and integrating by parts

$$\int_M d\rho \wedge d^c\rho \wedge \omega = -\int_M d\rho \wedge d^c\rho \wedge dd^c\varphi = \int_M d\varphi \wedge d^c\rho \wedge (\omega_\psi - \omega_\varphi).$$

By the Schwarz inequality

$$\left|\int_M d\varphi \wedge d^c\rho \wedge \omega_\psi\right| \le \left(\int_M d\varphi \wedge d^c\varphi \wedge \omega_\psi\right)^{1/2} \left(\int_M d\rho \wedge d^c\rho \wedge \omega_\psi\right)^{1/2} = 0$$

by (5.25) and, similarly, $\int_M d\varphi \wedge d^c\rho \wedge \omega_\varphi = 0$. Therefore $d\rho \wedge d^c\rho \wedge \omega = 0$.

For $n > 2$ the proof is similar but one has to use an appropriate inductive procedure: in the same way as before one shows for $l = 0, 1, \ldots, n-1$ that

$$d\rho \wedge d^c\rho \wedge \omega_\varphi^j \wedge \omega_\psi^k \wedge \omega^l = 0$$

if $j + k + l = n - 1$ (see [Bl03b] for details). □

The Monge–Ampère measure $(dd^c u)^n$ can be defined also for some not locally bounded psh u: for example if u is bounded outside a compact set (see [Dem93]). However, there is no uniqueness in this more general class.

Exercise 5.20 Show that

$$\varphi(z) := \log|z| - g(z), \quad \psi(z) := \log||z|| - g(z), \quad z \in \mathbb{C}^n,$$

where

$$|z| = \sqrt{|z_1^2| + \dots |z_n|^2}, \quad ||z|| = \max\{|z_1|, \dots, |z_n|\}, \quad g(z) = \frac{1}{2}\log(1 + |z|^2),$$

define admissible φ, ψ on $\mathbb{P}^n$ (with the Fubini–Study metric $\omega = dd^c \log |Z|$) such that $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ but $\varphi - \psi \neq const.$

Closely analyzing the proof of Theorem 5.19 one can get the quantitative estimate

$$\int_M d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge \omega^{n-1} \leq C \left(\int_M (\psi - \varphi)(\mathcal{M}(\varphi) - \mathcal{M}(\psi)) \right)^{2^{1-n}}, \tag{5.26}$$

where C is a constant depending only on n and upper bounds of $||\varphi||_\infty$, $||\psi||_\infty$ and V. The following Poincaré–Sobolev inequality on compact Riemannian manifolds M of real dimension m

$$||v||^2_{2m/(m-2)} \leq C_M \left(\left(\int_M v \right)^2 + ||Dv||_2^2 \right), \quad v \in W^{1,2}(M),$$

is more difficult to prove than (5.13) (see [Siu87, p.140]; the proof uses an isoperimetric inequality). This combined with (5.26) immediately gives the following stability of weak solutions whose Monge–Ampère measures have densities in L^1

$$||\varphi - \psi||_{2n/(n-1)} \leq C||f - g||_1^{2^{-n}},$$

provided that $\int_M \varphi\omega^n = \int_M \psi\omega^n$, where $\mathcal{M}(\varphi) = f\omega^n$, $\mathcal{M}(\psi) = g\omega^n$, and C depends only on M and on upper bounds for $||\varphi||_\infty$ and $||\psi||_\infty$.

For the proof of the existence part of Theorem 5.18 we will need a uniform stability.

Theorem 5.21 [Kol03] *Assume that $\varphi, \psi \in C(M)$ are admissible and that $\mathcal{M}(\varphi) = f\omega^n$, $\mathcal{M}(\psi) = g\omega^n$ for some $f, g \in C(M)$ with $||f - g||_\infty \leq 1/2$. Let φ, ψ be normalized by* $\max_M(\varphi - \psi) = \max_M(\psi - \varphi)$. *Then*

$$\operatorname*{osc}_M(\varphi - \psi) \leq C||f - g||_\infty^{1/n}, \tag{5.27}$$

where C depends only on M and on upper bounds for $||f||_\infty$, $||g||_\infty$.

Proof. First assume that we have proved the theorem for smooth, strongly admissible φ, ψ. From this and Theorem 5.3 we can easily deduce Theorem 5.18: any nonnegative $f \in C(M)$ with $\int_M f\omega^n = V$ can be uniformly approximated by positive $f_j \in C^\infty(M)$ with $\int_M f_j\omega^n = V$ and the existence part of Theorem 5.18 follows from the continuity of the Monge–Ampère operator for uniform sequences. Then obviously (5.27) will also hold for nonsmooth φ, ψ. It is thus enough to consider $\varphi, \psi \in C^\infty(M)$ with $f, g > 0$.

By Theorem 5.6 we may assume that $-C_1 \le \varphi, \psi \le 0$. Without loss of generality we may replace the normalizing condition $\max_M(\varphi - \psi) = \max_M(\psi - \varphi)$ with the normalizing inequalities

$$0 < \max_M(\varphi - \psi) \le 2 \max_M(\psi - \varphi) \le 4 \max_M(\psi - \varphi) \tag{5.28}$$

and then by the Sard theorem we may assume that 0 is the regular value $\varphi - \psi$ (we will only need that the boundaries of the sets $\{\varphi < \psi\}$ and $\{\psi < \varphi\}$ have volume zero). We will need the following comparison principle.

Proposition 5.22 *If $\varphi, \psi \in C(M)$ are admissible then*

$$\int_{\{\psi<\varphi\}} \mathcal{M}(\varphi) \le \int_{\{\psi<\varphi\}} \mathcal{M}(\psi).$$

Proof. It is a repetition of the proof for psh functions in domains in $\mathbb{C}^n$ (see [Ceg88, p. 43]). For $\varepsilon > 0$ let $\varphi_\varepsilon := \max\{\varphi, \psi + \varepsilon\}$. Then $\varphi_\varepsilon = \psi + \varepsilon$ in a neighborhood of the boundary of $\{\psi < \varphi\}$ and by the Stokes theorem

$$\int_{\{\psi<\varphi\}} \mathcal{M}(\varphi_\varepsilon) = \int_{\{\psi<\varphi\}} \mathcal{M}(\psi).$$

But φ_ε decreases to φ in $\{\psi < \varphi\}$ as ε decreases to 0 and we get the result from the weak convergence $\mathcal{M}(\varphi_\varepsilon) \to \mathcal{M}(\varphi)$. □

Proof of Theorem 5.21, (continued) Set $\delta := ||f - g||_\infty$. We may assume that $\int_{\{\psi<\varphi\}}(f + g)\omega^n \le V$ (otherwise replace φ with ψ). Then

$$\int_{\{\psi<\varphi\}} f\omega^n \le \frac{1+\delta}{2} V \le \frac{3}{4} V.$$

We can find $h \in C^\infty(M)$ such that $0 < h \le C_2$, $\int_M h\omega^n = V$ and $h \ge f + 1/C_3$ in $\{\psi < \varphi\}$ (here we use the fact that the boundary of $\{\psi < \varphi\}$ has volume zero, and thus $\int_{\mathrm{int}\{\psi\ge\varphi\}} f\omega^n \ge V/4$). Since $||f||_\infty$ is under control, we will get

$$h^{1/n} \ge f^{1/n} + 1/C_4 \quad \text{in } \{\psi < \varphi\}.$$

By Theorem 5.3 there is an admissible $\rho \in C^\infty(M)$ such that $(\omega + dd^c\rho)^n = h\omega^n$ and $-C_5 \le \rho \le -C_1$.

Let a be such that $0 < a < \max_M(\varphi - \psi)$. Then

$$\emptyset \ne \{\psi < \varphi - a\} \subset E := \{\psi < (1-t)\varphi + t\rho\} \subset \{\psi < \varphi\},$$

where $t = a/C_5 \le 1$. Using Proposition 5.22 and the concavity of (5.18) we get

$$\int_E g\omega^n \ge \int_E \big(\omega + (1-t)dd^c\varphi + tdd^c\rho\big)^n \ge \int_E \Big((1-t)f^{1/n} + th^{1/n}\Big)^n \omega^n$$
$$\ge \int_E \Big(f^{1/n} + t/C_4\Big)^n \omega^n$$
$$\ge \int_E f\omega^n + \frac{t^n}{C_4^n}\mathrm{vol}\,(E).$$

On the other hand, we have $g \le f + \delta$ and therefore

$$\int_E g\omega^n \le \int_E f\omega^n + \delta \mathrm{vol}\,(E).$$

Hence $a \le C_4 C_5 \delta^{1/n}$ and the estimate follows, since by (5.28)

$$\operatorname*{osc}_M(\varphi - \psi) \le 3 \max_M(\varphi - \psi).$$

□

Note that in the proof of Theorem 5.21, contrary to Theorem 5.19, we have heavily relied on Theorem 5.3 (in the construction of ρ).

From Theorems 5.3 and 5.13 we get the following regularity in the nondegenerate ($f > 0$) and degenerate ($f \ge 0$) case.

Theorem 5.23 *Let $\varphi \in C(M)$ be admissible and assume that $\mathcal{M}(\varphi) = f\omega^n$. Then*
i) $f \in C^\infty$, $f > 0 \Longrightarrow \varphi \in C^\infty$;
ii) $f^{1/(n-1)} \in C^{1,1} \Longrightarrow \Delta\varphi \in L^\infty \Longrightarrow \varphi \in C^{1,\alpha}$, $\alpha < 1$.

Part IV
Geodesics in the Space of Kähler Metrics

Chapter 6
The Riemannian Space of Kähler Metrics

Boris Kolev

Abstract The goal of this lecture is to describe the Riemannian structure on the space of Kähler metrics (in a fixed cohomology class) on a compact complex manifold which was proposed by Mabuchi [Mab87] and further studied by Semmes [Sem92], Donaldson [Don99], Chen and Calabi [Che00, CC02]. The lecture starts by explaining some of the difficulties encountered in this infinite dimensional setting.

6.1 Introduction

The goal of this lecture is to introduce the Riemannian structure on the space of Kähler metrics (in a fixed cohomology class) on a compact complex manifold which was proposed by Mabuchi [Mab87]. This space is a Fréchet manifold where the extension of classical Riemannian geometry leads to extra difficulties. This is why the first part of this lecture is devoted to the Riemannian geometry of diffeomorphisms groups of a compact manifold as a way to illustrate the inherent difficulties which may arise.

I will then introduce the Mabuchi metric on the space of Kähler metrics and its geodesic flow following Donaldson's paper [Don97]. I will explain the equivalence between the boundary problem for geodesics and the complex homogenous Monge–Ampère equation as discovered by Semmes [Sem92]. Finally, I will introduce the Aubin–Mabuchi functional and its relation with the problem of extremal Kähler metrics.

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Let us emphasize that in this lecture, we implicitly suppose that *geodesics are smooth* even if we are still not sure of the existence of even C^2 geodesics. I insist on the fact that this regularity problem, which is the main difficulty of the theory, will not be discussed here and will be delayed to other lectures of this workshop. As such, this exposition may appear excessively formal. However, the goal of this introductory lecture is to present the *big picture* of the *underlying geometry*, leaving the discussion on technical and difficult analytical points to more specialized lectures of this workshop.

Nota Bene. These notes are written by Boris Kolev and revised by Georges Dloussky after the lecture delivered by Boris Kolev in Marseille, March 2009. There is no claim for any originality. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

6.2 Diffeomorphisms Groups

Before entering into the core of the subject of the Riemannian structure introduced by Mabuchi [Mab87] on the space of Kähler metrics, I will discuss infinite dimensional Riemannian geometry and illustrate the main differences with the finite dimensional case by some known results for the diffeomorphisms group of a compact manifold.

6.2.1 Differentiable Structure

Let M be a smooth, compact manifold and $\mathrm{Diff}(M)$ its diffeomorphisms group. We start by the formal definition of its tangent space at the unit element. Let φ_t $(t \in (-\varepsilon, \varepsilon))$ be a smooth path in $\mathrm{Diff}(M)$ such that $\varphi_0 = Id$. Then

$$\frac{\partial \varphi}{\partial t}(0, x) = X(x) \in T_x M$$

and hence $T_{Id}\mathrm{Diff}(M)$ can be identified with $\mathrm{Vect}(M)$, the space of smooth vector fields on M. More generally, if $\varphi_0 = \varphi$ is any fixed diffeomorphism, then

$$\frac{\partial \varphi}{\partial t}(0, x) = X(x) \in T_{\varphi(x)} M$$

and $T_\varphi \mathrm{Diff}(M)$ can be identified with the vector space of smooth sections *above* φ. That is

$$T_\varphi \mathrm{Diff}(M) = \left\{ X_\varphi \in C^\infty(M, TM);\ \pi \circ X_\varphi(x) = \varphi(x) \right\},$$

where $\pi : TM \to M$ is the canonical projection.

Notice that Vect(M), the space of smooth sections of the tangent bundle, is a Fréchet vector space[1] but not a Banach space. This Fréchet space is used as a model to describe a differentiable structure on Diff(M). To build an atlas on Diff(M) with charts taking values in the Fréchet space Vect(M), we choose an embedding of M into $\mathbb{R}^N$ for some N (which furnishes also a Riemannian metric on M). The subset $C^\infty(M, M)$ is closed in $C^\infty(M, \mathbb{R}^N)$ and we can show that Diff(M) is open in $C^\infty(M, M)$ [Mil84]. Let exp be the Riemannian exponential map on M. Then the map

$$X \in \mathrm{Vect}(M) \mapsto f; \quad f(x) = \exp_x(X(x))$$

is a homeomorphism from a neighborhood of 0 in Vect(M) onto a neighborhood of the identity in Diff(M). By translating this open set everywhere on Diff(M) we get a smooth atlas on Diff(M). One can show that the resulting differentiable structure does not depend on the particular embedding of M [Mil84].

Since the vector space Vect(M) on which the charts are builded is locally convex, we deduce that Diff(M) is locally arcwise connected.

Remark 6.1 *It was discovered in the seventies by Epstein and Hermann that the group* Diff(M) *(or more precisely its identity component* $\mathrm{Diff}^0(M)$*) is* simple*. Indeed, it was shown by Epstein [Eps70] that the commutator subgroup*

$$[\mathrm{Diff}^0(M), \mathrm{Diff}^0(M)]$$

is simple and independently by Hermann [Her71] and Thurston [Thu74] (see also [Mat74, Eps84]) that

$$\mathrm{Diff}^0(M) = [\mathrm{Diff}^0(M), \mathrm{Diff}^0(M)],$$

which is the difficult part of the proof.

6.2.2 Fréchet–Lie Group Structure

Equipped with this differentiable structure, Diff(M) is a *Fréchet–Lie group*. Composition and inversion are smooth maps (in the weak sense of *Gâteaux*).

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[1] A topological vector space E has a canonical *uniform structure*. When this structure is *complete* and when the topology of E may be given by a countable family of *semi-norms*, we say that E is a *Fréchet vector space*. A Fréchet space is a Banach space if and only if it is locally bounded.

Remark 6.2 *Notice that the same construction for the group of C^k-diffeomorphisms leads to a Banach manifold structure on* $\mathrm{Diff}^k(M)$ *but does not make it a Lie group. Neither the composition, nor the inversion are differentiable for this structure [EM70].*

Remark 6.3 *The Lie group algebra bracket on* $\mathrm{Vect}(M)$ *differs form the usual bracket of vector fields by a sign. Indeed, the usual bracket of two vector fields X, Y is the derivative of the* adjoint action *of* $\exp(-sX)$ *rather than* $\exp(sX)$.

To each vector field $X \in \mathrm{Vect}(M)$ corresponds a one parameter subgroup in $\mathrm{Diff}(M)$ (the group is *regular* as defined by Milnor [Mil84]). However, the Lie group exponential is not a local diffeomorphism near the origin, in contrast to what happens for a finite dimensional Lie group. Indeed, every diffeomorphisms φ_t in a one parameter subgroup has a square root $\psi_t = \varphi_{t/2}$. If the group exponential was locally surjective near the origin, every diffeomorphism sufficiently close to the identity (for the smooth topology) would have a square root, which is not the case.[2]

To finish this section, let us cite some remarkable subgroups of $\mathrm{Diff}(M)$ which are themselves Fréchet–Lie groups.

Example 6.4 *The subgroup* $\mathrm{SDiff}_\mu(M)$*, of diffeomorphisms M which preserve a volume form μ. Its Lie algebra is the Lie subalgebra of divergence free vector fields*

$$\mathrm{SVect}_\mu(M) = \{X \in \mathrm{Vect}(M);\ L_X\mu = (\mathrm{div}\, X)\mu = 0\}$$

where the divergence of a vector field X is defined by

$$L_X\mu = d(i_X\mu) := (\mathrm{div}\, X)\mu.$$

Example 6.5 *The subgroup* $\mathrm{Symp}_\omega(M)$*, of diffeomorphisms of M which preserve a symplectic form ω. Its Lie algebra is the Lie subalgebra of symplectic vector fields*

$$\mathrm{SpVect}_\omega(M) = \{X \in \mathrm{Vect}(M);\ L_X\omega = 0\}\,.$$

6.2.3 Riemannian Metrics

A *right-invariant* Riemannian metric on $\mathrm{Diff}(M)$ may be constructed by choosing an inner product on the Lie algebra $\mathrm{Vect}(M)$ and translating it

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[2] For $\mathrm{Diff}(\mathbb{S}^1)$, for instance, it is easy to build diffeomorphisms arbitrary close to the identity and which have exactly one periodic orbit of period $2n$. But the number of periodic orbits of even period of the square of a homeomorphism is necessarily even. Therefore, such a diffeomorphism cannot have a square root [Mil84].

to each tangent space. We must however notice that, even if each of these inner products is continuous and positive definite, it produces only a *pre-Hilbertian* structure on each tangent space. In particular it does not provide an isomorphism with the cotangent space. Therefore, we call such a structure a *weak Riemannian metric*.

Example 6.6 *Fix a Riemannian metric g on M. We define the L^2 inner product on $T_{Id}\mathrm{Diff}(M) = \mathrm{Vect}(M)$ by*

$$< u, v >_{L^2} = \int_M g(u(x), v(x))\, d\mu_g,$$

for all $u, v \in \mathrm{Vect}(M)$, where $d\mu_g$ is the Riemannian volume element. More generally, for each $u \circ \varphi$, $v \circ \varphi \in T_\varphi \mathrm{Diff}(M)$, the same formula defines an inner product on $T_\varphi \mathrm{Diff}(M)$. These inner products generate a right invariant (weak) Riemannian metric on $\mathrm{Diff}(M)$.

Example 6.7 *Similarly, the H^1 metric on $\mathrm{Diff}(M)$ is defined by the introduction of the inner product*

$$< u, v >_{H^1} = \int_M \left[g(u(x), v(x)) + g((\nabla u)(x), (\nabla v)(x)) \right]\, d\mu_g.$$

on $\mathrm{Vect}(M)$.

The existence of a symmetric covariant derivative which preserves a given weak Riemannian metric is not granted automatically like in the finite dimensional case. For a weak Riemannian metric, the *Koszul formula*, which defines the *Levi–Civita connection* in finite dimension, only ensures the uniqueness of such a connection but *not its existence*.

Remark 6.8 *The interest for geodesic flows on the diffeomorphisms groups goes back to Arnold [Arn66] in the sixties. He showed that the Euler equation of fluid mechanics could, under certain hypothesis, be interpreted as the geodesic equation for the L^2 metric on the volume-preserving diffeomorphisms group. It should be however emphasized that, contrary to the finite dimensional case, where the* local existence of geodesics *is automatically ensured by the Cauchy–Lipschitz theorem, this is no more true for a weak Riemannian metric on a Fréchet manifolds.*

6.2.4 The Riemannian Semi-Distance

On a connected Riemannian manifold of finite dimension, the lower bound of C^1-piecewise paths length between two points

$$d_g(x,y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\|_g \, dt$$

defines a *distance* on the manifold M (i.e $d_g(x,y) > 0$ if $x \neq y$). This fact results essentially from the observation that, in finite dimension, there exists, around each point x_0, a local chart V and a constant $C > 0$ such that

$$\|\cdot\|_{g(x)} \geq C(x_0, V) \|\cdot\|_{Euclid},$$

for all x in V. For a *weak Riemannian metric* on a Fréchet manifold, this estimate is however meaningless. The only thing granted, in that case, is that d_g is a *semi-distance*: it is non-negative, symmetric and satisfies the triangle inequality. Worse, this function may even *vanish identically* as has been established in [MM05].

Theorem 6.9 (Michor & Mumford 2005) *The* Riemannian semi-distance d_{L^2} *induced by the* L^2 *Riemannian metric on the diffeomorphisms group* $\mathrm{Diff}^0(M)$ *of a compact manifold vanishes identically.*

Proof (Sketch of proof). We introduce the set $\mathcal{E}(M)$ of all diffeomorphisms $\varphi \in \mathrm{Diff}^0(M)$ with the following property : for each $\varepsilon > 0$ there exists a smooth curve $\varphi(t,\cdot)$ from the identity to φ_1 in $\mathrm{Diff}(M)$ such that

$$E(\varphi_t) = \int_0^1 \|u(t)\|_{L^2}^2 \, dt \leq \varepsilon,$$

where $u(t) = \dot\varphi_t \circ \varphi_t^{-1}$ is the Eulerian velocity. By the Cauchy–Schwarz inequality, we have

$$L(\varphi_t)^2 \leq E(\varphi_t)$$

where

$$L(\varphi_t) = \int_0^1 \|u(t)\|_{L^2} \, dt,$$

and it suffices therefore to show that $\mathcal{E} = \mathrm{Diff}^0(M)$. One first checks that $\mathcal{E}(M)$ is a normal subgroup of $\mathrm{Diff}^0(M)$. But since $\mathrm{Diff}^0(M)$ is simple, it is enough to show that $\mathcal{E}(M)$ is not reduced to $\{Id\}$. This is the difficult part of the proof. The trick consists in adapting the following construction, given for a diffeomorphism of the real line, to the case of a general compact manifold.

Let $0 < \varepsilon < 1/2$ and

$$f_\varepsilon(z) = \max(0, \min(1, z)) \star G_\varepsilon$$

where G_ε is the regularizing kernel of total weight 1 and of support $[-\varepsilon, \varepsilon]$. We set

$$\varphi_\varepsilon(t,x) = x + f_\varepsilon(t - \lambda x), \qquad \lambda = 1 - \varepsilon.$$

Notice that $0 \le f'_\varepsilon(z) \le 1$ and thus $\varphi_\varepsilon(t,\cdot)$ is a diffeomorphism for each t. We get

$$\begin{aligned} E(\varphi_\varepsilon) &= \int_0^1 \|u(t)\|^2_{L^2}\, dt \\ &= \int_0^1 \int_{\mathbb{R}} \left(\partial_t \varphi_\varepsilon \circ \varphi_\varepsilon^{-1}\right)^2 dx\, dt \\ &= \int_0^1 \int_{\mathbb{R}} \frac{1}{\lambda} f'_\varepsilon(z)^2 \left(1 - \lambda f'_\varepsilon(z)\right) dz\, dt \\ &\le \frac{1}{\lambda} \int_{-\varepsilon}^{1+\varepsilon} \left(1 - \lambda f'_\varepsilon(z)\right) dz. \end{aligned}$$

But this last integral equals $3\varepsilon/(1-\varepsilon)$. □

However, by a slight modification of the L^2 metric, we can obtain a positive distance. Let $c > 0$ and set

$$< u, v >_{L^2_c} := \int_M \left[g(u(x), v(x)) + c\,\mathrm{div}(u)\,\mathrm{div}(v)\right] d\mu_g.$$

Proposition 6.10 (Michor & Mumford 2005) *The L^2_c metric induces a positive distance on* $\mathrm{Diff}^0(M)$*.*

Proof. Let $\varphi_1 \in \mathrm{Diff}^0(M)$ be a diffeomorphism such that $\varphi_1 \neq Id$. We are going to show that the L^2_c length of every path $\varphi(t,\cdot)$ from Id to φ_1 is bounded below by a positive constant independent of the path. Notice first that if $\varphi_1 \neq Id$, there exists two functions $f, \rho \in C^\infty(M)$ such that

$$\int_M f(\varphi_1^{-1}(y))\rho(y)\, d\mu_g \neq \int_M f(y)\rho(y)\, d\mu_g.$$

Let

$$a(t) = \int_M f(\psi_t(y))\rho(y)\, d\mu_g,$$

where $\psi_t = \varphi_t^{-1}$. We have

$$\dot a = \int_M \left[d_{\psi_t(y)} f.\left(\dot\psi_t(y)\right)\right]\rho(y)\, d\mu_g = -\int_M \left[d_y(f\circ\psi_t).u_t(y)\right]\rho(y)\, d\mu_g$$

where $u = \dot\varphi_t \circ \varphi_t^{-1}$ is the Eulerian velocity of the path φ_t. Using the following formula

$$\mathrm{div}((f\circ\psi)\rho u) = (f\circ\psi)\,\mathrm{div}(\rho u) + \rho\, d(f\circ\psi).u,$$

we get

$$\dot{a} = \int_M (f \circ \psi) \operatorname{div}(\rho u) \, d\mu_g$$

and thus

$$|\dot{a}| \leq \max_M \big(|f| \big) \int_M |\operatorname{div}(\rho u)| \, d\mu_g \leq C \, \|u\|_{L^2_c} \, ,$$

where C is a positive constant which *depends only* on f and ρ. Therefore we have

$$C \int_0^1 \|u\|_{L^2_c} \, dt \geq |a(1) - a(0)| > 0,$$

for all path φ_t. Since $|a(1) - a(0)|$ is independent of the path, the proof is complete. □

Since $\|u\|_{L^2_c} \leq \|u\|_{H^1}$ on $\mathrm{Vect}(M)$ and $\|u\|_{L^2_c} = \|u\|_{L^2}$ on $\mathrm{SVect}(M)$, we have the following corollary.

Corollary 6.11 *The H^1 metric induces a positive distance on* $\mathrm{Diff}^0(M)$ *and the L^2 metric induces a positive distance on* $\mathrm{SDiff}^0(M)$.

6.2.5 Hamiltonian Diffeomorphisms

The group $\mathrm{Symp}(M)$, of symplectic diffeomorphisms of a compact, symplectic manifold (M, ω), is a Fréchet Lie group. Its Lie algebra is the space of symplectic vector fields on M (i.e vector fields X such that $L_X \omega = 0$). This group is *locally arcwise connected* and, given $\varphi \in \mathrm{Symp}^0(M)$, there exists a smooth curve φ_t of symplectic diffeomorphisms joining φ to the identity. Let

$$X_t = \dot{\varphi}_t \circ \varphi_t^{-1}$$

be the Eulerian velocity of this path. We have

$$d i_{X_t} \omega = L_{X_t} \omega = 0.$$

If each of these 1-forms is exact, we can find a smooth family of functions $h_t : M \to \mathbb{R}$ such that:

$$i_{X_t} \omega = d h_t.$$

We say then that φ is a *Hamiltonian diffeomorphism* and that h_t generates the Hamiltonian isotopy φ_t. The set of Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M)$.

Remark 6.12 *If $H^1(M, \mathbb{R}) = 0$ then* $\mathrm{Ham}(M) = \mathrm{Symp}^0(M)$.

Proposition 6.13 *The set* $\mathrm{Ham}(M)$ *is a normal subgroup of* $\mathrm{Symp}(M)$ *and is arcwise connected.*

Proof. 1. If h_t and k_t generate Hamiltonian isotopies joining respectively φ and ψ to the identity, one can check that $h_t + k_t \circ \varphi_t^{-1}$ and $-h_t \circ \varphi_t$ generate Hamiltonian isotopies joining respectively $\varphi \circ \psi$ and φ^{-1} to the identity.
2. Let $\psi \in \mathrm{Symp}(M)$ and φ_t be a Hamiltonian isotopy. Then $\psi^{-1} \circ \varphi_t \circ \psi$ is a Hamiltonian isotopy with corresponding Hamiltonian $h_t \circ \psi$.
3. The fact that $\mathrm{Ham}(M)$ is arcwise connected is obvious.

The Lie algebra of $\mathrm{Ham}(M)$ can be identified with the *Lie sub-algebra* of Hamiltonian vector fields in $\mathrm{SpVect}(M)$. We have moreover

$$[X_f, X_g] = X_{\{f,g\}}$$

where $\{f, g\}$ is the *Poisson bracket* of f, g which is defined by

$$\{f, g\}\,\omega^n = \omega(X_f, X_g)\omega^n = n\, df \wedge dg \wedge \omega^{n-1}.$$

The mapping $h \mapsto X_h$ leads to the identification of the Lie algebra of $\mathrm{Ham}(M)$ with the Lie algebra $C^\infty(M)$ (endowed with the Poisson bracket) modulo constant functions.

There is a natural inner product on $C^\infty(M)$

$$< f, g >= \frac{1}{n!} \int_M f g\, \omega^n.$$

which is invariant by the adjoint action of the group $\mathrm{Symp}(M)$ on $\mathrm{Ham}(M)$. This inner product induces a bi-invariant, weak Riemannian metric on $\mathrm{Ham}(M)$.

We have the following *orthogonal* decomposition:

$$C^\infty(M) = C_0^\infty(M) \oplus \mathbb{R}$$

where $C_0^\infty(M)$ is the Lie sub-algebra of functions with vanishing integral, which can therefore be identified with the Lie algebra of $\mathrm{Ham}(M)$.

6.2.6 Quantomorphisms

When $[\omega] \in H^2(M, \mathbb{R})$ is an *integral* cohomology class, one can find a principal bundle $p : P \to M$ with structure group $\mathbb{U}(1)$ and connection α such that:

$$d\alpha = -2i\pi p^*\omega.$$

We say then that the symplectic manifold (M, ω) is *quantizable*. This theorem is attributed to A. Weil [Wei58] and was rediscovered in the seventies by Kostant, Kirillov and Souriau (see [Kos70, Bry08, Woo92]).

The diffeomorphisms group of P which preserve α (i.e $\widetilde{\varphi}^*\alpha = \alpha$) is called the group of *quantomorphisms* and denoted $\mathrm{Quant}(P)$.

$\mathrm{Quant}(P)$ is also a Fréchet Lie group. Its Lie algebra is the subspace of vector fields $\widetilde{X}$ on P such that:

$$L_{\widetilde{X}}\alpha = 0.$$

For such a vector field, one can show that there exists a unique function $H \in C^\infty(M)$ such that:

$$\alpha(\widetilde{X}) = H \circ p, \qquad i_{\widetilde{X}} d\alpha = -d(H \circ p),$$

which shows that the Lie algebra of $\mathrm{Quant}(P)$ can be identified with $C^\infty(M)$.

Every quantomorphisms of P induces a symplectomorphism on M and the mapping

$$\widetilde{p} : \mathrm{Quant}(P) \to \mathrm{Symp}(M)$$

is a group morphism.

Its kernel is isomorphic to the structure group $\mathbb{U}(1)$. To each isotopy in $\mathrm{Quant}(P)$ corresponds a Hamiltonian isotopy in $\mathrm{Symp}(M)$ and conversely. Therefore, $\widetilde{p}(\mathrm{Quant}^0(P)) = \mathrm{Ham}(M)$. Since, moreover $\mathbb{U}(1)$ is in the center of $\mathrm{Quant}(M)$, we get that

$$\mathrm{Quant}(P) = \mathrm{Ham}(M) \times \mathbb{U}(1).$$

6.3 The Space of Kähler Metrics

Suppose now that (M, J) is a *compact* complex manifold of dimension n which admits a Kähler metric ω_0. Then each other Kähler metric on M *in the same cohomology class* as ω_0 can be written as

$$\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi.$$

Let $\mathcal{H}$ be the space of *Kähler potentials*

$$\mathcal{H} = \left\{ \phi \in C^\infty(M)\, ; \omega_\phi = \omega_0 + i\partial\bar{\partial}\phi > 0 \right\}.$$

Notice that $\mathcal{H}$ is a convex open subset of the Fréchet vector space $C^\infty(M)$. It is therefore itself a Fréchet manifold, which is moreover parallelizable : $T\mathcal{H} = \mathcal{H} \times C^\infty(M)$. Each tangent space is identified with $C^\infty(M)$.

Two Kähler potentials define the same metric if and only if they differ by an additive constant (since M is compact). We set

$$\mathcal{H}_0 = \mathcal{H}/\mathbb{R}$$

where $\mathbb{R}$ acts on $\mathcal{H}$ by addition. $\mathcal{H}_0$ is therefore the *space of Kähler metrics on M in the cohomology class* $[\omega_0]$.

Remark 6.14 *In the case where $[\omega_0]$ is integral, there exists a holomorphic line bundle $L \to M$ whose first Chern class is $[\omega_0]$ (see [Wei58] or [Kos70]). In that case, we can identify $\mathcal{H}$ with the space of* Hermitian metrics *on L, with positive curvature. If h_0 is one such metric whose curvature is $-2i\pi\omega_0$, then the curvature of $e^{2\pi\phi}h_0$ is $-2i\pi(\omega_0 + i\partial\bar{\partial}\phi)$.*

6.3.1 The Mabuchi Metric

Definition 6.15 *The* Mabuchi metric *(introduced in [Mab87]) is the L^2 Riemannian metric on $\mathcal{H}$. It is defined by*

$$< \psi_1, \psi_2 >_\phi = \int_M \psi_1 \psi_2 \, d\mu_\phi$$

where $\phi \in \mathcal{H}$, $\psi_1, \psi_2 \in C^\infty(M)$.

Geodesics between two points ϕ_0, ϕ_1 in $\mathcal{H}$ are defined as the extremals of the *Energy functional*, given by

$$E(\phi) = \frac{1}{2} \int_0^1 \int_M \dot{\phi_t}^2 \, d\mu_{\phi_t} \, dt.$$

where $\phi = \phi_t$ is a path in $\mathcal{H}$ joining ϕ_0 and ϕ_1. The geodesic equation is obtained by computing the Euler–Lagrange equation, $\delta E = 0$, for the Energy functional E (with fixed end points).

Recall first (see Lemma 6.31 in the Appendix) that when the Kähler potential is submitted to a variation $\delta\phi$, the variation of the volume is

$$\delta(d\mu_\phi) = \frac{1}{2}(\triangle \delta\phi) d\mu_\phi,$$

where the Laplacian is defined here as the trace of the Hessian

$$\triangle \psi = \operatorname{tr} \nabla d\psi.$$

Let $\phi_{s,t}$ be a variation of ϕ_t in $\mathcal{H}$, that is $\phi_{0,t} = \phi_t$, $\phi_{s,0} = \phi_0$ and $\phi_{s,1} = \phi_1$ and let $\psi_t = (\partial_s \phi)_{0,t}$. We have then

$$\begin{aligned}
d_\phi E.\psi &= \frac{1}{2}\int_0^1 \int_M \left(2\dot{\phi}_t\dot{\psi}_t d\mu_\phi + \frac{1}{2}\dot{\phi}^2(\triangle\psi)d\mu_\phi\right) dt \\
&= \frac{1}{2}\int_0^1 \int_M \left(-\psi_t(2\ddot{\phi} + \dot{\phi}\triangle\dot{\phi}) + \frac{1}{2}\triangle(\dot{\phi_t}^2)\psi_t\right) d\mu_\phi\, dt \\
&= -\int_0^1 \int_M \psi_t \left(\ddot{\phi} - \frac{1}{2}\left\|\operatorname{grad}\dot{\phi}\right\|_\phi^2\right) d\mu_\phi\, dt.
\end{aligned}$$

Hence, the geodesic equation reduces to

$$\ddot{\phi} = \frac{1}{2}\left\|\operatorname{grad}\dot{\phi}\right\|_\phi^2 \tag{6.1}$$

where the gradient is relative to the metric ω_ϕ.

Remark 6.16 *In [Don99], Donaldson uses the parametrization of $\mathcal{H}$ by*

$$\omega_0 + i\overline{\partial}\partial\phi$$

rather than

$$\omega_0 + i\partial\bar{\partial}\phi$$

as in [Mab87, CC02]. This comes to change ϕ into $-\phi$ and leads to a geodesic equation which differs from (6.1) by a sign. In the sequel, we will use the $\partial\bar{\partial}$ convention.

On a Riemannian manifold of finite dimension, the local expression of the Levi–Civita connection is obtained by the *polarization* of the geodesic equation. The same procedure may be applied in the more general setting of a weak Riemannian metric. We define the covariant derivative of the vector field ψ_t along the path ϕ_t in $\mathcal{H}$ by the formula

$$\frac{D\psi}{Dt} = \partial_t\psi - \frac{1}{2} < \operatorname{grad}\psi, \operatorname{grad}\dot{\phi} >_\phi .$$

This covariant derivative is symmetric by its very definition, that is

$$\frac{D}{Ds}\partial_t\phi = \frac{D}{Dt}\partial_s\phi,$$

for every family $\phi = \phi_{s,t}$ and it can be checked directly that it preserves the metric, that is

$$\frac{d}{dt} < \psi_1, \psi_2 >_\phi = < \frac{D\psi_1}{Dt}, \psi_2) >_\phi + < \psi_1, \frac{D\psi_2}{Dt} >_\phi .$$

Remark 6.17 *It has been established by Chen in [Che00] that the Mabuchi metric induces a* positive distance *on $\mathcal{H}$ (what he calls a* path length space*). This answers a question raised by Donaldson in [Don99]. See also [Cla10] for a proof of the more general result that the L^2 metric on the space of Riemannian metrics on a given closed manifold gives rise to a positive distance on this space.*

6.3.2 The Boundary Problem and the Complex Monge–Ampère Equation

There are two problems related to the geodesic equation (6.1) on the (weak) Riemannian manifold $\mathcal{H}$:

1. The first one is the *initial value problem*: given $\phi_0 \in \mathcal{H}$ and $\psi_0 \in C^\infty(M)$, is there a function ϕ_t defined on some interval $[0,\varepsilon)$, solution of (6.1) and such that:
$$\phi(0) = \phi_0 \qquad \text{and} \qquad \dot{\phi}(0) = \psi_0$$
2. The second one is the *boundary value problem*. Given ϕ_1, ϕ_2 distinct, is there a function $\phi(t)$ defined on $[0,1]$, solution of (6.1) and such that $\phi(0) = \phi_0$ and $\phi(1) = \phi_1$?

In this section, we intend to explain how the second problem can be reformulated as a *homogenous complex Monge–Ampère equation*. This was first noticed by Semmes in [Sem92].

For each path ϕ_t, $t \in [0,1]$ in $\mathcal{H}$, let

$$\Phi(x,t,s) = \phi_t(x), \qquad x \in M, \quad e^{t+is} \in A = [1,e] \times \mathbb{S}^1.$$

In other words, we associate to each path ϕ_t defined on $M \times [0,1]$, a *radial function* Φ on $M \times A$ and conversely. We consider the annulus A as a Riemann surface with boundary and use the complex coordinate $z = t + is$ to parameterize the annulus A. Let Ω_0 be the *pull-back* of ω_0 and

$$\Omega_\Phi = \Omega_0 + i\partial\bar{\partial}\Phi,$$

which is a $(1,1)$ form on $M \times A$.

Proposition 6.18 (Semmes) *The path ϕ_t is a geodesic in $\mathcal{H}$ if and only if the associated radial function Φ on $M \times A$ is solution of the homogenous complex Monge–Ampère equation $\Omega_\Phi^{n+1} = 0$.*

Proof. Let ∂_M and $\bar{\partial}_M$ the complex exterior derivatives on M. We have

$$\partial\Phi = \partial_M\Phi + \frac{\partial\Phi}{\partial z}dz, \qquad \bar{\partial}\Phi = \bar{\partial}_M\Phi + \frac{\partial\Phi}{\partial\bar{z}}d\bar{z}.$$

Therefore

$$\partial\bar{\partial}\Phi = \partial_M\bar{\partial}_M\Phi - \bar{\partial}_M\left(\frac{\partial\Phi}{\partial z}\right)\wedge dz + \partial_M\left(\frac{\partial\Phi}{\partial\bar{z}}\right)\wedge d\bar{z} + \left(\frac{\partial^2\Phi}{\partial z\partial\bar{z}}\right)dz\wedge d\bar{z}.$$

Hence, Ω_Φ can be written as $\omega_\phi + iX$ where

$$X = -\bar{\partial}_M\left(\frac{\partial\Phi}{\partial z}\right)\wedge dz + \partial_M\left(\frac{\partial\Phi}{\partial\bar{z}}\right)\wedge d\bar{z} + \left(\frac{\partial^2\Phi}{\partial z\partial\bar{z}}\right)dz\wedge d\bar{z}.$$

Notice that ω_ϕ and X commute (since they are 2-forms) and that $X^3 = 0$. Therefore

$$\Omega_\Phi^{n+1} = \omega_\phi^{n+1} + i(n+1)\omega_\phi^n\wedge X - \frac{n(n+1)}{2}\omega_\phi^{n-1}\wedge X\wedge X$$

Let

$$\alpha = \bar{\partial}_M\left(\frac{\partial\Phi}{\partial z}\right), \qquad \beta = \partial_M\left(\frac{\partial\Phi}{\partial\bar{z}}\right), \qquad c = \frac{\partial^2\Phi}{\partial z\partial\bar{z}},$$

we have

$$X = -\alpha\wedge dz + \beta\wedge d\bar{z} + c\,dz\wedge d\bar{z}, \qquad X\wedge X = 2\alpha\wedge\beta\wedge dz\wedge d\bar{z}$$

and

$$\begin{aligned}
&\omega_\phi^{n+1} = 0\\
&\omega_\phi^n\wedge X = c\,\omega_\phi^n\wedge(dz\wedge d\bar{z})\\
&\omega_\phi^{n-1}\wedge X\wedge X = 2\left(\alpha\wedge\beta\wedge\omega_\phi^{n-1}\right)\wedge(dz\wedge d\bar{z}).
\end{aligned}$$

Therefore, $\Omega_\Phi^{n+1} = 0$ if and only if

$$ic\,\omega_\phi^n\wedge(dz\wedge d\bar{z}) - n\left(\alpha\wedge\beta\wedge\omega_\phi^{n-1}\right)\wedge(dz\wedge d\bar{z}) = 0.$$

Notice that for every real smooth function f on M, we have

$$\bar{\partial}_M f = \omega_\phi(-iP\,\mathrm{grad}\,f,\cdot), \qquad \partial_M f = \omega_\phi(iQ\,\mathrm{grad}\,f,\cdot),$$

where

$$P = \frac{1}{2}(I - iJ), \qquad Q = \frac{1}{2}(I + iJ)$$

and that

$$\omega_\phi(-iP\,\mathrm{grad}\,f, iQ\,\mathrm{grad}\,f) = \frac{i}{2}\left\|\mathrm{grad}\,f\right\|^2.$$

Now, by its very definition

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial \bar{z}} = \dot{\phi}, \qquad \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = \ddot{\phi}.$$

Therefore, using Lemma 6.29 in the Appendix, we get

$$\alpha \wedge \beta \wedge \omega_\phi^{n-1} = \frac{i}{2n} \left\| \operatorname{grad} \dot{\phi} \right\|^2 \omega_\phi^n$$

and therefore $\Omega_\Phi^{n+1} = 0$ if and only if

$$\ddot{\phi} = \frac{1}{2} \left\| \operatorname{grad} \dot{\phi} \right\|^2.$$

6.3.3 The Aubin–Mabuchi Functional

Each tangent space $T_\phi \mathcal{H}$ admits the following orthogonal decomposition

$$T_\phi \mathcal{H} = \{\psi \in C^\infty(M);\ \alpha_\phi(\psi) = 0\} \oplus \mathbb{R},$$

where α is the 1-form defined on $\mathcal{H}$ by

$$\alpha_\phi(\psi) = \int_M \psi \, d\mu_\phi.$$

Lemma 6.19 *The 1-form α is closed. Therefore, there exists a unique function I defined on the convex open set $\mathcal{H}$, such that $\alpha = dI$ and $I(0) = 0$. It is called the* Aubin–Mabuchi functional *and can be expressed as*

$$I(\phi) = \sum_{p=0}^{n} \frac{1}{(p+1)!(n-p)!} \int_M \phi \, \omega_0^{n-p} \wedge (i\partial\bar{\partial}\phi)^p. \tag{6.2}$$

Remark 6.20 *In the sequel, we will make use of the following notation, as in finite dimensional geometry. Given a functional $\mathcal{F}$ defined on $\mathcal{H}$, we define*

$$\psi \cdot \mathcal{F} := \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(\phi_t),$$

where ϕ_t is a path on $\mathcal{H}$ such that $\phi_0 = \phi$ and $\dot{\phi}_0 = \psi$.

Proof. Let ψ_1, ψ_2 be two vectors in $C^\infty(M)$. We have

$$(d\alpha)_\phi(\psi_1,\psi_2) = \psi_1 \cdot \alpha(\psi_2) - \psi_2 \cdot \alpha(\psi_1) = \frac{1}{2}\int_M (\psi_2 \triangle \psi_1 - \psi_1 \triangle \psi_2)\, d\mu_\phi = 0$$

since the Laplacian is self-adjoint. This shows that α is closed. Moreover, since $\mathcal{H}$ is convex, there exists a unique function I defined on $\mathcal{H}$ such that $\alpha = dI$ and $I(0) = 0$. This function I is defined by

$$I(\phi) = \int_0^1 \alpha_{t\phi}(\phi)\, dt, \qquad \phi \in \mathcal{H}.$$

An elementary computation, using the fact that

$$d\mu_{t\phi} = \frac{1}{n!}\sum_{p=0}^{n} \binom{n}{p} t^p \omega_0^{n-p} \wedge (i\partial\bar{\partial}\phi)^p,$$

shows that this last integral is precisely equal to (6.2), which ends the proof.

Proposition 6.21 *The Aubin–Mabuchi functional is* affine *along geodesics in* $\mathcal{H}$.

Proof. Let ϕ_t be a geodesic. We have

$$\frac{d^2}{dt^2} I(\phi_t) = \frac{d}{dt}\int_M \dot{\phi}\, d\mu_\phi = \int_M \left(\ddot{\phi} + \frac{1}{2}\dot{\phi}\triangle\dot{\phi}\right) d\mu_\phi = 0,$$

as the integrant in the last integral is a divergence when ϕ_t is a geodesic. □

Since for each $\phi \in \mathcal{H}$, the path $\phi_t = \phi + t$ in $\mathcal{H}$ is a geodesic, we obtain in particular that

$$I(\phi + t) = t\left(\int_M d\mu_\phi\right) + I(\phi). \tag{6.3}$$

Therefore, given $\phi \in \mathcal{H}$, there exists a unique constant $c \in \mathbb{R}$ such that $I(\phi + c) = 0$. The restriction of the Mabuchi metric to the fiber $I^{-1}(0)$ induces a Riemannian structure on the quotient space $\mathcal{H}_0 = \mathcal{H}/\mathbb{R}$ and allows to write

$$\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$$

as a product of Riemannian manifolds.

Remark 6.22 *The Aubin–Mabuchi functional I may be used to formulate geometrically a problem posed by Calabi in [Cal56] and solved by Yau in [Yau78]:* find a Kähler metric ω_ϕ of prescribed volume form Ω in a given cohomology class $[\omega_0]$. *A volume form Ω gives a positive linear functional*

$\lambda : C^\infty(M) \to \mathbb{R}$. *Finding a solution to the problem is the same as finding a function* ϕ *in* $\mathcal{H}_0$ *such that* $dI_\phi = \lambda$. *There is a necessary condition:* $\lambda(1) = Vol_{\omega_0}(M)$. *The problem reduces to the minimization of the linear functional* λ *on the convex set*

$$\{\phi \in \mathcal{H};\ I(\phi) \leq 0\}.$$

6.3.4 Mabuchi K-Energy

There is another interesting problem, also going back to Calabi [Cal82]: find an "extremal Kähler metric" in $\mathcal{H}$, that is a metric of *constant scalar curvature*. This problem can be formulated, like the one for the prescribed volume form, as a variational problem on $\mathcal{H}$. In [Mab87], Mabuchi introduced a new functional, called the "K-energy" on $\mathcal{H}$ by specifying, as was done for I, its first variation. Let's define

$$\beta_\phi(\psi) = \int_M \psi S(\phi)\, d\mu_\phi$$

where $S(\phi)$ is the scalar curvature of ω_ϕ.

Lemma 6.23 *The 1-form* β *is closed. Therefore, there exists a function* K*, called the* K-energy*, defined on the open convex set* $\mathcal{H}$ *and such that* $\beta = dK$.

Proof. Let ψ_1, ψ_2 be two vectors in $C^\infty(M)$. We then have (see Lemmas 6.31 and 6.32 in Appendix B for the variations of the volume element and the scalar curvature and see Remark 6.20 for the definition of the notation $\psi_1 \cdot \beta(\psi_2)$)

$$\psi_1 \cdot \beta(\psi_2) = \int_M \psi_2 \left(\dot{S}(\phi) d\mu_\phi + S(\phi)(d\dot{\mu_\phi})\right)$$

$$= \int_M \psi_2 \left(\frac{1}{2}\triangle^2\psi_1 - < \operatorname{Hess}\psi_1, \operatorname{Ric} > + \frac{1}{2}S(\phi)\triangle\psi_1\right) d\mu_\phi.$$

Now, introducing the vector field

$$R(X) := \operatorname{Ric}(X, \cdot)^\sharp,$$

we get

$$\operatorname{div} R(\operatorname{grad}\psi_1) = -\delta \operatorname{Ric}(\operatorname{grad}\psi_1) + < \operatorname{Hess}\psi_1, \operatorname{Ric} >$$

where $\delta h = -\operatorname{tr}_{12}\nabla h$. But, from the *second Bianchi identity*, we have

$$\delta \operatorname{Ric} + \frac{1}{2}dS = 0,$$

and hence

$$\psi_2 < \operatorname{Hess}\psi_1, \operatorname{Ric} > = \operatorname{div}(\psi_2 R(\operatorname{grad}\psi_1)) - \operatorname{Ric}(\operatorname{grad}\psi_1, \operatorname{grad}\psi_2) \\ - \frac{1}{2}\psi_2 < \operatorname{grad} S, \operatorname{grad}\psi_1 >$$

Therefore, after some integrations by parts, we get

$$\psi_1 \cdot \beta(\psi_2) = \int_M \Big\{ \frac{1}{2} \triangle\psi_1 \triangle\psi_2 + \operatorname{Ric}(\operatorname{grad}\psi_1, \operatorname{grad}\psi_2) \\ - \frac{1}{2} S(\phi) < \operatorname{grad}\psi_1, \operatorname{grad}\psi_2 > \Big\} d\mu_\phi.$$

Since this expression is symmetric in ψ_1, ψ_2, we have

$$d\beta(\psi_1, \psi_2) = \psi_1 \cdot \beta(\psi_2) - \psi_2 \cdot \beta(\psi_1) = 0.$$

□

The K-energy functional is related to the problem of finding a metric of *constant scalar curvature* in a given cohomology class. Indeed, if ϕ is a critical points of K on $\mathcal{H}_0$ (identified with the submanifold $I = 0$), then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int_M \psi(S(\phi) - \lambda)\, d\mu_\phi = 0,$$

for all $\psi \in C^\infty(M)$. In other words, ω_ϕ has constant scalar curvature.

Let $\bar\partial_{TM}$ be the $\bar\partial$-operator on the holomorphic bundle TM identified with $T^{1,0}M$. Since the *Chern connection* and the *Levi–Civita connection* of any Kähler metric on a complex manifold M coincide (see [Mor07] for more details) we have

$$\bar\partial_{TM} Y = \nabla^{0,1} Y = \frac{1}{2}\left(\nabla Y + J(\nabla Y)J\right),$$

for every section Y of TM. To each Kähler potential ϕ, we introduce a linear operator D_ϕ on $C^\infty(M)$ with values in the space of *symmetric endomorphisms* of TM and defined by

$$D_\phi \psi = \bar\partial_{TM} \operatorname{grad}\psi.$$

Remark 6.24 *The Hessian* $\operatorname{Hess}\psi = \nabla d\psi$ *can be decomposed into the sum of a* J*-symmetric and a* J*-alternate bilinear form. More precisely, we have*

$$\operatorname{Hess}\psi(X, Y) =< \nabla^{1,0}_X \operatorname{grad}\psi, Y > + < \nabla^{0,1}_X \operatorname{grad}\psi, Y > .$$

The $(1,1)$*-form corresponding to the* J*-symmetric part,* $< \nabla_X^{1,0} \operatorname{grad}\psi, Y >$*, is precisely* $i\partial\bar{\partial}\psi$ *whereas*

$$< \nabla_X^{0,1} \operatorname{grad}\psi, Y > = < D_\phi \psi(X), Y >$$

is the component of the complex Hessian*, which for flat space is given in local coordinates by*

$$\frac{\partial^2 \psi}{\partial z^\alpha \partial z^\beta}.$$

Proposition 6.25 (Mabuchi) *The* K*-energy functional is convex along geodesics in* $\mathcal{H}$*. More precisely, we have*

$$\frac{d^2}{dt^2} K(\phi_t) = \frac{d}{dt} \int_M \dot{\phi} S(\phi_t)\, d\mu_{\phi_t} = \int_M \left\| D_{\phi_t} \dot{\phi} \right\|^2 \, d\mu_{\phi_t},$$

if ϕ_t *is a geodesic in* $\mathcal{H}$*.*

Proof. Using the following formula (which is also true on any manifold equipped with a symmetric linear connection ∇)

$$\operatorname{tr}(\nabla X_1 \circ \nabla X_2) = \operatorname{div}(\nabla_{X_2} X_1) - X_2 \cdot \operatorname{div}(X_1) + \operatorname{Ric}(X_2, X_1),$$

we get that

$$\int_M \operatorname{tr}\left[(D_\phi \psi_1)(D_\phi \psi_2)\right] d\mu_\phi = \int_M \psi_2 L_\phi(\psi_1)\, d\mu_\phi$$

where

$$L_\phi(\psi) = \frac{1}{2} \triangle^2 \psi - \frac{1}{2} < \operatorname{grad}\psi, \operatorname{grad} S(\phi) > - < \operatorname{Ric}(\phi), \operatorname{Hess}_\phi \psi > .$$

Moreover, we have

$$\frac{d}{dt} \int_M \dot{\phi} S(\phi_t)\, d\mu_{\phi_t} = \int_M \left(\ddot{\phi} S + \dot{\phi} \dot{S} + \frac{1}{2} \dot{\phi} S \triangle \dot{\phi} \right) d\mu_{\phi_t}$$

with

$$\ddot{\phi} = \frac{1}{2} \left\| \operatorname{grad} \dot{\phi} \right\|^2$$

and (see Lemma 6.32 in Appendix)

$$\dot{S} = L_\phi(\dot{\phi}) + \frac{1}{2} < \operatorname{grad}\dot{\phi}, \operatorname{grad} S(\phi) > .$$

Therefore, after some divergence transformations, we get

$$\frac{d}{dt}\int_M \dot{\phi} S(\phi_t)\, d\mu_{\phi_t} = \int_M \left(\dot{\phi} L_\phi(\dot{\phi})\right)\, d\mu_{\phi_t} = \int_M \left\| D_{\phi_t}\dot{\phi}\right\|^2\, d\mu_{\phi_t}.$$

□

Corollary 6.26 *Let ϕ_0, ϕ_1 be two critical points of K in $\mathcal{H}_0$ which can be joined by a geodesic. Then, there exists a holomorphic diffeomorphism f of M such that $f^*(\omega_1) = \omega_0$*

Proof. Let ϕ_t be a path in $\mathcal{H}$ and

$$X_t = -\frac{1}{2}\operatorname{grad}_{\omega(t)}\dot{\phi}_t$$

where $\omega(t) = \omega_0 + i\partial\bar{\partial}\phi_t$. Using Cartan formula we obtain

$$L_{X_t}\omega(t) = di_{X_t}\omega(t) = \frac{1}{2}dJ^t d\dot{\phi} = -i\partial\bar{\partial}\dot{\phi}.$$

Therefore, denoting by f_t the flow of X_t, we get

$$\frac{d}{dt}f_t^*\Big(\omega(t)\Big) = f_t^*\Big(L_{X_t}\omega(t) + \frac{d}{dt}\omega(t)\Big) = 0.$$

Now, let ϕ_t be a geodesic connecting ϕ_0 and ϕ_1. From the convexity of K along geodesics, we get that dK must vanishes on ϕ_t and thus that $D_{\phi_t}\dot{\phi} = 0$. In other words, the vector field $\operatorname{grad}\dot{\phi}$ is holomorphic for each t. Let f_t be the corresponding holomorphic flow. We have

$$f_t^*(\omega_t) = \omega_0$$

which ends the proof. □

6.3.5 $\mathcal{H}$ as a Symmetric Space

The curvature tensor of the Mabuchi metric on $\mathcal{H}$ (a weak Riemannian metric on a Fréchet manifold) is defined as follows. Consider a 2-parameters family $\phi(s,t) \in \mathcal{H}$ and a vector field $\psi(s,t) \in C^\infty(M)$ defined along ϕ. The curvature is given by

$$R_\phi(\phi_s, \phi_t)\psi = (D_s D_t - D_t D_s)\,\psi,$$

where ϕ_s, ϕ_t denote s and t derivatives of ϕ respectively and

$$D_t\psi = \psi_t + \Gamma_\phi(\phi_t, \psi) = \psi_t - \frac{1}{2} < \operatorname{grad}\psi, \operatorname{grad}\phi_t >_\phi$$

is the covariant derivative of ψ.

Proposition 6.27 *The curvature tensor on $\mathcal{H}$ can be expressed as*

$$R_\phi(\phi_s, \phi_t)\psi = -\frac{1}{4}\left\{\{\phi_s, \phi_t\}, \psi\right\}$$

where $\{\psi_1, \psi_2\}$ is the Poisson bracket *associated with the symplectic structure ω_ϕ. Furthermore, the covariant derivative $D_r R(\phi_s, \phi_t)$ of the curvature tensor R vanishes.*

Proof. We have

$$D_s D_t \psi = \psi_{st} + \frac{d}{ds}\big(\Gamma_\phi(\phi_t, \psi)\big) + \Gamma_\phi(\phi_s, \psi_t) + \Gamma_\phi(\phi_s, \Gamma_\phi(\phi_t, \psi)).$$

The calculus gives

$$\frac{d}{ds}\left(\Gamma_\phi(\phi_t, \psi)\right) = \Gamma_\phi(\phi_{st}, \psi) + \Gamma_\phi(\phi_t, \psi_s) + \frac{1}{2} i\partial\bar{\partial}\phi_s(\operatorname{grad}\phi_t, X_\psi)$$

where $X_\psi = J \operatorname{grad}\psi$ is the Hamiltonian of ψ relatively to the symplectic structure ω_ϕ. Moreover

$$i\partial\bar{\partial}\phi_s(\operatorname{grad}\phi_t, X_\psi) = \frac{1}{2}\left\{\operatorname{Hess}\phi_s(\operatorname{grad}\phi_t, \operatorname{grad}\psi) + \operatorname{Hess}\phi_s(X_{\phi_t}, X_\psi)\right\}$$

and hence

$$\begin{aligned}\frac{d}{ds}\left(\Gamma_\phi(\phi_t, \psi)\right) = \Gamma_\phi(\phi_{st}, \psi) &+ \Gamma_\phi(\phi_t, \psi_s) + \Gamma_\phi(\phi_t, \Gamma_\phi(\phi_s, \psi)) \\ &- \frac{1}{4}\operatorname{Hess}(\psi)(\operatorname{grad}\phi_s, \operatorname{grad}\phi_t) + \frac{1}{4}\omega_\phi(\nabla_{X_{\phi_t}} X_{\phi_s}, X_\psi),\end{aligned}$$

We get therefore

$$\begin{aligned} D_s D_t \psi - D_t D_s \psi &= \frac{1}{4}\left(\omega_\phi(\nabla_{X_{\phi_t}} X_{\phi_s}, X_\psi) - \omega_\phi(\nabla_{X_{\phi_s}} X_{\phi_t}, X_\psi)\right) \\ &= \frac{1}{4}\omega_\phi([X_{\phi_t}, X_{\phi_s}], X_\psi) \\ &= \frac{1}{4}\omega_\phi(X_{\{\phi_t, \phi_s\}}, X_\psi) \\ &= \frac{1}{4}\left\{\{\phi_t, \phi_s\}, \psi\right\} \\ &= -\frac{1}{4}\left\{\{\phi_s, \phi_t\}, \psi\right\}. \end{aligned}$$

To establish that the curvature tensor is parallel, we start to check that

$$D_t \{\psi_1, \psi_2\} = \{D_t\psi_1, \psi_2\} + \{\psi_1, D_t\psi_2\}. \tag{6.4}$$

Then, taking a 3-parameters family $\phi(r, s, t)$ in $\mathcal{H}$, we get

$$\begin{aligned}&(D_r R)(\phi_s, \phi_t)\psi \\ &\quad = D_r (R(\phi_s, \phi_t)\psi) - R(D_r\phi_s, \phi_t)\psi - R(\phi_s, D_r\phi_t)\psi - R(\phi_s\phi_t)D_r\psi = 0\end{aligned}$$

by virtue of relation (6.4) and the expression of the curvature tensor.

In other words, $\mathcal{H}$ is a *locally symmetric space*: $\nabla R = 0$. In finite dimension, such a manifold is characterized by the fact that in the neighbourhood of each point x, the (local) *geodesic symmetry* of center $x : \exp_x(X) \mapsto \exp_x(-X)$ is an isometry (see [Hel01]). If moreover, each of these local isometries can be extended globally, we say that the manifold is a *symmetric space*. In that case, the isometry group acts transitively. A symmetric space is also a homogeneous space G/H.

The best example of a symmetric space is a compact Lie group G endowed with a bi-invariant Riemannian metric. In that case, the curvature tensor is given by

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z].$$

and the sectional curvatures by

$$\sigma(X,Y) = \frac{1}{4}\left\|[X,Y]\right\|^2,$$

where X, Y are orthonormal vectors in $\mathfrak{g}$. To such a symmetric space corresponds a (non compact) *dual symmetric space*. This dual is defined as the homogeneous space

$$H = G^{\mathbb{C}}/G$$

where $G^{\mathbb{C}}$ is the *complexification* of G. The curvature tensor of H is given by

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z].$$

and the sectional curvatures by

$$\sigma(X,Y) = -\frac{1}{4}\left\|[X,Y]\right\|^2,$$

where X, Y are orthonormal vectors in $\mathfrak{g}$.

Example 6.28 *When $G = \mathrm{SO}(3)$ (with constant sectional curvature +1), then $G^{\mathbb{C}} = \mathrm{PSL}(2,\mathbb{C})$ and H is the hyperbolic space $\mathbb{H}^3$ with constant sectional curvature -1.*

In [Don99], Donaldson has applied this construction, at least formally, for the space $\mathcal{H}$ of Kähler potentials. In that case, the role of the "compact symmetric group" is played by the symplectomorphisms group $\mathrm{Symp}(M)$ and the *dual symmetric space* is played by $\mathcal{H}$. However, in that case, the "complexification" of the structure group $\mathrm{Symp}(M)$ is not played by a group but by a *parallelizable principal fiber bundle*, with base $\mathcal{H}$, which parameterizes all complex structures on M.

A Appendix

A.1 Linear Algebra on a Symplectic Vector Space

In this section, (E, ω) is a symplectic vector space of dimension $2n$, i.e. a vector space E equipped with a *non-degenerate* skew-symmetric bilinear form ω.

Lemma 6.29 *Let α, β be two linear functionals on E, then*

$$\alpha \wedge \beta \wedge \omega^{n-1} = \frac{1}{n}\omega(a,b)\omega^n,$$

where a, b are the vectors defined by $i_a\,\omega = \alpha$ and $i_b\,\omega = \beta$ respectively.

Proof. We have

$$\begin{aligned} 0 &= i_b\,(\alpha \wedge \omega^n) \\ &= \alpha(b)\omega^n - n\alpha \wedge (i_b\,\omega) \wedge \omega^{n-1} \\ &= \omega(a,b)\omega^n - n\alpha \wedge \beta \wedge \omega^{n-1}. \end{aligned}$$

□

Let Ω be a skew-symmetric, bilinear form, defined on E. We define the trace of Ω relatively to ω as

$$\mathrm{tr}_\omega\,\Omega = \omega^{ik}\Omega_{ki}$$

where (ω^{ij}) is the inverse of the matrix (ω_{ij}).

Lemma 6.30 *Let Ω be a skew-symmetric bilinear form on E, then*

$$\Omega \wedge \omega^{n-1} = \frac{1}{2n}(\mathrm{tr}_\omega\,\Omega)\omega^n.$$

Proof. It suffices to show the lemma for $\Omega = \alpha \wedge \beta$. But then, Lemma 6.29 gives us

$$\alpha \wedge \beta \wedge \omega^{n-1} = \frac{1}{n}\omega(a,b)\omega^n$$

and a straightforward computation shows that

$$\operatorname{tr}_\omega(\alpha \wedge \beta) = 2\omega(a, b)$$

which ends the proof.

A.2 Classical Formulas in Kähler Geometry

Let (z^α) be a system of complex coordinates on a Kähler manifold M of complex dimension n. The metric g is represented in this chart by

$$ds^2 = 2g_{\alpha\bar{\beta}}\, dz^\alpha d\bar{z}^\beta$$

and the Kähler form ω is represented by

$$\omega = i g_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta.$$

The volume element $d\mu$ is defined by

$$d\mu = \frac{1}{n!}\omega^n.$$

The complex Laplacian of a function ψ is defined as

$$\Box\psi = g^{\alpha\bar{\beta}}\, \frac{\partial^2\psi}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2}\, \operatorname{tr}_\omega(i\partial\bar{\partial}\psi) = \frac{1}{2}\, \triangle\psi,$$

where $\triangle\psi$ is the Riemannian Laplacian, defined as the trace of the Hessian $\nabla d\psi$.

Lemma 6.31 *If the Kähler metric is submitted to an infinitesimal variation $\delta\phi$ the variation of the volume is*

$$\delta(d\mu) = \frac{1}{2}(\triangle\delta\phi)\, d\mu$$

Proof. Let $\omega_s = \omega + is\partial\bar{\partial}\delta\phi$. We have

$$\left.\frac{d}{ds}\right|_{s=0} (\omega_s^n) = n(i\partial\bar{\partial}\delta\phi) \wedge \omega^{n-1} = \frac{1}{2}\, \operatorname{tr}_\omega(i\partial\bar{\partial}\delta\phi)\, \omega^n,$$

by virtue of Lemma 6.30 and the last expression is precisely

$$\frac{1}{2}(\triangle\delta\phi)\omega^n.$$

□

On a Kähler manifold, the *Ricci tensor*

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\,[\xi \mapsto R(X,\xi)Y]$$

is J-invariant. The corresponding $(1,1)$ form, ρ, called the *Ricci form* can be expressed (see [Mor07] for instance) in local coordinates by

$$\rho = i\partial\bar{\partial}\log d,$$

where $d = \det(g_{\alpha\bar{\beta}})$ is the determinant of the $n\times n$ complex matrix $(g_{\alpha\bar{\beta}})$. The scalar curvature S is defined as the trace of the Ricci tensor, or equivalently

$$S = \mathrm{tr}_\omega\,\rho.$$

Lemma 6.32 *If the Kähler metric is submitted to an infinitesimal variation $\delta\phi$ the variation of the scalar curvature is*

$$\delta S = \frac{1}{2}\triangle^2\delta\phi - < \mathrm{Hess}\,\delta\phi, \mathrm{Ric} > .$$

Proof. Let $\delta\omega = i\partial\bar{\partial}\delta\phi$. We have

$$\delta\rho = i\partial\bar{\partial}\left(\frac{1}{2}\triangle\delta\phi\right)$$

and

$$\delta S = - < \delta\omega, \rho > + \mathrm{tr}_\omega(\delta\rho) = - < i\partial\bar{\partial}\delta\phi, \rho > + \frac{1}{2}\triangle^2\delta\phi.$$

Now

$$< i\partial\bar{\partial}\delta\phi, \rho > = < \mathrm{Hess}\,\delta\phi, \mathrm{Ric} >,$$

which ends the proof. □

Chapter 7
Monge–Ampère Equations on Complex Manifolds with Boundary

Sébastien Boucksom

Abstract We survey the proofs of two fundamental results on the resolution of Monge–Ampère equations on complex manifolds with boundary. The first result guarantees the existence of smooth solutions to non-degenerate complex Monge–Ampère equations admitting subsolutions, it is a continuation of results due to Caffarelli–Kohn–Nirenberg–Spruck, Guan and Błocki. The second result shows the existence of almost C^2 solutions to degenerate complex Monge–Ampère equations admitting subsolutions, and yields as a special case X.X.Chen's result on the existence of almost C^2 geodesics in the space of Kähler metrics.

7.1 Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension $\dim X =: n$. Let ω_0, ω_1 be two Kähler forms in the cohomology class of ω, which can be written as $\omega_j = \omega + dd^c u_j$ for some $u_j \in C^\infty(X)$, $j = 0, 1$ by the dd^c-lemma. If we let $A \subset \mathbb{C}$ be a closed annulus then the functions u_0, u_1 induce a radially symmetric smooth function u on the boundary of $M := X \times A$ such that $\omega + dd^c u > 0$ on each X-slice. As we saw in Kolev's lectures (cf. [Kol, Proposition 6]) it has been observed by Semmes and Donaldson that there exists a (smooth) geodesic

$$\omega_t = \omega + dd^c u_t,\ 0 \leq t \leq 1$$

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joining ω_0 to ω_1 in the space of Kähler metrics cohomologous to ω iff the equations

$$(\omega + dd^c v)^{n+1} = 0,\ v|_{\partial M} = u, \tag{7.1}$$

i.e. the Dirichlet problem for a degenerate complex Monge–Ampère equation on the complex manifold with boundary M, admits a solution $v \in C^\infty(M)$ such that $\omega + dd^c v$ is furthermore *positive on X-slices* in the sense that

$$(\omega + dd^c v)|_{X\times z} > 0 \text{ for each } z \in A. \tag{7.2}$$

Here ω is identified with its pull back to M, a semipositive $(1,1)$-form on which is however *not* a Kähler form (since it vanishes in the A-directions).

The existence of $v \in C^\infty(M)$ satisfying both (7.1) and (7.2) is a very difficult analytic problem. Indeed Donaldson showed in [Don02, Theorem 2] that even for the simpler case where $M = X \times D$ with $D \subset \mathbb{C}$ a closed disc there always exist boundary data $u \in C^\infty(\partial M)$ with $\omega + dd^c u > 0$ on X-slices such that no smooth solution v to both (7.1) and (7.2) exists. Note however that the boundary data $u \in C^\infty(\partial M)$ is a priori not radially symmetric in Donaldson's construction.

It is easily seen that (7.1) and (7.2) imply that $\omega + dd^c v \geq 0$ on M, i.e. v is ω-plurisubharmonic (ω-psh for short) on the product space $M = X \times A$. The maximum principle for the Monge–Ampère operator shows that there exists at most one continuous ω-psh function v on M satisfying (7.1) (in the weak sense of Bedford–Taylor, cf. Corollary 7.7). The problem at hand therefore splits in two parts:

1. Show that (7.1) admits a smooth ω-psh solution $v \in C^\infty(M)$.
2. Show that this solution must satisfy $\omega + dd^c v > 0$ on X-slices.

More generally let $M = X \times S$ with S a compact Riemann surface with boundary. Then M is a complex manifold with boundary such that ∂M is Levi flat, and M has the property that any smooth function φ on ∂M such that $\omega + dd^c\varphi > 0$ on X-slices admits a smooth extension $\widetilde{\varphi}$ to M such that $\eta := \omega + dd^c\widetilde{\varphi} > 0$ on M (cf. Proposition 7.10 below). Since an ω-psh function v satisfies (7.1) iff $\psi := v - \widetilde{\varphi}$ satisfies

$$(\eta + dd^c\psi)^m = 0 \text{ and } \psi|_{\partial M} = 0 \tag{7.3}$$

we can trade the semipositive form ω for an actual Kähler form η.

We are going to present essentially self-contained proofs of the following general results, which contain as special cases [CKNS85] and [Che00]:

Theorem A. *Let (M,η) be an m-dimensional compact Kähler manifold with boundary. Given $\varphi \in C^\infty(\partial M)$ and a smooth positive volume form μ there exists a smooth η-psh solution ψ to the Dirichlet problem. In that case, the solution ψ is furthermore unique.*

$$\begin{cases} (\eta + dd^c\psi)^m = \mu \\ \psi|_{\partial M} = \varphi \end{cases}$$

iff there exists a subsolution, i.e. a smooth η-psh function $\widetilde{\varphi}$ such that

$$\begin{cases} (\eta + dd^c\widetilde{\varphi})^m \geq \mu \\ \widetilde{\varphi}|_{\partial M} = \varphi \end{cases}$$

Theorem B. *Let (M, η) be an m-dimensional compact Kähler manifold with boundary. Let $\varphi \in C^\infty(\partial M)$ and assume that φ admits a smooth η-psh extension $\widetilde{\varphi} \in C^\infty(M)$. Then we have:*

(i) There exists a unique Lipschitz continuous η-psh function ψ such that

$$\begin{cases} (\eta + dd^c\psi)^m = 0 \\ \psi|_{\partial M} = \varphi \end{cases}$$

(ii) If we assume furthermore that ∂M is weakly pseudoconcave *then $dd^c\psi$ has L^∞_{loc} coefficients.*

See Definition 7.4 for the definition of weakly pseudoconcave in this setting. Note that $dd^c\psi \in L^\infty_{\mathrm{loc}}$ is equivalent to ψ having bounded Laplacian on M (since ψ is quasi-psh). It implies that $\psi \in C^{1,\alpha}(M)$ for each $\alpha < 1$ by usual elliptic regularity but it is however a priori weaker than $\psi \in C^{1,1}(M)$, which means that the full *real* Hessian of ψ is bounded (cf. Sect. 7.2.3).

History of the results. We claim no originality in the proofs of Theorems A and B, which as we shall see are a combination of techniques and ideas from [Yau78, CKNS85, Gua98, Che00, Bl09a, Bl09b, PS09]. But since the history of these two results happens to be somewhat complicated we find it worthwhile to discuss it here in some detail.

Let us first consider the case where $M \subset \mathbb{C}^m$ is a smooth bounded domain (and $\eta = dd^c|z|^2$ for instance).

If ∂M is furthermore assumed to be strictly pseudoconvex then a subsolution always exists (cf. Proposition 7.10) and Theorem A was proved in that case in [CKNS85, Theorem 1.1]. A special case of [BT76] Theorem D proves (i) of Theorem B. The solution ψ was furthermore shown in [BT76, Theorem D] to be locally $C^{1,1}$ in the interior when M is a ball (cf. [GZ09] in this volume), and [Kry89] shows that $\psi \in C^{1,1}$ up to the boundary in the general case. We refer to Delarue's lecture in this volume [Del09] for a presentation of Krylov's results, which rely on completely different probabilistic tools in the setting of optimal control. It is interesting to note that $\psi \in C^{1,1}(M)$ up to boundary is not even known for a ball in $\mathbb{C}^2$ using barrier arguments.

For a general smooth bounded domain Theorem A was obtained in [Gua98] by improving the barrier arguments of [CKNS85], and Theorem B might follow as well from [Kry89].

Let us now consider the case where (M, η) is an arbitrary compact Kähler manifold with boundary. The first general results were obtained in [Che00] by combining techniques from [Yau78] (which settled the analogue of Theorem A when M has no boundary) and [Gua98] with a blow-up argument.

In fact [Che00] proved (ii) of Theorem B when $M = X \times S$ is a product of a compact Kähler manifold X with a compact Riemann surface with boundary S, but the product structure only turns out to matter near the boundary, so that [Che00] basically contains the proof of Theorem B when ∂M is Levi flat as was observed in [PS07, Lemma 1]. However there appears to be a small difficulty in X.X. Chen's argument which will be discussed in Sect. 7.3.2. This difficulty was subsequently settled in [Bl09b], which also provided a proof of Theorem A in the general case. We add here the minor observation that the proof goes through in the weakly pseudoconcave case as well.

Finally part (i) of Theorem B is a direct consequence of Theorem A combined with Błocki's gradient estimate [Bl09a]. It is a special case of [PS09, Theorem 2].

Nota Bene. What follows is an expanded set of notes written by Sébastien Boucksom, after the lecture he delivered in Marseille, March 2009. A first draft of these notes had originally been written by Benoît Claudon and Philippe Eyssidieux.

7.2 Preliminaries

7.2.1 Complex Manifolds with Boundary

We recall the following definitions.

Definition 7.1 *A* complex manifold with boundary M *is a C^∞-manifold with boundary endowed with a system of coordinate patches*

$$\Psi_j : U_j \simeq \{z \in B,\, r_j(z) \le 0\}$$

where B denotes the open unit ball in $\mathbb{C}^m$, r_j is a local defining function*, i.e. a smooth function on $\overline{B}$ with $dr_j \neq 0$ along $\{r_j = 0\}$ and $\Psi_j \circ \Psi_i^{-1}$ is holomorphic on $\Psi_i(U_i \cap U_j) \cap \{r_i < 0\}$.*

The *holomorphic tangent bundle* of ∂M is the largest complex subbundle of T_M contained in $T_{\partial M}$, i.e.

$$T^h_{\partial M} = T_{\partial M} \cap JT_{\partial M}$$

where $J : T_M \to T_M$ denotes the complex structure of M.

We will use the following elementary calculus lemma.

Lemma 7.2 *Let r be a smooth function defined near 0 in $\mathbb{R}^d$ with coordinates $t_1, ..., t_d$. Assume that $r_{t_d}(0) = -1$ and $r_{t_i}(0) = 0$ for all $i < d$. Then the restrictions of $t_1, ..., t_{d-1}$ to $N := \{r = 0\}$ yield local coordinates on N and for each smooth function v near 0 in $\mathbb{R}^d$ we have*

$$(v|_N)_{t_i|_N}(0) = v_{t_i}(0) + v_{t_p}(0) r_{t_i}(0)$$

and

$$(v|N)_{t_i|_N t_j|_N}(0) = v_{t_i t_j}(0) + v_{t_p}(0) r_{t_i t_j}(0)$$

for all $i, j < d$

Lemma 7.3 *Let M be a complex manifold with boundary and let r be a local defining function of ∂M. If v is a smooth function on M such that $v|_{\partial M} \equiv 0$ and ν is a local vector field that is normal to ∂M then we have*

$$dd^c v|_{T^h_{\partial M}} = \frac{\nu \cdot v}{\nu \cdot r} dd^c r|_{T^h_{\partial M}}.$$

Proof. We can choose the local coordinates $z_1, ..., z_m$ at a given point $0 \in \partial M$ such that

$$T^h_{\partial M,0} = \mathrm{Vect}\left(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_{m-1}}\right)$$

and $\nu = \partial/\partial x_m$ at 0. Then the result directly follows from Lemma 7.2. □

Given an outward pointing normal ν to ∂M (i.e. $\nu \cdot r > 0$ along ∂M) the Hermitian form $L_{\partial M,\nu}$ on $T^h_{\partial M}$ defined by $(\nu \cdot r)^{-1} dd^c r|_{T^h_{\partial M}}$ is thus independent of the choice of r and is called the *Levi form* of ∂M (with respect to ν).

Definition 7.4 *If M is a complex manifold with boundary then ∂M is said to be weakly (resp. strictly) pseudoconcave (resp. pseudoconvex) if the Levi form $L_{\partial M,\nu}$ of ∂M (with respect to an outward pointing normal) satisfies $L_{\partial M,\nu} \leq 0$ (resp. < 0, ≥ 0, > 0).*

7.2.2 Maximum Principles

We first state a simple version of the maximum principle for complex Monge–Ampère equations.

Proposition 7.5 *Let (M, η) be a compact Kähler manifold with boundary and let $\psi_1, \psi_2 \in C^\infty(M)$ be two strictly η-psh functions such that*

(i) $\psi_1 \leq \psi_2$ *on* ∂M.
(ii) $(\eta + dd^c\psi_1)^m \geq (\eta + dd^c\psi_2)^m$.

Then we have $\psi_1 \leq \psi_2$ *on* M.

Proof. The following argument is basically due to Calabi [Cal55] (compare also [CKNS85] Lemma 1.1, [Bl09b] Proposition 2.1). We write

$$0 \leq (\eta + dd^c\psi_1)^m - (\eta + dd^c\psi_2)^m = dd^c(\psi_1 - \psi_2) \wedge T \tag{7.4}$$

with

$$T := \sum_{j=0}^{m-1} (\eta + dd^c\psi_1)^j \wedge (\eta + dd^c\psi_2)^{m-1-j},$$

which is a (strictly) positive form of bidegree $(1,1)$. We thus see that $u := \psi_2 - \psi_1$ satisfies $Lu \geq 0$ where $Lu := dd^c u \wedge T$ is a second order elliptic operator, and the result follows from the usual (linear) maximum principle.

In order to get uniqueness in Theorem B we prove the following version of the comparison principle.

Proposition 7.6 *Let* $\varphi, \psi \in C^0(M)$ *be two* η*-psh functions such that* $\varphi \leq \psi$ *on* ∂M. *Then we have*

$$\int_{\{\psi<\varphi\}} (\eta + dd^c\varphi)^m \leq \int_{\{\psi<\varphi\}} (\eta + dd^c\psi)^m.$$

Here the Monge–Ampère measures are defined in the sense of Bedford–Taylor.

Proof. Let $\delta > 0$ and set $\Omega := \{\psi < \varphi - \delta\}$. For each $\varepsilon > 0$ set

$$\varphi_\varepsilon := \max(\varphi - \delta, \psi + \varepsilon).$$

We then have $\varphi_\varepsilon = \psi + \varepsilon$ in a neighbourhood of $\partial\Omega$, thus

$$\int_\Omega (\eta + dd^c\varphi_\varepsilon)^m = \int_\Omega (\eta + dd^c\psi)^m.$$

Indeed we have $\partial\Omega \cap \partial M = \emptyset$ since $\varphi \leq \psi$ on ∂M so the result follows from Stokes' theorem since

$$(\eta + dd^c\varphi_\varepsilon)^m - (\eta + dd^c\psi)^m$$

is exact by (7.4).

On the other hand φ_ε decreases to $\varphi - \delta$ as $\varepsilon \to 0$ thus Bedford–Taylor's monotone continuity theorem for the Monge–Ampère operator implies that

$$(\eta + dd^c\varphi_\varepsilon)^m \to (\eta + dd^c\varphi)^m$$

in the weak topology and we get

$$\int_\Omega (\eta + dd^c\varphi)^m \le \liminf_{\varepsilon\to 0} \int_\Omega (\eta + dd^c\varphi_\varepsilon)^m.$$

We have thus proved

$$\int_{\{\psi<\varphi-\delta\}} (\eta + dd^c\varphi)^m \le \int_{\{\psi<\varphi-\delta\}} (\eta + dd^c\psi)^m$$

for each $\delta > 0$ and the result follows by monotone convergence. □

Corollary 7.7 *Let $\psi \in C^0(M)$ be an η-psh function such that $(\eta+dd^c\psi)^m = 0$. Then for every continuous η-psh function φ on M we have*

$$\sup_M(\varphi - \psi) = \sup_{\partial M}(\varphi - \psi).$$

Proof. Upon adding a constant we may assume that $\sup_{\partial M}(\varphi - \psi) = 0$ and we have to show that $\varphi \le \psi$. For each $0 < \delta \ll 1$ we have

$$\begin{aligned}\int_{\{\psi<(1-\delta)\varphi\}} (\delta\eta)^m &\le \int_{\{\psi<(1-\delta)\varphi\}} (\eta + (1-\delta)dd^c\varphi)^m \\ &\le \int_{\{\psi<(1-\delta)\varphi\}} (\eta + dd^c\psi)^m = 0\end{aligned}$$

by the comparison principle, thus $\psi \ge (1-\delta)\varphi$ holds for each $\delta > 0$ and the result follows.

7.2.3 Elliptic Regularity

In this section we quickly recall some facts about second order linear elliptic PDE's. Let Δ be the Laplacian (with respect to a Riemannian metric g) locally near $0 \in \mathbb{R}^d$ and let u be a distribution near 0. By [GT83] we have

$$\Delta u \in L^p \Longrightarrow u \in L^p_2 \tag{7.5}$$

for each $1 \le p < +\infty$. Here L^p_k denotes the Sobolev space of functions whose derivatives of order at most k belongs to L^p (locally). We thus have

$$L^\infty_2 = C^{1,1}.$$

Note however that (7.5) fails in general when $p = \infty$. Indeed the following example can be found in [GT83]:

$$u(x, y) := |x||y| \log(|x| + |y|)$$

has bounded Laplacian near $0 \in \mathbb{R}^2$ but $\frac{\partial^2 u}{\partial x \partial y}$ is not locally bounded near 0. On the other hand we have

Lemma 7.8 *Suppose that $\Delta u \in L^\infty$ locally. Then we have $u \in C^{1,\alpha}$ for each $\alpha < 1$.*

Proof. By (7.5) we have $u \in L^p_2$ for each finite $p \geq 1$. But

$$L^p_2 \subset C^{1,\alpha}$$

for each $\alpha < 1 - d/p$ by Sobolev's embedding theorem and the result follows.

7.2.4 Miscellanea

Recall that the *trace* $\mathrm{tr}_\beta(\alpha)$ of a $(1,1)$-form α with respect to a positive $(1,1)$-form β is defined as the sum of the eigenvalues of α with respect to β. It satisfies

$$\mathrm{tr}_\beta(\alpha) = m \frac{\alpha \wedge \beta^{m-1}}{\beta^m}$$

We will use the following reformulation of the arithmetico-geometric inequality.

Proposition 7.9 *Let α, β be two positive $(1,1)$-forms on M. Then we have*

$$\frac{1}{m} \mathrm{tr}_\beta(\alpha) \geq \left(\frac{\alpha^m}{\beta^m}\right)^{1/m}.$$

Finally we state a general extension result.

Proposition 7.10 *Let (X, ω) be a compact Kähler manifold (without boundary) and let Y be a compact complex manifold with boundary such that there exists a smooth strictly psh function χ on Y with $\chi|_{\partial Y} = 0$.*

Then every $\varphi \in C^\infty(X \times \partial Y)$ which is a strictly ω-psh on X-slices admits a strictly ω-psh extension in $\widetilde{\varphi} \in C^\infty(X \times Y)$.

Note that the condition on Y is also necessary (apply the extension property to $\varphi = 0$ and X a point) and implies that ∂Y is strictly pseudoconvex.

Proof. Let U be an open neighbourhood of ∂Y in Y with a smooth retraction $\rho : U \to \partial Y$ and let $0 \leq \vartheta \leq 1$ be a smooth function with compact support in

U such that $\vartheta \equiv 1$ on a neighbourhood of ∂Y. Then $\psi(x, y) := \vartheta(y)\varphi(x, \rho(y))$ is smooth and strictly ω-psh on X-slices. Thus setting $\widetilde{\varphi} := \psi + C\chi$ with $C \gg 1$ yields the desired extension of φ.

7.3 A Priori Estimates

Let (M, η) be a compact Kähler manifold with boundary of complex dimension m and denote by Δ the analyst's Laplacian with respect to η so that

$$\Delta u = \mathrm{tr}_\eta(dd^c u)$$

for every function u.

The following a priori estimate is the key to the proof of Theorem A.

Theorem 7.11 *Let (M, η) be a compact Kähler manifold with boundary of complex dimension m. Then for every $A > 0$ there exists $C, \alpha > 0$ such that the following holds: given a non-positive function $F \in C^\infty(M)$ with*

$$\|F\|_{C^2(M)} \leq A$$

and a smooth η-psh function ψ on M with

$$(\eta + dd^c\psi)^m = e^F \eta^m,\ \psi|_{\partial M} \equiv 0$$

the a priori estimate $\|\psi\|_{C^{2+\alpha}(M)} \leq C$ holds.

7.3.1 A Series of Lemma

Let (M, η) be a compact Kähler manifold with boundary. In this whole section we let ψ be a smooth η-psh function such that

$$(\eta + dd^c\psi)^m = e^F \eta^m \text{ and } \psi|_{\partial M} = 0$$

with $F \in C^\infty(M)$ such that

(i) $-A_0 \leq F \leq 0$
(ii) $\sup_M |\nabla F| \leq A_1$
(iii) $\Delta F \geq -A_2$

for some given $A_0, A_1, A_2 > 0$.

In this section we are going to show in a series of lemma following [CKNS85, Gua98, Che00, Bl09b] that there exists $C > 0$ only depending on A_0, A_1, A_2 (and even only on A_1, A_2 when ∂M is pseudoconcave) such that

$$\sup_M (|\psi| + |\Delta\psi|) \leq C. \tag{7.6}$$

We will explain in the next section how to deduce Theorem 7.11 as a consequence of a general result on fully non-linear second order elliptic PDE's ([CKNS85, Theorem 1]) by adding a result of [Bl09b].

Remark 7.12 *Let us fix some notation and terminology. We fix once and for all a vector field ν on M which is outward pointing, of unit length and orthogonal to $T_{\partial M}$ (with respect to η) at every point of ∂M. We also fix a finite cover of ∂M by coordinate half-balls $B^{(\alpha)}$ with complex coordinates $z^{(\alpha)}$ and defining function r_α for $\partial M \cap B_\alpha$ such that the half-balls B'_α of radius half that of B_α still cover ∂M. We will write as usual*

$$\eta_\psi := \eta + dd^c\psi$$

and denote by tr_ψ and Δ_ψ the trace and Laplacian with respect to η_ψ, so that $\Delta_\psi u = tr_\psi(dd^c u)$.

When we say for instance that a constant depends only on A_0 we mean that it only depends on A_0 together with the background data η, ν and $(B^{(\alpha)}, z^{(\alpha)}, r_\alpha)$.

Remark 7.13 *The following situation occurs several times below. Given point $0 \in \partial M$ we will want to choose an adapted data B, r, z where B is a coordinate half-ball B, r is a defining function for $\partial M \cap B$ and the coordinates z on B are centered at 0 and satisfy*

$$r = -x_m + \Re\left(\sum_{1\leq j,k\leq m} a_{jk} z_j \overline{z}_k\right) + O\left|z|^3\right) \tag{7.7}$$

near 0. We will then estimate the value at 0 of certain partial derivatives of ψ in the z-coordinates in a way that only depends on, say, A_0. In order to ensure the uniformity with respect to the choice of the data B, r, z the latter will implicitely have been constructed as follows. We choose α such that 0 belongs to B'_α, we let B be the translate of B'_α centered at 0 and let w be the translate of $z^{(\alpha)}$. The Taylor series expansion of r in the w-coordinates writes

$$r = \Re\left(\sum_{1\leq j\leq m} c_j w_j + \sum_{1\leq j,k\leq m} b_{jk} w_j w_k + \sum_{1\leq j,k\leq m} a_{jk} w_j \overline{w}_k\right) + O\left(|w|^3\right)$$

and we first perform a linear change of coordinates $w \mapsto w'$ to arrange that $c_m = -1$ and $c_j = 0$ for $j < m$ in the w'-coordinates, and next set

$$z_m := w'_m - \sum_{1 \le j,k \le m} b_{jk} w'_j w'_k$$

in order to kill the b_{jk}*'s. It is now clear that any quantity which is uniform with respect to certain derivatives in the* z*-coordinates of a given background function independent of* ψ *will also be uniform with respect to certain derivatives of the same function in the original coordinates* $z^{(\alpha)}$*.*

Lemma 7.14 *There exists* $C > 0$ *independent of* ψ *such that*

$$\sup_M |\psi| + \sup_{\partial M} |\nabla \psi| \le C.$$

Proof. The inequality $\psi \ge 0$ follows from the maximum principle for complex Monge–Ampère equations (Proposition 7.5) since we have $(\eta + dd^c\psi)^m \le \eta^m$ and $\psi = 0$ on ∂M (recall that we have assumed $F \le 0$).

On the other hand let $h \in C^\infty(M)$ be the unique function on M such that

$$\Delta h = -m \text{ and } h|_{\partial M} = 0.$$

Then

$$\Delta(\psi - h) = \Delta\psi + m = \mathrm{tr}_\eta(\eta + dd^c\psi)$$

is non-negative and $\psi - h = 0$ on ∂M thus $\psi \le h$ as a consequence of the maximum principle, this time for subharmonic functions. Now $0 \le \psi \le h$ and $h = 0$ on ∂M shows that

$$\sup_{\partial M} |\nabla \psi| \le \sup_{\partial M} |\nabla h|$$

and the result follows. □

Lemma 7.15 *There exists a constant* $C > 0$ *only depending on* A_2 *such that*

$$\sup_M |\Delta \psi| \le C(1 + \sup_{\partial M} |\Delta\psi|).$$

This result is a rather direct consequence of Yau's estimates (compare [Che00, Corollary 1]).

Proof. Yau's famous pointwise inequality ([Yau78] p. 350 (2.18) and (2.20)) states that $(\eta + dd^c\psi)^m = e^F \eta^m$ implies

$$\begin{aligned} &e^{B\psi}\Delta_\psi\left(e^{-B\psi}(m + \Delta\psi)\right) \\ &\quad \ge -Bm(m + \Delta\psi) + Be^{-F/m-1}(m + \Delta\psi)^{1+1/m-1} + \Delta F - B \end{aligned}$$

where $B > 0$ is a lower bound for the holomorphic bisectional curvature of η and Δ_ψ denotes the Laplacian with respect to the Kähler form $\eta + dd^c\psi$. Let $x_0 \in M$ be a point where $e^{-B\psi}(m + \Delta\psi)$ achieves its maximum. If x_0 lies on ∂M then we get

$$\sup_M (m + \Delta\psi) \le e^{B(\sup_M \psi - \inf_M \psi)} \sup_{\partial M} (m + \Delta\psi),$$

where the oscillation $\sup_M \psi - \inf_M \psi$ is bounded in terms of η by Lemma 7.14. If x_0 lies in the interior of M then Yau's inequality implies that

$$Bm(m + \Delta\psi) + B(m + \Delta\psi)^{1+1/m-1} - A_2 - B \le 0$$

at x_0, using that $F \le 0$. It follows that $0 < (m + \Delta\psi)(x_0)$ is bounded above in terms of B and A_2 and the result follows using Lemma 7.14 again to bound the oscillation of ψ.

Lemma 7.16 *There exists $\varepsilon > 0$ only depending on A_0 such that*

$$(\eta + dd^c\psi)|_{T^h_{\partial M}} \ge \varepsilon\eta|_{T^h_{\partial M}}.$$

When ∂M is weakly pseudoconcave we can even take $\varepsilon = 1$.

The first assertion was proved in [CKNS85, pp. 221–223] in the strictly pseudoconvex case, and their argument was extended to the general case in [Gua98, pp.694–696]. In the product case of [Che00] the trivial independence of ε on A_0 was implicit and was made explicit in the Levi flat case in [Bl09b, Theorem 3.2']. Here we add the easy observation that the result holds in the weakly pseudoconcave case as well.

Proof. Let $0 \in \partial M$ and choose an adapted data B, r, z as in Remark 7.13.

We have $\psi \ge 0$ by Lemma 7.14 and $\psi|_{\partial M} = 0$ thus $dd^c\psi$ is equal to a non-positive multiple of $dd^c r$ by Lemma 7.3 and we see that

$$\eta_\psi|_{T^h_{\partial M}} \ge \eta|_{T^h_{\partial M}}$$

if $dd^c r|_{T^h_{\partial M}} \le 0$, which settles the second assertion.

We now consider the first assertion and we first claim that it is enough to show the existence of $\varepsilon > 0$ only depending on A_0 such that

$$\eta_\psi|_{T^h_{\partial M}} \ge \varepsilon dd^c r|_{T^h_{\partial M}}. \tag{7.8}$$

Indeed, Lemma 7.3 implies on the one hand that

$$(\eta_\psi - \eta)|_{T^h_{\partial M}} = dd^c\psi|_{T^h_{\partial M}} = \frac{\nu \cdot \psi}{\nu \cdot r} dd^c r|_{T^h_{\partial M}}.$$

On the other hand, there exists $C > 0$ independent of A, A_0 such that

$$-C \leq \frac{\nu \cdot \psi}{\nu \cdot r} \leq 0 \text{ on } \partial M$$

by Lemma 7.14. We thus see that (7.8) implies

$$\eta_\psi|_{T^h_{\partial M}} \geq -C^{-1}\varepsilon(\eta_\psi - \eta)|_{T^h_{\partial M}}$$

and the desired result follows easily.

In order to show (7.8) it is enough to concentrate on vectors

$$v = \sum_{j<m} v_j \frac{\partial}{\partial z_j} \in T^h_{\partial M,0}$$

such that $\sum |v_j|^2 = 1$ and $dd^c r(v) > 0$. After possibly performing a unitary change of coordinates we may thus assume that $v = \partial/\partial z_1$, since the unitary change of coordinates will preserve uniformity with respect to the background data (cf. Remark 7.13).

We now follow the local barrier argument of [CKNS85, pp. 221–223] and [Gua98, pp. 694–696].

Step 1: Choice of a Kähler potential. There exists a locally defined smooth function τ such $dd^c\tau = \eta$ and $\tau(0) = 0$. As a first step we claim that τ may be chosen so as to satisfy

$$\tau|_{\partial M} = \Re\left(\sum_{j=2}^{m} c_j z_1 \overline{z}_j\right) + O\left(|z_2|^2 + ... + |z_m|^2\right) \tag{7.9}$$

for some $c_j \in \mathbb{C}$. Indeed note that we can use the restriction of $(z_1, ..., z_{m-1}, y_m)$ to ∂M as local coordinates for the boundary. By Lemma 7.2 we have

$$(v|_{\partial M})_{z_1|_{\partial M}\overline{z}_j|_{\partial M}}(0) = v_{z_1\overline{z}_j}(0) + \delta_{1j}a_{11}v_{x_m}(0) \tag{7.10}$$

for every $j < m$ and every smooth function v on M, with a_{11} as in (7.7). Applying (7.10) to $v := x_m$ shows that there exists $b \in \mathbb{C}$ such that

$$x_m|_{\partial M} = a_{11}|z_1|^2 + y_m\Re(bz_1) + O\left(y_m|z_1|^2 + |z_2|^2 + ... + |z_{m-1}|^2 + y_m^2\right).$$

Since we assume $a_{11} = (dd^c r)(v) > 0$ it follows that the following holds near 0 on ∂M:

(i) $|z_1|^2$ writes as the real part of a complex linear combination of $z_m, z_1 z_m$ and $z_1\overline{z}_m$ modulo $O\left(|z_2|^2 + ... + |z_m|^2\right)$

(ii) $z_1|z_1|^2$ writes as the real part of a complex linear combination of z_1z_m and $z_1\overline{z}_m$ modulo $O\left(|z_2|^2 + ... + |z_m|^2\right)$
(iii) $|z_1|^4$, $y_m|z_1|^2$ and $z_j|z_1|^2$ are $O\left(|z_2|^2 + ... + |z_m|^2\right)$ for $j = 2, ..., m-1$.

As a consequence, Taylor series considerations show that any smooth real-valued function v on ∂M near 0 writes as the real part of a complex linear combination of

$$z_1, z_2, ..., z_m, z_1^2, z_1z_2, ..., z_1z_m, z_1\overline{z}_2, ..., z_1\overline{z}_m \text{ and } z_1^3$$

modulo $O\left(|z_2|^2 + ... + |z_m|^2\right)$. Since we can add to τ the real part of a holomorphic polynomial in $z_1, ..., z_m$ without changing the condition $dd^c\tau = \eta$, the above fact applied to $v := \tau|_{\partial M}$ shows that τ may indeed be chosen so as to satisfy (7.9).

Step 2: Choice of a barrier function. Let us now consider the barrier function

$$b(z_1, ..., z_m) := -\varepsilon_1 x_m + \varepsilon_2|z|^2 + \frac{1}{2\mu}\sum_{j=2}^{m}|c_jz_1 + \mu z_j|^2 \tag{7.11}$$

with $\varepsilon_1, \varepsilon_2, \mu > 0$ and let $B \subset M$ be a coordinate half-ball centered at 0. We are going to show that we may choose the radius α of B, $\varepsilon_1, \varepsilon_2$ and μ in terms of A_0 only such that

$$b \geq \tau + \psi \text{ on } B. \tag{7.12}$$

By (7.7) there exists $C > 0$ such that

$$|z_m| \geq x_m \geq \Re\left(\sum_{1\leq j,k\leq m} a_{jk}z_j\overline{z}_k\right) - C|z|^3 \text{ on } B.$$

Since we have $|z| = \alpha$ on $\partial B - B \cap \partial M$ and $a_{11} > 0$ we may thus shrink the radius α of B so that there exists $\beta > 0$ with

$$\sum_{j=2}^{m}|z_j|^2 \geq \beta \text{ on } \partial B - B \cap \partial M. \tag{7.13}$$

On the other hand $r = 0$ on ∂M so there exists $C > 0$ such that

$$-\varepsilon_1 x_m + \varepsilon_2|z|^2 \geq 0 \text{ on } \partial M \cap B \tag{7.14}$$

as soon as $\varepsilon_2 \geq C\varepsilon_1$.

Having fixed such a choice of B we next claim that there exists $\mu > 0$ independent of ψ such that for any $\varepsilon_1, \varepsilon_2$ with $\varepsilon_2 \geq C\varepsilon_1$ we have

$$\tau + \psi \leq b \text{ on } \partial B. \tag{7.15}$$

Indeed this holds on $B \cap \partial M$ by (7.9) and (7.11) since $\psi|_{\partial M} = 0$ and $-\varepsilon_1 x_m + \varepsilon_2 |z|^2 \geq 0$. On the other hand since $\sup \psi$ is under control by Lemma 7.14 the claim also holds on $\partial B - B \cap \partial M$ by (7.13).

Next we pick $\varepsilon_2 > 0$ only depending on μ and A_0 such that

$$(dd^c b)^m \leq e^{-A_0} \eta^m \leq e^F \eta^m = (dd^c(\tau + \psi))^m \text{ on } B, \tag{7.16}$$

which is possible since

$$\left(dd^c \sum_{j=2}^{m} |c_j z_1 + \mu z_j|^2 \right)^m = 0$$

thus $(dd^c b)^m = O(\varepsilon_2)$. Finally we choose ε_1 such that $\varepsilon_2 \geq C\varepsilon_1$. By (7.15) and (7.16) we finally get $\tau + \psi \leq b$ on B as desired by the maximum principle (Proposition 7.5).

Step 3: Conclusion. Since we have $\tau + \psi \leq b$ on B and $b(0) = (\tau + \psi)(0) = 0$ it follows that

$$(\tau + \psi)_{x_m}(0) \leq b_{x_m}(0) = -\varepsilon_1.$$

On the other hand (7.9) and (7.10) yield

$$\tau_{z_1 \bar{z}_1}(0) + \tau_{x_m}(0) a_1 = 0,$$

and $\psi|_{\partial M} = 0$ and (7.10) similarly imply

$$\psi_{z_1 \bar{z}_1}(0) + \psi_{x_m}(0) a_1 = 0.$$

Putting all this together we thus obtain

$$\eta_\psi(v) = (\tau + \psi)_{z_1 \bar{z}_1}(0) \geq \varepsilon_1 a_1 = \varepsilon_1 (dd^c r)(v)$$

which shows that (7.8) holds and concludes the proof. □

Lemma 7.17 *There exists $C = C(A_0, A_1)$ (resp. $C = C(A_1)$ when ∂M is weakly pseudoconcave) such that*

$$\sup_{\partial M} |\nabla^2 \psi| \leq C(1 + \sup_M |\nabla \psi|^2).$$

This result was proved in [CKNS85, Gua98, Che00] (see also [PS09] Lemma 1).

Proof. Let $0 \in \partial M$ and choose adapted data B, r, z as in Remark 7.13. We set for convenience

$$t_1 = y_1, t_2 = x_2, ..., t_{2m-1} = y_m, t_{2m} = x_m.$$

Let $D_1, ..., D_{2m}$ be the dual basis of

$$dt_1, ..., dt_{2m-1}, -dr,$$

so that

$$D_j = \frac{\partial}{\partial t_j} - \frac{r_{t_j}}{r_{x_m}} \frac{\partial}{\partial x_m} \tag{7.17}$$

for $j < 2m$ and

$$D_{2m} = -\frac{1}{r_{x_m}} \frac{\partial}{\partial x_m}. \tag{7.18}$$

Note that the D_j's commute and are tangent to ∂M for $j < 2m$, so that we have a trivial control on the tangent-tangent derivatives

$$D_i D_j \psi(0) = 0 \text{ for } i, j < 2m.$$

We set

$$K := \sup_M |\nabla \psi|$$

and note that there exists $C_0 > 0$ only depending on r such that

$$|D_j \psi| \leq C_0 K, \; j = 1, ..., 2m. \tag{7.19}$$

In the next two steps we are going to show the existence of $C = C(A_1)$ such that the normal-tangent derivatives of ψ satisfy

$$|D_j D_{2m} \psi(0)| \leq C(1 + K)$$

for $j < 2m$.

In the third step we will show how this combines with Lemma 7.16 to get $C = C(A_1, A_0)$ (resp. $C = C(A_1)$ in the weakly pseudoconcave case) such that

$$|D_{2m}^2 \psi(0)| \leq C(1 + K^2).$$

Step 1: Construction of a barrier function. As in Lemma 7.14 we let h be the unique function on M such that $h|_{\partial M} = 0$ and $\Delta h = -m$ and we introduce the barrier function

$$b := \psi + \varepsilon h - \mu r^2$$

on a coordinate half-ball $B \subset M$ centered at 0. We claim that we may choose B, ε and μ independently of ψ such that

$$b \geq 0 \text{ on } B \tag{7.20}$$

and

$$\Delta_\psi b \leq -\frac{1}{2} \operatorname{tr}_\psi(\eta) \text{ on } B. \tag{7.21}$$

Since

$$dd^c(r^2) = 2r dd^c r + 2dr \wedge d^c r$$

there exists C_1 only depending on h and r such that

$$\Delta_\psi b \leq m - \operatorname{tr}_\psi(\eta) + C_1(\varepsilon - 2\mu r) \operatorname{tr}_\psi(\eta) - 2\mu \operatorname{tr}_\psi(dr \wedge d^c r). \tag{7.22}$$

Using $dr \neq 0$ at 0 and $\eta > 0$ we may choose B small enough so that

$$dr \wedge d^c r \wedge \eta^{m-1} \geq \gamma \eta^m$$

on B for some $\gamma > 0$, hence

$$dr \wedge d^c r \wedge \eta^{m-1} \geq \gamma \eta_\psi^m$$

since

$$\eta_\psi^m = f \eta^m \leq \eta^m$$

by assumption. If we choose $\mu > 0$ such that $m\gamma\mu \geq 2$ we then get

$$(\frac{1}{4}\eta + 2\mu dr \wedge d^c r)^m \geq \gamma^{-1} dr \wedge d^c r \wedge \eta^{m-1} \geq \eta_\psi^m$$

and it follows from the arithmetico-geometric inequality (Proposition 7.9) that

$$\operatorname{tr}_\psi(\frac{1}{4}\eta + 2\mu dr \wedge d^c r) \geq m.$$

By (7.22) we obtain

$$\Delta_\psi b \leq (C_1(\varepsilon - 2\mu r) - 3/4) \operatorname{tr}_\psi(\eta).$$

Since $r(0) = 0$ we may assume upon shrinking B and choosing $\varepsilon > 0$ small enough with respect to μ and C_1 that $C_1(\varepsilon - 2\mu r) \leq 1/4$ on B and we get (7.21).

Let us now show how to obtain (7.20). Since $\psi \geq 0$ it is enough to guarantee

$$\varepsilon h \geq \mu r^2. \tag{7.23}$$

But we have $\Delta h = -m$ thus we get $\Delta(h+cr) \le 0$ on the whole of M if $c > 0$ is small enough and the maximum principle implies $h + cr \ge 0$ on M since $(h+cr)|_{\partial M} = 0$. We thus see that (7.23) holds if $-c\varepsilon r \ge \mu r^2$, i.e. $r \ge -c\varepsilon\mu^{-1}$, which will hold on B upon possibly shrinking it further in terms of ε, μ only.

Step 2: Bounding the normal-tangent derivatives. Let $j < 2m$ and consider the tangential vector field D_j. We claim that there exist μ_1, μ_2 only depending on A_1 such that

$$v := K(\mu_1 b + \mu_2 |z|^2) \pm D_j\psi \tag{7.24}$$

satisfies

$$v \ge 0 \text{ on } \partial B \tag{7.25}$$

and

$$\Delta_\psi v \le 0 \text{ on } B. \tag{7.26}$$

We first take care of (7.25). On the one hand we have $D_j\psi = b = 0$ on $B \cap \partial M$ thus $v \ge 0$ on $B \cap \partial M$. On the other hand on $\partial B - B \cap \partial M$ we have $|z|^2 = \alpha^2$ where $\alpha > 0$ denotes the radius of B. Since $b \ge 0$ on B it follows that $v \ge 0$ on ∂B as soon as $K\mu_2\alpha^2 \ge |\psi_{u_j}|$, which holds as soon as $\mu_2 \ge C_0\alpha^{-2}$ by (7.19).

Having fixed such a choice of μ_2 we now show how to choose μ_1 so that (7.26) holds. Applying D_j to

$$\log \frac{(\eta + dd^c\psi)^m}{\eta^m} = F$$

yields

$$\operatorname{tr}_\psi(D_j\eta + D_j dd^c\psi) = D_jF + \operatorname{tr}_\eta(D_j\eta) \tag{7.27}$$

where D_j acts on $(1,1)$-forms componentwise (in the $dz_k \wedge d\bar{z}_l$ basis). Since we have $\operatorname{tr}_\psi(\eta) \ge m$ by the arithmetico-geometric inequality (Proposition 7.9) we thus see that there exists $C = C(A_1)$ such that

$$|\operatorname{tr}_\psi(D_j dd^c\psi)| \le C \operatorname{tr}_\psi(\eta). \tag{7.28}$$

By (7.17) we have

$$D_j = \frac{\partial}{\partial t_j} + a\frac{\partial}{\partial x_m}$$

with $a := -r_{t_j}/r_{x_m}$ and an easy computation yields

$$dd^c(D_j\psi) = D_j dd^c\psi + \psi_{x_m} dd^c a + 2da \wedge d^c\psi_{x_m}$$

thus

$$\Delta_\psi(D_j\psi) = \operatorname{tr}_\psi(D_j dd^c\psi) + \psi_{x_m}\Delta_\psi a + 2\operatorname{tr}_\psi(da \wedge d^c\psi_{x_m}).$$

Now on the one hand one easily checks that

$$d^c\psi_{x_m} = d^c(i_{\partial/\partial x_m} d\psi) = -i_{\partial/\partial y_m} d^c d\psi = i_{\partial/\partial y_m} dd^c\psi$$

where i denotes the contraction operator. On the other hand applying $i_{\partial/\partial y_m}$ to the trivial relation

$$da \wedge \eta_\psi^m = 0$$

yields

$$\mathrm{tr}_\psi \left(da \wedge (i_{\partial/\partial y_m}\eta_\psi)\right) = a_{y_m}$$

so we get

$$\mathrm{tr}_\psi(da \wedge d^c\psi_{x_m}) + \mathrm{tr}_\psi(da \wedge i_{\partial/\partial y_m}\eta) = a_{y_m}.$$

Putting all this together we obtain

$$\Delta_\psi(D_j\psi) = \mathrm{tr}_\psi(D_j dd^c\psi) + \psi_{x_m}\Delta_\psi a + 2a_{y_m} - 2\,\mathrm{tr}_\psi(da \wedge i_{\partial/\partial y_m}\eta)$$

which combines with (7.28) to yield

$$|\Delta_\psi(D_j\psi)| \leq C(1+K)\,\mathrm{tr}_\psi(\eta) \tag{7.29}$$

for some $C = C(A_1)$.

By (7.21) we thus get

$$\Delta_\psi v \leq \left(-\frac{K}{2}\mu_1 + C(1+K)\right)\eta \wedge \eta_\psi^{m-1} \text{on } B$$

for some $C = C(A_1)$ and we may thus choose $\mu_1 = \mu_1(A_1)$ such that (7.26) holds.

Now (7.25) and (7.26) imply $v \geq 0$ on B by the maximum principle hence $D_{2m}v(0) \geq 0$ since $v(0) = 0$. In other words we have proved that

$$|D_{2m}D_j(0)| \leq CK\,(1 + D_{2m}b(0))$$

for some $C = C(A_1)$. But the gradient of $b = \psi + \varepsilon h - \mu r^2$ at 0 is bounded by a constant C independent of ψ by Lemma 7.14, and we thus get

$$|D_{2m}D_j\psi(0)| \leq C(1+K)$$

for some $C = C(A_1)$ as desired.

Step 3: Bounding the normal-normal derivatives. In this last step we are going to show that there exists a constant $C = C(A_1, A_0)$ (resp. $C = C(A_1)$ in the weakly pseudoconcave case) such that

$$\left|D_{2m}^2\psi(0)\right| \leq C(1+K^2).$$

By the bound on $D_i D_j(0)$ and $D_{2m} D_j \psi(0)$ for $i, j < 2m$ it is equivalent to show that

$$|\psi_{z_m \overline{z}_m}(0)| \leq C(1 + K^2) \tag{7.30}$$

and we already know that

$$\left|\psi_{z_j \overline{z}_m}(0)\right| \leq C(1 + K) \tag{7.31}$$

for all $j < m$ and $C = C(A_1)$. If we write

$$\eta = i \sum_{j,k} \eta_{jk} dz_j \wedge d\overline{z}_k$$

then expanding out the determinant thus yields

$$\left|\det(\eta_{jk} + \psi_{z_j \overline{z}_k})_{1 \leq j,k \leq m} - (\eta_{mm} + \psi_{z_m \overline{z}_m}) \det(\eta_{jk} + \psi_{z_j \overline{z}_k})_{1 \leq j,k < m}\right| \leq C(1 + K^2)$$

at 0.

Now on the one hand the equation $(\eta + dd^c \psi)^m = f \eta^m$ shows that

$$0 \leq \det(\eta_{jk} + \psi_{z_j \overline{z}_k})_{1 \leq j,k \leq m} \leq \det(\eta_{jk}) \leq C.$$

On the other hand since $T^h_{\partial M}$ is spanned by $\frac{\partial}{\partial z_j}$, $j = 1, ..., m - 1$ at 0 Lemma 7.16 yields $\varepsilon > 0$ only depending on A_0 (resp. $\varepsilon = 1$ in the weakly pseudoconcave case) such that

$$\det(\eta_{jk} + \psi_{z_j \overline{z}_k})_{1 \leq j,k < m} \geq \varepsilon$$

and Lemma 7.17 follows. □

Following [Che00] Sect. 3.2 we now use a blow-up argument to show:

Lemma 7.18 *There exists $C > 0$ only depending on A_0, A_1, A_2 (resp.on A_1, A_2 in the weakly pseudoconcave case) such that $\sup_M |\nabla \psi| \leq C$.*

Proof. By Lemma 7.15 and Lemma 7.17 we have

$$\sup_M |\Delta \psi| \leq C(1 + \sup_M |\nabla \psi|^2) \tag{7.32}$$

for some $C > 0$ only depending on A_0, A_1, A_2.

Assume by contradiction that the result fails. Then there exists sequences $x_j \in M$ and ψ_j such that

$$|\nabla \psi_j(x_j)| = \sup_M |\nabla \psi_j| =: C_j \to +\infty.$$

We may assume that $x_j \to x_\infty \in M$. Pick a coordinate half-ball B centered at x_∞ and set

$$\widetilde{\psi_j}(z) := \psi_j(x_j + C_j^{-1} z) \tag{7.33}$$

which satisfies

$$|\nabla \widetilde{\psi_j}(0)| = 1 \tag{7.34}$$

and

$$\sup_B |\Delta \widetilde{\psi_j}| \leq C. \tag{7.35}$$

by (7.32). By (7.35) $\widetilde{\psi_j}$ stays in a compact subset of $C^1(B)$, so we can assume that $\widetilde{\psi_j} \to \rho$ in $C^1(B)$, and (7.34) implies $|\nabla \rho(0)| = 1$, so that ρ is non-constant. Now there are two cases.

If $x_\infty \in \partial M$ then Lemma 7.14 implies $\rho \equiv 0$ and contradicts the fact that ρ is non-constant.

If $x_\infty \notin \partial M$ then (7.33) makes sense on a ball of size C_j, which shows that ρ is actually defined on the whole of $\mathbb{C}^m$. Since

$$dd^c \widetilde{\psi_j} \geq C_j^{-2} dd^c \psi_j \geq -C_j^{-2} \eta$$

we also see that ρ is psh on $\mathbb{C}^m$. On the other hand ρ is uniformly bounded above on $\mathbb{C}^m$ by Lemma 7.14, and these two properties imply that ρ is constant (for instance because ρ extends as a psh function on the complex projective space $\mathbb{P}^m$), which contradicts again $|\nabla \rho(0)| = 1$. □

7.3.2 Proof of Theorem 7.11

By Lemma 7.15, Lemma 7.17 and Lemma 7.18 there exists $C > 0$ only depending on A such that

$$\eta + dd^c \psi \leq C \eta.$$

Since we also have $(\eta + dd^c \psi)^m \geq e^{-A} \eta^m$ it follows upon possibly enlarging C that

$$C^{-1} \eta \leq \eta + dd^c \psi \leq C \eta.$$

This means that the Monge–Ampère equation

$$(\eta + dd^c \psi)^m = e^F \eta^m$$

is elliptic with ellipticity constants that are uniform with respect A, and one would like to conclude that the a priori bound on the complex Hessian $dd^c \psi$ should yield an a priori bound on the $C^{2+\alpha}$ norm of ψ for some $\alpha > 0$ by an Evans-Krylov-type result.

Indeed [Ev82] yields *inner* $C^{2+\alpha}$ estimates for solutions of uniformly elliptic fully non-linear second order PDE's as soon as a C^2 bound is available. Similarly [CKNS85, Theorem 1] yields $C^{2+\alpha}$ estimates *up to the boundary* in a similar situation.

However the difficulty in our case (which seems to have been overlooked in [Che00] Remark 1) is that we only have an a priori bound on $\Delta\psi$, which doesn't provide a bound on the C^2-norm in general.

A similar situation occurs in the proof of the Aubin–Yau theorem, i.e. for the analogue of Theorem A when M has no boundary. The solution adopted in [Siu87] was to reprove the Evans–Krylov theorem in the complex case, replacing the C^2-norm by a control on the complex Hessian. In principle it should be possible to follow the same path in our situation, i.e. to try and adapt the proof of [CKNS85, Theorem 1] to the complex case. However another difficulty occurs since it is assumed in [CKNS85, p. 231] that M is locally a half-plane near a given point of ∂M, which is of course trivial in the real case but cannot hold in the complex case unless ∂M is Levi flat. We thus see that a different strategy has to be proposed in the general case where no assumption is made on ∂M.

The way out of this difficulty is provided by the following result ([Bl09b, Theorem 3.4]).

Lemma 7.19 *Let* (M, η) *be a compact Kähler manifold with boundary. Given* $A > 0$ *there exists* $C > 0$ *such that the following holds. Let* ψ *be a smooth* η*-psh function such that*

$$(\eta + dd^c\psi)^m = e^F\eta^m \text{ and } \psi|_{\partial M} = 0$$

with $F \in C^\infty(M)$*. If*

(i) $A^{-1}\eta \le \eta_\psi \le A\eta$
(ii) $\|F\|_{C^2(M)} \le A$
(iii) $\|\psi\|_{C^1(M)} \le A$

then

$$\sup_M |\nabla^2\psi| \le C(1 + \sup_{\partial M}|\nabla\psi|).$$

Proof. We only sketch the argument, referring to [Bl09b, Theorem 3.4] for computational details.

Let first D be a local vector field on M with constant coefficients with respect to one of the given coordinate patches and of norm (with respect to η) at most 1. Applying D to

$$\log\frac{(\eta + dd^c\psi)^m}{\eta^m} = F \tag{7.36}$$

yields

$$\Delta(D\psi) \ge -C$$

with $C = C(A)$ by (i) and (ii). Similarly, applying D^2 to (7.36) implies

$$\Delta(D^2\psi) \geq -C$$

with $C = C(A)$ by (i) and (ii).

Now there exists $C > 0$ only depending on η such that

$$\sup_M |\nabla^2\psi| \leq C \sup_M D^2\psi \tag{7.37}$$

for some globally defined vector field D of length at most 1.

Somewhat tedious computations using (i) and (iii) then show that

$$\Delta\left(|\nabla\psi|^2\right) \geq C^{-1}|\nabla^2\psi|^2 - C$$

and

$$\Delta\left(D^2\psi\right) \geq -C(1 + |\nabla^2\psi|)$$

with $C = C(A)$, so that

$$\Delta\left(|\nabla\psi|^2 + D^2\psi\right) \geq C^{-1}|\nabla^2\psi|^2 - C(1 + |\nabla\psi|^2) \text{ on } M. \tag{7.38}$$

Now the result follows as in Lemma 7.15: pick $x_0 \in M$ at which $|\nabla\psi|^2 + D^2\psi$ achieves its maximum. If x_0 belongs to ∂M then we're done. Otherwise (7.38) yields

$$|\nabla^2\psi|(x_0) \leq C$$

for some $C = C(A)$. But we have

$$\sup_M D^2\psi \leq D^2\psi(x_0) + C \leq C(|\nabla^2\psi(x_0)| + 1)$$

and we infer $\sup_M |\nabla^2\psi| \leq C$ by (7.37). □

7.4 Proof of Theorem A and Theorem B

In this section we explain how to deduce Theorems A and B from the a priori estimates obtained in the previous section. The proof relies on the continuity method and we won't give much details for the standard parts of the procedure, referring for example to [Bl05b, Sect. 2] in this volume.

7.4.1 Proof of Theorem A

Uniqueness follows from Proposition 7.5.

Upon replacing η with $\eta + dd^c\widetilde{\varphi}$ we may assume that $\widetilde{\varphi} = 0$ (and in particular $\varphi = 0$). If we write $\mu = e^F\eta^m$ with $F \in C^\infty(M)$ we thus have $F \leq 0$.

We follow the continuity method: consider the set I of all $t \in [0,1]$ such that there exists a smooth (strictly) η-psh function ψ_t on M with

1. $\psi_t|_{\partial M} = 0$.
2. $(\eta + dd^c\psi_t)^m = ((1-t)e^F + t)\eta^m$.

Note that ψ_t is unique by the maximum principle (Proposition 7.5). The set I is non-empty since it contains 1 (with $\psi_1 \equiv 0$).

Since the linearization of the operator

$$\psi \mapsto \log \frac{(\eta + dd^c\psi)^m}{\eta^m}$$

at a given smooth strictly η-psh function ψ is equal to Δ_ψ, it follows from standard elliptic regularity and the inverse function theorem applied in appropriate Sobolev spaces that I is open.

On the other hand, the C^2 norm of

$$\log\left((1-t)e^F + t\right)$$

is clearly bounded independently of t thus, Theorem 7.11 yields $C > 0$ and $\alpha > 0$ such that $\|\psi_t\|_{C^{2+\alpha}(M)} \leq C$ for all $t \in I$. The usual compactness and elliptic bootstrapping argument using Schauder's estimates therefore shows that I is closed and we conclude that $I = [0,1]$, so that $0 \in I$ as desired.

7.4.2 Proof of Theorem B

We use the same strategy as in [PS09] Sect. 4.2.

Uniqueness follows from Corollary 7.7. Now let $\widetilde{\varphi}$ be the given η-psh extension of $\varphi \in C^\infty(\partial M)$ and set $\vartheta := \eta + dd^c\widetilde{\varphi}$. Here ϑ is merely semipositive so we cannot directly replace η with ϑ.

However we have $(1-t)\vartheta + t\eta > 0$ for each $t > 0$ and

$$((1-t)\vartheta + t\eta)^m \geq t^m\eta^m,$$

so by Theorem A there exists a unique smooth $((1-t)\vartheta + t\eta)$-psh function ψ_t on M such that

$$((1-t)\vartheta + t\eta + dd^c\psi_t)^m = t^m\eta^m \text{ and } \psi_t|_{\partial M} = 0. \tag{7.39}$$

In what follows $C > 0$ denotes a constant independent of t. Since $(1-t)\vartheta + t\eta \leq C\eta$ we have in particular $dd^c\psi_t \geq -C\eta$ and we get

$$\sup_M |\psi_t| + \sup_{\partial M} |\nabla\psi_t| \leq C \tag{7.40}$$

by replacing h with the solution of $\Delta h = -C$, $h|_{\partial M} = 0$ in the proof of Lemma 7.14. Now observe that (7.39) rewrites in terms of

$$\rho_t := (1-t)\widetilde{\varphi} + \psi_t$$

as

$$(\eta + dd^c\rho_t)^m = e^{F_t}\eta^m \text{ and } \rho_t|_{\partial M} = (1-t)\varphi. \tag{7.41}$$

with $F_t :\equiv m\log t$ (and thus $\nabla F_t = \Delta F_t \equiv 0$).

We claim that we have

$$\sup_M |\nabla\rho_t| \leq C,$$

and

$$\sup_M \Delta\rho_t \leq C$$

when ∂M is furthermore weakly pseudoconcave.

The bound on $\nabla\rho_t$ follows directly from (the proof of) Blocki's gradient estimate ([Bl09a] Theorem 1) since $\nabla\rho_t$ is bounded on ∂M by (7.40).

Assume that ∂M is weakly pseudoconcave. Then (the proof of) Lemma 7.16 and Lemma 7.17 shows that

$$\sup_{\partial M} \Delta\rho_t \leq C$$

and we infer $\sup_M \Delta\rho_t \leq C$ by Yau's inequality just as in Lemma 7.15.

Let us now conclude the proof of Theorem B. Since

$$\sup_M(|\psi_t| + |\nabla\psi_t|) \leq C$$

the ψ_t's stay in a compact subset of $C^0(M)$. If $\psi = \lim_{t_j\to 0_+}\psi_{t_j}$ is any limit point in $C^0(M)$ then ψ is η-psh and satisfies

$$(\eta + dd^c\psi)^m = 0 \text{ and } \psi|_{\partial M} = 0$$

by continuity of the Monge–Ampère operator in the topology of uniform convergence ([BT76]). Since $\sup_M |\nabla\psi_t| \leq C$ we also get that ψ is Lipschitz continuous which proves (i) of Theorem B.

If we assume furthermore that ∂M is weakly pseudoconcave then we have

$$\sup_M |\Delta\psi_t| \leq C$$

thus $\Delta\psi \in L^\infty$.

Since ψ is η-psh, it follows that $dd^c\psi$ has L^∞_{loc} coefficients. □

Chapter 8
Bergman Geodesics

Robert Berman and Julien Keller

Abstract The aim of this survey is to review the results of Phong–Sturm and Berndtsson on the convergence of Bergman geodesics towards geodesic segments in the space of positively curved metrics on an ample line bundle. As previously shown by Mabuchi, Semmes and Donaldson the latter geodesics may be described as solutions to the Dirichlet problem for a homogeneous complex Monge–Ampère equation. We emphasize in particular the relation between the convergence of the Bergman geodesics and semi-classical asymptotics for Berezin–Toeplitz quantization. Some extension to Wess–Zumino–Witten type equations are also briefly discussed.

8.1 Introduction

Let $L \to X$ be an ample line bundle on a smooth projective manifold X of complex dimension n and denote by $\mathcal{H}_\infty$ the space of all (smooth) Hermitian metrics h on L with positive curvature form. Fixing a reference metric h_0 with curvature form ω_0, any other metric may be written as $h_\phi = e^{-\phi} h_0$ with curvature form $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$, using the convention which makes the curvature form a *real* $2-$form. Hence, it will be convenient to make the identification

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$$\mathcal{H}_\infty = \{\phi \in C^\infty(X) : \omega_\phi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

realizing $\mathcal{H}_\infty$ as a subspace of the space of all, say continuous, ω_0-psh functions ϕ on X, i.e. $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi \geq 0$ in the sense of currents. The space $\mathcal{H}_\infty$ can be equipped with a Riemannian metric (Cf. the work of T. Mabuchi, S. Semmes and S.K. Donaldson): for any tangent vector at ϕ, which may be identified with a smooth function ψ on X

$$\|\psi\|_\phi^2 = \int_X |\psi|^2 \omega_\phi^n.$$

Hence one can speak, at least at the formal level, of geodesics in $\mathcal{H}_\infty$. Moreover, the geodesic curvature $c(t) = c(\phi_t)$ is given by the expression

$$c(\phi_t) = \ddot{\phi}_t - |\bar{\partial}_x \dot{\phi}_t|^2_{\omega_\phi} \tag{8.1}$$

which hence vanishes precisely when ϕ_t is a geodesic, see Sect. 6.3.1. By complexifying the variable t, a geodesic ϕ_t connecting two elements in $\mathcal{H}_\infty$ may be obtained by solving the Dirichlet problem for a complex Monge–Ampère operator in the (x,t)-variables.

The previous setup extends immediately to the "transcendental" setting where ω_0 is any fixed Kähler form on a compact complex manifold X, i.e. without assuming that $2\pi\omega_0$ is an *integral* class. Then $\mathcal{H}_\infty$ is by definition the open convex space in the cohomology class $[\omega_0]$ consisting of all ω_0-psh functions.

On the other hand, when $2\pi\omega_0$ is an *integer* class, i.e. ω_0 is the curvature of a metric h_0 on an ample line bundle $L \to X$, the setup may be "quantized" as follows. For any given positive integer k, the infinite dimensional space $\mathcal{H}_\infty$ is replaced by a certain *finite-dimensional* symmetric space $\mathcal{H}_k$: the space of *Bergman metrics at level* k. By definition, these metrics are pull-backs of Fubini–Study type metrics by the embeddings of the manifold into the projective space $\mathbb{P}H^0(L^k)^\vee$ (Cf. Sect. 8.2.3). A result of T. Bouche and G. Tian asserts that any element ϕ in $\mathcal{H}_\infty$ can be seen as a canonical limit of a sequence of elements $P_k(\phi)$ in the spaces $\mathcal{H}_k$. The symmetric spaces $\mathcal{H}_k$ come equipped with an intrinsic Riemannian structure. In particular, any two elements in $\mathcal{H}_k$ may be connected by a unique *Bergman* geodesic (at level k). Hence, it is natural to ask if the whole Bergman geodesic ψ_t connecting $P_k(\phi_0)$ and $P_k(\phi_1)$ in $\mathcal{H}_k$ converges to the geodesic ϕ_t in $\mathcal{H}_\infty$ when k tends to infinity (and not only its end points)? The question was answered in the affirmative by Phong–Sturm [PS06]. Their result was subsequently refined by Berndtsson [Bern09b] (see also the very recent work [Bern09c]) using a completely different approach.

The aim of the present notes is to survey these two latter results on convergence of segments of Bergman geodesics. There is also a stronger convergence result in the setting of toric varieties due to J. Song and S.

Zelditch that will not be discussed here. For a general introduction to the circle of ideas in Kähler geometry surrounding all these results see the recent survey [PS08]. Let us just briefly mention that an important feature of Kähler geometry is that several important functionals (Lagrangians) on $\mathcal{H}_\infty$, whose critical points yield canonical Kähler metrics, turn out to be convex along geodesics in $\mathcal{H}_\infty$ and $\mathcal{H}_k$. In particular, the use of geodesic *segments* in $\mathcal{H}_k$ is underlying in Donaldson's seminal work [Don01] to prove *uniqueness* (up to automorphisms) of constant scalar curvature metrics in $\mathcal{H}_\infty$ (essentially by connecting any given two such metrics by a geodesic segment). On the other hand geodesic *rays* in $\mathcal{H}_k$ are closely related to the still unsolved existence problem for canonical Kähler metrics, the so-called Yau–Tian–Donaldson conjecture. The point is to study "properness" i.e. the growth of energy functional along the geodesic rays which turns out to be related to algebro-geometric notions of stability (notably asymptotic Chow–Mumford stability as conjectured by Yau [Yau87] long time ago).

The organization of the survey is as follows. We begin by briefly recalling the "quantum formalism" as it offers a suggestive description of the results to be discussed. Then the key steps in the proofs of first Phong–Sturm and then Berndtsson's results are indicated essentially following arguments in the original papers. In Sect. 8.3.2 it is explained how to deduce a slightly weaker version of Berndtsson's convergence result using asymptotic formulas for products of Toeplitz operators. These formulas are well-known in the context of Berezin–Toeplitz quantization of Kähler manifolds. This latter approach is actually analytically far more involved than Berndtsson elegant curvature estimate, but hopefully it may shed some new light on Berndtsson's convergence result and its relation to quantization. Some extension to Wess–Zumino–Witten type equations are also briefly discussed in the last section.

8.1.1 The "Quantum Formalism"

The state space of a *classical* physical system is mathematically described by a symplectic manifold X equipped with a symplectic form ω. An "observable" on the state space (X, ω) is just a real-valued function on X. From this point of view *quantization* is the art of associating a Hilbert space $H(X, \omega)$ (the "quantum state space") to (X, ω) and Hermitian operators on $H(X, \omega)$ to real valued functions on X. Moreover, the quantizations should come in families parametrized by a small parameter $\hbar$ ("Planck's constant") and in the limit $\hbar \to 0$ the classical setting should emerge from the quantum one, in a suitable sense (the "correspondence principle"). One possibility to make this latter principle more precise is to demand that the non-commutative C^*-algebra of all (bounded) operators on $H(X, \omega)$ should induce a deformation (in the parameter $\hbar$) of the commutative C^*-algebra $C^\infty(X, \mathbb{C})$ (this is the subject of *deformation quantization*). See for example [AE05, Gut00] for a general survey on quantization.

As shown by Berezin, Cahen, Gutt, Rawnsley and others any positively curved metric ϕ on a line bundle $L \to X$ induces a quantization with $\hbar = 1/k$, where k is a positive integer. If $\omega = \omega_\phi$ the quantization (at level k) of (X, ω_ϕ) is obtained by letting $H(X, \omega) := H^0(X, L^{\otimes k})$ equipped with the Hermitian metric $Hilb(k\phi)$:

$$\langle s, \bar{s} \rangle_{k\phi} = Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|^2_{h_0^k} e^{-k\phi} \frac{\omega_\phi^n}{n!}.$$

To any complex-valued function f one associates the *Toeplitz operator* $T_f^{(k)}$ *on* $H^0(X, L^{\otimes k})$ *with symbol* f. It is defined by the corresponding quadratic form on the Hilbert space $H^0(X, L^{\otimes k})$:

$$\left\langle T_f^{(k)} s, s' \right\rangle_{k\phi} := \langle f s, s' \rangle_{k\phi}, \tag{8.2}$$

In other words,

$$T_f^{(k)} = P_k f \cdot, \;\; P_k : \mathcal{C}^\infty(X, L^{\otimes k}) \to H^0(X, L^{\otimes k}),$$

where P_k is the orthogonal projection induced by the Hilbert space structure. In particular, $T_{\bar{f}}^{(k)} = (T_f^{(k)})^*$ and hence $T_f^{(k)}$ is Hermitian if f is real. It turns out that there is an asymptotic expansion (in operator norm) [Eng02, Schl00, MM06]

$$T_f^{(k)} T_g^{(k)} - T_{fg}^{(k)} = \frac{1}{k} T_{c_1(f,g)}^{(k)} + \frac{1}{k^2} T_{c_2(f,g)}^{(k)} + ... \tag{8.3}$$

where $c_i(f, g)$ is a bi-differential operator (where we have written out $c_0(f, g) = fg$. The corresponding formal induced star product on symbols: $f * g := fg + c_1(f, g)h + ...$ is usually called the *Berezin–Toeplitz star product*. In particular,

$$[T_f^{(k)}, T_g^{(k)}] = \frac{1}{k} T_{c_1(f,g) - c_1(g,f)} + O(1/k^2),$$

where $c_1(f, g) - c_1(g, f)$ is the Poisson bracket on $C^\infty(X, \mathbb{C})$ induced by the symplectic form ω (compare with formula (8.26)). Moreover, as shown in [BG81, Berm06], if f is real-valued and $\sigma(T_f^{(k)})$ denotes the spectrum of the operator $T_f^{(k)}$, then

$$\frac{1}{k^n} \sum_{\lambda_i^{(k)} \in \sigma(T_f^{(k)})} \delta_{\lambda_i^{(k)}} \to f_* (\omega_\phi)^n / n! \tag{8.4}$$

In particular, setting $f = 1$ and integrating over $\mathbb{R}$ gives the asymptotic Riemann-Roch formula

$$N_k := \dim H^0(X, L^{\otimes k}) = k^n \int_X \frac{\omega_\phi^n}{n!} + O(k^{n-1}), \tag{8.5}$$

which is consistent with the "correspondence principle", since it identifies the leading asymptotics of the *dimension* of the *quantum* state space with the *volume* of the *classical* phase space. All of these results may be deduced from the asymptotic properties of the *Bergman kernel* $K_k(x, y)$, i.e. the integral kernel of the orthogonal projection P_k. These asymptotics were obtained by Catlin and Zelditch [Cat99, Ze98] using the micro-local analysis of Boutet de Monvel–Sjöstrand. There are by now several approaches to these asymptotics; see the review article [Ze09] for an introduction and references. In particular, there is an asymptotic expansion of the point-wise norm $\rho(k\phi)(x)$ of $K_k(x, x)$ (also called the "distortion function" or "density of states function"):

$$k^{-n}\rho(k\phi)(x) = (2\pi)^{-n}(1 + k^{-1}b_1(x) + k^{-2}b_2(x) + ...) \tag{8.6}$$

which holds in the $\mathcal{C}^\infty$-topology and where the coefficients b_i depend polynomially on ϕ and its derivatives. Finally, it should be pointed out that twisting $L^{\otimes k}$ with an Hermitian holomorphic line bundle L' only has the effect of changing the expression for the coefficients in the expansions above.

8.2 The Results of D.H. Phong and J. Sturm

8.2.1 The Homogeneous Monge–Ampère Equation and Geodesics

Given a smooth metric ϕ on $L \to X$ its Monge–Ampère is the form $(\omega_\phi)^n/n!$ of maximal degree on X. In particular, $MA(\phi) \geq 0$ if ϕ has semi-positive curvature, i.e. if $\omega_\phi \geq 0$. As shown in the seminal work of Bedford–Taylor $MA(\phi)$ is naturally defined as (positive) measure for any ϕ which is locally bounded with $\omega_\phi \geq 0$ in the sense of currents.

Given $\phi_0, \phi_1 \in \mathcal{H}_\infty$ the geodesic ϕ_t from the introduction may be obtained as follows from a complex point of view. Firstly, let us complexify the variable t to take values in the strip $[0, 1] + \sqrt{-1}[0, 2\pi]$ which we will identify, as a complex manifold, with the closure of an annulus A in $\mathbb{C}$. Then pull-back ϕ_0 and ϕ_1 to S^1-invariant functions on the boundary of $M := X \times A$. Pulling back ω_0 from X induces a semi-positive form $\pi^*\omega_0$ on M. Now denote by $\Phi = \phi_t(\cdot)$ the function on $\overline{M}$ obtained as the unique solution of the following Dirichlet problem: $\Phi \in C^0(\overline{M})$, where $\Phi_{\partial M}$ coincides with the given data above and

$$MA_{(x,t)}\Phi := (\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi(x,t))^{n+1} = 0, \ \ (x,t) \in M, \tag{8.7}$$

with $\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi(x,t) \geq 0$. As shown by Chen [Che00] (see also [Bl09b]) the solution Φ is unique and almost C^2-smooth, in the sense that $\partial\bar{\partial}\Phi(x,t)$ has locally bounded coefficients, see Chap. 7. Note that expanding gives the relation

$$MA_{x,t}(\Phi) = c(\phi_t)(dt \wedge d\bar{t} \wedge MA_x(\phi_t)), \tag{8.8}$$

where $c(\phi_t)$ is the geodesic curvature in $\mathcal{H}_\infty$ (compare with formula (8.1)). Hence, if (i) ϕ_t is in $C^\infty(X)$ and (ii) $\partial_x\bar{\partial}_x\phi_t > 0$ for all t, then (8.7) is equivalent to

$$c(\phi_t) := \ddot{\phi}_t - |\bar{\partial}_x\dot{\phi}_t|^2_{\omega_\phi} = 0 \tag{8.9}$$

i.e. ϕ_t is a geodesic in n $\mathcal{H}_\infty$. However, it should be pointed out that it is still not known whether any of the two conditions above hold in general. Hence, the "geodesic" ϕ_t obtained above is a path in the *closure* of $\mathcal{H}_\infty$.

Remark 8.1 *The situation becomes considerably simpler in the* toric *setting, i.e. when the real n-torus T^n acts with an open dense orbit on X and equivariantly on $L \to X$ and $\mathcal{H}_\infty$ is replaced by the space $\mathcal{H}_\infty^{T^n}$ of all T^n-invariant metrics. In this setting $\phi(x)$ may be represented by a convex function on $\mathbb{R}^n_x$. For any t we may consider the Legendre transform $\psi_t(p) = \phi_t^*(p)$ of $\phi_t(x)$ which is a convex function on the dual real vector space $(\mathbb{R}^n)^*$. Then equation (8.9) is equivalent to the equation*

$$\ddot{\psi}_t(p) = 0,$$

i.e. ψ_t is simply the affine *interpolation of ψ_0 and ψ_1 [G99, Theorem 3]. An important conceptual feature of the fiber-wise Legendre transformation is that the transform function ψ_t satisfies a differential equation depending only on the t-variable.*

8.2.2 A Canonical Functional

It is important to remark that one can identify the measure valued operator $\phi \mapsto MA(\phi)$ on the space $\mathcal{H}_\infty$ with a differential one-form MA on $\mathcal{H}_\infty$, using the fact that $\mathcal{H}_\infty$ is a convex subset of the affine space $C^\infty(X)$. As observed by Mabuchi this one-form is actually *exact* [Mab87]. Equivalently, there exists a functional $\mathcal{E} : \mathcal{H}_\infty \to \mathbb{R}$ (the "primitive" of MA) such that $d\mathcal{E}_{|\phi} = MA(\phi)$ or equivalently

$$\frac{d}{dt}\mathcal{E}(\phi_t) = \int_X \frac{d}{dt}\phi_t MA_x(\phi_t). \tag{8.10}$$

for any curve ϕ_t in $\mathcal{H}_\infty$ (see [Aub84, Sect. III] and [Yau87, Sect. 2]). If one imposes the normalization $\mathcal{E}(0) = 0$ the functional $\mathcal{E}$ is hence uniquely determined. Working now on $X \times A$, a direct computation shows that

$$\partial_t \bar{\partial}_t \mathcal{E}(\Phi(x,t)) = \int_{t \in A} MA_{x,t}(\Phi), \tag{8.11}$$

just using Leibniz rule and the relation (8.8). Hence, it follows directly from the homogeneous Monge–Ampère equation (8.7) that the following proposition holds:

Proposition 8.2 *The following properties of the functional $\mathcal{E}$ hold:*

- *If the metric $\Phi(z,t)$ on $\pi^* L \to X \times A$ has semi-positive curvature, then $t \mapsto \mathcal{E}(\Phi(.,t))$ is subharmonic with respect to t.*
- *If ϕ_t is a geodesic in $\mathcal{H}_\infty$, Φ the induced function on $\overline{M}$, then $\mathcal{E}(\Phi(.,t))$ is affine with respect to t real.*

8.2.3 Quantization Scheme

Let $H^0(kL) = H^0(X, L^{\otimes k})$ be the space of all global holomorphic sections of $kL := L^{\otimes k}$ over X. We will denote by N_k the dimension of this complex vector space of finite dimension (since M is compact). Let $\mathcal{H}_k$ be the set of all Hermitian metrics H on the vector space $H^0(kL)$, the *Bergman space at level* k. The map $A \mapsto A^* A$ clearly yields an isomorphism

$$GL(N_k, \mathbb{C})/U(N_k) \simeq \mathcal{H}_k \tag{8.12}$$

turning $\mathcal{H}_k$ into a symmetric space. We can define two natural maps:

- ("quantization"). The 'Hilbert' map

$$Hilb_k : \mathcal{H}_\infty \to \mathcal{H}_k$$

  such that

$$Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|^2_{h_0^k} e^{-k\phi} MA(\phi).$$

- ("dequantization"). The injective map 'Fubini–Study',

$$FS_k : \mathcal{H}_k \to \mathcal{H}_\infty$$

  such that

$$FS_k(H) = \frac{1}{k} \log \left(\sum_{i=1}^{N_k} |s_i^H|^2_{h_0^k} \right),$$

  where (s_i^H) is an H-orthonormal basis[1] of holomorphic sections of $H^0(kL)$.

[1] Note that this definition is independent of the choice of the basis.

We will often identify $\mathcal{H}_k$ with its image in $\mathcal{H}_\infty$ under FS_k and call it the space of *Bergman type metrics* of order k. Geometrically, $FS_k(H)$ is just the scaled pull-back of the Fubini–Study metric on $\mathcal{O}(1) \to \mathbb{P}H^0(kL)$, induced by H under the Kodaira embedding $x \mapsto [s_1^H(x) : ... : s_{N_k}^H(x)]$. More invariantly, this is the natural "evaluation map" $x \mapsto \mathbb{P}H^0(kL)^\vee$ composed with the isomorphism between $\mathbb{P}H^0(kL)^*$ and $\mathbb{P}H^0(kL)$ determined by H.

Next, we recall the following fundamental approximation result first proved by Tian [Tian90], Bouche [Bou90] (see [Ru98] for the issue of smooth convergence).

Proposition 8.3 *Let ϕ be an element in $\mathcal{H}_\infty$, i.e. a smooth metric on L with positive curvature. When $k \to \infty$ the composed maps $P_k := FS_k \circ Hilb_k$ approximate the identity. More precisely,*

$$FS_k \circ Hilb_k(k\phi) \to \phi \tag{8.13}$$

in the $\mathcal{C}^\infty$-topology.

In fact, if ϕ is only assumed continuous then a simply approximation arguments gives $\mathcal{C}^0$-convergence. The previous proposition is a direct consequence of the leading asymptotics of $\rho(k\phi)(x)$ since

$$\rho(k\phi)(x) := \sum_{i=1}^{N_k} |s_i^{Hilb_k(k\phi)}(x)|^2_{h_0^k} e^{-k\phi(x)} = e^{kFS_k(Hilb_k(k\phi))(x)} e^{-k\phi(x)}$$

so that taking log, dividing by k and letting $k \to \infty$ proves the proposition.

8.2.4 Geodesics in the Bergman Spaces

Let us fix k and let $H_0, H_1 \in \mathcal{H}_k$ be two Bergman metrics. By standard linear algebra there exist numbers λ_i with $1 \leq i \leq N_k$ and bases $(s_i^{H_0})$ and $(s_i^{H_1})$, orthonormal with respect to H_0 and H_1 respectively, such that

$$s_i^{H_1} = s_i^{H_0} e^{\lambda_i/2}$$

The geodesic H_t in $\mathcal{H}_k$ (with respect to the Riemann structure induced by the isomorphism (8.12)) between H_0 and H_1 may then be concretely obtained in the following way: H_t is the Hermitian metric such that

$$s_i^{H_t} = s_i^{H_0} e^{t\lambda_i/2}$$

is H_t-orthonormal.

We are now ready to state Phong and Sturm's result [PS05, PS06].

Theorem 8.4 (Phong–Sturm, 2005) *Let ϕ_t be the unique $C^{1,1}$ geodesic from ϕ_0 to ϕ_1 in $\overline{\mathcal{H}_\infty}$. Let $H_t^{(k)}$ be a Bergman geodesic curve in $\mathcal{H}_k$ such that $H_0^{(k)} = Hillb_k(k\phi_0)$ and $H_1^{(k)} = Hilb_k(k\phi_1)$. Then $\Phi^{(k)} := FS_k(H_t^{(k)})$, identified with a metric over $X \times A$, satisfies*

$$(\sup_{k \geq l} \Phi^{(k)})^{*usc} \to \Phi$$

uniformly over $X \times A$ as $l \to \infty$., with Φ a solution of the homogeneous complex Monge–Ampère equation (8.7).

Here, one has defined the upper-envelope of a bounded function $u : X \times [0,1] \to \mathbb{R}$, by setting

$$u^{*usc}(z) = \lim_{\epsilon \to 0} \sup_{|z'-z|<\epsilon} u(z').$$

Recall that a sequence of plurisubharmonic functions u_k which are locally uniformly bounded, $(\sup u_k)^{*usc}$ is still plurisubharmonic and equal to $\sup u_k$ almost everywhere. The proof of Phong–Sturm uses the result established by Chen [Che00] concerning the existence and regularity of the geodesic Φ in the (closure of) the space of Kähler potentials $\mathcal{H}_\infty$. More precisely, it is the $\mathcal{C}^0$-regularity of Φ which is needed (this fact immediately gives uniform convergence in the theorem above, by Dini's lemma). As recently observed in [BD09] this latter regularity can also be obtained by adapting the approach of Bedford–Taylor for pseudoconvex domains in $\mathbb{C}^n$ to the present situation.

8.2.4.1 The Proof of Theorem 8.4

We keep the notation $\Phi^{(k)} := FS_k(H^{(k)})$ for the metric over $X \times A$ induced by rotational symmetry from $FS_k(H_t^{(k)})$ on X. The two main ingredients in the proof of Phong and Sturm are as follows. Firstly the following uniform estimate in k on $X \times A$:

$$\left| \frac{\partial}{\partial t} \Phi^{(k)} \right| \leq C. \tag{8.14}$$

Secondly, the "volume estimate"

$$MA_{x,t}(\Phi^{(k)}) \to 0, \tag{8.15}$$

weakly on the interior of $X \times A$. The estimate (8.14) is used to control the boundary behaviour of $\Phi^{(k)}$ (by Proposition 8.3, convergence towards Φ at the boundary is already clear). Moreover, from the convergence (8.15), any limit point of $\Phi^{(k)}$ satisfies the homogeneous Monge–Ampère equation in the interior of $X \times A$. By adapting the pluripotential results of Bedford–Taylor for pseudoconvex domains in $\mathbb{C}^n$ to their situation, Phong–Sturm finally conclude the proof of Theorem 8.4. Finally, the last difficulty is to establish a suitable uniqueness result for "rough" solutions of the Dirichlet problem (8.7).

The proof of (8.14) uses the explicit formula

$$\frac{\partial}{\partial t}\Phi^{(k)} = \sum_{j=1}^{N_k} \lambda_j |s_J^{H_t^{(k)}}|^2_{h_0^k} e^{-FS_k(H_t^{(k)})}$$

to reduce the estimate to a uniform bound on $\frac{2}{k} \max |\lambda_j|$ (see [PS06, Lemma 1]). Note that the *upper* bound in (8.14) (without the absolute values) is a direct consequence of the convexity of the map $t \mapsto \Phi^{(k)}$ for t real, combined with the uniform bounds at the end points $t = 0, 1$ furnished by Proposition 8.3.

Now, the estimate (8.15) can be proved by first noting that, by the second point in Proposition 8.2, it is a consequence of the fact that

$$\int_{X\times A} MA_{x,t}(\Phi^{(k)}) \to 0 \tag{8.16}$$

when $k \to \infty$. But one has by (8.10) and (8.11),

$$\int_{X\times A} MA_{x,t}(\Phi^{(k)}) = \int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA((\Phi^{(k)}_{t=1})) - \int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=0} MA(\Phi^{(k)}_{t=0}). \tag{8.17}$$

If we let $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$, we can write

$$\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA(FS_k(\Phi^{(k)}_{t=1})) = \frac{1}{k}\int_X \sum_{j=1}^{N_k} \lambda_j |s_j^{H_1^{(k)}}|^2_{h_0^k} e^{-FS_k(H_1^{(k)})} MA(FS_k(H_1^{(k)}))$$
$$= \frac{1}{k}\int_X \sum_{j=1}^{N_k} \lambda_j |s_j^{H_1^{(k)}}|^2_{h_0^k} e^{-k\phi_1} \frac{e^{-kFS_k(H_1^{(k)})}}{e^{-\phi_1}} \frac{MA(FS_k(H_1^{(k)}))}{\omega^n_{\phi_1}} \omega^n_{\phi_1}.$$

But for $i = 0, 1$, Proposition 8.3 gives

$$\frac{MA_x(FS_k(H_i^{(k)}))}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_i)^n} = 1 + O(1/k), \qquad \frac{e^{-FS_k(H_i^{(k)})}}{e^{-\phi_i}} = 1 + O(1/k).$$

Combining this with the bound (8.14) gives

$$\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA(FS_k(\Phi^{(k)}_{t=1})) = \frac{1}{k}\sum_{j=1}^{N_k}\int_X \lambda_j |s_j^{Hilb_k(e^{-k\phi_1})}|^2_{h_0^k} e^{-k\phi_1}\omega^n_{\phi_1} + O\left(\frac{1}{k}\right)$$
$$= \frac{1}{k}\sum_{j=1}^{N_k}\lambda_j + O\left(\frac{1}{k}\right).$$

Repeating the argument for $t = 0$ also gives

$$\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=0} MA(FS_k(\Phi^{(k)}_{t=1})) = \frac{1}{k}\sum_{j=1}^{N_k}\lambda_j + O\left(\frac{1}{k}\right).$$

All in all this proves (8.16) and hence finishes the proof of (8.15).

8.3 The Results of B. Berndtsson

In [Bern09b], B. Berndtsson develops a different approach that we discuss now. He considers not the spaces $H^0(kL)$ but instead the spaces $H^0(kL + K_X)$ of all holomorphic n-forms with values in kL. We now redefine $N_k = \dim H^0(X, kL + K_X)$ and

$$\mathcal{H}_k = \{\text{smooth hermitian metrics on } H^0(kL + K_X)\},$$

and also

$$Hilb_k : \mathcal{H}_\infty \to \mathcal{H}_k$$

by

$$Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|^2_{h_0^k} e^{-k\phi} dz \wedge d\bar{z}.$$

A technical difficulty is to redefine the Fubini–Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\infty$ in this setting. We are lead to tensorize $\frac{1}{k}\log(\sum |S_i^H|^2)$, seen as a metric over $L + \frac{1}{k}K_X$, by a metric from $-\frac{1}{k}K_X$. This later metric is fixed a priori, and we will hence be able to control it uniformly when $k \to +\infty$. Finally, we continue to denote $H^{(k)}(t) = H_t^{(k)}$ the Bergman geodesic in $\mathcal{H}_k$ between $H_0^{(k)}$ and $H_1^{(k)}$ defined as in the previous section.
As it turns out the introduction of the canonical line bundle in the Bergman space will considerably simplify the estimates. The reason is that the corresponding L^2-estimates for the $\bar{\partial}$-equation of Hörmander and Kodaira are sharp in this setting.

Theorem 8.5 (Berndtsson, 2006) *Given two metrics $\phi_0, \phi_1 \in \mathcal{H}_\infty$, there exists $\Phi = \phi_t \in C^0(X \times A)$ such that $FS_k(H^{(k)}) \to \Phi$ in the $C^0(X \times A)$ topology. More precisely,*

$$\sup_{X \times A} |FS_k(H^{(k)}) - \Phi| \leq C \log k/k.$$

Moreover,

$$\Phi = \sup_{\Psi}\{\Psi : \Psi \leq \Phi \text{ on } \partial(X \times A)\}, \tag{8.18}$$

where the sup is taken over all strictly positively curved smooth metrics Ψ on the pulled back line bundle $\pi^* L \to X \times A$, and Φ is solution of the homogeneous complex Monge–Ampère equation (8.7).

As very recently shown by Berndtsson in [Bern09c] a simple modification of the proof of Theorem 8.5 shows that it is more generally valid for $kL + L'$ where L' is any Hermitian holomorphic line bundle equipped with a smooth metric (possibly depending on ϕ). In particular, it applies to the setting of Phong–Sturm. For simplicity we will mainly stick to the case of $kL + K_X$.

Remark 8.6 *The relation to continuous geodesics or more precisely continuous solutions of the Dirichlet problem (8.7) is not explicitly discussed in [Bern09b], but is essentially well-known for any manifold M with boundary (here $M := X \times A$). Indeed, by using a family version of Richberg's classical approximation result, the sup defining Φ may be taken over all Ψ which are only assumed to be continuous (and with semi-positive curvature current). Then, by solving* local *Dirichlet problems on any small ball in the interior of M, one sees that $MA(\Phi) = 0$ in the interior of M, by following Bedford–Taylor. Conversely, any* continuous *solution of the global Dirichlet problem on M is necessarily* maximal *in M by the maximum principle for the Monge–Ampère operator:*

$$\Phi \geq \Psi \text{ on } \partial M \ \Rightarrow \Phi \geq \Psi \text{ on } M$$

if Ψ is continuous with semi-positive curvature (this part is elementary and proved exactly as in the $\mathbb{C}^n$-setting, see for example Lemma 3.7.2 in [Klimbook]). In particular, the solution is unique *and of the form (8.18). It should be pointed out that, as opposed to Phong–Sturm's proof, the proof of Berndtsson does not rely on any regularity results concerning the solution of the Dirichlet problem (8.7). In fact, it gives a new interesting "constructive" proof of the C^0-regularity in this setting.*

The key point in the proof of Berndtsson's theorem is the following "quantized maximum principle" that we shall explain in Sect. 8.3.1.

Proposition 8.7 (Quantized maximum principle) *If ψ_t is smooth and plurisubharmonic on $X \times A$ and if $Hilb_k(k\psi_t) \geq H_t^{(k)}$ for $t \in \partial A$ with $H_t^{(k)}$*

geodesic in $\mathcal{H}_k$, then

$$Hilb_k(k\psi_t) \geq H_t^{(k)}$$

for all $t \in A$.

We explain now how Proposition 8.7 implies the convergence of the sequence of Bergman geodesics. We set $\phi_t^{(k)} = FS_k(H_t^{(k)})$ which is positively curved over $X \times A$, as follows immediately from its explicit expression. Let us show that the following two inequalities hold:

$$\phi_t^{(k)} \leq \phi_t + O(\log k/k), \tag{8.19}$$

$$\phi_t \leq \phi_t^{(k)} + O(1/k). \tag{8.20}$$

For (8.19), we notice that on $\partial(X \times A)$, Proposition 8.3 gives

$$\phi_t^{(k)} := FS_k(H_t^{(k)}) = FS_k(Hilb_k(k\phi_i)) \leq \phi_i + O(\log k/k)$$

with $i = 0$ or $i = 1$. Now, from the extremal definition of ϕ_t, we get (8.19) on all of $X \times A$, using that $FS_k(H_t^{(k)})$ has semi-positive curvature as a metric over $X \times A$. Note that the upper bound $O(\log k/k)$ is actually sharp.

For (8.20), we will now use Proposition 8.7. On $\partial(X \times A)$, since $Hilb_k(k\phi(.,t)) = H_t^{(k)}$, and because any given candidate Ψ for the sup defining Φ is positively curved over $X \times A$, we obtain on all of $X \times A$,

$$Hilb_k(k\Psi) \geq H_t^{(k)}.$$

which implies

$$FS_k(Hilb_k(k\Psi)) \leq \phi_t^{(k)}.$$

Finally, we obtain (8.20) if we can prove that for any smooth metric ψ on L with semi-positive curvature form

$$\psi \leq FS_k(k\psi) + \frac{c}{k} \tag{8.21}$$

with a uniform constant c independent of ψ. This estimate is a well-known consequence of the celebrated Ohsawa–Takegoshi theorem. We briefly recall a weak version of this latter result in the following

Proposition 8.8 (Ohsawa–Takegoshi) *Let $L \to X$ be an ample line bundle, ψ a smooth metric on L such that $\omega_\psi \geq 0$ and fix $x \in X$. Then, for any $k \gg 0$, there exists a holomorphic section $s_k \in H^0(kL + K_X)$ such that $|s_k(x)|^2_{k\psi} = 1$ and $\int_X |s_k|^2_{k\psi}\omega^n \leq C$ with C independent of ψ, x and s_k.*

The estimate (8.21) comes by using that

$$FS_k(Hilb_k(k\psi))(x) - \psi \geq \frac{1}{k}\log\left(\frac{|s_k(x)|^2_{k\psi}}{\int_X |s_k|^2_{k\psi}\omega^n}\right)$$

for any section s_k and in particular for the one, depending on x, furnished by the proposition.

8.3.1 Curvature of Direct Image Bundles

We will next turn to the proof of the crucial "quantized" maximum principle of Berndtsson. The starting point of its proof is the following geometric description of a geodesic $H_t^{(k)}$ in $\mathcal{H}_k$. It may be suggestively described in the "quantization" terminology. A curve $\phi_t \in \mathcal{H}_\infty$ gives rise to a family (X, ω_{ϕ_t}) of Kähler manifolds fibred over $[0,1]_t$. Its quantization is hence a family of Hermitian vector spaces $(H^0(kL + K_X), Hilb_k(k\phi_t))$ fibred over $[0,1]_t$. This is equivalent to say that, by complexifying the parameter t so that it lives in the disk annulus A of $\mathbb{C}$, we arrive at the following suggestive statement: *the quantization of a curve ϕ_t in $\mathcal{H}_\infty$ is a holomorphic Hermitian vector bundle $(E, \mathbf{H})$ over A, which is holomorphically isomorphic to $H^0(kL + K_X) \times A$.*

Similarly, any curve $H_t \in \mathcal{H}_k$ gives rise to a vector bundle $(E, \mathbf{H})$ over A. As observed by Berndtsson H_t is a geodesic in $\mathcal{H}_k$ precisely when the curvature $\Theta_E(\mathbf{H})$ of the vector bundle vanishes,

$$\Theta_E(\mathbf{H}) = 0 \in End(E).$$

Recall the following general definition of curvature [GH78]: If H_t is a family of Hermitian matrices locally representing an Hermitian metric on a holomorphic vector bundle $E \to A$, then the (Chern) connection form $\theta(H_t)$ is the following local matrix valued $(0,1)$-form on A :

$$\theta(H_t) = -H_t^{-1}\partial_t H.$$

Its curvature (which in our convention is "real") is the following local matrix valued real $(1,1)$-form on A :

$$\Theta_E(\mathbf{H}) = \Theta_E(H_t) = \sqrt{-1}\bar{\partial}_t\theta(H_t) = -\sqrt{-1}\bar{\partial}_t(H_t^{-1}\partial_t H)$$

defining a global $(1,1)$-form on A with values in $End(E)$. In our case the base A is one-dimensional and E is holomorphically trivial with fiber $H^0(kL + K_X)$. Hence we may and will identify $\Theta_E(H_t)$ with an Hermitian operator on $H^0(kL + K_X)$ for any $t \in A$.

We next state a fundamental result of B. Berndtsson [Bern09b] about the curvature of the vector bundle obtained by quantizing a curve ϕ_t in $\mathcal{H}_\infty$.

Theorem 8.9 *If Ψ is a metric on $\pi^*L \to X \times A$ such that $\psi_t := \Psi_t(\cdot) \in \mathcal{H}_\infty$, then*

$$\langle \sqrt{-1}\Theta_E\left(\mathbf{Hilb_k}(k\psi_t)\right)s, s\rangle \quad \geq \quad k\langle c(\psi_t)s, s\rangle$$

in terms of the geodesic curvature $c(\psi_t)$ of ψ_t (compare with formula (8.8)) and where the inner product is with respect to $\mathbf{Hilb_k}(k\psi_t)$. In particular, if Ψ is positively curved over $X \times A$ then

$$\sqrt{-1}\Theta_E(\mathbf{Hilb_k}(k\psi_t)) \geq 0$$

as an Hermitian form.

The Proof of Theorem 8.9

The proof of Berndtsson theorem[2] takes advantage of the fact that the hermitian holomorphic vector bundle E over A is a subbundle of the (infinite dimensional) hermitian holomorphic vector bundle F, where $F_t := \mathcal{C}^\infty(X, kL + K_X)$ whose fiber at t consist of all *smooth* sections. If one endows F with the holomorphic structure defined by the operator $\bar{\partial}_t$ (the $\bar{\partial}$-operator along the base) then E clearly becomes an Hermitian *holomorphic* subbundle of F. A simple calculation gives the connection "matrix" (i.e. a linear operator)

$$(\theta_F)_t := -\frac{\partial \Psi}{\partial t} = -u_t\cdot, \qquad u_t := \dot{\psi}_t$$

Moreover, well-known formulas for induced connections on holomorphic subbundles (see p. 78 in [GH78]) give

$$(\theta_F)_t = P_t(\theta_E) = -T^{(k)}_{u_t},$$

where P_t is the orthogonal projection $F_t \to E_t$, using the notation of Toeplitz operators in the introduction. Moreover, the curvature on the subbundle E may be expressed as (see [GH78])

$$\Theta_E = P\Theta_F - B(u)^*B(u) =_{|t} P_t(\dot{u}_t\cdot) - B(u_t)^*B(u_t) \tag{8.22}$$

where $B(u)$ is the linear operator ("second fundamental form") $B(u_t)s = u_ts - P_t(u_ts)$. We next formulate the key estimate of Berndtsson in the following result.

Lemma 8.10 *The following inequality holds for any smooth function u and smooth metric ψ on L with positive curvature:*

---

[2] And in particular the proof of the more general curvature estimate in [Bern09b].

$$\langle B(u)^*B(u)s,s\rangle_{k\psi} \leq \left\langle |\bar{\partial}_x u|^2_{\omega_\psi} s,s\right\rangle_{k\psi}$$

for all $s \in H^0(X, kL + K_X)$.

Proof. First note that

$$\langle B^*Bs,s\rangle_{k\psi} = \langle Bs,Bs\rangle_{k\psi} := \|us - P(us)\|^2_{k\psi}.$$

But since P is the orthogonal projection onto the kernel of $\bar{\partial}_x$, the smooth section $v := us - P(us)$ of $kL + K_X$ is the element with minimal norm of the following inhomogeneous $\bar{\partial}$-equation

$$\bar{\partial}_x v = \bar{\partial}_x(us).$$

Hence, by the L^2-estimates of Hörmander–Kodaira

$$\|v\|^2_{k\psi} \leq \left\|\bar{\partial}_x(su)\right\|^2_{k\psi,\omega_\psi}$$

where $\bar{\partial}_x(su)$ is an $(n,1)$-form with values in kL and where ω_ψ is used to measure the $(0,1)$-part in the usual way. □

To conclude the proof of Theorem 8.9 recall that $u_t = \dot{\psi}_t$. Hence formula (8.22) combined with the previous lemma proves the first statement of Theorem since $c(\psi_t) = \ddot{\psi}_t - |\bar{\partial}_x\dot{\psi}_t|^2_{\omega_\psi}$.

The proof of Proposition 8.7, is now a direct consequence of Theorem 8.9 and the following lemma (compare with [CS93]).

Lemma 8.11 *Let E be a holomorphic vector bundle on a smooth domain $\overline{D}$ in $\mathbb{C}$ and let H_0, H_1 be two hermitian metrics on E that extend continuously to $\overline{D}$. Assume that the curvature of H_0 is flat and the curvature of H_1 is semi-positive. If, $H_0 \leq H_1$ on ∂D, then $H_0 \leq H_1$ in D.*

Proof. For the sake of completeness we give a simple proof of the lemma, which was explained to us by Bo Berndtsson. First note that we may assume that the metric H_1 has *strictly* positive curvature (since otherwise we can just approximate H_1 with such metrics and then pass to the limit in the resulting inequality). To simplify the notation we further assume that E is a holomorphically trivial vector bundle and identify the total space of E with the set of all pairs (t,w) in $D \times \mathbb{C}^N$ (the general argument is essentially the same). On the latter space (minus the zero-section) we consider the following function:

$$f(z,w) := \log\left(\frac{|w|^2_{H_0(t)}}{|w|^2_{H_1(t)}}\right).$$

Since f is 0-homogenous in w there is a point (w_0,t_0) where its maximum is attained. To conclude the proof of the lemma it is hence enough to prove that the point (w_0, t_0) is such that t_0 is in the boundary of D. Assume for a contradiction that this is not the case. Then t_0 is an interior point and hence

$$\frac{\partial^2}{\partial t \partial \bar{t}} f(t, w_t) \leq 0$$

for any local holomorphic curve $t \mapsto (t, w_t)$ passing through (t_0, w_0). Identifying any such curve with a local holomorphic section s_t of E we can always find such a curve so that $D^{1,0}_{H_1} s_t = 0$ for $t = t_0$, where $D^{1,0}_H = \frac{\partial}{\partial t} - \theta(H)$ denotes the $(1,0)$-component of the Chern connection determined by H. Showing that

$$\frac{\partial^2}{\partial t \partial \bar{t}} f(t, s_t) > 0, \;\; t = t_0 \tag{8.23}$$

will hence yield the desired contradiction. In order to obtain this inequality, we first note that an application of Leibniz' rule gives the following general formula:

$$\frac{\partial^2}{\partial t \partial \bar{t}} (|s_t|^2_H) = -\sqrt{-1} \left\langle \Theta_H s_t, s_t \right\rangle_H + \left\langle D^{1,0}_H s_t, D^{1,0}_H s_t \right\rangle_H .$$

By assumption this is non-negative when $H = H_0$ for *any* section s_t and a further calculation also reveals that

$$\frac{\partial^2}{\partial t \partial \bar{t}} \left(\log(|s_t|^2_{H_0} \right) \geq 0$$

since $\Theta_{H_0} \leq 0$. Next, since we have assumed that $D^{1,0}_{H_1} s_t = 0$ for $t = t_0$ we get

$$\frac{\partial^2}{\partial t \partial \bar{t}} \left(-\log(|s_t|^2_{H_1} \right) = \frac{\left\langle \Theta_{H_1} s_t, s_t \right\rangle_{H_1}}{\left\langle s_t, s_t \right\rangle_{H_1}} + 0 > 0$$

at $t = t_0$ which finishes the proof of Inequality (8.23) and hence of the lemma. □

8.3.2 Concluding Remarks

8.3.2.1 The Correspondence Principle and Curvature Asymptotics

In the spirit of the "correspondence principle" referred to in the introduction, one may rewrite the lower bound in Theorem 8.9 as the inequality

$$\sqrt{-1} \Theta_E \left(\mathbf{Hilb_k}(k\psi_t) \right) \geq T^{(k)}_{c(\psi_t)} \tag{8.24}$$

between two Hermitian operators on $H^0(X, kL + K_X)$, i.e. *"the quantization of the geodesic curvature of a curve ψ_t is always smaller than the curvature of the quantization of ψ_t!"* (see the discussion in [Bern09]).

In fact, using essentially well-known asymptotic formulas one can show that one obtains an *equality* in (8.24), up to lower order terms in "Planck's constant" $\hbar = 1/k$. To see this first note that a simple computation gives, with notation as in Lemma 8.10, the following expression in terms of Toeplitz operators:

$$-B(u)^*B(u) = (T_u^{(k)})^2 - T_{u^2}^{(k)}.$$

Now we can use the asymptotic expansion from the introduction:

$$T_f^{(k)}T_g^{(k)} - T_{fg}^{(k)} = \frac{1}{k}T_{c_1(f,g)}^{(k)} + O\left(\frac{1}{k^2}\right) \tag{8.25}$$

in operator norm, with the explicit formula

$$c_1(f,g) = \sqrt{-1}(\partial f \wedge \bar{\partial} g \wedge \omega_\phi^{n-1})/\omega_\phi^n. \tag{8.26}$$

(this follows for example from explicit formula for c_1^{BT} in [Eng02]). Setting $f = g = u$ shows that (8.24) is an asymptotic *equality*, i.e.

$$k^{-1}\sqrt{-1}\Theta\left(\mathbf{Hilb_k}(k\psi_t)\right) = T_{c(\psi_t)} + O(1/k) \tag{8.27}$$

in operator norm. In particular, $\sqrt{-1}\Theta(\mathbf{Hilb_k}(k\psi_t)) > 0$ for $k > k_0$ if Ψ is smooth with strictly positive curvature (where k_0 depends on Ψ). Hence, the previous asymptotics could be used as a substitute for the curvature estimate in Theorem 8.9 and in the proof[3] of Theorem 8.5. Since the asymptotics (8.25) are still valid with the same formula for $c_1(f,g)$ when kL is twisted by a Hermitian holomorphic line bundle L' this argument also gives uniform convergence in the setting of Phong–Sturm.

Note also that, combined with the asymptotics (8.4) for spectral measures, (8.27) implies the spectral asymptotics

$$\frac{1}{k^n}\sum_i \delta_{\lambda_i^{(k)}} \to c(\psi_t)_*(\omega_{\psi_t})^n/n!$$

converging in the weak star topology of measures, summing over all eigenvalues of the curvature $\sqrt{-1}\Theta\left(\mathbf{Hilb_k}(k\psi_t)\right)$. It should also be pointed out that asymptotics for curvature operators on very general direct image bundles have been announced by Ma–Zhang in [MZ07] (without explicitly using the relation to Toeplitz operator asymptotics).

[3] At least to get uniform convergence, but without the explicit rate of convergence.

8.3.2.2 Wess–Zumino–Witten Type Equations

It may be of some interest to point of that the proof of Theorem 8.5 extends almost word for word to the case when the geodesic is replaced by a solution to certain Wess–Zumino–Witten type equations (see [Don99] for the relation between geodesics and Wess–Zumino–Witten type equations). To setup the problem first consider the following generalization of the Dirichlet problem (8.7). Let us replace the annulus A with a general smooth domain D in $\mathbb{C}$ and let us assume given a continuous function $\Phi = \phi_t(x)$ on ∂D such that $\phi_t \in H_\infty$ for all $t \in \partial D$. Then there is a unique continuous extension of Φ to D such that

$$MA_{x,t}(\Phi) = 0, \;\; (t,x) \in X \times D \tag{8.28}$$

with $\pi^*\omega_0+\sqrt{-1}\partial\bar{\partial}\Phi \geq 0$ on $X\times D$ [Che00,BD09]. There is also a "quantized" version of this problem where one assumes given a continuous family $\mathbf{H} = H_t \in \mathcal{H}_k$ for $t \in \partial D$. Then there exists a unique continuous extension of $\mathbf{H}$ from ∂D to a *flat* metric on $E \to D$, i.e. such that

$$\Theta_E(\mathbf{H})_t = 0 \in \mathrm{End}(E) \;\; \forall t \in D$$

[CS93, Don92]. When D is the unit-disc this amounts to classical results of Birkhoff, Grothendieck, Wiener and Masni. Note that this latter equation is a Laplace type PDE in t for a matrix H_t, which is quadratic in the first derivatives of H_t. Now the proof of Theorem 8.5 may be repeated essentially word for word, showing that the FS_k-images of the flat extensions of $H_t^{(k)} := Hilb_k(k\phi_t)$ converge in the large k limit uniformly on $X \times D$ to a solution Φ of the Dirichlet problem (8.28) on $X \times D$. The only new ingredient needed is the observation that $\Phi^{(k)} := FS_k(H_t^{(k)})$ still satisfy the condition $\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi^{(k)} \geq 0$ on $X \times D$. To see this, first recall the following general fact. The curvature $\Theta_E(\mathbf{H})_t$ is semi-positive over D precisely when $\log \|s^*\|^2_{H_t^*}$ is plurisubharmonic in (t, s^*) on the total space of the dual bundle $E^* - \{0\}$, where H_t^* denotes the fiber-wise dual Hermitian metric.[4] Moreover, by definition

$$FS_k(H_t)(x) := \log \|\Lambda_x\|^2_{H_t^*\otimes h_0^*} \,,$$

where Λ_x is the holomorphic section of $E^* \otimes L_x^*$ naturally induced by the pointwise evaluation functional $ev_x : H^0(L) \to L_x$. In particular, $FS_k(H_t)(x)$ is $\pi^*\omega_0$-psh on $D \times X$ proving the observation above.

Finally, it seems natural to ask how to approximate solutions to the Dirichlet problem (8.28) when D is a domain in $\mathbb{C}^m$ for $m > 1$? As is it well-known one has also to impose the condition that D is *pseudoconvex* in

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[4]This characterization holds for any dimension of the base D if positivity in the sense of Griffith's is used, cf. [Dem99].

order to get a continuous solution [BD09]. The model case is when D is the unit ball. This leads one to look for a Monge–Ampère type equation for a metric $\mathbf{H}$ on a holomorphic vector bundle $E \to D$ over an m-dimensional base. As it turns out, some of the results above do generalize to this higher dimensional setting. Details will hopefully appear elsewhere in the future.

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LECTURE NOTES IN MATHEMATICS

Edited by J.-M. Morel, B. Teissier; P.K. Maini

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