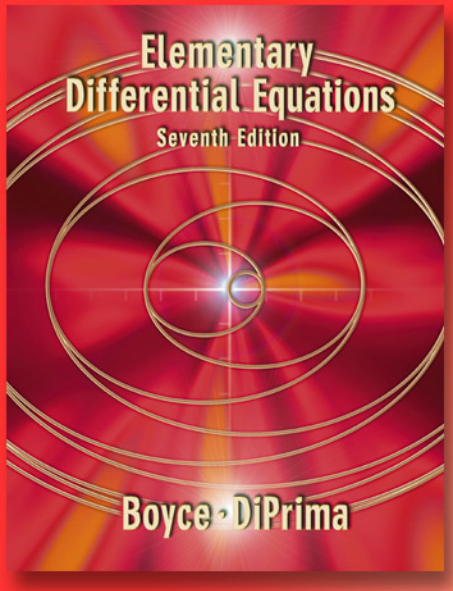
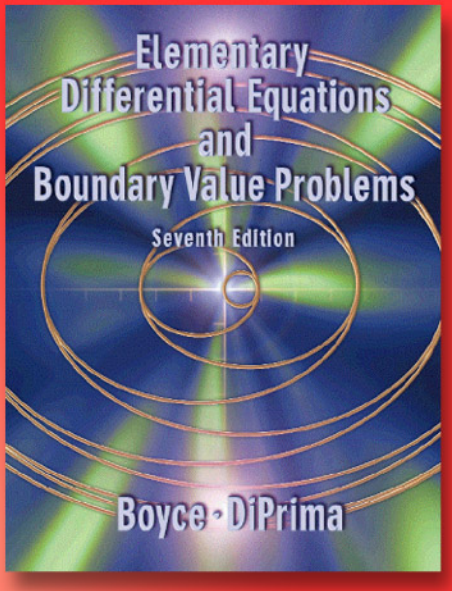






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Elementary Differential Equations and Boundary Value Problems

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Elementary Differential Equations and Boundary Value Problems



SEVENTH EDITION

Elementary Differential Equations and Boundary Value Problems

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To Elsa and Maureen

To Siobhan, James, Richard, Jr., Carolyn, and Ann

And to the next generation:
Charles, Aidan, Stephanie, Veronica, and Deirdre

The Authors

William E. Boyce received his B.A. degree in Mathematics from Rhodes College, and his M.S. and Ph.D. degrees in Mathematics from Carnegie-Mellon University. He is a member of the American Mathematical Society, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. He is currently the Edward P. Hamilton Distinguished Professor Emeritus of Science Education (Department of Mathematical Sciences) at Rensselaer. He is the author of numerous technical papers in boundary value problems and random differential equations and their applications. He is the author of several textbooks including two differential equations texts, and is the coauthor (with M.H. Holmes, J.G. Ecker, and W.L. Siegmann) of a text on using Maple to explore Calculus. He is also coauthor (with R.L. Borrelli and C.S. Coleman) of *Differential Equations Laboratory Workbook* (Wiley 1992), which received the EDUCOM Best Mathematics Curricular Innovation Award in 1993. Professor Boyce was a member of the NSF-sponsored CODEE (Consortium for Ordinary Differential Equations Experiments) that led to the widely-acclaimed *ODE Architect*. He has also been active in curriculum innovation and reform. Among other things, he was the initiator of the "Computers in Calculus" project at Rensselaer, partially supported by the NSF. In 1991 he received the William H. Wiley Distinguished Faculty Award given by Rensselaer.

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PREFACE

This edition, like its predecessors, is written from the viewpoint of the applied mathematician, whose interest in differential equations may be highly theoretical, intensely practical, or somewhere in between. We have sought to combine a sound and accurate (but not abstract) exposition of the elementary theory of differential equations with considerable material on methods of solution, analysis, and approximation that have proved useful in a wide variety of applications.

The book is written primarily for undergraduate students of mathematics, science, or engineering, who typically take a course on differential equations during their first or second year of study. The main prerequisite for reading the book is a working knowledge of calculus, gained from a normal two- or three-semester course sequence or its equivalent.

A Changing Learning Environment

The environment in which instructors teach, and students learn, differential equations has changed enormously in the past few years and continues to evolve at a rapid pace. Computing equipment of some kind, whether a graphing calculator, a notebook computer, or a desktop workstation is available to most students of differential equations. This equipment makes it relatively easy to execute extended numerical calculations, to generate graphical displays of a very high quality, and, in many cases, to carry out complex symbolic manipulations. A high-speed Internet connection offers an enormous range of further possibilities.

The fact that so many students now have these capabilities enables instructors, if they wish, to modify very substantially their presentation of the subject and their expectations of student performance. Not surprisingly, instructors have widely varying opinions as to how a course on differential equations should be taught under these circumstances. Nevertheless, at many colleges and universities courses on differential equations are becoming more visual, more quantitative, more project-oriented, and less formula-centered than in the past.

Mathematical Modeling

The main reason for solving many differential equations is to try to learn something about an underlying physical process that the equation is believed to model. It is basic to the importance of differential equations that even the simplest equations correspond to useful physical models, such as exponential growth and decay, spring-mass systems, or electrical circuits. Gaining an understanding of a complex natural process is usually accomplished by combining or building upon simpler and more basic models. Thus a thorough knowledge of these models, the equations that describe them, and their solutions, is the first and indispensable step toward the solution of more complex and realistic problems.

More difficult problems often require the use of a variety of tools, both analytical and numerical. We believe strongly that pencil and paper methods must be combined with effective use of a computer. Quantitative results and graphs, often produced by a computer, serve to illustrate and clarify conclusions that may be obscured by complicated analytical expressions. On the other hand, the implementation of an efficient numerical procedure typically rests on a good deal of preliminary analysis – to determine the qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover which ranges of the variables or parameters may require or merit special attention.

Thus, a student should come to realize that investigating a difficult problem may well require both analysis and computation; that good judgment may be required to determine which tool is best-suited for a particular task; and that results can often be presented in a variety of forms.

A Flexible Approach

To be widely useful a textbook must be adaptable to a variety of instructional strategies. This implies at least two things. First, instructors should have maximum flexibility to choose both the particular topics that they wish to cover and also the order in which they want to cover them. Second, the book should be useful to students having access to a wide range of technological capability.

With respect to content, we provide this flexibility by making sure that, so far as possible, individual chapters are independent of each other. Thus, after the basic parts of the first three chapters are completed (roughly Sections 1.1 through 1.3, 2.1 through 2.5, and 3.1 through 3.6) the selection of additional topics, and the order and depth in which they are covered, is at the discretion of the instructor. For example, while there is a good deal of material on applications of various kinds, especially in Chapters 2, 3, 9, and 10, most of this material appears in separate sections, so that an instructor can easily choose which applications to include and which to omit. Alternatively, an instructor who wishes to emphasize a systems approach to differential equations can take up Chapter 7 (Linear Systems) and perhaps even Chapter 9 (Nonlinear Autonomous Systems) immediately after Chapter 2. Or, while we present the basic theory of linear equations first in the context of a single second order equation (Chapter 3), many instructors have combined this material with the corresponding treatment of higher order equations (Chapter 4) or of linear systems (Chapter 7). Many other choices and

combinations are also possible and have been used effectively with earlier editions of this book.

With respect to technology, we note repeatedly in the text that computers are extremely useful for investigating differential equations and their solutions, and many of the problems are best approached with computational assistance. Nevertheless, the book is adaptable to courses having various levels of computer involvement, ranging from little or none to intensive. The text is independent of any particular hardware platform or software package. More than 450 problems are marked with the symbol ▶ to indicate that we consider them to be technologically intensive. These problems may call for a plot, or for substantial numerical computation, or for extensive symbolic manipulation, or for some combination of these requirements. Naturally, the designation of a problem as technologically intensive is a somewhat subjective judgment, and the ▶ is intended only as a guide. Many of the marked problems can be solved, at least in part, without computational help, and a computer can be used effectively on many of the unmarked problems.

From a student's point of view, the problems that are assigned as homework and that appear on examinations drive the course. We believe that the most outstanding feature of this book is the number, and above all the variety and range, of the problems that it contains. Many problems are entirely straightforward, but many others are more challenging, and some are fairly open-ended, and can serve as the basis for independent student projects. There are far more problems than any instructor can use in any given course, and this provides instructors with a multitude of possible choices in tailoring their course to meet their own goals and the needs of their students.

One of the choices that an instructor now has to make concerns the role of computing in the course. For instance, many more or less routine problems, such as those requesting the solution of a first or second order initial value problem, are now easy to solve by a computer algebra system. This edition includes quite a few such problems, just as its predecessors did. We do not state in these problems how they should be solved, because we believe that it is up to each instructor to specify whether their students should solve such problems by hand, with computer assistance, or perhaps both ways. Also, there are many problems that call for a graph of the solution. Instructors have the option of specifying whether they want an accurate computer-generated plot or a hand-drawn sketch, or perhaps both.

There are also a great many problems, as well as some examples in the text, that call for conclusions to be drawn about the solution. Sometimes this takes the form of asking for the value of the independent variable at which the solution has a certain property. Other problems ask for the effect of variations in a parameter, or for the determination of a critical value of a parameter at which the solution experiences a substantial change. Such problems are typical of those that arise in the applications of differential equations, and, depending on the goals of the course, an instructor has the option of assigning few or many of these problems.

Supplementary Materials

Three software packages that are widely used in differential equations courses are *Maple*, *Mathematica*, and *Matlab*. The books *Differential Equations with Maple*, *Differential Equations with Mathematica*, and *Differential Equations with Matlab* by K. R.

Coombes, B. R. Hunt, R. L. Lipsman, J. E. Osborn, and G. J. Stuck, all at the University of Maryland, provide detailed instructions and examples on the use of these software packages for the investigation and analysis of standard topics in the course.

For the first time, this text is available in an *Interactive Edition*, featuring an eBook version of the text linked to the award-winning *ODE Architect*. The interactive eBook links live elements in each chapter to *ODE Architect*'s powerful, yet easy-to-use, numerical differential equations solver and multimedia modules. The eBook provides a highly interactive environment in which students can construct and explore mathematical models using differential equations to investigate both real-world and hypothetical situations. A companion e-workbook that contains additional problems sets, called Explorations, provides background and opportunities for students to extend the ideas contained in each module. A stand-alone version of *ODE Architect* is also available.

There is a Student Solutions Manual, by Charles W. Haines of Rochester Institute of Technology, that contains detailed solutions to many of the problems in the book. A complete set of solutions, prepared by Josef Torok of Rochester Institute of Technology, is available to instructors via the Wiley website at **www.wiley.com/college/Boyce**.

Important Changes in the Seventh Edition

Readers who are familiar with the preceding edition will notice a number of modifications, although the general structure remains much the same. The revisions are designed to make the book more readable by students and more usable in a modern basic course on differential equations. Some changes have to do with content; for example, mathematical modeling, the ideas of stability and instability, and numerical approximations via Euler's method appear much earlier now than in previous editions. Other modifications are primarily organizational in nature. Most of the changes include new examples to illustrate the underlying ideas.

1. The first two sections of Chapter 1 are new and include an immediate introduction to some problems that lead to differential equations and their solutions. These sections also give an early glimpse of mathematical modeling, of direction fields, and of the basic ideas of stability and instability.

2. Chapter 2 now includes a new Section 2.7 on Euler's method of numerical approximation. Another change is that most of the material on applications has been consolidated into a single section. Finally, the separate section on first order homogeneous equations has been eliminated and this material has been placed in the problem set on separable equations instead.

3. Section 4.3 on the method of undetermined coefficients for higher order equations has been simplified by using examples rather than giving a general discussion of the method.

4. The discussion of eigenvalues and eigenvectors in Section 7.3 has been shortened by removing the material relating to diagonalization of matrices and to the possible shortage of eigenvectors when an eigenvalue is repeated. This material now appears in later sections of the same chapter where the information is actually used. Sections 7.7 and 7.8 have been modified to give somewhat greater emphasis to fundamental matrices and somewhat less to problems involving repeated eigenvalues.

5. An example illustrating the instabilities that can be encountered when dealing with stiff equations has been added to Section 8.5.

6. Section 9.2 has been streamlined by considerably shortening the discussion of autonomous systems in general and including instead two examples in which trajectories can be found by integrating a single first order equation.

7. There is a new section 10.1 on two-point boundary value problems for ordinary differential equations. This material can then be called on as the method of separation of variables is developed for partial differential equations. There are also some new three-dimensional plots of solutions of the heat conduction equation and of the wave equation.

As the subject matter of differential equations continues to grow, as new technologies become commonplace, as old areas of application are expanded, and as new ones appear on the horizon, the content and viewpoint of courses and their textbooks must also evolve. This is the spirit we have sought to express in this book.

William E. Boyce
Troy, New York
April, 2000

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It is a pleasure to offer my grateful appreciation to the many people who have generously assisted in various ways in the creation of this book.

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William E. Boyce

CONTENTS

Preface **vii**

Chapter 1 Introduction 1

1.1 Some Basic Mathematical Models; Direction Fields **1**
1.2 Solutions of Some Differential Equations **9**
1.3 Classification of Differential Equations **17**
1.4 Historical Remarks **23**

Chapter 2 First Order Differential Equations 29

2.1 Linear Equations with Variable Coefficients **29**
2.2 Separable Equations **40**
2.3 Modeling with First Order Equations **47**
2.4 Differences Between Linear and Nonlinear Equations **64**
2.5 Autonomous Equations and Population Dynamics **74**
2.6 Exact Equations and Integrating Factors **89**
2.7 Numerical Approximations: Euler's Method **96**
2.8 The Existence and Uniqueness Theorem **105**
2.9 First Order Difference Equations **115**

Chapter 3 Second Order Linear Equations 129

3.1 Homogeneous Equations with Constant Coefficients **129**
3.2 Fundamental Solutions of Linear Homogeneous Equations **137**
3.3 Linear Independence and the Wronskian **147**
3.4 Complex Roots of the Characteristic Equation **153**
3.5 Repeated Roots; Reduction of Order **160**
3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients **169**
3.7 Variation of Parameters **179**
3.8 Mechanical and Electrical Vibrations **186**
3.9 Forced Vibrations **200**

Chapter 4 Higher Order Linear Equations 209

4.1 General Theory of *n*th Order Linear Equations **209**
4.2 Homogeneous Equations with Constant Coeffients **214**

4.3 The Method of Undetermined Coefficients **222**
4.4 The Method of Variation of Parameters **226**

Chapter 5 Series Solutions of Second Order Linear Equations 231

5.1 Review of Power Series **231**
5.2 Series Solutions near an Ordinary Point, Part I **238**
5.3 Series Solutions near an Ordinary Point, Part II **249**
5.4 Regular Singular Points **255**
5.5 Euler Equations **260**
5.6 Series Solutions near a Regular Singular Point, Part I **267**
5.7 Series Solutions near a Regular Singular Point, Part II **272**
5.8 Bessel's Equation **280**

Chapter 6 The Laplace Transform 293

6.1 Definition of the Laplace Transform **293**
6.2 Solution of Initial Value Problems **299**
6.3 Step Functions **310**
6.4 Differential Equations with Discontinuous Forcing Functions **317**
6.5 Impulse Functions **324**
6.6 The Convolution Integral **330**

Chapter 7 Systems of First Order Linear Equations 339

7.1 Introduction **339**
7.2 Review of Matrices **348**
7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors **357**
7.4 Basic Theory of Systems of First Order Linear Equations **368**
7.5 Homogeneous Linear Systems with Constant Coefficients **373**
7.6 Complex Eigenvalues **384**
7.7 Fundamental Matrices **393**
7.8 Repeated Eigenvalues **401**
7.9 Nonhomogeneous Linear Systems **411**

Chapter 8 Numerical Methods 419

8.1 The Euler or Tangent Line Method **419**
8.2 Improvements on the Euler Method **430**

8.3 The Runge–Kutta Method **435**
8.4 Multistep Methods **439**
8.5 More on Errors; Stability **445**
8.6 Systems of First Order Equations **455**

Chapter 9 Nonlinear Differential Equations and Stability 459

9.1 The Phase Plane; Linear Systems **459**
9.2 Autonomous Systems and Stability **471**
9.3 Almost Linear Systems **479**
9.4 Competing Species **491**
9.5 Predator–Prey Equations **503**
9.6 Liapunov's Second Method **511**
9.7 Periodic Solutions and Limit Cycles **521**
9.8 Chaos and Strange Attractors; the Lorenz Equations **532**

Chapter 10 Partial Differential Equations and Fourier Series 541

10.1 Two-Point Boundary Valve Problems **541**
10.2 Fourier Series **547**
10.3 The Fourier Convergence Theorem **558**
10.4 Even and Odd Functions **564**
10.5 Separation of Variables; Heat Conduction in a Rod **573**
10.6 Other Heat Conduction Problems **581**
10.7 The Wave Equation; Vibrations of an Elastic String **591**
10.8 Laplace's Equation **604**
Appendix A. Derivation of the Heat Conduction Equation **614**
Appendix B. Derivation of the Wave Equation **617**

Chapter 11 Boundary Value Problems and Sturm–Liouville Theory 621

11.1 The Occurrence of Two Point Boundary Value Problems **621**
11.2 Sturm–Liouville Boundary Value Problems **629**
11.3 Nonhomogeneous Boundary Value Problems **641**
11.4 Singular Sturm–Liouville Problems **656**
11.5 Further Remarks on the Method of Separation of Variables: A Bessel Series Expansion **663**
11.6 Series of Orthogonal Functions: Mean Convergence **669**
Answers to Problems **679**
Index **737**

CHAPTER
1

Introduction

In this chapter we try in several different ways to give perspective to your study of differential equations. First, we use two problems to illustrate some of the basic ideas that we will return to and elaborate upon frequently throughout the remainder of the book. Later, we indicate several ways of classifying equations, in order to provide organizational structure for the book. Finally, we outline some of the major figures and trends in the historical development of the subject. The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. Nevertheless, it remains a dynamic field of inquiry today, with many interesting open questions.

1.1 Some Basic Mathematical Models; Direction Fields



Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it) you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.



Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric

circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1 A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

We begin by introducing letters to represent various quantities of possible interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction, that is, when the object is falling.

The physical law that governs the motion of objects is Newton's second law, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma. \tag{1}$$

In this equation m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second^2, and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite Eq. (1) in the form

$$F = m(dv/dt). \tag{2}$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. In the units we have chosen, g has been determined experimentally to be approximately equal to 9.8 m/sec^2 near the earth's surface. There is also a force due to air resistance, or drag, which is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv, where γ is a constant called the drag coefficient.

In writing an expression for the net force F we need to remember that gravity always acts in the downward (positive) direction, while drag acts in the upward (negative) direction, as shown in Figure 1.1.1. Thus

$$F = mg - \gamma v \tag{3}$$

and Eq. (2) then becomes

$$m\frac{dv}{dt} = mg - \gamma v. \tag{4}$$

Equation (4) is a mathematical model of an object falling in the atmosphere near sea level. Note that the model contains the three constants m, g, and γ. The constants m and γ depend very much on the particular object that is falling, and usually will be different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, the value of g is the same for all objects.

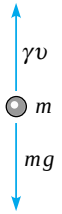


FIGURE 1.1.1 Free-body diagram of the forces on a falling object.

To solve Eq. (4) we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ, but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10$ kg and $\gamma = 2$ kg/sec. If the units for γ seem peculiar, remember that γv must have the units of force, namely, kg-m/sec^2. Then Eq. (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{5}$$

EXAMPLE 2 A Falling Object (continued)

Investigate the behavior of solutions of Eq. (5) without actually finding the solutions in question.

We will proceed by looking at Eq. (5) from a geometrical viewpoint. Suppose that v has a certain value. Then, by evaluating the right side of Eq. (5), we can find the corresponding value of dv/dt. For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments, or arrows, with slope 1.8 at several points on the line $v = 40$. Similarly, if $v = 50$, then $dv/dt = -0.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$. We obtain Figure 1.1.2 by proceeding in the same way with other values of v. Figure 1.1.2 is an example of what is called a **direction field** or sometimes a **slope field**.

The importance of Figure 1.1.2 is that each line segment is a tangent line to the graph of a solution of Eq. (5). Thus, even though we have not found any solutions, and

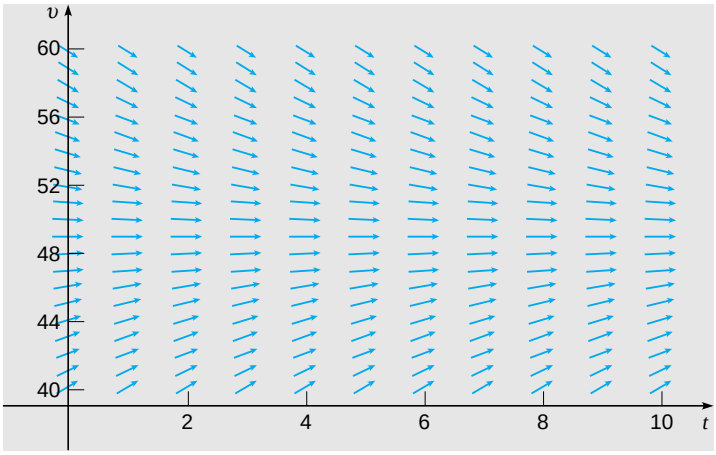


FIGURE 1.1.2 A direction field for Eq. (5).

no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to Eq. (5), we ask what value of v will cause dv/dt to be zero? The answer is $v = (5)(9.8) = 49$ m/sec.

In fact, the constant function $v(t) = 49$ is a solution of Eq. (5). To verify this statement, substitute $v(t) = 49$ into Eq. (5) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a balance between gravity and drag. In Figure 1.1.3 we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases.

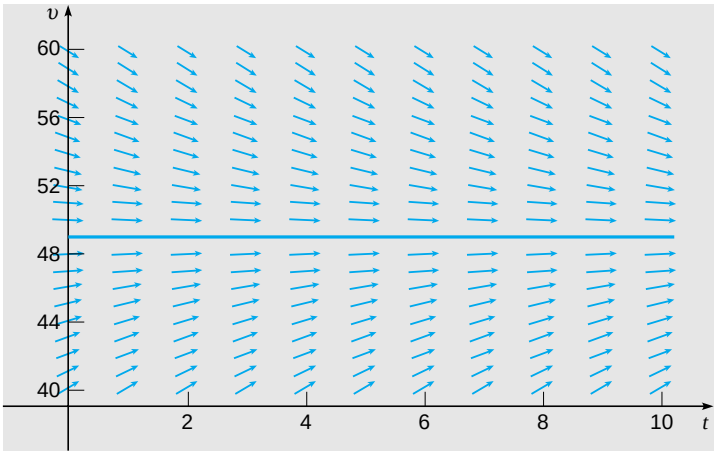


FIGURE 1.1.3 Direction field and equilibrium solution for Eq. (5).

The approach illustrated in Example 2 can be applied equally well to the more general Eq. (4), where the parameters m and γ are unspecified positive numbers. The results are essentially identical to those of Example 2. The equilibrium solution of Eq. (4) is $v(t) = mg/\gamma$. Solutions below the equilibrium solution increase with time, those above it decrease with time, and all other solutions approach the equilibrium solution as t becomes large.

Direction Fields. Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{6}$$

where f is a given function of the two variables t and y, sometimes referred to as the **rate function**. The equation in Example 2 is somewhat simpler in that in it f is a function of the dependent variable alone and not of the independent variable. A useful direction field for equations of the general form (6) can be constructed by evaluating f at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve Eq. (6), but merely evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function is a task for which a computer is well suited and you should usually use a computer to draw a direction field. All the direction fields shown in this book, such as the one in Figure 1.1.2, were computer-generated.

Field Mice and Owls. Now let us look at another quite different example. Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis [1] in a study of population growth. If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{7}$$

where the proportionality factor r is called the **rate constant** or **growth rate**. To be specific, suppose that time is measured in months and that the rate constant r has the value 0.5/month. Then each term in Eq. (7) has the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information

[1] A somewhat better model of population growth is discussed later in Section 2.5.

into the model, we must add another term to the differential equation (7), so that it becomes

$$\frac{dp}{dt} = 0.5p - 450. \tag{8}$$

Observe that the predation term is -450 rather than -15 because time is measured in months and the monthly predation rate is needed.

EXAMPLE 3

Investigate the solutions of Eq. (8) graphically.

A direction field for Eq. (8) is shown in Figure 1.1.4. For sufficiently large values of p it can be seen from the figure, or directly from Eq. (8) itself, that dp/dt is positive, so that solutions increase. On the other hand, for small values of p the opposite is the case. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in Eq. (8) and then solving for p, we find the equilibrium solution $p(t) = 900$ for which the growth term and the predation term in Eq. (8) are exactly balanced. The equilibrium solution is also shown in Figure 1.1.4.

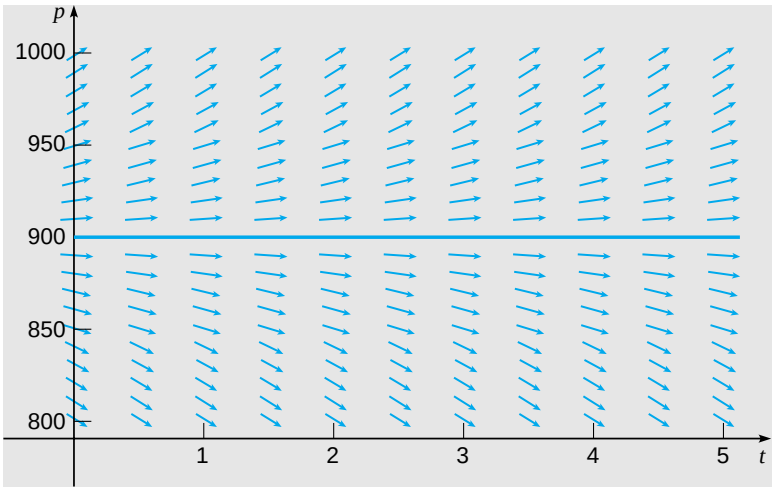


FIGURE 1.1.4 A direction field for Eq. (8).

Comparing this example with Example 2, we note that in both cases the equilibrium solution separates increasing from decreasing solutions. However, in Example 2 other solutions converge to, or are attracted by, the equilibrium solution, while in Example 3 other solutions diverge from, or are repelled by, the equilibrium solution. In both cases the equilibrium solution is very important in understanding how solutions of the given differential equation behave.

A more general version of Eq. (8) is

$$\frac{dp}{dt} = rp - k, \tag{9}$$

where the growth rate r and the predation rate k are unspecified. Solutions of this more general equation behave very much like those of Eq. (8). The equilibrium solution of Eq. (9) is $p(t) = k/r$. Solutions above the equilibrium solution increase, while those below it decrease.

You should keep in mind that both of the models discussed in this section have their limitations. The model (5) of the falling object ceases to be valid as soon as the object hits the ground, or anything else that stops or slows its fall. The population model (8) eventually predicts negative numbers of mice (if $p < 900$) or enormously large numbers (if $p > 900$). Both these predictions are unrealistic, so this model becomes unacceptable after a fairly short time interval.

Constructing Mathematical Models. In applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes, or models, the problem being investigated. In this section we have looked at two examples of this modeling process, one drawn from physics and the other from ecology. In constructing future mathematical models yourself, you should recognize that each problem is different, and that successful modeling is not a skill that can be reduced to the observance of a set of prescribed rules. Indeed, constructing a satisfactory model is sometimes the most difficult part of the problem. Nevertheless, it may be helpful to list some steps that are often part of the process:

1. Identify the independent and dependent variables and assign letters to represent them. The independent variable is often time.
2. Choose the units of measurement for each variable. In a sense the choice of units is arbitrary, but some choices may be much more convenient than others. For example, we chose to measure time in seconds in the falling object problem and in months in the population problem.
3. Articulate the basic principle that underlies or governs the problem you are investigating. This may be a widely recognized physical law, such as Newton's law of motion, or it may be a more speculative assumption that may be based on your own experience or observations. In any case, this step is likely not to be a purely mathematical one, but will require you to be familiar with the field in which the problem lies.
4. Express the principle or law in step 3 in terms of the variables you chose in step 1. This may be easier said than done. It may require the introduction of physical constants or parameters (such as the drag coefficient in Example 1) and the determination of appropriate values for them. Or it may involve the use of auxiliary or intermediate variables that must then be related to the primary variables.
5. Make sure that each term in your equation has the same physical units. If this is not the case, then your equation is wrong and you should seek to repair it. If the units agree, then your equation at least is dimensionally consistent, although it may have other shortcomings that this test does not reveal.
6. In the problems considered here the result of step 4 is a single differential equation, which constitutes the desired mathematical model. Keep in mind, though, that in more complex problems the resulting mathematical model may be much more complicated, perhaps involving a system of several differential equations, for example.

PROBLEMS

In each of Problems 1 through 6 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

1. $y' = 3 - 2y$
2. $y' = 2y - 3$
3. $y' = 3 + 2y$
4. $y' = -1 - 2y$
5. $y' = 1 + 2y$
6. $y' = y + 2$

In each of Problems 7 through 10 write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

7. All solutions approach $y = 3$.
8. All solutions approach $y = 2/3$.
9. All other solutions diverge from $y = 2$.
10. All other solutions diverge from $y = 1/3$.

In each of Problems 11 through 14 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$ and the behavior of their solutions is somewhat more complicated than for the equations in the text.

11. $y' = y(4 - y)$
12. $y' = -y(5 - y)$
13. $y' = y^2$
14. $y' = y(y - 2)^2$

15. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 gram of this chemical per gallon flows into the pond at a rate of 300 gal/min. The mixture flows out at the same rate so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.
    (a) Write a differential equation whose solution is the amount of chemical in the pond at any time.
    (b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?
16. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.
17. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/hr. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (hr)$^{-1}$.
    (a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.
    (b) How much of the drug is present in the bloodstream after a long time?
18. For small, slowly falling objects the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]
    (a) Write a differential equation for the velocity of a falling object of mass m if the drag force is proportional to the square of the velocity.
    (b) Determine the limiting velocity after a long time.
    (c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/sec.
    (d) Using the data in part (c), draw a direction field and compare it with Figure 1.1.3.

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *Amer. Math. Monthly 106*, 2 (1999), pp. 127–135.

In each of Problems 19 through 26 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right sides of these equations depend on t as well as y; therefore their solutions can exhibit more complicated behavior than those in the text.

- 19. $y' = -2 + t - y$
- 20. $y' = te^{-2t} - 2y$
- 21. $y' = e^{-t} + y$
- 22. $y' = t + 2y$
- 23. $y' = 3\sin t + 1 + y$
- 24. $y' = 2t - 1 - y^2$
- 25. $y' = -(2t + y)/2y$
- 26. $y' = y^3/6 - y - t^2/3$

1.2 Solutions of Some Differential Equations



In the preceding section we derived differential equations,

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k, \tag{2}$$

which model a falling object and a population of field mice preyed upon by owls, respectively. Both these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of Eqs. (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

EXAMPLE 1 Field Mice and Owls (continued)

Consider the equation

$$\frac{dp}{dt} = 0.5p - 450, \tag{4}$$

which describes the interaction of certain populations of field mice and owls [see Eq. (8) of Section 1.1]. Find solutions of this equation.

To solve Eq. (4) we need to find functions $p(t)$ that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite Eq. (4) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}, \tag{5}$$

or, if $p \neq 900$,

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{6}$$

Since, by the chain rule, the left side of Eq. (6) is the derivative of $\ln|p-900|$ with respect to t, it follows that

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}. \tag{7}$$

Then, by integrating both sides of Eq. (7), we obtain

$$\ln|p-900| = \frac{t}{2} + C, \tag{8}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of Eq. (8), we find that

$$|p-900| = e^C e^{t/2}, \tag{9}$$

or

$$p - 900 = \pm e^C e^{t/2}, \tag{10}$$

and finally

$$p = 900 + ce^{t/2}, \tag{11}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of Eq. (5) and that it is contained in the expression (11) if we allow c to take the value zero. Graphs of Eq. (11) for several values of c are shown in Figure 1.2.1.

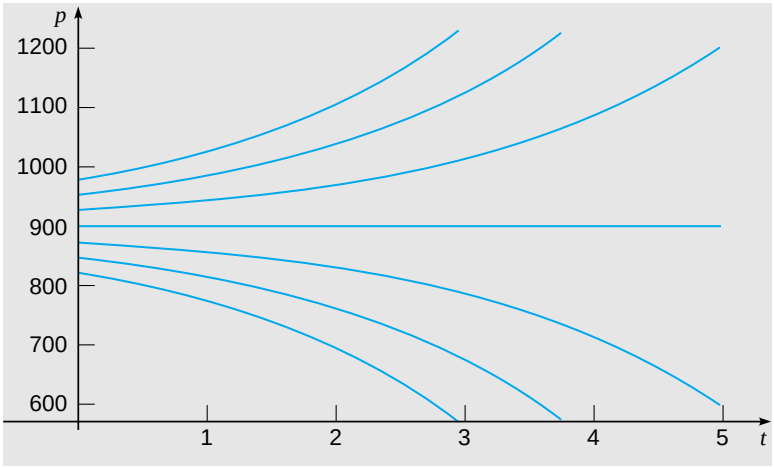


FIGURE 1.2.1 Graphs of Eq. (11) for several values of c.

Note that they have the character inferred from the direction field in Figure 1.1.4. For instance, solutions lying on either side of the equilibrium solution $p = 900$ tend to diverge from that solution.

In Example 1 we found infinitely many solutions of the differential equation (4), corresponding to the infinitely many values that the arbitrary constant c in Eq. (11)

might have. This is typical of what happens when you solve a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate an infinite family of solutions.

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this indirectly by specifying instead a point that must lie on the graph of the solution. For example, to determine the constant c in Eq. (11), we could require that the population have a given value at a certain time, such as the value 850 at time $t = 0$. In other words, the graph of the solution must pass through the point $(0, 850)$. Symbolically, we can express this condition as

$$p(0) = 850. \tag{12}$$

Then, substituting $t = 0$ and $p = 850$ into Eq. (11), we obtain

$$850 = 900 + c.$$

Hence $c = -50$, and by inserting this value in Eq. (11), we obtain the desired solution, namely,

$$p = 900 - 50e^{t/2}. \tag{13}$$

The additional condition (12) that we used to determine c is an example of an **initial condition**. The differential equation (4) together with the initial condition (12) form an **initial value problem**.

Now consider the more general problem consisting of the differential equation (3)

$$\frac{dy}{dt} = ay - b$$

and the initial condition

$$y(0) = y_0, \tag{14}$$

where y_0 is an arbitrary initial value. We can solve this problem by the same method as in Example 1. If $a \neq 0$ and $y \neq b/a$, then we can rewrite Eq. (3) as

$$\frac{dy/dt}{y - (b/a)} = a. \tag{15}$$

By integrating both sides, we find that

$$\ln|y - (b/a)| = at + C, \tag{16}$$

where C is arbitrary. Then, taking the exponential of both sides of Eq. (16) and solving for y, we obtain

$$y = (b/a) + ce^{at}, \tag{17}$$

where $c = \pm e^C$ is also arbitrary. Observe that $c = 0$ corresponds to the equilibrium solution $y = b/a$. Finally, the initial condition (14) requires that $c = y_0 - (b/a)$, so the solution of the initial value problem (3), (14) is

$$y = (b/a) + [y_0 - (b/a)]e^{at}. \tag{18}$$

The expression (17) contains all possible solutions of Eq. (3) and is called the **general solution**. The geometrical representation of the general solution (17) is an infinite family of curves, called **integral curves**. Each integral curve is associated with

a particular value of c, and is the graph of the solution corresponding to that value of c. Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point.

To relate the solution (18) to Eq. (2), which models the field mouse population, we need only replace a by the growth rate r and b by the predation rate k. Then the solution (18) becomes

$$p = (k/r) + [p_0 - (k/r)]e^{rt}, \tag{19}$$

where p_0 is the initial population of field mice. The solution (19) confirms the conclusions reached on the basis of the direction field and Example 1. If $p_0 = k/r$, then from Eq. (19) it follows that $p = k/r$ for all t; this is the constant, or equilibrium, solution. If $p_0 \neq k/r$, then the behavior of the solution depends on the sign of the coefficient $p_0 - (k/r)$ of the exponential term in Eq. (19). If $p_0 > k/r$, then p grows exponentially with time t; if $p_0 < k/r$, then p decreases and eventually becomes zero, corresponding to extinction of the field mouse population. Negative values of p, while possible for the expression (19), make no sense in the context of this particular problem.

To put the falling object equation (1) in the form (3), we must identify a with $-\gamma/m$ and b with $-g$. Making these substitutions in the solution (18), we obtain

$$v = (mg/\gamma) + [v_0 - (mg/\gamma)]e^{-\gamma t/m}, \tag{20}$$

where v_0 is the initial velocity. Again, this solution confirms the conclusions reached in Section 1.1 on the basis of a direction field. There is an equilibrium, or constant, solution $v = mg/\gamma$, and all other solutions tend to approach this equilibrium solution. The speed of convergence to the equilibrium solution is determined by the exponent $-\gamma/m$. Thus, for a given mass m the velocity approaches the equilibrium value faster as the drag coefficient γ increases.

EXAMPLE 2 A Falling Object (continued)

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/sec. Then the equation of motion (1) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{21}$$

Suppose this object is dropped from a height of 300 m. Find its velocity at any time t. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

The first step is to state an appropriate initial condition for Eq. (21). The word "dropped" in the statement of the problem suggests that the initial velocity is zero, so we will use the initial condition

$$v(0) = 0. \tag{22}$$

The solution of Eq. (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve Eq. (21) directly. First, rewrite the equation as

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}. \tag{23}$$

By integrating both sides we obtain

$$\ln|v - 49| = -\frac{t}{5} + C, \tag{24}$$

and then the general solution of Eq. (21) is

$$v = 49 + ce^{-t/5}, \tag{25}$$

where c is arbitrary. To determine c, we substitute $t = 0$ and $v = 0$ from the initial condition (22) into Eq. (25), with the result that $c = -49$. Then the solution of the initial value problem (21), (22) is

$$v = 49(1 - e^{-t/5}). \tag{26}$$

Equation (26) gives the velocity of the falling object at any positive time (before it hits the ground, of course).

Graphs of the solution (25) for several values of c are shown in Figure 1.2.2, with the solution (26) shown by the heavy curve. It is evident that all solutions tend to approach the equilibrium solution $v = 49$. This confirms the conclusions we reached in Section 1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

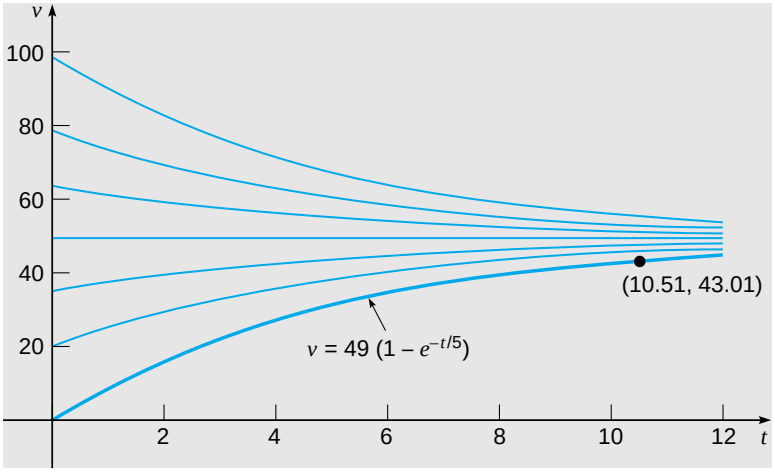


FIGURE 1.2.2 Graphs of the solution (25) for several values of c.

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance x the object has fallen is related to its velocity v by the equation $v = dx/dt$, or

$$\frac{dx}{dt} = 49(1 - e^{-t/5}). \tag{27}$$

Consequently,

$$x = 49t + 245e^{-t/5} + c, \tag{28}$$

where c is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$. From Eq. (28) it follows that $c = -245$, so the distance the object has fallen at time t is given by

$$x = 49t + 245e^{-t/5} - 245. \tag{29}$$

Let T be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in Eq. (29) we obtain the equation

$$49T + 245e^{-T/5} - 545 = 0. \tag{30}$$

The value of T satisfying Eq. (30) can be readily approximated by a numerical process using a scientific calculator or computer, with the result that $T \cong 10.51$ sec. At this time, the corresponding velocity v_T is found from Eq. (26) to be $v_T \cong 43.01$ m/sec.

Further Remarks on Mathematical Modeling. Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object the underlying physical principle (Newton's law of motion) is well-established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient γ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly, for example, by measuring the time of fall from a given height, and then calculating the value of γ that predicts this time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate r and the predation rate k depends on observations of actual populations, which may be subject to considerable variation. The assumption that r and k are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

Even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also be valid under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

PROBLEMS

1. Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.
   (a) $dy/dt = -y + 5, \quad y(0) = y_0$ (b) $dy/dt = -2y + 5, \quad y(0) = y_0$
   (c) $dy/dt = -2y + 10, \quad y(0) = y_0$

2. Follow the instructions for Problem 1 for the following initial value problems:
   (a) $dy/dt = y - 5, \qquad y(0) = y_0$
   (b) $dy/dt = 2y - 5, \qquad y(0) = y_0$
   (c) $dy/dt = 2y - 10, \qquad y(0) = y_0$
3. Consider the differential equation

$$dy/dt = -ay + b,$$

   where both a and b are positive numbers.
   (a) Solve the differential equation.
   (b) Sketch the solution for several different initial conditions.
   (c) Describe how the solutions change under each of the following conditions:
      i. a increases.
      ii. b increases.
      iii. Both a and b increase, but the ratio b/a remains the same.
4. Here is an alternative way to solve the equation

$$dy/dt = ay - b. \tag{i}$$

   (a) Solve the simpler equation

$$dy/dt = ay. \tag{ii}$$

   Call the solution $y_1(t)$.
   (b) Observe that the only difference between Eqs. (i) and (ii) is the constant $-b$ in Eq. (i). Therefore it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k so that $y = y_1(t) + k$ is a solution of Eq. (i).
   (c) Compare your solution from part (b) with the solution given in the text in Eq. (17).
   Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.6 in connection with second order equations.
5. Use the method of Problem 4 to solve the equation

$$dy/dt = -ay + b.$$

6. The field mouse population in Example 1 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

   (a) Find the time at which the population becomes extinct if $p(0) = 850$.
   (b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
   (c) Find the initial population p_0 if the population is to become extinct in 1 year.
7. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.
   (a) Find the rate constant r if the population doubles in 30 days.
   (b) Find r if the population doubles in N days.
8. The falling object in Example 2 satisfies the initial value problem

$$dv/dt = 9.8 - (v/5), \qquad v(0) = 0.$$

   (a) Find the time that must elapse for the object to reach 98% of its limiting velocity.
   (b) How far does the object fall in the time found in part (a)?
9. Modify Example 2 so that the falling object experiences no air resistance.
   (a) Write down the modified initial value problem.
   (b) Determine how long it takes the object to reach the ground.
   (c) Determine its velocity at the time of impact.

10. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.
    (a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.
    (b) Find an expression for the amount of thorium-234 present at any time t.
    (c) Find the time required for the thorium-234 to decay to one-half its original amount.
11. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that, for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.
12. Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.
13. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[3]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

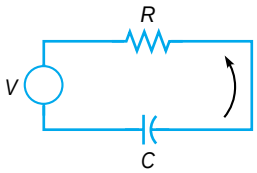


FIGURE 1.2.3 The electric circuit of Problem 13.

    (a) If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.
    (b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.
    (c) Suppose that $Q(t_1) = Q_L$ and that the battery is removed from the circuit at $t = t_1$. Find $Q(t)$ for $t > t_1$ and sketch its graph.

14. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 15 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/hr and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.
    (a) Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.
    (b) Solve the problem in part (a) for $Q(t)$. How much chemical is in the pond after 1 year?
    (c) At the end of 1 year the source of the chemical in the pond is removed and thereafter pure water flows into the pond and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.
    (d) Solve the initial value problem in part (c). How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?
    (e) How long does it take for $Q(t)$ to be reduced to 10 g?
    (f) Plot $Q(t)$ versus t for 3 years.
15. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a nontoxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering

[3]This equation results from Kirchhoff's laws, which are discussed later in Section 3.8.

system can take water from the pool, remove the dye, and return the water to the pool at a rate of 200 gal/min.
(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.
(b) Solve the problem in part (a).
(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hr. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hr?
(d) Find the time T at which the concentration of dye first reaches the value 0.02 g/gal.
(e) Find the flow rate that is sufficient to achieve the concentration 0.02 g/gal within 4 hr.

1.3 Classification of Differential Equations

The main purpose of this book is to discuss some of the properties of solutions of differential equations, and to describe some of the methods that have proved effective in finding solutions, or in some cases approximating them. To provide a framework for our presentation we describe here several useful ways of classifying differential equations.

Ordinary and Partial Differential Equations. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation**. In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation**.

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{1}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L; this equation is derived in Section 3.8. Typical examples of partial differential equations are the heat conduction equation

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \tag{2}$$

and the wave equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{3}$$

Here, α^2 and a^2 are certain physical constants. The heat conduction equation describes the conduction of heat in a solid body and the wave equation arises in a variety of problems involving wave motion in solids or fluids. Note that in both Eqs. (2) and (3) the dependent variable u depends on the two independent variables x and t.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example, the Lotka–Volterra, or predator–prey, equations are important in ecological modeling. They have the form

$$\begin{aligned} dx/dt &= ax - \alpha xy \\ dy/dt &= -cy + \gamma xy, \end{aligned} \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The constants a, α, c, and γ are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka–Volterra equations are examined in Section 9.5. It is not unusual in some areas of application to encounter systems containing a large number of equations.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first order equations, while Eq. (1) is a second order equation. Equations (2) and (3) are second order partial differential equations. More generally, the equation

$$F[t, u(t), u'(t), \ldots, u^{(n)}(t)] = 0 \tag{5}$$

is an ordinary differential equation of the nth order. Equation (5) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(t), u''(t), \ldots, u^{(n)}(t)$. Thus Eq. (5) is written as

$$F(t, y, y', \ldots, y^{(n)}) = 0. \tag{6}$$

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{7}$$

is a third order differential equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}). \tag{8}$$

We study only equations of the form (8). This is mainly to avoid the ambiguity that may arise because a single equation of the form (6) may correspond to several equations of the form (8). For example, the equation

$$y'^2 + ty' + 4y = 0 \tag{9}$$

leads to the two equations

$$y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \qquad \text{or} \qquad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}. \tag{10}$$

Linear and Nonlinear Equations. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is said to be **linear** if F is a linear function of the variables $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \tag{11}$$

Most of the equations you have seen thus far in this book are linear; examples are the equations in Sections 1.1 and 1.2 describing the falling object and the field mouse population. Similarly, in this section, Eq. (1) is a linear ordinary differential equation and Eqs. (2) and (3) are linear partial differential equations. An equation that is not of the form (11) is a **nonlinear** equation. Equation (7) is nonlinear because of the term yy'. Similarly, each equation in the system (4) is nonlinear because of the terms that involve the product xy.

A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle θ that an oscillating pendulum of length L makes with the vertical direction (see Figure 1.3.1) satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \tag{12}$$

whose derivation is outlined in Problem 29. The presence of the term involving $\sin\theta$ makes Eq. (12) nonlinear.

The mathematical theory and methods for solving linear equations are highly developed. In contrast, for nonlinear equations the theory is more complicated and methods of solution are less satisfactory. In view of this, it is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations. For example, for the pendulum, if the angle θ is small, then $\sin\theta \cong \theta$ and Eq. (12) can be approximated by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{13}$$

This process of approximating a nonlinear equation by a linear one is called **linearization** and it is an extremely valuable way to deal with nonlinear equations. Nevertheless, there are many physical phenomena that simply cannot be represented adequately

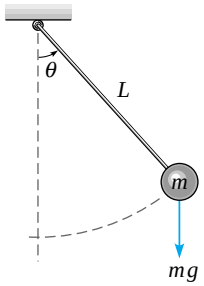


FIGURE 1.3.1 An oscillating pendulum.

by linear equations; to study these phenomena it is essential to deal with nonlinear equations.

In an elementary text it is natural to emphasize the simpler and more straightforward parts of the subject. Therefore the greater part of this book is devoted to linear equations and various methods for solving them. However, Chapters 8 and 9, as well as parts of Chapter 2, are concerned with nonlinear equations. Whenever it is appropriate, we point out why nonlinear equations are, in general, more difficult, and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

Solutions. A **solution** of the ordinary differential equation (8) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)] \tag{14}$$

for every t in $\alpha < t < \beta$. Unless stated otherwise, we assume that the function f of Eq. (8) is a real-valued function, and we are interested in obtaining real-valued solutions $y = \phi(t)$.

Recall that in Section 1.2 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

$$\frac{dp}{dt} = 0.5p - 450 \tag{15}$$

has the solution

$$p = 900 + ce^{t/2}, \tag{16}$$

where c is an arbitrary constant. It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a solution of a given equation, it is usually relatively easy to determine whether the function is actually a solution simply by substituting the function into the equation. For example, in this way it is easy to show that the function $y_1(t) = \cos t$ is a solution of

$$y'' + y = 0 \tag{17}$$

for all t. To confirm this, observe that $y_1'(t) = -\sin t$ and $y_1''(t) = -\cos t$; then it follows that $y_1''(t) + y_1(t) = 0$. In the same way you can easily show that $y_2(t) = \sin t$ is also a solution of Eq. (17). Of course, this does not constitute a satisfactory way to solve most differential equations because there are far too many possible functions for you to have a good chance of finding the correct one by a random choice. Nevertheless, it is important to realize that you can verify whether any proposed solution is correct by substituting it into the differential equation. For a problem of any importance this can be a very useful check and is one that you should make a habit of considering.

Some Important Questions. Although for the equations (15) and (17) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Does an equation of the form (8) always have a solution? The answer is "No." Merely writing down an equation of the form (8) does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of *existence* of a solution, and it is answered by theorems stating that under certain restrictions on the function f in Eq. (8), the equation always

has solutions. However, this is not a purely mathematical concern, for at least two reasons. If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematically as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

Second, if we assume that a given differential equation has at least one solution, the question arises as to how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of *uniqueness*. In general, solutions of differential equations contain one or more arbitrary constants of integration, as does the solution (16) of Eq. (15). Equation (16) represents an infinity of functions corresponding to the infinity of possible choices of the constant c. As we saw in Section 1.2, if p is specified at some time t, this condition will determine a value for c; even so, we have not yet ruled out the possibility that there may be other solutions of Eq. (15) that also have the prescribed value of p at the prescribed time t. The issue of uniqueness also has practical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

A third important question is: Given a differential equation of the form (8), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, while we discuss elementary methods that can be used to obtain solutions of certain relatively simple problems, it is also important to consider methods of a more general nature that can be applied to more difficult problems.

Computer Use in Differential Equations. A computer can be an extremely valuable tool in the study of differential equations. For many years computers have been used to execute numerical algorithms, such as those described in Chapter 8, to construct numerical approximations to solutions of differential equations. At the present time these algorithms have been refined to an extremely high level of generality and efficiency. A few lines of computer code, written in a high-level programming language and executed (often within a few seconds) on a relatively inexpensive computer, suffice to solve numerically a wide range of differential equations. More sophisticated routines are also readily available. These routines combine the ability to handle very large and complicated systems with numerous diagnostic features that alert the user to possible problems as they are encountered.

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more

illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The widespread availability of personal computers has brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are *Maple*,® *Mathematica*,® and MATLAB,® each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. In addition, *Maple* and *Mathematica* also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to delegate as many as possible of the routine (often repetitive) details to a computer, while you focus more attention on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require a calculator or computer. Good judgment is often needed in selecting a judicious combination.

PROBLEMS

In each of Problems 1 through 6 determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$
2. $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$
3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$
4. $\dfrac{dy}{dt} + ty^2 = 0$
5. $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin t$
6. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

In each of Problems 7 through 14 verify that the given function or functions is a solution of the differential equation.

7. $y'' - y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = \cosh t$
8. $y'' + 2y' - 3y = 0; \qquad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$
9. $ty' - y = t^2; \qquad y = 3t + t^2$

10. $y'''' + 4y''' + 3y = t; \qquad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3$
11. $2t^2y'' + 3ty' - y = 0, \quad t > 0; \qquad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$
12. $t^2y'' + 5ty' + 4y = 0, \quad t > 0; \qquad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2}\ln t$
13. $y'' + y = \sec t, \quad 0 < t < \pi/2; \qquad y = (\cos t)\ln\cos t + t\sin t$
14. $y' - 2ty = 1; \qquad y = e^{t^2}\displaystyle\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 15 through 18 determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

15. $y' + 2y = 0$
16. $y'' - y = 0$
17. $y'' + y' - 6y = 0$
18. $y''' - 3y'' + 2y' = 0$

In each of Problems 19 and 20 determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

19. $t^2y'' + 4ty' + 2y = 0$
20. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 21 through 24 determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

21. $u_{xx} + u_{yy} + u_{zz} = 0$
22. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$
23. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
24. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 25 through 28 verify that the given function or functions is a solution of the given partial differential equation.

25. $u_{xx} + u_{yy} = 0; \qquad u_1(x, y) = \cos x\cosh y, \quad u_2(x, y) = \ln(x^2 + y^2)$
26. $\alpha^2u_{xx} = u_t; \qquad u_1(x, t) = e^{-\alpha^2t}\sin x, \quad u_2(x, t) = e^{-\alpha^2\lambda^2t}\sin\lambda x, \quad \lambda$ a real constant
27. $a^2u_{xx} = u_{tt}; \qquad u_1(x, t) = \sin\lambda x\sin\lambda at, \quad u_2(x, t) = \sin(x - at), \quad \lambda$ a real constant
28. $\alpha^2u_{xx} = u_t; \qquad u = (\pi/t)^{1/2}e^{-x^2/4\alpha^2t}, \quad t > 0$
29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.
    (a) Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.
    (b) Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.
    (c) Simplify the result from part (b) to obtain Eq. (12) of the text.

1.4 Historical Remarks

Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of

the subject, and identify the most prominent early contributors. Other historical information is contained in footnotes scattered throughout the book and in the references listed at the end of the chapter.

The subject of differential equations originated in the study of calculus by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the seventeenth century. Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date from 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism, and did not begin to publish his results until 1687 with the appearance of his most famous book, *Philosophiae Naturalis Principia Mathematica*. While Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications in the eighteenth century, most notably by Euler. Newton classified first order differential equations according to the forms $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x,y)$. For the latter equation he developed a method of solution using infinite series when $f(x,y)$ is a polynomial in x and y. Newton's active research in mathematics ended in the early 1690s except for the solution of occasional challenge problems and the revision and publication of results obtained much earlier. He was appointed Warden of the British Mint in 1696 and resigned his professorship a few years later. He was knighted in 1705 and, upon his death, was buried in Westminster Abbey.

Leibniz was born in Leipzig and completed his doctorate in philosophy at the age of 20 at the University of Altdorf. Throughout his life he engaged in scholarly work in several different fields. He was mainly self-taught in mathematics, since his interest in this subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was very conscious of the power of good mathematical notation, and our notation for the derivative, dy/dx, and the integral sign are due to him. He discovered the method of separation of variables (Section 2.2) in 1691, the reduction of homogeneous equations to separable ones (Section 2.2, Problem 30) in 1691, and the procedure for solving first order linear equations (Section 2.1) in 1694. He spent his life as ambassador and adviser to several German royal families, which permitted him to travel widely and to carry on an extensive correspondence with other mathematicians, especially the Bernoulli brothers. In the course of this correspondence many problems in differential equations were solved during the latter part of the seventeenth century.

The brothers Jakob (1654–1705) and Johann (1667–1748) Bernoulli of Basel did much to develop methods of solving differential equations and to extend the range of their applications. Jakob became professor of mathematics at Basel in 1687, and Johann was appointed to the same position upon his brother's death in 1705. Both men were quarrelsome, jealous, and frequently embroiled in disputes, especially with each other. Nevertheless, both also made significant contributions to several areas of mathematics. With the aid of calculus they solved a number of problems in mechanics by formulating them as differential equations. For example, Jakob Bernoulli solved the differential equation $y' = [a^3/(b^2y - a^3)]^{1/2}$ in 1690 and in the same paper first used the term "integral" in the modern sense. In 1694 Johann Bernoulli was able to solve the equation $dy/dx = y/ax$. One problem to which both brothers contributed, and which led to much friction between them, was the *brachistochrone* problem (see

Problem 33 of Section 2.3). The brachistochrone problem was also solved by Leibniz and Newton in addition to the Bernoulli brothers. It is said, perhaps apocryphally, that Newton learned of the problem late in the afternoon of a tiring day at the Mint, and solved it that evening after dinner. He published the solution anonymously, but on seeing it, Johann Bernoulli exclaimed, "Ah, I know the lion by his paw."

Daniel Bernoulli (1700–1782), son of Johann, migrated to St. Petersburg as a young man to join the newly established St. Petersburg Academy, but returned to Basel in 1733 as professor of botany, and later, of physics. His interests were primarily in partial differential equations and their applications. For instance, it is his name that is associated with the Bernoulli equation in fluid mechanics. He was also the first to encounter the functions that a century later became known as Bessel functions (Section 5.8).

The greatest mathematician of the eighteenth century, Leonhard Euler (1707–1783), grew up near Basel and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. For the remainder of his life he was associated with the St. Petersburg Academy (1727–1741 and 1766–1783) and the Berlin Academy (1741–1766). Euler was the most prolific mathematician of all time; his collected works fill more than 70 large volumes. His interests ranged over all areas of mathematics and many fields of application. Even though he was blind during the last 17 years of his life, his work continued undiminished until the very day of his death. Of particular interest here is his formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Among other things, Euler identified the condition for exactness of first order differential equations (Section 2.6) in 1734–35, developed the theory of integrating factors (Section 2.6) in the same paper, and gave the general solution of homogeneous linear equations with constant coefficients (Sections 3.1, 3.4, 3.5, and 4.2) in 1743. He extended the latter results to nonhomogeneous equations in 1750–51. Beginning about 1750, Euler made frequent use of power series (Chapter 5) in solving differential equations. He also proposed a numerical procedure (Sections 2.7 and 8.1) in 1768–69, made important contributions in partial differential equations, and gave the first systematic treatment of the calculus of variations.

Joseph-Louis Lagrange (1736–1813) became professor of mathematics in his native Turin at the age of 19. He succeeded Euler in the chair of mathematics at the Berlin Academy in 1766, and moved on to the Paris Academy in 1787. He is most famous for his monumental work *Mécanique analytique*, published in 1788, an elegant and comprehensive treatise of Newtonian mechanics. With respect to elementary differential equations, Lagrange showed in 1762–65 that the general solution of an nth order linear homogeneous differential equation is a linear combination of n independent solutions (Sections 3.2, 3.3, and 4.1). Later, in 1774–75, he gave a complete development of the method of variation of parameters (Sections 3.7 and 4.4). Lagrange is also known for fundamental work in partial differential equations and the calculus of variations.

Pierre-Simon de Laplace (1749–1827) lived in Normandy as a boy but came to Paris in 1768 and quickly made his mark in scientific circles, winning election to the Académie des Sciences in 1773. He was preeminent in the field of celestial mechanics; his greatest work, *Traité de mécanique céleste*, was published in five volumes between 1799 and 1825. Laplace's equation is fundamental in many branches of mathematical physics, and Laplace studied it extensively in connection with gravitational attraction.

The Laplace transform (Chapter 6) is also named for him although its usefulness in solving differential equations was not recognized until much later.

By the end of the eighteenth century many elementary methods of solving ordinary differential equations had been discovered. In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on power series expansions (see Chapter 5). These methods find their natural setting in the complex plane. Consequently, they benefitted from, and to some extent stimulated, the more or less simultaneous development of the theory of complex analytic functions. Partial differential equations also began to be studied intensively, as their crucial role in mathematical physics became clear. In this connection a number of functions, arising as solutions of certain ordinary differential equations, occurred repeatedly and were studied exhaustively. Known collectively as higher transcendental functions, many of them are associated with the names of mathematicians, including Bessel, Legendre, Hermite, Chebyshev, and Hankel, among others.

The numerous differential equations that resisted solution by analytical means led to the investigation of methods of numerical approximation (see Chapter 8). By 1900 fairly effective numerical integration methods had been devised, but their implementation was severely restricted by the need to execute the computations by hand or with very primitive computing equipment. In the last 50 years the development of increasingly powerful and versatile computers has vastly enlarged the range of problems that can be investigated effectively by numerical methods. During the same period extremely refined and robust numerical integrators have been developed and are readily available. Versions appropriate for personal computers have brought the ability to solve a great many significant problems within the reach of individual students.

Another characteristic of differential equations in the twentieth century has been the creation of geometrical or topological methods, especially for nonlinear equations. The goal is to understand at least the qualitative behavior of solutions from a geometrical, as well as from an analytical, point of view. If more detailed information is needed, it can usually be obtained by using numerical approximations. An introduction to these geometrical methods appears in Chapter 9.

Within the past few years these two trends have come together. Computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Unexpected phenomena (Section 9.8), referred to by terms such as strange attractors, chaos, and fractals, have been discovered, are being intensively studied, and are leading to important new insights in a variety of applications. Although it is an old subject about which much is known, differential equations at the dawn of the twenty-first century remains a fertile source of fascinating and important unsolved problems.

REFERENCES

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For further reading in the history of mathematics see books such as those listed below:

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A useful historical appendix on the early development of differential equations appears in:

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An encyclopedic source of information about the lives and achievements of mathematicians of the past is:

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CHAPTER
2

First Order Differential Equations

This chapter deals with differential equations of first order,

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations. The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first order differential equations, introduce the idea of approximating a solution by numerical computation, and discuss some theoretical questions related to existence and uniqueness of solutions. The final section deals with first order difference equations, which have some important points of similarity with differential equations and are in some respects simpler to investigate.

2.1 Linear Equations with Variable Coefficients



If the function f in Eq. (1) depends linearly on the dependent variable y, then Eq. (1) is called a first order linear equation. In Sections 1.1 and 1.2 we discussed a restricted

type of first order linear equation in which the coefficients are constants. A typical example is

$$\frac{dy}{dt} = -ay + b, \tag{2}$$

where a and b are given constants. Recall that an equation of this form describes the motion of an object falling in the atmosphere. Now we want to consider the most general first order linear equation, which is obtained by replacing the coefficients a and b in Eq. (2) by arbitrary functions of t. We will usually write the general **first order linear equation** in the form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{3}$$

where p and g are given functions of the independent variable t.

Equation (2) can be solved by the straightforward integration method introduced in Section 1.2. That is, we rewrite the equation as

$$\frac{dy/dt}{y - (b/a)} = -a. \tag{4}$$

Then, by integration we obtain

$$\ln|y - (b/a)| = -at + C,$$

from which it follows that the general solution of Eq. (2) is

$$y = (b/a) + ce^{-at}, \tag{5}$$

where c is an arbitrary constant. For example, if $a = 2$ and $b = 3$, then Eq. (2) becomes

$$\frac{dy}{dt} + 2y = 3, \tag{6}$$

and its general solution is

$$y = \tfrac{3}{2} + ce^{-2t}. \tag{7}$$

Unfortunately, this direct method of solution cannot be used to solve the general equation (3), so we need to use a different method of solution for it. The method is due to Leibniz; it involves multiplying the differential equation (3) by a certain function $\mu(t)$, chosen so that the resulting equation is readily integrable. The function $\mu(t)$ is called an **integrating factor** and the main difficulty is to determine how to find it. To make the initial presentation as simple as possible, we will first use this method to solve Eq. (6), later showing how to extend it to other first order linear equations, including the general equation (3).

EXAMPLE 1

Solve Eq. (6),

$$\frac{dy}{dt} + 2y = 3,$$

by finding an integrating factor for this equation.

The first step is to multiply Eq. (6) by a function $\mu(t)$, as yet undetermined; thus

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = 3\mu(t). \tag{8}$$

The question now is whether we can choose $\mu(t)$ so that the left side of Eq. (8) is recognizable as the derivative of some particular expression. If so, then we can integrate Eq. (8), even though we do not know the function y. To guide our choice of the integrating factor $\mu(t)$, observe that the left side of Eq. (8) contains two terms and that the first term is part of the result of differentiating the product $\mu(t)y$. Thus, let us try to determine $\mu(t)$ so that the left side of Eq. (8) becomes the derivative of the expression $\mu(t)y$. If we compare the left side of Eq. (8) with the differentiation formula

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y, \tag{9}$$

we note that the first terms are identical and that the second terms also agree, provided we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt} = 2\mu(t). \tag{10}$$

Therefore our search for an integrating factor will be successful if we can find a solution of Eq. (10). Perhaps you can readily identify a function that satisfies Eq. (10): What function has a derivative that is equal to two times the original function? More systematically, rewrite Eq. (10) as

$$\frac{d\mu(t)/dt}{\mu(t)} = 2, \tag{11}$$

which is equivalent to

$$\frac{d}{dt}\ln|\mu(t)| = 2. \tag{12}$$

Then it follows that

$$\ln|\mu(t)| = 2t + C, \tag{13}$$

or

$$\mu(t) = ce^{2t}. \tag{14}$$

The function $\mu(t)$ given by Eq. (14) is the integrating factor for Eq. (6). Since we do not need the most general integrating factor, we will choose c to be one in Eq. (14) and use $\mu(t) = e^{2t}$.

Now we return to Eq. (6), multiply it by the integrating factor e^{2t}, and obtain

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 3e^{2t}. \tag{15}$$

By the choice we have made of the integrating factor, the left side of Eq. (15) is the derivative of $e^{2t}y$, so that Eq. (15) becomes

$$\frac{d}{dt}(e^{2t}y) = 3e^{2t}. \tag{16}$$

By integrating both sides of Eq. (16) we obtain

$$e^{2t}y = \tfrac{3}{2}e^{2t} + c, \tag{17}$$

where c is an arbitrary constant. Finally, on solving Eq. (17) for y, we have the general solution of Eq. (6), namely,

$$y = \tfrac{3}{2} + ce^{-2t}. \tag{18}$$

Of course, the solution (18) is the same as the solution (7) found earlier. Figure 2.1.1 shows the graph of Eq. (18) for several values of c. The solutions converge to the equilibrium solution $y = 3/2$, which corresponds to $c = 0$.

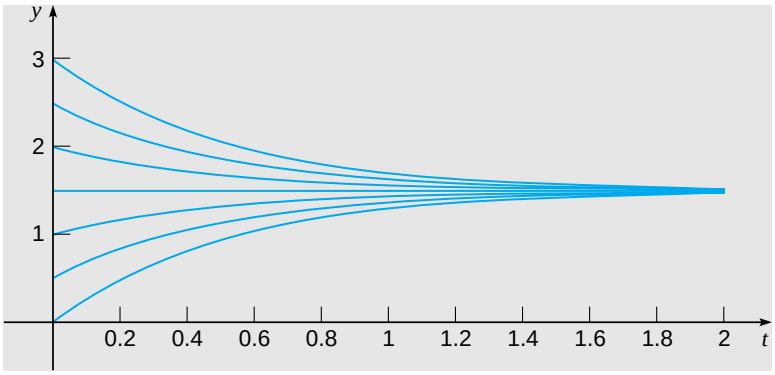


FIGURE 2.1.1 Integral curves of $y' + 2y = 3$.

Now that we have shown how the method of integrating factors works in this simple example, let us extend it to other classes of equations. We will do this in three stages. First, consider again Eq. (2), which we now write in the form

$$\frac{dy}{dt} + ay = b. \tag{19}$$

The derivation in Example 1 can now be repeated line for line, the only changes being that the coefficients 2 and 3 in Eq. (6) are replaced by a and b, respectively. The integrating factor is $\mu(t) = e^{at}$ and the solution is given by Eq. (5), which is the same as Eq. (18) with 2 replaced by a and 3 replaced by b.

The next stage is to replace the constant b by a given function $g(t)$, so that the differential equation becomes

$$\frac{dy}{dt} + ay = g(t). \tag{20}$$

The integrating factor depends only on the coefficient of y so for Eq. (20) the integrating factor is again $\mu(t) = e^{at}$. Multiplying Eq. (20) by $\mu(t)$, we obtain

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t),$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t). \tag{21}$$

By integrating both sides of Eq. (21) we find that

$$e^{at}y = \int e^{as}g(s)\,ds + c, \tag{22}$$

where c is an arbitrary constant. Note that we have used s to denote the integration variable to distinguish it from the independent variable t. By solving Eq. (22) for y we obtain the general solution

$$y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}. \tag{23}$$

For many simple functions $g(s)$ the integral in Eq. (23) can be evaluated and the solution y expressed in terms of elementary functions, as in the following examples. However, for more complicated functions $g(s)$, it may be necessary to leave the solution in the integral form given by Eq. (23).

EXAMPLE 2

Solve the differential equation

$$\frac{dy}{dt} + \tfrac{1}{2}y = 2 + t. \tag{24}$$

Sketch several solutions and find the particular solution whose graph contains the point $(0, 2)$.

In this case $a = 1/2$, so the integrating factor is $\mu(t) = e^{t/2}$. Multiplying Eq. (24) by this factor leads to the equation

$$\frac{d}{dt}(e^{t/2}y) = 2e^{t/2} + te^{t/2}. \tag{25}$$

By integrating both sides of Eq. (25), using integration by parts on the second term on the right side, we obtain

$$e^{t/2}y = 4e^{t/2} + 2te^{t/2} - 4e^{t/2} + c,$$

where c is an arbitrary constant. Thus

$$y = 2t + ce^{-t/2}. \tag{26}$$

To find the solution that passes through the initial point $(0, 2)$, we set $t = 0$ and $y = 2$ in Eq. (26), with the result that $2 = 0 + c$, so that $c = 2$. Hence the desired solution is

$$y = 2t + 2e^{-t/2}. \tag{27}$$

Graphs of the solution (26) for several values of c are shown in Figure 2.1.2. Observe that the solutions converge, not to a constant solution as in Example 1 and in the examples in Chapter 1, but to the solution $y = 2t$, which corresponds to $c = 0$.

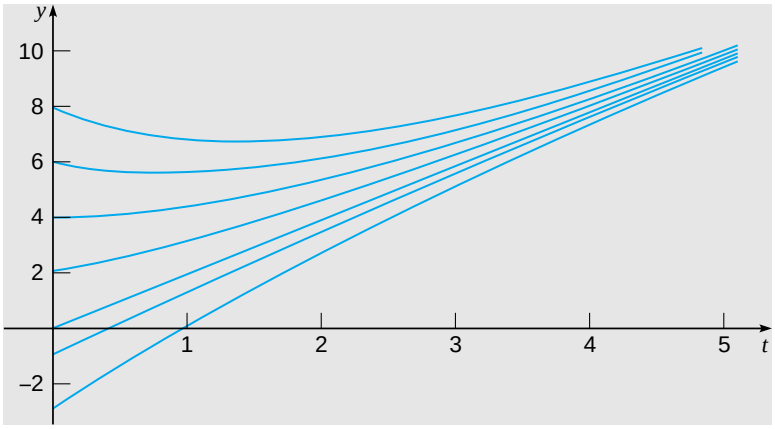


FIGURE 2.1.2 Integral curves of $y' + \frac{1}{2}y = 2 + t$.

EXAMPLE 3

Solve the differential equation

$$\frac{dy}{dt} - 2y = 4 - t \tag{28}$$

and sketch the graphs of several solutions. Find the initial point on the y-axis that separates solutions that grow large positively from those that grow large negatively as $t \to \infty$.

Since the coefficient of y is -2, the integrating factor for Eq. (28) is $\mu(t) = e^{-2t}$. Multiplying the differential equation by $\mu(t)$, we obtain

$$\frac{d}{dt}(e^{-2t}y) = 4e^{-2t} - te^{-2t}. \tag{29}$$

Then, by integrating both sides of this equation, we have

$$e^{-2t}y = -2e^{-2t} + \tfrac{1}{2}te^{-2t} + \tfrac{1}{4}e^{-2t} + c,$$

where we have used integration by parts on the last term in Eq. (29). Thus the general solution of Eq. (28) is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}. \tag{30}$$

Graphs of the solution (30) for several values of c are shown in Figure 2.1.3. The behavior of the solution for large values of t is determined by the term ce^{2t}. If $c \neq 0$, then the solution grows exponentially large in magnitude, with the same sign as c itself. Thus the solutions diverge as t becomes large. The boundary between solutions that ultimately grow positively from those that ultimately grow negatively occurs when $c = 0$. If we substitute $c = 0$ into Eq. (30) and then set $t = 0$, we find that $y = -7/4$. This is the separation point on the y-axis that was requested. Note that, for this initial value, the solution is $y = -\frac{7}{4} + \frac{1}{2}t$; it grows positively (but not exponentially).

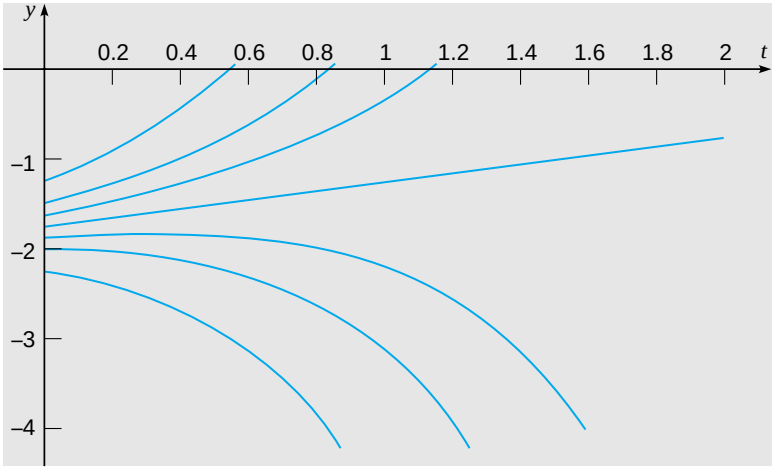


FIGURE 2.1.3 Integral curves of $y' - 2y = 4 - t$.

Examples 2 and 3 are special cases of Eq. (20),

$$\frac{dy}{dt} + ay = g(t),$$

whose solutions are given by Eq. (23),

$$y = e^{-at} \int e^{as} g(s)\, ds + ce^{-at}.$$

The solutions converge if $a > 0$, as in Example 2, and diverge if $a < 0$, as in Example 3. In contrast to the equations considered in Sections 1.1 and 1.2, however, Eq. (20) does not have an equilibrium solution.

The final stage in extending the method of integrating factors is to the general first order linear equation (3),

$$\frac{dy}{dt} + p(t)y = g(t),$$

where p and g are given functions. If we multiply Eq. (3) by an as yet undetermined function $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{31}$$

Following the same line of development as in Example 1, we see that the left side of Eq. (31) is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t). \tag{32}$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t),$$

and consequently

$$\ln \mu(t) = \int p(t)\,dt + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = \exp \int p(t)\,dt. \tag{33}$$

Note that $\mu(t)$ is positive for all t, as we assumed. Returning to Eq. (31), we have

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t). \tag{34}$$

Hence

$$\mu(t)y = \int \mu(s)g(s)\,ds + c,$$

so the general solution of Eq. (3) is

$$y = \frac{\int \mu(s)g(s)\,ds + c}{\mu(t)}. \tag{35}$$

Observe that, to find the solution given by Eq. (35), two integrations are required: one to obtain $\mu(t)$ from Eq. (33) and the other to obtain y from Eq. (35).

EXAMPLE 4

Solve the initial value problem

$$ty' + 2y = 4t^2, \tag{36}$$

$$y(1) = 2. \tag{37}$$

Rewriting Eq. (36) in the standard form (3), we have

$$y' + (2/t)y = 4t, \tag{38}$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve Eq. (38) we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp \int \frac{2}{t}\,dt = e^{2\ln|t|} = t^2. \tag{39}$$

On multiplying Eq. (38) by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = (t^2y)' = 4t^3,$$

and therefore

$$t^2y = t^4 + c,$$

where c is an arbitrary constant. It follows that

$$y = t^2 + \frac{c}{t^2} \tag{40}$$

is the general solution of Eq. (36). Integral curves of Eq. (36) for several values of c are shown in Figure 2.1.4. To satisfy the initial condition (37) it is necessary to choose $c = 1$; thus

$$y = t^2 + \frac{1}{t^2}, \qquad t > 0 \tag{41}$$

is the solution of the initial value problem (36), (37). This solution is shown by the heavy curve in Figure 2.1.4. Note that it becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. The function $y = t^2 + (1/t^2)$ for $t < 0$ is not part of the solution of this initial value problem.

This is the first example in which the solution fails to exist for some values of t. Again, this is due to the infinite discontinuity in $p(t)$ at $t = 0$, which restricts the solution to the interval $0 < t < \infty$.

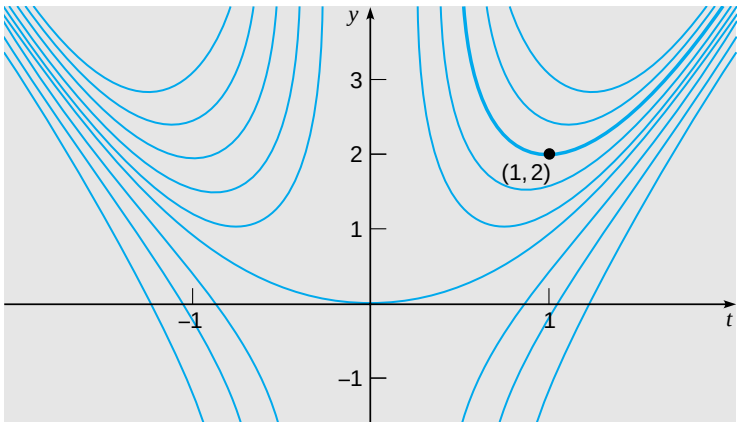


FIGURE 2.1.4 Integral curves of $ty' + 2y = 4t^2$.

Looking again at Figure 2.1.4, we see that some solutions (those for which $c > 0$) are asymptotic to the positive y-axis as $t \to 0$ from the right, while other solutions (for which $c < 0$) are asymptotic to the negative y-axis. The solution for which $c = 0$, namely, $y = t^2$, remains bounded and differentiable even at $t = 0$. If we generalize the initial condition (37) to

$$y(1) = y_0, \tag{42}$$

then $c = y_0 - 1$ and the solution (41) becomes

$$y = t^2 + \frac{y_0 - 1}{t^2}, \qquad t > 0. \tag{43}$$

As in Example 3, this is another instance where there is a critical initial value, namely, $y_0 = 1$, that separates solutions that behave in two quite different ways.

PROBLEMS

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.
(b) Based on an inspection of the direction field, describe how solutions behave for large t.
(c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \quad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \quad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20 find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \quad y(0) = 1$
14. $y' + 2y = te^{-2t}, \quad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2 \quad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \quad y(0) = 2$
18. $ty' + 2y = \sin t, \quad y(\pi/2) = 1$
19. $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$
20. $ty' + (t + 1)y = t, \quad y(\ln 2) = 1$

In each of Problems 21 and 22:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$
22. $2y' - y = e^{t/3}, \quad y(0) = a$

In each of Problems 23 and 24:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

23. $ty' + (t + 1)y = 2te^{-t}, \quad y(1) = a$
24. $ty' + 2y = (\sin t)/t, \quad y(-\pi/2) = a$
25. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

26. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

27. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

28. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

29. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

30. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 31 through 34 construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

31. All solutions have the limit 3 as $t \to \infty$.
32. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
33. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
34. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
35. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t)$ is identically zero, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.
(b) If $g(t)$ is not identically zero, assume that the solution is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left[\int p(t)\,dt\right]. \tag{iv}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (35) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.7 in connection with second order linear equations.

In each of Problems 36 and 37 use the method of Problem 35 to solve the given differential equation.

36. $y' - 2y = t^2e^{2t}$

37. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$

2.2 Separable Equations



In Sections 1.2 and 2.1 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x to denote the independent variable in this section rather than t for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations for which a direct integration process can be used.

To identify this class of equations we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. In the event that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the differential form

$$M(x)\,dx + N(y)\,dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be separated by the equals sign. The differential form (5) is also more symmetric and tends to diminish the distinction between independent and dependent variables.

EXAMPLE 1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

If we write Eq. (6) as

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Next, observe that the first term in Eq. (7) is the derivative of $-x^3/3$ and that the second term, by means of the chain rule, is the derivative with respect to x of $y - y^3/3$. Thus Eq. (7) can be written as

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0,$$

or

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0.$$

Therefore

$$-x^3 + 3y - y^3 = c, \tag{8}$$

where c is an arbitrary constant, is an equation for the integral curves of Eq. (6). A direction field and several integral curves are shown in Figure 2.2.1. An equation of the integral curve passing through a particular point (x_0, y_0) can be found by substituting x_0 and y_0 for x and y, respectively, in Eq. (8) and determining the corresponding value of c. Any differentiable function $y = \phi(x)$ that satisfies Eq. (8) is a solution of Eq. (6).

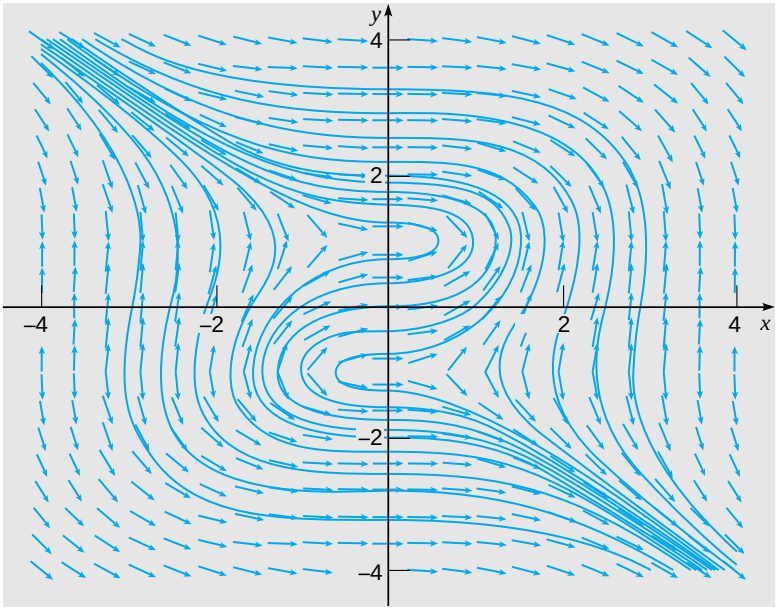


FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to Eq. (4), let H_1 and H_2 be any functions such that

$$H_1'(x) = M(x), \qquad H_2'(y) = N(y); \tag{9}$$

then Eq. (4) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{10}$$

According to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{11}$$

Consequently, Eq. (10) becomes

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0. \tag{12}$$

By integrating Eq. (12) we obtain

$$H_1(x) + H_2(y) = c, \tag{13}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies Eq. (13) is a solution of Eq. (4); in other words, Eq. (13) defines the solution implicitly rather than explicitly. The functions H_1 and H_2 are any antiderivatives of M and N, respectively. In practice, Eq. (13) is usually obtained from Eq. (5) by integrating the first term with respect to x and the second term with respect to y.

If, in addition to the differential equation, an initial condition

$$y(x_0) = y_0 \tag{14}$$

is prescribed, then the solution of Eq. (4) satisfying this condition is obtained by setting $x = x_0$ and $y = y_0$ in Eq. (13). This gives

$$c = H_1(x_0) + H_2(y_0). \tag{15}$$

Substituting this value of c in Eq. (13) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)\,ds, \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)\,ds,$$

we obtain

$$\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = 0. \tag{16}$$

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). You should bear in mind that the determination of an explicit formula for the solution requires that Eq. (16) be solved for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases one can resort to numerical methods to find approximate values of y for given values of x.

EXAMPLE 2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1, \tag{17}$$

and determine the interval in which the solution exists.

The differential equation can be written as

$$2(y-1)\,dy = (3x^2+4x+2)\,dx.$$

Integrating the left side with respect to y and the right side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in Eq. (18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{19}$$

To obtain the solution explicitly we must solve Eq. (19) for y in terms of x. This is a simple matter in this case, since Eq. (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in Eq. (20), so that we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (17). Note that if the plus sign is chosen by mistake in Eq. (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. The solution of the initial value problem and some other integral curves of the differential equation are shown in Figure 2.2.2. Observe that the boundary of the interval of validity of the solution (20) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

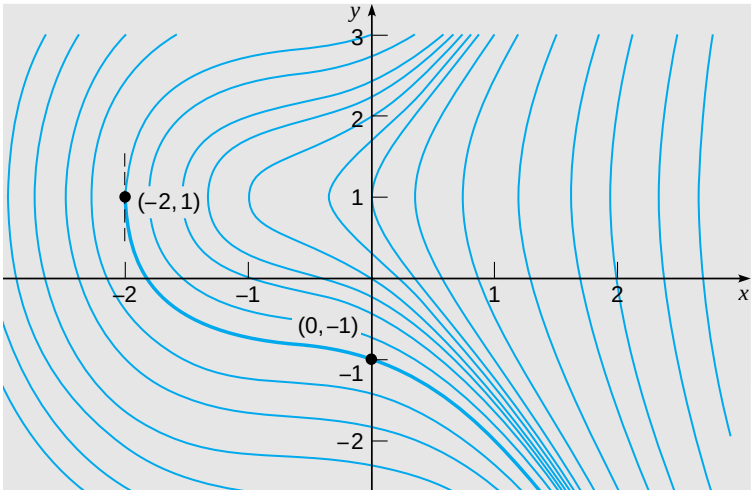


FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$.

EXAMPLE 3

Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2}, \qquad y(0) = 1. \tag{22}$$

Observe that $y = 0$ is a solution of the given differential equation. To find other solutions, assume that $y \neq 0$ and write the differential equation in the form

$$\frac{1 + 2y^2}{y}\, dy = \cos x\, dx. \tag{23}$$

Then, integrating the left side with respect to y and the right side with respect to x, we obtain

$$\ln|y| + y^2 = \sin x + c. \tag{24}$$

To satisfy the initial condition we substitute $x = 0$ and $y = 1$ in Eq. (24); this gives $c = 1$. Hence the solution of the initial value problem (22) is given implicitly by

$$\ln|y| + y^2 = \sin x + 1. \tag{25}$$

Since Eq. (25) is not readily solved for y as a function of x, further analysis of this problem becomes more delicate. One fairly evident fact is that no solution crosses the x-axis. To see this, observe that the left side of Eq. (25) becomes infinite if $y = 0$; however, the right side never becomes unbounded, so no point on the x-axis satisfies Eq. (25). Thus, for the solution of Eqs. (22) it follows that $y > 0$ always. Consequently, the absolute value bars in Eq. (25) can be dropped. It can also be shown that the interval of definition of the solution of the initial value problem (22) is the entire x-axis. Some integral curves of the given differential equation, including the solution of the initial value problem (22), are shown in Figure 2.2.3.

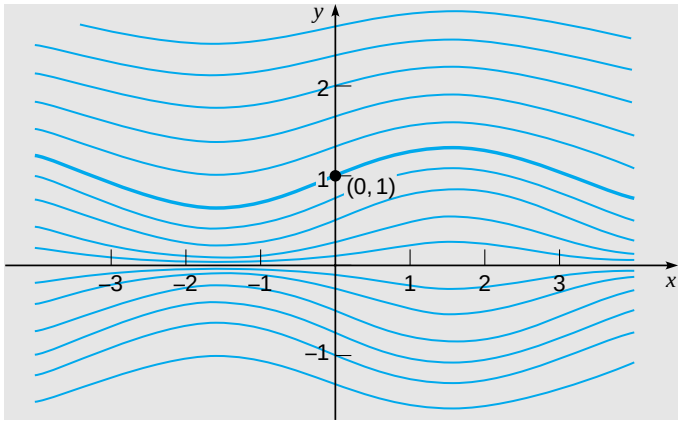


FIGURE 2.2.3 Integral curves of $y' = (y \cos x)/(1 + 2y^2)$.

The investigation of a first order nonlinear equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{26}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}, \tag{27}$$

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

$$dx/dt = G(x, y), \qquad dy/dt = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but, in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note: In Example 2 it was not difficult to solve explicitly for y as a function of x and to determine the exact interval in which the solution exists. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the terminology "solve the following differential equation" means to find the solution explicitly if it is convenient to do so, but otherwise to find an implicit formula for the solution.

PROBLEMS

In each of Problems 1 through 8 solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1 + x^3)$
3. $y' + y^2 \sin x = 0$
4. $y' = (3x^2 - 1)/(3 + 2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1 - y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^{y}}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.
(b) Plot the graph of the solution.
(c) Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = 2x/(y + x^2y), \quad y(0) = -2$
14. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
15. $y' = 2x/(1 + 2y), \quad y(2) = 0$
16. $y' = x(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2}$
17. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
18. $y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1$
19. $\sin 2x\,dx + \cos 3y\,dy = 0, \quad y(\pi/2) = \pi/3$
20. $y^2(1 - x^2)^{1/2}dy = \arcsin x\,dx, \quad y(0) = 0$

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically, or by plotting numerically generated approximations to the solutions. Try to form an opinion as to the advantages and disadvantages of each approach.

21. Solve the initial value problem

$$y' = (1 + 3x^2)/(3y^2 - 6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

22. Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

24. Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \qquad y(0) = 0$$

and determine where the solution attains its maximum value.

25. Solve the initial value problem

$$y' = 2\cos 2x/(3 + 2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

26. Solve the initial value problem

$$y' = 2(1 + x)(1 + y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

27. Consider the initial value problem

$$y' = ty(4 - y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.
(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

28. Consider the initial value problem

$$y' = ty(4 - y)/(1 + t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

29. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

Homogeneous Equations. If the right side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{i}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{ii}$$

thus Eq. (i) is homogeneous.
(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.
(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{iii}$$

Observe that Eq. (iii) is separable.
(d) Solve Eq. (iii) for v in terms of x.
(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).
(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (1) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.
(b) Solve the differential equation.
(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

▶ 31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

▶ 32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

▶ 33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

▶ 34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

▶ 35. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - y}$

▶ 36. $(x^2 + 3xy + y^2)\,dx - x^2 dy = 0$

▶ 37. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

▶ 38. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First Order Equations

ODE

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and

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their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Construction of the Model. This involves a translation of the physical situation into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton's laws, insect populations do not grow indefinitely as stated because of eventual limitations on their food supply, and heat transfer is affected by factors other than the temperature difference. Alternatively, one can adopt the point of view that the mathematical equations exactly describe the operation of a simplified physical model, which has been constructed (or conceived of) so as to embody the most important features of the actual process. Sometimes, the process of mathematical modeling involves the conceptual replacement of a discrete process by a continuous one. For instance, the number of members in an insect population changes by discrete amounts; however, if the population is large, it seems reasonable to consider it as a continuous variable and even to speak of its derivative.

Analysis of the Model. Once the problem has been formulated mathematically, one is often faced with the problem of solving one or more differential equations or, failing that, of finding out as much as possible about the properties of the solution. It may happen that this mathematical problem is quite difficult and, if so, further approximations may be indicated at this stage to make the problem mathematically tractable. For example, a nonlinear equation may be approximated by a linear one, or a slowly varying coefficient may be replaced by a constant. Naturally, any such approximations must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process under investigation. At the same time, an intimate knowledge of the physics of the problem may suggest reasonable mathematical approximations

that will make the mathematical problem more amenable to analysis. This interplay of understanding of physical phenomena and knowledge of mathematical techniques and their limitations is characteristic of applied mathematics at its best, and is indispensable in successfully constructing useful mathematical models of intricate physical processes.

Comparison with Experiment or Observation. Finally, having obtained the solution (or at least some information about it), you must interpret this information in the context in which the problem arose. In particular, you should always check that the mathematical solution appears physically reasonable. If possible, calculate the values of the solution at selected points and compare them with experimentally observed values. Or, ask whether the behavior of the solution after a long time is consistent with observations. Or, examine the solutions corresponding to certain special values of parameters in the problem. Of course, the fact that the mathematical solution appears to be reasonable does not guarantee it is correct. However, if the predictions of the mathematical model are seriously inconsistent with observations of the physical system it purports to describe, this suggests that either errors have been made in solving the mathematical problem, or the mathematical model itself needs refinement, or observations must be made with greater care.

The examples in this section are typical of applications in which first order differential equations arise.

EXAMPLE 1

Mixing

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water; see Figure 2.3.1. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L. Also find the flow rate that is required if the value of T is not to exceed 45 min.

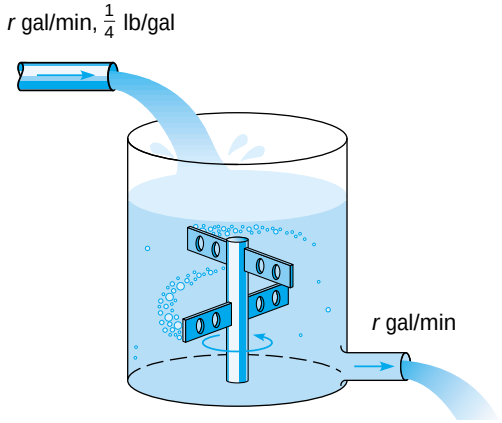


FIGURE 2.3.1 The water tank in Example 1.

We assume that salt is neither created nor destroyed in the tank. Therefore variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, dQ/dt, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}. \tag{1}$$

The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb/gal times the flow rate r gal/min, or $(r/4)$ lb/min. To find the rate at which salt leaves the tank we need to multiply the concentration of salt in the tank by the rate of outflow, r gal/min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is "well-stirred," the concentration throughout the tank is the same, namely, $[Q(t)/100]$ lb/gal. Therefore the rate at which salt leaves the tank is $[rQ(t)/100]$ lb/min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \tag{2}$$

The initial condition is

$$Q(0) = Q_0. \tag{3}$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb/gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can reach the same conclusion from a geometrical point of view by drawing a direction field for Eq. (2) for any positive value of r.

To solve the problem analytically note that Eq. (2) is both linear and separable. Rewriting it in the usual form for a linear equation, we have

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}. \tag{4}$$

Thus the integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}, \tag{5}$$

where c is an arbitrary constant. To satisfy the initial condition (3) we must choose $c = Q_0 - 25$. Therefore the solution of the initial value problem (2), (3) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} \tag{6}$$

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0e^{-rt/100}. \tag{7}$$

From Eq. (6) or (7), you can see that $Q(t) \to 25$ (lb) as $t \to \infty$, so the limiting value Q_L is 25, confirming our physical intuition. In interpreting the solution (7), note that the second term on the right side is the portion of the original salt that remains at time t, while the first term gives the amount of salt in the tank due to the action of the flow processes. Plots of the solution for $r = 3$ and for several values of Q_0 are shown in Figure 2.3.2.

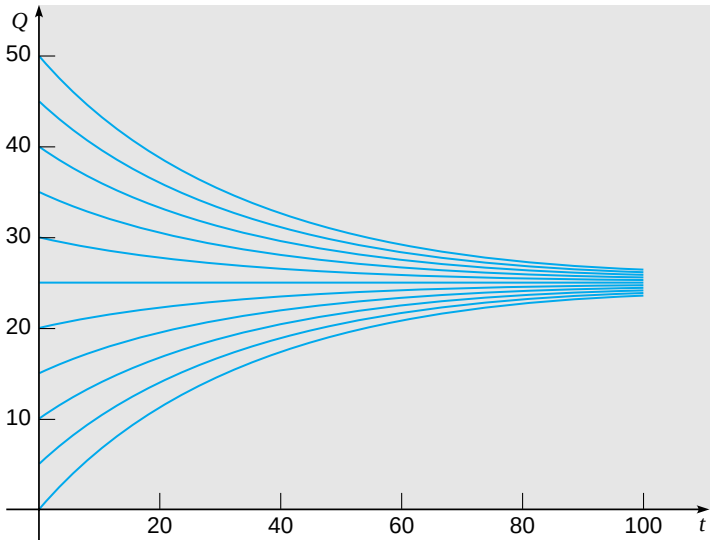


FIGURE 2.3.2 Solutions of the initial value problem (2), (3) for $r = 3$ and several values of Q_0.

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then Eq. (6) becomes

$$Q(t) = 25 + 25e^{-0.03t}. \tag{8}$$

Since 2% of 25 is 0.5, we wish to find the time T at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in Eq. (8) and solving for T, we obtain

$$T = (\ln 50)/0.03 \cong 130.4 \text{ (min)}. \tag{9}$$

To determine r so that $T = 45$, return to Eq. (6), set $t = 45$, $Q_0 = 50$, $Q(t) = 25.5$, and solve for r. The result is

$$r = (100/45) \ln 50 \cong 8.69 \text{ gal/min}. \tag{10}$$

Since this example is hypothetical, the validity of the model is not in question. If the flow rates are as stated, and if the concentration of salt in the tank is uniform, then the differential equation (1) is an accurate description of the flow process. While this particular example has no special significance, it is important that models of this kind are often used in problems involving a pollutant in a lake, or a drug in an organ of the body, for example, rather than a tank of salt water. In such cases the flow rates may not be easy to determine, or may vary with time. Similarly, the concentration may be far from uniform in some cases. Finally, the rates of inflow and outflow may be different, which means that the variation of the amount of liquid in the problem must also be taken into account.

EXAMPLE 2 Compound Interest

Suppose that a sum of money is deposited in a bank or money fund that pays interest at an annual rate r. The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as the interest rate. Financial institutions have various policies concerning compounding: some compound monthly, some weekly, some even daily. If we assume that compounding takes place *continuously*, then we can set up a simple initial value problem that describes the growth of the investment.

The rate of change of the value of the investment is dS/dt, and this quantity is equal to the rate at which interest accrues, which is the interest rate r times the current value of the investment $S(t)$. Thus

$$dS/dt = rS \tag{11}$$

is the differential equation that governs the process. Suppose that we also know the value of the investment at some particular time, say,

$$S(0) = S_0. \tag{12}$$

Then the solution of the initial value problem (11), (12) gives the balance $S(t)$ in the account at any time t. This initial value problem is readily solved, since the differential equation (11) is both linear and separable. Consequently, by solving Eqs. (11) and (12), we find that

$$S(t) = S_0 e^{rt}. \tag{13}$$

Thus a bank account with continuously compounding interest grows exponentially.

Let us now compare the results from this continuous model with the situation in which compounding occurs at finite time intervals. If interest is compounded once a year, then after t years

$$S(t) = S_0(1 + r)^t.$$

If interest is compounded twice a year, then at the end of 6 months the value of the investment is $S_0[1 + (r/2)]$, and at the end of 1 year it is $S_0[1 + (r/2)]^2$. Thus, after t years we have

$$S(t) = S_0\left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded m times per year, then

$$S(t) = S_0\left(1 + \frac{r}{m}\right)^{mt}. \tag{14}$$

The relation between formulas (13) and (14) is clarified if we recall from calculus that

$$\lim_{m\to\infty} S_0\left(1 + \frac{r}{m}\right)^{mt} = S_0 e^{rt}.$$

The same model applies equally well to more general investments in which dividends and perhaps capital gains can also accumulate, as well as interest. In recognition of this fact, we will from now on refer to r as the rate of return.

Table 2.3.1 shows the effect of changing the frequency of compounding for a return rate r of 8%. The second and third columns are calculated from Eq. (14) for quarterly and daily compounding, respectively, and the fourth column is calculated from Eq. (13) for continuous compounding. The results show that the frequency of compounding is not particularly important in most cases. For example, during a 10-year period the difference between quarterly and continuous compounding is \$17.50 per \$1000 invested, or less than \$2/year. The difference would be somewhat greater for higher rates of return and less for lower rates. From the first row in the table, we see that for the return rate $r = 8\%$, the annual yield for quarterly compounding is 8.24% and for daily or continuous compounding it is 8.33%.

TABLE 2.3.1 Growth of Capital at a Return Rate $r = 8\%$ for Several Modes of Compounding

	$S(t)/S(t_0)$ from Eq. (14)		$S(t)/S(t_0)$
Years	$m = 4$	$m = 365$	from Eq. (13)
1	1.0824	1.0833	1.0833
2	1.1717	1.1735	1.1735
5	1.4859	1.4918	1.4918
10	2.2080	2.2253	2.2255
20	4.8754	4.9522	4.9530
30	10.7652	11.0203	11.0232
40	23.7699	24.5239	24.5325

Returning now to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate k, then Eq. (11) is replaced by

$$dS/dt = rS + k,$$

or, in standard form,

$$dS/dt - rS = k, \tag{15}$$

where k is positive for deposits and negative for withdrawals.

Equation (15) is linear with the integrating factor e^{-rt}, so its general solution is

$$S(t) = ce^{rt} - (k/r),$$

where c is an arbitrary constant. To satisfy the initial condition (12) we must choose $c = S_0 + (k/r)$. Thus the solution of the initial value problem (15), (12) is

$$S(t) = S_0e^{rt} + (k/r)(e^{rt} - 1). \tag{16}$$

The first term in expression (16) is the part of $S(t)$ that is due to the return accumulated on the initial amount S_0, while the second term is the part that is due to the deposit or withdrawal rate k.

The advantage of stating the problem in this general way without specific values for S_0, r, or k lies in the generality of the resulting formula (16) for $S(t)$. With this formula we can readily compare the results of different investment programs or different rates of return.

For instance, suppose that one opens an individual retirement account (IRA) at age 25 and makes annual investments of \$2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the IRA at age 65? We have $S_0 = 0$, $r = 0.08$, and $k = \$2000$, and we wish to determine $S(40)$. From Eq. (16) we have

$$S(40) = (25{,}000)(e^{3.2} - 1) = \$588{,}313. \tag{17}$$

It is interesting to note that the total amount invested is \$80,000, so the remaining amount of \$508,313 results from the accumulated return on the investment. The balance after 40 years is also fairly sensitive to the assumed rate. For instance, $S(40) = \$508{,}948$ if $r = 0.075$ and $S(40) = \$681{,}508$ if $r = 0.085$.

Let us now examine the assumptions that have gone into the model. First, we have assumed that the return is compounded continuously and that additional capital is invested continuously. Neither of these is true in an actual financial situation. We have also assumed that the return rate r is constant for the entire period involved, whereas in fact it is likely to fluctuate considerably. Although we cannot reliably predict future rates, we can use expression (16) to determine the approximate effect of different rate projections. It is also possible to consider r and k in Eq. (15) to be functions of t rather than constants; in that case, of course, the solution may be much more complicated than Eq. (16).

The initial value problem (15), (12) and the solution (16) can also be used to analyze a number of other financial situations, including annuities, mortgages, and automobile loans among others.

EXAMPLE 3 Chemicals in a Pond

Consider a pond that initially contains 10 million gal of fresh water. Water containing an undesirable chemical flows into the pond at the rate of 5 million gal/year and the mixture in the pond flows out at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin 2t$ g/gal. Construct a mathematical model of this flow process and determine the amount of chemical in the pond at any time. Plot the solution and describe in words the effect of the variation in the incoming concentration.

Since the incoming and outgoing flows of water are the same, the amount of water in the pond remains constant at 10^7 gal. Let us denote time by t, measured in years, and the chemical by $Q(t)$, measured in grams. This example is similar to Example 1 and the same inflow/outflow principle applies. Thus

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out},$$

where "rate in" and "rate out" refer to the rates at which the chemical flows into and out of the pond, respectively. The rate at which the chemical flows in is given by

$$\text{rate in} = (5 \times 10^6) \text{ gal/yr } (2 + \sin 2t) \text{ g/gal}. \tag{18}$$

The concentration of chemical in the pond is $Q(t)/10^7$ g/gal, so the rate of flow out is

$$\text{rate out} = (5 \times 10^6) \text{ gal/yr}[Q(t)/10^7] \text{ g/gal} = Q(t)/2 \text{ g/yr}. \tag{19}$$

Thus we obtain the differential equation

$$\frac{dQ}{dt} = (5 \times 10^6)(2 + \sin 2t) - \frac{Q(t)}{2}, \tag{20}$$

where each term has the units of g/yr.

To make the coefficients more manageable, it is convenient to introduce a new dependent variable defined by $q(t) = Q(t)/10^6$ or $Q(t) = 10^6\, q(t)$. This means that $q(t)$ is measured in millions of grams, or megagrams. If we make this substitution in Eq. (20), then each term contains the factor 10^6, which can be cancelled. If we also transpose the term involving $q(t)$ to the left side of the equation, we finally have

$$\frac{dq}{dt} + \tfrac{1}{2}q = 10 + 5 \sin 2t. \tag{21}$$

Originally, there is no chemical in the pond, so the initial condition is

$$q(0) = 0. \tag{22}$$

Equation (21) is linear and, although the right side is a function of time, the coefficient of q is a constant. Thus the integrating factor is $e^{t/2}$. Multiplying Eq. (21) by this factor and integrating the resulting equation, we obtain the general solution

$$q(t) = 20 - \tfrac{40}{17}\cos 2t + \tfrac{10}{17}\sin 2t + ce^{-t/2}. \tag{23}$$

The initial condition (22) requires that $c = -300/17$, so the solution of the initial value problem (21), (22) is

$$q(t) = 20 - \tfrac{40}{17}\cos 2t + \tfrac{10}{17}\sin 2t - \tfrac{300}{17}e^{-t/2}. \tag{24}$$

A plot of the solution (24) is shown in Figure 2.3.3, along with the line $q = 20$. The exponential term in the solution is important for small t, but diminishes rapidly as t increases. Later, the solution consists of an oscillation, due to the $\sin 2t$ and $\cos 2t$ terms, about the constant level $q = 20$. Note that if the $\sin 2t$ term were not present in Eq. (21), then $q = 20$ would be the equilibrium solution of that equation.

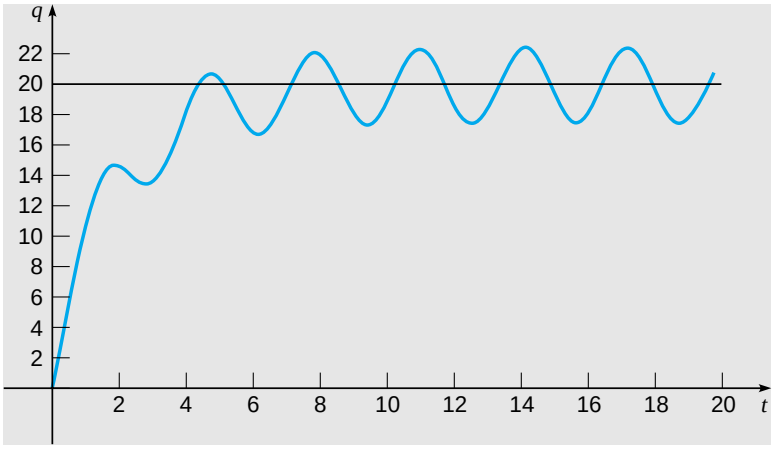


FIGURE 2.3.3 Solution of the initial value problem (21), (22).

Let us now consider the adequacy of the mathematical model itself for this problem. The model rests on several assumptions that have not yet been stated explicitly. In the first place, the amount of water in the pond is controlled entirely by the rates of flow in and out—none is lost by evaporation or by seepage into the ground, or gained by rainfall. Further, the same is also true of the chemical; it flows in and out of the pond, but none is absorbed by fish or other organisms living in the pond. In addition, we assume that the concentration of chemical in the pond is uniform throughout the pond. Whether the results obtained from the model are accurate depends strongly on the validity of these simplifying assumptions.

EXAMPLE 4 Escape Velocity

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find an expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude ξ above the surface of the earth, and the smallest initial velocity for which the body will not return to the earth; the latter is the **escape velocity**.

Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface; see Figure 2.3.4. The figure is drawn

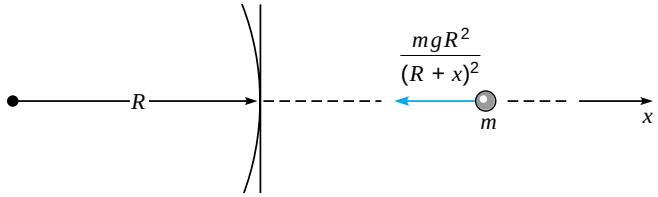


FIGURE 2.3.4 A body in the earth's gravitational field.

horizontally to remind you that gravity is directed toward the center of the earth, which is not necessarily downward from a perspective away from the earth's surface. The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = -k/(x + R)^2$, where k is a constant, R is the radius of the earth, and the minus sign signifies that $w(x)$ is directed in the negative x direction. We know that on the earth's surface $w(0)$ is given by $-mg$, where g is the acceleration due to gravity at sea level. Therefore $k = mgR^2$ and

$$w(x) = -\frac{mgR^2}{(R+x)^2}. \tag{25}$$

Since there are no other forces acting on the body, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \tag{26}$$

and the initial condition is

$$v(0) = v_0. \tag{27}$$

Unfortunately, Eq. (26) involves too many variables since it depends on t, x, and v. To remedy this situation we can eliminate t from Eq. (26) by thinking of x, rather than t, as the independent variable. Then we must express dv/dt in terms of dv/dx by the chain rule; hence

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and Eq. (26) is replaced by

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}. \tag{28}$$

Equation (28) is separable but not linear, so by separating the variables and integrating we obtain

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c. \tag{29}$$

Since $x = 0$ when $t = 0$, the initial condition (27) at $t = 0$ can be replaced by the condition that $v = v_0$ when $x = 0$. Hence $c = (v_0^2/2) - gR$ and

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}\,. \tag{30}$$

Note that Eq. (30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign if it is falling back to earth.

To determine the maximum altitude ξ that the body reaches, we set $v = 0$ and $x = \xi$ in Eq. (30) and then solve for ξ, obtaining

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}\,. \tag{31}$$

Solving Eq. (31) for v_0, we find the initial velocity required to lift the body to the altitude ξ, namely,

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}\,. \tag{32}$$

The escape velocity v_e is then found by letting $\xi \to \infty$. Consequently,

$$v_e = \sqrt{2gR}. \tag{33}$$

The numerical value of v_e is approximately 6.9 miles/sec or 11.1 km/sec.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.

PROBLEMS

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.
2. A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

▶ 5. A tank contains 100 gallons of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.
(a) Find the amount of salt in the tank at any time.
(b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.
(c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

6. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.
(a) Find the time T required for the original sum to double in value as a function of r.
(b) Determine T if $r = 7\%$.
(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

7. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.
(a) Determine the sum $S(t)$ accumulated at any time t.
(b) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.
(c) If $k =$ \$2000/year, determine the return rate r that must be obtained to have \$1 million available in 40 years.

▶ 8. Person A opens an IRA at age 25, contributes \$2000/year for 10 years, but makes no additional contributions thereafter. Person B waits until age 35 to open an IRA and contributes \$2000/year for 30 years. There is no initial investment in either case.
(a) Assuming a return rate of 8%, what is the balance in each IRA at age 65?
(b) For a constant, but unspecified, return rate r, determine the balance in each IRA at age 65 as a function of r.
(c) Plot the difference in the balances from part (b) for $0 \le r \le 0.10$.
(d) Determine the return rate for which the two IRA's have equal value at age 65.

9. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

10. A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
(a) Determine the maximum amount that this buyer can afford to borrow.
(b) Determine the total interest paid during the term of the mortgage.

11. How are the answers to Problem 10 changed if the term of the mortgage is 30 years?

12. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.
(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?
(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

13. A retired person has a sum $S(t)$ invested so as to draw interest at an annual rate r compounded continuously. Withdrawals for living expenses are made at a rate of k dollars/year; assume that the withdrawals are made continuously.
(a) If the initial value of the investment is S_0, determine $S(t)$ at any time.
(b) Assuming that S_0 and r are fixed, determine the withdrawal rate k_0 at which $S(t)$ will remain constant.
(c) If k exceeds the value k_0 found in part (b), then $S(t)$ will decrease and ultimately become zero. Find the time T at which $S(t) = 0$.
(d) Determine T if $r = 8\%$ and $k = 2k_0$.
(e) Suppose that a person retiring with capital S_0 wishes to withdraw funds at an annual rate k for not more than T years. Determine the maximum possible rate of withdrawal.
(f) How large an initial investment is required to permit an annual withdrawal of \$12,000 for 20 years, assuming an interest rate of 8%?

14. **Radiocarbon Dating.** An important tool in archeological research is radiocarbon dating. This is a means of determining the age of certain wood and plant remains, hence of animal or human bones or artifacts found buried at the same levels. The procedure was developed by the American chemist Willard Libby (1908–1980) in the early 1950s and resulted in his winning the Nobel prize for chemistry in 1960. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years[1]), measurable amounts of carbon-14 remain after many thousands of years. Libby showed that if even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, at least if this quantity is not too small. Present measurement techniques permit the use of this method for time periods up to about 50,000 years, after which the amount of carbon-14 remaining is only about 0.00236 of the original amount.
(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14.
(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.
(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

15. The population of mosquitoes in a certain area increases at a rate proportional to the current population and, in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, etc.) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

16. Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$dy/dt = (0.5 + \sin t)y/5.$$

[1] *McGraw-Hill Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

(a) If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.
(b) Suppose that the growth rate is replaced by its average value 1/10. Determine the doubling time τ in this case.
(c) Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ?
(d) Plot the solutions obtained in parts (a), (b), and (c) on a single set of axes.

► 17. Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \qquad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$ and k represents the rate of predation.
(a) Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between 1/2 and 1.
(b) Estimate the critical initial population y_c below which the population will become extinct.
(c) Choose other values of k and find the corresponding y_c for each one.
(d) Use the data you have found in parts (a) and (b) to plot y_c versus k.

18. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

19. Suppose that a room containing 1200 ft^3 of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke, containing 4% carbon monoxide, is introduced into the room at a rate of 0.1 ft^3/min, and the well-circulated mixture is allowed to leave the room at the same rate.
(a) Find an expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.
(b) Extended exposure to a carbon monoxide concentration as low as 0.00012 is harmful to the human body. Find the time τ at which this concentration is reached.

20. Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P. Note that the given assumptions neglect a number of factors that may, in some cases, be important; for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are not deposited evenly throughout the lake, but (usually) at isolated points around its periphery. The results below must be interpreted in the light of the neglect of such factors as these.
(a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \to \infty$?
(b) If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval T that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.
(c) Table 2.3.2 contains data[2] for several of the Great Lakes. Using these data determine

[2]This problem is based on R. H. Rainey, "Natural Displacement of Pollution from the Great Lakes," *Science 155* (1967), pp. 1242–1243; the information in the table was taken from that source.

TABLE 2.3.2 Volume and Flow Data for the Great Lakes

Lake	V (km$^3 \times 10^3$)	r (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

from part (b) the time T necessary to reduce the contamination of each of these lakes to 10% of the original value.

21. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/sec from the roof of a building 30 m high. Neglect air resistance.
    (a) Find the maximum height above the ground that the ball reaches.
    (b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.
    (c) Plot the graphs of velocity and position versus time.

22. Assume that the conditions are as in Problem 21 except that there is a force due to air resistance of $|v|/30$, where the velocity v is measured in m/sec.
    (a) Find the maximum height above the ground that the ball reaches.
    (b) Find the time that the ball hits the ground.
    (c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 21.

23. Assume that the conditions are as in Problem 21 except that there is a force due to air resistance of $v^2/1325$, where the velocity v is measured in m/sec.
    (a) Find the maximum height above the ground that the ball reaches.
    (b) Find the time that the ball hits the ground.
    (c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 21 and 22.

24. A sky diver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft, and opens the parachute after 10 sec of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in ft/sec.
    (a) Find the speed of the sky diver when the parachute opens.
    (b) Find the distance fallen before the parachute opens.
    (c) What is the limiting velocity v_L after the parachute opens?
    (d) Determine how long the sky diver is in the air after the parachute opens.
    (e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

25. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.
    (a) Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.
    (b) Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right],$$

$$x_m = \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right].$$

(c) Show that the quantity kv_0/mg is dimensionless.

26. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.
(a) Find the velocity $v(t)$ of the body at any time.
(b) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$, that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?
(c) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$, that is, as the mass approaches zero.

27. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes'[3] law $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.

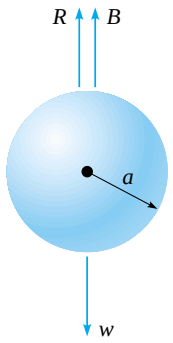


FIGURE 2.3.5 A body falling in a dense fluid.

(a) Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.
(b) In 1910 the American physicist R. A. Millikan (1868–1953) determined the charge on an electron by studying the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary ($v = 0$), and that w and B are as given above. Find a formula for e. Millikan was able to identify e as the charge on an electron and to determine that $e = 4.803 \times 10^{-10}$ esu.

▶ 28. A mass of 0.25 kg is dropped from rest in a medium offering a resistance of $0.2|v|$, where v is measured in m/sec.
(a) If the mass is dropped from a height of 30 m, find its velocity when it hits the ground.
(b) If the mass is to attain a velocity of no more than 10 m/sec, find the maximum height from which it can be dropped.
(c) Suppose that the resistive force is $k|v|$, where v is measured in m/sec and k is a constant. If the mass is dropped from a height of 30 m and must hit the ground with a velocity of no more than 10 m/sec, determine the coefficient of resistance k that is required.

[3]George Gabriel Stokes (1819–1903), professor at Cambridge, was one of the foremost applied mathematicians of the nineteenth century. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series, a subject of great interest and importance today.

29. Suppose that a rocket is launched straight up from the surface of the earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of the earth. Neglect air resistance.
(a) Find an expression for the velocity v in terms of the distance x from the surface of the earth.
(b) Find the time required for the rocket to go 240,000 miles (the approximate distance from the earth to the moon). Assume that $R = 4000$ miles.

30. Find the escape velocity for a body projected upward with an initial velocity v_0 from a point $x_0 = \xi R$ above the surface of the earth, where R is the radius of the earth and ξ is a constant. Neglect air resistance. Find the initial altitude from which the body must be launched in order to reduce the escape velocity to 85% of its value at the earth's surface.

▶ 31. Let $v(t)$ and $w(t)$ be the horizontal and vertical components of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$dv/dt = 0, \qquad dw/dt = -g.$$

(a) Show that

$$v = u\cos A, \qquad w = -gt + u\sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.
(b) Let $x(t)$ and $y(t)$, respectively, be the horizontal and vertical coordinates of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t.
(c) Let $g = 32$ ft/sec^2, $u = 125$ ft/sec, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A, that is, plot $x(t)$ and $y(t)$ parametrically.
(d) Suppose the outfield wall is at a distance L and has height H. Find a relation between u and A that must be satisfied if the ball is to clear the wall.
(e) Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part (d), find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/sec.
(f) For $L = 350$ and $H = 10$ find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

▶ 32. A more realistic model (than that in Problem 31) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$dv/dt = -rv, \qquad dw/dt = -g - rw,$$

where r is the coefficient of resistance.
(a) Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A.
(b) Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.
(c) Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A. How do the trajectories differ from those in Problem 31 with $r = 0$?
(d) Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 31(f).

33. **Brachistochrone Problem.** One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli, and by Isaac Newton, Gottfried Leibniz, and Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem it is convenient to take the origin as the upper point P and to orient the axes as shown in Figure 2.3.6. The lower point Q has coordinates (x_0, y_0). It is

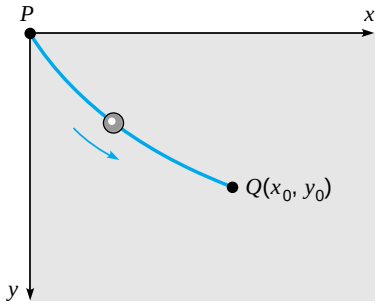


FIGURE 2.3.6 The brachistochrone.

then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2, \tag{i}$$

where k^2 is a certain positive constant to be determined later.
(a) Solve Eq. (i) for y'. Why is it necessary to choose the positive square root?
(b) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t. \tag{ii}$$

Show that the equation found in part (a) then takes the form

$$2k^2 \sin^2 t \, dt = dx. \tag{iii}$$

(c) Letting $\theta = 2t$, show that the solution of Eq. (iii) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin\theta)/2, \qquad y = k^2(1 - \cos\theta)/2. \tag{iv}$$

Equations (iv) are parametric equations of the solution of Eq. (i) that passes through $(0, 0)$. The graph of Eqs. (iv) is called a **cycloid**.
(d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$.

2.4 Differences Between Linear and Nonlinear Equations

Up to now, we have been primarily concerned with showing that first order differential equations can be used to investigate many different kinds of problems in the natural sciences, and with presenting methods of solving such equations if they are either linear or separable. Now it is time to turn our attention to some more general questions about differential equations and to explore in more detail some important ways in which nonlinear equations differ from linear ones.

Existence and Uniqueness of Solutions. So far, we have encountered a number of initial value problems, each of which had a solution and apparently only one solution. This raises the question of whether this is true of all initial value problems for first order

equations. In other words, does every initial value problem have exactly one solution? This may be an important question even for nonmathematicians. If you encounter an initial value problem in the course of investigating some physical problem, you might want to know that it has a solution before spending very much time and effort in trying to find it. Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions. For linear equations the answers to these questions are given by the following fundamental theorem.

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t) \tag{1}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0, \tag{2}$$

where y_0 is an arbitrary prescribed initial value.

Observe that Theorem 2.4.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and *uniqueness* of the solution of the initial value problem (1), (2). In addition, it states that the solution exists throughout any interval I containing the initial point t_0 in which the coefficients p and g are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous. Such points can often be identified at a glance.

The proof of this theorem is partly contained in the discussion in Section 2.1 leading to the formula [Eq. (35) in Section 2.1]

$$y = \frac{\int \mu(s)g(s)\,ds + c}{\mu(t)}, \tag{3}$$

where

$$\mu(t) = \exp \int p(s)\,ds. \tag{4}$$

The derivation in Section 2.1 shows that if Eq. (1) has a solution, then it must be given by Eq. (3). By looking slightly more closely at that derivation, we can also conclude that the differential equation (1) must indeed have a solution. Since p is continuous for $\alpha < t < \beta$, it follows that μ is defined in this interval and is a nonzero differentiable function. Upon multiplying Eq. (1) by $\mu(t)$ we obtain

$$[\mu(t)y]' = \mu(t)g(t). \tag{5}$$

Since both μ and g are continuous, the function μg is integrable, and Eq. (3) follows from Eq. (5). Further, the integral of μg is differentiable, so y as given by Eq. (3) exists and is differentiable throughout the interval $\alpha < t < \beta$. By substituting the expression for y in Eq. (3) into either Eq. (1) or Eq. (5), one can easily verify that this expression satisfies the differential equation throughout the interval $\alpha < t < \beta$. Finally, the initial

condition (2) determines the constant c uniquely, so there is only one solution of the initial value problem, thus completing the proof.

Equation (4) determines the integrating factor $\mu(t)$ only up to a multiplicative factor that depends on the lower limit of integration. If we choose this lower limit to be t_0, then

$$\mu(t) = \exp \int_{t_0}^{t} p(s)\, ds, \tag{6}$$

and it follows that $\mu(t_0) = 1$. Using the integrating factor given by Eq. (6) and choosing the lower limit of integration in Eq. (3) also to be t_0, we obtain the general solution of Eq. (1) in the form

$$y = \frac{\int_{t_0}^{t} \mu(s)g(s)\, ds + c}{\mu(t)}.$$

To satisfy the initial condition (2) we must choose $c = y_0$. Thus the solution of the initial value problem (1), (2) is

$$y = \frac{\int_{t_0}^{t} \mu(s)g(s)\, ds + y_0}{\mu(t)}, \tag{7}$$

where $\mu(t)$ is given by Eq. (6).

Turning now to nonlinear differential equations, we must replace Theorem 2.4.1 by a more general theorem, such as the following.

Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{8}$$

Observe that the hypotheses in Theorem 2.4.2 reduce to those in Theorem 2.4.1 if the differential equation is linear. For then $f(t, y) = -p(t)y + g(t)$ and $\partial f(t, y)/\partial y = -p(t)$, so the continuity of f and $\partial f/\partial y$ is equivalent to the continuity of p and g in this case. The proof of Theorem 2.4.1 was comparatively simple because it could be based on the expression (3) that gives the solution of an arbitrary linear equation. There is no corresponding expression for the solution of the differential equation (8), so the proof of Theorem 2.4.2 is much more difficult. It is discussed to some extent in Section 2.8 and in greater depth in more advanced books on differential equations.

Here we note that the conditions stated in Theorem 2.4.2 are sufficient to guarantee the existence of a unique solution of the initial value problem (8) in some interval $t_0 - h < t < t_0 + h$, but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function f. In fact, the existence of a solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

We now consider some examples.

EXAMPLE 1

Use Theorem 2.4.1 to find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \tag{9}$$

$$y(1) = 2 \tag{10}$$

has a unique solution.

Rewriting Eq. (9) in the standard form (1), we have

$$y' + (2/t)y = 4t, \tag{11}$$

so $p(t) = 2/t$ and $g(t) = 4t$. Thus, for this equation, g is continuous for all t, while p is continuous only for $t < 0$ or for $t > 0$. The interval $t > 0$ contains the initial point; consequently, Theorem 2.4.1 guarantees that the problem (9), (10) has a unique solution on the interval $0 < t < \infty$. In Example 4 of Section 2.1 we found the solution of this initial value problem to be

$$y = t^2 + \frac{1}{t^2}, \qquad t > 0. \tag{12}$$

Now suppose that the initial condition (10) is changed to $y(-1) = 2$. Then Theorem 2.4.1 asserts the existence of a unique solution for $t < 0$. As you can readily verify, the solution is again given by Eq. (12), but now on the interval $-\infty < t < 0$.

EXAMPLE 2

Apply Theorem 2.4.2 to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1. \tag{13}$$

Note that Theorem 2.4.1 is not applicable to this problem since the differential equation is nonlinear. To apply Theorem 2.4.2, observe that

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2}.$$

Thus both these functions are continuous everywhere except on the line $y = 1$. Consequently, a rectangle can be drawn about the initial point $(0, -1)$ in which both f and $\partial f/\partial y$ are continuous. Therefore Theorem 2.4.2 guarantees that the initial value problem has a unique solution in some interval about $x = 0$. However, even though the rectangle can be stretched infinitely far in both the positive and negative x directions, this does not necessarily mean that the solution exists for all x. Indeed, the initial value problem (13) was solved in Example 2 of Section 2.2 and the solution exists only for $x > -2$.

Now suppose we change the initial condition to $y(0) = 1$. The initial point now lies on the line $y = 1$ so no rectangle can be drawn about it within which f and $\partial f/\partial y$ are continuous. Consequently, Theorem 2.4.2 says nothing about possible solutions of this modified problem. However, if we separate the variables and integrate, as in Section 2.2, we find that

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

Further, if $x = 0$ and $y = 1$, then $c = -1$. Finally, by solving for y, we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}. \tag{14}$$

Equation (14) provides two functions that satisfy the given differential equation for $x > 0$ and also satisfy the initial condition $y(0) = 1$.

EXAMPLE 3

Consider the initial value problem

$$y' = y^{1/3}, \qquad y(0) = 0 \tag{15}$$

for $t \geq 0$. Apply Theorem 2.4.2 to this initial value problem, and then solve the problem.

The function $f(t, y) = y^{1/3}$ is continuous everywhere, but $\partial f/\partial y = y^{-2/3}/3$ is not continuous when $y = 0$. Thus Theorem 2.4.2 does not apply to this problem and no conclusion can be drawn from it. However, by the remark following Theorem 2.4.2 the continuity of f does assure the existence of solutions, but not their uniqueness.

To understand the situation more clearly, we must actually solve the problem, which is easy to do since the differential equation is separable. Thus we have

$$y^{-1/3}dy = dt,$$

so

$$\tfrac{3}{2}y^{2/3} = t + c$$

and

$$y = \left[\tfrac{2}{3}(t + c)\right]^{3/2}.$$

The initial condition is satisfied if $c = 0$, so

$$y = \phi_1(t) = \left(\tfrac{2}{3}t\right)^{3/2}, \qquad t \geq 0 \tag{16}$$

satisfies both of Eqs. (15). On the other hand, the function

$$y = \phi_2(t) = -\left(\tfrac{2}{3}t\right)^{3/2}, \qquad t \geq 0 \tag{17}$$

is also a solution of the initial value problem. Moreover, the function

$$y = \psi(t) = 0, \qquad t \geq 0 \tag{18}$$

is yet another solution. Indeed, it is not hard to show that, for an arbitrary positive t_0, the functions

$$y = \chi(t) = \begin{cases} 0, & \text{if } 0 \leq t < t_0, \\ \pm\left[\tfrac{2}{3}(t - t_0)\right]^{3/2}, & \text{if } t \geq t_0 \end{cases} \tag{19}$$

are continuous, differentiable (in particular at $t = t_0$), and are solutions of the initial value problem (15). Hence this problem has an infinite family of solutions; see Figure 2.4.1, where a few of these solutions are shown.

As already noted, the nonuniqueness of the solutions of the problem (15) does not contradict the existence and uniqueness theorem, since the theorem is not applicable if the initial point lies on the t-axis. If (t_0, y_0) is any point not on the t-axis, however, then the theorem guarantees that there is a unique solution of the differential equation $y' = y^{1/3}$ passing through (t_0, y_0).

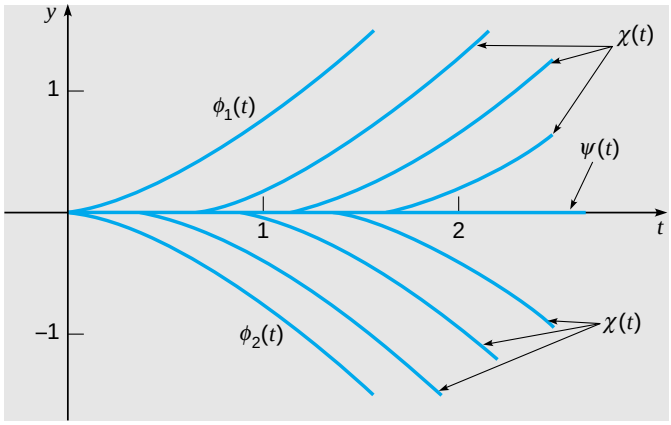


FIGURE 2.4.1 Several solutions of the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

Interval of Definition. According to Theorem 2.4.1, the solution of a linear equation (1),

$$y' + p(t)y = g(t),$$

subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about $t = t_0$ in which the functions p and g are continuous. Thus, vertical asymptotes or other discontinuities in the solution can occur only at points of discontinuity of p or g. For instance, the solutions in Example 1 (with one exception) are asymptotic to the y-axis, corresponding to the discontinuity at $t = 0$ in the coefficient $p(t) = 2/t$, but none of the solutions has any other point where it fails to exist and to be differentiable. The one exceptional solution shows that solutions may sometimes remain continuous even at points of discontinuity of the coefficients.

On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 2.4.2, the interval in which a solution exists may be difficult to determine. The solution $y = \phi(t)$ is certain to exist as long as the point $[t, \phi(t)]$ remains within a region in which the hypotheses of Theorem 2.4.2 are satisfied. This is what determines the value of h in that theorem. However, since $\phi(t)$ is usually not known, it may be impossible to locate the point $[t, \phi(t)]$ with respect to this region. In any case, the interval in which a solution exists may have no simple relationship to the function f in the differential equation $y' = f(t, y)$. This is illustrated by the following example.

EXAMPLE 4

Solve the initial value problem

$$y' = y^2, \qquad y(0) = 1, \tag{20}$$

and determine the interval in which the solution exists.

Theorem 2.4.2 guarantees that this problem has a unique solution since $f(t, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere. To find the solution we separate the variables and integrate with the result that

$$y^{-2}\, dy = dt \tag{21}$$

and

$$-y^{-1} = t + c.$$

Then, solving for y, we have

$$y = -\frac{1}{t+c}. \tag{22}$$

To satisfy the initial condition we must choose $c = -1$, so

$$y = \frac{1}{1-t} \tag{23}$$

is the solution of the given initial value problem. Clearly, the solution becomes unbounded as $t \to 1$; therefore, the solution exists only in the interval $-\infty < t < 1$. There is no indication from the differential equation itself, however, that the point $t = 1$ is in any way remarkable. Moreover, if the initial condition is replaced by

$$y(0) = y_0, \tag{24}$$

then the constant c in Eq. (22) must be chosen to be $c = -1/y_0$, and it follows that

$$y = \frac{y_0}{1 - y_0 t} \tag{25}$$

is the solution of the initial value problem with the initial condition (24). Observe that the solution (25) becomes unbounded as $t \to 1/y_0$, so the interval of existence of the solution is $-\infty < t < 1/y_0$ if $y_0 > 0$, and is $1/y_0 < t < \infty$ if $y_0 < 0$. This example illustrates another feature of initial value problems for nonlinear equations; namely, the singularities of the solution may depend in an essential way on the initial conditions as well as on the differential equation.

General Solution. Another way in which linear and nonlinear equations differ is in connection with the concept of a general solution. For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant. For instance, for the differential equation $y' = y^2$ in Example 4, the expression in Eq. (22) contains an arbitrary constant, but does not include all solutions of the differential equation. To show this, observe that the function $y = 0$ for all t is certainly a solution of the differential equation, but it cannot be obtained from Eq. (22) by assigning a value to c. In this example we might anticipate that something of this sort might happen because to rewrite the original differential equation in the form (21), we must require that y is not zero. However, the existence of "additional" solutions is not uncommon for nonlinear equations; a less obvious example is given in Problem 22. Thus we will use the term "general solution" only when discussing linear equations.

Implicit Solutions. Recall again that, for an initial value problem for a first order linear equation, Eq. (7) provides an explicit formula for the solution $y = \phi(t)$. As long as the necessary antiderivatives can be found, the value of the solution at any point can be determined merely by substituting the appropriate value of t into the equation. The

situation for nonlinear equations is much less satisfactory. Usually, the best that we can hope for is to find an equation

$$F(t, y) = 0 \tag{26}$$

involving t and y that is satisfied by the solution $y = \phi(t)$. Even this can be done only for differential equations of certain particular types, of which separable equations are the most important. The equation (26) is called an integral, or first integral, of the differential equation, and (as we have already noted) its graph is an integral curve, or perhaps a family of integral curves. Equation (26), assuming it can be found, defines the solution implicitly; that is, for each value of t we must solve Eq. (26) to find the corresponding value of y. If Eq. (26) is simple enough, it may be possible to solve it for y by analytical means and thereby obtain an explicit formula for the solution. However, more frequently this will not be possible, and you will have to resort to a numerical calculation to determine the value of y for a given value of t. Once several pairs of values of t and y have been calculated, it is often helpful to plot them and then to sketch the integral curve that passes through them. You should arrange for a computer to do this for you, if possible.

Examples 2, 3, and 4 are nonlinear problems in which it is easy to solve for an explicit formula for the solution $y = \phi(t)$. On the other hand, Examples 1 and 3 in Section 2.2 are cases in which it is better to leave the solution in implicit form, and to use numerical means to evaluate it for particular values of the independent variable. The latter situation is more typical; unless the implicit relation is quadratic in y, or has some other particularly simple form, it is unlikely that it can be solved exactly by analytical methods. Indeed, more often than not, it is impossible even to find an implicit expression for the solution of a first order nonlinear equation.

Graphical or Numerical Construction of Integral Curves. Because of the difficulty in obtaining exact analytic solutions of nonlinear differential equations, methods that yield approximate solutions or other qualitative information about solutions are of correspondingly greater importance. We have already described in Section 1.1 how the direction field of a differential equation can be constructed. The direction field can often show the qualitative form of solutions and can also be helpful in identifying regions of the ty-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigation. Graphical methods for first order equations are discussed further in Section 2.5. An introduction to numerical methods for first order equations is given in Section 2.7 and a systematic discussion of numerical methods appears in Chapter 8. However, it is not necessary to study the numerical algorithms themselves in order to use effectively one of the many software packages that generate and plot numerical approximations to solutions of initial value problems.

Summary. Linear equations have several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.

2. There is an expression for the solution, namely, Eq. (3) or Eq. (7). Moreover, although it involves two integrations, the expression is an explicit one for the solution $y = \phi(t)$ rather than an equation that defines ϕ implicitly.

3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t, then the solution also exists and is continuous for all t.

None of these statements is true, in general, of nonlinear equations. While a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as the differential equation.

PROBLEMS

In each of Problems 1 through 6 determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2$
2. $t(t-4)y'' + (t-2)y' + y = 0, \qquad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1$
5. $(4-t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \qquad y(2) = 3$

In each of Problems 7 through 12 state the region in the ty-plane where the hypotheses of Theorem 2.4.2 are satisfied. Thus there is a unique solution through each given initial point in this region.

7. $y' = \dfrac{t-y}{2t+5y}$
8. $y' = (1-t^2-y^2)^{1/2}$
9. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$
10. $y' = (t^2+y^2)^{3/2}$
11. $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$
12. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

13. $y' = -4t/y, \qquad y(0) = y_0$
14. $y' = 2ty^2, \qquad y(0) = y_0$
15. $y' + y^3 = 0, \qquad y(0) = y_0$
16. $y' = t^2/y(1+t^3), \qquad y(0) = y_0$

In each of Problems 17 through 20 draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases, and how their behavior depends on the initial value y_0 when $t = 0$.

17. $y' = ty(3-y)$
18. $y' = y(3-ty)$
19. $y' = -y(3-ty)$
20. $y' = t - 1 - y^2$

21. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.
    (a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

(b) Is there a solution that passes through the point (2, 1)? If so, find it.
(c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

22. (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + (t^2 + 4y)^{1/2}}{2}, \qquad y(2) = -1.$$

Where are these solutions valid?
(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
(c) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

23. (a) Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c.
(b) Show that $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$ for $t > 0$, but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part (b) is nonlinear, while that of part (a) is linear.

24. Show that if $y = \phi(t)$ is a solution of $y' + p(t)y = 0$, then $y = c\phi(t)$ is also a solution for any value of the constant c.

25. Let $y = y_1(t)$ be a solution of

$$y' + p(t)y = 0, \tag{i}$$

and let $y = y_2(t)$ be a solution of

$$y' + p(t)y = g(t). \tag{ii}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of Eq. (ii).

26. (a) Show that the solution (3) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \tag{i}$$

where c is an arbitrary constant. Identify the functions y_1 and y_2.
(b) Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \tag{ii}$$

corresponding to $g(t) = 0$.
(c) Show that y_2 is a solution of the full linear equation (1). We see later (for example, in Section 3.6) that solutions of higher order linear equations have a pattern similar to Eq. (i).

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

27. (a) Solve Bernoulli's equation when $n = 0$; when $n = 1$.
(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

In each of Problems 28 through 31 the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

28. $t^2y' + 2ty - y^3 = 0, \qquad t > 0$
29. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.
30. $y' = \epsilon y - \sigma y^3, \epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.
31. $dy/dt = (\Gamma \cos t + T)y - y^3$, where Γ and T are constants. This equation also occurs in the study of the stability of fluid flow.

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0.

32. Solve the initial value problem

$$y' + 2y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

33. Solve the initial value problem

$$y' + p(t)y = 0, \qquad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

2.5 Autonomous Equations and Population Dynamics





An important class of first order equations are those in which the independent variable does not appear explicitly. Such equations are called **autonomous** and have the form

$$dy/dt = f(y). \tag{1}$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Sections 1.1 and 1.2 we have already considered the special case of Eq. (1) in which $f(y) = ay + b$.

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, but our main object now is to show how geometric methods can be used to obtain important qualitative information directly from the differential equation, without solving the equation. Of fundamental importance in this effort are the concepts of stability and instability of solutions of differential equations. These ideas were introduced in

Chapter 1, are discussed further here, and will be examined in greater depth and in a more general setting in Chapter 9.

Exponential Growth. Let $y = \phi(t)$ be the population of the given species at time t. The simplest hypothesis concerning the variation of population is that the rate of change of y is proportional[4] to the current value of y, that is,

$$dy/dt = ry, \tag{2}$$

where the constant of proportionality r is called the **rate of growth or decline**, depending on whether it is positive or negative. Here, we assume that $r > 0$, so the population is growing.

Solving Eq. (2) subject to the initial condition

$$y(0) = y_0, \tag{3}$$

we obtain

$$y = y_0 e^{rt}. \tag{4}$$

Thus the mathematical model consisting of the initial value problem (2), (3) with $r > 0$ predicts that the population will grow exponentially for all time, as shown in Figure 2.5.1. Under ideal conditions Eq. (4) has been observed to be reasonably accurate for many populations, at least for limited periods of time. However, it is clear that such ideal conditions cannot continue indefinitely; eventually, limitations on space, food supply, or other resources will reduce the growth rate and bring an end to uninhibited exponential growth.

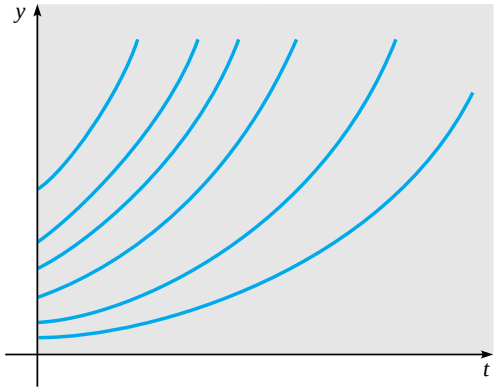


FIGURE 2.5.1 Exponential growth: y versus t for $dy/dt = ry$.

Logistic Growth. To take account of the fact that the growth rate actually depends on the population, we replace the constant r in Eq. (2) by a function $h(y)$ and thereby obtain the modified equation

$$dy/dt = h(y)y. \tag{5}$$

[4]It was apparently the British economist Thomas Malthus (1766–1834) who first observed that many biological populations increase at a rate proportional to the population. His first paper on populations appeared in 1798.

We now want to choose $h(y)$ so that $h(y) \cong r > 0$ when y is small, $h(y)$ decreases as y grows larger, and $h(y) < 0$ when y is sufficiently large. The simplest function having these properties is $h(y) = r - ay$, where a is also a positive constant. Using this function in Eq. (5), we obtain

$$dy/dt = (r - ay)y. \tag{6}$$

Equation (6) is known as the Verhulst[5] equation or the **logistic equation**. It is often convenient to write the logistic equation in the equivalent form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \tag{7}$$

where $K = r/a$. The constant r is called the **intrinsic growth rate**, that is, the growth rate in the absence of any limiting factors. The interpretation of K will become clear shortly.

We first seek solutions of Eq. (7) of the simplest possible type, that is, constant functions. For such a solution $dy/dt = 0$ for all t, so any constant solution of Eq. (7) must satisfy the algebraic equation

$$r(1 - y/K)y = 0.$$

Thus the constant solutions are $y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$. These solutions are called **equilibrium solutions** of Eq. (7) because they correspond to no change or variation in the value of y as t increases. In the same way, any equilibrium solutions of

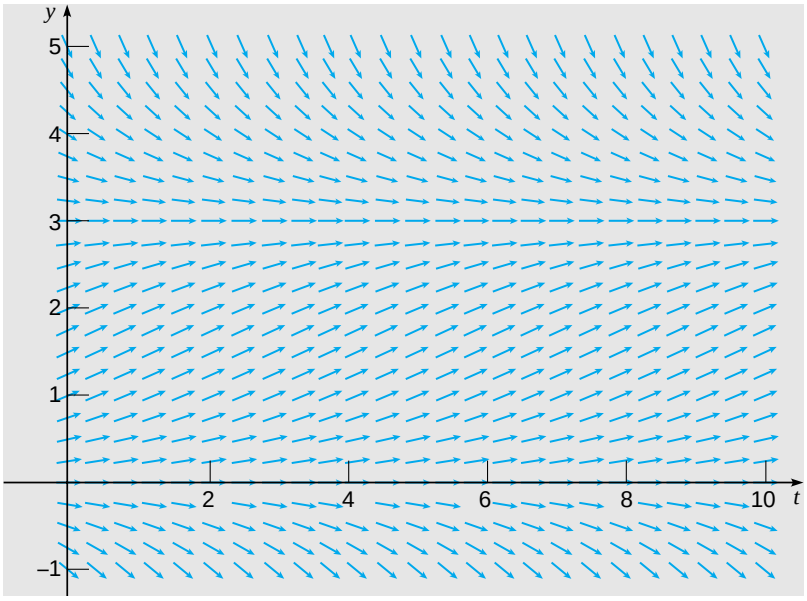


FIGURE 2.5.2 Direction field for $dy/dt = r(1 - y/K)y$ with $r = 1/2$ and $K = 3$.

[5]P. F. Verhulst (1804–1849) was a Belgian mathematician who introduced Eq. (6) as a model for human population growth in 1838. He referred to it as logistic growth; hence Eq. (6) is often called the logistic equation. He was unable to test the accuracy of his model because of inadequate census data, and it did not receive much attention until many years later. Reasonable agreement with experimental data was demonstrated by R. Pearl (1930) for *Drosophila melanogaster* (fruit fly) populations, and by G. F. Gause (1935) for *Paramecium* and *Tribolium* (flour beetle) populations.

the more general Eq. (1) can be found by locating the roots of $f(y) = 0$. The zeros of $f(y)$ are also called **critical points**.

To visualize other solutions of Eq. (7), let us draw a direction field for a typical case, as shown in Figure 2.5.2 when $r = 1/2$ and $K = 3$. Observe that the elements of the direction field have the same slope at all points on each particular horizontal line although the slope changes from one line to another. This property follows from the fact that the right side of the logistic equation does not depend on the independent variable t. Observe also that the equilibrium solutions $y = 0$ and $y = 3$ seem to be of particular importance. Other solutions appear to approach the solution $y = 3$ asymptotically as $t \to \infty$, whereas solutions on either side of $y = 0$ diverge from it.

To understand more clearly the nature of the solutions of Eq. (7) and to sketch their graphs quickly, we can start by drawing the graph of $f(y)$ versus y. In the case of Eq. (7), $f(y) = r(1 - y/K)y$, so the graph is the parabola shown in Figure 2.5.3. The intercepts are $(0, 0)$ and $(K, 0)$, corresponding to the critical points of Eq. (7), and the vertex of the parabola is $(K/2, rK/4)$. Observe that $dy/dt > 0$ for $0 < y < K$; therefore, y is an increasing function of t when y is in this interval; this is indicated by the rightward-pointing arrows near the y-axis in Figure 2.5.3, or by the upward-pointing arrows in Figure 2.5.4. Similarly, if $y > K$, then $dy/dt < 0$; hence y is decreasing, as indicated by the leftward-pointing arrow in Figure 2.5.3, or by the downward-pointing

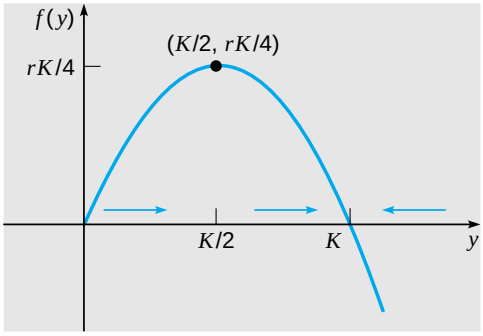


FIGURE 2.5.3 $f(y)$ versus y for $dy/dt = r(1 - y/K)y$.

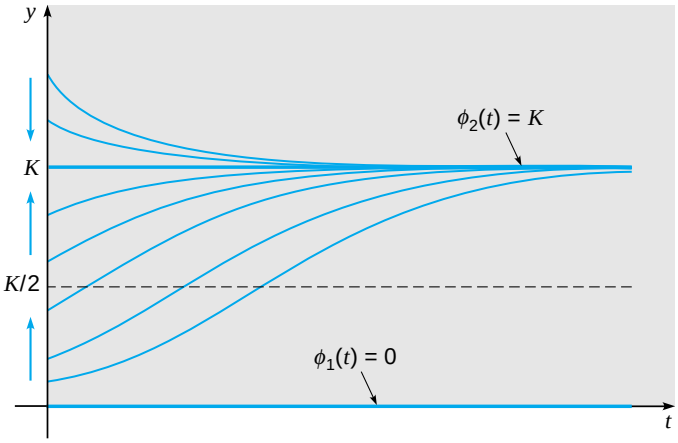


FIGURE 2.5.4 Logistic growth: y versus t for $dy/dt = r(1 - y/K)y$.

arrow in Figure 2.5.4. Further, note that if y is near zero or K, then the slope $f(y)$ is near zero, so the solution curves are relatively flat. They become steeper as the value of y leaves the neighborhood of zero or K. These observations mean that the graphs of solutions of Eq. (7) must have the general shape shown in Figure 2.5.4, regardless of the values of r and K.

To carry the investigation one step further, we can determine the concavity of the solution curves and the location of inflection points by finding d^2y/dt^2. From the differential equation (1) we obtain (using the chain rule)

$$\frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)\,f(y). \tag{8}$$

The graph of y versus t is concave up when $y'' > 0$, that is, when f and f' have the same sign. Similarly, it is concave down when $y'' < 0$, which occurs when f and f' have opposite signs. The signs of f and f' can be easily identified from the graph of $f(y)$ versus y. Inflection points may occur when $f'(y) = 0$.

In the case of Eq. (7) solutions are concave up for $0 < y < K/2$ where f is positive and increasing (see Figure 2.5.3), so that both f and f' are positive. Solutions are also concave up for $y > K$ where f is negative and decreasing (both f and f' are negative). For $K/2 < y < K$ solutions are concave down since here f is positive and decreasing, so f is positive but f' is negative. There is an inflection point whenever the graph of y versus t crosses the line $y = K/2$. The graphs in Figure 2.5.4 exhibit these properties.

Finally, recall that Theorem 2.4.2, the fundamental existence and uniqueness theorem, guarantees that two different solutions never pass through the same point. Hence, while solutions approach the equilibrium solution $y = K$ as $t \to \infty$, they do not attain this value at any finite time. Since K is the upper bound that is approached, but not exceeded, by growing populations starting below this value, it is natural to refer to K as the **saturation level**, or the **environmental carrying capacity**, for the given species.

A comparison of Figures 2.5.1 and 2.5.4 reveals that solutions of the nonlinear equation (6) are strikingly different from those of the linear equation (1), at least for large values of t. Regardless of the value of K, that is, no matter how small the nonlinear term in Eq. (6), solutions of that equation approach a finite value as $t \to \infty$, whereas solutions of Eq. (1) grow (exponentially) without bound as $t \to \infty$. Thus even a tiny nonlinear term in the differential equation has a decisive effect on the solution for large t.

In many situations it is sufficient to have the qualitative information about the solution $y = \phi(t)$ of Eq. (7) that is shown in Figure 2.5.4. This information was obtained entirely from the graph of $f(y)$ versus y, and without solving the differential equation (7). However, if we wish to have a more detailed description of logistic growth—for example, if we wish to know the value of the population at some particular time—then we must solve Eq. (7) subject to the initial condition (3). Provided that $y \neq 0$ and $y \neq K$, we can write Eq. (7) in the form

$$\frac{dy}{(1 - y/K)y} = r\,dt.$$

Using a partial fraction expansion on the left side, we have

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right) dy = r\, dt.$$

Then by integrating both sides we obtain

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c, \tag{9}$$

where c is an arbitrary constant of integration to be determined from the initial condition $y(0) = y_0$. We have already noted that if $0 < y_0 < K$, then y remains in this interval for all time. Thus in this case we can remove the absolute value bars in Eq. (9), and by taking the exponential of both sides, we find that

$$\frac{y}{1 - (y/K)} = Ce^{rt}, \tag{10}$$

where $C = e^c$. To satisfy the initial condition $y(0) = y_0$ we must choose $C = y_0/[1 - (y_0/K)]$. Using this value for C in Eq. (10) and solving for y, we obtain

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}. \tag{11}$$

We have derived the solution (11) under the assumption that $0 < y_0 < K$. If $y_0 > K$, then the details of dealing with Eq. (9) are only slightly different, and we leave it to you to show that Eq. (11) is also valid in this case. Finally, note that Eq. (11) also contains the equilibrium solutions $y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$ corresponding to the initial conditions $y_0 = 0$ and $y_0 = K$, respectively.

All the qualitative conclusions that we reached earlier by geometric reasoning can be confirmed by examining the solution (11). In particular, if $y_0 = 0$, then Eq. (11) requires that $y(t) = 0$ for all t. If $y_0 > 0$, and if we let $t \to \infty$ in Eq. (11), then we obtain

$$\lim_{t\to\infty} y(t) = y_0 K/y_0 = K.$$

Thus for each $y_0 > 0$ the solution approaches the equilibrium solution $y = \phi_2(t) = K$ asymptotically (in fact, exponentially) as $t \to \infty$. Therefore we say that the constant solution $\phi_2(t) = K$ is an **asymptotically stable solution** of Eq. (7), or that the point $y = K$ is an asymptotically stable equilibrium or critical point. This means that after a long time the population is close to the saturation level K regardless of the initial population size, as long as it is positive.

On the other hand, the situation for the equilibrium solution $y = \phi_1(t) = 0$ is quite different. Even solutions that start very near zero grow as t increases and, as we have seen, approach K as $t \to \infty$. We say that $\phi_1(t) = 0$ is an **unstable equilibrium solution** or that $y = 0$ is an unstable equilibrium or critical point. This means that the only way to guarantee that the solution remains near zero is to make sure its initial value is *exactly* equal to zero.

EXAMPLE 1

The logistic model has been applied to the natural growth of the halibut population in certain areas of the Pacific Ocean.[6] Let y, measured in kilograms, be the total mass, or biomass, of the halibut population at time t. The parameters in the logistic equation are estimated to have the values $r = 0.71/\text{year}$ and $K = 80.5 \times 10^6$ kg. If the initial biomass is $y_0 = 0.25K$, find the biomass 2 years later. Also find the time τ for which $y(\tau) = 0.75K$.

It is convenient to scale the solution (11) to the carrying capacity K; thus we write Eq. (11) in the form

$$\frac{y}{K} = \frac{y_0/K}{(y_0/K) + [1 - (y_0/K)]e^{-rt}}. \tag{12}$$

Using the data given in the problem, we find that

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-1.42}} \cong 0.5797.$$

Consequently, $y(2) \cong 46.7 \times 10^6$ kg.

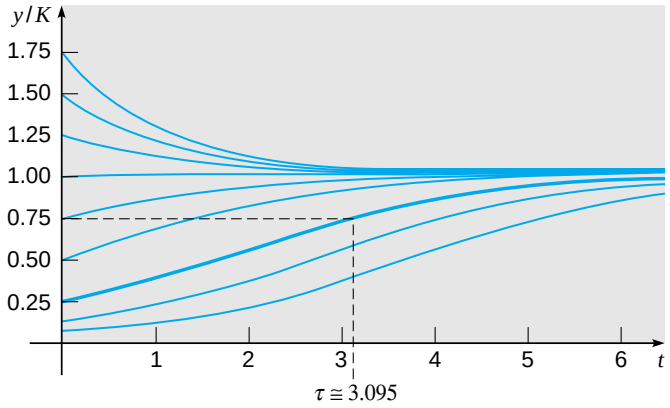


FIGURE 2.5.5 y/K versus t for population model of halibut in the Pacific Ocean.

To find τ we can first solve Eq. (12) for t. We obtain

$$e^{-rt} = \frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]};$$

hence

$$t = -\frac{1}{r} \ln \frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]}. \tag{13}$$

Using the given values of r and y_0/K and setting $y/K = 0.75$, we find that

$$\tau = -\frac{1}{0.71} \ln \frac{(0.25)(0.25)}{(0.75)(0.75)} = \frac{1}{0.71} \ln 9 \cong 3.095 \text{ years}.$$

[6] A good source of information on the population dynamics and economics involved in making efficient use of a renewable resource, with particular emphasis on fisheries, is the book by Clark (see the references at the end of this chapter). The parameter values used here are given on page 53 of this book and were obtained as a result of a study by M. S. Mohring.

The graphs of y/K versus t for the given parameter values and for several initial conditions are shown in Figure 2.5.5.

A Critical Threshold. We now turn to a consideration of the equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \tag{14}$$

where r and T are given positive constants. Observe that (except for replacing the parameter K by T) this equation differs from the logistic equation (7) only in the presence of the minus sign on the right side. However, as we will see, the solutions of Eq. (14) behave very differently from those of Eq. (7).

For Eq. (14) the graph of $f(y)$ versus y is the parabola shown in Figure 2.5.6. The intercepts on the y-axis are the critical points $y = 0$ and $y = T$, corresponding to the equilibrium solutions $\phi_1(t) = 0$ and $\phi_2(t) = T$. If $0 < y < T$, then $dy/dt < 0$, and y decreases as t increases. On the other hand, if $y > T$, then $dy/dt > 0$, and y grows as t increases. Thus $\phi_1(t) = 0$ is an asymptotically stable equilibrium solution and $\phi_2(t) = T$ is an unstable one. Further, $f'(y)$ is negative for $0 < y < T/2$ and positive for $T/2 < y < T$, so the graph of y versus t is concave up and concave down, respectively, in these intervals. Also, $f'(y)$ is positive for $y > T$, so the graph of y versus t is also concave up there. By making use of all of the information that we have obtained from Figure 2.5.6, we conclude that graphs of solutions of Eq. (14) for different values of y_0 must have the qualitative appearance shown in Figure 2.5.7. From this figure it is clear that as time increases, y either approaches zero or grows without bound, depending on whether the initial value y_0 is less than or greater than T. Thus T is a **threshold level**, below which growth does not occur.

We can confirm the conclusions that we have reached through geometric reasoning by solving the differential equation (14). This can be done by separating the variables and integrating, just as we did for Eq. (7). However, if we note that Eq. (14) can be obtained from Eq. (7) by replacing K by T and r by $-r$, then we can make the same substitutions in the solution (11) and thereby obtain

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}, \tag{15}$$

which is the solution of Eq. (14) subject to the initial condition $y(0) = y_0$.

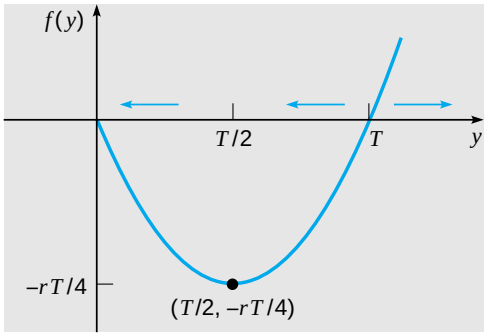


FIGURE 2.5.6 $f(y)$ versus y for $dy/dt = -r(1 - y/T)y$.

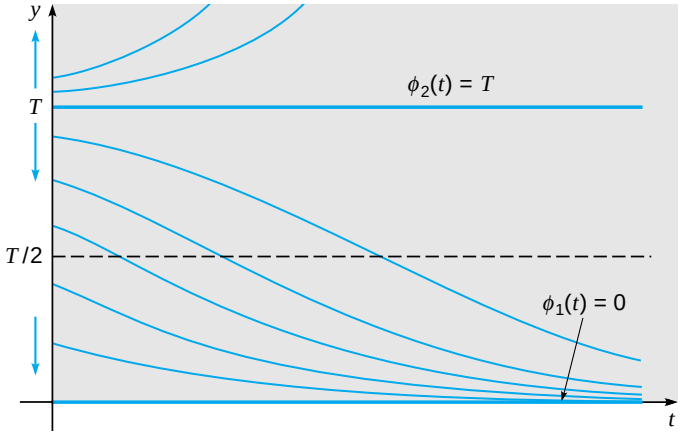


FIGURE 2.5.7 y versus t for $dy/dt = -r(1 - y/T)y$.

If $y_0 < T$, then it follows from Eq. (15) that $y \to 0$ as $t \to \infty$. This agrees with our qualitative geometric analysis. If $y_0 > T$, then the denominator on the right side of Eq. (15) is zero for a certain finite value of t. We denote this value by t^*, and calculate it from

$$y_0 - (y_0 - T)e^{rt^*} = 0,$$

which gives

$$t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}. \tag{16}$$

Thus, if the initial population y_0 is above the threshold T, the threshold model predicts that the graph of y versus t has a vertical asymptote at $t = t^*$; in other words, the population becomes unbounded in a finite time, which depends on the initial value y_0 and the threshold value T. The existence and location of this asymptote were not apparent from the geometric analysis, so in this case the explicit solution yields additional important qualitative, as well as quantitative, information.

The populations of some species exhibit the threshold phenomenon. If too few are present, the species cannot propagate itself successfully and the population becomes extinct. However, if a population larger than the threshold level can be brought together, then further growth occurs. Of course, the population cannot become unbounded, so eventually Eq. (14) must be modified to take this into account.

Critical thresholds also occur in other circumstances. For example, in fluid mechanics, equations of the form (7) or (14) often govern the evolution of a small disturbance y in a *laminar* (or smooth) fluid flow. For instance, if Eq. (14) holds and $y < T$, then the disturbance is damped out and the laminar flow persists. However, if $y > T$, then the disturbance grows larger and the laminar flow breaks up into a turbulent one. In this case T is referred to as the *critical amplitude*. Experimenters speak of keeping the disturbance level in a wind tunnel sufficiently low so that they can study laminar flow over an airfoil, for example.

The same type of situation can occur with automatic control devices. For example, suppose that y corresponds to the position of a flap on an airplane wing that is regulated by an automatic control. The desired position is $y = 0$. In the normal motion of the

plane the changing aerodynamic forces on the flap will cause it to move from its set position, but then the automatic control will come into action to damp out the small deviation and return the flap to its desired position. However, if the airplane is caught in a high gust of wind, the flap may be deflected so much that the automatic control cannot bring it back to the set position (this would correspond to a deviation greater than T). Presumably, the pilot would then take control and manually override the automatic system!

Logistic Growth with a Threshold. As we mentioned in the last subsection, the threshold model (14) may need to be modified so that unbounded growth does not occur when y is above the threshold T. The simplest way to do this is to introduce another factor that will have the effect of making dy/dt negative when y is large. Thus we consider

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y, \tag{17}$$

where $r > 0$ and $0 < T < K$.

The graph of $f(y)$ versus y is shown in Figure 2.5.8. In this problem there are three critical points: $y = 0$, $y = T$, and $y = K$, corresponding to the equilibrium solutions $\phi_1(t) = 0$, $\phi_2(t) = T$, and $\phi_3(t) = K$, respectively. From Figure 2.5.8 it is clear that $dy/dt > 0$ for $T < y < K$, and consequently y is increasing there. The reverse is true for $y < T$ and for $y > K$. Consequently, the equilibrium solutions $\phi_1(t)$ and $\phi_3(t)$ are asymptotically stable, and the solution $\phi_2(t)$ is unstable. Graphs of y versus t have the qualitative appearance shown in Figure 2.5.9. If y starts below the threshold T, then y declines to ultimate extinction. On the other hand, if y starts above T, then y eventually approaches the carrying capacity K. The inflection points on the graphs of y versus t in Figure 2.5.9 correspond to the maximum and minimum points, y_1 and y_2, respectively, on the graph of $f(y)$ versus y in Figure 2.5.8. These values can be obtained by differentiating the right side of Eq. (17) with respect to y, setting the result equal to zero, and solving for y. We obtain

$$y_{1,2} = (K + T \pm \sqrt{K^2 - KT + T^2})/3, \tag{18}$$

where the plus sign yields y_1 and the minus sign y_2.

A model of this general sort apparently describes the population of the passenger pigeon,[7] which was present in the United States in vast numbers until late in the nineteenth century. It was heavily hunted for food and for sport, and consequently its

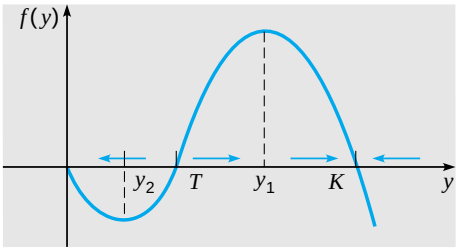


FIGURE 2.5.8 $f(y)$ versus y for $dy/dt = -r(1 - y/T)(1 - y/K)y$.

[7]See, for example, Oliver L. Austin, Jr., *Birds of the World* (New York: Golden Press, 1983), pp. 143–145.

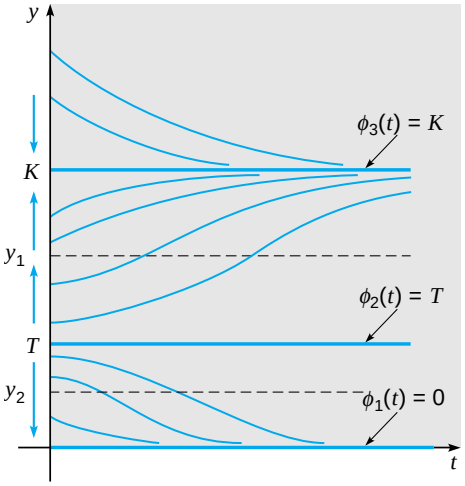


FIGURE 2.5.9 y versus t for $dy/dt = -r(1 - y/T)(1 - y/K)y$.

numbers were drastically reduced by the 1880s. Unfortunately, the passenger pigeon could apparently breed successfully only when present in a large concentration, corresponding to a relatively high threshold T. Although a reasonably large number of individual birds remained alive in the late 1880s, there were not enough in any one place to permit successful breeding, and the population rapidly declined to extinction. The last survivor died in 1914. The precipitous decline in the passenger pigeon population from huge numbers to extinction in scarcely more than three decades was one of the early factors contributing to a concern for conservation in this country.

PROBLEMS

Problems 1 through 6 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable.

1. $dy/dt = ay + by^2, \quad a > 0, \quad b > 0, \quad y_0 \geq 0$
2. $dy/dt = ay + by^2, \quad a > 0, \quad b > 0, \quad -\infty < y_0 < \infty$
3. $dy/dt = y(y - 1)(y - 2), \quad y_0 \geq 0$
4. $dy/dt = e^y - 1, \quad -\infty < y_0 < \infty$
5. $dy/dt = e^{-y} - 1, \quad -\infty < y_0 < \infty$
6. $dy/dt = -2(\arctan y)/(1 + y^2), \quad -\infty < y_0 < \infty$
7. **Semistable Equilibrium Solutions.** Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side depart from it (see Figure 2.5.10). In this case the equilibrium solution is said to be **semistable**.
   (a) Consider the equation

$$dy/dt = k(1 - y)^2, \tag{i}$$

   where k is a positive constant. Show that $y = 1$ is the only critical point, with the corresponding equilibrium solution $\phi(t) = 1$.
   (b) Sketch $f(y)$ versus y. Show that y is increasing as a function of t for $y < 1$ and also for $y > 1$. Thus solutions below the equilibrium solution approach it, while those above it grow farther away. Therefore $\phi(t) = 1$ is semistable.

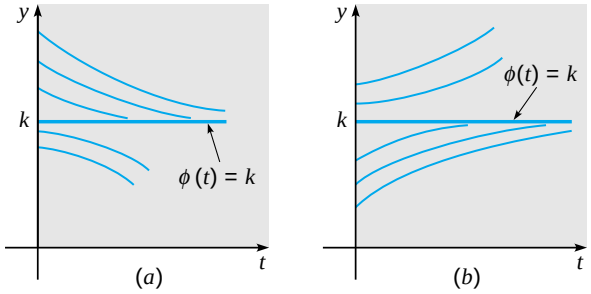


FIGURE 2.5.10 In both cases the equilibrium solution $\phi(t) = k$ is semistable. (*a*) $dy/dt \le 0$; (*b*) $dy/dt \ge 0$.

(c) Solve Eq. (i) subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part (b).

Problems 8 through 13 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable (see Problem 7).

8. $dy/dt = -k(y-1)^2, \quad k > 0, \quad -\infty < y_0 < \infty$
9. $dy/dt = y^2(y^2-1), \quad -\infty < y_0 < \infty$
10. $dy/dt = y(1-y^2), \quad -\infty < y_0 < \infty$
11. $dy/dt = ay - b\sqrt{y}, \quad a > 0, \quad b > 0, \quad y_0 \ge 0$
12. $dy/dt = y^2(4-y^2), \quad -\infty < y_0 < \infty$
13. $dy/dt = y^2(1-y)^2, \quad -\infty < y_0 < \infty$
14. Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point, that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.
15. Suppose that a certain population obeys the logistic equation $dy/dt = ry[1 - (y/K)]$.
    (a) If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.
    (b) If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.
16. Another equation that has been used to model population growth is the Gompertz equation:

$$dy/dt = ry \ln(K/y),$$

    where r and K are positive constants.
    (a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.
    (b) For $0 \le y \le K$ determine where the graph of y versus t is concave up and where it is concave down.
    (c) For each y in $0 < y \le K$ show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.
17. (a) Solve the Gompertz equation

$$dy/dt = ry \ln(K/y),$$

    subject to the initial condition $y(0) = y_0$.
    Hint: You may wish to let $u = \ln(y/K)$.

(b) For the data given in Example 1 in the text [$r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$], use the Gompertz model to find the predicted value of $y(2)$.
(c) For the same data as in part (b), use the Gompertz model to find the time τ at which $y(\tau) = 0.75K$.

18. A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.
(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$

where α is the coefficient of evaporation.
(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?
(c) Find a condition that must be satisfied if the pond is not to overflow.

19. Consider a cylindrical water tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole of area a in the bottom of the tank. From Torricelli's theorem in hydrodynamics it follows that the rate at which water flows through the hole is $\alpha a\sqrt{2gh}$, where h is the current depth of water in the tank, g is the acceleration due to gravity, and α is a contraction coefficient that satisfies $0.5 \le \alpha \le 1.0$.
(a) Show that the depth of water in the tank at any time satisfies the equation

$$dh/dt = (k - \alpha a\sqrt{2gh})/A.$$

(b) Determine the equilibrium depth h_e of water and show that it is asymptotically stable. Observe that h_e does not depend on A.

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

While it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level, and possibly even driven to extinction. Problems 20 and 21 explore some of the questions involved in formulating a rational strategy for managing the fishery.[8]

20. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: The more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$dy/dt = r(1 - y/K)y - Ey. \tag{i}$$

This equation is known as the **Schaefer model** after the biologist, M. B. Schaefer, who applied it to fish populations.
(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.
(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

[8] An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E; the graph of this function is known as the yield-effort curve.
(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

21. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$dy/dt = r(1 - y/K)y - h. \tag{i}$$

The assumption of a constant catch rate h may be reasonable when y is large, but becomes less so when y is small.
(a) If $h < rK/4$, show that Eq. (i) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.
(b) Show that y_1 is unstable and y_2 is asymptotically stable.
(c) From a plot of $f(y)$ versus y show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.
(d) If $h > rK/4$, show that y decreases to zero as t increases regardless of the value of y_0.
(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 7). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 20(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[9] Problems 22 through 24 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

22. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population, and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$dy/dt = \alpha y(1 - y), \qquad y(0) = y_0, \tag{i}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.
(a) Find the equilibrium points for the differential equation (i) and determine whether each is asymptotically stable, semistable, or unstable.
(b) Solve the initial value problem (i) and verify that the conclusions you reached in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

23. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease, but who exhibit no overt symptoms. Let x and y, respectively, denote

[9] A standard source is the book by Bailey listed in the references. The models in Problems 22 through 24 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

the proportion of susceptibles and carriers in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \tag{i}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = \alpha xy. \tag{ii}$$

(a) Determine y at any time t by solving Eq. (i) subject to the initial condition $y(0) = y_0$.
(b) Use the result of part (a) to find x at any time t by solving Eq. (ii) subject to the initial condition $x(0) = x_0$.
(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

24. Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t, and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt, the rate at which the number of susceptibles declines, is given by

$$dx/dt = -[\beta + \mu(t)]x; \tag{i}$$

the first term on the right side of Eq. (i) is the rate at which susceptibles contract smallpox, while the second term is the rate at which they die from all other causes. Also

$$dn/dt = -\nu\beta x - \mu(t)n, \tag{ii}$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right side are the death rates due to smallpox and to all other causes, respectively.
(a) Let $z = x/n$ and show that z satisfies the initial value problem

$$dz/dt = -\beta z(1 - \nu z), \qquad z(0) = 1. \tag{iii}$$

Observe that the initial value problem (iii) does not depend on $\mu(t)$.
(b) Find $z(t)$ by solving Eq. (iii).
(c) Bernoulli estimated that $\nu = \beta = \frac{1}{8}$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: Based on the model just described and using the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years 7 months. He therefore supported the inoculation program.

25. **Bifurcation Points.** In many physical problems some observable quantity, such as a velocity, waveform, or chemical reaction, depends on a parameter describing the physical state. As this parameter is increased, a critical value is reached at which the velocity, or waveform, or reaction suddenly changes its character. For example, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color

suddenly emerge in an originally quiescent fluid. In many such cases the mathematical analysis ultimately leads to an equation[10] of the form

$$dx/dt = (R - R_c)x - ax^3. \tag{i}$$

Here a and R_c are positive constants, and R is a parameter that may take on various values. For example, R may measure the amount of a certain chemical and x may measure a chemical reaction.

(a) If $R < R_c$, show that there is only one equilibrium solution $x = 0$ and that it is asymptotically stable.

(b) If $R > R_c$, show that there are three equilibrium solutions, $x = 0$ and $x = \pm\sqrt{(R - R_c)/a}$, and that the first solution is unstable while the other two are asymptotically stable.

(c) Draw a graph in the Rx-plane showing all equilibrium solutions and label each one as asymptotically stable or unstable.

The point $R = R_c$ is called a **bifurcation point**. For $R < R_c$ one observes the asymptotically stable equilibrium solution $x = 0$. However, this solution loses its stability as R passes through the value R_c, and for $R > R_c$ the asymptotically stable (and hence the observable) solutions are $x = \sqrt{(R - R_c)/a}$ and $x = -\sqrt{(R - R_c)/a}$. Because of the way in which the solutions branch at R_c, this type of bifurcation is called a pitchfork bifurcation; your sketch should suggest that this name is appropriate.

26. **Chemical Reactions.** A second order chemical reaction involves the interaction (collision) of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X; this is denoted by $P + Q \to X$. Suppose that p and q, where $p \neq q$, are the initial concentrations of P and Q, respectively, and let $x(t)$ be the concentration of X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t, and the rate at which the reaction occurs is given by the equation

$$dx/dt = \alpha(p - x)(q - x), \tag{i}$$

where α is a positive constant.

(a) If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and find $x(t)$ for any t.

(b) If the substances P and Q are the same, then $p = q$ and Eq. (i) is replaced by

$$dx/dt = \alpha(p - x)^2. \tag{ii}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of

[10] In fluid mechanics Eq. (i) arises in the study of the transition from laminar to turbulent flow; there it is often called the Landau equation. L. D. Landau (1908–1968) was a Russian physicist who received the Nobel prize in 1962 for his contributions to the understanding of condensed states, particularly liquid helium. He was also the coauthor, with E. M. Lifschitz, of a well-known series of physics textbooks.

equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \qquad 2xy = \frac{\partial \psi}{\partial y}. \tag{2}$$

Therefore the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x and calling upon the chain rule, we can write Eq. (3) in the equivalent form

$$\frac{d\psi}{dx} = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

Therefore

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant, is an equation that defines solutions of Eq. (1) implicitly.

In solving Eq. (1) the key step was the recognition that there is a function ψ that satisfies Eq. (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$

be given. Suppose that we can identify a function ψ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \qquad \frac{\partial \psi}{\partial y}(x, y) = N(x, y), \tag{7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. Then

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x, \phi(x)]$$

and the differential equation (6) becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0. \tag{8}$$

In this case Eq. (6) is said to be an **exact** differential equation. Solutions of Eq. (6), or the equivalent Eq. (8), are given implicitly by

$$\psi(x, y) = c, \tag{9}$$

where c is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, by recognizing the required function ψ. For more complicated equations it may not be possible to do this so easily. A systematic way of determining whether a given differential equation is exact is provided by the following theorem.

Theorem 2.6.1 Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular[11] region $R: \alpha < x < \beta, \gamma < y < \delta$. Then Eq. (6),

$$M(x, y) + N(x, y)y' = 0,$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \tag{10}$$

at each point of R. That is, there exists a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy Eq. (10).

The proof of this theorem has two parts. First, we show that if there is a function ψ such that Eqs. (7) are true, then it follows that Eq. (10) is satisfied. Computing M_y and N_x from Eqs. (7), we obtain

$$M_y(x, y) = \psi_{xy}(x, y), \qquad N_x(x, y) = \psi_{yx}(x, y). \tag{11}$$

Since M_y and N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This guarantees their equality, and Eq. (10) follows.

We now show that if M and N satisfy Eq. (10), then Eq. (6) is exact. The proof involves the construction of a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y).$$

Integrating the first of Eqs. (7) with respect to x, holding y constant, we find that

$$\psi(x, y) = \int M(x, y)\, dx + h(y). \tag{12}$$

The function h is an arbitrary function of y, playing the role of the arbitrary constant. Now we must show that it is always possible to choose $h(y)$ so that $\psi_y = N$. From Eq. (12)

$$\begin{aligned} \psi_y(x, y) &= \frac{\partial}{\partial y} \int M(x, y)\, dx + h'(y) \\ &= \int M_y(x, y)\, dx + h'(y). \end{aligned}$$

[11] It is not essential that the region be rectangular, only that it be simply connected. In two dimensions this means that the region has no holes in its interior. Thus, for example, rectangular or circular regions are simply connected, but an annular region is not. More details can be found in most books on advanced calculus.

Setting $\psi_y = N$ and solving for $h'(y)$, we obtain

$$h'(y) = N(x, y) - \int M_y(x, y)\, dx. \tag{13}$$

To determine $h(y)$ from Eq. (13), it is essential that, despite its appearance, the right side of Eq. (13) be a function of y only. To establish this fact, we can differentiate the quantity in question with respect to x, obtaining

$$N_x(x, y) - M_y(x, y),$$

which is zero on account of Eq. (10). Hence, despite its apparent form, the right side of Eq. (13) does not, in fact, depend on x, and a single integration then gives $h(y)$. Substituting for $h(y)$ in Eq. (12), we obtain as the solution of Eqs. (7)

$$\psi(x, y) = \int M(x, y)\, dx + \int \left[N(x, y) - \int M_y(x, y)\, dx \right] dy. \tag{14}$$

It should be noted that this proof contains a method for the computation of $\psi(x, y)$ and thus for solving the original differential equation (6). It is usually better to go through this process each time it is needed rather than to try to remember the result given in Eq. (14). Note also that the solution is obtained in implicit form; it may or may not be feasible to find the solution explicitly.

EXAMPLE 2

Solve the differential equation

$$(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0. \tag{15}$$

It is easy to see that

$$M_y(x, y) = \cos x + 2xe^y = N_x(x, y),$$

so the given equation is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = y \cos x + 2xe^y,$$
$$\psi_y(x, y) = \sin x + x^2 e^y - 1.$$

Integrating the first of these equations, we obtain

$$\psi(x, y) = y \sin x + x^2 e^y + h(y). \tag{16}$$

Setting $\psi_y = N$ gives

$$\psi_y(x, y) = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

Thus $h'(y) = -1$ and $h(y) = -y$. The constant of integration can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for $h(y)$ in Eq. (16) gives

$$\psi(x, y) = y \sin x + x^2 e^y - y.$$

Hence solutions of Eq. (15) are given implicitly by

$$y \sin x + x^2 e^y - y = c. \tag{17}$$

EXAMPLE 3

Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{18}$$

Here,

$$M_y(x, y) = 3x + 2y, \qquad N_x(x, y) = 2x + y;$$

since $M_y \neq N_x$, the given equation is not exact. To see that it cannot be solved by the procedure described previously, let us seek a function ψ such that

$$\psi_x(x, y) = 3xy + y^2, \qquad \psi_y(x, y) = x^2 + xy. \tag{19}$$

Integrating the first of Eqs. (19) gives

$$\psi(x,y) = \tfrac{3}{2}x^2y + xy^2 + h(y), \tag{20}$$

where h is an arbitrary function of y only. To try to satisfy the second of Eqs. (19) we compute ψ_y from Eq. (20) and set it equal to N, obtaining

$$\tfrac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

or

$$h'(y) = -\tfrac{1}{2}x^2 - xy. \tag{21}$$

Since the right side of Eq. (21) depends on x as well as y, it is impossible to solve Eq. (21) for $h(y)$. Thus there is no $\psi(x, y)$ satisfying both of Eqs. (19).

Integrating Factors. It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations in Section 2.1. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{22}$$

by a function μ and then try to choose μ so that the resulting equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{23}$$

is exact. By Theorem 2.6.1 Eq. (23) is exact if and only if

$$(\mu M)_y = (\mu N)_x. \tag{24}$$

Since M and N are given functions, Eq. (24) states that the integrating factor μ must satisfy the first order partial differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{25}$$

If a function μ satisfying Eq. (25) can be found, then Eq. (23) will be exact. The solution of Eq. (23) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies Eq. (22), since the integrating factor μ can be canceled out of Eq. (23).

A partial differential equation of the form (25) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (22). This possible nonuniqueness of the integrating factor is illustrated in Example 4.

Unfortunately, Eq. (25), which determines the integrating factor μ, is ordinarily at least as difficult to solve as the original equation (22). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both. Let us determine necessary conditions on M and N so that Eq. (22) has an integrating factor μ that depends on x only. Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y, \qquad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}.$$

Thus, if $(\mu M)_y$ is to equal $(\mu N)_x$, it is necessary that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{26}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (26), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (22) has an integrating factor depending only on y; see Problem 23.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{18}$$

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{27}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{28}$$

Hence

$$\mu(x) = x. \tag{29}$$

Multiplying Eq. (18) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{30}$$

The latter equation is exact and it is easy to show that its solutions are given implicitly by

$$x^3y + \tfrac{1}{2}x^2y^2 = c. \tag{31}$$

Solutions may also be readily found in explicit form since Eq. (31) is quadratic in y.

You may also verify that a second integrating factor of Eq. (18) is

$$\mu(x, y) = \frac{1}{xy(2x + y)},$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Determine whether or not each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. $(2x + 3) + (2y - 2)y' = 0$
2. $(2x + 4y) + (2x - 2y)y' = 0$
3. $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$
4. $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$
5. $\dfrac{dy}{dx} = -\dfrac{ax + by}{bx + cy}$
6. $\dfrac{dy}{dx} = -\dfrac{ax - by}{bx - cy}$
7. $(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$
8. $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$
9. $(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x)\,dx + (xe^{xy} \cos 2x - 3)\,dy = 0$
10. $(y/x + 6x)\,dx + (\ln x - 2)\,dy = 0, \qquad x > 0$
11. $(x \ln y + xy)\,dx + (y \ln x + xy)\,dy = 0; \qquad x > 0, \quad y > 0$
12. $\dfrac{x\,dx}{(x^2 + y^2)^{3/2}} + \dfrac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$

In each of Problems 13 and 14 solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x - y)\,dx + (2y - x)\,dy = 0, \qquad y(1) = 3$
14. $(9x^2 + y - 1)\,dx - (4y - x)\,dy = 0, \qquad y(1) = 0$

In each of Problems 15 and 16 find the value of b for which the given equation is exact and then solve it using that value of b.

15. $(xy^2 + bx^2y)\,dx + (x + y)x^2\,dy = 0$
16. $(ye^{2xy} + x)\,dx + bxe^{2xy}\,dy = 0$
17. Consider the exact differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

Find an implicit formula $\psi(x, y) = c$ for the solution analogous to Eq. (14) by first integrating the equation $\psi_y = N$, rather than $\psi_x = M$, as in the text.

18. Show that any separable equation,

$$M(x) + N(y)y' = 0,$$

is also exact.

Show that the equations in Problems 19 through 22 are not exact, but become exact when multiplied by the given integrating factor. Then solve the equations.

19. $x^2y^3 + x(1 + y^2)y' = 0, \qquad \mu(x, y) = 1/xy^3$
20. $\left(\dfrac{\sin y}{y} - 2e^{-x} \sin x\right)\,dx + \left(\dfrac{\cos y + 2e^{-x} \cos x}{y}\right)\,dy = 0, \qquad \mu(x, y) = ye^x$
21. $y\,dx + (2x - ye^y)\,dy = 0, \qquad \mu(x, y) = y$
22. $(x + 2)\sin y\,dx + x \cos y\,dy = 0, \qquad \mu(x, y) = xe^x$

23. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)\, dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

In each of Problems 25 through 31 find an integrating factor and solve the given equation.

25. $(3x^2y + 2xy + y^3)\, dx + (x^2 + y^2)\, dy = 0$
26. $y' = e^{2x} + y - 1$
27. $dx + (x/y - \sin y)\, dy = 0$
28. $y\, dx + (2xy - e^{-2y})\, dy = 0$
29. $e^x\, dx + (e^x \cot y + 2y \csc y)\, dy = 0$
30. $[4(x^3/y^2) + (3/y)]\, dx + [3(x/y^2) + 4y]\, dy = 0$
31. $\left(3x + \dfrac{6}{y}\right) + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$

Hint: See Problem 24.

32. Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x, y) = [xy(2x + y)]^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

2.7 Numerical Approximations: Euler's Method

Recall two important facts about the first order initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

First, if f and $\partial f/\partial y$ are continuous, then the initial value problem (1) has a unique solution $y = \phi(t)$ in some interval surrounding the initial point $t = t_0$. Second, it is usually not possible to find the solution ϕ by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to this statement, namely, differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means such as those considered in the first part of this chapter.

Therefore it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated differential equations. However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.

Another alternative is to compute approximate values of the solution $y = \phi(t)$ of the initial value problem (1) at a selected set of t-values. Ideally, the approximate solution values will be accompanied by error bounds that assure the level of accuracy of the approximations. Today there are numerous methods that produce numerical approximations to solutions of differential equations, and Chapter 8 is devoted to a fuller discussion of some of them. Here, we introduce the oldest and simplest such method, originated by Euler about 1768. It is called the **tangent line method** or the **Euler method**.

Let us consider how we might approximate the solution $y = \phi(t)$ of Eqs. (1) near $t = t_0$. We know that the solution passes through the initial point (t_0, y_0) and, from the differential equation, we also know that its slope at this point is $f(t_0, y_0)$. Thus we can write down an equation for the line tangent to the solution curve at (t_0, y_0), namely,

$$y = y_0 + f(t_0, y_0)(t - t_0). \tag{2}$$

The tangent line is a good approximation to the actual solution curve on an interval short enough so that the slope of the solution does not change appreciably from its value at the initial point; see Figure 2.7.1. Thus, if t_1 is close enough to t_0, we can approximate $\phi(t_1)$ by the value y_1 determined by substituting $t = t_1$ into the tangent line approximation at $t = t_0$; thus

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0). \tag{3}$$

To proceed further, we can try to repeat the process. Unfortunately, we do not know the value $\phi(t_1)$ of the solution at t_1. The best we can do is to use the approximate value y_1 instead. Thus we construct the line through (t_1, y_1) with the slope $f(t_1, y_1)$,

$$y = y_1 + f(t_1, y_1)(t - t_1). \tag{4}$$

To approximate the value of $\phi(t)$ at a nearby point t_2, we use Eq. (4) instead, obtaining

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1). \tag{5}$$

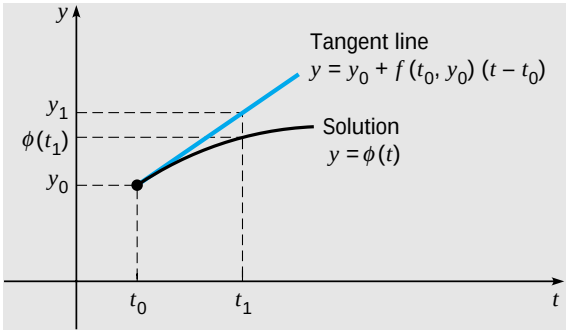


FIGURE 2.7.1 A tangent line approximation.

Continuing in this manner, we use the value of y calculated at each step to determine the slope of the approximation for the next step. The general expression for y_{n+1} in terms of t_n, t_{n+1}, and y_n is

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \qquad n = 0, 1, 2, \ldots. \tag{6}$$

If we introduce the notation $f_n = f(t_n, y_n)$, then we can rewrite Eq. (6) as

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n), \qquad n = 0, 1, 2, \ldots. \tag{7}$$

Finally, if we assume that there is a uniform step size h between the points $t_0, t_1, t_2, \ldots$, then $t_{n+1} = t_n + h$ for each n and we obtain Euler's formula in the form

$$y_{n+1} = y_n + f_n h, \qquad n = 0, 1, 2, \ldots. \tag{8}$$

To use Euler's method you simply evaluate Eq. (7) or Eq. (8) repeatedly, depending on whether or not the step size is constant, using the result of each step to execute the next step. In this way you generate a sequence of values $y_1, y_2, y_3, \ldots$ that approximate the values of the solution $\phi(t)$ at the points $t_1, t_2, t_3, \ldots$. If, instead of a sequence of points, you need an actual function to approximate the solution $\phi(t)$, then you can use the piecewise linear function constructed from the collection of tangent line segments. That is, let y be given by Eq. (2) in $[t_0, t_1]$, by Eq. (4) in $[t_1, t_2]$, and, in general, by

$$y = y_n + f(t_n, y_n)(t - t_n) \tag{9}$$

in $[t_n, t_{n+1}]$.

EXAMPLE 1

Consider the initial value problem

$$\frac{dy}{dt} = 3 + e^{-t} - \tfrac{1}{2}y, \qquad y(0) = 1. \tag{10}$$

Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of Eqs. (10) at $t = 0.1$, 0.2, 0.3, and 0.4. Compare them with the corresponding values of the actual solution of the initial value problem.

Proceeding as in Section 2.1, we find the solution of Eqs. (10) to be

$$y = \phi(t) = 6 - 2e^{-t} - 3e^{-t/2}. \tag{11}$$

To use Euler's method we note that in this case $f(t, y) = 3 + e^{-t} - y/2$. Using the initial values $t_0 = 0$ and $y_0 = 1$, we find that

$$f_0 = f(t_0, y_0) = f(0, 1) = 3 + e^0 - 0.5 = 3 + 1 - 0.5 = 3.5$$

and then, from Eq. (8) with $n = 0$,

$$y_1 = y_0 + f_0 h = 1 + (3.5)(0.1) = 1.35.$$

At the next step we have

$$f_1 = f(0.1, 1.35) = 3 + e^{-0.1} - (0.5)(1.35) \cong 3 + 0.904837 - 0.675 \cong 3.229837$$

and then

$$y_2 = y_1 + f_1 h \cong 1.35 + (3.229837)(0.1) \cong 1.672984.$$

Repeating the computation two more times, we obtain

$$f_2 \cong 2.982239, \qquad y_3 \cong 1.971208$$

and

$$f_3 \cong 2.755214, \qquad y_4 \cong 2.246729.$$

Table 2.7.1 shows these computed values, the corresponding values of the solution (11), and the differences between the two, which is the error in the numerical approximation.

TABLE 2.7.1 A Comparison of Exact Solution with Euler Method for $h = 0.1$ for $y' = 3 + e^{-t} - \frac{1}{2}y, \quad y(0) = 1$

t	Exact	Euler with $h = 0.1$	Error
0.0	1.0000	1.0000	0.0000
0.1	1.3366	1.3500	0.0134
0.2	1.6480	1.6730	0.0250
0.3	1.9362	1.9712	0.0350
0.4	2.2032	2.2467	0.0435

The purpose of Example 1 is to show you the details of implementing a few steps of Euler's method so that it will be clear exactly what computations are being executed. Of course, computations such as those in Example 1 are usually done on a computer. Some software packages include code for the Euler method, while others do not. In any case, it is easy to write a computer program to carry out the calculations required to produce results such as those in Table 2.7.1. The outline of such a program is given below; the specific instructions can be written in any high-level programming language.

The Euler Method

Step 1. **define** $f(t, y)$
Step 2. **input** initial values $t0$ and $y0$
Step 3. **input** step size h and number of steps n
Step 4. **output** $t0$ and $y0$
Step 5. **for** j from 1 to n **do**
Step 6. $k1 = f(t, y)$
$y = y + h * k1$
$t = t + h$
Step 7. **output** t and y
Step 8. **end**

The output of this algorithm can be numbers listed on the screen or printed on a printer, as in the third column of Table 2.7.1. Alternatively, the calculated results can be displayed in graphical form.

EXAMPLE 2

Consider again the initial value problem (10),

$$\frac{dy}{dt} = 3 + e^{-t} - \tfrac{1}{2}y, \qquad y(0) = 1.$$

Use Euler's method with various step sizes to calculate approximate values of the solution for $0 \le t \le 5$. Compare the calculated results with the corresponding values of the exact solution (11),

$$y = \phi(t) = 6 - 2e^{-t} - 3e^{-t/2}.$$

We used step sizes $h = 0.1$, 0.05, 0.025, and 0.01, corresponding respectively to 50, 100, 200, and 500 steps to go from $t = 0$ to $t = 5$. The results of these calculations, along with the values of the exact solution, are presented in Table 2.7.2. All computed entries are rounded to four decimal places, although more digits were retained in the intermediate calculations.

TABLE 2.7.2 A Comparison of Exact Solution with Euler Method for Several Step Sizes h for $y' = 3 + e^{-t} - \frac{1}{2}y, \quad y(0) = 1$

t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	3.4446	3.5175	3.4805	3.4624	3.4517
2.0	4.6257	4.7017	4.6632	4.6443	4.6331
3.0	5.2310	5.2918	5.2612	5.2460	5.2370
4.0	5.5574	5.6014	5.5793	5.5683	5.5617
5.0	5.7403	5.7707	5.7555	5.7479	5.7433

What conclusions can we draw from the data in Table 2.7.2? In the first place, for a fixed value of t the computed approximate values become more accurate as the step size h decreases. This is what we would expect, of course, but it is encouraging that the data confirm our expectations. For example, for $t = 1$ the approximate value with $h = 0.1$ is too large by about 2%, while the value with $h = 0.01$ is too large by only 0.2%. In this case, reducing the step size by a factor of 10 (and performing 10 times as many computations) also reduces the error by a factor of about 10. A second observation from Table 2.7.2 is that, for a fixed step size h, the approximations become more accurate as t increases. For instance, for $h = 0.1$ the error for $t = 5$ is only about 0.5% compared with 2% for $t = 1$. An examination of data at intermediate points not recorded in Table 2.7.2 would reveal where the maximum error occurs for a given step size and how large it is.

All in all, Euler's method seems to work rather well for this problem. Reasonably good results are obtained even for a moderately large step size $h = 0.1$ and the approximation can be improved by decreasing h.

Let us now look at another example.

EXAMPLE 3

Consider the initial value problem

$$\frac{dy}{dt} = 4 - t + 2y, \qquad y(0) = 1. \tag{12}$$

The general solution of this differential equation was found in Example 3 of Section 2.1, and the solution of the initial value problem (12) is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + \tfrac{11}{4}e^{2t}. \tag{13}$$

Use Euler's method with several step sizes to find approximate values of the solution on the interval $0 \le t \le 5$. Compare the results with the corresponding values of the solution (13).

Using the same range of step sizes as in Example 2, we obtain the results presented in Table 2.7.3.

TABLE 2.7.3 A Comparison of Exact Solution with Euler Method for Several Step Sizes h for $y' = 4 - t + 2y, \quad y(0) = 1$

t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	19.06990	15.77728	17.25062	18.10997	18.67278
2.0	149.3949	104.6784	123.7130	135.5440	143.5835
3.0	1109.179	652.5349	837.0745	959.2580	1045.395
4.0	8197.884	4042.122	5633.351	6755.175	7575.577
5.0	60573.53	25026.95	37897.43	47555.35	54881.32

The data in Table 2.7.3 again confirm our expectation that for a given value of t, accuracy improves as the step size h is reduced. For example, for $t = 1$ the percentage error diminishes from 17.3% when $h = 0.1$ to 2.1% when $h = 0.01$. However, the error increases fairly rapidly as t increases for a fixed h. Even for $h = 0.01$, the error at $t = 5$ is 9.4%, and it is much greater for larger step sizes. Of course, the accuracy that is needed depends on the purpose for which the results are intended, but the errors in Table 2.7.3 are too large for most scientific or engineering applications. To improve the situation, one might either try even smaller step sizes or else restrict the computations to a rather short interval away from the initial point. Nevertheless, it is clear that Euler's method is much less effective in this example than in Example 2.

To understand better what is happening in these examples, let us look again at Euler's method for the general initial value problem (1)

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0,$$

whose solution we denote by $\phi(t)$. Recall that a first order differential equation has an infinite family of solutions, indexed by an arbitrary constant c, and that the initial condition picks out one member of this infinite family by determining the value of c. Thus $\phi(t)$ is the member of the infinite family of solutions that satisfies the initial condition $\phi(t_0) = y_0$.

At the first step Euler's method uses the tangent line approximation to the graph of $y = \phi(t)$ passing through the initial point (t_0, y_0) and this produces the approximate value y_1 at t_1. Usually $y_1 \neq \phi(t_1)$, so at the second step Euler's method uses the tangent line approximation not to $y = \phi(t)$, but to a nearby solution $y = \phi_1(t)$ that passes through the point (t_1, y_1). So it is at each following step. Euler's method uses a succession of tangent line approximations to a sequence of different solutions $\phi(t), \phi_1(t), \phi_2(t), \ldots$ of the differential equation. At each step the tangent line is constructed to the solution passing through the point determined by the result of the preceding step, as shown in Figure 2.7.2. The quality of the approximation after many steps depends strongly on the behavior of the set of solutions that pass through the points (t_n, y_n) for $n = 1, 2, 3, \ldots$.

In Example 2 the general solution of the differential equation is

$$y = 6 - 2e^{-t} + ce^{-t/2} \tag{14}$$

and the solution of the initial value problem (10) corresponds to $c = -3$. This family of solutions is a converging family since the term involving the arbitrary constant c approaches zero as $t \to \infty$. It does not matter very much which solutions we are approximating by tangent lines in the implementation of Euler's method, since all the solutions are getting closer and closer to each other as t increases.

On the other hand, in Example 3 the general solution of the differential equation is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}, \tag{15}$$

and this is a diverging family. Note that solutions corresponding to two nearby values of c separate arbitrarily far as t increases. In Example 3 we are trying to follow the solution for $c = 11/4$, but in the use of Euler's method we are actually at each step following another solution that separates from the desired one faster and faster as t increases. This explains why the errors in Example 3 are so much larger than those in Example 2.

In using a numerical procedure such as the Euler method, one must always keep in mind the question of whether the results are accurate enough to be useful. In the preceding examples, the accuracy of the numerical results could be ascertained directly by a comparison with the solution obtained analytically. Of course, usually the analytical solution is not available if a numerical procedure is to be employed, so

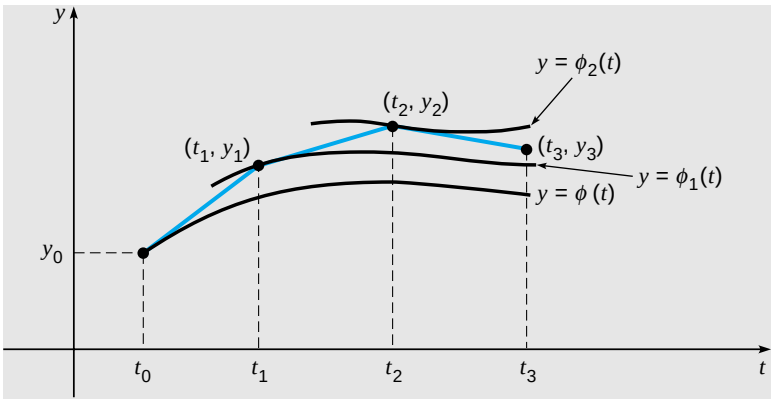


FIGURE 2.7.2 The Euler method.

what is needed are bounds for, or at least estimates of, the error that do not require a knowledge of the exact solution. In Chapter 8 we present some information on the analysis of errors and also discuss several algorithms that are computationally more efficient than the Euler method. However, the best that we can expect, or hope for, from a numerical approximation is that it reflect the behavior of the actual solution. Thus a member of a diverging family of solutions will always be harder to approximate than a member of a converging family. Finally, remember that drawing a direction field is often a helpful first step in understanding the behavior of differential equations and their solutions.

PROBLEMS

Many of the problems in this section call for fairly extensive numerical computations. The amount of computing that it is reasonable for you to do depends strongly on the type of computing equipment that you have. A few steps of the requested calculations can be carried out on almost any pocket calculator, or even by hand if necessary. To do more, you will find at least a programmable calculator desirable, while for some problems a computer may be needed.

Remember also that numerical results may vary somewhat depending on how your program is constructed, and how your computer executes arithmetic steps, rounds off, and so forth. Minor variations in the last decimal place may be due to such causes and do not necessarily indicate that something is amiss. Answers in the back of the book are recorded to six digits in most cases, although more digits were retained in the intermediate calculations.

In each of Problems 1 through 4:

(a) Find approximate values of the solution of the given initial value problem at $t = 0.1$, 0.2, 0.3, and 0.4 using the Euler method with $h = 0.1$.
(b) Repeat part (a) with $h = 0.05$. Compare the results with those found in (a).
(c) Repeat part (a) with $h = 0.025$. Compare the results with those found in (a) and (b).
(d) Find the solution $y = \phi(t)$ of the given problem and evaluate $\phi(t)$ at $t = 0.1$, 0.2, 0.3, and 0.4. Compare these values with the results of (a), (b), and (c).

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 2y - 1, \quad y(0) = 1$
3. $y' = 0.5 - t + 2y, \quad y(0) = 1$
4. $y' = 3\cos t - 2y, \quad y(0) = 0$

In each of Problems 5 through 10 draw a direction field for the given differential equation and state whether you think that the solutions are converging or diverging.

5. $y' = 5 - 3\sqrt{y}$
6. $y' = y(3 - ty)$
7. $y' = (4 - ty)/(1 + y^2)$
8. $y' = -ty + 0.1y^3$
9. $y' = t^2 + y^2$
10. $y' = (y^2 + 2ty)/(3 + t^2)$

In each of Problems 11 through 14 use Euler's method to find approximate values of the solution of the given initial value problem at $t = 0.5$, 1, 1.5, 2, 2.5, and 3:

(a) With $h = 0.1$.
(b) With $h = 0.05$.
(c) With $h = 0.025$.
(d) With $h = 0.01$.

11. $y' = 5 - 3\sqrt{y}, \quad y(0) = 2$
12. $y' = y(3 - ty), \quad y(0) = 0.5$
13. $y' = (4 - ty)/(1 + y^2), \quad y(0) = -2$
14. $y' = -ty + 0.1y^3, \quad y(0) = 1$
15. Consider the initial value problem

$$y' = 3t^2/(3y^2 - 4), \quad y(1) = 0.$$

(a) Use the Euler formula (6) with $h = 0.1$ to obtain approximate values of the solution at $t = 1.2$, 1.4, 1.6, and 1.8.
(b) Repeat part (a) with $h = 0.05$.
(c) Compare the results of parts (a) and (b). Note that they are reasonably close for $t = 1.2$, 1.4, and 1.6, but are quite different for $t = 1.8$. Also note (from the differential equation) that the line tangent to the solution is parallel to the y-axis when $y = \pm 2/\sqrt{3} \cong \pm 1.155$. Explain how this might cause such a difference in the calculated values.

16. Consider the initial value problem

$$y' = t^2 + y^2, \qquad y(0) = 1.$$

Use Euler's method with $h = 0.1$, 0.05, 0.025, and 0.01 to explore the solution of this problem for $0 \le t \le 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field in Problem 9?

17. Consider the initial value problem

$$y' = (y^2 + 2ty)/(3 + t^2), \qquad y(1) = 2.$$

Use Euler's method with $h = 0.1$, 0.05, 0.025, and 0.01 to explore the solution of this problem for $1 \le t \le 3$. What is your best estimate of the value of the solution at $t = 2.5$? At $t = 3$? Are your results consistent with the direction field in Problem 10?

18. Consider the initial value problem

$$y' = -ty + 0.1y^3, \qquad y(0) = \alpha,$$

where α is a given number.
(a) Draw a direction field for the differential equation (or reexamine the one from Problem 8). Observe that there is a critical value of α in the interval $2 \le \alpha \le 3$ that separates converging solutions from diverging ones. Call this critical value α_0.
(b) Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

19. Consider the initial value problem

$$y' = y^2 - t^2, \qquad y(0) = \alpha,$$

where α is a given number.
(a) Draw a direction field for the differential equation. Observe that there is a critical value of α in the interval $0 \le \alpha \le 1$ that separates converging solutions from diverging ones. Call this critical value α_0.
(b) Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

20. **Convergence of Euler's Method.** It can be shown that, under suitable conditions on f, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \qquad y(t_0) = y_0.$$

(a) Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.
(b) Using the Euler formula, show that

$$y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, \ldots.$$

(c) Noting that $y_1 = (1 + h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \tag{i}$$

for each positive integer n.

(d) Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.
Hint: $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

In each of Problems 21 through 23 use the technique discussed in Problem 20 to show that the approximation obtained by the Euler method converges to the exact solution at any fixed point as $h \to 0$.

21. $y' = y, \qquad y(0) = 1$
22. $y' = 2y - 1, \qquad y(0) = 1 \qquad$ *Hint:* $y_1 = (1 + 2h)/2 + 1/2$
23. $y' = \frac{1}{2} - t + 2y, \qquad y(0) = 1 \qquad$ *Hint:* $y_1 = (1 + 2h) + t_1/2$

2.8 The Existence and Uniqueness Theorem

In this section we discuss the proof of Theorem 2.4.2, the fundamental existence and uniqueness theorem for first order initial value problems. This theorem states that under certain conditions on $f(t, y)$, the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

has a unique solution in some interval containing the point t_0.

In some cases (for example, if the differential equation is linear) the existence of a solution of the initial value problem (1) can be established directly by actually solving the problem and exhibiting a formula for the solution. However, in general, this approach is not feasible because there is no method of solving the differential equation that applies in all cases. Therefore, for the general case it is necessary to adopt an indirect approach that demonstrates the existence of a solution of Eqs. (1), but usually does not provide a practical means of finding it. The heart of this method is the construction of a sequence of functions that converges to a limit function satisfying the initial value problem, although the members of the sequence individually do not. As a rule, it is impossible to compute explicitly more than a few members of the sequence; therefore, the limit function can be determined only in rare cases. Nevertheless, under the restrictions on $f(t, y)$ stated in Theorem 2.4.2, it is possible to show that the sequence in question converges and that the limit function has the desired properties. The argument is fairly intricate and depends, in part, on techniques and results that are usually encountered for the first time in a course on advanced calculus. Consequently, we do not go into all of the details of the proof here; we do, however, indicate its main features and point out some of the difficulties involved.

First of all, we note that it is sufficient to consider the problem in which the initial point (t_0, y_0) is the origin; that is, the problem

$$y' = f(t, y), \qquad y(0) = 0. \tag{2}$$

If some other initial point is given, then we can always make a preliminary change of variables, corresponding to a translation of the coordinate axes, that will take the given point (t_0, y_0) into the origin. The existence and uniqueness theorem can now be stated in the following way.

Theorem 2.8.1 If f and $\partial f/\partial y$ are continuous in a rectangle $R: |t| \le a,\ |y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (2).

To prove this theorem it is necessary to transform the initial value problem (2) into a more convenient form. If we suppose temporarily that there is a function $y = \phi(t)$ that satisfies the initial value problem, then $f[t, \phi(t)]$ is a continuous function of t only. Hence we can integrate $y' = f(t, y)$ from the initial point $t = 0$ to an arbitrary value of t, obtaining

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds, \tag{3}$$

where we have made use of the initial condition $\phi(0) = 0$. We also denote the dummy variable of integration by s.

Since Eq. (3) contains an integral of the unknown function ϕ, it is called an **integral equation**. This integral equation is not a formula for the solution of the initial value problem, but it does provide another relation satisfied by any solution of Eqs. (2). Conversely, suppose that there is a continuous function $y = \phi(t)$ that satisfies the integral equation (3); then this function also satisfies the initial value problem (2). To show this, we first substitute zero for t in Eq. (3), which shows that the initial condition is satisfied. Further, since the integrand in Eq. (3) is continuous, it follows from the fundamental theorem of calculus that $\phi'(t) = f[t, \phi(t)]$. Therefore the initial value problem and the integral equation are equivalent in the sense that any solution of one is also a solution of the other. It is more convenient to show that there is a unique solution of the integral equation in a certain interval $|t| \le h$. The same conclusion will then hold also for the initial value problem.

One method of showing that the integral equation (3) has a unique solution is known as the **method of successive approximations**, or Picard's[12] **iteration method**. In using this method, we start by choosing an initial function ϕ_0, either arbitrarily or to approximate in some way the solution of the initial value problem. The simplest choice is

$$\phi_0(t) = 0; \tag{4}$$

then ϕ_0 at least satisfies the initial condition in Eqs. (2), although presumably not the differential equation. The next approximation ϕ_1 is obtained by substituting $\phi_0(s)$ for $\phi(s)$ in the right side of Eq. (3), and calling the result of this operation $\phi_1(t)$. Thus

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]\, ds. \tag{5}$$

[12]Charles-Émile Picard (1856–1914), except for Henri Poincaré, perhaps the most distinguished French mathematician of his generation, was appointed professor at the Sorbonne before the age of 30. He is known for important theorems in complex variables and algebraic geometry as well as differential equations. A special case of the method of successive approximations was first published by Liouville in 1838. However, the method is usually credited to Picard, who established it in a general and widely applicable form in a series of papers beginning in 1890.

Similarly, ϕ_2 is obtained from ϕ_1:

$$\phi_2(t) = \int_0^t f[s, \phi_1(s)]\, ds, \tag{6}$$

and, in general,

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds. \tag{7}$$

In this manner we generate the sequence of functions $\{\phi_n\} = \phi_0, \phi_1, \ldots, \phi_n, \ldots$. Each member of the sequence satisfies the initial condition, but in general none satisfies the differential equation. However, if at some stage, say for $n = k$, we find that $\phi_{k+1}(t) = \phi_k(t)$, then it follows that ϕ_k is a solution of the integral equation (3). Hence ϕ_k is also a solution of the initial value problem (2), and the sequence is terminated at this point. In general, this does not occur, and it is necessary to consider the entire infinite sequence.

To establish Theorem 2.8.1 four principal questions must be answered:

1. Do all members of the sequence $\{\phi_n\}$ exist, or may the process break down at some stage?
2. Does the sequence converge?
3. What are the properties of the limit function? In particular, does it satisfy the integral equation (3), and hence the initial value problem (2)?
4. Is this the only solution, or may there be others?

We first show how these questions can be answered in a specific and relatively simple example, and then comment on some of the difficulties that may be encountered in the general case.

EXAMPLE 1

Solve the initial value problem

$$y' = 2t(1 + y), \qquad y(0) = 0. \tag{8}$$

by the method of successive approximations.

Note first that if $y = \phi(t)$, then the corresponding integral equation is

$$\phi(t) = \int_0^t 2s[1 + \phi(s)]\, ds. \tag{9}$$

If the initial approximation is $\phi_0(t) = 0$, it follows that

$$\phi_1(t) = \int_0^t 2s[1 + \phi_0(s)]\, ds = \int_0^t 2s\, ds = t^2. \tag{10}$$

Similarly,

$$\phi_2(t) = \int_0^t 2s[1 + \phi_1(s)]\, ds = \int_0^t 2s[1 + s^2]\, ds = t^2 + \frac{t^4}{2} \tag{11}$$

and

$$\phi_3(t) = \int_0^t 2s[1 + \phi_2(s)]\, ds = \int_0^t 2s\left[1 + s^2 + \frac{s^4}{2}\right] ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2 \cdot 3}. \tag{12}$$

Equations (10), (11), and (12) suggest that

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} \tag{13}$$

for each $n \geq 1$, and this result can be established by mathematical induction. Equation (13) is certainly true for $n = 1$; see Eq. (10). We must show that if it is true for $n = k$, then it also holds for $n = k + 1$. We have

$$\begin{aligned}\phi_{k+1}(t) &= \int_0^t 2s[1 + \phi_k(s)]\, ds \\ &= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2!} + \cdots + \frac{s^{2k}}{k!}\right) ds \\ &= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2k+2}}{(k+1)!}, \end{aligned} \tag{14}$$

and the inductive proof is complete.

A plot of the first four iterates, $\phi_1(t), \ldots, \phi_4(t)$ is shown in Figure 2.8.1. As k increases, the iterates seem to remain close over a gradually increasing interval, suggesting eventual convergence to a limit function.

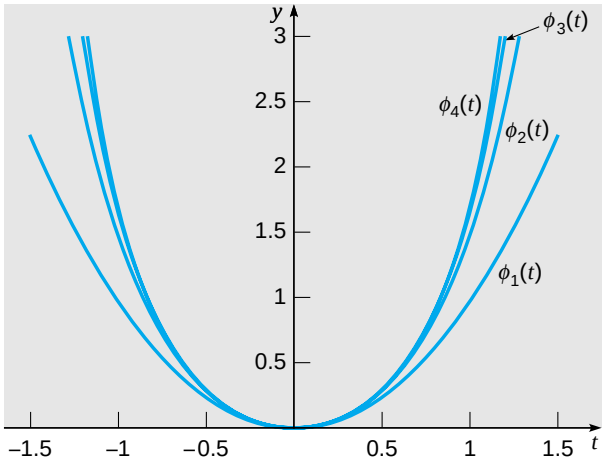


FIGURE 2.8.1 Plots of $\phi_1(t), \ldots, \phi_4(t)$ for Example 1.

It follows from Eq. (13) that $\phi_n(t)$ is the nth partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}; \tag{15}$$

hence $\lim_{n\to\infty} \phi_n(t)$ exists if and only if the series (15) converges. Applying the ratio test, we see that, for each t,

$$\left|\frac{t^{2k+2}}{(k+1)!}\,\frac{k!}{t^{2k}}\right| = \frac{t^2}{k+1} \to 0 \qquad \text{as} \quad k \to \infty; \tag{16}$$

thus the series (15) converges for all t, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$. Further, since the series (15) is a Taylor series, it can be differentiated or integrated term by term as long as t remains within the interval of convergence, which in this case is the entire t-axis. Therefore, we can verify by direct computation that $\phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$ is a solution of the integral equation (9). Alternatively, by substituting $\phi(t)$ for y in Eqs. (8), we can verify that this function satisfies the initial value problem. In this example it is also possible, from the series (15), to identify ϕ in terms of elementary functions, namely, $\phi(t) = e^{t^2} - 1$. However, this is not necessary for the discussion of existence and uniqueness.

Explicit knowledge of $\phi(t)$ does make it possible to visualize the convergence of the sequence of iterates more clearly by plotting $\phi(t) - \phi_k(t)$ for various values of k. Figure 2.8.2 shows this difference for $k = 1, \ldots, 4$. This figure clearly shows the gradually increasing interval over which successive iterates provide a good approximation to the solution of the initial value problem.

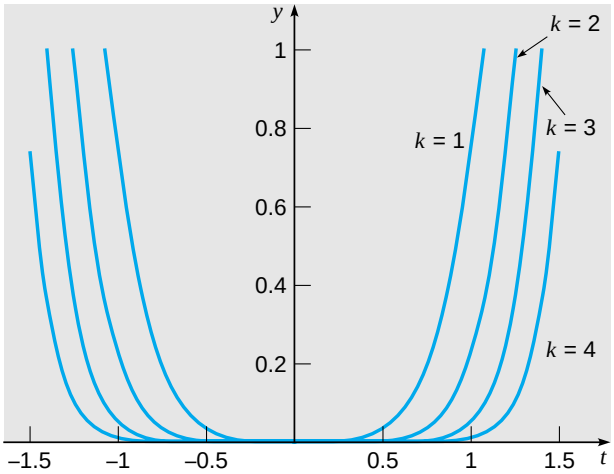


FIGURE 2.8.2 Plots of $\phi(t) - \phi_k(t)$ for Example 1 for $k = 1, \ldots, 4$.

Finally, to deal with the question of uniqueness, let us suppose that the initial value problem has two solutions ϕ and ψ. Since ϕ and ψ both satisfy the integral equation (9), we have by subtraction that

$$\phi(t) - \psi(t) = \int_0^t 2s[\phi(s) - \psi(s)]\, ds.$$

Taking absolute values of both sides, we have, if $t > 0$,

$$|\phi(t) - \psi(t)| = \left| \int_0^t 2s[\phi(s) - \psi(s)]\, ds \right| \le \int_0^t 2s|\phi(s) - \psi(s)|\, ds.$$

If we restrict t to lie in the interval $0 \le t \le A/2$, where A is arbitrary, then $2t \le A$, and

$$|\phi(t) - \psi(t)| \le A \int_0^t |\phi(s) - \psi(s)|\, ds. \tag{17}$$

It is now convenient to introduce the function U defined by

$$U(t) = \int_0^t |\phi(s) - \psi(s)|\, ds. \tag{18}$$

Then it follows at once that

$$U(0) = 0, \tag{19}$$

$$U(t) \ge 0, \qquad \text{for} \quad t \ge 0. \tag{20}$$

Further, U is differentiable, and $U'(t) = |\phi(t) - \psi(t)|$. Hence, by Eq. (17),

$$U'(t) - AU(t) \le 0. \tag{21}$$

Multiplying Eq. (21) by the positive quantity e^{-At} gives

$$[e^{-At}U(t)]' \le 0. \tag{22}$$

Then, upon integrating Eq. (22) from zero to t and using Eq. (19), we obtain

$$e^{-At}U(t) \le 0 \qquad \text{for} \quad t \ge 0.$$

Hence $U(t) \le 0$ for $t \ge 0$, and in conjunction with Eq. (20), this requires that $U(t) = 0$ for each $t \ge 0$. Thus $U'(t) \equiv 0$, and therefore $\psi(t) \equiv \phi(t)$, which contradicts the original hypothesis. Consequently, there cannot be two different solutions of the initial value problem for $t \ge 0$. A slight modification of this argument leads to the same conclusion for $t \le 0$.

Returning now to the general problem of solving the integral equation (3), let us consider briefly each of the questions raised earlier:

1. Do all members of the sequence $\{\phi_n\}$ exist? In the example f and $\partial f/\partial y$ were continuous in the whole ty-plane, and each member of the sequence could be explicitly calculated. In contrast, in the general case, f and $\partial f/\partial y$ are assumed to be continuous only in the rectangle R: $|t| \le a$, $|y| \le b$ (see Figure 2.8.3). Furthermore, the members of the sequence cannot as a rule be explicitly determined. The danger is that at some stage, say for $n = k$, the graph of $y = \phi_k(t)$ may contain points that lie outside of the rectangle R. Hence at the next stage—in the computation of $\phi_{k+1}(t)$—it would be necessary to evaluate $f(t, y)$ at points where it is not known to be continuous or even to exist. Thus the calculation of $\phi_{k+1}(t)$ might be impossible.

To avoid this danger it may be necessary to restrict t to a smaller interval than $|t| \le a$. To find such an interval we make use of the fact that a continuous function on a closed bounded region is bounded. Hence f is bounded on R; thus there exists a positive number M such that

$$|f(t, y)| \le M, \qquad (t, y) \text{ in } R. \tag{23}$$

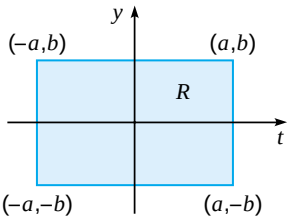


FIGURE 2.8.3 Region of definition for Theorem 2.8.1.

We have mentioned before that

$$\phi_n(0) = 0$$

for each n. Since $f[t, \phi_k(t)]$ is equal to $\phi'_{k+1}(t)$, the maximum absolute slope of the graph of the equation $y = \phi_{k+1}(t)$ is M. Since this graph contains the point $(0, 0)$ it must lie in the wedge-shaped shaded region in Figure 2.8.4. Hence the point $[t, \phi_{k+1}(t)]$ remains in R at least as long as R contains the wedge-shaped region, which is for $|t| \le b/M$. We hereafter consider only the rectangle D: $|t| \le h$, $|y| \le b$, where h is equal either to a or to b/M, whichever is smaller. With this restriction, all members of the sequence $\{\phi_n(t)\}$ exist. Note that if $b/M < a$, then a larger value of h can be obtained by finding a better bound for $|f(t, y)|$, provided that M is not already equal to the maximum value of $|f(t, y)|$.

2. Does the sequence $\{\phi_n(t)\}$ converge? As in the example, we can identify $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$ as the nth partial sum of the series

$$\phi_1(t) + \sum_{k=1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)]. \tag{24}$$

The convergence of the sequence $\{\phi_n(t)\}$ is established by showing that the series (24) converges. To do this, it is necessary to estimate the magnitude $|\phi_{k+1}(t) - \phi_k(t)|$ of the general term. The argument by which this is done is

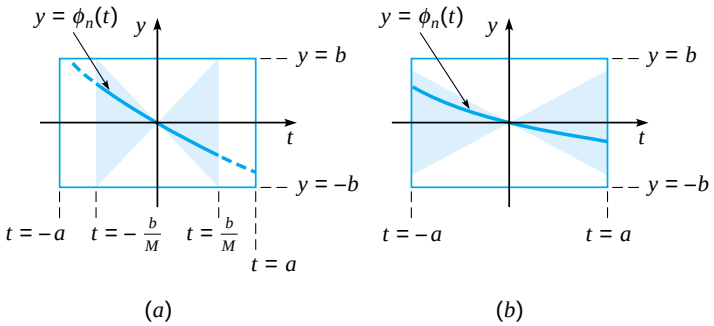


FIGURE 2.8.4 Regions in which successive iterates lie. (*a*) $b/M < a$; (*b*) $b/M > a$.

indicated in Problems 15 through 18 and will be omitted here. Assuming that the sequence converges, we denote the limit function by ϕ, so that

$$\phi(t) = \lim_{n\to\infty} \phi_n(t). \tag{25}$$

3. What are the properties of the limit function ϕ? In the first place, we would like to know that ϕ is continuous. This is not, however, a necessary consequence of the convergence of the sequence $\{\phi_n(t)\}$, even though each member of the sequence is itself continuous. Sometimes a sequence of continuous functions converges to a limit function that is discontinuous. A simple example of this phenomenon is given in Problem 13. One way to show that ϕ is continuous is to show not only that the sequence $\{\phi_n\}$ converges, but also that it converges in a certain manner, known as uniform convergence. We do not take up this question here but note only that the argument referred to in paragraph 2 is sufficient to establish the uniform convergence of the sequence $\{\phi_n\}$ and, hence, the continuity of the limit function ϕ in the interval $|t| \le h$.

Now let us return to Eq. (7),

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds.$$

Allowing n to approach ∞ on both sides, we obtain

$$\phi(t) = \lim_{n\to\infty} \int_0^t f[s, \phi_n(s)]\, ds. \tag{26}$$

We would like to interchange the operations of integrating and taking the limit on the right side of Eq. (26) so as to obtain

$$\phi(t) = \int_0^t \lim_{n\to\infty} f[s, \phi_n(s)]\, ds. \tag{27}$$

In general, such an interchange is not permissible (see Problem 14, for example), but once again the fact that the sequence $\{\phi_n(t)\}$ converges uniformly is sufficient to allow us to take the limiting operation inside the integral sign. Next, we wish to take the limit inside the function f, which would give

$$\phi(t) = \int_0^t f[s, \lim_{n\to\infty} \phi_n(s)]\, ds \tag{28}$$

and hence

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds. \tag{29}$$

The statement that $\lim_{n\to\infty} f[s, \phi_n(s)] = f[s, \lim_{n\to\infty} \phi_n(s)]$ is equivalent to the statement that f is continuous in its second variable, which is known by hypothesis. Hence Eq. (29) is valid and the function ϕ satisfies the integral equation (3). Therefore ϕ is also a solution of the initial value problem (2).

4. Are there other solutions of the integral equation (3) beside $y = \phi(t)$? To show the uniqueness of the solution $y = \phi(t)$, we can proceed much as in the example.

First, assume the existence of another solution $y = \psi(t)$. It is then possible to show (see Problem 19) that the difference $\phi(t) - \psi(t)$ satisfies the inequality

$$|\phi(t) - \psi(t)| \le A \int_0^t |\phi(s) - \psi(s)|\, ds \tag{30}$$

for $0 \le t \le h$ and a suitable positive number A. From this point the argument is identical to that given in the example, and we conclude that there is no solution of the initial value problem (2) other than the one generated by the method of successive approximations.

PROBLEMS

In each of Problems 1 and 2 transform the given initial value problem into an equivalent problem with the initial point at the origin.

1. $dy/dt = t^2 + y^2, \quad y(1) = 2$
2. $dy/dt = 1 - y^3, \quad y(-1) = 3$

In each of Problems 3 through 6 let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

(a) Determine $\phi_n(t)$ for an arbitrary value of n.
(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.
(c) Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.
(d) Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \ldots, 4$. For each of $\phi_1(t), \ldots, \phi_4(t)$ estimate the interval in which it is a reasonably good approximation to the actual solution.

3. $y' = 2(y + 1), \quad y(0) = 0$
4. $y' = -y - 1, \quad y(0) = 0$
5. $y' = -y/2 + t, \quad y(0) = 0$
6. $y' = y + 1 - t, \quad y(0) = 0$

In each of Problems 7 and 8 let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

(a) Determine $\phi_n(t)$ for an arbitrary value of n.
(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

7. $y' = ty + 1, \quad y(0) = 0$
8. $y' = t^2 y - t, \quad y(0) = 0$

In each of Problems 9 and 10 let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_3(t)$.
(b) Plot $\phi_1(t), \ldots, \phi_3(t)$ and observe whether the iterates appear to be converging.

9. $y' = t^2 + y^2, \quad y(0) = 0$
10. $y' = 1 - y^3, \quad y(0) = 0$

In each of Problems 11 and 12 let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_4(t)$, or (if necessary) Taylor approximations to these iterates. Keep terms up to order six.
(b) Plot the functions you found in part (a) and observe whether they appear to be converging.

11. $y' = -\sin y + 1, \quad y(0) = 0$
12. $y' = (3t^2 + 4t + 2)/2(y - 1), \quad y(0) = 0$
13. Let $\phi_n(x) = x^n$ for $0 \le x \le 1$ and show that

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

This example shows that a sequence of continuous functions may converge to a limit function that is discontinuous.

14. Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}$, $0 \le x \le 1$.
(a) Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \le x \le 1$; hence

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = 0.$$

(b) Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,dx \ne \int_a^b \lim_{n\to\infty} \phi_n(x)\,dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.

In Problems 15 through 18 we indicate how to prove that the sequence $\{\phi_n(t)\}$, defined by Eqs. (4) through (7), converges.

15. If $\partial f/\partial y$ is continuous in the rectangle D, show that there is a positive constant K such that

$$|f(t,y_1) - f(t,y_2)| \le K|y_1 - y_2|,$$

where (t,y_1) and (t,y_2) are any two points in D having the same t coordinate. This inequality is known as a Lipschitz condition.
Hint: Hold t fixed and use the mean value theorem on f as a function of y only. Choose K to be the maximum value of $|\partial f/\partial y|$ in D.

16. If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 15 to show that

$$|f[t,\phi_n(t)] - f[t,\phi_{n-1}(t)]| \le K|\phi_n(t) - \phi_{n-1}(t)|.$$

17. (a) Show that if $|t| \le h$, then

$$|\phi_1(t)| \le M|t|,$$

where M is chosen so that $|f(t,y)| \le M$ for (t,y) in D.
(b) Use the results of Problem 16 and part (a) of Problem 17 to show that

$$|\phi_2(t) - \phi_1(t)| \le \frac{MK|t|^2}{2}.$$

(c) Show, by mathematical induction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{MK^{n-1}|t|^n}{n!} \le \frac{MK^{n-1}h^n}{n!}.$$

18. Note that

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)].$$

(a) Show that

$$|\phi_n(t)| \le |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

(b) Use the results of Problem 17 to show that

$$|\phi_n(t)| \le \frac{M}{K}\left[Kh + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!}\right].$$

(c) Show that the sum in part (b) converges as $n \to \infty$ and, hence, the sum in part (a) also converges as $n \to \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

19. In this problem we deal with the question of uniqueness of the solution of the integral equation (3),

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds.$$

(a) Suppose that ϕ and ψ are two solutions of Eq. (3). Show that, for $t \ge 0$,

$$\phi(t) - \psi(t) = \int_0^t \{f[s, \phi(s)] - f[s, \psi(s)]\}\, ds.$$

(b) Show that

$$|\phi(t) - \psi(t)| \le \int_0^t |f[s, \phi(s)] - f[s, \psi(s)]|\, ds.$$

(c) Use the result of Problem 15 to show that

$$|\phi(t) - \psi(t)| \le K \int_0^t |\phi(s) - \psi(s)|\, ds,$$

where K is an upper bound for $|\partial f/\partial y|$ in D. This is the same as Eq. (30), and the rest of the proof may be constructed as indicated in the text.

2.9 First Order Difference Equations



While a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more natural. For instance, the continuous model of compound interest used in Section 2.3 is only an approximation to the actual discrete process. Similarly, sometimes population growth may be described more accurately by a discrete than by a continuous model. This is true, for example, of species whose generations do not overlap and that propagate at regular intervals, such as at particular times of the calendar year. Then the population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_n in the preceding year, that is,

$$y_{n+1} = f(n, y_n), \qquad n = 0, 1, 2, \ldots. \tag{1}$$

Equation (1) is called a **first order difference equation**. It is first order because the value of y_{n+1} depends on the value of y_n, but not earlier values y_{n-1}, y_{n-2}, and so forth. As for differential equations, the difference equation (1) is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation (1) is

a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. In addition to the difference equation itself, there may also be an **initial condition**

$$y_0 = \alpha \tag{2}$$

that prescribes the value of the first term of the solution sequence.

We now assume temporarily that the function f in Eq. (1) depends only on y_n, but not n. In this case

$$y_{n+1} = f(y_n), \qquad n = 0, 1, 2, \ldots. \tag{3}$$

If y_0 is given, then successive terms of the solution can be found from Eq. (3). Thus

$$y_1 = f(y_0),$$

and

$$y_2 = f(y_1) = f[f(y_0)].$$

The quantity $f[f(y_0)]$ is called the second iterate of the difference equation and is sometimes denoted by $f^2(y_0)$. Similarly, the third iterate y_3 is given by

$$y_3 = f(y_2) = f\{f[f(y_0)]\} = f^3(y_0),$$

and so on. In general, the nth iterate y_n is

$$y_n = f(y_{n-1}) = f^n(y_0).$$

This procedure is referred to as iterating the difference equation. It is often of primary interest to determine the behavior of y_n as $n \to \infty$, in particular, whether y_n approaches a limit, and if so, to find it.

Solutions for which y_n has the same value for all n are called **equilibrium solutions**. They are frequently of special importance, just as in the study of differential equations. If equilibrium solutions exist, one can find them by setting y_{n+1} equal to y_n in Eq. (3) and solving the resulting equation

$$y_n = f(y_n) \tag{4}$$

for y_n.

Linear Equations. Suppose that the population of a certain species in a given region in year $n + 1$, denoted by y_{n+1}, is a positive multiple ρ_n of the population y_n in year n; that is,

$$y_{n+1} = \rho_n y_n, \qquad n = 0, 1, 2, \ldots. \tag{5}$$

Note that the reproduction rate ρ_n may differ from year to year. The difference equation (5) is linear and can easily be solved by iteration. We obtain

$$\begin{aligned} y_1 &= \rho_0 y_0, \\ y_2 &= \rho_1 y_1 = \rho_1 \rho_0 y_0, \end{aligned}$$

and, in general,

$$y_n = \rho_{n-1} \cdots \rho_0 y_0, \qquad n = 1, 2, \ldots. \tag{6}$$

Thus, if the initial population y_0 is given, then the population of each succeeding generation is determined by Eq. (6). Although for a population problem ρ_n is intrinsically

positive, the solution (6) is also valid if ρ_n is negative for some or all values of n. Note, however, that if ρ_n is zero for some n, then y_{n+1} and all succeeding values of y are zero; in other words, the species has become extinct.

If the reproduction rate ρ_n has the same value ρ for each n, then the difference equation (5) becomes

$$y_{n+1} = \rho y_n \tag{7}$$

and its solution is

$$y_n = \rho^n y_0. \tag{8}$$

Equation (7) also has an equilibrium solution, namely, $y_n = 0$ for all n, corresponding to the initial value $y_0 = 0$. The limiting behavior of y_n is easy to determine from Eq. (8). In fact,

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & \text{if } |\rho| < 1; \\ y_0, & \text{if } \rho = 1; \\ \text{does not exist}, & \text{otherwise.} \end{cases} \tag{9}$$

In other words, the equilibrium solution $y_n = 0$ is asymptotically stable for $|\rho| < 1$ and unstable if $|\rho| > 1$.

Now we will modify the population model represented by Eq. (5) to include the effect of immigration or emigration. If b_n is the net increase in population in year n due to immigration, then the population in year $n + 1$ is the sum of those due to natural reproduction and those due to immigration. Thus

$$y_{n+1} = \rho y_n + b_n, \qquad n = 0, 1, 2, \ldots, \tag{10}$$

where we are now assuming that the reproduction rate ρ is constant. We can solve Eq. (10) by iteration in the same manner as before. We have

$$\begin{aligned} y_1 &= \rho y_0 + b_0, \\ y_2 &= \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1, \\ y_3 &= \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2, \end{aligned}$$

and so forth. In general, we obtain

$$y_n = \rho^n y_0 + \rho^{n-1} b_0 + \cdots + \rho b_{n-2} + b_{n-1} = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j. \tag{11}$$

Note that the first term on the right side of Eq. (11) represents the descendants of the original population, while the other terms represent the population in year n resulting from immigration in all preceding years.

In the special case where $b_n = b$ for all n, the difference equation is

$$y_{n+1} = \rho y_n + b, \tag{12}$$

and from Eq. (11) its solution is

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1})b. \tag{13}$$

If $\rho \neq 1$, we can write this solution in the more compact form

$$y_n = \rho^n y_0 + \frac{1 - \rho^n}{1 - \rho} b, \tag{14}$$

where again the two terms on the right side are the effects of the original population and of immigration, respectively. By rewriting Eq. (14) as

$$y_n = \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}, \tag{15}$$

the long-time behavior of y_n is more evident. It follows from Eq. (15) that $y_n \to b/(1-\rho)$ if $|\rho| < 1$ and that y_n has no limit if $|\rho| > 1$ or if $\rho = -1$. The quantity $b/(1-\rho)$ is an equilibrium solution of Eq. (12), as can readily be seen directly from that equation. Of course, Eq. (14) is not valid for $\rho = 1$. To deal with that case, we must return to Eq. (13) and let $\rho = 1$ there. It follows that

$$y_n = y_0 + nb, \tag{16}$$

so in this case y_n becomes unbounded as $n \to \infty$.

The same model also provides a framework for solving many problems of a financial character. For such problems y_n is the account balance in the nth time period, $\rho_n = 1 + r_n$, where r_n is the interest rate for that period, and b_n is the amount deposited or withdrawn. The following is a typical example.

EXAMPLE 1

A recent college graduate takes out a $10,000 loan to purchase a car. If the interest rate is 12%, what monthly payment is required to pay off the loan in 4 years?

The relevant difference equation is Eq. (12), where y_n is the loan balance outstanding in the nth month, $\rho = 1 + r$ is the interest rate per month, and b is the monthly payment. Note that b must be negative and $\rho = 1.01$, corresponding to a monthly interest rate of 1%.

The solution of the difference equation (12) with this value for ρ and the initial condition $y_0 = 10{,}000$ is given by Eq. (15), that is,

$$y_n = (1.01)^n (10{,}000 + 100b) - 100b. \tag{17}$$

The payment b needed to pay off the loan in 4 years is found by setting $y_{48} = 0$ and solving for b. This gives

$$b = -100 \frac{(1.01)^{48}}{(1.01)^{48} - 1} = -263.34. \tag{18}$$

The total amount paid on the loan is 48 times b or $12,640.32. Of this amount $10,000 is repayment of the principal and the remaining $2640.32 is interest.

Nonlinear Equations. Nonlinear difference equations are much more complicated and have much more varied solutions than linear equations. We will restrict our attention to a single equation, the logistic difference equation

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{k} \right), \tag{19}$$

which is analogous to the logistic differential equation

$$\frac{dy}{dt} = ry \left(1 - \frac{y}{K} \right) \tag{20}$$

that was discussed in Section 2.5. Note that if the derivative dy/dt in Eq. (20) is replaced by the difference $(y_{n+1} - y_n)/h$, then Eq. (20) reduces to Eq. (19) with $\rho = 1 + hr$ and $k = (1 + hr)K/hr$. To simplify Eq. (19) a little more, we can scale the variable y_n by introducing the new variable $u_n = y_n/k$. Then Eq. (19) becomes

$$u_{n+1} = \rho u_n(1 - u_n), \tag{21}$$

where ρ is a positive parameter.

We begin our investigation of Eq. (21) by seeking the equilibrium, or constant solutions. These can be found by setting u_{n+1} equal to u_n in Eq. (21), which corresponds to setting dy/dt equal to zero in Eq. (20). The resulting equation is

$$u_n = \rho u_n - \rho u_n^2, \tag{22}$$

so it follows that the equilibrium solutions of Eq. (21) are

$$u_n = 0, \qquad u_n = \frac{\rho - 1}{\rho}. \tag{23}$$

The next question is whether the equilibrium solutions are asymptotically stable or unstable; that is, for an initial condition near one of the equilibrium solutions, does the resulting solution sequence approach or depart from the equilibrium solution? One way to examine this question is by approximating Eq. (21) by a linear equation in the neighborhood of an equilibrium solution. For example, near the equilibrium solution $u_n = 0$, the quantity u_n^2 is small compared to u_n itself, so we assume that we can neglect the quadratic term in Eq. (21) in comparison with the linear terms. This leaves us with the linear difference equation

$$u_{n+1} = \rho u_n, \tag{24}$$

which is presumably a good approximation to Eq. (21) for u_n sufficiently near zero. However, Eq. (24) is the same as Eq. (7), and we have already concluded, in Eq. (9), that $u_n \to 0$ as $n \to \infty$ if and only if $|\rho| < 1$, or since ρ must be positive, for $0 < \rho < 1$. Thus the equilibrium solution $u_n = 0$ is asymptotically stable for the linear approximation (24) for this set of ρ values, so we conclude that it is also asymptotically stable for the full nonlinear equation (21). This conclusion is correct, although our argument is not complete. What is lacking is a theorem stating that the solutions of the nonlinear equation (21) resemble those of the linear equation (24) near the equilibrium solution $u_n = 0$. We will not take time to discuss this issue here; the same question is treated for differential equations in Section 9.3.

Now consider the other equilibrium solution $u_n = (\rho - 1)/\rho$. To study solutions in the neighborhood of this point, we write

$$u_n = \frac{\rho - 1}{\rho} + v_n, \tag{25}$$

where we assume that v_n is small. By substituting from Eq. (25) in Eq. (21) and simplifying the resulting equation, we eventually obtain

$$v_{n+1} = (2 - \rho)v_n - \rho v_n^2. \tag{26}$$

Since v_n is small, we again neglect the quadratic term in comparison with the linear terms and thereby obtain the linear equation

$$v_{n+1} = (2 - \rho)v_n. \tag{27}$$

Referring to Eq. (9) once more, we find that $v_n \to 0$ as $n \to \infty$ for $|2 - \rho| < 1$, that is, for $1 < \rho < 3$. Therefore we conclude that, for this range of values of ρ, the equilibrium solution $u_n = (\rho - 1)/\rho$ is asymptotically stable.

Figure 2.9.1 contains the graphs of solutions of Eq. (21) for $\rho = 0.8$, $\rho = 1.5$, and $\rho = 2.8$, respectively. Observe that the solution converges to zero for $\rho = 0.8$ and to the nonzero equilibrium solution for $\rho = 1.5$ and $\rho = 2.8$. The convergence is monotone for $\rho = 0.8$ and $\rho = 1.5$ and is oscillatory for $\rho = 2.8$. While the graphs shown are for particular initial conditions, the graphs for other initial conditions are similar.

Another way of displaying the solution of a difference equation is shown in Figure 2.9.2. In each part of this figure the graphs of the parabola $y = \rho x(1 - x)$ and of the straight line $y = x$ are shown. The equilibrium solutions correspond to the points of intersection of these two curves. The piecewise linear graph consisting of successive vertical and horizontal line segments, sometimes called a stairstep diagram, represents the solution sequence. The sequence starts at the point u_0 on the x-axis. The vertical line segment drawn upward to the parabola at u_0 corresponds to the calculation of $\rho u_0(1 - u_0) = u_1$. This value is then transferred from the y-axis to the x-axis; this step is represented by the horizontal line segment from the parabola to the line $y = x$. Then the process is repeated over and over again. Clearly, the sequence converges to the origin in Figure 2.9.2*a* and to the nonzero equilibrium solution in the other two cases.

To summarize our results so far: The difference equation (21) has two equilibrium solutions, $u_n = 0$ and $u_n = (\rho - 1)/\rho$; the former is stable for $0 \le \rho < 1$, and the latter is stable for $1 < \rho < 3$. This can be depicted as shown in Figure 2.9.3. The parameter ρ is plotted on the horizontal axis and u on the vertical axis. The equilibrium solutions $u = 0$ and $u = (\rho - 1)/\rho$ are shown. The intervals in which each one is stable are indicated by the heavy portions of the curves. Note that the two curves intersect at $\rho = 1$, where there is an **exchange of stability** from one equilibrium solution to the other.

For $\rho > 3$ neither of the equilibrium solutions is stable, and the solutions of Eq. (21) exhibit increasing complexity as ρ increases. For ρ somewhat greater than 3 the sequence u_n rapidly approaches a steady oscillation of period 2; that is, u_n oscillates back and forth between two distinct values. For $\rho = 3.2$ a solution is shown in Figure 2.9.4. For n greater than about 20, the solution alternates between the values 0.5130 and 0.7995. The graph is drawn for the particular initial condition $u_0 = 0.3$, but it is similar for all other initial values between 0 and 1. Figure 2.9.4*b* also shows the same steady oscillation as a rectangular path that is traversed repeatedly in the clockwise direction.

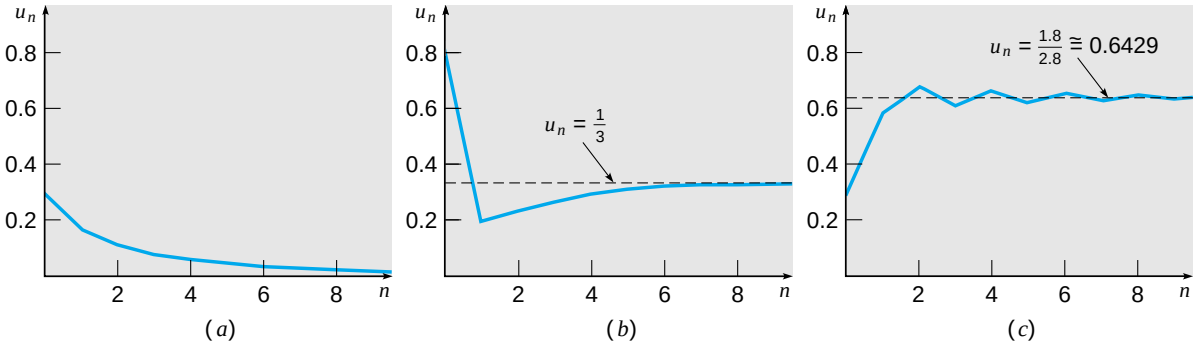


FIGURE 2.9.1 Solutions of $u_{n+1} = \rho u_n(1 - u_n)$: (*a*) $\rho = 0.8$; (*b*) $\rho = 1.5$; (*c*) $\rho = 2.8$.

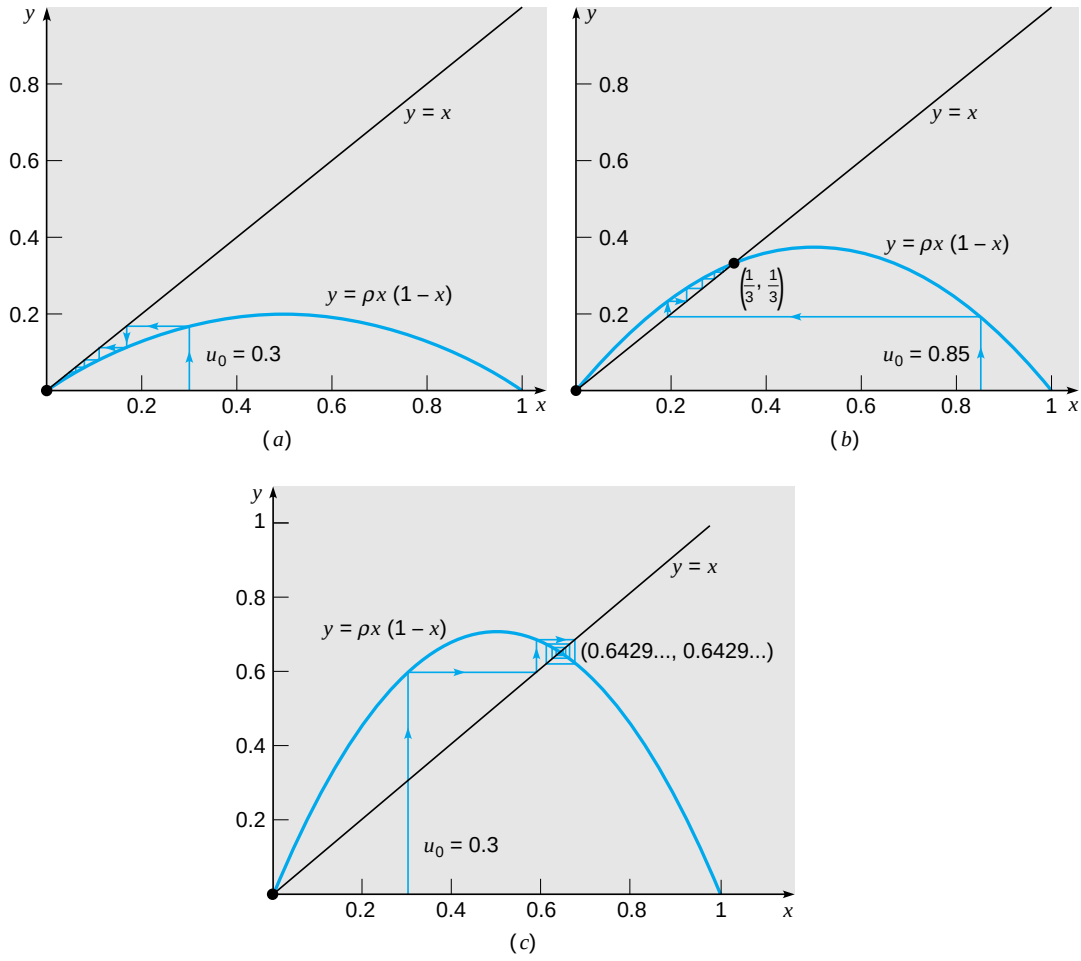


FIGURE 2.9.2 Iterates of $u_{n+1} = \rho u_n(1 - u_n)$. (*a*) $\rho = 0.8$; (*b*) $\rho = 1.5$; (*c*) $\rho = 2.8$.

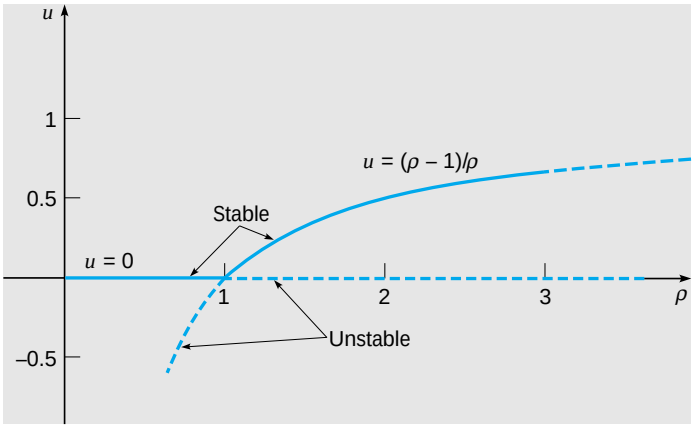
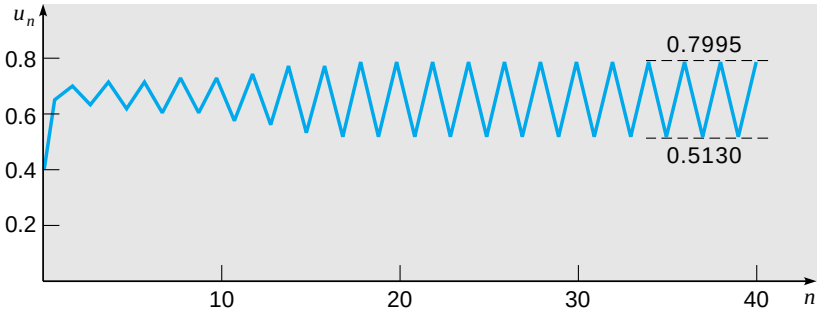
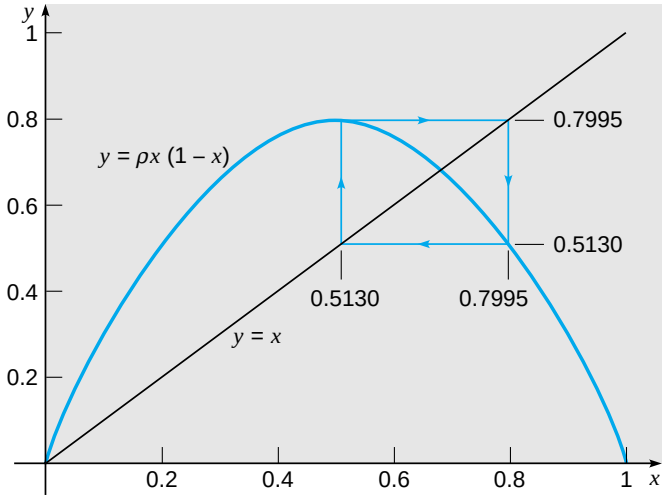


FIGURE 2.9.3 Exchange of stability for $u_{n+1} = \rho u_n(1 - u_n)$.



(*a*)

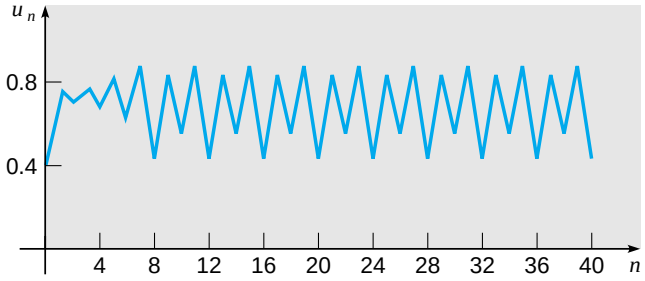


(*b*)

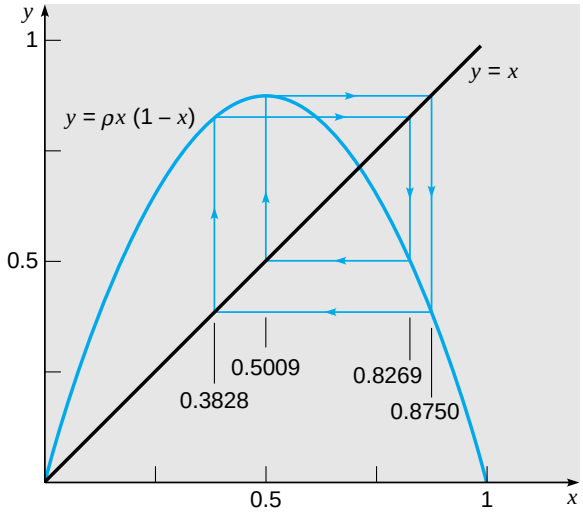
FIGURE 2.9.4 A solution of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.2$; period two. (*a*) u_n versus n; (*b*) a two-cycle.

At about $\rho = 3.449$ each state in the oscillation of period two separates into two distinct states, and the solution becomes periodic with period four; see Figure 2.9.5, which shows a solution of period four for $\rho = 3.5$. As ρ increases further, periodic solutions of period 8, 16, ... appear. The appearance of a new solution at a certain parameter value is called a **bifurcation**.

The ρ-values at which the successive period doublings occur approach a limit that is approximately 3.57. For $\rho > 3.57$ the solutions possess some regularity, but no discernible detailed pattern for most values of ρ. For example, a solution for $\rho = 3.65$ is shown in Figure 2.9.6. It oscillates between approximately 0.3 and 0.9, but its fine structure is unpredictable. The term **chaotic** is used to describe this situation. One of the features of chaotic solutions is extreme sensitivity to the initial conditions. This is illustrated in Figure 2.9.7, where two solutions of Eq. (21) for $\rho = 3.65$ are shown. One solution is the same as that in Figure 2.9.6 and has the initial value $u_0 = 0.3$, while the other solution has the initial value $u_0 = 0.305$. For about 15 iterations the two solutions



(*a*)



(*b*)

FIGURE 2.9.5 A solution of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.5$; period four. (*a*) u_n versus n; (*b*) a four-cycle.

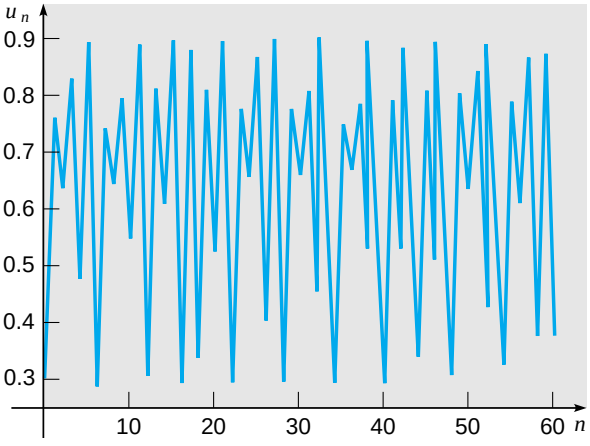


FIGURE 2.9.6 A solution of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.65$; a chaotic solution.

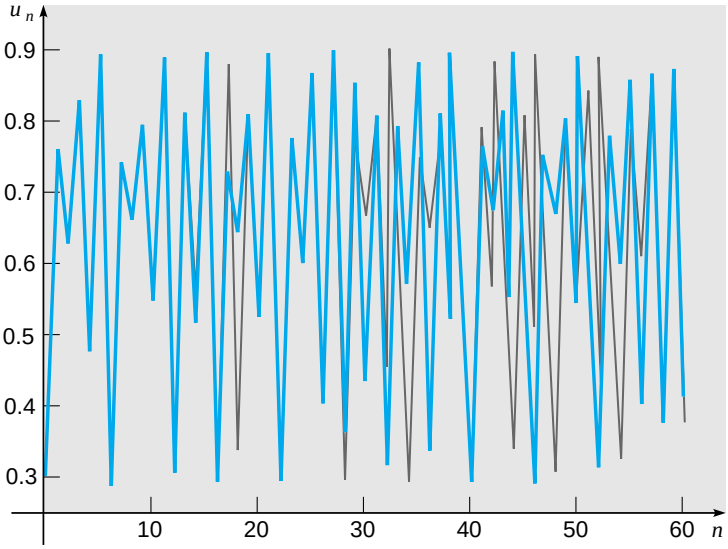


FIGURE 2.9.7 Two solutions of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.65$; $u_0 = 0.3$ and $u_0 = 0.305$.

remain close and are hard to distinguish from each other in the figure. After that, while they continue to wander about in approximately the same set of values, their graphs are quite dissimilar. It would certainly not be possible to use one of these solutions to estimate the value of the other for values of n larger than about 15.

It is only in the last several years that chaotic solutions of difference and differential equations have become widely known. Equation (20) was one of the first instances of mathematical chaos to be found and studied in detail, by Robert May[13] in 1974. On the basis of his analysis of this equation as a model of the population of certain insect species, May suggested that if the growth rate ρ is too large, then it will be impossible to make effective long-range predictions about these insect populations. The occurrence of chaotic solutions in simple problems has stimulated an enormous amount of research in recent years, but many questions remain unanswered. It is increasingly clear, however, that chaotic solutions are much more common than suspected at first and may be a part of the investigation of a wide range of phenomena.

PROBLEMS

In each of Problems 1 through 6 solve the given difference equation in terms of the initial value y_0. Describe the behavior of the solution as $n \to \infty$.

1. $y_{n+1} = -0.9y_n$
2. $y_{n+1} = \dfrac{n+1}{n+2}y_n$
3. $y_{n+1} = \sqrt{\dfrac{n+3}{n+1}}y_n$
4. $y_{n+1} = (-1)^{n+1}y_n$
5. $y_{n+1} = 0.5y_n + 6$
6. $y_{n+1} = -0.5y_n + 6$

[13] R. M. May, "Biological Populations with Nonoverlapping Generations: Stable Points, Stable Cycles, and Chaos," *Science 186* (1974), pp. 645–647; "Biological Populations Obeying Difference Equations: Stable Points, Stable Cycles, and Chaos," *Journal of Theoretical Biology 51* (1975), pp. 511–524.

7. Find the effective annual yield of a bank account that pays interest at a rate of 7%, compounded daily; that is, find the ratio of the difference between the final and initial balances divided by the initial balance.
8. An investor deposits \$1000 in an account paying interest at a rate of 8% compounded monthly, and also makes additional deposits of \$25 per month. Find the balance in the account after 3 years.
9. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. What monthly payment rate is required to pay off the loan in 3 years? Compare your result with that of Problem 9 in Section 2.3.
10. A homebuyer wishes to take out a mortgage of \$100,000 for a 30-year period. What monthly payment is required if the interest rate is (a) 9%, (b) 10%, (c) 12%?
11. A homebuyer takes out a mortgage of \$100,000 with an interest rate of 9%. What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?
12. If the interest rate on a 20-year mortgage is fixed at 10% and if a monthly payment of \$1000 is the maximum that the buyer can afford, what is the maximum mortgage loan that can be made under these conditions?
13. A homebuyer wishes to finance the purchase with a \$95,000 mortgage with a 20-year term. What is the maximum interest rate the buyer can afford, if the monthly payment is not to exceed \$900?

The Logistic Difference Equation. Problems 14 through 19 deal with the difference equation (21), $u_{n+1} = \rho u_n(1 - u_n)$.

14. Carry out the details in the linear stability analysis of the equilibrium solution $u_n = (\rho - 1)/\rho$; that is, derive the difference equation (26) in the text for the perturbation v_n.
15. (a) For $\rho = 3.2$ plot or calculate the solution of the logistic equation (21) for several initial conditions, say, $u_0 = 0.2$, 0.4, 0.6, and 0.8. Observe that in each case the solution approaches a steady oscillation between the same two values. This illustrates that the long-term behavior of the solution is independent of the initial conditions.
    (b) Make similar calculations and verify that the nature of the solution for large n is independent of the initial condition for other values of ρ, such as 2.6, 2.8, and 3.4.
16. Assume that $\rho > 1$ in Eq. (21).
    (a) Draw a qualitatively correct stairstep diagram and thereby show that if $u_0 < 0$, then $u_n \to -\infty$ as $n \to \infty$.
    (b) In a similar way determine what happens as $n \to \infty$ if $u_0 > 1$.
17. The solutions of Eq. (21) change from convergent sequences to periodic oscillations of period two as the parameter ρ passes through the value 3. To see more clearly how this happens, carry out the following calculations.
    (a) Plot or calculate the solution for $\rho = 2.9$, 2.95, and 2.99, respectively, using an initial value u_0 of your choice in the interval (0, 1). In each case estimate how many iterations are required for the solution to get "very close" to the limiting value. Use any convenient interpretation of what "very close" means in the preceding sentence.
    (b) Plot or calculate the solution for $\rho = 3.01$, 3.05, and 3.1, respectively, using the same initial condition as in part (a). In each case estimate how many iterations are needed to reach a steady-state oscillation. Also find or estimate the two values in the steady-state oscillation.
18. By calculating or plotting the solution of Eq. (21) for different values of ρ, estimate the value of ρ for which the solution changes from an oscillation of period two to one of period four. In the same way estimate the value of ρ for which the solution changes from period four to period eight.
19. Let ρ_k be the value of ρ for which the solution of Eq. (21) changes from period 2^{k-1} to period 2^k. Thus, as noted in the text, $\rho_1 = 3$, $\rho_2 \cong 3.449$, and $\rho_3 \cong 3.544$.

(a) Using these values of ρ_1, ρ_2, and ρ_3, or those you found in Problem 18, calculate $(\rho_2 - \rho_1)/(\rho_3 - \rho_2)$.
(b) Let $\delta_n = (\rho_n - \rho_{n-1})/(\rho_{n+1} - \rho_n)$. It has been shown that δ_n approaches a limit δ as $n \to \infty$, where $\delta \cong 4.6692$ is known as the Feigenbaum number. Determine the percentage difference between the limiting value δ and δ_2, as calculated in part (a).
(c) Assume that $\delta_3 = \delta$ and use this relation to estimate ρ_4, the value of ρ at which solutions of period 16 appear.
(d) By plotting or calculating solutions near the value of ρ_4 found in part (c), try to detect the appearance of a period 16 solution.
(e) Observe that

$$\rho_n = \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + \cdots + (\rho_n - \rho_{n-1}).$$

Assuming that $(\rho_4 - \rho_3) = (\rho_3 - \rho_2)\delta^{-1}$, $(\rho_5 - \rho_4) = (\rho_3 - \rho_2)\delta^{-2}$, and so forth, express ρ_n as a geometric sum. Then find the limit of ρ_n as $n \to \infty$. This is an estimate of the value of ρ at which the onset of chaos occurs in the solution of the logistic equation (21).

PROBLEMS

Miscellaneous Problems One of the difficulties in solving first order equations is that there are several methods of solution, each of which can be used on a certain type of equation. It may take some time to become proficient in matching solution methods with equations. The following problems are presented so that you may have some practice in identifying the method or methods applicable to a given equation.

In each of Problems 1 through 32 solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

1. $\dfrac{dy}{dx} = \dfrac{x^3 - 2y}{x}$
2. $(x + y)\,dx - (x - y)\,dy = 0$
3. $\dfrac{dy}{dx} = \dfrac{2x + y}{3 + 3y^2 - x}, \qquad y(0) = 0$
4. $(x + e^y)\,dy - dx = 0$
5. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$
6. $x\dfrac{dy}{dx} + xy = 1 - y, \qquad y(1) = 0$
7. $\dfrac{dy}{dx} = \dfrac{x}{x^2y + y^3}$ *Hint:* Let $u = x^2$.
8. $x\dfrac{dy}{dx} + 2y = \dfrac{\sin x}{x}, \qquad y(2) = 1$
9. $\dfrac{dy}{dx} = -\dfrac{2xy + 1}{x^2 + 2y}$
10. $(3y^2 + 2xy)\,dx - (2xy + x^2)\,dy = 0$
11. $(x^2 + y)\,dx + (x + e^y)\,dy = 0$
12. $\dfrac{dy}{dx} + y = \dfrac{1}{1 + e^x}$
13. $x\,dy - y\,dx = (xy)^{1/2}\,dx$
14. $(x + y)\,dx + (x + 2y)\,dy = 0, \qquad y(2) = 3$
15. $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$
16. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{x^2}$
17. $\dfrac{dy}{dx} = e^{2x} + 3y$
18. $(2y + 3x)\,dx = -x\,dy$
19. $x\,dy - y\,dx = 2x^2y^2\,dy, \qquad y(1) = -2$
20. $y' = e^{x+y}$
21. $xy' = y + xe^{y/x}$
22. $\dfrac{dy}{dx} = \dfrac{x^2 - 1}{y^2 + 1}, \qquad y(-1) = 1$
23. $xy' + y - y^2e^{2x} = 0$
24. $2\sin y \cos x\,dx + \cos y \sin x\,dy = 0$

25. $\left(2\dfrac{x}{y} - \dfrac{y}{x^2+y^2}\right)dx + \left(\dfrac{x}{x^2+y^2} - \dfrac{x^2}{y^2}\right)dy = 0$

26. $(2y+1)\,dx + \left(\dfrac{x^2-y}{x}\right)dy = 0$

27. $(\cos 2y - \sin x)\,dx - 2\tan x \sin 2y\,dy = 0$

28. $\dfrac{dy}{dx} = \dfrac{3x^2 - 2y - y^3}{2x + 3xy^2}$

29. $\dfrac{dy}{dx} = \dfrac{2y + \sqrt{x^2 - y^2}}{2x}$

30. $\dfrac{dy}{dx} = \dfrac{y^3}{1 - 2xy^2}, \qquad y(0) = 1$

31. $(x^2y + xy - y)\,dx + (x^2y - 2x^2)\,dy = 0$

32. $\dfrac{dy}{dx} = -\dfrac{3x^2y + y^2}{2x^3 + 3xy}, \qquad y(1) = -2$

REFERENCES

Two books mentioned in Section 2.3 are:

Bailey, N. T. J., *The Mathematical Theory of Infectious Diseases and Its Applications* (2nd ed.) (New York: Hafner Press, 1975).

Clark, Colin W., *Mathematical Bioeconomics* (2nd ed.) (New York: Wiley-Interscience, 1990).

An introduction to population dynamics in general is:

Frauenthal, J. C., *Introduction to Population Modeling* (Boston: Birkhauser, 1980).

A fuller discussion of the proof of the fundamental existence and uniqueness theorem can be found in many more advanced books on differential equations. Two that are reasonably accessible to elementary readers are:

Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice Hall, 1961; New York: Dover, 1989).

Brauer, F., and Nohel, J., *The Qualitative Theory of Ordinary Differential Equations* (New York: Benjamin, 1969; New York: Dover, 1989).

A valuable compendium of methods for solving differential equations is:

Zwillinger, D., *Handbook of Differential Equations* (3rd ed.) (San Diego: Academic Press, 1998).

A general reference on difference equations is:

Mickens, R. E., *Difference Equations, Theory and Applications* (2nd ed.) (New York: Van Nostrand Reinhold, 1990).

An elementary treatment of chaotic solutions of difference equations is:

Devaney, R. L., *Chaos, Fractals, and Dynamics* (Reading, MA: Addison-Wesley, 1990).



CHAPTER
3

Second Order Linear Equations

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations. As an example, we discuss the oscillations of some basic mechanical and electrical systems at the end of the chapter.

3.1 Homogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we

will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and y'. In Eq. (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of Eq. (3), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide Eq. (4) by $P(t)$ and thereby obtain Eq. (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

In discussing Eq. (3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.[1]

If Eq. (1) is not of the form (3) or (4), then it is called **nonlinear**. Analytical investigations of nonlinear equations are relatively difficult, so we will have little to say about them in this book. Numerical or geometical approaches are often more appropriate, and these are discussed in Chapters 8 and 9. In addition, there are two special types of second order nonlinear equations that can be solved by a change of variables that reduces them to first order equations. This procedure is outlined in Problems 28 through 43.

An initial value problem consists of a differential equation such as Eq. (1), (3), or (4) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{6}$$

where y_0 and y_0' are given numbers. Observe that the initial conditions for a second order equation prescribe not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y_0' of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second order equation because, roughly speaking, two integrations are required to find a solution and each integration introduces an arbitrary constant. Presumably, two initial conditions will suffice to determine values for these two constants.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (3), or the term $G(t)$ in Eq. (4), is zero for all t. Otherwise, the equation is called **nonhomogeneous**. As a result, the term $g(t)$, or $G(t)$, is sometimes called the nonhomogeneous term. We begin our discussion with homogeneous equations, which we will write in the form

$$P(t)y'' + Q(t)y' + R(t)y = 0. \tag{7}$$

[1]There is a corresponding treatment of higher order linear equations in Chapter 4. If you wish, you may read the appropriate parts of Chapter 4 in parallel with Chapter 3.

Later, in Sections 3.6 and 3.7, we will show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation (4), or at least to express the solution in terms of an integral. Thus the problem of solving the homogeneous equation is the more fundamental one.

In this chapter we will concentrate our attention on equations in which the functions P, Q, and R are constants. In this case, Eq. (7) becomes

$$ay'' + by' + cy = 0, \tag{8}$$

where a, b, and c are given constants. It turns out that Eq. (8) can always be solved easily in terms of the elementary functions of calculus. On the other hand, it is usually much more difficult to solve Eq. (7) if the coefficients are not constants, and a treatment of that case is deferred until Chapter 5.

Before taking up Eq. (8), let us first gain some experience by looking at a simple, but typical, example. Consider the equation

$$y'' - y = 0, \tag{9}$$

which is just Eq. (8) with $a = 1$, $b = 0$, and $c = -1$. In words, Eq. (9) says that we seek a function with the property that the second derivative of the function is the same as the function itself. A little thought will probably produce at least one well-known function from calculus with this property, namely, $y_1(t) = e^t$, the exponential function. A little more thought may also produce a second function, $y_2(t) = e^{-t}$. Some further experimentation reveals that constant multiples of these two solutions are also solutions. For example, the functions $2e^t$ and $5e^{-t}$ also satisfy Eq. (9), as you can verify by calculating their second derivatives. In the same way, the functions $c_1y_1(t) = c_1e^t$ and $c_2y_2(t) = c_2e^{-t}$ satisfy the differential equation (9) for all values of the constants c_1 and c_2. Next, it is of paramount importance to notice that any sum of solutions of Eq. (9) is also a solution. In particular, since $c_1y_1(t)$ and $c_2y_2(t)$ are solutions of Eq. (9), so is the function

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^t + c_2e^{-t} \tag{10}$$

for any values of c_1 and c_2. Again, this can be verified by calculating the second derivative y'' from Eq. (10). We have $y' = c_1e^t - c_2e^{-t}$ and $y'' = c_1e^t + c_2e^{-t}$; thus y'' is the same as y, and Eq. (9) is satisfied.

Let us summarize what we have done so far in this example. Once we notice that the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of Eq. (9), it follows that the general linear combination (10) of these functions is also a solution. Since the coefficients c_1 and c_2 in Eq. (10) are arbitrary, this expression represents a doubly infinite family of solutions of the differential equation (9).

It is now possible to consider how to pick out a particular member of this infinite family of solutions that also satisfies a given set of initial conditions. For example, suppose that we want the solution of Eq. (9) that also satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = -1. \tag{11}$$

In other words, we seek the solution that passes through the point (0, 2) and at that point has the slope -1. First, we set $t = 0$ and $y = 2$ in Eq. (10); this gives the equation

$$c_1 + c_2 = 2. \tag{12}$$

Next, we differentiate Eq. (10) with the result that

$$y' = c_1 e^t - c_2 e^{-t}.$$

Then, setting $t = 0$ and $y' = -1$, we obtain

$$c_1 - c_2 = -1. \tag{13}$$

By solving Eqs. (12) and (13) simultaneously for c_1 and c_2 we find that

$$c_1 = \tfrac{1}{2}, \qquad c_2 = \tfrac{3}{2}. \tag{14}$$

Finally, inserting these values in Eq. (10), we obtain

$$y = \tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}, \tag{15}$$

the solution of the initial value problem consisting of the differential equation (9) and the initial conditions (11).

We now return to the more general equation (8),

$$ay'' + by' + cy = 0,$$

which has arbitrary (real) constant coefficients. Based on our experience with Eq. (9), let us also seek exponential solutions of Eq. (8). Thus we suppose that $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. By substituting these expressions for y, y', and y'' in Eq. (8), we obtain

$$(ar^2 + br + c)e^{rt} = 0,$$

or, since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0. \tag{16}$$

Equation (16) is called the **characteristic equation** for the differential equation (8). Its significance lies in the fact that if r is a root of the polynomial equation (16), then $y = e^{rt}$ is a solution of the differential equation (8). Since Eq. (16) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates. We consider the first case here, and the latter two cases in Sections 3.4 and 3.5.

Assuming that the roots of the characteristic equation (16) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of Eq. (8). Just as in the preceding example, it now follows that

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{17}$$

is also a solution of Eq. (8). To verify that this is so, we can differentiate the expression in Eq. (17); hence

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \tag{18}$$

and

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}. \tag{19}$$

Substituting these expressions for y, y', and y'' in Eq. (8) and rearranging terms, we obtain

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t}. \tag{20}$$

The quantity in each of the parentheses on the right side of Eq. (20) is zero because r_1 and r_2 are roots of Eq. (16); therefore, y as given by Eq. (17) is indeed a solution of Eq. (8), as we wished to verify.

Now suppose that we want to find the particular member of the family of solutions (17) that satisfies the initial conditions (6),

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

By substituting $t = t_0$ and $y = y_0$ in Eq. (17), we obtain

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \tag{21}$$

Similarly, setting $t = t_0$ and $y' = y_0'$ in Eq. (18) gives

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'. \tag{22}$$

On solving Eqs. (21) and (22) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \qquad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}. \tag{23}$$

Thus, no matter what initial conditions are assigned, that is, regardless of the values of t_0, y_0, and y_0' in Eqs. (6), it is always possible to determine c_1 and c_2 so that the initial conditions are satisfied; moreover, there is only one possible choice of c_1 and c_2 for each set of initial conditions. With the values of c_1 and c_2 given by Eq. (23), the expression (17) is the solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{24}$$

It is possible to show, on the basis of the fundamental theorem cited in the next section, that all solutions of Eq. (8) are included in the expression (17), at least for the case in which the roots of Eq. (16) are real and different. Therefore, we call Eq. (17) the general solution of Eq. (8). The fact that any possible initial conditions can be satisfied by the proper choice of the constants in Eq. (17) makes more plausible the idea that this expression does include all solutions of Eq. (8).

EXAMPLE 1

Find the general solution of

$$y'' + 5y' + 6y = 0. \tag{25}$$

We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of Eq. (25) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{26}$$

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = 3. \tag{27}$$

The general solution of the differential equation was found in Example 1 and is given by Eq. (26). To satisfy the first initial condition we set $t = 0$ and $y = 2$ in Eq. (26); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{28}$$

To use the second initial condition we must first differentiate Eq. (26). This gives $y' = -2c_1e^{-2t} - 3c_2e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{29}$$

By solving Eqs. (28) and (29) we find that $c_1 = 9$ and $c_2 = -7$. Using these values in the expression (26), we obtain the solution

$$y = 9e^{-2t} - 7e^{-3t} \tag{30}$$

of the initial value problem (27). The graph of the solution is shown in Figure 3.1.1.

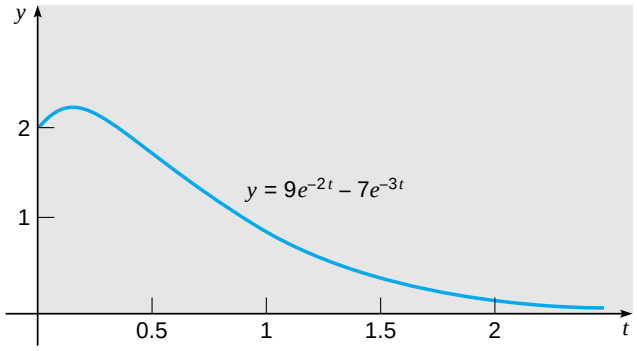


FIGURE 3.1.1 Solution of $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$.

EXAMPLE 3

Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}. \tag{31}$$

If $y = e^{rt}$, then the characteristic equation is

$$4r^2 - 8r + 3 = 0$$

and its roots are $r = 3/2$ and $r = 1/2$. Therefore the general solution of the differential equation is

$$y = c_1e^{3t/2} + c_2e^{t/2}. \tag{32}$$

Applying the initial conditions, we obtain the following two equations for c_1 and c_2:

$$c_1 + c_2 = 2, \qquad \tfrac{3}{2}c_1 + \tfrac{1}{2}c_2 = \tfrac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, and the solution of the initial value problem (31) is

$$y = -\tfrac{1}{2}e^{3t/2} + \tfrac{5}{2}e^{t/2}. \tag{33}$$

Figure 3.1.2 shows the graph of the solution.

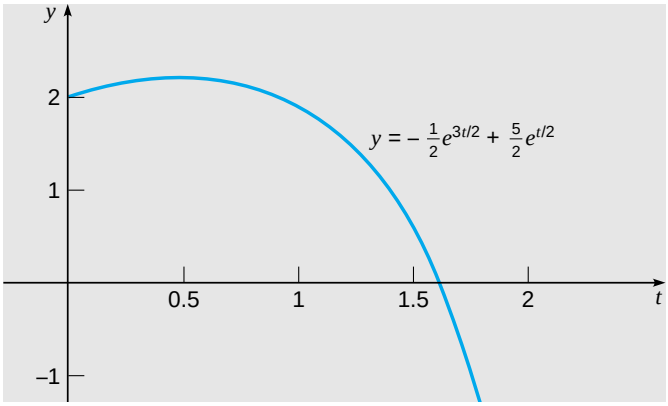


FIGURE 3.1.2 Solution of $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 0.5$.

EXAMPLE 4

The solution (30) of the initial value problem (27) initially increases (because its initial slope is positive) but eventually approaches zero (because both terms involve negative exponential functions). Therefore the solution must have a maximum point and the graph in Figure 3.1.1 confirms this. Determine the location of this maximum point.

One can estimate the coordinates of the maximum point from the graph, but to find them more precisely we seek the point where the solution has a horizontal tangent line. By differentiating the solution (30), $y = 9e^{-2t} - 7e^{-3t}$, with respect to t we obtain

$$y' = -18e^{-2t} + 21e^{-3t}. \tag{34}$$

Setting y' equal to zero and multiplying by e^{3t}, we find that the critical value t_c satisfies $e^t = 7/6$; hence

$$t_c = \ln(7/6) \cong 0.15415. \tag{35}$$

The corresponding maximum value y_M is given by

$$y_M = 9e^{-2t_c} - 7e^{-3t_c} = \frac{108}{49} \cong 2.20408. \tag{36}$$

In this example the initial slope is 3, but the solution of the given differential equation behaves in a similar way for any other positive initial slope. In Problem 26 you are asked to determine how the coordinates of the maximum point depend on the initial slope.

Returning to the equation $ay'' + by' + cy = 0$ with arbitrary coefficients, recall that when $r_1 \neq r_2$, its general solution (17) is the sum of two exponential functions. Therefore the solution has a relatively simple geometrical behavior: as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else grows rapidly (when at least one exponent is positive). These two cases are illustrated by the solutions of Examples 2 and 3, which are shown in Figures 3.1.1 and 3.1.2, respectively. There is also a third case that occurs less often; the solution approaches a constant when one exponent is zero and the other is negative.

PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1 e^{-t/2} + c_2 e^{-2t}$.
19. Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0, \quad y(0) = 2, \quad y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24 determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.

(c) Find the smallest value of β for which the solution has no minimum point.

26. Consider the initial value problem (see Example 4)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
(c) Determine the smallest value of β for which $y_m \geq 4$.
(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Find an equation of the form $ay'' + by' + cy = 0$ for which all solutions approach a multiple of e^{-t} as $t \to \infty$.

Equations with the Dependent Variable Missing. For a second order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first order equation for v, and a second is introduced in the integration for y. In each of Problems 28 through 33 use this substitution to solve the given equation.

28. $t^2y'' + 2ty' - 1 = 0, \quad t > 0$
29. $ty'' + y' = 1, \quad t > 0$
30. $y'' + t(y')^2 = 0$
31. $2t^2y'' + (y')^3 = 2ty', \quad t > 0$
32. $y'' + y' = e^{-t}$
33. $t^2y'' = (y')^2, \quad t > 0$

Equations with the Independent Variable Missing. If a second order differential equation has the form $y'' = f(y, y')$, then the independent variable t does not appear explicitly, but only through the dependent variable y. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation is not of the form of the first order equations discussed in Chapter 2. However, if we think of y as the independent variable, then by the chain rule $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$. Again, there are two arbitrary constants in the final result. In each of Problems 34 through 39 use this method to solve the given differential equation.

34. $yy'' + (y')^2 = 0$
35. $y'' + y = 0$
36. $y'' + y(y')^3 = 0$
37. $2y^2y'' + 2y(y')^2 = 1$
38. $yy'' - (y')^3 = 0$
39. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 39 the transformed equation is a Bernoulli equation. See Problem 27 in Section 2.4.

In each of Problems 40 through 43 solve the given initial value problem using the methods of Problems 28 through 39.

40. $y'y'' = 2, \quad y(0) = 1, \quad y'(0) = 2$
41. $y'' - 3y^2 = 0, \quad y(0) = 2, \quad y'(0) = 4$
42. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0, \quad y(1) = 2, \quad y'(1) = -1$
43. $y'y'' - t = 0, \quad y(1) = 2, \quad y'(1) = 1$

3.2 Fundamental Solutions of Linear Homogeneous Equations

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In

turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

In developing the theory of linear differential equations, it is helpful to introduce a differential operator notation. Let p and q be continuous functions on an open interval I, that is, for $\alpha < t < \beta$. The cases $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1 + t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1 + t)\sin 3t. \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator.

In this section we study the second order linear homogeneous equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With Eq. (2) we associate a set of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.

3. The solution ϕ is defined *throughout the interval* I where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For example, we found in Section 3.1 that the initial value problem

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \tfrac{1}{2}e^{t} + \tfrac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by Eq. (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

However, for most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first order and second order linear equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

EXAMPLE 1

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \qquad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

If the given differential equation is written in the form of Eq. (4), then $p(t) = 1/(t-3)$, $q(t) = -(t+3)/t(t-3)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$. Thus, this is the longest interval in which Theorem 3.2.1 guarantees that the solution exists.

EXAMPLE 2

Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are continuous in an open interval I containing t_0.

The function $y = \phi(t) = 0$ for all t in I certainly satisfies the differential equation and initial conditions. By the uniqueness part of Theorem 3.2.1 it is the only solution of the given problem.

[2] A proof of Theorem 3.2.1 may be found, for example, in Chapter 6, Section 8 of the book by Coddington listed in the references.

Let us now assume that y_1 and y_2 are two solutions of Eq. (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0, \tag{7}$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 **(Principle of Superposition)** If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any multiple of a solution of Eq. (2) is also a solution.

To prove Theorem 3.2.2 we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{8}$$

for y in Eq. (2). The result is

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2]. \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, y as given by Eq. (8) does satisfy the differential equation (2) and the proof of Theorem 3.2.2 is complete.

Theorem 3.2.2 states that, beginning with only two solutions of Eq. (2), we can construct a doubly infinite family of solutions by means of Eq. (8). The next question is whether all solutions of Eq. (2) are included in Eq. (8), or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in Eq. (8) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{9}$$

Upon solving Eqs. (9) for c_1 and c_2, we find that

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2 the expression (8) satisfies the initial conditions (3) as well as the differential equation (2).

In order for the expressions for c_1 and c_2 in Eqs. (10) or (11) to make sense, it is necessary that the denominators be nonzero. For both c_1 and c_2 the denominator is the same, namely, the determinant

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{12}$$

The determinant W is called the **Wronskian**[3] **determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W(y_1, y_2)(t_0)$ to stand for the expression on the right side of Eq. (12), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument suffices to establish the following result.

Theorem 3.2.3 Suppose that y_1 and y_2 are two solutions of Eq. (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at the point t_0 where the initial conditions (3),

$$y(t_0) = y_0, \qquad y'(t_0) = y_0',$$

are assigned. Then there is a choice of the constants c_1, c_2 for which $y = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation (2) and the initial conditions (3).

EXAMPLE 3

In Example 1 of Section 3.1 we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2.

The Wronskian of these two functions is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

[3] Wronskian determinants are named for Józef Maria Hoëné-Wronski (1776–1853), who was born in Poland but spent most of his life in France. Wronski was a gifted but troubled man, and his life was marked by frequent heated disputes with other individuals and institutions.

Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of t. One such initial value problem was solved in Example 2 of Section 3.1.

The next theorem justifies the term "general solution" that we introduced in Section 3.1 for the linear combination $c_1y_1 + c_2y_2$.

Theorem 3.2.4 If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of Eq. (2).

Let ϕ be any solution of Eq. (2). To prove the theorem we must show that ϕ is included in the linear combination $c_1y_1 + c_2y_2$; that is, for some choice of the constants c_1 and c_2, the linear combination is equal to ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; thus

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{13}$$

The function ϕ is certainly a solution of this initial value problem. On the other hand, since $W(y_1,y_2)(t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 so that $y = c_1y_1(t) + c_2y_2(t)$ is also a solution of the initial value problem (13). In fact, the proper values of c_1 and c_2 are given by Eqs. (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1y_1(t) + c_2y_2(t),$$

and therefore ϕ is included in the family of functions of $c_1y_1 + c_2y_2$. Finally, since ϕ is an *arbitrary* solution of Eq. (2), it follows that *every* solution of this equation is included in this family. This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that, as long as the Wronskian of y_1 and y_2 is not everywhere zero, the linear combination $c_1y_1 + c_2y_2$ contains all solutions of Eq. (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of Eq. (2). The solutions y_1 and y_2, with a nonzero Wronskian, are said to form a **fundamental set of solutions** of Eq. (2).

To restate the result of Theorem 3.2.4 in slightly different language: To find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition. Alternatively, the following example includes all those mentioned in Section 3.1, as well as many other problems of a similar type.

EXAMPLE 4

Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form (2). Show that they form a fundamental set of solutions if $r_1 \neq r_2$.

We calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\exp[(r_1 + r_2)t].$$

Since the exponential function is never zero, and since $r_2 - r_1 \neq 0$ by the statement of the problem, it follows that W is nonzero for every value of t. Consequently, y_1 and y_2 form a fundamental set of solutions.

EXAMPLE 5

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

We will show in Section 5.5 how to solve Eq. (14); see also Problem 38 in Section 3.4. However, at this stage we can verify by direct substitution that y_1 and y_2 are solutions of the differential equation. Since $y_1'(t) = \frac{1}{2}t^{-1/2}$ and $y_1''(t) = -\frac{1}{4}t^{-3/2}$, we have

$$2t^2(-\tfrac{1}{4}t^{-3/2}) + 3t(\tfrac{1}{2}t^{-1/2}) - t^{1/2} = (-\tfrac{1}{2} + \tfrac{3}{2} - 1)t^{1/2} = 0.$$

Similarly, $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$, so

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4 - 3 - 1)t^{-1} = 0.$$

Next we calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\tfrac{3}{2}t^{-3/2}. \tag{15}$$

Since $W \neq 0$ for $t > 0$, we conclude that y_1 and y_2 form a fundamental set of solutions there.

In several cases, we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question may arise as to whether or not a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 3.2.5 Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of Eq. (2) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of Eq. (2) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then y_1 and y_2 form a fundamental set of solutions of Eq. (2).

First observe that the *existence* of the functions y_1 and y_2 is assured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions we need only calculate their Wronskian at t_0:

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

Note that the difficult part of this proof, demonstrating the existence of a pair of solutions, is taken care of by reference to Theorem 3.2.1. Note also that Theorem 3.2.5 does not address the question of how to solve the specified initial value problems so as to find the functions y_1 and y_2 indicated in the theorem. Nevertheless, it may be reassuring to know that a fundamental set of solutions always exists.

EXAMPLE 6 Find the fundamental set of solutions specified by Theorem 3.2.5 for the differential equation

$$y'' - y = 0, \tag{16}$$

using the initial point $t_0 = 0$.

In Section 3.1 we noted that two solutions of Eq. (16) are $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. The Wronskian of these solutions is $W(y_1,y_2)(t) = -2 \neq 0$, so they form a fundamental set of solutions. However, they are not the fundamental solutions indicated by Theorem 3.2.5 because they do not satisfy the initial conditions mentioned in that theorem at the point $t = 0$.

To find the fundamental solutions specified by the theorem we need to find the solutions satisfying the proper initial conditions. Let us denote by $y_3(t)$ the solution of Eq. (16) that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{17}$$

The general solution of Eq. (16) is

$$y = c_1e^t + c_2e^{-t}, \tag{18}$$

and the initial conditions (17) are satisfied if $c_1 = 1/2$ and $c_2 = 1/2$. Thus

$$y_3(t) = \tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t} = \cosh t.$$

Similarly, if $y_4(t)$ satisfies the initial conditions

$$y(0) = 0, \qquad y'(0) = 1, \tag{19}$$

then

$$y_4(t) = \tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t} = \sinh t.$$

Since the Wronskian of y_3 and y_4 is

$$W(y_3, y_4)(t) = \cosh^2 t - \sinh^2 t = 1,$$

these functions also form a fundamental set of solutions, as stated by Theorem 3.2.5. Therefore, the general solution of Eq. (16) can be written as

$$y = k_1 \cosh t + k_2 \sinh t, \tag{20}$$

as well as in the form (18). We have used k_1 and k_2 for the arbitrary constants in Eq. (20) because they are not the same as the constants c_1 and c_2 in Eq. (18). One purpose of this example is to make clear that a given differential equation has more than one fundamental set of solutions; indeed, it has infinitely many. As a rule, you should choose the set that is most convenient.

We can summarize the discussion in this section as follows. To find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$ where $W \neq 0$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

PROBLEMS

In each of Problems 1 through 6 find the Wronskian of the given pair of functions.

1. $e^{2t}, \quad e^{-3t/2}$
2. $\cos t, \quad \sin t$
3. $e^{-2t}, \quad te^{-2t}$
4. $x, \quad xe^x$
5. $e^t \sin t, \quad e^t \cos t$
6. $\cos^2\theta, \quad 1 + \cos 2\theta$

In each of Problems 7 through 12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2$
8. $(t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$
9. $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1$
10. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \quad y(2) = 3, \quad y'(2) = 1$
11. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$
12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.
14. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.
16. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.
17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.
18. If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.
19. If $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ of u and v in terms of $W(f, g)$.
20. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.

In each of Problems 21 and 22 find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

21. $y'' + y' - 2y = 0, \qquad t_0 = 0$
22. $y'' + 4y' + 3y = 0, \qquad t_0 = 1$

In each of Problems 23 through 26 verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

23. $y'' + 4y = 0; \qquad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
24. $y'' - 2y' + y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = te^t$
25. $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \qquad y_1(x) = x, \quad y_2(x) = xe^x$
26. $(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \qquad y_1(x) = x, \quad y_2(x) = \sin x$
27. **Exact Equations.** The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be exact if it can be written in the form $[P(x)y']' + [f(x)y]' = 0$, where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is $P''(x) - Q'(x) + R(x) = 0$. It can be shown that this is also a sufficient condition.

In each of Problems 28 through 31 use the result of Problem 27 to determine whether the given equation is exact. If so, solve the equation.

28. $y'' + xy' + y = 0$
29. $y'' + 3x^2y' + xy = 0$
30. $xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$
31. $x^2y'' + xy' - y = 0, \quad x > 0$
32. **The Adjoint Equation.** If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$ can be written in the form $[\mu(x)P(x)y']' + [f(x)y]' = 0$. By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

In each of Problems 33 through 35 use the result of Problem 32 to find the adjoint of the given differential equation.

33. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation
34. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, Legendre's equation
35. $y'' - xy = 0$, Airy's equation
36. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
37. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 33 through 35 is self-adjoint.

3.3 Linear Independence and the Wronskian

The representation of the general solution of a second order linear homogeneous differential equation as a linear combination of two solutions whose Wronskian is not zero is intimately related to the concept of linear independence of two functions. This is a very important algebraic idea and has significance far beyond the present context; we briefly discuss it in this section.

We will refer to the following basic property of systems of linear homogeneous algebraic equations. Consider the two–by–two system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0, \\ a_{21}x_1 + a_{22}x_2 &= 0, \end{aligned} \tag{1}$$

and let $\Delta = a_{11}a_{22} - a_{12}a_{21}$ be the corresponding determinant of coefficients. Then $x = 0$, $y = 0$ is the only solution of the system (1) if and only if $\Delta \neq 0$. Further, the system (1) has nonzero solutions if and only if $\Delta = 0$.

Two functions f and g are said to be **linearly dependent** on an interval I if there exist two constants k_1 and k_2, not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0 \tag{2}$$

for all t in I. The functions f and g are said to be **linearly independent** on an interval I if they are not linearly dependent; that is, Eq. (2) holds for all t in I only if $k_1 = k_2 = 0$. In Section 4.1 these definitions are extended to an arbitrary number of functions. Although it may be difficult to determine whether a large set of functions is linearly independent or linearly dependent, it is usually easy to answer this question for a set of only two functions: they are linearly dependent if they are proportional to each other, and linearly independent otherwise. The following examples illustrate these definitions.

EXAMPLE 1 Determine whether the functions $\sin t$ and $\cos(t - \pi/2)$ are linearly independent or linearly dependent on an arbitrary interval.

The given functions are linearly dependent on any interval since

$$k_1 \sin t + k_2 \cos(t - \pi/2) = 0$$

for all t if we choose $k_1 = 1$ and $k_2 = -1$.

EXAMPLE 2

Show that the functions e^t and e^{2t} are linearly independent on any interval.

To establish this result we suppose that

$$k_1 e^t + k_2 e^{2t} = 0 \tag{3}$$

for all t in the interval; we must then show that $k_1 = k_2 = 0$. Choose two points t_0 and t_1 in the interval, where $t_1 \neq t_0$. Evaluating Eq. (3) at these points, we obtain

$$\begin{aligned} k_1 e^{t_0} + k_2 e^{2t_0} &= 0, \\ k_1 e^{t_1} + k_2 e^{2t_1} &= 0. \end{aligned} \tag{4}$$

The determinant of coefficients is

$$e^{t_0} e^{2t_1} - e^{2t_0} e^{t_1} = e^{t_0} e^{t_1} (e^{t_1} - e^{t_0}).$$

Since this determinant is not zero, it follows that the only solution of Eq. (4) is $k_1 = k_2 = 0$. Hence e^t and e^{2t} are linearly independent.

The following theorem relates linear independence and dependence to the Wronskian.

Theorem 3.3.1 If f and g are differentiable functions on an open interval I and if $W(f, g)(t_0) \neq 0$ for some point t_0 in I, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f, g)(t) = 0$ for every t in I.

To prove the first statement in Theorem 3.3.1, consider a linear combination $k_1 f(t) + k_2 g(t)$, and suppose that this expression is zero throughout the interval. Evaluating the expression and its derivative at t_0, we have

$$\begin{aligned} k_1 f(t_0) + k_2 g(t_0) &= 0, \\ k_1 f'(t_0) + k_2 g'(t_0) &= 0. \end{aligned} \tag{5}$$

The determinant of coefficients of Eqs. (5) is precisely $W(f, g)(t_0)$, which is not zero by hypothesis. Therefore, the only solution of Eqs. (5) is $k_1 = k_2 = 0$, so f and g are linearly independent.

The second part of Theorem 3.3.1 follows immediately from the first. Let f and g be linearly dependent, and suppose that the conclusion is false, that is, $W(f, g)$ is not everywhere zero in I. Then there is a point t_0 such that $W(f, g)(t_0) \neq 0$; by the first part of Theorem 3.3.1 this implies that f and g are linearly independent, which is a contradiction, thus completing the proof.

We can apply this result to the two functions $f(t) = e^t$ and $g(t) = e^{2t}$ discussed in Example 2. For any point t_0 we have

$$W(f, g)(t_0) = \begin{vmatrix} e^{t_0} & e^{2t_0} \\ e^{t_0} & 2e^{2t_0} \end{vmatrix} = e^{3t_0} \neq 0. \tag{6}$$

The functions e^t and e^{2t} are therefore linearly independent on any interval.

You should be careful not to read too much into Theorem 3.3.1. In particular, two functions f and g may be linearly independent even though $W(f, g)(t) = 0$ for every t in the interval I. This is illustrated in Problem 28.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 3.3.2 **(Abel's Theorem)**[4] If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{7}$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\, dt\right], \tag{8}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem we start by noting that y_1 and y_2 satisfy

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0, \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned} \tag{9}$$

If we multiply the first equation by $-y_2$, the second by y_1, and add the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{10}$$

Next, we let $W(t) = W(y_1, y_2)(t)$ and observe that

$$W' = y_1y_2'' - y_1''y_2. \tag{11}$$

Then we can write Eq. (10) in the form

$$W' + p(t)W = 0. \tag{12}$$

[4] The result in Theorem 3.3.2 was derived by the Norwegian mathematician Niels Henrik Abel (1802–1829) in 1827 and is known as Abel's formula. Abel also showed that there is no general formula for solving a quintic, or fifth degree, polynomial equation in terms of explicit algebraic operations on the coefficients, thereby resolving a question that had been open since the sixteenth century. His greatest contributions, however, were in analysis, particularly in the study of elliptic functions. Unfortunately, his work was not widely noticed until after his death. The distinguished French mathematician Legendre called it a "monument more lasting than bronze."

Equation (12) can be solved immediately since it is both a first order linear equation (Section 2.1) and a separable equation (Section 2.2). Thus

$$W(t) = c \exp\left[-\int p(t)\, dt\right], \tag{13}$$

where c is a constant. The value of c depends on which pair of solutions of Eq. (7) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t, which completes the proof of Theorem 3.3.2.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation.

EXAMPLE 3

In Example 5 of Section 3.2 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

Verify that the Wronskian of y_1 and y_2 is given by Eq. (13).

From the example just cited we know that $W(y_1, y_2)(t) = -(3/2)t^{-3/2}$. To use Eq. (13) we must write the differential equation (14) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$\begin{aligned} W(y_1, y_2)(t) &= c \exp\left[-\int \frac{3}{2t}\, dt\right] = c \exp\left(-\frac{3}{2}\ln t\right) \\ &= c\, t^{-3/2}. \end{aligned} \tag{15}$$

Equation (15) gives the Wronskian of any pair of solutions of Eq. (14). For the particular solutions given in this example we must choose $c = -3/2$.

A stronger version of Theorem 3.3.1 can be established if the two functions involved are solutions of a second order linear homogeneous differential equation.

Theorem 3.3.3 Let y_1 and y_2 be the solutions of Eq. (7),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then y_1 and y_2 are linearly dependent on I if and only if $W(y_1, y_2)(t)$ is zero for all t in I. Alternatively, y_1 and y_2 are linearly independent on I if and only if $W(y_1, y_2)(t)$ is never zero in I.

Of course, we know by Theorem 3.3.2 that $W(y_1, y_2)(t)$ is either everywhere zero or nowhere zero in I. In proving Theorem 3.3.3, observe first that if y_1 and y_2 are linearly

dependent, then $W(y_1, y_2)(t)$ is zero for all t in I by Theorem 3.3.1. It remains to prove the converse; that is, if $W(y_1, y_2)(t)$ is zero throughout I, then y_1 and y_2 are linearly dependent. Let t_0 be any point in I; then necessarily $W(y_1, y_2)(t_0) = 0$. Consequently, the system of equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0 \end{aligned} \tag{16}$$

for c_1 and c_2 has a nontrivial solution. Using these values of c_1 and c_2, let $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$. Then ϕ is a solution of Eq. (7), and by Eqs. (16) ϕ also satisfies the initial conditions

$$\phi(t_0) = 0, \qquad \phi'(t_0) = 0. \tag{17}$$

Therefore, by the uniqueness part of Theorem 3.2.1, or by Example 2 of Section 3.2, $\phi(t) = 0$ for all t in I. Since $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$ with c_1 and c_2 not both zero, this means that y_1 and y_2 are linearly dependent. The alternative statement of the theorem follows immediately.

We can now summarize the facts about fundamental sets of solutions, Wronskians, and linear independence in the following way. Let y_1 and y_2 be solutions of Eq. (7),

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then the following four statements are equivalent, in the sense that each one implies the other three:

1. The functions y_1 and y_2 are a fundamental set of solutions on I.
2. The functions y_1 and y_2 are linearly independent on I.
3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.
4. $W(y_1, y_2)(t) \neq 0$ for all t in I.

It is interesting to note the similarity between second order linear homogeneous differential equations and two-dimensional vector algebra. Two vectors **a** and **b** are said to be linearly dependent if there are two scalars k_1 and k_2, not both zero, such that $k_1\mathbf{a} + k_2\mathbf{b} = \mathbf{0}$; otherwise, they are said to be linearly independent. Let **i** and **j** be unit vectors directed along the positive x and y axes, respectively. Since $k_1\mathbf{i} + k_2\mathbf{j} = \mathbf{0}$ only if $k_1 = k_2 = 0$, the vectors **i** and **j** are linearly independent. Further, we know that any vector **a** with components a_1 and a_2 can be written as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, that is, as a linear combination of the two linearly independent vectors **i** and **j**. It is not difficult to show that any vector in two dimensions can be expressed as a linear combination of any two linearly independent two-dimensional vectors (see Problem 14). Such a pair of linearly independent vectors is said to form a basis for the vector space of two-dimensional vectors.

The term **vector space** is also applied to other collections of mathematical objects that obey the same laws of addition and multiplication by scalars that geometric vectors do. For example, it can be shown that the set of functions that are twice differentiable on the open interval I forms a vector space. Similarly, the set V of functions satisfying Eq. (7) also forms a vector space.

Since every member of V can be expressed as a linear combination of two linearly independent members y_1 and y_2, we say that such a pair forms a basis for V. This leads to the conclusion that V is two-dimensional; therefore, it is analogous in many respects to the space of geometric vectors in a plane. Later we find that the set of solutions of an

nth order linear homogeneous differential equation forms a vector space of dimension n, and that any set of n linearly independent solutions of the differential equation forms a basis for the space. This connection between differential equations and vectors constitutes a good reason for the study of abstract linear algebra.

PROBLEMS

In each of Problems 1 through 8 determine whether the given pair of functions is linearly independent or linearly dependent.

1. $f(t) = t^2 + 5t, \quad g(t) = t^2 - 5t$
2. $f(\theta) = \cos 3\theta, \quad g(\theta) = 4\cos^3\theta - 3\cos\theta$
3. $f(t) = e^{\lambda t}\cos\mu t, \quad g(t) = e^{\lambda t}\sin\mu t, \quad \mu \neq 0$
4. $f(x) = e^{3x}, \quad g(x) = e^{3(x-1)}$
5. $f(t) = 3t - 5, \quad g(t) = 9t - 15$
6. $f(t) = t, \quad g(t) = t^{-1}$
7. $f(t) = 3t, \quad g(t) = |t|$
8. $f(x) = x^3, \quad g(x) = |x|^3$
9. The Wronskian of two functions is $W(t) = t\sin^2 t$. Are the functions linearly independent or linearly dependent? Why?
10. The Wronskian of two functions is $W(t) = t^2 - 4$. Are the functions linearly independent or linearly dependent? Why?
11. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that c_1y_1 and c_2y_2 are also linearly independent solutions, provided that neither c_1 nor c_2 is zero.
12. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions. Conversely, if y_3 and y_4 are linearly independent solutions of the differential equation, show that y_1 and y_2 are also.
13. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, determine under what conditions the functions $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$ also form a linearly independent set of solutions.
14. (a) Prove that any two-dimensional vector can be written as a linear combination of $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j}$.
    (b) Prove that if the vectors $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ and $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j}$ are linearly independent, then any vector $\mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j}$ can be expressed as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Note that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $x_1y_2 - x_2y_1 \neq 0$. Why?

In each of Problems 15 through 18 find the Wronskian of two solutions of the given differential equation without solving the equation.

15. $t^2y'' - t(t+2)y' + (t+2)y = 0$
16. $(\cos t)y'' + (\sin t)y' - ty = 0$
17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation
18. $(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, Legendre's equation
19. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
20. If y_1 and y_2 are linearly independent solutions of $ty'' + 2y' + te^t y = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.
21. If y_1 and y_2 are linearly independent solutions of $t^2y'' - 2y' + (3+t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.
22. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?
23. If f, g, and h are differentiable functions, show that $W(fg, fh) = f^2W(g, h)$.

In Problems 24 through 26 assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

24. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.
25. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.
26. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.
27. Show that t and t^2 are linearly independent on $-1 < t < 1$; indeed, they are linearly independent on every interval. Show also that $W(t, t^2)$ is zero at $t = 0$. What can you conclude from this about the possibility that t and t^2 are solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$? Verify that t and t^2 are solutions of the equation $t^2y'' - 2ty' + 2y = 0$. Does this contradict your conclusion? Does the behavior of the Wronskian of t and t^2 contradict Theorem 3.3.2?
28. Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$, but are linearly independent on $-1 < t < 1$. Although f and g are linearly independent there, show that $W(f, g)$ is zero for all t in $-1 < t < 1$. Hence f and g cannot be solutions of an equation $y'' + p(t)y' + q(t)y = 0$ with p and q continuous on $-1 < t < 1$.

3.4 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

If the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5)

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in Eqs. (5) we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to accomplish this extension of the exponential function. Here we use a method based on infinite series; an alternative is outlined in Problem 28.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty. \tag{7}$$

If we now assume that we can substitute it for t in Eq. (7), then we have

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}, \end{aligned} \tag{8}$$

where we have separated the sum into its real and imaginary parts, making use of the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. The first series in Eq. (8) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i \sin t. \tag{9}$$

Equation (9) is known as Euler's formula and is an extremely important mathematical relationship. While our derivation of Eq. (9) is based on the unverified assumption that the series (7) can be used for complex as well as real values of the independent variable, our intention is to use this derivation only to make Eq. (9) seem plausible. We now put matters on a firm foundation by adopting Eq. (9) as the *definition* of e^{it}. In other words, whenever we write e^{it}, we mean the expression on the right side of Eq. (9).

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in Eq. (9) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i \sin t. \tag{10}$$

Further, if t is replaced by μt in Eq. (9), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos \mu t + i \sin \mu t. \tag{11}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp[(\lambda + i\mu)t]$ to satisfy

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t}. \tag{12}$$

Then, substituting for $e^{i\mu t}$ from Eq. (11), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t. \end{aligned} \tag{13}$$

We now take Eq. (13) as the definition of $\exp[(\lambda + i\mu)t]$. The value of the exponential function with a complex exponent is a complex number whose real and imaginary parts are given by the terms on the right side of Eq. (13). Observe that the real and

imaginary parts of $\exp[(\lambda + i\mu)t]$ are expressed entirely in terms of elementary real-valued functions. For example, the quantity in Eq. (6) has the value

$$e^{-3+6i} = e^{-3}\cos 6 + ie^{-3}\sin 6 \cong 0.0478041 - 0.0139113i.$$

With the definitions (9) and (13) it is straightforward to show that the usual laws of exponents are valid for the complex exponential function. It is also easy to verify that the differentiation formula

$$\frac{d}{dt}(e^{rt}) = re^{rt} \tag{14}$$

also holds for complex values of r.

Real-Valued Solutions. The functions $y_1(t)$ and $y_2(t)$, given by Eqs. (5) and with the meaning expressed by Eq. (13), are solutions of Eq. (1) when the roots of the characteristic equation (2) are complex numbers $\lambda \pm i\mu$. Unfortunately, the solutions y_1 and y_2 are complex-valued functions, whereas in general we would prefer to have real-valued solutions, if possible, because the differential equation itself has real coefficients. Such solutions can be found as a consequence of Theorem 3.2.2, which states that if y_1 and y_2 are solutions of Eq. (1), then any linear combination of y_1 and y_2 is also a solution. In particular, let us form the sum and then the difference of y_1 and y_2. We have

$$\begin{aligned} y_1(t) + y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) + e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2e^{\lambda t}\cos \mu t \end{aligned}$$

and

$$\begin{aligned} y_1(t) - y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) - e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2ie^{\lambda t}\sin \mu t. \end{aligned}$$

Hence, neglecting the constant multipliers 2 and $2i$, respectively, we have obtained a pair of real-valued solutions

$$u(t) = e^{\lambda t}\cos \mu t, \qquad v(t) = e^{\lambda t}\sin \mu t. \tag{15}$$

Observe that u and v are simply the real and imaginary parts, respectively, of y_1.

By direct computation you can show that the Wronskian of u and v is

$$W(u, v)(t) = \mu e^{2\lambda t}. \tag{16}$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. (Of course, if $\mu = 0$, then the roots are real and the discussion in this section is not applicable.) Consequently, if the roots of the characteristic equation are complex numbers $\lambda \pm i\mu$, with $\mu \neq 0$, then the general solution of Eq. (1) is

$$y = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t, \tag{17}$$

where c_1 and c_2 are arbitrary constants. Note that the solution (17) can be written down as soon as the values of λ and μ are known.

EXAMPLE 1

Find the general solution of

$$y'' + y' + y = 0. \tag{18}$$

The characteristic equation is

$$r^2 + r + 1 = 0,$$

and its roots are

$$r = \frac{-1 \pm (1-4)^{1/2}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus $\lambda = -1/2$ and $\mu = \sqrt{3}/2$, so the general solution of Eq. (18) is

$$y = c_1 e^{-t/2}\cos(\sqrt{3}t/2) + c_2 e^{-t/2}\sin(\sqrt{3}t/2). \tag{19}$$

EXAMPLE 2

Find the general solution of

$$y'' + 9y = 0. \tag{20}$$

The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$; thus $\lambda = 0$ and $\mu = 3$. The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t; \tag{21}$$

note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution.

EXAMPLE 3

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1. \tag{22}$$

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = 1/4 \pm 3i$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/4}\cos 3t + c_2 e^{t/4}\sin 3t. \tag{23}$$

To apply the first initial condition we set $t = 0$ in Eq. (23); this gives

$$y(0) = c_1 = -2.$$

For the second initial condition we must differentiate Eq. (23) and then set $t = 0$. In this way we find that

$$y'(0) = \tfrac{1}{4}c_1 + 3c_2 = 1,$$

from which $c_2 = 1/2$. Using these values of c_1 and c_2 in Eq. (23), we obtain

$$y = -2e^{t/4}\cos 3t + \tfrac{1}{2}e^{t/4}\sin 3t \tag{24}$$

as the solution of the initial value problem (22).

We will discuss the geometrical properties of solutions such as these more fully in Section 3.8, so we will be very brief here. Each of the solutions u and v in Eqs. (15) represents an oscillation, because of the trigonometric factors, and also either grows or

decays exponentially, depending on the sign of λ (unless $\lambda = 0$). In Example 1 we have $\lambda = -1/2 < 0$, so solutions are decaying oscillations. The graph of a typical solution of Eq. (18) is shown in Figure 3.4.1. On the other hand, $\lambda = 1/4 > 0$ in Example 3, so solutions of the differential equation (22) are growing oscillations. The graph of the solution (24) of the given initial value problem is shown in Figure 3.4.2. The intermediate case is illustrated in Example 2 in which $\lambda = 0$. In this case the solution neither grows nor decays exponentially, but oscillates steadily; a typical solution of Eq. (20) is shown in Figure 3.4.3.

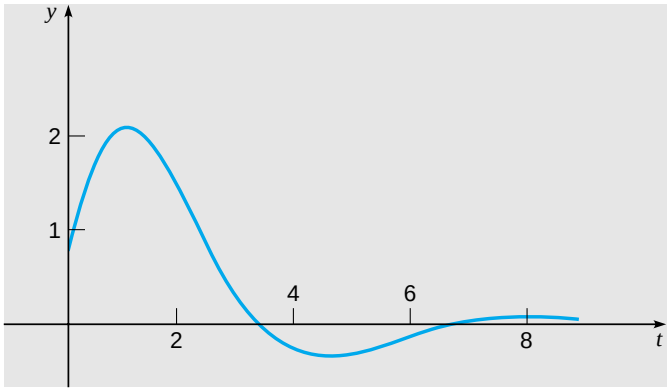


FIGURE 3.4.1 A typical solution of $y'' + y' + y = 0$.

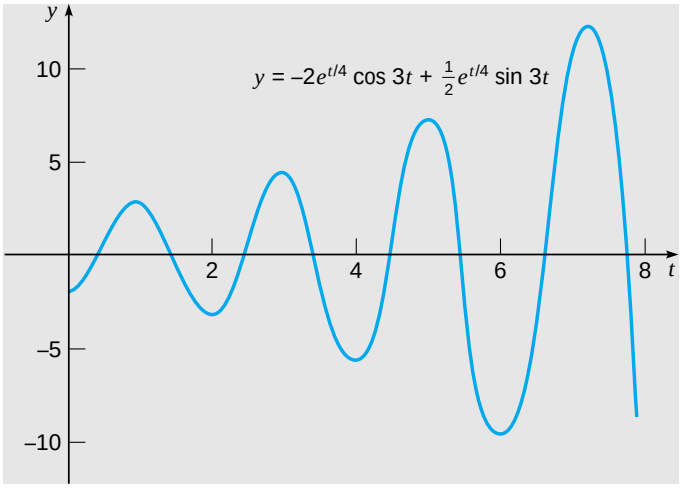


FIGURE 3.4.2 Solution of $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

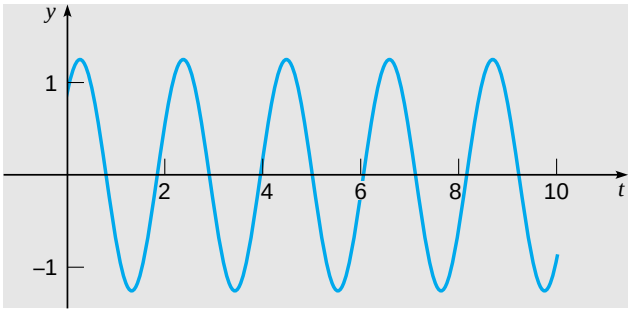


FIGURE 3.4.3 A typical solution of $y'' + 9y = 0$.

PROBLEMS

In each of Problems 1 through 6 use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-(\pi/2)i}$
5. 2^{1-i}
6. π^{-1+2i}

In each of Problems 7 through 16 find the general solution of the given differential equation.

7. $y'' - 2y' + 2y = 0$
8. $y'' - 2y' + 6y = 0$
9. $y'' + 2y' - 8y = 0$
10. $y'' + 2y' + 2y = 0$
11. $y'' + 6y' + 13y = 0$
12. $4y'' + 9y = 0$
13. $y'' + 2y' + 1.25y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
21. $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$
22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

▶ 23. Consider the initial value problem

$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ of this problem.
(b) Find the first time at which $|u(t)| = 10$.

▶ 24. Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$

(a) Find the solution $u(t)$ of this problem.
(b) Find the smallest T such that $|u(t)| \le 0.1$ for all $t > T$.

▶ 25. Consider the initial value problem

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \ge 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) Find α so that $y = 0$ when $t = 1$.

(c) Find, as a function of α, the smallest positive value of t for which $y = 0$.
(d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.

▶ 26. Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.
(c) Repeat part (b) for $a = 1/4$, $1/2$, and 2.
(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

27. Show that $W(e^{\lambda t}\cos \mu t, e^{\lambda t}\sin \mu t) = \mu e^{2\lambda t}$.

28. In this problem we outline a different derivation of Euler's formula.
(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{i}$$

for some constants c_1 and c_2. Why is this so?
(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.

29. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/2i.$$

30. If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.

31. If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt}e^{rt} = re^{rt}$$

for any complex number r.

32. Let the real-valued functions p and q be continuous on the open interval I, and let $y = \phi(t) = u(t) + iv(t)$ be a complex-valued solution of

$$y'' + p(t)y' + q(t)y = 0, \tag{i}$$

where u and v are real-valued functions. Show that u and v are also solutions of Eq. (i). *Hint:* Substitute $y = \phi(t)$ in Eq. (i) and separate into real and imaginary parts.

33. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.

Change of Variables. Often a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \tag{i}$$

can be put in a more suitable form for finding a solution by making a change of the independent and/or dependent variables. We explore these ideas in Problems 34 through 42. In particular, in Problem 34 we determine conditions under which Eq. (i) can be transformed into a differential equation with constant coefficients and thereby becomes easily solvable. Problems 35 through 42 give specific applications of this procedure.

34. In this problem we determine conditions on p and q such that Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable. Let

$x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the differential equation (i) becomes

$$\left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{ii}$$

(c) In order for Eq. (ii) to have constant coefficients, the coefficients of d^2y/dx^2 and of y must be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence

$$x = u(t) = \int [q(t)]^{1/2}\, dt. \tag{iii}$$

(d) With x chosen as in part (c) show that the coefficient of dy/dx in Eq. (ii) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \tag{iv}$$

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant. How must this result be modified if $q(t) < 0$?

In each of Problems 35 through 37 try to transform the given equation into one with constant coefficients by the method of Problem 34. If this is possible, find the general solution of the given equation.

35. $y'' + ty' + e^{-t^2}y = 0, \qquad -\infty < t < \infty$
36. $y'' + 3ty' + t^2y = 0, \qquad -\infty < t < \infty$
37. $ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty$
38. **Euler Equations.** An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0, \qquad t > 0,$$

where α and β are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transforms an Euler equation into an equation with constant coefficients. Euler equations are discussed in detail in Section 5.5.

In each of Problems 39 through 42 use the result of Problem 38 to solve the given equation for $t > 0$.

39. $t^2y'' + ty' + y = 0$
40. $t^2y'' + 4ty' + 2y = 0$
41. $t^2y'' + 3ty' + 1.25y = 0$
42. $t^2y'' - 4ty' - 6y = 0$

3.5 Repeated Roots; Reduction of Order

In earlier sections we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

are either real and different, or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case occurs when the discriminant $b^2 - 4ac$ is zero, and it follows from the quadratic formula that

$$r_1 = r_2 = -b/2a. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/2a} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE 1

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{5}$$

The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore one solution of Eq. (5) is $y_1(t) = e^{-2t}$. To find the general solution of Eq. (5) we need a second solution that is not a multiple of y_1. This second solution can be found in several ways (see Problems 20 through 22); here we use a method originated by D'Alembert[5] in the eighteenth century. Recall that since $y_1(t)$ is a solution of Eq. (1), so is $cy_1(t)$ for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is a solution of Eq. (1).

To carry out this program we substitute $y = v(t)y_1(t)$ in Eq. (1) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-2t}, \tag{6}$$

we have

$$y' = v'(t)e^{-2t} - 2v(t)e^{-2t} \tag{7}$$

and

$$y'' = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}. \tag{8}$$

By substituting the expressions in Eqs. (6), (7), and (8) in Eq. (5) and collecting terms, we obtain

$$[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)]e^{-2t} = 0,$$

which simplifies to

$$v''(t) = 0. \tag{9}$$

[5]Jean d'Alembert (1717–1783), a French mathematician, was a contemporary of Euler and Daniel Bernoulli, and is known primarily for his work in mechanics and differential equations. D'Alembert's principle in mechanics and d'Alembert's paradox in hydrodynamics are named for him, and the wave equation first appeared in his paper on vibrating strings in 1747. In his later years he devoted himself primarily to philosophy and to his duties as science editor of Diderot's Encyclopédie.

Therefore

$$v'(t) = c_1$$

and

$$v(t) = c_1 t + c_2, \tag{10}$$

where c_1 and c_2 are arbitrary constants. Finally, substituting for $v(t)$ in Eq. (6), we obtain

$$y = c_1 t e^{-2t} + c_2 e^{-2t}. \tag{11}$$

The second term on the right side of Eq. (11) corresponds to the original solution $y_1(t) = \exp(-2t)$, but the first term arises from a second solution, namely $y_2(t) = t\exp(-2t)$. These two solutions are obviously not proportional, but we can verify that they are linearly independent by calculating their Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} \\ &= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t} \neq 0. \end{aligned}$$

Therefore

$$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t} \tag{12}$$

form a fundamental set of solutions of Eq. (5), and the general solution of that equation is given by Eq. (11). Note that both $y_1(t)$ and $y_2(t)$ tend to zero as $t \to \infty$; consequently, all solutions of Eq. (5) behave in this way. The graph of a typical solution is shown in Figure 3.5.1.

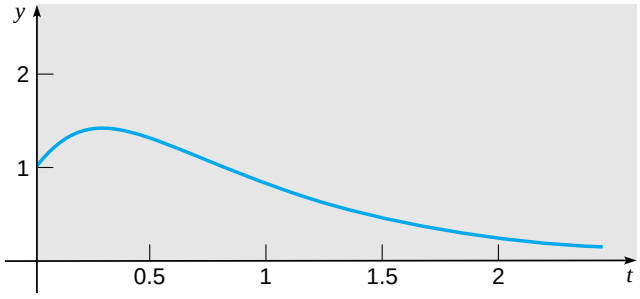


FIGURE 3.5.1 A typical solution of $y'' + 4y' + 4y = 0$.

The procedure used in Example 1 can be extended to a general equation whose characteristic equation has repeated roots. That is, we assume that the coefficients in Eq. (1) satisfy $b^2 - 4ac = 0$, in which case

$$y_1(t) = e^{-bt/2a}$$

is a solution. Then we assume that

$$y = v(t)y_1(t) = v(t)e^{-bt/2a} \tag{13}$$

and substitute in Eq. (1) to determine $v(t)$. We have

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a} \tag{14}$$

and

$$y'' = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}. \tag{15}$$

Then, by substituting in Eq. (1), we obtain

$$\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\}e^{-bt/2a} = 0. \tag{16}$$

Canceling the factor $\exp(-bt/2a)$, which is nonzero, and rearranging the remaining terms, we find that

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0. \tag{17}$$

The term involving $v'(t)$is obviously zero. Further, the coefficient of $v(t)$ is $c - (b^2/4a)$, which is also zero because $b^2 - 4ac = 0$ in the problem that we are considering. Thus, just as in Example 1, Eq. (17) reduces to

$$v''(t) = 0;$$

therefore,

$$v(t) = c_1 t + c_2.$$

Hence, from Eq. (13), we have

$$y = c_1 te^{-bt/2a} + c_2 e^{-bt/2a}. \tag{18}$$

Thus y is a linear combination of the two solutions

$$y_1(t) = e^{-bt/2a}, \qquad y_2(t) = te^{-bt/2a}. \tag{19}$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\dfrac{b}{2a}e^{-bt/2a} & \left(1 - \dfrac{bt}{2a}\right)e^{-bt/2a} \end{vmatrix} = e^{-bt/a}. \tag{20}$$

Since $W(y_1, y_2)(t)$ is never zero, the solutions y_1 and y_2 given by Eq. (19) are a fundamental set of solutions. Further, Eq. (18) is the general solution of Eq. (1) when the roots of the characteristic equation are equal. In other words, in this case, there is one exponential solution corresponding to the repeated root, while a second solution is obtained by multiplying the exponential solution by t.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{3}. \tag{21}$$

The characteristic equation is

$$r^2 - r + 0.25 = 0,$$

so the roots are $r_1 = r_2 = 1/2$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/2} + c_2 t e^{t/2}. \tag{22}$$

The first initial condition requires that

$$y(0) = c_1 = 2.$$

To satisfy the second initial condition, we first differentiate Eq. (22) and then set $t = 0$. This gives

$$y'(0) = \tfrac{1}{2}c_1 + c_2 = \tfrac{1}{3},$$

so $c_2 = -2/3$. Thus, the solution of the initial value problem is

$$y = 2e^{t/2} - \tfrac{2}{3}te^{t/2}. \tag{23}$$

The graph of this solution is shown in Figure 3.5.2.

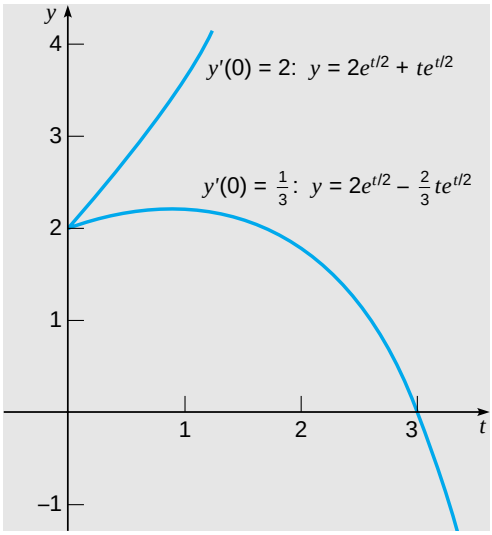


FIGURE 3.5.2 Solutions of $y'' - y' + 0.25y = 0$, $y(0) = 2$, with $y'(0) = 1/3$ and with $y'(0) = 2$, respectively.

Let us now modify the initial value problem (21) by changing the initial slope; to be specific, let the second initial condition be $y'(0) = 2$. The solution of this modified problem is

$$y = 2e^{t/2} + te^{t/2}$$

and its graph is also shown in Figure 3.5.2. The graphs shown in this figure suggest that there is a critical initial slope, with a value between $\frac{1}{3}$ and 2, that separates solutions that grow positively from those that ultimately grow negatively. In Problem 16 you are asked to determine this critical initial slope.

The geometrical behavior of solutions is similar in this case to that when the roots are real and different. If the exponents are either positive or negative, then the magnitude of the solution grows or decays accordingly; the linear factor t has little influence. A decaying solution is shown in Figure 3.5.1 and growing solutions in Figure 3.5.2. However, if the repeated root is zero, then the differential equation is $y'' = 0$ and the general solution is a linear function of t.

Summary. We can now summarize the results that we have obtained for second order linear homogeneous equations with constant coefficients,

$$ay'' + by' + cy = 0. \tag{1}$$

Let r_1 and r_2 be the roots of the corresponding characteristic polynomial

$$ar^2 + br + c = 0. \tag{2}$$

If r_1 and r_2 are real but not equal, then the general solution of the differential equation (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{24}$$

If r_1 and r_2 are complex conjugates $\lambda \pm i\mu$, then the general solution is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t. \tag{25}$$

If $r_1 = r_2$, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}. \tag{26}$$

Reduction of Order. It is worth noting that the procedure used earlier in this section for equations with constant coefficients is more generally applicable. Suppose we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0. \tag{27}$$

To find a second solution, let

$$y = v(t)y_1(t); \tag{28}$$

then

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

Substituting for y, y', and y'' in Eq. (27) and collecting terms, we find that

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0. \tag{29}$$

Since y_1 is a solution of Eq. (27), the coefficient of v in Eq. (29) is zero, so that Eq. (29) becomes

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{30}$$

Despite its appearance, Eq. (30) is actually a first order equation for the function v' and can be solved either as a first order linear equation or as a separable equation. Once v' has been found, then v is obtained by an integration. Finally, y is determined from

Eq. (28). This procedure is called the method of reduction of order because the crucial step is the solution of a first order differential equation for v' rather than the original second order equation for y. Although it is possible to write down a formula for $v(t)$, we will instead illustrate how this method works by an example.

EXAMPLE 3

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \tag{31}$$

find a second linearly independent solution.

We set $y = v(t)t^{-1}$; then

$$y' = v't^{-1} - vt^{-2}, \qquad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting for y, y', and y'' in Eq. (31) and collecting terms, we obtain

$$\begin{aligned} &2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ &\quad = 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ &\quad = 2tv'' - v' = 0. \end{aligned} \tag{32}$$

Note that the coefficient of v is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq. (32) and solving for $v'(t)$, we find that

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \tfrac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \tfrac{2}{3}ct^{1/2} + kt^{-1}, \tag{33}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have $y_2(t) = t^{1/2}$.

PROBLEMS

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $4y'' + 12y' + 9y = 0$
5. $y'' - 2y' + 10y = 0$
6. $y'' - 6y' + 9y = 0$
7. $4y'' + 17y' + 4y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $25y'' - 20y' + 4y = 0$
10. $2y'' + 2y' + y = 0$

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

11. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
13. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$
14. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.
(b) Determine where the solution has the value zero.
(c) Determine the coordinates (t_0, y_0) of the minimum point.
(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

17. Consider the initial value problem

$$4y'' + 4y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.
(b) Determine the coordinates (t_0, y_0) of the maximum point.
(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.
(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

19. If the roots of the characteristic equation are real, show that a solution of $ay'' + by' + cy = 0$ can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at}.
(b) Use Abel's formula [Eq. (8) of Section 3.3] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.
(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to show that a second solution is $y_2(t) = te^{-at}$.

21. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1t)$ and $\exp(r_2t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2t) - \exp(r_1t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed and use L'Hospital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

22. (a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2e^{rt}. \qquad \text{(i)}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \qquad \text{(ii)}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t \exp(r_1 t)$ is also a solution of $L[y] = 0$.

In each of Problems 23 through 30 use the method of reduction of order to find a second solution of the given differential equation.

23. $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \qquad y_1(t) = t^2$
24. $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \qquad y_1(t) = t$
25. $t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$
26. $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \qquad y_1(t) = t$
27. $xy'' - y' + 4x^3y = 0, \quad x > 0; \qquad y_1(x) = \sin x^2$
28. $(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$
29. $x^2y'' - (x - 0.1875)y = 0, \quad x > 0; \qquad y_1(x) = x^{1/4}e^{2\sqrt{x}}$
30. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$
31. The differential equation

$$xy'' - (x + N)y' + Ny = 0,$$

where N is a nonnegative integer, has been discussed by several authors.[6] One reason it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1(x) = e^x$.

(b) Show that a second solution has the form $y_2(x) = ce^x \int x^N e^{-x}\, dx$. Calculate $y_2(x)$ for $N = 1$ and $N = 2$; convince yourself that, with $c = -1/N!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

Note that $y_2(x)$ is exactly the first $N + 1$ terms in the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

32. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

33. The method of Problem 20 can be extended to second order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use Abel's formula [Eq. (8) of Section 3.3] to determine y_2.

In each of Problems 34 through 37 use the method of Problem 33 to find a second independent solution of the given equation.

34. $t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$
35. $ty'' - y' + 4t^3y = 0, \quad t > 0; \qquad y_1(t) = \sin(t^2)$

[6] T. A. Newton, "On Using a Differential Equation to Generate Polynomials," *American Mathematical Monthly 81* (1974), pp. 592–601. Also see the references given there.

36. $(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$
37. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$

Behavior of Solutions as $t \to \infty$. Problems 38 through 40 are concerned with the behavior of solutions in the limit as $t \to \infty$.

38. If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.
39. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 38 is no longer true, but that all solutions are bounded as $t \to \infty$.
    (b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 38 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y_0'$.
40. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t \sin t)y = 0$$

for any value of the constant k. If $0 < k < 2$, show that $1 - k\cos t \sin t > 0$ and $k\sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 38. Thus we observe a not unusual situation in the theory of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. Use the substitution introduced in Problem 38 in Section 3.4 to solve each of the equations in Problems 41 and 42.

41. $t^2y'' - 3ty' + 4y = 0, \qquad t > 0$
42. $t^2y'' + 2ty' + 0.25y = 0, \qquad t > 0$

3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{1}$$

where p, q, and g are given (continuous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{2}$$

in which $g(t) = 0$ and p and q are the same as in Eq. (1), is called the homogeneous equation corresponding to Eq. (1). The following two results describe the structure of solutions of the nonhomogeneous equation (1) and provide a basis for constructing its general solution.

Theorem 3.6.1 If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (2), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{3}$$

where c_1 and c_2 are certain constants.

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t), \qquad L[Y_2](t) = g(t). \tag{4}$$

Subtracting the second of these equations from the first, we have

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \tag{5}$$

However,

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2],$$

so Eq. (5) becomes

$$L[Y_1 - Y_2](t) = 0. \tag{6}$$

Equation (6) states that $Y_1 - Y_2$ is a solution of Eq. (2). Finally, since all solutions of Eq. (2) can be expressed as linear combinations of a fundamental set of solutions by Theorem 3.2.4, it follows that the solution $Y_1 - Y_2$ can be so written. Hence Eq. (3) holds and the proof is complete.

Theorem 3.6.2 The general solution of the nonhomogeneous equation (1) can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{7}$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation (2), c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (1).

The proof of Theorem 3.6.2 follows quickly from the preceding theorem. Note that Eq. (3) holds if we identify Y_1 with an arbitrary solution ϕ of Eq. (1) and Y_2 with the specific solution Y. From Eq. (3) we thereby obtain

$$\phi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t), \tag{8}$$

which is equivalent to Eq. (7). Since ϕ is an arbitrary solution of Eq. (1), the expression on the right side of Eq. (7) includes all solutions of Eq. (1); thus it is natural to call it the general solution of Eq. (1).

In somewhat different words, Theorem 3.6.2 states that to solve the nonhomogeneous equation (1), we must do three things:

1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is frequently called the complementary solution and may be denoted by $y_c(t)$.
2. Find some single solution $Y(t)$ of the nonhomogeneous equation. Often this solution is referred to as a particular solution.
3. Add together the functions found in the two preceding steps.

We have already discussed how to find $y_c(t)$, at least when the homogeneous equation (2) has constant coefficients. Therefore, in the remainder of this section and in the next, we will focus on finding a particular solution $Y(t)$ of the nonhomogeneous equation (1). There are two methods that we wish to discuss. They are known as the method of undetermined coefficients and the method of variation of parameters, respectively. Each has some advantages and some possible shortcomings.

Method of Undetermined Coefficients. The method of undetermined coefficients requires that we make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified. We then substitute the assumed expression into Eq. (1) and attempt to determine the coefficients so as to satisfy that equation. If we are successful, then we have found a solution of the differential equation (1) and can use it for the particular solution $Y(t)$. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again.

The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of $Y(t)$. Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. For this reason, this method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications. However, the algebraic details may become tedious and a computer algebra system can be very helpful in practical applications. We will illustrate the method of undetermined coefficients by several simple examples and then summarize some rules for using it.

EXAMPLE 1

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}. \tag{9}$$

We seek a function Y such that the combination $Y''(t) - 3Y'(t) - 4Y(t)$ is equal to $3e^{2t}$. Since the exponential function reproduces itself through differentiation, the most plausible way to achieve the desired result is to assume that $Y(t)$ is some multiple of e^{2t}, that is,

$$Y(t) = Ae^{2t},$$

where the coefficient A is yet to be determined. To find A we calculate

$$Y'(t) = 2Ae^{2t}, \qquad Y''(t) = 4Ae^{2t},$$

and substitute for y, y', and y'' in Eq. (9). We obtain

$$(4A - 6A - 4A)e^{2t} = 3e^{2t}.$$

Hence $-6Ae^{2t}$ must equal $3e^{2t}$, so $A = -1/2$. Thus a particular solution is

$$Y(t) = -\tfrac{1}{2}e^{2t}. \tag{10}$$

EXAMPLE 2

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t. \tag{11}$$

By analogy with Example 1, let us first assume that $Y(t) = A\sin t$, where A is a constant to be determined. On substituting in Eq. (11) and rearranging the terms, we obtain

$$-5A\sin t - 3A\cos t = 2\sin t,$$

or

$$(2 + 5A)\sin t + 3A\cos t = 0. \tag{12}$$

The functions $\sin t$ and $\cos t$ are linearly independent, so Eq. (12) can hold on an interval only if the coefficients $2 + 5A$ and $3A$ are both zero. These contradictory requirements mean that there is no choice of the constant A that makes Eq. (12) true for all t. Thus we conclude that our assumption concerning $Y(t)$ is inadequate. The appearance of the cosine term in Eq. (12) suggests that we modify our original assumption to include a cosine term in $Y(t)$, that is,

$$Y(t) = A\sin t + B\cos t,$$

where A and B are to be determined. Then

$$Y'(t) = A\cos t - B\sin t, \qquad Y''(t) = -A\sin t - B\cos t.$$

By substituting these expressions for y, y', and y'' in Eq. (11) and collecting terms, we obtain

$$(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t. \tag{13}$$

To satisfy Eq. (13) we must match the coefficients of $\sin t$ and $\cos t$ on each side of the equation; thus A and B must satisfy the equations

$$-5A + 3B = 2, \qquad -3A - 5B = 0.$$

Hence $A = -5/17$ and $B = 3/17$, so a particular solution of Eq. (11) is

$$Y(t) = -\tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t.$$

The method illustrated in the preceding examples can also be used when the right side of the equation is a polynomial. Thus, to find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1, \tag{14}$$

we initially assume that $Y(t)$ is a polynomial of the same degree as the nonhomogeneous term, that is, $Y(t) = At^2 + Bt + C$.

To summarize our conclusions up to this point: if the nonhomogeneous term $g(t)$ in Eq. (1) is an exponential function $e^{\alpha t}$, then assume that $Y(t)$ is proportional to the same exponential function; if $g(t)$ is $\sin\beta t$ or $\cos\beta t$, then assume that $Y(t)$ is a linear combination of $\sin\beta t$ and $\cos\beta t$; if $g(t)$ is a polynomial, then assume that $Y(t)$ is a polynomial of like degree. The same principle extends to the case where $g(t)$ is a product of any two, or all three, of these types of functions, as the next example illustrates.

EXAMPLE 3

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t. \tag{15}$$

In this case we assume that $Y(t)$ is the product of e^t and a linear combination of $\cos 2t$ and $\sin 2t$, that is,

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

The algebra is more tedious in this example, but it follows that

$$Y'(t) = (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t$$

and

$$Y''(t) = (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t.$$

By substituting these expressions in Eq. (15), we find that A and B must satisfy

$$10A + 2B = 8, \qquad 2A - 10B = 0.$$

Hence $A = 10/13$ and $B = 2/13$; therefore a particular solution of Eq. (15) is

$$Y(t) = \tfrac{10}{13}e^t \cos 2t + \tfrac{2}{13}e^t \sin 2t.$$

Now suppose that $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$, and suppose that Y_1 and Y_2 are solutions of the equations

$$ay'' + by' + cy = g_1(t) \tag{16}$$

and

$$ay'' + by' + cy = g_2(t), \tag{17}$$

respectively. Then $Y_1 + Y_2$ is a solution of the equation

$$ay'' + by' + cy = g(t). \tag{18}$$

To prove this statement, substitute $Y_1(t) + Y_2(t)$ for y in Eq. (18) and make use of Eqs. (16) and (17). A similar conclusion holds if $g(t)$ is the sum of any finite number of terms. The practical significance of this result is that for an equation whose nonhomogeneous function $g(t)$ can be expressed as a sum, one can consider instead several simpler equations and then add the results together. The following example is an illustration of this procedure.

EXAMPLE 4

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \tag{19}$$

By splitting up the right side of Eq. (19), we obtain the three equations

$$y'' - 3y' - 4y = 3e^{2t},$$
$$y'' - 3y' - 4y = 2\sin t,$$

and

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Solutions of these three equations have been found in Examples 1, 2, and 3, respectively. Therefore a particular solution of Eq. (19) is their sum, namely,

$$Y(t) = -\tfrac{1}{2}e^{2t} + \tfrac{3}{17}\cos t - \tfrac{5}{17}\sin t + \tfrac{10}{13}e^{t}\cos 2t + \tfrac{2}{13}e^{t}\sin 2t.$$

The procedure illustrated in these examples enables us to solve a fairly large class of problems in a reasonably efficient manner. However, there is one difficulty that sometimes occurs. The next example illustrates how it arises.

EXAMPLE 5

Find a particular solution of

$$y'' + 4y = 3\cos 2t. \tag{20}$$

Proceeding as in Example 2, we assume that $Y(t) = A\cos 2t + B\sin 2t$. By substituting in Eq. (20), we then obtain

$$(4A - 4A)\cos 2t + (4B - 4B)\sin 2t = 3\cos 2t. \tag{21}$$

Since the left side of Eq. (21) is zero, there is no choice of A and B that satisfies this equation. Therefore, there is no particular solution of Eq. (20) of the assumed form. The reason for this possibly unexpected result becomes clear if we solve the homogeneous equation

$$y'' + 4y = 0 \tag{22}$$

that corresponds to Eq. (20). A fundamental set of solutions of Eq. (22) is $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. Thus our assumed particular solution of Eq. (20) is actually a solution of the homogeneous equation (22); consequently, it cannot possibly be a solution of the nonhomogeneous equation (20).

To find a solution of Eq. (20) we must therefore consider functions of a somewhat different form. The simplest functions, other than $\cos 2t$ and $\sin 2t$ themselves, that when differentiated lead to $\cos 2t$ and $\sin 2t$ are $t\cos 2t$ and $t\sin 2t$. Hence we assume that $Y(t) = At\cos 2t + Bt\sin 2t$. Then, upon calculating $Y'(t)$ and $Y''(t)$, substituting them into Eq. (20), and collecting terms, we find that

$$-4A\sin 2t + 4B\cos 2t = 3\cos 2t.$$

Therefore $A = 0$ and $B = 3/4$, so a particular solution of Eq. (20) is

$$Y(t) = \tfrac{3}{4}t\sin 2t.$$

The fact that in some circumstances a purely oscillatory forcing term leads to a solution that includes a linear factor t as well as an oscillatory factor is important in some applications; see Section 3.9 for a further discussion.

The outcome of Example 5 suggests a modification of the principle stated previously: If the assumed form of the particular solution duplicates a solution of the corresponding homogeneous equation, then modify the assumed particular solution by multiplying it by t. Occasionally, this modification will be insufficient to remove all duplication with the solutions of the homogeneous equation, in which case it is necessary to multiply

by t a second time. For a second order equation, it will never be necessary to carry the process further than this.

Summary. We now summarize the steps involved in finding the solution of an initial value problem consisting of a nonhomogeneous equation of the form

$$ay'' + by' + cy = g(t), \tag{23}$$

where the coefficients a, b, and c are constants, together with a given set of initial conditions:

1. Find the general solution of the corresponding homogeneous equation.
2. Make sure that the function $g(t)$ in Eq. (23) belongs to the class of functions discussed in this section, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the next section).
3. If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation
$$ay'' + by' + cy = g_i(t),$$
where i runs from 1 to n.
4. For the ith subproblem assume a particular solution $Y_i(t)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (found in step 1), then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. See Table 3.6.1.
5. Find a particular solution $Y_i(t)$ for each of the subproblems. Then the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (23).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

TABLE 3.6.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t$ $+(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0$, 1, or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

For some problems this entire procedure is easy to carry out by hand, but in many cases it requires considerable algebra. Once you understand clearly how the method works, a computer algebra system can be of great assistance in executing the details.

The method of undetermined coefficients is self-correcting in the sense that if one assumes too little for $Y(t)$, then a contradiction is soon reached that usually points the way to the modification that is needed in the assumed form. On the other hand, if one assumes too many terms, then some unnecessary work is done and some coefficients turn out to be zero, but at least the correct answer is obtained.

Proof of the Method of Undetermined Coefficients. In the preceding discussion we have described the method of undetermined coefficients on the basis of several examples. To prove that the procedure always works as stated, we now give a general argument, in which we consider several cases corresponding to different forms for the nonhomogeneous term $g(t)$.

$g(t) = P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$. In this case Eq. (23) becomes

$$ay'' + by' + cy = a_0t^n + a_1t^{n-1} + \cdots + a_n. \tag{24}$$

To obtain a particular solution we assume that

$$Y(t) = A_0t^n + A_1t^{n-1} + \cdots + A_{n-2}t^2 + A_{n-1}t + A_n. \tag{25}$$

Substituting in Eq. (24), we obtain

$$\begin{aligned} a[n(n-1)A_0t^{n-2} + \cdots + 2A_{n-2}] + b(nA_0t^{n-1} + \cdots + A_{n-1}) \\ + c(A_0t^n + A_1t^{n-1} + \cdots + A_n) = a_0t^n + \cdots + a_n. \end{aligned} \tag{26}$$

Equating the coefficients of like powers of t gives

$$\begin{aligned} cA_0 &= a_0, \\ cA_1 + nbA_0 &= a_1, \\ &\vdots \\ cA_n + bA_{n-1} + 2aA_{n-2} &= a_n. \end{aligned}$$

Provided that $c \neq 0$, the solution of the first equation is $A_0 = a_0/c$, and the remaining equations determine $A_1, \ldots, A_n$ successively. If $c = 0$, but $b \neq 0$, then the polynomial on the left side of Eq. (26) is of degree $n-1$, and we cannot satisfy Eq. (26). To be sure that $aY''(t) + bY'(t)$ is a polynomial of degree n, we must choose $Y(t)$ to be a polynomial of degree $n+1$. Hence we assume that

$$Y(t) = t(A_0t^n + \cdots + A_n).$$

There is no constant term in this expression for $Y(t)$, but there is no need to include such a term since a constant is a solution of the homogeneous equation when $c = 0$. Since $b \neq 0$, we have $A_0 = a_0/b(n+1)$, and the other coefficients $A_1, \ldots, A_n$ can be determined similarly. If both c and b are zero, we assume that

$$Y(t) = t^2(A_0t^n + \cdots + A_n).$$

The term $aY''(t)$ gives rise to a term of degree n, and we can proceed as before. Again the constant and linear terms in $Y(t)$ are omitted, since in this case they are both solutions of the homogeneous equation.

$g(t) = e^{\alpha t} P_n(t)$. The problem of determining a particular solution of

$$ay'' + by' + cy = e^{\alpha t} P_n(t) \tag{27}$$

can be reduced to the preceding case by a substitution. Let

$$Y(t) = e^{\alpha t} u(t);$$

then

$$Y'(t) = e^{\alpha t}[u'(t) + \alpha u(t)]$$

and

$$Y''(t) = e^{\alpha t}[u''(t) + 2\alpha u'(t) + \alpha^2 u(t)].$$

Substituting for y, y', and y'' in Eq. (27), canceling the factor $e^{\alpha t}$, and collecting terms, we obtain

$$au''(t) + (2a\alpha + b)u'(t) + (a\alpha^2 + b\alpha + c)u(t) = P_n(t). \tag{28}$$

The determination of a particular solution of Eq. (28) is precisely the same problem, except for the names of the constants, as solving Eq. (24). Therefore, if $a\alpha^2 + b\alpha + c$ is not zero, we assume that $u(t) = A_0 t^n + \cdots + A_n$; hence a particular solution of Eq. (27) is of the form

$$Y(t) = e^{\alpha t}(A_0 t^n + A_1 t^{n-1} + \cdots + A_n). \tag{29}$$

On the other hand, if $a\alpha^2 + b\alpha + c$ is zero, but $2a\alpha + b$ is not, we must take $u(t)$ to be of the form $t(A_0 t^n + \cdots + A_n)$. The corresponding form for $Y(t)$ is t times the expression on the right side of Eq. (29). Note that if $a\alpha^2 + b\alpha + c$ is zero, then $e^{\alpha t}$ is a solution of the homogeneous equation. If both $a\alpha^2 + b\alpha + c$ and $2a\alpha + b$ are zero (and this implies that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation), then the correct form for $u(t)$ is $t^2(A_0 t^n + \cdots + A_n)$. Hence $Y(t)$ is t^2 times the expression on the right side of Eq. (29).

$g(t) = e^{\alpha t} P_n(t) \cos \beta t$ or $e^{\alpha t} P_n(t) \sin \beta t$. These two cases are similar, so we consider only the latter. We can reduce this problem to the preceding one by noting that, as a consequence of Euler's formula, $\sin \beta t = (e^{i\beta t} - e^{-i\beta t})/2i$. Hence $g(t)$ is of the form

$$g(t) = P_n(t)\frac{e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}}{2i}$$

and we should choose

$$Y(t) = e^{(\alpha+i\beta)t}(A_0 t^n + \cdots + A_n) + e^{(\alpha-i\beta)t}(B_0 t^n + \cdots + B_n),$$

or equivalently,

$$Y(t) = e^{\alpha t}(A_0 t^n + \cdots + A_n)\cos \beta t + e^{\alpha t}(B_0 t^n + \cdots + B_n)\sin \beta t.$$

Usually, the latter form is preferred. If $\alpha \pm i\beta$ satisfy the characteristic equation corresponding to the homogeneous equation, we must, of course, multiply each of the polynomials by t to increase their degrees by one.

If the nonhomogeneous function involves both $\cos \beta t$ and $\sin \beta t$, it is usually convenient to treat these terms together, since each one individually may give rise to the

same form for a particular solution. For example, if $g(t) = t\sin t + 2\cos t$, the form for $Y(t)$ would be

$$Y(t) = (A_0 t + A_1)\sin t + (B_0 t + B_1)\cos t,$$

provided that $\sin t$ and $\cos t$ are not solutions of the homogeneous equation.

PROBLEMS

In each of Problems 1 through 12 find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\sin 2t$
3. $y'' - 2y' - 3y = -3te^{-t}$
4. $y'' + 2y' = 3 + 4\sin 2t$
5. $y'' + 9y = t^2 e^{3t} + 6$
6. $y'' + 2y' + y = 2e^{-t}$
7. $2y'' + 3y' + y = t^2 + 3\sin t$
8. $y'' + y = 3\sin 2t + t\cos 2t$
9. $u'' + \omega_0^2 u = \cos\omega t, \quad \omega^2 \neq \omega_0^2$
10. $u'' + \omega_0^2 u = \cos\omega_0 t$
11. $y'' + y' + 4y = 2\sinh t$ *Hint:* $\sinh t = (e^t - e^{-t})/2$
12. $y'' - y' - 2y = \cosh 2t$ *Hint:* $\cosh t = (e^t + e^{-t})/2$

In each of Problems 13 through 18 find the solution of the given initial value problem.

13. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
15. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
16. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$
18. $y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

In each of Problems 19 through 26:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

19. $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$
20. $y'' + y = t(1 + \sin t)$
21. $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$
22. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$
23. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$
24. $y'' + 4y = t^2\sin 2t + (6t + 7)\cos 2t$
25. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$
26. $y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$
27. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

28. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

29. Follow the instructions in Problem 28 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \pi/2, \\ 0, & t > \pi/2 \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Behavior of Solutions as $t \to \infty$. In Problems 30 and 31 we continue the discussion started with Problems 38 through 40 of Section 3.5. Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{i}$$

where a, b, and c are positive.

30. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?
31. If $g(t) = d$, a constant, show that every solution of Eq. (i) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?
32. In this problem we indicate an alternate procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{i}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

In each of Problems 33 through 36 use the method of Problem 32 to solve the given differential equation.

33. $y'' - 3y' - 4y = 3e^{2t}$ (see Example 1)
34. $2y'' + 3y' + y = t^2 + 3\sin t$ (see Problem 7)
35. $y'' + 2y' + y = 2e^{-t}$ (see Problem 6)
36. $y'' + 2y' = 3 + 4\sin 2t$ (see Problem 4)

3.7 Variation of Parameters

In this section we describe another method of finding a particular solution of a nonhomogeneous equation. This method, known as **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about

[7] R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201; also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal 8* (1977), pp. 132–141, for a more general discussion of factoring operators.

the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires that we evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find a particular solution of

$$y'' + 4y = 3\csc t. \tag{1}$$

Observe that this problem does not fall within the scope of the method of undetermined coefficients because the nonhomogeneous term $g(t) = 3\csc t$ involves a quotient (rather than a sum or a product) of $\sin t$ or $\cos t$. Therefore, we need a different approach. Observe also that the homogeneous equation corresponding to Eq. (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of Eq. (2) is

$$y_c(t) = c_1 \cos 2t + c_2 \sin 2t. \tag{3}$$

The basic idea in the method of variation of parameters is to replace the constants c_1 and c_2 in Eq. (3) by functions $u_1(t)$ and $u_2(t)$, respectively, and then to determine these functions so that the resulting expression

$$y = u_1(t) \cos 2t + u_2(t) \sin 2t \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2 we need to substitute for y from Eq. (4) in Eq. (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and u_2. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.

Returning now to Eq. (4), we differentiate it and rearrange the terms, thereby obtaining

$$y' = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t + u_1'(t) \cos 2t + u_2'(t) \sin 2t. \tag{5}$$

Keeping in mind the possibility of choosing a second condition on u_1 and u_2, let us require the last two terms on the right side of Eq. (5) to be zero; that is, we require that

$$u_1'(t) \cos 2t + u_2'(t) \sin 2t = 0. \tag{6}$$

It then follows from Eq. (5) that

$$y' = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t. \tag{7}$$

Although the ultimate effect of the condition (6) is not yet clear, at the very least it has simplified the expression for y'. Further, by differentiating Eq. (7), we obtain

$$y'' = -4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t. \tag{8}$$

Then, substituting for y and y'' in Eq. (1) from Eqs. (4) and (8), respectively, we find that u_1 and u_2 must satisfy

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t. \tag{9}$$

Summarizing our results to this point, we want to choose u_1 and u_2 so as to satisfy Eqs. (6) and (9). These equations can be viewed as a pair of linear *algebraic* equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$. Equations (6) and (9) can be solved in various ways. For example, solving Eq. (6) for $u_2'(t)$, we have

$$u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}. \tag{10}$$

Then, substituting for $u_2'(t)$ in Eq. (9) and simplifying, we obtain

$$u_1'(t) = -\frac{3\csc t \sin 2t}{2} = -3\cos t. \tag{11}$$

Further, putting this expression for $u_1'(t)$ back in Eq. (10) and using the double angle formulas, we find that

$$u_2'(t) = \frac{3\cos t\cos 2t}{\sin 2t} = \frac{3(1-2\sin^2 t)}{2\sin t} = \frac{3}{2}\csc t - 3\sin t. \tag{12}$$

Having obtained $u_1'(t)$ and $u_2'(t)$, the next step is to integrate so as to obtain $u_1(t)$ and $u_2(t)$. The result is

$$u_1(t) = -3\sin t + c_1 \tag{13}$$

and

$$u_2(t) = \tfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2. \tag{14}$$

Finally, on substituting these expressions in Eq. (4), we have

$$y = -3\sin t\cos 2t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3\cos t\sin 2t + c_1\cos 2t + c_2\sin 2t,$$

or

$$y = 3\sin t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t + c_1\cos 2t + c_2\sin 2t. \tag{15}$$

The terms in Eq. (15) involving the arbitrary constants c_1 and c_2 are the general solution of the corresponding homogeneous equation, while the remaining terms are a particular solution of the nonhomogeneous equation (1). Therefore Eq. (15) is the general solution of Eq. (1).

In the preceding example the method of variation of parameters worked well in determining a particular solution, and hence the general solution, of Eq. (1). The next

question is whether this method can be applied effectively to an arbitrary equation. Therefore we consider

$$y'' + p(t)y' + q(t)y = g(t), \tag{16}$$

where p, q, and g are given continuous functions. As a starting point, we assume that we know the general solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{17}$$

of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{18}$$

This is a major assumption because so far we have shown how to solve Eq. (18) only if it has constant coefficients. If Eq. (18) has coefficients that depend on t, then usually the methods described in Chapter 5 must be used to obtain $y_c(t)$.

The crucial idea, as illustrated in Example 1, is to replace the constants c_1 and c_2 in Eq. (17) by functions $u_1(t)$ and $u_2(t)$, respectively; this gives

$$y = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{19}$$

Then we try to determine $u_1(t)$ and $u_2(t)$ so that the expression in Eq. (19) is a solution of the nonhomogeneous equation (16) rather than the homogeneous equation (18). Thus we differentiate Eq. (19), obtaining

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t). \tag{20}$$

As in Example 1, we now set the terms involving $u_1'(t)$ and $u_2'(t)$ in Eq. (20) equal to zero; that is, we require that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{21}$$

Then, from Eq. (20), we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{22}$$

Further, by differentiating again, we obtain

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t). \tag{23}$$

Now we substitute for y, y', and y'' in Eq. (16) from Eqs. (19), (22), and (23), respectively. After rearranging the terms in the resulting equation we find that

$$\begin{aligned} u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ + u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \end{aligned} \tag{24}$$

Each of the expressions in square brackets in Eq. (24) is zero because both y_1 and y_2 are solutions of the homogeneous equation (18). Therefore Eq. (24) reduces to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{25}$$

Equations (21) and (25) form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. They correspond exactly to Eqs. (6) and (9) in Example 1.

By solving the system (21), (25) we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}, \tag{26}$$

where $W(y_1,y_2)$ is the Wronskian of y_1 and y_2. Note that division by W is permissible since y_1 and y_2 are a fundamental set of solutions, and therefore their Wronskian is nonzero. By integrating Eqs. (26) we find the desired functions $u_1(t)$ and $u_2(t)$, namely,

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_2. \tag{27}$$

Finally, substituting from Eq. (27) in Eq. (19) gives the general solution of Eq. (16). We state the result as a theorem.

Theorem 3.7.1 If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation (18) corresponding to the nonhomogeneous equation (16),

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (16) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt, \tag{28}$$

and the general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t), \tag{29}$$

as prescribed by Theorem 3.6.2.

By examining the expression (28) and reviewing the process by which we derived it, we can see that there may be two major difficulties in using the method of variation of parameters. As we have mentioned earlier, one is the determination of $y_1(t)$ and $y_2(t)$, a fundamental set of solutions of the homogeneous equation (18), when the coefficients in that equation are not constants. The other possible difficulty is in the evaluation of the integrals appearing in Eq. (28). This depends entirely on the nature of the functions y_1, y_2, and g. In using Eq. (28), be sure that the differential equation is exactly in the form (16); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that Eq. (28) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions.

PROBLEMS

In each of Problems 1 through 4 use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 5 through 12 find the general solution of the given differential equation. In Problems 11 and 12 g is an arbitrary continuous function.

5. $y'' + y = \tan t, \quad 0 < t < \pi/2$
6. $y'' + 9y = 9\sec^2 3t, \quad 0 < t < \pi/6$
7. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$
8. $y'' + 4y = 3\csc 2t, \quad 0 < t < \pi/2$
9. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$
10. $y'' - 2y' + y = e^t/(1 + t^2)$
11. $y'' - 5y' + 6y = g(t)$
12. $y'' + 4y = g(t)$

In each of Problems 13 through 20 verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20 g is an arbitrary continuous function.

13. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$
14. $t^2y'' - t(t + 2)y' + (t + 2)y = 2t^3, \quad t > 0; \quad y_1(t) = t, \quad y_2(t) = te^t$
15. $ty'' - (1 + t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t, \quad y_2(t) = e^t$
16. $(1 - t)y'' + ty' - y = 2(t - 1)^2e^{-t}, \quad 0 < t < 1; \quad y_1(t) = e^t, \quad y_2(t) = t$
17. $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$
18. $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2} \sin x, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$
19. $(1 - x)y'' + xy' - y = g(x), \quad 0 < x < 1; \quad y_1(x) = e^x, \quad y_2(x) = x$
20. $x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$

21. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{i}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \quad u(t_0) = y_0, \quad u'(t_0) = y_0', \tag{ii}$$

$$L[v] = g(t), \quad v(t_0) = 0, \quad v'(t_0) = 0, \tag{iii}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known.

22. By choosing the lower limit of integration in Eq. (28) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)\, ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0.$$

Thus Y can be identified with v in Problem 21.

23. (a) Use the result of Problem 22 to show that the solution of the initial value problem

$$y'' + y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0 \tag{i}$$

is

$$y = \int_{t_0}^{t} \sin(t - s)g(s)\, ds. \tag{ii}$$

(b) Find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'.$$

24. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D - a)(D - b)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a and b are real numbers with $a \neq b$.

25. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Note that the roots of the characteristic equation are $\lambda \pm i\mu$.

26. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D - a)^2 y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a is any real number.

27. By combining the results of Problems 24 through 26, show that the solution of the initial value problem

$$L[y] = (aD^2 + bD + c)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a, b, and c are constants, has the form

$$y = \phi(t) = \int_{t_0}^{t} K(t - s)g(s)\, ds. \tag{i}$$

The function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. Thinking of $g(t)$ as the input to the problem and $\phi(t)$ as the output, it follows from Eq. (i) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in Eq. (i) is called the **convolution** of K and g, and K is referred to as the **kernel**.

28. The method of reduction of order (Section 3.5) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{i}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies Eq. (i) if v is a solution of

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t). \tag{ii}$$

Equation (ii) is a first order linear equation for v'. Solving this equation, integrating the result, and then multiplying by $y_1(t)$ lead to the general solution of Eq. (i).

In each of Problems 29 through 32 use the method outlined in Problem 28 to solve the given differential equation.

29. $t^2y'' - 2ty' + 2y = 4t^2, \quad t > 0; \qquad y_1(t) = t$
30. $t^2y'' + 7ty' + 5y = t, \quad t > 0; \qquad y_1(t) = t^{-1}$
31. $ty'' - (1 + t)y' + y = t^2e^{2t}, \quad t > 0; \qquad y_1(t) = 1 + t$ (see Problem 15)
32. $(1 - t)y'' + ty' - y = 2(t - 1)^2e^{-t}, \quad 0 < t < 1; \qquad y_1(t) = e^t$ (see Problem 16)

3.8 Mechanical and Electrical Vibrations



One of the reasons that second order linear equations with constant coefficients are worth studying is that they serve as mathematical models of some important physical processes. Two important areas of application are in the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an initial value problem of the form

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

This illustrates a fundamental relationship between mathematics and physics: *Many physical problems may have the same mathematical model*. Thus, once we know how to solve the initial value problem (1), it is only necessary to make appropriate interpretations of the constants a, b, and c, and the functions y and g, to obtain solutions of different physical problems.

We will study the motion of a mass on a spring in detail because an understanding of the behavior of this simple system is the first step in the investigation of more complex vibrating systems. Further, the principles involved are common to many problems. Consider a mass m hanging on the end of a vertical spring of original length l, as shown in Figure 3.8.1. The mass causes an elongation L of the spring in the downward (positive) direction. There are two forces acting at the point where the mass is attached to the spring; see Figure 3.8.2. The gravitational force, or weight of the mass, acts downward and has magnitude mg, where g is the acceleration due to gravity. There is also a force F_s, due to the spring, that acts upward. If we assume that the elongation L of the spring is small, the spring force is very nearly proportional to L; this is known

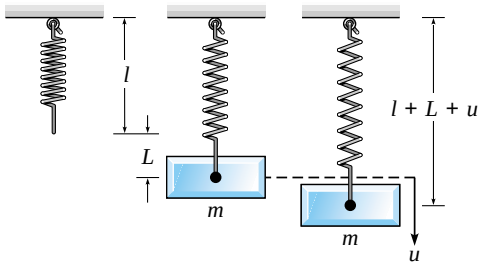


FIGURE 3.8.1 A spring–mass system.

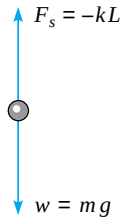


FIGURE 3.8.2 Force diagram for a spring–mass system.

as Hooke's[8] law. Thus we write $F_s = -kL$, where the constant of proportionality k is called the spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction. Since the mass is in equilibrium, the two forces balance each other, which means that

$$mg - kL = 0. \tag{2}$$

For a given weight $w = mg$, one can measure L and then use Eq. (2) to determine k. Note that k has the units of force/length.

In the corresponding dynamic problem we are interested in studying the motion of the mass when it is acted on by an external force or is initially displaced. Let $u(t)$, measured positive downward, denote the displacement of the mass from its equilibrium position at time t; see Figure 3.8.1. Then $u(t)$ is related to the forces acting on the mass through Newton's law of motion,

$$mu''(t) = f(t), \tag{3}$$

where u'' is the acceleration of the mass and f is the net force acting on the mass. Observe that both u and f are functions of time. In determining f there are four separate forces that must be considered:

1. The weight $w = mg$ of the mass always acts downward.

2. The spring force F_s is assumed to be proportional to the total elongation $L + u$ of the spring and always acts to restore the spring to its natural position. If $L + u > 0$, then the spring is extended, and the spring force is directed upward. In this case

$$F_s = -k(L + u). \tag{4}$$

On the other hand, if $L + u < 0$, then the spring is compressed a distance $|L + u|$, and the spring force, which is now directed downward, is given by $F_s = k|L + u|$. However, when $L + u < 0$, it follows that $|L + u| = -(L + u)$, so F_s is again given by Eq. (4). Thus, regardless of the position of the mass, the force exerted by the spring is always expressed by Eq. (4).

3. The damping or resistive force F_d always acts in the direction opposite to the direction of motion of the mass. This force may arise from several sources: resistance from the air or other medium in which the mass moves, internal energy dissipation due to the extension or compression of the spring, friction between the mass and the guides (if any) that constrain its motion to one dimension, or a mechanical device (dashpot) that imparts a resistive force to the mass. In any case, we assume that the resistive force is proportional to the speed $|du/dt|$ of the mass; this is usually referred to as viscous damping. If $du/dt > 0$, then u is increasing, so the mass is moving downward. Then F_d is directed upward and is given by

$$F_d(t) = -\gamma u'(t), \tag{5}$$

where γ is a positive constant of proportionality known as the damping constant. On the other hand, if $du/dt < 0$, then u is decreasing, the mass is moving upward, and F_d is directed downward. In this case, $F_d = \gamma|u'(t)|$; since $|u'(t)| = -u'(t)$,

[8]Robert Hooke (1635–1703) was an English scientist with wide-ranging interests. His most important book, *Micrographia*, was published in 1665 and described a variety of microscopical observations. Hooke first published his law of elastic behavior in 1676 as an anagram: *ceiiinosssttuv*; in 1678 he gave the solution *ut tensio sic vis*, which means, roughly, "as the force so is the displacement."

it follows that $F_d(t)$ is again given by Eq. (5). Thus, regardless of the direction of motion of the mass, the damping force is always expressed by Eq. (5).

The damping force may be rather complicated and the assumption that it is modeled adequately by Eq. (5) may be open to question. Some dashpots do behave as Eq. (5) states, and if the other sources of dissipation are small, it may be possible to neglect them altogether, or to adjust the damping constant γ to approximate them. An important benefit of the assumption (5) is that it leads to a linear (rather than a nonlinear) differential equation. In turn, this means that a thorough analysis of the system is straightforward, as we will show in this section and the next.

4. An applied external force $F(t)$ is directed downward or upward as $F(t)$ is positive or negative. This could be a force due to the motion of the mount to which the spring is attached, or it could be a force applied directly to the mass. Often the external force is periodic.

Taking account of these forces, we can now rewrite Newton's law (3) as

$$\begin{aligned} mu''(t) &= mg + F_s(t) + F_d(t) + F(t) \\ &= mg - k[L + u(t)] - \gamma u'(t) + F(t). \end{aligned} \tag{6}$$

Since $mg - kL = 0$ by Eq. (2), it follows that the equation of motion of the mass is

$$mu''(t) + \gamma u'(t) + ku(t) = F(t), \tag{7}$$

where the constants m, γ, and k are positive. Note that Eq. (7) has the same form as Eq. (1).

It is important to understand that Eq. (7) is only an approximate equation for the displacement $u(t)$. In particular, both Eqs. (4) and (5) should be viewed as approximations for the spring force and the damping force, respectively. In our derivation we have also neglected the mass of the spring in comparison with the mass of the attached body.

The complete formulation of the vibration problem requires that we specify two initial conditions, namely, the initial position u_0 and the initial velocity v_0 of the mass:

$$u(0) = u_0, \qquad u'(0) = v_0. \tag{8}$$

It follows from Theorem 3.2.1 that these conditions give a mathematical problem that has a unique solution. This is consistent with our physical intuition that if the mass is set in motion with a given initial displacement and velocity, then its position will be determined uniquely at all future times. The position of the mass is given (approximately) by the solution of Eq. (7) subject to the prescribed initial conditions (8).

EXAMPLE 1

A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is displaced an additional 6 in. in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/sec. Under the assumptions discussed in this section, formulate the initial value problem that governs the motion of the mass.

The required initial value problem consists of the differential equation (7) and initial conditions (8), so our task is to determine the various constants that appear in these equations. The first step is to choose the units of measurement. Based on the statement of the problem, it is natural to use the English rather than the metric system of units. The

only time unit mentioned is the second, so we will measure t in seconds. On the other hand, both the foot and inch appear in the statement as units of length. It is immaterial which one we use, but having made a choice, it is important to be consistent. To be definite, let us measure the displacement u in feet.

Since nothing is said in the statement of the problem about an external force, we assume that $F(t) = 0$. To determine m note that

$$m = \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft/ sec}^2} = \frac{1}{8} \frac{\text{lb-sec}^2}{\text{ft}}.$$

The damping coefficient γ is determined from the statement that $\gamma u'$ is equal to 6 lb when u' is 3 ft/sec. Therefore

$$\gamma = \frac{6 \text{ lb}}{3 \text{ ft/sec}} = 2 \frac{\text{lb-sec}}{\text{ft}}.$$

The spring constant k is found from the statement that the mass stretches the spring by 2 in., or by 1/6 ft. Thus

$$k = \frac{4 \text{ lb}}{1/6 \text{ ft}} = 24 \frac{\text{lb}}{\text{ft}}.$$

Consequently, Eq. (7) becomes

$$\tfrac{1}{8} u'' + 2u' + 24u = 0,$$

or

$$u'' + 16u' + 192u = 0. \tag{9}$$

The initial conditions are

$$u(0) = \tfrac{1}{2}, \qquad u'(0) = 0. \tag{10}$$

The second initial condition is implied by the word "released" in the statement of the problem, which we interpret to mean that the mass is set in motion with no initial velocity.

Undamped Free Vibrations. If there is no external force, then $F(t) = 0$ in Eq. (7). Let us also suppose that there is no damping, so that $\gamma = 0$; this is an idealized configuration of the system, seldom (if ever) completely attainable in practice. However, if the actual damping is very small, the assumption of no damping may yield satisfactory results over short to moderate time intervals. Then the equation of motion (7) reduces to

$$mu'' + ku = 0. \tag{11}$$

The general solution of Eq. (11) is

$$u = A \cos \omega_0 t + B \sin \omega_0 t, \tag{12}$$

where

$$\omega_0^2 = k/m. \tag{13}$$

The arbitrary constants A and B can be determined if initial conditions of the form (8) are given.

In discussing the solution of Eq. (11) it is convenient to rewrite Eq. (12) in the form

$$u = R\cos(\omega_0 t - \delta), \tag{14}$$

or

$$u = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t. \tag{15}$$

By comparing Eq. (15) with Eq. (12), we find that A, B, R, and δ are related by the equations

$$A = R\cos\delta, \qquad B = R\sin\delta. \tag{16}$$

Thus

$$R = \sqrt{A^2 + B^2}, \qquad \tan\delta = B/A. \tag{17}$$

In calculating δ care must be taken to choose the correct quadrant; this can be done by checking the signs of $\cos\delta$ and $\sin\delta$ in Eqs. (16).

The graph of Eq. (14), or the equivalent Eq. (12), for a typical set of initial conditions is shown in Figure 3.8.3. The graph is a displaced cosine wave that describes a periodic, or simple harmonic, motion of the mass. The **period** of the motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{1/2}. \tag{18}$$

The circular frequency $\omega_0 = \sqrt{k/m}$, measured in radians per unit time, is called the **natural frequency** of the vibration. The maximum displacement R of the mass from equilibrium is the **amplitude** of the motion. The dimensionless parameter δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.

Note that the motion described by Eq. (14) has a constant amplitude that does not diminish with time. This reflects the fact that in the absence of damping there is no way for the system to dissipate the energy imparted to it by the initial displacement and velocity. Further, for a given mass m and spring constant k, the system always vibrates at the same frequency ω_0, regardless of the initial conditions. However, the initial conditions do help to determine the amplitude of the motion. Finally, observe from Eq. (18) that T increases as m increases, so larger masses vibrate more slowly. On the other hand, T decreases as k increases, which means that stiffer springs cause the system to vibrate more rapidly.

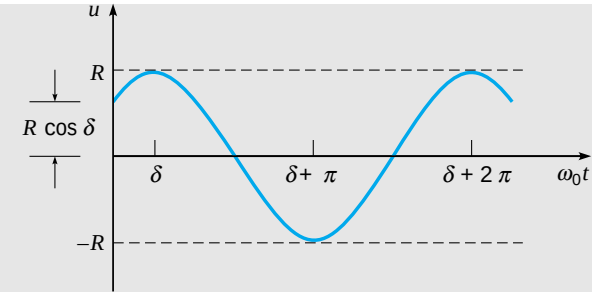


FIGURE 3.8.3 Simple harmonic motion; $u = R\cos(\omega_0 t - \delta)$.

EXAMPLE 2

Suppose that a mass weighing 10 lb stretches a spring 2 in. If the mass is displaced an additional 2 in. and is then set in motion with an initial upward velocity of 1 ft/sec, determine the position of the mass at any later time. Also determine the period, amplitude, and phase of the motion.

The spring constant is $k = 10$ lb/2 in. $= 60$ lb/ft and the mass is $m = w/g = 10/32$ lb-sec^2/ft. Hence the equation of motion reduces to

$$u'' + 192u = 0, \tag{19}$$

and the general solution is

$$u = A\cos(8\sqrt{3}t) + B\sin(8\sqrt{3}t).$$

The solution satisfying the initial conditions $u(0) = 1/6$ ft and $u'(0) = -1$ ft/sec is

$$u = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t). \tag{20}$$

The natural frequency is $\omega_0 = \sqrt{192} \cong 13.856$ rad/sec, so the period is $T = 2\pi/w_0 \cong 0.45345$ sec. The amplitude R and phase δ are found from Eqs. (17). We have

$$R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576}, \quad \text{so} \quad R \cong 0.18162 \text{ ft}.$$

The second of Eqs. (17) yields $\tan\delta = -\sqrt{3}/4$. There are two solutions of this equation, one in the second quadrant and one in the fourth. In the present problem $\cos\delta > 0$ and $\sin\delta < 0$, so δ is in the fourth quadrant, namely,

$$\delta = -\arctan(\sqrt{3}/4) \cong -0.40864 \text{ rad}.$$

The graph of the solution (20) is shown in Figure 3.8.4.

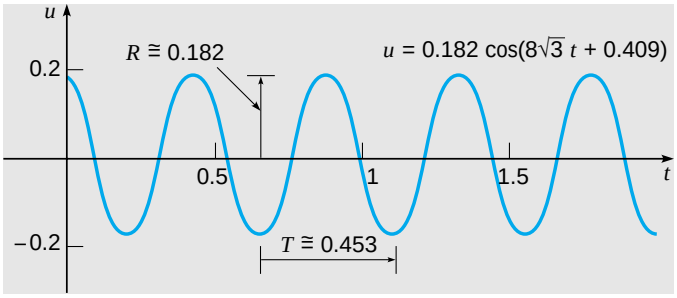


FIGURE 3.8.4 An undamped free vibration; $u'' + 192u = 0$, $u(0) = 1/6$, $u'(0) = -1$.

Damped Free Vibrations. If we include the effect of damping, the differential equation governing the motion of the mass is

$$mu'' + \gamma u' + ku = 0. \tag{21}$$

We are especially interested in examining the effect of variations in the damping coefficient γ for given values of the mass m and spring constant k. The roots of the corresponding characteristic equation are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right). \tag{22}$$

Depending on the sign of $\gamma^2 - 4km$, the solution u has one of the following forms:

$$\gamma^2 - 4km > 0, \quad u = Ae^{r_1 t} + Be^{r_2 t}; \tag{23}$$

$$\gamma^2 - 4km = 0, \quad u = (A + Bt)e^{-\gamma t/2m}; \tag{24}$$

$$\gamma^2 - 4km < 0, \quad u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \quad \mu = \frac{(4km - \gamma^2)^{1/2}}{2m} > 0. \tag{25}$$

Since m, γ, and k are positive, $\gamma^2 - 4km$ is always less than γ^2. Hence, if $\gamma^2 - 4km \geq 0$, then the values of r_1 and r_2 given by Eq. (22) are *negative*. If $\gamma^2 - 4km < 0$, then the values of r_1 and r_2 are complex, but with *negative* real part. Thus, in all cases, the solution u tends to zero as $t \to \infty$; this occurs regardless of the values of the arbitrary constants A and B, that is, regardless of the initial conditions. This confirms our intuitive expectation, namely, that damping gradually dissipates the energy initially imparted to the system, and consequently the motion dies out with increasing time.

The most important case is the third one, which occurs when the damping is small. If we let $A = R\cos\delta$ and $B = R\sin\delta$ in Eq. (25), then we obtain

$$u = Re^{-\gamma t/2m}\cos(\mu t - \delta). \tag{26}$$

The displacement u lies between the curves $u = \pm Re^{-\gamma t/2m}$; hence it resembles a cosine wave whose amplitude decreases as t increases. A typical example is sketched in Figure 3.8.5. The motion is called a damped oscillation, or a damped vibration. The amplitude factor R depends on m, γ, k, and the initial conditions.

Although the motion is not periodic, the parameter μ determines the frequency with which the mass oscillates back and forth; consequently, μ is called the **quasi frequency**. By comparing μ with the frequency ω_0 of undamped motion, we find that

$$\frac{\mu}{\omega_0} = \frac{(4km - \gamma^2)^{1/2}/2m}{\sqrt{k/m}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \cong 1 - \frac{\gamma^2}{8km}. \tag{27}$$

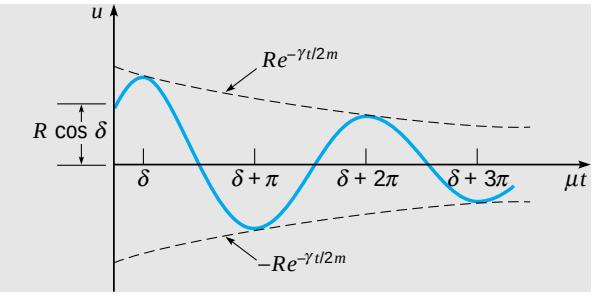


FIGURE 3.8.5 Damped vibration; $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$.

The last approximation is valid when $\gamma^2/4km$ is small; we refer to this situation as "small damping." Thus, the effect of small damping is to reduce slightly the frequency of the oscillation. By analogy with Eq. (18) the quantity $T_d = 2\pi/\mu$ is called the **quasi period**. It is the time between successive maxima or successive minima of the position of the mass, or between successive passages of the mass through its equilibrium position while going in the same direction. The relation between T_d and T is given by

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1 + \frac{\gamma^2}{8km}\right), \tag{28}$$

where again the last approximation is valid when $\gamma^2/4km$ is small. Thus, small damping increases the quasi period.

Equations (27) and (28) reinforce the significance of the dimensionless ratio $\gamma^2/4km$. It is not the magnitude of γ alone that determines whether damping is large or small, but the magnitude of γ^2 compared to $4km$. When $\gamma^2/4km$ is small, then we can neglect the effect of damping in calculating the quasi frequency and quasi period of the motion. On the other hand, if we want to study the detailed motion of the mass for all time, then we can *never* neglect the damping force, no matter how small.

As $\gamma^2/4km$ increases, the quasi frequency μ decreases and the quasi period T_d increases. In fact, $\mu \to 0$ and $T_d \to \infty$ as $\gamma \to 2\sqrt{km}$. As indicated by Eqs. (23), (24), and (25), the nature of the solution changes as γ passes through the value $2\sqrt{km}$. This value is known as **critical damping**, while for larger values of γ the motion is said to be **overdamped**. In these cases, given by Eqs. (24) and (23), respectively, the mass creeps back to its equilibrium position, but does not oscillate about it, as for small γ. Two typical examples of critically damped motion are shown in Figure 3.8.6, and the situation is discussed further in Problems 21 and 22.

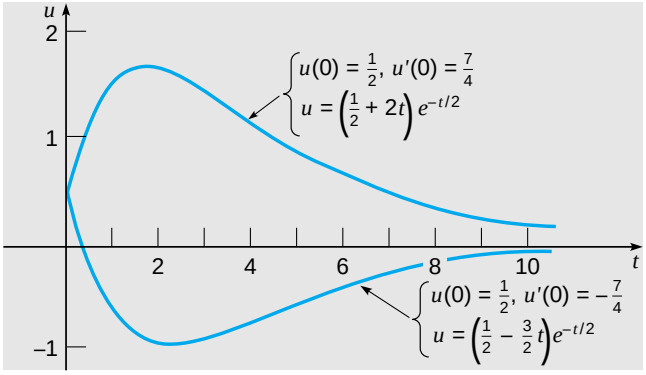


FIGURE 3.8.6 Critically damped motions: $u'' + u' + 0.25u = 0$; $u = (A + Bt)e^{-t/2}$.

EXAMPLE 3

The motion of a certain spring–mass system is governed by the differential equation

$$u'' + 0.125u' + u = 0, \tag{29}$$

where u is measured in feet and t in seconds. If $u(0) = 2$ and $u'(0) = 0$, determine the position of the mass at any time. Find the quasi frequency and the quasi period, as well

as the time at which the mass first passes through its equilibrium position. Also find the time τ such that $|u(t)| < 0.1$ for all $t > \tau$.

The solution of Eq. (29) is

$$u = e^{-t/16}\left[A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\right].$$

To satisfy the initial conditions we must choose $A = 2$ and $B = 2/\sqrt{255}$; hence the solution of the initial value problem is

$$\begin{aligned} u &= e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) \\ &= \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right), \end{aligned} \tag{30}$$

where $\tan\delta = 1/\sqrt{255}$, so $\delta \cong 0.06254$. The displacement of the mass as a function of time is shown in Figure 3.8.7. For the purpose of comparison we also show the motion if the damping term is neglected.

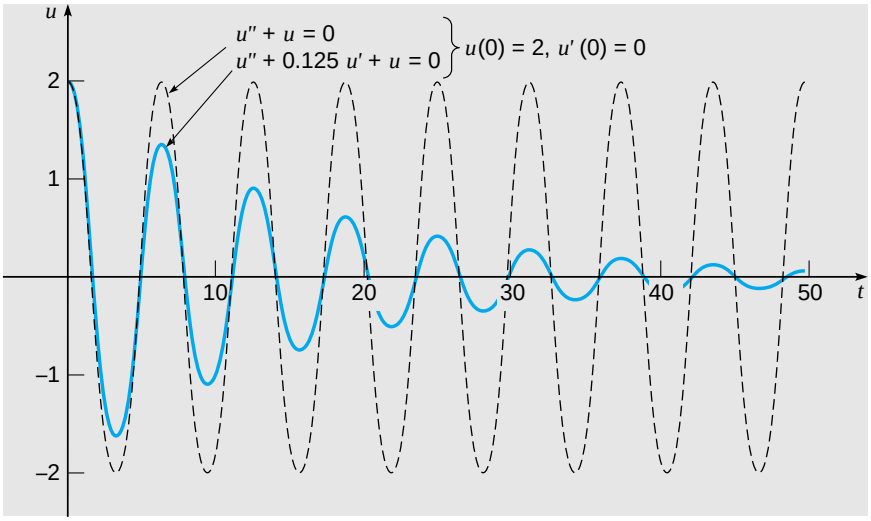


FIGURE 3.8.7 Vibration with small damping (solid curve) and with no damping (dashed curve).

The quasi frequency is $\mu = \sqrt{255}/16 \cong 0.998$ and the quasi period is $T_d = 2\pi/\mu \cong 6.295$ sec. These values differ only slightly from the corresponding values (1 and 2π, respectively) for the undamped oscillation. This is evident also from the graphs in Figure 3.8.7, which rise and fall almost together. The damping coefficient is small in this example, only one-sixteenth of the critical value, in fact. Nevertheless, the amplitude of the oscillation is reduced rather rapidly. Figure 3.8.8 shows the graph of the solution for $40 \le t \le 60$, together with the graphs of $u = \pm 0.1$. From the

graph it appears that τ is about 47.5 and by a more precise calculation we find that $\tau \cong 47.5149$ sec.

To find the time at which the mass first passes through its equilibrium position, we refer to Eq. (30) and set $\sqrt{255}t/16 - \delta$ equal to $\pi/2$, the smallest positive zero of the cosine function. Then, by solving for t, we obtain

$$t = \frac{16}{\sqrt{255}}\left(\frac{\pi}{2} + \delta\right) \cong 1.637 \text{ sec}.$$

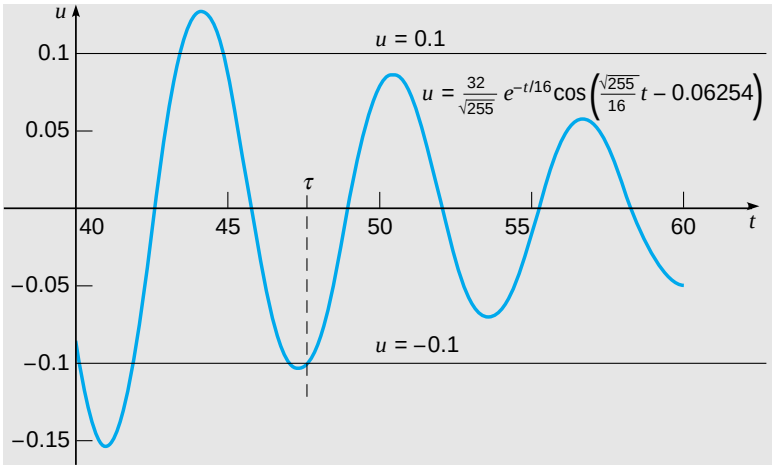


FIGURE 3.8.8 Solution of Example 3; determination of τ.

Electric Circuits. A second example of the occurrence of second order linear differential equations with constant coefficients is as a model of the flow of electric current in the simple series circuit shown in Figure 3.8.9. The current I, measured in amperes, is a function of time t. The resistance R (ohms), the capacitance C (farads), and the inductance L (henrys) are all positive and are assumed to be known constants. The impressed voltage E (volts) is a given function of time. Another physical quantity that enters the discussion is the total charge Q (coulombs) on the capacitor at time t. The relation between charge Q and current I is

$$I = dQ/dt. \tag{31}$$

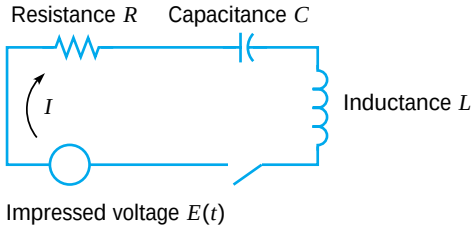


FIGURE 3.8.9 A simple electric circuit.

The flow of current in the circuit is governed by Kirchhoff's[9] second law: *In a closed circuit, the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.*

According to the elementary laws of electricity, we know that:

The voltage drop across the resistor is IR.

The voltage drop across the capacitor is Q/C.

The voltage drop across the inductor is $L\,dI/dt$.

Hence, by Kirchhoff's law,

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{32}$$

The units have been chosen so that 1 volt = 1 ohm · 1 ampere = 1 coulomb/1 farad = 1 henry · 1 ampere/1 second.

Substituting for I from Eq. (31), we obtain the differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{33}$$

for the charge Q. The initial conditions are

$$Q(t_0) = Q_0, \qquad Q'(t_0) = I(t_0) = I_0. \tag{34}$$

Thus we must know the charge on the capacitor and the current in the circuit at some initial time t_0.

Alternatively, we can obtain a differential equation for the current I by differentiating Eq. (33) with respect to t, and then substituting for dQ/dt from Eq. (31). The result is

$$LI'' + RI' + \frac{1}{C}I = E'(t), \tag{35}$$

with the initial conditions

$$I(t_0) = I_0, \qquad I'(t_0) = I_0'. \tag{36}$$

From Eq. (32) it follows that

$$I_0' = \frac{E(t_0) - RI_0 - (I/C)Q_0}{L}. \tag{37}$$

Hence I_0' is also determined by the initial charge and current, which are physically measurable quantities.

The most important conclusion from this discussion is that the flow of current in the circuit is described by an initial value problem of precisely the same form as the one that describes the motion of a spring–mass system. This is a good example of the unifying role of mathematics: Once you know how to solve second order linear equations with constant coefficients, you can interpret the results either in terms of mechanical vibrations, electric circuits, or any other physical situation that leads to the same problem.

[9]Gustav Kirchhoff (1824–1887), professor at Breslau, Heidelberg, and Berlin, was one of the leading physicists of the nineteenth century. He discovered the basic laws of electric circuits about 1845 while still a student at Königsberg. He is also famous for fundamental work in electromagnetic absorption and emission and was one of the founders of spectroscopy.

PROBLEMS

In each of Problems 1 through 4 determine ω_0, R, and δ so as to write the given expression in the form $u = R\cos(\omega_0 t - \delta)$.

1. $u = 3\cos 2t + 4\sin 2t$
2. $u = -\cos t + \sqrt{3}\sin t$
3. $u = 4\cos 3t - 2\sin 3t$
4. $u = -2\cos \pi t - 3\sin \pi t$

5. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position u of the mass at any time t. Plot u versus t. Find the frequency, period, and amplitude of the motion.
6. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?
7. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/sec, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.
8. A series circuit has a capacitor of 0.25×10^{-6} farad and an inductor of 1 henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q on the capacitor at any time t.
9. A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne-sec/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Plot u versus t. Determine the quasi frequency and the quasi period. Determine the ratio of the quasi period to the period of the corresponding undamped motion. Also find the time τ such that $|u(t)| < 0.05$ cm for all $t > \tau$.
10. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in./sec, find its position u at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|u(t)| < 0.01$ in. for all $t > \tau$.
11. A spring is stretched 10 cm by a force of 3 newtons. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 newtons when the velocity of the mass is 5 m/sec. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/sec, determine its position u at any time t. Find the quasi frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.
12. A series circuit has a capacitor of 10^{-5} farad, a resistor of 3×10^2 ohms, and an inductor of 0.2 henry. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge Q on the capacitor at any time t.
13. A certain vibrating system satisfies the equation $u'' + \gamma u' + u = 0$. Find the value of the damping coefficient γ for which the quasi period of the damped motion is 50% greater than the period of the corresponding undamped motion.
14. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.
15. Show that the solution of the initial value problem

$$mu'' + \gamma u' + ku = 0, \qquad u(t_0) = u_0, \quad u'(t_0) = u_0'$$

can be expressed as the sum $u = v + w$, where v satisfies the initial conditions $v(t_0) = u_0$, $v'(t_0) = 0$, w satisfies the initial conditions $w(t_0) = 0$, $w'(t_0) = u_0'$, and both v and w satisfy the same differential equation as u. This is another instance of superposing solutions of simpler problems to obtain the solution of a more general problem.

16. Show that $A\cos\omega_0 t + B\sin\omega_0 t$ can be written in the form $r\sin(\omega_0 t - \theta)$. Determine r and θ in terms of A and B. If $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$, determine the relationship among R, r, δ, and θ.
17. A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with coefficient γ. Determine the value of γ for which the system is critically damped; be sure to give the units for γ.
18. If a series circuit has a capacitor of $C = 0.8\times10^{-6}$ farad and an inductor of $L = 0.2$ henry, find the resistance R so that the circuit is critically damped.
19. Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
Hint: Determine all possible values of t for which $u = 0$.
20. Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is critically damped and the initial conditions are $u(0) = u_0$, $u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$, but that u is never zero. If u_0 is positive, determine a condition on v_0 that will assure that the mass passes through its equilibrium position after it is released.
21. **Logarithmic Decrement.** (a) For the damped oscillation described by Eq. (26), show that the time between successive maxima is $T_d = 2\pi/\mu$.
(b) Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/2m)$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by Δ.
(c) Show that $\Delta = \pi\gamma/m\mu$. Since m, μ, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the determination of the viscosity experimentally. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.
22. Referring to Problem 21, find the logarithmic decrement of the system in Problem 10.
23. For the system in Problem 17 suppose that $\Delta = 3$ and $T_d = 0.3$ sec. Referring to Problem 21, determine the value of the damping coefficient γ.
24. The position of a certain spring–mass system satisfies the initial value problem

$$\tfrac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \quad u'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.
25. Consider the initial value problem

$$u'' + \gamma u' + u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

We wish to explore how long a time interval is required for the solution to become "negligible" and how this interval depends on the damping coefficient γ. To be more precise, let us seek the time τ such that $|u(t)| < 0.01$ for all $t > \tau$. Note that critical damping for this problem occurs for $\gamma = 2$.
(a) Let $\gamma = 0.25$ and determine τ, or at least estimate it fairly accurately from a plot of the solution.
(b) Repeat part (a) for several other values of γ in the interval $0 < \gamma < 1.5$. Note that τ steadily decreases as γ increases for γ in this range.
(c) Obtain a graph of τ versus γ by plotting the pairs of values found in parts (a) and (b). Is the graph a smooth curve?
(d) Repeat part (b) for values of γ between 1.5 and 2. Show that τ continues to decrease until γ reaches a certain critical value γ_0, after which τ increases. Find γ_0 and the corresponding minimum value of τ to two decimal places.

(e) Another way to proceed is to write the solution of the initial value problem in the form (26). Neglect the cosine factor and consider only the exponential factor and the amplitude R. Then find an expression for τ as a function of γ. Compare the approximate results obtained in this way with the values determined in parts (a), (b), and (d).

26. Consider the initial value problem

$$mu'' + \gamma u' + ku = 0, \qquad u(0) = u_0, \quad u'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.

(a) Solve the initial value problem.

(b) Write the solution in the form $u(t) = R\exp(-\gamma t/2m)\cos(\mu t - \delta)$. Determine R in terms of m, γ, k, u_0, and v_0.

(c) Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

27. A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion and determine the period of the motion.

Hint: Use Archimedes' principle: An object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

▶ 28. The position of a certain undamped spring–mass system satisfies the initial value problem

$$u'' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. This plot is known as a phase plot and the uu'-plane is called the phase plane. Observe that a closed curve in the phase plane corresponds to a periodic solution $u(t)$. What is the direction of motion on the phase plot as t increases?

▶ 29. The position of a certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u in the phase plane (see Problem 28). Identify several corresponding points on the curves in parts (b) and (c). What is the direction of motion on the phase plot as t increases?

30. In the absence of damping the motion of a spring–mass system satisfies the initial value problem

$$mu'' + ku = 0, \qquad u(0) = a, \quad u'(0) = b.$$

(a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.

(b) Solve the given initial value problem.

(c) Using the solution in part (b), determine the total energy in the system at any time t. Your result should confirm the principle of conservation of energy for this system.

31. Suppose that a mass m slides without friction on a horizontal surface. The mass is attached to a spring with spring constant k, as shown in Figure 3.8.10, and is also subject to viscous air resistance with coefficient γ. Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies Eq. (21). How does the derivation of the equation of motion in this case differ from the derivation given in the text?

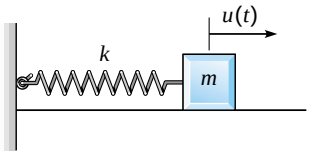


FIGURE 3.8.10 A spring–mass system.

▶ 32. In the spring–mass system of Problem 31, suppose that the spring force is not given by Hooke's law but instead satisfies the relation

$$F_s = -(ku + \epsilon u^3),$$

where $k > 0$ and ϵ is small but may be of either sign. The spring is called a hardening spring if $\epsilon > 0$ and a softening spring if $\epsilon < 0$. Why are these terms appropriate?
(a) Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies the differential equation

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0.$$

Suppose that the initial conditions are

$$u(0) = 0, \quad u'(0) = 1.$$

In the remainder of this problem assume that $m = 1$, $k = 1$, and $\gamma = 0$.
(b) Find $u(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.
(c) Let $\epsilon = 0.1$. Plot (a numerical approximation to) the solution. Does the motion appear to be periodic? Estimate the amplitude and period.
(d) Repeat part (c) for $\epsilon = 0.2$ and $\epsilon = 0.3$.
(e) Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.
(f) Repeat parts (c), (d), and (e) for negative values of ϵ.

3.9 Forced Vibrations



Consider now the case in which a periodic external force, say $F_0 \cos \omega t$ with $\omega > 0$, is applied to a spring–mass system. Then the equation of motion is

$$mu'' + \gamma u' + ku = F_0 \cos \omega t. \tag{1}$$

First suppose that there is no damping; then Eq. (1) reduces to

$$mu'' + ku = F_0 \cos \omega t. \tag{2}$$

If $\omega_0 = \sqrt{k/m} \neq \omega$, then the general solution of Eq. (2) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \tag{3}$$

The constants c_1 and c_2 are determined by the initial conditions. The resulting motion is, in general, the sum of two periodic motions of different frequencies (ω_0 and ω) and amplitudes. There are two particularly interesting cases.

Beats. Suppose that the mass is initially at rest, so that $u(0) = 0$ and $u'(0) = 0$. Then it turns out that the constants c_1 and c_2 in Eq. (3) are given by

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \qquad c_2 = 0, \tag{4}$$

and the solution of Eq. (2) is

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t). \tag{5}$$

This is the sum of two periodic functions of different periods but the same amplitude. Making use of the trigonometric identities for $\cos(A \pm B)$ with $A = (\omega_0 + \omega)t/2$ and $B = (\omega_0 - \omega)t/2$, we can write Eq. (5) in the form

$$u = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}\right] \sin \frac{(\omega_0 + \omega)t}{2}. \tag{6}$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega$ is much greater than $|\omega_0 - \omega|$. Consequently, $\sin(\omega_0 + \omega)t/2$ is a rapidly oscillating function compared to $\sin(\omega_0 - \omega)t/2$. Thus the motion is a rapid oscillation with frequency $(\omega_0 + \omega)/2$, but with a slowly varying sinusoidal amplitude

$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}.$$

This type of motion, possessing a periodic variation of amplitude, exhibits what is called a **beat**. Such a phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are sounded simultaneously. In this case the periodic variation of amplitude is quite apparent to the unaided ear. In electronics the variation of the amplitude with

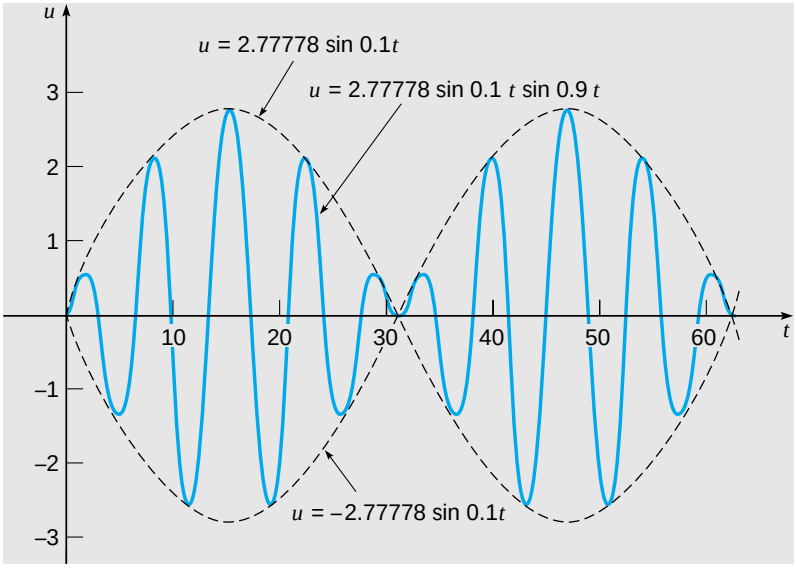


FIGURE 3.9.1 A beat; solution of $u'' + u = 0.5 \cos 0.8t$, $u(0) = 0$, $u'(0) = 0$; $u = 2.77778 \sin 0.1t \sin 0.9t$.

time is called **amplitude modulation**. The graph of u as given by Eq. (6) in a typical case is shown in Figure 3.9.1.

Resonance. As a second example, consider the case $\omega = \omega_0$; that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $F_0 \cos \omega t$ is a solution of the homogeneous equation. In this case the solution of Eq. (2) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \tag{7}$$

Because of the term $t \sin \omega_0 t$, the solution (7) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of c_1 and c_2; see Figure 3.9.2 for a typical example. Of course, in reality unbounded oscillations do not occur. As soon as u becomes large, the mathematical model on which Eq. (1) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that u be small. If damping is included in the model, the predicted motion remains bounded; however, the response to the input function $F_0 \cos \omega t$ may be quite large if the damping is small and ω is close to ω_0. This phenomenon is known as **resonance**.

Resonance can be either good or bad depending on the circumstances. It must be taken very seriously in the design of structures, such as buildings or bridges, where it can produce instabilities possibly leading to the catastrophic failure of the structure. For example, soldiers traditionally break step when crossing a bridge to eliminate the *periodic* force of their marching that could resonate with a natural frequency of the bridge. Another example occurred in the design of the high-pressure fuel turbopump for the space shuttle main engine. The turbopump was unstable and could not be operated over 20,000 rpm as compared to the design speed of 39,000 rpm. This difficulty led to a shutdown of the space shuttle program for 6 months at an estimated cost of

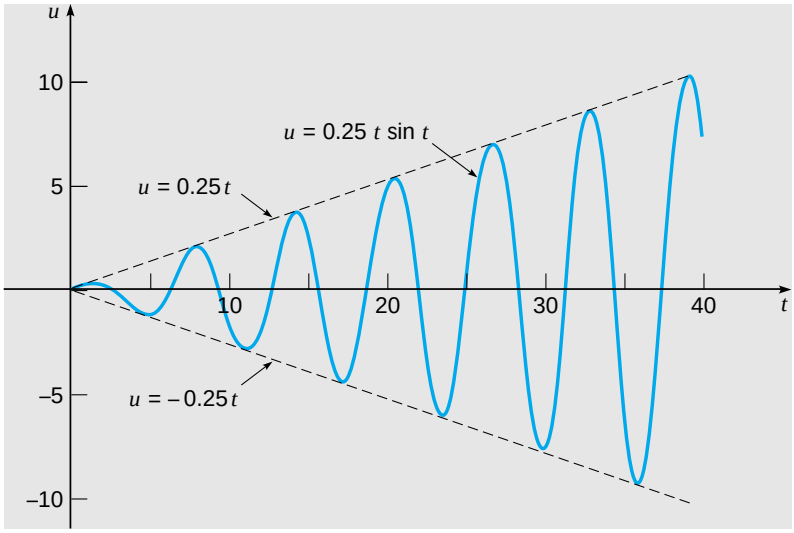


FIGURE 3.9.2 Resonance; solution of $u'' + u = 0.5 \cos t$, $u(0) = 0$, $u'(0) = 0$; $u = 0.25t \sin t$.

$500,000/day.[10] On the other hand, resonance can be put to good use in the design of instruments, such as seismographs, intended to detect weak periodic incoming signals.

Forced Vibrations with Damping. The motion of the spring–mass system with damping and the forcing function $F_0 \cos \omega t$ can be determined in a straightforward manner, although the computations are rather lengthy. The solution of Eq. (1) is

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t} + R\cos(\omega t - \delta), \quad r_1 \neq r_2 \tag{8}$$

where

$$R = \frac{F_0}{\Delta}, \qquad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \qquad \sin\delta = \frac{\gamma\omega}{\Delta}, \tag{9}$$

and

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{10}$$

In Eq. (8) r_1 and r_2 are the roots of the characteristic equation associated with Eq. (1); they may be either real and negative or complex conjugates with negative real part. In either case, both $\exp(r_1 t)$ and $\exp(r_2 t)$ approach zero as $t \to \infty$. Hence, as $t \to \infty$,

$$u \to U(t) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \delta). \tag{11}$$

For this reason $u_c(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is called the **transient solution**; $U(t)$, which represents a steady oscillation with the same frequency as the external force, is called the **steady-state solution** or the **forced response**. The transient solution enables us to satisfy whatever initial conditions are imposed; with increasing time the energy put into the system by the initial displacement and velocity is dissipated through the damping force, and the motion then becomes the response of the system to the external force. Without damping, the effect of the initial conditions would persist for all time.

It is interesting to investigate how the amplitude R of the steady-state oscillation depends on the frequency ω of the external force. For low-frequency excitation, that is, as $\omega \to 0$, it follows from Eqs. (9) and (10) that $R \to F_0/k$. At the other extreme, for very high-frequency excitation, Eqs. (9) and (10) imply that $R \to 0$ as $\omega \to \infty$. At an intermediate value of ω the amplitude may have a maximum. To find this maximum point, we can differentiate R with respect to ω and set the result equal to zero. In this way we find that the maximum amplitude occurs when $\omega = \omega_{\max}$, where

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2km}\right). \tag{12}$$

Note that $\omega_{\max} < \omega_0$ and that $\omega_{\max}$ is close to ω_0 when γ is small. The maximum value of R is

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right), \tag{13}$$

[10] F. Ehrich and D. Childs, "Self-Excited Vibration in High-Performance Turbomachinery," *Mechanical Engineering* (May 1984), p. 66.

where the last expression is an approximation for small γ. If $\gamma^2/2km > 1$, then $\omega_{\max}$ as given by Eq. (12) is imaginary; in this case the maximum value of R occurs for $\omega = 0$ and R is a monotone decreasing function of ω. For small γ it follows from Eq. (13) that $R_{\max} \cong F_0/\gamma\omega_0$. Thus, for small γ, the maximum response is much larger than the amplitude F_0 of the external force, and the smaller the value of γ, the larger the ratio $R_{\max}/F_0$. Figure 3.9.3 contains some representative graphs of Rk/F_0 versus ω/ω_0 for several values of γ.

The phase angle δ also depends in an interesting way on ω. For ω near zero, it follows from Eqs. (9) and (10) that $\cos\delta \cong 1$ and $\sin\delta \cong 0$. Thus $\delta \cong 0$, and the response is nearly in phase with the excitation, meaning that they rise and fall together, and in particular, assume their respective maxima nearly together and their respective minima nearly together. For $\omega = \omega_0$, we find that $\cos\delta = 0$ and $\sin\delta = 1$, so $\delta = \pi/2$. In this case the response lags behind the excitation by $\pi/2$; that is, the peaks of the response occur $\pi/2$ later than the peaks of the excitation, and similarly for the valleys. Finally, for ω very large, we have $\cos\delta \cong -1$ and $\sin\delta \cong 0$. Thus $\delta \cong \pi$, so that the response is nearly out of phase with the excitation; this means that the response is minimum when the excitation is maximum, and vice versa.

In Figure 3.9.4 we show the graph of the solution of the initial value problem

$$u'' + 0.125u' + u = 3\cos 2t, \qquad u(0) = 2, \quad u'(0) = 0.$$

The graph of the forcing function is also shown for comparison. (The unforced motion of this system is shown in Figure 3.8.7.) Observe that the initial transient motion decays as t increases, that the amplitude of the steady forced response is approximately 1, and that the phase difference between the excitation and response is approximately π.

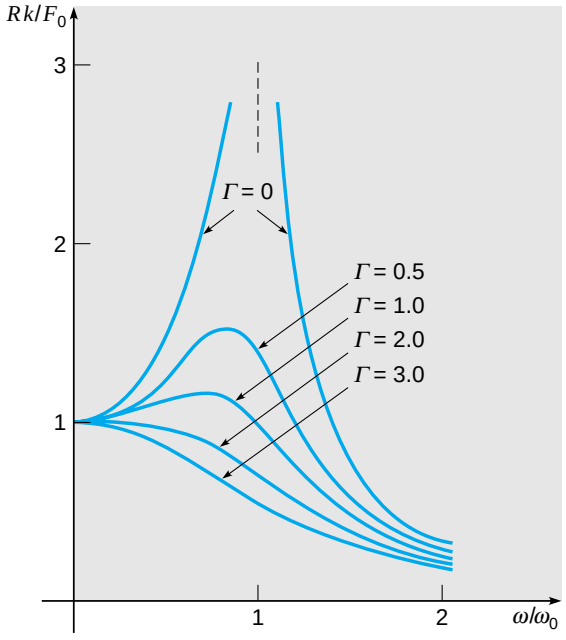


FIGURE 3.9.3 Forced vibration with damping: amplitude of steady-state response versus frequency of driving force; $\Gamma = \gamma^2/m^2\omega_0^2$.

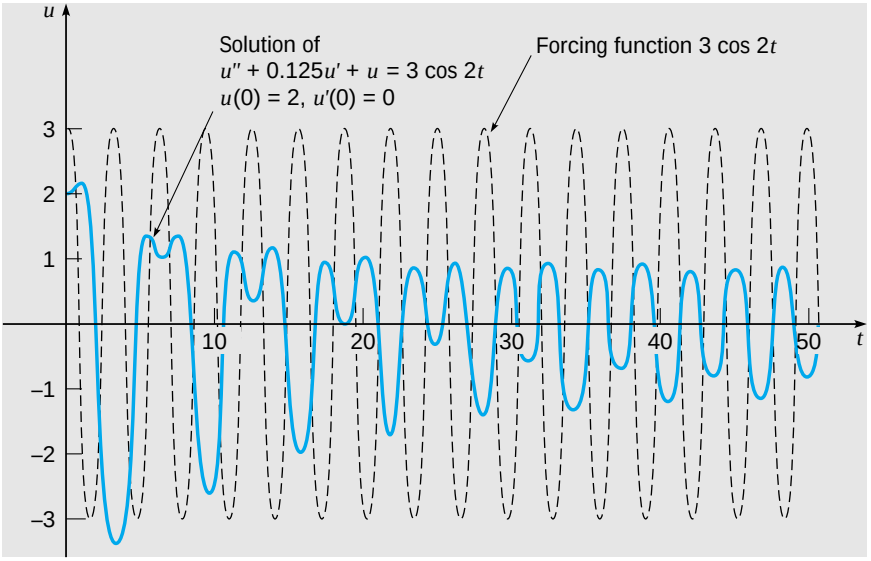


FIGURE 3.9.4 A forced vibration with damping; solution of $u'' + 0.125u' + u = 3\cos 2t$, $u(0) = 2$, $u'(0) = 0$.

More precisely, we find that $\Delta = \sqrt{145}/4 \cong 3.0104$, so $R = F_0/\Delta \cong 0.9965$. Further, $\cos\delta = -3/\Delta \cong -0.9965$ and $\sin\delta = 1/4\Delta \cong 0.08305$, so that $\delta \cong 3.0585$. Thus the calculated values of R and δ are close to the values estimated from the graph.

PROBLEMS

In each of Problems 1 through 4 write the given expression as a product of two trigonometric functions of different frequencies.

1. $\cos 9t - \cos 7t$
2. $\sin 7t - \sin 6t$
3. $\cos \pi t + \cos 2\pi t$
4. $\sin 3t + \sin 4t$
5. A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb, formulate the initial value problem describing the motion of the mass.
6. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.
7. (a) Find the solution of Problem 5.
   (b) Plot the graph of the solution.
   (c) If the given external force is replaced by a force $4\sin\omega t$ of frequency ω, find the value of ω for which resonance occurs.
8. (a) Find the solution of the initial value problem in Problem 6.
   (b) Identify the transient and steady-state parts of the solution.
   (c) Plot the graph of the steady-state solution.
   (d) If the given external force is replaced by a force $2\cos\omega t$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.

9. If an undamped spring–mass system with a mass that weighs 6 lb and a spring constant 1 lb/in. is suddenly set in motion at $t = 0$ by an external force of $4\cos 7t$ lb, determine the position of the mass at any time and draw a graph of the displacement versus t.
10. A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8\sin 8t$ lb. If the mass is pulled down 3 in. and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.
11. A spring is stretched 6 in. by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb-sec/ft and is acted on by an external force of $4\cos 2t$ lb.
    (a) Determine the steady-state response of this system.
    (b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.
12. A spring–mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $3\cos 3t - 2\sin 3t$ N, determine the steady-state response. Express your answer in the form $R\cos(\omega t - \delta)$.
13. Furnish the details in determining when the steady-state response given by Eq. (11) is maximum; that is, show that $\omega_{\max}^2$ and $R_{\max}$ are given by Eqs. (12) and (13), respectively.
14. (a) Show that the phase of the forced response of Eq. (1) satisfies $\tan\delta = \gamma\omega/m(\omega_0^2 - \omega^2)$.
    (b) Plot the phase δ as a function of the forcing frequency ω for the forced response of $u'' + 0.125u' + u = 3\cos\omega t$.
15. Find the solution of the initial value problem

$$u'' + u = F(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately and match the solutions in the different intervals by requiring that u and u' be continuous functions of t.

16. A series circuit has a capacitor of 0.25×10^{-6} farad, a resistor of 5×10^3 ohms, and an inductor of 1 henry. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ sec, at $t = 0.01$ sec, and at any time t. Also determine the limiting charge as $t \to \infty$.
17. Consider a vibrating system described by the initial value problem

$$u'' + \tfrac{1}{4}u' + 2u = 2\cos\omega t, \qquad u(0) = 0, \quad u'(0) = 2.$$

    (a) Determine the steady-state part of the solution of this problem.
    (b) Find the amplitude A of the steady-state solution in terms of ω.
    (c) Plot A versus ω.
    (d) Find the maximum value of A and the frequency ω for which it occurs.
18. Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

    (a) Find the solution $u(t)$ for $\omega \neq 1$.
    (b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

19. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos\omega t, \qquad u(0) = 1, \quad u'(0) = 1.$$

(a) Find the solution for $\omega \neq 1$.
(b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 18, that is, describe the effect of the nonzero initial conditions.

20. For the initial value problem in Problem 18 plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$; that is, draw the phase plot of the solution for these values of ω. Use a t interval that is long enough so the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 21 through 23 deal with the initial value problem

$$u'' + 0.125u' + u = F(t), \qquad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

(a) Plot the given forcing function $F(t)$ versus t and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 1$.
(b) Draw the phase plot of the solution, that is, plot u' versus u.

21. $F(t) = 3\cos(0.3t)$

22. $F(t) = 3\cos t$

23. $F(t) = 3\cos 3t$

24. A spring–mass system with a hardening spring (Problem 32 of Section 3.8) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$u'' + u + \tfrac{1}{5}u^3 = \cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(a) Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?
(b) Plot the solution for several values of ω between 1/2 and 2. Describe how the solution changes as ω increases.

25. Suppose that the system of Problem 24 is modified to include a damping term and that the resulting initial value problem is

$$u'' + \tfrac{1}{5}u' + u + \tfrac{1}{5}u^3 = \cos\omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(a) Plot a computer-generated solution of the given problem for several values of ω between 1/2 and 2 and estimate the amplitude R of the steady response in each case.
(b) Using the data from part (a), plot the graph of R versus ω. For what frequency ω is the amplitude greatest?
(c) Compare the results of parts (a) and (b) with the corresponding results for the linear spring.

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CHAPTER
4

Higher Order Linear Equations

The theoretical structure and methods of solution developed in the preceding chapter for second order linear equations extend directly to linear equations of third and higher order. In this chapter we briefly review this generalization, taking particular note of those instances where new phenomena may appear, due to the greater variety of situations that can occur for equations of higher order.

4.1 General Theory of *n*th Order Linear Equations

An nth order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t). \tag{1}$$

We assume that the functions $P_0, \ldots, P_n$ and G are continuous real-valued functions on some interval I: $\alpha < t < \beta$, and that P_0 is nowhere zero in this interval. Then, dividing Eq. (1) by $P_0(t)$, we obtain

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t). \tag{2}$$

The linear differential operator L of order n defined by Eq. (2) is similar to the second order operator introduced in Chapter 3. The mathematical theory associated with Eq. (2) is completely analogous to that for the second order linear equation; for this reason we

simply state the results for the nth order problem. The proofs of most of the results are also similar to those for the second order equation and are usually left as exercises.

Since Eq. (2) involves the nth derivative of y with respect to t, it will, so to speak, require n integrations to solve Eq. (2). Each of these integrations introduces an arbitrary constant. Hence we can expect that, to obtain a unique solution, it is necessary to specify n initial conditions,

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3}$$

where t_0 may be any point in the interval I and $y_0, y_0', \ldots, y_0^{(n-1)}$ is any set of prescribed real constants. That there does exist such a solution and that it is unique are assured by the following existence and uniqueness theorem, which is similar to Theorem 3.2.1.

Theorem 4.1.1 If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3). This solution exists throughout the interval I.

We will not give a proof of this theorem here. However, if the coefficients $p_1, \ldots, p_n$ are constants, then we can construct the solution of the initial value problem (2), (3) much as in Chapter 3; see Sections 4.2 through 4.4. Even though we may find a solution in this case, we do not know that it is unique without the use of Theorem 4.1.1. A proof of the theorem can be found in Ince (Section 3.32) or Coddington (Chapter 6).

The Homogeneous Equation. As in the corresponding second order problem, we first discuss the homogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \tag{4}$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), then it follows by direct computation that the linear combination

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t), \tag{5}$$

where $c_1, \ldots, c_n$ are arbitrary constants, is also a solution of Eq. (4). It is then natural to ask whether every solution of Eq. (4) can be expressed as a linear combination of $y_1, \ldots, y_n$. This will be true if, regardless of the initial conditions (3) that are prescribed, it is possible to choose the constants $c_1, \ldots, c_n$ so that the linear combination (5) satisfies the initial conditions. Specifically, for any choice of the point t_0 in I, and for any choice of $y_0, y_0', \ldots, y_0^{(n-1)}$, we must be able to determine $c_1, \ldots, c_n$ so that the equations

$$\begin{aligned} c_1y_1(t_0) + \cdots + c_ny_n(t_0) &= y_0 \\ c_1y_1'(t_0) + \cdots + c_ny_n'(t_0) &= y_0' \\ &\vdots \\ c_1y_1^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned} \tag{6}$$

are satisfied. Equations (6) can be solved uniquely for the constants $c_1, \ldots, c_n$, provided that the determinant of coefficients is not zero. On the other hand, if the determinant of coefficients is zero, then it is always possible to choose values of $y_0, y_0', \ldots, y_0^{(n-1)}$

such that Eqs. (6) do not have a solution. Hence a necessary and sufficient condition for the existence of a solution of Eqs. (6) for arbitrary values of $y_0, y_0', \ldots, y_0^{(n-1)}$ is that the Wronskian

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{7}$$

is not zero at $t = t_0$. Since t_0 can be any point in the interval I, it is necessary and sufficient that $W(y_1, y_2, \ldots, y_n)$ be nonzero at every point in the interval. Just as for the second order linear equation, it can be shown that if $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), then $W(y_1, y_2, \ldots, y_n)$ is either zero for every t in the interval I or else is never zero there; see Problem 20. Hence we have the following theorem.

Theorem 4.1.2 If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), and if $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point in I, then every solution of Eq. (4) can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.

A set of solutions $y_1, \ldots, y_n$ of Eq. (4) whose Wronskian is nonzero is referred to as a **fundamental set of solutions**. The existence of a fundamental set of solutions can be demonstrated in precisely the same way as for the second order linear equation (see Theorem 3.2.5). Since all solutions of Eq. (4) are of the form (5), we use the term **general solution** to refer to an arbitrary linear combination of any fundamental set of solutions of Eq. (4).

The discussion of linear dependence and independence given in Section 3.3 can also be generalized. The functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on I if there exists a set of constants $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0 \tag{8}$$

for all t in I. The functions $f_1, \ldots, f_n$ are said to be **linearly independent** on I if they are not linearly dependent there. If $y_1, \ldots, y_n$ are solutions of Eq. (4), then it can be shown that a necessary and sufficient condition for them to be linearly independent is that $W(y_1, \ldots, y_n)(t_0) \neq 0$ for some t_0 in I (see Problem 25). Hence a fundamental set of solutions of Eq. (4) is linearly independent, and a linearly independent set of n solutions of Eq. (4) forms a fundamental set of solutions.

The Nonhomogeneous Equation. Now consider the nonhomogeneous equation (2),

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t).$$

If Y_1 and Y_2 are any two solutions of Eq. (2), then it follows immediately from the linearity of the operator L that

$$L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0.$$

Hence the difference of any two solutions of the nonhomogeneous equation (2) is a solution of the homogeneous equation (4). Since any solution of the homogeneous

equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \ldots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{9}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (9) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

The method of reduction of order (Section 3.5) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n-1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6 determine intervals in which solutions are sure to exist.

1. $y^{\text{iv}} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{\text{iv}} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{\text{iv}} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2-4)y^{\text{vi}} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10 determine whether the given set of functions is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16 verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \quad 1, \quad \cos t, \quad \sin t$
12. $y^{\text{iv}} + y'' = 0; \quad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \quad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{\text{iv}} + 2y''' + y'' = 0; \quad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \quad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \quad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2],$$

where y_1 and y_2 are n times differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if $y_1, y_2, \ldots, y_n$ are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny,$$

where $a_0, a_1, \ldots, a_n$ are real constants.
(a) Find $L[t^n]$.
(b) Find $L[e^{rt}]$.
(c) Determine four solutions of the equation $y^{\text{iv}} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

20. In this problem we show how to generalize Theorem 3.3.2 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.
(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.
(b) Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\, dt\right].$$

It follows that W is either always zero or nowhere zero on I.
(d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish Abel's formula,

$$W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right],$$

for this case.

In each of Problems 21 through 24 use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. $y''' + 2y'' - y' - 3y = 0$
22. $y^{\text{iv}} + y = 0$
23. $ty''' + 2y'' - y' + ty = 0$
24. $t^2y^{\text{iv}} + ty''' + y'' - 4y = 0$

25. The purpose of this problem is to show that if $W(y_1, \ldots, y_n)(t_0) \neq 0$ for some t_0 in an interval I, then $y_1, \ldots, y_n$ are linearly independent on I, and if they are linearly independent and solutions of

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{i}$$

on I, then $W(y_1, \ldots, y_n)$ is nowhere zero in I.

(a) Suppose that $W(y_1, \ldots, y_n)(t_0) \neq 0$, and suppose that

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0 \tag{ii}$$

for all t in I. By writing the equations corresponding to the first $n-1$ derivatives of Eq. (ii) at t_0, show that $c_1 = \cdots = c_n = 0$. Therefore, $y_1, \ldots, y_n$ are linearly independent.
(b) Suppose that $y_1, \ldots, y_n$ are linearly independent solutions of Eq. (i). If $W(y_1, \ldots, y_n)(t_0) = 0$ for some t_0, show that there is a nonzero solution of Eq. (i) satisfying the initial conditions

$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0.$$

Since $y = 0$ is a solution of this initial value problem, the uniqueness part of Theorem 4.1.1 yields a contradiction. Thus W is never zero.

26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

In each of Problems 27 and 28 use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$
28. $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real constants. From our knowledge of second order linear equations with constant coefficients it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r) \tag{2}$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n. \tag{3}$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1).

A polynomial of degree n has n zeros,[1] say $r_1, r_2, \ldots, r_n$, some of which may be equal; hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n). \tag{4}$$

Real and Unequal Roots. If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ of Eq. (1). If these functions are linearly independent, then the general solution of Eq. (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}. \tag{5}$$

One way to establish the linear independence of $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ is to evaluate their Wronskian determinant. Another way is outlined in Problem 40.

EXAMPLE 1

Find the general solution of

$$y'''' + y''' - 7y'' - y' + 6y = 0. \tag{6}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = -2, \qquad y'''(0) = -1 \tag{7}$$

and plot its graph.

Assuming that $y = e^{rt}$, we must determine r by solving the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{8}$$

The roots of this equation are $r_1 = 1$, $r_2 = -1$, $r_3 = 2$, and $r_4 = -3$. Therefore the general solution of Eq. (6) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}. \tag{9}$$

The initial conditions (7) require that $c_1, \ldots, c_4$ satisfy the four equations

$$\begin{aligned} c_1 + c_2 + \ c_3 + \ \ c_4 &= \ \ 1, \\ c_1 - c_2 + 2c_3 - \ 3c_4 &= \ \ 0, \\ c_1 + c_2 + 4c_3 + \ 9c_4 &= -2, \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1. \end{aligned} \tag{10}$$

By solving this system of four linear algebraic equations, we find that

$$c_1 = 11/8, \qquad c_2 = 5/12, \qquad c_3 = -2/3, \qquad c_4 = -1/8.$$

Therefore the solution of the initial value problem is

$$y = \tfrac{11}{8}e^t + \tfrac{5}{12}e^{-t} - \tfrac{2}{3}e^{2t} - \tfrac{1}{8}e^{-3t}. \tag{11}$$

The graph of the solution is shown in Figure 4.2.1.

[1] An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

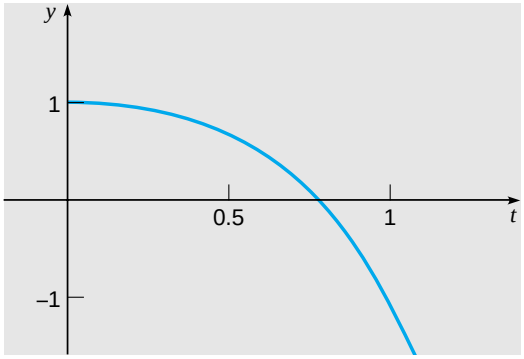


FIGURE 4.2.1 Solution of the initial value problem of Example 1.

As Example 1 illustrates, the procedure for solving an nth order linear differential equation with constant coefficients depends on finding the roots of a corresponding nth degree polynomial equation. If initial conditions are prescribed, then a system of n linear algebraic equations must be solved to determine the proper values of the constants $c_1, \ldots, c_n$. While each of these tasks becomes much more complicated as n increases, they can often be handled without difficulty with a calculator or computer.

For third and fourth degree polynomials there are formulas,[2] analogous to the formula for quadratic equations but more complicated, that give exact expressions for the roots. Root-finding algorithms are readily available on calculators and computers. Sometimes they are included in the differential equation solver, so that the process of factoring the characteristic polynomial is hidden and the solution of the differential equation is produced automatically.

If you are faced with the need to factor the characteristic polynomial by hand, here is one result that is sometimes helpful. Suppose that the polynomial

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{12}$$

has integer coefficients. If $r = p/q$ is a rational root, where p and q have no common factors, then p must be a factor of a_n and q must be a factor of a_0. For example, in Eq. (8) the factors of a_0 are ± 1 and the factors of a_n are ± 1, ± 2, ± 3, and ± 6. Thus, the only possible rational roots of this equation are ± 1, ± 2, ± 3, and ± 6. By testing these possible roots, we find that 1, -1, 2, and -3 are actual roots. In this case there are no other roots, since the polynomial is of fourth degree. If some of the roots are irrational or complex, as is usually the case, then this process will not find them, but at least the degree of the polynomial can be reduced by dividing out the factors corresponding to the rational roots.

If the roots of the characteristic equation are real and different, we have seen that the general solution (5) is simply a sum of exponential functions. For large values of t the

[2]The method for solving the cubic equation was apparently discovered by Scipione dal Ferro (1465–1526) about 1500, although it was first published in 1545 by Girolamo Cardano (1501–1576) in his *Ars Magna*. This book also contains a method for solving quartic equations that Cardano attributes to his pupil Ludovico Ferrari (1522–1565). The question of whether analogous formulas exist for the roots of higher degree equations remained open for more than two centuries, until in 1826 Niels Abel showed that no general solution formulas can exist for polynomial equations of degree five or higher. A more general theory was developed by Evariste Galois (1811–1832) in 1831, but unfortunately it did not become widely known for several decades.

solution will be dominated by the term corresponding to the algebraically largest root. If this root is positive, then solutions will become exponentially unbounded, while if it is negative, then solutions will tend exponentially to zero. Finally, if the largest root is zero, then solutions will approach a nonzero constant as t becomes large. Of course, for certain initial conditions the coefficient of the otherwise dominant term will be zero; then the nature of the solution for large t is determined by the next largest root.

Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, \ldots, a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of Eq. (1) is still of the form (4). However, just as for the second order equation (Section 3.4), we can replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t \tag{13}$$

obtained as the real and imaginary parts of $e^{(\lambda+i\mu)t}$. Thus, even though some of the roots of the characteristic equation are complex, it is still possible to express the general solution of Eq. (1) as a linear combination of real-valued solutions.

EXAMPLE 2

Find the general solution of

$$y^{\text{iv}} - y = 0. \tag{14}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 7/2, \qquad y'(0) = -4, \qquad y''(0) = 5/2, \qquad y'''(0) = -2 \tag{15}$$

and draw its graph.

Substituting e^{rt} for y, we find that the characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore the roots are $r = 1, -1, i, -i$, and the general solution of Eq. (14) is

$$y = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t.$$

If we impose the initial conditions (15), we find that

$$c_1 = 0, \qquad c_2 = 3, \qquad c_3 = 1/2, \qquad c_4 = -1;$$

thus the solution of the given initial value problem is

$$y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t. \tag{16}$$

The graph of this solution is shown in Figure 4.2.2.

Observe that the initial conditions (15) cause the coefficient c_1 of the exponentially growing term in the general solution to be zero. Therefore this term is absent in the solution (16), which describes an exponential decay to a steady oscillation, as Figure 4.2.2 shows. However, if the initial conditions are changed slightly, then c_1 is likely to be nonzero and the nature of the solution changes enormously. For example, if the first three initial conditions remain the same, but the value of $y'''(0)$ is changed from -2 to $-15/8$, then the solution of the initial value problem becomes

$$y = \tfrac{1}{32}e^t + \tfrac{95}{32}e^{-t} + \tfrac{1}{2}\cos t - \tfrac{17}{16}\sin t. \tag{17}$$

The coefficients in Eq. (17) differ only slightly from those in Eq. (16), but the exponentially growing term, even with the relatively small coefficient of 1/32, completely dominates the solution by the time t is larger than about 4 or 5. This is clearly seen in Figure 4.2.3, which shows the graphs of the two solutions (16) and (17).

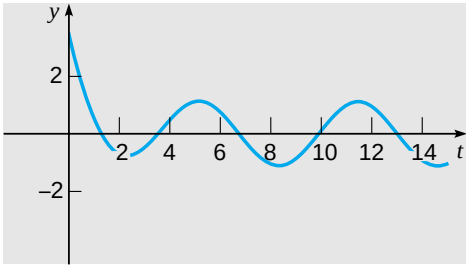


FIGURE 4.2.2 A plot of the solution (16).

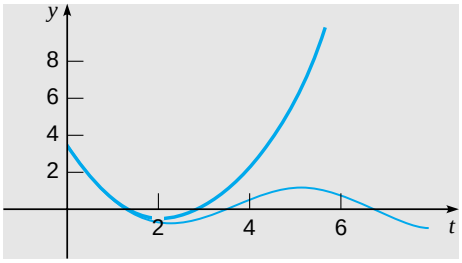


FIGURE 4.2.3 Plots of the solutions (16) (light curve) and (17) (heavy curve).

Repeated Roots. If the roots of the characteristic equation are not distinct, that is, if some of the roots are repeated, then the solution (5) is clearly not the general solution of Eq. (1). Recall that if r_1 is a repeated root for the second order linear equation $a_0y'' + a_1y' + a_2y = 0$, then the two linearly independent solutions are e^{r_1t} and te^{r_1t}. For an equation of order n, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity s (where $s \le n$), then

$$e^{r_1t}, \quad te^{r_1t}, \quad t^2e^{r_1t}, \quad \ldots, \quad t^{s-1}e^{r_1t} \tag{18}$$

are corresponding solutions of Eq. (1).

If a complex root $\lambda + i\mu$ is repeated s times, the complex conjugate $\lambda - i\mu$ is also repeated s times. Corresponding to these $2s$ complex-valued solutions, we can find $2s$ real-valued solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}$, $te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions:

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad te^{\lambda t}\cos\mu t, \quad te^{\lambda t}\sin\mu t,$$
$$\ldots, t^{s-1}e^{\lambda t}\cos\mu t, \quad t^{s-1}e^{\lambda t}\sin\mu t.$$

Hence the general solution of Eq. (1) can always be expressed as a linear combination of n real-valued solutions. Consider the following example.

EXAMPLE 3

Find the general solution of

$$y^{\text{iv}} + 2y'' + y = 0. \tag{19}$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0.$$

The roots are $r = i, i, -i, -i$, and the general solution of Eq. (19) is

$$y = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t.$$

In determining the roots of the characteristic equation it may be necessary to compute the cube roots, or fourth roots, or even higher roots of a (possibly complex) number. This can usually be done most conveniently by using Euler's formula $e^{it} = \cos t + i \sin t$ and the algebraic laws given in Section 3.4. This is illustrated in the following example.

EXAMPLE 4

Find the general solution of

$$y^{\text{iv}} + y = 0. \tag{20}$$

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation we must compute the fourth roots of -1. Now -1, thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π. Thus

$$-1 = \cos \pi + i \sin \pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π. Thus

$$-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi + 2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4 + m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0$, 1, 2, and 3; they are

$$\frac{1+i}{\sqrt{2}}, \qquad \frac{-1+i}{\sqrt{2}}, \qquad \frac{-1-i}{\sqrt{2}}, \qquad \frac{1-i}{\sqrt{2}}.$$

It is easy to verify that for any other value of m we obtain one of these four roots. For example, corresponding to $m = 4$, we obtain $(1 + i)/\sqrt{2}$. The general solution of Eq. (20) is

$$y = e^{t/\sqrt{2}}\left(c_1 \cos \frac{t}{\sqrt{2}} + c_2 \sin \frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3 \cos \frac{t}{\sqrt{2}} + c_4 \sin \frac{t}{\sqrt{2}}\right). \tag{21}$$

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. For instance, it may be difficult to determine whether two roots are equal, or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants $a_0, a_1, \ldots, a_n$ in Eq. (1) are complex numbers, the solution of Eq. (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6 express the given complex number in the form $R(\cos\theta + i \sin\theta) = Re^{i\theta}$.

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. -3
4. $-i$
5. $\sqrt{3} - i$
6. $-1 - i$

In each of Problems 7 through 10 follow the procedure illustrated in Example 4 to determine the indicated roots of the given complex number.

7. $1^{1/3}$
8. $(1-i)^{1/2}$
9. $1^{1/4}$
10. $[2(\cos\pi/3 + i\sin\pi/3)]^{1/2}$

In each of Problems 11 through 28 find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{iv} - 4y''' + 4y'' = 0$
15. $y^{vi} + y = 0$
16. $y^{iv} - 5y'' + 4y = 0$
17. $y^{vi} - 3y^{iv} + 3y'' - y = 0$
18. $y^{vi} - y'' = 0$
19. $y^{v} - 3y^{iv} + 3y''' - 3y'' + 2y' = 0$
20. $y^{iv} - 8y' = 0$
21. $y^{viii} + 8y^{iv} + 16y = 0$
22. $y^{iv} + 2y'' + y = 0$
23. $y''' - 5y'' + 3y' + y = 0$
24. $y''' + 5y'' + 6y' + 2y = 0$
25. $18y''' + 21y'' + 14y' + 4y = 0$
26. $y^{iv} - 7y''' + 6y'' + 30y' - 36y = 0$
27. $12y^{iv} + 31y''' + 75y'' + 37y' + 5y = 0$
28. $y^{iv} + 6y''' + 17y'' + 22y' + 14y = 0$

In each of Problems 29 through 36 find the solution of the given initial value problem and plot its graph. How does the solution behave as $t \to \infty$?

29. $y''' + y' = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$
30. $y^{iv} + y = 0; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
31. $y^{iv} - 4y''' + 4y'' = 0; \quad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$
32. $y''' - y'' + y' - y = 0; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$
33. $2y^{iv} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
34. $4y''' + y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
35. $6y''' + 5y'' + y' = 0; \quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$
36. $y^{iv} + 6y''' + 17y'' + 22y' + 14y = 0; \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0, \quad y'''(0) = 3$
37. Show that the general solution of $y^{iv} - y = 0$ can be written as

$$y = c_1\cos t + c_2\sin t + c_3\cosh t + c_4\sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

38. Consider the equation $y^{iv} - y = 0$.
(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.
(b) Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.
(c) Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.
39. Consider the spring–mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.
(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \tag{i}$$

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1:

$$u_1^{iv} + 7u_1'' + 6u_1 = 0. \tag{ii}$$

Find the general solution of Eq. (ii).

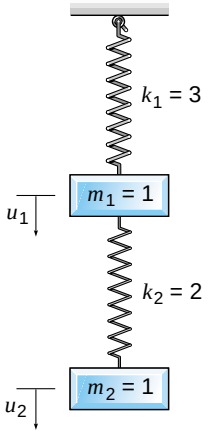


FIGURE 4.2.4 A two degree of freedom spring–mass system.

(c) Suppose that the initial conditions are

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0. \tag{iii}$$

Use the first of Eqs. (i) and the initial conditions (iii) to obtain values for $u_1''(0)$ and $u_1'''(0)$. Then show that the solution of Eq. (ii) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2\cos t$.

(d) Now suppose that the initial conditions are

$$u_1(0) = -2, \qquad u_1'(0) = 0, \qquad u_2(0) = 1, \qquad u_2'(0) = 0. \tag{iv}$$

Proceed as in part (c) to show that the corresponding solutions are $u_1(t) = -2\cos\sqrt{6}t$ and $u_2(t) = \cos\sqrt{6}t$.

(e) Observe that the solutions obtained in parts (c) and (d) describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up and vice versa. For other initial conditions, the motion of the masses is a combination of these two modes.

40. In this problem we outline one way to show that if $r_1, \ldots, r_n$ are all real and different, then $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty \tag{i}$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to t, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2 - r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n - r_{n-1}) \cdots (r_n - r_1)e^{(r_n - r_{n-1})t} = 0.$$

Hence $c_n = 0$ and therefore

$$c_1 e^{r_1 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0.$$

(d) Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \cdots = c_1 = 0$. Thus the functions $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent.

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. While the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator L is applied to a polynomial $A_0 t^m + A_1 t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, or a cosine function $\cos \beta t$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. Hence, if $g(t)$ is a sum of polynomials, exponentials, sines, and cosines, or products of such functions, we can expect that it is possible to find $Y(t)$ by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined so that Eq. (1) is satisfied.

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic polynomial equation may have multiplicity greater than 2. Consequently, terms proposed for the nonhomogeneous part of the solution may need to be multiplied by higher powers of t to make them different from terms in the solution of the corresponding homogeneous equation.

EXAMPLE 1

Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{2}$$

The characteristic polynomial for the homogeneous equation corresponding to Eq. (2) is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3,$$

so the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t. \tag{3}$$

To find a particular solution $Y(t)$ of Eq. (2), we start by assuming that $Y(t) = Ae^t$. However, since e^t, te^t, and $t^2 e^t$ are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3 e^t$,

where A is an undetermined coefficient. To find the correct value for A, we differentiate $Y(t)$ three times, substitute for y and its derivatives in Eq. (2), and collect terms in the resulting equation. In this way we obtain

$$6Ae^t = 4e^t.$$

Thus $A = \frac{2}{3}$ and the particular solution is

$$Y(t) = \tfrac{2}{3}t^3e^t. \tag{4}$$

The general solution of Eq. (2) is the sum of $y_c(t)$ from Eq. (3) and $Y(t)$ from Eq. (4).

EXAMPLE 2

Find a particular solution of the equation

$$y^{\text{iv}} + 2y'' + y = 3\sin t - 5\cos t. \tag{5}$$

The general solution of the homogeneous equation was found in Example 3 of Section 4.2, namely,

$$y_c(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t, \tag{6}$$

corresponding to the roots $r = i, i, -i$, and $-i$ of the characteristic equation. Our initial assumption for a particular solution is $Y(t) = A\sin t + B\cos t$, but we must multiply this choice by t^2 to make it different from all solutions of the homogeneous equation. Thus our final assumption is

$$Y(t) = At^2\sin t + Bt^2\cos t.$$

Next, we differentiate $Y(t)$ four times, substitute into the differential equation (4), and collect terms, obtaining finally

$$-8A\sin t - 8B\cos t = 3\sin t - 5\cos t.$$

Thus $A = -\frac{3}{8}$, $B = \frac{5}{8}$, and the particular solution of Eq. (4) is

$$Y(t) = -\tfrac{3}{8}t^2\sin t + \tfrac{5}{8}t^2\cos t. \tag{7}$$

If $g(t)$ is a sum of several terms, it is often easier in practice to compute separately the particular solution corresponding to each term in $g(t)$. As for the second order equation, the particular solution of the complete problem is the sum of the particular solutions of the individual component problems. This is illustrated in the following example.

EXAMPLE 3

Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}. \tag{8}$$

First we solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$, and the roots are $0, \pm 2$; hence

$$y_c(t) = c_1 + c_2e^{2t} + c_3e^{-2t}.$$

We can write a particular solution of Eq. (8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3\cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $Y_1(t)$ of the first equation is $A_0t + A_1$, but since a constant is a solution of the homogeneous equation, we multiply by t. Thus

$$Y_1(t) = t(A_0t + A_1).$$

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t,$$

and there is no need to modify this initial choice since $\cos t$ and $\sin t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that

$$Y_3(t) = Ete^{-2t}.$$

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$

The amount of algebra required to calculate the coefficients may be quite substantial for higher order equations, especially if the nonhomogeneous term is even moderately complicated. A computer algebra system can be extremely helpful in executing these algebraic calculations.

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8 determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{\text{iv}} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{\text{iv}} - 4y'' = t^2 + e^t$
6. $y^{\text{iv}} + 2y'' + y = 3 + \cos 2t$
7. $y^{\text{vi}} + y''' = t$
8. $y^{\text{iv}} + y''' = \sin 2t$

In each of Problems 9 through 12 find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$
10. $y^{\text{iv}} + 2y'' + y = 3t + 4, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t, \quad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}$
12. $y^{\text{iv}} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}, \quad y(0) = 3, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 2$

In each of Problems 13 through 18 determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{\text{iv}} - 2y'' + y = e^t + \sin t$
16. $y^{\text{iv}} + 4y'' = \sin 2t + te^t + 4$
17. $y^{\text{iv}} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{\text{iv}} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
19. Consider the nonhomogeneous nth order linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t),$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0 t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces the preceding equation to the form

$$k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_n u = b_0 t^m + \cdots + b_m,$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22 we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt. Then, for example, e^{-t} is a solution of $(D+1)y = 0$; the differential operator $D+1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. Similarly, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D-3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

20. Show that linear differential operators with constant coefficients obey the commutative law, that is,

$$(D-a)(D-b)\,f = (D-b)(D-a)\,f$$

for any twice differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

21. Consider the problem of finding the form of the particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \tag{i}$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator $(D-2)(D+1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D-2)^4(D+1)^3 Y = 0. \tag{ii}$$

Thus Y is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$\begin{aligned} Y(t) = {} & c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} \\ & + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}, \end{aligned} \tag{iii}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.

(c) Observe that e^{2t}, te^{2t}, t^2e^{2t}, and e^{-t} are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose c_1, c_2, c_3, and c_5 to be zero in Eq. (iii), so that

$$Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}. \tag{iv}$$

This is the form of the particular solution Y of Eq. (i). The values of the coefficients c_4, c_6, and c_7 can be found by substituting from Eq. (iv) in the differential equation (i).

Summary. Suppose that

$$L(D)y = g(t), \tag{i}$$

where $L(D)$ is a linear differential operator with constant coefficients, and $g(t)$ is a sum or product of exponential, polynomial, or sinusoidal terms. To find the form of the particular solution of Eq. (i), you can proceed as follows.

(a) Find a differential operator $H(D)$ with constant coefficients that annihilates $g(t)$, that is, an operator such that $H(D)g(t) = 0$.
(b) Apply $H(D)$ to Eq. (i), obtaining

$$H(D)L(D)y = 0, \tag{ii}$$

which is a homogeneous equation of higher order.
(c) Solve Eq. (ii).
(d) Eliminate from the solution found in step (c) the terms that also appear in the solution of $L(D)y = 0$. The remaining terms constitute the correct form of the particular solution of Eq. (i).

22. Use the method of annihilators to find the form of the particular solution $Y(t)$ for each of the equations in Problems 13 through 18. Do not evaluate the coefficients.

4.4 The Method of Variation of Parameters

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{1}$$

is a direct extension of the method for the second order differential equation (see Section 3.7). As before, to use the method of variation of parameters, it is first necessary to solve the corresponding homogeneous differential equation. In general, this may be difficult unless the coefficients are constants. However, the method of variation of parameters is still more general than the method of undetermined coefficients in that it leads to an expression for the particular solution for *any* continuous function g, whereas the method of undetermined coefficients is restricted in practice to a limited class of functions g.

Suppose then that we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation. Then the general solution of the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t). \tag{2}$$

The method of variation of parameters for determining a particular solution of Eq. (1) rests on the possibility of determining n functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t). \tag{3}$$

Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that Y satisfy Eq. (1). The other $n-1$ conditions are chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining Y if we must solve high order differential equations for $u_1, \ldots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \ldots, u_n$. From Eq. (3) we obtain

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n), \tag{4}$$

where we have omitted the independent variable t on which each function in Eq. (4) depends. Thus the first condition that we impose is that

$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0. \tag{5}$$

Continuing this process in a similar manner through $n-1$ derivatives of Y gives

$$Y^{(m)} = u_1y_1^{(m)} + u_2y_2^{(m)} + \cdots + u_ny_n^{(m)}, \qquad m = 0, 1, 2, \ldots, n-1, \tag{6}$$

and the following $n-1$ conditions on the functions $u_1, \ldots, u_n$:

$$u_1'y_1^{(m-1)} + u_2'y_2^{(m-1)} + \cdots + u_n'y_n^{(m-1)} = 0, \qquad m = 1, 2, \ldots, n-1. \tag{7}$$

The nth derivative of Y is

$$Y^{(n)} = (u_1y_1^{(n)} + \cdots + u_ny_n^{(n)}) + (u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)}). \tag{8}$$

Finally, we impose the condition that Y must be a solution of Eq. (1). On substituting for the derivatives of Y from Eqs. (6) and (8), collecting terms, and making use of the fact that $L[y_i] = 0,\ i = 1, 2, \ldots, n$, we obtain

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = g. \tag{9}$$

Equation (9), coupled with the $n-1$ equations (7), gives n simultaneous linear nonhomogeneous algebraic equations for $u_1', u_2', \ldots, u_n'$:

$$\begin{aligned}
y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0,\\
y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0,\\
y_1''u_1' + y_2''u_2' + \cdots + y_n''u_n' &= 0,\\
&\vdots\\
y_1^{(n-1)}u_1' + \cdots + y_n^{(n-1)}u_n' &= g.
\end{aligned} \tag{10}$$

The system (10) is a linear algebraic system for the unknown quantities $u_1', \ldots, u_n'$. By solving this system and then integrating the resulting expressions, you can obtain the coefficients $u_1, \ldots, u_n$. A sufficient condition for the existence of a solution of the system of equations (10) is that the determinant of coefficients is nonzero for each value of t. However, the determinant of coefficients is precisely $W(y_1, y_2, \ldots, y_n)$, and it is nowhere zero since $y_1, \ldots, y_n$ are linearly independent solutions of the homogeneous

equation. Hence it is possible to determine $u_1', \ldots, u_n'$. Using Cramer's rule, we find that the solution of the system of equations (10) is

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \qquad m = 1, 2, \ldots, n. \tag{11}$$

Here $W(t) = W(y_1, y_2, \ldots, y_n)(t)$ and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 0, 1)$. With this notation a particular solution of Eq. (1) is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\, ds, \tag{12}$$

where t_0 is arbitrary. While the procedure is straightforward, the algebraic computations involved in determining $Y(t)$ from Eq. (12) become more and more complicated as n increases. In some cases the calculations may be simplified to some extent by using Abel's identity (Problem 20 of Section 4.1),

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right].$$

The constant c can be determined by evaluating W at some convenient point.

EXAMPLE 1

Given that $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = e^{-t}$ are solutions of the homogeneous equation corresponding to

$$y''' - y'' - y' + y = g(t), \tag{13}$$

determine a particular solution of Eq. (13) in terms of an integral.

We use Eq. (12). First, we have

$$W(t) = W(e^t, te^t, e^{-t})(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix}.$$

Factoring e^t from each of the first two columns and e^{-t} from the third column, we obtain

$$W(t) = e^t \begin{vmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{vmatrix}.$$

Then, by subtracting the first row from the second and third rows, we have

$$W(t) = e^t \begin{vmatrix} 1 & t & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{vmatrix}.$$

Finally, evaluating the latter determinant by minors associated with the first column, we find that

$$W(t) = 4e^t.$$

Next,

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix}.$$

Using minors associated with the first column, we obtain

$$W_1(t) = \begin{vmatrix} te^t & e^{-t} \\ (t+1)e^t & -e^{-t} \end{vmatrix} = -2t - 1.$$

In a similar way

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = -\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = 2,$$

and

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}.$$

Substituting these results in Eq. (12), we have

$$\begin{aligned} Y(t) &= e^t \int_{t_0}^{t} \frac{g(s)(-1-2s)}{4e^s}\,ds + te^t \int_{t_0}^{t} \frac{g(s)(2)}{4e^s}\,ds + e^{-t} \int_{t_0}^{t} \frac{g(s)e^{2s}}{4e^s}\,ds \\ &= \frac{1}{4}\int_{t_0}^{t} \left\{e^{t-s}[-1+2(t-s)] + e^{-(t-s)}\right\} g(s)\,ds. \end{aligned}$$

PROBLEMS

In each of Problems 1 through 6 use the method of variation of parameters to determine the general solution of the given differential equation.

1. $y''' + y' = \tan t, \quad 0 < t < \pi$
2. $y''' - y' = t$
3. $y''' - 2y'' - y' + 2y = e^{4t}$
4. $y''' + y' = \sec t, \quad -\pi/2 < t < \pi/2$
5. $y''' - y'' + y' - y = e^{-t}\sin t$
6. $y^{\text{iv}} + 2y'' + y = \sin t$

In each of Problems 7 and 8 find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

7. $y''' - y'' + y' - y = \sec t, \quad -\pi/2 < t < \pi/2$
8. $y''' - y' = \csc t, \quad 0 < t < \pi$

In each of Problems 9 through 12 find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + y' = \sec t, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -2$
10. $y^{\text{iv}} + 2y'' + y = \sin t, \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 1$
11. $y''' - y'' + y' - y = \sec t, \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = 1$
12. $y''' - y' = \csc t, \quad y(\pi/2) = 2, \quad y'(\pi/2) = 1, \quad y''(\pi/2) = -1$
13. Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

14. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

15. Find a formula involving integrals for a particular solution of the differential equation

$$y^{\text{iv}} - y = g(t).$$

Hint: The functions $\sin t$, $\cos t$, $\sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

16. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

17. Find a formula involving integrals for a particular solution of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = g(x), \qquad x > 0.$$

Hint: Verify that x, x^2, and x^3 are solutions of the homogeneous equation.

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CHAPTER
5

Series Solutions of Second Order Linear Equations

Finding the general solution of a linear differential equation rests on determining a fundamental set of solutions of the homogeneous equation. So far, we have given a systematic procedure for constructing fundamental solutions only if the equation has constant coefficients. To deal with the much larger class of equations having variable coefficients it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: We assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

5.1 Review of Power Series

In this chapter we discuss the use of power series to construct fundamental sets of solutions of second order linear differential equations whose coefficients are functions of the independent variable. We begin by summarizing very briefly the pertinent results about power series that we need. Readers who are familiar with power series may go on to Section 5.2. Those who need more details than are presented here should consult a book on calculus.

1. A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and not for others.

2. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge absolutely at a point x if the series

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

converges. It can be shown that if the series converges absolutely, then the series also converges; however, the converse is not necessarily true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test. If $a_n \neq 0$, and if for a fixed value of x

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L|x-x_0|,$$

then the power series converges absolutely at that value of x if $|x-x_0| < 1/L$, and diverges if $|x-x_0| > 1/L$. If $|x-x_0| = 1/L$, the test is inconclusive.

EXAMPLE 1

For which values of x does the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(x-2)^n$$

converge?

To test for convergence we use the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n} \right| = |x-2| \lim_{n\to\infty} \frac{n+1}{n} = |x-2|.$$

According to statement 3 the series converges absolutely for $|x-2| < 1$, or $1 < x < 3$, and diverges for $|x-2| > 1$. The values of x corresponding to $|x-2| = 1$ are $x = 1$ and $x = 3$. The series diverges for each of these values of x since the nth term of the series does not approach zero as $n \to \infty$.

4. If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x-x_0| < |x_1-x_0|$; and if it diverges at $x = x_1$, it diverges for $|x-x_0| > |x_1-x_0|$.

5. There is a nonnegative number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. For a series that converges only at x_0, we define ρ to be zero; for a series that

converges for all x, we say that ρ is infinite. If $\rho > 0$, then the interval $|x - x_0| < \rho$ is called the **interval of convergence**; it is indicated by the hatched lines in Figure 5.1.1. The series may either converge or diverge when $|x - x_0| = \rho$.

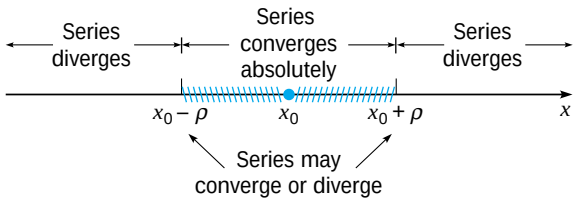


FIGURE 5.1.1 The interval of convergence of a power series.

EXAMPLE 2

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}.$$

We apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(x+1)^n} \right| = \frac{|x+1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+1|}{2}.$$

Thus the series converges absolutely for $|x + 1| < 2$, or $-3 < x < 1$, and diverges for $|x + 1| > 2$. The radius of convergence of the power series is $\rho = 2$. Finally, we check the endpoints of the interval of convergence. At $x = 1$ the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. At $x = -3$ we have

$$\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges, but does not converge absolutely. The series is said to converge conditionally at $x = -3$. To summarize, the given power series converges for $-3 \le x < 1$, and diverges otherwise. It converges absolutely for $-3 < x < 1$, and has a radius of convergence 2.

If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x - x_0| < \rho$, $\rho > 0$, then the following are true for $|x - x_0| < \rho$.

6. The series can be added or subtracted termwise and

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n.$$

7. The series can be formally multiplied and

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$. Further, if $g(x_0) \neq 0$, the series can be formally divided and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n.$$

In most cases the coefficients d_n can be most easily obtained by equating coefficients in the equivalent relation

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x-x_0)^n &= \left[\sum_{n=0}^{\infty} d_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_kb_{n-k}\right)(x-x_0)^n.\end{aligned}$$

Also, in the case of division, the radius of convergence of the resulting power series may be less than ρ.

8. The function f is continuous and has derivatives of all orders for $|x-x_0| < \rho$. Further, f', f'', ... can be computed by differentiating the series termwise; that is,

$$\begin{aligned}f'(x) &= a_1 + 2a_2(x-x_0) + \cdots + na_n(x-x_0)^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \\ f''(x) &= 2a_2 + 6a_3(x-x_0) + \cdots + n(n-1)a_n(x-x_0)^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2},\end{aligned}$$

and so forth, and each of the series converges absolutely for $|x-x_0| < \rho$.

9. The value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The series is called the Taylor[1] series for the function f about $x = x_0$.

10. If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ for each x, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$. In particular, if $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for each x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

[1]Brook Taylor (1685–1731) was the leading English mathematician in the generation following Newton. In 1715 he published a general statement of the expansion theorem that is named for him, a result that is fundamental in all branches of analysis. He was also one of the founders of the calculus of finite differences, and was the first to recognize the existence of singular solutions of differential equations.

A function f that has a Taylor series expansion about $x = x_0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be **analytic** at $x = x_0$. According to statements 6 and 7, if f and g are analytic at x_0, then $f \pm g$, $f \cdot g$, and f/g [provided that $g(x_0) \neq 0$] are analytic at $x = x_0$.

Shift of Index of Summation. The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable. Thus it is immaterial which letter is used for the index of summation. For example,

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}.$$

Just as we make changes of the variable of integration in a definite integral, we find it convenient to make changes of summation indices in calculating series solutions of differential equations. We illustrate by several examples how to shift the summation index.

EXAMPLE 3

Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$.

Let $m = n - 2$; then $n = m + 2$ and $n = 2$ corresponds to $m = 0$. Hence

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}. \tag{1}$$

By writing out the first few terms of each of these series, you can verify that they contain precisely the same terms. Finally, in the series on the right side of Eq. (1), we can replace the dummy index m by n, obtaining

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}. \tag{2}$$

In effect, we have shifted the index upward by 2, and compensated by starting to count at a level 2 lower than originally.

EXAMPLE 4

Write the series

$$\sum_{n=2}^{\infty} (n+2)(n+1) a_n (x - x_0)^{n-2} \tag{3}$$

as a series whose generic term involves $(x - x_0)^n$ rather than $(x - x_0)^{n-2}$.

Again, we shift the index by 2 so that n is replaced by $n + 2$ and start counting 2 lower. We obtain

$$\sum_{n=0}^{\infty} (n+4)(n+3) a_{n+2} (x - x_0)^n. \tag{4}$$

You can readily verify that the terms in the series (3) and (4) are exactly the same.

EXAMPLE 5

Write the expression

$$x^2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \tag{5}$$

as a series whose generic term involves x^{r+n}.

First take the x^2 inside the summation, obtaining

$$\sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1}. \tag{6}$$

Next, shift the index down by 1 and start counting 1 higher. Thus

$$\sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1} = \sum_{n=1}^{\infty} (r+n-1) a_{n-1} x^{r+n}. \tag{7}$$

Again, you can easily verify that the two series in Eq. (7) are identical, and that both are exactly the same as the expression (5).

EXAMPLE 6

Assume that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \tag{8}$$

for all x, and determine what this implies about the coefficients a_n.

We want to use statement 10 to equate corresponding coefficients in the two series. In order to do this, we must first rewrite Eq. (8) so that the series display the same power of x in their generic terms. For instance, in the series on the left side of Eq. (8), we can replace n by $n+1$ and start counting 1 lower. Thus Eq. (8) becomes

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n. \tag{9}$$

According to statement 10 we conclude that

$$(n+1) a_{n+1} = a_n, \qquad n = 0, 1, 2, 3, \ldots$$

or

$$a_{n+1} = \frac{a_n}{n+1}, \qquad n = 0, 1, 2, 3, \ldots \tag{10}$$

Hence, choosing successive values of n in Eq. (10), we have

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!},$$

and so forth. In general,

$$a_n = \frac{a_0}{n!}, \qquad n = 1, 2, 3, \ldots. \tag{11}$$

Thus the relation (8) determines all the following coefficients in terms of a_0. Finally, using the coefficients given by Eq. (11), we obtain

$$\sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x,$$

where we have followed the usual convention that $0! = 1$.

PROBLEMS

In each of Problems 1 through 8 determine the radius of convergence of the given power series.

1. $\sum_{n=0}^{\infty}(x-3)^n$
2. $\sum_{n=0}^{\infty}\frac{n}{2^n}x^n$
3. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$
4. $\sum_{n=0}^{\infty}2^n x^n$
5. $\sum_{n=1}^{\infty}\frac{(2x+1)^n}{n^2}$
6. $\sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n}$
7. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2(x+2)^n}{3^n}$
8. $\sum_{n=1}^{\infty}\frac{n!x^n}{n^n}$

In each of Problems 9 through 16 determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

9. $\sin x, \quad x_0 = 0$
10. $e^x, \quad x_0 = 0$
11. $x, \quad x_0 = 1$
12. $x^2, \quad x_0 = -1$
13. $\ln x, \quad x_0 = 1$
14. $\frac{1}{1+x}, \quad x_0 = 0$
15. $\frac{1}{1-x}, \quad x_0 = 0$
16. $\frac{1}{1-x}, \quad x_0 = 2$
17. Given that $y = \sum_{n=0}^{\infty} nx^n$, compute y' and y'' and write out the first four terms of each series as well as the coefficient of x^n in the general term.
18. Given that $y = \sum_{n=0}^{\infty} a_n x^n$, compute y' and y'' and write out the first four terms of each series as well as the coefficient of x^n in the general term. Show that if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1. Show that $a_{n+2} = a_n/(n+2)(n+1)$, $n = 0, 1, 2, 3, \ldots$.

In each of Problems 19 and 20 verify the given equation.

19. $\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$
20. $\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1})x^k$

In each of Problems 21 through 27 rewrite the given expression as a sum whose generic term involves x^n.

21. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
22. $\sum_{n=0}^{\infty} a_n x^{n+2}$
23. $x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$
24. $(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
25. $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$
26. $\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n$
27. $x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

28. Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

5.2 Series Solutions near an Ordinary Point, Part I



In Chapter 3 we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by x. It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{1}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

A wide class of problems in mathematical physics leads to equations of the form (1) having polynomial coefficients; for example, the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions P, Q, and R are polynomials. However, as we will see, the method of solution is also applicable when P, Q, and R are general analytic functions.

For the present, then, suppose that P, Q, and R are polynomials, and that they have no common factors. Suppose also that we wish to solve Eq. (1) in the neighborhood of a point x_0. The solution of Eq. (1) in an interval containing x_0 is closely associated with the behavior of P in that interval.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follows that there is an interval about x_0 in which $P(x)$ is never zero. In that interval we can divide Eq. (1) by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions. Hence, according to the existence and uniqueness Theorem 3.2.1, there exists in that interval a unique solution of Eq. (1) that also satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$ for arbitrary values of y_0 and y_0'. In this and the following section we discuss the solution of Eq. (1) in the neighborhood of an ordinary point.

On the other hand, if $P(x_0) = 0$, then x_0 is called a **singular point** of Eq. (1). In this case at least one of $Q(x_0)$ and $R(x_0)$ is not zero. Consequently, at least one of the

coefficients p and q in Eq. (2) becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this case. Sections 5.4 through 5.8 deal with finding solutions of Eq. (1) in the neighborhood of a singular point.

We now take up the problem of solving Eq. (1) in the neighborhood of an ordinary point x_0. We look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{3}$$

and assume that the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$. While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate them numerically or to plot their graphs.

The most practical way to determine the coefficients a_n is to substitute the series (3) and its derivatives for y, y', and y'' in Eq. (1). The following examples illustrate this process. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence. The differential equations in these examples are also of considerable importance in their own right.

EXAMPLE 1

Find a series solution of the equation

$$y'' + y = 0, \qquad -\infty < x < \infty. \tag{4}$$

As we know, two linearly independent solutions of this equation are $\sin x$ and $\cos x$, so series methods are not needed to solve this equation. However, this example illustrates the use of power series in a relatively simple case. For Eq. (4), $P(x) = 1$, $Q(x) = 0$, and $R(x) = 1$; hence every point is an ordinary point.

We look for a solution in the form of a power series about $x_0 = 0$,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n, \tag{5}$$

and assume that the series converges in some interval $|x| < \rho$.

Differentiating term by term yields

$$y' = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}, \tag{6}$$

$$y'' = 2a_2 + \cdots + n(n-1)a_nx^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}. \tag{7}$$

Substituting the series (5) and (7) for y and y'' in Eq. (4) gives

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n = 0.$$

To combine the two series we need to rewrite at least one of them so that both series display the same generic term. Thus, in the first sum, we shift the index of summation by replacing n by $n+2$ and starting the sum at 0 rather than 2. We obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

For this equation to be satisfied for all x, the coefficient of each power of x must be zero; hence, we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0, \qquad n = 0, 1, 2, 3, \ldots. \tag{8}$$

Equation (8) is referred to as a **recurrence relation**. The successive coefficients can be evaluated one by one by writing the recurrence relation first for $n = 0$, then for $n = 1$, and so forth. In this example Eq. (8) relates each coefficient to the second one before it. Thus the even-numbered coefficients $(a_0, a_2, a_4, \ldots)$ and the odd-numbered ones $(a_1, a_3, a_5, \ldots)$ are determined separately. For the even-numbered coefficients we have

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}, \qquad a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4!}, \qquad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \ldots.$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!}a_0, \qquad k = 1, 2, 3, \ldots. \tag{9}$$

We can prove Eq. (9) by mathematical induction. First observe that it is true for $k = 1$. Next, assume that it is true for an arbitrary value of k and consider the case $k+1$. We have

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} = -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!}a_0 = \frac{(-1)^{k+1}}{(2k+2)!}a_0.$$

Hence Eq. (9) is also true for $k+1$ and consequently it is true for all positive integers k.

Similarly, for the odd-numbered coefficients

$$a_3 = -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!}, \qquad a_5 = -\frac{a_3}{5\cdot 4} = +\frac{a_1}{5!}, \qquad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7!}, \ldots,$$

and in general, if $n = 2k+1$, then[2]

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1, \qquad k = 1, 2, 3, \ldots. \tag{10}$$

Substituting these coefficients into Eq. (5), we have

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 +$$

[2]The result given in Eq. (10) and other similar formulas in this chapter can be proved by an induction argument resembling the one just given for Eq. (9). We assume that the results are plausible and omit the inductive argument hereafter.

$$+\cdots+\frac{(-1)^n a_0}{(2n)!}x^{2n}+\frac{(-1)^n a_1}{(2n+1)!}x^{2n+1}+\cdots$$

$$=a_0\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+\cdots\right]$$

$$+a_1\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{(-1)^n}{(2n+1)!}x^{2n+1}+\cdots\right]$$

$$=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}. \tag{11}$$

Now that we have formally obtained two series solutions of Eq. (4), we can test them for convergence. Using the ratio test, it is easy to show that each of the series in Eq. (11) converges for all x, and this justifies retroactively all the steps used in obtaining the solutions. Indeed, we recognize that the first series in Eq. (11) is exactly the Taylor series for $\cos x$ about $x = 0$ and the second is the Taylor series for $\sin x$ about $x = 0$. Thus, as expected, we obtain the solution $y = a_0 \cos x + a_1 \sin x$.

Notice that no conditions are imposed on a_0 and a_1; hence they are arbitrary. From Eqs. (5) and (6) we see that y and y' evaluated at $x = 0$ are a_0 and a_1, respectively. Since the initial conditions $y(0)$ and $y'(0)$ can be chosen arbitrarily, it follows that a_0 and a_1 should be arbitrary until specific initial conditions are stated.

Figures 5.2.1 and 5.2.2 show how the partial sums of the series in Eq. (11) approximate $\cos x$ and $\sin x$. As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each x in this interval the accuracy of the approximation improves. However, you should always remember that a truncated power series provides only a local approximation of the solution in a neighborhood of the initial point $x = 0$; it cannot adequately represent the solution for large $|x|$.

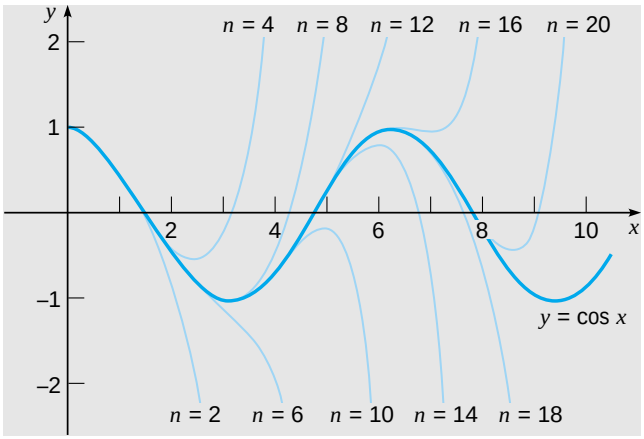


FIGURE 5.2.1 Polynomial approximations to $\cos x$. The value of n is the degree of the approximating polynomial.

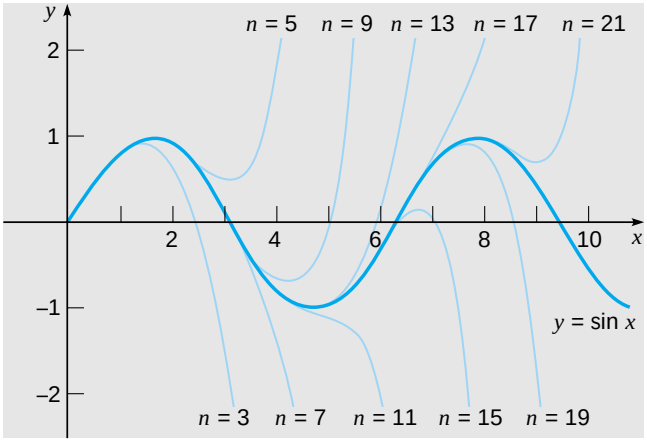


FIGURE 5.2.2 Polynomial approximations to $\sin x$. The value of n is the degree of the approximating polynomial.

In Example 1 we knew from the start that $\sin x$ and $\cos x$ form a fundamental set of solutions of Eq. (4). However, if we had not known this and had simply solved Eq. (4) using series methods, we would still have obtained the solution (11). In recognition of the fact that the differential equation (4) often occurs in applications we might decide to give the two solutions of Eq. (11) special names; perhaps,

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Then we might ask what properties these functions have. For instance, it follows at once from the series expansions that $C(0) = 1$, $S(0) = 0$, $C(-x) = C(x)$, and $S(-x) = -S(x)$. It is also easy to show that

$$S'(x) = C(x), \qquad C'(x) = -S(x).$$

Moreover, by calculating with the infinite series[3] we can show that the functions $C(x)$ and $S(x)$ have all the usual analytical and algebraic properties of the cosine and sine functions, respectively.

Although you probably first saw the sine and cosine functions defined in a more elementary manner in terms of right triangles, it is interesting that these functions can be defined as solutions of a certain simple second order linear differential equation. To be precise, the function $\sin x$ can be defined as the unique solution of the initial value problem $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$; similarly, $\cos x$ can be defined as the unique solution of the initial value problem $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Many other functions that are important in mathematical physics are also defined as solutions of certain initial value problems. For most of these functions there is no simpler or more elementary way to approach them.

[3]Such an analysis is given in Section 24 of K. Knopp, *Theory and Applications of Infinite Series* (New York: Hafner, 1951).

EXAMPLE 2

Find a series solution in powers of x of Airy's[4] equation

$$y'' - xy = 0, \qquad -\infty < x < \infty. \tag{12}$$

For this equation $P(x) = 1$, $Q(x) = 0$, and $R(x) = -x$; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n, \tag{13}$$

and that the series converges in some interval $|x| < \rho$. The series for y'' is given by Eq. (7); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{14}$$

Substituting the series (13) and (14) for y and y'' in Eq. (12), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{15}$$

Next, we shift the index of summation in the series on the right side of this equation by replacing n by $n-1$ and starting the summation at 1 rather than zero. Thus we have

$$2 \cdot 1a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Again, for this equation to be satisfied for all x it is necessary that the coefficients of like powers of x be equal; hence $a_2 = 0$, and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1} \qquad \text{for} \qquad n = 1, 2, 3, \ldots. \tag{16}$$

Since a_{n+2} is given in terms of a_{n-1}, the a's are determined in steps of three. Thus a_0 determines a_3, which in turn determines $a_6, \ldots$; a_1 determines a_4, which in turn determines $a_7, \ldots$; and a_2 determines a_5, which in turn determines $a_8, \ldots$. Since $a_2 = 0$, we immediately conclude that $a_5 = a_8 = a_{11} = \cdots = 0$.

For the sequence $a_0, a_3, a_6, a_9, \ldots$ we set $n = 1, 4, 7, 10, \ldots$ in the recurrence relation:

$$a_3 = \frac{a_0}{2 \cdot 3}, \qquad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \qquad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \ldots.$$

For this sequence of coefficients it is convenient to write a formula for a_{3n}, $n = 1, 2, 3, \ldots$. The preceding results suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)(3n-1)(3n)}, \qquad n = 1, 2, \ldots.$$

For the sequence $a_1, a_4, a_7, a_{10}, \ldots$, we set $n = 2, 5, 8, 11, \ldots$ in the recurrence relation:

$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \ldots.$$

[4]Sir George Airy (1801–1892), an English astronomer and mathematician, was director of the Greenwich Observatory from 1835 to 1881. One reason Airy's equation is of interest is that for x negative the solutions are oscillatory, similar to trigonometric functions, and for x positive they are monotonic, similar to hyperbolic functions. Can you explain why it is reasonable to expect such behavior?

In the same way as before we find that

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3)(3n-2)(3n)(3n+1)}, \qquad n = 1, 2, 3, \ldots.$$

The general solution of Airy's equation is

$$\begin{aligned} y &= a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right] \\ &\quad + a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right] \\ &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} \right]. \end{aligned} \tag{17}$$

Having obtained these two series solutions, we can now investigate their convergence. Because of the rapid growth of the denominators of the terms in the series (17), we might expect these series to have a large radius of convergence. Indeed, it is easy to use the ratio test to show that both these series converge for all x; see Problem 20.

Assuming for the moment that the series do converge for all x, let y_1 and y_2 denote the functions defined by the expressions in the first and second sets of brackets, respectively, in Eq. (17). Then, by first choosing $a_0 = 1$, $a_1 = 0$ and then $a_0 = 0$, $a_1 = 1$, it follows that y_1 and y_2 are individually solutions of Eq. (12). Notice that y_1 satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ and that y_2 satisfies the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$. Thus $W(y_1, y_2)(0) = 1 \neq 0$, and consequently y_1 and y_2 are linearly independent. Hence the general solution of Airy's equation is

$$y = a_0 y_1(x) + a_1 y_2(x), \qquad -\infty < x < \infty.$$

In Figures 5.2.3 and 5.2.4, respectively, we show the graphs of the solutions y_1 and y_2 of Airy's equation, as well as graphs of several partial sums of the two series in Eq. (17). Again, the partial sums provide local approximations to the solutions in a neighborhood of the origin. While the quality of the approximation improves as the number of terms increases, no polynomial can adequately represent y_1 and y_2 for large $|x|$. A practical way to estimate the interval in which a given partial sum is reasonably accurate is to compare the graphs of that partial sum and the next one, obtained by including one more term. As soon as the graphs begin to separate noticeably, one can be confident that the original partial sum is no longer accurate. For example, in Figure 5.2.3 the graphs for $n = 24$ and $n = 27$ begin to separate at about $x = -9/2$. Thus, beyond this point, the partial sum of degree 24 is worthless as an approximation to the solution.

Observe that both y_1 and y_2 are monotone for $x > 0$ and oscillatory for $x < 0$. One can also see from the figures that the oscillations are not uniform, but decay in amplitude and increase in frequency as the distance from the origin increases. In contrast to Example 1, the solutions y_1 and y_2 of Airy's equation are not elementary functions that you have already encountered in calculus. However, because of their importance in some physical applications, these functions have been extensively studied and their properties are well known to applied mathematicians and scientists.

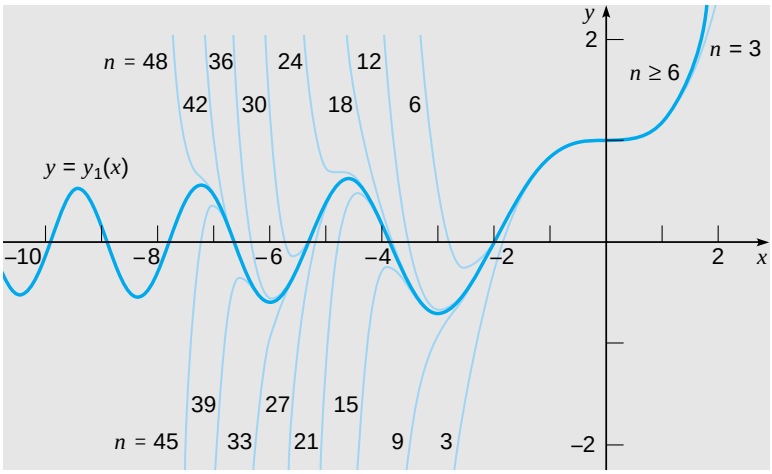


FIGURE 5.2.3 Polynomial approximations to the solution $y_1(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

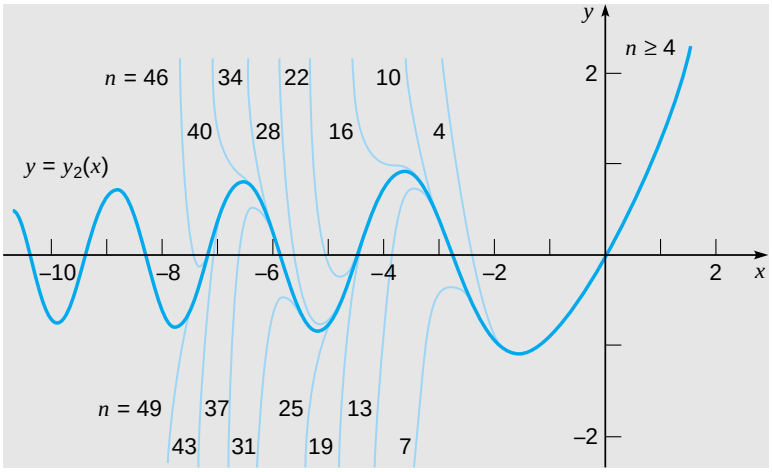


FIGURE 5.2.4 Polynomial approximations to the solution $y_2(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

EXAMPLE 3

Find a solution of Airy's equation in powers of $x - 1$.

The point $x = 1$ is an ordinary point of Eq. (12), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

where we assume that the series converges in some interval $|x - 1| < \rho$. Then

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting for y and y'' in Eq. (12), we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty}a_n(x-1)^n. \tag{18}$$

Now to equate the coefficients of like powers of $(x-1)$ we must express x, the coefficient of y in Eq. (12), in powers of $x-1$; that is, we write $x = 1 + (x-1)$. Note that this is precisely the Taylor series for x about $x = 1$. Then Eq. (18) takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n &= [1+(x-1)]\sum_{n=0}^{\infty}a_n(x-1)^n \\ &= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1}.\end{aligned}$$

Shifting the index of summation in the second series on the right gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n.$$

Equating coefficients of like powers of $x-1$, we obtain

$$\begin{aligned}2a_2 &= a_0,\\ (3\cdot 2)a_3 &= a_1 + a_0,\\ (4\cdot 3)a_4 &= a_2 + a_1,\\ (5\cdot 4)a_5 &= a_3 + a_2,\\ &\vdots\end{aligned}$$

The general recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \qquad \text{for} \qquad n \geq 1. \tag{19}$$

Solving for the first few coefficients a_n in terms of a_0 and a_1, we find that

$$a_2 = \frac{a_0}{2},\ a_3 = \frac{a_1}{6} + \frac{a_0}{6},\ a_4 = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12},\ a_5 = \frac{a_3}{20} + \frac{a_2}{20} = \frac{a_0}{30} + \frac{a_1}{120}.$$

Hence

$$\begin{aligned}y = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\right]\\ + a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\right].\end{aligned} \tag{20}$$

In general, when the recurrence relation has more than two terms, as in Eq. (19), the determination of a formula for a_n in terms a_0 and a_1 will be fairly complicated, if not impossible. In this example such a formula is not readily apparent. Lacking such a formula, we cannot test the two series in Eq. (20) for convergence by direct methods

such as the ratio test. However, even without knowing the formula for a_n we shall see in Section 5.3 that it is possible to establish that the series in Eq. (20) converge for all x, and further define functions y_3 and y_4 that are linearly independent solutions of the Airy equation (12). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for $-\infty < x < \infty$.

It is worth emphasizing, as we saw in Example 3, that if we look for a solution of Eq. (1) of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then the coefficients $P(x)$, $Q(x)$, and $R(x)$ in Eq. (1) must also be expressed in powers of $x - x_0$. Alternatively, we can make the change of variable $x - x_0 = t$, obtaining a new differential equation for y as a function of t, and then look for solutions of this new equation of the form $\sum_{n=0}^{\infty} a_n t^n$. When we have finished the calculations, we replace t by $x - x_0$ (see Problem 19).

In Examples 2 and 3 we have found two sets of solutions of Airy's equation. The functions y_1 and y_2 defined by the series in Eq. (17) are linearly independent solutions of Eq. (12) for all x, and this is also true for the functions y_3 and y_4 defined by the series in Eq. (20). According to the general theory of second order linear equations each of the first two functions can be expressed as a linear combination of the latter two functions and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient a_n in terms of a_0 and a_1. What is essential is that we can determine *as many coefficients as we want.* Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package one can also produce plots such as those shown in the figures in this section.

PROBLEMS

In each of Problems 1 through 14 solve the given differential equation by means of a power series about the given point x_0. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' - xy' - y = 0, \quad x_0 = 0$
3. $y'' - xy' - y = 0, \quad x_0 = 1$
4. $y'' + k^2x^2y = 0, \quad x_0 = 0,\ k$ a constant
5. $(1 - x)y'' + y = 0, \quad x_0 = 0$
6. $(2 + x^2)y'' - xy' + 4y = 0, \quad x_0 = 0$
7. $y'' + xy' + 2y = 0, \quad x_0 = 0$
8. $xy'' + y' + xy = 0, \quad x_0 = 1$
9. $(1 + x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$
10. $(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$
11. $(3 - x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$
12. $(1 - x)y'' + xy' - y = 0, \quad x_0 = 0$
13. $2y'' + xy' + 3y = 0, \quad x_0 = 0$
14. $2y'' + (x + 1)y' + 3y = 0, \quad x_0 = 2$

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.
(b) Plot the four-term and the five-term approximations to the solution on the same axes.
(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

15. $y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = 1;$ see Problem 2
16. $(2 + x^2)y'' - xy' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 3;$ see Problem 6
17. $y'' + xy' + 2y = 0, \quad y(0) = 4, \quad y'(0) = -1;$ see Problem 7
18. $(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, \quad y'(0) = 2;$ see Problem 12
19. By making the change of variable $x - 1 = t$ and assuming that y is a power series in t, find two linearly independent series solutions of

$$y'' + (x - 1)^2 y' + (x^2 - 1)y = 0$$

in powers of $x - 1$. Show that you obtain the same result directly by assuming that y is a Taylor series in powers of $x - 1$ and also expressing the coefficient $x^2 - 1$ in powers of $x - 1$.
20. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converge for all x; see Eq. (17) of the text.
21. **The Hermite Equation.** The equation

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.
(a) Find the first four terms in each of two linearly independent solutions about $x = 0$.
(b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0$, 2, 4, 6, 8, and 10. Note that each polynomial is determined only up to a multiplicative constant.
(c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), \ldots, H_5(x)$.
22. Consider the initial value problem $y' = \sqrt{1 - y^2}$, $y(0) = 0$.
(a) Show that $y = \sin x$ is the solution of this initial value problem.
(b) Look for a solution of the initial value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

In each of Problems 23 through 28 plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

23. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 2
24. $(2 + x^2)y'' - xy' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 6
25. $y'' + xy' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 7
26. $(4 - x^2)y'' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 10
27. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 4
28. $(1 - x)y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

[5]Charles Hermite (1822–1901) was an influential French analyst and algebraist. He introduced the Hermite functions in 1864, and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

5.3 Series Solutions near an Ordinary Point, Part II



In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that Eq. (1) does have a solution $y = \phi(x)$, and that ϕ has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{2}$$

which converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the a_n can be determined by directly substituting the series (2) for y in Eq. (1).

Let us now consider how we might justify the statement that if x_0 is an ordinary point of Eq. (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of Eq. (1) of the form (2). By differentiating Eq. (2) m times and setting x equal to x_0 it follows that

$$m!a_m = \phi^{(m)}(x_0).$$

Hence, to compute a_n in the series (2), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$ from the differential equation (1).

Suppose that $y = \phi(x)$ is a solution of Eq. (1) satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$. Then $a_0 = y_0$ and $a_1 = y_0'$. If we are solely interested in finding a solution of Eq. (1) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \ldots$, we turn to Eq. (1). Since ϕ is a solution of Eq. (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which P is nonvanishing we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{3}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Setting x equal to x_0 in Eq. (3) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0).$$

Hence a_2 is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{4}$$

To determine a_3 we differentiate Eq. (3) and then set x equal to x_0, obtaining

$$\begin{aligned} 3!a_3 = \phi'''(x_0) &= -[p\phi'' + (p' + q)\phi' + q'\phi]\Big|_{x=x_0} \\ &= -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0. \end{aligned} \tag{5}$$

Substituting for a_2 from Eq. (4) gives a_3 in terms of a_1 and a_0. Since P, Q, and R are polynomials and $P(x_0) \neq 0$, all of the derivatives of p and q exist at x_0. Hence, we can

continue to differentiate Eq. (3) indefinitely, determining after each differentiation the successive coefficients $a_4, a_5, \ldots$ by setting x equal to x_0.

Notice that the important property that we used in determining the a_n was that we could compute infinitely many derivatives of the functions p and q. It might seem reasonable to relax our assumption that the functions p and q are ratios of polynomials, and simply require that they be infinitely differentiable in the neighborhood of x_0. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are *analytic* at x_0; that is, they have Taylor series expansions that converge to them in some interval about the point x_0:

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \tag{6}$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n(x - x_0)^n. \tag{7}$$

With this idea in mind we can generalize the definitions of an ordinary point and a singular point of Eq. (1) as follows: If the functions $p = Q/P$ and $q = R/P$ are analytic at x_0, then the point x_0 is said to be an **ordinary point** of the differential equation (1); otherwise, it is a **singular point**.

Now let us turn to the question of the interval of convergence of the series solution. One possibility is actually to compute the series solution for each problem and then to apply one of the tests for convergence of an infinite series to determine its radius of convergence. However, the question can be answered at once for a wide class of problems by the following theorem.

Theorem 5.3.1 If x_0 is an ordinary point of the differential equation (1),

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of Eq. (1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x), \tag{8}$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are linearly independent series solutions that are analytic at x_0. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Notice from the form of the series solution that $y_1(x) = 1 + b_2(x - x_0)^2 + \cdots$ and $y_2(x) = (x - x_0) + c_2(x - x_0)^2 + \cdots$. Hence y_1 is the solution satisfying the initial conditions $y_1(x_0) = 1, y_1'(x_0) = 0$, and y_2 is the solution satisfying the initial conditions $y_2(x_0) = 0$, $y_2'(x_0) = 1$. Also note that while the calculation of the coefficients by successively differentiating the differential equation is excellent in theory, it is usually not a practical computational procedure. Rather, one should substitute the series (2) for

y in the differential equation (1) and determine the coefficients so that the differential equation is satisfied, as in the examples in the preceding section.

We will not prove this theorem, which in a slightly more general form is due to Fuchs.[6] What is important for our purposes is that there is a series solution of the form (2), and that the radius of convergence of the series solution cannot be less than the smaller of the radii of convergence of the series for p and q; hence we need only determine these.

This can be done in one of two ways. Again, one possibility is simply to compute the power series for p and q, and then to determine the radii of convergence by using one of the convergence tests for infinite series. However, there is an easier way when P, Q, and R are polynomials. It is shown in the theory of functions of a complex variable that the ratio of two polynomials, say Q/P, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$. Further, if we assume that any factors common to Q and P have been canceled, then the radius of convergence of the power series for Q/P about the point x_0 is precisely the distance from x_0 to the nearest zero of P. In determining this distance we must remember that $P(x) = 0$ may have complex roots, and these must also be considered.

EXAMPLE 1

What is the radius of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x = 0$?

One way to proceed is to find the Taylor series in question, namely,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

Then it can be verified by the ratio test that $\rho = 1$. Another approach is to note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance from 0 to i or to $-i$ in the complex plane is 1, the radius of convergence of the power series about $x = 0$ is 1.

EXAMPLE 2

What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? About $x = 1$?

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions $x = 1 \pm i$. The distance from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ in the complex plane is $\sqrt{2}$; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$.

The distance from $x = 1$ to either $x = 1 + i$ or $x = 1 - i$ is 1; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n (x-1)^n$ about $x = 1$ is 1.

[6] Immanuel Lazarus Fuchs (1833–1902) was a student and later a professor at the University of Berlin. He proved the result of Theorem 5.3.1 in 1866. His most important research was on singular points of linear differential equations. He recognized the significance of regular singular points (Section 5.4), and equations whose only singularities, including the point at infinity, are regular singular points are known as Fuchsian equations.

According to Theorem 5.3.1 the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of x and $x - 1$, respectively, since in each problem $P(x) = 1$ and hence is never zero.

A series solution may converge for a wider range of x than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

EXAMPLE 3

Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of P, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ at least, and possibly for larger values of x. Indeed, it can be shown that if α is a positive integer, one of the series solutions terminates after a finite number of terms and hence converges not just for $|x| < 1$ but for all x. For example, if $\alpha = 1$, the polynomial solution is $y = x$. See Problems 22 through 29 at the end of this section for a more complete discussion of the Legendre equation.

EXAMPLE 4

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0 \tag{9}$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

Again P, Q, and R are polynomials, and P has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1 + \frac{1}{4}} = \sqrt{5}/2$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n (x + \frac{1}{2})^n$ converges at least for $|x + \frac{1}{2}| < \sqrt{5}/2$.

An interesting observation that we can make about Eq. (9) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all x, we know from Theorem 3.2.1 that there exists a unique solution of the initial value problem on $-\infty < x < \infty$. On the other hand, Theorem 5.3.1 only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0$, $a_1 = y_0'$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all x.

EXAMPLE 5

Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x. Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x. Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x.

PROBLEMS

In each of Problems 1 through 4 determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{\text{iv}}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial value problem.

1. $y'' + xy' + y = 0; \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + (\sin x)y' + (\cos x)y = 0; \quad y(0) = 0, \quad y'(0) = 1$
3. $x^2y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0$
4. $y'' + x^2y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$

In each of Problems 5 through 8 determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

5. $y'' + 4y' + 6xy = 0; \quad x_0 = 0, \quad x_0 = 4$
6. $(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4, \quad x_0 = 0$
7. $(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$
8. $xy'' + y = 0; \quad x_0 = 1$
9. Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 14 of Section 5.2.
10. **The Chebyshev Equation.** The Chebyshev[7] differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.
(a) Determine two linearly independent solutions in powers of x for $|x| < 1$.
(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems requiring a polynomial approximation to a function defined on $-1 \le x \le 1$.
(c) Find a polynomial solution for each of the cases $\alpha = n = 0$, 1, 2, and 3.

For each of the differential equations in Problems 11 through 14 find the first four nonzero terms in each of two linearly independent power series solutions about the origin. What do you expect the radius of convergence to be for each solution?

11. $y'' + (\sin x)y = 0$
12. $e^x y'' + xy = 0$
13. $(\cos x)y'' + xy' - 2y = 0$
14. $e^{-x}y'' + \ln(1 + x)y' - xy = 0$
15. Suppose that you are told that x and x^2 are solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point?
Hint: Use Theorem 3.2.1, and note the values of x and x^2 at $x = 0$.

[7] Pafnuty L. Chebyshev (1821–1894), professor at Petersburg University for 35 years and the most influential nineteenth century Russian mathematician, founded the so-called "Petersburg school," which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 16 through 21 solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

16. $y' - y = 0$
17. $y' - xy = 0$
18. $y' = e^{x^2}y$, three terms only
19. $(1 - x)y' = y$
20. $y' - y = x^2$
21. $y' + xy = 1 + x$

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that it is necessary to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1 + \gamma)$ where $\gamma \ge 0$ leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

22. Show that two linearly independent solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \times \frac{\alpha(\alpha - 2)(\alpha - 4)\cdots(\alpha - 2m + 2)(\alpha + 1)(\alpha + 3)\cdots(\alpha + 2m - 1)}{(2m)!}x^{2m},$$

$$y_2(x) = x + \sum_{m=1}^{\infty}(-1)^m \times \frac{(\alpha - 1)(\alpha - 3)\cdots(\alpha - 2m + 1)(\alpha + 2)(\alpha + 4)\cdots(\alpha + 2m)}{(2m + 1)!}x^{2m+1}.$$

23. Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0$, 2, and 4. Show that, if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1$, 3, and 5.
24. The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.
    (a) Using the results of Problem 23, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.
    (b) Plot the graphs of $P_0(x), \ldots, P_5(x)$ for $-1 \le x \le 1$.
    (c) Find the zeros of $P_0(x), \ldots, P_5(x)$.
25. It can be shown that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n - 2k)!}{k!(n - k)!(n - 2k)!}x^{n-2k},$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$. By observing the form of $P_n(x)$ for n even and n odd, show that $P_n(-1) = (-1)^n$.

26. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \qquad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

27. Show that for $n = 0, 1, 2, 3$ the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

This formula, known as Rodrigues' (1794–1851) formula, is true for all positive integers n.

28. Show that the Legendre equation can also be written as

$$[(1 - x^2)y']' = -\alpha(\alpha + 1)y.$$

Then it follows that $[(1 - x^2)P_n'(x)]' = -n(n+1)P_n(x)$ and $[(1 - x^2)P_m'(x)]' = -m(m+1)P_m(x)$. By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, and then integrating by parts, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad \text{if} \qquad n \neq m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the preceding integral is $2/(2n+1)$.

29. Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, P_2, \ldots, P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of Problem 28, show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x)P_k(x)\,dx.$$

5.4 Regular Singular Points



In this section we will consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a singular point x_0. Recall that if the functions P, Q, and R are polynomials having no common factors, the singular points of Eq. (1) are the points for which $P(x) = 0$.

EXAMPLE 1

Determine the singular points and ordinary points of the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{2}$$

The point $x = 0$ is a singular point since $P(x) = x^2$ is zero there. All other points are ordinary points of Eq. (2).

EXAMPLE 2

Determine the singular points and ordinary points of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{3}$$

where α is a constant.

The singular points are the zeros of $P(x) = 1 - x^2$, namely, the points $x = \pm 1$. All other points are ordinary points.

Unfortunately, if we attempt to use the methods of the preceding two sections to solve Eq. (1) in the neighborhood of a singular point x_0, we find that these methods fail. This is because the solution of Eq. (1) is often not analytic at x_0, and consequently cannot be represented by a Taylor series in powers of $x - x_0$. Instead, we must use a more general series expansion.

Since the singular points of a differential equation are usually few in number, we might ask whether we can simply ignore them, especially since we already know how to construct solutions about ordinary points. However, this is not feasible because the singular points determine the principal features of the solution to a much larger extent than one might at first suspect. In the neighborhood of a singular point the solution often becomes large in magnitude or experiences rapid changes in magnitude. Thus the behavior of a physical system modeled by a differential equation frequently is most interesting in the neighborhood of a singular point. Often geometrical singularities in a physical problem, such as corners or sharp edges, lead to singular points in the corresponding differential equation. Thus, while at first we might want to avoid the few points where a differential equation is singular, it is precisely at these points that it is necessary to study the solution most carefully.

As an alternative to analytical methods, one can consider the use of numerical methods, which are discussed in Chapter 8. However, these methods are ill-suited for the study of solutions near a singular point. Thus, even if one adopts a numerical approach, it is advantageous to combine it with the analytical methods of this chapter in order to examine the behavior of solutions near singular points.

Without any additional information about the behavior of Q/P and R/P in the neighborhood of the singular point, it is impossible to describe the behavior of the solutions of Eq. (1) near $x = x_0$. It may be that there are two linearly independent solutions of Eq. (1) that remain bounded as $x \to x_0$, or there may be only one with the other becoming unbounded as $x \to x_0$, or they may both become unbounded as $x \to x_0$. To illustrate these possibilities consider the following examples.

EXAMPLE 3

The differential equation

$$x^2y'' - 2y = 0 \tag{4}$$

has a singular point at $x = 0$. It can be easily verified by direct substitution that $y_1(x) = x^2$ and $y_2(x) = 1/x$ are linearly independent solutions of Eq. (4) for $x > 0$ or

$x < 0$. Thus in any interval not containing the origin the general solution of Eq. (4) is $y = c_1x^2 + c_2x^{-1}$. The only solution of Eq. (4) that is bounded as $x \to 0$ is $y = c_1x^2$. Indeed, this solution is analytic at the origin even though if Eq. (4) is put in the standard form, $y'' - (2/x^2)y = 0$, the function $q(x) = -2/x^2$ is not analytic at $x = 0$, and Theorem 5.3.1 is not applicable. On the other hand, notice that the solution $y_2(x) = x^{-1}$ does not have a Taylor series expansion about the origin (is not analytic at $x = 0$); therefore, the method of Section 5.2 would fail in this case.

EXAMPLE 4

The differential equation

$$x^2y'' - 2xy' + 2y = 0 \tag{5}$$

also has a singular point at $x = 0$. It can be verified that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of Eq. (5), and both are analytic at $x = 0$. Nevertheless, it is still not proper to pose an initial value problem with initial conditions at $x = 0$. It is impossible to satisfy arbitrary initial conditions at $x = 0$, since any linear combination of x and x^2 is zero at $x = 0$.

It is also possible to construct a differential equation with a singular point x_0 such that every (nonzero) solution of the differential equation becomes unbounded as $x \to x_0$. Even if Eq. (1) does not have a solution that remains bounded as $x \to x_0$, it is often important to determine how the solutions of Eq. (1) behave as $x \to x_0$; for example, does $y \to \infty$ in the same manner as $(x - x_0)^{-1}$ or $|x - x_0|^{-1/2}$, or in some other manner?

Our goal is to extend the method already developed for solving Eq. (1) near an ordinary point so that it also applies to the neighborhood of a singular point x_0. To do this in a reasonably simple manner, it is necessary to restrict ourselves to cases in which the singularities in the functions Q/P and R/P at $x = x_0$ are not too severe, that is, to what we might call "weak singularities." At this point it is not clear exactly what is an acceptable singularity. However, as we develop the method of solution you will see that the appropriate conditions (see also Section 5.7, Problem 21) to distinguish "weak singularities" are

$$\lim_{x\to x_0} (x - x_0)\frac{Q(x)}{P(x)} \quad \text{is finite} \tag{6}$$

and

$$\lim_{x\to x_0} (x - x_0)^2\frac{R(x)}{P(x)} \quad \text{is finite.} \tag{7}$$

This means that the singularity in Q/P can be no worse than $(x - x_0)^{-1}$ and the singularity in R/P can be no worse than $(x - x_0)^{-2}$. Such a point is called a **regular singular point** of Eq. (1). For more general functions than polynomials, x_0 is a regular singular point of Eq. (1) if it is a singular point, and if both[9]

$$(x - x_0)\frac{Q(x)}{P(x)} \qquad \text{and} \qquad (x - x_0)^2\frac{R(x)}{P(x)} \tag{8}$$

[9]The functions given in Eq. (8) may not be defined at x_0, in which case their values at x_0 are to be assigned as their limits as $x \to x_0$.

have convergent Taylor series about x_0, that is, if the functions in Eq. (8) are analytic at $x = x_0$. Equations (6) and (7) imply that this will be the case when P, Q, and R are polynomials. Any singular point of Eq. (1) that is not a regular singular point is called an **irregular singular point** of Eq. (1).

In the following sections we discuss how to solve Eq. (1) in the neighborhood of a regular singular point. A discussion of the solutions of differential equations in the neighborhood of irregular singular points is more complicated and may be found in more advanced books.

EXAMPLE 5

In Example 2 we observed that the singular points of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

are $x = \pm 1$. Determine whether these singular points are regular or irregular singular points.

We consider the point $x = 1$ first and also observe that on dividing by $1 - x^2$ the coefficients of y' and y are $-2x/(1 - x^2)$ and $\alpha(\alpha + 1)/(1 - x^2)$, respectively. Thus we calculate

$$\lim_{x\to 1}(x - 1)\frac{-2x}{1 - x^2} = \lim_{x\to 1}\frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \lim_{x\to 1}\frac{2x}{1 + x} = 1$$

and

$$\begin{aligned}\lim_{x\to 1}(x - 1)^2\frac{\alpha(\alpha + 1)}{1 - x^2} &= \lim_{x\to 1}\frac{(x - 1)^2\alpha(\alpha + 1)}{(1 - x)(1 + x)} \\ &= \lim_{x\to 1}\frac{(x - 1)(-\alpha)(\alpha + 1)}{1 + x} = 0.\end{aligned}$$

Since these limits are finite, the point $x = 1$ is a regular singular point. It can be shown in a similar manner that $x = -1$ is also a regular singular point.

EXAMPLE 6

Determine the singular points of the differential equation

$$2x(x - 2)^2y'' + 3xy' + (x - 2)y = 0$$

and classify them as regular or irregular.

Dividing the differential equation by $2x(x - 2)^2$, we have

$$y'' + \frac{3}{2(x - 2)^2}y' + \frac{1}{2x(x - 2)}y = 0,$$

so $p(x) = Q(x)/P(x) = 3/2(x - 2)^2$ and $q(x) = R(x)/P(x) = 1/2x(x - 2)$. The singular points are $x = 0$ and $x = 2$. Consider $x = 0$. We have

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{2(x - 2)^2} = 0,$$

$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x(x - 2)} = 0.$$

Since these limits are finite, $x = 0$ is a regular singular point. For $x = 2$ we have

$$\lim_{x\to 2}(x-2)p(x) = \lim_{x\to 2}(x-2)\frac{3}{2(x-2)^2} = \lim_{x\to 2}\frac{3}{2(x-2)},$$

so the limit does not exist; hence $x = 2$ is an irregular singular point.

EXAMPLE 7

Determine the singular points of

$$\left(x-\frac{\pi}{2}\right)^2 y'' + (\cos x)y' + (\sin x)y = 0$$

and classify them as regular or irregular.

The only singular point is $x = \pi/2$. To study it we consider the functions

$$\left(x-\frac{\pi}{2}\right)p(x) = \left(x-\frac{\pi}{2}\right)\frac{Q(x)}{P(x)} = \frac{\cos x}{x-\pi/2}$$

and

$$\left(x-\frac{\pi}{2}\right)^2 q(x) = \left(x-\frac{\pi}{2}\right)^2\frac{R(x)}{P(x)} = \sin x.$$

Starting from the Taylor series for $\cos x$ about $x = \pi/2$, we find that

$$\frac{\cos x}{x-\pi/2} = -1 + \frac{(x-\pi/2)^2}{3!} - \frac{(x-\pi/2)^4}{5!} + \cdots,$$

which converges for all x. Similarly, $\sin x$ is analytic at $x = \pi/2$. Therefore, we conclude that $\pi/2$ is a regular singular point for this equation.

PROBLEMS

In each of Problems 1 through 18 find all singular points of the given equation and determine whether each one is regular or irregular.

1. $xy'' + (1-x)y' + xy = 0$
2. $x^2(1-x)^2y'' + 2xy' + 4y = 0$
3. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$
4. $x^2(1-x^2)y'' + (2/x)y' + 4y = 0$
5. $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$
6. $x^2y'' + xy' + (x^2-\nu^2)y = 0$, Bessel equation
7. $(x+3)y'' - 2xy' + (1-x^2)y = 0$
8. $x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0$
9. $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$
10. $x(3-x)y'' + (x+1)y' - 2y = 0$
11. $(x^2+x-2)y'' + (x+1)y' + 2y = 0$
12. $xy'' + e^xy' + (3\cos x)y = 0$
13. $y'' + (\ln|x|)y' + 3xy = 0$
14. $x^2y'' + 2(e^x-1)y' + (e^{-x}\cos x)y = 0$
15. $x^2y'' - 3(\sin x)y' + (1+x^2)y = 0$
16. $xy'' + y' + (\cot x)y = 0$
17. $(\sin x)y'' + xy' + 4y = 0$
18. $(x\sin x)y'' + 3y' + xy = 0$

In each of Problems 19 and 20 show that the point $x = 0$ is a regular singular point. In each problem try to find solutions of the form $\sum_{n=0}^{\infty} a_n x^n$. Show that there is only one nonzero solution of this form in Problem 19 and that there are no nonzero solutions of this form in Problem 20. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

19. $2xy'' + 3y' + xy = 0$
20. $2x^2y'' + 3xy' - (1+x)y = 0$

21. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point x_0 is finite. In more advanced work in differential equations it is often necessary to discuss the point at infinity. This is done by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$. Show that for the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$ the point at infinity is an ordinary point if

$$\frac{1}{P(1/\xi)}\left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4 P(1/\xi)}$$

have Taylor series expansions about $\xi = 0$. Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both

$$\frac{\xi}{P(1/\xi)}\left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2 P(1/\xi)}$$

do have such expansions.

In each of Problems 22 through 27 use the results of Problem 21 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

22. $y'' + y = 0$
23. $x^2y'' + xy' - 4y = 0$
24. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, Legendre equation
25. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel equation
26. $y'' - 2xy' + \lambda y = 0$, Hermite equation
27. $y'' - xy = 0$, Airy equation

5.5 Euler Equations

The simplest example of a differential equation having a regular singular point is the **Euler equation**, or the equidimensional equation,

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0, \tag{1}$$

where α and β are real constants. It is easy to show that $x = 0$ is a regular singular point of Eq. (1). Because the solution of the Euler equation is typical of the solutions of all differential equations with a regular singular point, it is worthwhile considering this equation in detail before discussing the more general problem.

In any interval not including the origin, Eq. (1) has a general solution of the form $y = c_1y_1(x) + c_2y_2(x)$, where y_1 and y_2 are linearly independent. For convenience we first consider the interval $x > 0$, extending our results later to the interval $x < 0$. First, we note that $(x^r)' = rx^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$. Hence, if we assume that Eq. (1) has a solution of the form

$$y = x^r, \tag{2}$$

then we obtain

$$\begin{aligned} L[x^r] &= x^2(x^r)'' + \alpha x(x^r)' + \beta x^r \\ &= x^r[r(r-1) + \alpha r + \beta]. \end{aligned} \tag{3}$$

If r is a root of the quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0, \tag{4}$$

then $L[x^r]$ is zero, and $y = x^r$ is a solution of Eq. (1). The roots of Eq. (4) are

$$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}, \tag{5}$$

and $F(r) = (r - r_1)(r - r_2)$. As for second order linear equations with constant coefficients, it is necessary to consider separately the cases in which the roots are real and different, real but equal, and complex conjugates. Indeed, the entire discussion in this section is similar to the treatment of second order linear equations with constant coefficients in Chapter 3 with e^{rx} replaced by x^r; see also Problem 23.

Real, Distinct Roots. If $F(r) = 0$ has real roots r_1 and r_2, with $r_1 \neq r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions of Eq. (1). Since $W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2-1}$ is nonvanishing for $r_1 \neq r_2$ and $x > 0$, it follows that the general solution of Eq. (1) is

$$y = c_1 x^{r_1} + c_2 x^{r_2}, \qquad x > 0. \tag{6}$$

Note that if r is not a rational number, then x^r is defined by $x^r = e^{r \ln x}$.

EXAMPLE 1

Solve

$$2x^2y'' + 3xy' - y = 0, \qquad x > 0. \tag{7}$$

Substituting $y = x^r$ in Eq. (7) gives

$$x^r[2r(r-1) + 3r - 1] = x^r(2r^2 + r - 1) = x^r(2r-1)(r+1) = 0.$$

Hence $r_1 = \frac{1}{2}$ and $r_2 = -1$, so the general solution of Eq. (7) is

$$y = c_1 x^{1/2} + c_2 x^{-1}, \qquad x > 0. \tag{8}$$

Equal Roots. If the roots r_1 and r_2 are equal, then we obtain only one solution $y_1(x) = x^{r_1}$ of the assumed form. A second solution can be obtained by the method of reduction of order, but for the purpose of our future discussion we consider an alternative method. Since $r_1 = r_2$, it follows that $F(r) = (r - r_1)^2$. Thus in this case not only does $F(r_1) = 0$ but also $F'(r_1) = 0$. This suggests differentiating Eq. (3) with respect to r and then setting r equal to r_1. Differentiating Eq. (3) with respect to r gives

$$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}[x^r F(r)].$$

Substituting for $F(r)$, interchanging differentiation with respect to x and with respect to r, and noting that $\partial(x^r)/\partial r = x^r \ln x$, we obtain

$$L[x^r \ln x] = (r - r_1)^2 x^r \ln x + 2(r - r_1)x^r. \tag{9}$$

The right side of Eq. (9) is zero for $r = r_1$; consequently,

$$y_2(x) = x^{r_1} \ln x, \qquad x > 0 \tag{10}$$

is a second solution of Eq. (1). It is easy to show that $W(x^{r_1}, x^{r_1} \ln x) = x^{2r_1-1}$. Hence x^{r_1} and $x^{r_1} \ln x$ are linearly independent for $x > 0$, and the general solution of Eq. (1) is

$$y = (c_1 + c_2 \ln x)x^{r_1}, \qquad x > 0. \tag{11}$$

EXAMPLE 2

Solve

$$x^2y'' + 5xy' + 4y = 0, \qquad x > 0. \tag{12}$$

Substituting $y = x^r$ in Eq. (12) gives

$$x^r[r(r-1) + 5r + 4] = x^r(r^2 + 4r + 4) = 0.$$

Hence $r_1 = r_2 = -2$, and

$$y = x^{-2}(c_1 + c_2 \ln x), \qquad x > 0. \tag{13}$$

Complex Roots. Finally, suppose that the roots r_1 and r_2 are complex conjugates, say, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\mu \neq 0$. We must now explain what is meant by x^r when r is complex. Remembering that

$$x^r = e^{r \ln x} \tag{14}$$

when $x > 0$ and r is real, we can use this equation to *define* x^r when r is complex. Then

$$\begin{aligned} x^{\lambda+i\mu} = e^{(\lambda+i\mu)\ln x} &= e^{\lambda \ln x}e^{i\mu \ln x} = x^{\lambda}e^{i\mu \ln x} \\ &= x^{\lambda}[\cos(\mu \ln x) + i \sin(\mu \ln x)], \qquad x > 0. \end{aligned} \tag{15}$$

With this definition of x^r for complex values of r, it can be verified that the usual laws of algebra and the differential calculus hold, and hence x^{r_1} and x^{r_2} are indeed solutions of Eq. (1). The general solution of Eq. (1) is

$$y = c_1x^{\lambda+i\mu} + c_2x^{\lambda-i\mu}. \tag{16}$$

The disadvantage of this expression is that the functions $x^{\lambda+i\mu}$ and $x^{\lambda-i\mu}$ are complex-valued. Recall that we had a similar situation for the second order differential equation with constant coefficients when the roots of the characteristic equation were complex. In the same way as we did then we observe that the real and imaginary parts of $x^{\lambda+i\mu}$, namely,

$$x^{\lambda} \cos(\mu \ln x) \qquad \text{and} \qquad x^{\lambda} \sin(\mu \ln x), \tag{17}$$

are also solutions of Eq. (1). A straightforward calculation shows that

$$W[x^{\lambda} \cos(\mu \ln x), x^{\lambda} \sin(\mu \ln x)] = \mu x^{2\lambda-1}.$$

Hence these solutions are also linearly independent for $x > 0$, and the general solution of Eq. (1) is

$$y = c_1x^{\lambda} \cos(\mu \ln x) + c_2x^{\lambda} \sin(\mu \ln x), \qquad x > 0. \tag{18}$$

EXAMPLE 3

Solve

$$x^2y'' + xy' + y = 0. \tag{19}$$

Substituting $y = x^r$ in Eq. (19) gives

$$x^r[r(r-1) + r + 1] = x^r(r^2 + 1) = 0.$$

Hence $r = \pm i$, and the general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x), \qquad x > 0. \tag{20}$$

Now let us consider the qualitative behavior of the solutions of Eq. (1) near the singular point $x = 0$. This depends entirely on the nature of the exponents r_1 and r_2. First, if r is real and positive, then $x^r \to 0$ as x tends to zero through positive values. On the other hand, if r is real and negative, then x^r becomes unbounded, while if $r = 0$, then $x^r = 1$. These possibilities are shown in Figure 5.5.1 for various values of r. If r is complex, then a typical solution is $x^\lambda \cos(\mu \ln x)$. This function becomes unbounded or approaches zero if λ is negative or positive, respectively, and also oscillates more and more rapidly as $x \to 0$. This behavior is shown in Figures 5.5.2 and 5.5.3 for selected values of λ and μ. If $\lambda = 0$, the oscillation is of constant amplitude. Finally, if there are repeated roots, then one solution is of the form $x^r \ln x$, which tends to zero if $r > 0$ and becomes unbounded if $r \le 0$. An example of each case is shown in Figure 5.5.4.

The extension of the solutions of Eq. (1) into the interval $x < 0$ can be carried out in a relatively straightforward manner. The difficulty lies in understanding what is meant by x^r when x is negative and r is not an integer; similarly, $\ln x$ has not been defined for $x < 0$. The solutions of the Euler equation that we have given for $x > 0$ can be shown to be valid for $x < 0$, but in general they are complex-valued. Thus in Example 1 the solution $x^{1/2}$ is imaginary for $x < 0$.

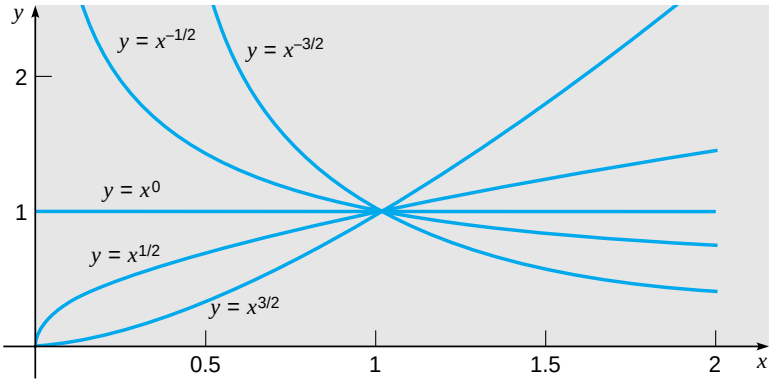


FIGURE 5.5.1 Solutions of an Euler equation; real roots.

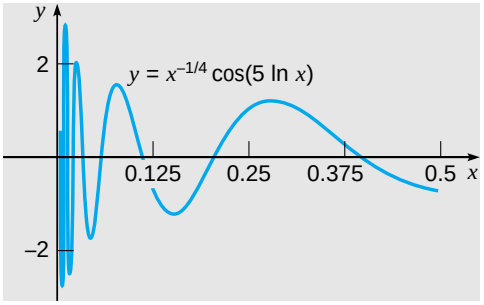


FIGURE 5.5.2 Solution of an Euler equation; complex roots with negative real part.

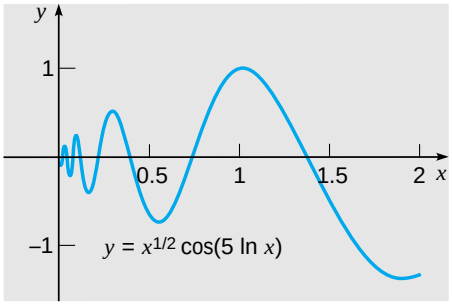


FIGURE 5.5.3 Solution of an Euler equation; complex roots with positive real part.

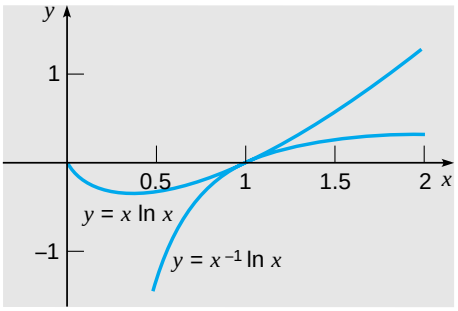


FIGURE 5.5.4 Solutions of an Euler equation; repeated roots.

It is always possible to obtain real-valued solutions of the Euler equation (1) in the interval $x < 0$ by making the following change of variable. Let $x = -\xi$, where $\xi > 0$, and let $y = u(\xi)$. Then we have

$$\frac{dy}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = -\frac{du}{d\xi}, \qquad \frac{d^2y}{dx^2} = \frac{d}{d\xi}\left(-\frac{du}{d\xi}\right)\frac{d\xi}{dx} = \frac{d^2u}{d\xi^2}. \tag{21}$$

Thus Eq. (1), for $x < 0$, takes the form

$$\xi^2\frac{d^2u}{d\xi^2} + \alpha\xi\frac{du}{d\xi} + \beta u = 0, \qquad \xi > 0. \tag{22}$$

But this is exactly the problem that we have just solved; from Eqs. (6), (11), and (18) we have

$$u(\xi) = \begin{cases} c_1\xi^{r_1} + c_2\xi^{r_2} \\ (c_1 + c_2\ln\xi)\xi^{r_1} \\ c_1\xi^{\lambda}\cos(\mu\ln\xi) + c_2\xi^{\lambda}\sin(\mu\ln\xi), \end{cases} \tag{23}$$

depending on whether the zeros of $F(r) = r(r-1) + \alpha r + \beta$ are real and different, real and equal, or complex conjugates. To obtain u in terms of x we replace ξ by $-x$ in Eqs. (23).

We can combine the results for $x > 0$ and $x < 0$ by recalling that $|x| = x$ when $x > 0$ and that $|x| = -x$ when $x < 0$. Thus we need only replace x by $|x|$ in Eqs. (6),

(11), and (18) to obtain real-valued solutions valid in any interval not containing the origin (also see Problems 30 and 31). These results are summarized in the following theorem.

Theorem 5.5.1 The general solution of the Euler equation (1),

$$x^2y'' + \alpha xy' + \beta y = 0,$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

If the roots are real and different, then

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{24}$$

If the roots are real and equal, then

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}. \tag{25}$$

If the roots are complex, then

$$y = |x|^{\lambda}[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)], \tag{26}$$

where $r_1, r_2 = \lambda \pm i\mu$.

The solutions of an Euler equation of the form

$$(x - x_0)^2y'' + \alpha(x - x_0)y' + \beta y = 0 \tag{27}$$

are similar to those given in Theorem 5.5.1. If one looks for solutions of the form $y = (x - x_0)^r$, then the general solution is given by either Eq. (24), (25), or (26) with x replaced by $x - x_0$. Alternatively, we can reduce Eq. (27) to the form of Eq. (1) by making the change of independent variable $t = x - x_0$.

The situation for a general second order differential equation with a regular singular point is analogous to that for an Euler equation. We consider that problem in the next section.

PROBLEMS

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' + 3xy' + 5y = 0$
5. $x^2y'' - xy' + y = 0$
6. $(x-1)^2y'' + 8(x-1)y' + 12y = 0$
7. $x^2y'' + 6xy' - y = 0$
8. $2x^2y'' - 4xy' + 6y = 0$
9. $x^2y'' - 5xy' + 9y = 0$
10. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$
11. $x^2y'' + 2xy' + 4y = 0$
12. $x^2y'' - 4xy' + 4y = 0$

In each of Problems 13 through 16 find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

13. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$
14. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$
15. $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$
16. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$

17. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \to 0$.
18. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.
19. Find γ so that the solution of the initial value problem $x^2y'' - 2y = 0, y(1) = 1, y'(1) = \gamma$ is bounded as $x \to 0$.
20. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \to \infty$.
21. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that
    (a) All solutions approach zero as $x \to 0$.
    (b) All solutions are bounded as $x \to 0$.
    (c) All solutions approach zero as $x \to \infty$.
    (d) All solutions are bounded as $x \to \infty$.
    (e) All solutions are bounded both as $x \to 0$ and as $x \to \infty$.
22. Using the method of reduction of order, show that if r_1 is a repeated root of $r(r-1) + \alpha r + \beta = 0$, then x^{r_1} and $x^{r_1} \ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.
23. **Transformation to a Constant Coefficient Equation.** The Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ can be reduced to an equation with constant coefficients by a change of the independent variable. Let $x = e^z$, or $z = \ln x$, and consider only the interval $x > 0$.
    (a) Show that
    $$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}.$$
    (b) Show that the Euler equation becomes
    $$\frac{d^2y}{dz^2} + (\alpha - 1)\frac{dy}{dz} + \beta y = 0.$$
    Letting r_1 and r_2 denote the roots of $r^2 + (\alpha - 1)r + \beta = 0$, show that
    (c) If r_1 and r_2 are real and different, then
    $$y = c_1e^{r_1z} + c_2e^{r_2z} = c_1x^{r_1} + c_2x^{r_2}.$$
    (d) If r_1 and r_2 are real and equal, then
    $$y = (c_1 + c_2z)e^{r_1z} = (c_1 + c_2 \ln x)x^{r_1}.$$
    (e) If r_1 and r_2 are complex conjugates, $r_1 = \lambda + i\mu$, then
    $$y = e^{\lambda z}[c_1 \cos(\mu z) + c_2 \sin(\mu z)] = x^{\lambda}[c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].$$

In each of Problems 24 through 29 use the method of Problem 23 to solve the given equation for $x > 0$.

24. $x^2y'' - 2y = 0$
25. $x^2y'' - 3xy' + 4y = \ln x$
26. $x^2y'' + 7xy' + 5y = x$
27. $x^2y'' - 2xy' + 2y = 3x^2 + 2 \ln x$
28. $x^2y'' + xy' + 4y = \sin(\ln x)$
29. $3x^2y'' + 12xy' + 9y = 0$
30. Show that if $L[y] = x^2y'' + \alpha xy' + \beta y$, then
    $$L[(-x)^r] = (-x)^r F(r)$$
    for all $x < 0$, where $F(r) = r(r-1) + \alpha r + \beta$. Hence conclude that if $r_1 \neq r_2$ are roots of $F(r) = 0$, then linearly independent solutions of $L[y] = 0$ for $x < 0$ are $(-x)^{r_1}$ and $(-x)^{r_2}$.
31. Suppose that x^{r_1} and x^{r_2} are solutions of an Euler equation for $x > 0$, where $r_1 \neq r_2$, and r_1 is an integer. According to Eq. (24) the general solution in any interval not containing the origin is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$. Show that the general solution can also be written as

$y = k_1 x^{r_1} + k_2 |x|^{r_2}$.
Hint: Show by a proper choice of constants that the expressions are identical for $x > 0$, and by a different choice of constants that they are identical for $x < 0$.

5.6 Series Solutions near a Regular Singular Point, Part I



We now consider the question of solving the general second order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point $x = x_0$. For convenience we assume that $x_0 = 0$. If $x_0 \neq 0$, the equation can be transformed into one for which the regular singular point is at the origin by letting $x - x_0$ equal t.

The fact that $x = 0$ is a regular singular point of Eq. (1) means that $xQ(x)/P(x) = xp(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0$, and are analytic at $x = 0$. Thus they have convergent power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

on some interval $|x| < \rho$ about the origin, where $\rho > 0$. To make the quantities $xp(x)$ and $x^2q(x)$ appear in Eq. (1), it is convenient to divide Eq. (1) by $P(x)$ and then to multiply by x^2, obtaining

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{3}$$

or

$$x^2y'' + x(p_0 + p_1x + \cdots + p_nx^n + \cdots)y' + (q_0 + q_1x + \cdots + q_nx^n + \cdots)y = 0. \tag{4}$$

If all of the coefficients p_n and q_n are zero except possibly

$$p_0 = \lim_{x\to 0} \frac{xQ(x)}{P(x)} \qquad \text{and} \qquad q_0 = \lim_{x\to 0} \frac{x^2R(x)}{P(x)}, \tag{5}$$

then Eq. (4) reduces to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0, \tag{6}$$

which was discussed in the preceding section. In general, of course, some of the p_n and q_n, $n \geq 1$, are not zero. However, the essential character of solutions of Eq. (4) is identical to that of solutions of the Euler equation (6). The presence of the terms $p_1x + \cdots + p_nx^n + \cdots$ and $q_1x + \cdots + q_nx^n + \cdots$ merely complicates the calculations.

We restrict our discussion primarily to the interval $x > 0$. The interval $x < 0$ can be treated, just as for the Euler equation, by making the change of variable $x = -\xi$ and then solving the resulting equation for $\xi > 0$.

Since the coefficients in Eq. (4) are "Euler coefficients" times power series, it is natural to seek solutions in the form of "Euler solutions" times power series. Thus we assume that

$$y = x^r(a_0 + a_1x + \cdots + a_nx^n + \cdots) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}, \tag{7}$$

where $a_0 \neq 0$. In other words, r is the exponent of the first term in the series and a_0 is its coefficient. As part of the solution we have to determine:

1. The values of r for which Eq. (1) has a solution of the form (7).

2. The recurrence relation for the coefficients a_n.

3. The radius of convergence of the series $\sum_{n=0}^{\infty} a_nx^n$.

The general theory is due to Frobenius[10] and is fairly complicated. Rather than trying to present this theory we simply assume in this and the next two sections that there does exist a solution of the stated form. In particular, we assume that any power series in an expression for a solution has a nonzero radius of convergence, and concentrate on showing how to determine the coefficients in such a series. To illustrate the method of Frobenius we first consider an example.

EXAMPLE 1

Solve the differential equation

$$2x^2y'' - xy' + (1 + x)y = 0. \tag{8}$$

It is easy to show that $x = 0$ is a regular singular point of Eq. (8). Further, $xp(x) = -1/2$ and $x^2q(x) = (1 + x)/2$. Thus $p_0 = -1/2$, $q_0 = 1/2$, $q_1 = 1/2$, and all other p's and q's are zero. Then, from Eq. (6), the Euler equation corresponding to Eq. (8) is

$$2x^2y'' - xy' + y = 0. \tag{9}$$

To solve Eq. (8) we assume that there is a solution of the form (7). Then y' and y'' are given by

$$y' = \sum_{n=0}^{\infty} a_n(r + n)x^{r+n-1} \tag{10}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(r + n)(r + n - 1)x^{r+n-2}. \tag{11}$$

By substituting the expressions for y, y', and y'' in Eq. (8) we obtain

$$\begin{aligned} 2x^2y'' - xy' + (1 + x)y = &\sum_{n=0}^{\infty} 2a_n(r + n)(r + n - 1)x^{r+n} \\ &- \sum_{n=0}^{\infty} a_n(r + n)x^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n+1}. \end{aligned} \tag{12}$$

[10] Ferdinand Georg Frobenius (1849–1917) was (like Fuchs) a student and eventually a professor at the University of Berlin. He showed how to construct series solutions about regular singular points in 1874. His most distinguished work, however, was in algebra where he was one of the foremost early developers of group theory.

The last term in Eq. (12) can be written as $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$, so by combining the terms in Eq. (12) we obtain

$$2x^2y'' - xy' + (1+x)y = a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty}\{[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\}x^{r+n} = 0. \quad (13)$$

If Eq. (13) is to be satisfied for all x, the coefficient of each power of x in Eq. (13) must be zero. From the coefficient of x^r we obtain, since $a_0 \neq 0$,

$$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0. \quad (14)$$

Equation (14) is called the **indicial equation** for Eq. (8). Note that it is exactly the polynomial equation we would obtain for the Euler equation (9) associated with Eq. (8). The roots of the indicial equation are

$$r_1 = 1, \qquad r_2 = 1/2. \quad (15)$$

These values of r are called the **exponents at the singularity** for the regular singular point $x = 0$. They determine the qualitative behavior of the solution (7) in the neighborhood of the singular point.

Now we return to Eq. (13) and set the coefficient of x^{r+n} equal to zero. This gives the relation

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0 \quad (16)$$

or

$$a_n = -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1} = -\frac{a_{n-1}}{[(r+n)-1][2(r+n)-1]}, \qquad n \geq 1. \quad (17)$$

For each root r_1 and r_2 of the indicial equation we use the recurrence relation (17) to determine a set of coefficients $a_1, a_2, \ldots$. For $r = r_1 = 1$, Eq. (17) becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n}, \qquad n \geq 1.$$

Thus

$$a_1 = -\frac{a_0}{3 \cdot 1},$$
$$a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)},$$

and

$$a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}.$$

In general we have

$$a_n = \frac{(-1)^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}a_0, \qquad n \geq 1. \quad (18)$$

Hence, if we omit the constant multiplier a_0, one solution of Eq. (8) is

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}\right], \qquad x > 0. \tag{19}$$

To determine the radius of convergence of the series in Eq. (19) we use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)} = 0$$

for all x. Thus the series converges for all x.

Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly. From Eq. (17) we have

$$a_n = -\frac{a_{n-1}}{2n(n-\frac{1}{2})} = -\frac{a_{n-1}}{n(2n-1)}, \qquad n \geq 1.$$

Hence

$$\begin{aligned} a_1 &= -\frac{a_0}{1\cdot 1}, \\ a_2 &= -\frac{a_1}{2\cdot 3} = \frac{a_0}{(1\cdot 2)(1\cdot 3)}, \\ a_3 &= -\frac{a_2}{3\cdot 5} = -\frac{a_0}{(1\cdot 2\cdot 3)(1\cdot 3\cdot 5)}, \end{aligned}$$

and in general

$$a_n = \frac{(-1)^n}{n![1\cdot 3\cdot 5\cdots(2n-1)]}a_0, \qquad n \geq 1. \tag{20}$$

Again omitting the constant multiplier a_0, we obtain the second solution

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n![1\cdot 3\cdot 5\cdots(2n-1)]}\right], \qquad x > 0. \tag{21}$$

As before, we can show that the series in Eq. (21) converges for all x. Since the leading terms in the series solutions y_1 and y_2 are x and $x^{1/2}$, respectively, it follows that the solutions are linearly independent. Hence the general solution of Eq. (8) is

$$y = c_1 y_1(x) + c_2 y_2(x), \qquad x > 0.$$

The preceding example illustrates that if $x = 0$ is a regular singular point, then sometimes there are two solutions of the form (7) in the neighborhood of this point. Similarly, if there is a regular singular point at $x = x_0$, then there may be two solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{22}$$

that are valid near $x = x_0$. However, just as an Euler equation may not have two solutions of the form $y = x^r$, so a more general equation with a regular singular point may not have two solutions of the form (7) or (22). In particular, we show in the next section that if the roots r_1 and r_2 of the indicial equation are equal, or differ by an

integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22); if r_1 and r_2 differ by an integer, this solution corresponds to the larger value of r. If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.7 and 5.8.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of x. However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If P, Q, and R are polynomials, it is often much better to work directly with Eq. (1) than with Eq. (3). This avoids the necessity of expressing $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ as power series. For example, it is more convenient to consider the equation

$$x(1+x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2y'' + \frac{2x}{1+x}y' + \frac{x^2}{1+x}y = 0,$$

which would entail expanding $2x/(1+x)$ and $x^2/(1+x)$ in power series.

PROBLEMS

In each of Problems 1 through 10 show that the given differential equation has a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
3. $xy'' + y = 0$
4. $xy'' + y' - y = 0$
5. $3x^2y'' + 2xy' + x^2y = 0$
6. $x^2y'' + xy' + (x-2)y = 0$
7. $xy'' + (1-x)y' - y = 0$
8. $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$
9. $x^2y'' - x(x+3)y' + (x+3)y = 0$
10. $x^2y'' + (x^2 + \frac{1}{4})y = 0$
11. The Legendre equation of order α is

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

The solution of this equation near the ordinary point $x = 0$ was discussed in Problems 22 and 23 of Section 5.3. In Example 5 of Section 5.4 it was shown that $x = \pm 1$ are regular singular points. Determine the indicial equation and its roots for the point $x = 1$. Find a series solution in powers of $x - 1$ for $x - 1 > 0$.
Hint: Write $1 + x = 2 + (x-1)$ and $x = 1 + (x-1)$. Alternatively, make the change of variable $x - 1 = t$ and determine a series solution in powers of t.

12. The Chebyshev equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant; see Problem 10 of Section 5.3.
(a) Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.
(b) Find two linearly independent solutions about $x = 1$.

13. The Laguerre[11] differential equation is

$$xy'' + (1 - x)y' + \lambda y = 0.$$

Show that $x = 0$ is a regular singular point. Determine the indicial equation, its roots, the recurrence relation, and one solution ($x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized this polynomial is known as the Laguerre polynomial, $L_m(x)$.

14. The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0.$$

Show that $x = 0$ is a regular singular point; that the roots of the indicial equation are $r_1 = r_2 = 0$; and that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Show that the series converges for all x. The function J_0 is known as the Bessel function of the first kind of order zero.

15. Referring to Problem 14, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.
Hint: If $y_2(x) = J_0(x)v(x)$, then

$$y_2(x) = J_0(x) \int \frac{dx}{x[J_0(x)]^2}.$$

Find the first term in the series expansion of $1/x[J_0(x)]^2$.

16. The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

(a) Show that $x = 0$ is a regular singular point; that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$; and that one solution for $x > 0$ is

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!\, n!\, 2^{2n}}.$$

Show that the series converges for all x. The function J_1 is known as the Bessel function of the first kind of order one.

(b) Show that it is impossible to determine a second solution of the form

$$x^{-1} \sum_{n=0}^{\infty} b_n x^n, \qquad x > 0.$$

5.7 Series Solutions near a Regular Singular Point, Part II

Now let us consider the general problem of determining a solution of the equation

$$L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{1}$$

[11] Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879.

where

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

and both series converge in an interval $|x| < \rho$ for some $\rho > 0$. The point $x = 0$ is a regular singular point, and the corresponding Euler equation is

$$x^2 y'' + p_0 x y' + q_0 y = 0. \tag{3}$$

We seek a solution of Eq. (1) for $x > 0$ and assume that it has the form

$$y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \tag{4}$$

where $a_0 \neq 0$, and we have written $y = \phi(r, x)$ to emphasize that ϕ depends on r as well as x. It follows that

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \qquad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}. \tag{5}$$

Then, substituting from Eqs. (2), (4), and (5) in Eq. (1) gives

$$\begin{aligned} &a_0 r(r-1)x^r + a_1(r+1)r x^{r+1} + \cdots + a_n(r+n)(r+n-1)x^{r+n} + \cdots \\ &\quad + (p_0 + p_1 x + \cdots + p_n x^n + \cdots) \\ &\quad \times [a_0 r x^r + a_1(r+1)x^{r+1} + \cdots + a_n(r+n)x^{r+n} + \cdots] \\ &\quad + (q_0 + q_1 x + \cdots + q_n x^n + \cdots) \\ &\quad \times (a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots) = 0. \end{aligned}$$

Multiplying the infinite series together and then collecting terms, we obtain

$$\begin{aligned} &a_0 F(r) x^r + [a_1 F(r+1) + a_0(p_1 r + q_1)]x^{r+1} \\ &\quad + \{a_2 F(r+2) + a_0(p_2 r + q_2) + a_1[p_1(r+1) + q_1]\}x^{r+2} \\ &\quad + \cdots + \{a_n F(r+n) + a_0(p_n r + q_n) + a_1[p_{n-1}(r+1) + q_{n-1}] \\ &\quad + \cdots + a_{n-1}[p_1(r+n-1) + q_1]\}x^{r+n} + \cdots = 0, \end{aligned}$$

or in a more compact form,

$$\begin{aligned} L[\phi](r, x) &= a_0 F(r) x^r \\ &\quad + \sum_{n=1}^{\infty} \left\{ F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0, \end{aligned} \tag{6}$$

where

$$F(r) = r(r-1) + p_0 r + q_0. \tag{7}$$

For Eq. (6) to be satisfied identically the coefficient of each power of x must be zero.

Since $a_0 \neq 0$, the term involving x^r yields the equation $F(r) = 0$. This equation is called the **indicial equation**; note that it is exactly the equation we would obtain in looking for solutions $y = x^r$ of the Euler equation (3). Let us denote the roots of the indicial equation by r_1 and r_2 with $r_1 \geq r_2$ if the roots are real. If the roots are complex, the designation of the roots is immaterial. Only for these values of r can we expect to find solutions of Eq. (1) of the form (4). The roots r_1 and r_2 are called the

exponents at the singularity; they determine the qualitative nature of the solution in the neighborhood of the singular point.

Setting the coefficient of x^{r+n} in Eq. (6) equal to zero gives the **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \qquad n \geq 1. \tag{8}$$

Equation (8) shows that, in general, a_n depends on the value of r and all the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$. It also shows that we can successively compute $a_1, a_2, \ldots, a_n, \ldots$ in terms of a_0 and the coefficients in the series for $xp(x)$ and $x^2q(x)$ provided that $F(r+1), F(r+2), \ldots, F(r+n), \ldots$ are not zero. The only values of r for which $F(r) = 0$ are $r = r_1$ and $r = r_2$; since $r_1 \geq r_2$, it follows that $r_1 + n$ is not equal to r_1 or r_2 for $n \geq 1$. Consequently, $F(r_1 + n) \neq 0$ for $n \geq 1$. Hence we can always determine one solution of Eq. (1) in the form (4), namely,

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \qquad x > 0. \tag{9}$$

Here we have introduced the notation $a_n(r_1)$ to indicate that a_n has been determined from Eq. (8) with $r = r_1$. To specify the arbitrary constant in the solution we have taken a_0 to be 1.

If r_2 is not equal to r_1, and $r_1 - r_2$ is not a positive integer, then $r_2 + n$ is not equal to r_1 for any value of $n \geq 1$; hence $F(r_2 + n) \neq 0$, and we can also obtain a second solution

$$y_2(x) = x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right], \qquad x > 0. \tag{10}$$

Just as for the series solutions about ordinary points discussed in Section 5.3, the series in Eqs. (9) and (10) converge at least in the interval $|x| < \rho$ where the series for both $xp(x)$ and $x^2q(x)$ converge. Within their radii of convergence, the power series $1 + \sum_{n=1}^{\infty} a_n(r_1)x^n$ and $1 + \sum_{n=1}^{\infty} a_n(r_2)x^n$ define functions that are analytic at $x = 0$. Thus the singular behavior, if there is any, of the solutions y_1 and y_2 is due to the factors x^{r_1} and x^{r_2} that multiply these two analytic functions. Next, to obtain real-valued solutions for $x < 0$, we can make the substitution $x = -\xi$ with $\xi > 0$. As we might expect from our discussion of the Euler equation, it turns out that we need only replace x^{r_1} in Eq. (9) and x^{r_2} in Eq. (10) by $|x|^{r_1}$ and $|x|^{r_2}$, respectively. Finally, note that if r_1 and r_2 are complex numbers, then they are necessarily complex conjugates and $r_2 \neq r_1 + N$. Thus, in this case we can always find two series solutions of the form (4); however, they are complex-valued functions of x. Real-valued solutions can be obtained by taking the real and imaginary parts of the complex-valued solutions. The exceptional cases in which $r_1 = r_2$ or $r_1 - r_2 = N$, where N is a positive integer, require more discussion and will be considered later in this section.

It is important to realize that r_1 and r_2, the exponents at the singular point, are easy to find and that they determine the qualitative behavior of the solutions. To calculate r_1 and r_2 it is only necessary to solve the quadratic indicial equation

$$r(r-1) + p_0 r + q_0 = 0, \tag{11}$$

whose coefficients are given by

$$p_0 = \lim_{x\to 0} xp(x), \qquad q_0 = \lim_{x\to 0} x^2 q(x). \tag{12}$$

Note that these are exactly the limits that must be evaluated in order to classify the singularity as a regular singular point; thus they have usually been determined at an earlier stage of the investigation.

Further, if $x = 0$ is a regular singular point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13}$$

where the functions P, Q, and R are polynomials, then $xp(x) = xQ(x)/P(x)$ and $x^2q(x) = x^2R(x)/P(x)$. Thus

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)}, \qquad q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)}. \tag{14}$$

Finally, the radii of convergence for the series in Eqs. (9) and (10) are at least equal to the distance from the origin to the nearest zero of P other than $x = 0$ itself.

EXAMPLE 1

Discuss the nature of the solutions of the equation

$$2x(1+x)y'' + (3+x)y' - xy = 0$$

near the singular points.

This equation is of the form (13) with $P(x) = 2x(1+x)$, $Q(x) = 3+x$, and $R(x) = -x$. The points $x = 0$ and $x = -1$ are the only singular points. The point $x = 0$ is a regular singular point, since

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{3+x}{2x(1+x)} = \frac{3}{2},$$
$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{-x}{2x(1+x)} = 0.$$

Further, from Eq. (14), $p_0 = \frac{3}{2}$ and $q_0 = 0$. Thus the indicial equation is $r(r-1) + \frac{3}{2}r = 0$, and the roots are $r_1 = 0$, $r_2 = -\frac{1}{2}$. Since these roots are not equal and do not differ by an integer, there are two linearly independent solutions of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n \qquad \text{and} \qquad y_2(x) = |x|^{-1/2}\left[1 + \sum_{n=1}^{\infty} a_n(-\tfrac{1}{2})x^n\right]$$

for $0 < |x| < \rho$. A lower bound for the radius of convergence of each series is 1, the distance from $x = 0$ to $x = -1$, the other zero of $P(x)$. Note that the solution y_1 is bounded as $x \to 0$, indeed is analytic there, and that the second solution y_2 is unbounded as $x \to 0$.

The point $x = -1$ is also a regular singular point, since

$$\lim_{x\to -1} (x+1)\frac{Q(x)}{P(x)} = \lim_{x\to -1} \frac{(x+1)(3+x)}{2x(1+x)} = -1,$$
$$\lim_{x\to -1} (x+1)^2\frac{R(x)}{P(x)} = \lim_{x\to -1} \frac{(x+1)^2(-x)}{2x(1+x)} = 0.$$

In this case $p_0 = -1$, $q_0 = 0$, so the indicial equation is $r(r-1) - r = 0$. The roots of the indicial equation are $r_1 = 2$ and $r_2 = 0$. Corresponding to the larger root there is a solution of the form

$$y_1(x) = (x+1)^2\left[1 + \sum_{n=1}^{\infty} a_n(2)(x+1)^n\right].$$

The series converges at least for $|x+1| < 1$ and y_1 is an analytic function there. Since the two roots differ by a positive integer, there may or may not be a second solution of the form

$$y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x+1)^n.$$

We cannot say more without further analysis.

Observe that no complicated calculations were required to discover the information about the solutions presented in this example. All that was needed was to evaluate a few limits and solve two quadratic equations.

We now consider the cases in which the roots of the indicial equation are equal, or differ by a positive integer, $r_1 - r_2 = N$. As we have shown earlier, there is always one solution of the form (9) corresponding to the larger root r_1 of the indicial equation. By analogy with the Euler equation, we might expect that if $r_1 = r_2$, then the second solution contains a logarithmic term. This may also be true if the roots differ by an integer.

Equal Roots. The method of finding the second solution is essentially the same as the one we used in finding the second solution of the Euler equation (see Section 5.5) when the roots of the indicial equation were equal. We consider r to be a continuous variable and determine a_n as a function of r by solving the recurrence relation (8). For this choice of $a_n(r)$ for $n \geq 1$, Eq. (6) reduces to

$$L[\phi](r, x) = a_0 F(r) x^r = a_0 (r - r_1)^2 x^r, \tag{15}$$

since r_1 is a repeated root of $F(r)$. Setting $r = r_1$ in Eq. (15), we find that $L[\phi](r_1, x) = 0$; hence, as we already know, $y_1(x)$ given by Eq. (9) is one solution of Eq. (1). But more important, it also follows from Eq. (15), just as for the Euler equation, that

$$\begin{aligned} L\left[\frac{\partial \phi}{\partial r}\right](r_1, x) &= a_0 \frac{\partial}{\partial r}[x^r (r - r_1)^2]\Big|_{r=r_1} \\ &= a_0[(r - r_1)^2 x^r \ln x + 2(r - r_1)x^r]\Big|_{r=r_1} = 0. \end{aligned} \tag{16}$$

Hence, a second solution of Eq. (1) is

$$\begin{aligned} y_2(x) = \frac{\partial \phi(r, x)}{\partial r}\Big|_{r=r_1} &= \frac{\partial}{\partial r}\left\{x^r\left[a_0 + \sum_{n=1}^{\infty} a_n(r)x^n\right]\right\}\Bigg|_{r=r_1} \\ &= (x^{r_1} \ln x)\left[a_0 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right] + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n \\ &= y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n, \qquad x > 0, \end{aligned} \tag{17}$$

where $a_n'(r_1)$ denotes da_n/dr evaluated at $r = r_1$.

It may turn out that it is difficult to determine $a_n(r)$ as a function of r from the recurrence relation (8) and then to differentiate the resulting expression with respect to r. An alternative is simply to assume that y has the *form* of Eq. (17), that is,

$$y = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n, \qquad x > 0, \tag{18}$$

where $y_1(x)$ has already been found. The coefficients b_n are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of x equal to zero. A third possibility is to use the method of reduction of order to find $y_2(x)$ once $y_1(x)$ is known.

Roots Differing by an Integer. For this case the derivation of the second solution is considerably more complicated and will not be given here. The form of this solution is stated in Eq. (24) in the following theorem. The coefficients $c_n(r_2)$ in Eq. (24) are given by

$$c_n(r_2) = \frac{d}{dr}[(r - r_2)a_n(r)]\Big|_{r=r_2}, \qquad n = 1, 2, \ldots, \tag{19}$$

where $a_n(r)$ is determined from the recurrence relation (8) with $a_0 = 1$. Further, the coefficient a in Eq. (24) is

$$a = \lim_{r\to r_2}(r - r_2)a_N(r). \tag{20}$$

If $a_N(r_2)$ is finite, then $a = 0$ and there is no logarithmic term in y_2. A full derivation of the formulas (19), (20) may be found in Coddington (Chapter 4).

In practice the best way to determine whether a is zero in the second solution is simply to try to compute the a_n corresponding to the root r_2 and to see whether it is possible to determine $a_N(r_2)$. If so, there is no further problem. If not, we must use the form (24) with $a \neq 0$.

When $r_1 - r_2 = N$, there are again three ways to find a second solution. First, we can calculate a and $c_n(r_2)$ directly by substituting the expression (24) for y in Eq. (1). Second, we can calculate $c_n(r_2)$ and a of Eq. (24) using the formulas (19) and (20). If this is the planned procedure, then in calculating the solution corresponding to $r = r_1$ be sure to obtain the general formula for $a_n(r)$ rather than just $a_n(r_1)$. The third alternative is to use the method of reduction of order.

Theorem 5.7.1 Consider the differential equation (1),

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0,$$

where $x = 0$ is a regular singular point. Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either of the intervals $-\rho < x < 0$ or $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \tag{21}$$

where the $a_n(r_1)$ are given by the recurrence relation (8) with $a_0 = 1$ and $r = r_1$.

If $r_1 - r_2$ is not zero or a positive integer, then in either of the intervals $-\rho < x < 0$ or $0 < x < \rho$, there exists a second linearly independent solution of the form

$$y_2(x) = |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right]. \tag{22}$$

The $a_n(r_2)$ are also determined by the recurrence relation (8) with $a_0 = 1$ and $r = r_2$. The power series in Eqs. (21) and (22) converge at least for $|x| < \rho$.

If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n. \tag{23}$$

If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right]. \tag{24}$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, $c_n(r_2)$, and the constant a can be determined by substituting the form of the series solutions for y in Eq. (1). The constant a may turn out to be zero, in which case there is no logarithmic term in the solution (24). Each of the series in Eqs. (23) and (24) converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$.

PROBLEMS

In each of Problems 1 through 12 find all the regular singular points of the given differential equation. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$
2. $x^2y'' - x(2+x)y' + (2+x^2)y = 0$
3. $x(x-1)y'' + 6x^2y' + 3y = 0$
4. $y'' + 4xy' + 6y = 0$
5. $x^2y'' + 3(\sin x)y' - 2y = 0$
6. $2x(x+2)y'' + y' - xy = 0$
7. $x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0$
8. $(x+1)^2y'' + 3(x^2-1)y' + 3y = 0$
9. $x^2(1-x)y'' - (1+x)y' + 2xy = 0$
10. $(x-2)^2(x+2)y'' + 2xy' + 3(x-2)y = 0$
11. $(4-x^2)y'' + 2xy' + 3y = 0$
12. $x(x+3)^2y'' - 2(x+3)y' - xy = 0$

In each of Problems 13 through 17:

(a) Show that $x = 0$ is a regular singular point of the given differential equation.
(b) Find the exponents at the singular point $x = 0$.

(c) Find the first three nonzero terms in each of two linearly independent solutions about $x = 0$.

13. $xy'' + y' - y = 0$
14. $xy'' + 2xy' + 6e^x y = 0$; see Problem 1
15. $x(x-1)y'' + 6x^2 y' + 3y = 0$; see Problem 3
16. $xy'' + y = 0$
17. $x^2 y'' + (\sin x)y' - (\cos x)y = 0$
18. Show that

$$(\ln x)y'' + \tfrac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$. Determine the roots of the indicial equation at $x = 1$. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$. What would you expect the radius of convergence of the series to be?

19. In several problems in mathematical physics (for example, the Schrödinger equation for a hydrogen atom) it is necessary to study the differential equation

$$x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0, \tag{i}$$

where α, β, and γ are constants. This equation is known as the **hypergeometric equation**.
(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are 0 and $1 - \gamma$.
(b) Show that $x = 1$ is a regular singular point, and that the roots of the indicial equation are 0 and $\gamma - \alpha - \beta$.
(c) Assuming that $1 - \gamma$ is not a positive integer, show that in the neighborhood of $x = 0$ one solution of (i) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots.$$

What would you expect the radius of convergence of this series to be?
(d) Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right].$$

(e) Show that the point at infinity is a regular singular point, and that the roots of the indicial equation are α and β. See Problem 21 of Section 5.4.

20. Consider the differential equation

$$x^3 y'' + \alpha x y' + \beta y = 0,$$

where α and β are real constants and $\alpha \neq 0$.
(a) Show that $x = 0$ is an irregular singular point.
(b) By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear, and consequently there is only one formal solution of the assumed form.
(c) Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of β/α show that the formal series solution has a zero radius of convergence, and so does not represent an actual solution in any interval.

21. Consider the differential equation

$$y'' + \frac{\alpha}{x^s}y' + \frac{\beta}{x^t}y = 0, \tag{i}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and s and t are positive integers that for the moment are arbitrary.

(a) Show that if $s > 1$ or $t > 2$, then the point $x = 0$ is an irregular singular point.

(b) Try to find a solution of Eq. (i) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \qquad x > 0. \tag{ii}$$

Show that if $s = 2$ and $t = 2$, then there is only one possible value of r for which there is a formal solution of Eq. (i) of the form (ii).

(c) Show that if $s = 1$ and $t = 3$, then there are no solutions of Eq. (i) of the form (ii).

(d) Show that the maximum values of s and t for which the indicial equation is quadratic in r [and hence we can hope to find two solutions of the form (ii)] are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a "weak singularity," or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution we should point out that while it is sometimes possible to obtain a formal series solution of the form (ii) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 20 for an example.

5.8 Bessel's Equation



In this section we consider three special cases of Bessel's[12] equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{1}$$

where ν is a constant, which illustrate the theory discussed in Section 5.7. It is easy to show that $x = 0$ is a regular singular point. For simplicity we consider only the case $x > 0$.

Bessel Equation of Order Zero. This example illustrates the situation in which the roots of the indicial equation are equal. Setting $\nu = 0$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + x^2y = 0. \tag{2}$$

Substituting

$$y = \phi(r, x) = a_0x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \tag{3}$$

[12] Friedrich Wilhelm Bessel (1784–1846) embarked on a career in business as a youth, but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of Eq. (1). He is also famous for making the first accurate determination (1838) of the distance from the earth to a star.

we obtain

$$\begin{aligned} L[\phi](r,x) &= \sum_{n=0}^{\infty} a_n[(r+n)(r+n-1)+(r+n)]x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0[r(r-1)+r]x^r + a_1[(r+1)r+(r+1)]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}\{a_n[(r+n)(r+n-1)+(r+n)]+a_{n-2}\}x^{r+n} = 0. \end{aligned} \tag{4}$$

The roots of the indicial equation $F(r) = r(r-1)+r = 0$ are $r_1 = 0$ and $r_2 = 0$; hence we have the case of equal roots. The recurrence relation is

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \qquad n \geq 2. \tag{5}$$

To determine $y_1(x)$ we set r equal to 0. Then from Eq. (4) it follows that for the coefficient of x^{r+1} to be zero we must choose $a_1 = 0$. Hence from Eq. (5), $a_3 = a_5 = a_7 = \cdots = 0$. Further,

$$a_n(0) = -a_{n-2}(0)/n^2, \qquad n = 2, 4, 6, 8, \ldots,$$

or letting $n = 2m$, we obtain

$$a_{2m}(0) = -a_{2m-2}(0)/(2m)^2, \qquad m = 1, 2, 3, \ldots .$$

Thus

$$a_2(0) = -\frac{a_0}{2^2}, \quad a_4(0) = \frac{a_0}{2^4 2^2}, \quad a_6(0) = -\frac{a_0}{2^6(3\cdot 2)^2},$$

and, in general,

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots. \tag{6}$$

Hence

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \qquad x > 0. \tag{7}$$

The function in brackets is known as the Bessel function of the first kind of order zero and is denoted by $J_0(x)$. It follows from Theorem 5.7.1 that the series converges for all x, and that J_0 is analytic at $x = 0$. Some of the important properties of J_0 are discussed in the problems. Figure 5.8.1 shows the graphs of $y = J_0(x)$ and some of the partial sums of the series (7).

To determine $y_2(x)$ we will calculate $a_n'(0)$. The alternative procedure in which we simply substitute the form (23) of Section 5.7 in Eq. (2) and then determine the b_n is discussed in Problem 10. First we note from the coefficient of x^{r+1} in Eq. (4) that $(r+1)^2 a_1(r) = 0$. It follows that not only does $a_1(0) = 0$ but also $a_1'(0) = 0$. It is easy to deduce from the recurrence relation (5) that $a_3'(0) = a_5'(0) = \cdots = a_{2n+1}'(0) = \cdots = 0$; hence we need only compute $a_{2m}'(0)$, $m = 1, 2, 3, \ldots$. From Eq. (5) we have

$$a_{2m}(r) = -a_{2m-2}(r)/(r+2m)^2, \qquad m = 1, 2, 3, \ldots.$$

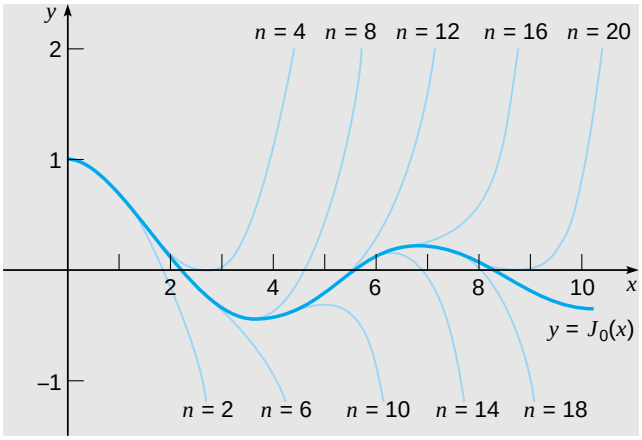


FIGURE 5.8.1 Polynomial approximations to $J_0(x)$. The value of n is the degree of the approximating polynomial.

By solving this recurrence relation we obtain

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2(r+4)^2\cdots(r+2m-2)^2(r+2m)^2}, \qquad m = 1, 2, 3, \ldots. \tag{8}$$

The computation of $a'_{2m}(r)$ can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n},$$

and if x is not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Applying this result to $a_{2m}(r)$ from Eq. (8) we find that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right),$$

and, setting r equal to 0, we obtain

$$a'_{2m}(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right]a_{2m}(0).$$

Substituting for $a_{2m}(0)$ from Eq. (6), and letting

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}, \tag{9}$$

we obtain, finally,

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots.$$

The second solution of the Bessel equation of order zero is found by setting $a_0 = 1$ and substituting for $y_1(x)$ and $a'_{2m}(0) = b_{2m}(0)$ in Eq. (23) of Section 5.7. We obtain

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \qquad x > 0. \tag{10}$$

Instead of y_2, the second solution is usually taken to be a certain linear combination of J_0 and y_2. It is known as the Bessel function of the second kind of order zero and is denoted by Y_0. Following Copson (Chapter 12), we define[13]

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2) J_0(x)]. \tag{11}$$

Here γ is a constant, known as the Euler–Máscheroni (1750–1800) constant; it is defined by the equation

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) \cong 0.5772. \tag{12}$$

Substituting for $y_2(x)$ in Eq. (11), we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right], \qquad x > 0. \tag{13}$$

The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

Note that $J_0(x) \to 1$ as $x \to 0$ and that $Y_0(x)$ has a logarithmic singularity at $x = 0$; that is, $Y_0(x)$ behaves as $(2/\pi)\ln x$ when $x \to 0$ through positive values. Thus if we are interested in solutions of Bessel's equation of order zero that are finite at the origin, which is often the case, we must discard Y_0. The graphs of the functions J_0 and Y_0 are shown in Figure 5.8.2.

It is interesting to note from Figure 5.8.2 that for x large both $J_0(x)$ and $Y_0(x)$ are oscillatory. Such a behavior might be anticipated from the original equation; indeed it

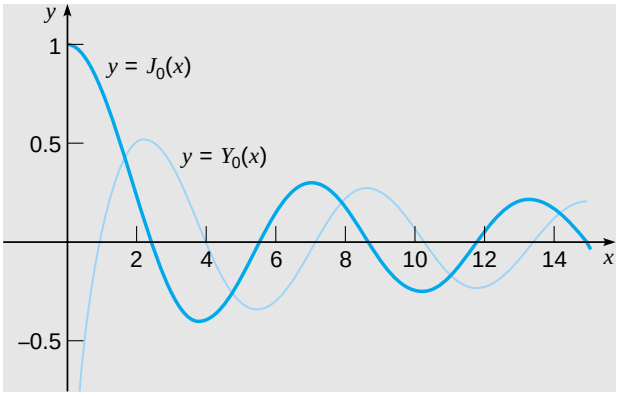


FIGURE 5.8.2 The Bessel functions of order zero.

[13]Other authors use other definitions for Y_0. The present choice for Y_0 is also known as the Weber (1842–1913) function.

is true for the solutions of the Bessel equation of order ν. If we divide Eq. (1) by x^2, we obtain

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

For x very large it is reasonable to suspect that the terms $(1/x)y'$ and $(\nu^2/x^2)y$ are small and hence can be neglected. If this is true, then the Bessel equation of order ν can be approximated by

$$y'' + y = 0.$$

The solutions of this equation are $\sin x$ and $\cos x$; thus we might anticipate that the solutions of Bessel's equation for large x are similar to linear combinations of $\sin x$ and $\cos x$. This is correct insofar as the Bessel functions are oscillatory; however, it is only partly correct. For x large the functions J_0 and Y_0 also decay as x increases; thus the equation $y'' + y = 0$ does not provide an adequate approximation to the Bessel equation for large x, and a more delicate analysis is required. In fact, it is possible to show that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \qquad \text{as} \qquad x \to \infty, \tag{14}$$

and that

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) \qquad \text{as} \qquad x \to \infty. \tag{15}$$

These asymptotic approximations, as $x \to \infty$, are actually very good. For example, Figure 5.8.3 shows that the asymptotic approximation (14) to $J_0(x)$ is reasonably accurate for all $x \geq 1$. Thus to approximate $J_0(x)$ over the entire range from zero to infinity, one can use two or three terms of the series (7) for $x \leq 1$ and the asymptotic approximation (14) for $x \geq 1$.

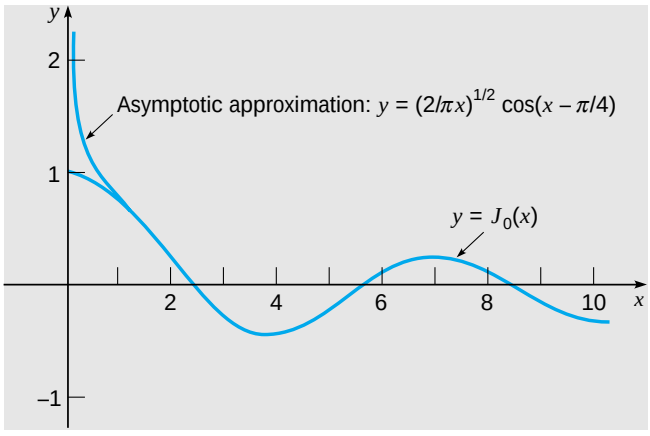


FIGURE 5.8.3 Asymptotic approximation to $J_0(x)$.

Bessel Equation of Order One-Half. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting $\nu = \frac{1}{2}$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0. \tag{16}$$

If we substitute the series (3) for $y = \phi(r, x)$, we obtain

$$\begin{aligned} L[\phi](r,x) &= \sum_{n=0}^{\infty} \left[(r+n)(r+n-1) + (r+n) - \tfrac{1}{4}\right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= (r^2 - \tfrac{1}{4})a_0 x^r + \left[(r+1)^2 - \tfrac{1}{4}\right] a_1 x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \left\{\left[(r+n)^2 - \tfrac{1}{4}\right] a_n + a_{n-2}\right\} x^{r+n} = 0. \end{aligned} \tag{17}$$

The roots of the indicial equation are $r_1 = \frac{1}{2}$, $r_2 = -\frac{1}{2}$; hence the roots differ by an integer. The recurrence relation is

$$\left[(r+n)^2 - \tfrac{1}{4}\right] a_n = -a_{n-2}, \qquad n \geq 2. \tag{18}$$

Corresponding to the larger root $r_1 = \frac{1}{2}$ we find from the coefficient of x^{r+1} in Eq. (17) that $a_1 = 0$. Hence, from Eq. (18), $a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$. Further, for $r = \frac{1}{2}$,

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \qquad n = 2, 4, 6 \ldots,$$

or letting $n = 2m$, we obtain

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \qquad m = 1, 2, 3, \ldots.$$

By solving this recurrence relation we find that

$$a_2 = -\frac{a_0}{3!}, \qquad a_4 = \frac{a_0}{5!}, \ldots$$

and, in general,

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \qquad m = 1, 2, 3, \ldots.$$

Hence, taking $a_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!}\right] = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}, \qquad x > 0. \tag{19}$$

The power series in Eq. (19) is precisely the Taylor series for $\sin x$; hence one solution of the Bessel equation of order one-half is $x^{-1/2}\sin x$. The Bessel function of the first kind of order one-half, $J_{1/2}$, is defined as $(2/\pi)^{1/2}y_1$. Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \qquad x > 0. \tag{20}$$

Corresponding to the root $r_2 = -\frac{1}{2}$ it is possible that we may have difficulty in computing a_1 since $N = r_1 - r_2 = 1$. However, from Eq. (17) for $r = -\frac{1}{2}$ the coefficients of x^r and x^{r+1} are both zero regardless of the choice of a_0 and a_1. Hence a_0 and a_1 can be chosen arbitrarily. From the recurrence relation (18) we obtain a set of even-numbered coefficients corresponding to a_0 and a set of odd-numbered coefficients corresponding to a_1. Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for $r = -\frac{1}{2}$,

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \qquad n = 1, 2, \ldots.$$

Hence

$$\begin{aligned} y_2(x) &= x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\ &= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \qquad x > 0. \end{aligned} \tag{21}$$

The constant a_1 simply introduces a multiple of $y_1(x)$. The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution for which $a_0 = (2/\pi)^{1/2}$ and $a_1 = 0$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad x > 0. \tag{22}$$

The general solution of Eq. (16) is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$.

By comparing Eqs. (20) and (22) with Eqs. (14) and (15) we see that, except for a phase shift of $\pi/4$, the functions $J_{-1/2}$ and $J_{1/2}$ resemble J_0 and Y_0, respectively, for large x. The graphs of $J_{1/2}$ and $J_{-1/2}$ are shown in Figure 5.8.4.

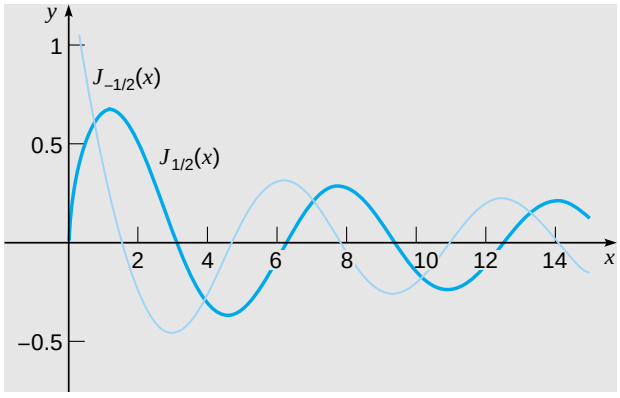


FIGURE 5.8.4 The Bessel functions $J_{1/2}$ and $J_{-1/2}$.

Bessel Equation of Order One. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting $\nu = 1$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + (x^2 - 1)y = 0. \tag{23}$$

If we substitute the series (3) for $y = \phi(r, x)$ and collect terms as in the preceding cases, we obtain

$$L[\phi](r, x) = a_0(r^2 - 1)x^r + a_1[(r + 1)^2 - 1]x^{r+1} + \sum_{n=2}^{\infty}\{[(r + n)^2 - 1]a_n + a_{n-2}\}x^{r+n} = 0. \tag{24}$$

The roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. The recurrence relation is

$$[(r + n)^2 - 1]a_n(r) = -a_{n-2}(r), \qquad n \geq 2. \tag{25}$$

Corresponding to the larger root $r = 1$ the recurrence relation becomes

$$a_n = -\frac{a_{n-2}}{(n + 2)n}, \qquad n = 2, 3, 4, \ldots.$$

We also find from the coefficient of x^{r+1} in Eq. (24) that $a_1 = 0$; hence from the recurrence relation $a_3 = a_5 = \cdots = 0$. For even values of n, let $n = 2m$; then

$$a_{2m} = -\frac{a_{2m-2}}{(2m + 2)(2m)} = -\frac{a_{2m-2}}{2^2(m + 1)m}, \qquad m = 1, 2, 3, \ldots.$$

By solving this recurrence relation we obtain

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m + 1)!m!}, \qquad m = 1, 2, 3, \ldots. \tag{26}$$

The Bessel function of the first kind of order one, denoted by J_1, is obtained by choosing $a_0 = 1/2$. Hence

$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m + 1)!m!}. \tag{27}$$

The series converges absolutely for all x, so the function J_1 is analytic everywhere.

In determining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in Eq. (28) below is rather complicated, but the first few coefficients can be found fairly easily. According to Theorem 5.7.1 we assume that

$$y_2(x) = aJ_1(x)\ln x + x^{-1}\left[1 + \sum_{n=1}^{\infty}c_n x^n\right], \qquad x > 0. \tag{28}$$

Computing $y_2'(x)$, $y_2''(x)$, substituting in Eq. (23), and making use of the fact that J_1 is a solution of Eq. (23) give

$$2axJ_1'(x) + \sum_{n=0}^{\infty}[(n - 1)(n - 2)c_n + (n - 1)c_n - c_n]x^{n-1} + \sum_{n=0}^{\infty}c_n x^{n+1} = 0, \tag{29}$$

where $c_0 = 1$. Substituting for $J_1(x)$ from Eq. (27), shifting the indices of summation in the two series, and carrying out several steps of algebra give

$$-c_1 + [0 \cdot c_2 + c_0]x + \sum_{n=2}^{\infty} [(n^2-1)c_{n+1} + c_{n-1}]x^n = -a\left[x + \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)x^{2m+1}}{2^{2m}(m+1)!\,m!}\right]. \tag{30}$$

From Eq. (30) we observe first that $c_1 = 0$, and $a = -c_0 = -1$. Further, since there are only odd powers of x on the right, the coefficient of each even power of x on the left must be zero. Thus, since $c_1 = 0$, we have $c_3 = c_5 = \cdots = 0$. Corresponding to the odd powers of x we obtain the recurrence relation [let $n = 2m + 1$ in the series on the left side of Eq. (30)]

$$[(2m+1)^2 - 1]c_{2m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!\,m!}, \qquad m = 1, 2, 3, \ldots . \tag{31}$$

When we set $m = 1$ in Eq. (31), we obtain

$$(3^2 - 1)c_4 + c_2 = (-1)3/(2^2 \cdot 2!).$$

Notice that c_2 can be selected *arbitrarily*, and then this equation determines c_4. Also notice that in the equation for the coefficient of x, c_2 appeared multiplied by 0, and that equation was used to determine a. That c_2 is arbitrary is not surprising, since c_2 is the coefficient of x in the expression $x^{-1}[1 + \sum_{n=1}^{\infty} c_n x^n]$. Consequently, c_2 simply generates a multiple of J_1, and y_2 is only determined up to an additive multiple of J_1. In accord with the usual practice we choose $c_2 = 1/2^2$. Then we obtain

$$c_4 = \frac{-1}{2^4 \cdot 2}\left[\frac{3}{2} + 1\right] = \frac{-1}{2^4 2!}\left[\left(1 + \frac{1}{2}\right) + 1\right] = \frac{(-1)}{2^4 \cdot 2!}(H_2 + H_1).$$

It is possible to show that the solution of the recurrence relation (31) is

$$c_{2m} = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m} m!(m-1)!}, \qquad m = 1, 2, \ldots$$

with the understanding that $H_0 = 0$. Thus

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m!(m-1)!}\, x^{2m}\right], \qquad x > 0. \tag{32}$$

The calculation of $y_2(x)$ using the alternative procedure [see Eqs. (19) and (20) of Section 5.7] in which we determine the $c_n(r_2)$ is slightly easier. In particular the latter procedure yields the general formula for c_{2m} without the necessity of solving a recurrence relation of the form (31) (see Problem 11). In this regard the reader may also wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 10.

The second solution of Eq. (23), the Bessel function of the second kind of order one, Y_1, is usually taken to be a certain linear combination of J_1 and y_2. Following Copson (Chapter 12), Y_1 is defined as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)\, J_1(x)], \tag{33}$$

where γ is defined in Eq. (12). The general solution of Eq. (23) for $x > 0$ is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that while J_1 is analytic at $x = 0$, the second solution Y_1 becomes unbounded in the same manner as $1/x$ as $x \to 0$. The graphs of J_1 and Y_1 are shown in Figure 5.8.5.

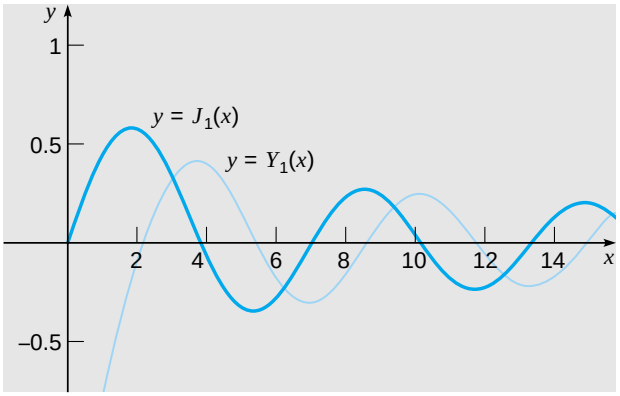


FIGURE 5.8.5 The Bessel functions J_1 and Y_1.

PROBLEMS

In each of Problems 1 through 4 show that the given differential equation has a regular singular point at $x = 0$, and determine two linearly independent solutions for $x > 0$.

1. $x^2y'' + 2xy' + xy = 0$
2. $x^2y'' + 3xy' + (1 + x)y = 0$
3. $x^2y'' + xy' + 2xy = 0$
4. $x^2y'' + 4xy' + (2 + x)y = 0$
5. Find two linearly independent solutions of the Bessel equation of order $\frac{3}{2}$,

$$x^2y'' + xy' + (x^2 - \tfrac{9}{4})y = 0, \qquad x > 0.$$

6. Show that the Bessel equation of order one-half,

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0, \qquad x > 0,$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

7. Show directly that the series for $J_0(x)$, Eq. (7), converges absolutely for all x.
8. Show directly that the series for $J_1(x)$, Eq. (27), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

9. Consider the Bessel equation of order ν,

$$x^2y'' + xy' + (x^2 - \nu^2) = 0, \qquad x > 0.$$

Take ν real and greater than zero.
(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are ν and $-\nu$.
(b) Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^\nu \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)} \left(\frac{x}{2}\right)^{2m}\right].$$

(c) If 2ν is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\cdots(m-1-\nu)(m-\nu)} \left(\frac{x}{2}\right)^{2m}\right].$$

Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.
(d) Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution provided only that ν is not an integer.

10. In this section we showed that one solution of Bessel's equation of order zero,

$$L[y] = x^2y'' + xy' + x^2y = 0,$$

is J_0, where $J_0(x)$ is given by Eq. (7) with $a_0 = 1$. According to Theorem 5.7.1 a second solution has the form ($x > 0$)

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

(a) Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \tag{i}$$

(b) Substituting the series representation for $J_0(x)$ in Eq. (i), show that

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}. \tag{ii}$$

(c) Note that only even powers of x appear on the right side of Eq. (ii). Show that $b_1 = b_3 = b_5 = \cdots = 0$, $b_2 = 1/2^2(1!)^2$, and that

$$(2n)^2 b_{2n} + b_{2n-2} = -2(-1)^n(2n)/2^{2n}(n!)^2, \qquad n = 2, 3, 4, \ldots.$$

Deduce that

$$b_4 = -\frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right) \qquad \text{and} \qquad b_6 = \frac{1}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

The general solution of the recurrence relation is $b_{2n} = (-1)^{n+1}H_n/2^{2n}(n!)^2$. Substituting for b_n in the expression for $y_2(x)$ we obtain the solution given in Eq. (10).

11. Find a second solution of Bessel's equation of order one by computing the $c_n(r_2)$ and a of Eq. (24) of Section 5.7 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use Eq. (24) of this

section to show that $a_1(-1)$ and $a_1'(-1)$ are 0. Then show that $c_1(-1) = 0$ and, from the recurrence relation, that $c_n(-1) = 0$ for $n = 3, 5, \ldots$. Finally, use Eq. (25) to show that

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+1)(r+3)^2 \cdots (r+2m-1)^2(r+2m+1)},$$

for $m = 1, 2, 3, \ldots$, and calculate

$$c_{2m}(-1) = (-1)^{m+1}(H_m + H_{m-1})/2^{2m} m!(m-1)!.$$

12. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of

$$x^2 y'' + (\alpha^2 \beta^2 x^{2\beta} + \tfrac{1}{4} - \nu^2 \beta^2) y = 0, \qquad x > 0$$

is given by $y = x^{1/2} f(\alpha x^\beta)$ where $f(\xi)$ is a solution of the Bessel equation of order ν.

13. Using the result of Problem 12 show that the general solution of the Airy equation

$$y'' - xy = 0, \qquad x > 0$$

is $y = x^{1/2}[c_1 f_1(\frac{2}{3} i x^{3/2}) + c_2 f_2(\frac{2}{3} i x^{3/2})]$ where $f_1(\xi)$ and $f_2(\xi)$ are linearly independent solutions of the Bessel equation of order one-third.

14. It can be shown that J_0 has infinitely many zeros for $x > 0$. In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.8.1). Let λ_j, $j = 1, 2, 3, \ldots$, denote the zeros of J_0; it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that $y = J_0(\lambda_j x)$ satisfies the differential equation

$$y'' + \frac{1}{x} y' + \lambda_j^2 y = 0, \qquad x > 0.$$

Hence show that

$$\int_0^1 x J_0(\lambda_i x) J_0(\lambda_j x)\, dx = 0 \qquad \text{if} \qquad \lambda_i \neq \lambda_j.$$

This important property of $J_0(\lambda_i x)$, known as the orthogonality property, is useful in solving boundary value problems.

Hint: Write the differential equation for $J_0(\lambda_i x)$. Multiply it by $x J_0(\lambda_j x)$ and subtract it from $x J_0(\lambda_i x)$ times the differential equation for $J_0(\lambda_j x)$. Then integrate from 0 to 1.

REFERENCES

Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice Hall, 1961; New York: Dover, 1989).

Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable* (Oxford: Oxford University, 1935).

Proofs of Theorems 5.3.1 and 5.7.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, or Chapters 3 and 4 of:

Rainville, E. D., *Intermediate Differential Equations* (2nd ed.) (New York: Macmillan, 1964).

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 21 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of:

Coddington, E. A., and Levinson, N., *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955).

More complete discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. A text dealing with special functions such as the Legendre polynomials and the Bessel functions is:

Hochstadt, H., *Special Functions of Mathematical Physics* (New York: Holt, 1961).

An excellent compilation of formulas, graphs, and tables of Bessel functions, Legendre functions, and other special functions of mathematical physics may be found in:

Abramowitz, M., and Stegun, I. A. (eds.), *Handbook of Mathematical Functions* (New York: Dover, 1965); originally published by the National Bureau of Standards, Washington, DC, 1964.

CHAPTER
6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those arising in engineering applications.

6.1 Definition of the Laplace Transform

Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt, \tag{1}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. It is possible that $\alpha = -\infty$ or $\beta = \infty$, or both. The relation (1) transforms the function f into another function F, which is called the transform of f. The general idea in using an integral transform to solve a differential equation is as follows: Use the relation (1) to transform a problem for an unknown function f into a simpler problem for F, then solve this simpler problem to find F, and finally recover the desired function f from its transform F. This last step is known as "inverting the transform."

There are several integral transforms that are useful in applied mathematics, but in this chapter we consider only the Laplace[1] transform. This transform is defined in the following way. Let $f(t)$ be given for $t \geq 0$, and suppose that f satisfies certain conditions to be stated a little later. Then the Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{2}$$

The Laplace transform makes use of the kernel $K(s,t) = e^{-st}$. Since the solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.

Since the Laplace transform is defined by an integral over the range from zero to infinity, it is useful to review some basic facts about such integrals. In the first place, an integral over an unbounded interval is called an **improper integral**, and is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\, dt = \lim_{A\to\infty} \int_a^A f(t)\, dt, \tag{3}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Let $f(t) = e^{ct}$, $t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}\, dt = \lim_{A\to\infty} \int_0^A e^{ct}\, dt = \lim_{A\to\infty} \frac{e^{ct}}{c}\bigg|_0^A$$
$$= \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1).$$

It follows that the improper integral converges if $c < 0$, and diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1, and the integral again diverges.

EXAMPLE 2

Let $f(t) = 1/t$, $t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

[1]The Laplace transform is named for the eminent French mathematician P. S. Laplace, who studied the relation (2) in 1782. However, the techniques described in this chapter were not developed until a century or more later. They are due mainly to Oliver Heaviside (1850–1925), an innovative but unconventional English electrical engineer, who made significant contributions to the development and application of electromagnetic theory.

EXAMPLE 3

Let $f(t) = t^{-p}$, $t \geq 1$, where p is a real constant and $p \neq 1$; the case $p = 1$ was considered in Example 2. Then

$$\int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\int_1^A t^{-p}\,dt = \lim_{A\to\infty}\frac{1}{1-p}(A^{1-p}-1).$$

As $A \to \infty$, $A^{1-p} \to 0$ if $p > 1$, but $A^{1-p} \to \infty$ if $p < 1$. Hence $\int_1^\infty t^{-p}\,dt$ converges for $p > 1$, but (incorporating the result of Example 2) diverges for $p \leq 1$. These results are analogous to those for the infinite series $\sum_{n=1}^{\infty} n^{-p}$.

Before discussing the possible existence of $\int_a^\infty f(t)\,dt$, it is helpful to define certain terms. A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Figure 6.1.1.

If f is piecewise continuous on the interval $a \leq t \leq A$, then it can be shown that $\int_a^A f(t)\,dt$ exists. Hence, if f is piecewise continuous for $t \geq a$, then $\int_a^A f(t)\,dt$ exists for each $A > a$. However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)\,dt$, as the preceding examples show.

If f cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)\,dt$ may be difficult to apply. Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem, which is analogous to a similar theorem for infinite series.

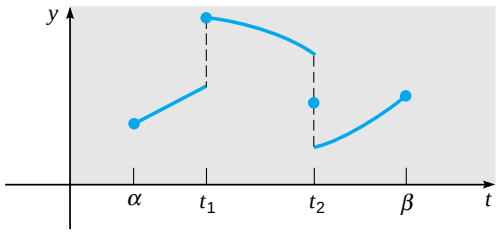


FIGURE 6.1.1 A piecewise continuous function.

Theorem 6.1.1 If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The proof of this result from the calculus will not be given here. It is made plausible, however, by comparing the areas represented by $\int_M^\infty g(t)\,dt$ and $\int_M^\infty |f(t)|\,dt$. The functions most useful for comparison purposes are e^{ct} and t^{-p}, which were considered in Examples 1, 2, and 3.

We now return to a consideration of the Laplace transform $\mathcal{L}\{f(t)\}$ or $F(s)$, which is defined by Eq. (2) whenever this improper integral converges. In general, the parameter s may be complex, but for our discussion we need consider only real values of s. The foregoing discussion of integrals indicates that the Laplace transform F of a function f exists if f satisfies certain conditions, such as those stated in the following theorem.

Theorem 6.1.2 Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (2), exists for $s > a$.

To establish this theorem it is necessary to show only that the integral in Eq. (2) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt. \tag{4}$$

The first integral on the right side of Eq. (4) exists by hypothesis (1) of the theorem; hence the existence of $F(s)$ depends on the convergence of the second integral. By hypothesis (2) we have, for $t \geq M$,

$$|e^{-st} f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t},$$

and thus, by Theorem 6.1.1, $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t}\,dt$ converges. Referring to Example 1 with c replaced by $a - s$, we see that this latter integral converges when $a - s < 0$, which establishes Theorem 6.1.2.

Unless the contrary is specifically stated, in this chapter we deal only with functions satisfying the conditions of Theorem 6.1.2. Such functions are described as piecewise continuous, and of **exponential order** as $t \to \infty$. The Laplace transforms of some important elementary functions are given in the following examples.

EXAMPLE 4

Let $f(t) = 1, t \geq 0$. Then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \frac{1}{s}, \qquad s > 0.$$

EXAMPLE 5

Let $f(t) = e^{at}, t \geq 0$. Then

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt \\ &= \frac{1}{s-a}, \qquad s > a.\end{aligned}$$

EXAMPLE 6

Let $f(t) = \sin at, t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at\, dt, \qquad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at\, dt,$$

upon integrating by parts we obtain

$$\begin{aligned}F(s) &= \lim_{A\to\infty} \left[-\frac{e^{-st}\cos at}{a}\bigg|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at\, dt \right] \\ &= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt.\end{aligned}$$

A second integration by parts then yields

$$\begin{aligned}F(s) &= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\, dt \\ &= \frac{1}{a} - \frac{s^2}{a^2}F(s).\end{aligned}$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\, dt \\ &= c_1\int_0^\infty e^{-st} f_1(t)\, dt + c_2\int_0^\infty e^{-st} f_2(t)\, dt;\end{aligned}$$

hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \tag{5}$$

Equation (5) is a statement of the fact that the Laplace transform is a *linear operator*. This property is of paramount importance, and we make frequent use of it later.

PROBLEMS

In each of Problems 1 through 4 sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2 + t, & 1 < t \le 2 \\ 6 - t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3 - t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) t
   (b) t^2
   (c) t^n, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10 find the Laplace transform of the given function; a and b are real constants.

7. $\cosh bt$
8. $\sinh bt$
9. $e^{at}\cosh bt$
10. $e^{at}\sinh bt$

In each of Problems 11 through 14 recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and $\sin bt = (e^{ibt} - e^{-ibt})/2i$. Assuming that the necessary elementary integration formulas extend to this case, find the Laplace transform of the given function; a and b are real constants.

11. $\sin bt$
12. $\cos bt$
13. $e^{at}\sin bt$
14. $e^{at}\cos bt$

In each of Problems 15 through 20, using integration by parts, find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. te^{at}
16. $t\sin at$
17. $t\cosh at$
18. $t^n e^{at}$
19. $t^2\sin at$
20. $t^2\sinh at$

In each of Problems 21 through 24 determine whether the given integral converges or diverges.

21. $\displaystyle\int_0^\infty (t^2+1)^{-1}\,dt$
22. $\displaystyle\int_0^\infty te^{-t}\,dt$
23. $\displaystyle\int_1^\infty t^{-2}e^{t}\,dt$
24. $\displaystyle\int_0^\infty e^{-t}\cos t\,dt$

25. Suppose that f and f' are continuous for $t \ge 0$, and of exponential order as $t \to \infty$. Show by integration by parts that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.
26. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx. \tag{i}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show that for $p > 0$

$$\Gamma(p+1) = p\Gamma(p).$$

(b) Show that $\Gamma(1) = 1$.

(c) If p is a positive integer n, show that

$$\Gamma(n+1) = n!.$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

(d) Show that for $p > 0$

$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length, say, $0 < p \le 1$. It is possible to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Find $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{11}{2})$.

27. Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 26, show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p\,dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx$$
$$= \Gamma(p+1)/s^{p+1}, \qquad s > 0.$$

(b) Let p be a positive integer n in (a); show that

$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx, \qquad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/2s^{3/2}, \qquad s > 0.$$

6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform in this connection rests primarily on the fact that the transform

of f' is related in a simple way to the transform of f. The relationship is expressed in the following theorem.

Theorem 6.2.1 Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{1}$$

To prove this theorem we consider the integral

$$\int_0^A e^{-st} f'(t)\, dt.$$

If f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1, t_2, \ldots, t_n$. Then we can write this integral as

$$\int_0^A e^{-st} f'(t)\, dt = \int_0^{t_1} e^{-st} f'(t)\, dt + \int_{t_1}^{t_2} e^{-st} f'(t)\, dt + \cdots + \int_{t_n}^A e^{-st} f'(t)\, dt.$$

Integrating each term on the right by parts yields

$$\int_0^A e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_n}^A + s\left[\int_0^{t_1} e^{-st} f(t)\, dt + \int_{t_1}^{t_2} e^{-st} f(t)\, dt + \cdots + \int_{t_n}^A e^{-st} f(t)\, dt\right].$$

Since f is continuous, the contributions of the integrated terms at $t_1, t_2, \ldots, t_n$ cancel. Combining the integrals gives

$$\int_0^A e^{-st} f'(t)\, dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\, dt.$$

As $A \to \infty$, $e^{-sA} f(A) \to 0$ whenever $s > a$. Hence, for $s > a$,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

which establishes the theorem.

If f' and f'' satisfy the same conditions that are imposed on f and f', respectively, in Theorem 6.2.1, then it follows that the Laplace transform of f'' also exists for $s > a$ and is given by

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{2}$$

Indeed, provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the nth derivative $f^{(n)}$ can be derived by successive applications of this theorem. The result is given in the following corollary.

Corollary 6.2.2 Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous, and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a,

and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, ..., $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{3}$$

We now show how the Laplace transform can be used to solve initial value problems. It is most useful for problems involving nonhomogeneous differential equations, as we will demonstrate in later sections of this chapter. However, we begin by looking at some homogeneous equations, which are a bit simpler. For example, consider the differential equation

$$y'' - y' - 2y = 0 \tag{4}$$

and the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{5}$$

This problem is easily solved by the methods of Section 3.1. The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0, \tag{6}$$

and consequently the general solution of Eq. (4) is

$$y = c_1e^{-t} + c_2e^{2t}. \tag{7}$$

To satisfy the initial conditions (5) we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$; hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solution of the initial value problem (4) and (5) is

$$y = \phi(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t}. \tag{8}$$

Now let us solve the same problem by using the Laplace transform. To do this we must assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation (4), we obtain

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0, \tag{9}$$

where we have used the linearity of the transform to write the transform of a sum as the sum of the separate transforms. Upon using the corollary to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$, we find that Eq. (9) becomes

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0,$$

or

$$(s^2 - s - 2)Y(s) + (1-s)y(0) - y'(0) = 0, \tag{10}$$

where $Y(s) = \mathcal{L}\{y\}$. Substituting for $y(0)$ and $y'(0)$ in Eq. (10) from the initial conditions (5), and then solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}. \tag{11}$$

We have thus obtained an expression for the Laplace transform $Y(s)$ of the solution $y = \phi(t)$ of the given initial value problem. To determine the function ϕ we must find the function whose Laplace transform is $Y(s)$, as given by Eq. (11).

This can be done most easily by expanding the right side of Eq. (11) in partial fractions. Thus we write

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}, \tag{12}$$

where the coefficients a and b are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

$$s - 1 = a(s+1) + b(s-2),$$

an equation that must hold for all s. In particular, if we set $s = 2$, then it follows that $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these values for a and b, respectively, we have

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}. \tag{13}$$

Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{3}e^{2t}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Laplace transform,

$$y = \phi(t) = \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t}$$

has the transform (13) and is therefore the solution of the initial value problem (4), (5). Of course, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order linear equation with constant coefficients,

$$ay'' + by' + cy = f(t). \tag{14}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{15}$$

where $F(s)$ is the transform of $f(t)$. By solving Eq. (15) for $Y(s)$ we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{16}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose transform is $Y(s)$.

Even at this early stage of our discussion we can point out some of the essential features of the transform method. In the first place, the transform $Y(s)$ of the unknown function $y = \phi(t)$ is found by solving an *algebraic equation* rather than a *differential equation*, Eq. (10) rather than Eq. (4), or in general Eq. (15) rather than Eq. (14). This is the key to the usefulness of Laplace transforms for solving linear, constant coefficient, ordinary differential equations—the problem is reduced from a differential equation to an algebraic one. Next, the solution satisfying given initial conditions is automatically found, so that the task of determining appropriate values for the arbitrary constants in the general solution does not arise. Further, as indicated in Eq. (15), nonhomogeneous equations are handled in exactly the same way as homogeneous ones; it is not necessary to solve the corresponding homogeneous equation first. Finally, the method can be applied in the same way to higher order equations, as long as we assume that the solution satisfies the conditions of the corollary for the appropriate value of n.

Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the use of a partial fraction expansion of $Y(s)$ to determine $\phi(t)$ requires us to factor this polynomial, the use of Laplace transforms does not avoid the necessity of finding roots of the characteristic equation. For equations of higher than second order this may be a difficult algebraic problem, particularly if the roots are irrational or complex.

The main difficulty that occurs in solving initial value problems by the transform technique lies in the problem of determining the function $y = \phi(t)$ corresponding to the transform $Y(s)$. This problem is known as the inversion problem for the Laplace transform; $\phi(t)$ is called the inverse transform corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as inverting the transform. We also use the notation $\mathcal{L}^{-1}\{Y(s)\}$ to denote the inverse transform of $Y(s)$. There is a general formula for the inverse Laplace transform, but its use requires a knowledge of the theory of functions of a complex variable, and we do not consider it in this book. However, it is still possible to develop many important properties of the Laplace transform, and to solve many interesting problems, without the use of complex variables.

In solving the initial value problem (4), (5) we did not consider the question of whether there may be functions other than the one given by Eq. (8) that also have the transform (13). In fact, it can be shown that if f is a continuous function with the Laplace transform F, then there is no other continuous function having the same transform. In other words, there is essentially a one-to-one correspondence between functions and their Laplace transforms. This fact suggests the compilation of a table, such as Table 6.2.1, giving the transforms of functions frequently encountered, and vice versa. The entries in the second column of Table 6.2.1 are the transforms of those in the first column. Perhaps more important, the functions in the first column are the inverse transforms of those in the second column. Thus, for example, if the transform of the solution of a differential equation is known, the solution itself can often be found merely by looking it up in the table. Some of the entries in Table 6.2.1 have been used as examples, or appear as problems in Section 6.1, while others will be developed later in the chapter. The third column of the table indicates where the derivation of the given transforms may be found. While Table 6.2.1 is sufficient for the examples and problems in this book, much larger tables are also available (see the list of references at the end of the chapter). Transforms and inverse transforms can also be readily obtained electronically by using a computer algebra system.

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms,

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{17}$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property stated previously there is no other continuous function f having the same transform. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}; \tag{18}$$

that is, the inverse Laplace transform is also a linear operator.

In many problems it is convenient to make use of this property by decomposing a given transform into a sum of functions whose inverse transforms are already known or can be found in the table. Partial fraction expansions are particularly useful in this

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)\,f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

connection, and a general result covering many cases is given in Problem 38. Other useful properties of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of solving initial value problems by means of the Laplace transform and partial fraction expansions, consider the following examples.

EXAMPLE 1

Find the solution of the differential equation

$$y'' + y = \sin 2t, \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

We assume that this initial value problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21) we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$\begin{aligned} a + c &= 2, \qquad & b + d &= 1, \\ 4a + c &= 8, \qquad & 4b + d &= 6. \end{aligned}$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \tfrac{5}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{24}$$

EXAMPLE 2

Find the solution of the initial value problem

$$y^{\text{iv}} - y = 0, \tag{25}$$

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 0. \tag{26}$$

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1},$$

and it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{28}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28) we obtain the pair of equations

$$2(a + b) = 1, \qquad 2(-a + b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{29}$$

and from lines 7 and 5 of Table 6.2.1 the solution of the initial value problem (25), (26) is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{30}$$

The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.8. A vibrating spring–mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{31}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation describing an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{32}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$ we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{33}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously in Section 3.8 that Eq. (31) for the spring–mass system and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

PROBLEMS

In each of Problems 1 through 10 find the inverse Laplace transform of the given function.

1. $\dfrac{3}{s^2+4}$
2. $\dfrac{4}{(s-1)^3}$
3. $\dfrac{2}{s^2+3s-4}$
4. $\dfrac{3s}{s^2-s-6}$
5. $\dfrac{2s+2}{s^2+2s+5}$
6. $\dfrac{2s-3}{s^2-4}$
7. $\dfrac{2s+1}{s^2-2s+2}$
8. $\dfrac{8s^2-4s+12}{s(s^2+4)}$
9. $\dfrac{1-2s}{s^2+4s+5}$
10. $\dfrac{2s-3}{s^2+2s+10}$

In each of Problems 11 through 23 use the Laplace transform to solve the given initial value problem.

11. $y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$
13. $y'' - 2y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' - 2y = 0; \quad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = -1$
17. $y^{\text{iv}} - 4y''' + 6y'' - 4y' + y = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
18. $y^{\text{iv}} - y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{\text{iv}} - 4y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \quad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t; \quad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t}; \quad y(0) = 2, \quad y'(0) = -1$

In each of Problems 24 through 26 find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3.

24. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \quad y(0) = 1, \quad y'(0) = 0$

25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

27. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.
(a) Using the Taylor series for $\sin t$,

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \qquad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \ne 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero J_0 has the Taylor series (see Section 5.8)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1,$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/4s}, \qquad s > 0.$$

Problems 28 through 36 are concerned with differentiation of the Laplace transform.

28. Let

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.
(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.
(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 29 through 34 use the result of Problem 28 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

29. te^{at}
30. $t^2 \sin bt$
31. t^n
32. $t^n e^{at}$
33. $te^{at} \sin bt$
34. $te^{at} \cos bt$
35. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Expanding $(1 + s^2)^{-1/2}$ in a binomial series valid for $s > 1$ and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$, and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.8 that the second solution of this equation becomes unbounded as $t \to 0$.

36. For each of the following initial value problems use the results of Problem 28 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.

(a) $y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)

(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \quad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

37. Suppose that

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

38. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = P(s)/Q(s),$$

where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \ldots, r_n$ and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$

where the coefficients $A_1, \ldots, A_n$ must be determined.

(a) Show that

$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \ldots, n. \tag{ii}$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.

(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function**, or **Heaviside function**. This function will be denoted by u_c, and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0. \tag{1}$$

The graph of $y = u_c(t)$ is shown in Figure 6.3.1. The step can also be negative. For instance, Figure 6.3.2 shows the graph $y = 1 - u_c(t)$.

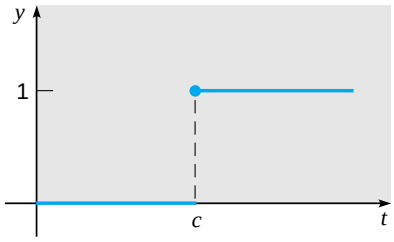


FIGURE 6.3.1 Graph of $y = u_c(t)$.

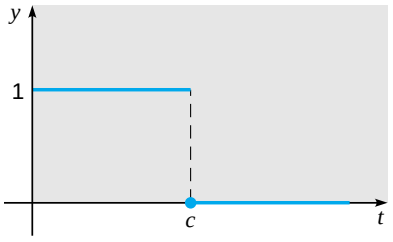


FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

From the definition of $u_c(t)$ in Eq. (1) we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < \infty. \end{cases}$$

Thus the equation $y = h(t)$ has the graph shown in Figure 6.3.3. This function can be thought of as a rectangular pulse.

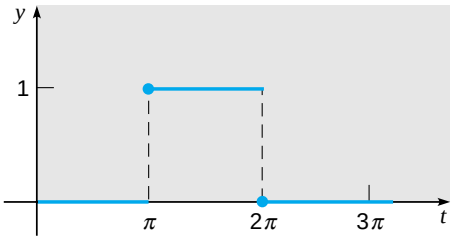


FIGURE 6.3.3 Graph of $y = u_\pi(t) - u_{2\pi}(t)$.

The Laplace transform of u_c is easily determined:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt$$
$$= \frac{e^{-cs}}{s}, \qquad s > 0. \tag{2}$$

For a given function f, defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction; see Figure 6.3.4. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)\, f(t-c).$$

The unit step function is particularly important in transform use because of the following relation between the transform of $f(t)$ and that of its translation $u_c(t)\, f(t-c)$.

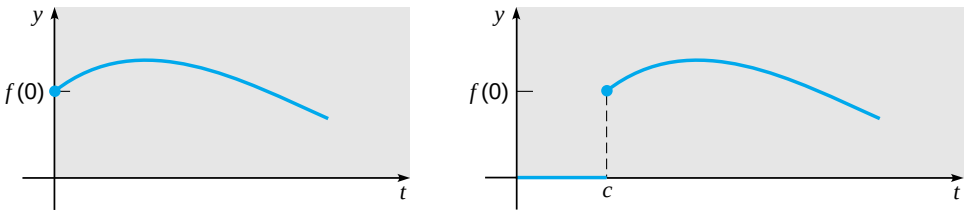


FIGURE 6.3.4 A translation of the given function. (*a*) $y = f(t)$; (*b*) $y = u_c(t)\, f(t-c)$.

Theorem 6.3.1 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{3}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)\, f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{4}$$

Theorem 6.3.1 simply states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1 it is sufficient to compute the transform of $u_c(t)\, f(t-c)$:

$$\begin{aligned}\mathcal{L}\{u_c(t)\, f(t-c)\} &= \int_0^\infty e^{-st} u_c(t)\, f(t-c)\, dt \\ &= \int_c^\infty e^{-st} f(t-c)\, dt.\end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned}\mathcal{L}\{u_c(t)\, f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s} f(\xi)\, d\xi = e^{-cs}\int_0^\infty e^{-s\xi} f(\xi)\, d\xi \\ &= e^{-cs}F(s).\end{aligned}$$

Thus Eq. (3) is established; Eq. (4) follows by taking the inverse transform of both sides of Eq. (3).

A simple example of this theorem occurs if we take $f(t) = 1$. Recalling that $\mathcal{L}\{1\} = 1/s$, we immediately have from Eq. (3) that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$. This result agrees with that of Eq. (2). Examples 2 and 3 illustrate further how Theorem 6.3.1 can be used in the calculation of transforms and inverse transforms.

EXAMPLE 2

If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 6.3.5.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \pi/4, \\ \cos(t - \pi/4), & t \ge \pi/4. \end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t)\cos(t - \pi/4),$$

and

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos(t-\pi/4)\} \\ &= \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}.\end{aligned}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1 + se^{-\pi s/4}}{s^2+1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

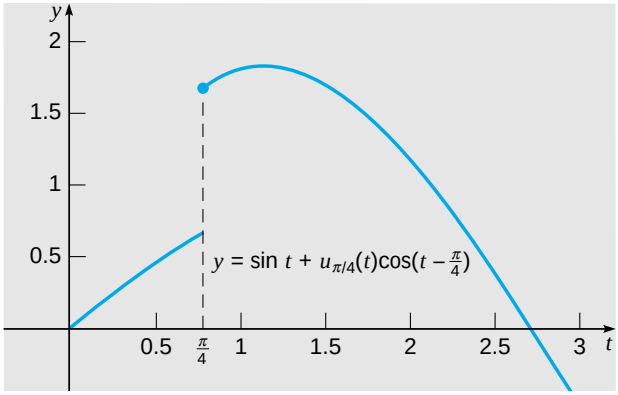


FIGURE 6.3.5 Graph of the function in Example 2.

EXAMPLE 3

Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$
$$= t - u_2(t)(t-2).$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

The following theorem contains another very useful property of Laplace transforms that is somewhat analogous to that given in Theorem 6.3.1.

Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \qquad s > a + c. \tag{5}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{6}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. The proof of this theorem requires merely the evaluation of $\mathcal{L}\{e^{ct} f(t)\}$. Thus

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt = \int_0^\infty e^{-(s-c)t} f(t)\, dt$$
$$= F(s-c),$$

which is Eq. (5). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \le Ke^{at}$; hence $|e^{ct} f(t)| \le Ke^{(a+c)t}$. Equation (6) follows by taking the inverse transform of Eq. (5), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 4.

EXAMPLE 4

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2 + 1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t} \sin t.$$

The results of this section are often useful in solving differential equations, particularly those having discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6 sketch the graph of the given function on the interval $t \ge 0$.

1. $u_1(t) + 2u_3(t) - 6u_4(t)$
2. $(t-3)u_2(t) - (t-2)u_3(t)$
3. $f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $f(t-1)u_2(t)$, where $f(t) = 2t$
6. $f(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12 find the Laplace transform of the given function.

7. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$
8. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$
9. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$
10. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
11. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
12. $f(t) = t - u_1(t)(t-1), \quad t \ge 0$

In each of Problems 13 through 18 find the inverse Laplace transform of the given function.

13. $F(s) = \dfrac{3!}{(s-2)^4}$
14. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
15. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
16. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
17. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
18. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

19. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

In each of Problems 20 through 23 use the results of Problem 19 to find the inverse Laplace transform of the given function.

20. $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

21. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

22. $F(s) = \dfrac{1}{9s^2-12s+3}$

23. $F(s) = \dfrac{e^2e^{-4s}}{2s-1}$

In each of Problems 24 through 27 find the Laplace transform of the given function. In Problem 27 assume that term-by-term integration of the infinite series is permissible.

24. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

25. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

26. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)$

27. $f(t) = 1 + \sum_{k=1}^{\infty}(-1)^k u_k(t)$. See Figure 6.3.6.

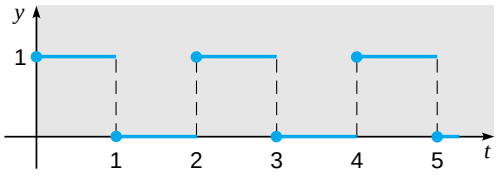


FIGURE 6.3.6 A square wave.

28. Let f satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}.$$

In each of Problems 29 through 32, use the result of Problem 28 to find the Laplace transform of the given function.

29. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
Compare with Problem 27.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
See Figure 6.3.7.

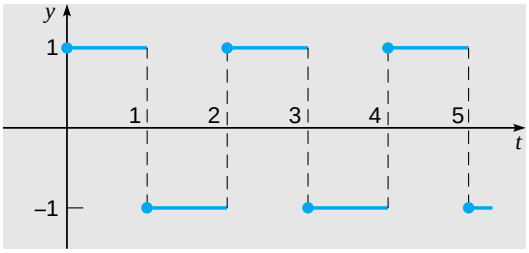


FIGURE 6.3.7 A square wave.

31. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t)$.
See Figure 6.3.8.

32. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t)$.
See Figure 6.3.9.

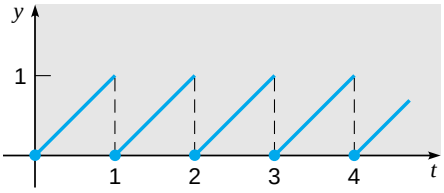


FIGURE 6.3.8 A sawtooth wave.

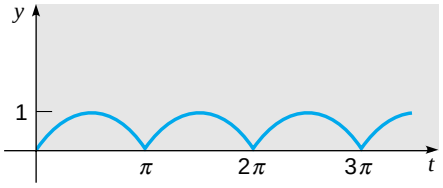


FIGURE 6.3.9 A rectified sine wave.

33. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 24. Sketch the graph of $y = f(t)$.
(b) Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.
(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

34. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.
(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 33(c) and then using the result of Problem 28.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\,dt,$$

where f is the function in Problem 30, and then using Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0. \tag{3}$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \le t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

The Laplace transform of Eq. (1) is

$$\begin{aligned} 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) &= \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} \\ &= (e^{-5s} - e^{-20s})/s. \end{aligned}$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y = \phi(t)$ it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \tag{5}$$

where

$$H(s) = 1/s(2s^2 + s + 2). \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Observe that we have used Theorem 6.3.1 to write the inverse transforms of $e^{-5s}H(s)$ and $e^{-20s}H(s)$, respectively. Finally, to determine $h(t)$ we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} \\ &= \frac{1/2}{s} - \left(\frac{1}{2}\right) \frac{(s + \frac{1}{4}) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}, \end{aligned} \tag{9}$$

so that, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \tfrac{1}{2} - \tfrac{1}{2}[e^{-t/4}\cos(\sqrt{15}\,t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}\,t/4)]. \tag{10}$$

In Figure 6.4.1 the graph of $y = \phi(t)$ from Eqs. (7) and (10) shows that the solution consists of three distinct parts. For $0 < t < 5$ the differential equation is

$$2y'' + y' + 2y = 0 \tag{11}$$

and the initial conditions are given by Eq. (3). Since the initial conditions impart no energy to the system, and since there is no external forcing, the system remains at rest; that is, $y = 0$ for $0 < t < 5$. This can be confirmed by solving Eq. (11) subject to the initial conditions (3). In particular, evaluating the solution and its derivative at $t = 5$, or more precisely, as t approaches 5 from below, we have

$$y(5) = 0, \qquad y'(5) = 0. \tag{12}$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1, \tag{13}$$

whose solution is the sum of a constant (the response to the constant forcing function) and a damped oscillation (the solution of the corresponding homogeneous equation). The plot in Figure 6.4.1 shows this behavior clearly for the interval $5 \le t \le 20$. An expression for this portion of the solution can be found by solving the differential equation (13) subject to the initial conditions (12). Finally, for $t > 20$ the differential equation becomes Eq. (11) again, and the initial conditions are obtained by evaluating the solution of Eqs. (13), (12) and its derivative at $t = 20$. These values are

$$y(20) \cong 0.50162, \qquad y'(20) \cong 0.01125. \tag{14}$$

The initial value problem (11), (14) contains no external forcing, so its solution is a damped oscillation about $y = 0$, as can be seen in Figure 6.4.1.

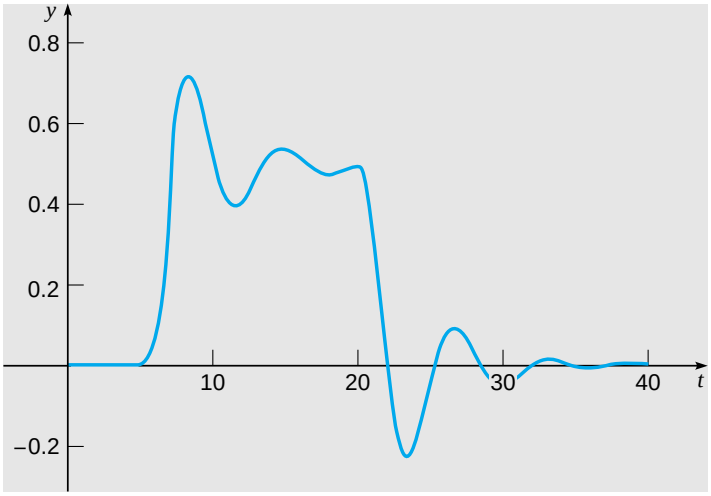


FIGURE 6.4.1 Solution of the initial value problem (1), (2), (3).

While it may be helpful to visualize the solution shown in Figure 6.4.1 as composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others having discontinuous forcing functions.

The effect of the discontinuity in the forcing function can be seen if we examine the solution $\phi(t)$ of Example 1 more closely. According to the existence and uniqueness Theorem 3.2.1 the solution ϕ and its first two derivatives are continuous except possibly at the points $t = 5$ and $t = 20$ where g is discontinuous. This can also be seen at once from Eq. (7). One can also show by direct computation from Eq. (7) that ϕ and ϕ' are continuous even at $t = 5$ and $t = 20$. However, if we calculate ϕ'', we find that

$$\lim_{t\to 5-} \phi''(t) = 0, \qquad \lim_{t\to 5+} \phi''(t) = 1/2.$$

Consequently, $\phi''(t)$ has a jump of $1/2$ at $t = 5$. In a similar way one can show that $\phi''(t)$ has a jump of $-1/2$ at $t = 20$. Thus the jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ on the left side of the equation.

Consider now the general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{15}$$

where p and q are continuous on some interval $\alpha < t < \beta$, but g is only piecewise continuous there. If $y = \psi(t)$ is a solution of Eq. (15), then ψ and ψ' are continuous on $\alpha < t < \beta$, but ψ'' has jump discontinuities at the same points as g. Similar remarks apply to higher order equations; the highest derivative of the solution appearing in the differential equation has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous even at those points.

EXAMPLE 2

Describe the qualitative nature of the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 0, \qquad y'(0) = 0, \tag{17}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10, \end{cases} \tag{18}$$

and then find the solution.

In this example the forcing function has the graph shown in Figure 6.4.2 and is known as ramp loading. It is relatively easy to identify the general form of the solution. For $t < 5$ the solution is simply $y = 0$. On the other hand, for $t > 10$ the solution has the form

$$y = c_1 \cos 2t + c_2 \sin 2t + 1/4. \tag{19}$$

The constant $1/4$ is a particular solution of the nonhomogeneous equation while the other two terms are the general solution of the corresponding homogeneous equation. Thus the solution (19) is a simple harmonic oscillation about $y = 1/4$. Similarly, in the intermediate range $5 < t < 10$, the solution is an oscillation about a certain linear function. In an engineering context, for example, we might be interested in knowing the amplitude of the eventual steady oscillation.

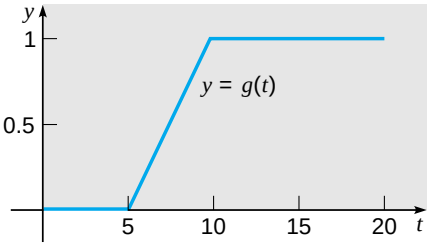


FIGURE 6.4.2 Ramp loading; $y = g(t)$ from Eq. (18).

To solve the problem it is convenient to write

$$g(t) = [u_5(t)(t-5) - u_{10}(t)(t-10)]/5, \tag{20}$$

as you may verify. Then we take the Laplace transform of the differential equation and use the initial conditions, thereby obtaining

$$(s^2 + 4)Y(s) = (e^{-5s} - e^{-10s})/5s^2,$$

or

$$Y(s) = (e^{-5s} - e^{-10s})H(s)/5, \tag{21}$$

where

$$H(s) = 1/s^2(s^2 + 4). \tag{22}$$

Thus the solution of the initial value problem (16), (17), (18) is

$$y = \phi(t) = [u_5(t)h(t-5) - u_{10}(t)h(t-10)]/5, \tag{23}$$

where $h(t)$ is the inverse transform of $H(s)$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}, \tag{24}$$

and it then follows from lines 3 and 5 of Table 6.2.1 that

$$h(t) = \tfrac{1}{4}t - \tfrac{1}{8}\sin 2t. \tag{25}$$

The graph of $y = \phi(t)$ is shown in Figure 6.4.3. Observe that it has the qualitative form that we indicated earlier. To find the amplitude of the eventual steady oscillation it is sufficient to locate one of the maximum or minimum points for $t > 10$. Setting the derivative of the solution (23) equal to zero, we find that the first maximum is located approximately at (10.642, 0.2979), so the amplitude of the oscillation is approximately 0.0479.

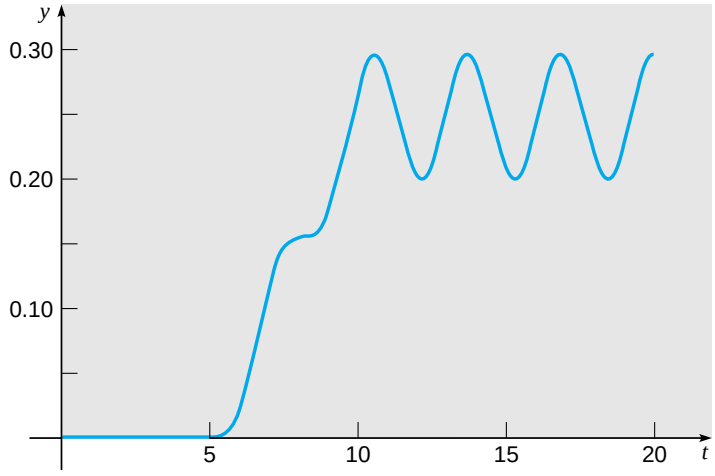


FIGURE 6.4.3 Solution of the initial value problem (16), (17), (18).

Note that in this example the forcing function g is continuous but g' is discontinuous at $t = 5$ and $t = 10$. It follows that the solution ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at $t = 5$ and at $t = 10$ that match the discontinuities in g' at those points.

PROBLEMS

In each of Problems 1 through 13 find the solution of the given initial value problem. Draw the graphs of the solution and of the forcing function; explain how they are related.

1. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & \pi/2 \le t < \infty \end{cases}$

2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \quad y'(0) = 1; \quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \text{ and } t \ge 2\pi \end{cases}$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$
4. $y'' + 4y = \sin t + u_{\pi}(t)\sin(t - \pi); \quad y(0) = 0, \quad y'(0) = 0$
5. $y'' + 3y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0; \quad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$
6. $y'' + 3y' + 2y = u_2(t); \quad y(0) = 0, \quad y'(0) = 1$
7. $y'' + y = u_{3\pi}(t); \quad y(0) = 1, \quad y'(0) = 0$
8. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2); \quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = g(t); \quad y(0) = 0, \quad y'(0) = 1; \quad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$
10. $y'' + y' + \frac{5}{4}y = g(t); \quad y(0) = 0, \quad y'(0) = 0; \quad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$
11. $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$
12. $y^{\text{iv}} - y = u_1(t) - u_2(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. $y^{\text{iv}} + 5y'' + 4y = 1 - u_{\pi}(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.
15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.
16. A certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + u = kg(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.
(a) Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.
(b) Solve the initial value problem.
(c) Plot the solution for $k = 1/2$, $k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on k.
(d) Find, to two decimal places, the smallest value of k for which the solution $u(t)$ reaches the value 2.
(e) Suppose $k = 2$. Find the time τ after which $|u(t)| < 0.1$ for all $t > \tau$.
17. Modify the problem in Example 2 in the text by replacing the given forcing function $g(t)$ by

$$f(t) = [u_5(t)(t - 5) - u_{5+k}(t)(t - 5 - k)]/k.$$

(a) Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?
(b) Solve the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

(c) The solution in part (b) depends on k, but for sufficiently large t the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.
18. Consider the initial value problem

$$y'' + \tfrac{1}{3}y' + 4y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4 - k \le t < 4 + k \\ 0, & 0 \le t < 4 - k \quad \text{and} \quad t \ge 4 + k \end{cases}$$

and $0 < k < 4$.

(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of k. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on k.
(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.
(c) Plot the solution for $k = 2$, $k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on k.

Resonance and Beats. In Section 3.9 we observed that an undamped harmonic oscillator (such as a spring–mass system), with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 19 through 23 we explore the effect of some nonsinusoidal periodic forcing functions.

▶ 19. Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.
(b) Find the solution of the initial value problem.
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

▶ 20. Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f(t)$ is the same as in Problem 19.
(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.
(b) Estimate the amplitude and frequency of the steady-state part of the solution.
(c) Compare the results of part (b) with those from Section 3.9 for a sinusoidally forced oscillator.

▶ 21. Consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).
(b) Find the solution of the initial value problem.
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

▶ 22. Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the same as in Problem 21.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.
(b) Estimate the amplitude and frequency of the steady-state part of the solution.
(c) Compare the results of part (b) with those from Problem 20 and from Section 3.9 for a sinusoidally forced oscillator.

▶ 23. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 19 except that the frequency of the forcing term has been increased somewhat.
(a) Find the solution of this initial value problem.
(b) Let $n \geq 33$ and plot the solution for $0 \leq t \leq 90$ or longer. Your plot should show a clearly recognizable beat.
(c) From the graph in part (b) estimate the "slow period" and the "fast period" for this oscillator.
(d) For a sinusoidally forced oscillator in Section 3.9 it was shown that the "slow frequency" is given by $|\omega - \omega_0|/2$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly the "fast frequency" is $(\omega + \omega_0)/2$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part (c)?

6.5 Impulse Functions

In some applications it is necessary to deal with phenomena of an impulsive nature, for example, voltages or forces of large magnitude that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{1}$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ and is otherwise zero.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt, \tag{2}$$

or, since $g(t) = 0$ outside of the interval $(t_0 - \tau, t_0 + \tau)$,

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\, dt, \tag{3}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force, $I(\tau)$ is the total **impulse** of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$. Similarly, if y is the current in an electric circuit and $g(t)$ is the time derivative of the voltage, then $I(\tau)$ represents the total voltage impressed on the circuit during the interval $(t_0 - \tau, t_0 + \tau)$.

In particular, let us suppose that t_0 is zero, and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau, \\ 0, & t \le -\tau \quad \text{or} \quad t \ge \tau, \end{cases} \tag{4}$$

where τ is a small positive constant (see Figure 6.5.1). According to Eq. (2) or (3) it follows immediately that in this case $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$. Now let us idealize the forcing function d_τ by prescribing it to act over shorter and shorter time intervals; that is, we require that $\tau \to 0$, as indicated in Figure 6.5.2. As a result of this limiting operation we obtain

$$\lim_{\tau \to 0} d_\tau(t) = 0, \qquad t \neq 0. \tag{5}$$

Further, since $I(\tau) = 1$ for each $\tau \neq 0$, it follows that

$$\lim_{\tau \to 0} I(\tau) = 1. \tag{6}$$

Equations (5) and (6) can be used to define an idealized **unit impulse function** δ, which imparts an impulse of magnitude one at $t = 0$, but is zero for all values of t other than zero. That is, the "function" δ is defined to have the properties

$$\delta(t) = 0, \qquad t \neq 0; \tag{7}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{8}$$

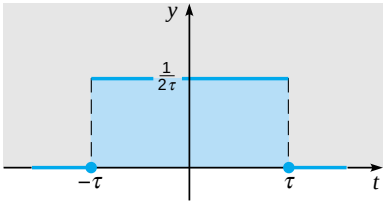


FIGURE 6.5.1 Graph of $y = d_\tau(t)$.

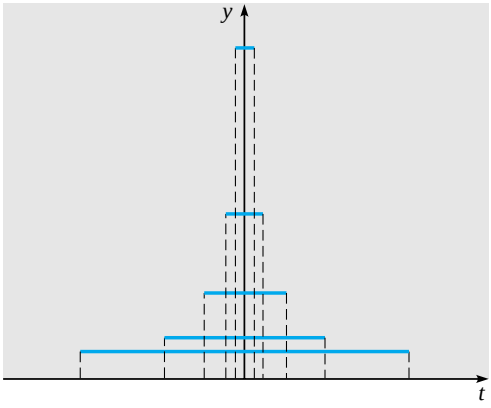


FIGURE 6.5.2 Graphs of $y = d_\tau(t)$ as $\tau \to 0$.

There is no ordinary function of the kind studied in elementary calculus that satisfies both of Eqs. (7) and (8). The "function" δ, defined by those equations, is an example of what are known as generalized functions and is usually called the Dirac[2] **delta function**. Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From Eqs. (7) and (8) it follows that

$$\delta(t - t_0) = 0, \qquad t \neq t_0; \tag{9}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1. \tag{10}$$

The delta function does not satisfy the conditions of Theorem 6.1.2, but its Laplace transform can nevertheless be formally defined. Since $\delta(t)$ is defined as the limit of $d_\tau(t)$ as $\tau \to 0$, it is natural to define the Laplace transform of δ as a similar limit of the transform of d_τ. In particular, we will assume that $t_0 > 0$, and define $\mathcal{L}\{\delta(t - t_0)\}$ by the equation

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0} \mathcal{L}\{d_\tau(t - t_0)\}. \tag{11}$$

To evaluate the limit in Eq. (11) we first observe that if $\tau < t_0$, which must eventually be the case as $\tau \to 0$, then $t_0 - \tau > 0$. Since $d_\tau(t - t_0)$ is nonzero only in the interval from $t_0 - \tau$ to $t_0 + \tau$, we have

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \int_0^{\infty} e^{-st} d_\tau(t - t_0)\, dt \\ &= \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0)\, dt. \end{aligned}$$

Substituting for $d_\tau(t - t_0)$ from Eq. (4), we obtain

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st}\, dt = -\frac{1}{2s\tau} e^{-st}\Big|_{t=t_0-\tau}^{t=t_0+\tau} \\ &= \frac{1}{2s\tau} e^{-st_0}(e^{s\tau} - e^{-s\tau}) \end{aligned}$$

or

$$\mathcal{L}\{d_\tau(t - t_0)\} = \frac{\sinh s\tau}{s\tau} e^{-st_0}. \tag{12}$$

The quotient $(\sinh s\tau)/s\tau$ is indeterminate as $\tau \to 0$, but its limit can be evaluated by L'Hospital's rule. We obtain

$$\lim_{\tau \to 0} \frac{\sinh s\tau}{s\tau} = \lim_{\tau \to 0} \frac{s \cosh s\tau}{s} = 1.$$

Then from Eq. (11) it follows that

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}. \tag{13}$$

[2]Paul A. M. Dirac (1902–1984), English mathematical physicist, received his Ph.D. from Cambridge in 1926 and was professor of mathematics there until 1969. He was awarded the Nobel prize in 1933 (with Erwin Schrödinger) for fundamental work in quantum mechanics. His most celebrated result was the relativistic equation for the electron, published in 1928. From this equation he predicted the existence of an "anti-electron," or positron, which was first observed in 1932. Following his retirement from Cambridge, Dirac moved to the United States and held a research professorship at Florida State University.

Equation (13) defines $\mathcal{L}\{\delta(t-t_0)\}$ for any $t_0 > 0$. We extend this result, to allow t_0 to be zero, by letting $t_0 \to 0$ on the right side of Eq. (13); thus

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} e^{-st_0} = 1. \tag{14}$$

In a similar way it is possible to define the integral of the product of the delta function and any continuous function f. We have

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\, dt. \tag{15}$$

Using the definition (4) of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\, dt &= \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)\, dt \\ &= \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) = f(t^*), \end{aligned}$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence, $t^* \to t_0$ as $\tau \to 0$, and it follows from Eq. (15) that

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0). \tag{16}$$

It is often convenient to introduce the delta function when working with impulse problems, and to operate formally on it as though it were a function of the ordinary kind. This is illustrated in the example below. It is important to realize, however, that the ultimate justification of such procedures must rest on a careful analysis of the limiting operations involved. Such a rigorous mathematical theory has been developed, but we do not discuss it here.

EXAMPLE 1

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \tag{17}$$

$$y(0) = 0, \qquad y'(0) = 0. \tag{18}$$

This initial value problem arises from the study of the same electrical circuit or mechanical oscillator as in Example 1 of Section 6.4. The only difference is in the forcing term.

To solve the given problem we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2} \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}. \tag{19}$$

By Theorem 6.3.2 or from line 9 of Table 6.2.1

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right\} = \tfrac{4}{\sqrt{15}}\, e^{-t/4} \sin \tfrac{\sqrt{15}}{4} t. \tag{20}$$

Hence, by Theorem 6.3.1, we have

$$y = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{2}{\sqrt{15}}\, u_5(t) e^{-(t-5)/4} \sin \tfrac{\sqrt{15}}{4}(t-5), \tag{21}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \tfrac{2}{\sqrt{15}}\, e^{-(t-5)/4} \sin \tfrac{\sqrt{15}}{4}(t-5), & t \geq 5. \end{cases} \tag{22}$$

The graph of Eq. (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous, and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$ and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

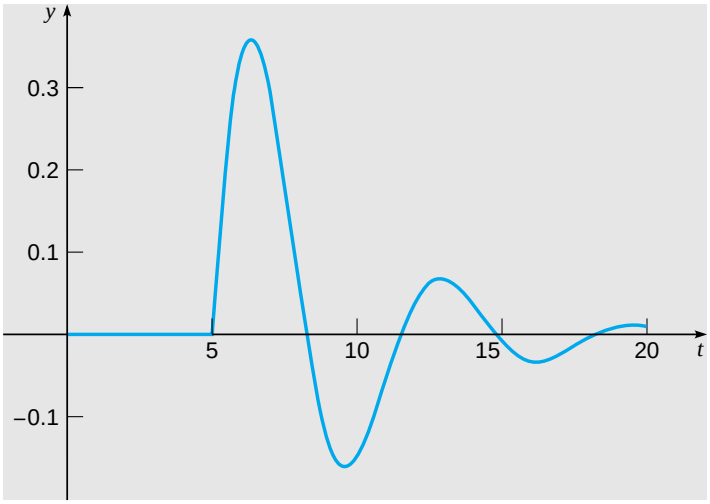


FIGURE 6.5.3 Solution of the initial value problem (17), (18).

PROBLEMS

In each of Problems 1 through 12 find the solution of the given initial value problem and draw its graph.

1. $y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t); \quad y(0) = 0, \quad y'(0) = 1/2$
4. $y'' - y = -20\delta(t - 3); \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \quad y(0) = 0, \quad y'(0) = 0$
6. $y'' + 4y = \delta(t - 4\pi); \quad y(0) = 1/2, \quad y'(0) = 0$
7. $y'' + y = \delta(t - 2\pi)\cos t; \quad y(0) = 0, \quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4); \quad y(0) = 0, \quad y'(0) = 0$

9. $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$
10. $2y'' + y' + 4y = \delta(t - \pi/6)\sin t; \qquad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$
12. $y^{\text{iv}} - y = \delta(t - 1); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. Consider again the system in Example 1 in the text in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle, that is, when the response first returns to equilibrium moving in the positive direction.
    (a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.
    (b) Solve the resulting initial value problem and plot its solution to confirm that it behaves in the specified manner.
14. Consider the initial value problem
$$y'' + \gamma y' + y = \delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$
    where γ is the damping coefficient (or resistance).
    (a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.
    (b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.
    (c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).
    (d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?
15. Consider the initial value problem
$$y'' + \gamma y' + y = k\delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$
    where k is the magnitude of an impulse at $t = 1$ and γ is the damping coefficient (or resistance).
    (a) Let $\gamma = \frac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1.
    (b) Repeat part (a) for $\gamma = \frac{1}{4}$.
    (c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?
16. Consider the initial value problem
$$y'' + y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$
    where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \le 1$.
    (a) Find the solution $y = \phi(t, k)$ of the initial value problem.
    (b) Calculate $\lim_{k\to 0} \phi(t, k)$ from the solution found in part (a).
    (c) Observe that $\lim_{k\to 0} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim_{k\to 0} \phi(t, k)$?
    (d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that
$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$
For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.
(b) Test your prediction by finding the solution and drawing its graph.
(c) Determine what happens after the sequence of impulses ends.

17. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi)$

18. $f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)$

19. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi/2)$

20. $f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi/2)$

21. $f(t) = \sum_{k=1}^{15} \delta[t - (2k-1)\pi]$

22. $f(t) = \sum_{k=1}^{40} (-1)^{k+1} \delta(t - 11k/4)$

23. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.
(a) Try to predict the nature of the solution without solving the problem.
(b) Test your prediction by finding the solution and drawing its graph.
(c) Determine what happens after the sequence of impulses ends.

24. Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k-1)\pi], \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

25. (a) By the method of variation of parameters show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t - \pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t - \pi)$ and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are known as convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in Eq. (2).

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \qquad \text{(commutative law)} \tag{4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \tag{5}$$

$$(f * g) * h = f * (g * h) \qquad \text{(associative law)} \tag{6}$$

$$f * 0 = 0 * f = 0. \tag{7}$$

The proofs of these properties are left to the reader. However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t-\tau)\cdot 1\,d\tau = \int_0^t f(t-\tau)\,d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned}(f * 1)(t) &= \int_0^t \cos(t-\tau)\,d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t.\end{aligned}$$

Clearly, $(f * 1)(t) \neq f(t)$. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

Convolution integrals arise in various applications in which the behavior of the system at time t depends not only on its state at time t, but on its past history as well. Systems of this kind are sometimes called hereditary systems and occur in such diverse fields as neutron transport, viscoelasticity, and population dynamics.

Turning now to the proof of Theorem 6.6.1, we note first that if

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\eta} g(\eta)\, d\eta,$$

then

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \int_0^\infty e^{-s\eta} g(\eta)\, d\eta. \tag{8}$$

Since the integrand of the first integral does not depend on the integration variable of the second, we can write $F(s)G(s)$ as an iterated integral,

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)\, d\xi. \tag{9}$$

This expression can be put into a more convenient form by introducing new variables of integration. First let $\xi = t - \eta$, for fixed η. Then the integral with respect to ξ in Eq. (9) is transformed into one with respect to t; hence

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_\eta^\infty e^{-st} f(t-\eta)\, dt. \tag{10}$$

Next let $\eta = \tau$; then Eq. (10) becomes

$$F(s)G(s) = \int_0^\infty g(\tau)\, d\tau \int_\tau^\infty e^{-st} f(t-\tau)\, dt. \tag{11}$$

The integral on the right side of Eq. (11) is carried out over the shaded wedge-shaped region extending to infinity in the $t\tau$-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we finally obtain

$$F(s)G(s) = \int_0^\infty e^{-st}\, dt \int_0^t f(t-\tau)g(\tau)\, d\tau, \tag{12}$$

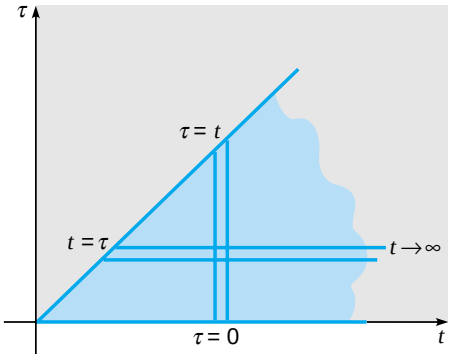


FIGURE 6.6.1 Region of integration in $F(s)G(s)$.

or

$$F(s)G(s) = \int_0^\infty e^{-st}h(t)\,dt$$
$$= \mathcal{L}\{h(t)\}, \tag{13}$$

where $h(t)$ is defined by Eq. (2). This completes the proof of Theorem 6.6.1.

EXAMPLE 1

Find the inverse transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}. \tag{14}$$

It is convenient to think of $H(s)$ as the product of s^{-2} and $a/(s^2+a^2)$, which, according to lines 3 and 5 of Table 6.2.1, are the transforms of t and $\sin at$, respectively. Hence, by Theorem 6.6.1, the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\,d\tau = \frac{at - \sin at}{a^2}. \tag{15}$$

You can verify that the same result is obtained if $h(t)$ is written in the alternate form

$$h(t) = \int_0^t \tau \sin a(t-\tau)\,d\tau,$$

which confirms Eq. (2) in this case. Of course, $h(t)$ can also be found by expanding $H(s)$ in partial fractions.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 3, \qquad y'(0) = -1. \tag{17}$$

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2Y(s) - 3s + 1 + 4Y(s) = G(s),$$

or

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{18}$$

Observe that the first and second terms on the right side of Eq. (18) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{19}$$

Then, using lines 5 and 6 of Table 6.2.1 and Theorem 6.6.1, we obtain

$$y = 3\cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \tag{20}$$

If a specific forcing function g is given, then the integral in Eq. (20) can be evaluated (by numerical means, if necessary).

Example 2 illustrates the power of the convolution integral as a tool for writing the solution of an initial value problem in terms of an integral. In fact, it is possible to proceed in much the same way in more general problems. Consider the problem consisting of the differential equation

$$ay'' + by' + cy = g(t), \tag{21}$$

where a, b, and c are real constants and g is a given function, together with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_0'. \tag{22}$$

The transform approach yields some important insights concerning the structure of the solution of any problem of this type.

The initial value problem (21), (22) is often referred to as an input–output problem. The coefficients a, b, and c describe the properties of some physical system, and $g(t)$ is the input to the system. The values y_0 and y_0' describe the initial state, and the solution y is the output at time t.

By taking the Laplace transform of Eq. (21) and using the initial conditions (22), we obtain

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s).$$

If we let

$$\Phi(s) = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c}, \qquad \Psi(s) = \frac{G(s)}{as^2 + bs + c}, \tag{23}$$

then we can write

$$Y(s) = \Phi(s) + \Psi(s). \tag{24}$$

Consequently,

$$y = \phi(t) + \psi(t), \tag{25}$$

where $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Observe that $y = \phi(t)$ is a solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{26}$$

obtained from Eqs. (21) and (22) by setting $g(t)$ equal to zero. Similarly, $y = \psi(t)$ is the solution of

$$ay'' + by' + cy = g(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{27}$$

in which the initial values y_0 and y_0' are each replaced by zero.

Once specific values of a, b, and c are given, we can find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ by using Table 6.2.1, possibly in conjunction with a translation or a partial fraction expansion. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$ it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{28}$$

where $H(s) = (as^2 + bs + c)^{-1}$. The function H is known as the **transfer function**[3] and depends only on the properties of the system under consideration; that is, $H(s)$ is determined entirely by the coefficients a, b, and c. On the other hand, $G(s)$ depends only on the external excitation $g(t)$ that is applied to the system. By the convolution theorem we can write

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)\,d\tau, \tag{29}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

To obtain a better understanding of the significance of $h(t)$, we consider the case in which $G(s) = 1$; consequently, $g(t) = \delta(t)$ and $\Psi(s) = H(s)$. This means that $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{30}$$

obtained from Eq. (27) by replacing $g(t)$ by $\delta(t)$. Thus $h(t)$ is the response of the system to a unit impulse applied at $t = 0$, and it is natural to call $h(t)$ the **impulse response** of the system. Equation (29) then says that $\psi(t)$ is the convolution of the impulse response and the forcing function.

Referring to Example 2, we note that in that case the transfer function is $H(s) = 1/(s^2 + 4)$, and the impulse response is $h(t) = (\sin 2t)/2$. Also, the first two terms on the right side of Eq. (20) constitute the function $\phi(t)$, the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

PROBLEMS

1. Establish the commutative, distributive, and associative properties of the convolution integral.
   (a) $f * g = g * f$
   (b) $f * (g_1 + g_2) = f * g_1 + f * g_2$
   (c) $f * (g * h) = (f * g) * h$
2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.
3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 7 find the Laplace transform of the given function.

4. $f(t) = \displaystyle\int_0^t (t-\tau)^2 \cos 2\tau\, d\tau$
5. $f(t) = \displaystyle\int_0^t e^{-(t-\tau)} \sin\tau\, d\tau$
6. $f(t) = \displaystyle\int_0^t (t-\tau)e^{\tau}\, d\tau$
7. $f(t) = \displaystyle\int_0^t \sin(t-\tau)\cos\tau\, d\tau$

In each of Problems 8 through 11 find the inverse Laplace transform of the given function by using the convolution theorem.

8. $F(s) = \dfrac{1}{s^4(s^2+1)}$
9. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$
10. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$
11. $F(s) = \dfrac{G(s)}{s^2+1}$

[3]This terminology arises from the fact that $H(s)$ is the ratio of the transforms of the output and the input of the problem (27).

In each of Problems 12 through 19 express the solution of the given initial value problem in terms of a convolution integral.

12. $y'' + \omega^2 y = g(t); \qquad y(0) = 0, \quad y'(0) = 1$
13. $y'' + 2y' + 2y = \sin \alpha t; \qquad y(0) = 0, \quad y'(0) = 0$
14. $4y'' + 4y' + 17y = g(t); \qquad y(0) = 0, \quad y'(0) = 0$
15. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \qquad y(0) = 1, \quad y'(0) = -1$
16. $y'' + 4y' + 4y = g(t); \qquad y(0) = 2, \quad y'(0) = -3$
17. $y'' + 3y' + 2y = \cos \alpha t; \qquad y(0) = 1, \quad y'(0) = 0$
18. $y^{\text{iv}} - y = g(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
19. $y^{\text{iv}} + 5y'' + 4y = g(t); \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
20. Consider the equation

$$\phi(t) + \int_0^t k(t - \xi)\phi(\xi)\, d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

21. Consider the Volterra integral equation (see Problem 20)

$$\phi(t) + \int_0^t (t - \xi)\phi(\xi)\, d\xi = \sin 2t.$$

(a) Show that if u is a function such that $u''(t) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

(b) Show that the given integral equation is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t; \qquad u(0) = 0, \quad u'(0) = 0.$$

(c) Solve the given integral equation by using the Laplace transform.
(d) Solve the initial value problem of part (b) and verify that the solution is the same as that obtained in part (c).

22. **The Tautochrone.** A problem of interest in the history of mathematics is that of finding the *tautochrone*—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoulli's solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved.

The geometrical configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}. \tag{i}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b - y}}\, dy. \tag{ii}$$

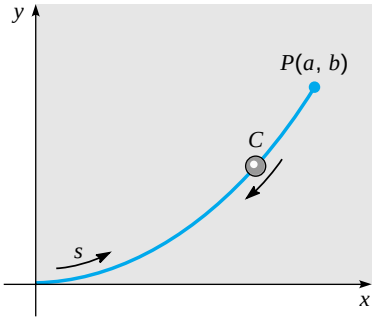


FIGURE 6.6.2 The tautochrone.

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of Eq. (ii) in this case and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}}\,\frac{T_0}{\sqrt{s}}; \tag{iii}$$

then show that

$$f(y) = \sqrt{\frac{2g}{\pi}}\,\frac{T_0}{\sqrt{y}}. \tag{iv}$$

Hint: See Problem 27 of Section 6.1.

(b) Combining Eqs. (i) and (iv), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{v}$$

where $\alpha = gT_0^2/\pi^2$.

(c) Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve Eq. (v), and show that

$$x = \alpha(\theta + \sin\theta), \qquad y = \alpha(1 - \cos\theta). \tag{vi}$$

Equations (vi) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

REFERENCES

The books listed below contain additional information on the Laplace transform and its applications:

Churchill, R. V., *Operational Mathematics* (3rd ed.) (New York: McGraw-Hill, 1971).

Doetsch, G., Nader, W. (tr.), *Introduction to the Theory and Application of the Laplace Transform* (New York: Springer-Verlag, 1974).

Kaplan, W., *Operational Methods for Linear Systems* (Reading, MA: Addison-Wesley, 1962).

Kuhfittig, P. K. F., *Introduction to the Laplace Transform* (New York: Plenum, 1978).

Miles, J. W., *Integral Transforms in Applied Mathematics* (London: Cambridge University Press, 1971).

Rainville, E. D., *The Laplace Transform: An Introduction* (New York: Macmillan, 1963).

Each of the books just mentioned contains a table of transforms. Extensive tables are also available; see, for example:

Erdelyi, A. (ed.), *Tables of Integral Transforms* (Vol. 1) (New York: McGraw-Hill, 1954).

Roberts, G. E., and Kaufman, H., *Table of Laplace Transforms* (Philadelphia: Saunders, 1966).

A further discussion of generalized functions can be found in:

Lighthill, M. J., *Fourier Analysis and Generalized Functions* (London: Cambridge University Press, 1958).

CHAPTER
7

Systems of First Order Linear Equations

There are many physical problems that involve a number of separate elements linked together in some manner. For example, electrical networks have this character, as do some problems in mechanics or in other fields. In these and similar cases the corresponding mathematical problem consists of a *system* of two or more differential equations, which can always be written as first order equations. In this chapter we focus on systems of first order *linear* equations, utilizing some of the elementary aspects of linear algebra to unify the presentation.

7.1 Introduction

Systems of simultaneous ordinary differential equations arise naturally in problems involving several dependent variables, each of which is a function of a single independent variable. We will denote the independent variable by t, and let $x_1, x_2, x_3, \ldots$ represent dependent variables that are functions of t. Differentiation with respect to t will be denoted by a prime.

For example, consider the spring–mass system in Figure 7.1.1. The two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by the three springs whose constants are k_1, k_2, and k_3,

respectively. Using arguments similar to those in Section 3.8, we find the following equations for the coordinates x_1 and x_2 of the two masses:

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= k_2(x_2 - x_1) - k_1x_1 + F_1(t) \\ &= -(k_1 + k_2)x_1 + k_2x_2 + F_1(t), \\ m_2\frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2(x_2 - x_1) + F_2(t) \\ &= k_2x_1 - (k_2 + k_3)x_2 + F_2(t). \end{aligned} \tag{1}$$

A derivation of Eqs. (1) is outlined in Problem 17.

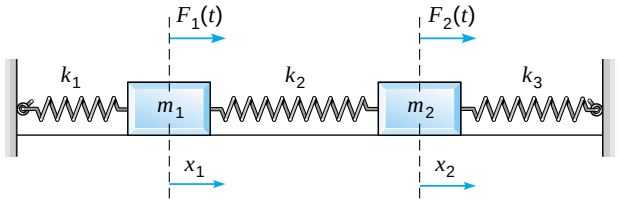


FIGURE 7.1.1 A two degrees of freedom spring–mass system.

Next, consider the parallel *LRC* circuit shown in Figure 7.1.2. Let V be the voltage drop across the capacitor and I the current through the inductor. Then, referring to Section 3.8 and to Problem 18 of this section, we can show that the voltage and current are described by the system of equations

$$\begin{aligned} \frac{dI}{dt} &= \frac{V}{L}, \\ \frac{dV}{dt} &= -\frac{I}{C} - \frac{V}{RC}, \end{aligned} \tag{2}$$

where L is the inductance, C the capacitance, and R the resistance.

As a final example, we mention the predator–prey problem, one of the fundamental problems of mathematical ecology, which is discussed in more detail in Section 9.5. Let $H(t)$ and $P(t)$ denote the populations at time t of two species, one of which (P) preys on the other (H). For example, $P(t)$ and $H(t)$ may be the number of foxes and rabbits, respectively, in a woods, or the number of bass and redear (which are eaten by bass) in a pond. Without the prey the predators will decrease, and without the predator

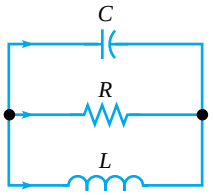


FIGURE 7.1.2 A parallel *LRC* circuit.

the prey will increase. A mathematical model showing how an ecological balance can be maintained when both are present was proposed about 1925 by Lotka and Volterra. The model consists of the system of differential equations

$$\begin{aligned} dH/dt &= a_1 H - b_1 HP, \\ dP/dt &= -a_2 P + b_2 HP, \end{aligned} \tag{3}$$

known as the predator–prey equations. In Eqs. (3) the coefficient a_1 is the birth rate of the population H; similarly, a_2 is the death rate of the population P. The HP terms in the two equations model the interaction of the two populations. The number of encounters between predator and prey is assumed to be proportional to the product of the populations. Since any such encounter tends to be good for the predator and bad for the prey, the sign of the HP term is negative in the first equation and positive in the second. The coefficients b_1 and b_2 are the coefficients of interaction between predator and prey.

Another reason for the importance of systems of first order equations is that equations of higher order can always be transformed into such systems. This is usually required if a numerical approach is planned, because almost all codes for generating approximate numerical solutions of differential equations are written for systems of first order equations. The following example illustrates how easy it is to make the transformation.

EXAMPLE 1

The motion of a certain spring–mass system (see Example 3 of Section 3.8) is described by the second order differential equation

$$u'' + 0.125u' + u = 0. \tag{4}$$

Rewrite this equation as a system of first order equations.

Let $x_1 = u$ and $x_2 = u'$. Then it follows that $x_1' = x_2$. Further, $u'' = x_2'$. Then, by substituting for u, u', and u'' in Eq. (4), we obtain

$$x_2' + 0.125x_2 + x_1 = 0.$$

Thus x_1 and x_2 satisfy the following system of two first order differential equations:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 - 0.125x_2. \end{aligned} \tag{5}$$

The general equation of motion of a spring–mass system,

$$mu'' + \gamma u' + ku = F(t), \tag{6}$$

can be transformed to a system of first order equations in the same manner. If we let $x_1 = u$ and $x_2 = u'$, and proceed as in Example 1, we quickly obtain the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -(k/m)x_1 - (\gamma/m)x_2 + F(t)/m. \end{aligned} \tag{7}$$

To transform an arbitrary nth order equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) \tag{8}$$

into a system of n first order equations we extend the method of Example 1 by introducing the variables $x_1, x_2, \ldots, x_n$ defined by

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots, \quad x_n = y^{(n-1)}. \tag{9}$$

It then follows immediately that

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \end{aligned} \tag{10}$$

and from Eq. (8)

$$x_n' = F(t, x_1, x_2, \ldots, x_n). \tag{11}$$

Equations (10) and (11) are a special case of the more general system

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n), \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n). \end{aligned} \tag{12}$$

In a similar way the system (1) can be reduced to a system of four first order equations of the form (12), while the systems (2) and (3) are already in this form. In fact, systems of the form (12) include almost all cases of interest, so much of the more advanced theory of differential equations is devoted to such systems.

The system (12) is said to have a **solution** on the interval $I: \alpha < t < \beta$ if there exists a set of n functions

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \quad \ldots, \quad x_n = \phi_n(t), \tag{13}$$

that are differentiable at all points in the interval I and that satisfy the system of equations (12) at all points in this interval. In addition to the given system of differential equations there may also be given initial conditions of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0, \tag{14}$$

where t_0 is a specified value of t in I, and $x_1^0, \ldots, x_n^0$ are prescribed numbers. The differential equations (12) and initial conditions (14) together form an initial value problem.

A solution (13) can be viewed as a set of parametric equations in an n-dimensional space. For a given value of t, Eqs. (13) give values for the coordinates $x_1, \ldots, x_n$ of a point in the space. As t changes, the coordinates in general also change. The collection of points corresponding to $\alpha < t < \beta$ form a curve in the space. It is often helpful to think of the curve as the trajectory or path of a particle moving in accordance with the system of differential equations (12). The initial conditions (14) determine the starting point of the moving particle.

The following conditions on $F_1, F_2, \ldots, F_n$, which are easily checked in specific problems, are sufficient to assure that the initial value problem (12), (14) has a unique solution. Theorem 7.1.1 is analogous to Theorem 2.4.2, the existence and uniqueness theorem for a single first order equation.

Theorem 7.1.1 Let each of the functions $F_1, \ldots, F_n$ and the partial derivatives $\partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n$ be continuous in a region R of $tx_1x_2\cdots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_n < \beta_n$, and let the point $(t_0, x_1^0, x_2^0, \ldots, x_n^0)$ be in R. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system of differential equations (12) that also satisfies the initial conditions (14).

The proof of this theorem can be constructed by generalizing the argument in Section 2.8, but we do not give it here. However, note that in the hypotheses of the theorem nothing is said about the partial derivatives of $F_1, \ldots, F_n$ with respect to the independent variable t. Also, in the conclusion, the length $2h$ of the interval in which the solution exists is not specified exactly, and in some cases may be very short. Finally, the same result can be established on the basis of somewhat weaker, but more complicated hypotheses, so the theorem as stated is not the most general one known, and the given conditions are sufficient, but not necessary, for the conclusion to hold.

If each of the functions $F_1, \ldots, F_n$ in Eqs. (12) is a linear function of the dependent variables $x_1, \ldots, x_n$, then the system of equations is said to be **linear**; otherwise, it is **nonlinear**. Thus the most general system of n first order linear equations has the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{15}$$

If each of the functions $g_1(t), \ldots, g_n(t)$ is zero for all t in the interval I, then the system (15) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**. Observe that the systems (1) and (2) are both linear, but the system (3) is nonlinear. The system (1) is nonhomogeneous unless $F_1(t) = F_2(t) = 0$, while the system (2) is homogeneous. For the linear system (15) the existence and uniqueness theorem is simpler and also has a stronger conclusion. It is analogous to Theorems 2.4.1 and 3.2.1.

Theorem 7.1.2 If the functions $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval $I: \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (15) that also satisfies the initial conditions (14), where t_0 is any point in I and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.

Note that in contrast to the situation for a nonlinear system the existence and uniqueness of the solution of a linear system are guaranteed throughout the interval in which the hypotheses are satisfied. Furthermore, for a linear system the initial values $x_1^0, \ldots, x_n^0$ at $t = t_0$ are completely arbitrary, whereas in the nonlinear case the initial point must lie in the region R defined in Theorem 7.1.1.

The rest of this chapter is devoted to systems of linear first order equations (nonlinear systems are included in the discussion in Chapters 8 and 9). Our presentation makes use of matrix notation and assumes that you have some familiarity with the properties of matrices. The basic facts about matrices are summarized in Sections 7.2 and 7.3 and some more advanced material is reviewed as needed in later sections.

PROBLEMS

In each of Problems 1 through 4 transform the given equation into a system of first order equations.

1. $u'' + 0.5u' + 2u = 0$
2. $u'' + 0.5u' + 2u = 3\sin t$
3. $t^2u'' + tu' + (t^2 - 0.25)u = 0$
4. $u'''' - u = 0$
5. Consider the initial value problem $u'' + p(t)u' + q(t)u = g(t)$, $u(0) = u_0$, $u'(0) = u_0'$. Transform this problem into an initial value problem for two first order equations.
6. Reduce the system (1) to a system of first order equations of the form (12).
7. Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x_1' = -2x_1 + x_2, \qquad x_2' = x_1 - 2x_2.$$

(a) Solve the first equation for x_2 and substitute into the second equation, thereby obtaining a second order equation for x_1. Solve this equation for x_1 and then determine x_2 also.
(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$.
(c) Sketch the curve, for $t \geq 0$, given parametrically by the expressions for x_1 and x_2 obtained in part (b).

In each of Problems 8 through 12 proceed as in Problem 7 to transform the given system into a single equation of second order. Then find x_1 and x_2 that also satisfy the given initial conditions. Finally, sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

8. $x_1' = 3x_1 - 2x_2, \quad x_1(0) = 3$
   $x_2' = 2x_1 - 2x_2, \quad x_2(0) = \frac{1}{2}$
9. $x_1' = 1.25x_1 + 0.75x_2, \quad x_1(0) = -2$
   $x_2' = 0.75x_1 + 1.25x_2, \quad x_2(0) = 1$
10. $x_1' = x_1 - 2x_2, \quad x_1(0) = -1$
    $x_2' = 3x_1 - 4x_2, \quad x_2(0) = 2$
11. $x_1' = 2x_2, \quad x_1(0) = 3$
    $x_2' = -2x_1, \quad x_2(0) = 4$
12. $x_1' = -0.5x_1 + 2x_2, \quad x_1(0) = -2$
    $x_2' = -2x_1 - 0.5x_2, \quad x_2(0) = 2$
13. Transform Eqs. (2) for the parallel circuit into a single second order equation.
14. Show that if a_{11}, a_{12}, a_{21}, and a_{22} are constants with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$x_1' = a_{11}x_1 + a_{12}x_2 + g_1(t), \qquad x_1(0) = x_1^0$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + g_2(t), \qquad x_2(0) = x_2^0$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?
15. Consider the linear homogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y,$$
$$y' = p_{21}(t)x + p_{22}(t)y.$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition.
16. Let $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ be any two solutions of the linear nonhomogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y + g_1(t),$$
$$y' = p_{21}(t)x + p_{22}(t)y + g_2(t).$$

Show that $x = x_1(t) - x_2(t)$, $y = y_1(t) - y_2(t)$ is a solution of the corresponding homogeneous system.

17. Equations (1) can be derived by drawing a free-body diagram showing the forces acting on each mass. Figure 7.1.3*a* shows the situation when the displacements x_1 and x_2 of the two masses are both positive (to the right) and $x_2 > x_1$. Then springs 1 and 2 are elongated and spring 3 is compressed, giving rise to forces as shown in Figure 7.1.3*b*. Use Newton's law ($F = ma$) to derive Eqs. (1).

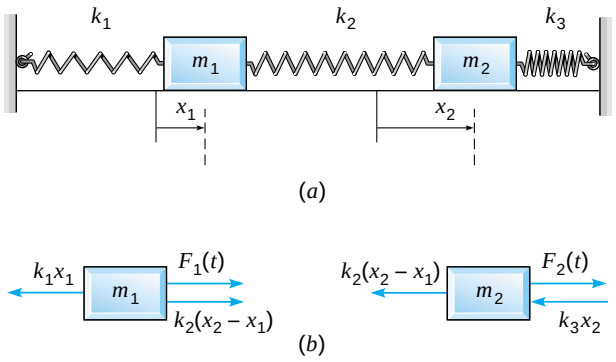


FIGURE 7.1.3 (*a*) The displacements x_1 and x_2 are both positive. (*b*) The free-body diagram for the spring–mass system.

Electric Circuits. The theory of electric circuits, such as that shown in Figure 7.1.2, consisting of inductors, resistors, and capacitors, is based on Kirchhoff's laws: (1) The net flow of current through each node (or junction) is zero, (2) the net voltage drop around each closed loop is zero. In addition to Kirchhoff's laws we also have the relation between the current I in amperes through each circuit element and the voltage drop V in volts across the element; namely,

$$V = RI, \qquad R = \text{resistance in ohms};$$

$$C\frac{dV}{dt} = I, \qquad C = \text{capacitance in farads}^1;$$

$$L\frac{dI}{dt} = V, \qquad L = \text{inductance in henrys}.$$

Kirchhoff's laws and the current–voltage relation for each circuit element provide a system of algebraic and differential equations from which the voltage and current throughout the circuit can be determined. Problems 18 through 20 illustrate the procedure just described.

18. Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the current through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.
    (a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0. \tag{i}$$

    In a similar way, show that

$$V_2 - V_3 = 0. \tag{ii}$$

[1]Actual capacitors typically have capacitances measured in microfarads. We use farad as the unit for numerical convenience.

(b) Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0. \tag{iii}$$

(c) Use the current–voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \qquad V_2 = RI_2, \qquad LI_3' = V_3. \tag{iv}$$

(d) Eliminate V_2, V_3, I_1, and I_2 among Eqs. (i) through (iv) to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \qquad LI_3' = V_1. \tag{v}$$

Observe that if we omit the subscripts in Eqs. (v), then we have the system (2) of the text.

19. Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 18 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\frac{dI}{dt} = -I - V, \qquad \frac{dV}{dt} = 2I - V.$$

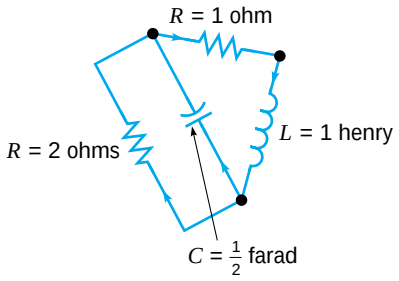


FIGURE 7.1.4 The circuit in Problem 19.

20. Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 18 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \qquad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

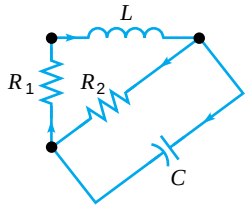


FIGURE 7.1.5 The circuit in Problem 20.

21. Consider the two interconnected tanks shown in Figure 7.1.6. Tank 1 initially contains 30 gal of water and 25 oz of salt, while Tank 2 initially contains 20 gal of water and 15 oz

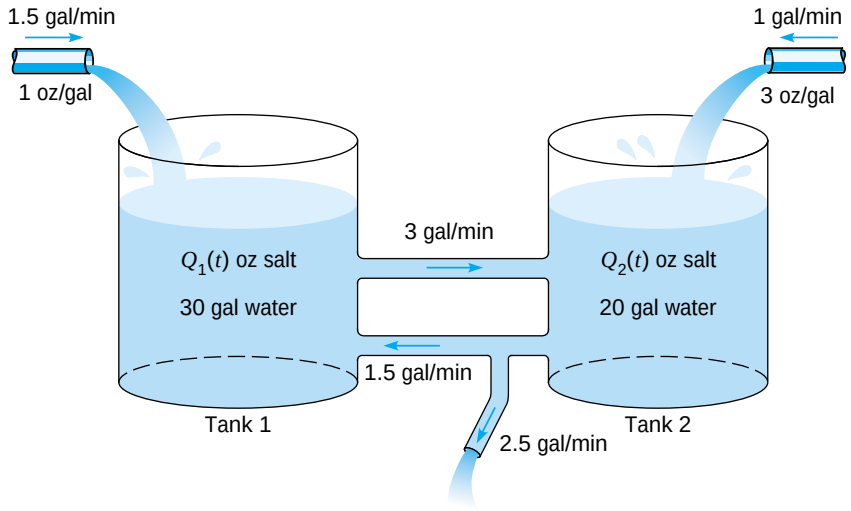


FIGURE 7.1.6 Two interconnected tanks (Problem 21).

of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

(b) Find the values of Q_1 and Q_2 for which the system is in equilibrium, that is, does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?

(c) Let $x_1 = Q_1(t) - Q_1^E$ and $x_2 = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.

22. Consider two interconnected tanks similar to those in Figure 7.1.6. Tank 1 initially contains 60 gal of water and Q_1^0 oz of salt, while Tank 2 initially contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, of which half flows into Tank 2 while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves the tank at a rate of 3 gal/min, of which 1 gal/min flows back into Tank 1, while the rest leaves the system.

(a) Draw a diagram that depicts the flow process described above. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process.

(b) Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

(c) Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

(d) Describe which equilibrium states are possible for this system for various values of q_1 and q_2.

7.2 Review of Matrices

For both theoretical and computational reasons it is advisable to bring to bear some of the results of matrix theory[2] on the initial value problem for a system of linear differential equations. For reference purposes this section and the next are devoted to a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra. We assume, however, that you are familiar with determinants and how to evaluate them.

We designate matrices by boldfaced capitals **A**, **B**, **C**, . . . , occasionally using boldfaced Greek capitals **Φ**, **Ψ**, ... A matrix **A** consists of a rectangular array of numbers, or elements, arranged in m rows and n columns, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{1}$$

We speak of **A** as an $m \times n$ matrix. Although later in the chapter we will often assume that the elements of certain matrices are real numbers, in this section we assume that the elements of matrices may be complex numbers. The element lying in the ith row and jth column is designated by a_{ij}, the first subscript identifying its row and the second its column. Sometimes the notation (a_{ij}) is used to denote the matrix whose generic element is a_{ij}.

Associated with each matrix **A** is the matrix $\mathbf{A}^T$, known as the **transpose** of **A**, and obtained from **A** by interchanging the rows and columns of **A**. Thus, if $\mathbf{A} = (a_{ij})$, then $\mathbf{A}^T = (a_{ji})$. Also, we will denote by $\overline{a}_{ij}$ the complex conjugate of a_{ij}, and by $\overline{\mathbf{A}}$ the matrix obtained from **A** by replacing each element a_{ij} by its conjugate $\overline{a}_{ij}$. The matrix $\overline{\mathbf{A}}$ is called the **conjugate** of **A**. It will also be necessary to consider the transpose of the conjugate matrix $\overline{\mathbf{A}}^T$. This matrix is called the **adjoint** of **A** and will be denoted by $\mathbf{A}^*$.

For example, let

$$\mathbf{A} = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}.$$

Then

$$\mathbf{A}^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}, \qquad \overline{\mathbf{A}} = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix},$$

$$\mathbf{A}^* = \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix}.$$

[2]The properties of matrices were first extensively explored in 1858 in a paper by the English algebraist Arthur Cayley (1821–1895), although the word matrix was introduced by his good friend James Sylvester (1814–1897) in 1850. Cayley did some of his best mathematical work while practicing law from 1849 to 1863; he then became professor of mathematics at Cambridge, a position he held for the rest of his life. After Cayley's groundbreaking work, the development of matrix theory proceeded rapidly, with significant contributions by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.

We are particularly interested in two somewhat special kinds of matrices: square matrices, which have the same number of rows and columns, that is, $m = n$; and vectors (or column vectors), which can be thought of as $n \times 1$ matrices, or matrices having only one column. Square matrices having n rows and n columns are said to be of order n. We denote (column) vectors by boldfaced lowercase letters, $\mathbf{x}$, $\mathbf{y}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, ... The transpose $\mathbf{x}^T$ of an $n \times 1$ column vector is a $1 \times n$ row vector, that is, the matrix consisting of one row whose elements are the same as the elements in the corresponding positions of $\mathbf{x}$.

Properties of Matrices.

1. **Equality.** Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be equal if all corresponding elements are equal, that is, if $a_{ij} = b_{ij}$ for each i and j.

2. **Zero.** The symbol $\mathbf{0}$ will be used to denote the matrix (or vector), each of whose elements is zero.

3. **Addition.** The sum of two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as the matrix obtained by adding corresponding elements:

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}). \tag{2}$$

With this definition it follows that matrix addition is commutative and associative, so that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \qquad \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}. \tag{3}$$

4. **Multiplication by a Number.** The product of a matrix $\mathbf{A}$ by a complex number α is defined as follows:

$$\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij}). \tag{4}$$

The distributive laws

$$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}, \qquad (\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A} \tag{5}$$

are satisfied for this type of multiplication. In particular, the negative of $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined by

$$-\mathbf{A} = (-1)\mathbf{A}. \tag{6}$$

5. **Subtraction.** The difference $\mathbf{A} - \mathbf{B}$ of two $m \times n$ matrices is defined by

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}). \tag{7}$$

Thus

$$\mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}), \tag{8}$$

which is similar to Eq. (2).

6. **Multiplication.** The product $\mathbf{AB}$ of two matrices is defined whenever the number of columns of the first factor is the same as the number of rows in the second. If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ and $n \times r$ matrices, respectively, then the product $\mathbf{C} = \mathbf{AB}$ is an $m \times r$ matrix. The element in the ith row and jth column of $\mathbf{C}$ is found by multiplying each element of the ith row of $\mathbf{A}$ by the corresponding element of the jth column of $\mathbf{B}$ and then adding the resulting products. Symbolically,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{9}$$

By direct calculation it can be shown that matrix multiplication satisfies the associative law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{10}$$

and the distributive law

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}. \tag{11}$$

However, in general, matrix multiplication is not commutative. For both products **AB** and **BA** to exist and to be of the same size, it is necessary that **A** and **B** be square matrices of the same order. Even in that case the two products are usually unequal, so that, in general

$$\mathbf{AB} \neq \mathbf{BA}. \tag{12}$$

EXAMPLE 1

To illustrate the multiplication of matrices, and also the fact that matrix multiplication is not necessarily commutative, consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$

From the definition of multiplication given in Eq. (9) we have

$$\mathbf{AB} = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+0+1 \\ 0+2-2 & 0-2+1 & 0+0-1 \\ 4+1+2 & 2-1-1 & -2+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}.$$

Similarly, we find that

$$\mathbf{BA} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix}.$$

Clearly, $\mathbf{AB} \neq \mathbf{BA}$.

7. **Multiplication of Vectors.** Matrix multiplication also applies as a special case if the matrices **A** and **B** are $1 \times n$ and $n \times 1$ row and column vectors, respectively. Denoting these vectors by $\mathbf{x}^T$ and **y** we have

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i. \tag{13}$$

This is the extension to n dimensions of the familiar dot product from physics and calculus. The result of Eq. (13) is a (complex) number, and it follows directly from Eq. (13) that

$$\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}, \qquad \mathbf{x}^T(\mathbf{y}+\mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}, \qquad (\alpha\mathbf{x})^T\mathbf{y} = \alpha(\mathbf{x}^T\mathbf{y}) = \mathbf{x}^T(\alpha\mathbf{y}). \tag{14}$$

There is another vector product that is also defined for any two vectors having the same number of components. This product, denoted by $(\mathbf{x}, \mathbf{y})$, is called the **scalar** or **inner product**, and is defined by

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y}_i, \tag{15}$$

The scalar product is also a (complex) number, and by comparing Eqs. (13) and (15) we see that

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \overline{\mathbf{y}}. \tag{16}$$

Thus, if all of the elements of $\mathbf{y}$ are real, then the two products (13) and (15) are identical. From Eq. (15) it follows that

$$\begin{aligned} (\mathbf{x}, \mathbf{y}) &= \overline{(\mathbf{y}, \mathbf{x})}, & (\mathbf{x}, \mathbf{y} + \mathbf{z}) &= (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}), \\ (\alpha\mathbf{x}, \mathbf{y}) &= \alpha(\mathbf{x}, \mathbf{y}), & (\mathbf{x}, \alpha\mathbf{y}) &= \overline{\alpha}(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{17}$$

Note that even if the vector $\mathbf{x}$ has elements with nonzero imaginary parts, the scalar product of $\mathbf{x}$ with itself yields a nonnegative real number,

$$(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} x_i \overline{x}_i = \sum_{i=1}^{n} |x_i|^2. \tag{18}$$

The nonnegative quantity $(\mathbf{x}, \mathbf{x})^{1/2}$, often denoted by $\|\mathbf{x}\|$, is called the **length**, or **magnitude**, of $\mathbf{x}$. If $(\mathbf{x}, \mathbf{y}) = 0$, then the two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal**. For example, the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of three-dimensional vector geometry form an orthogonal set. On the other hand, if some of the elements of $\mathbf{x}$ are not real, then the matrix product

$$\mathbf{x}^T \mathbf{x} = \sum_{i=1}^{n} x_i^2 \tag{19}$$

may not be a real number.

For example, let

$$\mathbf{x} = \begin{pmatrix} i \\ -2 \\ 1+i \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 2-i \\ i \\ 3 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{x}^T \mathbf{y} &= (i)(2-i) + (-2)(i) + (1+i)(3) = 4 + 3i, \\ (\mathbf{x}, \mathbf{y}) &= (i)(2+i) + (-2)(-i) + (1+i)(3) = 2 + 7i, \\ \mathbf{x}^T \mathbf{x} &= (i)^2 + (-2)^2 + (1+i)^2 = 3 + 2i, \\ (\mathbf{x}, \mathbf{x}) &= (i)(-i) + (-2)(-2) + (1+i)(1-i) = 7. \end{aligned}$$

8. **Identity.** The multiplicative identity, or simply the identity matrix $\mathbf{I}$, is given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{20}$$

From the definition of matrix multiplication we have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \tag{21}$$

for any (square) matrix $\mathbf{A}$. Hence the commutative law does hold for square matrices if one of the matrices is the identity.

9. **Inverse.** The matrix $\mathbf{A}$ is said to be **nonsingular** or **invertible** if there is another matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is the identity. If there is such a $\mathbf{B}$, it can be shown that there is only one. It is called the multiplicative inverse, or simply the inverse, of $\mathbf{A}$, and we write $\mathbf{B} = \mathbf{A}^{-1}$. Then

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{22}$$

Matrices that do not have an inverse are called **singular** or **noninvertible**.

There are various ways to compute $\mathbf{A}^{-1}$ from $\mathbf{A}$, assuming that it exists. One way involves the use of determinants. Associated with each element a_{ij} of a given matrix is the minor M_{ij}, which is the determinant of the matrix obtained by deleting the ith row and jth column of the original matrix, that is, the row and column containing a_{ij}. Also associated with each element a_{ij} is the cofactor C_{ij} defined by the equation

$$C_{ij} = (-1)^{i+j} M_{ij}. \tag{23}$$

If $\mathbf{B} = \mathbf{A}^{-1}$, then it can be shown that the general element b_{ij} is given by

$$b_{ij} = \frac{C_{ji}}{\det \mathbf{A}}. \tag{24}$$

While Eq. (24) is not an efficient way[3] to calculate $\mathbf{A}^{-1}$, it does suggest a condition that $\mathbf{A}$ must satisfy for it to have an inverse. In fact, the condition is both necessary and sufficient: $\mathbf{A}$ is nonsingular if and only if $\det \mathbf{A} \neq 0$. If $\det \mathbf{A} = 0$, then $\mathbf{A}$ is singular.

Another and usually better way to compute $\mathbf{A}^{-1}$ is by means of elementary row operations. There are three such operations:

1. Interchange of two rows.

2. Multiplication of a row by a nonzero scalar.

3. Addition of any multiple of one row to another row.

Any nonsingular matrix $\mathbf{A}$ can be transformed into the identity $\mathbf{I}$ by a systematic sequence of these operations. It is possible to show that if the same sequence of operations is then performed on $\mathbf{I}$, it is transformed into $\mathbf{A}^{-1}$. The transformation of a matrix by a sequence of elementary row operations is referred to as **row reduction** or **Gaussian elimination**. The following example illustrates the calculation of an inverse matrix in this way.

[3]For large n the number of multiplications required to evaluate $\mathbf{A}^{-1}$ by Eq. (24) is proportional to $n!$. If one uses more efficient methods, such as the row reduction procedure described later, the number of multiplications is proportional only to n^3. Even for small values of n (such as $n = 4$), determinants are not an economical tool in calculating inverses, and row reduction methods are preferred.

EXAMPLE 2

Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

The matrix **A** can be transformed into **I** by the following sequence of operations. The result of each step appears below the statement.

(a) Obtain zeros in the off-diagonal positions in the first column by adding (-3) times the first row to the second row and adding (-2) times the first row to the third row.

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 5 \\ 0 & 4 & 5 \end{pmatrix}$$

(b) Obtain a 1 in the diagonal position in the second column by multiplying the second row by $\frac{1}{2}$.

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{5}{2} \\ 0 & 4 & 5 \end{pmatrix}$$

(c) Obtain zeros in the off-diagonal positions in the second column by adding the second row to the first row and adding (-4) times the second row to the third row.

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & -5 \end{pmatrix}$$

(d) Obtain a 1 in the diagonal position in the third column by multiplying the third row by $(-\frac{1}{5})$.

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

(e) Obtain zeros in the off-diagonal positions in the third column by adding $(-\frac{3}{2})$ times the third row to the first row and adding $(-\frac{5}{2})$ times the third row to the second row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we perform the same sequence of operations in the same order on **I**, we obtain the following sequence of matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -2 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ 4 & -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{pmatrix}, \quad \begin{pmatrix} \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

The last of these matrices is $\mathbf{A}^{-1}$, a result that can be verified by direct multiplication with the original matrix $\mathbf{A}$.

This example is made slightly simpler by the fact that the original matrix $\mathbf{A}$ had a 1 in the upper left corner ($a_{11} = 1$). If this is not the case, then the first step is to produce a 1 there by multiplying the first row by $1/a_{11}$, as long as $a_{11} \neq 0$. If $a_{11} = 0$, then the first row must be interchanged with some other row to bring a nonzero element into the upper left position before proceeding.

Matrix Functions. We sometimes need to consider vectors or matrices whose elements are functions of a real variable t. We write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \tag{25}$$

respectively.

The matrix $\mathbf{A}(t)$ is said to be continuous at $t = t_0$ or on an interval $\alpha < t < \beta$ if each element of $\mathbf{A}$ is a continuous function at the given point or on the given interval. Similarly, $\mathbf{A}(t)$ is said to be differentiable if each of its elements is differentiable, and its derivative $d\mathbf{A}/dt$ is defined by

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt}\right); \tag{26}$$

that is, each element of $d\mathbf{A}/dt$ is the derivative of the corresponding element of $\mathbf{A}$. In the same way the integral of a matrix function is defined as

$$\int_a^b \mathbf{A}(t)\, dt = \left(\int_a^b a_{ij}(t)\, dt\right). \tag{27}$$

For example, if

$$\mathbf{A}(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix},$$

then

$$\mathbf{A}'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}, \qquad \int_0^{\pi} \mathbf{A}(t)\, dt = \begin{pmatrix} 2 & \pi^2/2 \\ \pi & 0 \end{pmatrix}.$$

Many of the rules of elementary calculus extend easily to matrix functions; in particular,

$$\frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{d\mathbf{A}}{dt}, \qquad \text{where } \mathbf{C} \text{ is a constant matrix;} \tag{28}$$

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}; \tag{29}$$

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}. \tag{30}$$

In Eqs. (28) and (30) care must be taken in each term to avoid interchanging the order of multiplication. The definitions expressed by Eqs. (26) and (27) also apply as special cases to vectors.

PROBLEMS

1. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$, find

   (a) $2\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - 4\mathbf{B}$
   (c) $\mathbf{AB}$ (d) $\mathbf{BA}$

2. If $\mathbf{A} = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$, find

   (a) $\mathbf{A} - 2\,\mathbf{B}$ (b) $3\mathbf{A} + \mathbf{B}$
   (c) $\mathbf{AB}$ (d) $\mathbf{BA}$

3. If $\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$, find

   (a) $\mathbf{A}^T$ (b) $\mathbf{B}^T$
   (c) $\mathbf{A}^T + \mathbf{B}^T$ (d) $(\mathbf{A} + \mathbf{B})^T$

4. If $\mathbf{A} = \begin{pmatrix} 3-2i & 1+i \\ 2-i & -2+3i \end{pmatrix}$, find

   (a) $\mathbf{A}^T$ (b) $\overline{\mathbf{A}}$ (c) $\mathbf{A}^*$

5. If $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, verify that $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$.

6. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, verify that

   (a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
   (c) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

7. Prove each of the following laws of matrix algebra:
   (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (b) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
   (c) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$ (d) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
   (e) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (f) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

8. If $\mathbf{x} = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$, find

   (a) $\mathbf{x}^T\mathbf{y}$ (b) $\mathbf{y}^T\mathbf{y}$
   (c) $(\mathbf{x}, \mathbf{y})$ (d) $(\mathbf{y}, \mathbf{y})$

9. If $\mathbf{x} = \begin{pmatrix} 1-2i \\ i \\ 2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 3-i \\ 1+2i \end{pmatrix}$, show that

   (a) $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$ (b) $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$

In each of Problems 10 through 19 either compute the inverse of the given matrix, or else show that it is singular.

10. $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

11. $\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

13. $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$

15. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

16. $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

19. $\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{pmatrix}$

20. Prove that if there are two matrices $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$, then $\mathbf{B} = \mathbf{C}$. This shows that a matrix $\mathbf{A}$ can have only one inverse.

21. If $\mathbf{A}(t) = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix}$ and $\mathbf{B}(t) = \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix}$, find

   (a) $\mathbf{A} + 3\mathbf{B}$ (b) $\mathbf{AB}$

   (c) $d\mathbf{A}/dt$ (d) $\displaystyle\int_0^1 \mathbf{A}(t)\, dt$

In each of Problems 22 through 24 verify that the given vector satisfies the given differential equation.

22. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$

23. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t, \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t$

24. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

In each of Problems 25 and 26 verify that the given matrix satisfies the given differential equation.

25. $\boldsymbol{\Psi}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \boldsymbol{\Psi}, \qquad \boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$

26. $\boldsymbol{\Psi}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \boldsymbol{\Psi}, \qquad \boldsymbol{\Psi}(t) = \begin{pmatrix} e^{t} & e^{-2t} & e^{3t} \\ -4e^{t} & -e^{-2t} & 2e^{3t} \\ -e^{t} & -e^{-2t} & e^{3t} \end{pmatrix}$

7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors



In this section we review some results from linear algebra that are important for the solution of systems of linear differential equations. Some of these results are easily proved and others are not; since we are interested simply in summarizing some useful information in compact form, we give no indication of proofs in either case. All the results in this section depend on some basic facts about the solution of systems of linear algebraic equations.

Systems of Linear Algebraic Equations. A set of n simultaneous linear algebraic equations in n variables,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned} \tag{1}$$

can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2}$$

where the $n \times n$ matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are given, and the components of $\mathbf{x}$ are to be determined. If $\mathbf{b} = \mathbf{0}$, the system is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

If the coefficient matrix $\mathbf{A}$ is nonsingular, that is, if $\det \mathbf{A}$ is not zero, then there is a unique solution of the system (2). Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and the solution can be found by multiplying each side of Eq. (2) on the left by $\mathbf{A}^{-1}$; thus

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3}$$

In particular, the homogeneous problem $\mathbf{A}\mathbf{x} = \mathbf{0}$, corresponding to $\mathbf{b} = \mathbf{0}$ in Eq. (2), has only the trivial solution $\mathbf{x} = \mathbf{0}$.

On the other hand, if $\mathbf{A}$ is singular, that is, if $\det \mathbf{A}$ is zero, then solutions of Eq. (2) either do not exist, or do exist but are not unique. Since $\mathbf{A}$ is singular, $\mathbf{A}^{-1}$ does not exist, so Eq. (3) is no longer valid. The homogeneous system

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{4}$$

has (infinitely many) nonzero solutions in addition to the trivial solution. The situation for the nonhomogeneous system (2) is more complicated. This system has no solution unless the vector $\mathbf{b}$ satisfies a certain further condition. This condition is that

$$(\mathbf{b}, \mathbf{y}) = 0, \tag{5}$$

for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. If condition (5) is met, then the system (2) has (infinitely many) solutions. Each of these solutions has the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}, \tag{6}$$

where $\mathbf{x}^{(0)}$ is a particular solution of Eq. (2), and $\boldsymbol{\xi}$ is any solution of the homogeneous system (4). Note the resemblance between Eq. (6) and the solution of a nonhomogeneous linear differential equation. The proofs of some of the preceding statements are outlined in Problems 25 through 29.

The results in the preceding paragraph are important as a means of classifying the solutions of linear systems. However, for solving particular systems it is generally best to use row reduction to transform the system into a much simpler one from which the solution(s), if there are any, can be written down easily. To do this efficiently we can form the **augmented matrix**

$$\mathbf{A} \,|\, \mathbf{b} = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right) \tag{7}$$

by adjoining the vector $\mathbf{b}$ to the coefficient matrix $\mathbf{A}$ as an additional column. The dashed line replaces the equals sign and is said to partition the augmented matrix. We now perform row operations on the augmented matrix so as to transform $\mathbf{A}$ into a triangular matrix, that is, a matrix whose elements below the main diagonal are all zero. Once this is done, it is easy to see whether the system has solutions, and to find them if it does. Observe that elementary row operations on the augmented matrix (7) correspond to legitimate operations on the equations in the system (1). The following examples illustrate the process.

EXAMPLE 1

Solve the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7, \\ -x_1 + x_2 - 2x_3 &= -5, \\ 2x_1 - x_2 - x_3 &= 4. \end{aligned} \tag{8}$$

The augmented matrix for the system (8) is

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \tag{9}$$

We now perform row operations on the matrix (9) with a view to introducing zeros in the lower left part of the matrix. Each step is described and the result recorded below.

(a) Add the first row to the second row and add (-2) times the first row to the third row.

$$\begin{pmatrix} 1 & -2 & 3 & | & 7 \\ 0 & -1 & 1 & | & 2 \\ 0 & 3 & -7 & | & -10 \end{pmatrix}$$

(b) Multiply the second row by -1.

$$\begin{pmatrix} 1 & -2 & 3 & | & 7 \\ 0 & 1 & -1 & | & -2 \\ 0 & 3 & -7 & | & -10 \end{pmatrix}$$

(c) Add (-3) times the second row to the third row.

$$\begin{pmatrix} 1 & -2 & 3 & | & 7 \\ 0 & 1 & -1 & | & -2 \\ 0 & 0 & -4 & | & -4 \end{pmatrix}$$

(d) Divide the third row by -4.

$$\begin{pmatrix} 1 & -2 & 3 & | & 7 \\ 0 & 1 & -1 & | & -2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

The matrix obtained in this manner corresponds to the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7, \\ x_2 - x_3 &= -2, \\ x_3 &= 1, \end{aligned} \tag{10}$$

which is equivalent to the original system (8). Note that the coefficients in Eqs. (10) form a triangular matrix. From the last of Eqs. (10) we have $x_3 = 1$, from the second equation $x_2 = -2 + x_3 = -1$, and from the first equation $x_1 = 7 + 2x_2 - 3x_3 = 2$. Thus we obtain

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

which is the solution of the given system (8). Incidentally, since the solution is unique, we conclude that the coefficient matrix is nonsingular.

EXAMPLE 2

Discuss solutions of the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1, \\ -x_1 + x_2 - 2x_3 &= b_2, \\ 2x_1 - x_2 + 3x_3 &= b_3 \end{aligned} \tag{11}$$

for various values of b_1, b_2, and b_3.

Observe that the coefficients in the system (11) are the same as those in the system (8) except for the coefficient of x_3 in the third equation. The augmented matrix for the system (11) is

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right). \tag{12}$$

By performing steps (a), (b), and (c) as in Example 1 we transform the matrix (12) into

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{array}\right). \tag{13}$$

The equation corresponding to the third row of the matrix (13) is

$$b_1 + 3b_2 + b_3 = 0; \tag{14}$$

thus the system (11) has no solution unless the condition (14) is satisfied by b_1, b_2, and b_3. It is possible to show that this condition is just Eq. (5) for the system (11).

Let us now assume that $b_1 = 2$, $b_2 = 1$, and $b_3 = -5$, in which case Eq. (14) is satisfied. Then the first two rows of the matrix (13) correspond to the equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2, \\ x_2 - x_3 &= -3. \end{aligned} \tag{15}$$

To solve the system (15) we can choose one of the unknowns arbitrarily and then solve for the other two. Letting $x_3 = \alpha$, where α is arbitrary, it then follows that

$$\begin{aligned} x_2 &= \alpha - 3, \\ x_1 &= 2(\alpha - 3) - 3\alpha + 2 = -\alpha - 4. \end{aligned}$$

If we write the solution in vector notation, we have

$$\mathbf{x} = \begin{pmatrix} -\alpha - 4 \\ \alpha - 3 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}. \tag{16}$$

It is easy to verify that the second term on the right side of Eq. (16) is a solution of the nonhomogeneous system (11), while the first term is the most general solution of the homogeneous system corresponding to (11).

Row reduction is also useful in solving homogeneous systems, and systems in which the number of equations is different from the number of unknowns.

Linear Independence. A set of k vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ is said to be **linearly dependent** if there exists a set of (complex) numbers $c_1, \ldots, c_k$, at least one of which is nonzero, such that

$$c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}. \tag{17}$$

In other words $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are linearly dependent if there is a linear relation among them. On the other hand, if the only set $c_1, \ldots, c_k$ for which Eq. (17) is satisfied is $c_1 = c_2 = \cdots = c_k = 0$, then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are said to be **linearly independent**.

Consider now a set of n vectors, each of which has n components. Let $x_{ij} = x_i^{(j)}$ be the ith component of the vector $\mathbf{x}^{(j)}$, and let $\mathbf{X} = (x_{ij})$. Then Eq. (17) can be written as

$$\begin{pmatrix} x_1^{(1)}c_1 + \cdots + x_1^{(n)}c_n \\ \vdots \qquad\qquad \vdots \\ x_n^{(1)}c_1 + \cdots + x_n^{(n)}c_n \end{pmatrix} = \begin{pmatrix} x_{11}c_1 + \cdots + x_{1n}c_n \\ \vdots \qquad\qquad \vdots \\ x_{n1}c_1 + \cdots + x_{nn}c_n \end{pmatrix} = \mathbf{Xc} = \mathbf{0}. \tag{18}$$

If $\det \mathbf{X} \neq 0$, then the only solution of Eq. (18) is $\mathbf{c} = \mathbf{0}$, but if $\det \mathbf{X} = 0$, there are nonzero solutions. Thus the set of vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ is linearly independent if and only if $\det \mathbf{X} \neq 0$.

EXAMPLE 3

Determine whether the vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix} \tag{19}$$

are linearly independent or linearly dependent. If linearly dependent, find a linear relation among them.

To determine whether $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ are linearly dependent we compute $\det(x_{ij})$, whose columns are the components of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$, respectively. Thus

$$\det(x_{ij}) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix},$$

and an elementary calculation shows that it is zero. Thus $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ are linearly dependent, and there are constants c_1, c_2, and c_3 such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}. \tag{20}$$

Equation (20) can also be written in the form

$$\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{21}$$

and solved by means of elementary row operations starting from the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array}\right). \tag{22}$$

We proceed as in Examples 1 and 2.

(a) Add (-2) times the first row to the second row, and add the first row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right)$$

(b) Divide the second row by -3; then add (-5) times the second row to the third row.

$$\begin{pmatrix} 1 & 2 & -4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Thus we obtain the equivalent system

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0, \\ c_2 - 3c_3 &= 0. \end{aligned} \tag{23}$$

From the second of Eqs. (23) we have $c_2 = 3c_3$, and from the first we obtain $c_1 = 4c_3 - 2c_2 = -2c_3$. Thus we have solved for c_1 and c_2 in terms of c_3, with the latter remaining arbitrary. If we choose $c_3 = -1$ for convenience, then $c_1 = 2$ and $c_2 = -3$. In this case the desired relation (20) becomes

$$2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0}.$$

Frequently, it is useful to think of the columns (or rows) of a matrix $\mathbf{A}$ as vectors. These column (or row) vectors are linearly independent if and only if $\det \mathbf{A} \neq 0$. Further, if $\mathbf{C} = \mathbf{AB}$, then it can be shown that $\det \mathbf{C} = (\det \mathbf{A})(\det \mathbf{B})$. Therefore, if the columns (or rows) of both $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, then the columns (or rows) of $\mathbf{C}$ are also linearly independent.

Now let us extend the concepts of linear dependence and independence to a set of vector functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ defined on an interval $\alpha < t < \beta$. The vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are said to be linearly dependent on $\alpha < t < \beta$ if there exists a set of constants $c_1, \ldots, c_k$, not all of which are zero, such that $c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0}$ for all t in the interval. Otherwise, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are said to be linearly independent. Note that if $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are linearly dependent on an interval, they are linearly dependent at each point in the interval. However, if $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are linearly independent on an interval, they may or may not be linearly independent at each point; they may, in fact, be linearly dependent at each point, but with different sets of constants at different points. See Problem 14 for an example.

Eigenvalues and Eigenvectors. The equation

$$\mathbf{Ax} = \mathbf{y} \tag{24}$$

can be viewed as a linear transformation that maps (or transforms) a given vector $\mathbf{x}$ into a new vector $\mathbf{y}$. Vectors that are transformed into multiples of themselves are important in many applications.[4] To find such vectors we set $\mathbf{y} = \lambda\mathbf{x}$, where λ is a scalar proportionality factor, and seek solutions of the equations

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{25}$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{26}$$

[4]For example, this problem is encountered in finding the principal axes of stress or strain in an elastic body, and in finding the modes of free vibration in a conservative system with a finite number of degrees of freedom.

The latter equation has nonzero solutions if and only if λ is chosen so that

$$\Delta(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{27}$$

Values of λ that satisfy Eq. (27) are called **eigenvalues** of the matrix $\mathbf{A}$, and the nonzero solutions of Eq. (25) or (26) that are obtained by using such a value of λ are called the **eigenvectors** corresponding to that eigenvalue.

If $\mathbf{A}$ is a 2×2 matrix, then Eq. (26) has the form

$$\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

and Eq. (27) becomes

$$\Delta(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0. \tag{29}$$

The following example illustrates how eigenvalues and eigenvectors are found.

EXAMPLE 4

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}. \tag{30}$$

The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0. \tag{32}$$

Thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

To find the eigenvectors we return to Eq. (31) and replace λ by each of the eigenvalues in turn. For $\lambda = 2$ we have

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

Hence each row of this vector equation leads to the condition $x_1 - x_2 = 0$, so x_1 and x_2 are equal, but their value is not determined. If $x_1 = c$, then $x_2 = c$ also and the eigenvector $\mathbf{x}^{(1)}$ is

$$\mathbf{x}^{(1)} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c \neq 0. \tag{34}$$

Usually, we will drop the arbitrary constant c when finding eigenvectors; thus instead of Eq. (34) we write

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{35}$$

and remember that any nonzero multiple of this vector is also an eigenvector. We say that $\mathbf{x}^{(1)}$ is the eigenvector corresponding to the eigenvalue $\lambda_1 = 2$.

Now setting $\lambda = -1$ in Eq. (31), we obtain

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{36}$$

Again we obtain a single condition on x_1 and x_2, namely, $4x_1 - x_2 = 0$. Thus the eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \tag{37}$$

or any nonzero multiple of this vector.

As Example 4 illustrates, eigenvectors are determined only up to an arbitrary nonzero multiplicative constant; if this constant is specified in some way, then the eigenvectors are said to be **normalized**. In Example 4, we set the constant equal to 1, but any other nonzero value could also have been used. Sometimes it is convenient to normalize an eigenvector $\mathbf{x}$ by choosing the constant so that $(\mathbf{x}, \mathbf{x}) = 1$.

Equation (27) is a polynomial equation of degree n in λ, so there are n eigenvalues $\lambda_1, \ldots, \lambda_n$, some of which may be repeated. If a given eigenvalue appears m times as a root of Eq. (27), then that eigenvalue is said to have **multiplicity** m. Each eigenvalue has at least one associated eigenvector, and an eigenvalue of multiplicity m may have q linearly independent eigenvectors, where

$$1 \le q \le m. \tag{38}$$

Examples show that q may be any integer in this interval. If all the eigenvalues of a matrix $\mathbf{A}$ are **simple** (have multiplicity one), then it is possible to show that the n eigenvectors of $\mathbf{A}$, one for each eigenvalue, are linearly independent. On the other hand, if $\mathbf{A}$ has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors associated with $\mathbf{A}$, since for a repeated eigenvalue we may have $q < m$. As we will see in Section 7.8, this fact may lead to complications later on in the solution of systems of differential equations.

EXAMPLE 5

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{39}$$

The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = 0. \tag{41}$$

The roots of Eq. (41) are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -1$. Thus 2 is a simple eigenvalue, and -1 is an eigenvalue of multiplicity 2.

To find the eigenvector $\mathbf{x}^{(1)}$ corresponding to the eigenvalue λ_1 we substitute $\lambda = 2$ in Eq. (40); this gives the system

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{42}$$

We can reduce this to the equivalent system

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{43}$$

by elementary row operations. Solving this system we obtain the eigenvector

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{44}$$

For $\lambda = -1$, Eqs. (40) reduce immediately to the single equation

$$x_1 + x_2 + x_3 = 0. \tag{45}$$

Thus values for two of the quantities x_1, x_2, x_3 can be chosen arbitrarily and the third is determined from Eq. (45). For example, if $x_1 = 1$ and $x_2 = 0$, then $x_3 = -1$, and

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{46}$$

is an eigenvector. Any nonzero multiple of $\mathbf{x}^{(2)}$ is also an eigenvector, but a second independent eigenvector can be found by making another choice of x_1 and x_2; for instance, $x_1 = 0$ and $x_2 = 1$. Again $x_3 = -1$ and

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{47}$$

is an eigenvector linearly independent of $\mathbf{x}^{(2)}$. Therefore in this example two linearly independent eigenvectors are associated with the double eigenvalue.

An important special class of matrices, called **self-adjoint** or **Hermitian** matrices, are those for which $\mathbf{A}^* = \mathbf{A}$; that is, $\overline{a}_{ji} = a_{ij}$. Hermitian matrices include as a subclass real symmetric matrices, that is, matrices that have real elements and for which $\mathbf{A}^T = \mathbf{A}$. The eigenvalues and eigenvectors of Hermitian matrices always have the following useful properties:

1. All eigenvalues are real.

2. There always exists a full set of n linearly independent eigenvectors, regardless of the multiplicities of the eigenvalues.

3. If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues, then $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$. Thus, if all eigenvalues are simple, then the associated eigenvectors form an orthogonal set of vectors.

4. Corresponding to an eigenvalue of multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal. Thus the full set of n eigenvectors can always be chosen to be orthogonal as well as linearly independent.

Example 5 above involves a real symmetric matrix and illustrates properties **1**, **2**, and **3**, but the choice we have made for $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ does not illustrate property **4**. However, it is always possible to choose an $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ so that $(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = 0$. For example, in Example 5 we could have chosen

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

as the eigenvectors associated with the eigenvalue $\lambda = -1$. These eigenvectors are orthogonal to each other as well as to the eigenvector $\mathbf{x}^{(1)}$ corresponding to the eigenvalue $\lambda = 2$. The proofs of statements **1** and **3** above are outlined in Problems 32 and 33.

PROBLEMS

In each of Problems 1 through 5 either solve the given set of equations, or else show that there is no solution.

1. $$\begin{aligned} x_1 \quad - \quad x_3 &= 0 \\ 3x_1 + \ x_2 + \ x_3 &= 1 \\ -x_1 + \ x_2 + 2x_3 &= 2 \end{aligned}$$

2. $$\begin{aligned} x_1 + 2x_2 - \ x_3 &= 1 \\ 2x_1 + \ x_2 + \ x_3 &= 1 \\ x_1 - \ x_2 + 2x_3 &= 1 \end{aligned}$$

3. $$\begin{aligned} x_1 + 2x_2 - \ x_3 &= \ 2 \\ 2x_1 + \ x_2 + \ x_3 &= \ 1 \\ x_1 - \ x_2 + 2x_3 &= -1 \end{aligned}$$

4. $$\begin{aligned} x_1 + 2x_2 - \ x_3 &= 0 \\ 2x_1 + \ x_2 + \ x_3 &= 0 \\ x_1 - \ x_2 + 2x_3 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 \quad - \quad x_3 &= 0 \\ 3x_1 + \ x_2 + \ x_3 &= 0 \\ -x_1 + \ x_2 + 2x_3 &= 0 \end{aligned}$$

In each of Problems 6 through 10 determine whether the given set of vectors is linearly independent. If linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space, but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

6. $\mathbf{x}^{(1)} = (1, 1, 0)$, $\mathbf{x}^{(2)} = (0, 1, 1)$, $\mathbf{x}^{(3)} = (1, 0, 1)$
7. $\mathbf{x}^{(1)} = (2, 1, 0)$, $\mathbf{x}^{(2)} = (0, 1, 0)$, $\mathbf{x}^{(3)} = (-1, 2, 0)$
8. $\mathbf{x}^{(1)} = (1, 2, 2, 3)$, $\mathbf{x}^{(2)} = (-1, 0, 3, 1)$, $\mathbf{x}^{(3)} = (-2, -1, 1, 0)$, $\mathbf{x}^{(4)} = (-3, 0, -1, 3)$
9. $\mathbf{x}^{(1)} = (1, 2, -1, 0)$, $\mathbf{x}^{(2)} = (2, 3, 1, -1)$, $\mathbf{x}^{(3)} = (-1, 0, 2, 2)$, $\mathbf{x}^{(4)} = (3, -1, 1, 3)$
10. $\mathbf{x}^{(1)} = (1, 2, -2)$, $\mathbf{x}^{(2)} = (3, 1, 0)$, $\mathbf{x}^{(3)} = (2, -1, 1)$, $\mathbf{x}^{(4)} = (4, 3, -2)$
11. Suppose that the vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ each have n components, where $n < m$. Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent.

In each of Problems 12 and 13 determine whether the given set of vectors is linearly independent for $-\infty < t < \infty$. If linearly dependent, find the linear relation among them. As in Problems 6 through 10 the vectors are written as row vectors to save space.

12. $\mathbf{x}^{(1)}(t) = (e^{-t}, 2e^{-t})$, $\mathbf{x}^{(2)}(t) = (e^{-t}, e^{-t})$, $\mathbf{x}^{(3)}(t) = (3e^{-t}, 0)$

13. $\mathbf{x}^{(1)}(t) = (2\sin t, \sin t), \qquad \mathbf{x}^{(2)}(t) = (\sin t, 2\sin t)$

14. Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

In each of Problems 15 through 24 find all eigenvalues and eigenvectors of the given matrix.

15. $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

17. $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

18. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

19. $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

20. $\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

23. $\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}$

24. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Problems 25 through 29 deal with the problem of solving $\mathbf{Ax} = \mathbf{b}$ when $\det \mathbf{A} = 0$.

25. Suppose that, for a given matrix $\mathbf{A}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. Show that there is also a nonzero vector $\mathbf{y}$ such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.
26. Show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
27. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{Ax} = \mathbf{b}$ has solutions. Show that $(\mathbf{b}, \mathbf{y}) = 0$, where $\mathbf{y}$ is any solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Verify that this statement is true for the set of equations in Example 2.
Hint: Use the result of Problem 26.
28. Suppose that $\det \mathbf{A} = 0$, and that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Show that if $\boldsymbol{\xi}$ is a solution of $\mathbf{A}\boldsymbol{\xi} = \mathbf{0}$ and α is any constant, then $\mathbf{x} = \mathbf{x}^{(0)} + \alpha\boldsymbol{\xi}$ is also a solution of $\mathbf{Ax} = \mathbf{b}$.
29. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{y}$ is a solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Show that if $(\mathbf{b}, \mathbf{y}) = 0$ for every such $\mathbf{y}$, then $\mathbf{Ax} = \mathbf{b}$ has solutions. Note that this is the converse of Problem 27; the form of the solution is given by Problem 28.
30. Prove that $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.
31. Prove that if $\mathbf{A}$ is Hermitian, then $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{Ay})$, where $\mathbf{x}$ and $\mathbf{y}$ are any vectors.
32. In this problem we show that the eigenvalues of a Hermitian matrix $\mathbf{A}$ are real. Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ.
    (a) Show that $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$. *Hint:* See Problem 31.
    (b) Show that $\lambda(\mathbf{x}, \mathbf{x}) = \bar{\lambda}(\mathbf{x}, \mathbf{x})$. *Hint:* Recall that $\mathbf{Ax} = \lambda\mathbf{x}$.
    (c) Show that $\lambda = \bar{\lambda}$; that is, the eigenvalue λ is real.
33. Show that if λ_1 and λ_2 are eigenvalues of a Hermitian matrix $\mathbf{A}$, and if $\lambda_1 \ne \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
Hint: Use the results of Problems 31 and 32 to show that $(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$.

7.4 Basic Theory of Systems of First Order Linear Equations

The general theory of a system of n first order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1}$$

closely parallels that of a single linear equation of nth order. The discussion in this section therefore follows the same general lines as that in Sections 3.2, 3.3, and 4.1. To discuss the system (1) most effectively, we write it in matrix notation. That is, we consider $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ to be components of a vector $\mathbf{x} = \boldsymbol{\phi}(t)$; similarly, $g_1(t), \ldots, g_n(t)$ are components of a vector $\mathbf{g}(t)$, and $p_{11}(t), \ldots, p_{nn}(t)$ are elements of an $n \times n$ matrix $\mathbf{P}(t)$. Equation (1) then takes the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{2}$$

The use of vectors and matrices not only saves a great deal of space and facilitates calculations but also emphasizes the similarity between systems of equations and single (scalar) equations.

A vector $\mathbf{x} = \boldsymbol{\phi}(t)$ is said to be a solution of Eq. (2) if its components satisfy the system of equations (1). Throughout this section we assume that $\mathbf{P}$ and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$; that is, each of the scalar functions $p_{11}, \ldots, p_{nn}, g_1, \ldots, g_n$ is continuous there. According to Theorem 7.1.2, this is sufficient to guarantee the existence of solutions of Eq. (2) on the interval $\alpha < t < \beta$.

It is convenient to consider first the homogeneous equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{3}$$

obtained from Eq. (2) by setting $\mathbf{g}(t) = \mathbf{0}$. Once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous equation (2); this is taken up in Section 7.9. We use the notation

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}, \ldots \tag{4}$$

to designate specific solutions of the system (3). Note that $x_{ij}(t) = x_i^{(j)}(t)$ refers to the ith component of the jth solution $\mathbf{x}^{(j)}(t)$. The main facts about the structure of solutions of the system (3) are stated in Theorems 7.4.1 to 7.4.4. They closely resemble the corresponding theorems in Sections 3.2, 3.3, and 4.1; some of the proofs are left to the reader as exercises.

Theorem 7.4.1 If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (3), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

This is the principle of superposition; it is proved simply by differentiating $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ and using the fact that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ satisfy Eq. (3). By repeated application of

Theorem 7.4.1 we reach the conclusion that if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are solutions of Eq. (3), then

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) \tag{5}$$

is also a solution for any constants $c_1, \ldots, c_k$. As an example, it can be verified that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \tag{6}$$

satisfy the equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{7}$$

According to Theorem 7.4.1

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \\ &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \end{aligned} \tag{8}$$

also satisfies Eq. (7).

As we indicated previously, by repeatedly applying Theorem 7.4.1, it follows that every finite linear combination of solutions of Eq. (3) is also a solution. The question now arises as to whether all solutions of Eq. (3) can be found in this way. By analogy with previous cases it is reasonable to expect that for a system of the form (3) of nth order it is sufficient to form linear combinations of n properly chosen solutions. Therefore let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be n solutions of the nth order system (3), and consider the matrix $\mathbf{X}(t)$ whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$:

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}. \tag{9}$$

Recall from Section 7.3 that the columns of $\mathbf{X}(t)$ are linearly independent for a given value of t if and only if $\det \mathbf{X} \neq 0$ for that value of t. This determinant is called the Wronskian of the n solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ and is also denoted by $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$; that is,

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) = \det \mathbf{X}(t). \tag{10}$$

The solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are then linearly independent at a point if and only if $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ is not zero there.

Theorem 7.4.2 If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (3) for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system (3) can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$,

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t), \tag{11}$$

in exactly one way.

Before proving Theorem 7.4.2, note that according to Theorem 7.4.1 all expressions of the form (11) are solutions of the system (3), while by Theorem 7.4.2 all solutions of Eq. (3) can be written in the form (11). If the constants $c_1, \ldots, c_n$ are thought of as arbitrary, then Eq. (11) includes all solutions of the system (3), and it is customary to call it the **general solution.** Any set of solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ of Eq. (3), which is linearly independent at each point in the interval $\alpha < t < \beta$, is said to be a **fundamental set of solutions** for that interval.

To prove Theorem 7.4.2, we will show, given any solution $\boldsymbol{\phi}$ of Eq. (3), that $\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ for suitable values of $c_1, \ldots, c_n$. Let $t = t_0$ be some point in the interval $\alpha < t < \beta$ and let $\boldsymbol{\xi} = \boldsymbol{\phi}(t_0)$. We now wish to determine whether there is any solution of the form $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ that also satisfies the same initial condition $\mathbf{x}(t_0) = \boldsymbol{\xi}$. That is, we wish to know whether there are values of $c_1, \ldots, c_n$ such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) = \boldsymbol{\xi}, \tag{12}$$

or in scalar form

$$\begin{aligned} c_1x_{11}(t_0) + \cdots + c_nx_{1n}(t_0) &= \xi_1, \\ &\vdots \\ c_1x_{n1}(t_0) + \cdots + c_nx_{nn}(t_0) &= \xi_n. \end{aligned} \tag{13}$$

The necessary and sufficient condition that Eqs. (13) possess a unique solution $c_1, \ldots, c_n$ is precisely the nonvanishing of the determinant of coefficients, which is the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ evaluated at $t = t_0$. The hypothesis that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent throughout $\alpha < t < \beta$ guarantees that $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ is not zero at $t = t_0$, and therefore there is a (unique) solution of Eq. (3) of the form $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ that also satisfies the initial condition (12). By the uniqueness part of Theorem 7.1.2 this solution is identical to $\boldsymbol{\phi}(t)$, and hence $\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$, as was to be proved.

Theorem 7.4.3 If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of Eq. (3) on the interval $\alpha < t < \beta$, then in this interval $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero or else never vanishes.

The significance of Theorem 7.4.3 lies in the fact that it relieves us of the necessity of examining $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ at all points in the interval of interest, and enables us to determine whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions merely by evaluating their Wronskian at any convenient point in the interval.

Theorem 7.4.3 is proved by first establishing that the Wronskian of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ satisfies the differential equation (see Problem 2)

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W. \tag{14}$$

Hence W is an exponential function, and the conclusion of the theorem follows immediately. The expression for W obtained by solving Eq. (14) is known as Abel's formula; note the analogy with Eq. (8) of Section 3.3.

Alternatively, Theorem 7.4.3 can be established by showing that if n solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ of Eq. (3) are linearly dependent at one point $t = t_0$, then they must

be linearly dependent at each point in $\alpha < t < \beta$ (see Problem 8). Consequently, if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent at one point, they must be linearly independent at each point in the interval.

The next theorem states that the system (3) always has at least one fundamental set of solutions.

Theorem 7.4.4 Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix};$$

further let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be the solutions of the system (3) satisfying the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \ldots, \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}, \tag{15}$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions of the system (3).

To prove this theorem, note that the existence and uniqueness of the solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ mentioned in Theorem 7.4.4 are assured by Theorem 7.1.2. It is not hard to see that the Wronskian of these solutions is equal to 1 when $t = t_0$; therefore $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are a fundamental set of solutions.

Once one fundamental set of solutions has been found, other sets can be generated by forming (independent) linear combinations of the first set. For theoretical purposes the set given by Theorem 7.4.4 is usually the simplest.

To summarize, any set of n linearly independent solutions of the system (3) constitutes a fundamental set of solutions. Under the conditions given in this section, such fundamental sets always exist, and every solution of the system (3) can be represented as a linear combination of any fundamental set of solutions.

PROBLEMS

1. Using matrix algebra, prove the statement following Theorem 7.4.1 for an arbitrary value of the integer k.
2. In this problem we outline a proof of Theorem 7.4.3 in the case $n = 2$. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be solutions of Eq. (3) for $\alpha < t < \beta$, and let W be the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
   (a) Show that

$$\frac{dW}{dt} = \begin{vmatrix} \dfrac{dx_1^{(1)}}{dt} & \dfrac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \dfrac{dx_2^{(1)}}{dt} & \dfrac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

   (b) Using Eq. (3), show that

$$\frac{dW}{dt} = (p_{11} + p_{22})W.$$

   (c) Find $W(t)$ by solving the differential equation obtained in part (b). Use this expression to obtain the conclusion stated in Theorem 7.4.3.

(d) Generalize this procedure so as to prove Theorem 7.4.3 for an arbitrary value of n.

3. Show that the Wronskians of two fundamental sets of solutions of the system (3) can differ at most by a multiplicative constant.
Hint: Use Eq. (14).

4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \tag{i}$$

corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -q(t)x_1 - p(t)x_2. \end{aligned} \tag{ii}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eqs. (ii), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (i), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant.
Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

5. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$ of this equation and the general solution $\mathbf{x}^{(c)}$ of the corresponding homogeneous equation.

6. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.
(a) Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
(b) In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?
(c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?
(d) Find this system of equations and verify the conclusions of part (c).

7. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, and answer the same questions as in Problem 6.

The following two problems indicate an alternative derivation of Theorem 7.4.2.

8. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent on the interval (α, β) if they are linearly dependent at any point in it.
Hint: There are constants $c_1, \ldots, c_m$ such that $c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}$. Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$ and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

9. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}$ is continuous on $\alpha < t < \beta$.
(a) Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form

$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.
Hint: Use the result of Problem 11 of Section 7.3, and also Problem 8 above.
(b) Show that the expression for the solution $\mathbf{z}(t)$ in part (a) is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$ and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

7.5 Homogeneous Linear Systems with Constant Coefficients



We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients; that is, systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Unless stated otherwise, we will assume further that all the elements of $\mathbf{A}$ are real (rather than complex) numbers.

If $n = 1$, then the system reduces to a single first order equation

$$\frac{dx}{dt} = ax, \tag{2}$$

whose solution is $x = ce^{at}$. In Section 2.5 we noted that $x = 0$ is the only equilibrium solution if $a \neq 0$. Other solutions approach $x = 0$ if $a < 0$ and in this case we say that $x = 0$ is an asymptotically stable equilibrium solution. On the other hand, if $a > 0$, then $x = 0$ is unstable, since other solutions depart from it. For higher order systems the situation is somewhat analogous, but more complicated. Equilibrium solutions are found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$. We assume that $\det \mathbf{A} \neq 0$, so $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution. An important question is whether other solutions approach this equilibrium solution or depart from it as t increases; in other words, is $\mathbf{x} = \mathbf{0}$ asymptotically stable or unstable? Or are there still other possibilities?

The case $n = 2$ is particularly important and lends itself to visualization in the x_1x_2-plane, called the **phase plane**. By evaluating $\mathbf{A}\mathbf{x}$ at a large number of points and plotting the resulting vectors one obtains a direction field of tangent vectors to solutions of the system of differential equations. A qualitative understanding of the behavior of solutions can usually be gained from a direction field. More precise information results from including in the plot some solution curves, or trajectories. A plot that shows a representative sample of trajectories for a given system is called a **phase portrait**. Examples of direction fields and phase portraits occur later in this section.

To construct the general solution of the system (1) we proceed by analogy with the treatment of second order linear equations in Section 3.1. Thus we seek solutions of Eq. (1) of the form

$$\mathbf{x} = \boldsymbol{\xi} e^{rt}, \tag{3}$$

where the exponent r and the constant vector $\boldsymbol{\xi}$ are to be determined. Substituting from Eq. (3) for $\mathbf{x}$ in the system (1) gives

$$r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt}.$$

Upon canceling the nonzero scalar factor e^{rt} we obtain $\mathbf{A}\boldsymbol{\xi} = r\boldsymbol{\xi}$, or

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{4}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Thus, to solve the system of differential equations (1), we must solve the system of algebraic equations (4). This latter problem is precisely the one that determines the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Therefore the vector $\mathbf{x}$ given by Eq. (3) is a solution of Eq. (1) provided that r is an eigenvalue and $\boldsymbol{\xi}$ an associated eigenvector of the coefficient matrix $\mathbf{A}$.

The following two examples illustrate the solution procedure in the case of 2×2 coefficient matrices. We also show how to construct the corresponding phase portraits. Later in the section we return to a further discussion of the general $n \times n$ system.

EXAMPLE 1

Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{5}$$

Plot a direction field and determine the qualitative behavior of solutions. Then find the general solution and draw several trajectories.

A direction field for this system is shown in Figure 7.5.1. From this figure it is easy to see that a typical solution departs from the neighborhood of the origin and ultimately has a slope of approximately 2 in either the first or third quadrant.

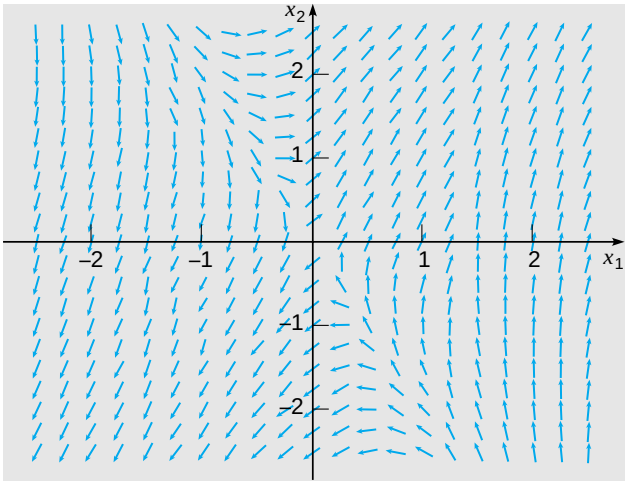


FIGURE 7.5.1 Direction field for the system (5).

To find solutions explicitly we assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, and substitute for $\mathbf{x}$ in Eq. (5). We are led to the system of algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6}$$

Equations (6) have a nontrivial solution if and only if the determinant of coefficients is zero. Thus allowable values of r are found from the equation

$$\begin{aligned} \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} &= (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = 0. \end{aligned} \tag{7}$$

Equation (7) has the roots $r_1 = 3$ and $r_2 = -1$; these are the eigenvalues of the coefficient matrix in Eq. (5). If $r = 3$, then the system (6) reduces to the single equation

$$-2\xi_1 + \xi_2 = 0. \tag{8}$$

Thus $\xi_2 = 2\xi_1$ and the eigenvector corresponding to $r_1 = 3$ can be taken as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{9}$$

Similarly, corresponding to $r_2 = -1$, we find that $\xi_2 = -2\xi_1$, so the eigenvector is

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{10}$$

The corresponding solutions of the differential equation are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}. \tag{11}$$

The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t}, \tag{12}$$

which is never zero. Hence the solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set, and the general solution of the system (5) is

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \end{aligned} \tag{13}$$

where c_1 and c_2 are arbitrary constants.

To visualize the solution (13) it is helpful to consider its graph in the x_1x_2-plane for various values of the constants c_1 and c_2. We start with $\mathbf{x} = c_1\mathbf{x}^{(1)}(t)$, or in scalar form

$$x_1 = c_1e^{3t}, \qquad x_2 = 2c_1e^{3t}.$$

By eliminating t between these two equations, we see that this solution lies on the straight line $x_2 = 2x_1$; see Figure 7.5.2*a*. This is the line through the origin in the direction of the eigenvector $\boldsymbol{\xi}^{(1)}$. If we look on the solution as the trajectory of a moving particle, then the particle is in the first quadrant when $c_1 > 0$ and in the third quadrant when $c_1 < 0$. In either case the particle departs from the origin as t increases. Next consider $\mathbf{x} = c_2\mathbf{x}^{(2)}(t)$, or

$$x_1 = c_2e^{-t}, \qquad x_2 = -2c_2e^{-t}.$$

This solution lies on the line $x_2 = -2x_1$, whose direction is determined by the eigenvector $\boldsymbol{\xi}^{(2)}$. The solution is in the fourth quadrant when $c_2 > 0$ and in the second quadrant when $c_2 < 0$, as shown in Figure 7.5.2*a*. In both cases the particle moves toward the origin as t increases. The solution (13) is a combination of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$. For large t the term $c_1\mathbf{x}^{(1)}(t)$ is dominant and the term $c_2\mathbf{x}^{(2)}(t)$ becomes negligible. Thus all solutions for which $c_1 \neq 0$ are asymptotic to the line $x_2 = 2x_1$ as $t \to \infty$. Similarly, all solutions for which $c_2 \neq 0$ are asymptotic to the line $x_2 = -2x_1$ as $t \to -\infty$. The graphs of several solutions are shown in Figure 7.5.2*a*. The pattern of trajectories in this figure is typical of all second order systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which the eigenvalues are real and of opposite signs. The origin is called a **saddle point** in this case. Saddle points are always unstable because almost all trajectories depart from them as t increases.

In the preceding paragraph we have described how to draw by hand a qualitatively correct sketch of the trajectories of a system such as Eq. (5), once the eigenvalues and eigenvectors have been determined. However, to produce a detailed and accurate drawing, such as Figure 7.5.2*a* and other figures that appear later in this chapter, a computer is extremely helpful, if not indispensable.

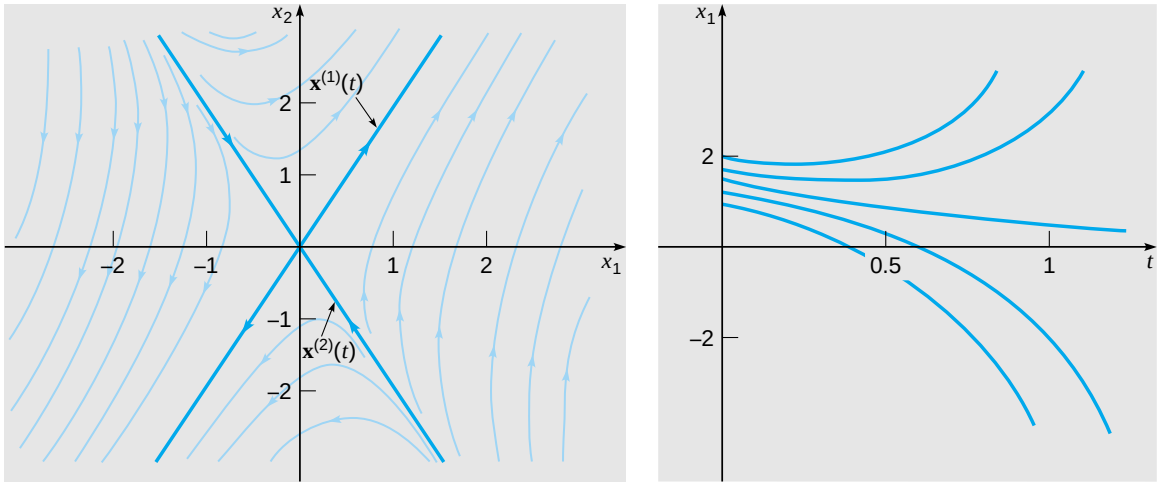


FIGURE 7.5.2 (*a*) Trajectories of the system (5); the origin is a saddle point. (*b*) Plots of x_1 versus t for the system (5).

As an alternative to Figure 7.5.2*a* one can also plot x_1 or x_2 as a function of t; some typical plots of x_1 versus t are shown in Figure 7.5.2*b*, and those of x_2 versus t are similar. For certain initial conditions it follows that $c_1 = 0$ in Eq. (13), so that $x_1 = c_2 e^{-t}$ and $x_1 \to 0$ as $t \to \infty$. One such graph is shown in Figure 7.5.2*b*, corresponding to a trajectory that approaches the origin in Figure 7.5.2*a*. For most initial conditions, however, $c_1 \neq 0$ and x_1 is given by $x_1 = c_1 e^{3t} + c_2 e^{-t}$. Then the presence of the positive exponential term causes x_1 to grow exponentially in magnitude as t increases. Several graphs of this type are shown in Figure 7.5.2*b*, corresponding to trajectories that depart from the neighborhood of the origin in Figure 7.5.2*a*. It is important to understand the relation between parts (*a*) and (*b*) of Figure 7.5.2 and other similar figures that appear later, since one may want to visualize solutions either in the x_1x_2-plane or as functions of the independent variable t.

EXAMPLE 2

Consider the system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}. \tag{14}$$

Draw a direction field for this system; then find its general solution and plot several trajectories in the phase plane.

The direction field for the system (14) in Figure 7.5.3 shows clearly that all solutions approach the origin. To find the solutions assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$; then we obtain the algebraic system

$$\begin{pmatrix} -3 - r & \sqrt{2} \\ \sqrt{2} & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

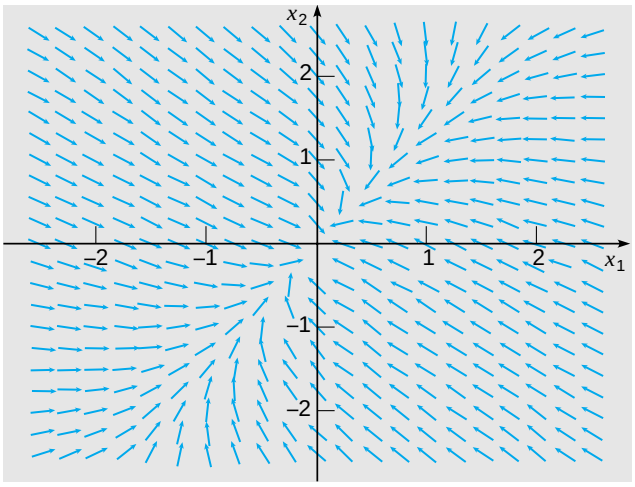


FIGURE 7.5.3 Direction field for the system (14).

The eigenvalues satisfy

$$\begin{aligned}(-3-r)(-2-r)-2 &= r^2+5r+4\\ &= (r+1)(r+4)=0,\end{aligned} \tag{16}$$

so $r_1=-1$ and $r_2=-4$. For $r=-1$, Eq. (15) becomes

$$\begin{pmatrix}-2 & \sqrt{2}\\ \sqrt{2} & -1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{17}$$

Hence $\xi_2=\sqrt{2}\,\xi_1$ and the eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to the eigenvalue $r_1=-1$ can be taken as

$$\boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}. \tag{18}$$

Similarly, corresponding to the eigenvalue $r_2=-4$, we have $\xi_1=-\sqrt{2}\,\xi_2$, so the eigenvector is

$$\boldsymbol{\xi}^{(2)}=\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}. \tag{19}$$

Thus a fundamental set of solutions of the system (14) is

$$\mathbf{x}^{(1)}(t)=\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}e^{-t},\qquad \mathbf{x}^{(2)}(t)=\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}e^{-4t}, \tag{20}$$

and the general solution is

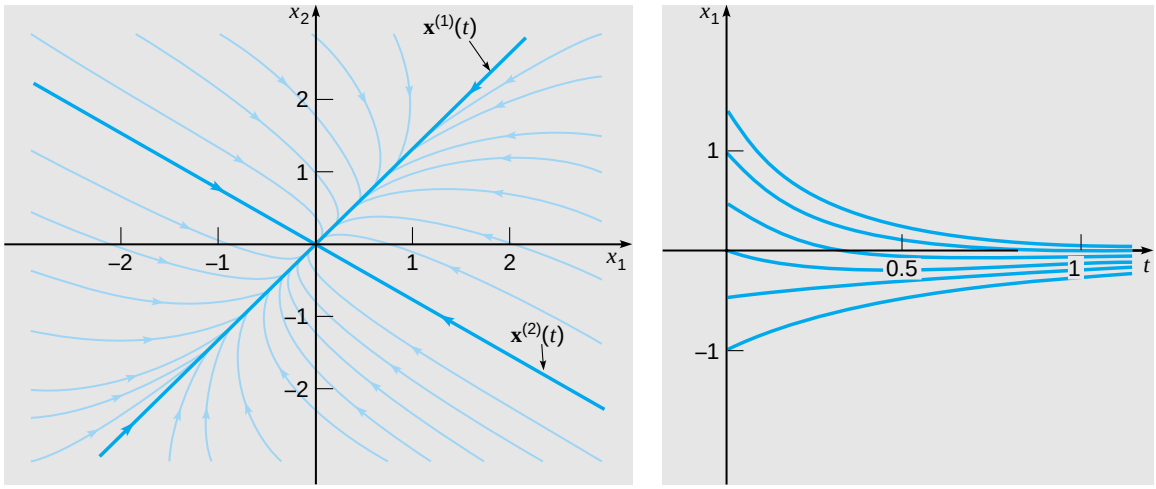
$$\mathbf{x}=c_1\mathbf{x}^{(1)}(t)+c_2\mathbf{x}^{(2)}=c_1\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}e^{-t}+c_2\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}e^{-4t}. \tag{21}$$

Graphs of the solution (21) for several values of c_1 and c_2 are shown in Figure 7.5.4*a*. The solution $\mathbf{x}^{(1)}(t)$ approaches the origin along the line $x_2=\sqrt{2}\,x_1$, while the solution $\mathbf{x}^{(2)}(t)$ approaches the origin along the line $x_1=-\sqrt{2}\,x_2$. The directions of

these lines are determined by the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, respectively. In general, we have a combination of these two fundamental solutions. As $t \to \infty$, the solution $\mathbf{x}^{(2)}(t)$ is negligible compared to $\mathbf{x}^{(1)}(t)$. Thus, unless $c_1 = 0$, the solution (21) approaches the origin tangent to the line $x_2 = \sqrt{2}x_1$. The pattern of trajectories shown in Figure 7.5.4*a* is typical of all second order systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which the eigenvalues are real, different, and of the same sign. The origin is called a **node** for such a system. If the eigenvalues were positive rather than negative, then the trajectories would be similar but traversed in the outward direction. Nodes are asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.

Although Figure 7.5.4*a* was computer-generated, a qualitatively correct sketch of the trajectories can be drawn quickly by hand, based on a knowledge of the eigenvalues and eigenvectors.

Some typical plots of x_1 versus t are shown in Figure 7.5.4*b*. Observe that each of the graphs approaches the t-axis asymptotically as t increases, corresponding to a trajectory that approaches the origin in Figure 7.5.2*a*. The behavior of x_2 as a function of t is similar.



(*a*) (*b*)

FIGURE 7.5.4 (*a*) Trajectories of the system (14); the origin is a node. (*b*) Plots of x_1 versus t for the system (14).

The two preceding examples illustrate the two main cases for 2×2 systems having eigenvalues that are real and different: Either the eigenvalues have opposite signs (Example 1) or the same sign (Example 2). The other possibility is that zero is an eigenvalue, but in this case it follows that $\det \mathbf{A} = 0$, which violates the assumption made at the beginning of this section.

Returning to the general system (1) we proceed as in the examples. To find solutions of the differential equation (1) we must find the eigenvalues and eigenvectors of $\mathbf{A}$

from the associated algebraic system (4). The eigenvalues $r_1, \ldots, r_n$ (which need not all be different) are roots of the nth degree polynomial equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{22}$$

The nature of the eigenvalues and the corresponding eigenvectors determines the nature of the general solution of the system (1). If we assume that $\mathbf{A}$ is a real-valued matrix, there are three possibilities for the eigenvalues of $\mathbf{A}$:

1. All eigenvalues are real and different from each other.

2. Some eigenvalues occur in complex conjugate pairs.

3. Some eigenvalues are repeated.

If the eigenvalues are all real and different, as in the two preceding examples, then associated with each eigenvalue r_i is a real eigenvector $\boldsymbol{\xi}^{(i)}$ and the set of n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ is linearly independent. The corresponding solutions of the differential system (1) are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}. \tag{23}$$

To show that these solutions form a fundamental set, we evaluate their Wronskian:

$$\begin{aligned} W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) &= \begin{vmatrix} \xi_1^{(1)} e^{r_1 t} & \cdots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \cdots & \xi_n^{(n)} e^{r_n t} \end{vmatrix} \\ &= e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{vmatrix}. \end{aligned} \tag{24}$$

First, we observe that the exponential function is never zero. Next, since the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are linearly independent, the determinant in the last term of Eq. (24) is nonzero. As a consequence, the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t)$ is never zero; hence $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions. Thus the general solution of Eq. (1) is

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}. \tag{25}$$

If $\mathbf{A}$ is real and symmetric (a special case of Hermitian matrices), recall from Section 7.3 that all the eigenvalues $r_1, \ldots, r_n$ must be real. Further, even if some of the eigenvalues are repeated, there is always a full set of n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ that are linearly independent (in fact, orthogonal). Hence the corresponding solutions of the differential system (1) given by Eq. (23) again form a fundamental set of solutions and the general solution is again given by Eq. (25). The following example illustrates this case.

EXAMPLE 3

Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}. \tag{26}$$

Observe that the coefficient matrix is real and symmetric. The eigenvalues and eigenvectors of this matrix were found in Example 5 of Section 7.3, namely,

$$r_1 = 2, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \tag{27}$$

$$r_2 = -1, \qquad r_3 = -1; \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{28}$$

Hence a fundamental set of solutions of Eq. (26) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t} \tag{29}$$

and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}. \tag{30}$$

This example illustrates the fact that even though an eigenvalue ($r = -1$) has multiplicity 2, it may still be possible to find two linearly independent eigenvectors $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ and, as a consequence, to construct the general solution (30).

The behavior of the solution (30) depends critically on the initial conditions. For large t the first term on the right side of Eq. (30) is the dominant one; therefore, if $c_1 \neq 0$, all components of $\mathbf{x}$ become unbounded as $t \to \infty$. On the other hand, for certain initial points c_1 will be zero. In this case, the solution involves only the negative exponential terms and $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$. The initial points that cause c_1 to be zero are precisely those that lie in the plane determined by the eigenvectors $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ corresponding to the two negative eigenvalues. Thus, solutions that start in this plane approach the origin as $t \to \infty$, while all other solutions become unbounded.

If some of the eigenvalues occur in complex conjugate pairs, then there are still n linearly independent solutions of the form (23), provided that all the eigenvalues are different. Of course, the solutions arising from complex eigenvalues are complex-valued. However, as in Section 3.4, it is possible to obtain a full set of real-valued solutions. This is discussed in Section 7.6.

More serious difficulties can occur if an eigenvalue is repeated. In this event the number of corresponding linearly independent eigenvectors may be smaller than the multiplicity of the eigenvalue. If so, the number of linearly independent solutions of the form $\boldsymbol{\xi} e^{rt}$ will be smaller than n. To construct a fundamental set of solutions it is then necessary to seek additional solutions of another form. The situation is somewhat analogous to that for an nth order linear equation with constant coefficients; a repeated root of the characteristic equation gave rise to solutions of the form $e^{rt}, te^{rt}, t^2e^{rt}, \ldots$. The case of repeated eigenvalues is treated in Section 7.8.

Finally, if $\mathbf{A}$ is complex, then complex eigenvalues need not occur in conjugate pairs, and the eigenvectors are normally complex-valued even though the associated

eigenvalue may be real. The solutions of the differential equation (1) are still of the form (23), provided that the eigenvalues are distinct, but in general all the solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6 find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$. Also draw a direction field and plot a few trajectories of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8 find the general solution of the given system of equations. Also draw a direction field and a few of the trajectories. In each of these problems the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

7. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 9 through 14 find the general solution of the given system of equations.

9. $\mathbf{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \mathbf{x}$

11. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}$

13. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 15 through 18 solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

15. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

16. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

17. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

18. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$

19. The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second order Euler equation (Section 5.5). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that $t > 0$.

20. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

21. $t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$

23. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 24 through 27 the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Sketch a phase portrait of the system.
(b) Sketch the trajectory passing through the initial point $(2, 3)$.
(c) For the trajectory in part (b) sketch the graphs of x_1 versus t and of x_2 versus t on the same set of axes.

24. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

25. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

26. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

27. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

28. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent, and then showing that this leads to a contradiction.
(a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, note that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.
(b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.
(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 is not zero; suppose that $c_1 \neq 0$. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.
(d) Modify the argument of part (c) in case c_1 is zero but c_2 is not.
(e) Carry out a similar argument for the case in which the order n is equal to 3; note that the procedure can be extended to cover an arbitrary value of n.

29. Consider the equation

$$ay'' + by' + cy = 0, \tag{i}$$

where a, b, and c are constants. In Chapter 3 it was shown that the general solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{ii}$$

(a) Transform Eq. (i) into a system of first order equations by letting $x_1 = y$, $x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Find the equation that determines the eigenvalues of the coefficient matrix $\mathbf{A}$ in part (a). Note that this equation is just the characteristic equation (ii) of Eq. (i).

30. The two-tank system of Problem 21 in Section 7.1 leads to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix},$$

where x_1 and x_2 are the deviations of the salt levels Q_1 and Q_2 from their respective equilibria.

(a) Find the solution of the given initial value problem.

(b) Plot x_1 versus t and x_2 versus t on the same set of axes.

(c) Find the time T such that $|x_1(t)| \le 0.5$ and $|x_2(t)| \le 0.5$ for all $t \ge T$.

31. Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}.$$

(a) Solve the system for $\alpha = 0.5$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(b) Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(c) In parts (a) and (b) solutions of the system exhibit two quite different types of behavior. Find the eigenvalues of the coefficient matrix in terms of α and determine the value of α between 0.5 and 2 where the transition from one type of behavior to the other occurs. This critical value of α is called a **bifurcation point**.

Electric Circuits. Problems 32 and 33 are concerned with the electric circuit described by the system of differential equations in Problem 20 of Section 7.1:

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{CR_2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}. \tag{i}$$

32. (a) Find the general solution of Eq. (i) if $R_1 = 1$ ohm, $R_2 = \frac{3}{5}$ ohm, $L = 2$ henrys, and $C = \frac{2}{3}$ farad.

(b) Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ regardless of the initial values $I(0)$ and $V(0)$.

33. Consider the preceding system of differential equations (i).

(a) Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different.

(b) If the condition found in part (a) is satisfied, show that both eigenvalues are negative. Then show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ regardless of the initial conditions.

(c) If the condition found in part (a) is not satisfied, then the eigenvalues are either complex or repeated. Do you think that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ in these cases as well? *Hint:* In part (c) one approach is to change the system (i) into a single second order equation. We also discuss complex and repeated eigenvalues in Sections 7.6 and 7.8.

7.6 Complex Eigenvalues



In this section we consider again a system of n linear homogeneous equations with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where the coefficient matrix $\mathbf{A}$ is real-valued. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, then it follows as in Section 7.5 that r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$. Recall that the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0, \tag{2}$$

and that the corresponding eigenvectors satisfy

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{3}$$

If $\mathbf{A}$ is real, then the coefficients in the polynomial equation (2) for r are real, and any complex eigenvalues must occur in conjugate pairs. For example, if $r_1 = \lambda + i\mu$, where λ and μ are real, is an eigenvalue of $\mathbf{A}$, then so is $r_2 = \lambda - i\mu$. Further, the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates. To see that this is so, suppose that r_1 and $\boldsymbol{\xi}^{(1)}$ satisfy

$$(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}. \tag{4}$$

On taking the complex conjugate of this equation, and noting that $\mathbf{A}$ and $\mathbf{I}$ are real-valued, we obtain

$$(\mathbf{A} - \overline{r}_1\mathbf{I})\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0}, \tag{5}$$

where $\overline{r}_1$ and $\overline{\boldsymbol{\xi}}^{(1)}$ are the complex conjugates of r_1 and $\boldsymbol{\xi}^{(1)}$, respectively. In other words, $r_2 = \overline{r}_1$ is also an eigenvalue, and $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}}^{(1)}$ is a corresponding eigenvector. The corresponding solutions

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}}^{(1)} e^{\overline{r}_1 t} \tag{6}$$

of the differential equation (1) are then complex conjugates of each other. Therefore, as in Section 3.4, we can find two real-valued solutions of Eq. (1) corresponding to the eigenvalues r_1 and r_2 by taking the real and imaginary parts of $\mathbf{x}^{(1)}(t)$ or $\mathbf{x}^{(2)}(t)$ given by Eq. (6).

Let us write $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real; then we have

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos \mu t + i \sin \mu t). \end{aligned} \tag{7}$$

Upon separating $\mathbf{x}^{(1)}(t)$ into its real and imaginary parts, we obtain

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos \mu t - \mathbf{b}\sin \mu t) + ie^{\lambda t}(\mathbf{a}\sin \mu t + \mathbf{b}\cos \mu t). \tag{8}$$

If we write $\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$, then the vectors

$$\begin{aligned} \mathbf{u}(t) &= e^{\lambda t}(\mathbf{a}\cos \mu t - \mathbf{b}\sin \mu t), \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a}\sin \mu t + \mathbf{b}\cos \mu t) \end{aligned} \tag{9}$$

are real-valued solutions of Eq. (1). It is possible to show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent solutions (see Problem 27).

For example, suppose that $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that $r_3, \ldots, r_n$ are all real and distinct. Let the corresponding eigenvectors be $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, $\boldsymbol{\xi}^{(2)} = \mathbf{a} - i\mathbf{b}$, $\boldsymbol{\xi}^{(3)}, \ldots, \boldsymbol{\xi}^{(n)}$. Then the general solution of Eq. (1) is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}^{(3)}e^{r_3 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}, \tag{10}$$

where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are given by Eqs. (9). We emphasize that this analysis applies only if the coefficient matrix $\mathbf{A}$ in Eq. (1) is real, for it is only then that complex eigenvalues and eigenvectors must occur in conjugate pairs.

The following examples illustrate the case $n = 2$, both to simplify the algebraic calculations and to permit easy visualization of the solutions in the phase plane.

EXAMPLE 1

Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x} \tag{11}$$

and display them graphically.

A direction field for the system (11) is shown in Figure 7.6.1. This plot suggests that the trajectories in the phase plane spiral clockwise toward the origin.

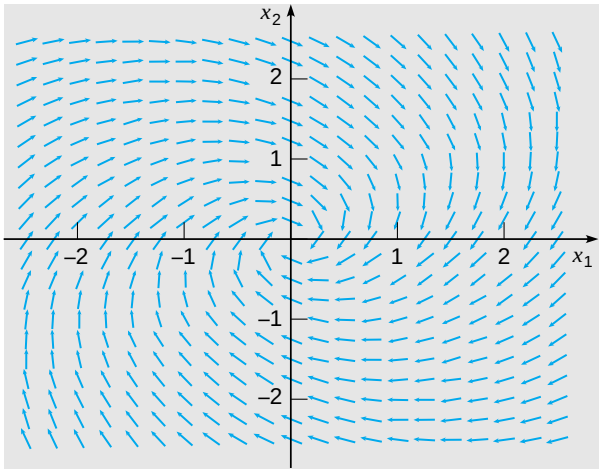


FIGURE 7.6.1 A direction field for the system (11).

To find a fundamental set of solutions we assume that

$$\mathbf{x} = \boldsymbol{\xi} e^{rt} \tag{12}$$

and obtain the set of linear algebraic equations

$$\begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

for the eigenvalues and eigenvectors of $\mathbf{A}$. The characteristic equation is

$$\begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \tfrac{5}{4} = 0; \tag{14}$$

therefore the eigenvalues are $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. From Eq. (13) a straightforward calculation shows that the corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{15}$$

Hence a fundamental set of solutions of the system (11) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}. \tag{16}$$

To obtain a set of real-valued solutions, we must find the real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. In fact,

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}. \tag{17}$$

Hence

$$\mathbf{u}(t) = e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \mathbf{v}(t) = e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \tag{18}$$

is a set of real-valued solutions. To verify that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, we compute their Wronskian:

$$\begin{aligned} W(\mathbf{u}, \mathbf{v})(t) &= \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} \\ &= e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}. \end{aligned}$$

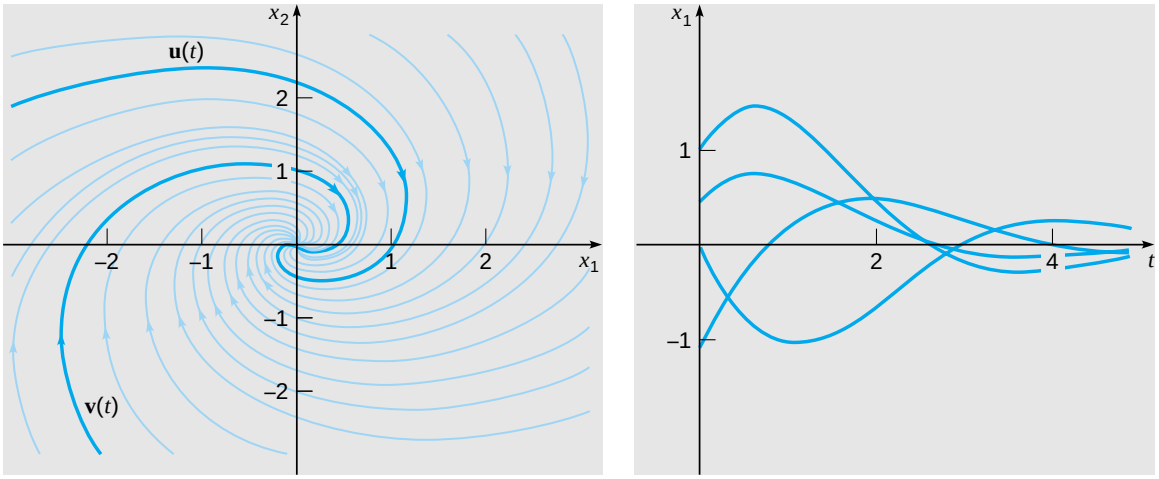
Since the Wronskian is never zero, it follows that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ constitute a fundamental set of (real-valued) solutions of the system (11).

The graphs of the solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are shown in Figure 7.6.2*a*. Since

$$\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the graphs of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ pass through the points $(1, 0)$ and $(0, 1)$, respectively. Other solutions of the system (11) are linear combinations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and graphs of a few of these solutions are also shown in Figure 7.6.2*a*. In all cases the solution approaches the origin along a spiral path as $t \to \infty$; this is due to the fact that the solutions (18) are products of decaying exponential and sine or cosine factors. Some typical graphs of x_1 versus t are shown in Figure 7.6.2*b*; each one represents a decaying oscillation in time.

Figure 7.6.2*a* is typical of all second order systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose eigenvalues are complex with negative real part. The origin is called a **spiral point** and is asymptotically stable because all trajectories approach it as t increases. For a system whose eigenvalues have a positive real part the trajectories are similar to those in Figure 7.6.2*a*, but the direction of motion is away from the origin and the trajectories become unbounded. In this case, the origin is unstable. If the real part of the eigenvalues is zero, then the trajectories neither approach the origin nor become unbounded but instead traverse



(*a*) (*b*)

FIGURE 7.6.2 (*a*) Trajectories of the system (11); the origin is a spiral point. (*b*) Plots of x_1 versus t for the system (11).

repeatedly a closed curve about the origin. In this case the origin is called a **center** and is said to be stable, but not asymptotically stable. In all three cases the direction of motion may be either clockwise, as in this example, or counterclockwise, depending on the elements of the coefficient matrix **A**.

EXAMPLE 2

The electric circuit shown in Figure 7.6.3 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}, \tag{19}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. These equations were derived in Problem 19 of Section 7.1. Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$ at any time.

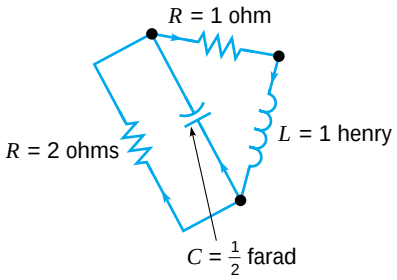


FIGURE 7.6.3 The circuit in Example 2.

Assuming that

$$\begin{pmatrix} I \\ V \end{pmatrix} = \boldsymbol{\xi} e^{rt}, \tag{20}$$

we obtain the algebraic equations

$$\begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{21}$$

The eigenvalues are determined from the condition

$$\begin{vmatrix} -1-r & -1 \\ 2 & -1-r \end{vmatrix} = r^2 + 2r + 3 = 0; \tag{22}$$

thus $r_1 = -1 + \sqrt{2}\,i$ and $r_2 = -1 - \sqrt{2}\,i$. The corresponding eigenvectors are then found from Eq. (21), namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2}\,i \end{pmatrix}. \tag{23}$$

The complex-valued solution corresponding to r_1 and $\boldsymbol{\xi}^{(1)}$ is

$$\begin{aligned} \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} e^{(-1+\sqrt{2}\,i)t} \\ &= \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} e^{-t}(\cos\sqrt{2}\,t + i\sin\sqrt{2}\,t) \\ &= e^{-t}\begin{pmatrix} \cos\sqrt{2}\,t \\ \sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin\sqrt{2}\,t \\ -\sqrt{2}\cos\sqrt{2}\,t \end{pmatrix}. \end{aligned} \tag{24}$$

The real and imaginary parts of this solution form a pair of linearly independent real-valued solutions of Eq. (19):

$$\mathbf{u}(t) = e^{-t}\begin{pmatrix} \cos\sqrt{2}\,t \\ \sqrt{2}\sin\sqrt{2}\,t \end{pmatrix}, \qquad \mathbf{v}(t) = e^{-t}\begin{pmatrix} \sin\sqrt{2}\,t \\ -\sqrt{2}\cos\sqrt{2}\,t \end{pmatrix}. \tag{25}$$

Hence the general solution of Eqs. (19) is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos\sqrt{2}\,t \\ \sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin\sqrt{2}\,t \\ -\sqrt{2}\cos\sqrt{2}\,t \end{pmatrix}. \tag{26}$$

Upon imposing the initial conditions

$$\begin{pmatrix} I \\ V \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \tag{27}$$

we find that

$$c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \tag{28}$$

Thus $c_1 = 2$ and $c_2 = -\sqrt{2}$. The solution of the stated problem is then given by Eq. (26) with these values of c_1 and c_2. The graph of the solution is shown in Figure 7.6.4. The trajectory spirals counterclockwise and rapidly approaches the origin, due to the factor e^{-t}.

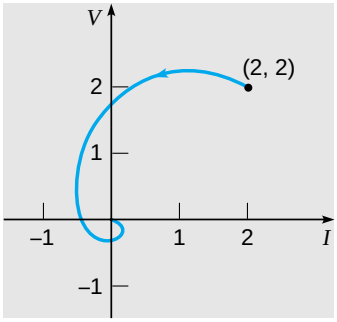


FIGURE 7.6.4 Solution of the initial value problem in Example 2.

EXAMPLE 3

The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x} \tag{29}$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behavior of the trajectories in the phase plane changes markedly.

The behavior of the trajectories is controlled by the eigenvalues of the coefficient matrix. The characteristic equation is

$$r^2 - \alpha r + 4 = 0, \tag{30}$$

so the eigenvalues are

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}. \tag{31}$$

From Eq. (31) it follows that the eigenvalues are complex conjugates for $-4 < \alpha < 4$ and are real otherwise. Thus two critical values are $\alpha = -4$ and $\alpha = 4$, where the eigenvalues change from real to complex, or vice versa. For $\alpha < -4$ both eigenvalues are negative, so all trajectories approach the origin, which is an asymptotically stable node. For $\alpha > 4$ both eigenvalues are positive, so the origin is again a node, this time unstable; all trajectories (except $\mathbf{x} = \mathbf{0}$) become unbounded. In the intermediate range, $-4 < \alpha < 4$, the eigenvalues are complex and the trajectories are spirals. However, for $-4 < \alpha < 0$ the real part of the eigenvalues is negative, the spirals are directed inward, and the origin is asymptotically stable, while for $0 < \alpha < 4$ the opposite is the case and the origin is unstable. Thus $\alpha = 0$ is also a critical value where the direction of the spirals changes from inward to outward. For this value of α the origin is a center and the trajectories are closed curves about the origin, corresponding to solutions that are periodic in time. The other critical values, $\alpha = \pm 4$, yield eigenvalues that are real and equal. In this case the origin is again a node, but the phase portrait differs somewhat from those in Section 7.5. We take up this case in Section 7.8.

For second order systems with real coefficients we have now completed our description of the three main cases that can occur.

1. Eigenvalues have opposite signs; $\mathbf{x} = \mathbf{0}$ is a saddle point.

2. Eigenvalues have the same sign but are unequal; $\mathbf{x} = \mathbf{0}$ is a node.

3. Eigenvalues are complex with nonzero real part; $\mathbf{x} = \mathbf{0}$ is a spiral point.

Other possibilities are of less importance and occur as transitions between two of the cases just listed. For example, a zero eigenvalue occurs during the transition between a saddle point and a node. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points. Finally, real and equal eigenvalues appear during the transition between nodes and spiral points.

PROBLEMS

In each of Problems 1 through 8 express the general solution of the given system of equations in terms of real-valued functions. In each of Problems 1 through 6 also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10 find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.
(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane.
(c) For your trajectory in part (b) draw the graphs of x_1 versus t and of x_2 versus t.
(d) For your trajectory in part (b) draw the corresponding graph in three-dimensional tx_1x_2-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20 the coefficient matrix contains a parameter α. In each of these problems:

(a) Determine the eigenvalues in terms of α.
(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.
(c) Draw a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

17. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

18. $\mathbf{x}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} \alpha & 10 \\ -1 & -4 \end{pmatrix} \mathbf{x}$

20. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}$

In each of Problems 21 and 22 solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

21. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 23 and 24:

(a) Find the eigenvalues of the given system.
(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane. Also draw the trajectories in the x_1x_3- and x_2x_3-planes.
(c) For the initial point in part (b) draw the corresponding trajectory in $x_1x_2x_3$-space.

23. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}$

24. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \mathbf{x}$

25. Consider the electric circuit shown in Figure 7.6.5. Suppose that $R_1 = R_2 = 4$ ohms, $C = \frac{1}{2}$ farad, and $L = 8$ henrys.
(a) Show that this circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. *Hint:* See Problem 18 of Section 7.1.
(b) Find the general solution of Eqs. (i) in terms of real-valued functions.
(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ amperes and $V(0) = 3$ volts.
(d) Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

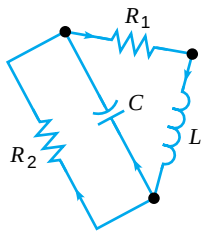


FIGURE 7.6.5 The circuit in Problem 25.

26. The electric circuit shown in Figure 7.6.6 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. These differential equations were derived in Problem 18 of Section 7.1.

(a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.

(b) Suppose that $R = 1$ ohm, $C = \frac{1}{2}$ farad, and $L = 1$ henry. Find the general solution of the system (i) in this case.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ amperes and $V(0) = 1$ volt.

(d) For the circuit of part (b) determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

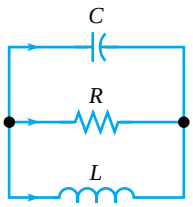


FIGURE 7.6.6 The circuit in Problem 26.

27. In this problem we indicate how to show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$, as given by Eqs. (9), are linearly independent. Let $r_1 = \lambda + i\mu$ and $\bar{r}_1 = \lambda - i\mu$ be a pair of conjugate eigenvalues of the coefficient matrix $\mathbf{A}$ of Eq. (1); let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{a} - i\mathbf{b}$ be the corresponding eigenvectors. Recall that it was stated in Section 7.3 that if $r_1 \neq \bar{r}_1$, then $\boldsymbol{\xi}^{(1)}$ and $\bar{\boldsymbol{\xi}}^{(1)}$ are linearly independent.

(a) First we show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Consider the equation $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\boldsymbol{\xi}^{(1)}$ and $\bar{\boldsymbol{\xi}}^{(1)}$, and then show that $(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$.

(b) Show that $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ and then that $c_1 = 0$ and $c_2 = 0$. Consequently, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

(c) To show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, where t_0 is an arbitrary point. Rewrite this equation in terms of $\mathbf{a}$ and $\mathbf{b}$, and then proceed as in part (b) to show that $c_1 = 0$ and $c_2 = 0$. Hence $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent at the arbitrary point t_0. Therefore they are linearly independent at every point and on every interval.

28. A mass m on a spring with constant k satisfies the differential equation (see Section 3.8)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

(a) Let $x_1 = u$ and $x_2 = u'$; show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}\mathbf{x}.$$

(b) Find the eigenvalues of the matrix for the system in part (a).

(c) Sketch several trajectories of the system. Choose one of your trajectories and sketch the corresponding graphs of x_1 versus t and of x_2 versus t. Sketch both graphs on one set of axes.
(d) What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring–mass system?

▶ 29. Consider the two-mass, three-spring system shown in Figure 7.1.1, whose equations of motion are given in Eqs. (1) of Section 7.1. If there are no external forces, and if the masses and spring constants are equal and of unit magnitude, then the equations of motion are

$$x_1'' = -2x_1 + x_2, \qquad x_2'' = x_1 - 2x_2.$$

(a) Transform the system into a system of four first order equations by letting $y_1 = x_1$, $y_2 = x_1'$, $y_3 = x_2$, and $y_4 = x_2'$.
(b) Find the eigenvalues of the coefficient matrix for the system in part (a).
(c) Solve the system in part (a) subject to the initial conditions $\mathbf{y}^T(0) = (2, 1, 2, 1)$. Describe the physical motion of the spring–mass system that corresponds to this solution.
(d) Solve the system in part (a) subject to the initial conditions $\mathbf{y}^T(0) = (2, \sqrt{3}, -2, -\sqrt{3})$. Describe the physical motion of the spring–mass system that corresponds to this solution.
(e) Observe that the spring–mass system has two natural modes of oscillation in this case. How are the natural frequencies related to the eigenvalues of the coefficient matrix? Do you think that there might be a third natural mode of oscillation with a different frequency?

7.7 Fundamental Matrices

The structure of the solutions of systems of linear differential equations can be further illuminated by introducing the idea of a fundamental matrix. Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{1}$$

on some interval $\alpha < t < \beta$. Then the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \tag{2}$$

whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** for the system (1). Note that a fundamental matrix is nonsingular since its columns are linearly independent vectors.

EXAMPLE 1

Find a fundamental matrix for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{3}$$

In Example 1 of Section 7.5 we found that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

are linearly independent solutions of Eq. (3). Thus a fundamental matrix for the system (3) is

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{4}$$

The solution of an initial value problem can be written very compactly in terms of a fundamental matrix. The general solution of Eq. (1) is

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{5}$$

or, in terms of $\Psi(t)$,

$$\mathbf{x} = \Psi(t)\mathbf{c}, \tag{6}$$

where $\mathbf{c}$ is a constant vector with arbitrary components $c_1, \ldots, c_n$. For an initial value problem consisting of the differential equation (1) and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0, \tag{7}$$

where t_0 is a given point in $\alpha < t < \beta$ and $\mathbf{x}^0$ is a given initial vector, it is only necessary to choose the vector $\mathbf{c}$ in Eq. (6) so as to satisfy the initial condition (7). Hence $\mathbf{c}$ must satisfy

$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0. \tag{8}$$

Therefore, since $\Psi(t_0)$ is nonsingular,

$$\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0 \tag{9}$$

and

$$\mathbf{x} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0 \tag{10}$$

is the solution of the initial value problem (1), (7). We emphasize, however, that to solve a given initial value problem one would ordinarily solve Eq. (8) by row reduction and then substitute for $\mathbf{c}$ in Eq. (6), rather than compute $\Psi^{-1}(t_0)$ and use Eq. (10).

Recall that each column of the fundamental matrix Ψ is a solution of Eq. (1). It follows that Ψ satisfies the matrix differential equation

$$\Psi' = \mathbf{P}(t)\Psi. \tag{11}$$

This relation is readily confirmed by comparing the two sides of Eq. (11) column by column.

Sometimes it is convenient to make use of the special fundamental matrix, denoted by $\Phi(t)$, whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ designated in Theorem 7.4.4. Besides the differential equation (1) these vectors satisfy the initial conditions

$$\mathbf{x}^{(j)}(t_0) = \mathbf{e}^{(j)}, \tag{12}$$

where $\mathbf{e}^{(j)}$ is the unit vector, defined in Theorem 7.4.4, with a one in the jth position and zeros elsewhere. Thus $\mathbf{\Phi}(t)$ has the property that

$$\mathbf{\Phi}(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}. \tag{13}$$

We will always reserve the symbol $\mathbf{\Phi}$ to denote the fundamental matrix satisfying the initial condition (13) and use $\mathbf{\Psi}$ when an arbitrary fundamental matrix is intended. In terms of $\mathbf{\Phi}(t)$ the solution of the initial value problem (1), (7) is even simpler in appearance; since $\mathbf{\Phi}^{-1}(t_0) = \mathbf{I}$, it follows from Eq. (10) that

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0. \tag{14}$$

While the fundamental matrix $\mathbf{\Phi}(t)$ is often more complicated than $\mathbf{\Psi}(t)$, it is especially helpful if the same system of differential equations is to be solved repeatedly subject to many different initial conditions. This corresponds to a given physical system that can be started from many different initial states. If the fundamental matrix $\mathbf{\Phi}(t)$ has been determined, then the solution for each set of initial conditions can be found simply by matrix multiplication, as indicated by Eq. (14). The matrix $\mathbf{\Phi}(t)$ thus represents a transformation of the initial conditions $\mathbf{x}^0$ into the solution $\mathbf{x}(t)$ at an arbitrary time t. By comparing Eqs. (10) and (14) it is clear that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$.

EXAMPLE 2

For the system (3),

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x},$$

in Example 1, find the fundamental matrix $\mathbf{\Phi}$ such that $\mathbf{\Phi}(0) = \mathbf{I}$.

The columns of $\mathbf{\Phi}$ are solutions of Eq. (3) that satisfy the initial conditions

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{15}$$

Since the general solution of Eq. (3) is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

we can find the solution satisfying the first set of these initial conditions by choosing $c_1 = c_2 = \frac{1}{2}$; similarly, we obtain the solution satisfying the second set of initial conditions by choosing $c_1 = \frac{1}{4}$ and $c_2 = -\frac{1}{4}$. Hence

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}. \tag{16}$$

Note that the elements of $\mathbf{\Phi}(t)$ are more complicated than those of the fundamental matrix $\mathbf{\Psi}(t)$ given by Eq. (4); however, it is now easy to determine the solution corresponding to any set of initial conditions.

The Matrix $\exp(\boldsymbol{At})$**.** Recall that the solution of the scalar initial value problem

$$x' = ax, \qquad x(0) = x_0, \tag{17}$$

where a is a constant, is

$$x = x_0 \exp(at). \tag{18}$$

Now consider the corresponding initial value problem for an $n \times n$ system, namely,

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0, \tag{19}$$

where $\mathbf{A}$ is a constant matrix. Applying the results of this section to the problem (19), we can write its solution as

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0, \tag{20}$$

where $\mathbf{\Phi}(0) = \mathbf{I}$. Comparing the problems (17) and (19), and their solutions, suggests that the matrix $\mathbf{\Phi}(t)$ might have an exponential character. We now explore this possibility.

The scalar exponential function $\exp(at)$ can be represented by the power series

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}, \tag{21}$$

which converges for all t. Let us now replace the scalar a by the $n \times n$ constant matrix $\mathbf{A}$, and consider the corresponding series

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots. \tag{22}$$

Each term in the series (22) is an $n \times n$ matrix. It is possible to show that each element of this matrix sum converges for all t as $n \to \infty$. Thus the series (22) defines as its sum a new matrix, which we denote by $\exp(\mathbf{A}t)$; that is,

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}, \tag{23}$$

analogous to the expansion (21) of the scalar function $\exp(at)$.

By differentiating the series (23) term by term we obtain

$$\frac{d}{dt}[\exp(\mathbf{A}t)] = \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^{n-1}}{(n-1)!} = \mathbf{A}\left[\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}\right] = \mathbf{A}\exp(\mathbf{A}t). \tag{24}$$

Thus $\exp(\mathbf{A}t)$ satisfies the differential equation

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A}\exp(\mathbf{A}t). \tag{25}$$

Further, when $t = 0$, $\exp(\mathbf{A}t)$ satisfies the initial condition

$$\exp(\mathbf{A}t)\Big|_{t=0} = \mathbf{I}. \tag{26}$$

The fundamental matrix $\mathbf{\Phi}$ satisfies the same initial value problem as $\exp(\mathbf{A}t)$, namely,

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \qquad \mathbf{\Phi}(0) = \mathbf{I}. \tag{27}$$

Thus we can identify $\exp(\mathbf{A}t)$ with the fundamental matrix $\mathbf{\Phi}(t)$ and we can write the solution of the initial value problem (19) in the form

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}^0, \tag{28}$$

which is analogous to the solution (18) of the initial value problem (17).

In order to justify more conclusively the use of $\exp(\mathbf{A}t)$ for the sum of the series (22), we should demonstrate that this matrix function does indeed have the properties we associate with the exponential function. One way to do this is outlined in Problem 15.

Diagonalizable Matrices. The basic reason why a system of linear (algebraic or differential) equations presents some difficulty is that the equations are usually *coupled*. In other words, some or all of the equations involve more than one, typically all, of the unknown variables. Hence the equations in the system must be solved *simultaneously*. In contrast, if each equation involves only a single variable, then each equation can be solved independently of all the others, which is a much easier task. This observation suggests that one way to solve a system of equations might be by transforming it into an equivalent *uncoupled* system in which each equation contains only one unknown variable. This corresponds to transforming the coefficient matrix $\mathbf{A}$ into a *diagonal* matrix.

Eigenvectors are useful in accomplishing such a transformation. Suppose that the $n \times n$ matrix $\mathbf{A}$ has a full set of n linearly independent eigenvectors. Recall that this will certainly be the case if the eigenvalues of $\mathbf{A}$ are all different, or if $\mathbf{A}$ is Hermitian. Letting $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ denote these eigenvectors and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues, form the matrix $\mathbf{T}$ whose columns are the eigenvectors, that is,

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}. \tag{29}$$

Since the columns of $\mathbf{T}$ are linearly independent vectors, $\det \mathbf{T} \neq 0$; hence $\mathbf{T}$ is nonsingular and $\mathbf{T}^{-1}$ exists. A straightforward calculation shows that the columns of the matrix $\mathbf{AT}$ are just the vectors $\mathbf{A}\boldsymbol{\xi}^{(1)}, \ldots, \mathbf{A}\boldsymbol{\xi}^{(n)}$. Since $\mathbf{A}\boldsymbol{\xi}^{(k)} = \lambda_k \boldsymbol{\xi}^{(k)}$, it follows that

$$\mathbf{AT} = \begin{pmatrix} \lambda_1\xi_1^{(1)} & \cdots & \lambda_n\xi_1^{(n)} \\ \vdots & & \vdots \\ \lambda_1\xi_n^{(1)} & \cdots & \lambda_n\xi_n^{(n)} \end{pmatrix} = \mathbf{TD}, \tag{30}$$

where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{31}$$

is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$. From Eq. (30) it follows that

$$\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}. \tag{32}$$

Thus, if the eigenvalues and eigenvectors of $\mathbf{A}$ are known, $\mathbf{A}$ can be transformed into a diagonal matrix by the process shown in Eq. (32). This process is known as a **similarity**

transformation, and Eq. (32) is summed up in words by saying that **A** is **similar** to the diagonal matrix **D**. Alternatively, we may say that **A** is **diagonalizable**. Observe that a similarity transformation leaves the eigenvalues of **A** unchanged and transforms its eigenvectors into the coordinate vectors $\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(n)}$.

If **A** is Hermitian, then the determination of $\mathbf{T}^{-1}$ is very simple. We choose the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ of **A** so that they are normalized by $(\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}^{(i)}) = 1$ for each i, as well as orthogonal. Then it is easy to verify that $\mathbf{T}^{-1} = \mathbf{T}^{*}$; in other words, the inverse of **T** is the same as its adjoint (the transpose of its complex conjugate).

Finally, we note that if **A** has fewer than n linearly independent eigenvectors, then there is no matrix **T** such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. In this case, **A** is not similar to a diagonal matrix, and is not diagonalizable.

EXAMPLE 3

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}. \tag{33}$$

Find the similarity transformation matrix **T** and show that **A** can be diagonalized.

In Example 1 of Section 7.5 we found that the eigenvalues and eigenvectors of **A** are

$$r_1 = 3, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{34}$$

Thus the transformation matrix **T** and its inverse $\mathbf{T}^{-1}$ are

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}; \qquad \mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}. \tag{35}$$

Consequently, you can check that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{D}. \tag{36}$$

Now let us turn again to the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{37}$$

where **A** is a constant matrix. In Sections 7.5 and 7.6 we have described how to solve such a system by starting from the assumption that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$. Now we provide another viewpoint, one based on diagonalizing the coefficient matrix **A**.

According to the results stated just above, it is possible to diagonalize **A** whenever **A** has a full set of n linearly independent eigenvectors. Let $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ be eigenvectors of **A** corresponding to the eigenvalues $r_1, \ldots, r_n$ and form the transformation matrix **T** whose columns are $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$. Then, defining a new dependent variable **y** by the relation

$$\mathbf{x} = \mathbf{T}\mathbf{y}, \tag{38}$$

we have from Eq. (37) that

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y}. \tag{39}$$

Multiplying by $\mathbf{T}^{-1}$, we then obtain

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y}, \tag{40}$$

or, using Eq. (32),

$$\mathbf{y}' = \mathbf{D}\mathbf{y}. \tag{41}$$

Recall that $\mathbf{D}$ is the diagonal matrix with the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ along the diagonal. A fundamental matrix for the system (41) is the diagonal matrix (see Problem 16)

$$\mathbf{Q}(t) = \exp(\mathbf{D}t) = \begin{pmatrix} e^{r_1 t} & 0 & \cdots & 0 \\ 0 & e^{r_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{r_n t} \end{pmatrix}. \tag{42}$$

A fundamental matrix $\mathbf{\Psi}$ for the system (37) is then found from $\mathbf{Q}$ by the transformation (38),

$$\mathbf{\Psi} = \mathbf{T}\mathbf{Q}; \tag{43}$$

that is,

$$\mathbf{\Psi}(t) = \begin{pmatrix} \xi_1^{(1)} e^{r_1 t} & \cdots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \cdots & \xi_n^{(n)} e^{r_n t} \end{pmatrix}. \tag{44}$$

Equation (44) is the same result that was obtained in Section 7.5. This diagonalization procedure does not afford any computational advantage over the method of Section 7.5, since in either case it is necessary to calculate the eigenvalues and eigenvectors of the coefficient matrix in the system of differential equations. Nevertheless, it is noteworthy that the problem of solving a system of differential equations and the problem of diagonalizing a matrix are mathematically the same.

EXAMPLE 4

Consider again the system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{45}$$

where $\mathbf{A}$ is given by Eq. (33). Using the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (35), you can reduce the system (45) to the diagonal system

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = \mathbf{D}\mathbf{y}. \tag{46}$$

Obtain a fundamental matrix for the system (46) and then transform it to obtain a fundamental matrix for the original system (45).

By multiplying $\mathbf{D}$ repeatedly with itself, we find that

$$\mathbf{D}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^3 = \begin{pmatrix} 27 & 0 \\ 0 & -1 \end{pmatrix}, \ldots. \tag{47}$$

Therefore it follows from Eq. (23) that $\exp(\mathbf{D}t)$ is a diagonal matrix with the entries e^{3t} and e^{-t} on the diagonal, that is,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{48}$$

Finally, we obtain the required fundamental matrix $\boldsymbol{\Psi}(t)$ by multiplying $\mathbf{T}$ and $\exp(\mathbf{D}t)$:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{49}$$

Observe that this fundamental matrix is the same as the one found in Example 1.

PROBLEMS

In each of Problems 1 through 10 find a fundamental matrix for the given system of equations. In each case also find the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(0) = \mathbf{I}$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$
2. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} \mathbf{x}$
3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$
4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$
5. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$
6. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$
7. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$
8. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$
9. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$
10. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$
11. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

by using the fundamental matrix $\boldsymbol{\Phi}(t)$ found in Problem 3.

12. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

by using the fundamental matrix $\boldsymbol{\Phi}(t)$ found in Problem 6.

13. Show that $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)$, where $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Psi}(t)$ are as defined in this section.
14. The fundamental matrix $\boldsymbol{\Phi}(t)$ for the system (3) was found in Example 2. Show that $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s) = \boldsymbol{\Phi}(t+s)$ by multiplying $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}(s)$.
15. Let $\boldsymbol{\Phi}(t)$ denote the fundamental matrix satisfying $\boldsymbol{\Phi}' = \mathbf{A}\boldsymbol{\Phi}$, $\boldsymbol{\Phi}(0) = \mathbf{I}$. In the text we also denoted this matrix by $\exp(\mathbf{A}t)$. In this problem we show that $\boldsymbol{\Phi}$ does indeed have the principal algebraic properties associated with the exponential function.
    (a) Show that $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s) = \boldsymbol{\Phi}(t+s)$; that is, $\exp(\mathbf{A}t)\exp(\mathbf{A}s) = \exp[\mathbf{A}(t+s)]$.
    Hint: Show that if s is fixed and t is variable, then both $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)$ and $\boldsymbol{\Phi}(t+s)$ satisfy the initial value problem $\mathbf{Z}' = \mathbf{A}\mathbf{Z}$, $\mathbf{Z}(0) = \boldsymbol{\Phi}(s)$.
    (b) Show that $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(-t) = \mathbf{I}$; that is, $\exp(\mathbf{A}t)\exp[\mathbf{A}(-t)] = \mathbf{I}$. Then show that $\boldsymbol{\Phi}(-t) = \boldsymbol{\Phi}^{-1}(t)$.
    (c) Show that $\boldsymbol{\Phi}(t-s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s)$.

16. Show that if $\mathbf{A}$ is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$, then $\exp(\mathbf{A}t)$ is also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \ldots, \exp(a_n t)$.
17. The method of successive approximations (see Section 2.8) can also be applied to systems of equations. For example, consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0, \tag{i}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}^0$ a prescribed vector.
(a) Assuming that a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ exists, show that it must satisfy the integral equation

$$\boldsymbol{\phi}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\, ds. \tag{ii}$$

(b) Start with the initial approximation $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}^0$. Substitute this expression for $\boldsymbol{\phi}(s)$ in the right side of Eq. (ii) and obtain a new approximation $\boldsymbol{\phi}^{(1)}(t)$. Show that

$$\boldsymbol{\phi}^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0. \tag{iii}$$

(c) Repeat this process and thereby obtain a sequence of approximations $\boldsymbol{\phi}^{(0)}$, $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}, \ldots, \boldsymbol{\phi}^{(n)}, \ldots$. Use an inductive argument to show that

$$\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0. \tag{iv}$$

(d) Let $n \to \infty$ and show that the solution of the initial value problem (i) is

$$\boldsymbol{\phi}(t) = \exp(\mathbf{A}t)\mathbf{x}^0. \tag{v}$$

7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

with a discussion of the case in which the matrix $\mathbf{A}$ has a repeated eigenvalue. Recall that in Section 7.3 we noted that a repeated eigenvalue with multiplicity k may be associated with fewer than k linearly independent eigenvectors. The following example illustrates this possibility.

EXAMPLE 1

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{2}$$

The eigenvalues r and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = 0. \tag{4}$$

Thus the two eigenvalues are $r_1 = 2$ and $r_2 = 2$; that is, the eigenvalue 2 has multiplicity 2.

To determine the eigenvectors we must return to Eq. (3) and use for r the value 2. This gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

Hence we obtain the single condition $\xi_1 + \xi_2 = 0$, which determines ξ_2 in terms of ξ_1, or vice versa. Thus the eigenvector corresponding to the eigenvalue $r = 2$ is

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{6}$$

or any nonzero multiple of this vector. Observe that there is only one linearly independent eigenvector associated with the double eigenvalue.

Returning to the system (1), suppose that $r = \rho$ is a k-fold root of the determinantal equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{7}$$

Then ρ is an eigenvalue of multiplicity k of the matrix $\mathbf{A}$. In this event, there are two possibilities: either there are k linearly independent eigenvectors corresponding to the eigenvalue ρ, or else there are fewer than k such eigenvectors.

In the first case, let $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(k)}$ be k linearly independent eigenvectors associated with the eigenvalue ρ of multiplicity k. Then $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{\rho t}, \ldots, \mathbf{x}^{(k)}(t) = \boldsymbol{\xi}^{(k)} e^{\rho t}$ are k linearly independent solutions of Eq. (1). Thus in this case it makes no difference that the eigenvalue $r = \rho$ is repeated; there is still a fundamental set of solutions of Eq. (1) of the form $\boldsymbol{\xi} e^{rt}$. This case always occurs if the coefficient matrix $\mathbf{A}$ is Hermitian.

However, if the coefficient matrix is not Hermitian, then there may be fewer than k independent eigenvectors corresponding to an eigenvalue ρ of multiplicity k, and if so, there will be fewer than k solutions of Eq. (1) of the form $\boldsymbol{\xi} e^{\rho t}$ associated with this eigenvalue. Therefore, to construct the general solution of Eq. (1) it is necessary to find other solutions of a different form. By analogy with previous results for linear equations of order n, it is natural to seek additional solutions involving products of polynomials and exponential functions. We first consider an example.

EXAMPLE 2

Find a fundamental set of solutions of

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \tag{8}$$

and draw a phase portrait for this system.

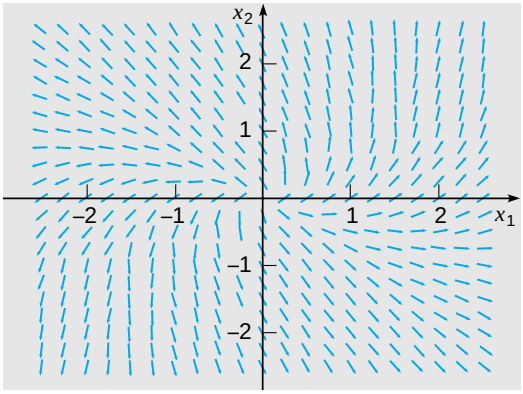


FIGURE 7.8.1 A direction field for the system (8).

A direction field for the system (8) is shown in Figure 7.8.1. From this figure it appears that all nonzero solutions depart from the origin.

To solve the system observe that the coefficient matrix $\mathbf{A}$ is the same as the matrix in Example 1. Thus we know that $r = 2$ is a double eigenvalue and that it has only a single corresponding eigenvector, which we may take as $\boldsymbol{\xi}^T = (1, -1)$. Thus one solution of the system (8) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \tag{9}$$

but there is no second solution of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$.

Based on the procedure used for second order linear equations in Section 3.5, it may be natural to attempt to find a second solution of the system (8) of the form

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t}, \tag{10}$$

where $\boldsymbol{\xi}$ is a constant vector to be determined. Substituting for $\mathbf{x}$ in Eq. (8) gives

$$2\boldsymbol{\xi} t e^{2t} + \boldsymbol{\xi} e^{2t} - \mathbf{A}\boldsymbol{\xi} t e^{2t} = \mathbf{0}. \tag{11}$$

For Eq. (11) to be satisfied for all t it is necessary for the coefficients of te^{2t} and e^{2t} each to be zero. From the term in e^{2t} we find that

$$\boldsymbol{\xi} = \mathbf{0}. \tag{12}$$

Hence there is no nonzero solution of the system (8) of the form (10).

Since Eq. (11) contains terms in both te^{2t} and e^{2t}, it appears that in addition to $\boldsymbol{\xi} te^{2t}$ the second solution must contain a term of the form $\boldsymbol{\eta} e^{2t}$; in other words, we need to assume that

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}, \tag{13}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are constant vectors. Upon substituting this expression for $\mathbf{x}$ in Eq. (8) we obtain

$$2\boldsymbol{\xi} t e^{2t} + (\boldsymbol{\xi} + 2\boldsymbol{\eta}) e^{2t} = \mathbf{A}(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}). \tag{14}$$

Equating coefficients of te^{2t} and e^{2t} on each side of Eq. (14) gives the conditions

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \tag{15}$$

and

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi} \tag{16}$$

for the determination of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Equation (15) is satisfied if $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = 2$, that is, $\boldsymbol{\xi}^T = (1, -1)$. Since $\det(\mathbf{A} - 2\mathbf{I})$ is zero, we might expect that Eq. (16) cannot be solved. However, this is not necessarily true, since for some vectors $\boldsymbol{\xi}$ it is possible to solve Eq. (16). In fact, the augmented matrix for Eq. (16) is

$$\begin{pmatrix} -1 & -1 & | & 1 \\ 1 & 1 & | & -1 \end{pmatrix}.$$

The second row of this matrix is proportional to the first, so the system is solvable. We have

$$-\eta_1 - \eta_2 = 1,$$

so if $\eta_1 = k$, where k is arbitrary, then $\eta_2 = -k - 1$. If we write

$$\boldsymbol{\eta} = \begin{pmatrix} k \\ -1 - k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{17}$$

then by substituting for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in Eq. (13) we obtain

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}. \tag{18}$$

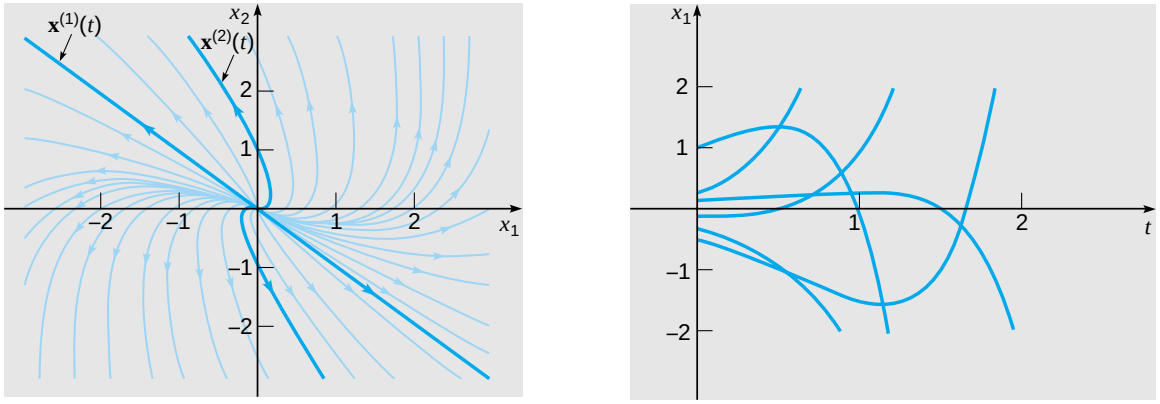
The last term in Eq. (18) is merely a multiple of the first solution $\mathbf{x}^{(1)}(t)$, and may be ignored, but the first two terms constitute a new solution:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}. \tag{19}$$

An elementary calculation shows that $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = -e^{4t}$, and therefore $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions of the system (8). The general solution is

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]. \end{aligned} \tag{20}$$

The graph of the solution (20) is a little more difficult to analyze than in some of the previous examples. It is clear that $\mathbf{x}$ becomes unbounded as $t \to \infty$ and that $\mathbf{x} \to \mathbf{0}$ as $t \to -\infty$. It is possible to show that as $t \to -\infty$, all solutions approach the origin tangent to the line $x_2 = -x_1$ determined by the eigenvector. Similarly, as $t \to \infty$, each trajectory is asymptotic to a line of slope -1. The trajectories of the system (8) are shown in Figure 7.8.2*a* and some typical plots of x_1 versus t are shown in Figure 7.8.2*b*. The pattern of trajectories in this figure is typical of second order systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with equal eigenvalues and only one independent eigenvector. The origin is called an **improper node** in this case. If the eigenvalues are negative, then the trajectories are similar, but are traversed in the inward direction. An improper node is asymptotically stable or unstable, depending on whether the eigenvalues are negative or positive.



(*a*) (*b*)

FIGURE 7.8.2 (*a*) Trajectories of the system (8); the origin is an improper node. (*b*) Plots of x_1 versus t for the system (8).

One difference between a system of two first order equations and a single second order equation is evident from the preceding example. Recall that for a second order linear equation with a repeated root r_1 of the characteristic equation, a term $ce^{r_1 t}$ in the second solution is not required since it is a multiple of the first solution. On the other hand, for a system of two first order equations, the term $\boldsymbol{\eta} e^{r_1 t}$ of Eq. (13) with $r_1 = 2$ is not a multiple of the first solution $\boldsymbol{\xi} e^{r_1 t}$, so the term $\boldsymbol{\eta} e^{r_1 t}$ must be retained.

Example 2 is entirely typical of the general case when there is a double eigenvalue and a single associated eigenvector. Consider again the system (1), and suppose that $r = \rho$ is a double eigenvalue of $\mathbf{A}$, but that there is only one corresponding eigenvector $\boldsymbol{\xi}$. Then one solution [similar to Eq. (9)] is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{\rho t}, \tag{21}$$

where $\boldsymbol{\xi}$ satisfies

$$(\mathbf{A} - \rho \mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{22}$$

By proceeding as in Example 2 we find that a second solution [similar to Eq. (19)] is

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t}, \tag{23}$$

where $\boldsymbol{\xi}$ satisfies Eq. (22) and $\boldsymbol{\eta}$ is determined from

$$(\mathbf{A} - \rho \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{24}$$

Even though $\det(\mathbf{A} - \rho \mathbf{I}) = 0$, it can be shown that it is always possible to solve Eq. (24) for $\boldsymbol{\eta}$. The vector $\boldsymbol{\eta}$ is called a **generalized eigenvector** corresponding to the eigenvalue ρ.

Fundamental Matrices. As explained in Section 7.7, fundamental matrices are formed by arranging linearly independent solutions in columns. Thus, for example, a

fundamental matrix for the system (8) can be formed from the solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ from Eqs. (9) and (19), respectively:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix}. \tag{25}$$

The matrix $\boldsymbol{\Phi}$ that satisfies $\boldsymbol{\Phi}(0) = \mathbf{I}$ can also be readily found from the relation $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$. For Eq. (8) we have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \boldsymbol{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \tag{26}$$

and then

$$\begin{aligned} \boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) &= e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}. \end{aligned} \tag{27}$$

The latter matrix is also the exponential matrix $\exp(\mathbf{A}t)$.

Jordan Forms. An $n \times n$ matrix $\mathbf{A}$ can be diagonalized as discussed in Section 7.7 only if it has a full complement of n linearly independent eigenvectors. If there is a shortage of eigenvectors (because of repeated eigenvalues), then $\mathbf{A}$ can always be transformed into a nearly diagonal matrix called its Jordan[5] form, which has the eigenvalues of $\mathbf{A}$ on the main diagonal, ones in certain positions on the diagonal above the main diagonal, and zeros elsewhere.

Consider again the matrix $\mathbf{A}$ given by Eq. (2). Form the transformation matrix $\mathbf{T}$ with the single eigenvector $\boldsymbol{\xi}$ from Eq. (6) in its first column and the generalized eigenvector $\boldsymbol{\eta}$ from Eq. (17) with $k = 0$ in the second column. Then $\mathbf{T}$ and its inverse are given by

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{28}$$

As you can verify, it follows that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \mathbf{J}. \tag{29}$$

The matrix $\mathbf{J}$ in Eq. (29) is the Jordan form of $\mathbf{A}$. It is typical of all Jordan forms in that it has a 1 above the main diagonal in the column corresponding to the eigenvector that is lacking (and is replaced in $\mathbf{T}$ by the generalized eigenvector).

If we start again from Eq. (1),

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (28), produces the system

$$\mathbf{y}' = \mathbf{J}\mathbf{y}, \tag{30}$$

where $\mathbf{J}$ is given by Eq. (29). In scalar form the system (30) is

$$y_1' = 2y_1 + y_2, \quad y_2' = 2y_2. \tag{31}$$

[5]Camille Jordan (1838–1921), professor at the École Polytechnique and the Collège de France, made important contributions to analysis, topology, and especially to algebra. The Jordan form of a matrix appeared in his influential book *Traité des substitutions et des équations algébriques*, published in 1870.

These equations can be solved readily in reverse order. In this way we obtain

$$y_2 = c_1 e^{2t}, \quad y_1 = c_1 t e^{2t} + c_2 e^{2t}. \tag{32}$$

Thus two independent solutions of the system (30) are

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \quad \mathbf{y}^{(2)} = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t} \tag{33}$$

and the corresponding fundamental matrix is

$$\widehat{\mathbf{\Psi}}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \tag{34}$$

Since $\widehat{\mathbf{\Psi}}(0) = \mathbf{I}$, we can also indentify the matrix in Eq. (34) as $\exp(\mathbf{J}t)$. The same result can be reached by calculating powers of $\mathbf{J}$ and substituting them into the exponential series (see Problems 19 through 21). To obtain a fundamental matrix for the original system we now form the product

$$\mathbf{\Psi}(t) = \mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -e^{2t} - te^{2t} \end{pmatrix}, \tag{35}$$

which is the same as the fundamental matrix given in Eq. (25).

PROBLEMS

In each of Problems 1 through 6 find the general solution of the given system of equations. In each of Problems 1 through 4 also draw a direction field, sketch a few trajectories, and describe how the solutions behave as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 7 through 10 find the solution of the given initial value problem. Draw the trajectory of the solution in the x_1x_2-plane and also the graph of x_1 versus t.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

9. $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 11 and 12 find the solution of the given initial value problem. Draw the corresponding trajectory in $x_1x_2x_3$-space and also draw the graph of x_1 versus t.

▶ 11. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

▶ 12. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 13 and 14 solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

13. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

14. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}$

15. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 38 in Section 3.5.

16. Consider again the electric circuit in Problem 26 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

(a) Show that the eigenvalues are real and equal if $L = 4R^2C$.
(b) Suppose that $R = 1$ ohm, $C = 1$ farad, and $L = 4$ henrys. Suppose also that $I(0) = 1$ ampere and $V(0) = 2$ volts. Find $I(t)$ and $V(t)$.

Eigenvalues of Multiplicity 3. If the matrix $\mathbf{A}$ has an eigenvalue of multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.

17. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\mathbf{x}. \tag{i}$$

(a) Show that $r = 2$ is an eigenvalue of multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (i). There is no other solution of the purely exponential form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$.
(c) To find a second solution assume that $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Since $\boldsymbol{\xi}$ has already been found in part (a), solve the second equation for $\boldsymbol{\eta}$. Neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$, since it leads only to a multiple of the first solution $\mathbf{x}^{(1)}$. Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (i).
(d) To find a third solution assume that $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} t e^{2t} + \boldsymbol{\zeta} e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations are the same as in part (c), so solve the third equation for $\boldsymbol{\zeta}$, again neglecting the multiple of $\boldsymbol{\xi}^{(1)}$ that appears. Then write down a third solution $\mathbf{x}^{(3)}(t)$ of the system (i).
(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).
(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column, and the generalized eigenvectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third columns. Then find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

18. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}. \tag{i}$$

(a) Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$, and that there are only two linearly independent eigenvectors, which we may take as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}. \tag{ii}$$

Find two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of Eq. (i).
(b) To find a third solution assume that $\mathbf{x} = \boldsymbol{\xi} t e^t + \boldsymbol{\eta} e^t$; then show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{iii}$$

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{iv}$$

(c) Show that $\boldsymbol{\xi} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$, where c_1 and c_2 are arbitrary constants, is the most general solution of Eq. (iii). Show that in order to solve Eq. (iv) it is necessary that $c_1 = c_2$.
(d) It is convenient to choose $c_1 = c_2 = 2$. For this choice show that

$$\boldsymbol{\xi} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \tag{v}$$

where we have dropped the multiples of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ that appear in $\boldsymbol{\eta}$. Use the results given in Eqs. (v) to find a third linearly independent solution $\mathbf{x}^{(3)}(t)$ of Eq. (i).
(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).
(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and with the eigenvector $\boldsymbol{\xi}$ and the generalized eigenvector $\boldsymbol{\eta}$ from Eqs. (v) in the other two columns. Find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

19. Let $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is an arbitrary real number.

    (a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

    (b) Use an inductive argument to show that $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

    (c) Determine $\exp(\mathbf{J}t)$.

    (d) Use $\exp(\mathbf{J}t)$ to solve the initial value problem $\mathbf{x}' = \mathbf{J}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

20. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

    where λ is an arbitrary real number.

    (a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

    (b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

    (c) Determine $\exp(\mathbf{J}t)$.

    (d) Observe that if you choose $\lambda = 1$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 18(f). Using the matrix $\mathbf{T}$ from Problem 18(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 1$. Is the resulting matrix the same as the fundamental matrix $\mathbf{\Psi}(t)$ in Problem 18(e)? If not, explain the discrepancy.

21. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

    where λ is an arbitrary real number.

    (a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

    (b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & [n(n-1)/2]\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

    (c) Determine $\exp(\mathbf{J}t)$.

    (d) Observe that if you choose $\lambda = 2$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 17(f). Using the matrix $\mathbf{T}$ from Problem 17(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 2$. Observe that the resulting matrix is the same as the fundamental matrix $\mathbf{\Psi}(t)$ in Problem 17(e).

7.9 Nonhomogeneous Linear Systems

In this section we turn to the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{1}$$

where the $n \times n$ matrix $\mathbf{P}(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. By the same argument as in Section 3.6 (see also Problem 16 in this section) the general solution of Eq. (1) can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t), \tag{2}$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system (1). We will briefly describe several methods for determining $\mathbf{v}(t)$.

Diagonalization. We begin with systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \tag{3}$$

where $\mathbf{A}$ is an $n \times n$ diagonalizable constant matrix. By diagonalizing the coefficient matrix $\mathbf{A}$, as indicated in Section 7.7, we can transform Eq. (3) into a system of equations that is readily solvable.

Let $\mathbf{T}$ be the matrix whose columns are the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$, and define a new dependent variable $\mathbf{y}$ by

$$\mathbf{x} = \mathbf{T}\mathbf{y}. \tag{4}$$

Then substituting for $\mathbf{x}$ in Eq. (3), we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

By multiplying by $\mathbf{T}^{-1}$ it follows that

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \tag{5}$$

where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$ and where $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$, arranged in the same order as the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ that appear as columns of $\mathbf{T}$. Equation (5) is a system of *n uncoupled* equations for $y_1(t), \ldots, y_n(t)$; as a consequence, the equations can be solved separately. In scalar form Eq. (5) has the form

$$y_j'(t) = r_j y_j(t) + h_j(t), \qquad j = 1, \ldots, n, \tag{6}$$

where $h_j(t)$ is a certain linear combination of $g_1(t), \ldots, g_n(t)$. Equation (6) is a first order linear equation and can be solved by the methods of Section 2.1. In fact, we have

$$y_j(t) = e^{r_j t}\int_{t_0}^{t} e^{-r_j s}h_j(s)\,ds + c_j e^{r_j t}, \qquad j = 1, \ldots, n, \tag{7}$$

where the c_j are arbitrary constants. Finally, the solution $\mathbf{x}$ of Eq. (3) is obtained from Eq. (4). When multiplied by the transformation matrix $\mathbf{T}$, the second term on the right side of Eq. (7) produces the general solution of the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, while the first term on the right side of Eq. (7) yields a particular solution of the nonhomogeneous system (3).

EXAMPLE 1

Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{8}$$

Proceeding as in Section 7.5, we find that the eigenvalues of the coefficient matrix are $r_1 = -3$ and $r_2 = -1$, and the corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{9}$$

Thus the general solution of the homogeneous system is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{10}$$

Before writing down the matrix $\mathbf{T}$ of eigenvectors we recall that eventually we must find $\mathbf{T}^{-1}$. The coefficient matrix $\mathbf{A}$ is real and symmetric, so we can use the result stated at the end of Section 7.3: $\mathbf{T}^{-1}$ is simply the adjoint or (since $\mathbf{T}$ is real) the transpose of $\mathbf{T}$, provided that the eigenvectors of $\mathbf{A}$ are normalized so that $(\boldsymbol{\xi}, \boldsymbol{\xi}) = 1$. Hence, upon normalizing $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, we have

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{11}$$

Letting $\mathbf{x} = \mathbf{T}\mathbf{y}$ and substituting for $\mathbf{x}$ in Eq. (8), we obtain the following system of equations for the new dependent variable $\mathbf{y}$:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}. \tag{12}$$

Thus

$$\begin{aligned} y_1' + 3y_1 &= \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t, \\ y_2' + y_2 &= \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t. \end{aligned} \tag{13}$$

Each of Eqs. (13) is a first order linear equation, and so can be solved by the methods of Section 2.1. In this way we obtain

$$\begin{aligned} y_1 &= \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left[\left(\frac{t}{3}\right) - \frac{1}{9}\right] + c_1 e^{-3t}, \\ y_2 &= \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}. \end{aligned} \tag{14}$$

Finally, we write the solution in terms of the original variables:

$$\begin{aligned} \mathbf{x} = \mathbf{T}\mathbf{y} &= \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} (c_1/\sqrt{2})e^{-3t} + [(c_2/\sqrt{2}) + \frac{1}{2}]e^{-t} + t - \frac{4}{3} + te^{-t} \\ -(c_1/\sqrt{2})e^{-3t} + [(c_2/\sqrt{2}) - \frac{1}{2}]e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix} \end{aligned}$$

$$= k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \tag{15}$$

where $k_1 = c_1/\sqrt{2}$ and $k_2 = c_2/\sqrt{2}$. The first two terms on the right side of Eq. (15) form the general solution of the homogeneous system corresponding to Eq. (8). The remaining terms are a particular solution of the nonhomogeneous system.

If the coefficient matrix $\mathbf{A}$ in Eq. (3) is not diagonalizable (due to repeated eigenvalues and a shortage of eigenvectors), it can nevertheless be reduced to a Jordan form $\mathbf{J}$ by a suitable transformation matrix $\mathbf{T}$ involving both eigenvectors and generalized eigenvectors. In this case the differential equations for $y_1, \ldots, y_n$ are not totally uncoupled since some rows of $\mathbf{J}$ have two nonzero elements, an eigenvalue in the diagonal position and a 1 in the adjacent position to the right. However, the equations for $y_1, \ldots, y_n$ can still be solved consecutively, starting with y_n. Then the solution of the original system (3) can be found by the relation $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Undetermined Coefficients. A second way to find a particular solution of the nonhomogeneous system (1) is the method of undetermined coefficients. To make use of this method, one assumes the form of the solution with some or all of the coefficients unspecified, and then seeks to determine these coefficients so as to satisfy the differential equation. As a practical matter, this method is applicable only if the coefficient matrix $\mathbf{P}$ is a constant matrix, and if the components of $\mathbf{g}$ are polynomial, exponential, or sinusoidal functions, or sums or products of these. In these cases the correct form of the solution can be predicted in a simple and systematic manner. The procedure for choosing the form of the solution is substantially the same as that given in Section 3.6 for linear second order equations. The main difference is illustrated by the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic equation. In this situation, rather than assuming a solution of the form $\mathbf{a}te^{\lambda t}$, it is necessary to use $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, where $\mathbf{a}$ and $\mathbf{b}$ are determined by substituting into the differential equation.

EXAMPLE 2

Use the method of undetermined coefficients to find a particular solution of

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{16}$$

This is the same system of equations as in Example 1. To use the method of undetermined coefficients, we write $\mathbf{g}(t)$ in the form

$$\mathbf{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t. \tag{17}$$

Then we assume that

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}, \tag{18}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are vectors to be determined. Observe that $r = -1$ is an eigenvalue of the coefficient matrix, and therefore we must include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$ in the

assumed solution. By substituting Eq. (18) into Eq. (16) and collecting terms, we obtain the following algebraic equations for **a**, **b**, **c**, and **d**:

$$\begin{aligned} \mathbf{A}\mathbf{a} &= -\mathbf{a}, \\ \mathbf{A}\mathbf{b} &= \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\ \mathbf{A}\mathbf{c} &= -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \\ \mathbf{A}\mathbf{d} &= \mathbf{c}. \end{aligned} \tag{19}$$

From the first of Eqs. (19) we see that **a** is an eigenvector of **A** corresponding to the eigenvalue $r = -1$. Thus $\mathbf{a}^T = (\alpha, \alpha)$, where α is any nonzero constant. Then we find that the second of Eqs. (19) can be solved only if $\alpha = 1$ and that in this case

$$\mathbf{b} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

for any constant k. The simplest choice is $k = 0$, from which $\mathbf{b}^T = (0, -1)$. Then the third and fourth of Eqs. (19) yield $\mathbf{c}^T = (1, 2)$ and $\mathbf{d}^T = (-\frac{4}{3}, -\frac{5}{3})$, respectively. Finally, from Eq. (18) we obtain the particular solution

$$\mathbf{v}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{21}$$

The particular solution (21) is not identical to the one contained in Eq. (15) of Example 1 because the term in e^{-t} is different. However, if we choose $k = \frac{1}{2}$ in Eq. (20), then $\mathbf{b}^T = (\frac{1}{2}, -\frac{1}{2})$ and the two particular solutions agree.

Variation of Parameters. Now let us turn to more general problems in which the coefficient matrix is not constant or not diagonalizable. Let

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{22}$$

where $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on $\alpha < t < \beta$. Assume that a fundamental matrix $\mathbf{\Psi}(t)$ for the corresponding homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{23}$$

has been found. We use the method of variation of parameters to construct a particular solution, and hence the general solution, of the nonhomogeneous system (22).

Since the general solution of the homogeneous system (23) is $\mathbf{\Psi}(t)\mathbf{c}$, it is natural to proceed as in Section 3.7, and to seek a solution of the nonhomogeneous system (22) by replacing the constant vector **c** by a vector function $\mathbf{u}(t)$. Thus, we assume that

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t), \tag{24}$$

where $\mathbf{u}(t)$ is a vector to be found. Upon differentiating **x** as given by Eq. (24) and requiring that Eq. (22) be satisfied, we obtain

$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t). \tag{25}$$

Since $\mathbf{\Psi}(t)$ is a fundamental matrix, $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$; hence Eq. (25) reduces to

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t). \tag{26}$$

Recall that $\mathbf{\Psi}(t)$ is nonsingular on any interval where $\mathbf{P}$ is continuous. Hence $\mathbf{\Psi}^{-1}(t)$ exists, and therefore

$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t). \tag{27}$$

Thus for $\mathbf{u}(t)$ we can select any vector from the class of vectors that satisfy Eq. (27); these vectors are determined only up to an arbitrary additive constant (vector); therefore we denote $\mathbf{u}(t)$ by

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\, ds + \mathbf{c}, \tag{28}$$

where the constant vector $\mathbf{c}$ is arbitrary. Finally, substituting for $\mathbf{u}(t)$ in Eq. (24) gives the solution $\mathbf{x}$ of the system (22):

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\, ds. \tag{29}$$

Since $\mathbf{c}$ is arbitrary, any initial condition at a point $t = t_0$ can be satisfied by an appropriate choice of $\mathbf{c}$. Thus, every solution of the system (22) is contained in the expression given by Eq. (29); it is therefore the general solution of Eq. (22). Note that the first term on the right side of Eq. (29) is the general solution of the corresponding homogeneous system (23), and the second term is a particular solution of Eq. (22) itself.

Now let us consider the initial value problem consisting of the differential equation (22) and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0. \tag{30}$$

We can write the solution of this problem most conveniently if we choose for the particular solution in Eq. (29) the specific one that is equal to the zero vector when $t = t_0$. This can be done by using t_0 as the lower limit of integration in Eq. (29), so that the general solution of the differential equation takes the form

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t) \int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\, ds. \tag{31}$$

The initial condition (30) can also be satisfied provided that

$$\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0. \tag{32}$$

Therefore

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 + \mathbf{\Psi}(t) \int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\, ds \tag{33}$$

is the solution of the given initial value problem. Again, while it is helpful to use $\mathbf{\Psi}^{-1}$ to write the solutions (29) and (33), it is usually better in particular cases to solve the necessary equations by row reduction rather than to calculate $\mathbf{\Psi}^{-1}$ and substitute into Eqs. (29) and (33).

The solution (33) takes a slightly simpler form if we use the fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(t_0) = \mathbf{I}$. In this case we have

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\, ds. \tag{34}$$

Equation (34) can be simplified further if the coefficient matrix $\mathbf{P}(t)$ is a constant matrix (see Problem 17).

EXAMPLE 3

Use the method of variation of parameters to find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{35}$$

This is the same system of equations as in Examples 1 and 2.

The general solution of the corresponding homogeneous system was given in Eq. (10). Thus

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \tag{36}$$

is a fundamental matrix. Then the solution $\mathbf{x}$ of Eq. (35) is given by $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{37}$$

Solving Eq. (37) by row reduction, we obtain

$$\begin{aligned} u_1' &= e^{2t} - \tfrac{3}{2}te^{3t}, \\ u_2' &= 1 + \tfrac{3}{2}te^{t}. \end{aligned}$$

Hence

$$\begin{aligned} u_1(t) &= \tfrac{1}{2}e^{2t} - \tfrac{1}{2}te^{3t} + \tfrac{1}{6}e^{3t} + c_1, \\ u_2(t) &= t + \tfrac{3}{2}te^{t} - \tfrac{3}{2}e^{t} + c_2, \end{aligned}$$

and

$$\begin{aligned} \mathbf{x} &= \mathbf{\Psi}(t)\mathbf{u}(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \\ &\quad + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \end{aligned} \tag{38}$$

which is the same as the solution obtained previously.

Each of the methods for solving nonhomogeneous equations has some advantages and disadvantages. The method of undetermined coefficients requires no integration, but is limited in scope and may entail the solution of several sets of algebraic equations. The method of diagonalization requires finding the inverse of the transformation matrix and the solution of a set of uncoupled first order linear equations, followed by a matrix multiplication. Its main advantage is that for Hermitian coefficient matrices the inverse of the transformation matrix can be written down without calculation, a feature that is more important for large systems. Variation of parameters is the most general method. On the other hand, it involves the solution of a set of linear algebraic equations with variable coefficients, followed by an integration and a matrix multiplication, so it may also be the most complicated from a computational viewpoint. For many small systems

with constant coefficients, such as the one in the examples in this section, there may be little reason to select one of these methods over another. Keep in mind, however, that the method of diagonalization is slightly more complicated if the coefficient matrix is not diagonalizable, but only reducible to a Jordan form, and the method of undetermined coefficients is practical only for the kinds of nonhomogeneous terms mentioned earlier.

For initial value problems for linear systems with constant coefficients, the Laplace transform is often an effective tool also. Since it is used in essentially the same way as described in Chapter 6 for single scalar equations, we do not give any details here.

PROBLEMS

In each of Problems 1 through 12 find the general solution of the given system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0$

6. $\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \quad t > 0$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^t$

8. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t$

9. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$

11. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \quad 0 < t < \pi$

12. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}, \quad \frac{\pi}{2} < t < \pi$

13. The electric circuit shown in Figure 7.9.1 is described by the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}I(t), \tag{i}$$

where x_1 is the current through the inductor, x_2 is the voltage drop across the capacitor, and $I(t)$ is the current supplied by the external source.

(a) Determine a fundamental matrix $\mathbf{\Psi}(t)$ for the homogeneous system corresponding to Eq. (i). Refer to Problem 25 of Section 7.6.

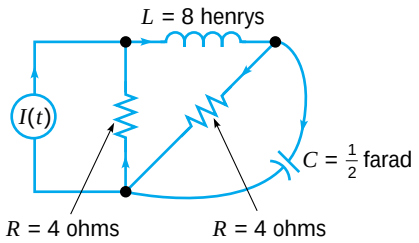


FIGURE 7.9.1 The circuit in Problem 13.

(b) If $I(t) = e^{-t/2}$, determine the solution of the system (i) that also satisfies the initial conditions $\mathbf{x}(0) = \mathbf{0}$.

In each of Problems 14 and 15 verify that the given vector is the general solution of the corresponding homogeneous system, and then solve the nonhomogeneous system. Assume that $t > 0$.

14. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 - t^2 \\ 2t \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$

15. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$

16. Let $\mathbf{x} = \boldsymbol{\phi}(t)$ be the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, and let $\mathbf{x} = \mathbf{v}(t)$ be some particular solution of the same system. By considering the difference $\boldsymbol{\phi}(t) - \mathbf{v}(t)$, show that $\boldsymbol{\phi}(t) = \mathbf{u}(t) + \mathbf{v}(t)$, where $\mathbf{u}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

17. Consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{x}^0.$$

(a) By referring to Problem 15(c) in Section 7.7, show that

$$\mathbf{x} = \boldsymbol{\Phi}(t)\mathbf{x}^0 + \int_0^t \boldsymbol{\Phi}(t - s)\mathbf{g}(s)\, ds.$$

(b) Show also that

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}^0 + \int_0^t \exp[\mathbf{A}(t - s)]\mathbf{g}(s)\, ds.$$

Compare these results with those of Problem 27 in Section 3.7.

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Further information on matrices and linear algebra is available in any introductory book on the subject. The following is a representative sample:

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CHAPTER
8

Numerical Methods

Up to this point we have discussed methods for solving differential equations by using analytical techniques such as integration or series expansions. Usually, the emphasis was on finding an exact expression for the solution. Unfortunately, there are many important problems in engineering and science, especially nonlinear ones, to which these methods either do not apply or are very complicated to use. In this chapter we discuss an alternative approach, the use of numerical approximation methods to obtain an accurate approximation to the solution of an initial value problem. The procedures described here can be executed easily on personal computers as well as on some pocket calculators.

8.1 The Euler or Tangent Line Method

To discuss the development and use of numerical procedures we will concentrate mainly on the first order initial value problem consisting of the differential equation

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

and the initial condition

$$y(t_0) = y_0. \tag{2}$$

We assume that the functions f and f_y are continuous on some rectangle in the ty-plane containing the point (t_0, y_0). Then, by Theorem 2.4.2, there exists a unique solution

$y = \phi(t)$ of the given problem in some interval about t_0. If Eq. (1) is nonlinear, then the interval of existence of the solution may be difficult to determine and may have no simple relationship to the function f. However, in all our discussions we assume that there is a unique solution of the initial value problem (1), (2) in the interval of interest.

In Section 2.7 we described the oldest and simplest numerical approximation method, namely, the Euler or tangent line method. This method is expressed by the equation

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \qquad n = 0, 1, 2, \ldots. \tag{3}$$

If the step size has a uniform value h and if we denote $f(t_n, y_n)$ by f_n, then Eq. (3) simplifies to

$$y_{n+1} = y_n + f_n h, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Euler's method consists of repeatedly evaluating Eq. (3) or (4), using the result of each step to execute the next step. In this way you obtain a sequence of values $y_0, y_1, y_2, \ldots, y_n, \ldots$ that approximate the values of the solution $\phi(t)$ at the points $t_0, t_1, t_2, \ldots, t_n, \ldots$.

In Section 2.7 we pointed out that a computer program for Euler's method has the structure given below. The specific instructions can be written in any convenient programming language.

The Euler Method

Step 1. **define** $f(t, y)$
Step 2. **input** initial values $t0$ and $y0$
Step 3. **input** step size h and number of steps n
Step 4. **output** $t0$ and $y0$
Step 5. **for** j from 1 to n **do**
Step 6. $k1 = f(t, y)$
$y = y + h * k1$
$t = t + h$
Step 7. **output** t and y
Step 8. **end**

Some examples of Euler's method appear in Section 2.7. As another example, consider the initial value problem

$$y' = 1 - t + 4y, \tag{5}$$

$$y(0) = 1. \tag{6}$$

Equation (5) is a first order linear equation, and it is easily verified that the solution satisfying the initial condition (6) is

$$y = \phi(t) = \tfrac{1}{4}t - \tfrac{3}{16} + \tfrac{19}{16}e^{4t}. \tag{7}$$

Since the exact solution is known, we do not need numerical methods to solve the initial value problem (5), (6). On the other hand, the availability of the exact solution makes it easy to determine the accuracy of any numerical procedure that we use on this problem. We will use this problem throughout the chapter to illustrate and compare different numerical methods. The solutions of Eq. (5) diverge rather rapidly from each other, so we should expect that it will be fairly difficult to approximate the solution (7) well over any considerable interval. Indeed, this is the reason for choosing this particular problem; it will be relatively easy to observe the benefits of using more accurate methods.

EXAMPLE 1

Using the Euler formula (4) and step sizes $h = 0.05$, 0.025, 0.01, and 0.001, determine approximate values of the solution $y = \phi(t)$ of the problem (5), (6) on the interval $0 \le t \le 2$.

The indicated calculations were carried out on a computer and some of the results are shown in Table 8.1.1. Their accuracy is not particularly impressive. For $h = 0.01$ the percentage error is 3.85% at $t = 0.5$, 7.49% at $t = 1.0$, and 14.4% at $t = 2.0$. The corresponding percentage errors for $h = 0.001$ are 0.40%, 0.79%, and 1.58%, respectively. Observe that if $h = 0.001$, then it requires 2,000 steps to traverse the interval from $t = 0$ to $t = 2$. Thus considerable computation is needed to obtain even reasonably good accuracy for this problem using the Euler method. When we discuss other numerical approximation methods later in this chapter, we will find that it is possible to obtain comparable or better accuracy with much larger step sizes and many fewer computational steps.

TABLE 8.1.1 A Comparison of Results for the Numerical Solution of $y' = 1 - t + 4y$, $y(0) = 1$ Using the Euler Method for Different Step Sizes h

t	$h = 0.05$	$h = 0.025$	$h = 0.01$	$h = 0.001$	Exact
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5475000	1.5761188	1.5952901	1.6076289	1.6090418
0.2	2.3249000	2.4080117	2.4644587	2.5011159	2.5053299
0.3	3.4333560	3.6143837	3.7390345	3.8207130	3.8301388
0.4	5.0185326	5.3690304	5.6137120	5.7754845	5.7942260
0.5	7.2901870	7.9264062	8.3766865	8.6770692	8.7120041
1.0	45.588400	53.807866	60.037126	64.382558	64.897803
1.5	282.07187	361.75945	426.40818	473.55979	479.25919
2.0	1745.6662	2432.7878	3029.3279	3484.1608	3540.2001

To begin to investigate the errors in using numerical approximations and also to suggest ways to construct more accurate algorithms it is helpful to mention some alternative ways to look at the Euler method.

First, let us write the differential equation (1) at the point $t = t_n$ in the form

$$\frac{d\phi}{dt}(t_n) = f[t_n, \phi(t_n)]. \tag{8}$$

Then we approximate the derivative in Eq. (8) by the corresponding (forward) difference quotient, obtaining

$$\frac{\phi(t_{n+1}) - \phi(t_n)}{t_{n+1} - t_n} \cong f[t_n, \phi(t_n)]. \tag{9}$$

Finally, if we replace $\phi(t_{n+1})$ and $\phi(t_n)$ by their approximate values y_{n+1} and y_n, respectively, and solve for y_{n+1}, we obtain the Euler formula (3).

Another point of view is to write the problem as an integral equation, as follows. Since $y = \phi(t)$ is a solution of the initial value problem (1), (2), by integrating from t_n to t_{n+1} we obtain

$$\int_{t_n}^{t_{n+1}} \phi'(t)\, dt = \int_{t_n}^{t_{n+1}} f[t, \phi(t)]\, dt$$

or

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f[t, \phi(t)]\, dt. \tag{10}$$

The integral in Eq. (10) is represented geometrically as the area under the curve in Figure 8.1.1 between $t = t_n$ and $t = t_{n+1}$. If we approximate the integral by replacing $f[t, \phi(t)]$ by its value $f[t_n, \phi(t_n)]$ at $t = t_n$, then we are approximating the actual area by the area of the shaded rectangle. In this way we obtain

$$\begin{aligned}\phi(t_{n+1}) &\cong \phi(t_n) + f[t_n, \phi(t_n)](t_{n+1} - t_n)\\ &= \phi(t_n) + hf[t_n, \phi(t_n)].\end{aligned} \tag{11}$$

Finally, to obtain an approximation y_{n+1} for $\phi(t_{n+1})$ we make a second approximation by replacing $\phi(t_n)$ by its approximate value y_n in Eq. (11). This gives the Euler formula $y_{n+1} = y_n + hf(t_n, y_n)$. A more accurate algorithm can be obtained by approximating the integral more accurately. This is discussed in Section 8.2.

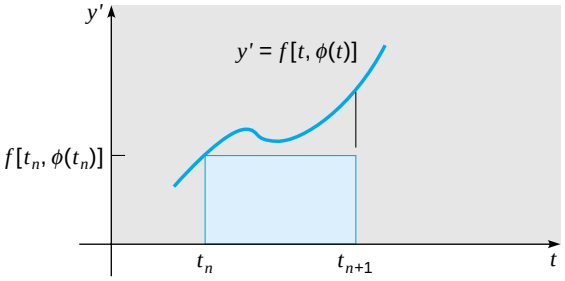


FIGURE 8.1.1 Integral derivation of the Euler method.

A third approach is to assume that the solution $y = \phi(t)$ has a Taylor series about the point t_n. Then

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \phi''(t_n)\frac{h^2}{2!} + \cdots$$

or

$$\phi(t_{n+1}) = \phi(t_n) + f[t_n, \phi(t_n)]h + \phi''(t_n)\frac{h^2}{2!} + \cdots. \tag{12}$$

If the series is terminated after the first two terms, and $\phi(t_{n+1})$ and $\phi(t_n)$ are replaced by their approximate values y_{n+1} and y_n, we again obtain the Euler formula (4). If more terms in the series are retained, a more accurate formula is obtained. Further, by using a Taylor series with a remainder it is possible to estimate the magnitude of the error in the formula. This is discussed later in this section.

The Backward Euler Formula. A variation on the Euler formula can be obtained by approximating the derivative in Eq. (8) by the backward difference quotient $[\phi(t_n) - \phi(t_{n-1})]/h$ instead of the forward difference quotient used in Eq. (9). In this way we obtain

$$\phi(t_n) - \phi(t_{n-1}) \cong hf(t_n, y_n)$$

or

$$y_n = y_{n-1} + hf(t_n, y_n).$$

Stepping the index up from n to $n + 1$, we obtain the backward Euler formula

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}). \tag{13}$$

Assuming that y_n is known and y_{n+1} is to be calculated, observe that Eq. (13) does not provide an explicit formula for y_{n+1}. Rather, it is an equation that implicitly defines y_{n+1} and must be solved to determine the value of y_{n+1}. How difficult this is depends entirely on the nature of the function f.

EXAMPLE 2

Use the backward Euler formula (13) and step sizes $h = 0.05$, 0.025, 0.01, and 0.001 to find approximate values of the solution of the initial value problem (5), (6) on the interval $0 \le t \le 2$.

For this problem the backward Euler formula (13) becomes

$$y_{n+1} = y_n + h(1 - t_{n+1} + 4y_{n+1}).$$

We will show the first two steps in detail so that it will be clear how the method works. At the first step we have

$$y_1 = y_0 + h(1 - t_1 + 4y_1) = 1 + (0.05)(1 - 0.05 + 4y_1).$$

Solving this equation for y_1, we obtain

$$y_1 = 1.0475/0.8 = 1.309375.$$

Observe that because the differential equation is linear, the implicit equation for y_1 is also linear and therefore easy to solve. Next,

$$y_2 = y_1 + h(1 - t_2 + 4y_2) = 1.309375 + (0.05)(1 - 0.1 + 4y_2),$$

which leads to

$$y_2 = 1.354375/0.8 = 1.69296875.$$

Continuing the computations on a computer, we obtain the results shown in Table 8.1.2. The values given by the backward Euler method are uniformly too large for this problem, whereas the values obtained from the Euler method were too small. In this problem the errors are somewhat larger for the backward Euler method than for the Euler method, although for small values of h the differences are insignificant. Since the backward Euler method appears to be no more accurate than the Euler method, and is somewhat more complicated, a natural question is why it should even be mentioned. The answer is that it is the simplest example of a class of methods known as backward differentiation formulas that are very useful for certain types of differential equations. We will return to this issue later in this chapter.

TABLE 8.1.2 Results for the Numerical Solution of $y' = 1 - t + 4y$, $y(0) = 1$ Using the Backward Euler Method for Different Step Sizes h

t	$h = 0.05$	$h = 0.025$	$h = 0.01$	$h = 0.001$	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.6929688	1.6474374	1.6236638	1.6104634	1.6090418
0.2	2.7616699	2.6211306	2.5491368	2.5095731	2.5053299
0.3	4.4174530	4.0920886	3.9285724	3.8396379	3.8301388
0.4	6.9905516	6.3209569	5.9908303	5.8131282	5.7942260
0.5	10.996956	9.7050002	9.0801473	8.7472667	8.7120041
1.0	103.06171	80.402761	70.452395	65.419964	64.897803
1.5	959.44236	661.00731	542.12432	485.05825	479.25919
2.0	8934.0696	5435.7294	4172.7228	3597.4478	3540.2001

Errors in Numerical Approximations. The use of a numerical procedure, such as the Euler formula, to solve an initial value problem raises a number of questions that must be answered before the approximate numerical solution can be accepted as satisfactory. One of these is the question of **convergence**. That is, as the step size h tends to zero, do the values of the numerical solution $y_1, y_2, \ldots, y_n, \ldots$ approach the corresponding values of the actual solution? If we assume that the answer is affirmative, there remains the important practical question of how rapidly the numerical approximation converges to the solution. In other words, how small a step size is needed in order to guarantee a given level of accuracy? We want to use a step size that is small enough to assure the required accuracy, but not too small. An unnecessarily small step size slows down the calculations, makes them more expensive, and in some cases may even cause a loss of accuracy.

There are two fundamental sources of error in solving an initial value problem numerically. Let us first assume that our computer is such that we can carry out all computations with complete accuracy; that is, we can retain an infinite number of decimal places. The difference E_n between the solution $y = \phi(t)$ of the initial value problem (1), (2) and its numerical approximation is given by

$$E_n = \phi(t_n) - y_n, \tag{14}$$

and is known as the **global truncation error**. It arises from two causes: First, at each step we use an approximate formula to determine y_{n+1}; second, the input data at each step are only approximately correct since in general $\phi(t_n)$ is not equal to y_n. If we assume that $y_n = \phi(t_n)$, then the only error in going one step is due to the use of an approximate formula. This error is known as the **local truncation error** e_n.

The second fundamental source of error is that we carry out the computations in arithmetic with only a finite number of digits. This leads to a **round-off error** R_n defined by

$$R_n = y_n - Y_n, \tag{15}$$

where Y_n is the value *actually computed* from the given numerical method.

The absolute value of the total error in computing $\phi(t_n)$ is given by

$$|\phi(t_n) - Y_n| = |\phi(t_n) - y_n + y_n - Y_n|. \tag{16}$$

Making use of the triangle inequality, $|a+b| \le |a| + |b|$, we obtain from Eq. (16)

$$\begin{aligned} |\phi(t_n) - Y_n| &\le |\phi(t_n) - y_n| + |y_n - Y_n| \\ &\le |E_n| + |R_n|. \end{aligned} \tag{17}$$

Thus the total error is bounded by the sum of the absolute values of the truncation and round-off errors. For the numerical procedures discussed in this book it is possible to obtain useful estimates of the truncation error. However, we limit our discussion primarily to the local truncation error, which is somewhat simpler. The round-off error is clearly more random in nature. It depends on the type of computer used, the sequence in which the computations are carried out, the method of rounding off, and so forth. While an analysis of round-off error is beyond the scope of this book, it is possible to say more about it than one might at first expect (see, for example, Henrici). Some of the dangers from round-off error are discussed in Problems 25 through 27 and in Section 8.5.

Local Truncation Error for the Euler Method. Let us assume that the solution $y = \phi(t)$ of the initial value problem (1), (2) has a continuous second derivative in the interval of interest. To assure this, we can assume that f, f_t, and f_y are continuous. Observe that if f has these properties and if ϕ is a solution of the initial value problem (1), (2), then

$$\phi'(t) = f[t, \phi(t)],$$

and by the chain rule

$$\begin{aligned} \phi''(t) &= f_t[t, \phi(t)] + f_y[t, \phi(t)]\phi'(t) \\ &= f_t[t, \phi(t)] + f_y[t, \phi(t)]\, f[t, \phi(t)]. \end{aligned} \tag{18}$$

Since the right side of this equation is continuous, ϕ'' is also continuous.

Then, making use of a Taylor polynomial with a remainder to expand ϕ about t_n, we obtain

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \tfrac{1}{2}\phi''(\bar{t}_n)h^2, \tag{19}$$

where $\bar{t}_n$ is some point in the interval $t_n < \bar{t}_n < t_n + h$. Subtracting Eq. (4) from Eq. (19), and noting that $\phi(t_n + h) = \phi(t_{n+1})$ and $\phi'(t_n) = f[t_n, \phi(t_n)]$, we find that

$$\phi(t_{n+1}) - y_{n+1} = [\phi(t_n) - y_n] + h\{f[t_n, \phi(t_n)] - f(t_n, y_n)\} + \tfrac{1}{2}\phi''(\bar{t}_n)h^2. \tag{20}$$

To compute the local truncation error we apply Eq. (20) to the true solution $\phi(t)$, that is, we take y_n to be $\phi(t_n)$. Then we immediately see from Eq. (20) that the local truncation error e_{n+1} is

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = \tfrac{1}{2}\phi''(\bar{t}_n)h^2. \tag{21}$$

Thus the local truncation error for the Euler method is proportional to the square of the step size h and the proportionality factor depends on the second derivative of the solution ϕ. The expression given by Eq. (21) depends on n and, in general, is different for each step. A uniform bound, valid on an interval $[a, b]$, is given by

$$|e_n| \le Mh^2/2, \tag{22}$$

where M is the maximum of $|\phi''(t)|$ on the interval $[a, b]$. Since Eq. (22) is based on a consideration of the worst possible case, that is, the largest possible value of $|\phi''(t)|$,

it may well be a considerable overestimate of the actual local truncation error in some parts of the interval $[a, b]$. One use of Eq. (22) is to choose a step size that will result in a local truncation error no greater than some given tolerance level. For example, if the local truncation error must be no greater than ϵ, then from Eq. (22) we have

$$h \leq \sqrt{2\epsilon/M}. \tag{23}$$

The primary difficulty in using any of Eqs. (21), (22), or (23) lies in estimating $|\phi''(t)|$ or M. However, the central fact expressed by these equations is that the local truncation error is proportional to h^2. Thus, if h is reduced by a factor of $\frac{1}{2}$, then the error is reduced by $\frac{1}{4}$, and so forth.

More important than the local truncation error is the global truncation error E_n. The analysis for estimating E_n is more difficult than that for e_n. However, knowing the local truncation error, we can make an *intuitive* estimate of the global truncation error at a fixed $\bar{t} > t_0$ as follows. Suppose that we take n steps in going from t_0 to $\bar{t} = t_0 + nh$. In each step the error is at most $Mh^2/2$; thus the error in n steps is at most $nMh^2/2$. Noting that $n = (\bar{t} - t_0)/h$, we find that the global truncation error for the Euler method in going from t_0 to $\bar{t}$ is bounded by

$$n\frac{Mh^2}{2} = (\bar{t} - t_0)\frac{Mh}{2}. \tag{24}$$

This argument is not complete since it does not take into account the effect that an error at one step will have in succeeding steps. Nevertheless, it can be shown that the global truncation error in using the Euler method on a finite interval is no greater than a constant times h; see Problem 23 for more details. The Euler method is called a first order method because its global truncation error is proportional to the first power of the step size.

Because it is more accessible, we will hereafter use the local truncation error as our principal measure of the accuracy of a numerical method, and for comparing different methods. If we have *a priori* information about the solution of the given initial value problem, we can use the result (21) to obtain more precise information about how the local truncation error varies with t. As an example, consider the illustrative problem

$$y' = 1 - t + 4y, \qquad y(0) = 1 \tag{25}$$

on the interval $0 \leq t \leq 2$. Let $y = \phi(t)$ be the solution of the initial value problem (25). Then, as noted previously,

$$\phi(t) = (4t - 3 + 19e^{4t})/16$$

and therefore

$$\phi''(t) = 19e^{4t}.$$

Equation (21) then states that

$$e_{n+1} = \frac{19e^{4\bar{t}_n}h^2}{2}, \qquad t_n < \bar{t}_n < t_n + h. \tag{26}$$

The appearance of the factor 19 and the rapid growth of e^{4t} explain why the results in this section with $h = 0.05$ were not very accurate.

For instance, the error in the first step is

$$e_1 = \phi(t_1) - y_1 = \frac{19e^{4\bar{t}_0}(0.0025)}{2}, \qquad 0 < \bar{t}_0 < 0.05.$$

It is clear that e_1 is positive and, since $e^{4\bar{t}_0} < e^{0.2}$, we have

$$e_1 \le \frac{19e^{0.2}(0.0025)}{2} \cong 0.02901. \tag{27}$$

Note also that $e^{4\bar{t}_0} > 1$; hence $e_1 > 19(0.0025)/2 = 0.02375$. The actual error is 0.02542. It follows from Eq. (26) that the error becomes progressively worse with increasing t; this is also clearly shown by the results in Table 8.1.1. Similar computations for bounds for the local truncation error give

$$1.0617 \cong \frac{19e^{3.8}(0.0025)}{2} \le e_{20} \le \frac{19e^{4}(0.0025)}{2} \cong 1.2967 \tag{28}$$

in going from 0.95 to 1.0 and

$$57.96 \cong \frac{19e^{7.8}(0.0025)}{2} \le e_{40} \le \frac{19e^{8}(0.0025)}{2} \cong 70.80 \tag{29}$$

in going from 1.95 to 2.0.

These results indicate that for this problem the local truncation error is about 2500 times larger near $t = 2$ than near $t = 0$. Thus, to reduce the local truncation error to an acceptable level throughout $0 \le t \le 2$, one must choose a step size h based on an analysis near $t = 2$. Of course, this step size will be much smaller than necessary near $t = 0$. For example, to achieve a local truncation error of 0.01 for this problem, we need a step size of about 0.00059 near $t = 2$ and a step size of about 0.032 near $t = 0$. The use of a uniform step size that is smaller than necessary over much of the interval results in more calculations than necessary, more time consumed, and possibly more danger of unacceptable round-off errors.

Another approach is to keep the local truncation error approximately constant throughout the interval by gradually reducing the step size as t increases. In the example problem we would need to reduce h by a factor of about 50 in going from $t = 0$ to $t = 2$. A method that provides for variations in the step size is called **adaptive**. All modern computer codes for solving differential equations have the capability of adjusting the step size as needed. We will return to this question in the next section.

PROBLEMS

In each of Problems 1 through 6 find approximate values of the solution of the given initial value problem at $t = 0.1$, 0.2, 0.3, and 0.4.

(a) Use the Euler method with $h = 0.05$.
(b) Use the Euler method with $h = 0.025$.
(c) Use the backward Euler method with $h = 0.05$.
(d) Use the backward Euler method with $h = 0.025$.

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
3. $y' = 2y - 3t, \quad y(0) = 1$
4. $y' = 2t + e^{-ty}, \quad y(0) = 1$
5. $y' = \dfrac{y^2 + 2ty}{3 + t^2}, \quad y(0) = 0.5$
6. $y' = (t^2 - y^2)\sin y, \quad y(0) = -1$

In each of Problems 7 through 12 find approximate values of the solution of the given initial value problem at $t = 0.5$, 1.0, 1.5, and 2.0.

(a) Use the Euler method with $h = 0.025$.
(b) Use the Euler method with $h = 0.0125$.
(c) Use the backward Euler method with $h = 0.025$.
(d) Use the backward Euler method with $h = 0.0125$.

7. $y' = 0.5 - t + 2y, \quad y(0) = 1$
8. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
9. $y' = \sqrt{t + y}, \quad y(0) = 3$
10. $y' = 2t + e^{-ty}, \quad y(0) = 1$
11. $y' = (4 - ty)/(1 + y^2), \quad y(0) = -2$
12. $y' = (y^2 + 2ty)/(3 + t^2), \quad y(0) = 0.5$
13. Complete the calculations leading to the entries in columns three and four of Table 8.1.1.
14. Complete the calculations leading to the entries in columns three and four of Table 8.1.2.
15. Using three terms in the Taylor series given in Eq. (12) and taking $h = 0.1$, determine approximate values of the solution of the illustrative example $y' = 1 - t + 4y$, $y(0) = 1$ at $t = 0.1$ and 0.2. Compare the results with those using the Euler method and with the exact values.

    Hint: If $y' = f(t, y)$, what is y''?

In each of Problems 16 and 17 estimate the local truncation error for the Euler method in terms of the solution $y = \phi(t)$. Obtain a bound for e_{n+1} in terms of t and $\phi(t)$ that is valid on the interval $0 \le t \le 1$. By using a formula for the solution obtain a more accurate error bound for e_{n+1}. For $h = 0.1$ compute a bound for e_1 and compare it with the actual error at $t = 0.1$. Also compute a bound for the error e_4 in the fourth step.

16. $y' = 2y - 1, \quad y(0) = 1$
17. $y' = \frac{1}{2} - t + 2y, \quad y(0) = 1$

In each of Problems 18 through 21 obtain a formula for the local truncation error for the Euler method in terms of t and the solution ϕ.

18. $y' = t^2 + y^2, \quad y(0) = 1$
19. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
20. $y' = \sqrt{t + y}, \quad y(1) = 3$
21. $y' = 2t + e^{-ty}, \quad y(0) = 1$
22. Consider the initial value problem

$$y' = \cos 5\pi t, \qquad y(0) = 1.$$

    (a) Determine the solution $y = \phi(t)$ and draw a graph of $y = \phi(t)$ for $0 \le t \le 1$.
    (b) Determine approximate values of $\phi(t)$ at $t = 0.2$, 0.4, and 0.6 using the Euler method with $h = 0.2$. Draw a broken-line graph for the approximate solution and compare it with the graph of the exact solution.
    (c) Repeat the computation of part (b) for $0 \le t \le 0.4$, but take $h = 0.1$.
    (d) Show by computing the local truncation error that neither of these step sizes is sufficiently small. Determine a value of h to ensure that the local truncation error is less than 0.05 throughout the interval $0 \le t \le 1$. That such a small value of h is required results from the fact that $\max |\phi''(t)|$ is large.
23. In this problem we discuss the global truncation error associated with the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. Assuming that the functions f and f_y are continuous in a region R of the ty-plane that includes the point (t_0, y_0), it can be shown that there exists a constant L such that $|f(t, y) - f(t, \tilde{y}| < L|y - \tilde{y}|$, where (t, y) and $(t, \tilde{y})$ are any two points in R with the same t coordinate (see Problem 15 of Section 2.8). Further, we assume that f_t is continuous, so the solution ϕ has a continuous second derivative.

(a) Using Eq. (20) show that

$$|E_{n+1}| \le |E_n| + h|f[t_n, \phi(t_n)] - f(t_n, y_n)| + \tfrac{1}{2}h^2|\phi''(\bar{t}_n)| \le \alpha|E_n| + \beta h^2, \qquad \text{(i)}$$

where $\alpha = 1 + hL$ and $\beta = \max|\phi''(t)|/2$ on $t_0 \le t \le t_n$.
(b) Accepting without proof that if $E_0 = 0$, and if $|E_n|$ satisfies Eq. (i), then $|E_n| \le \beta h^2(\alpha^n - 1)/(\alpha - 1)$ for $\alpha \ne 1$, show that

$$|E_n| \le \frac{(1 + hL)^n - 1}{L}\beta h. \qquad \text{(ii)}$$

Equation (ii) gives a bound for $|E_n|$ in terms of h, L, n, and β. Notice that for a fixed h, this error bound increases with increasing n; that is, the error bound increases with distance from the starting point t_0.
(c) Show that $(1 + hL)^n \le e^{nhL}$; hence

$$|E_n| \le \frac{e^{nhL} - 1}{L}\beta h = \frac{e^{(t_n - t_0)L} - 1}{L}\beta h.$$

For a fixed point $\bar{t} = t_0 + nh$ [that is, nh is constant and $h = (\bar{t} - t_0)/n$] this error bound is of the form of a constant times h and approaches zero as $h \to 0$. Also note that for $nhL = (\bar{t} - t_0)L$ small the right side of the preceding equation is approximately $nh^2\beta = (\bar{t} - t_0)\beta h$, which was obtained in Eq. (24) by an intuitive argument.

24. Derive an expression analogous to Eq. (21) for the local truncation error for the backward Euler formula.
Hint: Construct a suitable Taylor approximation to $\phi(t)$ about $t = t_{n+1}$.

▶ 25. Using a step size $h = 0.05$ and the Euler method, but retaining only three digits throughout the computations, determine approximate values of the solution at $t = 0.1$, 0.2, 0.3, and 0.4 for each of the following initial value problems.
(a) $y' = 1 - t + 4y, \qquad y(0) = 1$
(b) $y' = 3 + t - y, \qquad y(0) = 1$
(c) $y' = 2y - 3t, \qquad y(0) = 1$
Compare the results with those obtained in Example 1 and in Problems 1 and 3. The small differences between some of those results rounded to three digits and the present results are due to round-off error. The round-off error would become important if the computation required many steps.

26. The following problem illustrates a danger that occurs because of round-off error when nearly equal numbers are subtracted, and the difference then multiplied by a large number. Evaluate the quantity

$$1000 \cdot \begin{vmatrix} 6.010 & 18.04 \\ 2.004 & 6.000 \end{vmatrix}$$

as follows.
(a) First round each entry in the determinant to two digits.
(b) First round each entry in the determinant to three digits.
(c) Retain all four digits. Compare this value with the results in parts (a) and (b).

27. The distributive law $a(b - c) = ab - ac$ does not hold, in general, if the products are rounded off to a smaller number of digits. To show this in a specific case take $a = 0.22$, $b = 3.19$, and $c = 2.17$. After each multiplication round off the last digit.

8.2 Improvements on the Euler Method

Since for many problems the Euler method requires a very small step size to produce sufficiently accurate results, much effort has been devoted to the development of more efficient methods. In the next three sections we will discuss some of these methods. Consider the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

and let $y = \phi(t)$ denote its solution. Recall from Eq. (10) of Section 8.1 that by integrating the given differential equation from t_n to t_{n+1} we obtain

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f[t, \phi(t)]\, dt. \tag{2}$$

The Euler formula

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{3}$$

is obtained by replacing $f[t, \phi(t)]$ in Eq. (2) by its approximate value $f(t_n, y_n)$ at the left endpoint of the interval of integration.

Improved Euler Formula. A better approximate formula can be obtained if the integrand in Eq. (2) is approximated more accurately. One way to do this is to replace the integrand by the average of its values at the two endpoints, namely, $\{f[t_n, \phi(t_n)] + f[t_{n+1}, \phi(t_{n+1})]\}/2$. This is equivalent to approximating the area under the curve in Figure 8.2.1 between $t = t_n$ and $t = t_{n+1}$ by the area of the shaded trapezoid. Further, we replace $\phi(t_n)$ and $\phi(t_{n+1})$ by their respective approximate values y_n and y_{n+1}. In this way we obtain from Eq. (2)

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h. \tag{4}$$

Since the unknown y_{n+1} appears as one of the arguments of f on the right side of Eq. (4), this equation defines y_{n+1} implicitly rather than explicitly. Depending on the nature of the function f, it may be fairly difficult to solve Eq. (4) for y_{n+1}. This

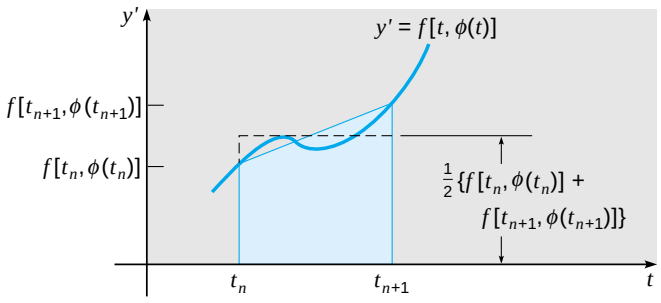


FIGURE 8.2.1 Derivation of the improved Euler method.

difficulty can be overcome by replacing y_{n+1} on the right side of Eq. (4) by the value obtained using the Euler formula (3). Thus

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f[t_n + h, y_n + hf(t_n, y_n)]}{2} h$$
$$= y_n + \frac{f_n + f(t_n + h, y_n + hf_n)}{2} h, \tag{5}$$

where t_{n+1} has been replaced by $t_n + h$.

Equation (5) gives an explicit formula for computing y_{n+1}, the approximate value of $\phi(t_{n+1})$, in terms of the data at t_n. This formula is known as the **improved Euler formula** or the **Heun formula**. The improved Euler formula is an example of a two-stage method; that is, we first calculate $y_n + hf_n$ from the Euler formula and then use this result to calculate y_{n+1} from Eq. (5). The improved Euler formula (5) does represent an improvement over the Euler formula (3) because the local truncation error in using Eq. (5) is proportional to h^3, while for the Euler method it is proportional to h^2. This error estimate for the improved Euler formula is established in Problem 14. It can also be shown that for a finite interval the global truncation error for the improved Euler formula is bounded by a constant times h^2, so this method is a second order method. Note that this greater accuracy is achieved at the expense of more computational work, since it is now necessary to evaluate $f(t, y)$ twice in order to go from t_n to t_{n+1}.

If $f(t, y)$ depends only on t and not on y, then solving the differential equation $y' = f(t, y)$ reduces to integrating $f(t)$. In this case the improved Euler formula (5) becomes

$$y_{n+1} - y_n = \frac{h}{2}[f(t_n) + f(t_n + h)], \tag{6}$$

which is just the trapezoid rule for numerical integration.

EXAMPLE 1

Use the improved Euler formula (5) to calculate approximate values of the solution of the initial value problem

$$y' = 1 - t + 4y, \qquad y(0) = 1. \tag{7}$$

To make clear exactly what computations are required, we show a couple of steps in detail. For this problem $f(t, y) = 1 - t + 4y$; hence

$$f_n = 1 - t_n + 4y_n$$

and

$$f(t_n + h, y_n + hf_n) = 1 - (t_n + h) + 4(y_n + hf_n).$$

Further, $t_0 = 0$, $y_0 = 1$, and $f_0 = 1 - t_0 + 4y_0 = 5$. If $h = 0.025$, then

$$f(t_0 + h, y_0 + hf_0) = 1 - 0.025 + 4[1 + (0.025)(5)] = 5.475.$$

Then, from Eq. (5),

$$y_1 = 1 + (0.5)(5 + 5.475)(0.025) = 1.1309375. \tag{8}$$

At the second step we must calculate

$$f_1 = 1 - 0.025 + 4(1.1309375) = 5.49875,$$
$$y_1 + hf_1 = 1.1309375 + (0.025)(5.49875) = 1.26840625,$$

and

$$f(t_2, y_1 + hf_1) = 1 - 0.05 + 4(1.26840625) = 6.023625.$$

Then, from Eq. (5),

$$y_2 = 1.1309375 + (0.5)(5.49875 + 6.023625)(0.025) = 1.2749671875. \quad (9)$$

Further results for $0 \le t \le 2$ obtained by using the improved Euler method with $h = 0.025$ and $h = 0.01$ are given in Table 8.2.1. To compare the results of the improved Euler method with those of the Euler method, note that the improved Euler method requires two evaluations of f at each step while the Euler method requires only one. This is significant because typically most of the computing time in each step is spent in evaluating f, so counting these evaluations is a reasonable way to estimate the total computing effort. Thus, for a given step size h, the improved Euler method requires twice as many evaluations of f as the Euler method. Alternatively, the improved Euler method for step size h requires the same number of evaluations of f as the Euler method with step size $h/2$.

TABLE 8.2.1 A Comparison of Results Using the Euler and Improved Euler Methods for the Initial Value Problem $y' = 1 - t + 4y$, $y(0) = 1$

	Euler		Improved Euler		
t	$h = 0.01$	$h = 0.001$	$h = 0.025$	$h = 0.01$	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5952901	1.6076289	1.6079462	1.6088585	1.6090418
0.2	2.4644587	2.5011159	2.5020619	2.5047828	2.5053299
0.3	3.7390345	3.8207130	3.8228282	3.8289146	3.8301388
0.4	5.6137120	5.7754845	5.7796888	5.7917911	5.7942260
0.5	8.3766865	8.6770692	8.6849039	8.7074637	8.7120041
1.0	60.037126	64.82558	64.497931	64.830722	64.897803
1.5	426.40818	473.55979	474.83402	478.51588	479.25919
2.0	3029.3279	3484.1608	3496.6702	3532.8789	3540.2001

By referring to Table 8.2.1 you can see that the improved Euler method with $h = 0.025$ gives much better results than the Euler method with $h = 0.01$. Note that to reach $t = 2$ with these step sizes the improved Euler method requires 160 evaluations of f while the Euler method requires 200. More noteworthy is that the improved Euler method with $h = 0.025$ is also slightly more accurate than the Euler method with $h = 0.001$ (2000 evaluations of f). In other words, with something like one-twelfth of the computing effort, the improved Euler method yields results for this problem that are comparable to, or a bit better than, those generated by the Euler method. This illustrates that, compared to the Euler method, the improved Euler method is clearly more efficient, yielding substantially better results or requiring much less total computing effort, or both.

The percentage errors at $t = 2$ for the improved Euler method are 1.22% for $h = 0.025$ and 0.21% for $h = 0.01$.

A computer program for the Euler method can be readily modified to implement the improved Euler method instead. All that is required is to replace Step 6 in the algorithm in Section 8.1 by the following:

Step 6. $k1 = f(t, y)$
$k2 = f(t + h, y + h * k1)$
$y = y + (h/2) * (k1 + k2)$
$t = t + h$

Variation of Step Size. In Section 8.1 we mentioned the possibility of adjusting the step size as a calculation proceeds so as to maintain the local truncation error at a more or less constant level. The goal is to use no more steps than necessary, and at the same time to keep some control over the accuracy of the approximation. Here we will describe how this can be done. Suppose that after n steps we have reached the point (t_n, y_n). We choose a step size h and calculate y_{n+1}. Next we need to estimate the error we have made in calculating y_{n+1}. In the absence of knowing the actual solution the best that we can do is to use a more accurate method and repeat the calculation starting from (t_n, y_n). For example, if we used the Euler method for the original calculation, we might repeat with the improved Euler method. Then the difference between the two calculated values is an estimate e_{n+1}^{est} of the error in using the original method. If the estimated error is different from the error tolerance ϵ, then we adjust the step size and repeat the calculation. The key to making this adjustment efficiently is knowing how the local truncation error e_{n+1} depends on the step size h. For the Euler method the local truncation error is proportional to h^2, so to bring the estimated error down (or up) to the tolerance level ϵ we must multiply the original step size by the factor $\sqrt{\epsilon/e_{n+1}^{\text{est}}}$.

To illustrate this procedure consider the example problem (7),

$$y' = 1 - t + 4y, \qquad y(0) = 1.$$

You can verify that after one step with $h = 0.1$ we obtain the values 1.5 and 1.595 from the Euler method and the improved Euler method, respectively. Thus the estimated error in using the Euler method is 0.095. If we have chosen an error tolerance of 0.05, for instance, then we need to adjust the step size downward by the factor $\sqrt{0.05/0.095} \cong 0.73$. Rounding downward to be conservative, let us choose the adjusted step size $h = 0.07$. Then, from the Euler formula we obtain

$$y_1 = 1 + (0.07)\, f(0, 1) = 1.35 \cong \phi(0.07).$$

Using the improved Euler method we obtain $y_1 = 1.39655$, so the estimated error in using the Euler formula is 0.04655, which is slightly less than the specified tolerance. The actual error, based on a comparison with the solution itself, is somewhat greater, namely, 0.05122.

The same procedure can be followed at each step of the calculation, thereby keeping the local truncation error approximately constant throughout the entire numerical process. Modern adaptive codes for solving differential equations adjust the step size as they proceed in very much this way, although they usually use more accurate formulas than the Euler and improved Euler formulas. Consequently, they are able to achieve both efficiency and accuracy by using very small steps only where they are really needed.

PROBLEMS

In each of Problems 1 through 6 find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4. Compare the results with those obtained by the Euler method and the backward Euler method in Section 8.1 and with the exact solution (if available).

(a) Use the improved Euler method with $h = 0.05$.
(b) Use the improved Euler method with $h = 0.025$.
(c) Use the improved Euler method with $h = 0.0125$.

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
3. $y' = 2y - 3t, \quad y(0) = 1$
4. $y' = 2t + e^{-ty}, \quad y(0) = 1$
5. $y' = \dfrac{y^2 + 2ty}{3 + t^2}, \quad y(0) = 0.5$
6. $y' = (t^2 - y^2)\sin y, \quad y(0) = -1$

In each of Problems 7 through 12 find approximate values of the solution of the given initial value problem at $t = 0.5, 1.0, 1.5$, and 2.0.

(a) Use the improved Euler method with $h = 0.025$.
(b) Use the improved Euler method with $h = 0.0125$.

7. $y' = 0.5 - t + 2y, \quad y(0) = 1$
8. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
9. $y' = \sqrt{t + y}, \quad y(0) = 3$
10. $y' = 2t + e^{-ty}, \quad y(0) = 1$
11. $y' = (4 - ty)/(1 + y^2), \quad y(0) = -2$
12. $y' = (y^2 + 2ty)/(3 + t^2), \quad y(0) = 0.5$
13. Complete the calculations leading to the entries in columns four and five of Table 8.2.1.
14. In this problem we establish that the local truncation error for the improved Euler formula is proportional to h^3. If we assume that the solution ϕ of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ has derivatives that are continuous through the third order (f has continuous second partial derivatives), it follows that

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3,$$

where $t_n < \bar{t}_n < t_n + h$. Assume that $y_n = \phi(t_n)$.
(a) Show that for y_{n+1} as given by Eq. (5)

$$\begin{aligned} e_{n+1} &= \phi(t_{n+1}) - y_{n+1} \\ &= \frac{\phi''(t_n)h - \{f[t_n + h, y_n + hf(t_n, y_n)] - f(t_n, y_n)\}}{2!}h \\ &\qquad + \frac{\phi'''(\bar{t}_n)h^3}{3!}. \end{aligned} \tag{i}$$

(b) Making use of the facts that $\phi''(t) = f_t[t, \phi(t)] + f_y[t, \phi(t)]\phi'(t)$, and that the Taylor approximation with a remainder for a function $F(t, y)$ of two variables is

$$\begin{aligned} F(a + h, b + k) = F(a, b) &+ F_t(a, b)h + F_y(a, b)k \\ &+ \frac{1}{2!}(h^2 F_{tt} + 2hkF_{ty} + k^2 F_{yy})\Big|_{x=\xi, y=\eta} \end{aligned}$$

where ξ lies between a and $a + h$ and η lies between b and $b + k$, show that the first term on the right side of Eq. (i) is proportional to h^3 plus higher order terms. This is the desired result.

(c) Show that if $f(t, y)$ is linear in t and y, then $e_{n+1} = \phi'''(\bar{t}_n)h^3/6$, where $t_n < \bar{t}_n < t_{n+1}$. itfontHint: What are f_{tt}, f_{ty}, and f_{yy}?

15. Consider the improved Euler method for solving the illustrative initial value problem $y' = 1 - t + 4y$, $y(0) = 1$. Using the result of Problem 14(c) and the exact solution of the initial value problem, determine e_{n+1} and a bound for the error at any step on $0 \le t \le 1$. Compare this error with the one obtained in Eq. (26) of Section 8.1 using the Euler method. Also obtain a bound for e_1 for $h = 0.1$ and compare it with Eq. (27) of Section 8.1.

In each of Problems 16 and 17 use the actual solution $\phi(t)$ to determine e_{n+1} and a bound for e_{n+1} at any step on $0 \le t \le 1$ for the improved Euler method for the given initial value problem. Also obtain a bound for e_1 for $h = 0.1$ and compare it with the similar estimate for the Euler method and with the actual error using the improved Euler method.

16. $y' = 2y - 1, \quad y(0) = 1$

17. $y' = 0.5 - t + 2y, \quad y(0) = 1$

In each of Problems 18 through 21 carry out one step of the Euler method and of the improved Euler method using the step size $h = 0.1$. Suppose that a local truncation error no greater than 0.0025 is required. Estimate the step size that is needed for the Euler method to satisfy this requirement at the first step.

18. $y' = 0.5 - t + 2y, \quad y(0) = 1$

19. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$

20. $y' = \sqrt{t + y}, \quad y(0) = 3$

21. $y' = (y^2 + 2ty)/(3 + t^2), \quad y(0) = 0.5$

22. The **modified Euler formula** for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ is given by

$$y_{n+1} = y_n + hf[t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf(t_n, y_n)].$$

Following the procedure outlined in Problem 14, show that the local truncation error in the modified Euler formula is proportional to h^3.

In each of Problems 23 through 26 use the modified Euler formula of Problem 22 with $h = 0.05$ to compute approximate values of the solution of the given initial value problem at $t = 0.1$, 0.2, 0.3, and 0.4. Compare the results with those obtained in Problems 1 through 4.

23. $y' = 3 + t - y, \quad y(0) = 1$

24. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$

25. $y' = 2y - 3t, \quad y(0) = 1$

26. $y' = 2t + e^{-ty}, \quad y(0) = 1$

27. Show that the modified Euler formula of Problem 22 is identical to the improved Euler formula of Eq. (5) for $y' = f(t, y)$ if f is linear in both t and y.

8.3 The Runge–Kutta Method

In preceding sections we have introduced the Euler formula, the backward Euler formula, and the improved Euler formula as ways to solve the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

numerically. The local truncation errors for these methods are proportional to h^2, h^2, and h^3, respectively. The Euler and improved Euler methods belong to what is now called the Runge–Kutta[1] class of methods.

In this section we discuss the method originally developed by Runge and Kutta. This method is now called the classic fourth order four-stage Runge–Kutta method, but it is often referred to simply as *the* Runge–Kutta method, and we will follow this practice for brevity. This method has a local truncation error that is proportional to h^5. Thus it is two orders of magnitude more accurate than the improved Euler method and three orders of magnitude better than the Euler method. It is relatively simple to use and is sufficiently accurate to handle many problems efficiently. This is especially true of adaptive Runge–Kutta methods in which a provision is made to vary the step size as needed.

The Runge–Kutta formula involves a weighted average of values of $f(t, y)$ at different points in the interval $t_n \le t \le t_{n+1}$. It is given by

$$y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right), \tag{2}$$

where

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_{n1}), \\ k_{n3} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_{n2}), \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned} \tag{3}$$

The sum $(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})/6$ can be interpreted as an average slope. Note that k_{n1} is the slope at the left end of the interval, k_{n2} is the slope at the midpoint using the Euler formula to go from t_n to $t_n + h/2$, k_{n3} is a second approximation to the slope at the midpoint, and finally k_{n4} is the slope at $t_n + h$ using the Euler formula and the slope k_{n3} to go from t_n to $t_n + h$.

While in principle it is not difficult to show that Eq. (2) differs from the Taylor expansion of the solution ϕ by terms that are proportional to h^5, the algebra is rather lengthy.[2] Thus we will accept the fact that the local truncation error in using Eq. (2) is proportional to h^5 and that for a finite interval the global truncation error is at most a constant times h^4.

Clearly the Runge–Kutta formula, Eqs. (2) and (3), is more complicated than any of the formulas discussed previously. This is of relatively little significance, however, since it is not hard to write a computer program to implement this method. Such a program has the same structure as the algorithm for the Euler method outlined in

[1] Carl David Runge (1856–1927), German mathematician and physicist, worked for many years in spectroscopy. The analysis of data led him to consider problems in numerical computation, and the Runge–Kutta method originated in his paper on the numerical solution of differential equations in 1895. The method was extended to systems of equations in 1901 by M. Wilhelm Kutta (1867–1944). Kutta was a German mathematician and aerodynamicist who is also well known for his important contributions to classical airfoil theory.

[2] See, for example, Chapter 3 of the book by Henrici listed in the references.

Section 8.1. To be specific, the lines in Step 6 in the Euler algorithm must be replaced by the following:

Step 6.
$$k1 = f(t, y)$$
$$k2 = f(t + 0.5 * h, y + 0.5 * h * k1)$$
$$k3 = f(t + 0.5 * h, y + 0.5 * h * k2)$$
$$k4 = f(t + h, y + h * k3)$$
$$y = y + (h/6) * (k1 + 2 * k2 + 2 * k3 + k4)$$
$$t = t + h$$

Note that if f does not depend on y, then

$$k_{n1} = f(t_n), \qquad k_{n2} = k_{n3} = f(t_n + h/2), \qquad k_{n4} = f(t_n + h), \tag{4}$$

and Eq. (2) reduces to

$$y_{n+1} - y_n = \frac{h}{6}[f(t_n) + 4f(t_n + h/2) + f(t_n + h)]. \tag{5}$$

Equation (5) can be identified as Simpson's (1710–1761) rule for the approximate evaluation of the integral of $y' = f(t)$. The fact that Simpson's rule has an error proportional to h^5 is consistent with the local truncation error in the Runge–Kutta formula.

EXAMPLE 1

Use the Runge–Kutta method to calculate approximate values of the solution $y = \phi(t)$ of the initial value problem

$$y' = 1 - t + 4y, \qquad y(0) = 1. \tag{6}$$

Taking $h = 0.2$ we have

$$k_{01} = f(0, 1) = 5; \qquad hk_{01} = 1.0,$$
$$k_{02} = f(0 + 0.1, 1 + 0.5) = 6.9; \qquad hk_{02} = 1.38,$$
$$k_{03} = f(0 + 0.1, 1 + 0.69) = 7.66; \qquad hk_{03} = 1.532,$$
$$k_{04} = f(0 + 0.2, 1 + 1.532) = 10.928.$$

Thus

$$y_1 = 1 + \frac{0.2}{6}[5 + 2(6.9) + 2(7.66) + 10.928]$$
$$= 1 + 1.5016 = 2.5016.$$

Further results using the Runge–Kutta method with $h = 0.2$, $h = 0.1$, and $h = 0.05$ are given in Table 8.3.1. Note that the Runge–Kutta method yields a value at $t = 2$ that differs from the exact solution by only 0.122% if the step size is $h = 0.1$, and by only 0.00903% if $h = 0.05$. In the latter case the error is less than one part in ten thousand, and the calculated value at $t = 2$ is correct to four digits.

For comparison, note that both the Runge–Kutta method with $h = 0.05$ and the improved Euler method with $h = 0.025$ require 160 evaluations of f to reach $t = 2$. The improved Euler method yields a result at $t = 2$ that is in error by 1.22%. While this error may be acceptable for some purposes, it is more than 135 times the error yielded by the Runge–Kutta method with comparable computing effort. Note also that the Runge–Kutta method with $h = 0.2$, or 40 evaluations of f, produces a value at $t = 2$

with an error of 1.40%, which is only slightly greater than the error in the improved Euler method with $h = 0.025$, or 160 evaluations of f. Thus we see again that a more accurate algorithm is more efficient; it produces better results with similar effort, or similar results with less effort.

TABLE 8.3.1 A Comparison of Results for the Numerical Solution of the Initial Value Problem $y' = 1 - t + 4y$, $y(0) = 1$

	Improved Euler	Runge–Kutta			
t	$h = 0.025$	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.6079462		1.6089333	1.6090338	1.6090418
0.2	2.5020619	2.5016000	2.5050062	2.5053060	2.5053299
0.3	3.8228282		3.8294145	3.8300854	3.8301388
0.4	5.7796888	5.7776358	5.7927853	5.7941198	5.7942260
0.5	8.6849039		8.7093175	8.7118060	8.7120041
1.0	64.497931	64.441579	64.858107	64.894875	64.897803
1.5	474.83402		478.81928	479.22674	479.25919
2.0	3496.6702	3490.5574	3535.8667	3539.8804	3540.2001

PROBLEMS

In each of Problems 1 through 6 find approximate values of the solution of the given initial value problem at $t = 0.1$, 0.2, 0.3, and 0.4. Compare the results with those obtained by using other methods and with the exact solution (if available).

(a) Use the Runge–Kutta method with $h = 0.1$.
(b) Use the Runge–Kutta method with $h = 0.05$.

1. $y' = 3 + t - y$, $y(0) = 1$
2. $y' = 5t - 3\sqrt{y}$, $y(0) = 2$
3. $y' = 2y - 3t$, $y(0) = 1$
4. $y' = 2t + e^{-ty}$, $y(0) = 1$
5. $y' = \dfrac{y^2 + 2ty}{3 + t^2}$, $y(0) = 0.5$
6. $y' = (t^2 - y^2)\sin y$, $y(0) = -1$

In each of Problems 7 through 12 find approximate values of the solution of the given initial value problem at $t = 0.5$, 1.0, 1.5, and 2.0. Compare the results with those given by other methods.

(a) Use the Runge–Kutta method with $h = 0.1$.
(b) Use the Runge–Kutta method with $h = 0.05$.

7. $y' = 0.5 - t + 2y$, $y(0) = 1$
8. $y' = 5t - 3\sqrt{y}$, $y(0) = 2$
9. $y' = \sqrt{t + y}$, $y(0) = 3$
10. $y' = 2t + e^{-ty}$, $y(0) = 1$
11. $y' = (4 - ty)/(1 + y^2)$, $y(0) = -2$
12. $y' = (y^2 + 2ty)/(3 + t^2)$, $y(0) = 0.5$
13. Confirm the results in Table 8.3.1 by executing the indicated computations.
14. Consider the initial value problem

$$y' = t^2 + y^2, \qquad y(0) = 1.$$

(a) Draw a direction field for this equation.

(b) Use the Runge–Kutta or other methods to find approximate values of the solution at $t = 0.8$, 0.9, and 0.95. Choose a small enough step size so that you believe your results are accurate to at least four digits.
(c) Try to extend the calculations in part (b) to obtain an accurate approximation to the solution at $t = 1$. If you encounter difficulties in doing this, explain why you think this happens. The direction field in part (a) may be helpful.

15. Consider the initial value problem

$$y' = 3t^2/(3y^2 - 4), \qquad y(0) = 0.$$

(a) Draw a direction field for this equation.
(b) Estimate how far the solution can extended to the right. Let t_M be the right endpoint of the interval of existence of this solution. What happens at t_M to prevent the solution from continuing farther?
(c) Use the Runge–Kutta method with various step sizes to determine an approximate value of t_M.
(d) If you continue the computation beyond t_M, you can continue to generate values of y. What significance, if any, do these values have?
(e) Suppose that the initial condition is changed to $y(0) = 1$. Repeat parts (b) and (c) for this problem.

8.4 Multistep Methods

In previous sections we have discussed numerical procedures for solving the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0, \tag{1}$$

in which data at the point $t = t_n$ are used to calculate an approximate value of the solution $\phi(t_{n+1})$ at the next mesh point $t = t_{n+1}$. In other words, the calculated value of ϕ at any mesh point depends only on the data at the preceding mesh point. Such methods are called **one-step methods**. However, once approximate values of the solution $y = \phi(t)$ have been obtained at a few points beyond t_0, it is natural to ask whether we can make use of some of this information, rather than just the value at the last point, to calculate the value of $\phi(t)$ at the next point. Specifically, if y_1 at t_1, y_2 at t_2, ..., y_n at t_n are known, how can we use this information to determine y_{n+1} at t_{n+1}? Methods that use information at more than the last mesh point are referred to as **multistep methods**. In this section we will describe two types of multistep methods, Adams[3] methods and backward differentiation formulas. Within each type one can achieve various levels of accuracy, depending on the number of preceding data points that are used. For simplicity we will assume throughout our discussion that the step size h is constant.

[3]John Couch Adams (1819–1892), English astronomer, is most famous as codiscoverer with Joseph Leverrier of the planet Neptune in 1846. Adams was also extremely skilled at computation; his procedure for numerical integration of differential equations appeared in 1883 in a book with Francis Bashforth on capillary action.

Adams Methods. Recall that

$$\phi(t_{n+1}) - \phi(t_n) = \int_{t_n}^{t_{n+1}} \phi'(t)\, dt, \tag{2}$$

where $\phi(t)$ is the solution of the initial value problem (1). The basic idea of an Adams method is to approximate $\phi'(t)$ by a polynomial $P_k(t)$ of degree $k-1$ and to use the polynomial to evaluate the integral on the right side of Eq. (2). The coefficients in $P_k(t)$ are determined by using k previously calculated data points. For example, suppose that we wish to use a first degree polynomial $P_2(t) = At + B$. Then we need only the two data points (t_n, y_n) and (t_{n-1}, y_{n-1}). Since P_2 is to be an approximation to ϕ', we require that $P_2(t_n) = f(t_n, y_n)$ and that $P_2(t_{n-1}) = f(t_{n-1}, y_{n-1})$. Recall that we denote $f(t_j, y_j)$ by f_j for an integer j. Then A and B must satisfy the equations

$$\begin{aligned} At_n + B &= f_n, \\ At_{n-1} + B &= f_{n-1}. \end{aligned} \tag{3}$$

Solving for A and B, we obtain

$$A = \frac{f_n - f_{n-1}}{h}, \qquad B = \frac{f_{n-1}t_n - f_n t_{n-1}}{h}. \tag{4}$$

Replacing $\phi'(t)$ by $P_2(t)$ and evaluating the integral in Eq. (2), we find that

$$\phi(t_{n+1}) - \phi(t_n) = \frac{A}{2}(t_{n+1}^2 - t_n^2) + B(t_{n+1} - t_n).$$

Finally, we replace $\phi(t_{n+1})$ and $\phi(t_n)$ by y_{n+1} and y_n, respectively, and carry out some algebraic simplification. For a constant step size h we obtain

$$y_{n+1} = y_n + \tfrac{3}{2}hf_n - \tfrac{1}{2}hf_{n-1}. \tag{5}$$

Equation (5) is the second order Adams–Bashforth formula. It is an explicit formula for y_{n+1} in terms of y_n and y_{n-1} and has a local truncation error proportional to h^3.

We note in passing that the first order Adams–Bashforth formula, based on the polynomial $P_1(t) = f_n$ of degree zero, is just the original Euler formula.

More accurate Adams formulas can be obtained by following the procedure outlined above, but using a higher degree polynomial and correspondingly more data points. For example, suppose that a polynomial $P_4(t)$ of degree three is used. The coefficients are determined from the four points (t_n, y_n), (t_{n-1}, y_{n-1}), (t_{n-2}, y_{n-2}), and (t_{n-3}, y_{n-3}). Substituting this polynomial for $\phi'(t)$ in Eq. (2), evaluating the integral, and simplifying the result, we eventually obtain the fourth order Adams–Bashforth formula, namely,

$$y_{n+1} = y_n + (h/24)(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}). \tag{6}$$

The local truncation error of this fourth order formula is proportional to h^5.

A variation on the derivation of the Adams–Bashforth formulas gives another set of formulas called the Adams–Moulton[4] formulas. To see the difference, let us again consider the second order case. Again we use a first degree polynomial $Q_2(t) = \alpha t + \beta$,

[4] Forest Ray Moulton (1872–1952) was an American astronomer and administrator of science. While calculating ballistics trajectories during World War I, he devised substantial improvements in the Adams formula.

but we determine the coefficients by using the points (t_n, y_n) and (t_{n+1}, y_{n+1}). Thus α and β must satisfy

$$\begin{aligned} \alpha t_n + \beta &= f_n, \\ \alpha t_{n+1} + \beta &= f_{n+1}, \end{aligned} \tag{7}$$

and it follows that

$$\alpha = \frac{f_{n+1} - f_n}{h}, \qquad \beta = \frac{f_n t_{n+1} - f_{n+1} t_n}{h}. \tag{8}$$

Substituting $Q_2(t)$ for $\phi'(t)$ in Eq. (2) and simplifying, we obtain

$$y_{n+1} = y_n + \tfrac{1}{2} h f_n + \tfrac{1}{2} h f(t_{n+1}, y_{n+1}), \tag{9}$$

which is the second order Adams–Moulton formula. We have written $f(t_{n+1}, y_{n+1})$ in the last term to emphasize that the Adams–Moulton formula is implicit, rather than explicit, since the unknown y_{n+1} appears on both sides of the equation. The local truncation error for the second order Adams–Moulton formula is proportional to h^3.

The first order Adams–Moulton formula is just the backward Euler formula, as you might anticipate by analogy with the first order Adams–Bashforth formula.

More accurate higher order formulas can be obtained by using an approximating polynomial of higher degree. The fourth order Adams–Moulton formula, with a local truncation error proportional to h^5, is

$$y_{n+1} = y_n + (h/24)(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}). \tag{10}$$

Observe that this is also an implicit formula because y_{n+1} appears in f_{n+1}.

Although both the Adams–Bashforth and Adams–Moulton formulas of the same order have local truncation errors proportional to the same power of h, the Adams–Moulton formulas of moderate order are in fact considerably more accurate. For example, for the fourth order formulas (6) and (10), the proportionality constant for the Adams–Moulton formula is less than 1/10 of the proportionality constant for the Adams–Bashforth formula. Thus the question arises: Should one use the explicit (and faster) Adams–Bashforth formula, or the more accurate but implicit (and slower) Adams–Moulton formula? The answer depends on whether by using the more accurate formula one can increase the step size, and therefore reduce the number of steps, enough to compensate for the additional computations required at each step.

In fact, numerical analysts have attempted to achieve both simplicity and accuracy by combining the two formulas in what is called a **predictor–corrector method**. Once y_{n-3}, y_{n-2}, y_{n-1}, and y_n are known, we can compute f_{n-3}, f_{n-2}, f_{n-1}, and f_n, and then use the Adams–Bashforth (predictor) formula (6) to obtain a first value for y_{n+1}. Then we compute f_{n+1} and use the Adams–Moulton (corrector) formula (10), which is no longer implicit, to obtain an improved value of y_{n+1}. We can, of course, continue to use the corrector formula (10) if the change in y_{n+1} is too large. However, if it is necessary to use the corrector formula more than once or perhaps twice, it means that the step size h is too large and should be reduced.

In order to use any of the multistep methods it is necessary first to calculate a few y_j by some other method. For example, the fourth order Adams–Moulton method requires values for y_1 and y_2, while the fourth order Adams–Bashforth method also requires a value for y_3. One way to proceed is to use a one-step method of comparable accuracy to calculate the necessary starting values. Thus, for a fourth order multistep method,

one might use the fourth order Runge–Kutta method to calculate the starting values. This is the method used in the next example.

Another approach is to use a low order method with a very small h to calculate y_1, and then to increase gradually both the order and the step size until enough starting values have been determined.

EXAMPLE 1

Consider again the initial value problem

$$y' = 1 - t + 4y, \qquad y(0) = 1. \tag{11}$$

With a step size of $h = 0.1$ determine an approximate value of the solution $y = \phi(t)$ at $t = 0.4$ using the fourth order Adams–Bashforth formula, the fourth order Adams–Moulton formula, and the predictor–corrector method.

For starting data we use the values of y_1, y_2, and y_3 found from the Runge–Kutta method. These are tabulated in Table 8.3.1. Next, calculating the corresponding values of $f(t, y)$, we obtain

$$\begin{aligned} y_0 &= 1, & f_0 &= 5, \\ y_1 &= 1.6089333, & f_1 &= 7.3357332, \\ y_2 &= 2.5050062, & f_2 &= 10.820025, \\ y_3 &= 3.8294145, & f_3 &= 16.017658. \end{aligned}$$

Then from the Adams–Bashforth formula, Eq. (6), we find that $y_4 = 5.7836305$. The exact value of the solution at $t = 0.4$, correct through eight digits, is 5.7942260, so the error is -0.010595.

The Adams–Moulton formula, Eq. (10), leads to the equation

$$y_4 = 4.9251275 + 0.15y_4,$$

from which it follows that $y_4 = 5.7942676$ with an error of only 0.0000416.

Finally, using the result from the Adams–Bashforth formula as a predicted value of $\phi(0.4)$, we can then use Eq. (10) as a corrector. Corresponding to the predicted value of y_4 we find that $f_4 = 23.734522$. Hence, from Eq. (10), the corrected value of y_4 is 5.7926721. This result is in error by -0.0015539.

Observe that the Adams–Bashforth method is the simplest and fastest of these methods, since it involves only the evaluation of a single explicit formula. It is also the least accurate. Using the Adams–Moulton formula as a corrector increases the amount of calculation that is required, but the method is still explicit. In this problem the error in the corrected value of y_4 is reduced by approximately a factor of 7 when compared to the error in the predicted value. The Adams–Moulton method alone yields by far the best result, with an error that is about 1/40 as large as the error from the predictor–corrector method. Remember, however, that the Adams–Moulton method is implicit, which means that an equation must be solved at each step. In the problem considered here this equation is linear, so the solution is quickly found, but in other problems this part of the procedure may be much more time-consuming.

The Runge–Kutta method with $h = 0.1$ gives $y_4 = 5.7927853$ with an error of -0.0014407; see Table 8.3.1. Thus for this problem the Runge–Kutta method is comparable in accuracy to the predictor–corrector method.

Backward Differentiation Formulas. Another type of multistep method arises by using a polynomial $P_k(t)$ to approximate the solution $\phi(t)$ of the initial value problem (1) rather than its derivative $\phi'(t)$, as in the Adams methods. We then differentiate $P_k(t)$ and set $P_k'(t_{n+1})$ equal to $f(t_{n+1}, y_{n+1})$ to obtain an implicit formula for y_{n+1}. These are called **backward differentiation formulas**.

The simplest case uses a first degree polynomial $P_1(t) = At + B$. The coefficients are chosen to match the computed values of the solution y_n and y_{n+1}. Thus A and B must satisfy

$$\begin{aligned} At_n + B &= y_n, \\ At_{n+1} + B &= y_{n+1}. \end{aligned} \tag{12}$$

Since $P_1'(t) = A$, the requirement that

$$P_1'(t_{n+1}) = f(t_{n+1}, y_{n+1})$$

is just

$$A = f(t_{n+1}, y_{n+1}). \tag{13}$$

Another expression for A comes from subtracting the first of Eqs. (12) from the second, which gives

$$A = (y_{n+1} - y_n)/h.$$

Substituting this value of A into Eq. (13) and rearranging terms, we obtain the first order backward differentiation formula

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}). \tag{14}$$

Note that Eq. (14) is just the backward Euler formula that we first saw in Section 8.1.

By using higher order polynomials and correspondingly more data points one can obtain backward differentiation formulas of any order. The second order formula is

$$y_{n+1} = \tfrac{1}{3}\left[4y_n - y_{n-1} + 2hf(t_{n+1}, y_{n+1})\right] \tag{15}$$

and the fourth order formula is

$$y_{n+1} = \tfrac{1}{25}\left[48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf(t_{n+1}, y_{n+1})\right]. \tag{16}$$

These formulas have local truncation errors proportional to h^3 and h^5, respectively.

EXAMPLE 2

Use the fourth order backward differentiation formula with $h = 0.1$ and the data given in Example 1 to determine an approximate value of the solution $y = \phi(t)$ at $t = 0.4$ for the initial value problem (11).

Using Eq. (16) with $n = 3$, $h = 0.1$, and with $y_0, \ldots, y_3$ given in Example 1, we obtain the equation

$$y_4 = 4.6837842 + 0.192y_4.$$

Thus

$$y_4 = 5.7967626.$$

Comparing the calculated value with the exact value $\phi(0.4) = 5.7942260$, we find that the error is 0.0025366. This is somewhat better than the result using the Adams–Bashforth method, but not as good as the result using the predictor–corrector method, and not nearly as good as the result using the Adams–Moulton method.

A comparison between one-step and multistep methods must take several factors into consideration. The fourth order Runge–Kutta method requires four evaluations of f at each step, while the fourth order Adams–Bashforth method (once past the starting values) requires only one and the predictor–corrector method only two. Thus, for a given step size h, the latter two methods may well be considerably faster than Runge–Kutta. However, if Runge–Kutta is more accurate and therefore can use fewer steps, then the difference in speed will be reduced and perhaps eliminated. The Adams–Moulton and backward differentiation formulas also require that the difficulty in solving the implicit equation at each step be taken into account. All multistep methods have the possible disadvantage that errors in earlier steps can feed back into later calculations with unfavorable consequences. On the other hand, the underlying polynomial approximations in multistep methods make it easy to approximate the solution at points between the mesh points, should this be desirable. Multistep methods have become popular largely because it is relatively easy to estimate the error at each step and to adjust the order or the step size to control it. For a further discussion of such questions as these see the books listed at the end of the chapter; in particular, Shampine (1994) is an authoritative source.

PROBLEMS

In each of Problems 1 through 6 determine an approximate value of the solution at $t = 0.4$ and $t = 0.5$ using the specified method. For starting values use the values given by the Runge–Kutta method; see Problems 1 through 6 of Section 8.3. Compare the results of the various methods with each other and with the actual solution (if available).

(a) Use the fourth order predictor–corrector method with $h = 0.1$. Use the corrector formula once at each step.
(b) Use the fourth order Adams–Moulton method with $h = 0.1$.
(c) Use the fourth order backward differentiation method with $h = 0.1$.

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
3. $y' = 2y - 3t, \quad y(0) = 1$
4. $y' = 2t + e^{-ty}, \quad y(0) = 1$
5. $y' = \dfrac{y^2 + 2ty}{3 + t^2}, \quad y(0) = 0.5$
6. $y' = (t^2 - y^2)\sin y, \quad y(0) = -1$

In each of Problems 7 through 12 find approximate values of the solution of the given initial value problem at $t = 0.5, 1.0, 1.5$, and 2.0, using the specified method. For starting values use the values given by the Runge–Kutta method; see Problems 7 through 12 in Section 8.3. Compare the results of the various methods with each other and with the actual solution (if available).

(a) Use the fourth order predictor–corrector method with $h = 0.05$. Use the corrector formula once at each step.
(b) Use the fourth order Adams–Moulton method with $h = 0.05$.
(c) Use the fourth order backward differentiation method with $h = 0.05$.

7. $y' = 0.5 - t + 2y, \quad y(0) = 1$
8. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
9. $y' = \sqrt{t + y}, \quad y(0) = 3$
10. $y' = 2t + e^{-ty}, \quad y(0) = 1$

11. $y' = (4 - ty)/(1 + y^2), \qquad y(0) = -2$
12. $y' = (y^2 + 2ty)/(3 + t^2), \qquad y(0) = 0.5$
13. Show that the first order Adams–Bashforth method is the Euler method, and that the first order Adams–Moulton method is the backward Euler method.
14. Show that the third order Adams–Bashforth formula is

$$y_{n+1} = y_n + (h/12)(23 f_n - 16 f_{n-1} + 5 f_{n-2}).$$

15. Show that the third order Adams–Moulton formula is

$$y_{n+1} = y_n + (h/12)(5 f_{n+1} + 8 f_n - f_{n-1}).$$

16. Derive the second order backward differentiation formula given by Eq. (15) in the text.

8.5 More on Errors; Stability

In Section 8.1 we discussed some ideas related to the errors that can occur in a numerical solution of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

In this section we continue that discussion and also point out some other difficulties that can arise. Some of the points that we wish to make are fairly difficult to treat in detail, so we will illustrate them by means of examples.

Truncation and Round-off Errors. Recall that for the Euler method we showed that the local truncation error is proportional to h^2 and that for a finite interval the global truncation error is at most a constant times h. In general, for a method of order p, the local truncation error is proportional to h^{p+1} and the global truncation error on a finite interval is bounded by a constant times h^p. To achieve high accuracy we normally use a numerical procedure for which p is fairly large, perhaps four or higher. As p increases, the formula used in computing y_{n+1} normally becomes more complicated, and hence more calculations are required at each step; however, this is usually not a serious problem unless $f(t, y)$ is very complicated or the calculation must be repeated very many times. If the step size h is decreased, the global truncation error is decreased by the same factor raised to the power p. However, as we mentioned in Section 8.1, if h is very small, a great many steps will be required to cover a fixed interval, and the global round-off error may be larger than the global truncation error. The situation is shown schematically in Figure 8.5.1. We assume that the round-off error R_n is proportional to the number of computations performed and therefore is inversely proportional to the step size h. On the other hand, the truncation error E_n is proportional to a positive power of h. From Eq. (17) of Section 8.1 we know that the total error is bounded by $|E_n| + |R_n|$; hence we wish to choose h so as to minimize this quantity. The optimum value of h occurs when the rate of increase of the truncation error (as h increases) is balanced by the rate of decrease of the round-off error, as indicated in Figure 8.5.1.

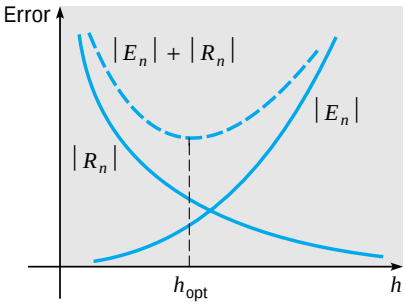


FIGURE 8.5.1 The dependence of truncation and round-off errors on the step size h.

EXAMPLE 1

Consider the example problem

$$y' = 1 - t + 4y, \qquad y(0) = 1. \tag{2}$$

Using the Euler method with various step sizes, calculate approximate values for the solution $\phi(t)$ at $t = 0.5$ and $t = 1$. Try to determine the optimum step size.

Keeping only four digits in order to shorten the calculations, we obtain the data shown in Table 8.5.1. The first two columns are the step size h and the number of steps N required to traverse the interval $0 \le t \le 1$. Then $y_{N/2}$ and y_N are approximations to $\phi(0.5) = 8.712$ and $\phi(1) = 64.90$, respectively. These quantities appear in the third and fifth columns. The fourth and sixth columns display the differences between the calculated values and the actual value of the solution.

TABLE 8.5.1 Approximations to the Solution of the Initial Value Problem $y' = 1 - t + 4y$, $y(0) = 1$ Using the Euler Method with Different Step Sizes

h	N	$y_{N/2}$	Error	y_N	Error
0.01	100	8.390	0.322	60.12	4.78
0.005	200	8.551	0.161	62.51	2.39
0.002	500	8.633	0.079	63.75	1.15
0.001	1000	8.656	0.056	63.94	0.96
0.0008	1250	8.636	0.076	63.78	1.12
0.000625	1600	8.616	0.096	64.35	0.55
0.0005	2000	8.772	0.060	64.00	0.90
0.0004	2500	8.507	0.205	63.40	1.50
0.00025	4000	8.231	0.481	56.77	8.13

For relatively large step sizes the round-off error is much less than the global truncation error. Consequently, the total error is approximately the same as the global truncation error, which for the Euler method is bounded by a constant times h. Thus, as the step size is reduced, the error is reduced proportionally. The first three lines in Table 8.5.1 show this type of behavior. For $h = 0.001$ the error has been further reduced, but much less than proportionally; this indicates that round-off error is becoming important. As h is reduced still more, the error begins to fluctuate and further improvements in

accuracy become problematical. For values of h less than 0.0005 the error is clearly increasing, which indicates that round-off error is now the dominant part of the total error.

These results can also be expressed in terms of the number of steps N. For N less than about 1000 accuracy is improved by taking more steps, while for N greater than about 2000 using more steps has an adverse effect. Thus for this problem it is best to use an N somewhere between 1000 and 2000. For the calculations shown in Table 8.5.1 the best result at $t = 0.5$ occurs for $N = 1000$ while at $t = 1.0$ the best result is for $N = 1600$.

One should be careful not to read too much into the results shown in Example 1. The optimum ranges for h and N depend on the differential equation, the numerical method that is used, and the number of digits that are retained in the calculation. Nevertheless, it is generally true that if too many steps are required in a calculation, then eventually round-off error is likely to accumulate to the point that it seriously degrades the accuracy of the procedure. For many problems this is not a concern: For them any of the fourth order methods we have discussed in Sections 8.3 and 8.4 will produce good results with a number of steps far less than the level at which round-off error becomes important. For some problems, however, round-off error does become vitally important. For such problems the choice of method may be crucial. This is also one reason why modern codes provide a means of adjusting the step size as they go along, using a larger step size wherever possible, and a very small step size only where necessary.

Vertical Asymptotes. As a second example, consider the problem of determining the solution $y = \phi(t)$ of

$$y' = t^2 + y^2, \qquad y(0) = 1. \tag{3}$$

Since the differential equation is nonlinear, the existence and uniqueness theorem (Theorem 2.4.2) guarantees only that there is a solution in *some* interval about $t = 0$. Suppose that we try to compute a solution of the initial value problem on the interval $0 \le t \le 1$ using different numerical procedures.

If we use the Euler method with $h = 0.1$, 0.05, and 0.01, we find the following approximate values at $t = 1$: 7.189548, 12.32093, and 90.75551, respectively. The large differences among the computed values are convincing evidence that we should use a more accurate numerical procedure—the Runge–Kutta method, for example. Using the Runge–Kutta method with $h = 0.1$ we find the approximate value 735.0991 at $t = 1$, which is quite different from those obtained using the Euler method. Repeating the calculations using step sizes of $h = 0.05$ and $h = 0.01$, we obtain the interesting information listed in Table 8.5.2.

TABLE 8.5.2 Calculation of the Solution of the Initial Value Problem $y' = t^2 + y^2$, $y(0) = 1$ Using the Runge–Kutta Method

h	$t = 0.90$	$t = 1.0$
0.1	14.02182	735.0991
0.05	14.27117	1.75863×10^5
0.01	14.30478	2.0913×10^{2893}
0.001	14.30486	

While the values at $t = 0.90$ are reasonable and we might well believe that the solution has a value of about 14.305 at $t = 0.90$, it is clear that something strange is happening between $t = 0.9$ and $t = 1.0$. To help determine what is happening let us turn to some analytical approximations to the solution of the initial value problem (3). Note that on $0 \le t \le 1$

$$y^2 \le t^2 + y^2 \le 1 + y^2. \tag{4}$$

This suggests that the solution $y = \phi_1(t)$ of

$$y' = 1 + y^2, \qquad y(0) = 1 \tag{5}$$

and the solution $y = \phi_2(t)$ of

$$y' = y^2, \qquad y(0) = 1 \tag{6}$$

are upper and lower bounds, respectively, for the solution $y = \phi(t)$ of the original problem, since all these solutions pass through the same initial point. Indeed, it can be shown (for example, by the iteration method of Section 2.8) that $\phi_2(t) \le \phi(t) \le \phi_1(t)$ as long as these functions exist. The important thing to note is that we *can solve* Eqs. (5) and (6) for ϕ_1 and ϕ_2 by separation of variables. We find that

$$\phi_1(t) = \tan\left(t + \frac{\pi}{4}\right), \qquad \phi_2(t) = \frac{1}{1-t}. \tag{7}$$

Thus $\phi_2(t) \to \infty$ as $t \to 1$, and $\phi_1(t) \to \infty$ as $t \to \pi/4 \cong 0.785$. These calculations show that the solution of the original initial value problem exists at least for $0 \le t < \pi/4$ and at most for $0 \le t < 1$. The solution of the problem (3) has a vertical asymptote for some t in $\pi/4 \le t \le 1$ and thus does not exist on the entire interval $0 \le t \le 1$.

Our numerical calculations, however, suggest that we can go beyond $t = \pi/4$, and probably beyond $t = 0.9$. Assuming that the solution of the initial value problem exists at $t = 0.9$ and has the value 14.305, we can obtain a more accurate appraisal of what happens for larger t by considering the initial value problems (5) and (6) with $y(0) = 1$ replaced by $y(0.9) = 14.305$. Then we obtain

$$\phi_1(t) = \tan(t + 0.60100), \qquad \phi_2(t) = 1/(0.96991 - t), \tag{8}$$

where only five decimal places have been kept. Thus $\phi_1(t) \to \infty$ as $t \to \pi/2 - 0.60100 \cong 0.96980$ and $\phi_2(t) \to \infty$ as $t \to 0.96991$. We conclude that the asymptote of the solution of the initial value problem (3) lies between these two values. This example illustrates the sort of information that can be obtained by a judicious combination of analytical and numerical work.

Stability. The concept of stability is associated with the possibility that small errors that are introduced in the course of a mathematical procedure may die out as the procedure continues. Conversely, instability occurs if small errors tend to increase, perhaps without bound. For example, in Section 2.5 we identified equilibrium solutions of a differential equation as (asymptotically) stable or unstable, depending on whether solutions that were initially near the equilibrium solution tended to approach it or to depart from it as t increases. Somewhat more generally, the solution of an initial value problem is asymptotically stable if initially nearby solutions tend to approach the given solution, and unstable if they tend to depart from it. Visually, in an asymptotically stable problem the graphs of solutions will come together, while in an unstable problem they will separate.

If we are solving an initial value problem numerically, the best that we can hope for is that the numerical approximation will mimic the behavior of the actual solution. We cannot make an unstable problem into a stable one merely by solving it numerically. However, it may well happen that a numerical procedure will introduce instabilities that were not part of the original problem, and this can cause trouble in approximating the solution. Avoidance of such instabilities may require us to place restrictions on the step size h.

To illustrate what can happen in the simplest possible context consider the differential equation

$$dy/dt = ry, \tag{9}$$

where r is a constant. Suppose that in solving this equation we have reached the point (t_n, y_n). Let us compare the exact solution of Eq. (9) that passes through this point, namely,

$$y = y_n \exp[r(t - t_n)], \tag{10}$$

with numerical approximations obtained from the Euler formula

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{11}$$

and from the backward Euler formula

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}). \tag{12}$$

From the Euler formula (11) we obtain

$$y_{n+1} = y_n + hry_n = y_n(1 + rh). \tag{13}$$

Similarly, from the backward Euler formula (12) we have

$$y_{n+1} = y_n + hry_{n+1},$$

or

$$y_{n+1} = \frac{y_n}{1 - rh} = y_n[1 + rh + (rh)^2 + \cdots]. \tag{14}$$

Finally, evaluating the solution (10) at $t_n + h$, we find that

$$y_{n+1} = y_n \exp(rh) = y_n \left[1 + rh + \frac{(rh)^2}{2} + \cdots\right]. \tag{15}$$

Comparing Eqs. (13), (14), and (15) we see that the errors in both the Euler formula and in the backward Euler formula are of order h^2, as the theory predicts.

Now suppose that we change the value y_n to $y_n + \delta$. Think, if you wish, of δ as the error that has accumulated by the time we reach $t = t_n$. In going one more step to t_{n+1}, the question is whether this error increases or decreases.

For the exact solution (15) the change in y_{n+1} due to the change δ in y_n is just $\delta \exp(rh)$. This quantity is less than δ if $\exp(rh) < 1$, that is, if $r < 0$. This confirms our conclusion in Chapter 2 that Eq. (9) is asymptotically stable if $r < 0$ and unstable if $r > 0$.

For the backward Euler method the change in y_{n+1} in Eq. (14) due to δ is $\delta/(1 - rh)$. For $r \leq 0$ this quantity is always nonnegative and never greater than 1. Thus, if the differential equation is stable, then so is the backward Euler method for an arbitrary step size h.

On the other hand, for the Euler method, the change in y_{n+1} in Eq. (13) due to δ is $\delta(1 + rh)$. If we recall that $r \le 0$ and require that $|1 + rh| \le 1$, then we find that h must satisfy $h \le 2/|r|$. Thus the Euler method is not stable for this problem unless h is sufficiently small.

The restriction on the step size h in using the Euler method in the preceding example is rather mild unless $|r|$ is quite large. Nonetheless, the example illustrates that it may be necessary to restrict h in order to achieve stability in the numerical method, even though the initial value problem itself is stable for all values of h. Problems for which a much smaller step size is needed for stability than for accuracy are called **stiff**. The backward differentiation formulas described in Section 8.4 (of which the backward Euler formula is the lowest order example) are the most popular formulas for solving stiff problems. The following example illustrates the kind of instabilities that can occur in trying to approximate the solution of a stiff problem.

EXAMPLE 2 A Stiff Problem

Consider the initial value problem

$$y' = -100y + 100t + 1, \qquad y(0) = 1. \tag{16}$$

Find numerical approximations to the solution for $0 \le t \le 1$ using the Euler, backward Euler, and Runge–Kutta methods. Compare the numerical results with the exact solution.

Since the differential equation is linear, it is easy to solve, and the solution of the initial value problem (16) is

$$y = \phi(t) = e^{-100t} + t. \tag{17}$$

Some values of the solution $\phi(t)$, correct to six decimal places, are given in the second column of Table 8.5.3, and a graph of the solution is shown in Figure 8.5.2. There is a thin layer (sometimes called a boundary layer) to the right of $t = 0$ in which the exponential term is significant and the solution varies rapidly. Once past this layer, however, $\phi(t) \cong t$ and the graph of the solution is essentially a straight line. The width of the boundary layer is somewhat arbitrary, but it is certainly small. At $t = 0.1$, for example, $\exp(-100t) \cong 0.000045$.

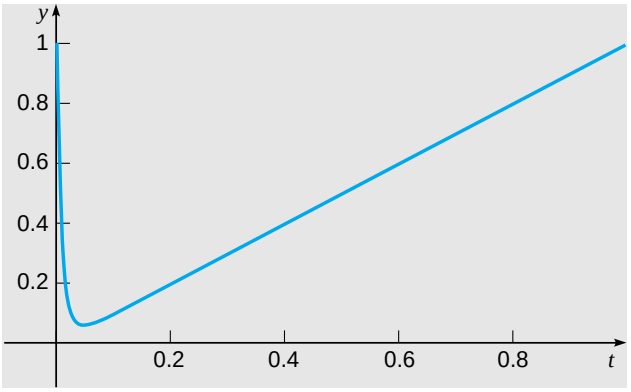


FIGURE 8.5.2 The solution of the initial value problem (16).

If we plan to approximate the solution (17) numerically, we might intuitively expect that a small step size will be needed only in the boundary layer. To make this expectation a bit more precise recall from Section 8.1 that the local truncation errors for the Euler and backward Euler methods are proportional to $\phi''(t)$. For this problem $\phi''(t) = 10^4 e^{-100t}$, which varies from a value of 10^4 at $t = 0$ to nearly zero for $t > 0.2$. Thus a very small step size is needed for accuracy near $t = 0$, but a much larger step size is adequate once t is a little larger.

On the other hand, the stability analysis in Eqs. (9) through (15) also applies to this problem. Since $r = -100$ for Eq. (16), it follows that for stability we need $h < 0.02$ for the Euler method, but there is no corresponding restriction for the backward Euler method.

Some results using the Euler method are shown in columns 3 and 4 of Table 8.5.3. The values obtained for $h = 0.025$ are worthless due to instability, while those for $h = 0.01666\ldots$ are reasonably accurate for $t \geq 0.2$. However, comparable accuracy for this range of t is obtained for $h = 0.1$ using the backward Euler method, as shown by the results in column 7 of the table.

The situation is not improved by using instead of the Euler method a more accurate one, such as Runge–Kutta. For this problem the Runge–Kutta method is unstable for $h = 0.033\ldots$ but stable for $h = 0.025$, as shown by the results in columns 5 and 6 in Table 8.5.3.

The results given in the table for $t = 0.05$ and for $t = 0.1$ show that in the boundary layer a smaller step size is needed to obtain an accurate approximation. You are invited to explore this matter further in Problem 3.

TABLE 8.5.3 Numerical Approximations to the Solution of the Initial Value Problem $y' = -100y + 100t + 1$, $y(0) = 1$

t	Exact	Euler 0.025	Euler 0.0166...	Runge–Kutta 0.0333...	Runge–Kutta 0.025	Backward Euler 0.1
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.05	0.056738	2.300000	−0.246296		0.470471	
0.1	0.100045	5.162500	0.187792	10.6527	0.276796	0.190909
0.2	0.200000	25.8289	0.207707	111.559	0.231257	0.208264
0.4	0.400000	657.241	0.400059	1.24×10^4	0.400977	0.400068
0.6	0.600000	1.68×10^4	0.600000	1.38×10^6	0.600031	0.600001
0.8	0.800000	4.31×10^5	0.800000	1.54×10^8	0.800001	0.800000
1.0	1.000000	1.11×10^7	1.000000	1.71×10^{10}	1.000000	1.000000

As a final example, consider the problem of determining two linearly independent solutions of the second order linear equation

$$y'' - 10\pi^2 y = 0 \tag{18}$$

for $t > 0$. The generalization of numerical techniques for the first order equations to higher order equations or to systems of equations is discussed in Section 8.6, but that is not needed for the present discussion. Two linearly independent solutions of Eq. (18) are $\phi_1(t) = \cosh \sqrt{10}\pi t$ and $\phi_2(t) = \sinh \sqrt{10}\pi t$. The first solution, $\phi_1(t) = \cosh \sqrt{10}\pi t$, is generated by the initial conditions $\phi_1(0) = 1, \phi_1'(0) = 0$; the second solution, $\phi_2(t) =$

$\sinh\sqrt{10}\pi t$, is generated by the initial conditions $\phi_2(0) = 0$, $\phi_2'(0) = \sqrt{10}\pi$. While analytically we can tell the difference between $\cosh\sqrt{10}\pi t$ and $\sinh\sqrt{10}\pi t$, for large t we have $\cosh\sqrt{10}\pi t \sim e^{\sqrt{10}\pi t}/2$ and $\sinh\sqrt{10}\pi t \sim e^{\sqrt{10}\pi t}/2$; numerically these two functions look exactly the same if only a fixed number of digits are retained. For example, correct to eight significant figures, we find that for $t = 1$

$$\sinh\sqrt{10}\pi = \cosh\sqrt{10}\pi = 10{,}315.894.$$

If the calculations are carried out on a machine that carries only eight digits, the two solutions ϕ_1 and ϕ_2 are identical at $t = 1$ and indeed for all $t > 1$. Thus, even though the solutions are linearly independent, their numerical tabulation would show that they are the same because we can retain only a finite number of digits. This phenomenon is called **numerical dependence**.

For the present problem we can partially circumvent this difficulty by computing, instead of $\sinh\sqrt{10}\pi t$ and $\cosh\sqrt{10}\pi t$, the linearly independent solutions $\phi_3(t) = e^{\sqrt{10}\pi t}$ and $\phi_4(t) = e^{-\sqrt{10}\pi t}$ corresponding to the initial conditions $\phi_3(0) = 1$, $\phi_3'(0) = \sqrt{10}\pi$ and $\phi_4(0) = 1$, $\phi_4'(0) = -\sqrt{10}\pi$, respectively. The solution ϕ_3 grows exponentially while ϕ_4 decays exponentially. Even so, we encounter difficulty in calculating ϕ_4 correctly on a large interval. The reason is that at each step of the calculation for ϕ_4 we introduce truncation and round-off errors. Thus at any point t_n the data to be used in going to the next point are not precisely the values of $\phi_4(t_n)$ and $\phi_4'(t_n)$. The solution of the initial value problem with these data at t_n involves not only $e^{-\sqrt{10}\pi t}$ but also $e^{\sqrt{10}\pi t}$. Because the error in the data at t_n is small, the latter function appears with a very small coefficient. Nevertheless, since $e^{-\sqrt{10}\pi t}$ tends to zero and $e^{\sqrt{10}\pi t}$ grows very rapidly, the latter eventually dominates, and the calculated solution is simply a multiple of $e^{\sqrt{10}\pi t} = \phi_3(t)$.

To be specific, suppose that we use the Runge–Kutta method to calculate the solution $y = \phi_4(t) = e^{-\sqrt{10}\pi t}$ of the initial value problem

$$y'' - 10\pi^2 y = 0, \qquad y(0) = 1, \qquad y'(0) = -\sqrt{10}\pi.$$

(The Runge–Kutta method for second order systems is described in Section 8.6.) Using single-precision (eight-digit) arithmetic with a step size $h = 0.01$, we obtain the results in Table 8.5.4. It is clearly evident from these results that the numerical solution begins to deviate significantly from the exact solution for $t > 0.5$, and soon differs from it by many orders of magnitude. The reason is the presence in the numerical solution of a small component of the exponentially growing solution $\phi_3(t) = e^{\sqrt{10}\pi t}$. With eight-digit arithmetic we can expect a round-off error of the order of 10^{-8} at each step. Since $e^{\sqrt{10}\pi t}$ grows by a factor of 3.7×10^{21} from $t = 0$ to $t = 5$, an error of order 10^{-8} near $t = 0$ can produce an error of order 10^{13} at $t = 5$ even if no further errors are introduced in the intervening calculations. The results given in Table 8.5.4 demonstrate that this is exactly what happens.

Equation (18) is highly unstable and the behavior shown in this example is typical of unstable problems. One can track a solution accurately for a while, and the interval can be extended by using smaller step sizes or more accurate methods, but eventually the instability in the problem itself takes over and leads to large errors.

TABLE 8.5.4 Exact Solution of $y'' - 10\pi^2 y = 0$, $y(0) = 1$, $y'(0) = -\sqrt{10}\pi$ and Numerical Solution Using the Runge–Kutta Method with $h = 0.01$

	y	
t	Numerical	Exact
0.0	1.0	1.0
0.25	8.3439×10^{-2}	8.3438×10^{-2}
0.5	6.9631×10^{-3}	6.9620×10^{-3}
0.75	5.9390×10^{-4}	5.8089×10^{-4}
1.0	2.0437×10^{-4}	4.8469×10^{-5}
1.5	2.2394×10^{-2}	3.3744×10^{-7}
2.0	3.2166	2.3492×10^{-9}
2.5	4.6202×10^{2}	1.6356×10^{-11}
3.0	6.6363×10^{4}	1.1386×10^{-13}
3.5	9.5322×10^{6}	7.9272×10^{-16}
4.0	1.3692×10^{9}	5.5189×10^{-18}
4.5	1.9667×10^{11}	3.8422×10^{-20}
5.0	2.8249×10^{13}	2.6750×10^{-22}

Some Comments on Numerical Methods. In this chapter we have introduced several numerical methods for approximating the solution of an initial value problem. We have tried to emphasize some important ideas while maintaining a reasonable level of complexity. For one thing, we have always used a uniform step size, whereas production codes currently in use provide for varying the step size as the calculation proceeds.

There are several considerations that must be taken into account in choosing step sizes. Of course, one is accuracy; too large a step size leads to an inaccurate result. Normally, an error tolerance is prescribed in advance and the step size at each step must be consistent with this requirement. As we have seen, the step size must also be chosen so that the method is stable. Otherwise, small errors will grow and soon render the results worthless. Finally, for implicit methods an equation must be solved at each step and the method used to solve the equation may impose additional restrictions on the step size.

In choosing a method one must also balance the considerations of accuracy and stability against the amount of time required to execute each step. An implicit method, such as the Adams–Moulton method, requires more calculations for each step, but if its accuracy and stability permit a larger step size (and consequently fewer steps), then this may more than compensate for the additional calculations. The backward differentiation formulas of moderate order, say, four, are highly stable and are therefore indicated for stiff problems, for which stability is the controlling factor.

Some current production codes also permit the order of the method to be varied, as well as the step size, as the calculation proceeds. The error is estimated at each step and the order and step size are chosen to satisfy the prescribed error tolerance. In practice, Adams methods up to order twelve and backward differentiation formulas up to order five are in use. Higher order backward differentiation formulas are unsuitable due to a lack of stability.

Finally, we note that the smoothness of the function f, that is, the number of continuous derivatives that it possesses, is a factor in choosing the order of the method

to be used. High order methods lose some of their accuracy if f is not smooth to a corresponding order.

PROBLEMS

1. To obtain some idea of the possible dangers of small errors in the initial conditions, such as those due to round-off, consider the initial value problem

$$y' = t + y - 3, \qquad y(0) = 2.$$

(a) Show that the solution is $y = \phi_1(t) = 2 - t$.
(b) Suppose that in the initial condition a mistake is made and 2.001 is used instead of 2. Determine the solution $y = \phi_2(t)$ in this case, and compare the difference $\phi_2(t) - \phi_1(t)$ at $t = 1$ and as $t \to \infty$.

2. Consider the initial value problem

$$y' = t^2 + e^y, \qquad y(0) = 0. \tag{i}$$

Using the Runge–Kutta method with step size h, we obtain the results in Table 8.5.5. These results suggest that the solution has a vertical asymptote between $t = 0.9$ and $t = 1.0$.
(a) Show that for $0 \le t \le 1$ the solution $y = \phi(t)$ of the problem (i) satisfies

$$\phi_2(t) \le \phi(t) \le \phi_1(t), \tag{ii}$$

where $y = \phi_1(t)$ is the solution of

$$y' = 1 + e^y, \qquad y(0) = 0 \tag{iii}$$

and $y = \phi_2(t)$ is the solution of

$$y' = e^y, \qquad y(0) = 0. \tag{iv}$$

(b) Determine $\phi_1(t)$ and $\phi_2(t)$. Then show that $\phi(t) \to \infty$ for some t between $t = \ln 2 \cong 0.69315$ and $t = 1$.
(c) Solve the differential equations $y' = e^y$ and $y' = 1 + e^y$, respectively, with the initial condition $y(0.9) = 3.4298$. Use the results to show that $\phi(t) \to \infty$ when $t \cong 0.932$.

TABLE 8.5.5 Calculation of the Solution of the Initial Value Problem $y' = t^2 + e^y$, $y(0) = 0$ Using the Runge–Kutta Method

h	$t = 0.90$	$t = 1.0$
0.02	3.42985	$> 10^{38}$
0.01	3.42982	$> 10^{38}$

3. Consider again the initial value problem (16) from Example 2. Investigate how small a step size h must be chosen in order that the error at $t = 0.05$ and at $t = 0.1$ is less than 0.0005.
(a) Use the Euler method.
(b) Use the backward Euler method.
(c) Use the Runge–Kutta method.

4. Consider the initial value problem

$$y' = -10y + 2.5t^2 + 0.5t, \qquad y(0) = 4.$$

(a) Find the solution $y = \phi(t)$ and draw its graph for $0 \le t \le 5$.
(b) The stability analysis in the text suggests that for this problem the Euler method is stable only for $h < 0.2$. Confirm that this is true by applying the Euler method to this problem for $0 \le t \le 5$ with step sizes near 0.2.

(c) Apply the Runge–Kutta method to this problem for $0 \le t \le 5$ with various step sizes. What can you conclude about the stability of this method?
(d) Apply the backward Euler method to this problem for $0 \le t \le 5$ with various step sizes. What step size is needed in order that the error at $t = 5$ is less than 0.01?

In each of Problems 5 and 6

(a) Find a formula for the solution of the initial value problem, and note that it is independent of λ.
(b) Use the Runge–Kutta method with $h = 0.01$ to compute approximate values of the solution for $0 \le t \le 1$ for various values of λ such as $\lambda = 1$, 10, 20, and 50.
(c) Explain the differences, if any, between the exact solution and the numerical approximations.

▶ 5. $y' - \lambda y = 1 - \lambda t, \qquad y(0) = 0$

▶ 6. $y' - \lambda y = 2t - \lambda t^2, \qquad y(0) = 0$

8.6 Systems of First Order Equations

In the preceding sections we discussed numerical methods for solving initial value problems associated with a first order differential equation. These methods can also be applied to a system of first order equations. Since a higher order equation can always be reduced to a system of first order equations, it is sufficient to deal with systems of first order equations alone. For simplicity we consider a system of two first order equations

$$x' = f(t, x, y), \qquad y' = g(t, x, y), \tag{1}$$

with the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0. \tag{2}$$

The functions f and g are assumed to satisfy the conditions of Theorem 7.1.1 so that The initial value problem (1), (2) has a unique solution in some interval of the t-axis containing the point t_0. We wish to determine approximate values $x_1, x_2, \ldots, x_n, \ldots$ and $y_1, y_2, \ldots, y_n, \ldots$ of the solution $x = \phi(t)$, $y = \psi(t)$ at the points $t_n = t_0 + nh$ with $n = 1, 2, \ldots$.

In vector notation the initial value problem (1), (2) can be written as

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3}$$

where $\mathbf{x}$ is the vector with components x and y, $\mathbf{f}$ is the vector function with components f and g, and $\mathbf{x}_0$ is the vector with components x_0 and y_0. The methods of the previous sections can be readily generalized to handle systems of two (or more) equations. All that is needed (formally) is to replace the scalar variable x by the vector $\mathbf{x}$ and the scalar function f by the vector function $\mathbf{f}$ in the appropriate equations. For example, the Euler formula becomes

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n, \tag{4}$$

or, in component form,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} f(t_n, x_n, y_n) \\ g(t_n, x_n, y_n) \end{pmatrix}. \tag{5}$$

The initial conditions are used to determine $\mathbf{f}_0$, which is the vector tangent to the graph of the solution $\mathbf{x} = \boldsymbol{\phi}(t)$ in the xy-plane. We move in the direction of this tangent vector for a time step h in order to find the next point $\mathbf{x}_1$. Then we calculate a new tangent vector $\mathbf{f}_1$, move along it for a time step h to find $\mathbf{x}_2$, and so forth.

In a similar way the Runge–Kutta method can be extended to a system. For the step from t_n to t_{n+1} we have

$$\mathbf{x}_{n+1} = \mathbf{x}_n + (h/6)(\mathbf{k}_{n1} + 2\mathbf{k}_{n2} + 2\mathbf{k}_{n3} + \mathbf{k}_{n4}), \tag{6}$$

where

$$\begin{aligned} \mathbf{k}_{n1} &= \mathbf{f}(t_n, \mathbf{x}_n), \\ \mathbf{k}_{n2} &= \mathbf{f}[t_n + (h/2), \mathbf{x}_n + (h/2)\mathbf{k}_{n1}], \\ \mathbf{k}_{n3} &= \mathbf{f}[t_n + (h/2), \mathbf{x}_n + (h/2)\mathbf{k}_{n2}], \\ \mathbf{k}_{n4} &= \mathbf{f}(t_n + h, \mathbf{x}_n + h\mathbf{k}_{n3}). \end{aligned} \tag{7}$$

The formulas for the Adams–Moulton predictor–corrector method as it applies to the initial value problem (1), (2) are given in Problem 9.

The vector equations (3), (4), (6), and (7) are, in fact, valid in any number of dimensions. All that is needed is to interpret the vectors as having n components rather than two.

EXAMPLE 1

Determine approximate values of the solution $x = \phi(t)$, $y = \psi(t)$ of the initial value problem

$$x' = x - 4y, \qquad y' = -x + y, \tag{8}$$

$$x(0) = 1, \qquad y(0) = 0, \tag{9}$$

at the point $t = 0.2$. Use the Euler method with $h = 0.1$ and the Runge–Kutta method with $h = 0.2$. Compare the results with the values of the exact solution:

$$\phi(t) = \frac{e^{-t} + e^{3t}}{2}, \qquad \psi(t) = \frac{e^{-t} - e^{3t}}{4}. \tag{10}$$

Let us first use the Euler method. For this problem $f_n = x_n - 4y_n$ and $g_n = -x_n + y_n$; hence

$$f_0 = 1 - (4)(0) = 1, \qquad g_0 = -1 + 0 = -1.$$

Then, from the Euler formulas (4) and (5) we obtain

$$x_1 = 1 + (0.1)(1) = 1.1, \qquad y_1 = 0 + (0.1)(-1) = -0.1.$$

At the next step

$$f_1 = 1.1 - (4)(-0.1) = 1.5, \qquad g_1 = -1.1 + (-0.1) = -1.2.$$

Consequently,

$$x_2 = 1.1 + (0.1)(1.5) = 1.25, \qquad y_2 = -0.1 + (0.1)(-1.2) = -0.22.$$

The values of the exact solution, correct to eight digits, are $\phi(0.2) = 1.3204248$ and $\psi(0.2) = -0.25084701$. Thus the values calculated from the Euler method are in error by about 0.0704 and 0.0308, respectively, corresponding to percentage errors of about 5.3% and 12.3%.

Now let us use the Runge–Kutta method to approximate $\phi(0.2)$ and $\psi(0.2)$. With $h = 0.2$ we obtain the following values from Eqs. (7):

$$\mathbf{k}_{01} = \begin{pmatrix} f(1, 0) \\ g(1, 0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$\mathbf{k}_{02} = \begin{pmatrix} f(1.1, -0.1) \\ g(1.1, -0.1) \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.2 \end{pmatrix};$$

$$\mathbf{k}_{03} = \begin{pmatrix} f(1.15, -0.12) \\ g(1.15, -0.12) \end{pmatrix} = \begin{pmatrix} 1.63 \\ -1.27 \end{pmatrix};$$

$$\mathbf{k}_{04} = \begin{pmatrix} f(1.326, -0.254) \\ g(1.326, -0.254) \end{pmatrix} = \begin{pmatrix} 2.342 \\ -1.580 \end{pmatrix}.$$

Then, substituting these values in Eq. (6), we obtain

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{0.2}{6} \begin{pmatrix} 9.602 \\ -7.52 \end{pmatrix} = \begin{pmatrix} 1.3200667 \\ -0.25066667 \end{pmatrix}.$$

These values of x_1 and y_1 are in error by about 0.000358 and 0.000180, respectively, with percentage errors much less than one-tenth of 1%.

This example again illustrates the great gains in accuracy that are obtainable by using a more accurate approximation method, such as the Runge–Kutta method. In the calculations we have just outlined, the Runge–Kutta method requires only twice as many function evaluations as the Euler method, but the error in the Runge–Kutta method is about 200 times less than the Euler method.

PROBLEMS

In each of Problems 1 through 6 determine approximate values of the solution $x = \phi(t)$, $y = \psi(t)$ of the given initial value problem at $t = 0.2$, 0.4, 0.6, 0.8, and 1.0. Compare the results obtained by different methods and different step sizes.

(a) Use the Euler method with $h = 0.1$.
(b) Use the Runge–Kutta method with $h = 0.2$.
(c) Use the Runge–Kutta method with $h = 0.1$.

1. $x' = x + y + t, \quad y' = 4x - 2y; \qquad x(0) = 1, \quad y(0) = 0$
2. $x' = 2x + ty, \quad y' = xy; \qquad x(0) = 1, \quad y(0) = 1$
3. $x' = -tx - y - 1, \quad y' = x; \qquad x(0) = 1, \quad y(0) = 1$
4. $x' = x - y + xy, \quad y' = 3x - 2y - xy; \qquad x(0) = 0, \quad y(0) = 1$
5. $x' = x(1 - 0.5x - 0.5y), \quad y' = y(-0.25 + 0.5x); \qquad x(0) = 4, \quad y(0) = 1$
6. $x' = \exp(-x + y) - \cos x, \quad y' = \sin(x - 3y); \qquad x(0) = 1, \quad y(0) = 2$
7. Consider the example problem $x' = x - 4y$, $y' = -x + y$ with the initial conditions $x(0) = 1$ and $y(0) = 0$. Use the Runge–Kutta method to solve this problem on the interval $0 \le t \le 1$. Start with $h = 0.2$ and then repeat the calculation with step sizes $h = 0.1, 0.05, \ldots$, each half as long as in the preceding case. Continue the process until

the first five digits of the solution at $t = 1$ are unchanged for successive step sizes. Determine whether these digits are accurate by comparing them with the exact solution given in Eqs. (10) in the text.

▶ 8. Consider the initial value problem

$$x'' + t^2 x' + 3x = t, \qquad x(0) = 1, \quad x'(0) = 2.$$

Convert this problem to a system of two first order equations and determine approximate values of the solution at $t = 0.5$ and $t = 1.0$ using the Runge–Kutta method with $h = 0.1$.

▶ 9. Consider the initial value problem $x' = f(t, x, y)$ and $y' = g(t, x, y)$ with $x(t_0) = x_0$ and $y(t_0) = y_0$. The generalization of the Adams–Moulton predictor–corrector method of Section 8.4 is

$$x_{n+1} = x_n + \tfrac{1}{24}h(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}),$$
$$y_{n+1} = y_n + \tfrac{1}{24}h(55g_n - 59g_{n-1} + 37g_{n-2} - 9g_{n-3})$$

and

$$x_{n+1} = x_n + \tfrac{1}{24}h(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}),$$
$$y_{n+1} = y_n + \tfrac{1}{24}h(9g_{n+1} + 19g_n - 5g_{n-1} + g_{n-2}).$$

Determine an approximate value of the solution at $t = 0.4$ for the example initial value problem $x' = x - 4y$, $y' = -x + y$ with $x(0) = 1$, $y(0) = 0$. Take $h = 0.1$. Correct the predicted value once. For the values of $x_1, \ldots, y_3$ use the values of the exact solution rounded to six digits: $x_1 = 1.12883$, $x_2 = 1.32042$, $x_3 = 1.60021$, $y_1 = -0.110527$, $y_2 = -0.250847$, and $y_3 = -0.429696$.

REFERENCES

There are many books of varying degrees of sophistication dealing with numerical analysis in general and the numerical solution of ordinary differential equations in particular. Among these are:

Ascher, Uri M., and Petzold, Linda R., *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations* (Philadelphia: Society for Industrial and Applied Mathematics, 1998).

Gear, C. William, *Numerical Initial Value Problems in Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice Hall, 1971).

Henrici, Peter, *Discrete Variable Methods in Ordinary Differential Equations* (New York: Wiley, 1962).

Shampine, Lawrence F., *Numerical Solution of Ordinary Differential Equations* (New York: Chapman and Hall, 1994).

A detailed exposition of Adams predictor–corrector methods, including practical guidelines for implementation, may be found in:

Shampine, L. F., and Gordon, M. K., *Computer Solution of Ordinary Differential Equations: The Initial Value Problem* (San Francisco: Freeman, 1975).

Many books on numerical analysis have chapters on differential equations. For example, at an elementary level, see:

Burden, R. L., and Faires, J. D., *Numerical Analysis* (6th ed.) (Pacific Grove, CA: Brooks/Cole, 1997).

CHAPTER
9

Nonlinear Differential Equations and Stability

There are many differential equations, especially nonlinear ones, that are not susceptible to analytical solution in any reasonably convenient manner. Numerical methods, such as those discussed in the preceding chapter, provide one means of dealing with these equations. Another approach, presented in this chapter, is geometrical in character and leads to a qualitative understanding of the behavior of solutions rather than detailed quantitative information.

9.1 The Phase Plane: Linear Systems



Since many differential equations cannot be solved conveniently by analytical methods, it is important to consider what qualitative[1] information can be obtained about their solutions without actually solving the equations. The questions that we consider in this chapter are associated with the idea of stability of a solution, and the methods that we employ are basically geometrical. Both the concept of stability and the use of

[1]The qualitative theory of differential equations was created by Henri Poincaré (1854–1912) in several major papers between 1880 and 1886. Poincaré was professor at the University of Paris and is generally considered the leading mathematician of his time. He made fundamental discoveries in several different areas of mathematics, including complex function theory, partial differential equations, and celestial mechanics. In a series of papers beginning in 1894 he initiated the use of modern methods in topology. In differential equations he was a pioneer in the use of asymptotic series, one of the most powerful tools of contemporary applied mathematics. Among other things, he used asymptotic expansions to obtain solutions about irregular singular points, thereby extending the work of Fuchs and Frobenius discussed in Chapter 5.

geometric analysis were introduced in Chapter 1 and used in Section 2.5 for first order autonomous equations

$$dy/dt = f(y). \tag{1}$$

In this chapter we refine the ideas and extend the discussion to systems of equations.

We start with a consideration of the simplest system, namely, a second order linear homogeneous system with constant coefficients. Such a system has the form

$$d\mathbf{x}/dt = \mathbf{A}\mathbf{x}, \tag{2}$$

where $\mathbf{A}$ is a 2×2 constant matrix and $\mathbf{x}$ is a 2×1 vector. Systems of this kind were solved in Sections 7.5 through 7.8. Recall that if we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, then by substitution for $\mathbf{x}$ in Eq. (2) we find that

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{3}$$

Thus r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$. The eigenvalues are the roots of the polynomial equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0 \tag{4}$$

and the eigenvectors are determined from Eq. (3) up to an arbitrary multiplicative constant.

In Section 2.5 we found that points where the right side of Eq. (1) is zero are of special importance. Such points correspond to constant solutions, or **equilibrium solutions**, of Eq. (1), and are often called **critical points**. Similarly, for the system (2), points where $\mathbf{A}\mathbf{x} = \mathbf{0}$ correspond to equilibrium (constant) solutions, and again they are called critical points. We will assume that $\mathbf{A}$ is nonsingular, or that $\det \mathbf{A} \neq 0$. It follows that $\mathbf{x} = \mathbf{0}$ is the only critical point of the system (2).

Recall that a solution of Eq. (2) is a vector function $\mathbf{x} = \boldsymbol{\phi}(t)$ that satisfies the differential equation. Such a function can be viewed as a parametric representation for a curve in the x_1x_2-plane. It is often useful to regard this curve as the path, or **trajectory**, traversed by a moving particle whose velocity $d\mathbf{x}/dt$ is specified by the differential equation. The x_1x_2-plane itself is called the **phase plane** and a representative set of trajectories is referred to as a **phase portrait**.

In analyzing the system (2) several different cases must be considered, depending on the nature of the eigenvalues of $\mathbf{A}$. These cases also occurred in Sections 7.5 through 7.8, where we were primarily interested in finding a convenient formula for the general solution. Now our main goal is to characterize the differential equation according to the geometric pattern formed by its trajectories. In each case we discuss the behavior of the trajectories in general and illustrate it with an example. It is important that you become familiar with the types of behavior that the trajectories have for each case, because these are the basic ingredients of the qualitative theory of differential equations.

CASE 1 **Real Unequal Eigenvalues of the Same Sign.** The general solution of Eq. (2) is

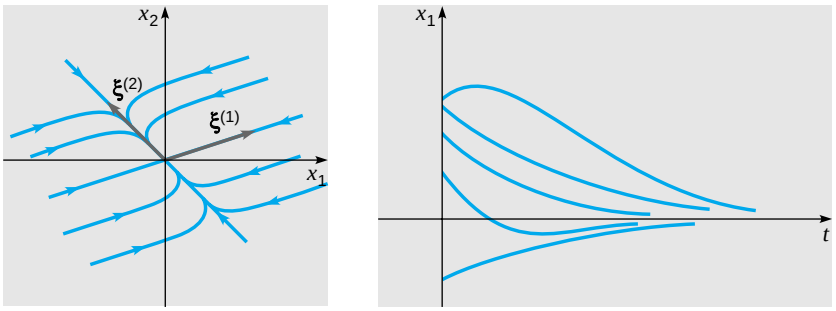
$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2t}, \tag{5}$$

where r_1 and r_2 are either both positive or both negative. Suppose first that $r_1 < r_2 < 0$, and that the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are as shown in Figure 9.1.1*a*. It follows from Eq. (5) that $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$ regardless of the values of c_1 and c_2; in other words, all solutions approach the critical point at the origin as $t \to \infty$. If the solution starts at an

initial point on the line through $\boldsymbol{\xi}^{(1)}$, then $c_2 = 0$. Consequently, the solution remains on the line through $\boldsymbol{\xi}^{(1)}$ for all t, and approaches the origin as $t \to \infty$. Similarly, if the initial point is on the line through $\boldsymbol{\xi}^{(2)}$, then the solution approaches the origin along that line. In the general situation, it is helpful to rewrite Eq. (5) in the form

$$\mathbf{x} = e^{r_2 t}[c_1 \boldsymbol{\xi}^{(1)} e^{(r_1 - r_2)t} + c_2 \boldsymbol{\xi}^{(2)}]. \tag{6}$$

Observe that $r_1 - r_2 < 0$. Therefore, as long as $c_2 \neq 0$, the term $c_1 \boldsymbol{\xi}^{(1)} \exp[(r_1 - r_2)t]$ is negligible compared to $c_2 \boldsymbol{\xi}^{(2)}$ for t sufficiently large. Thus, as $t \to \infty$, the trajectory not only approaches the origin, it also tends toward the line through $\boldsymbol{\xi}^{(2)}$. Hence all solutions approach the critical point tangent to $\boldsymbol{\xi}^{(2)}$ except for those solutions that start exactly on the line through $\boldsymbol{\xi}^{(1)}$. Several trajectories are sketched in Figure 9.1.1*a*. Some typical graphs of x_1 versus t are shown in Figure 9.1.1*b*, illustrating that all solutions exhibit exponential decay in time. The behavior of x_2 versus t is similar. This type of critical point is called a **node** or a **nodal sink**.



(*a*) (*b*)

FIGURE 9.1.1 A node; $r_1 < r_2 < 0$. (*a*) The phase plane. (*b*) x_1 versus t.

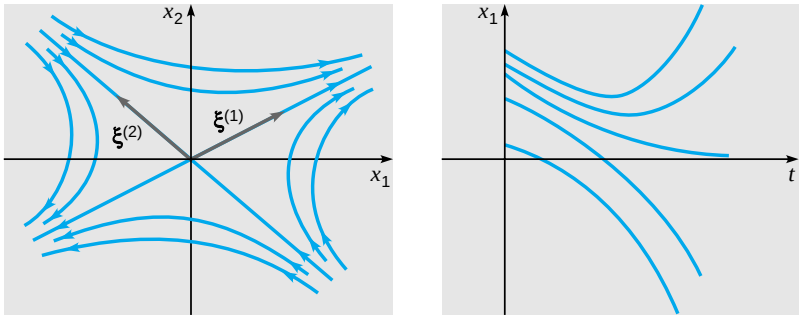
If r_1 and r_2 are both positive, and $0 < r_2 < r_1$, then the trajectories have the same pattern as in Figure 9.1.1*a*, but the direction of motion is away from, rather than toward, the critical point at the origin. In this case x_1 and x_2 grow exponentially as functions of t. Again the critical point is called a node, or (often) a **nodal source**.

An example of a node occurs in Example 2 of Section 7.5, and its trajectories are shown in Figure 7.5.4.

CASE 2 **Real Eigenvalues of Opposite Sign.** The general solution of Eq. (2) is

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t}, \tag{7}$$

where $r_1 > 0$ and $r_2 < 0$. Suppose that the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are as shown in Figure 9.1.2*a*. If the solution starts at an initial point on the line through $\boldsymbol{\xi}^{(1)}$, then it follows that $c_2 = 0$. Consequently, the solution remains on the line through $\boldsymbol{\xi}^{(1)}$ for all t, and since $r_1 > 0$, $\|\mathbf{x}\| \to \infty$ as $t \to \infty$. If the solution starts at an initial point on the line through $\boldsymbol{\xi}^{(2)}$, then the situation is similar except that $\|\mathbf{x}\| \to 0$ as $t \to \infty$ because $r_2 < 0$. Solutions starting at other initial points follow trajectories such as those shown in Figure 9.1.2*a*. The positive exponential is the dominant term in Eq. (7) for large t, so eventually all these solutions approach infinity asymptotic to the line determined



(*a*) (*b*)

FIGURE 9.1.2 A saddle point; $r_1 > 0$, $r_2 < 0$. (*a*) The phase plane. (*b*) x_1 versus t.

by the eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to the positive eigenvalue r_1. The only solutions that approach the critical point at the origin are those that start precisely on the line determined by $\boldsymbol{\xi}^{(2)}$. Figure 9.1.2*b* shows some typical graphs of x_1 versus t. For certain initial conditions the positive exponential term is absent from the solution, so $x_1 \to 0$ as $t \to \infty$. For all other initial conditions the positive exponential term eventually dominates and causes x_1 to become unbounded. The behavior of x_2 is similar. The origin is called a **saddle point** in this case.

A specific example of a saddle point is in Example 1 of Section 7.5 whose trajectories are shown in Figure 7.5.2.

CASE 3 **Equal Eigenvalues.** We now suppose that $r_1 = r_2 = r$. We consider the case in which the eigenvalues are negative; if they are positive, the trajectories are similar but the direction of motion is reversed. There are two subcases, depending on whether the repeated eigenvalue has two independent eigenvectors or only one.

(a) Two independent eigenvectors. The general solution of Eq. (2) is

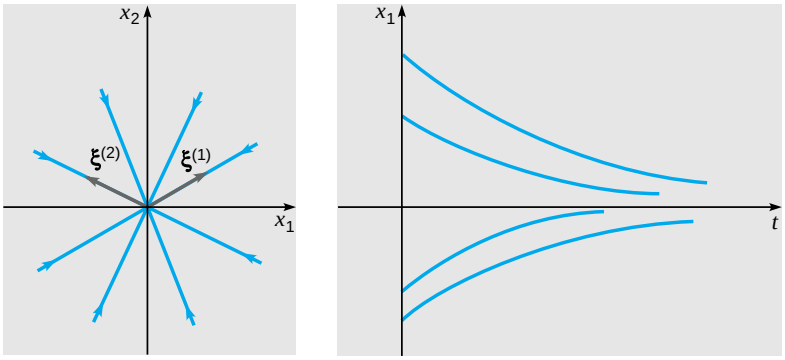
$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{rt} + c_2\boldsymbol{\xi}^{(2)}e^{rt}, \tag{8}$$

where $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$are the independent eigenvectors. The ratio x_2/x_1 is independent of t, but depends on the components of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, and on the arbitrary constants c_1 and c_2. Thus every trajectory lies on a straight line through the origin, as shown in Figure 9.1.3*a*. Typical graphs of x_1 or x_2 versus t are shown in Figure 9.1.3*b*. The critical point is called a **proper node**, or sometimes a **star point**.

(b) One independent eigenvector. As shown in Section 7.8, the general solution of Eq. (2) in this case is

$$\mathbf{x} = c_1\boldsymbol{\xi}e^{rt} + c_2(\boldsymbol{\xi}te^{rt} + \boldsymbol{\eta}e^{rt}), \tag{9}$$

where $\boldsymbol{\xi}$ is the eigenvector and $\boldsymbol{\eta}$ is the generalized eigenvector associated with the repeated eigenvalue. For large t the dominant term in Eq. (9) is $c_2\boldsymbol{\xi}te^{rt}$. Thus, as $t \to \infty$, every trajectory approaches the origin tangent to the line through the eigenvector. This is true even if $c_2 = 0$, for then the solution $\mathbf{x} = c_1\boldsymbol{\xi}e^{rt}$ lies on this line. Similarly, for large negative t the term $c_2\boldsymbol{\xi}te^{rt}$ is again the dominant one, so as $t \to -\infty$, each trajectory is asymptotic to a line parallel to $\boldsymbol{\xi}$.



(*a*) (*b*)

FIGURE 9.1.3 A proper node, two independent eigenvectors; $r_1 = r_2 < 0$. (*a*) The phase plane. (*b*) x_1 versus t.

The orientation of the trajectories depends on the relative positions of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. One possible situation is shown in Figure 9.1.4*a*. To locate the trajectories it is helpful to write the solution (9) in the form

$$\mathbf{x} = [(c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + c_2\boldsymbol{\xi}t]e^{rt} = \mathbf{y}e^{rt}, \tag{10}$$

where $\mathbf{y} = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + c_2\boldsymbol{\xi}t$. Observe that the vector $\mathbf{y}$ determines the direction of $\mathbf{x}$, whereas the scalar quantity e^{rt} affects only the magnitude of $\mathbf{x}$. Also note that, for fixed values of c_1 and c_2, the expression for $\mathbf{y}$ is a vector equation of the straight line through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ and parallel to $\boldsymbol{\xi}$.

To sketch the trajectory corresponding to a given pair of values of c_1 and c_2, you can proceed in the following way. First, draw the line given by $(c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + c_2\boldsymbol{\xi}t$ and note the direction of increasing t on this line. Two such lines are shown in Figure 9.1.4*a*, one for $c_2 > 0$ and the other for $c_2 < 0$. Next, note that the given trajectory passes through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ when $t = 0$. Further, as t increases, the direction of the vector $\mathbf{x}$ given by Eq. (10) follows the direction of increasing t on the line, but the magnitude of $\mathbf{x}$ rapidly decreases and approaches zero because of the decaying exponential factor e^{rt}. Finally, as t decreases toward $-\infty$ the direction of $\mathbf{x}$ is determined by points on the corresponding part of the line and the magnitude of $\mathbf{x}$ approaches infinity. In this way we obtain the heavy trajectories in Figure 9.1.4*a*. A few other trajectories are lightly sketched as well to help complete the diagram. Typical graphs of x_1 versus t are shown in Figure 9.1.4*b*.

The other possible situation is shown in Figure 9.1.4*c*, where the relative orientation of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is reversed. As indicated in the figure, this results in a reversal in the orientation of the trajectories.

If $r_1 = r_2 > 0$, you can sketch the trajectories by following the same procedure. In this event the trajectories are traversed in the outward direction, and the orientation of the trajectories with respect to that of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is also reversed.

When a double eigenvalue has only a single independent eigenvector, the critical point is called an **improper** or **degenerate node**. A specific example of this case is Example 2 in Section 7.8; the trajectories are shown in Figure 7.8.2.

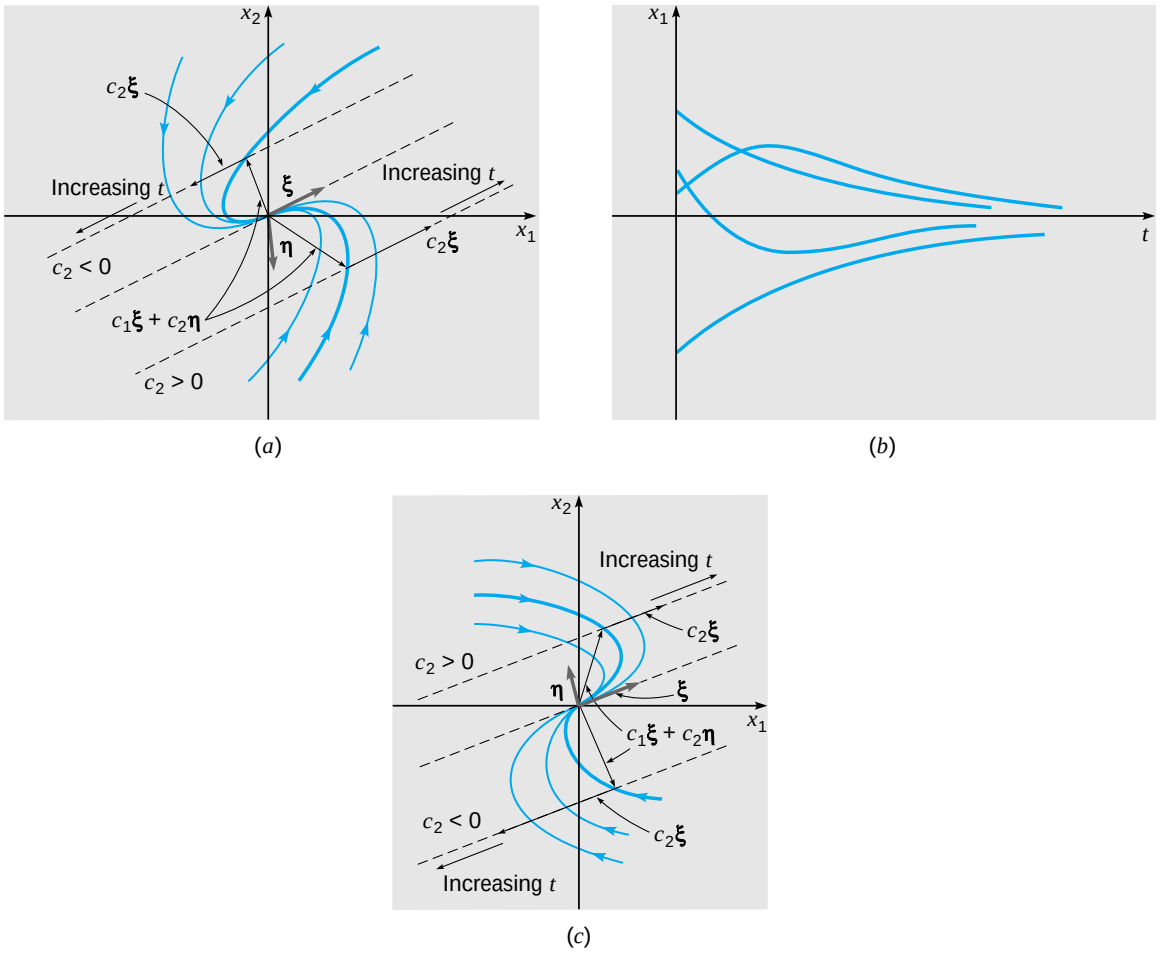


FIGURE 9.1.4 An improper node, one independent eigenvector; $r_1 = r_2 < 0$. (*a*) The phase plane. (*b*) x_1 versus t. (*c*) The phase plane.

CASE 4 **Complex Eigenvalues.** Suppose that the eigenvalues are $\lambda \pm i\mu$, where λ and μ are real, $\lambda \neq 0$, and $\mu > 0$. It is possible to write down the general solution in terms of the eigenvalues and eigenvectors, as shown in Section 7.6. However, we proceed in a different way.

Systems having the eigenvalues $\lambda \pm i\mu$ are typified by

$$\mathbf{x}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{x} \tag{11}$$

or, in scalar form,

$$x_1' = \lambda x_1 + \mu x_2, \qquad x_2' = -\mu x_1 + \lambda x_2. \tag{12}$$

We introduce the polar coordinates r, θ given by

$$r^2 = x_1^2 + x_2^2, \qquad \tan\theta = x_2/x_1.$$

By differentiating these equations we obtain

$$rr' = x_1x_1' + x_2x_2', \qquad (\sec^2\theta)\theta' = (x_1x_2' - x_2x_1')/x_1^2. \tag{13}$$

Substituting from Eqs. (12) in the first of Eqs. (13), we find that

$$r' = \lambda r, \tag{14}$$

and hence

$$r = ce^{\lambda t}, \tag{15}$$

where c is a constant. Similarly, substituting from Eqs. (12) in the second of Eqs. (13), and using the fact that $\sec^2\theta = r^2/x_1^2$, we have

$$\theta' = -\mu. \tag{16}$$

Hence

$$\theta = -\mu t + \theta_0, \tag{17}$$

where θ_0 is the value of θ when $t = 0$.

Equations (15) and (17) are parametric equations in polar coordinates of the trajectories of the system (11). Since $\mu > 0$, it follows from Eq. (17) that θ decreases as t increases, so the direction of motion on a trajectory is clockwise. As $t \to \infty$, we see from Eq. (15) that $r \to 0$ if $\lambda < 0$ and $r \to \infty$ if $\lambda > 0$. Thus the trajectories are spirals, which approach or recede from the origin depending on the sign of λ. Both possibilities are shown in Figure 9.1.5, along with some typical graphs of x_1 versus t. The critical point is called a **spiral point** in this case. Frequently, the terms **spiral sink** and **spiral source**, respectively, are used to refer to spiral points whose trajectories approach, or depart from, the critical point.

More generally, it is possible to show that for any system with complex eigenvalues $\lambda \pm i\mu$, where $\lambda \neq 0$, the trajectories are always spirals. They are directed inward or outward, respectively, depending on whether λ is negative or positive. They may be elongated and skewed with respect to the coordinate axes, and the direction of motion may be either clockwise or counterclockwise. While a detailed analysis is moderately difficult, it is easy to obtain a general idea of the orientation of the trajectories directly from the differential equations. Suppose that

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{18}$$

has complex eigenvalues $\lambda \pm i\mu$, and look at the point (0, 1) on the positive y-axis. At this point it follows from Eqs. (18) that $dx/dt = b$ and $dy/dt = d$. Depending on the signs of b and d, one can infer the direction of motion and the approximate orientation of the trajectories. For instance, if both b and d are negative, then the trajectories cross the positive y-axis so as to move down and into the second quadrant. If $\lambda < 0$ also, then the trajectories must be inward-pointing spirals resembling the one in Figure 9.1.6. Another case was given in Example 1 of Section 7.6, whose trajectories are shown in Figure 7.6.2.

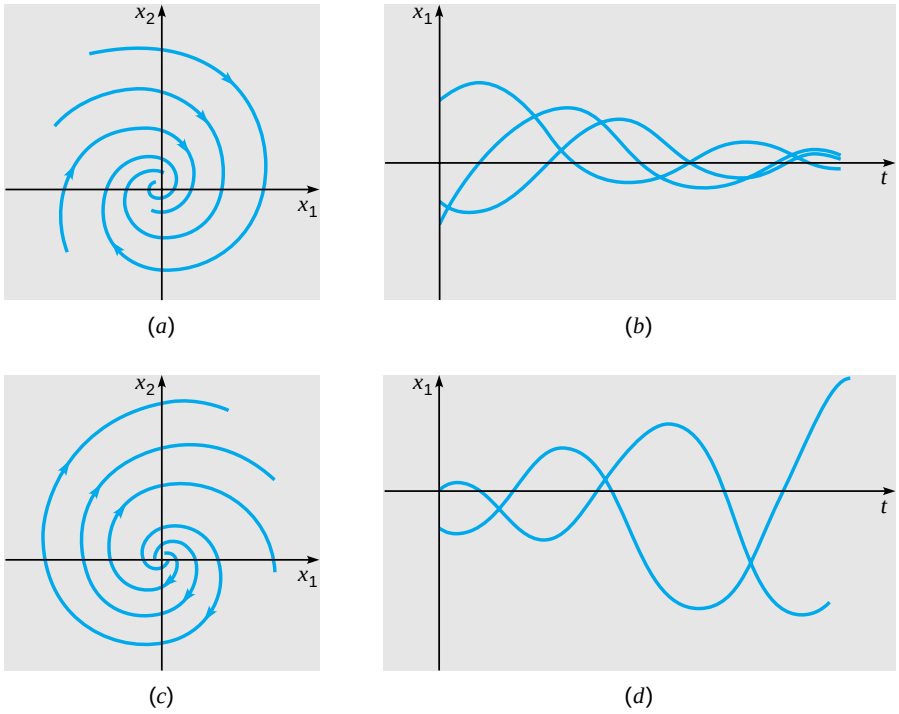


FIGURE 9.1.5 A spiral point; $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$. (*a*) $\lambda < 0$, the phase plane. (*b*) $\lambda < 0$, x_1 versus t. (*c*) $\lambda > 0$, the phase plane. (*d*) $\lambda > 0$, x_1 versus t.

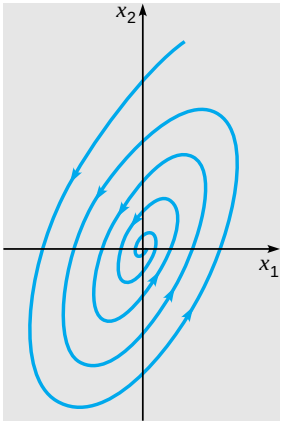


FIGURE 9.1.6 A spiral point; $r = \lambda \pm i\mu$ with $\lambda < 0$.

CASE 5 **Pure Imaginary Eigenvalues.** In this case $\lambda = 0$ and the system (11) reduces to

$$\mathbf{x}' = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \mathbf{x} \tag{19}$$

with eigenvalues $\pm i\mu$. Using the same argument as in Case 4, we find that

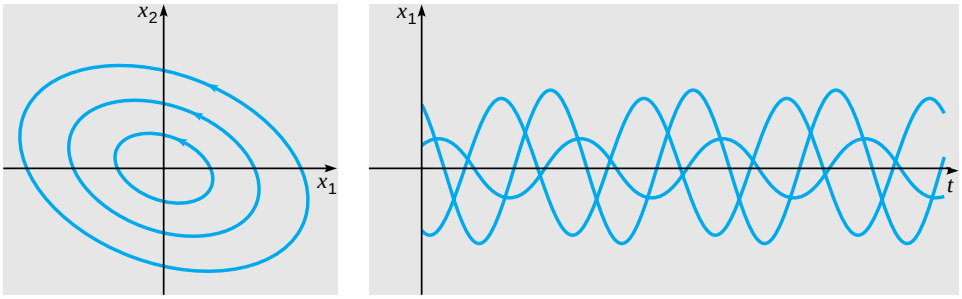
$$r' = 0, \qquad \theta' = -\mu, \tag{20}$$

and consequently,

$$r = c, \qquad \theta = -\mu t + \theta_0, \tag{21}$$

where c and θ_0 are constants. Thus the trajectories are circles, with center at the origin, that are traversed clockwise if $\mu > 0$ and counterclockwise if $\mu < 0$. A complete circuit about the origin is made in a time interval of length $2\pi/\mu$, so all solutions are periodic with period $2\pi/\mu$. The critical point is called a **center**.

In general, when the eigenvalues are pure imaginary, it is possible to show (see Problem 19) that the trajectories are ellipses centered at the origin. A typical situation is shown in Figure 9.1.7, which also shows some typical graphs of x_1 versus t.



(*a*) (*b*)

FIGURE 9.1.7 A center; $r_1 = i\mu, r_2 = -i\mu$. (*a*) The phase plane. (*b*) x_1 versus t.

By reflecting on these five cases and by examining the corresponding figures, we can make several observations:

1. After a long time each individual trajectory exhibits one of only three types of behavior. As $t \to \infty$, each trajectory either approaches infinity, approaches the critical point $\mathbf{x} = \mathbf{0}$, or repeatedly traverses a closed curve, corresponding to a periodic solution, that surrounds the critical point.
2. Viewed as a whole, the pattern of trajectories in each case is relatively simple. To be more specific, through each point (x_0, y_0) in the phase plane there is only one trajectory; thus the trajectories do not cross each other. Do not be misled by the figures, in which it sometimes appears that many trajectories pass through the critical point $\mathbf{x} = \mathbf{0}$. In fact, the only solution passing through the origin is the equilibrium solution $\mathbf{x} = \mathbf{0}$. The other solutions that appear to pass through the origin actually only approach this point as $t \to \infty$ or $t \to -\infty$.
3. In each case the set of all trajectories is such that one of three situations occurs.
   (a) All trajectories approach the critical point $\mathbf{x} = \mathbf{0}$ as $t \to \infty$. This is the case if the eigenvalues are real and negative or complex with negative real part. The origin is either a nodal or a spiral sink.
   (b) All trajectories remain bounded but do not approach the origin as $t \to \infty$. This is the case if the eigenvalues are pure imaginary. The origin is a center.

(c) Some trajectories, and possibly all trajectories except $\mathbf{x} = \mathbf{0}$, tend to infinity as $t \to \infty$. This is the case if at least one of the eigenvalues is positive or if the eigenvalues have positive real part. The origin is either a nodal source, a spiral source, or a saddle point.

The situations described in 3(a), (b), and (c) above illustrate the concepts of asymptotic stability, stability, and instability, respectively, of the equilibrium solution $\mathbf{x} = \mathbf{0}$ of the system (2). The precise definitions of these terms are given in Section 9.2, but their basic meaning should be clear from the geometrical discussion in this section. The information that we have obtained about the system (2) is summarized in Table 9.1.1. Also see Problems 20 and 21.

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	Center	Stable

The analysis in this section applies only to second order systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose solutions are represented geometrically as curves in the phase plane. A similar, though more complicated, analysis can be carried out for an nth order system, with an $n \times n$ coefficient matrix $\mathbf{A}$, whose solutions are curves in an n-dimensional phase space. The cases that can occur in higher order systems are essentially combinations of those we have seen in two dimensions. For instance, in a third order system with a three-dimensional phase space, one possibility is that solutions in a certain plane may spiral toward the origin, while other solutions may tend to infinity along a line transverse to this plane. This would be the case if the coefficient matrix has two complex eigenvalues with negative real part and one positive real eigenvalue. However, because of their complexity, we will not discuss systems of higher than second order.

PROBLEMS

For each of the systems in Problems 1 through 12:

(a) Find the eigenvalues and eigenvectors.

(b) Classify the critical point (0, 0) as to type and determine whether it is stable, asymptotically stable, or unstable.

(c) Sketch several trajectories in the phase plane and also sketch some typical graphs of x_1 versus t.

(d) Use a computer to plot accurately the curves requested in part (c).

► 1. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

► 2. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

3. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}$

5. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}$

6. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

7. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

8. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \mathbf{x}$

9. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

10. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

11. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x}$

12. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 16 determine the critical point $\mathbf{x} = \mathbf{x}^0$, and then classify its type and examine its stability by making the transformation $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$.

13. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

14. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

15. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

16. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \alpha \\ -\gamma \end{pmatrix}; \qquad \alpha, \beta, \gamma, \delta > 0$

17. The equation of motion of a spring–mass system with damping (see Section 3.8) is

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0,$$

where m, c, and k are positive. Write this second order equation as a system of two first order equations for $x = u$, $y = du/dt$. Show that $x = 0$, $y = 0$ is a critical point, and analyze the nature and stability of the critical point as a function of the parameters m, c, and k. A similar analysis can be applied to the electric circuit equation (see Section 3.8)

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

18. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and suppose that $\mathbf{A}$ has one zero eigenvalue.
(a) Show that $\mathbf{x} = \mathbf{0}$ is a critical point, and that, in addition, every point on a certain straight line through the origin is also a critical point.
(b) Let $r_1 = 0$ and $r_2 \neq 0$, and let $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ be corresponding eigenvectors. Show that the trajectories are as indicated in Figure 9.1.8. What is the direction of motion on the trajectories?

19. In this problem we indicate how to show that the trajectories are ellipses when the eigenvalues are pure imaginary. Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{i}$$

(a) Show that the eigenvalues of the coefficient matrix are pure imaginary if and only if

$$a_{11} + a_{22} = 0, \qquad a_{11}a_{22} - a_{12}a_{21} > 0. \tag{ii}$$

(b) The trajectories of the system (i) can be found by converting Eqs. (i) into the single equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}. \tag{iii}$$

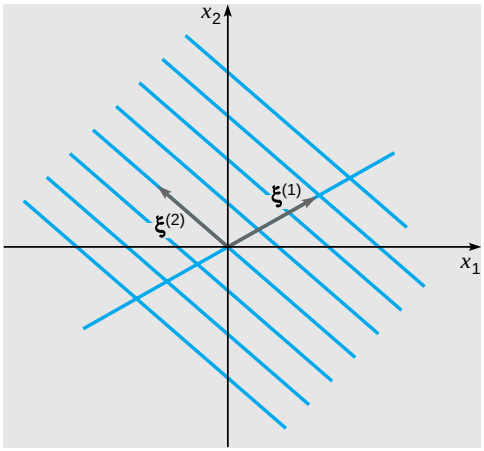


FIGURE 9.1.8 Nonisolated critical points; $r_1 = 0, r_2 \neq 0$. Every point on the line through $\boldsymbol{\xi}^{(1)}$ is a critical point.

Use the first of Eqs. (ii) to show that Eq. (iii) is exact.

(c) By integrating Eq. (iii) show that

$$a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = k, \tag{iv}$$

where k is a constant. Use Eqs. (ii) to conclude that the graph of Eq. (iv) is always an ellipse.

Hint: What is the discriminant of the quadratic form in Eq. (iv)?

20. Consider the linear system

$$dx/dt = a_{11}x + a_{12}y, \qquad dy/dt = a_{21}x + a_{22}y,$$

where $a_{11}, \ldots, a_{22}$ are real constants. Let $p = a_{11} + a_{22}$, $q = a_{11}a_{22} - a_{12}a_{21}$, and $\Delta = p^2 - 4q$. Show that the critical point (0, 0) is a

(a) Node if $q > 0$ and $\Delta \geq 0$;
(b) Saddle point if $q < 0$;
(c) Spiral point if $p \neq 0$ and $\Delta < 0$;
(d) Center if $p = 0$ and $q > 0$.

Hint: These conclusions can be obtained by studying the eigenvalues r_1 and r_2. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point (0, 0) is

(a) Asymptotically stable if $q > 0$ and $p < 0$;
(b) Stable if $q > 0$ and $p = 0$;
(c) Unstable if $q < 0$ or $p > 0$.

Notice that results (a), (b), and (c), together with the fact that $q \neq 0$, show that the critical point is asymptotically stable if, and only if, $q > 0$ and $p < 0$. The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

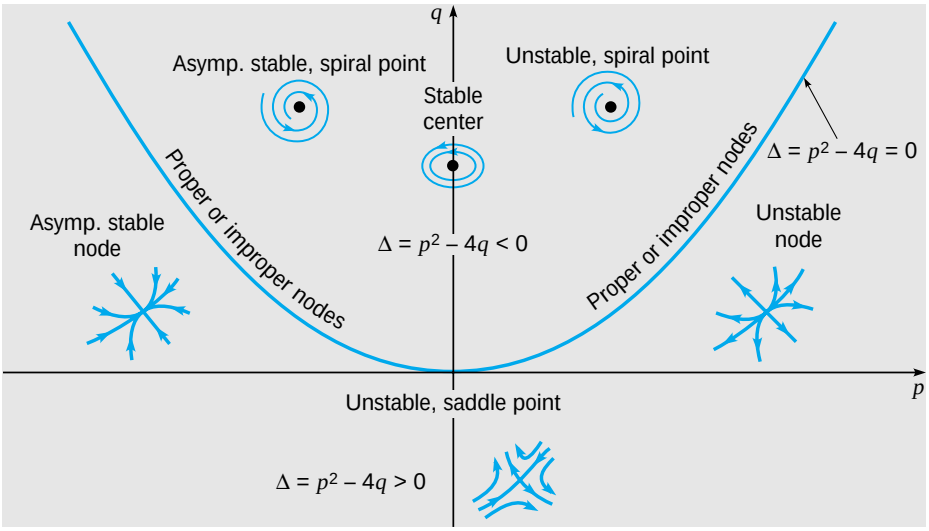


FIGURE 9.1.9 Stability diagram.

9.2 Autonomous Systems and Stability

ODE

In this section we begin to draw together and to expand on the geometrical ideas introduced in Section 2.5 for certain first order equations and in Section 9.1 for second order linear homogeneous systems with constant coefficients. These ideas concern the qualitative study of differential equations and the concept of stability, an idea that will be defined precisely later in this section.

ODE

Autonomous Systems. We are concerned with systems of two simultaneous differential equations of the form

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y). \tag{1}$$

ODE

We assume that the functions F and G are continuous and have continuous partial derivatives in some domain D of the xy-plane. If (x_0, y_0) is a point in this domain, then by Theorem 7.1.1 there exists a unique solution $x = \phi(t)$, $y = \psi(t)$ of the system (1) satisfying the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0. \tag{2}$$

The solution is defined in some time interval I that contains the point t_0.

Frequently, we will write the initial value problem (1), (2) in the vector form

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}^0, \tag{3}$$

where $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{f(x)} = F(x, y)\mathbf{i} + G(x, y)\mathbf{j}$, and $\mathbf{x}^0 = x_0\mathbf{i} + y_0\mathbf{j}$. In this case the solution is expressed as $\mathbf{x} = \boldsymbol{\phi}(t)$, where $\boldsymbol{\phi}(t) = \phi(t)\mathbf{i} + \psi(t)\mathbf{j}$. As usual, we interpret a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ as a curve traced by a moving point in the xy-plane, the phase plane.

Observe that the functions F and G in Eqs. (1) do not depend on the independent variable t, but only on the dependent variables x and y. A system with this property is said to be **autonomous**. The system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{4}$$

where $\mathbf{A}$ is a constant matrix, is a simple example of a two-dimensional autonomous system. On the other hand, if one or more of the elements of the coefficient matrix $\mathbf{A}$ is a function of the independent variable t, then the system is nonautonomous. The distinction between autonomous and nonautonomous systems is important because the geometric qualitative analysis in Section 9.1 can be effectively extended to two-dimensional autonomous systems in general, but is not nearly as useful for nonautonomous systems.

In particular, the autonomous system (1) has an associated direction field that is independent of time. Consequently, there is only one trajectory passing through each point (x_0, y_0) in the phase plane. In other words, all solutions that satisfy an initial condition of the form (2) lie on the same trajectory, regardless of the time t_0 at which they pass through (x_0, y_0). Thus, just as for the constant coefficient linear system (4), a single phase portrait simultaneously displays important qualitative information about all solutions of the system (1). We will see this fact confirmed repeatedly in this chapter.

Autonomous systems occur frequently in applications. Physically, an autonomous system is one whose configuration, including physical parameters and external forces or effects, is independent of time. The response of the system to given initial conditions is then independent of the time at which the conditions are imposed.

Stability and Instability. The concepts of stability, asymptotic stability, and instability have already been mentioned several times in this book. It is now time to give a precise mathematical definition of these concepts, at least for autonomous systems of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{5}$$

In the following definitions, and elsewhere, we use the notation $\|\mathbf{x}\|$ to designate the length, or magnitude, of the vector $\mathbf{x}$.

The points, if any, where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are called **critical points** of the autonomous system (5). At such points $\mathbf{x}' = \mathbf{0}$ also, so critical points correspond to constant, or equilibrium, solutions of the system of differential equations. A critical point $\mathbf{x}^0$ of the system (5) is said to be **stable** if, given any $\epsilon > 0$, there is a $\delta > 0$ such that every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system (1), which at $t = 0$ satisfies

$$\|\boldsymbol{\phi}(0) - \mathbf{x}^0\| < \delta, \tag{6}$$

exists for all positive t and satisfies

$$\|\boldsymbol{\phi}(t) - \mathbf{x}^0\| < \epsilon \tag{7}$$

for all $t \geq 0$. This is illustrated geometrically in Figures 9.2.1*a* and 9.2.1*b*. These mathematical statements say that all solutions that start "sufficiently close" (that is, within the distance δ) to $\mathbf{x}^0$ stay "close" (within the distance ϵ) to $\mathbf{x}^0$. Note that in Figure 9.2.1*a* the trajectory is within the circle $\|\mathbf{x} - \mathbf{x}^0\| = \delta$ at $t = 0$ and, while it soon passes outside of this circle, it remains within the circle $\|\mathbf{x} - \mathbf{x}^0\| = \epsilon$ for all $t \geq 0$. However, the trajectory of the solution does not have to approach the critical point $\mathbf{x}^0$ as $t \to \infty$, as illustrated in Figure 9.2.1*b*. A critical point that is not stable is said to be **unstable**.

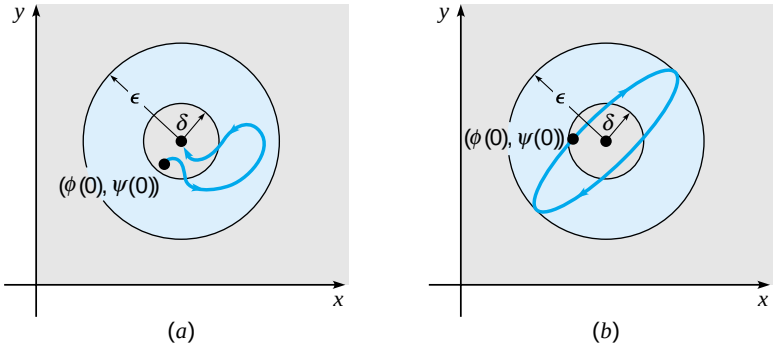


FIGURE 9.2.1 (*a*) Asymptotic stability. (*b*) Stability.

A critical point $\mathbf{x}^0$ is said to be **asymptotically stable** if it is stable and if there exists a δ_0, with $0 < \delta_0 < \delta$, such that if a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ satisfies

$$\|\boldsymbol{\phi}(0) - \mathbf{x}^0\| < \delta_0, \tag{8}$$

then

$$\lim_{t\to\infty} \boldsymbol{\phi}(t) = \mathbf{x}^0. \tag{9}$$

Thus trajectories that start "sufficiently close" to $\mathbf{x}^0$ must not only stay "close" but must eventually approach $\mathbf{x}^0$ as $t \to \infty$. This is the case for the trajectory in Figure 9.2.1*a* but not for the one in Figure 9.2.1*b*. Note that asymptotic stability is a stronger property than stability, since a critical point must be stable before we can even talk about whether it might be asymptotically stable. On the other hand, the limit condition (9), which is an essential feature of asymptotic stability, does not by itself imply even ordinary stability. Indeed, examples can be constructed in which all of the trajectories approach $\mathbf{x}^0$ as $t \to \infty$, but for which $\mathbf{x}^0$ is not a stable critical point. Geometrically, all that is needed is a family of trajectories having members that start arbitrarily close to $\mathbf{x}^0$, then depart an arbitrarily large distance before eventually approaching $\mathbf{x}^0$ as $t \to \infty$.

While we specified originally that the system (5) is of second order, the definitions just given are independent of the order of the system. If you interpret the vectors in Eqs. (5) through (9) as n-dimensional, then the definitions of stability, asymptotic stability, and instability apply also to nth order systems. These definitions can be made more concrete by interpreting them in terms of a specific physical problem.

The Oscillating Pendulum. The concepts of asymptotic stability, stability, and instability can be easily visualized in terms of an oscillating pendulum. Consider the configuration shown in Figure 9.2.2, in which a mass m is attached to one end of a rigid, but weightless, rod of length L. The other end of the rod is supported at the origin O, and the rod is free to rotate in the plane of the paper. The position of the pendulum is described by the angle θ between the rod and the downward vertical direction, with the counterclockwise direction taken as positive. The gravitational force mg acts downward, while the damping force $c|d\theta/dt|$, where c is positive, is always opposite to the direction of motion. We assume that θ and $d\theta/dt$ are both positive. The equation of motion can be quickly derived from the principle of angular momentum,

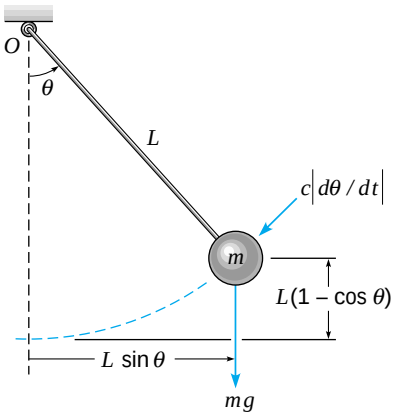


FIGURE 9.2.2 An oscillating pendulum.

which states that the time rate of change of angular momentum about any point is equal to the moment of the resultant force about that point. The angular momentum about the origin is $mL^2(d\theta/dt)$, so the governing equation is

$$mL^2\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta. \tag{10}$$

The factors L and $L\sin\theta$ on the right side of Eq. (10) are the moment arms of the resistive force and of the gravitational force, respectively, while the minus signs are due to the fact that the two forces tend to make the pendulum rotate in the clockwise (negative) direction. You should verify, as an exercise, that the same equation is obtained for the other three possible sign combinations of θ and $d\theta/dt$.

By straightforward algebraic operations we can write Eq. (10) in the standard form

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0, \tag{11}$$

or

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin\theta = 0, \tag{12}$$

where $\gamma = c/mL$ and $\omega^2 = g/L$. To convert Eq. (12) to a system of two first order equations we let $x = \theta$ and $y = d\theta/dt$; then

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2\sin x - \gamma y. \tag{13}$$

Since γ and ω^2 are constants, the system (13) is an autonomous system of the form (1).

The critical points of Eqs. (13) are found by solving the equations

$$y = 0, \qquad -\omega^2\sin x - \gamma y = 0.$$

We obtain $y = 0$ and $x = \pm n\pi$, where n is an integer. These points correspond to two physical equilibrium positions, one with the mass directly below the point of support ($\theta = 0$) and the other with the mass directly above the point of support ($\theta = \pi$). Our intuition suggests that the first is stable and the second is unstable.

More precisely, if the mass is slightly displaced from the lower equilibrium position, it will oscillate back and forth with gradually decreasing amplitude, eventually approaching the equilibrium position as the initial potential energy is dissipated by the damping force. This type of motion illustrates *asymptotic stability.*

On the other hand, if the mass is slightly displaced from the upper equilibrium position, it will rapidly fall, under the influence of gravity, and will ultimately approach the lower equilibrium position in this case also. This type of motion illustrates *instability.* In practice, it is impossible to maintain the pendulum in its upward equilibrium position for any length of time without an external constraint mechanism since the slightest perturbation will cause the mass to fall.

Finally, consider the ideal situation in which the damping coefficient c (or γ) is zero. In this case, if the mass is displaced slightly from its lower equilibrium position, it will oscillate indefinitely with constant amplitude about the equilibrium position. Since there is no dissipation in the system, the mass will remain near the equilibrium position, but will not approach it asymptotically. This type of motion is *stable*, but not asymptotically stable. In general, this motion is impossible to achieve experimentally because the slightest degree of air resistance or friction at the point of support will eventually cause the pendulum to approach its rest position.

These three types of motion are illustrated schematically in Figure 9.2.3. Solutions of the pendulum equations are discussed in more detail in the next section.

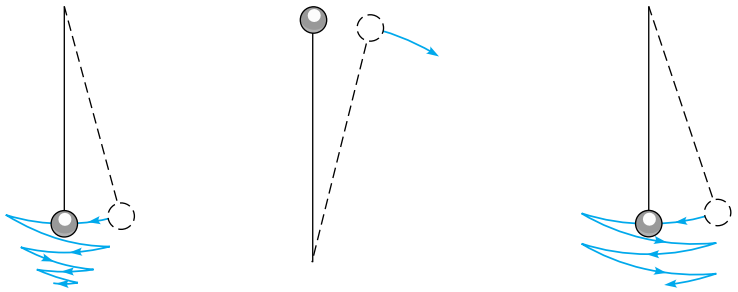


FIGURE 9.2.3 Qualitative motion of a pendulum. (*a*) With air resistance. (*b*) With or without air resistance. (*c*) Without air resistance.

Determination of Trajectories The trajectories of a two-dimensional autonomous system can sometimes be found by solving a related first order differential equation. From Eqs. (1) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)}, \tag{14}$$

which is a first order equation in the variables x and y. Observe that such a reduction is not usually possible if F and G depend also on t. If Eq. (14) can be solved by any of the methods of Chapter 2, and if we write solutions (implicitly) in the form

$$H(x, y) = c, \tag{15}$$

then Eq. (15) is an equation for the trajectories of the system (14). In other words, the trajectories lie on the level curves of $H(x, y)$. Keep in mind that there is no general

way of solving Eq. (14) to obtain the function H, so this approach is applicable only in special cases.

EXAMPLE 1

Find the trajectories of the system

$$dx/dt = y, \qquad dy/dt = x. \tag{16}$$

In this case, Eq. (14) becomes

$$\frac{dy}{dx} = \frac{x}{y}. \tag{17}$$

This equation is separable since it can be written as

$$y\,dy = x\,dx,$$

and its solutions are given by

$$H(x, y) = y^2 - x^2 = c, \tag{18}$$

where c is arbitrary. Therefore the trajectories of the system (16) are the hyperbolas shown in Figure 9.2.4. The direction of motion on the trajectories can be inferred from the fact that both dx/dt and dy/dt are positive in the first quadrant. The only critical point is the saddle point at the origin.

Another way to obtain the trajectories is to solve the system (16) by the methods of Section 7.5. We omit the details, but the result is

$$x = c_1e^t + c_2e^{-t}, \quad y = c_1e^t - c_2e^{-t}.$$

Eliminating t between these two equations again leads to Eq. (18).

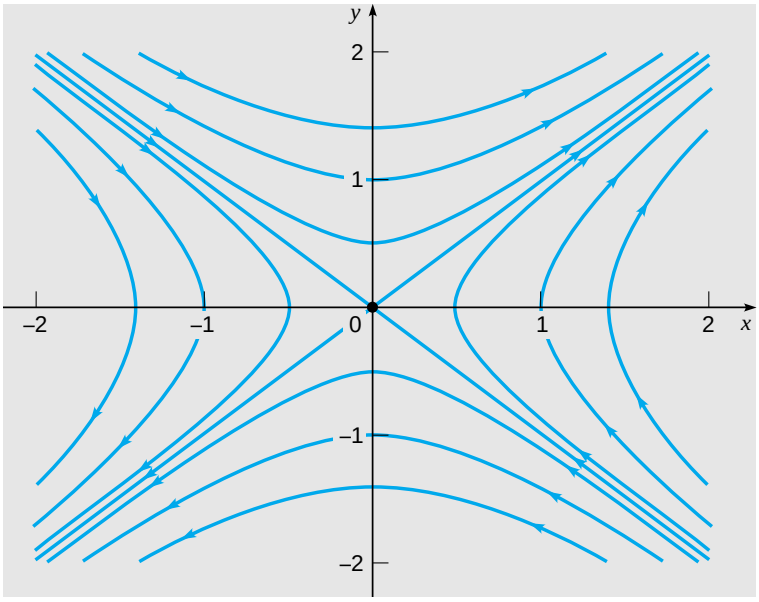


FIGURE 9.2.4 Trajectories of the system (16).

EXAMPLE 2

Find the trajectories of the system

$$\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = 12 - 3x^2. \tag{19}$$

From the equations

$$4 - 2y = 0, \quad 12 - 3x^2 = 0$$

we find that the critical points of the system (19) are the points $(-2, 2)$ and $(2, 2)$. To determine the trajectories note that for this system Eq. (14) becomes

$$\frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}. \tag{20}$$

Separating the variables in Eq. (20) and integrating, we find that solutions satisfy

$$H(x, y) = 4y - y^2 - 12x + x^3 = c, \tag{21}$$

where c is an arbitrary constant. A computer plotting routine is helpful in displaying the level curves of $H(x, y)$, some of which are shown in Figure 9.2.5. The direction of motion on the trajectories can be determined by drawing a direction field for the system (19), or by evaluating dx/dt and dy/dt at one or two selected points. From Figure 9.2.5 you can see that the critical point $(2, 2)$ is a saddle point while the point $(-2, 2)$ is a center. Observe that one trajectory leaves the saddle point (at $t = -\infty$), loops around the center, and returns to the saddle point (at $t = +\infty$).

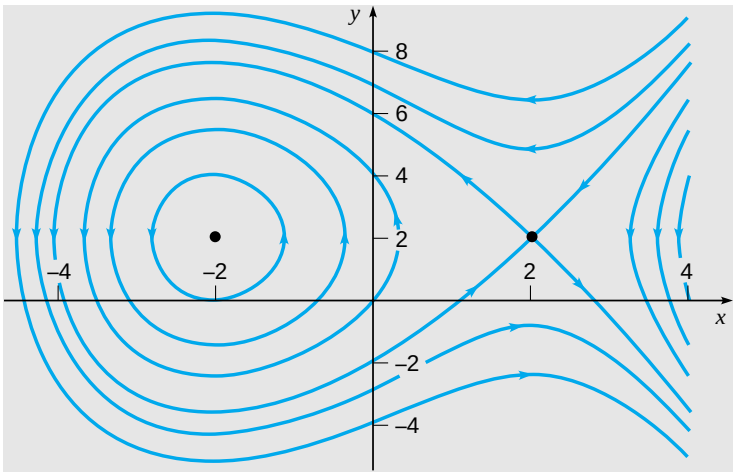


FIGURE 9.2.5 Trajectories of the system (19).

PROBLEMS

In each of Problems 1 through 4 sketch the trajectory corresponding to the solution satisfying the specified initial conditions, and indicate the direction of motion for increasing t.

1. $dx/dt = -x, \quad dy/dt = -2y; \quad x(0) = 4, \quad y(0) = 2$
2. $dx/dt = -x, \quad dy/dt = 2y; \quad x(0) = 4, \quad y(0) = 2$ and $x(0) = 4, \quad y(0) = 0$
3. $dx/dt = -y, \quad dy/dt = x; \quad x(0) = 4, \quad y(0) = 0$ and $x(0) = 0, \quad y(0) = 4$
4. $dx/dt = ay, \quad dy/dt = -bx, \quad a > 0, \quad b > 0; \quad x(0) = \sqrt{a}, \quad y(0) = 0$

For each of the systems in Problems 5 through 14:

(a) Find all the critical points (equilibrium solutions).

(b) Use a computer to draw a direction field and phase portrait for the system.

(c) From the plot(s) in part (b) determine whether each critical point is asymptotically stable, stable, or unstable, and classify it as to type.

5. $dx/dt = x - xy, \quad dy/dt = y + 2xy$
6. $dx/dt = 1 + 2y, \quad dy/dt = 1 - 3x^2$
7. $dx/dt = x - x^2 - xy, \quad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$
8. $dx/dt = -(x - y)(1 - x - y), \quad dy/dt = x(2 + y)$
9. $dx/dt = y(2 - x - y), \quad dy/dt = -x - y - 2xy$
10. $dx/dt = (2 + x)(y - x), \quad dy/dt = y(2 + x - x^2)$
11. $dx/dt = -x + 2xy, \quad dy/dt = y - x^2 - y^2$
12. $dx/dt = y, \quad dy/dt = x - \frac{1}{6}x^3 - \frac{1}{5}y$
13. $dx/dt = (2 + x)(y - x), \quad dy/dt = (4 - x)(y + x)$
14. The van der Pol equation: $dx/dt = y, \quad dy/dt = (1 - x^2)y - x$

In each of Problems 15 through 22:

(a) Find an equation of the form $H(x, y) = c$ satisfied by the trajectories.

(b) Plot several level curves of the function H. These are trajectories of the given system. Indicate the direction of motion on each trajectory.

15. $dx/dt = 2y, \quad dy/dt = 8x$
16. $dx/dt = 2y, \quad dy/dt = -8x$
17. $dx/dt = y, \quad dy/dt = 2x + y$
18. $dx/dt = -x + y, \quad dy/dt = -x - y$
19. $dx/dt = -x + y + x^2, \quad dy/dt = y - 2xy$
20. $dx/dt = 2x^2y - 3x^2 - 4y, \quad dy/dt = -2xy^2 + 6xy$
21. Undamped pendulum: $dx/dt = y, \quad dy/dt = -\sin x$
22. Duffing's equation: $dx/dt = y, \quad dy/dt = -x + (x^3/6)$
23. Given that $x = \phi(t)$, $y = \psi(t)$ is a solution of the autonomous system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y)$$

for $\alpha < t < \beta$, show that $x = \Phi(t) = \phi(t - s)$, $y = \Psi(t) = \psi(t - s)$ is a solution for $\alpha + s < t < \beta + s$ for any real number s.

24. Prove that for the system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y)$$

there is at most one trajectory passing through a given point (x_0, y_0).
Hint: Let C_0 be the trajectory generated by the solution $x = \phi_0(t)$, $y = \psi_0(t)$, with $\phi_0(t_0) = x_0$, $\psi_0(t_0) = y_0$, and let C_1 be the trajectory generated by the solution $x = \phi_1(t)$, $y = \psi_1(t)$, with $\phi_1(t_1) = x_0$, $\psi_1(t_1) = y_0$. Use the fact that the system is autonomous and also the existence and uniqueness theorem to show that C_0 and C_1 are the same.

25. Prove that if a trajectory starts at a noncritical point of the system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y),$$

then it cannot reach a critical point (x_0, y_0) in a finite length of time.
Hint: Assume the contrary; that is, assume that the solution $x = \phi(t)$, $y = \psi(t)$ satisfies $\phi(a) = x_0$, $\psi(a) = y_0$. Then use the fact that $x = x_0$, $y = y_0$ is a solution of the given system satisfying the initial condition $x = x_0$, $y = y_0$ at $t = a$.

26. Assuming that the trajectory corresponding to a solution $x = \phi(t)$, $y = \psi(t)$, $-\infty < t < \infty$, of an autonomous system is closed, show that the solution is periodic.

Hint: Since the trajectory is closed, there exists at least one point (x_0, y_0) such that $\phi(t_0) = x_0$, $\psi(t_0) = y_0$ and a number $T > 0$ such that $\phi(t_0 + T) = x_0$, $\psi(t_0 + T) = y_0$. Show that $x = \Phi(t) = \phi(t + T)$ and $y = \Psi(t) = \psi(t + T)$ is a solution and then use the existence and uniqueness theorem to show that $\Phi(t) = \phi(t)$ and $\Psi(t) = \psi(t)$ for all t.

9.3 Almost Linear Systems



In Section 9.1 we gave an informal description of the stability properties of the equilibrium solution $\mathbf{x} = \mathbf{0}$ of the two-dimensional linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{1}$$



The results are summarized in Table 9.1.1. Recall that we required that $\det \mathbf{A} \neq 0$, so that $\mathbf{x} = \mathbf{0}$ is the only critical point of the system (1). Now that we have defined the concepts of asymptotic stability, stability, and instability more precisely, we can restate these results in the following theorem.

Theorem 9.3.1 The critical point $\mathbf{x} = \mathbf{0}$ of the linear system (1) is asymptotically stable if the eigenvalues r_1, r_2 are real and negative or have negative real part; stable, but not asymptotically stable, if r_1 and r_2 are pure imaginary; unstable if r_1 and r_2 are real and either is positive, or if they have positive real part.

It is apparent from this theorem or from Table 9.1.1 that the eigenvalues r_1, r_2 of the coefficient matrix $\mathbf{A}$ determine the type of critical point at $\mathbf{x} = \mathbf{0}$ and its stability characteristics. In turn, the values of r_1 and r_2 depend on the coefficients in the system (1). When such a system arises in some applied field, the coefficients usually result from the measurements of certain physical quantities. Such measurements are often subject to small uncertainties, so it is of interest to investigate whether small changes (perturbations) in the coefficients can affect the stability or instability of a critical point and/or significantly alter the pattern of trajectories.

Recall that the eigenvalues r_1, r_2 are the roots of the polynomial equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{2}$$

It is possible to show that *small* perturbations in some or all the coefficients are reflected in *small* perturbations in the eigenvalues. The most sensitive situation occurs when $r_1 = i\mu$ and $r_2 = -i\mu$, that is, when the critical point is a center and the trajectories are closed curves surrounding it. If a slight change is made in the coefficients, then the eigenvalues r_1 and r_2 will take on new values $r_1' = \lambda' + i\mu'$ and $r_2' = \lambda' - i\mu'$, where λ' is small in magnitude and $\mu' \cong \mu$ (see Figure 9.3.1). If $\lambda' \neq 0$, which almost always occurs, then the trajectories of the perturbed system are spirals, rather than closed curves. The system is asymptotically stable if $\lambda' < 0$, but unstable if $\lambda' > 0$. Thus, in the case of a center, small perturbations in the coefficients may well change a stable system into an unstable one, and in any case may be expected to alter radically the pattern of trajectories in the phase plane (see Problem 25).

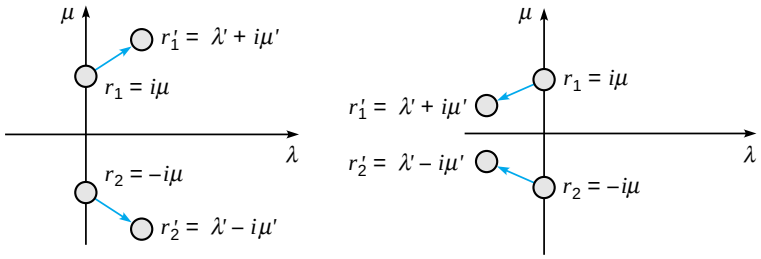


FIGURE 9.3.1 Schematic perturbation of $r_1 = i\mu, r_2 = -i\mu$.

Another slightly less sensitive case occurs if the eigenvalues r_1 and r_2 are equal; in this case the critical point is a node. Small perturbations in the coefficients will normally cause the two equal roots to separate (bifurcate). If the separated roots are real, then the critical point of the perturbed system remains a node, but if the separated roots are complex conjugates, then the critical point becomes a spiral point. These two possibilities are shown schematically in Figure 9.3.2. In this case the stability or instability of the system is not affected by small perturbations in the coefficients, but the trajectories may be altered considerably (see Problem 26).

In all other cases the stability or instability of the system is not changed, nor is the type of critical point altered, by sufficiently small perturbations in the coefficients of the system. For example, if r_1 and r_2 are real, negative, and unequal, then a *small* change in the coefficients will not change the sign of r_1 and r_2 nor allow them to coalesce. Thus the critical point remains an asymptotically stable node.

Now let us consider a nonlinear two-dimensional autonomous system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{3}$$

Our main object is to investigate the behavior of trajectories of the system (3) near a critical point $\mathbf{x}^0$. We will seek to do this by approximating the nonlinear system (3) by an appropriate linear system, whose trajectories are easy to describe. The crucial question is whether the trajectories of the linear system are good approximations to those of the nonlinear system. Of course, we also need to know how to find the approximating linear system.

It is convenient to choose the critical point to be the origin. This involves no loss of generality, since if $\mathbf{x}^0 \neq \mathbf{0}$, it is always possible to make the substitution $\mathbf{u} = \mathbf{x} - \mathbf{x}^0$ in Eq. (3). Then $\mathbf{u}$ will satisfy an autonomous system with a critical point at the origin.

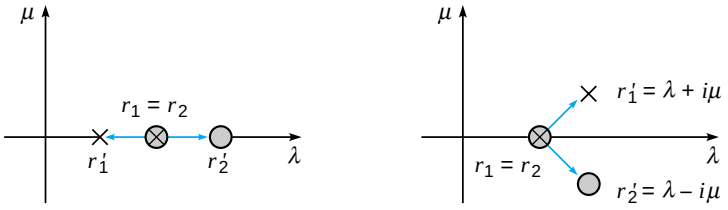


FIGURE 9.3.2 Schematic perturbation of $r_1 = r_2$.

First, let us consider what it means for a nonlinear system (3) to be "close" to a linear system (1). Accordingly, suppose that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}), \tag{4}$$

and that $\mathbf{x} = \mathbf{0}$ is an **isolated** critical point of the system (4). This means that there is some circle about the origin within which there are no other critical points. In addition, we assume that $\det \mathbf{A} \neq 0$, so that $\mathbf{x} = \mathbf{0}$ is also an isolated critical point of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. For the nonlinear system (4) to be close to the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we must assume that $\mathbf{g}(\mathbf{x})$ is small. More precisely, we assume that the components of $\mathbf{g}$ have continuous first partial derivatives and satisfy the limit condition

$$\|\mathbf{g}(\mathbf{x})\|/\|\mathbf{x}\| \to 0 \qquad \text{as} \quad \mathbf{x} \to \mathbf{0}; \tag{5}$$

that is, $\|\mathbf{g}\|$ is small in comparison to $\|\mathbf{x}\|$ itself near the origin. Such a system is called an **almost linear system** in the neighborhood of the critical point $\mathbf{x} = \mathbf{0}$.

It may be helpful to express the condition (5) in scalar form. If we let $\mathbf{x}^T = (x, y)$, then $\|\mathbf{x}\| = (x^2 + y^2)^{1/2} = r$. Similarly, if $\mathbf{g}^T(\mathbf{x}) = (g_1(x, y), g_2(x, y))$, then $\|\mathbf{g(x)}\| = [g_1^2(x, y) + g_2^2(x, y)]^{1/2}$. Then it follows that condition (5) is satisfied if and only if

$$g_1(x, y)/r \to 0, \qquad g_2(x, y)/r \to 0 \qquad \text{as} \quad r \to 0. \tag{6}$$

EXAMPLE 1

Determine whether the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix} \tag{7}$$

is almost linear in the neighborhood of the origin.

Observe that the system (7) is of the form (4), that (0, 0) is a critical point, and that $\det \mathbf{A} \neq 0$. It is not hard to show that the other critical points of Eqs. (7) are (0, 2), (1, 0), and (0.5, 0.5); consequently, the origin is an isolated critical point. In checking the condition (6) it is convenient to introduce polar coordinates by letting $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\begin{aligned} \frac{g_1(x, y)}{r} = \frac{-x^2 - xy}{r} &= \frac{-r^2\cos^2\theta - r^2\sin\theta\cos\theta}{r} \\ &= -r(\cos^2\theta + \sin\theta\cos\theta) \to 0 \end{aligned}$$

as $r \to 0$. In a similar way you can show that $g_2(x, y)/r \to 0$ as $r \to 0$. Hence the system (7) is almost linear near the origin.

EXAMPLE 2

The motion of a pendulum is described by the system [see Eq. (13) of Section 9.2]

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y. \tag{8}$$

The critical points are $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, . . . , so the origin is an isolated critical point of this system. Show that the system is almost linear near the origin.

To compare Eqs. (8) with Eq. (4) we must rewrite the former so that the linear and nonlinear terms are clearly identified. If we write $\sin x = x + (\sin x - x)$ and substitute this expression in the second of Eqs. (8), we obtain the equivalent system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \omega^2 \begin{pmatrix} 0 \\ \sin x - x \end{pmatrix}. \tag{9}$$

On comparing Eq. (9) with Eq. (4) we see that $g_1(x, y) = 0$ and $g_2(x, y) = -\omega^2(\sin x - x)$. From the Taylor series for $\sin x$ we know that $\sin x - x$ behaves like $-x^3/3! = -(r^3 \cos^3\theta)/3!$ when x is small. Consequently, $(\sin x - x)/r \to 0$ as $r \to 0$. Thus the conditions (6) are satisfied and the system (9) is almost linear near the origin.

Let us now return to the general nonlinear system (3), which we write in the scalar form

$$x' = F(x, y), \qquad y' = G(x, y). \tag{10}$$

The system (10) is almost linear in the neighborhood of a critical point (x_0, y_0) whenever the functions F and G have continuous partial derivatives up to order two. To show this, use Taylor expansions about the point (x_0, y_0) to write $F(x, y)$ and $G(x, y)$ in the form

$$\begin{aligned} F(x, y) &= F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1(x, y), \\ G(x, y) &= G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + \eta_2(x, y), \end{aligned}$$

where $\eta_1(x, y)/[(x - x_0)^2 + (y - y_0)^2]^{1/2} \to 0$ as $(x, y) \to (x_0, y_0)$, and similarly for η_2. Note that $F(x_0, y_0) = G(x_0, y_0) = 0$, and that $dx/dt = d(x - x_0)/dt$ and $dy/dt = d(y - y_0)/dt$. Then the system (10) reduces to

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{pmatrix}, \tag{11}$$

or, in vector notation,

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x}^0)\mathbf{u} + \boldsymbol{\eta}(\mathbf{x}), \tag{12}$$

where $\mathbf{u} = (x - x_0, y - y_0)^T$ and $\boldsymbol{\eta} = (\eta_1, \eta_2)^T$.

The significance of this result is twofold. First, if the functions F and G are twice differentiable, then the system (10) is almost linear and it is unnecessary to resort to the limiting process used in Examples 1 and 2. Second, the linear system that approximates the nonlinear system (10) near (x_0, y_0) is given by the linear part of Eqs. (11) or (12), namely,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{13}$$

where $u_1 = x - x_0$ and $u_2 = y - y_0$. Equation (13) provides a simple and general method for finding the linear system corresponding to an almost linear system near a given critical point.

EXAMPLE 3

Use Eq. (13) to find the linear system corresponding to the pendulum equations (8) near the origin; near the critical point $(\pi, 0)$.

In this case

$$F(x, y) = y, \qquad G(x, y) = -\omega^2 \sin x - \gamma y; \tag{14}$$

since these functions are differentiable as many times as necessary, the system (8) is almost linear near each critical point. The derivatives of F and G are

$$F_x = 0, \qquad F_y = 1, \qquad G_x = -\omega^2 \cos x, \qquad G_y = -\gamma. \tag{15}$$

Thus, at the origin the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{16}$$

which agrees with Eq. (9).

Similarly, evaluating the partial derivatives in Eq. (15) at $(\pi, 0)$, we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

where $u = x - \pi$, $v = y$. This is the linear system corresponding to Eqs. (8) near the point $(\pi, 0)$.

We now return to the almost linear system (4). Since the nonlinear term $\mathbf{g}(\mathbf{x})$ is small compared to the linear term $\mathbf{Ax}$ when $\mathbf{x}$ is small, it is reasonable to hope that the trajectories of the linear system (1) are good approximations to those of the nonlinear system (4), at least near the origin. This turns out to be true in many (but not all) cases, as the following theorem states.

Theorem 9.3.2 Let r_1 and r_2 be the eigenvalues of the linear system (1) corresponding to the almost linear system (4). Then the type and stability of the critical point (0, 0) of the linear system (1) and the almost linear system (4) are as shown in Table 9.3.1.

At this stage, the proof of Theorem 9.3.2 is too difficult to give, so we will accept the results without proof. The statements for asymptotic stability and instability follow as a consequence of a result discussed in Section 9.6, and a proof is sketched in Problems 10 to 12 of that section. Essentially, Theorem 9.3.2 says that for small $\mathbf{x}$ (or $\mathbf{x} - \mathbf{x}^0$) the nonlinear terms are also small and do not affect the stability and type of critical point as determined by the linear terms except in two sensitive cases: r_1 and r_2 pure imaginary, and r_1 and r_2 real and equal. Recall that earlier in this section we stated that small perturbations in the coefficients of the linear system (1), and hence in the eigenvalues r_1 and r_2, can alter the type and stability of the critical point only in these two sensitive cases. It is reasonable to expect that the small nonlinear term in Eq. (4) might have a similar substantial effect, at least in these two sensitive cases. This is so, but the main significance of Theorem 9.3.2 is that in *all other cases* the small nonlinear term does not alter the type or stability of the critical point. Thus, except in the two sensitive cases, the type and stability of the critical point of the nonlinear system (4) can be determined from a study of the much simpler linear system (1).

TABLE 9.3.1 Stability and Instability Properties of Linear and Almost Linear Systems

	Linear System		Almost Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Even if the critical point is of the same type as that of the linear system, the trajectories of the almost linear system may be considerably different in appearance from those of the corresponding linear system, except very near the critical point. However, it can be shown that the slopes at which trajectories "enter" or "leave" the critical point are given correctly by the linear equation.

Damped Pendulum. We continue the discussion of the damped pendulum begun in Examples 2 and 3. Near the origin the nonlinear equations (8) are approximated by the linear system (16), whose eigenvalues are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}. \tag{18}$$

The nature of the solutions of Eqs. (8) and (16) depends on the sign of $\gamma^2 - 4\omega^2$ as follows.

1. If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real, unequal, and negative. The critical point $(0, 0)$ is an asymptotically stable node of the linear system (16) and of the almost linear system (8).
2. If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are real, equal, and negative. The critical point $(0, 0)$ is an asymptotically stable (proper or improper) node of the linear system (16). It may be either an asymptotically stable node or spiral point of the almost linear system (8).
3. If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part. The critical point $(0, 0)$ is an asymptotically stable spiral point of the linear system (16) and of the almost linear system (8).

Thus the critical point $(0, 0)$ is a spiral point of the system (8) if the damping is small and a node if the damping is large enough. In either case, the origin is asymptotically stable.

Let us now consider the case $\gamma^2 - 4\omega^2 < 0$, corresponding to small damping, in more detail. The direction of motion on the spirals near $(0, 0)$ can be obtained directly from Eqs. (8). Consider the point at which a spiral intersects the positive y-axis ($x = 0$ and $y > 0$). At such a point it follows from Eqs. (8) that $dx/dt > 0$. Thus the point (x, y) on the trajectory is moving to the right, so the direction of motion on the spirals is clockwise.

The behavior of the pendulum near the critical points $(\pm n\pi, 0)$, with n even, is the same as its behavior near the origin. We expect this on physical grounds since all these critical points correspond to the downward equilibrium position of the pendulum. The conclusion can be confirmed by repeating the analysis carried out above for the origin. Figure 9.3.3 shows the clockwise spirals at a few of these critical points.

Now let us consider the critical point $(\pi, 0)$. Here the nonlinear equations (8) are approximated by the linear system (17), whose eigenvalues are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}. \tag{19}$$

One eigenvalue (r_1) is positive and the other (r_2) is negative. Therefore, regardless of the amount of damping, the critical point $x = \pi$, $y = 0$ is an unstable saddle point of both the linear system (17) and of the almost linear system (8).

To examine the behavior of trajectories near the saddle point $(\pi, 0)$ in more detail we write down the general solution of Eqs. (17), namely,

$$\begin{pmatrix} u \\ v \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ r_1 \end{pmatrix} e^{r_1 t} + C_2 \begin{pmatrix} 1 \\ r_2 \end{pmatrix} e^{r_2 t}, \tag{20}$$

where C_1 and C_2 are arbitrary constants. Since $r_1 > 0$ and $r_2 < 0$, it follows that the solution that approaches zero as $t \to \infty$ corresponds to $C_1 = 0$. For this solution $v/u = r_2$, so the slope of the entering trajectories is negative; one lies in the second quadrant ($C_2 < 0$) and the other lies in the fourth quadrant ($C_2 > 0$). For $C_2 = 0$ we obtain the pair of trajectories "exiting" from the saddle point. These trajectories have slope $r_1 > 0$; one lies in the first quadrant ($C_1 > 0$) and the other lies in the third quadrant ($C_1 < 0$).

The situation is the same at other critical points $(n\pi, 0)$ with n odd. These all correspond to the upward equilibrium position of the pendulum, so we expect them to be unstable. The analysis at $(\pi, 0)$ can be repeated to show that they are saddle points oriented in the same way as the one at $(\pi, 0)$. Diagrams of the trajectories in the neighborhood of two saddle points are shown in Figure 9.3.4.

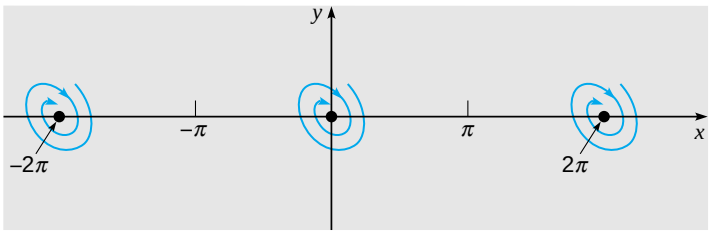


FIGURE 9.3.3 Asymptotically stable spiral points for the damped pendulum.

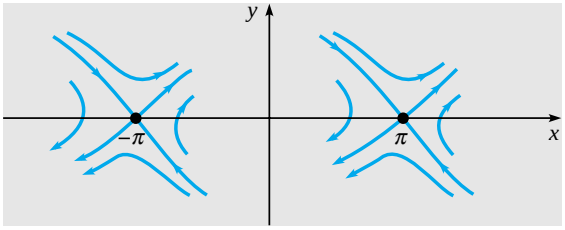


FIGURE 9.3.4 Unstable saddle points for the damped pendulum.

EXAMPLE 4

The equations of motion of a certain pendulum are

$$dx/dt = y, \qquad dy/dt = -9\sin x - \tfrac{1}{5}y, \tag{21}$$

where $x = \theta$ and $y = d\theta/dt$. Draw a phase portrait for this system and explain how it shows the possible motions of the pendulum.

By plotting the trajectories starting at various initial points in the phase plane we obtain the phase portrait shown in Figure 9.3.5. As we have seen, the critical points (equilibrium solutions) are the points $(n\pi, 0)$, where $n = 0, \pm1, \pm2, \ldots$. Even values of n, including zero, correspond to the downward position of the pendulum, while odd values of n correspond to the upward position. Near each of the asymptotically stable critical points the trajectories are clockwise spirals that represent a decaying oscillation about the downward equilibrium position. The wavy horizontal portions of the trajectories that occur for larger values of $|y|$ represent whirling motions of the pendulum. Note that a whirling motion cannot continue indefinitely, no matter how large $|y|$ is; eventually the angular velocity is so much reduced by the damping term that the pendulum can no longer go over the top, and instead begins to oscillate about its downward position.

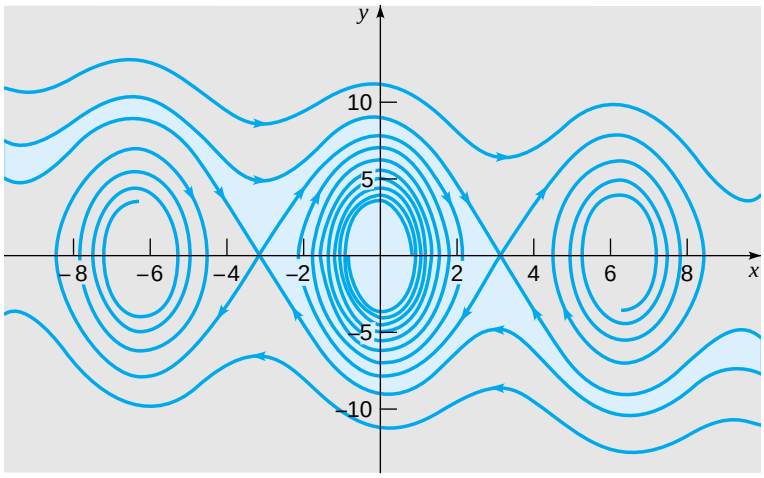


FIGURE 9.3.5 Phase portrait for the damped pendulum of Example 4.

The trajectories that enter the saddle points separate the phase plane into regions. Such a trajectory is called a **separatrix**. Each region contains exactly one of the asymptotically stable spiral points. The initial conditions on θ and $d\theta/dt$ determine the position of an initial point (x, y) in the phase plane. The subsequent motion of the pendulum is represented by the trajectory passing through the initial point as it spirals toward the asymptotically stable critical point in that region. The set of all initial points from which trajectories approach a given asymptotically stable critical point is called the **basin of attraction** or the **region of asymptotic stability** for that critical point. Each asymptotically stable critical point has its own basin of attraction, which is bounded by the separatrices through the neighboring unstable saddle points. The basin of attraction for the origin is shown in blue in Figure 9.3.5. Note that it is mathematically possible (but physically unrealizable) to choose initial conditions on a separatrix so that the resulting motion leads to a balanced pendulum in a vertically upward position of unstable equilibrium.

An important difference between nonlinear autonomous systems and the linear systems discussed in Section 9.1 is illustrated by the pendulum equations. Recall that the linear system (1) has only the single critical point $\mathbf{x} = \mathbf{0}$ if $\det \mathbf{A} \neq 0$. Thus, if the origin is asymptotically stable, then not only do trajectories that start close to the origin approach it, but, in fact, every trajectory approaches the origin. In this case the critical point $\mathbf{x} = \mathbf{0}$ is said to be **globally asymptotically stable**. This property of linear systems is not, in general, true for nonlinear systems. For nonlinear systems an important question is to determine (or to estimate) the basin of attraction for each asymptotically stable critical point.

PROBLEMS

In each of Problems 1 through 4, verify that (0, 0) is a critical point, show that the system is almost linear, and discuss the type and stability of the critical point (0, 0) by examining the corresponding linear system.

1. $dx/dt = x - y^2, \qquad dy/dt = x - 2y + x^2$
2. $dx/dt = -x + y + 2xy, \qquad dy/dt = -4x - y + x^2 - y^2$
3. $dx/dt = (1 + x)\sin y, \qquad dy/dt = 1 - x - \cos y$
4. $dx/dt = x + y^2, \qquad dy/dt = x + y$

In each of Problems 5 through 16:

(a) Determine all critical points of the given system of equations.

(b) Find the corresponding linear system near each critical point.

(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?

(d) Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.

5. $dx/dt = (2 + x)(y - x), \qquad dy/dt = (4 - x)(y + x)$
6. $dx/dt = x - x^2 - xy, \qquad dy/dt = 3y - xy - 2y^2$
7. $dx/dt = 1 - y, \qquad dy/dt = x^2 - y^2$
8. $dx/dt = x - x^2 - xy, \qquad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$
9. $dx/dt = -(x - y)(1 - x - y), \qquad dy/dt = x(2 + y)$
10. $dx/dt = x + x^2 + y^2, \qquad dy/dt = y - xy$
11. $dx/dt = 2x + y + xy^3, \qquad dy/dt = x - 2y - xy$

12. $dx/dt = (1 + x)\sin y, \quad dy/dt = 1 - x - \cos y$
13. $dx/dt = x - y^2, \quad dy/dt = y - x^2$
14. $dx/dt = 1 - xy, \quad dy/dt = x - y^3$
15. $dx/dt = -2x - y - x(x^2 + y^2), \quad dy/dt = x - y + y(x^2 + y^2)$
16. $dx/dt = y + x(1 - x^2 - y^2), \quad dy/dt = -x + y(1 - x^2 - y^2)$
17. Consider the autonomous system

$$dx/dt = y, \qquad dy/dt = x + 2x^3.$$

(a) Show that the critical point (0, 0) is a saddle point.
(b) Sketch the trajectories for the corresponding linear system by integrating the equation for dy/dx. Show from the parametric form of the solution that the only trajectory on which $x \to 0$, $y \to 0$ as $t \to \infty$ is $y = -x$.
(c) Determine the trajectories for the nonlinear system by integrating the equation for dy/dx. Sketch the trajectories for the nonlinear system that correspond to $y = -x$ and $y = x$ for the linear system.

18. Consider the autonomous system

$$dx/dt = x, \qquad dy/dt = -2y + x^3.$$

(a) Show that the critical point (0, 0) is a saddle point.
(b) Sketch the trajectories for the corresponding linear system and show that the trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$ is given by $x = 0$.
(c) Determine the trajectories for the nonlinear system for $x \neq 0$ by integrating the equation for dy/dx. Show that the trajectory corresponding to $x = 0$ for the linear system is unaltered, but that the one corresponding to $y = 0$ is $y = x^3/5$. Sketch several of the trajectories for the nonlinear system.

19. The equation of motion of an undamped pendulum is $d^2\theta/dt^2 + \omega^2 \sin\theta = 0$, where $\omega^2 = g/L$. Let $x = \theta$, $y = d\theta/dt$ to obtain the system of equations

$$dx/dt = y, \qquad dy/dt = -\omega^2 \sin x.$$

(a) Show that the critical points are $(\pm n\pi, 0)$, $n = 0, 1, 2, \ldots$, and that the system is almost linear in the neighborhood of each critical point.
(b) Show that the critical point (0, 0) is a (stable) center of the corresponding linear system. Using Theorem 9.3.2 what can be said about the nonlinear system? The situation is similar at the critical points $(\pm 2n\pi, 0)$, $n = 1, 2, 3, \ldots$. What is the physical interpretation of these critical points?
(c) Show that the critical point $(\pi, 0)$ is an (unstable) saddle point of the corresponding linear system. What conclusion can you draw about the nonlinear system? The situation is similar at the critical points $[\pm(2n - 1)\pi, 0]$, $n = 1, 2, 3, \ldots$. What is the physical interpretation of these critical points?
(d) Choose a value for ω^2 and plot a few trajectories of the nonlinear system in the neighborhood of the origin. Can you now draw any further conclusion about the nature of the critical point at (0, 0) for the nonlinear system?
(e) Using the value of ω^2 from part (d) draw a phase portrait for the pendulum. Compare your plot with Figure 9.3.5 for the damped pendulum.

20. (a) By solving the equation for dy/dx, show that the equation of the trajectories of the undamped pendulum of Problem 19 can be written as

$$\tfrac{1}{2}y^2 + \omega^2(1 - \cos x) = c, \tag{i}$$

where c is a constant of integration.
(b) Multiply Eq. (i) by mL^2. Then express the result in terms of θ to obtain

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos\theta) = E, \tag{ii}$$

where $E = mL^2c$.

(c) Show that the first term in Eq. (ii) is the kinetic energy of the pendulum and that the second term is the potential energy due to gravity. Thus the total energy E of the pendulum is constant along any trajectory; its value is determined by the initial conditions.

▶ 21. The motion of a certain undamped pendulum is described by the equations

$$dx/dt = y, \qquad dy/dt = -4\sin x.$$

If the pendulum is set in motion with an angular displacement A and no initial velocity, then the initial conditions are $x(0) = A$, $y(0) = 0$.
(a) Let $A = 0.25$ and plot x versus t. From the graph estimate the amplitude R and period T of the resulting motion of the pendulum.
(b) Repeat part (a) for $A = 0.5$, 1.0, 1.5, and 2.0.
(c) How do the amplitude and period of the pendulum's motion depend on the initial position A? Draw a graph to show each of these relationships. Can you say anything about the limiting value of the period as $A \to 0$?
(d) Let $A = 4$ and plot x versus t. Explain why this graph differs from those in parts (a) and (b). For what value of A does the transition take place?

▶ 22. Consider again the pendulum equations (see Problem 21)

$$dx/dt = y, \qquad dy/dt = -4\sin x.$$

If the pendulum is set in motion from its downward equilibrium position with angular velocity v, then the initial conditions are $x(0) = 0$, $y(0) = v$.
(a) Plot x versus t for $v = 2$ and also for $v = 5$. Explain the differing motions of the pendulum that these two graphs represent.
(b) There is a critical value of v, which we denote by v_c, such that one type of motion occurs for $v < v_c$ and the other for $v > v_c$. Estimate the value of v_c.

▶ 23. This problem extends Problem 22 to a damped pendulum. The equations of motion are

$$dx/dt = y, \qquad dy/dt = -4\sin x - \gamma y,$$

where γ is the damping coefficient, with the initial conditions $x(0) = 0$, $y(0) = v$.
(a) For $\gamma = 1/4$ plot x versus t for $v = 2$ and for $v = 5$. Explain these plots in terms of the motions of the pendulum that they represent. Also explain how they relate to the corresponding graphs in Problem 22(a).
(b) Estimate the critical value v_c of the initial velocity where the transition from one type of motion to the other occurs.
(c) Repeat part (b) for other values of γ and determine how v_c depends on γ.

24. Theorem 9.3.2 provides no information about the stability of a critical point of an almost linear system if that point is a center of the corresponding linear system. That this must be the case is illustrated by the systems

$$\begin{aligned} dx/dt &= y + x(x^2 + y^2), \\ dy/dt &= -x + y(x^2 + y^2) \end{aligned} \tag{i}$$

and

$$\begin{aligned} dx/dt &= y - x(x^2 + y^2), \\ dy/dt &= -x - y(x^2 + y^2). \end{aligned} \tag{ii}$$

(a) Show that (0, 0) is a critical point of each system and, furthermore, is a center of the corresponding linear system.
(b) Show that each system is almost linear.
(c) Let $r^2 = x^2 + y^2$, and note that $x\,dx/dt + y\,dy/dt = r\,dr/dt$. For system (ii) show that $dr/dt < 0$ and that $r \to 0$ as $t \to \infty$; hence the critical point is asymptotically stable. For system (i) show that the solution of the initial value problem for r with $r = r_0$ at $t = 0$ becomes unbounded as $t \to 1/2r_0^2$, and hence the critical point is unstable.

25. In this problem we show how small changes in the coefficients of a system of linear equations can affect a critical point that is a center. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

Show that the eigenvalues are $\pm i$ so that $(0, 0)$ is a center. Now consider the system

$$\mathbf{x}' = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix} \mathbf{x},$$

where $|\epsilon|$ is arbitrarily small. Show that the eigenvalues are $\epsilon \pm i$. Thus no matter how small $|\epsilon| \neq 0$ is, the center becomes a spiral point. If $\epsilon < 0$, the spiral point is asymptotically stable; if $\epsilon > 0$, the spiral point is unstable.

26. In this problem we show how small changes in the coefficients of a system of linear equations can affect the nature of a critical point when the eigenvalues are equal. Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}.$$

Show that the eigenvalues are $r_1 = -1$, $r_2 = -1$ so that the critical point $(0, 0)$ is an asymptotically stable node. Now consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -\epsilon & -1 \end{pmatrix} \mathbf{x},$$

where $|\epsilon|$ is arbitrarily small. Show that if $\epsilon > 0$, then the eigenvalues are $-1 \pm i\sqrt{\epsilon}$, so that the asymptotically stable node becomes an asymptotically stable spiral point. If $\epsilon < 0$, then the roots are $-1 \pm \sqrt{|\epsilon|}$, and the critical point remains an asymptotically stable node.

▶ 27. In this problem we derive a formula for the natural period of an undamped nonlinear pendulum [$c = 0$ in Eq. (10) of Section 9.2]. Suppose that the bob is pulled through a positive angle α and then released with zero velocity.

(a) We usually think of θ and $d\theta/dt$ as functions of t. However, if we reverse the roles of t and θ, we can regard t as a function of θ, and consequently also think of $d\theta/dt$ as a function of θ. Then derive the following sequence of equations:

$$\frac{1}{2}mL^2\frac{d}{d\theta}\left[\left(\frac{d\theta}{dt}\right)^2\right] = -mgL\sin\theta,$$

$$\frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 = mgL(\cos\theta - \cos\alpha),$$

$$dt = -\sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Why was the negative square root chosen in the last equation?

(b) If T is the natural period of oscillation, derive the formula

$$\frac{T}{4} = -\sqrt{\frac{L}{2g}}\int_\alpha^0 \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

(c) By using the identities $\cos\theta = 1 - 2\sin^2(\theta/2)$ and $\cos\alpha = 1 - 2\sin^2(\alpha/2)$, followed by the change of variable $\sin(\theta/2) = k\sin\phi$ with $k = \sin(\alpha/2)$, show that

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

The integral is called the **elliptic integral** of the first kind. Note that the period depends on the ratio L/g and also the initial displacement α through $k = \sin(\alpha/2)$.
(d) By evaluating the integral in the expression for T obtain values for T that you can compare with the graphical estimates you obtained in Problem 21.

28. A generalization of the damped pendulum equation discussed in the text, or a damped spring–mass system, is the Liénard equation

$$\frac{d^2x}{dt^2} + c(x)\frac{dx}{dt} + g(x) = 0.$$

If $c(x)$ is a constant and $g(x) = kx$, then this equation has the form of the linear pendulum equation [replace $\sin\theta$ with θ in Eq. (12) of Section 9.2]; otherwise the damping force $c(x)\,dx/dt$ and restoring force $g(x)$ are nonlinear. Assume that c is continuously differentiable, g is twice continuously differentiable, and $g(0) = 0$.
(a) Write the Liénard equation as a system of two first order equations by introducing the variable $y = dx/dt$.
(b) Show that (0, 0) is a critical point and that the system is almost linear in the neighborhood of (0, 0).
(c) Show that if $c(0) > 0$ and $g'(0) > 0$, then the critical point is asymptotically stable, and that if $c(0) < 0$ or $g'(0) < 0$, then the critical point is unstable.
Hint: Use Taylor series to approximate c and g in the neighborhood of $x = 0$.

9.4 Competing Species





In this section and the next we explore the application of phase plane analysis to some problems in population dynamics. These problems involve two interacting populations and are extensions of those discussed in Section 2.5, which dealt with a single population. While the equations discussed here are extremely simple, compared to the very complex relationships that exist in nature, it is still possible to acquire some insight into ecological principles from a study of these model problems.

Suppose that in some closed environment there are two similar species competing for a limited food supply; for example, two species of fish in a pond that do not prey on each other, but do compete for the available food. Let x and y be the populations of the two species at time t. As discussed in Section 2.5, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation. Thus

$$dx/dt = x(\epsilon_1 - \sigma_1 x), \tag{1a}$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y), \tag{1b}$$

respectively, where ϵ_1 and ϵ_2 are the growth rates of the two populations, and ϵ_1/σ_1 and ϵ_2/σ_2 are their saturation levels. However, when both species are present, each will impinge on the available food supply for the other. In effect, they reduce the growth rates and saturation populations of each other. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $\epsilon_1 - \sigma_1 x$ in Eq. (1a) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$, where α_1 is a measure of the degree to

which species y interferes with species x. Similarly, in Eq. (1b) we replace $\epsilon_2 - \sigma_2 y$ by $\epsilon_2 - \sigma_2 y - \alpha_2 x$. Thus we have the system of equations

$$\begin{aligned} dx/dt &= x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \\ dy/dt &= y(\epsilon_2 - \sigma_2 y - \alpha_2 x). \end{aligned} \tag{2}$$

The values of the positive constants ϵ_1, σ_1, α_1, ϵ_2, σ_2, and α_2 depend on the particular species under consideration and in general must be determined from observations. We are interested in solutions of Eqs. (2) for which x and y are nonnegative. In the following two examples we discuss two typical problems in some detail. At the end of the section we return to the general equations (2).

EXAMPLE 1

Discuss the qualitative behavior of solutions of the system

$$\begin{aligned} dx/dt &= x(1 - x - y), \\ dy/dt &= y(0.75 - y - 0.5x). \end{aligned} \tag{3}$$

We find the critical points by solving the system of algebraic equations

$$x(1 - x - y) = 0, \qquad y(0.75 - y - 0.5x) = 0. \tag{4}$$

There are four points that satisfy Eqs. (4), namely, (0, 0), (0, 0.75), (1, 0), and (0.5, 0.5); they correspond to equilibrium solutions of the system (3). The first three of these points involve the extinction of one or both species; only the last corresponds to the long-term survival of both species. Other solutions are represented as curves or trajectories in the xy-plane that describe the evolution of the populations in time. To begin to discover their qualitative behavior we can proceed in the following way.

A direction field for the system (3) in the positive quadrant is shown in Figure 9.4.1; the heavy dots in this figure are the critical points or equilibrium solutions. Based on the direction field it appears that the point $(0.5, 0.5)$ attracts other solutions and is therefore asymptotically stable, while the other three critical points are unstable. To confirm these conclusions we can look at the linear approximations near each critical point.

The system (3) is almost linear in the neighborhood of each critical point. There are two ways to obtain the linear system near a critical point (X, Y). First, we can use the substitution $x = X + u$, $y = Y + v$ in Eqs. (3), retaining only the terms that are linear in u and v. Alternatively, we can use Eq. (13) of Section 9.3, that is,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_x(X, Y) & F_y(X, Y) \\ G_x(X, Y) & G_y(X, Y) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{5}$$

where, for the system (3),

$$F(x, y) = x(1 - x - y), \qquad G(x, y) = y(0.75 - y - 0.5x). \tag{6}$$

Thus Eq. (5) becomes

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 - 2X - Y & -X \\ -0.5Y & 0.75 - 2Y - 0.5X \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{7}$$

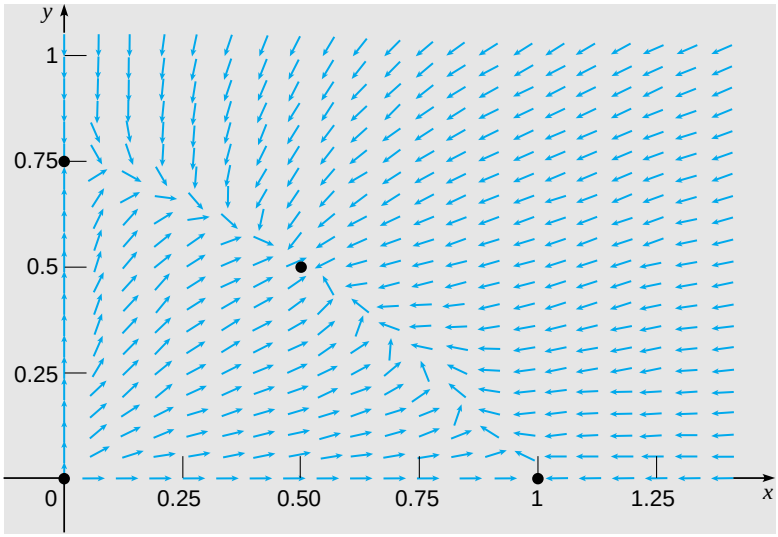


FIGURE 9.4.1 Critical points and direction field for the system (3).

$x = 0$, $y = 0$. This critical point corresponds to a state in which both species die as a result of their competition. By rewriting the system (3) in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^2 + xy \\ 0.5xy + y^2 \end{pmatrix}, \tag{8}$$

or by setting $X = Y = 0$ in Eq. (7), we see that near the origin the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

The eigenvalues and eigenvectors of the system (9) are

$$r_1 = 1, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = 0.75, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{10}$$

so the general solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{0.75t}. \tag{11}$$

Thus the origin is an unstable node of both the linear system (9) and the nonlinear system (8) or (3). In the neighborhood of the origin all trajectories are tangent to the y-axis except for one trajectory that lies along the x-axis.

$x = 1$, $y = 0$. This corresponds to a state in which species x survives the competition, but species y does not. The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{12}$$

Its eigenvalues and eigenvectors are

$$r_1 = -1, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = 0.25, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \tag{13}$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{0.25t}. \tag{14}$$

Since the eigenvalues have opposite signs, the point (1, 0) is a saddle point, and hence is an unstable equilibrium point of the linear system (12) and of the nonlinear system (3). The behavior of the trajectories near (1, 0) can be seen from Eq. (14). If $c_2 = 0$, then there is one pair of trajectories that approaches the critical point along the x-axis. All other trajectories depart from the neighborhood of (1, 0).

$x = 0$, $y = 0.75$. In this case species y survives but x does not. The analysis is similar to that for the point (1, 0). The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{15}$$

The eigenvalues and eigenvectors are

$$r_1 = 0.25, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}; \qquad r_2 = -0.75, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{16}$$

so the general solution of Eq. (15) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -3 \end{pmatrix} e^{0.25t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.75t}. \tag{17}$$

Thus the point (0, 0.75) is also a saddle point. All trajectories leave the neighborhood of this point except one pair that approaches along the y-axis.

$x = 0.5$, $y = 0.5$. This critical point corresponds to a mixed equilibrium state, or coexistence, in the competition between the two species. The eigenvalues and eigenvectors of the corresponding linear system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{18}$$

are

$$\begin{aligned} r_1 &= (-2 + \sqrt{2})/4 \cong -0.146, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}; \\ r_2 &= (-2 - \sqrt{2})/4 \cong -0.854, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \end{aligned} \tag{19}$$

Therefore the general solution of Eq. (18) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-0.146t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{-0.854t}. \tag{20}$$

Since both eigenvalues are negative, the critical point (0.5, 0.5) is an asymptotically stable node of the linear system (18) and of the nonlinear system (3). All trajectories approach the critical point as $t \to \infty$. One pair of trajectories approaches the critical

point along the line with slope $\sqrt{2}/2$ determined from the eigenvector $\boldsymbol{\xi}^{(2)}$. All other trajectories approach the critical point tangent to the line with slope $-\sqrt{2}/2$ determined from the eigenvector $\boldsymbol{\xi}^{(1)}$.

A phase portrait for the system (3) is shown in Figure 9.4.2. By looking closely at the trajectories near each critical point you can see that they behave in the manner predicted by the linear system near that point. In addition, note that the quadratic terms on the right side of Eqs. (3) are all negative. Since for x and y large and positive these terms are the dominant ones, it follows that far from the origin in the first quadrant both x' and y' are negative, that is, the trajectories are directed inward. Thus all trajectories that start at a point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$ eventually approach the point (0.5, 0.5).

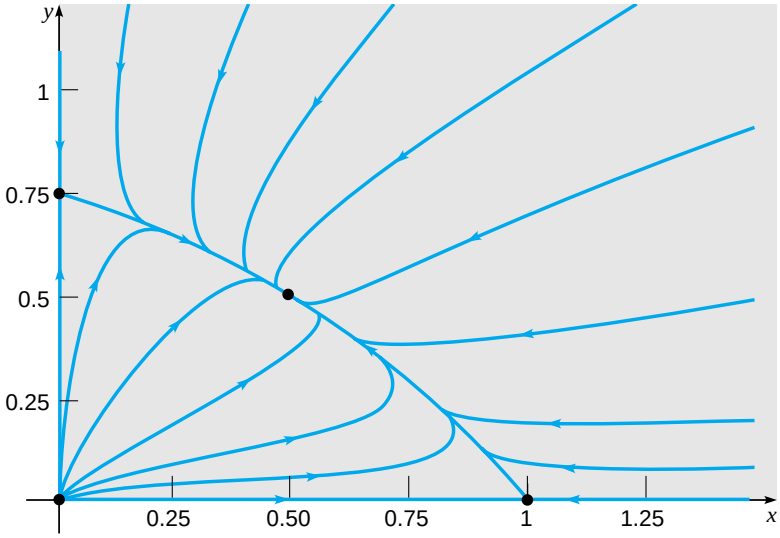


FIGURE 9.4.2 A phase portrait of the system (3).

EXAMPLE 2

Discuss the qualitative behavior of the solutions of the system

$$\begin{aligned} dx/dt &= x(1 - x - y), \\ dy/dt &= y(0.5 - 0.25y - 0.75x), \end{aligned} \tag{21}$$

when x and y are nonnegative. Observe that this system is also a special case of the system (2) for two competing species.

Once again, there are four critical points, namely, (0, 0), (1, 0), (0, 2), and (0.5, 0.5), corresponding to equilibrium solutions of the system (21). Figure 9.4.3 shows a direction field for the system (21), together with the four critical points. From the direction field it appears that the mixed equilibrium solution (0.5, 0.5) is a saddle point, and therefore unstable, while the points (1, 0) and (0, 2) are asymptotically stable. Thus, for competition described by Eqs. (21), one species will eventually overwhelm the other and drive it to extinction. The surviving species is determined by the initial state of the system. To confirm these conclusions we can look at the linear approximations near each critical point.

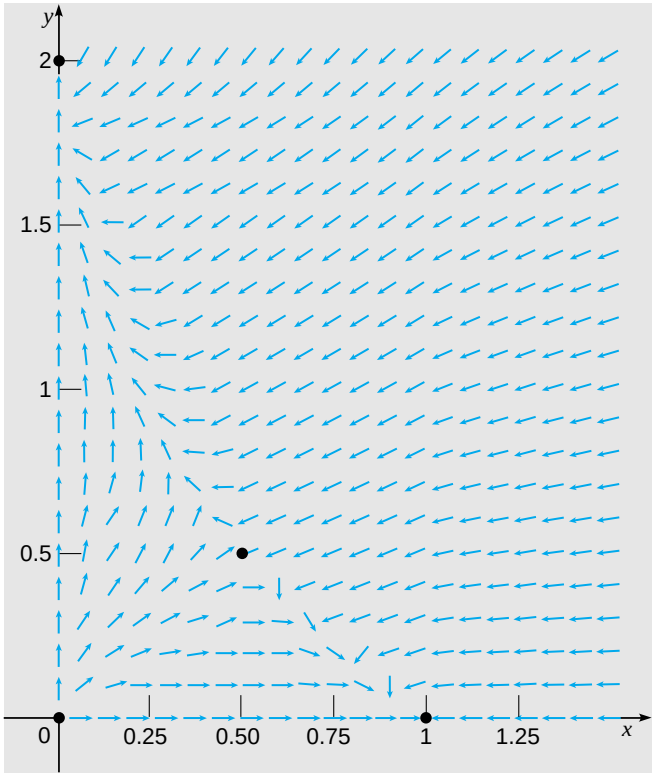


FIGURE 9.4.3 Critical points and direction field for the system (21).

$x = 0$, $y = 0$. Neglecting the nonlinear terms in Eqs. (21), we obtain the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{22}$$

which is valid near the origin. The eigenvalues and eigenvectors of the system (22) are

$$r_1 = 1, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = 0.5, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{23}$$

so the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{0.5t}. \tag{24}$$

Therefore the origin is an unstable node of the linear system (22) and also of the nonlinear system (21). All trajectories leave the origin tangent to the y-axis except for one trajectory that lies along the x-axis.

$x = 1$, $y = 0$. The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{25}$$

Its eigenvalues and eigenvectors are

$$r_1 = -1, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = -0.25, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \tag{26}$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{-0.25t}. \tag{27}$$

The point (1, 0) is an asymptotically stable node of the linear system (25) and of the nonlinear system (21). If the initial values of x and y are sufficiently close to (1, 0), then the interaction process will lead ultimately to that state, that is, to the survival of species x and the extinction of species y. There is one pair of trajectories that approaches the critical point along the x-axis. All other trajectories approach (1, 0) tangent to the line with slope $-3/4$ that is determined by the eigenvector $\boldsymbol{\xi}^{(2)}$.

$x = 0$, $y = 2$. The analysis in this case is similar to that for the point (1, 0). The appropriate linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{28}$$

The eigenvalues and eigenvectors of this system are

$$r_1 = -1, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \qquad r_2 = -0.5, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{29}$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.5t}. \tag{30}$$

Thus the critical point (0, 2) is an asymptotically stable node of both the linear system (28) and the nonlinear system (21). All trajectories approach the critical point along the y-axis except for one trajectory that approaches along the line with slope 3.

$x = 0.5$, $y = 0.5$. The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.375 & -0.125 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{31}$$

The eigenvalues and eigenvectors are

$$\begin{aligned} r_1 &= \frac{-5 + \sqrt{57}}{16} \cong 0.1594, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ (-3 - \sqrt{57})/8 \end{pmatrix} \cong \begin{pmatrix} 1 \\ -1.3187 \end{pmatrix}, \\ r_2 &= \frac{-5 - \sqrt{57}}{16} \cong -0.7844, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ (-3 + \sqrt{57})/8 \end{pmatrix} \cong \begin{pmatrix} 1 \\ 0.5687 \end{pmatrix}, \end{aligned} \tag{32}$$

so the general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1.3187 \end{pmatrix} e^{0.1594t} + c_2 \begin{pmatrix} 1 \\ 0.5687 \end{pmatrix} e^{-0.7844t}. \tag{33}$$

Since the eigenvalues are of opposite sign, the critical point (0.5, 0.5) is a saddle point, and therefore is unstable, as we had surmised earlier. One pair of trajectories approaches

the critical point as $t \to \infty$; the others depart from it. As they approach the critical point, the entering trajectories are tangent to the line with slope $(\sqrt{57} - 3)/8 \cong 0.5687$ determined from the eigenvector $\boldsymbol{\xi}^{(2)}$.

A phase portrait for the system (21) is shown in Figure 9.4.4. Near each of the critical points the trajectories of the nonlinear system behave as predicted by the corresponding linear approximation. Of particular interest is the pair of trajectories that enter the saddle point. These trajectories form a separatrix that divides the first quadrant into two basins of attraction. Trajectories starting above the separatrix ultimately approach the node at (0, 2), while trajectories starting below the separatrix approach the node at Bad math construct!(1,0)Bad math construct!. If the initial state lies precisely on the separatrix, then the solution (x, y) will approach the saddle point as $t \to \infty$. However, the slightest perturbation as one follows this trajectory will dislodge the point (x, y) from the separatrix and cause it to approach one of the nodes instead. Thus, in practice, one species will survive the competition and the other will not.

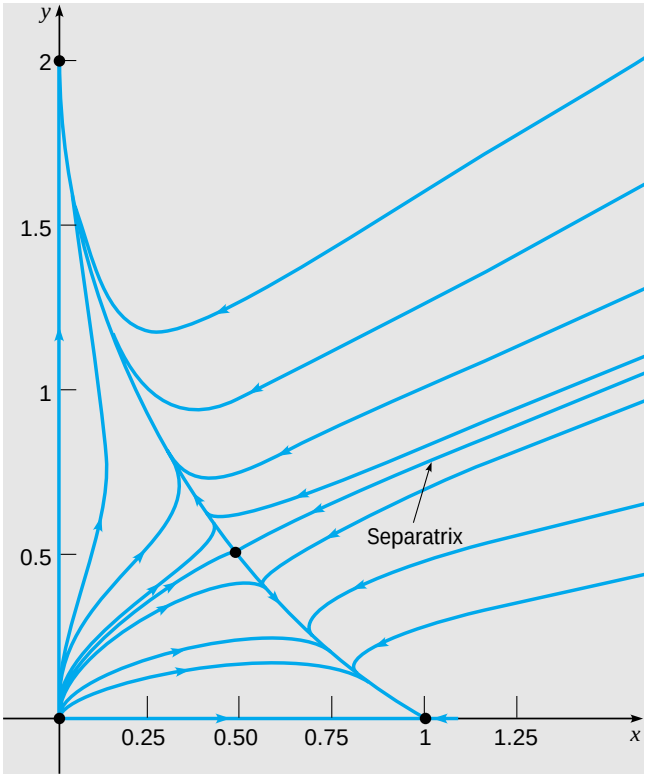


FIGURE 9.4.4 A phase portrait of the system (21).

Examples 1 and 2 show that in some cases the competition between two species leads to an equilibrium state of coexistence, while in other cases the competition results in the eventual extinction of one of the species. To understand more clearly how and why this happens, and to learn how to predict which situation will occur, it is useful to look

again at the general system (2). There are four cases to be considered, depending on the relative orientation of the lines

$$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0 \quad \text{and} \quad \epsilon_2 - \sigma_2 y - \alpha_2 x = 0, \tag{34}$$

as shown in Figure 9.4.5. These lines are called the x and y **nullclines**, respectively, because x' is zero on the first and y' is zero on the second. Let (X, Y) denote any critical point in any one of the four cases. As in Examples 1 and 2 the system (2) is almost linear in the neighborhood of this point because the right side of each differential equation is a quadratic polynomial. To study the system (2) in the neighborhood of this critical point we can look at the corresponding linear system obtained from Eq. (13) of Section 9.3,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon_1 - 2\sigma_1 X - \alpha_1 Y & -\alpha_1 X \\ -\alpha_2 Y & \epsilon_2 - 2\sigma_2 Y - \alpha_2 X \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{35}$$

We now use Eq. (35) to determine the conditions under which the model described by Eqs. (2) permits the coexistence of the two species x and y. Of the four possible cases shown in Figure 9.4.5 coexistence is possible only in cases (*c*) and (*d*). In these cases the nonzero values of X and Y are readily obtained by solving the algebraic equations (34); the result is

$$X = \frac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \qquad Y = \frac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}. \tag{36}$$

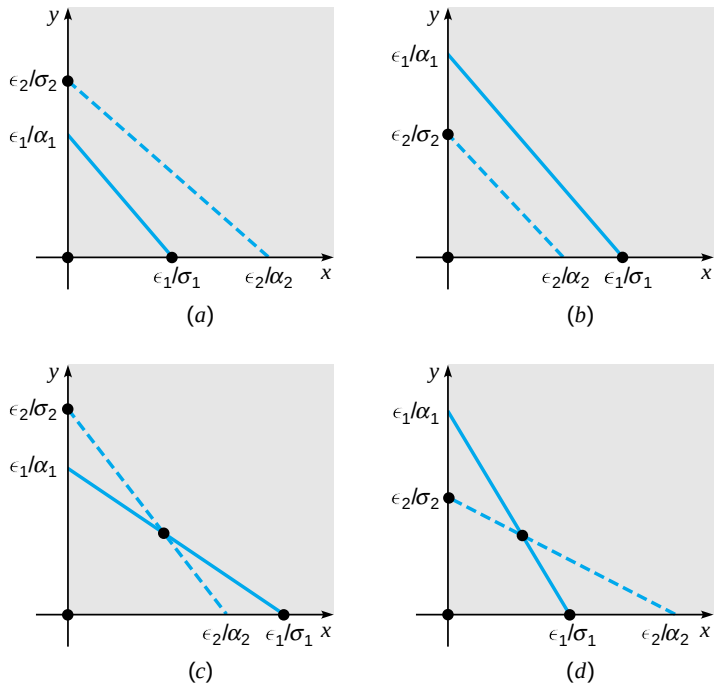


FIGURE 9.4.5 The various cases for the competing species system (2).

Further, since $\epsilon_1 - \sigma_1 X - \alpha_1 Y = 0$ and $\epsilon_2 - \sigma_2 Y - \alpha_2 X = 0$, Eq. (35) immediately reduces to

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sigma_1 X & -\alpha_1 X \\ -\alpha_2 Y & -\sigma_2 Y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{37}$$

The eigenvalues of the system (37) are found from the equation

$$r^2 + (\sigma_1 X + \sigma_2 Y)r + (\sigma_1\sigma_2 - \alpha_1\alpha_2)XY = 0. \tag{38}$$

Thus

$$r_{1,2} = \frac{-(\sigma_1 X + \sigma_2 Y) \pm \sqrt{(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY}}{2}. \tag{39}$$

If $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$, then the radicand of Eq. (39) is positive and greater than $(\sigma_1 X + \sigma_2 Y)^2$. Thus the eigenvalues are real and of opposite sign. Consequently, the critical point (X, Y) is an (unstable) saddle point, and coexistence is not possible. This is the case in Example 2, where $\sigma_1 = 1$, $\alpha_1 = 1$, $\sigma_2 = 0.25$, $\alpha_2 = 0.75$, and $\sigma_1\sigma_2 - \alpha_1\alpha_2 = -0.5$.

On the other hand, if $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$, then the radicand of Eq. (39) is less than $(\sigma_1 X + \sigma_2 Y)^2$. Thus the eigenvalues are real, negative, and unequal, or complex with negative real part. A straightforward analysis of the radicand of Eq. (39) shows that the eigenvalues cannot be complex (see Problem 7). Thus the critical point is an asymptotically stable node, and sustained coexistence is possible. This is illustrated by Example 1, where $\sigma_1 = 1$, $\alpha_1 = 1$, $\sigma_2 = 1$, $\alpha_2 = 0.5$, and $\sigma_1\sigma_2 - \alpha_1\alpha_2 = 0.5$.

Let us relate this result to Figures 9.4.5*c* and 9.4.5*d*. In Figure 9.4.5*c* we have

$$\frac{\epsilon_1}{\sigma_1} > \frac{\epsilon_2}{\alpha_2} \quad \text{or} \quad \epsilon_1\alpha_2 > \epsilon_2\sigma_1 \quad \text{and} \quad \frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1} \quad \text{or} \quad \epsilon_2\alpha_1 > \epsilon_1\sigma_2. \tag{40}$$

These inequalities, coupled with the condition that X and Y given by Eqs. (36) be positive, yield the inequality $\sigma_1\sigma_2 < \alpha_1\alpha_2$. Hence in this case the critical point is a saddle point. On the other hand, in Figure 9.4.5*d* we have

$$\frac{\epsilon_1}{\sigma_1} < \frac{\epsilon_2}{\alpha_2} \quad \text{or} \quad \epsilon_1\alpha_2 < \epsilon_2\sigma_1 \quad \text{and} \quad \frac{\epsilon_2}{\sigma_2} < \frac{\epsilon_1}{\alpha_1} \quad \text{or} \quad \epsilon_2\alpha_1 < \epsilon_1\sigma_2. \tag{41}$$

Now the condition that X and Y are positive yields $\sigma_1\sigma_2 > \alpha_1\alpha_2$. Hence the critical point is asymptotically stable. For this case we can also show that the other critical points $(0, 0)$, $(\epsilon_1/\sigma_1, 0)$, and $(0, \epsilon_2/\sigma_2)$ are unstable. Thus for any positive initial values of x and y the two populations approach the equilibrium state of coexistence given by Eqs. (36).

Equations (2) provide the biological interpretation of the result that coexistence occurs or not depending on whether $\sigma_1\sigma_2 - \alpha_1\alpha_2$ is positive or negative. The σ's are a measure of the inhibitory effect the growth of each population has on itself, while the α's are a measure of the inhibiting effect the growth of each population has on the other species. Thus, when $\sigma_1\sigma_2 > \alpha_1\alpha_2$, interaction (competition) is "weak" and the species can coexist; when $\sigma_1\sigma_2 < \alpha_1\alpha_2$, interaction (competition) is "strong" and the species cannot coexist—one must die out.

PROBLEMS

Each of Problems 1 through 6 can be interpreted as describing the interaction of two species with populations x and y. In each of these problems carry out the following steps.

(a) Draw a direction field and describe how solutions seem to behave.

(b) Find the critical points.

(c) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.

(d) Sketch the trajectories in the neighborhood of each critical point.

(e) Compute and plot enough trajectories of the given system to show clearly the behavior of the solutions.

(f) Determine the limiting behavior of x and y as $t \to \infty$ and interpret the results in terms of the populations of the two species.

1. $dx/dt = x(1.5 - x - 0.5y)$
   $dy/dt = y(2 - y - 0.75x)$
2. $dx/dt = x(1.5 - x - 0.5y)$
   $dy/dt = y(2 - 0.5y - 1.5x)$
3. $dx/dt = x(1.5 - 0.5x - y)$
   $dy/dt = y(2 - y - 1.125x)$
4. $dx/dt = x(1.5 - 0.5x - y)$
   $dy/dt = y(0.75 - y - 0.125x)$
5. $dx/dt = x(1 - x - y)$
   $dy/dt = y(1.5 - y - x)$
6. $dx/dt = x(1 - x + 0.5y)$
   $dy/dt = y(2.5 - 1.5y + 0.25x)$
7. Show that

$$(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY = (\sigma_1 X - \sigma_2 Y)^2 + 4\alpha_1\alpha_2 XY.$$

Hence conclude that the eigenvalues given by Eq. (39) can never be complex.

8. Two species of fish that compete with each other for food, but do not prey on each other, are bluegill and redear. Suppose that a pond is stocked with bluegill and redear and let x and y be the populations of bluegill and redear, respectively, at time t. Suppose further that the competition is modeled by the equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y),$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

(a) If $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?
(b) If $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?

9. Consider the competition between bluegill and redear mentioned in Problem 8. Suppose that $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, so, as shown in the text, there is a stable equilibrium point at which both species can coexist. It is convenient to rewrite the equations of Problem 8 in terms of the carrying capacities of the pond for bluegill ($B = \epsilon_1/\sigma_1$) in the absence of redear and for redear ($R = \epsilon_2/\sigma_2$) in the absence of bluegill.
(a) Show that the equations of Problem 8 take the form

$$\frac{dx}{dt} = \epsilon_1 x\left(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y\right), \qquad \frac{dy}{dt} = \epsilon_2 y\left(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x\right),$$

where $\gamma_1 = \alpha_1/\sigma_1$ and $\gamma_2 = \alpha_2/\sigma_2$. Determine the coexistence equilibrium point (X, Y) in terms of B, R, γ_1, and γ_2.
(b) Now suppose that a fisherman fishes only for bluegill with the effect that B is reduced. What effect does this have on the equilibrium populations? Is it possible, by fishing, to reduce the population of bluegill to such a level that they will die out?

10. Consider the system (2) in the text, and assume that $\sigma_1\sigma_2 - \alpha_1\alpha_2 = 0$.

(a) Find all the critical points of the system. Observe that the result depends on whether or not $\sigma_1\epsilon_2 - \alpha_2\epsilon_1$ is zero.
(b) If $\sigma_1\epsilon_2 - \alpha_2\epsilon_1 > 0$, classify each critical point and determine whether it is asymptotically stable, stable, or unstable. Note that Problem 5 is of this type. Then do the same if $\sigma_1\epsilon_2 - \alpha_2\epsilon_1 < 0$.
(c) Analyze the nature of the trajectories when $\sigma_1\epsilon_2 - \alpha_2\epsilon_1 = 0$.

11. Consider the system (3) in Example 1 of the text. Recall that this system has an asymptotically stable critical point at (0.5, 0.5), corresponding to the stable coexistence of the two population species. Now suppose that immigration or emigration occurs at the constant rates of δa and δb for the species x and y, respectively. In this case Eqs. (3) are replaced by

$$\begin{aligned} dx/dt &= x(1 - x - y) + \delta a, \\ dy/dt &= y(0.75 - y - 0.5x) + \delta b. \end{aligned} \tag{i}$$

The question is what effect this has on the location of the stable equilibrium point.
(a) To find the new critical point we must solve the equations

$$\begin{aligned} x(1 - x - y) + \delta a &= 0, \\ y(0.75 - y - 0.5x) + \delta b &= 0. \end{aligned} \tag{ii}$$

One way to proceed is to assume that x and y are given by power series in the parameter δ; thus

$$x = x_0 + x_1\delta + \cdots, \qquad y = y_0 + y_1\delta + \cdots. \tag{iii}$$

Substitute Eqs. (iii) into Eqs. (ii) and collect terms according to powers of δ.
(b) From the constant terms (the terms not involving δ) show that $x_0 = 0.5$ and $y_0 = 0.5$, thus confirming that in the absence of immigration or emigration the critical point is (0.5, 0.5).
(c) From the terms that are linear in δ show that

$$x_1 = 4a - 4b, \qquad y_1 = -2a + 4b. \tag{iv}$$

(d) Suppose that $a > 0$ and $b > 0$ so that immigration occurs for both species. Show that the resulting equilibrium solution may represent an increase in both populations, or an increase in one but a decrease in the other. Explain intuitively why this is a reasonable result.

▶ 12. Suppose that a certain pair of competing species are described by the system

$$dx/dt = x(4 - x - y), \qquad dy/dt = y(2 + 2\alpha - y - \alpha x),$$

where $\alpha > 0$ is a parameter.
(a) Find the critical points. Note that (2, 2) is a critical point for all values of α.
(b) Determine the nature of the critical point (2, 2) for $\alpha = 0.75$ and for $\alpha = 1.25$. There is a value of α between 0.75 and 1.25 where the nature of the critical point changes abruptly. Denote this value by α_0; it is called a bifurcation point.
(c) Find the approximate linear system near the point (2, 2) in terms of α.
(d) Find the eigenvalues of the linear system in part (c) as functions of α. Then determine the bifurcation point α_0.
(e) Draw phase portraits near (2, 2) for $\alpha = \alpha_0$ and for values of α slightly less than, and slightly greater than, α_0. Explain how the transition in the phase portrait takes place as α passes through α_0.

▶ 13. The system

$$x' = -y, \quad y' = -\gamma y - x(x - 0.15)(x - 2)$$

results from an approximation to the Hodgkin–Huxley[2] equations, which model the transmission of neural impulses along an axon.
(a) Find the critical points and classify them by investigating the approximate linear system near each one.
(b) Draw phase portraits for $\gamma = 0.8$ and for $\gamma = 1.5$.
(c) Consider the trajectory that leaves the critical point $(2, 0)$. Find the value of γ for which this trajectory ultimately approaches the origin as $t \to \infty$. Draw a phase portrait for this value of γ.

9.5 Predator–Prey Equations





In the preceding section we discussed a model of two species that interact by competing for a common food supply or other natural resource. In this section we investigate the situation in which one species (the predator) preys on the other species (the prey), while the prey lives on a different source of food. For example, consider foxes and rabbits in a closed forest: The foxes prey on the rabbits, the rabbits live on the vegetation in the forest. Other examples are bass in a lake as predators and redear as prey, or ladybugs as predators and aphids as prey. We emphasize again that a model involving only two species cannot fully describe the complex relationships among species that actually occur in nature. Nevertheless, the study of simple models is the first step toward an understanding of more complicated phenomena.

We will denote by x and y the populations of the prey and predator, respectively, at time t. In constructing a model of the interaction of the two species, we make the following assumptions:

1. In the absence of the predator the prey grows at a rate proportional to the current population; thus $dx/dt = ax$, $a > 0$, when $y = 0$.
2. In the absence of the prey the predator dies out; thus $dy/dt = -cy$, $c > 0$, when $x = 0$.
3. The number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth of the predator and to inhibit the growth of the prey. Thus the growth rate of the predator is increased by a term of the form γxy, while the growth rate of the prey is decreased by a term $-\alpha xy$, where γ and α are positive constants.

As a consequence of these assumptions, we are led to the equations

$$\begin{aligned} dx/dt &= ax - \alpha xy = x(a - \alpha y), \\ dy/dt &= -cy + \gamma xy = y(-c + \gamma x). \end{aligned} \tag{1}$$

The constants a, c, α, and γ are all positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, and α and γ are measures of the effect

[2]Alan L. Hodgkin (1914–1998) and Andrew F. Huxley (1917–) were awarded the Nobel prize in physiology and medicine in 1963 for their work on the excitation and transmission of neural impulses, first published in 1952, when they were at Cambridge University.

of the interaction between the two species. Equations (1) are known as the Lotka–Volterra equations. They were developed in papers by Lotka[3] in 1925 and by Volterra[4] in 1926. Although these are rather simple equations, they do characterize a wide class of problems. Ways of making them more realistic are discussed at the end of this section and in the problems. Our goal here is to determine the qualitative behavior of the solutions (trajectories) of the system (1) for arbitrary positive initial values of x and y. We do this first for a specific example and then return to the general equations (1) at the end of the section.

EXAMPLE 1

Discuss the solutions of the system

$$\begin{aligned} dx/dt &= x(1 - 0.5y) = x - 0.5xy, \\ dy/dt &= y(-0.75 + 0.25x) = -0.75y + 0.25xy \end{aligned} \tag{2}$$

for x and y positive.

The critical points of this system are the solutions of the algebraic equations

$$x(1 - 0.5y) = 0, \qquad y(-0.75 + 0.25x) = 0, \tag{3}$$

namely, the points $(0, 0)$ and $(3, 2)$. Figure 9.5.1 shows the critical points and a direction field for the system (2). From this figure we conclude tentatively that the trajectories in the first quadrant may be closed curves surrounding the critical point $(3, 2)$.

Next we examine the local behavior of solutions near each critical point. Near the origin we can neglect the nonlinear terms in Eqs. (2) to obtain the corresponding linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

The eigenvalues and eigenvectors of Eq. (4) are

$$r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = -0.75, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

so its general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-0.75t}. \tag{6}$$

Thus the origin is a saddle point of both the linear system (4) and of the nonlinear system (2), and therefore is unstable. One pair of trajectories enters the origin along the y-axis; all other trajectories depart from the neighborhood of the origin.

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[3]Alfred J. Lotka (1880–1949), an American biophysicist, was born in what is now the Ukraine, and was educated mainly in Europe. He is remembered chiefly for his formulation of the Lotka–Volterra equations. He was also the author, in 1924, of the first book on mathematical biology; it is now available as *Elements of Mathematical Biology* (New York: Dover, 1956).

[4]Vito Volterra (1860–1940), a distinguished Italian mathematician, held professorships at Pisa, Turin, and Rome. He is particularly famous for his work in integral equations and functional analysis. Indeed, one of the major classes of integral equations is named for him; see Problem 20 of Section 6.6. His theory of interacting species was motivated by data collected by a friend, D'Ancona, concerning fish catches in the Adriatic Sea. A translation of his 1926 paper can be found in an appendix to R. N. Chapman, *Animal Ecology with Special Reference to Insects* (New York: McGraw-Hill, 1931).

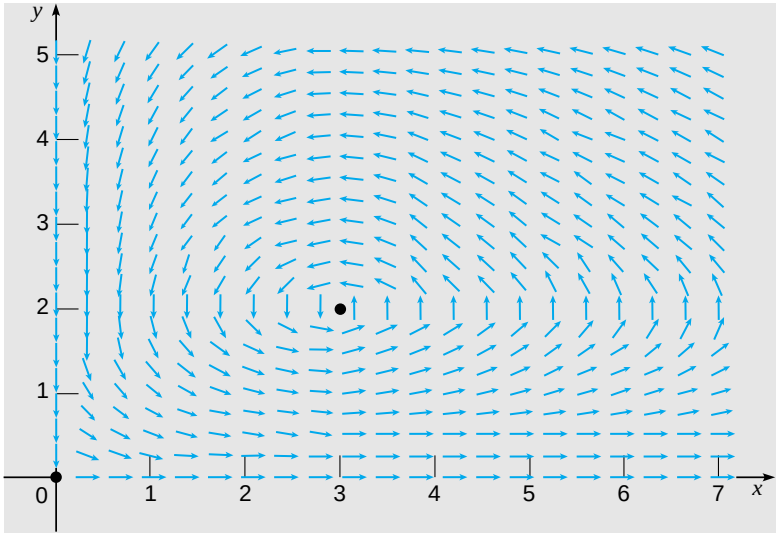


FIGURE 9.5.1 Critical points and direction field for the predator–prey system (2).

To examine the critical point (3, 2) we can either make the substitution

$$x = 3 + u, \qquad y = 2 + v \tag{7}$$

in Eqs. (2) and then neglect the nonlinear terms in u and v, or else refer to Eq. (13) of Section 9.3. In either case we obtain the linear system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{8}$$

The eigenvalues and eigenvectors of this system are

$$r_1 = \frac{\sqrt{3}\,i}{2}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -i/\sqrt{3} \end{pmatrix}; \qquad r_2 = -\frac{\sqrt{3}\,i}{2}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ i/\sqrt{3} \end{pmatrix}. \tag{9}$$

Since the eigenvalues are imaginary, the critical point (3, 2) is a center of the linear system (8) and is therefore a stable critical point for that system. Recall from Section 9.3 that this is one of the cases in which the behavior of the linear system may or may not carry over to the nonlinear system, so the nature of the point (3, 2) for the nonlinear system (2) cannot be determined from this information. The simplest way to find the trajectories of the linear system (8) is to divide the second of Eqs. (8) by the first so as to obtain the differential equation

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = \frac{0.5u}{-1.5v} = -\frac{u}{3v},$$

or

$$u\,du + 3v\,dv = 0. \tag{10}$$

Consequently,

$$u^2 + 3v^2 = k, \tag{11}$$

where k is an arbitrary nonnegative constant of integration. Thus the trajectories of the linear system (8) are ellipses centered at the critical point and elongated somewhat in the horizontal direction.

Now let us return to the nonlinear system (2). Dividing the second of Eqs. (2) by the first, we obtain

$$\frac{dy}{dx} = \frac{y(-0.75 + 0.25x)}{x(1 - 0.5y)}. \tag{12}$$

Equation (12) is a separable equation and can be put in the form

$$\frac{1 - 0.5y}{y}\,dy = \frac{-0.75 + 0.25x}{x}\,dx,$$

from which it follows that

$$0.75 \ln x + \ln y - 0.5y - 0.25x = c, \tag{13}$$

where c is a constant of integration. Although by using only elementary functions we cannot solve Eq. (13) explicitly for either variable in terms of the other, it is possible to show that the graph of the equation for a fixed value of c is a closed curve surrounding the critical point (3, 2). Thus the critical point is also a center of the nonlinear system (2) and the predator and prey populations exhibit a cyclic variation.

Figure 9.5.2 shows a phase portrait of the system (2). For some initial conditions the trajectory represents small variations in x and y about the critical point, and is almost elliptical in shape, as the linear analysis suggests. For other initial conditions the oscillations in x and y are more pronounced, and the shape of the trajectory is significantly different from an ellipse. Observe that the trajectories are traversed in the counterclockwise direction. The dependence of x and y on t for a typical set of initial conditions is shown in Figure 9.5.3. Note that x and y are periodic functions of t, as they must be since the trajectories are closed curves. Further, the oscillation of the predator population lags behind that of the prey. Starting from a state in which both predator and prey populations are relatively small, the prey first increase because there is little predation. Then the predators, with abundant food, increase in population also. This causes heavier predation and the prey tend to decrease. Finally, with a diminished food supply, the predator population also decreases, and the system returns to the original state.

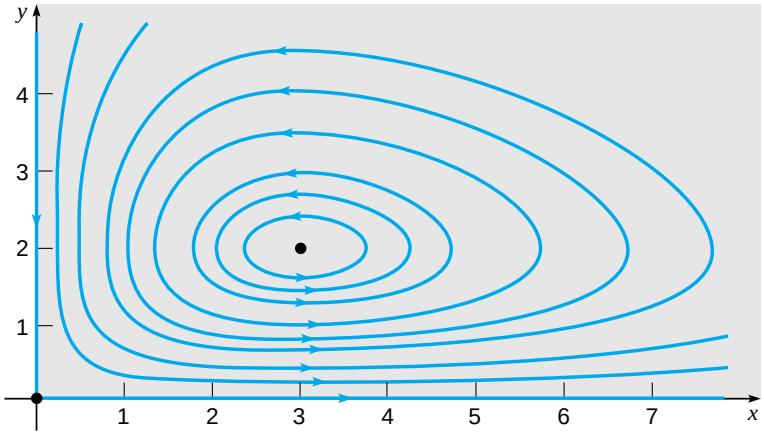


FIGURE 9.5.2 A phase portrait of the system (2).

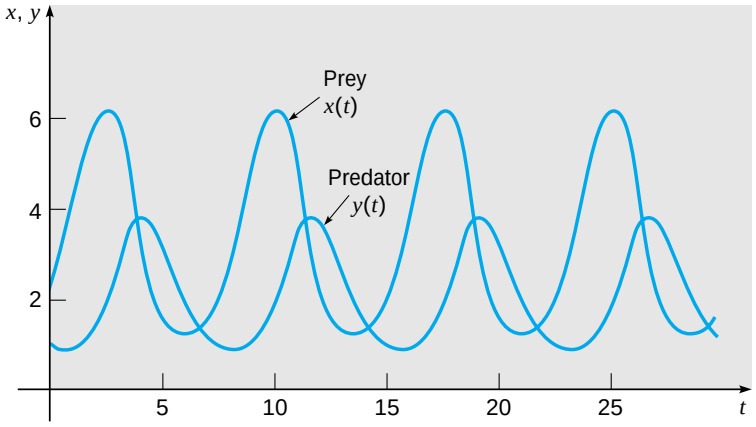


FIGURE 9.5.3 Variations of the prey and predator populations with time for the system (2).

The general system (1) can be analyzed in exactly the same way as in the example. The critical points of the system (1) are the solutions of

$$x(a - \alpha y) = 0, \qquad y(-c + \gamma x) = 0,$$

that is, the points $(0, 0)$ and $(c/\gamma, a/\alpha)$. We first examine the solutions of the corresponding linear system near each critical point.

In the neighborhood of the origin the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{14}$$

The eigenvalues and eigenvectors are

$$r_1 = a, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = -c, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{15}$$

so the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{at} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-ct}. \tag{16}$$

Thus the origin is a saddle point and hence unstable. Entrance to the saddle point is along the y-axis; all other trajectories depart from the neighborhood of the critical point.

Next consider the critical point $(c/\gamma, a/\alpha)$. If $x = (c/\gamma) + u$ and $y = (a/\alpha) + v$, then the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{17}$$

The eigenvalues of the system (17) are $r = \pm i\sqrt{ac}$, so the critical point is a (stable) center of the linear system. To find the trajectories of the system (17) we can divide the second equation by the first to obtain

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = -\frac{(\gamma a/\alpha)u}{(\alpha c/\gamma)v}, \tag{18}$$

or

$$\gamma^2 au\,du + \alpha^2 cv\,dv = 0. \tag{19}$$

Consequently,

$$\gamma^2 au^2 + \alpha^2 cv^2 = k, \tag{20}$$

where k is a nonnegative constant of integration. Thus the trajectories of the linear system (17) are ellipses, just as in the example.

Returning briefly to the nonlinear system (1), observe that it can be reduced to the single equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-c+\gamma x)}{x(a-\alpha y)}. \tag{21}$$

Equation (21) is separable and has the solution

$$a\ln y - \alpha y + c\ln x - \gamma x = C, \tag{22}$$

where C is a constant of integration. Again it is possible to show that the graph of Eq. (22), for fixed C, is a closed curve surrounding the critical point $(c/\gamma, a/\alpha)$. Thus this critical point is also a center for the general nonlinear system (1).

The cyclic variation of the predator and prey populations can be analyzed in more detail when the deviations from the point $(c/\gamma, a/\alpha)$ are small and the linear system (17) can be used. The solution of the system (17) can be written in the form

$$u = \frac{c}{\gamma} K\cos(\sqrt{ac}\,t + \phi), \qquad v = \frac{a}{\alpha}\sqrt{\frac{c}{a}}\,K\sin(\sqrt{ac}\,t + \phi), \tag{23}$$

where the constants K and ϕ are determined by the initial conditions. Thus

$$\begin{aligned} x &= \frac{c}{\gamma} + \frac{c}{\gamma} K\cos(\sqrt{ac}\,t + \phi), \\ y &= \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}}\,K\sin(\sqrt{ac}\,t + \phi). \end{aligned} \tag{24}$$

These equations are good approximations for the nearly elliptical trajectories close to the critical point $(c/\gamma, a/\alpha)$. We can use them to draw several conclusions about the cyclic variation of the predator and prey on such trajectories.

1. The sizes of the predator and prey populations vary sinusoidally with period $2\pi/\sqrt{ac}$. This period of oscillation is independent of the initial conditions.
2. The predator and prey populations are out of phase by one-quarter of a cycle. The prey leads and the predator lags, as explained in the example.
3. The amplitudes of the oscillations are Kc/γ for the prey and $a\sqrt{c}K/\alpha\sqrt{a}$ for the predator and hence depend on the initial conditions as well as on the parameters of the problem.
4. The average populations of predator and prey over one complete cycle are c/γ and a/α, respectively. These are the same as the equilibrium populations; see Problem 10.

Cyclic variations of predator and prey as predicted by Eqs. (1) have been observed in nature. One striking example is described by Odum (pp. 191–192); based on the records of the Hudson Bay Company of Canada, the abundance of lynx and snowshoe hare as indicated by the number of pelts turned in over the period 1845–1935 shows

a distinct periodic variation with period of 9 to 10 years. The peaks of abundance are followed by very rapid declines, and the peaks of abundance of the lynx and hare are out of phase, with that of the hare preceding that of the lynx by a year or more.

The Lotka–Volterra model of the predator–prey problem has revealed a cyclic variation that perhaps could have been anticipated. On the other hand, the use of the Lotka–Volterra model in other situations can lead to conclusions that are not intuitively obvious. An example that suggests a possible danger in using insecticides is given in Problem 12.

One criticism of the Lotka–Volterra equations is that in the absence of the predator the prey will grow without bound. This can be corrected by allowing for the natural inhibiting effect that an increasing population has on the growth rate of the population; for example, the first of Eqs. (1) can be modified so that when $y = 0$, it reduces to a logistic equation for x (see Problem 13). The most important consequence of this modification is that the critical point at $(c/\gamma, a/\alpha)$ moves to $(c/\gamma, a/\alpha - \sigma c/\alpha\gamma)$ and becomes an asymptotically stable point. It is either a node or a spiral point depending on the values of the parameters in the differential equations. In either case, other trajectories are no longer closed curves but approach the critical point as $t \to \infty$.

PROBLEMS

Each of Problems 1 through 5 can be interpreted as describing the interaction of two species with population densities x and y. In each of these problems carry out the following steps.

(a) Draw a direction field and describe how solutions seem to behave.

(b) Find the critical points.

(c) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.

(d) Sketch the trajectories in the neighborhood of each critical point.

(e) Draw a phase portrait for the system.

(f) Determine the limiting behavior of x and y as $t \to \infty$ and interpret the results in terms of the populations of the two species.

1. $dx/dt = x(1.5 - 0.5y)$
   $dy/dt = y(-0.5 + x)$

2. $dx/dt = x(1 - 0.5y)$
   $dy/dt = y(-0.25 + 0.5x)$

3. $dx/dt = x(1 - 0.5x - 0.5y)$
   $dy/dt = y(-0.25 + 0.5x)$

4. $dx/dt = x(1.125 - x - 0.5y)$
   $dy/dt = y(-1 + x)$

5. $dx/dt = x(-1 + 2.5x - 0.3y - x^2)$
   $dy/dt = y(-1.5 + x)$

6. In this problem we examine the phase difference between the cyclic variations of the predator and prey populations as given by Eqs. (24) of the text. Suppose we assume that $K > 0$ and that t is measured from the time that the prey population x is a maximum; then $\phi = 0$. Show that the predator population y is a maximum at $t = \pi/2\sqrt{ac} = T/4$, where T is the period of the oscillation. When is the prey population increasing most rapidly, decreasing most rapidly, a minimum? Answer the same questions for the predator population. Draw a typical elliptic trajectory enclosing the point $(c/\gamma, a/\alpha)$, and mark these points on it.

7. (a) Find the ratio of the amplitudes of the oscillations of the prey and predator populations about the critical point $(c/\gamma, a/\alpha)$, using the approximation (24), which is valid for small oscillations. Observe that the ratio is independent of the initial conditions.
   (b) Evaluate the ratio found in part (a) for the system (2).
   (c) Estimate the amplitude ratio for the solution of the nonlinear system (2) shown in Figure 9.5.3. Does the result agree with that obtained from the linear approximation?

(d) Determine the prey–predator amplitude ratio for other solutions of the system (2), that is, for solutions satisfying other initial conditions. Is the ratio independent of the initial conditions?

▶ 8. (a) Find the period of the oscillations of the prey and predator populations, using the approximation (24), which is valid for small oscillations. Note that the period is independent of the amplitude of the oscillations.
(b) For the solution of the nonlinear system (2) shown in Figure 9.5.3 estimate the period as well as possible. Is the result the same as for the linear approximation?
(c) Calculate other solutions of the system (2), that is, solutions satisfying other initial conditions, and determine their periods. Is the period the same for all initial conditions?

▶ 9. Consider the system

$$dx/dt = ax[1 - (y/2)], \quad dy/dt = by[-1 + (x/3)],$$

where a and b are positive constants. Observe that this system is the same as in the example in the text if $a = 1$ and $b = 0.75$. Suppose the initial conditions are $x(0) = 5$ and $y(0) = 2$.
(a) Let $a = 1$ and $b = 1$. Plot the trajectory in the phase plane and determine (or estimate) the period of the oscillation.
(b) Repeat part (a) for $a = 3$ and $a = 1/3$, with $b = 1$.
(c) Repeat part (a) for $b = 3$ and $b = 1/3$, with $a = 1$.
(d) Describe how the period and the shape of the trajectory depend on a and b.

▶ 10. The average sizes of the prey and predator populations are defined as

$$\overline{x} = \frac{1}{T}\int_A^{A+T} x(t)\,dt, \qquad \overline{y} = \frac{1}{T}\int_A^{A+T} y(t)\,dt,$$

respectively, where T is the period of a full cycle, and A is any nonnegative constant.
(a) Using the approximation (24), valid near the critical point, show that $\overline{x} = c/\gamma$ and $\overline{y} = a/\alpha$.
(b) For the solution of the nonlinear system (2) shown in Figure 9.5.3 estimate $\overline{x}$ and $\overline{y}$ as well as you can. Try to determine whether $\overline{x}$ and $\overline{y}$ are given by c/γ and a/α, respectively, in this case.
Hint: Consider how you might estimate the value of an integral even though you do not have a formula for the integrand.
(c) Calculate other solutions of the system (2), that is, solutions satisfying other initial conditions, and determine $\overline{x}$ and $\overline{y}$ for these solutions. Are the values of $\overline{x}$ and $\overline{y}$ the same for all solutions?

11. Suppose that the predator–prey equations (1) of the text describe foxes (y) and rabbits (x) in a forest. A trapping company is engaged in trapping foxes and rabbits for their pelts. Explain why it is reasonable for the company to conduct its operation so as to move the population of each species closer to the center $(c/\gamma, a/\alpha)$. When is it best to trap foxes? Rabbits? Rabbits and foxes? Neither?
Hint: See Problem 6. An intuitive argument is all that is required.

12. Suppose that an insect population x is controlled by a natural predator population y according to the model (1), so that there are small cyclic variations of the populations about the critical point $(c/\gamma, a/\alpha)$. Suppose that an insecticide is employed with the goal of reducing the population of the insects, and suppose that this insecticide is also toxic to the predators; indeed, suppose that the insecticide kills both prey and predator at rates proportional to their respective populations. Write down the modified differential equations, determine the new equilibrium point, and compare it to the original equilibrium point.

To ban insecticides on the basis of this simple model and counterintuitive result would certainly be ill-advised. On the other hand, it is also rash to ignore the possible genuine existence of a phenomenon suggested by such a model.

13. As mentioned in the text, one improvement in the predator–prey model is to modify the equation for the prey so that it has the form of a logistic equation in the absence of the predator. Thus in place of Eqs. (1) we consider the system

$$dx/dt = x(a - \sigma x - \alpha y), \qquad dy/dt = y(-c + \gamma x),$$

where a, σ, α, c, and γ are positive constants. Determine all critical points and discuss their nature and stability characteristics. Assume that $a/\sigma \gg c/\gamma$. What happens for initial data $x \neq 0$, $y \neq 0$?

9.6 Liapunov's Second Method

In Section 9.3 we showed how the stability of a critical point of an almost linear system can usually be determined from a study of the corresponding linear system. However, no conclusion can be drawn when the critical point is a center of the corresponding linear system. Examples of this situation are the undamped pendulum, Equations (1) and (2) below, and the predator–prey problem discussed in Section 9.5. Also, for an asymptotically stable critical point it may be important to investigate the basin of attraction; that is, the domain such that all solutions starting within that domain approach the critical point. Since the theory of almost linear systems is a local theory, it provides no information about this question.

In this section we discuss another approach, known as **Liapunov's**[5] **second method** or **direct method**. The method is referred to as a direct method because no knowledge of the solution of the system of differential equations is required. Rather, conclusions about the stability or instability of a critical point are obtained by constructing a suitable auxiliary function. The technique is a very powerful one that provides a more global type of information, for example, an estimate of the extent of the basin of attraction of a critical point. In addition, Liapunov's second method can also be used to study systems of equations that are not almost linear; however, we will not discuss such problems.

Basically, Liapunov's second method is a generalization of two physical principles for conservative systems, namely, (i) a rest position is stable if the potential energy is a local minimum, otherwise it is unstable, and (ii) the total energy is a constant during any motion. To illustrate these concepts, again consider the undamped pendulum (a conservative mechanical system), which is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \tag{1}$$

The corresponding system of first order equations is

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{g}{L}\sin x, \tag{2}$$

[5]Alexandr M. Liapunov (1857–1918), a student of Chebyshev at St. Petersburg, taught at the University of Kharkov from 1885 to 1901, when he became academician in applied mathematics at the St. Petersburg Academy of Sciences. In 1917 he moved to Odessa because of his wife's frail health. His research in stability encompassed both theoretical analysis and applications to various physical problems. His second method formed part of his most influential work, *General Problem of Stability of Motion*, published in 1892.

where $x = \theta$ and $y = d\theta/dt$. If we omit an arbitrary constant, the potential energy U is the work done in lifting the pendulum above its lowest position, namely,

$$U(x, y) = mgL(1 - \cos x); \tag{3}$$

see Figure 9.2.2. The critical points of the system (2) are $x = \pm n\pi$, $y = 0$, $n = 0, 1, 2, 3, \ldots$, corresponding to $\theta = \pm n\pi, d\theta/dt = 0$. Physically, we expect the points $x = 0$, $y = 0$; $x = \pm 2\pi$, $y = 0$; $\ldots$, corresponding to $\theta = 0, \pm 2\pi, \ldots$, to be stable, since for them the pendulum bob is vertical with the weight down; further, we expect the points $x = \pm\pi$, $y = 0$; $x = \pm 3\pi$, $y = 0$; $\ldots$, corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$, to be unstable, since for them the pendulum bob is vertical with the weight up. This agrees with statement (i), for at the former points U is a minimum equal to zero, and at the latter points U is a maximum equal to $2mgL$.

Next consider the total energy V, which is the sum of the potential energy U and the kinetic energy $\frac{1}{2}mL^2(d\theta/dt)^2$. In terms of x and y

$$V(x, y) = mgL(1 - \cos x) + \tfrac{1}{2}mL^2y^2. \tag{4}$$

On a trajectory corresponding to a solution $x = \phi(t)$, $y = \psi(t)$ of Eqs. (2), V can be considered as a function of t. The derivative of $V[\phi(t), \psi(t)]$ with respect to t is called the rate of change of V following the trajectory. By the chain rule

$$\begin{aligned}\frac{dV[\phi(t), \psi(t)]}{dt} &= V_x[\phi(t), \psi(t)]\,\frac{d\phi(t)}{dt} + V_y[\phi(t), \psi(t)]\,\frac{d\psi(t)}{dt} \\ &= (mgL\sin x)\,\frac{dx}{dt} + mL^2y\,\frac{dy}{dt},\end{aligned} \tag{5}$$

where it is understood that $x = \phi(t)$, $y = \psi(t)$. Finally, substituting in Eq. (5) for dx/dt and dy/dt from Eqs. (2) we find that $dV/dt = 0$. Hence V is a constant along any trajectory of the system (2), which is statement (ii).

It is important to note that at any point (x, y) the rate of change of V along the trajectory through that point was computed *without actually solving* the system (2). It is precisely this fact that allows us to use Liapunov's second method for systems whose solution we do not know, and is the main reason for its importance.

At the stable critical points, $x = \pm 2n\pi$, $y = 0$, $n = 0, 1, 2, \ldots$, the energy V is zero. If the initial state, say, (x_1, y_1), of the pendulum is sufficiently near a stable critical point, then the energy $V(x_1, y_1)$ is small, and the motion (trajectory) associated with this energy stays close to the critical point. It can be shown that if $V(x_1, y_1)$ is sufficiently small, then the trajectory is closed and contains the critical point. For example, suppose that (x_1, y_1) is near $(0, 0)$ and that $V(x_1, y_1)$ is very small. The equation of the trajectory with energy $V(x_1, y_1)$ is

$$V(x, y) = mgL(1 - \cos x) + \tfrac{1}{2}mL^2y^2 = V(x_1, y_1).$$

For x small we have $1 - \cos x = 1 - (1 - x^2/2! + \cdots) \cong x^2/2$. Thus the equation of the trajectory is approximately

$$\tfrac{1}{2}mgLx^2 + \tfrac{1}{2}mL^2y^2 = V(x_1, y_1),$$

or

$$\frac{x^2}{2V(x_1, y_1)/mgL} + \frac{y^2}{2V(x_1, y_1)/mL^2} = 1.$$

This is an ellipse enclosing the critical point $(0, 0)$; the smaller $V(x_1, y_1)$ is, the smaller are the major and minor axes of the ellipse. Physically, the closed trajectory corresponds to a solution that is periodic in time—the motion is a small oscillation about the equilibrium point.

If damping is present, however, it is natural to expect that the amplitude of the motion decays in time and that the stable critical point (center) becomes an asymptotically stable critical point (spiral point). See the phase portrait for the damped pendulum in Figure 9.3.5. This can almost be argued from a consideration of dV/dt. For the damped pendulum, the total energy is still given by Eq. (4), but now from Eqs. (8) of Section 9.3 $dx/dt = y$ and $dy/dt = -(g/L)\sin x - (c/Lm)y$. Substituting for dx/dt and dy/dt in Eq. (5) gives $dV/dt = -cLy^2 \leq 0$. Thus the energy is nonincreasing along any trajectory and, except for the line $y = 0$, the motion is such that the energy decreases. Hence each trajectory must approach a point of minimum energy—a stable equilibrium point. If $dV/dt < 0$ instead of $dV/dt \leq 0$, it is reasonable to expect that this would be true for all trajectories that start sufficiently close to the origin.

To pursue these ideas further, consider the autonomous system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y), \tag{6}$$

and suppose that the point $x = 0$, $y = 0$ is an asymptotically stable critical point. Then there exists some domain D containing $(0, 0)$ such that every trajectory that starts in D must approach the origin as $t \to \infty$. Suppose that there exists an "energy" function V such that $V \geq 0$ for (x, y) in D with $V = 0$ only at the origin. Since each trajectory in D approaches the origin as $t \to \infty$, then following any particular trajectory, V decreases to zero as t approaches infinity. The type of result we want to prove is essentially the converse: If, on every trajectory, V decreases to zero as t increases, then the trajectories must approach the origin as $t \to \infty$, and hence the origin is asymptotically stable. First, however, it is necessary to make several definitions.

Let V be defined on some domain D containing the origin. Then V is said to be **positive definite** on D if $V(0, 0) = 0$ and $V(x, y) > 0$ for all other points in D. Similarly, V is said to be **negative definite** on D if $V(0, 0) = 0$ and $V(x, y) < 0$ for all other points in D. If the inequalities $>$ and $<$ are replaced by $\geq$ and $\leq$, then V is said to be **positive semidefinite** and **negative semidefinite**, respectively. We emphasize that in speaking of a positive definite (negative definite, ...) function on a domain D containing the origin, the function must be zero at the origin in addition to satisfying the proper inequality at all other points in D.

EXAMPLE 1

The function

$$V(x, y) = \sin(x^2 + y^2)$$

is positive definite on $x^2 + y^2 < \pi/2$ since $V(0, 0) = 0$ and $V(x, y) > 0$ for $0 < x^2 + y^2 < \pi/2$. However, the function

$$V(x, y) = (x + y)^2$$

is only positive semidefinite since $V(x, y) = 0$ on the line $y = -x$.

We also want to consider the function

$$\dot{V}(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y), \tag{7}$$

where F and G are the same functions as in Eqs. (6). We choose this notation because $\dot{V}(x, y)$ can be identified as the rate of change of V along the trajectory of the system (6) that passes through the point (x, y). That is, if $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (6), then

$$\begin{aligned}\frac{dV[\phi(t), \psi(t)]}{dt} &= V_x[\phi(t), \psi(t)]\frac{d\phi(t)}{dt} + V_y[\phi(t), \psi(t)]\frac{d\psi(t)}{dt} \\ &= V_x(x, y)F(x, y) + V_y(x, y)G(x, y) \\ &= \dot{V}(x, y). \end{aligned} \tag{8}$$

The function $\dot{V}$ is sometimes referred to as the derivative of V with respect to the system (6).

We now state two Liapunov theorems, the first dealing with stability, the second with instability.

Theorem 9.6.1 Suppose that the autonomous system (6) has an isolated critical point at the origin. If there exists a function V that is continuous and has continuous first partial derivatives, is positive definite, and for which the function $\dot{V}$ given by Eq. (7) is negative definite on some domain D in the xy-plane containing $(0, 0)$, then the origin is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite, then the origin is a stable critical point.

Theorem 9.6.2 Let the origin be an isolated critical point of the autonomous system (6). Let V be a function that is continuous and has continuous first partial derivatives. Suppose that $V(0, 0) = 0$ and that in every neighborhood of the origin there is at least one point at which V is positive (negative). If there exists a domain D containing the origin such that the function $\dot{V}$ given by Eq. (7) is positive definite (negative definite) on D, then the origin is an unstable critical point.

The function V is called a **Liapunov function**. Before sketching geometrical arguments for Theorems 9.6.1 and 9.6.2, we note that the difficulty in using these theorems is that they tell us nothing about how to construct a Liapunov function, assuming that one exists. In cases where the autonomous system (6) represents a physical problem, it is natural to consider first the actual total energy function of the system as a possible Liapunov function. However, Theorems 9.6.1 and 9.6.2 are applicable in cases where the concept of physical energy is not pertinent. In such cases a judicious trial-and-error approach may be necessary.

Now consider the second part of Theorem 9.6.1, that is, the case $\dot{V} \leq 0$. Let $c \geq 0$ be a constant and consider the curve in the xy-plane given by $V(x, y) = c$. For $c = 0$ the curve reduces to the single point $x = 0$, $y = 0$. We assume that if $0 < c_1 < c_2$, then the curve $V(x, y) = c_1$ contains the origin and lies within the curve $V(x, y) = c_2$, as illustrated in Figure 9.6.1*a*. We show that a trajectory starting inside a closed curve $V(x, y) = c$ cannot cross to the outside. Thus, given a circle of radius ϵ about the

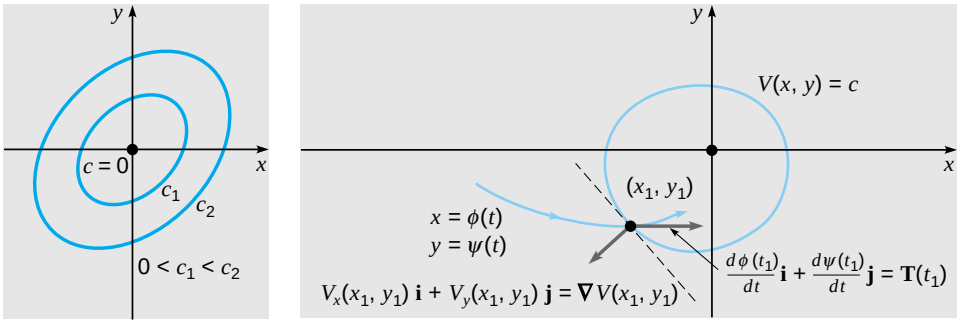


FIGURE 9.6.1 Geometrical interpretation of Liapunov's method.

origin, by taking c sufficiently small we can ensure that every trajectory starting inside the closed curve $V(x, y) = c$ stays within the circle of radius ϵ; indeed, it stays within the closed curve $V(x, y) = c$ itself. Thus the origin is a stable critical point.

To show this, recall from calculus that the vector

$$\nabla V(x, y) = V_x(x, y)\mathbf{i} + V_y(x, y)\mathbf{j}, \tag{9}$$

known as the gradient of V, is normal to the curve $V(x, y) = c$ and points in the direction of increasing V. In the present case, V increases outward from the origin, so ∇V points away from the origin, as indicated in Figure 9.6.1*b*. Next, consider a trajectory $x = \phi(t)$, $y = \psi(t)$ of the system (6), and recall that the vector $\mathbf{T}(t) = \phi'(t)\mathbf{i} + \psi'(t)\mathbf{j}$ is tangent to the trajectory at each point; see Figure 9.6.1*b*. Let $x_1 = \phi(t_1)$, $y_1 = \psi(t_1)$ be a point of intersection of the trajectory and a closed curve $V(x, y) = c$. At this point $\phi'(t_1) = F(x_1, y_1)$, $\psi'(t_1) = G(x_1, y_1)$, so from Eq. (7) we obtain

$$\begin{aligned}\dot{V}(x_1, y_1) &= V_x(x_1, y_1)\phi'(t_1) + V_y(x_1, y_1)\psi'(t_1)\\ &= [V_x(x_1, y_1)\mathbf{i} + V_y(x_1, y_1)\mathbf{j}] \cdot [\phi'(t_1)\mathbf{i} + \psi'(t_1)\mathbf{j}]\\ &= \nabla V(x_1, y_1) \cdot \mathbf{T}(t_1).\end{aligned} \tag{10}$$

Thus $\dot{V}(x_1, y_1)$ is the scalar product of the vector $\nabla V(x_1, y_1)$ and the vector $\mathbf{T}(t_1)$. Since $\dot{V}(x_1, y_1) \le 0$, it follows that the cosine of the angle between $\nabla V(x_1, y_1)$ and $\mathbf{T}(t_1)$ is also less than or equal to zero; hence the angle itself is in the range $[\pi/2, 3\pi/2]$. Thus the direction of motion on the trajectory is inward with respect to $V(x_1, y_1) = c$ or, at worst, tangent to this curve. Trajectories starting inside a closed curve $V(x_1, y_1) = c$ (no matter how small c is) cannot escape, so the origin is a stable point. If $\dot{V}(x_1, y_1) < 0$, then the trajectories passing through points on the curve are actually pointed inward. As a consequence, it can be shown that trajectories starting sufficiently close to the origin must approach the origin; hence the origin is asymptotically stable.

A geometric argument for Theorem 9.6.2 follows in a somewhat similar manner. Briefly, suppose that $\dot{V}$ is positive definite, and suppose that given any circle about the origin there is an interior point (x_1, y_1) at which $V(x_1, y_1) > 0$. Consider a trajectory that starts at (x_1, y_1). Along this trajectory it follows from Eq. (8) that V must increase, since $\dot{V}(x_1, y_1) > 0$; furthermore, since $V(x_1, y_1) > 0$, the trajectory cannot approach the origin because $V(0, 0) = 0$. This shows that the origin cannot be asymptotically

stable. By further exploiting the fact that $\dot{V}(x, y) > 0$, it is possible to show that the origin is an unstable point; however, we will not pursue this argument.

EXAMPLE 2

Use Theorem 9.6.1 to show that $(0, 0)$ is a stable critical point for the undamped pendulum equations (2). Also use Theorem 9.6.2 to show that $(\pi, 0)$ is an unstable critical point.

Let V be the total energy given by Eq. (4):

$$V(x, y) = mgL(1 - \cos x) + \tfrac{1}{2}mL^2y^2. \tag{4}$$

If we take D to be the domain $-\pi/2 < x < \pi/2$, $-\infty < y < \infty$, then V is positive there except at the origin, where it is zero. Thus V is positive definite on D. Further, as we have already seen,

$$\dot{V} = (mgL \sin x)(y) + (mL^2y)(-g \sin x)/L = 0$$

for all x and y. Thus $\dot{V}$ is negative semidefinite on D. Consequently, by the last statement in Theorem 9.6.1, the origin is a stable critical point for the undamped pendulum. Observe that this conclusion cannot be obtained from Theorem 9.3.2 because $(0, 0)$ is a center for the corresponding linear system.

Now consider the critical point $(\pi, 0)$. The Liapunov function given by Eq. (4) is no longer suitable because Theorem 9.6.2 calls for a function V for which $\dot{V}$ is either positive or negative definite. To analyze the point $(\pi, 0)$ it is convenient to move this point to the origin by the change of variables $x = \pi + u$, $y = v$. Then the differential equations (2) become

$$du/dt = v, \quad dv/dt = \frac{g}{L} \sin u, \tag{11}$$

and the critical point is $(0, 0)$ in the uv-plane. Consider the function

$$V(u, v) = v \sin u \tag{12}$$

and let D be the domain $-\pi/4 < u < \pi/4$, $-\infty < v < \infty$. Then

$$\dot{V} = (v \cos u)(v) + (\sin u)[(g/L) \sin u] = v^2 \cos u + (g/L) \sin^2 u \tag{13}$$

is positive definite in D. The only remaining question is whether there are points in every neighborhood of the origin where V itself is positive. From Eq. (12) we see that $V(u, v) > 0$ in the first quadrant (where both $\sin u$ and v are positive) and in the third quadrant (where both are negative). Thus the conditions of Theorem 9.6.2 are satisfied and the point $(0, 0)$ in the uv-plane, or the point $(\pi, 0)$ in the xy-plane, is unstable.

The damped pendulum equations are discussed in Problem 7.

From a practical point of view we are often more interested in the basin of attraction. The following theorem provides some information on this subject.

Theorem 9.6.3 Let the origin be an isolated critical point of the autonomous system (6). Let the function V be continuous and have continuous first partial derivatives. If there is a bounded domain D_K containing the origin where $V(x, y) < K$, V is positive definite,

and $\dot{V}$ is negative definite, then every solution of Eqs. (6) that starts at a point in D_K approaches the origin as t approaches infinity.

In other words, Theorem 9.6.3 says that if $x = \phi(t)$, $y = \psi(t)$ is the solution of Eqs. (6) for initial data lying in D_K, then (x, y) approaches the critical point (0, 0) as $t \to \infty$. Thus D_K gives a region of asymptotic stability; of course, it may not be the entire basin of attraction. This theorem is proved by showing that (i) there are no periodic solutions of the system (6) in D_K, and (ii) there are no other critical points in D_K. It then follows that trajectories starting in D_K cannot escape and, therefore, must tend to the origin as t tends to infinity.

Theorems 9.6.1 and 9.6.2 give sufficient conditions for stability and instability, respectively. However, these conditions are not necessary, nor does our failure to determine a suitable Liapunov function mean that there is not one. Unfortunately, there are no general methods for the construction of Liapunov functions; however, there has been extensive work on the construction of Liapunov functions for special classes of equations. An elementary algebraic result that is often useful in constructing positive definite or negative definite functions is stated without proof in the following theorem.

Theorem 9.6.4 The function

$$V(x, y) = ax^2 + bxy + cy^2 \tag{14}$$

is positive definite if, and only if,

$$a > 0 \quad \text{and} \quad 4ac - b^2 > 0, \tag{15}$$

and is negative definite if, and only if,

$$a < 0 \quad \text{and} \quad 4ac - b^2 > 0. \tag{16}$$

The use of Theorem 9.6.4 is illustrated in the following example.

EXAMPLE 3

Show that the critical point (0, 0) of the autonomous system

$$dx/dt = -x - xy^2, \qquad dy/dt = -y - x^2y \tag{17}$$

is asymptotically stable.

We try to construct a Liapunov function of the form (14). Then $V_x(x, y) = 2ax + by$, $V_y(x, y) = bx + 2cy$, so

$$\begin{aligned}\dot{V}(x, y) &= (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - x^2y) \\ &= -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + x^3y) + 2c(y^2 + x^2y^2)].\end{aligned}$$

If we choose $b = 0$, and a and c to be any positive numbers, then $\dot{V}$ is negative definite and V is positive definite by Theorem 9.6.4. Thus by Theorem 9.6.1 the origin is an asymptotically stable critical point.

EXAMPLE 4

Consider the system

$$\begin{aligned} dx/dt &= x(1 - x - y), \\ dy/dt &= y(0.75 - y - 0.5x). \end{aligned} \tag{18}$$

In Example 1 of Section 9.4 we found that this system models a certain pair of competing species, and that the critical point $(0.5, 0.5)$ is asymptotically stable. Confirm this conclusion by finding a suitable Liapunov function.

It is helpful to transform the point $(0.5, 0.5)$ to the origin. To this end let

$$x = 0.5 + u, \qquad y = 0.5 + v. \tag{19}$$

Then, substituting for x and y in Eqs. (18), we obtain the new system

$$\begin{aligned} du/dt &= -0.5u - 0.5v - u^2 - uv, \\ dv/dt &= -0.25u - 0.5v - 0.5uv - v^2. \end{aligned} \tag{20}$$

To keep the calculations relatively simple, consider the function $V(u, v) = u^2 + v^2$ as a possible Liapunov function. This function is clearly positive definite, so we only need to determine whether there is a region containing the origin in the uv-plane where the derivative $\dot{V}$ with respect to the system (20) is negative definite. We compute $\dot{V}(u, v)$ and find that

$$\begin{aligned} \dot{V}(u, v) &= V_u \frac{du}{dt} + V_v \frac{dv}{dt} \\ &= 2u(-0.5u - 0.5v - u^2 - uv) + 2v(-0.25u - 0.5v - 0.5uv - v^2), \end{aligned}$$

or

$$\dot{V}(u, v) = -[(u^2 + 1.5uv + v^2) + (2u^3 + 2u^2v + uv^2 + 2v^3)], \tag{21}$$

where we have collected together the quadratic and cubic terms. We want to show that the expression in square brackets in Eq. (21) is positive definite, at least for u and v sufficiently small. Observe that the quadratic terms can be written as

$$u^2 + 1.5uv + v^2 = 0.25(u^2 + v^2) + 0.75(u + v)^2, \tag{22}$$

so these terms are positive definite. On the other hand, the cubic terms in Eq. (18) may be of either sign. Thus we must show that, in some neighborhood of $u = 0$, $v = 0$, the cubic terms are smaller in magnitude than the quadratic terms; that is,

$$|2u^3 + 2u^2v + uv^2 + 2v^3| < 0.25(u^2 + v^2) + 0.75(u + v)^2. \tag{23}$$

To estimate the left side of Eq. (23) we introduce polar coordinates $u = r\cos\theta$, $v = r\sin\theta$. Then

$$\begin{aligned} |2u^3 + 2u^2v + uv^2 + 2v^3| &= r^3|2\cos^3\theta + 2\cos^2\theta\sin\theta + \cos\theta\sin^2\theta + 2\sin^3\theta| \\ &\le r^3[2|\cos^3\theta| + 2\cos^2\theta|\sin\theta| + |\cos\theta|\sin^2\theta + 2|\sin^3\theta|] \\ &\le 7r^3, \end{aligned}$$

since $|\sin\theta|, |\cos\theta| \le 1$. To satisfy Eq. (23) it is now certainly sufficient to satisfy the more stringent requirement

$$7r^3 < 0.25(u^2 + v^2) = 0.25r^2,$$

which yields $r < 1/28$. Thus, at least in this disk, the hypotheses of Theorem 9.6.1 are satisfied, so the origin is an asymptotically stable critical point of the system (20). The same is then true of the critical point (0.5, 0.5) of the original system (18).

If we refer to Theorem 9.6.3, the preceding argument also shows that the disk with center (0.5, 0.5) and radius 1/28 is a region of asymptotic stability for the system (18). This is a severe underestimate of the full basin of attraction, as the discussion in Section 9.4 shows. To obtain a better estimate of the actual basin of attraction from Theorem 9.6.3, one would have to estimate the terms in Eq. (23) more accurately, use a better (and presumably more complicated) Liapunov function, or both.

PROBLEMS

In each of Problems 1 through 4 construct a suitable Liapunov function of the form $ax^2 + cy^2$, where a and c are to be determined. Then show that the critical point at the origin is of the indicated type.

1. $dx/dt = -x^3 + xy^2, \quad dy/dt = -2x^2y - y^3;$ asymptotically stable
2. $dx/dt = -\frac{1}{2}x^3 + 2xy^2, \quad dy/dt = -y^3;$ asymptotically stable
3. $dx/dt = -x^3 + 2y^3, \quad dy/dt = -2xy^2;$ stable (at least)
4. $dx/dt = x^3 - y^3, \quad dy/dt = 2xy^2 + 4x^2y + 2y^3;$ unstable
5. Consider the system of equations

$$dx/dt = y - xf(x, y), \qquad dy/dt = -x - yf(x, y),$$

where f is continuous and has continuous first partial derivatives. Show that if $f(x, y) > 0$ in some neighborhood of the origin, then the origin is an asymptotically stable critical point, and if $f(x, y) < 0$ in some neighborhood of the origin, then the origin is an unstable critical point.

Hint: Construct a Liapunov function of the form $c(x^2 + y^2)$.

6. A generalization of the undamped pendulum equation is

$$d^2u/dt^2 + g(u) = 0, \tag{i}$$

where $g(0) = 0$, $g(u) > 0$ for $0 < u < k$, and $g(u) < 0$ for $-k < u < 0$; that is, $ug(u) > 0$ for $u \neq 0$, $-k < u < k$. Notice that $g(u) = \sin u$ has this property on $(-\pi/2, \pi/2)$.

(a) Letting $x = u$, $y = du/dt$, write Eq. (i) as a system of two equations, and show that $x = 0$, $y = 0$ is a critical point.

(b) Show that

$$V(x, y) = \tfrac{1}{2}y^2 + \int_0^x g(s)\, ds, \qquad -k < x < k \tag{ii}$$

is positive definite, and use this result to show that the critical point $(0, 0)$ is stable. Note that the Liapunov function V given by Eq. (ii) corresponds to the energy function $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$ for the case $g(u) = \sin u$.

7. By introducing suitable dimensionless variables, the system of nonlinear equations for the damped pendulum [Eqs. (8) of Section 9.3] can be written as

$$dx/dt = y, \qquad dy/dt = -y - \sin x.$$

(a) Show that the origin is a critical point.

(b) Show that while $V(x, y) = x^2 + y^2$ is positive definite, $\dot{V}(x, y)$ takes on both positive and negative values in any domain containing the origin, so that V is not a Liapunov function.

Hint: $x - \sin x > 0$ for $x > 0$ and $x - \sin x < 0$ for $x < 0$. Consider these cases with y positive but y so small that y^2 can be ignored compared to y.

(c) Using the energy function $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$ mentioned in Problem 6(b), show that the origin is a stable critical point. Note, however, that even though there is damping and we can expect that the origin is asymptotically stable, it is not possible to draw this conclusion using this Liapunov function.

(d) To show asymptotic stability it is necessary to construct a better Liapunov function than the one used in part (c). Show that $V(x, y) = \frac{1}{2}(x + y)^2 + x^2 + \frac{1}{2}y^2$ is such a Liapunov function, and conclude that the origin is an asymptotically stable critical point.

Hint: From Taylor's formula with a remainder it follows that $\sin x = x - \alpha x^3/3!$, where α depends on x but $0 < \alpha < 1$ for $-\pi/2 < x < \pi/2$. Then letting $x = r\cos\theta$, $y = r\sin\theta$, show that $\dot{V}(r\cos\theta, r\sin\theta) = -r^2[1 + h(r, \theta)]$, where $|h(r, \theta)| < 1$ if r is sufficiently small.

8. The Liénard equation (Problem 28 of Section 9.3) is

$$\frac{d^2u}{dt^2} + c(u)\frac{du}{dt} + g(u) = 0,$$

where g satisfies the conditions of Problem 6 and $c(u) \geq 0$. Show that the point $u = 0$, $du/dt = 0$ is a stable critical point.

9. (a) A special case of the Liénard equation of Problem 8 is

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + g(u) = 0,$$

where g satisfies the conditions of Problem 6. Letting $x = u, y = du/dt$, show that the origin is a critical point of the resulting system. This equation can be interpreted as describing the motion of a spring–mass system with damping proportional to the velocity and a nonlinear restoring force. Using the Liapunov function of Problem 6, show that the origin is a stable critical point, but note that even with damping we cannot conclude asymptotic stability using this Liapunov function.

(b) Asymptotic stability of the critical point $(0, 0)$ can be shown by constructing a better Liapunov function as was done in part (d) of Problem 7. However, the analysis for a general function g is somewhat sophisticated and we only mention that an appropriate form for V is

$$V(x, y) = \tfrac{1}{2}y^2 + Ayg(x) + \int_0^x g(s)\,ds,$$

where A is a positive constant to be chosen so that V is positive definite and $\dot{V}$ is negative definite. For the pendulum problem $[g(x) = \sin x]$ use V as given by the preceding equation with $A = \frac{1}{2}$ to show that the origin is asymptotically stable.

Hint: Use $\sin x = x - \alpha x^3/3!$ and $\cos x = 1 - \beta x^2/2!$ where α and β depend on x, but $0 < \alpha < 1$ and $0 < \beta < 1$ for $-\pi/2 < x < \pi/2$; let $x = r\cos\theta$, $y = r\sin\theta$, and show that $\dot{V}(r\cos\theta, r\sin\theta) = -\frac{1}{2}r^2[1 + \frac{1}{2}\sin 2\theta + h(r, \theta)]$, where $|h(r, \theta)| < \frac{1}{2}$ if r is sufficiently small. To show that V is positive definite use $\cos x = 1 - x^2/2 + \gamma x^4/4!$, where γ depends on x, but $0 < \gamma < 1$ for $-\pi/2 < x < \pi/2$.

In Problems 10 and 11 we will prove part of Theorem 9.3.2: If the critical point $(0, 0)$ of the almost linear system

$$dx/dt = a_{11}x + a_{12}y + F_1(x, y), \qquad dy/dt = a_{21}x + a_{22}y + G_1(x, y) \tag{i}$$

is an asymptotically stable critical point of the corresponding linear system

$$dx/dt = a_{11}x + a_{12}y, \qquad dy/dt = a_{21}x + a_{22}y, \tag{ii}$$

then it is an asymptotically stable critical point of the almost linear system (i). Problem 12 deals with the corresponding result for instability.

10. Consider the linear system (ii).
(a) Since $(0, 0)$ is an asymptotically stable critical point, show that $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. (See Problem 21 of Section 9.1.)
(b) Construct a Liapunov function $V(x, y) = Ax^2 + Bxy + Cy^2$ such that V is positive definite and $\dot{V}$ is negative definite. One way to ensure that $\dot{V}$ is negative definite is to choose A, B, and C so that $\dot{V}(x, y) = -x^2 - y^2$. Show that this leads to the result

$$A = -\frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}, \qquad B = \frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta},$$
$$C = -\frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta},$$

where $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$.
(c) Using the result of part (a) show that $A > 0$ and then show (several steps of algebra are required) that

$$4AC - B^2 = \frac{(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)(a_{11}a_{22} - a_{12}a_{21}) + 2(a_{11}a_{22} - a_{12}a_{21})^2}{\Delta^2} > 0.$$

Thus by Theorem 9.6.4, V is positive definite.

11. In this problem we show that the Liapunov function constructed in the preceding problem is also a Liapunov function for the almost linear system (i). We must show that there is some region containing the origin for which $\dot{V}$ is negative definite.
(a) Show that

$$\dot{V}(x, y) = -(x^2 + y^2) + (2Ax + By)F_1(x, y) + (Bx + 2Cy)G_1(x, y).$$

(b) Recall that $F_1(x, y)/r \to 0$ and $G_1(x, y)/r \to 0$ as $r = (x^2 + y^2)^{1/2} \to 0$. This means that given any $\epsilon > 0$ there exists a circle $r = R$ about the origin such that for $0 < r < R$, $|F_1(x, y)| < \epsilon r$, and $|G_1(x, y)| < \epsilon r$. Letting M be the maximum of $|2A|$, $|B|$, and $|2C|$, show by introducing polar coordinates that R can be chosen so that $\dot{V}(x, y) < 0$ for $r < R$.
Hint: Choose ϵ sufficiently small in terms of M.

12. In this problem we prove a part of Theorem 9.3.2 relating to instability.
(a) Show that if $a_{11} + a_{22} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$, then the critical point $(0, 0)$ of the linear system (ii) is unstable.
(b) The same result holds for the almost linear system (i). As in Problems 10 and 11 construct a positive definite function V such that $\dot{V}(x, y) = x^2 + y^2$ and hence is positive definite, and then invoke Theorem 9.6.2.

9.7 Periodic Solutions and Limit Cycles



In this section we discuss further the possible existence of periodic solutions of second order autonomous systems

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{1}$$

Such solutions satisfy the relation



$$\mathbf{x}(t + T) = \mathbf{x}(t) \tag{2}$$

for all t and for some nonnegative constant T called the period. The corresponding trajectories are *closed curves* in the phase plane. Periodic solutions often play an important role in physical problems because they represent phenomena that occur repeatedly. In many situations a periodic solution represents a "final state" toward which all "neighboring" solutions tend as the transients due to the initial conditions die out.

A special case of a periodic solution is a constant solution $\mathbf{x} = \mathbf{x}^0$, which corresponds to a critical point of the autonomous system. Such a solution is clearly periodic with any period. In this section, when we speak of a periodic solution, we mean a nonconstant periodic solution. In this case the period T is positive and is usually chosen as the smallest positive number for which Eq. (2) is valid.

Recall that the solutions of the linear autonomous system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{3}$$

are periodic if and only if the eigenvalues of $\mathbf{A}$ are pure imaginary. In this case the critical point at the origin is a center, as discussed in Section 9.1. We emphasize that if the eigenvalues of $\mathbf{A}$ are pure imaginary, then every solution of the linear system (3) is periodic; while if the eigenvalues are not pure imaginary, then there are no (nonconstant) periodic solutions. The predator–prey equations discussed in Section 9.5, although nonlinear, behave similarly: All solutions in the first quadrant are periodic. The following example illustrates a different way in which periodic solutions of nonlinear autonomous systems can occur.

EXAMPLE 1

Discuss the solutions of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{pmatrix}. \tag{4}$$

It is not difficult to show that $(0, 0)$ is the only critical point of the system (4), and also that the system is almost linear in the neighborhood of the origin. The corresponding linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

has eigenvalues $1 \pm i$. Therefore the origin is an unstable spiral point for both the linear system (5) and the nonlinear system (4). Thus any solution that starts near the origin in the phase plane will spiral away from the origin. Since there are no other critical points, we might think that all solutions of Eqs. (4) correspond to trajectories that spiral out to infinity. However, we now show that this is incorrect because far away from the origin the trajectories are directed inward.

It is convenient to introduce polar coordinates r and θ, where

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{6}$$

and $r \geq 0$. If we multiply the first of Eqs. (4) by x, the second by y, and add, we then obtain

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2 + y^2) - (x^2 + y^2)^2. \tag{7}$$

Since $r^2 = x^2 + y^2$ and $r(dr/dt) = x(dx/dt) + y(dy/dt)$, it follows from Eq. (7) that

$$r\frac{dr}{dt} = r^2(1 - r^2). \tag{8}$$

This is similar to the equations discussed in Section 2.5. The critical points (for $r \geq 0$) are the origin and the point $r = 1$, which corresponds to the unit circle in the phase plane. From Eq. (8) it follows that $dr/dt > 0$ if $r < 1$ and $dr/dt < 0$ if $r > 1$. Thus, inside the unit circle the trajectories are directed outward, while outside the unit circle they are directed inward. Apparently, the circle $r = 1$ is a limiting trajectory for this system.

To determine an equation for θ we multiply the first of Eqs. (4) by y, the second by x, and subtract, obtaining

$$y\frac{dx}{dt} - x\frac{dy}{dt} = x^2 + y^2. \tag{9}$$

Upon calculating dx/dt and dy/dt from Eqs. (6), we find that the left side of Eq. (9) is $-r^2(d\theta/dt)$, so Eq. (9) reduces to

$$\frac{d\theta}{dt} = -1. \tag{10}$$

The system of equations (8), (10) for r and θ is equivalent to the original system (4). One solution of the system (8), (10) is

$$r = 1, \qquad \theta = -t + t_0, \tag{11}$$

where t_0 is an arbitrary constant. As t increases, a point satisfying Eqs. (11) moves clockwise around the unit circle. Thus the autonomous system (4) has a periodic solution. Other solutions can be obtained by solving Eq. (8) by separation of variables; if $r \neq 0$ and $r \neq 1$, then

$$\frac{dr}{r(1 - r^2)} = dt. \tag{12}$$

Equation (12) can be solved by using partial fractions to rewrite the left side and then integrating. By performing these calculations we find that the solution of Eqs. (10) and (12) is

$$r = \frac{1}{\sqrt{1 + c_0 e^{-2t}}}, \qquad \theta = -t + t_0, \tag{13}$$

where c_0 and t_0 are arbitrary constants. The solution (13) also contains the solution (11), which is obtained by setting $c_0 = 0$ in the first of Eqs. (13).

The solution satisfying the initial conditions $r = \rho$, $\theta = \alpha$ at $t = 0$ is given by

$$r = \frac{1}{\sqrt{1 + [(1/\rho^2) - 1]e^{-2t}}}, \qquad \theta = -(t - \alpha). \tag{14}$$

If $\rho < 1$, then $r \to 1$ from the inside as $t \to \infty$; if $\rho > 1$, then $r \to 1$ from the outside as $t \to \infty$. Thus in all cases the trajectories spiral toward the circle $r = 1$ as $t \to \infty$. Several trajectories are shown in Figure 9.7.1.

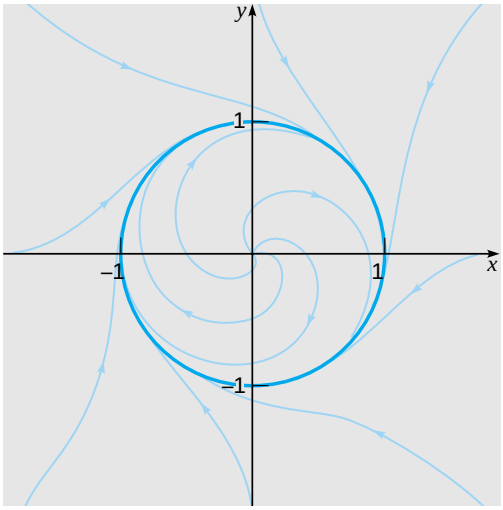


FIGURE 9.7.1 Trajectories of the system (4); a limit cycle.

In this example the circle $r = 1$ not only corresponds to periodic solutions of the system (4), but, in addition, other nonclosed trajectories spiral toward it as $t \to \infty$. In general, a closed trajectory in the phase plane such that other nonclosed trajectories spiral toward it, either from the inside or the outside, as $t \to \infty$, is called a **limit cycle**. Thus the circle $r = 1$ is a limit cycle for the system (4). If all trajectories that start near a closed trajectory (both inside and outside) spiral toward the closed trajectory as $t \to \infty$, then the limit cycle is **asymptotically stable**. Since the limiting trajectory is itself a periodic orbit rather than an equilibrium point, this type of stability is often called **orbital stability**. If the trajectories on one side spiral toward the closed trajectory, while those on the other side spiral away as $t \to \infty$, then the limit cycle is said to be **semistable**. If the trajectories on both sides of the closed trajectory spiral away as $t \to \infty$, then the closed trajectory is **unstable**. It is also possible to have closed trajectories that other trajectories neither approach nor depart from, for example, the periodic solutions of the predator–prey equations in Section 9.5. In this case the closed trajectory is **stable**.

In Example 1 the existence of an asymptotically stable limit cycle was established by solving the equations explicitly. Unfortunately, this is usually not possible, so it is worthwhile to know general theorems concerning the existence or nonexistence of limit cycles of nonlinear autonomous systems. In discussing these theorems it is convenient to rewrite the system (1) in the scalar form

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y). \tag{15}$$

Theorem 9.7.1 Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. A closed trajectory of the system (15) must necessarily enclose at least one critical (equilibrium) point. If it encloses only one critical point, the critical point cannot be a saddle point.

Although we omit the proof of this theorem, it is easy to show examples of it. One is given by Example 1 and Figure 9.7.1 in which the closed trajectory encloses the critical point $(0, 0)$, a spiral point. Another example is the system of predator–prey equations in Section 9.5; see Figure 9.5.2. Each closed trajectory surrounds the critical point $(3, 2)$; in this case the critical point is a center.

Theorem 9.7.1 is also useful in a negative sense. If a given region contains no critical points, then there can be no closed trajectory lying entirely in the region. The same conclusion is true if the region contains only one critical point, and this point is a saddle point. For instance, in Example 2 of Section 9.4, an example of competing species, the only critical point in the interior of the first quadrant is the saddle point $(0.5, 0.5)$. Therefore this system has no closed trajectory lying in the first quadrant.

A second result about the nonexistence of closed trajectories is given in the following theorem.

Theorem 9.7.2 Let the functions F and G have continuous first partial derivatives in a simply connected domain D of the xy-plane. If $F_x + G_y$ has the same sign throughout D, then there is no closed trajectory of the system (15) lying entirely in D.

A simply connected two-dimensional domain is one with no holes. Theorem 9.7.2 is a straightforward consequence of Green's theorem in the plane; see Problem 13. Note that if $F_x + G_y$ changes sign in the domain, then no conclusion can be drawn; there may or may not be closed trajectories in D.

To illustrate Theorem 9.7.2 consider the system (4). A routine calculation shows that

$$F_x(x, y) + G_y(x, y) = 2 - 4(x^2 + y^2) = 2(1 - 2r^2), \tag{16}$$

where, as usual, $r^2 = x^2 + y^2$. Hence $F_x + G_y$ is positive for $0 \le r < 1/\sqrt{2}$, so there is no closed trajectory in this circular disk. Of course, we showed in Example 1 that there is no closed trajectory in the larger region $r < 1$. This illustrates that the information given by Theorem 9.7.2 may not be the best possible result. Again referring to Eq. (16), note that $F_x + G_y < 0$ for $r > 1/\sqrt{2}$. However, the theorem is not applicable in this case because this annular region is not simply connected. Indeed, as shown in Example 1, it does contain a limit cycle.

The following theorem gives conditions that guarantee the existence of a closed trajectory.

Theorem 9.7.3 (Poincaré–Bendixson[6] Theorem). Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. Let D_1 be a bounded subdomain in D, and let R be the region that consists of D_1 plus its boundary (all points of R are in D). Suppose that R contains no critical point of the system (15). If there exists a constant t_0 such that $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (15) that exists and stays in R for all $t \ge t_0$, then either $x = \phi(t)$, $y = \psi(t)$ is a periodic

[6]Ivar Otto Bendixson (1861–1935) was a Swedish mathematician. This theorem appeared in a paper published by him in *Acta Mathematica* in 1901.

solution (closed trajectory), or $x = \phi(t)$, $y = \psi(t)$ spirals toward a closed trajectory as $t \to \infty$. In either case, the system (15) has a periodic solution in R.

Note that if R does contain a closed trajectory, then necessarily, by Theorem 9.7.1, this trajectory must enclose a critical point. However, this critical point cannot be in R. Thus R cannot be simply connected; it must have a hole.

As an application of the Poincaré–Bendixson theorem, consider again the system (4). Since the origin is a critical point, it must be excluded. For instance, we can consider the region R defined by $0.5 \le r \le 2$. Next, we must show that there is a solution whose trajectory stays in R for all t greater than or equal to some t_0. This follows immediately from Eq. (8). For $r = 0.5$, $dr/dt > 0$, so r increases, while for $r = 2$, $dr/dt < 0$, so r decreases. Thus any trajectory that crosses the boundary of R is entering R. Consequently, any solution of Eqs. (4) that starts in R at $t = t_0$ cannot leave but must stay in R for $t > t_0$. Of course, other numbers could be used instead of 0.5 and 2; all that is important is that $r = 1$ is included.

One should not infer from this discussion of the preceding theorems that it is easy to determine whether a given nonlinear autonomous system has periodic solutions or not; often it is not a simple matter at all. Theorems 9.7.1 and 9.7.2 are frequently inconclusive, while for Theorem 9.7.3 it is often difficult to determine a region R and a solution that always remains within it.

We close this section with another example of a nonlinear system that has a limit cycle.

EXAMPLE 2

The van der Pol (1889–1959) equation

$$u'' - \mu(1 - u^2)u' + u = 0, \tag{17}$$

where μ is a positive constant, describes the current u in a triode oscillator. Discuss the solutions of this equation.

If $\mu = 0$, Eq. (17) reduces to $u'' + u = 0$, whose solutions are sine or cosine waves of period 2π. For $\mu > 0$ the second term on the left side of Eq. (17) must also be considered. This is the resistance term, proportional to u', with a coefficient $-\mu(1 - u^2)$ that depends on u. For large u this term is positive and acts as usual to reduce the amplitude of the response. However, for small u the resistance term is negative and so causes the response to grow. This suggests that perhaps there is a solution of intermediate size that other solutions approach as t increases.

To analyze Eq. (17) more carefully we write it as a second order system by introducing the variables $x = u$, $y = u'$. Then it follows that

$$x' = y, \qquad y' = -x + \mu(1 - x^2)y. \tag{18}$$

The only critical point of the system (18) is the origin. Near the origin the corresponding linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{19}$$

whose eigenvalues are $(\mu \pm \sqrt{\mu^2 - 4})/2$. Thus the origin is an unstable spiral point for $0 < \mu < 2$ and an unstable node for $\mu \ge 2$. In all cases, a solution that starts near the origin grows as t increases.

With regard to periodic solutions Theorems 9.7.1 and 9.7.2 provide only partial information. From Theorem 9.7.1 we conclude that if there are closed trajectories, they must enclose the origin. Next, we calculate $F_x(x, y) + G_y(x, y)$, with the result that

$$F_x(x, y) + G_y(x, y) = \mu(1 - x^2). \tag{20}$$

Then, it follows from Theorem 9.7.2 that closed trajectories, if there are any, are not contained in the strip $|x| < 1$ where $F_x + G_y > 0$.

The application of the Poincaré–Bendixson theorem to this problem is not nearly as simple as for the preceding example. If we introduce polar coordinates, we find that the equation for the radial variable r is

$$r' = \mu(1 - r^2 \cos^2\theta) r \sin^2\theta. \tag{21}$$

Again, consider an annular region R given by $r_1 \le r \le r_2$, where r_1 is small and r_2 is large. When $r = r_1$, the linear term on the right side of Eq. (21) dominates, and $r' > 0$ except on the x-axis, where $\sin\theta = 0$ and consequently $r' = 0$ also. Thus, trajectories are entering R at every point on the circle $r = r_1$ except possibly for those on the x-axis, where the trajectories are tangent to the circle. When $r = r_2$, the cubic term on the right side of Eq. (21) is the dominant one. Thus $r' < 0$ except for points on the x-axis where $r' = 0$ and for points near the y-axis where $r^2 \cos^2\theta < 1$ and the linear term makes $r' > 0$. Thus, no matter how large a circle is chosen, there will be points on it (namely, the points on or near the y-axis) where trajectories are leaving R. Therefore, the Poincaré–Bendixson theorem is not applicable unless we consider more complicated regions.

It is possible to show, by a more intricate analysis, that the van der Pol equation does have a unique limit cycle. However, we will not follow this line of argument further, but turn instead to a different approach in which we plot numerically computed solutions. Experimental observations indicate that the van der Pol equation has a stable periodic solution whose period and amplitude depend on the parameter μ. By looking at graphs of trajectories in the phase plane and of u versus t we can gain some understanding of this periodic behavior.

Figure 9.7.2 shows two trajectories of the van der Pol equation in the phase plane for $\mu = 0.2$. The trajectory starting near the origin spirals outward in the clockwise

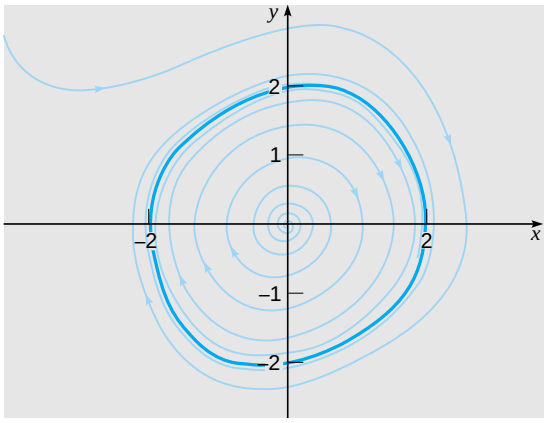


FIGURE 9.7.2 Trajectories of the van der Pol equation (17) for $\mu = 0.2$.

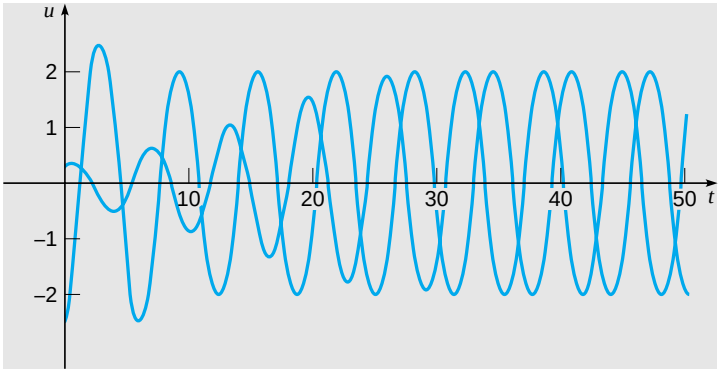


FIGURE 9.7.3 Plots of u versus t for the trajectories in Figure 9.7.2.

direction; this is consistent with the behavior of the linear approximation near the origin. The other trajectory passes through $(-3, 2)$ and spirals inward, again in the clockwise direction. Both trajectories approach a closed curve that corresponds to a stable periodic solution. In Figure 9.7.3 we show the plots of u versus t for the solutions corresponding to the trajectories in Figure 9.7.2. The solution that is initially smaller gradually increases in amplitude, while the larger solution gradually decays. Both solutions approach a stable periodic motion that corresponds to the limit cycle. Figure 9.7.3 also shows that there is a phase difference between the two solutions as they approach the limit cycle. The plots of u versus t are nearly sinusoidal in shape, consistent with the nearly circular limit cycle in this case.

Figures 9.7.4 and 9.7.5 show similar plots for the case $\mu = 1$. Trajectories again move clockwise in the phase plane, but the limit cycle is considerably different from a

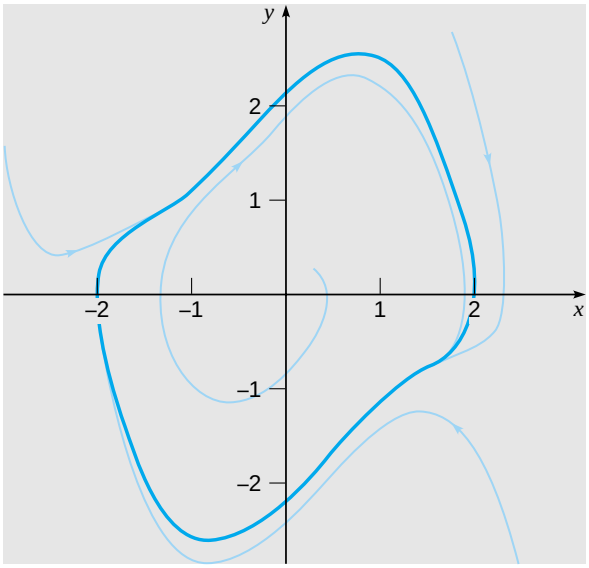


FIGURE 9.7.4 Trajectories of the van der Pol equation (17) for $\mu = 1$.

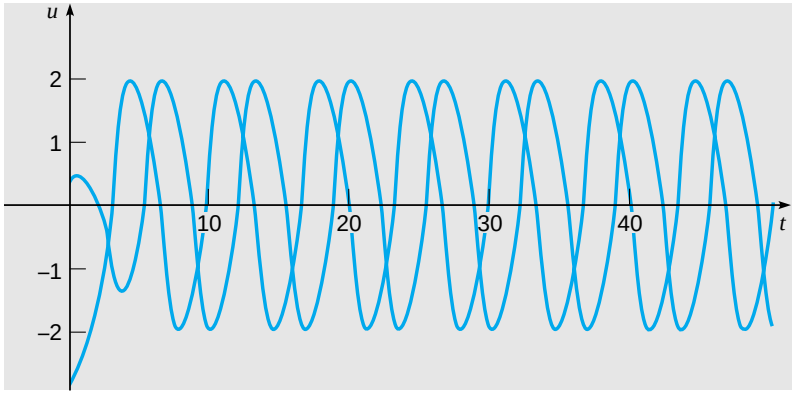


FIGURE 9.7.5 Plots of u versus t for the trajectories in Figure 9.7.4.

circle. The plots of u versus t tend more rapidly to the limiting oscillation, and again show a phase difference. The oscillations are somewhat less symmetric in this case, rising somewhat more steeply than they fall.

Figure 9.7.6 shows the phase plane for $\mu = 5$. The motion remains clockwise, and the limit cycle is even more elongated, especially in the y direction. In Figure 9.7.7 is a plot of u versus t. Although the solution starts far from the limit cycle, the limiting oscillation is virtually reached in a fraction of a period. Starting from one of its extreme values on the x-axis in the phase plane, the solution moves toward the other extreme position slowly at first, but once a certain point on the trajectory is reached, the remainder of the transition is completed very swiftly. The process is then repeated

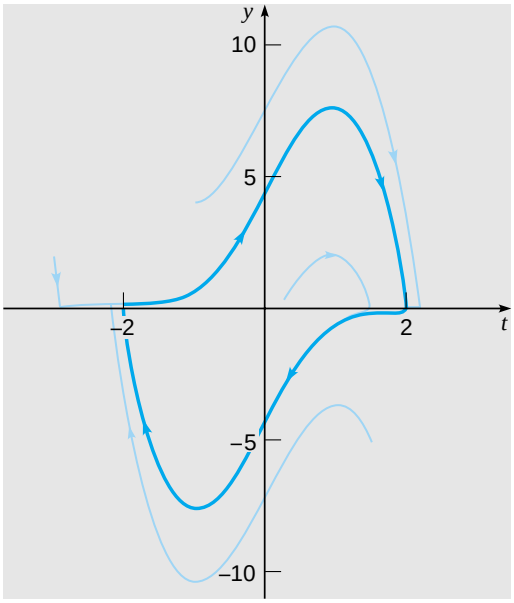


FIGURE 9.7.6 Trajectories of the van der Pol equation (17) for $\mu = 5$.

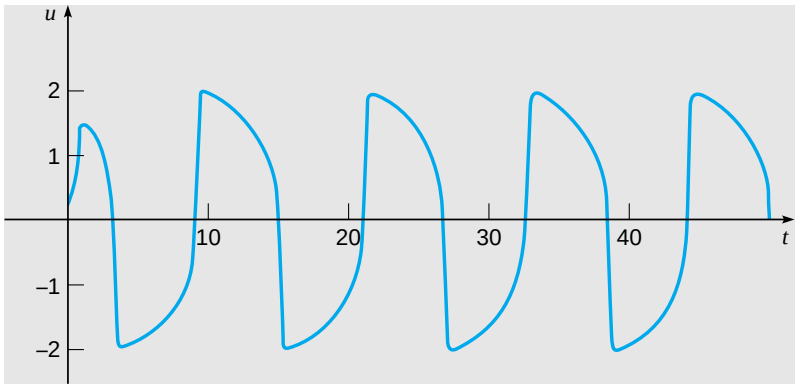


FIGURE 9.7.7 Plot of u versus t for the outward spiralling trajectory in Figure 9.7.6.

in the opposite direction. The waveform of the limit cycle, as shown in Figure 9.7.7, is quite different from a sine wave.

These graphs clearly show that, in the absence of external excitation, the van der Pol oscillator has a certain characteristic mode of vibration for each value of μ. The graphs of u versus t show that the amplitude of this oscillation changes very little with μ, but the period increases as μ increases. At the same time, the waveform changes from one that is very nearly sinusoidal to one that is much less smooth.

The presence of a single periodic motion that attracts all (nearby) solutions, that is, a stable limit cycle, is one of the characteristic phenomena associated with nonlinear differential equations.

PROBLEMS

In each of Problems 1 through 6 an autonomous system is expressed in polar coordinates. Determine all periodic solutions, all limit cycles, and determine their stability characteristics.

1. $dr/dt = r^2(1-r^2), \qquad d\theta/dt = 1$
2. $dr/dt = r(1-r)^2, \qquad d\theta/dt = -1$
3. $dr/dt = r(r-1)(r-3), \qquad d\theta/dt = 1$
4. $dr/dt = r(1-r)(r-2), \qquad d\theta/dt = -1$
5. $dr/dt = \sin \pi r, \qquad d\theta/dt = 1$
6. $dr/dt = r|r-2|(r-3), \qquad d\theta/dt = -1$
7. If $x = r\cos\theta$, $y = r\sin\theta$, show that $y(dx/dt) - x(dy/dt) = -r^2(d\theta/dt)$.
8. (a) Show that the system

$$dx/dt = -y + xf(r)/r, \qquad dy/dt = x + yf(r)/r$$

has periodic solutions corresponding to the zeros of $f(r)$. What is the direction of motion on the closed trajectories in the phase plane?

(b) Let $f(r) = r(r-2)^2(r^2-4r+3)$. Determine all periodic solutions and determine their stability characteristics.

9. Determine the periodic solutions, if any, of the system

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \qquad \frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2).$$

10. Using Theorem 9.7.2, show that the linear autonomous system

$$dx/dt = a_{11}x + a_{12}y, \qquad dy/dt = a_{21}x + a_{22}y$$

does not have a periodic solution (other than $x = 0$, $y = 0$) if $a_{11} + a_{22} \neq 0$.

In each of Problems 11 and 12 show that the given system has no periodic solutions other than constant solutions.

11. $dx/dt = x + y + x^3 - y^2, \qquad dy/dt = -x + 2y + x^2y + y^3/3$
12. $dx/dt = -2x - 3y - xy^2, \qquad dy/dt = y + x^3 - x^2y$
13. Prove Theorem 9.7.2 by completing the following argument. According to Green's theorem in the plane, if C is a sufficiently smooth simple closed curve, and if F and G are continuous and have continuous first partial derivatives, then

$$\int_C [F(x, y)\, dy - G(x, y)\, dx] = \iint_R [F_x(x, y) + G_y(x, y)]\, dA,$$

where C is traversed counterclockwise and R is the region enclosed by C. Assume that $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (15) that is periodic with period T. Let C be the closed curve given by $x = \phi(t)$, $y = \psi(t)$ for $0 \leq t \leq T$. Show that for this curve the line integral is zero. Then show that the conclusion of Theorem 9.7.2 must follow.

▶ 14. (a) By examining the graphs of u versus t in Figures 9.7.3, 9.7.5, and 9.7.7 estimate the period T of the van der Pol oscillator in these cases.
(b) Calculate and plot the graphs of solutions of the van der Pol equation for other values of the parameter μ. Estimate the period T in these cases also.
(c) Plot the estimated values of T versus μ. Describe how T depends on μ.

▶ 15. The equation

$$u'' - \mu(1 - \tfrac{1}{3}u'^2)u' + u = 0$$

is often called the Rayleigh equation.
(a) Write the Rayleigh equation as a system of two first order equations.
(b) Show that the origin is the only critical point of this system. Determine its type and whether it is stable or unstable.
(c) Let $\mu = 1$. Choose initial conditions and compute the corresponding solution of the system on an interval such as $0 \leq t \leq 20$ or longer. Plot u versus t and also plot the trajectory in the phase plane. Observe that the trajectory approaches a closed curve (limit cycle). Estimate the amplitude A and the period T of the limit cycle.
(d) Repeat part (c) for other values of μ, such as $\mu = 0.2, 0.5, 2$, and 5. In each case estimate the amplitude A and the period T.
(e) Describe how the limit cycle changes as μ increases. For example, make a table of values and/or plot A and T as functions of μ.

▶ 16. The system

$$x' = 3(x + y - \tfrac{1}{3}x^3 - k), \qquad y' = -\tfrac{1}{3}(x + 0.8y - 0.7)$$

is a special case of the Fitzhugh–Nagumo equations, which model the transmission of neural impulses along an axon. The parameter k is the external stimulus.
(a) For $k = 0$ show that there is one critical point. Find this point and show that it is an asymptotically stable spiral point. Repeat the analysis for $k = 0.5$ and show that the critical point is now an unstable spiral point. Draw a phase portrait for the system in each case.
(b) Find the value k_0 where the critical point changes from asymptotically stable to unstable. Draw a phase portrait for the system for $k = k_0$.
(c) For $k \geq k_0$ the system exhibits an asymptotically stable limit cycle. Plot x versus t for $k = k_0$ for several periods and estimate the value of the period T.
(d) The limit cycle actually exists for a small range of k below k_0. Let k_1 be the smallest value of k for which there is a limit cycle. Find k_1.

9.8 Chaos and Strange Attractors: The Lorenz Equations



In principle, the methods described in this chapter for second order autonomous systems can also be applied to higher order systems as well. In practice, there are several difficulties that arise in trying to do this. One problem is that there is simply a greater number of possible cases that can occur, and the number increases with the order of the system (and the dimension of the phase space). Another is the difficulty of graphing trajectories accurately in a phase space of higher than two dimensions; even in three dimensions it may not be easy to construct a clear and understandable plot of the trajectories, and it becomes more difficult as the number of variables increases. Finally, and this has only been clearly realized in the last few years, there are different and very complex phenomena that can occur, and do frequently occur, in systems of third and higher order that are not present in second order systems. Our goal in this section is to provide a brief introduction to some of these phenomena by discussing one particular third order autonomous system that has been intensively studied in recent years. In some respects the presentation here is similar to the treatment of the logistic difference equation in Section 2.9.

An important problem in meteorology, and in other applications of fluid dynamics, concerns the motion of a layer of fluid, such as the earth's atmosphere, that is warmer at the bottom than at the top; see Figure 9.8.1. If the vertical temperature difference ΔT is small, then there is a linear variation of temperature with altitude, but no significant motion of the fluid layer. However, if ΔT is large enough, then the warmer air rises, displacing the cooler air above it, and a steady convective motion results. If the temperature difference increases further, then eventually the steady convective flow breaks up and a more complex and turbulent motion ensues.

While investigating this phenomenon, Edward N. Lorenz[7] was led (by a process too involved to describe here) to the nonlinear autonomous third order system

$$\begin{aligned} dx/dt &= \sigma(-x + y), \\ dy/dt &= rx - y - xz, \\ dz/dt &= -bz + xy. \end{aligned} \tag{1}$$

Equations (1) are now commonly referred to as the Lorenz equations[8]. Observe that the second and third equations involve quadratic nonlinearities. However, except for being

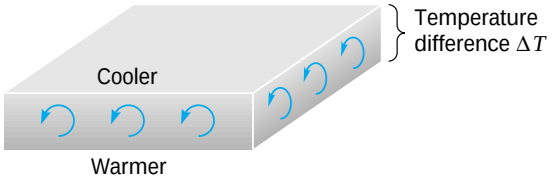


FIGURE 9.8.1 A layer of fluid heated from below.

[7]Edward N. Lorenz (1917–), American meteorologist, received his Ph.D. from the Massachusetts Institute of Technology in 1948 and has been associated with that institution throughout his scientific career. The Lorenz equations were first studied by him in a famous paper published in 1963 dealing with the stability of fluid flows in the atmosphere.

[8]A very thorough treatment of the Lorenz equations appears in the book by Sparrow listed in the references at the end of the chapter.

a third order system, superficially the Lorenz equations appear no more complicated than the competing species or predator–prey equations discussed in Sections 9.4 and 9.5. The variable x in Eqs. (1) is related to the intensity of the fluid motion, while the variables y and z are related to the temperature variations in the horizontal and vertical directions. The Lorenz equations also involve three parameters σ, r, and b, all of which are real and positive. The parameters σ and b depend on the material and geometrical properties of the fluid layer. For the earth's atmosphere reasonable values of these parameters are $\sigma = 10$ and $b = 8/3$; they will be assigned these values in much of what follows in this section. The parameter r, on the other hand, is proportional to the temperature difference ΔT, and our purpose is to investigate how the nature of the solutions of Eqs. (1) changes with r.

The first step in analyzing the Lorenz equations is to locate the critical points by solving the algebraic system

$$\begin{aligned} \sigma x - \sigma y &= 0, \\ rx - y - xz &= 0, \\ -bz + xy &= 0. \end{aligned} \tag{2}$$

From the first equation we have $y = x$. Then, eliminating y from the second and third equations, we obtain

$$x(r - 1 - z) = 0, \tag{3}$$

$$-bz + x^2 = 0. \tag{4}$$

One way to satisfy Eq. (3) is to choose $x = 0$. Then it follows that $y = 0$ and, from Eq. (4), $z = 0$. Alternatively, we can satisfy Eq. (3) by choosing $z = r - 1$. Then Eq. (4) requires that $x = \pm\sqrt{b(r-1)}$ and then $y = \pm\sqrt{b(r-1)}$ also. Observe that these expressions for x and y are real only when $r \geq 1$. Thus (0, 0, 0), which we will denote by P_1, is a critical point for all values of r, and it is the only critical point for $r < 1$. However, when $r > 1$, there are also two other critical points, namely, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$. We will denote the latter two points by P_2 and P_3, respectively. Note that all three critical points coincide when $r = 1$. As r increases through the value 1, the critical point P_1 at the origin *bifurcates* and the critical points P_2 and P_3 come into existence.

Next we will determine the local behavior of solutions in the neighborhood of each critical point. Although much of the following analysis can be carried out for arbitrary values of σ and b, we will simplify our work by using the values $\sigma = 10$ and $b = 8/3$. Near the origin (the critical point P_1) the approximating linear system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}$$

The eigenvalues are determined from the equation

$$\begin{vmatrix} -10-\lambda & 10 & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -8/3-\lambda \end{vmatrix} = -(8/3+\lambda)[\lambda^2 + 11\lambda - 10(r-1)] = 0. \tag{6}$$

Therefore

$$\lambda_1 = -\frac{8}{3}, \qquad \lambda_2 = \frac{-11 - \sqrt{81 + 40r}}{2}, \qquad \lambda_3 = \frac{-11 + \sqrt{81 + 40r}}{2}. \tag{7}$$

Note that all three eigenvalues are negative for $r < 1$; for example, when $r = 1/2$, the eigenvalues are $\lambda_1 = -8/3$, $\lambda_2 = -10.52494$, $\lambda_3 = -0.47506$. Hence the origin is asymptotically stable for this range of r both for the linear approximation (5) and for the original system (1). However, λ_3 changes sign when $r = 1$ and is positive for $r > 1$. The value $r = 1$ corresponds to the initiation of convective flow in the physical problem described earlier. The origin is unstable for $r > 1$; all solutions starting near the origin tend to grow except for those lying precisely in the plane determined by the eigenvectors associated with λ_1 and λ_2 [or, for the nonlinear system (1), in a certain surface tangent to this plane at the origin].

Next let us consider the neighborhood of the critical point $P_2(\sqrt{8(r-1)/3}, \sqrt{8(r-1)/3}, r-1)$ for $r > 1$. If u, v, and w are the perturbations from the critical point in the x, y, and z directions, respectively, then the approximating linear system is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{8}$$

The eigenvalues of the coefficient matrix of Eq. (8) are determined from the equation

$$3\lambda^3 + 41\lambda^2 + 8(r + 10)\lambda + 160(r - 1) = 0, \tag{9}$$

which is obtained by straightforward algebraic steps that are omitted here. The solutions of Eq. (9) depend on r in the following way:

For $1 < r < r_1 \cong 1.3456$ there are three negative real eigenvalues.

For $r_1 < r < r_2 \cong 24.737$ there are one negative real eigenvalue and two complex eigenvalues with negative real part.

For $r_2 < r$ there are one negative real eigenvalue and two complex eigenvalues with positive real part.

The same results are obtained for the critical point P_3. Thus there are several different situations.

For $0 < r < 1$ the only critical point is P_1 and it is asymptotically stable. All solutions approach this point (the origin) as $t \to \infty$.

For $1 < r < r_1$ the critical points P_2 and P_3 are asymptotically stable and P_1 is unstable. All nearby solutions approach one or the other of the points P_2 and P_3 exponentially.

For $r_1 < r < r_2$ the critical points P_2 and P_3 are asymptotically stable and P_1 is unstable. All nearby solutions approach one or the other of the points P_2 and P_3; most of them spiral inward to the critical point.

For $r_2 < r$ all three critical points are unstable. Most solutions near P_2 or P_3 spiral away from the critical point.

However, this is by no means the end of the story. Let us consider solutions for r somewhat greater than r_2. In this case P_1 has one positive eigenvalue and each of P_2 and P_3 has a pair of complex eigenvalues with positive real part. A trajectory can approach any one of the critical points only on certain highly restricted paths. The

slightest deviation from these paths causes the trajectory to depart from the critical point. Since none of the critical points is stable, one might expect that most trajectories will approach infinity for large t. However, it can be shown that all solutions remain bounded as $t \to \infty$; see Problem 5. In fact, it can be shown that all solutions ultimately approach a certain limiting set of points that has zero volume. Indeed, this is true not only for $r > r_2$ but for all positive values of r.

A plot of computed values of x versus t for a typical solution with $r > r_2$ is shown in Figure 9.8.2. Note that the solution oscillates back and forth between positive and negative values in a rather erratic manner. Indeed, the graph of x versus t resembles a random vibration, although the Lorenz equations are entirely deterministic and the solution is completely determined by the initial conditions. Nevertheless, the solution also exhibits a certain *regularity* in that the frequency and amplitude of the oscillations are essentially constant in time.

The solutions of the Lorenz equations are also extremely sensitive to perturbations in the initial conditions. Figure 9.8.3 shows the graphs of computed values of x versus t

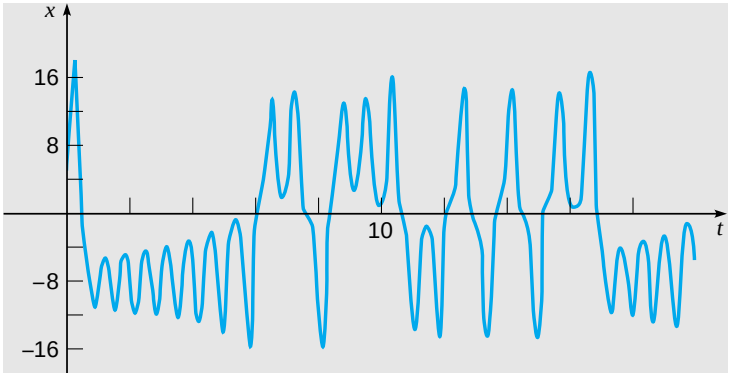


FIGURE 9.8.2 A plot of x versus t for the Lorenz equations (1) with $r = 28$; initial point is (5, 5, 5).

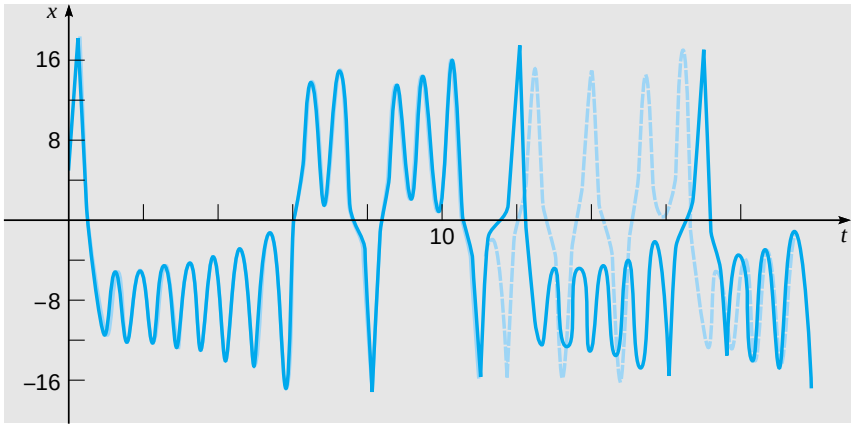


FIGURE 9.8.3 Plots of x versus t for two initially nearby solutions of Lorenz equations with $r = 28$; initial point is (5, 5, 5) for dashed curve and (5.01, 5, 5) for solid curve.

for the two solutions whose initial points are $(5, 5, 5)$ and $(5.01, 5, 5)$. The dashed graph is the same as the one in Figure 9.8.2, while the solid graph starts at a nearby point. The two solutions remain close until t is near 10, after which they are quite different and, indeed, seem to have no relation to each other. It was this property that particularly attracted the attention of Lorenz in his original study of these equations, and caused him to conclude that detailed long-range weather predictions are probably not possible.

The attracting set in this case, although of zero volume, has a rather complicated structure and is called a **strange attractor**. The term **chaotic** has come into general use to describe solutions such as those shown in Figures 9.8.2 and 9.8.3.

To determine how and when the strange attractor is created it is illuminating to investigate solutions for smaller values of r. For $r = 21$ solutions starting at three different initial points are shown in Figure 9.8.4. For the initial point $(3, 8, 0)$ the solution begins to converge to the point P_3 almost at once; see Figure 9.8.4*a*. For the second initial point $(5, 5, 5)$ there is a fairly short interval of transient behavior, after which the solution converges to P_2; see Figure 9.8.4*b*. However, as shown in Figure 9.8.4*c*, for the third initial point $(5, 5, 10)$ there is a much longer interval of transient chaotic behavior before the solution eventually converges to P_2. As r increases, the duration of the chaotic transient behavior also increases. When $r = r_3 \cong 24.06$, the chaotic transients appear to continue indefinitely and the strange attractor comes into being.

One can also show the trajectories of the Lorenz equations in the three-dimensional phase space, or at least projections of them in various planes. Figures 9.8.5 and 9.8.6

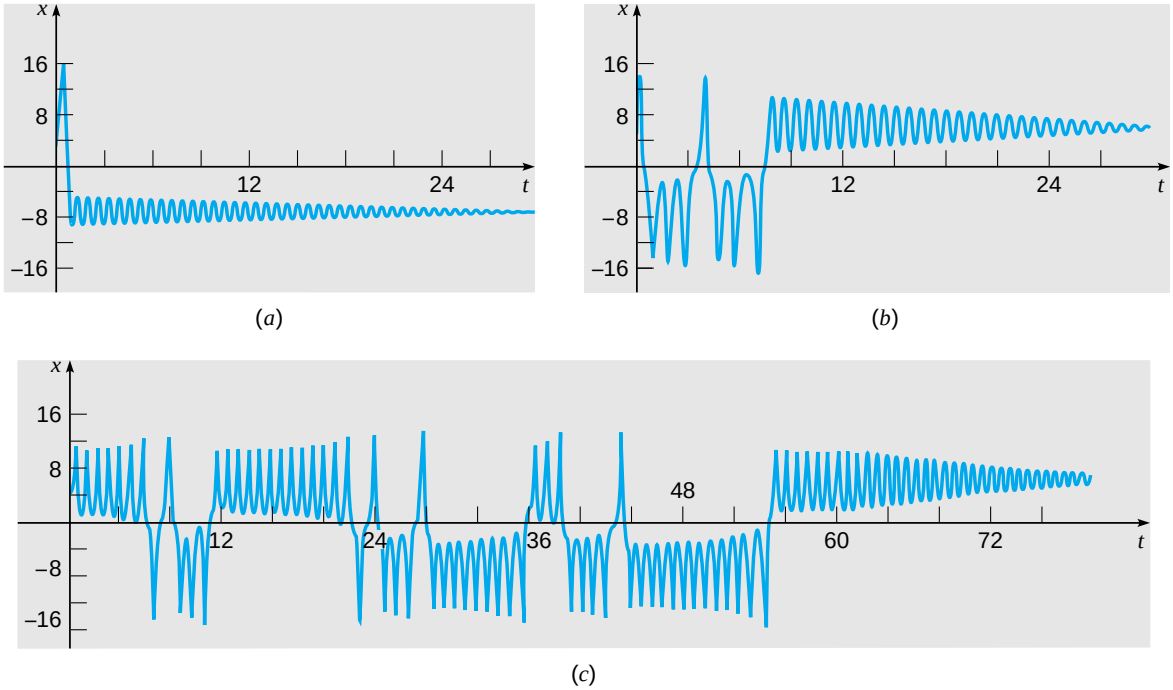


FIGURE 9.8.4 Plots of x versus t for three solutions of Lorenz equations with $r = 21$. (*a*) Initial point is (3, 8, 0). (*b*) Initial point is (5, 5, 5). (*c*) Initial point is (5, 5, 10).

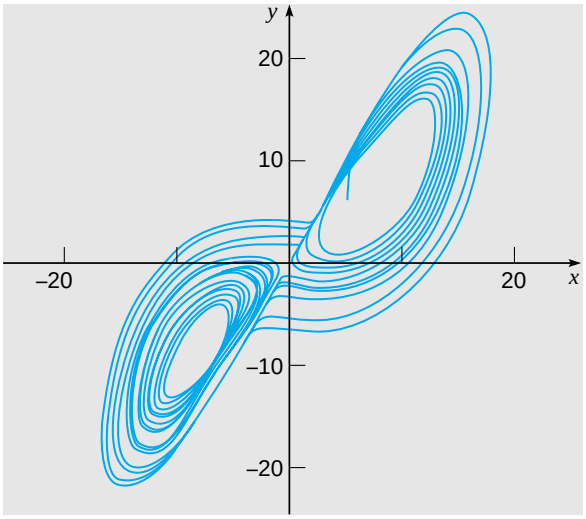


FIGURE 9.8.5 Projections of a trajectory of the Lorenz equations (with $r = 28$) in the xy-plane.

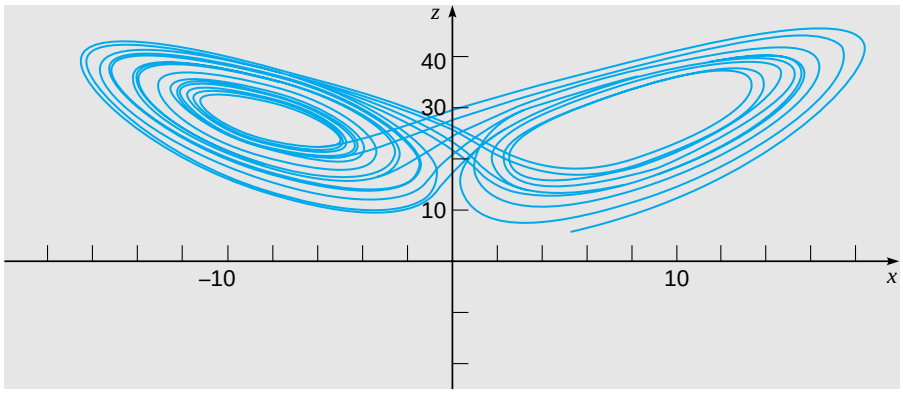


FIGURE 9.8.6 Projections of a trajectory of the Lorenz equations (with $r = 28$) in the xz-plane.

show projections in the xy- and xz-planes, respectively, of the trajectory starting at $(5, 5, 5)$. Observe that the graphs in these figures appear to cross over themselves repeatedly, but this cannot be true for the actual trajectories in three-dimensional space because of the general uniqueness theorem. The apparent crossings are due wholly to the two-dimensional character of the figures.

The sensitivity of solutions to perturbations of the initial data also has implications for numerical computations, such as those reported here. Different step sizes, different numerical algorithms, or even the execution of the same algorithm on different machines will introduce small differences in the computed solution, which eventually lead to large deviations. For example, the exact sequence of positive and negative loops in the calculated solution depends strongly on the precise numerical algorithm and its

implementation, as well as on the initial conditions. However, the general appearance of the solution and the structure of the attracting set is independent of all these factors.

Solutions of the Lorenz equations for other parameter ranges exhibit other interesting types of behavior. For example, for certain values of r greater than r_2, intermittent bursts of chaotic behavior separate long intervals of apparently steady periodic oscillation. For other ranges of r, solutions show the period-doubling property that we saw in Section 2.9 for the logistic difference equation. Some of these features are taken up in the problems.

Since about 1975 the Lorenz equations and other higher order autonomous systems have been studied intensively, and this is one of the most active areas of current mathematical research. Chaotic behavior of solutions appears to be much more common than was suspected at first, and many questions remain unanswered. Some of these are mathematical in nature, while others relate to the physical applications or interpretations of solutions.

PROBLEMS

Problems 1 through 3 ask you to fill in some of the details of the analysis of the Lorenz equations in the text.

1. (a) Show that the eigenvalues of the linear system (5), valid near the origin, are given by Eq. (7).
   (b) Determine the corresponding eigenvectors.
   (c) Determine the eigenvalues and eigenvectors of the system (5) in the case where $r = 28$.
2. (a) Show that the linear approximation valid near the critical point P_2 is given by Eq. (8).
   (b) Show that the eigenvalues of the system (8) satisfy Eq. (9).
   (c) For $r = 28$ solve Eq. (9) and thereby determine the eigenvalues of the system (8).
3. (a) By solving Eq. (9) numerically show that the real part of the complex roots changes sign when $r \cong 24.737$.
   (b) Show that a cubic polynomial $x^3 + Ax^2 + Bx + C$ has one real zero and two pure imaginary zeros only if $AB = C$.
   (c) By applying the result of part (b) to Eq. (9) show that the real part of the complex roots changes sign when $r = 470/19$.
4. Use the Liapunov function $V(x, y, z) = x^2 + \sigma y^2 + \sigma z^2$ to show that the origin is a globally asymptotically stable critical point for the Lorenz equations (1) if $r < 1$.
5. Consider the ellipsoid

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 = c > 0.$$

   (a) Calculate dV/dt along trajectories of the Lorenz equations (1).
   (b) Determine a sufficient condition on c so that every trajectory crossing $V(x, y, z) = c$ is directed inward.
   (c) Evaluate the condition found in part (b) for the case $\sigma = 10$, $b = 8/3$, $r = 28$.

In each of Problems 6 through 10 carry out the indicated investigations of the Lorenz equations.

6. For $r = 28$ plot x versus t for the cases shown in Figures 9.8.2 and 9.8.3. Do your graphs agree with those shown in the figures? Recall the discussion of numerical computation in the text.
7. For $r = 28$ plot the projections in the xy- and xz-planes, respectively, of the trajectory starting at the point $(5, 5, 5)$. Do the graphs agree with those in Figures 9.8.5 and 9.8.6?
8. (a) For $r = 21$ plot x versus t for the solutions starting at the initial points $(3, 8, 0)$, $(5, 5, 5)$, and $(5, 5, 10)$. Use a t interval of at least $0 \le t \le 30$. Compare your graphs with those in Figure 9.8.4.

(b) Repeat the calculation in part (a) for $r = 22, r = 23$, and $r = 24$. Increase the t interval as necessary so that you can determine when each solution begins to converge to one of the critical points. Record the approximate duration of the chaotic transient in each case. Describe how this quantity depends on the value of r.

(c) Repeat the calculations in parts (a) and (b) for values of r slightly greater than 24. Try to estimate the value of r for which the duration of the chaotic transient approaches infinity.

▶ 9. For certain r intervals, or windows, the Lorenz equations exhibit a period-doubling property similar to that of the logistic difference equation discussed in Section 2.9. Careful calculations may reveal this phenomenon.

(a) One period-doubling window contains the value $r = 100$. Let $r = 100$ and plot the trajectory starting at $(5, 5, 5)$ or some other initial point of your choice. Does the solution appear to be periodic? What is the period?

(b) Repeat the calculation in part (a) for slightly smaller values of r. When $r \cong 99.98$, you may be able to observe that the period of the solution doubles. Try to observe this result by performing calculations with nearby values of r.

(c) As r decreases further, the period of the solution doubles repeatedly. The next period doubling occurs at about $r = 99.629$. Try to observe this by plotting trajectories for nearby values of r.

▶ 10. Now consider values of r slightly larger than those in Problem 9.

(a) Plot trajectories of the Lorenz equations for values of r between 100 and 100.78. You should observe a steady periodic solution for this range of r values.

(b) Plot trajectories for values of r between 100.78 and 100.8. Determine as best you can how and when the periodic trajectory breaks up.

REFERENCES

There are many recent books that expand on the material treated in this chapter, for example:

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Brauer, F., and Nohel, J., *Qualitative Theory of Ordinary Differential Equations* (New York: Benjamin, 1969; New York: Dover, 1989).

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Pielou, E. C., *Mathematical Ecology* (New York: Wiley, 1977).

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Lorenz, E. N., "Deterministic Nonperiodic Flow," *Journal of the Atmospheric Sciences 20* (1963), pp. 130–141.

A very detailed treatment of the Lorenz equations is:

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CHAPTER
10

Partial Differential Equations and Fourier Series

In many important physical problems there are two or more independent variables, so that the corresponding mathematical models involve partial, rather than ordinary, differential equations. This chapter treats one important method for solving partial differential equations, a method known as separation of variables. Its essential feature is the replacement of the partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions. The first section of this chapter deals with some basic properties of boundary value problems for ordinary differential equations. The desired solution of the partial differential equation is then expressed as a sum, usually an infinite series, formed from solutions of the ordinary differential equations. In many cases we ultimately need to deal with a series of sines and/or cosines, so part of the chapter is devoted to a discussion of such series, which are known as Fourier series. With the necessary mathematical background in place, we then illustrate the use of separation of variables on a variety of problems arising from heat conduction, wave propagation, and potential theory.

10.1 Two-Point Boundary Value Problems

Up to this point in the book we have dealt with initial value problems, consisting of a differential equation together with suitable initial conditions at a given point. A typical example, which was discussed at length in Chapter 3, is the differential equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{1}$$

with the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2}$$

Physical applications often lead to another type of problem, one in which the value of the dependent variable y or its derivative is specified at two *different* points. Such conditions are called **boundary conditions** to distinguish them from initial conditions that specify the value of y and y' at the *same* point. A differential equation together with suitable boundary conditions form a **two-point boundary value problem**. A typical example is the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{3}$$

with the boundary conditions

$$y(\alpha) = y_0, \quad y(\beta) = y_1. \tag{4}$$

The natural occurrence of boundary value problems usually involves a space coordinate as the independent variable so we have used x rather than t in Eqs. (3) and (4). To solve the boundary value problem (3), (4) we need to find a function $y = \phi(x)$ that satisfies the differential equation (3) in the interval $\alpha < x < \beta$ and that takes on the specified values y_0 and y_1 at the endpoints of the interval. Usually, we seek first the general solution of the differential equation and then use the boundary conditions to determine the values of the arbitrary constants.

Boundary value problems can also be posed for nonlinear differential equations but we will restrict ourselves to a consideration of linear equations only. An important classification of linear boundary value problems is whether they are homogeneous or nonhomogeneous. If the function g has the value zero for each x, and if the boundary values y_0 and y_1 are also zero, then the problem (3), (4) is called **homogeneous**. Otherwise, the problem is **nonhomogeneous**.

Although the initial value problem (1), (2) and the boundary value problem (3), (4) may superficially appear to be quite similar, their solutions differ in some very important ways. Under mild conditions on the coefficients initial value problems are certain to have a unique solution. On the other hand, boundary value problems under similar conditions may have a unique solution, but may also have no solution or, in some cases, infinitely many solutions. In this respect, linear boundary value problems resemble systems of linear algebraic equations.

Let us recall some facts (see Section 7.3) about the system

$$\mathbf{Ax} = \mathbf{b}, \tag{5}$$

where $\mathbf{A}$ is a given $n \times n$ matrix, $\mathbf{b}$ is a given $n \times 1$ vector, and $\mathbf{x}$ is an $n \times 1$ vector to be determined. If $\mathbf{A}$ is nonsingular, then the system (5) has a unique solution for any $\mathbf{b}$. However, if $\mathbf{A}$ is singular, then the system (5) has no solution unless $\mathbf{b}$ satisfies a certain additional condition, in which case the system has infinitely many solutions. Now consider the corresponding homogeneous system

$$\mathbf{Ax} = \mathbf{0}, \tag{6}$$

obtained from the system (5) when $\mathbf{b} = \mathbf{0}$. The homogeneous system (6) always has the solution $\mathbf{x} = \mathbf{0}$. If $\mathbf{A}$ is nonsingular, then this is the only solution, but if $\mathbf{A}$ is singular, then there are infinitely many (nonzero) solutions. Note that it is impossible for the homogeneous system to have no solution. These results can also be stated in the following way: The nonhomogeneous system (5) has a unique solution if and only

if the homogeneous system (6) has only the solution $\mathbf{x} = \mathbf{0}$, and the nonhomogeneous system (5) has either no solution or infinitely many if and only if the homogeneous system (6) has nonzero solutions.

We now turn to some examples of linear boundary value problems that illustrate very similar behavior. A more general discussion of linear boundary value problems appears in Chapter 11.

EXAMPLE 1

Solve the boundary value problem

$$y'' + 2y = 0, \qquad y(0) = 1, \quad y(\pi) = 0. \tag{7}$$

The general solution of the differential equation (7) is

$$y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x. \tag{8}$$

The first boundary condition requires that $c_1 = 1$. The second boundary condition implies that $c_1 \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi = 0$, so $c_2 = -\cot \sqrt{2}\pi \cong -0.2762$. Thus the solution of the boundary value problem (7) is

$$y = \cos \sqrt{2}x - \cot \sqrt{2}\pi \sin \sqrt{2}x. \tag{9}$$

This example illustrates the case of a nonhomogeneous boundary value problem with a unique solution.

EXAMPLE 2

Solve the boundary value problem

$$y'' + y = 0, \qquad y(0) = 1, \quad y(\pi) = a, \tag{10}$$

where a is a given number.

The general solution of this differential equation is

$$y = c_1 \cos x + c_2 \sin x \tag{11}$$

and from the first boundary condition we find that $c_1 = 1$. The second boundary condition now requires that $-c_1 = a$. These two conditions on c_1 are incompatible if $a \neq -1$ so the problem has no solution in that case. However, if $a = -1$, then both boundary conditions are satisfied provided that $c_1 = 1$, regardless of the value of c_2. In this case there are infinitely many solutions of the form

$$y = \cos x + c_2 \sin x, \tag{12}$$

where c_2 remains arbitrary. This example illustrates that a nonhomogeneous boundary value problem may have no solution, and also that under special circumstances it may have infinitely many solutions.

Corresponding to the nonhomogeneous boundary value problem (3), (4) is the homogeneous problem consisting of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{13}$$

and the boundary conditions

$$y(\alpha) = 0, \quad y(\beta) = 0. \tag{14}$$

Observe that this problem has the solution $y = 0$ for all x regardless of the coefficients $p(x)$ and $q(x)$. This solution is often called the trivial solution and is rarely of interest. What we usually want to know is whether the problem has other, nonzero, solutions. Consider the following two examples.

EXAMPLE 3

Solve the boundary value problem

$$y'' + 2y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \tag{15}$$

The general solution of the differential equation is again given by Eq. (8),

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x.$$

The first boundary condition requires that $c_1 = 0$ and the second boundary condition leads to $c_2 \sin\sqrt{2}\pi = 0$. Since $\sin\sqrt{2}\pi \neq 0$, it follows that $c_2 = 0$ also. Consequently, $y = 0$ for all x is the only solution of the problem (15). This example illustrates that a homogeneous boundary value problem may have only the trivial solution $y = 0$.

EXAMPLE 4

Solve the boundary value problem

$$y'' + y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \tag{16}$$

The general solution is given by Eq. (11),

$$y = c_1 \cos x + c_2 \sin x,$$

and the first boundary condition requires that $c_1 = 0$. Since $\sin\pi = 0$, the second boundary condition is also satisfied regardless of the value of c_2. Thus the solution of the problem (16) is $y = c_2 \sin x$, where c_2 remains arbitrary. This example illustrates that a homogeneous boundary value problem may have infinitely many solutions.

Examples 1 through 4 illustrate (but of course do not prove) that there is the same relationship between homogeneous and nonhomogeneous linear boundary value problems as there is between homogeneous and nonhomogeneous linear algebraic systems. A nonhomogeneous boundary value problem (Example 1) has a unique solution and the corresponding homogeneous problem (Example 3) has only the trivial solution. Further, a nonhomogeneous problem (Example 2) has either no solution or infinitely many and the corresponding homogeneous problem (Example 4) has nontrivial solutions.

Eigenvalue Problems. Recall the matrix equation

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{17}$$

that was discussed in Section 7.3. Equation (17) has the solution $\mathbf{x} = \mathbf{0}$ for every value of λ but for certain values of λ, called eigenvalues, there are also other nonzero solutions, called eigenvectors. The situation is similar for boundary value problems.

Consider the problem consisting of the differential equation

$$y'' + \lambda y = 0, \tag{18}$$

together with the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0. \tag{19}$$

Observe that the problem (18), (19) is the same as the problems in Examples 3 and 4 if $\lambda = 2$ and $\lambda = 1$, respectively. Recalling the results of these examples, we note that for $\lambda = 2$, Eqs. (18), (19) have only the trivial solution $y = 0$, while for $\lambda = 1$, the problem (18), (19) has other, nontrivial, solutions. By extension of the terminology associated with Eq. (17) the values of λ for which nontrivial solutions of (18), (19) occur are called **eigenvalues** and the nontrivial solutions themselves are called **eigenfunctions**. Restating the results of Examples 3 and 4, we have found that $\lambda = 1$ is an eigenvalue of the problem (18), (19) and that $\lambda = 2$ is not. Further, any nonzero multiple of $\sin x$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

Let us now turn to the problem of finding other eigenvalues and eigenfunctions of the problem (18), (19). We need to consider separately the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, since the form of the solution of Eq. (18) is different in each of these cases. Suppose first that $\lambda > 0$. To avoid the frequent appearance of radical signs, it is convenient to let $\lambda = \mu^2$ and to rewrite Eq. (18) as

$$y'' + \mu^2 y = 0. \tag{20}$$

The characteristic polynomial equation for Eq. (20) is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$, so the general solution is

$$y = c_1 \cos \mu x + c_2 \sin \mu x. \tag{21}$$

Note that μ is nonzero (since $\lambda > 0$) and there is no loss of generality if we also assume that μ is positive. The first boundary condition requires that $c_1 = 0$ and then the second boundary condition reduces to

$$c_2 \sin \mu\pi = 0. \tag{22}$$

We are seeking nontrivial solutions so we must require that $c_2 \neq 0$. Consequently, $\sin \mu\pi$ must be zero and our task is to choose μ so that this will occur. We know that the sine function has the value zero at every integer multiple of π so we can choose μ to be any (positive) integer. The corresponding values of λ are the squares of the positive integers, so we have determined that

$$\lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 9, \ldots, \quad \lambda_n = n^2, \ldots \tag{23}$$

are eigenvalues of the problem (18), (19). The eigenfunctions are given by Eq. (21) with $c_1 = 0$, so they are just multiples of the functions $\sin nx$ for $n = 1, 2, 3, \ldots$. Observe that the constant c_2 in Eq. (21) is never determined, so eigenfunctions are determined only up to an arbitrary multiplicative constant [just as are the eigenvectors of the matrix problem (17)]. We will usually choose the multiplicative constant to be 1 and write the eigenfunctions as

$$y_1(x) = \sin x, \quad y_2(x) = \sin 2x, \ldots, \quad y_n(x) = \sin nx, \ldots, \tag{24}$$

remembering that multiples of these functions are also eigenfunctions.

Now let us suppose that $\lambda < 0$. If we let $\lambda = -\mu^2$, then Eq. (18) becomes

$$y'' - \mu^2 y = 0. \tag{25}$$

The characteristic equation for Eq. (25) is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$, so its general solution can be written as

$$y = c_1 \cosh \mu x + c_2 \sinh \mu x. \tag{26}$$

We have chosen the hyperbolic functions $\cosh \mu x$ and $\sinh \mu x$ as a fundamental set of solutions rather than the exponential functions $\exp(\mu x)$ and $\exp(-\mu x)$ for convenience in applying the boundary conditions. The first boundary condition requires that $c_1 = 0$ and then the second boundary condition gives $c_2 \sinh \mu\pi = 0$. Since $\mu \neq 0$, it follows that $\sinh \mu\pi \neq 0$, and therefore we must have $c_2 = 0$. Consequently, $y = 0$ and there are no nontrivial solutions for $\lambda < 0$. In other words, the problem (18), (19) has no negative eigenvalues.

Finally, consider the possibility that $\lambda = 0$. Then Eq. (18) becomes

$$y'' = 0, \tag{27}$$

and its general solution is

$$y = c_1 x + c_2. \tag{28}$$

The boundary conditions (19) can only be satisfied by choosing $c_1 = 0$ and $c_2 = 0$, so there is only the trivial solution $y = 0$ in this case as well. That is, $\lambda = 0$ is not an eigenvalue.

To summarize our results: We have shown that the problem (18), (19) has an infinite sequence of positive eigenvalues $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$ and that the corresponding eigenfunctions are proportional to $\sin nx$. Further, there are no other real eigenvalues. There remains the possibility that there might be some complex eigenvalues; recall that a matrix with real elements may very well have complex eigenvalues. In Problem 17 we outline an argument showing that the particular problem (18), (19) cannot have complex eigenvalues. Later, in Section 11.2, we discuss an important class of boundary value problems that includes (18), (19). One of the important properties of this class of problems is that all their eigenvalues are real.

In later sections of this chapter we will often encounter the problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(L) = 0, \tag{29}$$

which differs from the problem (18), (19) only in that the second boundary condition is imposed at an arbitrary point $x = L$ rather than at $x = \pi$. The solution process is exactly the same as before up to the step where the second boundary condition is applied. For the problem (29) this condition requires that

$$c_2 \sin \mu L = 0 \tag{30}$$

rather than Eq. (22), as in the former case. Consequently, μL must be an integer multiple of π, so $\mu = n\pi/L$, where n is a positive integer. Hence the eigenvalues and eigenfunctions of the problem (29) are given by

$$\lambda_n = n^2\pi^2/L^2, \quad y_n(x) = \sin(n\pi x/L), \quad n = 1, 2, 3, \ldots. \tag{31}$$

As usual, the eigenfunctions $y_n(x)$ are determined only up to an arbitrary multiplicative constant. In the same way as for (18), (19) you can show that the problem (29) has no eigenvalues or eigenfunctions other than those in Eq. (31).

The problems following this section explore to some extent the effect of different boundary conditions on the eigenvalues and eigenfunctions. A more systematic discussion of two-point boundary and eigenvalue problems appears in Chapter 11.

PROBLEMS

In each of Problems 1 through 10 either solve the given boundary value problem or else show that it has no solution.

1. $y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 1$
2. $y'' + 2y = 0, \quad y'(0) = 1, \quad y'(\pi) = 0$
3. $y'' + y = 0, \quad y(0) = 0, \quad y(L) = 0$
4. $y'' + y = 0, \quad y'(0) = 1, \quad y(L) = 0$
5. $y'' + y = x, \quad y(0) = 0, \quad y(\pi) = 0$
6. $y'' + 2y = x, \quad y(0) = 0, \quad y(\pi) = 0$
7. $y'' + 4y = \cos x, \quad y(0) = 0, \quad y(\pi) = 0$
8. $y'' + 4y = \sin x, \quad y(0) = 0, \quad y(\pi) = 0$
9. $y'' + 4y = \cos x, \quad y'(0) = 0, \quad y'(\pi) = 0$
10. $y'' + 3y = \cos x, \quad y'(0) = 0, \quad y'(\pi) = 0$

In each of Problems 11 through 16 find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

11. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$
12. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$
13. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$
14. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$
15. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$
16. $y'' - \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$
17. In this problem we outline a proof that the eigenvalues of the boundary value problem (18), (19) are real.
    (a) Write the solution of Eq. (18) as $y = k_1 \exp(i\mu x) + k_2 \exp(-i\mu x)$, where $\lambda = \mu^2$, and impose the boundary conditions (19). Show that nontrivial solutions exist if and only if
    $$\exp(i\mu\pi) - \exp(-i\mu\pi) = 0. \tag{i}$$
    (b) Let $\mu = \nu + i\sigma$ and use Euler's relation $\exp(i\nu\pi) = \cos(\nu\pi) + i\sin(\nu\pi)$ to determine the real and imaginary parts of Eq. (i).
    (c) By considering the equations found in part (b), show that $\sigma = 0$; hence μ is real and so is λ. Show also that $\nu = n$, where n is an integer.

10.2 Fourier Series

Later in this chapter you will find that you can solve many important problems involving partial differential equations provided that you can express a given function as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**[1];

[1]Fourier series are named for Joseph Fourier, who made the first systematic use, although not a completely rigorous investigation, of them in 1807 and 1811 in his papers on heat conduction. According to Riemann, when Fourier presented his first paper to the Paris Academy in 1807, stating that an arbitrary function could be expressed as a series of the form (1), the mathematician Lagrange was so surprised that he denied the possibility in the most definite terms. Although it turned out that Fourier's claim of generality was somewhat too strong, his results inspired a flood of important research that has continued to the present day. See Grattan-Guinness or Carslaw [Historical Introduction] for a detailed history of Fourier series.

they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

We begin with a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{1}$$

On the set of points where the series (1) converges, it defines a function f, whose value at each point is the sum of the series for that value of x. In this case the series (1) is said to be the Fourier series for f. Our immediate goals are to determine what functions can be represented as a sum of a Fourier series and to find some means of computing the coefficients in the series corresponding to a given function. The first term in the series (1) is written as $a_0/2$ rather than simply as a_0 to simplify a formula for the coefficients that we derive below. Besides their association with the method of separation of variables and partial differential equations, Fourier series are also useful in various other ways, such as in the analysis of mechanical or electrical systems acted on by periodic external forces.

Periodicity of the Sine and Cosine Functions. To discuss Fourier series it is necessary to develop certain properties of the trigonometric functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer. The first is their periodic character. A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x) \tag{2}$$

for every value of x. An example of a periodic function is shown in Figure 10.2.1. It follows immediately from the definition that if T is a period of f, then $2T$ is also a period, and so indeed is any integral multiple of T.

The smallest value of T for which Eq. (2) holds is called the **fundamental period** of f. In this connection it should be noted that a constant may be thought of as a periodic function with an arbitrary period, but no fundamental period.

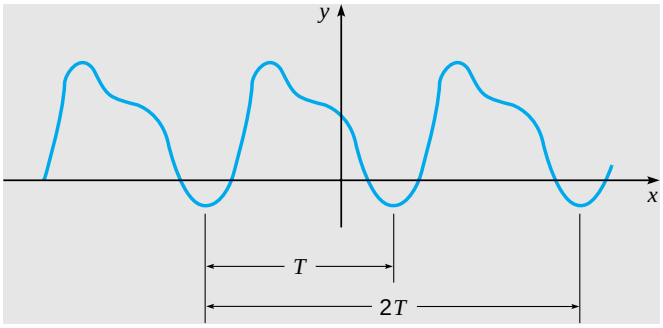


FIGURE 10.2.1 A periodic function.

If f and g are any two periodic functions with common period T, then their product fg and any linear combination $c_1 f + c_2 g$ are also periodic with period T. To prove the latter statement, let $F(x) = c_1 f(x) + c_2 g(x)$; then for any x

$$F(x+T) = c_1 f(x+T) + c_2 g(x+T) = c_1 f(x) + c_2 g(x) = F(x). \tag{3}$$

Moreover, it can be shown that the sum of any finite number, or even the sum of a convergent infinite series, of functions of period T is also periodic with period T.

In particular, the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, 3, \ldots$, are periodic with fundamental period $T = 2L/m$. To see this, recall that $\sin x$ and $\cos x$ have fundamental period 2π, and that $\sin \alpha x$ and $\cos \alpha x$ have fundamental period $2\pi/\alpha$. If we choose $\alpha = m\pi/L$, then the period T of $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ is given by $T = 2\pi L/m\pi = 2L/m$.

Note also that, since every positive integral multiple of a period is also a period, each of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ has the common period $2L$.

Orthogonality of the Sine and Cosine Functions. To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ we generalize the concept of orthogonality of vectors (see Section 7.2). The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u, v) = \int_\alpha^\beta u(x)v(x)\, dx. \tag{4}$$

The functions u and v are said to be **orthogonal** on $\alpha \le x \le \beta$ if their inner product is zero, that is, if

$$\int_\alpha^\beta u(x)v(x)\, dx = 0. \tag{5}$$

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \ldots$, form a mutually orthogonal set of functions on the interval $-L \le x \le L$. In fact, they satisfy the following orthogonality relations:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n, \\ L, & m = n; \end{cases} \tag{6}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0, \qquad \text{all } m, n; \tag{7}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n, \\ L, & m = n. \end{cases} \tag{8}$$

These results can be obtained by direct integration. For example, to derive Eq. (8), note that

$$\begin{aligned} \int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx &= \frac{1}{2}\int_{-L}^{L} \left[\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right] dx \\ &= \frac{1}{2}\frac{L}{\pi}\left\{\frac{\sin[(m-n)\pi x/L]}{m-n} - \frac{\sin[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L} \\ &= 0, \end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. Since m and n are positive, $m+n \neq 0$. On the other hand, if $m-n=0$, then $m=n$, and the integral must be evaluated in a different way. In this case

$$\begin{aligned}\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx &= \int_{-L}^{L}\left(\sin\frac{m\pi x}{L}\right)^2 dx \\ &= \frac{1}{2}\int_{-L}^{L}\left[1-\cos\frac{2m\pi x}{L}\right]dx \\ &= \frac{1}{2}\left\{x-\frac{\sin(2m\pi x/L)}{2m\pi/L}\right\}\Bigg|_{-L}^{L} \\ &= L.\end{aligned}$$

This establishes Eq. (8); Eqs. (6) and (7) can be verified by similar computations.

The Euler–Fourier Formulas. Now let us suppose that a series of the form (1) converges, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\right). \tag{9}$$

The coefficients a_m and b_m can be related to $f(x)$ as a consequence of the orthogonality conditions (6), (7), and (8). First multiply Eq. (9) by $\cos(n\pi x/L)$, where n is a *fixed* positive integer ($n>0$), and integrate with respect to x from $-L$ to L. Assuming that the integration can be legitimately carried out term by term,[2] we obtain

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx &+ \sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\ &+ \sum_{m=1}^{\infty} b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx.\end{aligned} \tag{10}$$

Keeping in mind that n is fixed whereas m ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only nonzero term on the right side of Eq. (10) is the one for which $m=n$ in the first summation. Hence

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad n=1,2,\ldots. \tag{11}$$

To determine a_0 we can integrate Eq. (9) from $-L$ to L, obtaining

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \frac{a_0}{2}\int_{-L}^{L}dx + \sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx + \sum_{m=1}^{\infty} b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &= La_0,\end{aligned} \tag{12}$$

since each integral involving a trigonometric function is zero. Thus

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n=0,1,2,\ldots. \tag{13}$$

[2]This is a nontrivial assumption, since not all convergent series with variable terms can be so integrated. For the special case of Fourier series, however, term-by-term integration can always be justified.

By writing the constant term in Eq. (9) as $a_0/2$, it is possible to compute all the a_n from Eq. (13). Otherwise, a separate formula would have to be used for a_0.

A similar expression for b_n may be obtained by multiplying Eq. (9) by $\sin(n\pi x/L)$, integrating termwise from $-L$ to L, and using the orthogonality relations (7) and (8); thus

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{14}$$

Equations (13) and (14) are known as the Euler–Fourier formulas for the coefficients in a Fourier series. Hence, if the series (9) converges to $f(x)$, and if the series can be integrated term by term, then the coefficients *must be given* by Eqs. (13) and (14).

Note that Eqs. (13) and (14) are explicit formulas for a_n and b_n in terms of f, and that the determination of any particular coefficient is independent of all the other coefficients. Of course, the difficulty in evaluating the integrals in Eqs. (13) and (14) depends very much on the particular function f involved.

Note also that the formulas (13) and (14) depend only on the values of $f(x)$ in the interval $-L \le x \le L$. Since each of the terms in the Fourier series (9) is periodic with period $2L$, the series converges for all x whenever it converges in $-L \le x \le L$, and its sum is also a periodic function with period $2L$. Hence $f(x)$ is determined for all x by its values in the interval $-L \le x \le L$.

It is possible to show (see Problem 27) that if g is periodic with period T, then every integral of g over an interval of length T has the same value. If we apply this result to the Euler–Fourier formulas (13) and (14), it follows that the interval of integration, $-L \le x \le L$, can be replaced, if it is more convenient to do so, by any other interval of length $2L$.

EXAMPLE 1

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x). \tag{15}$$

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus in this case $L = 2$ and the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{2} + b_m\sin\frac{m\pi x}{2}\right), \tag{16}$$

where the coefficients are computed from Eqs. (13) and (14) with $L = 2$. Substituting for $f(x)$ in Eq. (13) with $m = 0$, we have

$$a_0 = \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 1 + 1 = 2. \tag{17}$$

For $m > 0$, Eq. (13) yields

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{m\pi x}{2}\,dx.$$

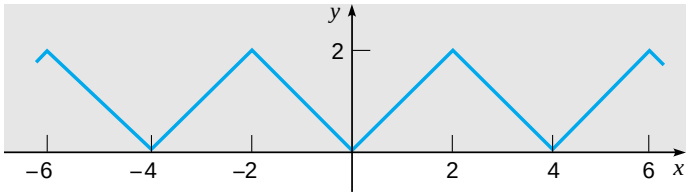


FIGURE 10.2.2 Triangular wave.

These integrals can be evaluated through integration by parts, with the result that

$$\begin{aligned} a_m &= \frac{1}{2}\left[-\frac{2}{m\pi}x\sin\frac{m\pi x}{2} - \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{-2}^{0} \\ &\qquad + \frac{1}{2}\left[\frac{2}{m\pi}x\sin\frac{m\pi x}{2} + \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{0}^{2} \\ &= \frac{1}{2}\left[-\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2\cos m\pi + \left(\frac{2}{m\pi}\right)^2\cos m\pi - \left(\frac{2}{m\pi}\right)^2\right] \\ &= \frac{4}{(m\pi)^2}(\cos m\pi - 1), \qquad m = 1, 2, \ldots \\ &= \begin{cases} -8/(m\pi)^2, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \end{aligned} \tag{18}$$

Finally, from Eq. (14) it follows in a similar way that

$$b_m = 0, \qquad m = 1, 2, \ldots. \tag{19}$$

By substituting the coefficients from Eqs. (17), (18), and (19) in the series (16) we obtain the Fourier series for f:

$$\begin{aligned} f(x) &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\cos(m\pi x/2)}{m^2} \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}. \end{aligned} \tag{20}$$

EXAMPLE 2

Let

$$f(x) = \begin{cases} 0, & -3 < x < -1, \\ 1, & -1 < x < 1, \\ 0, & 1 < x < 3 \end{cases} \tag{21}$$

and suppose that $f(x+6) = f(x)$; see Figure 10.2.3. Find the coefficients in the Fourier series for f.

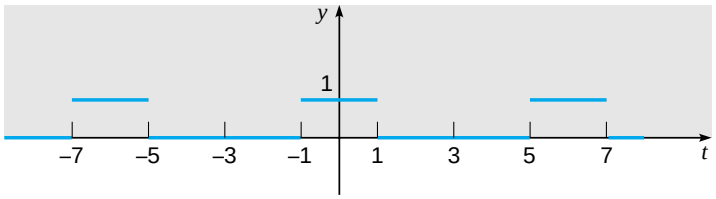


FIGURE 10.2.3 Graph of $f(x)$ in Example 2.

Since f has period 6, it follows that $L = 3$ in this problem. Consequently, the Fourier series for f has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3} \right), \tag{22}$$

where the coefficients a_n and b_n are given by Eqs. (13) and (14) with $L = 3$. We have

$$a_0 = \frac{1}{3} \int_{-3}^{3} f(x)\, dx = \frac{1}{3} \int_{-1}^{1} dx = \frac{2}{3}. \tag{23}$$

Similarly,

$$a_n = \frac{1}{3} \int_{-1}^{1} \cos \frac{n\pi x}{3}\, dx = \frac{1}{n\pi} \sin \frac{n\pi x}{3} \bigg|_{-1}^{1} = \frac{2}{n\pi} \sin \frac{n\pi}{3}, \qquad n = 1, 2, \ldots, \tag{24}$$

and

$$b_n = \frac{1}{3} \int_{-1}^{1} \sin \frac{n\pi x}{3}\, dx = -\frac{1}{n\pi} \cos \frac{n\pi x}{3} \bigg|_{-1}^{1} = 0, \qquad n = 1, 2, \ldots. \tag{25}$$

Thus the Fourier series for f is

$$\begin{aligned} f(x) &= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{3} \cos \frac{n\pi x}{3} \\ &= \frac{1}{3} + \frac{\sqrt{3}}{\pi} \left[\cos(\pi x/3) + \frac{\cos(2\pi x/3)}{2} - \frac{\cos(4\pi x/3)}{4} - \frac{\cos(5\pi x/3)}{5} + \cdots \right]. \end{aligned} \tag{26}$$

EXAMPLE 3

Consider again the function in Example 1 and its Fourier series (20). Investigate the speed with which the series converges. In particular, determine how many terms are needed so that the error is no greater than 0.01 for all x.

The mth partial sum in this series,

$$s_m(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{m} \frac{\cos(2n-1)\pi x/2}{(2n-1)^2}, \tag{27}$$

can be used to approximate the function f. The coefficients diminish as $(2n-1)^{-2}$, so the series converges fairly rapidly. This is borne out by Figure 10.2.4, where the partial sums for $m = 1$ and $m = 2$ are plotted. To investigate the convergence in more detail

we can consider the error $e_m(x) = f(x) - s_m(x)$. Figure 10.2.5 shows a plot of $|e_6(x)|$ versus x for $0 \le x \le 2$. Observe that $|e_6(x)|$ is greatest at the points $x = 0$ and $x = 2$ where the graph of $f(x)$ has corners. It is more difficult for the series to approximate the function near these points, resulting in a larger error there for a given n. Similar graphs are obtained for other values of m.

Once you realize that the maximum error always occurs at $x = 0$ or $x = 2$, you can obtain a *uniform* error bound for each m simply by evaluating $|e_m(x)|$ at one of these points. For example, for $m = 6$ we have $e_6(2) = 0.03370$, so $|e_6(x)| < 0.034$ for $0 \le x \le 2$, and consequently for all x. Table 10.2.1 shows corresponding data for other values of m; these data are plotted in Figure 10.2.6. From this information you can begin to estimate the number of terms that are needed in the series in order to achieve a given level of accuracy in the approximation. For example, to guarantee that $|e_m(x)| \le 0.01$ we need to choose $m = 21$.

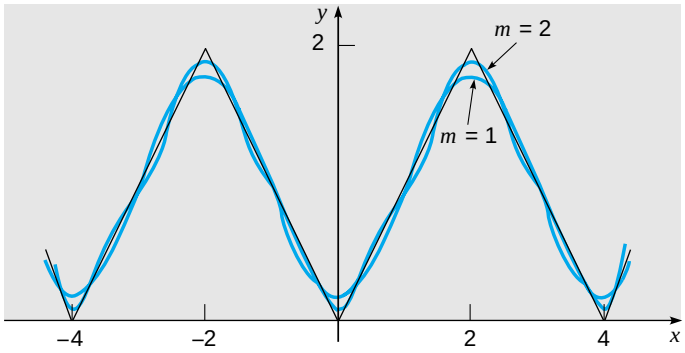


FIGURE 10.2.4 Partial sums in the Fourier series, Eq. (20), for the triangular wave.

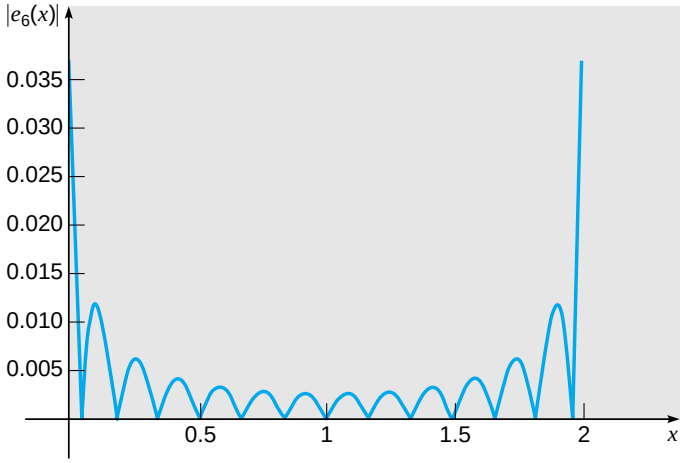


FIGURE 10.2.5 Plot of $|e_6(x)|$ versus x for the triangular wave.

TABLE 10.2.1 Values of the Error $e_m(2)$ for the Triangular Wave

m	$e_m(2)$
2	0.09937
4	0.05040
6	0.03370
10	0.02025
15	0.01350
20	0.01013
25	0.00810

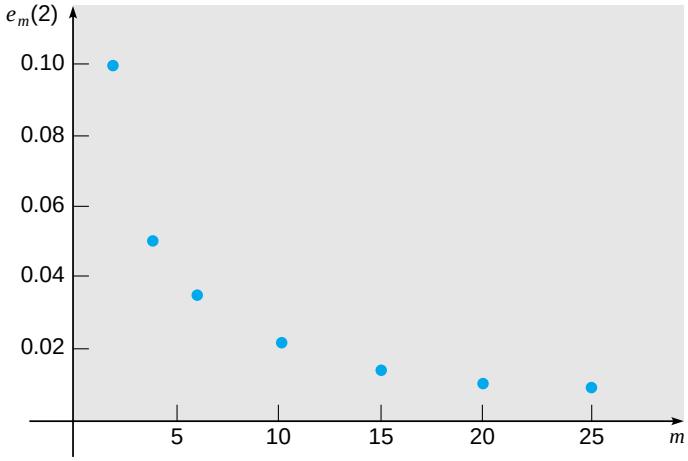


FIGURE 10.2.6 Plot of $e_m(2)$ versus m for the triangular wave.

In this book Fourier series appear mainly as a means of solving certain problems in partial differential equations. However, such series have much wider application in science and engineering, and in general are valuable tools in the investigation of periodic phenomena. A basic problem is to resolve an incoming signal into its harmonic components, which amounts to constructing its Fourier series representation. In some frequency ranges the separate terms correspond to different colors or to different audible tones. The magnitude of the coefficient determines the amplitude of each component. This process is referred to as spectral analysis.

PROBLEMS

In each of Problems 1 through 8 determine whether the given function is periodic. If so, find its fundamental period.

1. $\sin 5x$
2. $\cos 2\pi x$
3. $\sinh 2x$
4. $\sin \pi x/L$
5. $\tan \pi x$
6. x^2
7. $f(x) = \begin{cases} 0, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$
8. $f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

9. If $f(x) = -x$ for $-L < x < L$, and if $f(x + 2L) = f(x)$, find a formula for $f(x)$ in the interval $L < x < 2L$; in the interval $-3L < x < -2L$.
10. If $f(x) = \begin{cases} x + 1, & -1 < x < 0, \\ x, & 0 < x < 1, \end{cases}$ and if $f(x + 2) = f(x)$, find a formula for $f(x)$ in the interval $1 < x < 2$; in the interval $8 < x < 9$.
11. If $f(x) = L - x$ for $0 < x < 2L$, and if $f(x + 2L) = f(x)$, find a formula for $f(x)$ in the interval $-L < x < 0$.
12. Verify Eqs. (6) and (7) of the text by direct integration.

In each of Problems 13 through 18:

(a) Sketch the graph of the given function for three periods.
(b) Find the Fourier series for the given function.

13. $f(x) = -x, \quad -L \le x < L; \qquad f(x + 2L) = f(x)$
14. $f(x) = \begin{cases} 1, & -L \le x < 0, \\ 0, & 0 \le x < L; \end{cases} \qquad f(x + 2L) = f(x)$
15. $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \qquad f(x + 2\pi) = f(x)$
16. $f(x) = \begin{cases} x + 1, & -1 \le x < 0, \\ 1 - x, & 0 \le x < 1; \end{cases} \qquad f(x + 2) = f(x)$
17. $f(x) = \begin{cases} x + L, & -L \le x \le 0, \\ L, & 0 < x < L; \end{cases} \qquad f(x + 2L) = f(x)$
18. $f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ x, & -1 < x < 1, \\ 0, & 1 \le x < 2; \end{cases} \qquad f(x + 4) = f(x)$

In each of Problems 19 through 24:

(a) Sketch the graph of the given function for three periods.
(b) Find the Fourier series for the given function.
(c) Plot $s_m(x)$ versus x for $m = 5$, 10, and 20.
(d) Describe how the Fourier series seems to be converging.

19. $f(x) = \begin{cases} -1, & -2 \le x < 0, \\ 1, & 0 \le x < 2; \end{cases} \qquad f(x + 4) = f(x)$
20. $f(x) = x, \quad -1 \le x < 1; \qquad f(x + 2) = f(x)$
21. $f(x) = x^2/2, \quad -2 \le x \le 2; \qquad f(x + 4) = f(x)$
22. $f(x) = \begin{cases} x + 2, & -2 \le x < 0, \\ 2 - 2x, & 0 \le x < 2; \end{cases} \qquad f(x + 4) = f(x)$
23. $f(x) = \begin{cases} -\frac{1}{2}x, & -2 \le x < 0, \\ 2x - \frac{1}{2}x^2, & 0 \le x < 2; \end{cases} \qquad f(x + 4) = f(x)$
24. $f(x) = \begin{cases} 0, & -3 \le x \le 0, \\ x^2(3 - x), & 0 < x < 3; \end{cases} \qquad f(x + 6) = f(x)$
25. Consider the function f defined in Problem 21 and let $e_m(x) = f(x) - s_m(x)$. Plot $|e_m(x)|$ versus x for $0 \le x \le 2$ for several values of m. Find the smallest value of m for which $|e_m(x)| \le 0.01$ for all x.
26. Consider the function f defined in Problem 24 and let $e_m(x) = f(x) - s_m(x)$. Plot $|e_m(x)|$ versus x for $0 \le x \le 3$ for several values of m. Find the smallest value of m for which $|e_m(x)| \le 0.1$ for all x.
27. Suppose that g is an integrable periodic function with period T.

(a) If $0 \le a \le T$, show that

$$\int_0^T g(x)\,dx = \int_a^{a+T} g(x)\,dx.$$

Hint: Show first that $\int_0^a g(x)\,dx = \int_T^{a+T} g(x)\,dx$. Consider the change of variable $s = x - T$ in the second integral.

(b) Show that for any value of a, not necessarily in $0 \le a \le T$,

$$\int_0^T g(x)\,dx = \int_a^{a+T} g(x)\,dx.$$

(c) Show that for any values of a and b,

$$\int_a^{a+T} g(x)\,dx = \int_b^{b+T} g(x)\,dx.$$

28. If f is differentiable and is periodic with period T, show that f' is also periodic with period T. Determine whether

$$F(x) = \int_0^x f(t)\,dt$$

is always periodic.

29. In this problem we indicate certain similarities between three-dimensional geometric vectors and Fourier series.

(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be a set of mutually orthogonal vectors in three dimensions and let $\mathbf{u}$ be any three-dimensional vector. Show that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, \tag{i}$$

where

$$a_i = \frac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}, \qquad i = 1, 2, 3. \tag{ii}$$

Show that a_i can be interpreted as the projection of $\mathbf{u}$ in the direction of $\mathbf{v}_i$ divided by the length of $\mathbf{v}_i$.

(b) Define the inner product (u, v) by

$$(u, v) = \int_{-L}^{L} u(x)v(x)\,dx. \tag{iii}$$

Also let

$$\begin{aligned} \phi_n(x) &= \cos(n\pi x/L), \qquad n = 0, 1, 2, \ldots; \\ \psi_n(x) &= \sin(n\pi x/L), \qquad n = 1, 2, \ldots. \end{aligned} \tag{iv}$$

Show that Eq. (10) can be written in the form

$$(f, \phi_n) = \frac{a_0}{2}(\phi_0, \phi_n) + \sum_{m=1}^{\infty} a_m(\phi_m, \phi_n) + \sum_{m=1}^{\infty} b_m(\psi_m, \phi_n). \tag{v}$$

(c) Use Eq. (v) and the corresponding equation for (f, ψ_n) together with the orthogonality relations to show that

$$a_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}, \qquad n = 0, 1, 2, \ldots; \qquad b_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}, \qquad n = 1, 2, \ldots. \tag{vi}$$

Note the resemblance between Eqs. (vi) and Eq. (ii). The functions ϕ_n and ψ_n play a role for functions similar to that of the orthogonal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in three-dimensional

space. The coefficients a_n and b_n can be interpreted as projections of the function f onto the base functions ϕ_n and ψ_n.

Observe also that any vector in three dimensions can be expressed as a linear combination of three mutually orthogonal vectors. In a somewhat similar way any sufficiently smooth function defined on $-L \le x \le L$ can be expressed as a linear combination of the mutually orthogonal functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, that is, as a Fourier series.

10.3 The Fourier Convergence Theorem

In the preceding section we showed that if the Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right) \tag{1}$$

converges and thereby defines a function f, then f is periodic with period $2L$, and the coefficients a_m and b_m are related to $f(x)$ by the Euler–Fourier formulas:

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx, \qquad m = 0, 1, 2, \ldots; \tag{2}$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx, \qquad m = 1, 2, \ldots. \tag{3}$$

In this section we adopt a somewhat different point of view. Suppose that a function f is given. If this function is periodic with period $2L$ and integrable on the interval $[-L, L]$, then a set of coefficients a_m and b_m can be computed from Eqs. (2) and (3), and a series of the form (1) can be formally constructed. The question is whether this series converges for each value of x and, if so, whether its sum is $f(x)$. Examples have been discovered showing that the Fourier series corresponding to a function f may not converge to $f(x)$, or may even diverge. Functions whose Fourier series do not converge to the value of the function at isolated points are easily constructed, and examples will be presented later in this section. Functions whose Fourier series diverge at one or more points are more pathological, and we will not consider them in this book.

To guarantee convergence of a Fourier series to the function from which its coefficients were computed it is essential to place additional conditions on the function. From a practical point of view, such conditions should be broad enough to cover all situations of interest, yet simple enough to be easily checked for particular functions. Through the years several sets of conditions have been devised to serve this purpose.

Before stating a convergence theorem for Fourier series, we define a term that appears in the theorem. A function f is said to be **piecewise continuous** on an interval $a \le x \le b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ so that

1. f is continuous on each open subinterval $x_{i-1} < x < x_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

The graph of a piecewise continuous function is shown in Figure 10.3.1.

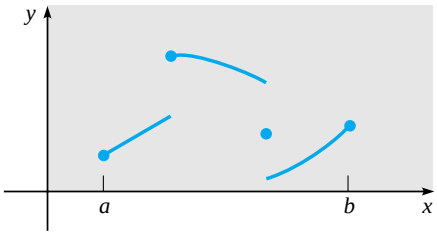


FIGURE 10.3.1 A piecewise continuous function.

The notation $f(c+)$ is used to denote the limit of $f(x)$ as $x \to c$ from the right; similarly, $f(c-)$ denotes the limit of $f(x)$ as x approaches c from the left.

Note that it is not essential that the function even be defined at the partition points x_i. For example, in the following theorem we assume that f' is piecewise continuous; but certainly f' does not exist at those points where f itself is discontinuous. It is also not essential that the interval be closed; it may also be open, or open at one end and closed at the other.

Theorem 10.3.1 Suppose that f and f' are piecewise continuous on the interval $-L \le x < L$. Further, suppose that f is defined outside the interval $-L \le x < L$ so that it is periodic with period $2L$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right), \tag{4}$$

whose coefficients are given by Eqs. (2) and (3). The Fourier series converges to $f(x)$ at all points where f is continuous, and to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous.

Note that $[f(x+) + f(x-)]/2$ is the mean value of the right- and left-hand limits at the point x. At any point where f is continuous, $f(x+) = f(x-) = f(x)$. Thus it is correct to say that the Fourier series converges to $[f(x+) + f(x-)]/2$ at all points. Whenever we say that a Fourier series converges to a function f, we always mean that it converges in this sense.

It should be emphasized that the conditions given in this theorem are only sufficient for the convergence of a Fourier series; they are by no means necessary. Neither are they the most general sufficient conditions that have been discovered. In spite of this, the proof of the theorem is fairly difficult and is not given here.[3]

To obtain a better understanding of the content of the theorem it is helpful to consider some classes of functions that fail to satisfy the assumed conditions. Functions that are not included in the theorem are primarily those with infinite discontinuities in the interval $[-L, L]$, such as $1/x^2$ as $x \to 0$, or $\ln|x - L|$ as $x \to L$. Functions having an infinite number of jump discontinuities in this interval are also excluded; however, such functions are rarely encountered.

[3]Proofs of the convergence of a Fourier series can be found in most books on advanced calculus. See, for example, Kaplan (Chapter 7) or Buck (Chapter 6).

It is noteworthy that a Fourier series may converge to a sum that is not differentiable, or even continuous, in spite of the fact that each term in the series (4) is continuous, and even differentiable infinitely many times. The example below is an illustration of this, as is Example 2 in Section 10.2.

EXAMPLE 1

Let

$$f(x) = \begin{cases} 0, & -L < x < 0, \\ L, & 0 < x < L, \end{cases} \tag{5}$$

and let f be defined outside this interval so that $f(x + 2L) = f(x)$ for all x. We will temporarily leave open the definition of f at the points $x = 0, \pm L$, except that its value must be finite. Find the Fourier series for this function and determine where it converges.

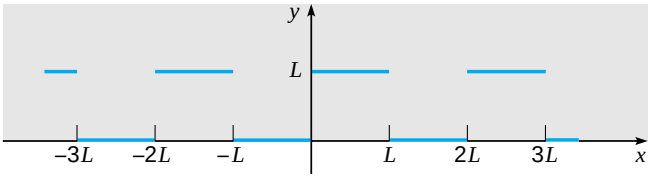


FIGURE 10.3.2 Square wave.

The equation $y = f(x)$ has the graph shown in Figure 10.3.2, extended to infinity in both directions. It can be thought of as representing a square wave. The interval $[-L, L]$ can be partitioned to give the two open subintervals $(-L, 0)$ and $(0, L)$. In $(0, L)$, $f(x) = L$ and $f'(x) = 0$. Clearly, both f and f' are continuous and furthermore have limits as $x \to 0$ from the right and as $x \to L$ from the left. The situation in $(-L, 0)$ is similar. Consequently, both f and f' are piecewise continuous on $[-L, L)$, so f satisfies the conditions of Theorem 10.3.1. If the coefficients a_m and b_m are computed from Eqs. (2) and (3), the convergence of the resulting Fourier series to $f(x)$ is assured at all points where f is continuous. Note that the values of a_m and b_m are the same regardless of the definition of f at its points of discontinuity. This is true because the value of an integral is unaffected by changing the value of the integrand at a finite number of points. From Eq. (2)

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_0^L dx = L;$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_0^L \cos\frac{m\pi x}{L}\,dx$$
$$= 0, \qquad m \neq 0.$$

Similarly, from Eq. (3),

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = \int_0^L \sin\frac{m\pi x}{L}\,dx$$
$$= \frac{L}{m\pi}(1 - \cos m\pi)$$
$$= \begin{cases} 0, & m \text{ even}; \\ 2L/m\pi, & m \text{ odd}. \end{cases}$$

Hence

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right)$$
$$= \frac{L}{2} + \frac{2L}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\sin(m\pi x/L)}{m}$$
$$= \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x/L}{2n-1}. \tag{6}$$

At the points $x = 0, \pm nL$, where the function f in the example is not continuous, all terms in the series after the first vanish and the sum is $L/2$. This is the mean value of the limits from the right and left, as it should be. Thus we might as well define f at these points to have the value $L/2$. If we choose to define it otherwise, the series still gives the value $L/2$ at these points, since none of the preceding calculations is altered in any detail; it simply does not converge to the function at those points unless f is defined to have this value. This illustrates the possibility that the Fourier series corresponding to a function may not converge to it at points of discontinuity unless the function is suitably defined at such points.

The manner in which the partial sums

$$s_n(x) = \frac{L}{2} + \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} + \cdots + \frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{L}\right), \qquad n = 1, 2, \ldots$$

of the Fourier series (6) converge to $f(x)$ is indicated in Figure 10.3.3, where L has been chosen to be one and the graph of $s_8(x)$ is plotted. The figure suggests that at points where f is continuous the partial sums do approach $f(x)$ as n increases. However, in the neighborhood of points of discontinuity, such as $x = 0$ and $x = L$, the partial sums do not converge smoothly to the mean value. Instead they tend to overshoot the mark at each end of the jump, as though they cannot quite accommodate themselves to the sharp turn required at this point. This behavior is typical of Fourier series at points of discontinuity, and is known as the Gibbs[4] phenomenon.

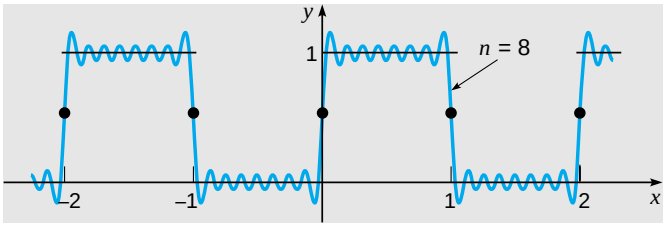


FIGURE 10.3.3 The partial sum $s_8(x)$ in the Fourier series, Eq. (6), for the square wave.

Additional insight is attained by considering the error $e_n(x) = f(x) - s_n(x)$. Figure 10.3.4 shows a plot of $|e_n(x)|$ versus x for $n = 8$ and for $L = 1$. The least upper bound of $|e_8(x)|$ is 0.5 and is approached as $x \to 0$ and as $x \to 1$. As n increases, the error

[4]The Gibbs phenomenon is named after Josiah Willard Gibbs (1839–1903), who is better known for his work on vector analysis and statistical mechanics. Gibbs was professor of mathematical physics at Yale, and one of the first American scientists to achieve an international reputation. Gibbs' phenomenon is discussed in more detail by Carslaw (Chapter 9).

decreases in the interior of the interval [where $f(x)$ is continuous] but the least upper bound does not diminish with increasing n. Thus one cannot uniformly reduce the error throughout the interval by increasing the number of terms.

Figures 10.3.3 and 10.3.4 also show that the series in this example converges more slowly than the one in Example 1 in Section 10.2. This is due to the fact that the coefficients in the series (6) are proportional only to $1/(2n-1)$.

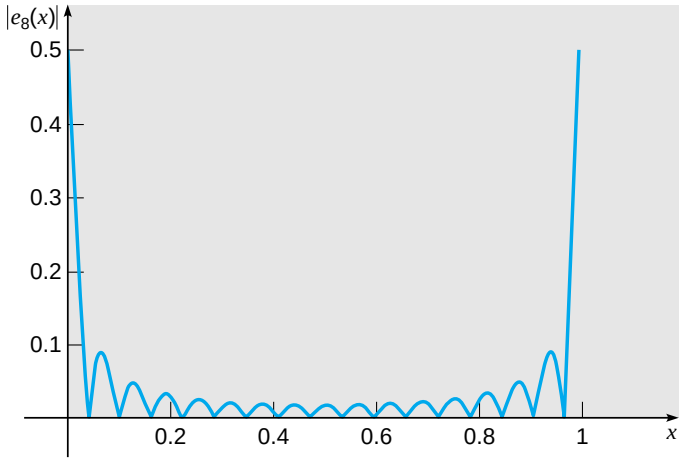


FIGURE 10.3.4 A plot of the error $|e_8(x)|$ versus x for the square wave.

PROBLEMS

In each of Problems 1 through 6 assume that the given function is periodically extended outside the original interval.

(a) Find the Fourier series for the extended function.

(b) Sketch the graph of the function to which the series converges for three periods.

1. $f(x) = \begin{cases} -1, & -1 \le x < 0, \\ 1, & 0 \le x < 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} L + x, & -L \le x < 0, \\ L - x, & 0 \le x < L \end{cases}$

4. $f(x) = 1 - x^2, \quad -1 \le x \le 1$

5. $f(x) = \begin{cases} 0, & -\pi \le x < -\pi/2, \\ 1, & -\pi/2 \le x < \pi/2, \\ 0, & \pi/2 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ x^2, & 0 \le x < 1 \end{cases}$

In each of Problems 7 through 12 assume that the given function is periodically extended outside the original interval.

(a) Find the Fourier series for the given function.

(b) Let $e_n(x) = f(x) - s_n(x)$. Find the least upper bound or the maximum value (if it exists) of $|e_n(x)|$ for $n = 10$, 20, and 40.

(c) If possible, find the smallest n for which $|e_n(x)| \le 0.01$ for all x.

▶ 7. $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \quad f(x + 2\pi) = f(x)$ (see Section 10.2, Problem 15)

8. $f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1; \end{cases}$ $f(x+2) = f(x)$ (see Section 10.2, Problem 16)
9. $f(x) = x, \quad -1 \le x < 1; \qquad f(x+2) = f(x)$ (see Section 10.2, Problem 20)
10. $f(x) = \begin{cases} x+2, & -2 \le x < 0, \\ 2-2x, & 0 \le x < 2; \end{cases}$ $f(x+4) = f(x)$ (see Section 10.2, Problem 22)
11. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ x^2, & 0 \le x < 1; \end{cases}$ $f(x+2) = f(x)$ (see Problem 6)
12. $f(x) = x - x^3, \quad -1 \le x < 1; \qquad f(x+2) = f(x)$

Periodic Forcing Terms. In this chapter we are concerned mainly with the use of Fourier series to solve boundary value problems for certain partial differential equations. However, Fourier series are also useful in many other situations where periodic phenomena occur. Problems 13 through 16 indicate how they can be employed to solve initial value problems with periodic forcing terms.

13. Find the solution of the initial value problem

$$y'' + \omega^2 y = \sin nt, \qquad y(0) = 0, \quad y'(0) = 0,$$

where n is a positive integer and $\omega^2 \neq n^2$. What happens if $\omega^2 = n^2$?

14. Find the formal solution of the initial value problem

$$y'' + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt, \qquad y(0) = 0, \quad y'(0) = 0,$$

where $\omega > 0$ is not equal to a positive integer. How is the solution altered if $\omega = m$, where m is a positive integer?

15. Find the formal solution of the initial value problem

$$y'' + \omega^2 y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where f is periodic with period 2π and

$$f(t) = \begin{cases} 1, & 0 < t < \pi; \\ 0, & t = 0, \pi, 2\pi; \\ -1, & \pi < t < 2\pi. \end{cases}$$

See Problem 1.

16. Find the formal solution of the initial value problem

$$y'' + \omega^2 y = f(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where f is periodic with period 2 and

$$f(t) = \begin{cases} 1 - t, & 0 \le t < 1; \\ -1 + t, & 1 \le t < 2. \end{cases}$$

See Problem 8.

17. Assuming that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{i}$$

show formally that

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This relation between a function f and its Fourier coefficients is known as Parseval's (1755–1836) equation. Parseval's equation is very important in the theory of Fourier series and is discussed further in Section 11.6.
Hint: Multiply Eq. (i) by $f(x)$, integrate from $-L$ to L, and use the Euler–Fourier formulas.

18. This problem indicates a proof of convergence of a Fourier series under conditions more restrictive than those in Theorem 10.3.1.
(a) If f and f' are piecewise continuous on $-L \le x < L$, and if f is periodic with period $2L$, show that na_n and nb_n are bounded as $n \to \infty$.
Hint: Use integration by parts.
(b) If f is continuous on $-L \le x \le L$ and periodic with period $2L$, and if f' and f'' are piecewise continuous on $-L \le x < L$, show that n^2a_n and n^2b_n are bounded as $n \to \infty$. Use this fact to show that the Fourier series for f converges at each point in $-L \le x \le L$. Why must f be continuous on the *closed* interval?
Hint: Again, use integration by parts.

Acceleration of Convergence. In the next problem we show how it is sometimes possible to improve the speed of convergence of a Fourier, or other infinite, series.

19. Suppose that we wish to calculate values of the function g, where

$$g(x) = \sum_{n=1}^{\infty} \frac{(2n-1)}{1+(2n-1)^2} \sin(2n-1)\pi x. \tag{i}$$

It is possible to show that this series converges, albeit rather slowly. However, observe that for large n the terms in the series (i) are approximately equal to $[\sin(2n-1)\pi x]/(2n-1)$ and that the latter terms are similar to those in the example in the text, Eq. (6).
(a) Show that

$$\sum_{n=1}^{\infty} [\sin(2n-1)\pi x]/(2n-1) = (\pi/2)[f(x) - \tfrac{1}{2}], \tag{ii}$$

where f is the square wave in the example with $L = 1$.
(b) Subtract Eq. (ii) from Eq. (i) and show that

$$g(x) = \frac{\pi}{2}[f(x) - \tfrac{1}{2}] - \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{(2n-1)[1+(2n-1)^2]}. \tag{iii}$$

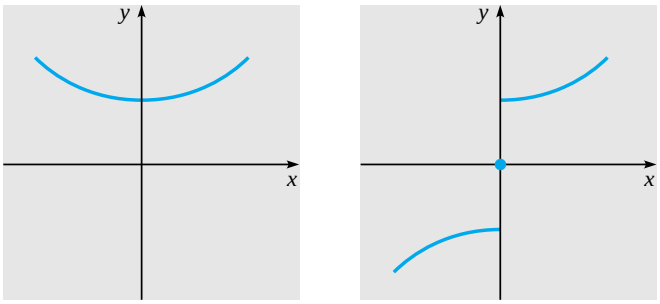
The series (iii) converges much faster than the series (i) and thus provides a better way to calculate values of $g(x)$.

10.4 Even and Odd Functions

Before looking at further examples of Fourier series it is useful to distinguish two classes of functions for which the Euler–Fourier formulas can be simplified. These are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the y-axis and the origin, respectively (see Figure 10.4.1).

Analytically, f is an **even function** if its domain contains the point $-x$ whenever it contains the point x, and if

$$f(-x) = f(x) \tag{1}$$



(*a*) (*b*)

FIGURE 10.4.1 (*a*) An even function. (*b*) An odd function.

for each x in the domain of f. Similarly, f is an **odd function** if its domain contains $-x$ whenever it contains x, and if

$$f(-x) = -f(x) \tag{2}$$

for each x in the domain of f. Examples of even functions are 1, x^2, $\cos nx$, $|x|$, and x^{2n}. The functions x, x^3, $\sin nx$, and x^{2n+1} are examples of odd functions. Note that according to Eq. (2), $f(0)$ must be zero if f is an odd function whose domain contains the origin. Most functions are neither even nor odd, for instance, e^x. Only one function, f identically zero, is both even and odd.

Elementary properties of even and odd functions include the following:

1. The sum (difference) and product (quotient) of two even functions are even.

2. The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even.

3. The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of two such functions is odd.[5]

The proofs of all these assertions are simple and follow directly from the definitions. For example, if both f_1 and f_2 are odd, and if $g(x) = f_1(x) + f_2(x)$, then

$$\begin{aligned} g(-x) &= f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) \\ &= -[f_1(x) + f_2(x)] = -g(x), \end{aligned} \tag{3}$$

so $f_1 + f_2$ is an odd function also. Similarly, if $h(x) = f_1(x)\,f_2(x)$, then

$$h(-x) = f_1(-x)\,f_2(-x) = [-f_1(x)][-f_2(x)] = f_1(x)\,f_2(x) = h(x), \tag{4}$$

so that $f_1 f_2$ is even.

Also of importance are the following two integral properties of even and odd functions:

4. If f is an even function, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx. \tag{5}$$

[5]These statements may need to be modified if either function vanishes identically.

5. If f is an odd function, then

$$\int_{-L}^{L} f(x)\,dx = 0. \tag{6}$$

These properties are intuitively clear from the interpretation of an integral in terms of area under a curve, and also follow immediately from the definitions. For example, if f is even, then

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx.$$

Letting $x = -s$ in the first term on the right side, and using Eq. (1), we obtain

$$\int_{-L}^{L} f(x)\,dx = -\int_{L}^{0} f(s)\,ds + \int_{0}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

The proof of the corresponding property for odd functions is similar.

Even and odd functions are particularly important in applications of Fourier series since their Fourier series have special forms, which occur frequently in physical problems.

Cosine Series. Suppose that f and f' are piecewise continuous on $-L \le x < L$, and that f is an even periodic function with period $2L$. Then it follows from properties 1 and 3 that $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd. As a consequence of Eqs. (5) and (6), the Fourier coefficients of f are then given by

$$\begin{aligned} a_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad && n = 0, 1, 2, \ldots; \\ b_n &= 0, && n = 1, 2, \ldots. \end{aligned} \tag{7}$$

Thus f has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

In other words, the Fourier series of any even function consists only of the even trigonometric functions $\cos(n\pi x/L)$ and the constant term; it is natural to call such a series a **Fourier cosine series**. From a computational point of view, observe that only the coefficients a_n, for $n = 0, 1, 2, \ldots$, need to be calculated from the integral formula (7). Each of the b_n, for $n = 1, 2, \ldots$, is automatically zero for any even function, and so does not need to be calculated by integration.

Sine Series. Suppose that f and f' are piecewise continuous on $-L \le x < L$, and that f is an odd periodic function of period $2L$. Then it follows from properties 2 and 3 that $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even. In this case the Fourier coefficients of f are

$$\begin{aligned} a_n &= 0, && n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad && n = 1, 2, \ldots, \end{aligned} \tag{8}$$

and the Fourier series for f is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Thus the Fourier series for any odd function consists only of the odd trigonometric functions $\sin(n\pi x/L)$; such a series is called a **Fourier sine series**. Again observe that only half of the coefficients need to be calculated by integration, since each a_n, for $n = 0, 1, 2, \ldots$, is zero for any odd function.

EXAMPLE 1

Let $f(x) = x$, $-L < x < L$, and let $f(-L) = f(L) = 0$. Let f be defined elsewhere so that it is periodic of period $2L$ (see Figure 10.4.2). The function defined in this manner is known as a sawtooth wave. Find the Fourier series for this function.

Since f is an odd function, its Fourier coefficients are, according to Eq. (8),

$$\begin{aligned} a_n &= 0, \qquad n = 0, 1, 2, \ldots; \\ b_n &= \frac{2}{L}\int_0^L x \sin \frac{n\pi x}{L}\, dx \\ &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left\{\sin \frac{n\pi x}{L} - \frac{n\pi x}{L}\cos \frac{n\pi x}{L}\right\}\Bigg|_0^L \\ &= \frac{2L}{n\pi}(-1)^{n+1}, \qquad n = 1, 2, \ldots. \end{aligned}$$

Hence the Fourier series for f, the sawtooth wave, is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}. \tag{9}$$

Observe that the periodic function f is discontinuous at the points $\pm L, \pm 3L, \ldots$, as shown in Figure 10.4.2. At these points the series (9) converges to the mean value of the left and right limits, namely, zero. The partial sum of the series (9) for $n = 9$ is shown in Figure 10.4.3. The Gibbs phenomenon (mentioned in Section 10.3) again occurs near the points of discontinuity.

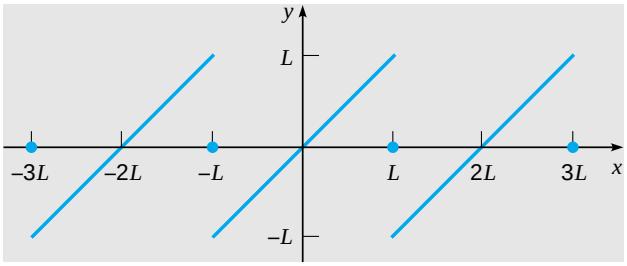


FIGURE 10.4.2 Sawtooth wave.

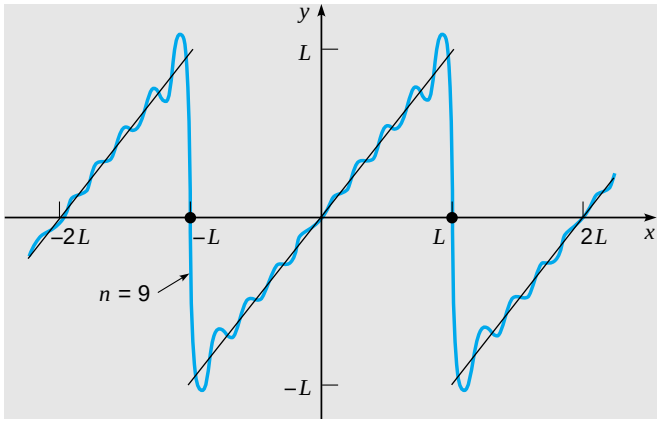


FIGURE 10.4.3 A partial sum in the Fourier series, Eq. (9), for the sawtooth wave.

Note that in this example $f(-L) = f(L) = 0$, as well as $f(0) = 0$. This is required if the function f is to be both odd and periodic with period $2L$. When we speak of constructing a sine series for a function defined on $0 \le x \le L$, it is understood that, if necessary, we must first redefine the function to be zero at $x = 0$ and $x = L$.

It is worthwhile to observe that the triangular wave function (Example 1 of Section 10.2) and the sawtooth wave function just considered are identical on the interval $0 \le x < L$. Therefore, their Fourier series converge to the same function, $f(x) = x$, on this interval. Thus, if it is required to represent the function $f(x) = x$ on $0 \le x < L$ by a Fourier series, it is possible to do this by *either a cosine series or a sine series*. In the former case f is extended as an *even* function into the interval $-L < x < 0$ and elsewhere periodically (the triangular wave). In the latter case f is extended into $-L < x < 0$ as an *odd* function, and elsewhere periodically (the sawtooth wave). If f is extended in any other way, the resulting Fourier series will still converge to x in $0 \le x < L$ but will involve both sine and cosine terms.

In solving problems in differential equations it is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on the interval $[0, L]$. As indicated previously for the function $f(x) = x$ several alternatives are available. Explicitly, we can:

1. Define a function g of period $2L$ so that

$$g(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(-x), & -L < x < 0. \end{cases} \tag{10}$$

The function g is thus the even periodic extension of f. Its Fourier series, which is a cosine series, represents f on $[0, L]$.

2. Define a function h of period $2L$ so that

$$h(x) = \begin{cases} f(x), & 0 < x < L, \\ 0, & x = 0, L, \\ -f(-x), & -L < x < 0. \end{cases} \tag{11}$$

The function h is thus the odd periodic extension of f. Its Fourier series, which is a sine series, also represents f on $(0, L)$.

3. Define a function k of period $2L$ so that

$$k(x) = f(x), \qquad 0 \le x \le L, \tag{12}$$

and let $k(x)$ be defined for $(-L, 0)$ in any way consistent with the conditions of Theorem 10.3.1. Sometimes it is convenient to define $k(x)$ to be zero for $-L < x < 0$. The Fourier series for k, which involves both sine and cosine terms, also represents f on $[0, L]$, regardless of the manner in which $k(x)$ is defined in $(-L, 0)$. Thus there are infinitely many such series, all of which converge to $f(x)$ in the original interval.

Usually, the form of the expansion to be used will be dictated (or at least suggested) by the purpose for which it is needed. However, if there is a choice as to the kind of Fourier series to be used, the selection can sometimes be based on the rapidity of convergence. For example, the cosine series for the triangular wave [Eq. (20) of Section 10.2] converges more rapidly than the sine series for the sawtooth wave [Eq. (9) in this section], although both converge to the same function for $0 \le x < L$. This is due to the fact that the triangular wave is a smoother function than the sawtooth wave and is therefore easier to approximate. In general, the more continuous derivatives possessed by a function over the entire interval $-\infty < x < \infty$, the faster its Fourier series will converge. See Problem 18 of Section 10.3.

EXAMPLE 2

Suppose that

$$f(x) = \begin{cases} 1 - x, & 0 < x \le 1, \\ 0, & 1 < x \le 2. \end{cases} \tag{13}$$

As indicated previously, we can represent f either by a cosine series or by a sine series. Sketch the graph of the sum of each of these series for $-6 \le x \le 6$.

In this example $L = 2$, so the cosine series for f converges to the even periodic extension of f of period 4, whose graph is sketched in Figure 10.4.4.

Similarly, the sine series for f converges to the odd periodic extension of f of period 4. The graph of this function is shown in Figure 10.4.5.

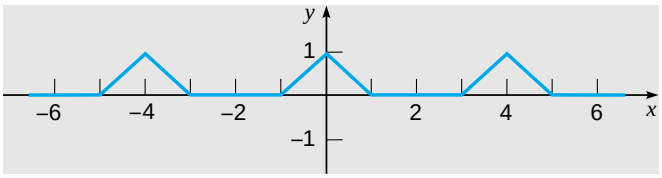


FIGURE 10.4.4 Even periodic extension of $f(x)$ given by Eq. (13).

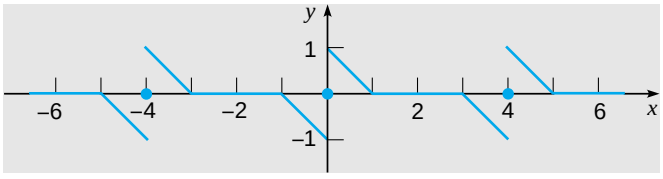


FIGURE 10.4.5 Odd periodic extension of $f(x)$ given by Eq. (13).

PROBLEMS

In each of Problems 1 through 6 determine whether the given function is even, odd, or neither.

1. $x^3 - 2x$
2. $x^3 - 2x + 1$
3. $\tan 2x$
4. $\sec x$
5. $|x|^3$
6. e^{-x}

In each of Problems 7 through 12 a function f is given on an interval of length L. In each case sketch the graphs of the even and odd extensions of f of period $2L$.

7. $f(x) = \begin{cases} x, & 0 \le x < 2, \\ 1, & 2 \le x < 3 \end{cases}$
8. $f(x) = \begin{cases} 0, & 0 \le x < 1, \\ x - 1, & 1 \le x < 2 \end{cases}$
9. $f(x) = 2 - x, \quad 0 < x < 2$
10. $f(x) = x - 3, \quad 0 < x < 4$
11. $f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 \le x < 2 \end{cases}$
12. $f(x) = 4 - x^2, \quad 0 < x < 1$
13. Prove that any function can be expressed as the sum of two other functions, one of which is even and the other odd. That is, for any function f, whose domain contains $-x$ whenever it contains x, show that there is an even function g and an odd function h such that $f(x) = g(x) + h(x)$.
    Hint: Assuming $f(x) = g(x) + h(x)$, what is $f(-x)$?
14. Find the coefficients in the cosine and sine series described in Example 2.

In each of Problems 15 through 22 find the required Fourier series for the given function and sketch the graph of the function to which the series converges over three periods.

15. $f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2; \end{cases}$ cosine series, period 4
    Compare with Example 1 and Problem 5 of Section 10.3.
16. $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & 1 \le x < 2; \end{cases}$ sine series, period 4
17. $f(x) = 1, \quad 0 \le x \le \pi;$ cosine series, period 2π
18. $f(x) = 1, \quad 0 < x < \pi;$ sine series, period 2π
19. $f(x) = \begin{cases} 0, & 0 < x < \pi, \\ 1, & \pi < x < 2\pi, \\ 2, & 2\pi < x < 3\pi \end{cases}$ sine series, period 6π
20. $f(x) = x, \quad 0 \le x < 1;$ series of period 1
21. $f(x) = L - x, \quad 0 \le x \le L;$ cosine series, period $2L$
    Compare with Example 1 of Section 10.2.
22. $f(x) = L - x, \quad 0 < x < L;$ sine series, period $2L$

In each of Problems 23 through 26:

(a) Find the required Fourier series for the given function.
(b) Sketch the graph of the function to which the series converges for three periods.
(c) Plot one or more partial sums of the series.

▶ 23. $f(x) = \begin{cases} x, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi; \end{cases}$ cosine series, period 4π

▶ 24. $f(x) = -x, \quad -\pi < x < 0;$ sine series, period 2π

▶ 25. $f(x) = 2 - x^2, \quad 0 < x < 2;$ sine series, period 4

▶ 26. $f(x) = x^2 - 2x, \quad 0 < x < 4;$ cosine series, period 8

In each of Problems 27 through 30 a function is given on an interval $0 < x < L$.

(a) Sketch the graphs of the even extension $g(x)$ and the odd extension $h(x)$ of the given function of period $2L$ over three periods.

(b) Find the Fourier cosine and sine series for the given function.
(c) Plot a few partial sums of each series.
(d) For each series investigate the dependence on n of the maximum error on $[0, L]$.

27. $f(x) = 3 - x, \quad 0 < x < 3$

28. $f(x) = \begin{cases} x, & 0 < x < 1, \\ 0, & 1 < x < 2 \end{cases}$

29. $f(x) = (4x^2 - 4x - 3)/4, \quad 0 < x < 2$

30. $f(x) = x^3 - 5x^2 + 5x + 1, \quad 0 < x < 3$

31. Prove that if f is an odd function, then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

32. Prove properties 2 and 3 of even and odd functions, as stated in the text.
33. Prove that the derivative of an even function is odd, and that the derivative of an odd function is even.
34. Let $F(x) = \int_0^x f(t)\,dt$. Show that if f is even, then F is odd, and that if f is odd, then F is even.
35. From the Fourier series for the square wave in Example 1 of Section 10.3, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

This relation between π and the odd positive integers was discovered by Leibniz in 1674.
36. From the Fourier series for the triangular wave (Example 1 of Section 10.2), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

37. Assume that f has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L), \qquad 0 \le x \le L.$$

(a) Show formally that

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2.$$

Compare this result with that of Problem 17 in Section 10.3. What is the corresponding result if f has a cosine series?
(b) Apply the result of part (a) to the series for the sawtooth wave given in Eq. (9), and thereby show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This relation was discovered by Euler about 1735.

More Specialized Fourier Series. Let f be a function originally defined on $0 \le x \le L$. In this section we have shown that it is possible to represent f either by a sine series or by a cosine series by constructing odd or even periodic extensions of f, respectively. Problems 38 through 40 concern some other more specialized Fourier series that converge to the given function f on $(0, L)$.

38. Let f be extended into $(L, 2L]$ in an arbitrary manner. Then extend the resulting function into $(-2L, 0)$ as an odd function and elsewhere as a periodic function of period $4L$ (see Figure 10.4.6). Show that this function has a Fourier sine series in terms of the functions $\sin(n\pi x/2L)$, $n = 1, 2, 3, \ldots$; that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2L),$$

where

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin(n\pi x/2L)\,dx.$$

This series converges to the original function on $(0, L)$.

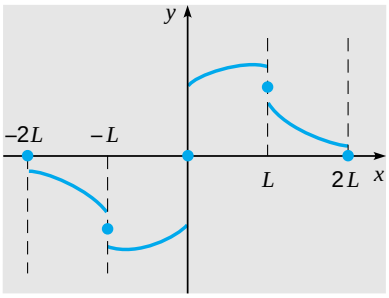


FIGURE 10.4.6 Graph of the function in Problem 38.

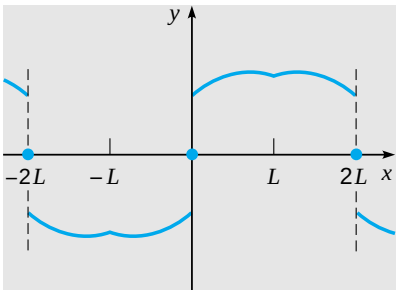


FIGURE 10.4.7 Graph of the function in Problem 39.

39. Let f first be extended into $(L, 2L)$ so that it is symmetric about $x = L$; that is, so as to satisfy $f(2L - x) = f(x)$ for $0 \le x < L$. Let the resulting function be extended into $(-2L, 0)$ as an odd function and elsewhere (see Figure 10.4.7) as a periodic function of period $4L$. Show that this function has a Fourier series in terms of the functions $\sin(\pi x/2L)$, $\sin(3\pi x/2L)$, $\sin(5\pi x/2L), \ldots$; that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2L},$$

where

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

This series converges to the original function on $(0, L]$.

40. How should f, originally defined on $[0, L]$, be extended so as to obtain a Fourier series involving only the functions $\cos(\pi x/2L)$, $\cos(3\pi x/2L)$, $\cos(5\pi x/2L), \ldots$? Refer to Problems 38 and 39. If $f(x) = x$ for $0 \le x \le L$, sketch the function to which the Fourier series converges for $-4L \le x \le 4L$.

10.5 Separation of Variables; Heat Conduction in a Rod

The basic partial differential equations of heat conduction, wave propagation, and potential theory that we discuss in this chapter are associated with three distinct types of physical phenomena: diffusive processes, oscillatory processes, and time independent or steady processes. Consequently, they are of fundamental importance in many branches of physics. They are also of considerable significance from a mathematical point of view. The partial differential equations whose theory is best developed and whose applications are most significant and varied are the linear equations of second order. All such equations can be classified into one of three categories; the heat conduction equation, the wave equation, and the potential equation, respectively, are prototypes of each of these categories. Thus a study of these three equations yields much information about more general second order linear partial differential equations.

During the last two centuries several methods have been developed for solving partial differential equations. The method of separation of variables is the oldest systematic method, having been used by D'Alembert, Daniel Bernoulli, and Euler in about 1750 in their investigations of waves and vibrations. It has been considerably refined and generalized in the meantime, and remains a method of great importance and frequent use today. To show how the method of separation of variables works we consider first a basic problem of heat conduction in a solid body. The mathematical study of heat conduction originated[6] about 1800, and it continues to command the attention of modern scientists. For example, the analysis of the dissipation and transfer of heat away from its sources in high-speed machinery is frequently an important technological problem.

Let us now consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = L$ denote the ends of the bar (see Figure 10.5.1). Suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered as constant on any given cross section. Then u is a function only of the axial coordinate x and the time t.

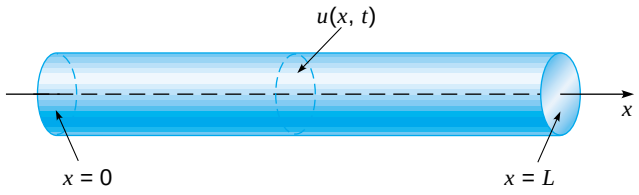


FIGURE 10.5.1 A heat-conducting solid bar.

[6]The first important investigation of heat conduction was carried out by Joseph Fourier (1768–1830) while he was serving as prefect of the department of Isère (Grenoble) from 1801 to 1815. He presented basic papers on the subject to the Academy of Sciences of Paris in 1807 and 1811. However, these papers were criticized by the referees (principally Lagrange) for lack of rigor and so were not published. Fourier continued to develop his ideas and eventually wrote one of the classics of applied mathematics, *Théorie analytique de la chaleur*, published in 1822.

The variation of temperature in the bar is governed by a partial differential equation whose derivation appears in Appendix A at the end of the chapter. The equation is called the **heat conduction equation**, and has the form

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \qquad t > 0, \tag{1}$$

where α^2 is a constant known as the **thermal diffusivity**. The parameter α^2 depends only on the material from which the bar is made, and is defined by

$$\alpha^2 = \kappa/\rho s, \tag{2}$$

where κ is the thermal conductivity, ρ is the density, and s is the specific heat of the material in the bar. The units of α^2 are (length)^2/time. Typical values of α^2 are given in Table 10.5.1.

TABLE 10.5.1 Values of the Thermal Diffusivity for Some Common Materials

Material	α^2(cm^2/sec)
Silver	1.71
Copper	1.14
Aluminum	0.86
Cast iron	0.12
Granite	0.011
Brick	0.0038
Water	0.00144

In addition, we assume that the initial temperature distribution in the bar is given; thus

$$u(x, 0) = f(x), \qquad 0 \le x \le L, \tag{3}$$

where f is a given function. Finally, we assume that the ends of the bar are held at fixed temperatures: the temperature T_1 at $x = 0$ and the temperature T_2 at $x = L$. However, it turns out that we need only consider the case where $T_1 = T_2 = 0$. We show in Section 10.6 how to reduce the more general problem to this special case. Thus in this section we will assume that u is always zero when $x = 0$ or $x = L$:

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t > 0. \tag{4}$$

The fundamental problem of heat conduction is to find $u(x, t)$ that satisfies the differential equation (1) for $0 < x < L$ and for $t > 0$, the initial condition (3) when $t = 0$, and the boundary conditions (4) at $x = 0$ and $x = L$.

The problem described by Eqs. (1), (3), and (4) is an initial value problem in the time variable t; an initial condition is given and the differential equation governs what happens later. However, with respect to the space variable x, the problem is a boundary value problem; boundary conditions are imposed at each end of the bar and the differential equation describes the evolution of the temperature in the interval between them. Alternatively, we can consider the problem as a boundary value problem in the xt-plane (see Figure 10.5.2). The solution $u(x, t)$ of Eq. (1) is sought in the semi-infinite strip $0 < x < L$, $t > 0$ subject to the requirement that $u(x, t)$ must assume a prescribed value at each point on the boundary of this strip.

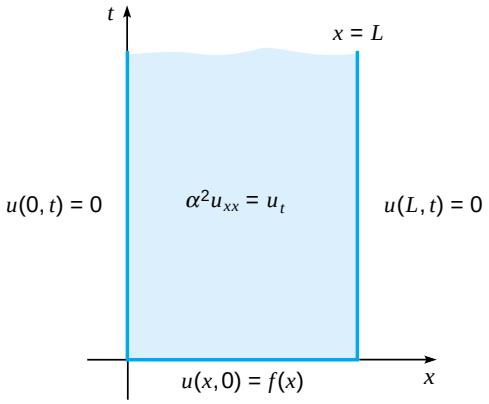


FIGURE 10.5.2 Boundary value problem for the heat conduction equation.

The heat conduction problem (1), (3), (4) is *linear* since u appears only to the first power throughout. The differential equation and boundary conditions are also *homogeneous*. This suggests that we might approach the problem by seeking solutions of the differential equation and boundary conditions, and then superposing them to satisfy the initial condition. The remainder of this section describes how this plan can be implemented.

One solution of the differential equation (1) that satisfies the boundary conditions (4) is the function $u(x, t) = 0$, but this solution does not satisfy the initial condition (3) except in the trivial case in which $f(x)$ is also zero. Thus our goal is to find other, nonzero, solutions of the differential equation and boundary conditions. To find the needed solutions, we start by making a basic assumption about the form of the solutions that has far-reaching, and perhaps unforeseen, consequences. The assumption is that $u(x, t)$ is a product of two other functions, one depending only on x and the other depending only on t; thus

$$u(x, t) = X(x)T(t). \tag{5}$$

Substituting from Eq. (5) for u in the differential equation (1) yields

$$\alpha^2 X''T = XT', \tag{6}$$

where primes refer to ordinary differentiation with respect to the independent variable, whether x or t. Equation (6) is equivalent to

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}, \tag{7}$$

in which the variables are separated; that is, the left side depends only on x and the right side only on t. For Eq. (7) to be valid for $0 < x < L$, $t > 0$, it is necessary that both sides of Eq. (7) be equal to the same constant. Otherwise, by keeping one independent variable (say, x) fixed and varying the other, one side (the left in this case) of Eq. (7) would remain unchanged while the other varied, thus violating the equality. If we call this separation constant $-\lambda$, then Eq. (7) becomes

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda. \tag{8}$$

Hence we obtain the following two ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \lambda X = 0, \tag{9}$$

$$T' + \alpha^2 \lambda T = 0. \tag{10}$$

We denote the separation constant by $-\lambda$ (rather than λ) because it turns out that it must be negative, and it is convenient to exhibit the minus sign explicitly.

The assumption (5) has led to the replacement of the partial differential equation (1) by the two ordinary differential equations (9) and (10). Each of these equations can be readily solved for *any* value of λ. The product of two solutions of Eq. (9) and (10), respectively, provides a solution of the partial differential equation (1). However, we are interested only in those solutions of Eq. (1) that also satisfy the boundary conditions (4). As we now show, this severely restricts the possible values of λ.

Substituting for $u(x, t)$ from Eq. (5) in the boundary condition at $x = 0$, we obtain

$$u(0, t) = X(0)T(t) = 0. \tag{11}$$

If Eq. (11) is satisfied by choosing $T(t)$ to be zero for all t, then $u(x, t)$ is zero for all x and t, and we have already rejected this possibility. Therefore Eq. (11) must be satisfied by requiring that

$$X(0) = 0. \tag{12}$$

Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0. \tag{13}$$

We now want to consider Eq. (9) subject to the boundary conditions (12) and (13). This is an eigenvalue problem and, in fact, is the same problem that we discussed in detail at the end of Section 10.1; see especially the paragraph following Eq. (29) in that section. The only difference is that the dependent variable there was called y rather than X. If we refer to the results obtained earlier [Eq. (31) of Section 10.1], the only nontrivial solutions of Eqs. (9), (12), and (13) are the eigenfunctions

$$X_n(x) = \sin(n\pi x/L), \quad n = 1, 2, 3, \ldots, \tag{14}$$

associated with the eigenvalues

$$\lambda_n = n^2\pi^2/L^2, \quad n = 1, 2, 3, \ldots. \tag{15}$$

Turning now to Eq. (10) for $T(t)$ and substituting $n^2\pi^2/L^2$ for λ, we have

$$T' + (n^2\pi^2\alpha^2/L^2)T = 0. \tag{16}$$

Thus $T(t)$ is proportional to $\exp(-n^2\pi^2\alpha^2 t/L^2)$. Hence, multiplying solutions of Eqs. (9) and (10) together, and neglecting arbitrary constants of proportionality, we conclude that the functions

$$u_n(x, t) = e^{-n^2\pi^2\alpha^2 t/L^2} \sin(n\pi x/L), \qquad n = 1, 2, 3, \ldots \tag{17}$$

satisfy the partial differential equation (1) and the boundary conditions (4) for each positive integer value of n. The functions u_n are sometimes called fundamental solutions of the heat conduction problem (1), (3), and (4).

It remains only to satisfy the initial condition (3),

$$u(x, 0) = f(x), \qquad 0 \le x \le L. \tag{18}$$

Recall that we have often solved initial value problems by forming linear combinations of a set of fundamental solutions, and then choosing the coefficients to satisfy the initial conditions. The analogous step in the present problem is to form a linear combination of the functions (17) and then to choose the coefficients to satisfy Eq. (18). The main difference now from earlier problems is that there are infinitely many functions (17), so a general linear combination of them is an infinite series. Thus we assume that

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}, \tag{19}$$

where the coefficients c_n are as yet undetermined. The individual terms in the series (19) satisfy the differential equation (1) and boundary conditions (4). We will assume that the infinite series of Eq. (19) converges and also satisfies Eqs. (1) and (4). To satisfy the initial condition (3) we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x). \tag{20}$$

In other words, we need to choose the coefficients c_n so that the series of sine functions in Eq. (20) converges to the initial temperature distribution $f(x)$ for $0 \le x \le L$. The series in Eq. (20) is just the Fourier sine series for f; according to Eq. (8) of Section 10.4 its coefficients are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{21}$$

Hence the solution of the heat conduction problem of Eqs. (1), (3), and (4) is given by the series in Eq. (19) with the coefficients computed from Eq. (21).

EXAMPLE 1

Find the temperature $u(x,t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20°C throughout, and whose ends are maintained at 0°C for all $t > 0$.

The temperature in the rod satisfies the heat conduction problem (1), (3), (4) with $L = 50$ and $f(x) = 20$ for $0 < x < 50$. Thus, from Eq. (19), the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/2500} \sin\frac{n\pi x}{50}, \tag{22}$$

where, from Eq. (21),

$$\begin{aligned} c_n &= \frac{4}{5}\int_0^{50} \sin\frac{n\pi x}{50}\,dx \\ &= \frac{40}{n\pi}(1-\cos n\pi) = \begin{cases} 80/n\pi, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases} \end{aligned} \tag{23}$$

Finally, by substituting for c_n in Eq. (22) we obtain

$$u(x,t) = \frac{80}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-n^2\pi^2\alpha^2 t/2500} \sin\frac{n\pi x}{50}. \tag{24}$$

The expression (24) for the temperature is moderately complicated, but the negative exponential factor in each term of the series causes the series to converge quite rapidly,

except for small values of t or α^2. Therefore accurate results can usually be obtained by using only a few terms of the series.

In order to display quantitative results let us measure t in seconds; then α^2 has the units of cm^2/sec. If we choose $\alpha^2 = 1$ for convenience, this corresponds to a rod of a material whose thermal properties are somewhere between copper and aluminum. The behavior of the solution can be seen from the graphs in Figures 10.5.3 through 10.5.5. In Figure 10.5.3 we show the temperature distribution in the bar at several different times. Observe that the temperature diminishes steadily as heat in the bar is lost through the endpoints. The way in which the temperature decays at a given point in the bar is indicated in Figure 10.5.4, where temperature is plotted against time for a few selected points in the bar. Finally, Figure 10.5.5 is a three-dimensional plot of u versus both x and t. Observe that the graphs in Figures 10.5.3 and 10.5.4 are obtained by intersecting the surface in Figure 10.5.5 by planes on which either t or x is constant. The slight waviness in Figure 10.5.5 at $t = 0$ results from using only a finite number of terms in the series for $u(x, t)$ and from the slow convergence of the series for $t = 0$.

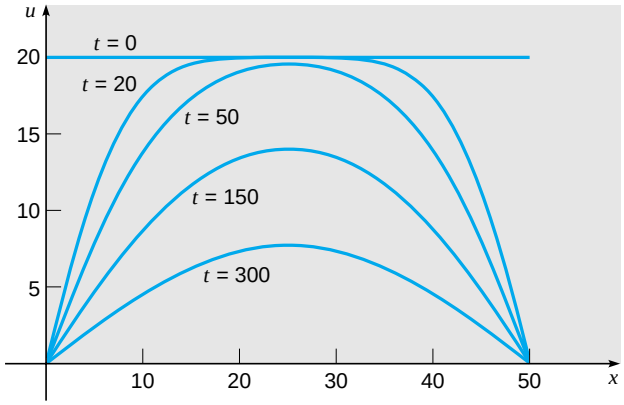


FIGURE 10.5.3 Temperature distributions at several times for the heat conduction problem of Example 1.

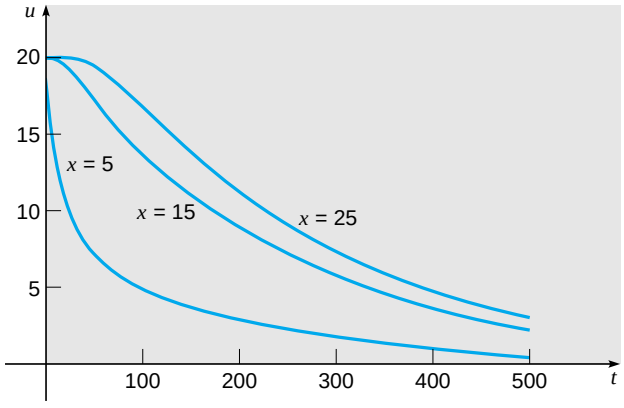


FIGURE 10.5.4 Dependence of temperature on time at several locations for the heat conduction problem of Example 1.

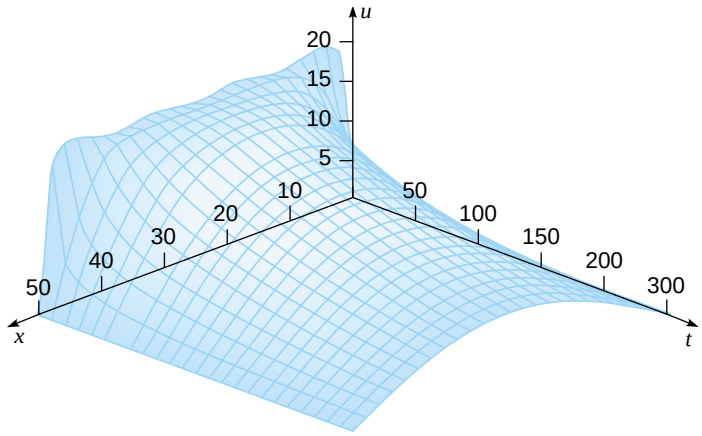


FIGURE 10.5.5 Plot of temperature u versus x and t for the heat conduction problem of Example 1.

A question with possible practical implications is to determine the time τ at which the entire bar has cooled to a specified temperature. For example, when is the temperature in the entire bar no greater than 1°C? Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point in the bar is always the center. Thus, τ is found by solving $u(25, t) = 1$ for t. Using one term in the series expansion (24), we obtain

$$\tau = \frac{2500}{\pi^2} \ln(80/\pi) \cong 820 \text{ sec}.$$

PROBLEMS

In each of Problems 1 through 6 determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. $xu_{xx} + u_t = 0$
2. $tu_{xx} + xu_t = 0$
3. $u_{xx} + u_{xt} + u_t = 0$
4. $[p(x)u_x]_x - r(x)u_{tt} = 0$
5. $u_{xx} + (x + y)u_{yy} = 0$
6. $u_{xx} + u_{yy} + xu = 0$
7. Find the solution of the heat conduction problem

$$\begin{aligned} &100u_{xx} = u_t, \qquad 0 < x < 1, \quad t > 0; \\ &u(0, t) = 0, \qquad u(1, t) = 0, \quad t > 0; \\ &u(x, 0) = \sin 2\pi x - \sin 5\pi x, \qquad 0 \le x \le 1. \end{aligned}$$

8. Find the solution of the heat conduction problem

$$\begin{aligned} &u_{xx} = 4u_t, \qquad 0 < x < 2, \quad t > 0; \\ &u(0, t) = 0, \qquad u(2, t) = 0, \quad t > 0; \\ &u(x, 0) = 2\sin(\pi x/2) - \sin \pi x + 4 \sin 2\pi x, \qquad 0 \le x \le 2. \end{aligned}$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. In each of Problems 9 through 12 find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

9. $u(x,0) = 50, \quad 0 < x < 40$

10. $u(x,0) = \begin{cases} x, & 0 \le x < 20, \\ 40 - x, & 20 \le x \le 40 \end{cases}$

11. $u(x,0) = \begin{cases} 0, & 0 \le x < 10, \\ 50, & 10 \le x \le 30, \\ 0, & 30 < x \le 40 \end{cases}$

12. $u(x,0) = x, \quad 0 < x < 40$

▶ 13. Consider again the rod in Problem 9. For $t = 5$ and $x = 20$ determine how many terms are needed to find the solution correct to three decimal places. A reasonable way to do this is to find n so that including one more term does not change the first three decimal places of $u(20, 5)$. Repeat for $t = 20$ and $t = 80$. Form a conclusion about the speed of convergence of the series for $u(x, t)$.

▶ 14. For the rod in Problem 9:
(a) Plot u versus x for $t = 5$, 10, 20, 40, 100, and 200. Put all the graphs on the same set of axes and thereby obtain a picture of the way in which the temperature distribution changes with time.
(b) Plot u versus t for $x = 5$, 10, 15, and 20.
(c) Draw a three-dimensional plot of u versus x and t.
(d) How long does it take for the entire rod to cool off to a temperature of no more than 1°C?

▶ 15. Follow the instructions in Problem 14 for the rod in Problem 10.

▶ 16. Follow the instructions in Problem 14 for the rod in Problem 11.

▶ 17. For the rod in Problem 12:
(a) Plot u versus x for $t = 5$, 10, 20, 40, 100, and 200.
(b) For each value of t used in part (a) estimate the value of x for which the temperature is greatest. Plot these values versus t to see how the location of the warmest point in the rod changes with time.
(c) Plot u versus t for $x = 10$, 20, and 30.
(d) Draw a three-dimensional plot of u versus x and t.
(e) How long does it take for the entire rod to cool off to a temperature of no more than 1°C?

▶ 18. Let a metallic rod 20 cm long be heated to a uniform temperature of 100°C. Suppose that at $t = 0$ the ends of the bar are plunged into an ice bath at 0°C, and thereafter maintained at this temperature, but that no heat is allowed to escape through the lateral surface. Find an expression for the temperature at any point in the bar at any later time. Determine the temperature at the center of the bar at time $t = 30$ sec if the bar is made of (a) silver, (b) aluminum, or (c) cast iron.

▶ 19. For the rod of Problem 18 find the time that will elapse before the center of the bar cools to a temperature of 5°C if the bar is made of (a) silver, (b) aluminum, or (c) cast iron.

20. In solving differential equations the computations can almost always be simplified by the use of *dimensionless variables*. Show that if the dimensionless variable $\xi = x/L$ is introduced, the heat conduction equation becomes

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{L^2}{\alpha^2}\frac{\partial u}{\partial t}, \qquad 0 < \xi < 1, \quad t > 0.$$

Since L^2/α^2 has the units of time, it is convenient to use this quantity to define a dimensionless time variable $\tau = (\alpha^2/L^2)t$. Then show that the heat conduction equation reduces to

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \tau}, \qquad 0 < \xi < 1, \quad \tau > 0.$$

21. Consider the equation

$$au_{xx} - bu_t + cu = 0, \tag{i}$$

where a, b, and c are constants.

(a) Let $u(x,t) = e^{\delta t} w(x,t)$, where δ is constant, and find the corresponding partial differential equation for w.
(b) If $b \neq 0$, show that δ can be chosen so that the partial differential equation found in part (a) has no term in w. Thus, by a change of dependent variable, it is possible to reduce Eq. (i) to the heat conduction equation.

22. The heat conduction equation in two space dimensions is

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

Assuming that $u(x,y,t) = X(x)Y(y)T(t)$, find ordinary differential equations satisfied by $X(x)$, $Y(y)$, and $T(t)$.

23. The heat conduction equation in two space dimensions may be expressed in terms of polar coordinates as

$$\alpha^2[u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta}] = u_t.$$

Assuming that $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

10.6 Other Heat Conduction Problems

In Section 10.5 we considered the problem consisting of the heat conduction equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0, \tag{1}$$

the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t > 0, \tag{2}$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq L. \tag{3}$$

We found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}, \tag{4}$$

where the coefficients c_n are the same as in the series

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}. \tag{5}$$

The series in Eq. (5) is just the Fourier sine series for f; according to Section 10.4 its coefficients are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{6}$$

Hence the solution of the heat conduction problem, Eqs. (1) to (3), is given by the series in Eq. (4) with the coefficients computed from Eq. (6).

We emphasize that at this stage the solution (4) must be regarded as a *formal* solution; that is, we obtained it without rigorous justification of the limiting processes involved.

Such a justification is beyond the scope of this book. However, once the series (4) has been obtained, it is possible to show that in $0 < x < L$, $t > 0$ it converges to a continuous function, that the derivatives u_{xx} and u_t can be computed by differentiating the series (4) term by term, and that the heat conduction equation (1) is indeed satisfied. The argument rests heavily on the fact that each term of the series (4) contains a negative exponential factor, and this results in relatively rapid convergence of the series. A further argument establishes that the function u given by Eq. (4) also satisfies the boundary and initial conditions; this completes the justification of the formal solution.

It is interesting to note that although f satisfies the conditions of the Fourier convergence theorem (Theorem 10.3.1), it may have points of discontinuity. In this case the initial temperature distribution $u(x, 0) = f(x)$ is discontinuous at one or more points. Nevertheless, the solution $u(x, t)$ is continuous for arbitrarily small values of $t > 0$. This illustrates the fact that heat conduction is a diffusive process that instantly smooths out any discontinuities that may be present in the initial temperature distribution. Finally, since f is bounded, it follows from Eq. (6) that the coefficients c_n are also bounded. Consequently, the presence of the negative exponential factor in each term of the series (4) guarantees that

$$\lim_{t\to\infty} u(x, t) = 0 \tag{7}$$

for all x regardless of the initial condition. This is in accord with the result expected from physical intuition.

We now consider two other problems of one-dimensional heat conduction that can be handled by the method developed in Section 10.5.

Nonhomogeneous Boundary Conditions. Suppose now that one end of the bar is held at a constant temperature T_1 and the other is maintained at a constant temperature T_2. Then the boundary conditions are

$$u(0, t) = T_1, \qquad u(L, t) = T_2, \qquad t > 0. \tag{8}$$

The differential equation (1) and the initial condition (3) remain unchanged.

This problem is only slightly more difficult, because of the nonhomogeneous boundary conditions, than the one in Section 10.5. We can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as in Section 10.5. The technique for doing this is suggested by the following physical argument.

After a long time, that is, as $t \to \infty$, we anticipate that a steady temperature distribution $v(x)$ will be reached, which is independent of the time t and the initial conditions. Since $v(x)$ must satisfy the equation of heat conduction (1), we have

$$v''(x) = 0, \qquad 0 < x < L. \tag{9}$$

Hence the steady-state temperature distribution is a linear function of x. Further, $v(x)$ must satisfy the boundary conditions

$$v(0) = T_1, \qquad v(L) = T_2, \tag{10}$$

which are valid even as $t \to \infty$. The solution of Eq. (9) satisfying Eqs. (10) is

$$v(x) = (T_2 - T_1)\frac{x}{L} + T_1. \tag{11}$$

Returning to the original problem, Eqs. (1), (3), and (8), we will try to express $u(x, t)$ as the sum of the steady-state temperature distribution $v(x)$ and another (transient) temperature distribution $w(x, t)$; thus we write

$$u(x, t) = v(x) + w(x, t). \tag{12}$$

Since $v(x)$ is given by Eq. (11), the problem will be solved provided we can determine $w(x, t)$. The boundary value problem for $w(x, t)$ is found by substituting the expression in Eq. (12) for $u(x, t)$ in Eqs. (1), (3), and (8).

From Eq. (1) we have

$$\alpha^2 (v + w)_{xx} = (v + w)_t;$$

it follows that

$$\alpha^2 w_{xx} = w_t, \tag{13}$$

since $v_{xx} = 0$ and $v_t = 0$. Similarly, from Eqs. (12), (8), and (10),

$$\begin{aligned} w(0, t) &= u(0, t) - v(0) = T_1 - T_1 = 0, \\ w(L, t) &= u(L, t) - v(L) = T_2 - T_2 = 0. \end{aligned} \tag{14}$$

Finally, from Eqs. (12) and (3),

$$w(x, 0) = u(x, 0) - v(x) = f(x) - v(x), \tag{15}$$

where $v(x)$ is given by Eq. (11). Thus the transient part of the solution to the original problem is found by solving the problem consisting of Eqs. (13), (14), and (15). This latter problem is precisely the one solved in Section 10.5 provided that $f(x) - v(x)$ is now regarded as the initial temperature distribution. Hence

$$u(x, t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}, \tag{16}$$

where

$$c_n = \frac{2}{L}\int_0^L \left[f(x) - (T_2 - T_1)\frac{x}{L} - T_1 \right] \sin\frac{n\pi x}{L}\, dx. \tag{17}$$

This is another case in which a more difficult problem is solved by reducing it to a simpler problem that has already been solved. The technique of reducing a problem with nonhomogeneous boundary conditions to one with homogeneous boundary conditions by subtracting the steady-state solution is capable of wide application.

EXAMPLE 1

Consider the heat conduction problem

$$u_{xx} = u_t, \qquad 0 < x < 30, \quad t > 0, \tag{18}$$

$$u(0, t) = 20, \qquad u(30, t) = 50, \qquad t > 0, \tag{19}$$

$$u(x, 0) = 60 - 2x, \qquad 0 < x < 30. \tag{20}$$

Find the steady-state temperature distribution and the boundary value problem that determines the transient distribution.

The steady-state temperature satisfies $v''(x) = 0$ and the boundary conditions $v(0) = 20$ and $v(30) = 50$. Thus $v(x) = 20 + x$. The transient distribution $w(x, t)$ satisfies the heat conduction equation

$$w_{xx} = w_t, \tag{21}$$

the homogeneous boundary conditions

$$w(0, t) = 0, \qquad w(30, t) = 0, \tag{22}$$

and the modified initial condition

$$w(x, 0) = 60 - 2x - (20 + x) = 40 - 3x. \tag{23}$$

Note that this problem is of the form (1), (2), (3) with $f(x) = 40 - 3x$, $\alpha^2 = 1$, and $L = 30$. Thus the solution is given by Eqs. (4) and (6).

Figure 10.6.1 shows a plot of the initial temperature distribution $60 - 2x$, the final temperature distribution $20 + x$, and the temperature at two intermediate times found by solving Eqs. (21) through (23). Note that the intermediate temperature satisfies the boundary conditions (19) for any $t > 0$. As t increases, the effect of the boundary conditions gradually moves from the ends of the bar toward its center.

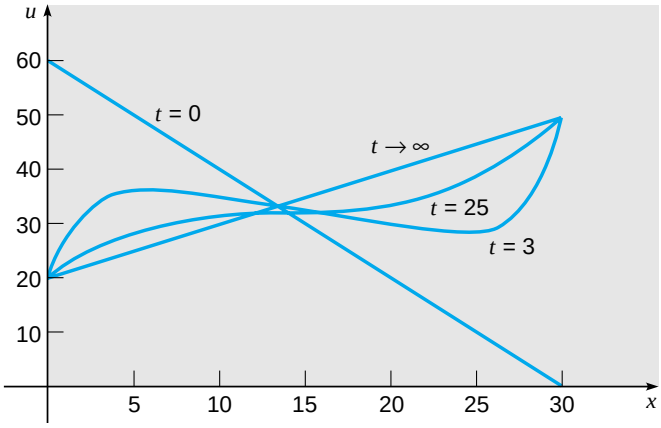


FIGURE 10.6.1 Temperature distributions at several times for the heat conduction problem of Example 1.

Bar with Insulated Ends. A slightly different problem occurs if the ends of the bar are insulated so that there is no passage of heat through them. According to Eq. (2) in Appendix A, the rate of flow of heat across a cross section is proportional to the rate of change of temperature in the x direction. Thus in the case of no heat flow the boundary conditions are

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0, \qquad t > 0. \tag{24}$$

The problem posed by Eqs. (1), (3), and (24) can also be solved by the method of separation of variables. If we let

$$u(x, t) = X(x)T(t), \tag{25}$$

and substitute for u in Eq. (1), then it follows as in Section 10.5 that

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda, \tag{26}$$

where λ is a constant. Thus we obtain again the two ordinary differential equations

$$X'' + \lambda X = 0, \tag{27}$$

$$T' + \alpha^2 \lambda T = 0. \tag{28}$$

For any value of λ a product of solutions of Eqs. (27) and (28) is a solution of the partial differential equation (1). However, we are interested only in those solutions that also satisfy the boundary conditions (24).

If we substitute for $u(x, t)$ from Eq. (25) in the boundary condition at $x = 0$, we obtain $X'(0)T(t) = 0$. We cannot permit $T(t)$ to be zero for all t, since then $u(x, t)$ would also be zero for all t. Hence we must have

$$X'(0) = 0. \tag{29}$$

Proceeding in the same way with the boundary condition at $x = L$, we find that

$$X'(L) = 0. \tag{30}$$

Thus we wish to solve Eq. (27) subject to the boundary conditions (29) and (30). It is possible to show that nontrivial solutions of this problem can exist only if λ is real. One way to show this is indicated in Problem 18; alternatively, one can appeal to a more general theory to be discussed later in Section 11.2. We will assume that λ is real and consider in turn the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

If $\lambda < 0$, it is convenient to let $\lambda = -\mu^2$, where μ is real and positive. Then Eq. (27) becomes $X'' - \mu^2 X = 0$ and its general solution is

$$X(x) = k_1 \sinh \mu x + k_2 \cosh \mu x. \tag{31}$$

In this case the boundary conditions can be satisfied only by choosing $k_1 = k_2 = 0$. Since this is unacceptable, it follows that λ cannot be negative; in other words, the problem (27), (29), (30) has no negative eigenvalues.

If $\lambda = 0$, then Eq. (27) is $X'' = 0$, and therefore

$$X(x) = k_1 x + k_2. \tag{32}$$

The boundary conditions (29) and (30) require that $k_1 = 0$ but do not determine k_2. Thus $\lambda = 0$ is an eigenvalue, corresponding to the eigenfunction $X(x) = 1$. For $\lambda = 0$ it follows from Eq. (28) that $T(t)$ is also a constant, which can be combined with k_2. Hence, for $\lambda = 0$, we obtain the constant solution $u(x, t) = k_2$.

Finally, if $\lambda > 0$, let $\lambda = \mu^2$, where μ is real and positive. Then Eq. (27) becomes $X'' + \mu^2 X = 0$, and consequently

$$X(x) = k_1 \sin \mu x + k_2 \cos \mu x. \tag{33}$$

The boundary condition (29) requires that $k_1 = 0$, and the boundary condition (30) requires that $\mu = n\pi/L$ for $n = 1, 2, 3, \ldots$, but leaves k_2 arbitrary. Thus the problem (27), (29), (30) has an infinite sequence of positive eigenvalues $\lambda = n^2\pi^2/L^2$ with the corresponding eigenfunctions $X(x) = \cos(n\pi x/L)$. For these values of λ the solutions $T(t)$ of Eq. (28) are proportional to $\exp(-n^2\pi^2\alpha^2 t/L^2)$.

Combining all these results, we have the following fundamental solutions for the problem (1), (3), and (24):

$$\begin{aligned} u_0(x,t) &= 1, \\ u_n(x,t) &= e^{-n^2\pi^2\alpha^2 t/L^2}\cos\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots, \end{aligned} \tag{34}$$

where arbitrary constants of proportionality have been dropped. Each of these functions satisfies the differential equation (1) and the boundary conditions (24). Because both the differential equation and boundary conditions are linear and homogeneous, any finite linear combination of the fundamental solutions satisfies them. We will assume that this is true for convergent infinite linear combinations of fundamental solutions as well. Thus, to satisfy the initial condition (3), we assume that $u(x,t)$ has the form

$$\begin{aligned} u(x,t) &= \frac{c_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} c_n u_n(x,t) \\ &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2}\cos\frac{n\pi x}{L}. \end{aligned} \tag{35}$$

The coefficients c_n are determined by the requirement that

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L} = f(x). \tag{36}$$

Thus the unknown coefficients in Eq. (35) must be the coefficients in the Fourier cosine series of period $2L$ for f. Hence

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients c_0, c_1, c_2, ..., the series (35) provides the solution to the heat conduction problem for a rod with insulated ends, Eqs. (1), (3), and (24).

It is worth observing that the solution (35) can also be thought of as the sum of a steady-state temperature distribution (given by the constant $c_0/2$), which is independent of time t, and a transient distribution (given by the rest of the infinite series) that vanishes in the limit as t approaches infinity. That the steady-state is a constant is consistent with the expectation that the process of heat conduction will gradually smooth out the temperature distribution in the bar as long as no heat is allowed to escape to the outside. The physical interpretation of the term

$$\frac{c_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx \tag{38}$$

is that it is the mean value of the original temperature distribution.

EXAMPLE 2

Find the temperature $u(x, t)$ in a metal rod of length 25 cm that is insulated on the ends as well as on the sides and whose initial temperature distribution is $u(x, 0) = x$ for $0 < x < 25$.

The temperature in the rod satisfies the heat conduction problem (1), (3), (24) with $L = 25$. Thus, from Eq. (35), the solution is

$$u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/625} \cos\frac{n\pi x}{25}, \tag{39}$$

where the coefficients are determined from Eq. (37). We have

$$c_0 = \frac{2}{25}\int_0^{25} x\, dx = 25 \tag{40}$$

and, for $n \geq 1$,

$$\begin{aligned} c_n &= \frac{2}{25}\int_0^{25} x \cos\frac{n\pi x}{25}\, dx \\ &= 50(\cos n\pi - 1)/(n\pi)^2 = \begin{cases} -100/(n\pi)^2, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases} \end{aligned} \tag{41}$$

Thus

$$u(x, t) = \frac{25}{2} - \frac{100}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2\alpha^2 t/625} \cos(n\pi x/25) \tag{42}$$

is the solution of the given problem.

For $\alpha^2 = 1$ Figure 10.6.2 shows plots of the temperature distribution in the bar at several times. Again the convergence of the series is rapid so that only a relatively few terms are needed to generate the graphs.

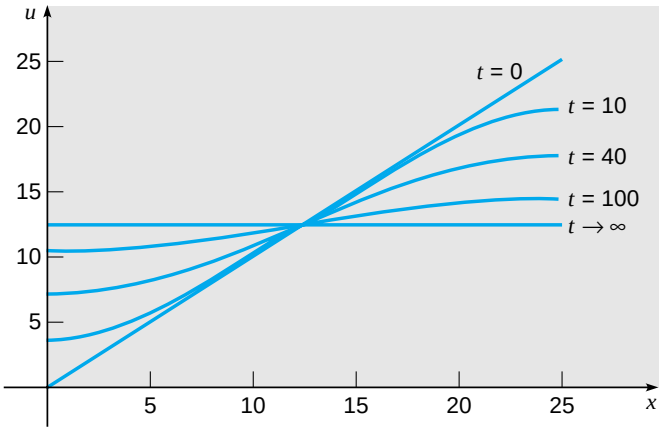


FIGURE 10.6.2 Temperature distributions at several times for the heat conduction problem of Example 2.

More General Problems. The method of separation of variables can also be used to solve heat conduction problems with other boundary conditions than those given by Eqs. (8) and Eqs. (24). For example, the left end of the bar might be held at a fixed temperature T while the other end is insulated. In this case the boundary conditions are

$$u(0,t) = T, \qquad u_x(L,t) = 0, \qquad t > 0. \tag{43}$$

The first step in solving this problem is to reduce the given boundary conditions to homogeneous ones by subtracting the steady-state solution. The resulting problem is solved by essentially the same procedure as in the problems previously considered. However, the extension of the initial function f outside of the interval $[0, L]$ is somewhat different from that in any case considered so far (see Problem 15).

A more general type of boundary condition occurs when the rate of heat flow through the end of the bar is proportional to the temperature. It is shown in Appendix A that the boundary conditions in this case are of the form

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(L,t) + h_2 u(L,t) = 0, \qquad t > 0, \tag{44}$$

where h_1 and h_2 are nonnegative constants. If we apply the method of separation of variables to the problem consisting of Eqs. (1), (3), and (44), we find that $X(x)$ must be a solution of

$$X'' + \lambda X = 0, \qquad X'(0) - h_1 X(0) = 0, \qquad X'(L) + h_2 X(L) = 0, \tag{45}$$

where λ is the separation constant. Once again it is possible to show that nontrivial solutions can exist only for certain nonnegative real values of λ, the eigenvalues, but these values are not given by a simple formula (see Problem 20). It is also possible to show that the corresponding solutions of Eqs. (45), the eigenfunctions, satisfy an orthogonality relation, and that one can satisfy the initial condition (3) by superposing solutions of Eqs. (45). However, the resulting series is not included in the discussion of this chapter. There is more general theory that covers such problems, and it is outlined in Chapter 11.

PROBLEMS

In each of Problems 1 through 8 find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

1. $u(0,t) = 10, \quad u(50,t) = 40$
2. $u(0,t) = 30, \quad u(40,t) = -20$
3. $u_x(0,t) = 0, \quad u(L,t) = 0$
4. $u_x(0,t) = 0, \quad u(L,t) = T$
5. $u(0,t) = 0, \quad u_x(L,t) = 0$
6. $u(0,t) = T, \quad u_x(L,t) = 0$
7. $u_x(0,t) - u(0,t) = 0, \quad u(L,t) = T$
8. $u(0,t) = T, \quad u_x(L,t) + u(L,t) = 0$

▶ 9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0°C while the end $x = 20$ is heated to 60°C, and both are thereafter maintained at those temperatures.
(a) Find the temperature distribution in the rod at any time t.
(b) Plot the initial temperature distribution, the final (steady-state) temperature distribution, and the temperature distributions at two representative intermediate times on the same set of axes.
(c) Plot u versus t for $x = 5$, 10, and 15.
(d) Determine the time interval that must elapse before the temperature at $x = 5$ cm comes (and remains) within 1% of its steady-state value.

▶ 10. (a) Let the ends of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100°C by an external heat source and that this situation is maintained until a steady-state results. Find this steady-state temperature distribution.

(b) At a time $t = 0$ [after the steady-state of part (a) has been reached] let the heat source be removed. At the same instant let the end $x = 0$ be placed in thermal contact with a reservoir at 20°C while the other end remains at 0°C. Find the temperature as a function of position and time.
(c) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(d) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1 degree of its limiting value?

▶ 11. Consider a rod of length 30 for which $\alpha^2 = 1$. Suppose the initial temperature distribution is given by $u(x, 0) = x(60 - x)/30$ and that the boundary conditions are $u(0, t) = 30$ and $u(30, t) = 0$.
(a) Find the temperature in the rod as a function of position and time.
(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(c) Plot u versus t for $x = 12$. Observe that u initially decreases, then increases for a while, and finally decreases to approach its steady-state value. Explain physically why this behavior occurs at this point.

▶ 12. Consider a uniform rod of length L with an initial temperature given by $u(x, 0) = \sin(\pi x/L)$, $0 \le x \le L$. Assume that both ends of the bar are insulated.
(a) Find the temperature $u(x, t)$.
(b) What is the steady-state temperature as $t \to \infty$?
(c) Let $\alpha^2 = 1$ and $L = 40$. Plot u versus x for several values of t. Also plot u versus t for several values of x.
(d) Describe briefly how the temperature in the rod changes as time progresses.

▶ 13. Consider a bar of length 40 cm whose initial temperature is given by $u(x, 0) = x(60 - x)/30$. Suppose that $\alpha^2 = 1/4$ cm^2/sec and that both ends of the bar are insulated.
(a) Find the temperature $u(x, t)$.
(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(c) Determine the steady-state temperature in the bar.
(d) Determine how much time must elapse before the temperature at $x = 40$ comes within 1 degree of its steady-state value.

▶ 14. Consider a bar 30 cm long that is made of a material for which $\alpha^2 = 1$ and whose ends are insulated. Suppose that the initial temperature is zero except for the interval $5 < x < 10$, where the initial temperature is 25°.
(a) Find the temperature $u(x, t)$.
(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(c) Plot $u(4, t)$ and $u(11, t)$ versus t. Observe that the points $x = 4$ and $x = 11$ are symmetrically located with respect to the initial temperature pulse, yet their temperature plots are significantly different. Explain physically why this is so.

15. Consider a uniform bar of length L having an initial temperature distribution given by $f(x)$, $0 \le x \le L$. Assume that the temperature at the end $x = 0$ is held at 0°C, while the end $x = L$ is insulated so that no heat passes through it.
(a) Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n(x, t) = e^{-(2n-1)^2\pi^2\alpha^2 t/4L^2} \sin[(2n - 1)\pi x/2L], \qquad n = 1, 2, 3, \ldots.$$

(b) Find a formal series expansion for the temperature $u(x, t)$,

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t),$$

that also satisfies the initial condition $u(x, 0) = f(x)$.
Hint: Even though the fundamental solutions involve only the odd sines, it is still possible

to represent f by a Fourier series involving only these functions. See Problem 39 of Section 10.4.

16. In the bar of Problem 15 suppose that $L = 30$, $\alpha^2 = 1$, and the initial temperature distribution is $f(x) = 30 - x$ for $0 < x < 30$.
(a) Find the temperature $u(x, t)$.
(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(c) How does the location x_m of the warmest point in the bar change as t increases? Draw a graph of x_m versus t.
(d) Plot the maximum temperature in the bar versus t.

17. Suppose that the conditions are as in Problems 15 and 16 except that the boundary condition at $x = 0$ is $u(0, t) = 40$.
(a) Find the temperature $u(x, t)$.
(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.
(c) Compare the plots you obtained in this problem with those from Problem 16. Explain how the change in the boundary condition at $x = 0$ causes the observed differences in the behavior of the temperature in the bar.

18. Consider the problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X'(L) = 0. \tag{i}$$

Let $\lambda = \mu^2$, where $\mu = \nu + i\sigma$ with ν and σ real. Show that if $\sigma \neq 0$, then the only solution of Eqs. (i) is the trivial solution $X(x) = 0$.
Hint: Use an argument similar to that in Problem 17 of Section 10.1.

19. The right end of a bar of length a with thermal conductivity κ_1 and cross-sectional area A_1 is joined to the left end of a bar of thermal conductivity κ_2 and cross-sectional area A_2. The composite bar has a total length L. Suppose that the end $x = 0$ is held at temperature zero, while the end $x = L$ is held at temperature T. Find the steady-state temperature in the composite bar, assuming that the temperature and rate of heat flow are continuous at $x = a$.
Hint: See Eq. (2) of Appendix A.

20. Consider the problem

$$\begin{aligned} &\alpha^2 u_{xx} = u_t, && 0 < x < L, \quad t > 0; \\ u(0, t) = 0, \qquad &u_x(L, t) + \gamma u(L, t) = 0, && t > 0; \\ &u(x, 0) = f(x), && 0 \le x \le L. \end{aligned} \tag{i}$$

(a) Let $u(x, t) = X(x)T(t)$ and show that

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(L) + \gamma X(L) = 0, \tag{ii}$$

and

$$T' + \lambda\alpha^2 T = 0,$$

where λ is the separation constant.
(b) Assume that λ is real, and show that problem (ii) has no nontrivial solutions if $\lambda \le 0$.
(c) If $\lambda > 0$, let $\lambda = \mu^2$ with $\mu > 0$. Show that problem (ii) has nontrivial solutions only if μ is a solution of the equation

$$\mu \cos \mu L + \gamma \sin \mu L = 0. \tag{iii}$$

(d) Rewrite Eq. (iii) as $\tan \mu L = -\mu/\gamma$. Then, by drawing the graphs of $y = \tan \mu L$ and $y = -\mu L/\gamma L$ for $\mu > 0$ on the same set of axes, show that Eq. (iii) is satisfied by infinitely many positive values of μ; denote these by $\mu_1, \mu_2, \ldots, \mu_n, \ldots$, ordered in increasing size.
(e) Determine the set of fundamental solutions $u_n(x, t)$ corresponding to the values μ_n found in part (d).

10.7 The Wave Equation: Vibrations of an Elastic String

A second partial differential equation occurring frequently in applied mathematics is the wave[7] equation. Some form of this equation, or a generalization of it, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. For example, the studies of acoustic waves, water waves, electromagnetic waves, and seismic waves are all based on this equation.

Perhaps the easiest situation to visualize occurs in the investigation of mechanical vibrations. Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the string (see Figure 10.7.1). The elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line. Suppose that the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane, and let $u(x, t)$ denote the vertical displacement experienced by the string at the point x at time t. If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then $u(x, t)$ satisfies the partial differential equation

$$a^2 u_{xx} = u_{tt} \tag{1}$$

in the domain $0 < x < L$, $t > 0$. Equation (1) is known as the **wave equation** and is derived in Appendix B at the end of the chapter. The constant coefficient a^2 appearing in Eq. (1) is given by

$$a^2 = T/\rho, \tag{2}$$

where T is the tension (force) in the string, and ρ is the mass per unit length of the string material. It follows that a has the units of length/time, that is, of velocity. In Problem 14 it is shown that a is the velocity of propagation of waves along the string.

To describe the motion of the string completely it is necessary also to specify suitable initial and boundary conditions for the displacement $u(x, t)$. The ends are assumed to remain fixed, and therefore the boundary conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t \geq 0. \tag{3}$$

Since the differential equation (1) is of second order with respect to t, it is plausible to prescribe two initial conditions. These are the initial position of the string,

$$u(x, 0) = f(x), \qquad 0 \leq x \leq L, \tag{4}$$

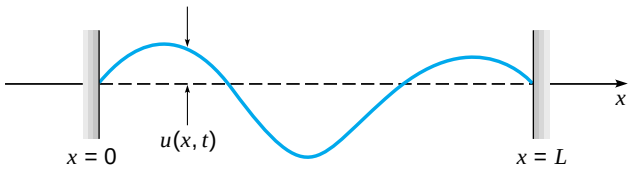


FIGURE 10.7.1 A vibrating string.

[7]The solution of the wave equation was one of the major mathematical problems of the mid-eighteenth century. The wave equation was first derived and studied by D'Alembert in 1746. It also attracted the attention of Euler (1748), Daniel Bernoulli (1753), and Lagrange (1759). Solutions were obtained in several different forms, and the merits of, and relations among, these solutions were argued, sometimes heatedly, in a series of papers extending over more than 25 years. The major points at issue concerned the nature of a function, and the kinds of functions that can be represented by trigonometric series. These questions were not resolved until the nineteenth century.

and its initial velocity,

$$u_t(x, 0) = g(x), \qquad 0 \le x \le L, \tag{5}$$

where f and g are given functions. In order for Eqs. (3), (4), and (5) to be consistent it is also necessary to require that

$$f(0) = f(L) = 0, \qquad g(0) = g(L) = 0. \tag{6}$$

The mathematical problem then is to determine the solution of the wave equation (1) that also satisfies the boundary conditions (3) and the initial conditions (4) and (5). Like the heat conduction problem of Sections 10.5 and 10.6, this problem is an initial value problem in the time variable t, and a boundary value problem in the space variable x. Alternatively, it can be considered as a boundary value problem in the semi-infinite strip $0 < x < L, t > 0$ of the xt-plane (see Figure 10.7.2). One condition is imposed at each point on the semi-infinite sides, and two are imposed at each point on the finite base.

It is important to realize that Eq. (1) governs a large number of other wave problems besides the transverse vibrations of an elastic string. For example, it is only necessary to interpret the function u and the constant a appropriately to have problems dealing with water waves in an ocean, acoustic or electromagnetic waves in the atmosphere, or elastic waves in a solid body. If more than one space dimension is significant, then Eq. (1) must be slightly generalized. The two-dimensional wave equation is

$$a^2(u_{xx} + u_{yy}) = u_{tt}. \tag{7}$$

This equation would arise, for example, if we considered the motion of a thin elastic sheet, such as a drumhead. Similarly, in three dimensions the wave equation is

$$a^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}. \tag{8}$$

In connection with the latter two equations, the boundary and initial conditions must also be suitably generalized.

We now solve some typical boundary value problems involving the one-dimensional wave equation.

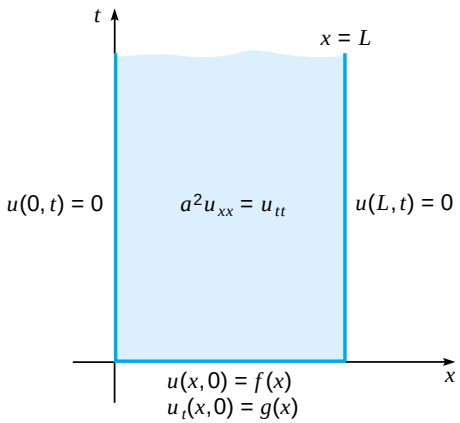


FIGURE 10.7.2 Boundary value problem for the wave equation.

Elastic String with Nonzero Initial Displacement. First suppose that the string is disturbed from its equilibrium position, and then released at time $t = 0$ with zero velocity to vibrate freely. Then the vertical displacement $u(x, t)$ must satisfy the wave equation (1),

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$

the boundary conditions (3),

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t \geq 0;$$

and the initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = 0, \qquad 0 \leq x \leq L, \tag{9}$$

where f is a given function describing the configuration of the string at $t = 0$.

The method of separation of variables can be used to obtain the solution of Eqs. (1), (3), and (9). Assuming that

$$u(x, t) = X(x)T(t) \tag{10}$$

and substituting for u in Eq. (1), we obtain

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda, \tag{11}$$

where λ is a separation constant. Thus we find that $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X'' + \lambda X = 0, \tag{12}$$

$$T'' + a^2 \lambda T = 0. \tag{13}$$

Further, by substituting from Eq. (10) for $u(x, t)$ in the boundary conditions (3) we find that $X(x)$ must satisfy the boundary conditions

$$X(0) = 0, \qquad X(L) = 0. \tag{14}$$

Finally, by substituting from Eq. (10) into the second of the initial conditions (9), we also find that $T(t)$ must satisfy the initial condition

$$T'(0) = 0. \tag{15}$$

Our next task is to determine $X(x)$, $T(t)$, and λ by solving Eq. (12) subject to the boundary conditions (14) and Eq. (13) subject to the initial condition (15).

The problem of solving the differential equation (12) subject to the boundary conditions (14) is *precisely the same problem* that arose in Section 10.5 in connection with the heat conduction equation. Thus we can use the results obtained there and at the end of Section 10.1: The problem (12) and (14) has nontrivial solutions if and only if λ is an eigenvalue,

$$\lambda = n^2\pi^2/L^2, \qquad n = 1, 2, \ldots, \tag{16}$$

and $X(x)$ is proportional to the corresponding eigenfunction $\sin(n\pi x/L)$.

Using the values of λ given by Eq. (16) in Eq. (13), we obtain

$$T'' + \frac{n^2\pi^2 a^2}{L^2} T = 0. \tag{17}$$

Therefore

$$T(t) = k_1 \cos\frac{n\pi at}{L} + k_2 \sin\frac{n\pi at}{L}, \tag{18}$$

where k_1 and k_2 are arbitrary constants. The initial condition (15) requires that $k_2 = 0$, so $T(t)$ must be proportional to $\cos(n\pi at/L)$.

Thus the functions

$$u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \qquad n = 1, 2, \ldots, \tag{19}$$

satisfy the partial differential equation (1), the boundary conditions (3), and the second initial condition (9). These functions are the fundamental solutions for the given problem.

To satisfy the remaining (nonhomogeneous) initial condition (9) we will consider a superposition of the fundamental solutions (19) with properly chosen coefficients. Thus we assume that $u(x,t)$ has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \tag{20}$$

where the constants c_n remain to be chosen. The initial condition $u(x,0) = f(x)$ requires that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x). \tag{21}$$

Consequently, the coefficients c_n must be the coefficients in the Fourier sine series of period $2L$ for f; hence

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{22}$$

Thus the formal solution of the problem of Eqs. (1), (3), and (9) is given by Eq. (20) with the coefficients calculated from Eq. (22).

For a fixed value of n the expression $\sin(n\pi x/L)\cos(n\pi at/L)$ in Eq. (19) is periodic in time t with the period $2L/na$; it therefore represents a vibratory motion of the string having this period, or having the frequency $n\pi a/L$. The quantities $\lambda a = n\pi a/L$ for $n = 1, 2, \ldots$ are the **natural frequencies** of the string, that is, the frequencies at which the string will freely vibrate. The factor $\sin(n\pi x/L)$ represents the displacement pattern occurring in the string when it is executing vibrations of the given frequency. Each displacement pattern is called a **natural mode** of vibration and is periodic in the space variable x; the spatial period $2L/n$ is called the **wavelength** of the mode of frequency $n\pi a/L$. Thus the eigenvalues $n^2\pi^2/L^2$ of the problem (12), (14) are proportional to the squares of the natural frequencies, and the eigenfunctions $\sin(n\pi x/L)$ give the natural modes. The first three natural modes are sketched in Figure 10.7.3. The total motion of the string, given by the function $u(x,t)$ of Eq. (20), is thus a combination of the natural modes of vibration, and is also a periodic function of time with period $2L/a$.

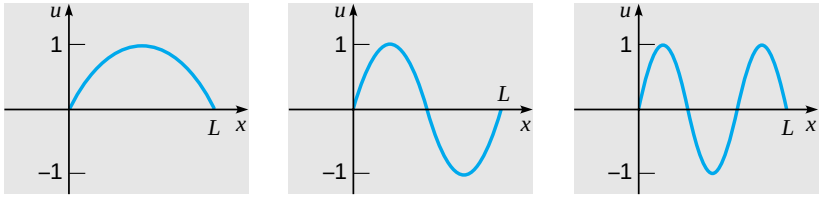


FIGURE 10.7.3 First three fundamental modes of vibration of an elastic string. (*a*) Frequency $= \pi a/L$, wavelength $= 2L$; (*b*) frequency $= 2\pi a/L$, wavelength $= L$; (*c*) frequency $= 3\pi a/L$, wavelength $= 2L/3$.

EXAMPLE 1

Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < 30, \quad t > 0 \tag{23}$$

Assume that the ends of the string are fixed, and that the string is set in motion with no initial velocity from the initial position

$$u(x, 0) = f(x) = \begin{cases} x/10, & 0 \le x \le 10, \\ (30 - x)/20, & 10 < x \le 30. \end{cases} \tag{24}$$

Find the displacement $u(x, t)$ of the string and describe its motion through one period.

The solution is given by Eq. (20) with $a = 2$ and $L = 30$, that is,

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{30} \cos \frac{2n\pi t}{30}, \tag{25}$$

where c_n is calculated from Eq. (22). Substituting from Eq. (24) into Eq. (22), we obtain

$$c_n = \frac{2}{30} \int_0^{10} \frac{x}{10} \sin \frac{n\pi x}{30}\, dx + \frac{2}{30} \int_{10}^{30} \frac{30 - x}{20} \sin \frac{n\pi x}{30}\, dx. \tag{26}$$

By evaluating the integrals in Eq. (26), we find that

$$c_n = \frac{9}{n^2\pi^2} \sin \frac{n\pi}{3}, \quad n = 1, 2, \ldots. \tag{27}$$

The solution (25), (27) gives the displacement of the string at any point x at any time t. The motion is periodic in time with period 30, so it is sufficient to analyze the solution for $0 \le t \le 30$.

The best way to visualize the solution is by a computer animation showing the dynamic behavior of the vibrating string. Here we indicate the motion of the string in Figures 10.7.4, 10.7.5, and 10.7.6. Plots of u versus x for $t = 0$, 4, 7.5, 11, and 15 are shown in Figure 10.7.4. Observe that the maximum initial displacement is positive and occurs at $x = 10$, while at $t = 15$, a half period later, the maximum displacement is negative and occurs at $x = 20$. The string then retraces its motion and returns to its original configuration at $t = 30$. Figure 10.7.5 shows the behavior of the points $x = 10$, 15, and 20 by plots of u versus t for these fixed values of x. The plots confirm that the motion is indeed periodic with period 30. Observe also that each interior point

on the string is motionless for one-third of each period. Figure 10.7.6 shows a three-dimensional plot of u versus both x and t, from which the overall nature of the solution is apparent. Of course, the curves in Figures 10.7.4 and 10.7.5 lie on the surface shown in Figure 10.7.6.

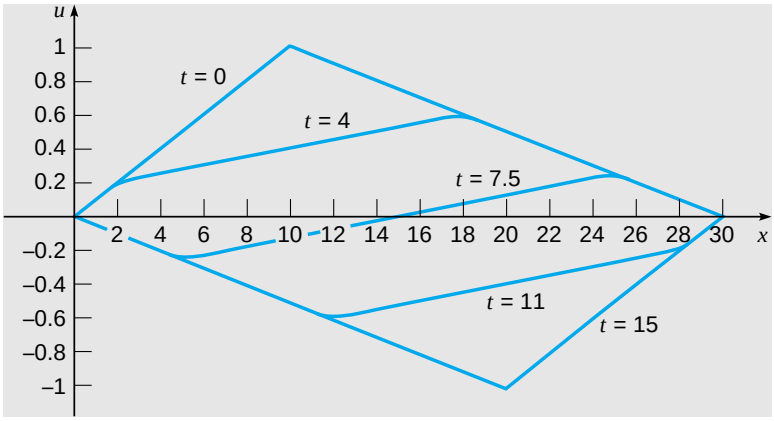


FIGURE 10.7.4 Plots of u versus x for fixed values of t for the string in Example 1.

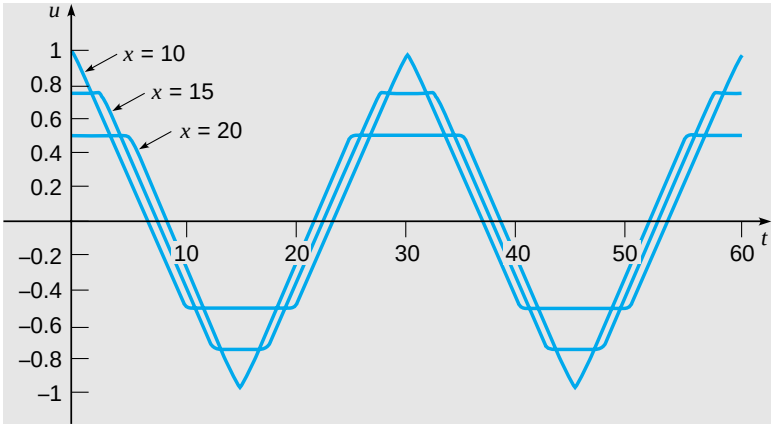


FIGURE 10.7.5 Plots of u versus t for fixed values of x for the string in Example 1.

Justification of the Solution. As in the heat conduction problem considered earlier, Eq. (20) with the coefficients c_n given by Eq. (22) is only a *formal* solution of Eqs. (1), (3), and (9). To ascertain whether Eq. (20) *actually* represents the solution of the given problem requires some further investigation. As in the heat conduction problem, it is tempting to try to show this directly by substituting Eq. (20) for $u(x, t)$ in Eqs. (1), (3), and (9). However, on formally computing u_{xx}, for example, we obtain

$$u_{xx}(x, t) = -\sum_{n=1}^{\infty} c_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} \cos\frac{n\pi at}{L};$$

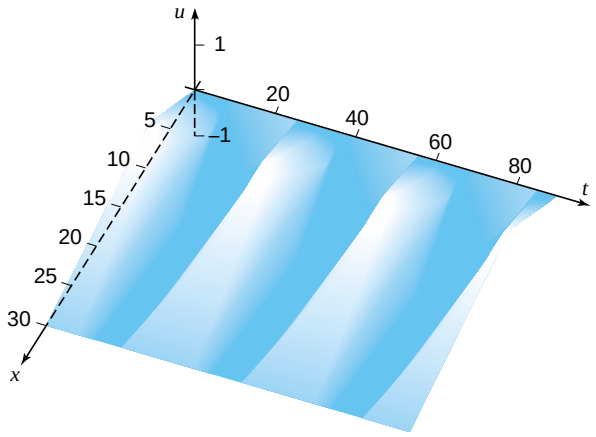


FIGURE 10.7.6 Plot of u versus x and t for the string in Example 1.

due to the presence of the n^2 factor in the numerator this series may not converge. This would not necessarily mean that the series (20) for $u(x, t)$ is incorrect, but only that the series (20) cannot be used to calculate u_{xx} and u_{tt}. A basic difference between solutions of the wave equation and those of the heat conduction equation is that the latter contain negative exponential terms that approach zero very rapidly with increasing n, which ensures the convergence of the series solution and its derivatives. In contrast, series solutions of the wave equation contain only oscillatory terms that do not decay with increasing n.

However, there is an alternative way to establish the validity of Eq. (20) indirectly. At the same time we will gain additional information about the structure of the solution. First we will show that Eq. (20) is equivalent to

$$u(x, t) = \tfrac{1}{2}[h(x - at) + h(x + at)], \tag{28}$$

where h is the function obtained by extending the initial data f into $(-L, 0)$ as an odd function, and to other values of x as a periodic function of period $2L$. That is,

$$h(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L < x < 0; \end{cases} \tag{29}$$

$$h(x + 2L) = h(x).$$

To establish Eq. (28) note that h has the Fourier series

$$h(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}. \tag{30}$$

Then, using the trigonometric identities for the sine of a sum or difference, we obtain

$$h(x - at) = \sum_{n=1}^{\infty} c_n \left(\sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} - \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

$$h(x + at) = \sum_{n=1}^{\infty} c_n \left(\sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} + \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

and Eq. (28) follows immediately upon adding the last two equations. From Eq. (28) we see that $u(x, t)$ is continuous for $0 < x < L$, $t > 0$ provided that h is continuous on the interval $(-\infty, \infty)$. This requires that f be continuous on the original interval $[0, L]$. Similarly, u is twice continuously differentiable with respect to either variable in $0 < x < L, t > 0$, provided that h is twice continuously differentiable on $(-\infty, \infty)$. This requires that f' and f'' be continuous on $[0, L]$. Furthermore, since h'' is the odd extension of f'', we must also have $f''(0) = f''(L) = 0$. However, since h' is the even extension of f', no further conditions are required on f'. Provided that these conditions are met, u_{xx} and u_{tt} can be computed from Eq. (28), and it is an elementary exercise to show that these derivatives satisfy the wave equation. Some of the details of the argument just indicated are given in Problems 20 and 21.

If some of the continuity requirements stated in the last paragraph are not met, then u is not differentiable at some points in the semi-infinite strip $0 < x < L, t > 0$, and thus is a solution of the wave equation only in a somewhat restricted sense. An important physical consequence of this observation is that if there are any discontinuities present in the initial data f, then they will be preserved in the solution $u(x, t)$ for all time. In contrast, in heat conduction problems initial discontinuities are instantly smoothed out (Section 10.6). Suppose that the initial displacement f has a jump discontinuity at $x = x_0$, $0 \le x_0 \le L$. Since h is a periodic extension of f, the same discontinuity is present in $h(\xi)$ at $\xi = x_0 + 2nL$ and at $\xi = -x_0 + 2nL$, where n is any integer. Thus $h(x - at)$ is discontinuous when $x - at = x_0 + 2nL$, or when $x - at = -x_0 + 2nL$. For a fixed x in $[0, L]$ the discontinuity that was originally at x_0 will reappear in $h(x - at)$ at the times $t = (x \pm x_0 - 2nL)/a$. Similarly, $h(x + at)$ is discontinuous at the point x at the times $t = (-x \pm x_0 + 2mL)/a$, where m is any integer. If we refer to Eq. (28), it then follows that the solution $u(x, t)$ is also discontinuous at the given point x at these times. Since the physical problem is posed for $t > 0$, only those values of m and n yielding positive values of t are of interest.

General Problem for the Elastic String. Let us modify the problem considered previously by supposing that the string is set in motion from its equilibrium position with a given velocity. Then the vertical displacement $u(x, t)$ must satisfy the wave equation (1),

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \qquad t > 0;$$

the boundary conditions (3),

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t \ge 0;$$

and the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = g(x), \qquad 0 \le x \le L, \tag{31}$$

where $g(x)$ is the initial velocity at the point x of the string.

The solution of this new problem can be obtained by following the procedure described above for the problem (1), (3), and (9). Upon separating variables, we find that the problem for $X(x)$ is exactly the same as before. Thus, once again $\lambda = n^2\pi^2/L^2$ and $X(x)$ is proportional to $\sin(n\pi x/L)$. The differential equation for $T(t)$ is again Eq. (17), but the associated initial condition is now

$$T(0) = 0. \tag{32}$$

corresponding to the first of the initial conditions (31). The general solution of Eq. (17) is given by Eq. (18), but now the initial condition (32) requires that $k_1 = 0$. Therefore $T(t)$ is now proportional to $\sin(n\pi at/L)$ and the fundamental solutions for the problem (1), (3), and (31) are

$$u_n(x,t) = \sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}, \quad n = 1, 2, 3, \ldots. \tag{33}$$

Each of the functions $u_n(x,t)$ satisfies the wave equation (1), the boundary conditions (3), and the first of the initial conditions (31). The main consequence of using the initial conditions (31) rather than (9) is that the time-dependent factor in $u_n(x,t)$ involves a sine rather than a cosine.

To satisfy the remaining (nonhomogeneous) initial condition we assume that $u(x,t)$ can be expressed as a linear combination of the fundamental solutions (33), that is,

$$u(x,t) = \sum_{n=1}^{\infty} k_n u_n(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}. \tag{34}$$

To determine the values of the coefficients k_n we differentiate Eq. (34) with respect to t, set $t = 0$, and use the second initial condition (31); this gives the equation

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} k_n \sin\frac{n\pi x}{L} = g(x). \tag{35}$$

Hence the quantities $(n\pi a/L)k_n$ are the coefficients in the Fourier sine series of period $2L$ for g; therefore

$$\frac{n\pi a}{L}k_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{36}$$

Thus Eq. (34), with the coefficients given by Eq. (36), constitutes a formal solution to the problem of Eqs. (1), (3), and (31). The validity of this formal solution can be established by arguments similar to those previously indicated for the solution of Eqs. (1), (3), and (9).

Finally, we turn to the problem consisting of the wave equation (1), the boundary conditions (3), and the general initial conditions (4), (5):

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \tag{37}$$

where $f(x)$ and $g(x)$ are the given initial position and velocity, respectively, of the string. Although this problem can be solved by separating variables, as in the cases discussed previously, it is important to note that it can also be solved simply by adding together the two solutions that we obtained above. To show that this is true, let $v(x,t)$ be the solution of the problem (1), (3), and (9), and let $w(x,t)$ be the solution of the problem (1), (3), and (31). Thus $v(x,t)$ is given by Eqs. (20) and (22), while $w(x,t)$ is given by Eqs. (34) and (36). Now let $u(x,t) = v(x,t) + w(x,t)$; what problem does $u(x,t)$ satisfy? First, observe that

$$a^2u_{xx} - u_{tt} = (a^2v_{xx} - v_{tt}) + (a^2w_{xx} - w_{tt}) = 0 + 0 = 0, \tag{38}$$

so $u(x,t)$ satisfies the wave equation (1). Next, we have

$$u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0, \quad u(L,t) = v(L,t) + w(L,t) = 0 + 0 = 0, \tag{39}$$

so $u(x,t)$ also satisfies the boundary conditions (3). Finally, we have

$$u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x) \tag{40}$$

and

$$u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x). \tag{41}$$

Thus $u(x,t)$ satisfies the general initial conditions (37).

We can restate the result we have just obtained in the following way. To solve the wave equation with the general initial conditions (37) you can solve instead the somewhat simpler problems with the initial conditions (9) and (31), respectively, and then add together the two solutions. This is another use of the principle of superposition.

PROBLEMS

Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$. In each of Problems 1 through 4 carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x,t)$ for the given initial position $f(x)$.
(b) Plot $u(x,t)$ versus x for $0 \le x \le 10$ and for several values of t between $t = 0$ and $t = 20$.
(c) Plot $u(x,t)$ versus t for $0 \le t \le 20$ and for several values of x.
(d) Construct an animation of the solution in time for at least one period.
(e) Describe the motion of the string in a few sentences.

▶ 1. $f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2, \\ 2(L-x)/L, & L/2 < x \le L \end{cases}$

▶ 2. $f(x) = \begin{cases} 4x/L, & 0 \le x \le L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L-x)/L, & 3L/4 \le x \le L \end{cases}$

▶ 3. $f(x) = 8x(L-x)^2/L^3$

▶ 4. $f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise} \end{cases}$

Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x,0) = g(x)$. In each of Problems 5 through 8 carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x,t)$ for the given $g(x)$.
(b) Plot $u(x,t)$ versus x for $0 \le x \le 10$ and for several values of t between $t = 0$ and $t = 20$.
(c) Plot $u(x,t)$ versus t for $0 \le t \le 20$ and for several values of x.
(d) Construct an animation of the solution in time for at least one period.
(e) Describe the motion of the string in a few sentences.

▶ 5. $g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2, \\ 2(L-x)/L, & L/2 < x \le L \end{cases}$

▶ 6. $g(x) = \begin{cases} 4x/L, & 0 \le x \le L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L-x)/L, & 3L/4 \le x \le L \end{cases}$

▶ 7. $g(x) = 8x(L-x)^2/L^3$

▶ 8. $g(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise} \end{cases}$

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x, t)$ in an elastic string of length L, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where f is a given function.
Hint: Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x, t) = \sin \lambda_n x \cos \lambda_n at,$$

where $\lambda_n = (2n - 1)\pi/2L$, $n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

▶ 10. Consider an elastic string of length L. The end $x = 0$ is held fixed while the end $x = L$ is free; thus the boundary conditions are $u(0, t) = 0$ and $u_x(L, t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the displacement $u(x, t)$.
(b) With $L = 10$ and $a = 1$ plot u versus x for $0 \le x \le 10$ and for several values of t. Pay particular attention to values of t between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.
(c) With $L = 10$ and $a = 1$ plot u versus t for several values of x.
(d) Construct an animation of the solution in time for at least one period.
(e) Describe the motion of the string in a few sentences.

▶ 11. Suppose that the string in Problem 10 is started instead from the initial position $f(x) = 8x(L - x)^2/L^3$. Follow the instructions in Problem 10 for this new problem.

12. Dimensionless variables can be introduced into the wave equation $a^2 u_{xx} = u_{tt}$ in the following manner. Let $s = x/L$ and show that the wave equation becomes

$$a^2 u_{ss} = L^2 u_{tt}.$$

Then show that L/a has the dimensions of time, and thus can be used as the unit on the time scale. Finally, let $\tau = at/L$ and show that the wave equation then reduces to

$$u_{ss} = u_{\tau\tau}.$$

Problems 13 and 14 indicate the form of the general solution of the wave equation and the physical significance of the constant a.

13. Show that the wave equation

$$a^2 u_{xx} = u_{tt}$$

can be reduced to the form $u_{\xi\eta} = 0$ by change of variables $\xi = x - at$, $\eta = x + at$. Show that $u(x, t)$ can be written as

$$u(x, t) = \phi(x - at) + \psi(x + at),$$

where ϕ and ψ are arbitrary functions.

14. Plot the value of $\phi(x - at)$ for $t = 0$, $1/a$, $2/a$, and t_0/a if $\phi(s) = \sin s$. Note that for any $t \neq 0$ the graph of $y = \phi(x - at)$ is the same as that of $y = \phi(x)$ when $t = 0$, but displaced a distance at in the positive x direction. Thus a represents the velocity at which a disturbance moves along the string. What is the interpretation of $\phi(x + at)$?

15. A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit length of 0.026 lb/ft.
(a) Find the velocity of propagation of transverse waves in the wire.

(b) Find the natural frequencies of vibration.
(c) If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?

16. A vibrating string moving in an elastic medium satisfies the equation

$$a^2u_{xx} - \alpha^2 u = u_{tt},$$

where α^2 is proportional to the coefficient of elasticity of the medium. Suppose that the string is fixed at the ends, and is released with no initial velocity from the initial position $u(x, 0) = f(x)$, $0 < x < L$. Find the displacement $u(x, t)$.

17. Consider the wave equation

$$a^2u_{xx} = u_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = 0, \qquad -\infty < x < \infty.$$

(a) Using the form of the solution obtained in Problem 13, show that ϕ and ψ must satisfy

$$\phi(x) + \psi(x) = f(x),$$
$$-\phi'(x) + \psi'(x) = 0.$$

(b) Solve the equations of part (a) for ϕ and ψ, and thereby show that

$$u(x, t) = \tfrac{1}{2}[f(x - at) + f(x + at)].$$

This form of the solution was obtained by D'Alembert in 1746.
Hint: Note that the equation $\psi'(x) = \phi'(x)$ is solved by choosing $\psi(x) = \phi(x) + c$.
(c) Let

$$f(x) = \begin{cases} 2, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that

$$f(x - at) = \begin{cases} 2, & -1 + at < x < 1 + at, \\ 0, & \text{otherwise.} \end{cases}$$

Also determine $f(x + at)$.
(d) Sketch the solution found in part (b) at $t = 0$, $t = 1/2a$, $t = 1/a$, and $t = 2/a$, obtaining the results shown in Figure 10.7.7. Observe that an initial displacement produces two waves moving in opposite directions away from the original location; each wave consists of one-half of the initial displacement.

18. Consider the wave equation

$$a^2u_{xx} = u_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = g(x), \qquad -\infty < x < \infty.$$

(a) Using the form of the solution obtained in Problem 13, show that

$$\phi(x) + \psi(x) = 0,$$
$$-a\phi'(x) + a\psi'(x) = g(x).$$

(b) Use the first equation of part (a) to show that $\psi'(x) = -\phi'(x)$. Then use the second equation to show that $-2a\phi'(x) = g(x)$, and therefore that

$$\phi(x) = -\frac{1}{2a}\int_{x_0}^{x} g(\xi)\,d\xi + \phi(x_0),$$

where x_0 is arbitrary. Finally, determine $\psi(x)$.

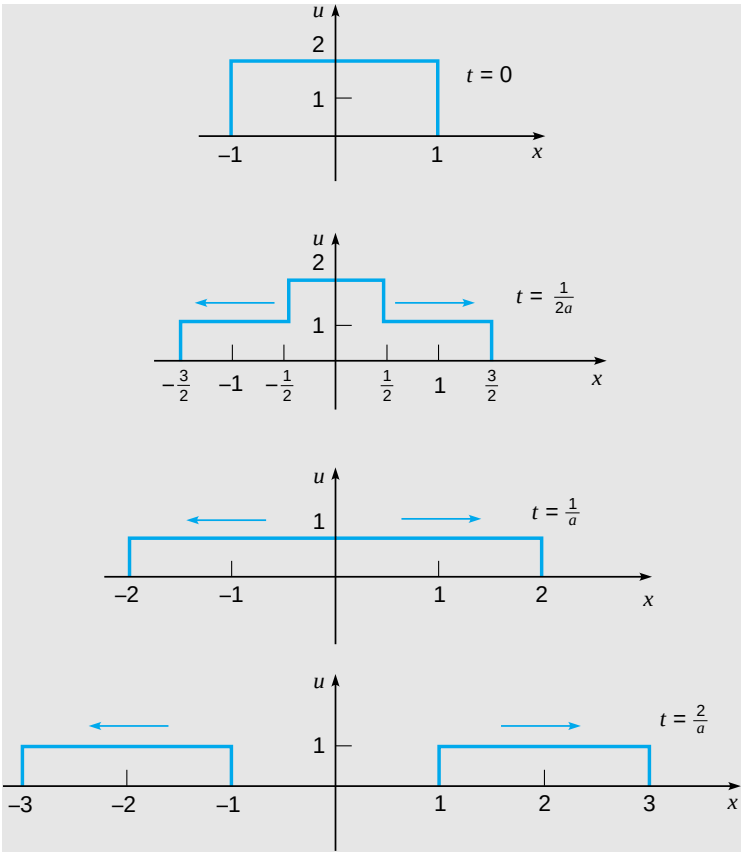


FIGURE 10.7.7 Propagation of initial disturbance in an infinite one-dimensional medium.

(c) Show that

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

19. By combining the results of Problems 17 and 18 show that the solution of the problem

$$a^2u_{xx} = u_{tt},$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad -\infty < x < \infty$$

is given by

$$u(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

Problems 20 and 21 indicate how the formal solution (20), (22) of Eqs. (1), (3), and (9) can be shown to constitute the actual solution of that problem.

20. By using the trigonometric identity $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ show that the solution (20) of the problem of Eqs. (1), (3), and (9) can be written in the form (28).
21. Let $h(\xi)$ represent the initial displacement in $[0, L]$, extended into $(-L, 0)$ as an odd function and extended elsewhere as a periodic function of period $2L$. Assuming that h, h', and h'' are all continuous, show by direct differentiation that $u(x,t)$ as given in

Eq. (28) satisfies the wave equation (1) and also the initial conditions (9). Note also that since Eq. (20) clearly satisfies the boundary conditions (3), the same is true of Eq. (28). Comparing Eq. (28) with the solution of the corresponding problem for the infinite string (Problem 17), we see that they have the same form provided that the initial data for the finite string, defined originally only on the interval $0 \le x \le L$, are extended in the given manner over the entire x-axis. If this is done, the solution for the infinite string is also applicable to the finite one.

22. The motion of a circular elastic membrane, such as a drumhead, is governed by the two-dimensional wave equation in polar coordinates

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = a^{-2}u_{tt}.$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

23. The total energy $E(t)$ of the vibrating string is given as a function of time by

$$E(t) = \int_0^L \left[\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2}T u_x^2(x,t)\right] dx; \tag{i}$$

the first term is the kinetic energy due to the motion of the string, and the second term is the potential energy created by the displacement of the string away from its equilibrium position.

For the displacement $u(x,t)$ given by Eq. (20), that is, for the solution of the string problem with zero initial velocity, show that

$$E(t) = \frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 c_n^2. \tag{ii}$$

Note that the right side of Eq. (ii) does not depend on t. Thus the total energy E is a constant, and therefore is *conserved* during the motion of the string.

Hint: Use Parseval's equation (Problem 37 of Section 10.4 and Problem 17 of Section 10.3), and recall that $a^2 = T/\rho$.

10.8 Laplace's Equation

One of the most important of all partial differential equations occurring in applied mathematics is that associated with the name of Laplace[8]: in two dimensions

$$u_{xx} + u_{yy} = 0, \tag{1}$$

and in three dimensions

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{2}$$

For example, in a two-dimensional heat conduction problem, the temperature $u(x, y, t)$ must satisfy the differential equation

$$\alpha^2(u_{xx} + u_{yy}) = u_t, \tag{3}$$

[8]Laplace's equation is named for Pierre-Simon de Laplace, who, beginning in 1782, studied its solutions extensively while investigating the gravitational attraction of arbitrary bodies in space. However, the equation first appeared in 1752 in a paper by Euler on hydrodynamics.

where α^2 is the thermal diffusivity. If a steady-state exists, u is a function of x and y only, and the time derivative vanishes; in this case Eq. (3) reduces to Eq. (1). Similarly, for the steady-state heat conduction problem in three dimensions, the temperature must satisfy the three-dimensional form of Laplace's equation. Equations (1) and (2) also occur in other branches of mathematical physics. In the consideration of electrostatic fields the electric potential function in a dielectric medium containing no electric charges must satisfy either Eq. (1) or Eq. (2), depending on the number of space dimensions involved. Similarly, the potential function of a particle in free space acted on only by gravitational forces satisfies the same equations. Consequently, Laplace's equation is often referred to as the **potential equation**. Another example arises in the study of the steady (time-independent), two-dimensional, inviscid, irrotational motion of an incompressible fluid, which centers about two functions, known as the velocity potential function and the stream function, both of which satisfy Eq. (1). In elasticity the displacements that occur when a perfectly elastic bar is twisted are described in terms of the so-called warping function, which also satisfies Eq. (1).

Since there is no time dependence in any of the problems mentioned previously, there are no initial conditions to be satisfied by the solutions of Eq. (1) or (2). They must, however, satisfy certain boundary conditions on the bounding curve or surface of the region in which the differential equation is to be solved. Since Laplace's equation is of second order, it might be plausible to expect that two boundary conditions would be required to determine the solution completely. This, however, is not the case. Recall that in the heat conduction problem for the finite bar (Sections 10.5 and 10.6) it was necessary to prescribe one condition at each end of the bar, that is, *one condition at each point of the boundary*. If we generalize this observation to multidimensional problems, it is then natural to prescribe one condition on the function u at each point on the boundary of the region in which a solution of Eq. (1) or (2) is sought. The most common boundary condition occurs when the value of u is specified at each boundary point; in terms of the heat conduction problem this corresponds to prescribing the temperature on the boundary. In some problems the value of the derivative, or rate of change, of u in the direction normal to the boundary is specified instead; the condition on the boundary of a thermally insulated body, for example, is of this type. It is entirely possible for more complicated boundary conditions to occur; for example, u might be prescribed on part of the boundary, and its normal derivative specified on the remainder. The problem of finding a solution of Laplace's equation that takes on given boundary values is known as a **Dirichlet problem**, in honor of P. G. L. Dirichlet.[9] In contrast, if the values of the normal derivative are prescribed on the boundary, the problem is said to be a **Neumann problem**, in honor of K. G. Neumann.[10] The Dirichlet and Neumann problems are also known as the first and second boundary value problems of potential theory, respectively.

Physically, it is plausible to expect that the types of boundary conditions just mentioned will be sufficient to determine the solution completely. Indeed, it is possible to establish the existence and uniqueness of the solution of Laplace's equation under the

[9]Peter Gustav Dirichlet (1805–1859) was a professor at Berlin and, after the death of Gauss, at Göttingen. In 1829 he gave the first set of conditions sufficient to guarantee the convergence of a Fourier series. The definition of function usually used today in elementary calculus is essentially the one given by Dirichlet in 1837. While he is best known for his work in analysis and differential equations, Dirichlet was also one of the leading number theorists of the nineteenth century.

[10]Karl Gottfried Neumann (1832–1925), professor at Leipzig, made contributions to differential equations, integral equations, and complex variables.

boundary conditions mentioned, provided the shape of the boundary and the functions appearing in the boundary conditions satisfy certain very mild requirements. However, the proofs of these theorems, and even their accurate statement, are beyond the scope of the present book. Our only concern will be solving some typical problems by means of separation of variables and Fourier series.

While the problems chosen as examples are capable of interesting physical interpretations (in terms of electrostatic potentials or steady-state temperature distributions, for instance), our purpose here is primarily to point out some of the features that may occur during their mathematical solution. It is also worth noting again that more complicated problems can sometimes be solved by expressing the solution as the sum of solutions of several simpler problems (see Problems 3 and 4).

Dirichlet Problem for a Rectangle. Consider the mathematical problem of finding the function u satisfying Laplace's equation (1),

$$u_{xx} + u_{yy} = 0,$$

in the rectangle $0 < x < a$, $0 < y < b$, and also satisfying the boundary conditions

$$\begin{aligned} &u(x, 0) = 0, \qquad u(x, b) = 0, \qquad 0 < x < a, \\ &u(0, y) = 0, \qquad u(a, y) = f(y), \qquad 0 \le y \le b, \end{aligned} \tag{4}$$

where f is a given function on $0 \le y \le b$ (see Figure 10.8.1).

To solve this problem we wish to construct a fundamental set of solutions satisfying the partial differential equation and the homogeneous boundary conditions; then we will superpose these solutions so as to satisfy the remaining boundary condition. Let us assume that

$$u(x, y) = X(x)Y(y) \tag{5}$$

and substitute for u in Eq. (1). This yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

where λ is the separation constant. Thus we obtain the two ordinary differential equations

$$X'' - \lambda X = 0, \tag{6}$$

$$Y'' + \lambda Y = 0. \tag{7}$$

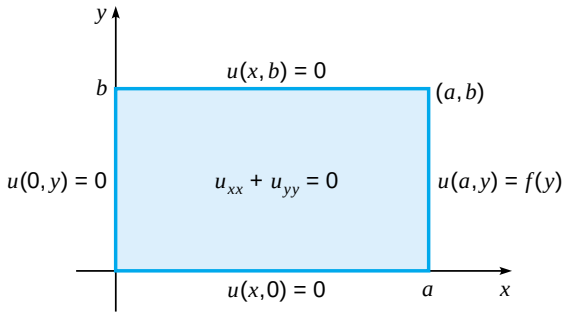


FIGURE 10.8.1 Dirichlet problem for a rectangle.

If we now substitute for u from Eq. (5) in each of the homogeneous boundary conditions, we find that

$$X(0) = 0 \tag{8}$$

and

$$Y(0) = 0, \qquad Y(b) = 0. \tag{9}$$

We will first determine the solution of the differential equation (7) subject to the boundary conditions (9). However, this problem is essentially identical to one encountered previously in Sections 10.1, 10.5, and 10.7. We conclude that there are nontrivial solutions if and only if λ is an eigenvalue, namely,

$$\lambda = (n\pi/b)^2, \qquad n = 1, 2, \ldots; \tag{10}$$

and $Y(y)$ is proportional to the corresponding eigenfunction $\sin(n\pi y/b)$. Next, we substitute from Eq. (10) for λ in Eq. (6), and solve this equation subject to the boundary condition (8). It is convenient to write the general solution of Eq. (6) as

$$X(x) = k_1 \cosh(n\pi x/b) + k_2 \sinh(n\pi x/b), \tag{11}$$

and the boundary condition (8) then requires that $k_1 = 0$. Therefore $X(x)$ must be proportional to $\sinh(n\pi x/b)$. Thus we obtain the fundamental solutions

$$u_n(x, y) = \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \qquad n = 1, 2, \ldots. \tag{12}$$

These functions satisfy the differential equation (1) and all the homogeneous boundary conditions for each value of n.

To satisfy the remaining nonhomogeneous boundary condition at $x = a$ we assume, as usual, that we can represent the solution $u(x, y)$ in the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n u_n(x, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}. \tag{13}$$

The coefficients c_n are determined by the boundary condition

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = f(y). \tag{14}$$

Therefore the quantities $c_n \sinh(n\pi a/b)$ must be the coefficients in the Fourier sine series of period $2b$ for f and are given by

$$c_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy. \tag{15}$$

Thus the solution of the partial differential equation (1) satisfying the boundary conditions (4) is given by Eq. (13) with the coefficients c_n computed from Eq. (15).

From Eqs. (13) and (15) we see that the solution contains the factor $\sinh(n\pi x/b)/\sinh(n\pi a/b)$. To estimate this quantity for large n we can use the approximation $\sinh \xi \cong e^{\xi}/2$, and thereby obtain

$$\frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} \cong \frac{\frac{1}{2}\exp(n\pi x/b)}{\frac{1}{2}\exp(n\pi a/b)} = \exp[-n\pi(a - x)/b].$$

Thus this factor has the character of a negative exponential; consequently, the series (13) converges quite rapidly unless $a - x$ is very small.

EXAMPLE 1

To illustrate these results, let $a = 3$, $b = 2$, and

$$f(y) = \begin{cases} y, & 0 \le y \le 1, \\ 2 - y, & 1 \le y \le 2. \end{cases} \tag{16}$$

By evaluating c_n from Eq. (15) we find that

$$c_n = \frac{8 \sin(n\pi/2)}{n^2 \pi^2 \sinh(3n\pi/2)}. \tag{17}$$

Then $u(x, y)$ is given by Eq. (13). Keeping 20 terms in the series we can plot u versus x and y, as shown in Figure 10.8.2. Alternatively, one can construct a contour plot showing level curves of $u(x, y)$; Figure 10.8.3 is such a plot, with an increment of 0.1 between adjacent curves.

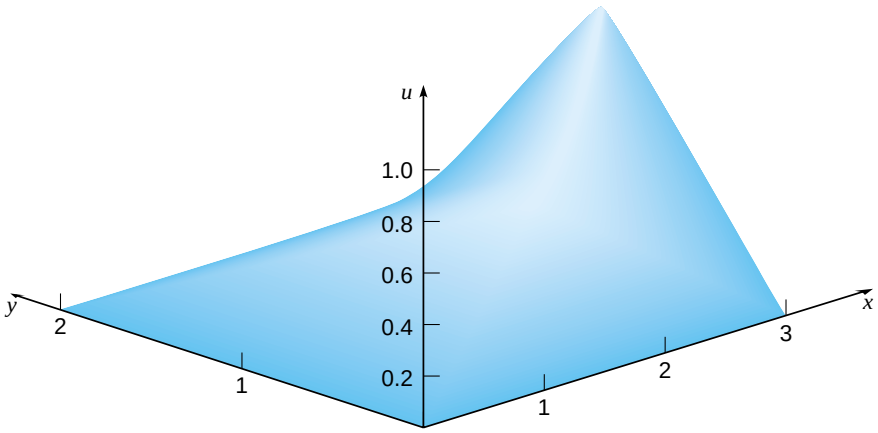


FIGURE 10.8.2 Plot of u versus x and y for Example 1.

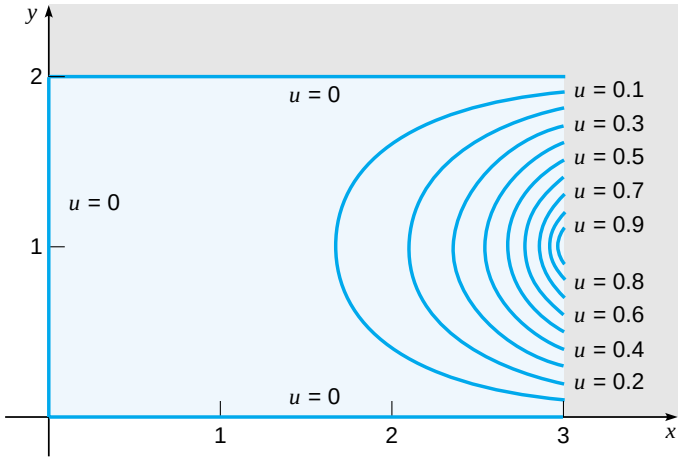


FIGURE 10.8.3 Level curves of $u(x, y)$ for Example 1.

Dirichlet Problem for a Circle. Consider the problem of solving Laplace's equation in a circular region $r < a$ subject to the boundary condition

$$u(a, \theta) = f(\theta), \tag{18}$$

where f is a given function on $0 \le \theta < 2\pi$ (see Figure 10.8.4). In polar coordinates Laplace's equation takes the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{19}$$

To complete the statement of the problem we note that for $u(r, \theta)$ to be single-valued, it is necessary that u be periodic in θ with period 2π. Moreover, we state explicitly that $u(r, \theta)$ must be bounded for $r \le a$, since this will become important later.

To apply the method of separation of variables to this problem we assume that

$$u(r, \theta) = R(r)\Theta(\theta), \tag{20}$$

and substitute for u in the differential equation (19). This yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda, \tag{21}$$

where λ is the separation constant. Thus we obtain the two ordinary differential equations

$$r^2R'' + rR' - \lambda R = 0, \tag{22}$$

$$\Theta'' + \lambda\Theta = 0. \tag{23}$$

In this problem there are no homogeneous boundary conditions; recall, however, that solutions must be bounded and also periodic in θ with period 2π. It is possible to show (Problem 9) that the periodicity condition requires that λ must be real. We will consider in turn the cases in which λ is negative, zero, and positive.

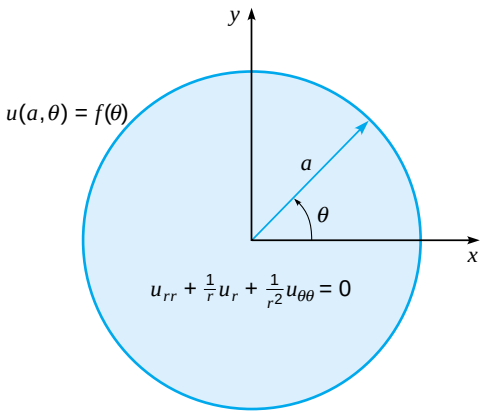


FIGURE 10.8.4 Dirichlet problem for a circle.

If $\lambda < 0$, let $\lambda = -\mu^2$, where $\mu > 0$. Then Eq. (23) becomes $\Theta'' - \mu^2\Theta = 0$, and consequently

$$\Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}. \tag{24}$$

Thus $\Theta(\theta)$ can be periodic only if $c_1 = c_2 = 0$, and we conclude that λ cannot be negative.

If $\lambda = 0$, then Eq. (23) becomes $\Theta'' = 0$, and thus

$$\Theta(\theta) = c_1 + c_2\theta. \tag{25}$$

For $\Theta(\theta)$ to be periodic we must have $c_2 = 0$, so that $\Theta(\theta)$ is a constant. Further, for $\lambda = 0$, Eq. (22) becomes

$$r^2R'' + rR' = 0. \tag{26}$$

This equation is of the Euler type, and has the solution

$$R(r) = k_1 + k_2 \ln r. \tag{27}$$

The logarithmic term cannot be accepted if $u(r, \theta)$ is to remain bounded as $r \to 0$; hence $k_2 = 0$. Thus, corresponding to $\lambda = 0$, we conclude that $u(r, \theta)$ must be a constant, that is, proportional to the solution

$$u_0(r, \theta) = 1. \tag{28}$$

Finally, if $\lambda > 0$, we let $\lambda = \mu^2$ where $\mu > 0$. Then Eqs. (22) and (23) become

$$r^2R'' + rR' - \mu^2R = 0 \tag{29}$$

and

$$\Theta'' + \mu^2\Theta = 0, \tag{30}$$

respectively. Equation (29) is an Euler equation and has the solution

$$R(r) = k_1 r^{\mu} + k_2 r^{-\mu}, \tag{31}$$

while Eq. (30) has the solution

$$\Theta(\theta) = c_1 \sin\mu\theta + c_2 \cos\mu\theta. \tag{32}$$

In order that Θ be periodic with period 2π it is necessary that μ be a positive integer n. With $\mu = n$ it follows that the solution $r^{-\mu}$ in Eq. (31) must be discarded since it becomes unbounded as $r \to 0$. Consequently, $k_2 = 0$ and the appropriate solutions of Eq. (19) are

$$u_n(r, \theta) = r^n \cos n\theta, \qquad v_n(r, \theta) = r^n \sin n\theta, \qquad n = 1, 2, \ldots. \tag{33}$$

These functions, together with $u_0(r, \theta) = 1$, form a set of fundamental solutions for the present problem.

In the usual way we now assume that u can be expressed as a linear combination of the fundamental solutions; that is,

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + k_n \sin n\theta). \tag{34}$$

The boundary condition (18) then requires that

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + k_n \sin n\theta) = f(\theta) \tag{35}$$

for $0 \le \theta < 2\pi$. The function f may be extended outside this interval so that it is periodic with period 2π, and therefore has a Fourier series of the form (35). Since the extended function has period 2π, we may compute its Fourier coefficients by integrating over any period of the function. In particular, it is convenient to use the original interval $(0, 2\pi)$; then

$$a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad n = 0, 1, 2, \ldots; \tag{36}$$

$$a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta, \qquad n = 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients, Eq. (34) represents the solution of the boundary value problem of Eqs. (18) and (19). Note that in this problem we needed both sine and cosine terms in the solution. This is because the boundary data were given on $0 \le \theta < 2\pi$ and have period 2π. As a consequence, the full Fourier series is required, rather than sine or cosine terms alone.

PROBLEMS

1. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= 0, \quad & u(a, y) &= 0, \quad & 0 < y < b,\\ u(x, 0) &= 0, \quad & u(x, b) &= g(x), \quad & 0 \le x \le a. \end{aligned}$$

(b) Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2,\\ a - x, & a/2 \le x \le a. \end{cases}$$

(c) For $a = 3$ and $b = 1$ plot u versus x for several values of y and also plot u versus y for several values of x.
(d) Plot u versus both x and y in three dimensions. Also draw a contour plot showing several level curves of $u(x, y)$ in the xy-plane.

2. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= 0, \quad & u(a, y) &= 0, \quad & 0 < y < b,\\ u(x, 0) &= h(x), \quad & u(x, b) &= 0, \quad & 0 \le x \le a. \end{aligned}$$

3. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= 0, \quad & u(a, y) &= f(y), \quad & 0 < y < b,\\ u(x, 0) &= h(x), \quad & u(x, b) &= 0, \quad & 0 \le x \le a. \end{aligned}$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a, y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x, 0) = h(x)$.
(b) Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - (y/b)$.
(c) Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x, u versus y, u versus both x and y, and a contour plot.

4. Show how to find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= k(y), \quad u(a, y) = f(y), \quad 0 < y < b, \\ u(x, 0) &= h(x), \quad u(x, b) = g(x), \quad 0 \le x \le a. \end{aligned}$$

Hint: See Problem 3.

5. Find the solution $u(r, \theta)$ of Laplace's equation

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = 0$$

outside the circle $r = a$, also satisfying the boundary condition

$$u(a, \theta) = f(\theta), \qquad 0 \le \theta < 2\pi,$$

on the circle. Assume that $u(r, \theta)$ is single-valued and bounded for $r > a$.

▶ 6. (a) Find the solution $u(r, \theta)$ of Laplace's equation in the semicircular region $r < a$, $0 < \theta < \pi$, also satisfying the boundary conditions

$$\begin{aligned} u(r, 0) = 0, \qquad u(r, \pi) = 0, \qquad 0 \le r < a, \\ u(a, \theta) = f(\theta), \qquad 0 \le \theta \le \pi. \end{aligned}$$

Assume that u is single-valued and bounded in the given region.
(b) Find the solution if $f(\theta) = \theta(\pi - \theta)$.
(c) Let $a = 2$ and plot the solution in several ways: u versus r, u versus θ, u versus both r and θ, and a contour plot.

7. Find the solution $u(r, \theta)$ of Laplace's equation in the circular sector $0 < r < a, 0 < \theta < \alpha$, also satisfying the boundary conditions

$$\begin{aligned} u(r, 0) = 0, \qquad u(r, \alpha) = 0, \qquad 0 \le r < a, \\ u(a, \theta) = f(\theta), \qquad 0 \le \theta \le \alpha. \end{aligned}$$

Assume that u is single-valued and bounded in the sector.

▶ 8. (a) Find the solution $u(x, y)$ of Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) = 0, \qquad u(a, y) = 0, \qquad y > 0, \\ u(x, 0) = f(x), \qquad 0 \le x \le a \end{aligned}$$

and the additional condition that $u(x, y) \to 0$ as $y \to \infty$.
(b) Find the solution if $f(x) = x(a - x)$.
(c) Let $a = 5$. Find the smallest value of y_0 for which $u(x, y) \le 0.1$ for all $y \ge y_0$.

9. Show that Eq. (23) has periodic solutions only if λ is real.
Hint: Let $\lambda = -\mu^2$ where $\mu = \nu + i\sigma$ with ν and σ real.

10. Consider the problem of finding a solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u_x(0, y) = 0, \quad u_x(a, y) = f(y), \quad 0 < y < b, \\ u_y(x, 0) = 0, \quad u_y(x, b) = 0, \qquad 0 \le x \le a. \end{aligned}$$

This is an example of a Neumann problem.
(a) Show that Laplace's equation and the homogeneous boundary conditions determine the fundamental set of solutions

$$\begin{aligned} u_0(x, y) &= c_0, \\ u_n(x, y) &= c_n \cosh(n\pi x/b) \cos(n\pi y/b), \qquad n = 1, 2, 3, \ldots. \end{aligned}$$

(b) By superposing the fundamental solutions of part (a), formally determine a function u also satisfying the nonhomogeneous boundary condition $u_x(a, y) = f(y)$. Note that when $u_x(a, y)$ is calculated, the constant term in $u(x, y)$ is eliminated, and there is no condition

from which to determine c_0. Furthermore, it must be possible to express f by means of a Fourier cosine series of period $2b$, which does not have a constant term. This means that

$$\int_0^b f(y)\,dy = 0$$

is a necessary condition for the given problem to be solvable. Finally, note that c_0 remains arbitrary, and hence the solution is determined only up to this additive constant. This is a property of all Neumann problems.

11. Find a solution $u(r, \theta)$ of Laplace's equation inside the circle $r = a$, also satisfying the boundary condition on the circle

$$u_r(a, \theta) = g(\theta), \qquad 0 \le \theta < 2\pi.$$

Note that this is a Neumann problem, and that its solution is determined only up to an arbitrary additive constant. State a necessary condition on $g(\theta)$ for this problem to be solvable by the method of separation of variables (see Problem 10).

▶ 12. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= 0, \quad u(a, y) = 0, \qquad 0 < y < b, \\ u_y(x, 0) &= 0, \quad u(x, b) = g(x), \quad 0 \le x \le a. \end{aligned}$$

Note that this is neither a Dirichlet nor a Neumann problem, but a mixed problem in which u is prescribed on part of the boundary and its normal derivative on the rest.

(b) Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

(c) Let $a = 3$ and $b = 1$. By drawing suitable plots compare this solution with the solution of Problem 1.

▶ 13. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= 0, \quad u(a, y) = f(y), \quad 0 < y < b, \\ u(x, 0) &= 0, \quad u_y(x, b) = 0, \qquad 0 \le x \le a. \end{aligned}$$

Hint: Eventually it will be necessary to expand $f(y)$ in a series making use of the functions $\sin(\pi y/2b)$, $\sin(3\pi y/2b)$, $\sin(5\pi y/2b), \ldots$ (see Problem 39 of Section 10.4).

(b) Find the solution if $f(y) = y(2b - y)$.

(c) Let $a = 3$ and $b = 2$; plot the solution in several ways.

▶ 14. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, also satisfying the boundary conditions

$$\begin{aligned} u_x(0, y) &= 0, \quad u_x(a, y) = 0, \qquad 0 < y < b, \\ u(x, 0) &= 0, \quad u(x, b) = g(x), \quad 0 \le x \le a. \end{aligned}$$

(b) Find the solution if $g(x) = 1 + x^2(x - a)^2$.

(c) Let $a = 3$ and $b = 2$; plot the solution in several ways.

15. By writing Laplace's equation in cylindrical coordinates r, θ, and z and then assuming that the solution is axially symmetric (no dependence on θ), we obtain the equation

$$u_{rr} + (1/r)u_r + u_{zz} = 0.$$

Assuming that $u(r, z) = R(r)Z(z)$, show that R and Z satisfy the equations

$$rR'' + R' + \lambda^2 rR = 0, \qquad Z'' - \lambda^2 Z = 0.$$

The equation for R is Bessel's equation of order zero with independent variable λr.

APPENDIX
A

Derivation of the Heat Conduction Equation. In this section we derive the differential equation that, to a first approximation at least, governs the conduction of heat in solids. It is important to understand that the mathematical analysis of a physical situation or process such as this ultimately rests on a foundation of empirical knowledge of the phenomenon involved. The mathematician must have a place to start, so to speak, and this place is furnished by experience. Consider a uniform rod insulated on the lateral surfaces so that heat can flow only in the axial direction. It has been demonstrated many times that if two parallel cross sections of the same area A and different temperatures T_1 and T_2, respectively, are separated by a small distance d, an amount of heat per unit time will pass from the warmer section to the cooler one. Moreover, this amount of heat is proportional to the area A, the temperature difference $|T_2 - T_1|$, and inversely proportional to the separation distance d. Thus

$$\text{Amount of heat per unit time} = \kappa A|T_2 - T_1|/d, \tag{1}$$

where the positive proportionality factor κ is called the thermal conductivity and depends primarily on the material[11] of the rod. The relation (1) is often called Fourier's law of heat conduction. We repeat that Eq. (1) is an empirical, not a theoretical, result and that it can be, and has often been, verified by careful experiment. It is the basis of the mathematical theory of heat conduction.

Now consider a straight rod of uniform cross section and homogeneous material, oriented so that the x-axis lies along the axis of the rod (see Figure 10.A.1). Let $x = 0$ and $x = L$ designate the ends of the bar.

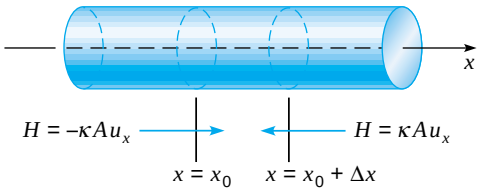


FIGURE 10.A.1 Conduction of heat in an element of a rod.

We will assume that the sides of the bar are perfectly insulated so that there is no passage of heat through them. We will also assume that the temperature u depends only on the axial position x and the time t, and not on the lateral coordinates y and z. In other words, we assume that the temperature remains constant on any cross section of the bar. This assumption is usually satisfactory when the lateral dimensions of the rod are small compared to its length.

The differential equation governing the temperature in the bar is an expression of a fundamental physical balance; the rate at which heat flows into any portion of the bar is equal to the rate at which heat is absorbed in that portion of the bar. The terms in the equation are called the flux (flow) term and the absorption term, respectively.

We will first calculate the flux term. Consider an element of the bar lying between the cross sections $x = x_0$ and $x = x_0 + \Delta x$, where x_0 is arbitrary and Δx is small. The

[11] Actually, κ also depends on the temperature, but if the temperature range is not too great, it is satisfactory to assume that κ is independent of temperature.

instantaneous rate of heat transfer $H(x_0, t)$ from left to right across the cross section $x = x_0$ is given by

$$\begin{aligned} H(x_0) &= -\lim_{d\to 0} \kappa A \frac{u(x_0 + d/2, t) - u(x_0 - d/2, t)}{d} \\ &= -\kappa A u_x(x_0, t). \end{aligned} \tag{2}$$

The minus sign appears in this equation since there will be a positive flow of heat from left to right only if the temperature is greater to the left of $x = x_0$ than to the right; in this case $u_x(x_0, t)$ is negative. In a similar manner, the rate at which heat passes from left to right through the cross section $x = x_0 + \Delta x$ is given by

$$H(x_0 + \Delta x, t) = -\kappa A u_x(x_0 + \Delta x, t). \tag{3}$$

The net rate at which heat flows into the segment of the bar between $x = x_0$ and $x = x_0 + \Delta x$ is thus given by

$$Q = H(x_0, t) - H(x_0 + \Delta x, t) = \kappa A[u_x(x_0 + \Delta x, t) - u_x(x_0, t)], \tag{4}$$

and the amount of heat entering this bar element in time Δt is

$$Q\,\Delta t = \kappa A[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]\,\Delta t. \tag{5}$$

Let us now calculate the absorption term. The average change in temperature Δu, in the time interval Δt, is proportional to the amount of heat $Q\,\Delta t$ introduced and inversely proportional to the mass Δm of the element. Thus

$$\Delta u = \frac{1}{s}\frac{Q\,\Delta t}{\Delta m} = \frac{Q\,\Delta t}{s\rho A\,\Delta x}, \tag{6}$$

where the constant of proportionality s is known as the specific heat of the material of the bar, and ρ is its density.[12] The average temperature change Δu in the bar element under consideration is the actual temperature change at some intermediate point $x = x_0 + \theta\,\Delta x$, where $0 < \theta < 1$. Thus Eq. (6) can be written as

$$u(x_0 + \theta\,\Delta x, t + \Delta t) - u(x_0 + \theta\,\Delta x, t) = \frac{Q\,\Delta t}{s\rho A\,\Delta x}, \tag{7}$$

or as

$$Q\,\Delta t = [u(x_0 + \theta\,\Delta x, t + \Delta t) - u(x_0 + \theta\,\Delta x, t)]s\rho A\,\Delta x. \tag{8}$$

To balance the flux and absorption terms, we equate the two expressions for $Q\,\Delta t$:

$$\begin{aligned} &\kappa A[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]\,\Delta t \\ &\qquad = s\rho A[u(x_0 + \theta\,\Delta x, t + \Delta t) - u(x_0 + \theta\,\Delta x, t)]\,\Delta x. \end{aligned} \tag{9}$$

On dividing Eq. (9) by $\Delta x\,\Delta t$ and then letting $\Delta x \to 0$ and $\Delta t \to 0$, we obtain the *heat conduction* or *diffusion* equation

$$\alpha^2 u_{xx} = u_t. \tag{10}$$

The quantity α^2 defined by

$$\alpha^2 = \kappa/\rho s \tag{11}$$

[12]The dependence of the density and specific heat on temperature is relatively small and will be neglected. Thus both ρ and s will be considered as constants.

is called the **thermal diffusivity**, and is a parameter depending only on the material of the bar. The units of α^2 are (length)^2/time. Typical values of α^2 are given in Table 10.5.1.

Several relatively simple conditions may be imposed at the ends of the bar. For example, the temperature at an end may be maintained at some constant value T. This might be accomplished by placing the end of the bar in thermal contact with some reservoir of sufficient size so that any heat that may flow between the bar and reservoir does not appreciably alter the temperature of the reservoir. At an end where this is done the boundary condition is

$$u = T. \tag{12}$$

Another simple boundary condition occurs if the end is insulated so that no heat passes through it. Recalling the expression (2) for the amount of heat crossing any cross section of the bar, we conclude that the condition of insulation is that this quantity vanish. Thus

$$u_x = 0 \tag{13}$$

is the boundary condition at an insulated end.

A more general type of boundary condition occurs if the rate of flow of heat through an end of the bar is proportional to the temperature there. Let us consider the end $x = 0$, where the rate of flow of heat from left to right is given by $-\kappa A u_x(0, t)$; see Eq. (2). Hence the rate of heat flow out of the bar (from right to left) at $x = 0$ is $\kappa A u_x(0, t)$. If this quantity is proportional to the temperature $u(0, t)$, then we obtain the boundary condition

$$u_x(0, t) - h_1 u(0, t) = 0, \qquad t > 0, \tag{14}$$

where h_1 is a nonnegative constant of proportionality. Note that $h_1 = 0$ corresponds to an insulated end, while $h_1 \to \infty$ corresponds to an end held at zero temperature.

If heat flow is taking place at the right end of the bar ($x = L$), then in a similar way we obtain the boundary condition

$$u_x(L, t) + h_2 u(L, t) = 0, \qquad t > 0, \tag{15}$$

where again h_2 is a nonnegative constant of proportionality.

Finally, to determine completely the flow of heat in the bar it is necessary to state the temperature distribution at one fixed instant, usually taken as the initial time $t = 0$. This initial condition is of the form

$$u(x, 0) = f(x), \qquad 0 \le x \le L. \tag{16}$$

The problem then is to determine the solution of the differential equation (10) subject to one of the boundary conditions (12) to (15) at each end, and to the initial condition (16) at $t = 0$.

Several generalizations of the heat equation (10) also occur in practice. First, the bar material may be nonuniform and the cross section may not be constant along the length of the bar. In this case, the parameters κ, ρ, s, and A may depend on the axial variable x. Going back to Eq. (2) we see that the rate of heat transfer from left to right across the cross section at $x = x_0$ is now given by

$$H(x_0, t) = -\kappa(x_0) A(x_0) u_x(x_0, t) \tag{17}$$

with a similar expression for $H(x_0 + \Delta x, t)$. If we introduce these quantities into Eq. (4) and eventually into Eq. (9), and proceed as before, we obtain the partial differential equation

$$(\kappa A u_x)_x = s\rho A u_t. \tag{18}$$

We will usually write Eq. (18) in the form

$$r(x)u_t = [p(x)u_x]_x, \tag{19}$$

where $p(x) = \kappa(x)A(x)$ and $r(x) = s(x)\rho(x)A(x)$. Note that both these quantities are intrinsically positive.

A second generalization occurs if there are other ways in which heat enters or leaves the bar. Suppose that there is a *source* that adds heat to the bar at a rate $G(x, t, u)$ per unit time per unit length, where $G(x, t, u) > 0$. In this case we must add the term $G(x, t, u)\,\Delta x\,\Delta t$ to the left side of Eq. (9), and this leads to the differential equation

$$r(x)u_t = [p(x)u_x]_x + G(x, t, u). \tag{20}$$

If $G(x, t, u) < 0$, then we speak of a *sink* that removes heat from the bar at the rate $G(x, t, u)$ per unit time per unit length. To make the problem tractable we must restrict the form of the function G. In particular, we assume that G is linear in u and that the coefficient of u does not depend on t. Thus we write

$$G(x, t, u) = F(x, t) - q(x)u. \tag{21}$$

The minus sign in Eq. (21) has been introduced so that certain equations that appear later will have their customary forms. Substituting from Eq. (21) into Eq. (20), we obtain

$$r(x)u_t = [p(x)u_x]_x - q(x)u + F(x, t). \tag{22}$$

This equation is sometimes called the generalized heat conduction equation. Boundary value problems for Eq. (22) will be discussed to some extent in Chapter 11.

Finally, if instead of a one-dimensional bar, we consider a body with more than one significant space dimension, then the temperature is a function of two or three space coordinates rather than of x alone. Considerations similar to those leading to Eq. (10) can be employed to derive the heat conduction equation in two dimensions,

$$\alpha^2(u_{xx} + u_{yy}) = u_t, \tag{23}$$

or in three dimensions,

$$\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_t. \tag{24}$$

The boundary conditions corresponding to Eqs. (12) and (13) for multidimensional problems correspond to a prescribed temperature distribution on the boundary, or to an insulated boundary. Similarly, the initial temperature distribution will in general be a function of x and y for Eq. (23), and a function of x, y, and z for Eq. (24).

APPENDIX B

Derivation of the Wave Equation. In this appendix we derive the wave equation in one space dimension as it applies to the transverse vibrations of an elastic string, or cable; the elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line. The same equation, however, with the variables properly interpreted, occurs in many other wave problems having only one significant space variable.

Consider a perfectly flexible elastic string stretched tightly between supports fixed at the same horizontal level (see Figure 10.B.1*a*). Let the x-axis lie along the string with the endpoints located at $x = 0$ and $x = L$. If the string is set in motion at some initial time $t = 0$ (by plucking, for example) and is thereafter left undisturbed, it will vibrate freely in a vertical plane provided that damping effects, such as air resistance, are neglected. To determine the differential equation governing this motion we will consider the forces acting on a small element of the string of length Δx lying between the points x and $x + \Delta x$ (see Figure 10.B.1*b*). We assume that the motion of the string is small, and as a consequence, each point on the string moves solely in a vertical line. We denote by $u(x, t)$ the vertical displacement of the point x at the time t. Let the tension in the string, which always acts in the tangential direction, be denoted by $T(x, t)$, and let ρ denote the mass per unit length of the string.

Newton's law, as it applies to the element Δx of the string, states that the net external force, due to the tension at the ends of the element, must be equal to the product of the mass of the element and the acceleration of its mass center. Since there is no horizontal acceleration, the horizontal components must satisfy

$$T(x + \Delta x, t)\cos(\theta + \Delta\theta) - T(x, t)\cos\theta = 0. \tag{1}$$

If we denote the horizontal component of the tension (see Figure 10.B.1*c*) by H, then Eq. (1) states that H is independent of x.

On the other hand, the vertical components satisfy

$$T(x + \Delta x, t)\sin(\theta + \Delta\theta) - T(x, t)\sin\theta = \rho\,\Delta x\, u_{tt}(\overline{x}, t), \tag{2}$$

where $\overline{x}$ is the coordinate of the center of mass of the element of the string under consideration. Clearly, $\overline{x}$ lies in the interval $x < \overline{x} < x + \Delta x$. The weight of the string,

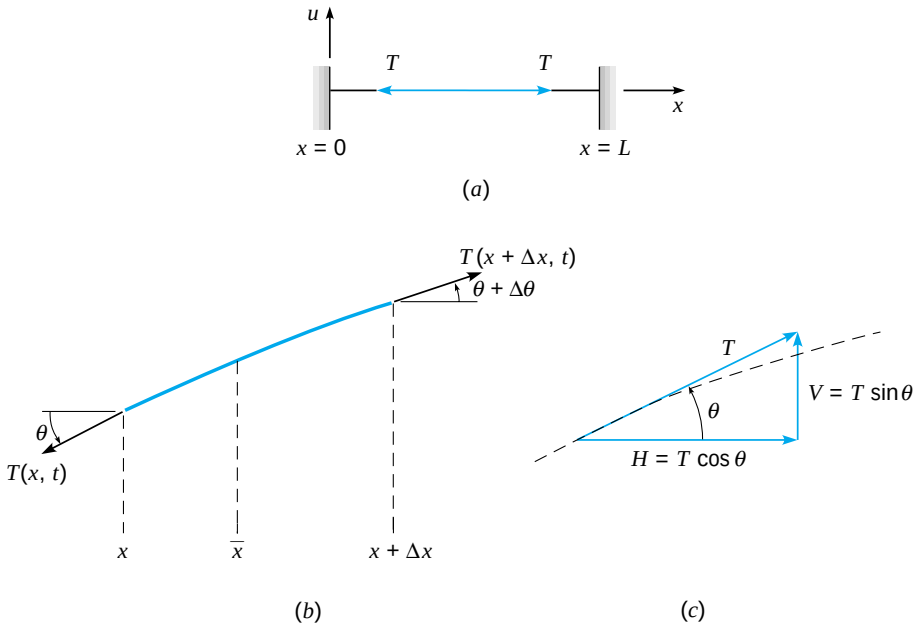


FIGURE 10.B.1 (*a*) An elastic string under tension. (*b*) An element of the displaced string. (*c*) Resolution of the tension T into components.

which acts vertically downward, is assumed to be negligible, and has been neglected in Eq. (2).

If the vertical component of T is denoted by V, then Eq. (2) can be written as

$$\frac{V(x+\Delta x,t)-V(x,t)}{\Delta x}=\rho u_{tt}(\overline{x},t).$$

Passing to the limit as $\Delta x \to 0$ gives

$$V_x(x,t)=\rho u_{tt}(x,t). \tag{3}$$

To express Eq. (3) entirely in terms of u we note that

$$V(x,t)=H(t)\tan\theta=H(t)u_x(x,t).$$

Hence Eq. (3) becomes

$$(Hu_x)_x=\rho u_{tt},$$

or, since H is independent of x,

$$Hu_{xx}=\rho u_{tt}. \tag{4}$$

For small motions of the string it is permissible to replace $H=T\cos\theta$ by T. Then Eq. (4) takes its customary form

$$a^2u_{xx}=u_{tt}, \tag{5}$$

where

$$a^2=T/\rho. \tag{6}$$

We will assume further that a^2 is a constant, although this is not required in our derivation, even for small motions. Equation (5) is called the wave equation for one space dimension. Since T has the dimension of force, and ρ that of mass/length, it follows that the constant a has the dimension of velocity. It is possible to identify a as the velocity with which a small disturbance (wave) moves along the string. According to Eq. (6) the wave velocity a varies directly with the tension in the string, but inversely with the density of the string material. These facts are in agreement with experience.

As in the case of the heat conduction equation, there are various generalizations of the wave equation (5). One important equation is known as the *telegraph equation* and has the form

$$u_{tt}+cu_t+ku=a^2u_{xx}+F(x,t), \tag{7}$$

where c and k are nonnegative constants. The terms cu_t, ku, and $F(x,t)$ arise from a viscous damping force, an elastic restoring force, and an external force, respectively. Note the similarity of Eq. (7), except for the term a^2u_{xx}, with the equation for the spring–mass system derived in Section 3.8; the additional term a^2u_{xx} arises from a consideration of internal elastic forces.

For a vibrating system with more than one significant space coordinate, it may be necessary to consider the wave equation in two dimensions,

$$a^2(u_{xx}+u_{yy})=u_{tt}, \tag{8}$$

or in three dimensions,

$$a^2(u_{xx}+u_{yy}+u_{zz})=u_{tt}. \tag{9}$$

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CHAPTER
11

Boundary Value Problems and Sturm-Liouville Theory

As a result of separating variables in a partial differential equation in Chapter 10 we repeatedly encountered the differential equation

$$X'' + \lambda X = 0, \qquad 0 < x < L$$

with the boundary conditions

$$X(0) = 0, \qquad X(L) = 0.$$

This boundary value problem is the prototype of a large class of problems that are important in applied mathematics. These problems are known as Sturm–Liouville boundary value problems. In this chapter we discuss the major properties of Sturm–Liouville problems and their solutions; in the process we are able to generalize somewhat the method of separation of variables for partial differential equations.

11.1 The Occurrence of Two-Point Boundary Value Problems

In Chapter 10 we described the method of separation of variables as a means of solving certain problems involving partial differential equations. The heat conduction problem consisting of the partial differential equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0 \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0 \tag{2}$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{3}$$

is typical of the problems considered there. A crucial part of the process of solving such problems is to find the eigenvalues and eigenfunctions of the differential equation

$$X'' + \lambda X = 0, \qquad 0 < x < L \tag{4}$$

with the boundary conditions

$$X(0) = 0, \qquad X(L) = 0 \tag{5}$$

or perhaps

$$X'(0) = 0, \qquad X'(L) = 0. \tag{6}$$

The sine or cosine functions that result from solving Eq. (4) subject to the boundary conditions (5) or (6) are used to expand the initial temperature distribution $f(x)$ in a Fourier series.

In this chapter we extend and generalize the results of Chapter 10. Our main goal is to show how the method of separation of variables can be used to solve problems somewhat more general than that of Eqs. (1), (2), and (3). We are interested in three types of generalizations.

First, we wish to consider more general partial differential equations, for example, the equation

$$r(x)u_t = [p(x)u_x]_x - q(x)u + F(x,t). \tag{7}$$

This equation can arise, as indicated in Appendix A of Chapter 10, in the study of heat conduction in a bar of variable material properties in the presence of heat sources. If p and r are constants, and if the source terms qu and F are zero, then Eq. (7) reduces to Eq. (1). The partial differential equation (7) also occurs in the investigation of other phenomena of a diffusive character.

A second generalization is to allow more general boundary conditions. In particular, we wish to consider boundary conditions of the form

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(L,t) + h_2 u(L,t) = 0. \tag{8}$$

Such conditions occur when the rate of heat flow through an end of the bar is proportional to the temperature there. Usually, h_1 and h_2 are nonnegative constants, but in some cases they may be negative or depend on t. The boundary conditions (2) are obtained in the limit as $h_1 \to \infty$ and $h_2 \to \infty$. The other important limiting case, $h_1 = h_2 = 0$, gives the boundary conditions for insulated ends.

The final generalization that we discuss in this chapter concerns the geometry of the region in which the problem is posed. The results of Chapter 10 are adequate only for a rather restricted class of problems, chiefly those in which the region of interest is rectangular or, in a few cases, circular. Later in this chapter we consider certain problems posed in a few other geometrical regions.

Let us consider the equation

$$r(x)u_t = [p(x)u_x]_x - q(x)u \tag{9}$$

obtained by setting the term $F(x,t)$ in Eq. (7) equal to zero. To separate the variables we assume that

$$u(x,t) = X(x)T(t), \tag{10}$$

and substitute for u in Eq. (9). We obtain

$$r(x)XT' = [p(x)X']'T - q(x)XT \tag{11}$$

or, upon dividing by $r(x)XT$,

$$\frac{T'}{T} = \frac{[p(x)X']'}{r(x)X} - \frac{q(x)}{r(x)} = -\lambda. \tag{12}$$

We have denoted the separation constant by $-\lambda$ in anticipation of the fact that usually it will turn out to be real and negative. From Eq. (12) we obtain the following two ordinary differential equations for X and T:

$$[p(x)X']' - q(x)X + \lambda r(x)X = 0, \tag{13}$$

$$T' + \lambda T = 0. \tag{14}$$

If we substitute from Eq. (10) for u in Eqs. (8) and assume that h_1 and h_2 are constants, then we obtain the boundary conditions

$$X'(0) - h_1X(0) = 0, \qquad X'(L) + h_2X(L) = 0. \tag{15}$$

To proceed further we need to solve Eq. (13) subject to the boundary conditions (15). While this is a more general linear homogeneous two-point boundary value problem than the problem consisting of the differential equation (4) and the boundary conditions (5) or (6), the solutions behave very much the same way. For every value of λ, the problem (13), (15) has the trivial solution $X(x) = 0$. For certain values of λ, called **eigenvalues**, there are also other nontrivial solutions, called **eigenfunctions**. These eigenfunctions form the basis for series solutions of a variety of problems in partial differential equations, such as the generalized heat conduction equation (9) subject to the boundary conditions (8) and the initial condition (3).

In this chapter we discuss some of the properties of solutions of two-point boundary value problems for second order linear equations. Sometimes we consider the general linear homogeneous equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{16}$$

investigated in Chapter 3. However, for most purposes it is better to discuss equations in which the first and second derivative terms are related as in Eq. (13). It is always possible to transform the general equation (16) so that the derivative terms appear as in Eq. (13) (see Problem 11).

Boundary value problems with higher order differential equations can also occur; in them the number of boundary conditions must equal the order of the differential equation. As a rule, the order of the differential equation is even, and half the boundary conditions are given at each end of the interval. It is also possible for a single boundary condition to involve values of the solution and/or its derivatives at both boundary points; for example,

$$y(0) - y(L) = 0. \tag{17}$$

The following example involves one boundary condition of the form (15) and is therefore more complicated than the problems in Section 10.1.

EXAMPLE 1

Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \tag{18}$$

$$y(0) = 0, \qquad y'(1) + y(1) = 0. \tag{19}$$

One place where this problem occurs is in the heat conduction problem in a bar of unit length. The boundary condition at $x = 0$ corresponds to a zero temperature there. The boundary condition at $x = 1$ corresponds to a rate of heat flow that is proportional to the temperature there, and units are chosen so that the constant of proportionality is 1 (see Appendix A of Chapter 10).

The solution of the differential equation may have one of several forms, depending on λ, so it is necessary to consider several cases. First, if $\lambda = 0$, the general solution of the differential equation is

$$y = c_1 x + c_2. \tag{20}$$

The two boundary conditions require that

$$c_2 = 0, \qquad 2c_1 + c_2 = 0, \tag{21}$$

respectively. The only solution of Eqs. (21) is $c_1 = c_2 = 0$, so the boundary value problem has no nontrivial solution in this case. Hence $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$, then the general solution of the differential equation (18) is

$$y = c_1 \sin \sqrt{\lambda}\, x + c_2 \cos \sqrt{\lambda}\, x, \tag{22}$$

where $\sqrt{\lambda} > 0$. The boundary condition at $x = 0$ requires that $c_2 = 0$; from the boundary condition at $x = 1$ we then obtain the equation

$$c_1(\sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda}) = 0.$$

For a nontrivial solution y we must have $c_1 \neq 0$, and thus λ must satisfy

$$\sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda} = 0. \tag{23}$$

Note that if λ is such that $\cos \sqrt{\lambda} = 0$, then $\sin \sqrt{\lambda} \neq 0$, and Eq. (23) is not satisfied. Hence we may assume that $\cos \sqrt{\lambda} \neq 0$; dividing Eq. (23) by $\cos \sqrt{\lambda}$, we obtain

$$\sqrt{\lambda} = -\tan \sqrt{\lambda}. \tag{24}$$

The solutions of Eq. (24) can be determined numerically. They can also be found approximately by sketching the graphs of $f(\sqrt{\lambda}) = \sqrt{\lambda}$ and $g(\sqrt{\lambda}) = -\tan \sqrt{\lambda}$ for $\sqrt{\lambda} > 0$ on the same set of axes, and identifying the points of intersection of the two curves (see Figure 11.1.1). The point $\sqrt{\lambda} = 0$ is specifically excluded from this argument because the solution (22) is valid only for $\sqrt{\lambda} \neq 0$. Despite the fact that the curves intersect there, $\lambda = 0$ is not an eigenvalue, as we have already shown. The first three positive solutions of Eq. (24) are $\sqrt{\lambda_1} \cong 2.029$, $\sqrt{\lambda_2} \cong 4.913$, and $\sqrt{\lambda_3} \cong 7.979$. As can be seen from Figure 11.1.1, the other roots are given with reasonable accuracy

by $\sqrt{\lambda_n} \cong (2n-1)\pi/2$ for $n = 4, 5, \ldots$, the precision of this estimate improving as n increases. Hence the eigenvalues are

$$\begin{aligned} \lambda_1 &\cong 4.116, \qquad \lambda_2 \cong 24.14, \\ \lambda_3 &\cong 63.66, \qquad \lambda_n \cong (2n-1)^2\pi^2/4 \qquad \text{for } n = 4, 5, \ldots. \end{aligned} \tag{25}$$

Finally, since $c_2 = 0$, the eigenfunction corresponding to the eigenvalue λ_n is

$$\phi_n(x, \lambda_n) = k_n \sin \sqrt{\lambda_n}\, x; \qquad n = 1, 2, \ldots, \tag{26}$$

where the constant k_n remains arbitrary.

Next consider $\lambda < 0$. In this case it is convenient to let $\lambda = -\mu$ so that $\mu > 0$. Then Eq. (14) becomes

$$y'' - \mu y = 0, \tag{27}$$

and its general solution is

$$y = c_1 \sinh \sqrt{\mu}\, x + c_2 \cosh \sqrt{\mu}\, x, \tag{28}$$

where $\sqrt{\mu} > 0$. Proceeding as in the previous case, we find that μ must satisfy the equation

$$\sqrt{\mu} = -\tanh \sqrt{\mu}. \tag{29}$$

From Figure 11.1.2 it is clear that the graphs of $f(\sqrt{\mu}) = \sqrt{\mu}$ and $g(\sqrt{\mu}) = -\tanh \sqrt{\mu}$ intersect only at the origin. Hence there are no positive values of $\sqrt{\mu}$ that satisfy Eq. (29), and hence the boundary value problem (18), (19) has no negative eigenvalues.

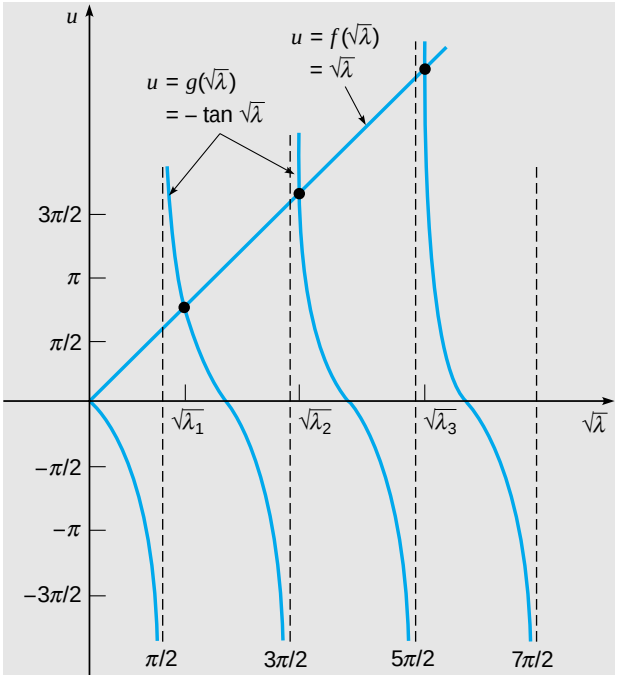


FIGURE 11.1.1 Graphical solution of $\sqrt{\lambda} = -\tan \sqrt{\lambda}$.

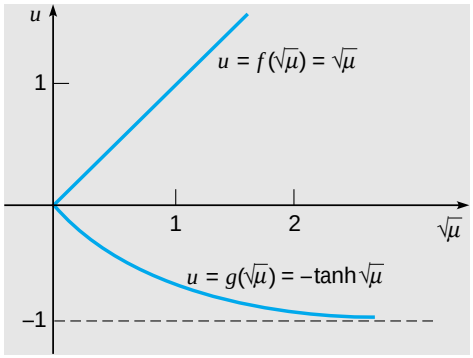


FIGURE 11.1.2 Graphical solution of $\sqrt{\mu} = -\tanh\sqrt{\mu}$.

Finally, it is necessary to consider the possibility that λ may be complex. It is possible to show by direct calculation that the problem (18), (19) has no complex eigenvalues. However, in Section 11.2 we consider in more detail a large class of problems that includes this example. One of the things we show there is that every problem in this class has only real eigenvalues. Therefore we omit the discussion of the nonexistence of complex eigenvalues here. Thus we conclude that all the eigenvalues and eigenfunctions of the problem (18), (19) are given by Eqs. (25) and (26).

PROBLEMS

In each of Problems 1 through 6 state whether the given boundary value problem is homogeneous or nonhomogeneous.

1. $y'' + 4y = 0, \quad y(-1) = 0, \quad y(1) = 0$
2. $[(1 + x^2)y']' + 4y = 0, \quad y(0) = 0, \quad y(1) = 1$
3. $y'' + 4y = \sin x, \quad y(0) = 0, \quad y(1) = 0$
4. $-y'' + x^2y = \lambda y, \quad y'(0) - y(0) = 0, \quad y'(1) + y(1) = 0$
5. $-[(1 + x^2)y']' = \lambda y + 1, \quad y(-1) = 0, \quad y(1) = 0$
6. $-y'' = \lambda(1 + x^2)y, \quad y(0) = 0, \quad y'(1) + 3y(1) = 0$

In each of Problems 7 through 10 determine the form of the eigenfunctions and the determinantal equation satisfied by the nonzero eigenvalues. Determine whether $\lambda = 0$ is an eigenvalue, and find approximate values for λ_1 and λ_2, the nonzero eigenvalues of smallest absolute value. Estimate λ_n for large values of n. Assume that all eigenvalues are real.

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$
8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0$
9. $y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y(1) + y'(1) = 0$
10. $y'' - \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) = 0$
11. Consider the general linear homogeneous second order equation

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{i}$$

We seek an integrating factor $\mu(x)$ such that, upon multiplying Eq. (i) by $\mu(x)$, the resulting equation can be written in the form

$$[\mu(x)P(x)y']' + \mu(x)R(x)y = 0. \tag{ii}$$

(a) By equating coefficients of y', show that μ must be a solution of

$$P\mu' = (Q - P')\mu. \tag{iii}$$

(b) Solve Eq. (iii) and thereby show that

$$\mu(x) = \frac{1}{P(x)} \exp \int_{x_0}^{x} \frac{Q(s)}{P(s)}\, ds. \tag{iv}$$

Compare this result with that of Problem 27 in Section 3.2.

In each of Problems 12 through 15 use the method of Problem 11 to transform the given equation into the form $[p(x)y']' + q(x)y = 0$.

12. $y'' - 2xy' + \lambda y = 0$, Hermite equation
13. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel equation
14. $xy'' + (1 - x)y' + \lambda y = 0$, Laguerre equation
15. $(1 - x^2)y'' - xy' + \alpha^2 y = 0$, Chebyshev equation

16. The equation

$$u_{tt} + cu_t + ku = a^2u_{xx} + F(x, t), \tag{i}$$

where $a^2 > 0$, $c \geq 0$, and $k \geq 0$ are constants, is known as the **telegraph equation**. It arises in the study of an elastic string under tension (see Appendix B of Chapter 10). Equation (i) also occurs in other applications. Assuming that $F(x, t) = 0$, let $u(x, t) = X(x)T(t)$, separate the variables in Eq. (i), and derive ordinary differential equations for X and T.

17. Consider the boundary value problem

$$y'' - 2y' + (1 + \lambda)y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

(a) Introduce a new dependent variable u by the relation $y = s(x)u$. Determine $s(x)$ so that the differential equation for u has no u' term.
(b) Solve the boundary value problem for u and thereby determine the eigenvalues and eigenfunctions of the original problem. Assume that all eigenvalues are real.
(c) Also solve the given problem directly (without introducing u).

18. Consider the boundary value problem

$$y'' + 4y' + (4 + 9\lambda)y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$$

(a) Determine, at least approximately, the real eigenvalues and the corresponding eigenfunctions by proceeding as in Problem 17(a, b).
(b) Also solve the given problem directly (without introducing a new variable).
Hint: In part (a) be sure to pay attention to the boundary conditions as well as the differential equation.

The differential equations in Problems 19 and 20 differ from those in previous problems in that the parameter λ multiplies the y' term as well as the y term. In each of these problems determine the real eigenvalues and the corresponding eigenfunctions.

19. $y'' + y' + \lambda(y' + y) = 0$,
$y'(0) = 0, \qquad y(1) = 0$

20. $x^2y'' - \lambda(xy' - y) = 0$,
$y(1) = 0, \qquad y(2) - y'(2) = 0$

21. Consider the problem

$$y'' + \lambda y = 0, \qquad 2y(0) + y'(0) = 0, \quad y(1) = 0.$$

(a) Find the determinantal equation satisfied by the positive eigenvalues. Show that there is an infinite sequence of such eigenvalues. Find λ_1 and λ_2. Then show that $\lambda_n \cong [(2n + 1)\pi/2]^2$ for large n.

(b) Find the determinantal equation satisfied by the negative eigenvalues. Show that there is exactly one negative eigenvalue and find its value.

22. Consider the problem

$$y'' + \lambda y = 0, \qquad \alpha y(0) + y'(0) = 0, \quad y(1) = 0,$$

where α is a given constant.
(a) Show that for all values of α there is an infinite sequence of positive eigenvalues.
(b) If $\alpha < 1$, show that all (real) eigenvalues are positive. Show that the smallest eigenvalue approaches zero as α approaches 1 from below.
(c) Show that $\lambda = 0$ is an eigenvalue only if $\alpha = 1$.
(d) If $\alpha > 1$, show that there is exactly one negative eigenvalue and that this eigenvalue decreases as α increases.

23. Consider the problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$$

Show that if ϕ_m and ϕ_n are eigenfunctions, corresponding to the eigenvalues λ_m and λ_n, respectively, with $\lambda_m \neq \lambda_n$, then

$$\int_0^L \phi_m(x)\phi_n(x)\, dx = 0.$$

Hint: Note that

$$\phi_m'' + \lambda_m \phi_m = 0, \qquad \phi_n'' + \lambda_n \phi_n = 0.$$

Multiply the first of these equations by ϕ_n, the second by ϕ_m, and integrate from 0 to L, using integration by parts. Finally, subtract one equation from the other.

24. In this problem we consider a higher order eigenvalue problem. In the study of transverse vibrations of a uniform elastic bar one is led to the differential equation

$$y^{\text{iv}} - \lambda y = 0,$$

where y is the transverse displacement and $\lambda = m\omega^2/EI$; m is the mass per unit length of the rod, E is Young's modulus, I is the moment of inertia of the cross section about an axis through the centroid perpendicular to the plane of vibration, and ω is the frequency of vibration. Thus for a bar whose material and geometric properties are given, the eigenvalues determine the natural frequencies of vibration. Boundary conditions at each end are usually one of the following types:

$$\begin{aligned} y = y' &= 0, && \text{clamped end,} \\ y = y'' &= 0, && \text{simply supported or hinged end,} \\ y'' = y''' &= 0, && \text{free end.} \end{aligned}$$

For each of the following three cases find the form of the eigenfunctions and the equation satisfied by the eigenvalues of this fourth order boundary value problem. Determine λ_1 and λ_2, the two eigenvalues of smallest magnitude. Assume that the eigenvalues are real and positive.
(a) $y(0) = y''(0) = 0, \qquad y(L) = y''(L) = 0$
(b) $y(0) = y''(0) = 0, \qquad y(L) = y'(L) = 0$
(c) $y(0) = y'(0) = 0, \qquad y''(L) = y'''(L) = 0$ (cantilevered bar)

25. This problem illustrates that the eigenvalue parameter sometimes appears in the boundary conditions as well as in the differential equation. Consider the longitudinal vibrations of a uniform straight elastic bar of length L. It can be shown that the axial displacement $u(x, t)$ satisfies the partial differential equation

$$(E/\rho)u_{xx} = u_{tt}; \qquad 0 < x < L, \quad t > 0, \tag{i}$$

where E is Young's modulus and ρ is the mass per unit volume. If the end $x = 0$ is fixed, then the boundary condition there is

$$u(0, t) = 0, \qquad t > 0. \tag{ii}$$

Suppose that the end $x = L$ is rigidly attached to a mass m but is otherwise unrestrained. We can obtain the boundary condition here by writing Newton's law for the mass. From the theory of elasticity it can be shown that the force exerted by the bar on the mass is given by $-EAu_x(L, t)$. Hence the boundary condition is

$$EAu_x(L, t) + mu_{tt}(L, t) = 0, \qquad t > 0. \tag{iii}$$

(a) Assume that $u(x, t) = X(x)T(t)$, and show that $X(x)$ and $T(t)$ satisfy the differential equations

$$X'' + \lambda X = 0, \tag{iv}$$

$$T'' + \lambda(E/\rho)T = 0. \tag{v}$$

(b) Show that the boundary conditions are

$$X(0) = 0, \qquad X'(L) - \gamma\lambda L X(L) = 0, \tag{vi}$$

where $\gamma = m/\rho AL$ is a dimensionless parameter that gives the ratio of the end mass to the mass of the rod.
Hint: Use the differential equation for $T(t)$ in simplifying the boundary condition at $x = L$.
(c) Determine the form of the eigenfunctions and the equation satisfied by the real eigenvalues of Eqs. (iv) and (vi). Find the first two eigenvalues λ_1 and λ_2 if $\gamma = 0.5$.

11.2 Sturm–Liouville Boundary Value Problems

We now consider two-point boundary value problems of the type obtained in Section 11.1 by separating the variables in a heat conduction problem for a bar of variable material properties and with a source term proportional to the temperature. This kind of problem also occurs in many other applications.

These boundary value problems are commonly associated with the names of Sturm and Liouville.[1] They consist of a differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0 \tag{1}$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1y(0) + a_2y'(0) = 0, \qquad b_1y(1) + b_2y'(1) = 0 \tag{2}$$

[1]Charles-François Sturm (1803–1855) and Joseph Liouville (1809–1882), in a series of papers in 1836 and 1837, set forth many properties of the class of boundary value problems associated with their names, including the results stated in Theorems 11.2.1 to 11.2.4. Sturm is also famous for a theorem on the number of real zeros of a polynomial, and in addition, did extensive work in physics and mechanics. Besides his own research in analysis, algebra, and number theory, Liouville was the founder, and for 39 years the editor, of the influential *Journal de mathématiques pures et appliquées*. One of his most important results was the proof (in 1844) of the existence of transcendental numbers.

at the endpoints. It is often convenient to introduce the linear homogeneous differential operator L defined by

$$L[y] = -[p(x)y']' + q(x)y. \tag{3}$$

Then the differential equation (1) can be written as

$$L[y] = \lambda r(x)y. \tag{4}$$

We assume that the functions p, p', q, and r are continuous on the interval $0 \le x \le 1$ and, further, that $p(x) > 0$ and $r(x) > 0$ at all points in $0 \le x \le 1$. These assumptions are necessary to render the theory as simple as possible while retaining considerable generality. It turns out that these conditions are satisfied in many significant problems in mathematical physics. For example, the equation $y'' + \lambda y = 0$, which arose repeatedly in the preceding chapter, is of the form (1) with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The boundary conditions (2) are said to be **separated**; that is, each involves only one of the boundary points. These are the most general separated boundary conditions that are possible for a second order differential equation.

Before proceeding to establish some of the properties of the Sturm–Liouville problem (1), (2), it is necessary to derive an identity, known as **Lagrange's identity**, which is basic to the study of linear boundary value problems. Let u and v be functions having continuous second derivatives on the interval $0 \le x \le 1$. Then[2]

$$\int_0^1 L[u]v\,dx = \int_0^1 [-(pu')'v + quv]\,dx.$$

Integrating the first term on the right side twice by parts, we obtain

$$\begin{aligned}\int_0^1 L[u]v\,dx &= -p(x)u'(x)v(x)\Big|_0^1 + p(x)u(x)v'(x)\Big|_0^1 + \int_0^1 [-u(pv')' + uqv]\,dx \\ &= -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 + \int_0^1 uL[v]\,dx.\end{aligned}$$

Hence, on transposing the integral on the right side, we have

$$\int_0^1 \{L[u]v - uL[v]\}\,dx = -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1, \tag{5}$$

which is Lagrange's identity.

Now let us suppose that the functions u and v in Eq. (5) also satisfy the boundary conditions (2). Then, if we assume that $a_2 \ne 0$ and $b_2 \ne 0$, the right side of Eq. (5) becomes

$$\begin{aligned}&-p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 \\ &= -p(1)[u'(1)v(1) - u(1)v'(1)] + p(0)[u'(0)v(0) - u(0)v'(0)] \\ &= -p(1)\left[-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)\right] + p(0)\left[-\frac{a_1}{a_2}u(0)v(0) + \frac{a_1}{a_2}u(0)v(0)\right] \\ &= 0.\end{aligned}$$

[2]For brevity we sometimes use the notation $\int_0^1 f\,dx$ rather than $\int_0^1 f(x)\,dx$ in this chapter.

The same result holds if either a_2 or b_2 is zero; the proof in this case is even simpler, and is left for you. Thus, if the differential operator L is defined by Eq. (3), and if the functions u and v satisfy the boundary conditions (2), Lagrange's identity reduces to

$$\int_0^1 \{L[u]v - uL[v]\}\, dx = 0. \tag{6}$$

Let us now write Eq. (6) in a slightly different way. In Eq. (4) of Section 10.2 we introduced the inner product (u, v) of two real-valued functions u and v on a given interval; using the interval $0 \le x \le 1$, we have

$$(u, v) = \int_0^1 u(x)v(x)\, dx. \tag{7}$$

In this notation Eq. (6) becomes

$$(L[u], v) - (u, L[v]) = 0. \tag{8}$$

In proving Theorem 11.2.1 below it is necessary to deal with complex-valued functions. By analogy with the definition in Section 7.2 for vectors, we define the inner product of two complex-valued functions on $0 \le x \le 1$ as

$$(u, v) = \int_0^1 u(x)\overline{v}(x)\, dx, \tag{9}$$

where $\overline{v}$ is the complex conjugate of v. Clearly, Eq. (9) coincides with Eq. (7) if $u(x)$ and $v(x)$ are real. It is important to know that Eq. (8) remains valid under the stated conditions if u and v are complex-valued functions and if the inner product (9) is used. To see this, one can start with the quantity $\int_0^1 L[u]\overline{v}\, dx$ and retrace the steps leading to Eq. (6), making use of the fact that $p(x)$, $q(x)$, a_1, a_2, b_1, and b_2 are all real quantities (see Problem 22).

We now consider some of the implications of Eq. (8) for the Sturm–Liouville boundary value problem (1), (2). We assume without proof [3] that this problem actually has eigenvalues and eigenfunctions. In Theorems 11.2.1 to 11.2.4 below, we state several of their important, but relatively elementary, properties. Each of these properties is illustrated by the basic Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0, \tag{10}$$

whose eigenvalues are $\lambda_n = n^2\pi^2$, with the corresponding eigenfunctions $\phi_n(x) = \sin n\pi x$.

Theorem 11.2.1 All the eigenvalues of the Sturm–Liouville problem (1), (2) are real.

To prove this theorem let us suppose that λ is a (possibly complex) eigenvalue of the problem (1), (2) and that ϕ is a corresponding eigenfunction, also possibly complex-valued. Let us write $\lambda = \mu + i\nu$ and $\phi(x) = U(x) + iV(x)$, where μ, ν, $U(x)$, and $V(x)$ are real. Then, if we let $u = \phi$ and also $v = \phi$ in Eq. (8), we have

$$(L[\phi], \phi) = (\phi, L[\phi]). \tag{11}$$

[3] The proof may be found, for example, in the references by Sagan (Chapter 5) or Birkhoff and Rota (Chapter 10).

However, we know that $L[\phi] = \lambda r\phi$, so Eq. (11) becomes

$$(\lambda r\phi, \phi) = (\phi, \lambda r\phi). \tag{12}$$

Writing out Eq. (12) in full, using the definition (9) of the inner product, we obtain

$$\int_0^1 \lambda r(x)\phi(x)\overline{\phi}(x)\,dx = \int_0^1 \phi(x)\overline{\lambda}\overline{r}(x)\overline{\phi}(x)\,dx. \tag{13}$$

Since $r(x)$ is real, Eq. (13) reduces to

$$(\lambda - \overline{\lambda}) \int_0^1 r(x)\phi(x)\overline{\phi}(x)\,dx = 0,$$

or

$$(\lambda - \overline{\lambda}) \int_0^1 r(x)[U^2(x) + V^2(x)]\,dx = 0. \tag{14}$$

The integrand in Eq. (14) is nonnegative and not identically zero. Since the integrand is also continuous, it follows that the integral is positive. Therefore, the factor $\lambda - \overline{\lambda} = 2i\nu$ must be zero. Hence $\nu = 0$ and λ is real, so the theorem is proved.

An important consequence of Theorem 11.2.1 is that in finding eigenvalues and eigenfunctions of a Sturm–Liouville boundary value problem, one need look only for real eigenvalues. Recall that this is what we did in Chapter 10. It is also possible to show that the eigenfunctions of the boundary value problem (1), (2) are real. A proof is sketched in Problem 23.

Theorem 11.2.2 If ϕ_1 and ϕ_2 are two eigenfunctions of the Sturm–Liouville problem (1), (2) corresponding to eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)\,dx = 0. \tag{15}$$

This theorem expresses the property of **orthogonality** of the eigenfunctions with respect to the weight function r. To prove the theorem we note that ϕ_1 and ϕ_2 satisfy the differential equations

$$L[\phi_1] = \lambda_1 r\phi_1 \tag{16}$$

and

$$L[\phi_2] = \lambda_2 r\phi_2, \tag{17}$$

respectively. If we let $u = \phi_1$, $v = \phi_2$, and substitute for $L[u]$ and $L[v]$ in Eq. (8), we obtain

$$(\lambda_1 r\phi_1, \phi_2) - (\phi_1, \lambda_2 r\phi_2) = 0,$$

or, using Eq. (9),

$$\lambda_1 \int_0^1 r(x)\phi_1(x)\overline{\phi}_2(x)\,dx - \overline{\lambda}_2 \int_0^1 \phi_1(x)\overline{r}(x)\overline{\phi}_2(x)\,dx = 0.$$

Because λ_2, $r(x)$, and $\phi_2(x)$ are real, this equation becomes

$$(\lambda_1 - \lambda_2) \int_0^1 r(x)\phi_1(x)\phi_2(x)\, dx = 0. \tag{18}$$

Since by hypothesis $\lambda_1 \neq \lambda_2$, it follows that ϕ_1 and ϕ_2 must satisfy Eq. (15), and the theorem is proved.

Theorem 11.2.3 The eigenvalues of the Sturm–Liouville problem (1), (2) are all simple; that is, to each eigenvalue there corresponds only one linearly independent eigenfunction. Further, the eigenvalues form an infinite sequence, and can be ordered according to increasing magnitude so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots.$$

Moreover, $\lambda_n \to \infty$ as $n \to \infty$.

The proof of this theorem is somewhat more advanced than those of the two previous theorems, and will be omitted. However, a proof that the eigenvalues are simple is indicated in Problem 20.

Again we note that all the properties stated in Theorems 11.2.1 to 11.2.3 are exemplified by the eigenvalues $\lambda_n = n^2\pi^2$ and eigenfunctions $\phi_n(x) = \sin n\pi x$ of the example problem (10). Clearly, the eigenvalues are real. The eigenfunctions satisfy the orthogonality relation

$$\int_0^1 \phi_m(x)\phi_n(x)\, dx = \int_0^1 \sin m\pi x \, \sin n\pi x \, dx = 0, \qquad m \neq n, \tag{19}$$

which was established in Section 10.2 by direct integration. Further, the eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$. Finally, to each eigenvalue there corresponds a single linearly independent eigenfunction.

We will now assume that the eigenvalues of the Sturm–Liouville problem (1), (2) are ordered as indicated in Theorem 11.2.3. Associated with the eigenvalue λ_n is a corresponding eigenfunction ϕ_n, determined up to a multiplicative constant. It is often convenient to choose the arbitrary constant multiplying each eigenfunction so as to satisfy the condition

$$\int_0^1 r(x)\phi_n^2(x)\, dx = 1, \qquad n = 1, 2, \ldots. \tag{20}$$

Equation (20) is called a normalization condition, and eigenfunctions satisfying this condition are said to be **normalized**. Indeed, in this case, the eigenfunctions are said to form an **orthonormal set** (with respect to the weight function r) since they already satisfy the orthogonality relation (15). It is sometimes useful to combine Eqs. (15) and (20) into a single equation. To this end we introduce the symbol δ_{mn}, known as the Kronecker (1823–1891) delta and defined by

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \tag{21}$$

Making use of the Kronecker delta, we can write Eqs. (15) and (20) as

$$\int_0^1 r(x)\phi_m(x)\phi_n(x)\,dx = \delta_{mn}. \tag{22}$$

EXAMPLE 1

Determine the normalized eigenfunctions of the problem (10):

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

The eigenvalues of this problem are $\lambda_1 = \pi^2$, $\lambda_2 = 4\pi^2, \ldots, \lambda_n = n^2\pi^2, \ldots$, and the corresponding eigenfunctions are $k_1 \sin \pi x$, $k_2 \sin 2\pi x, \ldots, k_n \sin n\pi x, \ldots$, respectively. In this case the weight function is $r(x) = 1$. To satisfy Eq. (20) we must choose k_n so that

$$\int_0^1 (k_n \sin n\pi x)^2\,dx = 1 \tag{23}$$

for each value of n. Since

$$k_n^2 \int_0^1 \sin^2 n\pi x\,dx = k_n^2 \int_0^1 (\tfrac{1}{2} - \tfrac{1}{2}\cos 2n\pi x)\,dx = \tfrac{1}{2}k_n^2,$$

Eq. (23) is satisfied if k_n is chosen to be $\sqrt{2}$ for each value of n. Hence the normalized eigenfunctions of the given boundary value problem are

$$\phi_n(x) = \sqrt{2}\sin n\pi x, \qquad n = 1, 2, 3, \ldots. \tag{24}$$

EXAMPLE 2

Determine the normalized eigenfunctions of the problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0. \tag{25}$$

In Example 1 of Section 11.1 we found that the eigenvalues λ_n satisfy the equation

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0, \tag{26}$$

and that the corresponding eigenfunctions are

$$\phi_n(x) = k_n \sin\sqrt{\lambda_n}\,x, \tag{27}$$

where k_n is arbitrary. We can determine k_n from the normalization condition (20). Since $r(x) = 1$ in this problem, we have

$$\begin{aligned}
\int_0^1 \phi_n^2(x)\,dx &= k_n^2 \int_0^1 \sin^2\sqrt{\lambda_n}\,x\,dx \\
&= k_n^2 \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\sqrt{\lambda_n}\,x\right)dx = k_n^2\left(\frac{x}{2} - \frac{\sin 2\sqrt{\lambda_n}\,x}{4\sqrt{\lambda_n}}\right)\Bigg|_0^1 \\
&= k_n^2\,\frac{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}{4\sqrt{\lambda_n}} = k_n^2\,\frac{\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{2\sqrt{\lambda_n}} \\
&= k_n^2\,\frac{1 + \cos^2\sqrt{\lambda_n}}{2},
\end{aligned}$$

where in the last step we have used Eq. (26). Hence, to normalize the eigenfunctions ϕ_n we must choose

$$k_n = \left(\frac{2}{1 + \cos^2 \sqrt{\lambda_n}} \right)^{1/2}. \tag{28}$$

The normalized eigenfunctions of the given problem are

$$\phi_n(x) = \frac{\sqrt{2} \sin \sqrt{\lambda_n}\, x}{(1 + \cos^2 \sqrt{\lambda_n})^{1/2}}; \qquad n = 1, 2, \ldots. \tag{29}$$

We now turn to the question of expressing a given function f as a series of eigenfunctions of the Sturm–Liouville problem (1), (2). We have already seen examples of such expansions in Sections 10.2 to 10.4. For example, it was shown there that if f is continuous and has a piecewise continuous derivative on $0 \le x \le 1$, and satisfies the boundary conditions $f(0) = f(1) = 0$, then f can be expanded in a Fourier sine series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x. \tag{30}$$

The functions $\sin n\pi x$, $n = 1, 2, \ldots$, are precisely the eigenfunctions of the boundary value problem (10). The coefficients b_n are given by

$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx \tag{31}$$

and the series (30) converges for each x in $0 \le x \le 1$. In a similar way f can be expanded in a Fourier cosine series using the eigenfunctions $\cos n\pi x$, $n = 0, 1, 2, \ldots$, of the boundary value problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$.

Now suppose that a given function f, satisfying suitable conditions, can be expanded in an infinite series of eigenfunctions of the more general Sturm–Liouville problem (1), (2). If this can be done, then we have

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{32}$$

where the functions $\phi_n(x)$ satisfy Eqs. (1), (2), and also the orthogonality condition (22). To compute the coefficients in the series (32) we multiply Eq. (32) by $r(x)\phi_m(x)$, where m is a fixed positive integer, and integrate from $x = 0$ to $x = 1$. Assuming that the series can be integrated term by term we obtain

$$\int_0^1 r(x) f(x) \phi_m(x) \, dx = \sum_{n=1}^{\infty} c_n \int_0^1 r(x) \phi_m(x) \phi_n(x) \, dx = \sum_{n=1}^{\infty} c_n \delta_{mn}. \tag{33}$$

Hence, using the definition of δ_{mn}, we have

$$c_m = \int_0^1 r(x) f(x) \phi_m(x) \, dx = (f, r\phi_m), \qquad m = 1, 2, \ldots. \tag{34}$$

The coefficients in the series (32) have thus been formally determined. Equation (34) has the same structure as the Euler–Fourier formulas for the coefficients in a Fourier series, and the eigenfunction series (32) also has convergence properties similar to those of Fourier series. The following theorem is analogous to Theorem 10.3.1.

Theorem 11.2.4 Let $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ be the normalized eigenfunctions of the Sturm–Liouville problem (1), (2):

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0,$$

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0.$$

Let f and f' be piecewise continuous on $0 \le x \le 1$. Then the series (32) whose coefficients c_m are given by Eq. (34) converges to $[f(x+) + f(x-)]/2$ at each point in the open interval $0 < x < 1$.

If f satisfies further conditions, then a stronger conclusion can be established. Suppose that, in addition to the hypotheses of Theorem 11.2.4, the function f is continuous on $0 \le x \le 1$. If $a_2 = 0$ in the first of Eqs. (2) [so that $\phi_n(0) = 0$], then assume that $f(0) = 0$. Similarly, if $b_2 = 0$ in the second of Eqs. (2), assume that $f(1) = 0$. Otherwise no boundary conditions need be prescribed for f. Then the series (32) converges to $f(x)$ at each point in the closed interval $0 \le x \le 1$.

EXAMPLE 3 Expand the function

$$f(x) = x, \qquad 0 \le x \le 1 \tag{35}$$

in terms of the normalized eigenfunctions $\phi_n(x)$ of the problem (25).

In Example 2 we found the normalized eigenfunctions to be

$$\phi_n(x) = k_n \sin \sqrt{\lambda_n}\, x, \tag{36}$$

where k_n is given by Eq. (28) and λ_n satisfies Eq. (26). To find the expansion for f in terms of the eigenfunctions ϕ_n we write

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{37}$$

where the coefficients are given by Eq. (34). Thus

$$c_n = \int_0^1 f(x)\phi_n(x)\, dx = k_n \int_0^1 x \sin \sqrt{\lambda_n}\, x\, dx.$$

Integrating by parts, we obtain

$$c_n = k_n \left(\frac{\sin \sqrt{\lambda_n}}{\lambda_n} - \frac{\cos \sqrt{\lambda_n}}{\sqrt{\lambda_n}} \right) = k_n \frac{2 \sin \sqrt{\lambda_n}}{\lambda_n},$$

where we have used Eq. (26) in the last step. Upon substituting for k_n from Eq. (28) we obtain

$$c_n = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n(1+\cos^2\sqrt{\lambda_n})^{1/2}}. \tag{38}$$

Thus

$$f(x) = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\,x}{\lambda_n(1+\cos^2\sqrt{\lambda_n})}. \tag{39}$$

Observe that although the right side of Eq. (39) is a series of sines, it is not included in the discussion of Fourier sine series in Section 10.4.

Self-adjoint Problems. Sturm–Liouville boundary value problems are of great importance in their own right, but they can also be viewed as belonging to a much more extensive class of problems that have many of the same properties. For example, there are many similarities between Sturm–Liouville problems and the algebraic system

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{40}$$

where the $n \times n$ matrix $\mathbf{A}$ is real symmetric or Hermitian. Comparing the results mentioned in Section 7.3 with those of this section, we note that in both cases the eigenvalues are real and the eigenfunctions or eigenvectors form an orthogonal set. Further, the eigenfunctions or eigenvectors can be used as the basis for expressing an essentially arbitrary function or vector, respectively, as a sum. The most important difference is that a matrix has only a finite number of eigenvalues and eigenvectors, while a Sturm–Liouville problem has infinitely many. It is interesting and of fundamental importance in mathematics that these seemingly different problems—the matrix problem (40) and the Sturm–Liouville problem (1), (2)—which arise in different ways, are actually part of a single underlying theory. This theory is usually referred to as linear operator theory, and is part of the subject of functional analysis.

We now point out some ways in which Sturm–Liouville problems can be generalized, while still preserving the main results of Theorems 11.2.1 to 11.2.4—the existence of a sequence of real eigenvalues tending to infinity, the orthogonality of the eigenfunctions, and the possibility of expressing an arbitrary function as a series of eigenfunctions. In making these generalizations it is essential that the crucial relation (8) remain valid.

Let us consider the boundary value problem consisting of the differential equation

$$L[y] = \lambda r(x)y, \qquad 0 < x < 1, \tag{41}$$

where

$$L[y] = P_n(x)\frac{d^n y}{dx^n} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y, \tag{42}$$

and n linear homogeneous boundary conditions at the endpoints. If Eq. (8) is valid for every pair of sufficiently differentiable functions that satisfy the boundary conditions, then the given problem is said to be **self-adjoint**. It is important to observe that Eq. (8) involves restrictions on both the differential equation and the boundary conditions. The differential operator L must be such that the same operator appears in both terms of

Eq (8). This requires that L be of even order. Further, a second order operator must have the form (3), a fourth order operator must have the form

$$L[y] = [p(x)y'']'' - [q(x)y']' + s(x)y, \tag{43}$$

and higher order operators must have an analogous structure. In addition, the boundary conditions must be such as to eliminate the boundary terms that arise during the integration by parts used in deriving Eq. (8). For example, in a second order problem this is true for the separated boundary conditions (2) and also in certain other cases, one of which is given in Example 4 below.

Let us suppose that we have a self-adjoint boundary value problem for Eq. (41), where $L[y]$ is given now by Eq. (43). We assume that p, q, r, and s are continuous on $0 \le x \le 1$, and that the derivatives of p and q indicated in Eq. (43) are also continuous. If in addition $p(x) > 0$ and $r(x) > 0$ for $0 \le x \le 1$, then there is an infinite sequence of real eigenvalues tending to $+\infty$, the eigenfunctions are orthogonal with respect to the weight function r, and an arbitrary function can be expressed as a series of eigenfunctions. However, the eigenfunctions may not be simple in these more general problems.

We turn now to the relation between Sturm–Liouville problems and Fourier series. We have noted previously that Fourier sine and cosine series can be obtained by using the eigenfunctions of certain Sturm–Liouville problems involving the differential equation $y'' + \lambda y = 0$. This raises the question of whether we can obtain a full Fourier series, including both sine and cosine terms, by choosing a suitable set of boundary conditions. The answer is provided by the following example, which also serves to illustrate the occurrence of nonseparated boundary conditions.

EXAMPLE 4

Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \tag{44}$$

$$y(-L) - y(L) = 0, \qquad y'(-L) - y'(L) = 0. \tag{45}$$

This is not a Sturm–Liouville problem because the boundary conditions are not separated. The boundary conditions (45) are called **periodic boundary conditions** since they require that y and y' assume the same values at $x = L$ as at $x = -L$. Nevertheless, it is straightforward to show that the problem (44), (45) is self-adjoint. A simple calculation establishes that $\lambda_0 = 0$ is an eigenvalue and that the corresponding eigenfunction is $\phi_0(x) = 1$. Further, there are additional eigenvalues $\lambda_1 = (\pi/L)^2$, $\lambda_2 = (2\pi/L)^2, \ldots, \lambda_n = (n\pi/L)^2, \ldots$. To each of these nonzero eigenvalues there correspond *two* linearly independent eigenfunctions; for example, corresponding to λ_n are the two eigenfunctions $\phi_n(x) = \cos(n\pi x/L)$ and $\psi_n(x) = \sin(n\pi x/L)$. This illustrates that the eigenvalues may not be simple when the boundary conditions are not separated. Further, if we seek to expand a given function f of period $2L$ in a series of eigenfunctions of the problem (44), (45), we obtain the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right),$$

which is just the Fourier series for f.

We will not give further consideration to problems that have nonseparated boundary conditions, nor will we deal with problems of higher than second order, except in a few problems. There is, however, one other kind of generalization that we do wish to discuss. That is the case in which the coefficients p, q, and r in Eq. (1) do not quite satisfy the rather strict continuity and positivity requirements laid down at the beginning of this section. Such problems are called singular Sturm–Liouville problems, and are the subject of Section 11.4.

PROBLEMS

In each of Problems 1 through 5 determine the normalized eigenfunctions of the given problem.

1. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$
2. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$
3. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0$
4. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) + y(1) = 0;$ see Section 11.1, Problem 8.
5. $y'' - 2y' + (1+\lambda)y = 0, \quad y(0) = 0, \quad y(1) = 0;$ see Section 11.1, Problem 17.

In each of Problems 6 through 9 find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n \phi_n(x)$ of the given function, using the normalized eigenfunctions of Problem 1.

6. $f(x) = 1, \quad 0 \le x \le 1$
7. $f(x) = x, \quad 0 \le x \le 1$
8. $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$
9. $f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$

In each of Problems 10 through 13 find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n \phi_n(x)$ of the given function, using the normalized eigenfunctions of Problem 4.

10. $f(x) = 1, \quad 0 \le x \le 1$
11. $f(x) = x, \quad 0 \le x \le 1$
12. $f(x) = 1 - x, \quad 0 \le x \le 1$
13. $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$

In each of Problems 14 through 18 determine whether the given boundary value problem is self-adjoint.

14. $y'' + y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 0$
15. $(1+x^2)y'' + 2xy' + y = 0, \quad y'(0) = 0, \quad y(1) + 2y'(1) = 0$
16. $y'' + y = \lambda y, \quad y(0) - y'(1) = 0, \quad y'(0) - y(1) = 0$
17. $(1+x^2)y'' + 2xy' + y = \lambda(1+x^2)y, \quad y(0) - y'(1) = 0, \quad y'(0) + 2y(1) = 0$
18. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$
19. Show that if the functions u and v satisfy Eqs. (2), and either $a_2 = 0$ or $b_2 = 0$, or both, then
$$p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 = 0.$$
20. In this problem we outline a proof of the first part of Theorem 11.2.3: that the eigenvalues of the Sturm–Liouville problem (1), (2) are simple. For a given λ suppose that ϕ_1 and ϕ_2 are two linearly independent eigenfunctions. Compute the Wronskian $W(\phi_1, \phi_2)(x)$ and use the boundary conditions (2) to show that $W(\phi_1, \phi_2)(0) = 0$. Then use Theorems 3.3.2 and 3.3.3 to conclude that ϕ_1 and ϕ_2 cannot be linearly independent as assumed.
21. Consider the Sturm–Liouville problem
$$-[p(x)y']' + q(x)y = \lambda r(x)y,$$
$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0,$$

where p, q, and r satisfy the conditions stated in the text.
(a) Show that if λ is an eigenvalue and ϕ a corresponding eigenfunction, then

$$\lambda \int_0^1 r\phi^2\,dx = \int_0^1 (p\phi'^2 + q\phi^2)\,dx + \frac{b_1}{b_2}p(1)\phi^2(1) - \frac{a_1}{a_2}p(0)\phi^2(0),$$

provided that $a_2 \neq 0$ and $b_2 \neq 0$. How must this result be modified if $a_2 = 0$ or $b_2 = 0$?
(b) Show that if $q(x) \geq 0$ and if b_1/b_2 and $-a_1/a_2$ are nonnegative, then the eigenvalue λ is nonnegative.
(c) Under the conditions of part (b) show that the eigenvalue λ is strictly positive unless $q(x) = 0$ for each x in $0 \leq x \leq 1$ and also $a_1 = b_1 = 0$.

22. Derive Eq. (8) using the inner product (9) and assuming that u and v are complex-valued functions.
Hint: Consider the quantity $\int_0^1 L[u]\overline{v}\,dx$, split u and v into real and imaginary parts, and proceed as in the text.

23. In this problem we indicate a proof that the eigenfunctions of the Sturm–Liouville problem (1), (2) are real.
(a) Let λ be an eigenvalue and ϕ a corresponding eigenfunction. Let $\phi(x) = U(x) + iV(x)$, and show that U and V are also eigenfunctions corresponding to λ.
(b) Using Theorem 11.2.3, or the result of Problem 20, show that U and V are linearly dependent.
(c) Show that ϕ must be real, apart from an arbitrary multiplicative constant that may be complex.

24. Consider the problem

$$x^2y'' = \lambda(xy' - y), \qquad y(1) = 0, \quad y(2) = 0.$$

Note that λ appears as a coefficient of y' as well as of y itself. It is possible to extend the definition of self-adjointness to this type of problem, and to show that this particular problem is not self-adjoint. Show that the problem has eigenvalues, but that none of them is real. This illustrates that in general nonself-adjoint problems may have eigenvalues that are not real.

Buckling of an Elastic Column. In an investigation of the buckling of a uniform elastic column of length L by an axial load P (Figure 11.2.1*a*) one is led to the differential equation

$$y^{\text{iv}} + \lambda y'' = 0, \qquad 0 < x < L. \tag{i}$$

The parameter λ is equal to P/EI, where E is Young's modulus and I is the moment of inertia of the cross section about an axis through the centroid perpendicular to the xy-plane. The boundary conditions at $x = 0$ and $x = L$ depend on how the ends of the column are supported. Typical boundary conditions are

$$y = y' = 0, \quad \text{clamped end;}$$
$$y = y'' = 0, \quad \text{simply supported (hinged) end.}$$

The bar shown in Figure 11.2.1*a* is simply supported at $x = 0$ and clamped at $x = L$. It is desired to determine the eigenvalues and eigenfunctions of Eq. (i) subject to suitable boundary conditions. In particular, the smallest eigenvalue λ_1 gives the load at which the column buckles, or can assume a curved equilibrium position, as shown in Figure 11.2.1*b*. The corresponding eigenfunction describes the configuration of the buckled column. Note that the differential equation (i) does not fall within the theory discussed in this section. It is possible to show, however, that in each of the cases given here all the eigenvalues are real and positive. Problems 25 and 26 deal with column buckling problems.

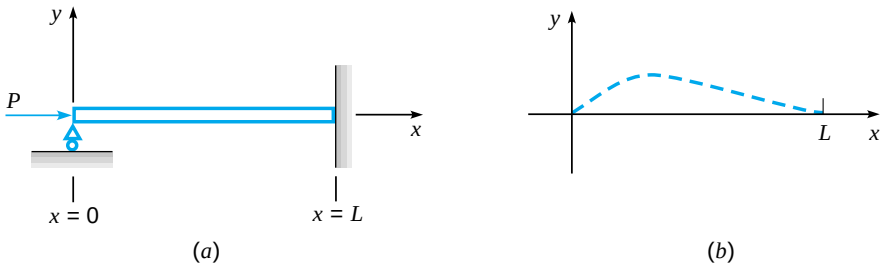


FIGURE 11.2.1 (*a*) A column under compression. (*b*) Shape of the buckled column.

25. For each of the following boundary conditions find the smallest eigenvalue (the buckling load) of $y^{iv} + \lambda y'' = 0$, and also find the corresponding eigenfunction (the shape of the buckled column).
    (a) $y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0$
    (b) $y(0) = y''(0) = 0, \quad y(L) = y'(L) = 0$
    (c) $y(0) = y'(0) = 0, \quad y(L) = y'(L) = 0$
26. In some buckling problems the eigenvalue parameter appears in the boundary conditions as well as in the differential equation. One such case occurs when one end of the column is clamped and the other end is free. In this case the differential equation $y^{iv} + \lambda y'' = 0$ must be solved subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(L) = 0, \qquad y'''(L) + \lambda y'(L) = 0.$$

Find the smallest eigenvalue and the corresponding eigenfunction.

11.3 Nonhomogeneous Boundary Value Problems

In this section we discuss how to solve nonhomogeneous boundary value problems, both for ordinary and partial differential equations. Most of our attention is directed toward problems in which the differential equation alone is nonhomogeneous, while the boundary conditions are homogeneous. We assume that the solution can be expanded in a series of eigenfunctions of a related homogeneous problem, and then determine the coefficients in this series so that the nonhomogeneous problem is satisfied. We first describe this method as it applies to boundary value problems for second order linear ordinary differential equations. Later we illustrate its use for partial differential equations by solving a heat conduction problem in a bar with variable material properties and in the presence of source terms.

Nonhomogeneous Sturm-Liouville Problems. Consider the boundary value problem consisting of the nonhomogeneous differential equation

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x), \tag{1}$$

where μ is a given constant and f is a given function on $0 \le x \le 1$, and the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0. \tag{2}$$

As in Section 11.2 we assume that p, p', q, and r are continuous on $0 \le x \le 1$ and that $p(x) > 0$ and $r(x) > 0$ there. We will solve the problem (1), (2) by making use of the eigenfunctions of the corresponding homogeneous problem consisting of the differential equation

$$L[y] = \lambda r(x) y \tag{3}$$

and the boundary conditions (2). Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ be the eigenvalues of this problem, and let $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ be the corresponding normalized eigenfunctions.

We now assume that the solution $y = \phi(x)$ of the nonhomogeneous problem (1), (2) can be expressed as a series of the form

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \tag{4}$$

From Eq. (34) of Section 11.2 we know that

$$b_n = \int_0^1 r(x)\phi(x)\phi_n(x)\, dx, \qquad n = 1, 2, \ldots. \tag{5}$$

However, since we do not know $\phi(x)$, we cannot use Eq. (5) to calculate b_n. Instead, we will try to determine b_n so that the problem (1), (2) is satisfied, and then use Eq. (4) to find $\phi(x)$. Note first that ϕ as given by Eq. (4) always satisfies the boundary conditions (2) since each ϕ_n does.

Now consider the differential equation that ϕ must satisfy. This is just Eq. (1) with y replaced by ϕ:

$$L[\phi](x) = \mu r(x)\phi(x) + f(x). \tag{6}$$

We substitute the series (4) into the differential equation (6) and attempt to determine b_n so that the differential equation is satisfied. The term on the left side of Eq. (6) becomes

$$\begin{aligned} L[\phi](x) &= L\left[\sum_{n=1}^{\infty} b_n \phi_n\right](x) = \sum_{n=1}^{\infty} b_n L[\phi_n](x) \\ &= \sum_{n=1}^{\infty} b_n \lambda_n r(x)\phi_n(x), \end{aligned} \tag{7}$$

where we have assumed that we can interchange the operations of summation and differentiation.

Note that the function r appears in Eq. (7) and also in the term $\mu r(x)\phi(x)$ in Eq. (6). This suggests that we rewrite the nonhomogeneous term in Eq. (6) as $r(x)[f(x)/r(x)]$ so that $r(x)$ also appears as a multiplier in this term. If the function f/r satisfies the conditions of Theorem 11.2.4, then

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{8}$$

where, using Eq. (5) with ϕ replaced by f/r,

$$c_n = \int_0^1 r(x)\frac{f(x)}{r(x)}\phi_n(x)\, dx = \int_0^1 f(x)\phi_n(x)\, dx, \qquad n = 1, 2, \ldots. \tag{9}$$

Upon substituting for $\phi(x)$, $L[\phi](x)$, and $f(x)$ in Eq. (6) from Eqs. (4), (7), and (8), respectively, we find that

$$\sum_{n=1}^{\infty} b_n \lambda_n r(x)\phi_n(x) = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + r(x) \sum_{n=1}^{\infty} c_n \phi_n(x).$$

After collecting terms and canceling the common nonzero factor $r(x)$ we have

$$\sum_{n=1}^{\infty} \big[(\lambda_n - \mu)b_n - c_n\big]\phi_n(x) = 0. \tag{10}$$

If Eq. (10) is to hold for each x in the interval $0 \le x \le 1$, then the coefficient of $\phi_n(x)$ must be zero for each n; see Problem 14 for a proof of this fact. Hence

$$(\lambda_n - \mu)b_n - c_n = 0, \qquad n = 1, 2, \ldots. \tag{11}$$

We must now distinguish two main cases, one of which also has two subcases.

First suppose that $\mu \neq \lambda_n$ for $n = 1, 2, 3, \ldots$; that is, μ is not equal to any eigenvalue of the corresponding homogeneous problem. Then

$$b_n = \frac{c_n}{\lambda_n - \mu}, \qquad n = 1, 2, 3, \ldots, \tag{12}$$

and

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x). \tag{13}$$

Equation (13), with c_n given by Eq. (9), is a formal solution of the nonhomogeneous boundary value problem (1), (2). Our argument does not prove that the series (13) converges. However, any solution of the boundary value problem (1), (2) clearly satisfies the conditions of Theorem 11.2.4; indeed, it satisfies the more stringent conditions given in the paragraph following that theorem. Thus it is reasonable to expect that the series (13) does converge at each point, and this fact can be established provided, for example, that f is continuous.

Now suppose that μ is equal to one of the eigenvalues of the corresponding homogeneous problem, say, $\mu = \lambda_m$; then the situation is quite different. In this event, for $n = m$ Eq. (11) has the form $0 \cdot b_m - c_m = 0$. Again we must consider two cases.

If $\mu = \lambda_m$ and $c_m \neq 0$, then it is impossible to solve Eq. (11) for b_m, and the nonhomogeneous problem (1), (2) has no solution.

If $\mu = \lambda_m$ and $c_m = 0$, then Eq. (11) is satisfied regardless of the value of b_m; in other words, b_m remains arbitrary. In this case the boundary value problem (1), (2) does have a solution, but it is not unique, since it contains an arbitrary multiple of the eigenfunction ϕ_m.

Since c_m is given by Eq. (9), the condition $c_m = 0$ means that

$$\int_0^1 f(x)\phi_m(x)\,dx = 0. \tag{14}$$

Thus, if $\mu = \lambda_m$, the nonhomogeneous boundary value problem (1), (2) can be solved only if f is orthogonal to the eigenfunction corresponding to the eigenvalue λ_m.

The results we have formally obtained are summarized in the following theorem.

Theorem 11.3.1 The nonhomogeneous boundary value problem (1), (2) has a unique solution for each continuous f whenever μ is different from all the eigenvalues of the corresponding homogeneous problem; the solution is given by Eq. (13), and the series converges for each x in $0 \le x \le 1$. If μ is equal to an eigenvalue λ_m of the corresponding homogeneous problem, then the nonhomogeneous boundary value problem has no solution unless f is orthogonal to ϕ_m; that is, unless the condition (14) holds. In that case, the solution is not unique and contains an arbitrary multiple of $\phi_m(x)$.

The main part of Theorem 11.3.1 is sometimes stated in the following way:

Theorem 11.3.2 For a given value of μ, either the nonhomogeneous problem (1), (2) has a unique solution for each continuous f (if μ is not equal to any eigenvalue λ_m of the corresponding homogeneous problem), or else the homogeneous problem (3), (2) has a nontrivial solution (the eigenfunction corresponding to λ_m).

This latter form of the theorem is known as the Fredholm[4] alternative theorem. This is one of the basic theorems of mathematical analysis and occurs in many different contexts. You may be familiar with it in connection with sets of linear algebraic equations where the vanishing or nonvanishing of the determinant of coefficients replaces the statements about μ and λ_m. See the discussion in Section 7.3.

EXAMPLE 1

Solve the boundary value problem

$$y'' + 2y = -x, \tag{15}$$

$$y(0) = 0, \qquad y(1) + y'(1) = 0. \tag{16}$$

This particular problem can be solved directly in an elementary way and has the solution

$$y = \frac{\sin\sqrt{2}\,x}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}} - \frac{x}{2}. \tag{17}$$

The method of solution described below illustrates the use of eigenfunction expansions, a method that can be employed in many problems not accessible by elementary procedures. To identify Eq. (15) with Eq. (1) it is helpful to write the former as

$$-y'' = 2y + x. \tag{18}$$

We seek the solution of the given problem as a series of normalized eigenfunctions ϕ_n of the corresponding homogeneous problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) + y'(1) = 0. \tag{19}$$

[4]The Swedish mathematician Erik Ivar Fredholm (1866–1927), professor at the University of Stockholm, established the modern theory of integral equations in a fundamental paper in 1903. Fredholm's work emphasized the similarities between integral equations and systems of linear algebraic equations. There are also many interrelations between differential and integral equations; for example, see Section 2.8 and Problem 21 of Section 6.6.

These eigenfunctions were found in Example 2 of Section 11.2, and are

$$\phi_n(x) = k_n \sin\sqrt{\lambda_n}\, x, \tag{20}$$

where

$$k_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2} \tag{21}$$

and λ_n satisfies

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0. \tag{22}$$

Recall that in Example 1 of Section 11.1 we found that

$$\lambda_1 \cong 4.116, \qquad \lambda_2 \cong 24.14,$$
$$\lambda_3 \cong 63.66, \qquad \lambda_n \cong (2n-1)^2\pi^2/4 \qquad \text{for } n = 4, 5, \ldots.$$

We assume that y is given by Eq. (4),

$$y = \sum_{n=1}^{\infty} b_n \phi_n(x),$$

and it follows that the coefficients b_n are found from Eq. (12),

$$b_n = \frac{c_n}{\lambda_n - 2},$$

where the c_n are the expansion coefficients of the nonhomogeneous term $f(x) = x$ in Eq. (18) in terms of the eigenfunctions ϕ_n. These coefficients were found in Example 3 of Section 11.2, and are

$$c_n = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n(1+\cos^2\sqrt{\lambda_n})^{1/2}}. \tag{23}$$

Putting everything together, we finally obtain the solution

$$y = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}}{\lambda_n(\lambda_n - 2)(1+\cos^2\sqrt{\lambda_n})} \sin\sqrt{\lambda_n}\, x. \tag{24}$$

While Eqs. (17) and (24) are quite different in appearance, they are actually two different expressions for the same function. This follows from the uniqueness part of Theorem 11.3.1 or 11.3.2 since $\lambda = 2$ is not an eigenvalue of the homogeneous problem (19). Alternatively, one can show the equivalence of Eqs. (17) and (24) by expanding the right side of Eq. (17) in terms of the eigenfunctions $\phi_n(x)$. For this problem it is fairly obvious that the formula (17) is more convenient than the series formula (24). However, we emphasize again that in other problems we may not be able to obtain the solution except by series (or numerical) methods.

Nonhomogeneous Heat Conduction Problems. To show how eigenfunction expansions can be used to solve nonhomogeneous problems for partial differential equations, let us consider the generalized heat conduction equation

$$r(x)u_t = [p(x)u_x]_x - q(x)u + F(x,t) \tag{25}$$

with the boundary conditions

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_2 u(1,t) = 0 \tag{26}$$

and the initial condition

$$u(x,0) = f(x). \tag{27}$$

This problem was previously discussed in Appendix A of Chapter 10 and in Section 11.1. In the latter section we let $u(x,t) = X(x)T(t)$ in the homogeneous equation obtained by setting $F(x,t) = 0$, and showed that $X(x)$ must be a solution of the boundary value problem

$$-[p(x)X']' + q(x)X = \lambda r(x)X, \tag{28}$$

$$X'(0) - h_1 X(0) = 0, \qquad X'(1) + h_2 X(1) = 0. \tag{29}$$

If we assume that p, q, and r satisfy the proper continuity requirements and that $p(x)$ and $r(x)$ are always positive, the problem (28), (29) is a Sturm–Liouville problem as discussed in Section 11.2. Thus we obtain a sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ and corresponding normalized eigenfunctions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$.

We will solve the given nonhomogeneous boundary value problem (25) to (27) by assuming that $u(x,t)$ can be expressed as a series of eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{30}$$

and then showing how to determine the coefficients $b_n(t)$. The procedure is basically the same as that used in the problem (1), (2) considered earlier, although it is more complicated in certain respects. For instance, the coefficients b_n must now depend on t, because otherwise u would be a function of x only. Note that the boundary conditions (26) are automatically satisfied by an expression of the form (30) because each $\phi_n(x)$ satisfies the boundary conditions (29).

Next we substitute from Eq. (30) for u in Eq. (25). From the first two terms on the right side of Eq. (25) we formally obtain

$$\begin{aligned}[p(x)u_x]_x - q(x)u &= \frac{\partial}{\partial x}\left[p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n'(x)\right] - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} b_n(t)\{[p(x)\phi_n'(x)]' - q(x)\phi_n(x)\}. \end{aligned} \tag{31}$$

Since $[p(x)\phi_n'(x)]' - q(x)\phi_n(x) = -\lambda_n r(x)\phi_n(x)$, we obtain finally

$$[p(x)u_x]_x - q(x)u = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x). \tag{32}$$

Now consider the term on the left side of Eq. (25). We have

$$\begin{aligned} r(x)u_t &= r(x)\frac{\partial}{\partial t}\sum_{n=1}^{\infty} b_n(t)\phi_n(x) \\ &= r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x). \end{aligned} \tag{33}$$

We must also express the nonhomogeneous term in Eq. (25) as a series of eigenfunctions. Once again, it is convenient to look at the ratio $F(x,t)/r(x)$ and to write

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x), \tag{34}$$

where the coefficients are given by

$$\begin{aligned} \gamma_n(t) &= \int_0^1 r(x)\frac{F(x,t)}{r(x)}\phi_n(x)\,dx \\ &= \int_0^1 F(x,t)\phi_n(x)\,dx, \qquad n = 1, 2, \ldots. \end{aligned} \tag{35}$$

Since $F(x,t)$ is given, we can consider the functions $\gamma_n(t)$ to be known.

Gathering all these results together, we substitute from Eqs. (32), (33), and (34) in Eq. (25), and find that

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) + r(x)\sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x). \tag{36}$$

To simplify Eq. (36) we cancel the common nonzero factor $r(x)$ from all terms, and write everything in one summation:

$$\sum_{n=1}^{\infty}\left[b_n'(t) + \lambda_n b_n(t) - \gamma_n(t)\right]\phi_n(x) = 0. \tag{37}$$

Once again, if Eq. (37) is to hold for all x in $0 < x < 1$, it is necessary that the quantity in square brackets be zero for each n (again see Problem 14). Hence $b_n(t)$ is a solution of the first order linear ordinary differential equation

$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t), \qquad n = 1, 2, \ldots, \tag{38}$$

where $\gamma_n(t)$ is given by Eq. (35). To determine $b_n(t)$ completely we must have an initial condition

$$b_n(0) = \alpha_n, \qquad n = 1, 2, \ldots \tag{39}$$

for Eq. (38). This we obtain from the initial condition (27). Setting $t = 0$ in Eq. (30) and using Eq. (27), we have

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = \sum_{n=1}^{\infty}\alpha_n\phi_n(x) = f(x). \tag{40}$$

Thus the initial values α_n are the coefficients in the eigenfunction expansion for $f(x)$. Therefore,

$$\alpha_n = \int_0^1 r(x) f(x) \phi_n(x)\, dx, \qquad n = 1, 2, \ldots. \tag{41}$$

Note that everything on the right side of Eq. (41) is known, so we can consider α_n as known.

The initial value problem (38), (39) is solved by the methods of Section 2.1. The integrating factor is $\mu(t) = \exp(\lambda_n t)$, and it follows that

$$b_n(t) = \alpha_n e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-s)} \gamma_n(s)\, ds, \qquad n = 1, 2, \ldots. \tag{42}$$

The details of this calculation are left to you. Note that the first term on the right side of Eq. (42) depends on the function f through the coefficients α_n, while the second depends on the nonhomogeneous term F through the coefficients $\gamma_n(s)$.

Thus an explicit solution of the boundary value problem (25) to (27) is given by Eq. (30),

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x),$$

where the coefficients $b_n(t)$ are determined from Eq. (42). The quantities α_n and $\gamma_n(s)$ in Eq. (42) are found in turn from Eqs. (41) and (35), respectively.

Summarizing, to use this method to solve a boundary value problem such as that given by Eqs. (25) to (27) we must

1. Find the eigenvalues λ_n and the normalized eigenfunctions ϕ_n of the homogeneous problem (28), (29).
2. Calculate the coefficients α_n and $\gamma_n(t)$ from Eqs. (41) and (35), respectively.
3. Evaluate the integral in Eq. (42) to determine $b_n(t)$.
4. Sum the infinite series (30).

Since any or all of these steps may be difficult, the entire process can be quite formidable. One redeeming feature is that often the series (30) converges rapidly, in which case only a very few terms may be needed to obtain an adequate approximation to the solution.

EXAMPLE 2

Find the solution of the heat conduction problem

$$u_t = u_{xx} + xe^{-t}, \tag{43}$$

$$u(0, t) = 0, \qquad u_x(1, t) + u(1, t) = 0, \tag{44}$$

$$u(x, 0) = 0. \tag{45}$$

Again we use the normalized eigenfunctions ϕ_n of the problem (19), and assume that u is given by Eq. (30),

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x).$$

The coefficients b_n are determined from the differential equation

$$b_n' + \lambda_n b_n = \gamma_n(t), \tag{46}$$

where λ_n is the nth eigenvalue of the problem (19) and $\gamma_n(t)$ is the nth expansion coefficient of the nonhomogeneous term xe^{-t} in terms of the eigenfunctions ϕ_n. Thus we have

$$\begin{aligned}\gamma_n(t) &= \int_0^1 xe^{-t}\phi_n(x)\,dx = e^{-t}\int_0^1 x\phi_n(x)\,dx \\ &= c_n e^{-t},\end{aligned} \tag{47}$$

where $c_n = \int_0^1 x\phi_n(x)\,dx$ is given by Eq. (23). The initial condition for Eq. (46) is

$$b_n(0) = 0 \tag{48}$$

since the initial temperature distribution (45) is zero everywhere. The solution of the initial value problem (46), (48) is

$$\begin{aligned}b_n(t) &= e^{-\lambda_n t}\int_0^t e^{\lambda_n s}c_n e^{-s}\,ds = c_n e^{-\lambda_n t}\,\frac{e^{(\lambda_n - 1)t} - 1}{\lambda_n - 1} \\ &= \frac{c_n}{\lambda_n - 1}(e^{-t} - e^{-\lambda_n t}).\end{aligned} \tag{49}$$

Thus the solution of the heat conduction problem (43) to (45) is given by

$$u(x,t) = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\,(e^{-t} - e^{-\lambda_n t})\sin\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 1)(1 + \cos^2\sqrt{\lambda_n})}. \tag{50}$$

The solution given by Eq. (50) is exact, but complicated. To judge whether a satisfactory approximation to the solution can be obtained by using only a few terms in this series, we must estimate its speed of convergence. First we split the right side of Eq. (50) into two parts:

$$u(x,t) = 4e^{-t}\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\,\sin\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 1)(1 + \cos^2\sqrt{\lambda_n})} - 4\sum_{n=1}^{\infty} \frac{e^{-\lambda_n t}\sin\sqrt{\lambda_n}\,\sin\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 1)(1 + \cos^2\sqrt{\lambda_n})}. \tag{51}$$

Recall from Example 1 in Section 11.1 that the eigenvalues λ_n are very nearly proportional to n^2. In the first series on the right side of Eq. (51) the trigonometric factors are all bounded as $n \to \infty$; therefore, this series converges similarly to the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ or $\sum_{n=1}^{\infty} n^{-4}$. Hence at most two or three terms are required to obtain an excellent approximation to this part of the solution. The second series contains the additional factor $e^{-\lambda_n t}$, so its convergence is even more rapid for $t > 0$; all terms after the first are almost surely negligible.

Further Discussion. Eigenfunction expansions can be used to solve a much greater variety of problems than the preceding discussion and examples may suggest. The following brief remarks are intended to indicate to some extent the scope of this method:

1. **Nonhomogeneous Time-Independent Boundary Conditions.** Suppose that the boundary conditions (26) in the heat conduction problem are replaced by

$$u(0, t) = T_1, \qquad u(1, t) = T_2. \tag{52}$$

Then we can proceed much as in Section 10.6 to reduce the problem to one with homogeneous boundary conditions; that is, we subtract from u a function v that is chosen to satisfy Eqs. (52). Then the difference $w = u - v$ satisfies a problem with homogeneous boundary conditions, but with modified forcing term and initial condition. This problem can be solved by the procedure described in this section.

2. **Nonhomogeneous Time-Dependent Boundary Conditions.** Suppose that the boundary conditions (26) are replaced by

$$u(0, t) = T_1(t), \qquad u(1, t) = T_2(t). \tag{53}$$

If T_1 and T_2 are differentiable functions, one can proceed just as in paragraph 1. In the same way it is possible to deal with more general boundary conditions involving both u and u_x. If T_1 and T_2 are not differentiable, however, then the method fails. A further refinement of the method of eigenfunction expansions can be used to cope with this type of problem, but the procedure is more complicated and we will not discuss it.

3. **Use of Functions Other Than Eigenfunctions.** One potential difficulty in using eigenfunction expansions is that the normalized eigenfunctions of the corresponding homogeneous problem must be found. For a differential equation with variable coefficients this may be difficult, if not impossible. In such a case it is sometimes possible to use other functions, such as eigenfunctions of a simpler problem, that satisfy the same boundary conditions. For instance, if the boundary conditions are

$$u(0, t) = 0, \qquad u(1, t) = 0, \tag{54}$$

then it may be convenient to replace the functions $\phi_n(x)$ in Eq. (30) by $\sin n\pi x$. These functions at least satisfy the correct boundary conditions, although in general they are not solutions of the corresponding homogeneous differential equation. Next we expand the nonhomogeneous term $F(x, t)$ in a series of the form (34), again with $\phi_n(x)$ replaced by $\sin n\pi x$, and then substitute for both u and F in Eq. (25). Upon collecting the coefficients of $\sin n\pi x$ for each n, we have an infinite set of linear first order differential equations from which to determine $b_1(t), b_2(t), \ldots$. The essential difference between this case and the one considered earlier is that now the equations for the functions $b_n(t)$ are *coupled*. Thus they cannot be solved one by one, as before, but must be dealt with simultaneously. In practice, the infinite system is replaced by an approximating finite system, from which approximations to a finite number of coefficients are calculated. Despite its complexity, this procedure has proved very useful in solving certain types of difficult problems.

4. **Higher Order Problems.** Boundary value problems for equations of higher than second order can often be solved by eigenfunction expansions. In some cases the procedure parallels almost exactly that for second order problems. However, a variety of complications can also arise, and we will not discuss such problems in this book.

Finally, we emphasize that the discussion in this section has been purely formal. Separate and sometimes elaborate arguments must be used to establish convergence of

eigenfunction expansions, or to justify some of the steps used, such as term-by-term differentiation of eigenfunction series.

There are also other altogether different methods for solving nonhomogeneous boundary value problems. One of these leads to a solution expressed as a definite integral rather than as an infinite series. This approach involves certain functions known as Green's functions, and for ordinary differential equations is the subject of Problems 28 through 36.

PROBLEMS

In each of Problems 1 through 5 solve the given problem by means of an eigenfunction expansion.

1. $y'' + 2y = -x, \quad y(0) = 0, \quad y(1) = 0$
2. $y'' + 2y = -x, \quad y(0) = 0, \quad y'(1) = 0;$ see Section 11.2, Problem 7.
3. $y'' + 2y = -x, \quad y'(0) = 0, \quad y'(1) = 0;$ see Section 11.2, Problem 3.
4. $y'' + 2y = -x, \quad y'(0) = 0, \quad y'(1) + y(1) = 0;$ see Section 11.2, Problem 11.
5. $y'' + 2y = -1 + |1 - 2x|, \quad y(0) = 0, \quad y(1) = 0$

In each of Problems 6 through 9 determine a formal eigenfunction series expansion for the solution of the given problem. Assume that f satisfies the conditions of Theorem 11.3.1. State the values of μ for which the solution exists.

6. $y'' + \mu y = -f(x), \quad y(0) = 0, \quad y'(1) = 0$
7. $y'' + \mu y = -f(x), \quad y'(0) = 0, \quad y(1) = 0$
8. $y'' + \mu y = -f(x), \quad y'(0) = 0, \quad y'(1) = 0$
9. $y'' + \mu y = -f(x), \quad y'(0) = 0, \quad y'(1) + y(1) = 0$

In each of Problems 10 through 13 determine whether there is any value of the constant a for which the problem has a solution. Find the solution for each such value.

10. $y'' + \pi^2 y = a + x, \quad y(0) = 0, \quad y(1) = 0$
11. $y'' + 4\pi^2 y = a + x, \quad y(0) = 0, \quad y(1) = 0$
12. $y'' + \pi^2 y = a, \quad y'(0) = 0, \quad y'(1) = 0$
13. $y'' + \pi^2 y = a - \cos \pi x, \quad y(0) = 0, \quad y(1) = 0$
14. Let $\phi_1, \ldots, \phi_n, \ldots$ be the normalized eigenfunctions of the differential equation (3) subject to the boundary conditions (2). If $\sum_{n=1}^{\infty} c_n \phi_n(x)$ converges to $f(x)$, where $f(x) = 0$ for each x in $0 \le x \le 1$, show that $c_n = 0$ for each n.
   Hint: Multiply by $r(x)\phi_m(x)$, integrate, and use the orthogonality property of the eigenfunctions.
15. Let L be a second order linear differential operator. Show that the solution $y = \phi(x)$ of the problem

$$L[y] = f(x),$$
$$a_1 y(0) + a_2 y'(0) = \alpha, \qquad b_1 y(1) + b_2 y'(1) = \beta$$

can be written as $y = u + v$, where $u = \phi_1(x)$ and $v = \phi_2(x)$ are solutions of the problems

$$L[u] = 0,$$
$$a_1 u(0) + a_2 u'(0) = \alpha, \qquad b_1 u(1) + b_2 u'(1) = \beta$$

and

$$L[v] = f(x),$$
$$a_1 v(0) + a_2 v'(0) = 0, \qquad b_1 v(1) + b_2 v'(1) = 0,$$

respectively.

16. Show that the problem

$$y'' + \pi^2 y = \pi^2 x, \qquad y(0) = 1, \quad y(1) = 0$$

has the solution

$$y = c_1 \sin \pi x + \cos \pi x + x.$$

Also show that this solution cannot be obtained by splitting the problem as suggested in Problem 15, since neither of the two subsidiary problems can be solved in this case.

17. Consider the problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = a, \quad y(1) = b.$$

Let $y = u + v$, where v is any twice differentiable function satisfying the boundary conditions (but not necessarily the differential equation). Show that u is a solution of the problem

$$u'' + p(x)u' + q(x)u = g(x), \qquad u(0) = 0, \quad u(1) = 0,$$

where $g(x) = -[v'' + p(x)v' + q(x)v]$, and is known once v is chosen. Thus nonhomogeneities can be transferred from the boundary conditions to the differential equation. Find a function v for this problem.

18. Using the method of Problem 17, transform the problem

$$y'' + 2y = 2 - 4x, \qquad y(0) = 1, \quad y(1) + y'(1) = -2$$

into a new problem in which the boundary conditions are homogeneous. Solve the latter problem by reference to Example 1 of the text.

In each of Problems 19 through 22 use eigenfunction expansions to find the solution of the given boundary value problem.

19. $u_t = u_{xx} - x, \qquad u(0,t) = 0, \quad u_x(1,t) = 0, \quad u(x,0) = \sin(\pi x/2);$
see Problem 2.
20. $u_t = u_{xx} + e^{-t}, \qquad u_x(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad u(x,0) = 1 - x;$
see Section 11.2, Problems 10 and 12.
21. $u_t = u_{xx} + 1 - |1 - 2x|, \qquad u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = 0;$
see Problem 5.
22. $u_t = u_{xx} + e^{-t}(1 - x), \qquad u(0,t) = 0, \quad u_x(1,t) = 0, \quad u(x,0) = 0;$
see Section 11.2, Problems 6 and 7.

23. Consider the boundary value problem

$$r(x)u_t = [p(x)u_x]_x - q(x)u + F(x),$$
$$u(0,t) = T_1, \qquad u(1,t) = T_2, \qquad u(x,0) = f(x).$$

(a) Let $v(x)$ be a solution of the problem

$$[p(x)v']' - q(x)v = -F(x), \qquad v(0) = T_1, \quad v(1) = T_2.$$

If $w(x,t) = u(x,t) - v(x)$, find the boundary value problem satisfied by w. Note that this problem can be solved by the method of this section.
(b) Generalize the procedure of part (a) to the case where u satisfies the boundary conditions

$$u_x(0,t) - h_1 u(0,t) = T_1, \qquad u_x(1,t) + h_2 u(1,t) = T_2.$$

In each of Problems 24 and 25 use the method indicated in Problem 23 to solve the given boundary value problem.

24. $u_t = u_{xx} - 2$,
$u(0,t) = 1, \quad u(1,t) = 0$,
$u(x,0) = x^2 - 2x + 2$

25. $u_t = u_{xx} - \pi^2 \cos \pi x$,
$u_x(0,t) = 0, \quad u(1,t) = 1$,
$u(x,0) = \cos(3\pi x/2) - \cos \pi x$

26. The method of eigenfunction expansions is often useful for nonhomogeneous problems related to the wave equation or its generalizations. Consider the problem

$$r(x)u_{tt} = [p(x)u_x]_x - q(x)u + F(x,t), \tag{i}$$
$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_2 u(1,t) = 0, \tag{ii}$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x). \tag{iii}$$

This problem can arise in connection with generalizations of the telegraph equation (Problem 16 in Section 11.1) or the longitudinal vibrations of an elastic bar (Problem 25 in Section 11.1).

(a) Let $u(x,t) = X(x)T(t)$ in the homogeneous equation corresponding to Eq. (i) and show that $X(x)$ satisfies Eqs. (28) and (29) of the text. Let λ_n and $\phi_n(x)$ denote the eigenvalues and normalized eigenfunctions of this problem.

(b) Assume that $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$, and show that $b_n(t)$ must satisfy the initial value problem

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t), \qquad b_n(0) = \alpha_n, \quad b_n'(0) = \beta_n,$$

where α_n, β_n, and $\gamma_n(t)$ are the expansion coefficients for $f(x)$, $g(x)$, and $F(x,t)/r(x)$ in terms of the eigenfunctions $\phi_1(x), \ldots, \phi_n(x), \ldots$.

27. In this problem we explore a little further the analogy between Sturm–Liouville boundary value problems and Hermitian matrices. Let $\mathbf{A}$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthogonal eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$.

Consider the nonhomogeneous system of equations

$$\mathbf{Ax} - \mu\mathbf{x} = \mathbf{b}, \tag{i}$$

where μ is a given real number and $\mathbf{b}$ is a given vector. We will point out a way of solving Eq. (i) that is analogous to the method presented in the text for solving Eqs. (1) and (2).

(a) Show that $\mathbf{b} = \sum_{i=1}^{n} b_i \boldsymbol{\xi}^{(i)}$, where $b_i = (\mathbf{b}, \boldsymbol{\xi}^{(i)})$.

(b) Assume that $\mathbf{x} = \sum_{i=1}^{n} a_i \boldsymbol{\xi}^{(i)}$ and show that for Eq. (i) to be satisfied, it is necessary that $a_i = b_i/(\lambda_i - \mu)$. Thus

$$\mathbf{x} = \sum_{i=1}^{n} \frac{(\mathbf{b}, \boldsymbol{\xi}^{(i)})}{\lambda_i - \mu} \boldsymbol{\xi}^{(i)}, \tag{ii}$$

provided that μ is not one of the eigenvalues of $\mathbf{A}$, $\mu \neq \lambda_i$ for $i = 1, \ldots, n$. Compare this result with Eq. (13).

Green's[5] Functions. Consider the nonhomogeneous system of algebraic equations

$$\mathbf{Ax} - \mu\mathbf{x} = \mathbf{b}, \tag{i}$$

[5]Green's functions are named after George Green (1793–1841) of England. He was almost entirely self-taught in mathematics, and made significant contributions to electricity and magnetism, fluid mechanics, and partial differential equations. His most important work was an essay on electricity and magnetism that was published privately in 1828. In this paper Green was the first to recognize the importance of potential functions. He introduced the functions now known as Green's functions as a means of solving boundary value problems, and developed the integral transformation theorems of which Green's theorem in the plane is a particular case. However, these results did not become widely known until Green's essay was republished in the 1850s through the efforts of William Thomson (Lord Kelvin).

where $\mathbf{A}$ is an $n \times n$ Hermitian matrix, μ is a given real number, and $\mathbf{b}$ is a given vector. Instead of using an eigenvector expansion as in Problem 27, we can solve Eq. (i) by computing the inverse matrix $(\mathbf{A} - \mu\mathbf{I})^{-1}$, which exists if μ is not an eigenvalue of $\mathbf{A}$. Then

$$\mathbf{x} = (\mathbf{A} - \mu\mathbf{I})^{-1}\mathbf{b}. \tag{ii}$$

Problems 28 through 36 indicate a way of solving nonhomogeneous boundary value problems that is analogous to using the inverse matrix for a system of linear algebraic equations. The Green's function plays a part similar to the inverse of the matrix of coefficients. This method leads to solutions expressed as definite integrals rather than as infinite series. Except in Problem 35 we will assume that $\mu = 0$ for simplicity.

28. (a) Show by the method of variation of parameters that the general solution of the differential equation

$$-y'' = f(x)$$

can be written in the form

$$y = \phi(x) = c_1 + c_2 x - \int_0^x (x - s) f(s)\, ds,$$

where c_1 and c_2 are arbitrary constants.
(b) Let $y = \phi(x)$ also be required to satisfy the boundary conditions $y(0) = 0$, $y(1) = 0$. Show that in this case

$$c_1 = 0, \qquad c_2 = \int_0^1 (1 - s) f(s)\, ds.$$

(c) Show that, under the conditions of parts (a) and (b), $\phi(x)$ can be written in the form

$$\phi(x) = \int_0^x s(1 - x) f(s)\, ds + \int_x^1 x(1 - s) f(s)\, ds.$$

(d) Defining

$$G(x, s) = \begin{cases} s(1 - x), & 0 \le s \le x, \\ x(1 - s), & x \le s \le 1, \end{cases}$$

show that the solution takes the form

$$\phi(x) = \int_0^1 G(x, s) f(s)\, ds.$$

The function $G(x, s)$ appearing under the integral sign is a Green's function. The usefulness of a Green's function solution rests on the fact that the Green's function is independent of the nonhomogeneous term in the differential equation. Thus, once the Green's function is determined, the solution of the boundary value problem for any nonhomogeneous term $f(x)$ is obtained by a single integration. Note further that no determination of arbitrary constants is required, since $\phi(x)$ as given by the Green's function integral formula automatically satisfies the boundary conditions.

29. By a procedure similar to that in Problem 28 show that the solution of the boundary value problem

$$-(y'' + y) = f(x), \qquad y(0) = 0, \quad y(1) = 0$$

is

$$y = \phi(x) = \int_0^1 G(x, s) f(s)\, ds,$$

where

$$G(x,s)=\begin{cases}\dfrac{\sin s\,\sin(1-x)}{\sin 1}, & 0\le s\le x,\\[2ex] \dfrac{\sin x\,\sin(1-s)}{\sin 1}, & x\le s\le 1.\end{cases}$$

30. It is possible to show that the Sturm–Liouville problem

$$L[y]=-[p(x)y']'+q(x)y=f(x), \tag{i}$$

$$a_1y(0)+a_2y'(0)=0,\qquad b_1y(1)+b_2y'(1)=0 \tag{ii}$$

has a Green's function solution

$$y=\phi(x)=\int_0^1 G(x,s)\,f(s)\,ds, \tag{iii}$$

provided that $\lambda=0$ is not an eigenvalue of $L[y]=\lambda y$ subject to the boundary conditions (ii). Further, $G(x,s)$ is given by

$$G(x,s)=\begin{cases}-y_1(s)y_2(x)/p(x)W(y_1,y_2)(x), & 0\le s\le x,\\ -y_1(x)y_2(s)/p(x)W(y_1,y_2)(x), & x\le s\le 1,\end{cases} \tag{iv}$$

where y_1 is a solution of $L[y]=0$ satisfying the boundary condition at $x=0$, y_2 is a solution of $L[y]=0$ satisfying the boundary condition at $x=1$, and $W(y_1,y_2)$ is the Wronskian of y_1 and y_2.

(a) Verify that the Green's function obtained in Problem 28 is given by formula (iv).
(b) Verify that the Green's function obtained in Problem 29 is given by formula (iv).
(c) Show that $p(x)W(y_1,y_2)(x)$ is a constant, by showing that its derivative is zero.
(d) Using Eq. (iv) and the result of part (c), show that $G(x,s)=G(s,x)$.
(e) Verify that $y=\phi(x)$ from Eq. (iii) with $G(x,s)$ given by Eq. (iv) satisfies the differential equation (i) and the boundary conditions (ii).

In each of Problems 31 through 34 solve the given boundary value problem by determining the appropriate Green's function and expressing the solution as a definite integral. Use Eqs. (i) to (iv) of Problem 30.

31. $-y''=f(x),\qquad y'(0)=0,\quad y(1)=0$
32. $-y''=f(x),\qquad y(0)=0,\quad y(1)+y'(1)=0$
33. $-(y''+y)=f(x),\qquad y'(0)=0,\quad y(1)=0$
34. $-y''=f(x),\qquad y(0)=0,\quad y'(1)=0$

35. Consider the boundary value problem

$$L[y]=-[p(x)y']'+q(x)y=\mu r(x)y+f(x), \tag{i}$$

$$a_1y(0)+a_2y'(0)=0,\qquad b_1y(1)+b_2y'(1)=0. \tag{ii}$$

According to the text, the solution $y=\phi(x)$ is given by Eq. (13), where c_n is defined by Eq. (9), provided that μ is not an eigenvalue of the corresponding homogeneous problem. In this case it can also be shown that the solution is given by a Green's function integral of the form

$$y=\phi(x)=\int_0^1 G(x,s,\mu)\,f(s)\,ds. \tag{iii}$$

Note that in this problem the Green's function also depends on the parameter μ.

(a) Show that if these two expressions for $\phi(x)$ are to be equivalent, then

$$G(x, s, \mu) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(s)}{\lambda_i - \mu}, \tag{iv}$$

where λ_i and ϕ_i are the eigenvalues and eigenfunctions, respectively, of Eqs. (3), (2) of the text. Again we see from Eq. (iv) that μ cannot be equal to any eigenvalue λ_i.
(b) Derive Eq. (iv) directly by assuming that $G(x, s, \mu)$ has the eigenfunction expansion

$$G(x, s, \mu) = \sum_{i=1}^{\infty} a_i(x, \mu)\phi_i(s); \tag{v}$$

determine $a_i(x, \mu)$ by multiplying Eq. (v) by $r(s)\phi_j(s)$ and integrating with respect to s from $s = 0$ to $s = 1$.
Hint: Show first that λ_i and ϕ_i satisfy the equation

$$\phi_i(x) = (\lambda_i - \mu) \int_0^1 G(x, s, \mu) r(s)\phi_i(s)\, ds. \tag{vi}$$

36. Consider the boundary value problem

$$-d^2y/ds^2 = \delta(s - x), \qquad y(0) = 0, \quad y(1) = 0,$$

where s is the independent variable, $s = x$ is a definite point in the interval $0 < s < 1$, and δ is the Dirac delta function (see Section 6.5). Show that the solution of this problem is the Green's function $G(x, s)$ obtained in Problem 28.

In solving the given problem, note that $\delta(s - x) = 0$ in the intervals $0 \le s < x$ and $x < s \le 1$. Note further that $-dy/ds$ experiences a jump of magnitude 1 as s passes through the value x.

This problem illustrates a general property, namely, that the Green's function $G(x, s)$ can be identified as the response at the point s to a unit impulse at the point x. A more general nonhomogeneous term f on $0 \le x \le 1$ can be regarded as a continuous distribution of impulses with magnitude $f(x)$ at the point x. The solution of a nonhomogeneous boundary value problem in terms of a Green's function integral can then be interpreted as the result of superposing the responses to the set of impulses represented by the nonhomogeneous term $f(x)$.

11.4 Singular Sturm–Liouville Problems

In the preceding sections of this chapter we considered Sturm–Liouville boundary value problems: the differential equation

$$L[y] = -[p(x)y']' + q(x)y = \lambda r(x)y, \qquad 0 < x < 1, \tag{1}$$

together with boundary conditions of the form

$$a_1y(0) + a_2y'(0) = 0, \tag{2}$$
$$b_1y(1) + b_2y'(1) = 0. \tag{3}$$

Until now, we have always assumed that the problem is regular; that is, p is differentiable, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ at all points in the *closed*

interval. However, there are also equations of physical interest in which some of these conditions are not satisfied.

For example, suppose that we wish to study Bessel's equation of order ν on the interval $0 < x < 1$. This equation is sometimes written in the form[6]

$$-(xy')' + \frac{\nu^2}{x}y = \lambda xy, \tag{4}$$

so that $p(x) = x$, $q(x) = \nu^2/x$, and $r(x) = x$. Thus $p(0) = 0$, $r(0) = 0$, and $q(x)$ is unbounded and hence discontinuous as $x \to 0$. However, the conditions imposed on regular Sturm–Liouville problems are met elsewhere in the interval.

Similarly, for Legendre's equation we have

$$-[(1 - x^2)y']' = \lambda y, \qquad -1 < x < 1, \tag{5}$$

where $\lambda = \alpha(\alpha + 1)$, $p(x) = 1 - x^2$, $q(x) = 0$, and $r(x) = 1$. Here the required conditions on p, q, and r are satisfied in the interval $0 \le x \le 1$ except at $x = 1$ where p is zero.

We use the term singular Sturm–Liouville problem to refer to a certain class of boundary value problems for the differential equation (1) in which the functions p, q, and r satisfy the conditions stated earlier on the open interval $0 < x < 1$, but at least one of these functions fails to satisfy them at one or both of the boundary points. We also prescribe suitable separated boundary conditions, of a kind to be described in more detail later in this section. Singular problems also occur if the interval is unbounded, for example, $0 \le x < \infty$. We do not consider this latter kind of singular problem in this book.

As an example of a singular problem on a finite interval, consider the equation

$$xy'' + y' + \lambda xy = 0, \tag{6}$$

or

$$-(xy')' = \lambda xy, \tag{7}$$

on the interval $0 < x < 1$, and suppose that $\lambda > 0$. This equation arises in the study of free vibrations of a circular elastic membrane, and is discussed further in Section 11.5. If we introduce the new independent variable t defined by $t = \sqrt{\lambda}\, x$, then

$$\frac{dy}{dx} = \sqrt{\lambda}\,\frac{dy}{dt}, \qquad \frac{d^2y}{dx^2} = \lambda\,\frac{d^2y}{dt^2}.$$

Hence Eq. (6) becomes

$$\frac{t}{\sqrt{\lambda}}\,\lambda\,\frac{d^2y}{dt^2} + \sqrt{\lambda}\,\frac{dy}{dt} + \lambda\frac{t}{\sqrt{\lambda}}y = 0,$$

or, if we cancel the common factor $\sqrt{\lambda}$ in each term,

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0. \tag{8}$$

[6]The substitution $t = \sqrt{\lambda}\, x$ reduces Eq. (4) to the standard form $t^2y'' + ty' + (t^2 - \nu^2)y = 0$.

Equation (8) is Bessel's equation of order zero (see Section 5.8). The general solution of Eq. (8) for $t > 0$ is

$$y = c_1 J_0(t) + c_2 Y_0(t);$$

hence the general solution of Eq. (7) for $x > 0$ is

$$y = c_1 J_0(\sqrt{\lambda}\, x) + c_2 Y_0(\sqrt{\lambda}\, x), \tag{9}$$

where J_0 and Y_0 denote the Bessel functions of the first and second kinds of order zero. From Eqs. (7) and (13) of Section 5.8 we have

$$J_0(\sqrt{\lambda}\, x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^m x^{2m}}{2^{2m}(m!)^2}, \qquad x > 0, \tag{10}$$

$$Y_0(\sqrt{\lambda}\, x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{\sqrt{\lambda}\, x}{2}\right) J_0(\sqrt{\lambda}\, x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m \lambda^m x^{2m}}{2^{2m}(m!)^2}\right], \qquad x > 0, \tag{11}$$

where $H_m = 1 + (1/2) + \cdots + (1/m)$, and $\gamma = \lim_{m\to\infty} (H_m - \ln m)$. The graphs of $y = J_0(x)$ and $y = Y_0(x)$ are given in Figure 5.8.2.

Suppose that we seek a solution of Eq. (7) that also satisfies the boundary conditions

$$y(0) = 0, \tag{12}$$

$$y(1) = 0, \tag{13}$$

which are typical of those we have met in other problems in this chapter. Since $J_0(0) = 1$ and $Y_0(x) \to -\infty$ as $x \to 0$, the condition $y(0) = 0$ can be satisfied only by choosing $c_1 = c_2 = 0$ in Eq. (9). Thus the boundary value problem (7), (12), (13) has only the trivial solution.

One interpretation of this result is that the boundary condition (12) at $x = 0$ is too restrictive for the differential equation (7). This illustrates the general situation, namely, that at a singular boundary point it is necessary to consider a modified type of boundary condition. In the present problem, suppose that we require only that the solution (9) and its derivative remain bounded. In other words, we take as the boundary condition at $x = 0$ the requirement

$$y, y' \text{ bounded as } x \to 0. \tag{14}$$

This condition can be satisfied by choosing $c_2 = 0$ in Eq. (9), so as to eliminate the unbounded solution Y_0. The second boundary condition, $y(1) = 0$, then yields

$$J_0(\sqrt{\lambda}) = 0. \tag{15}$$

It is possible to show[7] that Eq. (15) has an infinite set of discrete positive roots, which yield the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of the given problem. The corresponding eigenfunctions are

$$\phi_n(x) = J_0\left(\sqrt{\lambda_n}\, x\right), \tag{16}$$

[7]The function J_0 is well tabulated; the roots of Eq. (15) can be found in various tables, for example, those in Jahnke and Emde or Abramowitz and Stegun. The first three roots of Eq. (15) are $\sqrt{\lambda} = 2.405$, 5.520, and 8.654, respectively, to four significant figures; $\sqrt{\lambda_n} \cong (n - 1/4)\pi$ for large n.

determined only up to a multiplicative constant. The boundary value problem (7), (13), and (14) is an example of a singular Sturm–Liouville problem. This example illustrates that if the boundary conditions are relaxed in an appropriate way, then a singular Sturm–Liouville problem may have an infinite sequence of eigenvalues and eigenfunctions, just as a regular Sturm–Liouville problem does.

Because of their importance in applications, it is worthwhile to investigate singular boundary value problems a little further. There are two main questions that are of concern.

1. Precisely what type of boundary conditions can be allowed in a singular Sturm–Liouville problem?
2. To what extent do the eigenvalues and eigenfunctions of a singular problem share the properties of eigenvalues and eigenfunctions of regular Sturm–Liouville problems? In particular, are the eigenvalues real, are the eigenfunctions orthogonal, and can a given function be expanded as a series of eigenfunctions?

Both these questions can be answered by a study of the identity

$$\int_0^1 \{L[u]v - uL[v]\}\, dx = 0, \tag{17}$$

which played an essential part in the development of the theory of regular Sturm–Liouville problems. We therefore investigate the conditions under which this relation holds for singular problems, where the integral in Eq. (17) may now have to be examined as an improper integral. To be definite we consider the differential equation (1) and assume that $x = 0$ is a singular boundary point, but that $x = 1$ is not. The boundary condition (3) is imposed at the nonsingular boundary point $x = 1$, but we leave unspecified, for the moment, the boundary condition at $x = 0$. Indeed, our principal object is to determine what kinds of boundary conditions are allowable at a singular boundary point if Eq. (17) is to hold.

Since the boundary value problem under investigation is singular at $x = 0$, we choose $\epsilon > 0$ and consider the integral $\int_\epsilon^1 L[u]v\, dx$, instead of $\int_0^1 L[u]v\, dx$, as in Section 11.2. Afterwards we let ϵ approach zero. Assuming that u and v have at least two continuous derivatives on $\epsilon \le x \le 1$, and integrating twice by parts, we find that

$$\int_\epsilon^1 \{L[u]v - uL[v]\}\, dx = -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_\epsilon^1. \tag{18}$$

The boundary term at $x = 1$ is again eliminated if both u and v satisfy the boundary condition (3), and thus,

$$\int_\epsilon^1 \{L[u]v - uL[v]\}\, dx = p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)]. \tag{19}$$

Taking the limit as $\epsilon \to 0$ yields

$$\int_0^1 \{L[u]v - uL[v]\}\, dx = \lim_{\epsilon\to 0} p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)]. \tag{20}$$

Hence Eq. (17) holds if and only if, in addition to the assumptions stated previously,

$$\lim_{\epsilon\to 0} p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] = 0 \tag{21}$$

for every pair of functions u and v in the class under consideration. Equation (21) is therefore the criterion that determines what boundary conditions are allowable at $x = 0$ if that point is a singular boundary point. A similar condition applies at $x = 1$ if that boundary point is singular, namely,

$$\lim_{\epsilon \to 0} p(1-\epsilon)[u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)] = 0. \tag{22}$$

In summary, as in Section 11.2, a singular boundary value problem for Eq. (1) is said to be **self-adjoint** if Eq. (17) is valid, possibly as an improper integral, for each pair of functions u and v with the following properties: They are twice continuously differentiable on the open interval $0 < x < 1$, they satisfy a boundary condition of the form (2) at each regular boundary point, and they satisfy a boundary condition sufficient to ensure Eq. (21) if $x = 0$ is a singular boundary point, or Eq. (22) if $x = 1$ is a singular boundary point. If at least one boundary point is singular, then the differential equation (1), together with two boundary conditions of the type just described, are said to form a **singular Sturm–Liouville problem**.

For example, for Eq. (7) we have $p(x) = x$. If both u and v satisfy the boundary condition (14) at $x = 0$, it is clear that Eq. (21) will hold. Hence the singular boundary value problem, consisting of the differential equation (7), the boundary condition (14) at $x = 0$, and any boundary condition of the form (3) at $x = 1$, is self-adjoint.

The most striking difference between regular and singular Sturm–Liouville problems is that in a singular problem the eigenvalues may not be discrete. That is, the problem may have nontrivial solutions for every value of λ, or for every value of λ in some interval. In such a case the problem is said to have a **continuous spectrum**. It may happen that a singular problem has a mixture of discrete eigenvalues and also a continuous spectrum. Finally, it is possible that only a discrete set of eigenvalues exists, just as in the regular case discussed in Section 11.2. For example, this is true of the problem consisting of Eqs. (7), (13), and (14). In general, it may be difficult to determine which case actually occurs in a given problem.

A systematic discussion of singular Sturm–Liouville problems is quite sophisticated[8] indeed, requiring a substantial extension of the methods presented in this book. We restrict ourselves to some examples related to physical applications; in each of these examples it is known that there is an infinite set of discrete eigenvalues.

If a singular Sturm–Liouville problem does have only a discrete set of eigenvalues and eigenfunctions, then Eq. (17) can be used, just as in Section 11.2, to prove that the eigenvalues of such a problem are real, and that the eigenfunctions are orthogonal with respect to the weight function r. The expansion of a given function in terms of a series of eigenfunctions then follows as in Section 11.2.

Such expansions are useful, as in the regular case, for solving nonhomogeneous boundary value problems. The procedure is very similar to that described in Section 11.3. Some examples for ordinary differential equations are indicated in Problems 1 to 4, and some problems for partial differential equations appear in Section 11.5.

For instance, the eigenfunctions $\phi_n(x) = J_0(\sqrt{\lambda_n}\, x)$ of the singular Sturm–Liouville problem

$$-(xy')' = \lambda xy, \qquad 0 < x < 1,$$

$$y,\ y' \text{ bounded as } x \to 0, \qquad y(1) = 0$$

[8] See, for example, Chapter 5 of the book by Yosida.

satisfy the orthogonality relation

$$\int_0^1 x\phi_m(x)\phi_n(x)\,dx = 0, \qquad m \neq n \tag{23}$$

with respect to the weight function $r(x) = x$. Then, if f is a given function, we assume that

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_n}\,x). \tag{24}$$

Multiplying Eq. (24) by $x\,J_0(\sqrt{\lambda_m}\,x)$ and integrating term by term from $x = 0$ to $x = 1$ yield

$$\int_0^1 x f(x) J_0(\sqrt{\lambda_m}\,x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 x\,J_0(\sqrt{\lambda_m}\,x)\,J_0(\sqrt{\lambda_n}\,x)\,dx. \tag{25}$$

Because of the orthogonality condition (23), the right side of Eq. (25) collapses to a single term; hence

$$c_m = \frac{\displaystyle\int_0^1 x f(x) J_0(\sqrt{\lambda_m}\,x)\,dx}{\displaystyle\int_0^1 x\,J_0^2(\sqrt{\lambda_m}\,x)\,dx}, \tag{26}$$

which determines the coefficients in the series (24).

The convergence of the series (24) is established by an extension of Theorem 11.2.4 to cover this case. This theorem can also be shown to hold for other sets of Bessel functions, which are solutions of appropriate boundary value problems, for Legendre polynomials, and for solutions of a number of other singular Sturm–Liouville problems of considerable interest.

It must be emphasized that the singular problems mentioned here are not necessarily typical. In general, singular boundary value problems are characterized by continuous spectra, rather than by discrete sets of eigenvalues. The corresponding sets of eigenfunctions are therefore not denumerable, and series expansions of the type described in Theorem 11.2.4 do not exist. They are replaced by appropriate integral representations.

PROBLEMS

1. Find a formal solution of the nonhomogeneous boundary value problem

$$-(xy')' = \mu xy + f(x),$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y(1) = 0,$$

where f is a given continuous function on $0 \leq x \leq 1$, and μ is not an eigenvalue of the corresponding homogeneous problem.
Hint: Use a series expansion similar to those in Section 11.3.

2. Consider the boundary value problem

$$-(xy')' = \lambda xy,$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y'(1) = 0.$$

(a) Show that $\lambda_0 = 0$ is an eigenvalue of this problem corresponding to the eigenfunction $\phi_0(x) = 1$. If $\lambda > 0$, show formally that the eigenfunctions are given by $\phi_n(x) =$

$J_0(\sqrt{\lambda_n}\,x)$, where $\sqrt{\lambda_n}$ is the nth positive root (in increasing order) of the equation $J_0'(\sqrt{\lambda}) = 0$. It is possible to show that there is an infinite sequence of such roots.

(b) Show that if $m,\ n = 0, 1, 2, \ldots$, then

$$\int_0^1 x\phi_m(x)\phi_n(x)\,dx = 0, \qquad m \neq n.$$

(c) Find a formal solution to the nonhomogeneous problem

$$-(xy')' = \mu xy + f(x),$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y'(1) = 0,$$

where f is a given continuous function on $0 \leq x \leq 1$, and μ is not an eigenvalue of the corresponding homogeneous problem.

3. Consider the problem

$$-(xy')' + (k^2/x)y = \lambda xy,$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y(1) = 0,$$

where k is a positive integer.

(a) Using the substitution $t = \sqrt{\lambda}\,x$, show that the given differential equation reduces to Bessel's equation of order k (see Problem 9 of Section 5.8). One solution is $J_k(t)$; a second linearly independent solution, denoted by $Y_k(t)$, is unbounded as $t \to 0$.

(b) Show formally that the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the given problem are the squares of the positive zeros of $J_k(\sqrt{\lambda})$, and that the corresponding eigenfunctions are $\phi_n(x) = J_k(\sqrt{\lambda_n}\,x)$. It is possible to show that there is an infinite sequence of such zeros.

(c) Show that the eigenfunctions $\phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\phi_m(x)\phi_n(x)\,dx = 0, \qquad m \neq n.$$

(d) Determine the coefficients in the formal series expansion

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

(e) Find a formal solution of the nonhomogeneous problem

$$-(xy')' + (k^2/x)y = \mu xy + f(x),$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y(1) = 0,$$

where f is a given continuous function on $0 \leq x \leq 1$, and μ is not an eigenvalue of the corresponding homogeneous problem.

4. Consider Legendre's equation (see Problems 22 through 24 of Section 5.3)

$$-[(1 - x^2)y']' = \lambda y$$

subject to the boundary conditions

$$y(0) = 0, \qquad y,\ y' \text{ bounded as } x \to 1.$$

The eigenfunctions of this problem are the odd Legendre polynomials $\phi_1(x) = P_1(x) = x$, $\phi_2(x) = P_3(x) = (5x^3 - 3x)/2, \ldots, \phi_n(x) = P_{2n-1}(x), \ldots$ corresponding to the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 4 \cdot 3, \ldots, \lambda_n = 2n(2n-1), \ldots$.

(a) Show that

$$\int_0^1 \phi_m(x)\phi_n(x)\,dx = 0, \qquad m \neq n.$$

(b) Find a formal solution of the nonhomogeneous problem

$$-[(1-x^2)y']' = \mu y + f(x),$$
$$y(0) = 0, \qquad y,\ y' \text{ bounded as } x \to 1,$$

where f is a given continuous function on $0 \le x \le 1$, and μ is not an eigenvalue of the corresponding homogeneous problem.

5. The equation

$$(1-x^2)y'' - xy' + \lambda y = 0 \tag{i}$$

is Chebyshev's equation; see Problem 10 of Section 5.3.

(a) Show that Eq. (i) can be written in the form

$$-[(1-x^2)^{1/2}y']' = \lambda(1-x^2)^{-1/2}y, \qquad -1 < x < 1. \tag{ii}$$

(b) Consider the boundary conditions

$$y,\ y' \text{ bounded as } x \to -1, \qquad y,\ y' \text{ bounded as } x \to 1. \tag{iii}$$

Show that the boundary value problem (ii), (iii) is self-adjoint.

(c) It can be shown that the boundary value problem (ii), (iii) has the eigenvalues $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 4, \ldots, \lambda_n = n^2, \ldots$. The corresponding eigenfunctions are the Chebyshev polynomials $T_n(x)$: $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 1 - 2x^2, \ldots$. Show that

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}\,dx = 0, \qquad m \ne n. \tag{iv}$$

Note that this is a convergent improper integral.

11.5 Further Remarks on the Method of Separation of Variables: A Bessel Series Expansion

In this chapter we are interested in extending the method of separation of variables developed in Chapter 10 to a larger class of problems—to problems involving more general differential equations, more general boundary conditions, or different geometrical regions. We indicated in Section 11.3 how to deal with a class of more general differential equations or boundary conditions. Here we concentrate on problems posed in various geometrical regions, with emphasis on those leading to singular Sturm–Liouville problems when the variables are separated.

Because of its relative simplicity, as well as the considerable physical significance of many problems to which it is applicable, the method of separation of variables merits its important place in the theory and application of partial differential equations. However, this method does have certain limitations that should not be forgotten. In the first place, the problem must be linear so that the principle of superposition can be invoked to construct additional solutions by forming linear combinations of the fundamental solutions of an appropriate homogeneous problem.

As a practical matter, we must also be able to solve the ordinary differential equations, obtained after separating the variables, in a reasonably convenient manner. In some problems to which the method of separation of variables can be applied in principle, it

is of very limited practical value due to lack of information about the solutions of the ordinary differential equations that appear.

Furthermore, the geometry of the region involved in the problem is subject to rather severe restrictions. On one hand, a coordinate system must be employed in which the variables can be separated, and the partial differential equation replaced by a set of ordinary differential equations. For Laplace's equation there are about a dozen such coordinate systems; only rectangular, circular cylindrical, and spherical coordinates are likely to be familiar to most readers of this book. On the other hand, the boundary of the region of interest must consist of coordinate curves or surfaces, that is, curves or surfaces on which one variable remains constant. Thus, at an elementary level, one is limited to regions bounded by straight lines or circular arcs in two dimensions, or by planes, circular cylinders, circular cones, or spheres in three dimensions.

In three-dimensional problems the separation of variables in Laplace's operator $u_{xx} + u_{yy} + u_{zz}$ leads to the equation $X'' + \lambda X = 0$ in rectangular coordinates, to Bessel's equation in cylindrical coordinates, and to Legendre's equation in spherical coordinates. It is this fact that is largely responsible for the intensive study that has been made of these equations and the functions defined by them. It is also noteworthy that two of the three most important situations lead to singular, rather than regular, Sturm–Liouville problems. Thus, singular problems are by no means exceptional and may be of even greater interest than regular ones. The remainder of this section is devoted to an example involving an expansion of a given function as a series of Bessel functions.

The Vibrations of a Circular Elastic Membrane. In Section 10.7 [Eq. (7)] we noted that the transverse vibrations of a thin elastic membrane are governed by the two-dimensional wave equation

$$a^2(u_{xx} + u_{yy}) = u_{tt}. \tag{1}$$

To study the motion of a circular membrane it is convenient to write Eq. (1) in polar coordinates:

$$a^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_{tt}. \tag{2}$$

We will assume that the membrane has unit radius, that it is fixed securely around its circumference, and that initially it occupies a displaced position independent of the angular variable θ, from which it is released at time $t = 0$. Because of the circular symmetry of the initial and boundary conditions, it is natural to assume also that u is independent of θ; that is, u is a function of r and t only. In this event the differential equation (2) becomes

$$a^2\left(u_{rr} + \frac{1}{r}u_r\right) = u_{tt}, \qquad 0 < r < 1, \quad t > 0. \tag{3}$$

The boundary condition at $r = 1$ is

$$u(1, t) = 0, \qquad t \geq 0, \tag{4}$$

and the initial conditions are

$$u(r, 0) = f(r), \qquad 0 \leq r \leq 1, \tag{5}$$

$$u_t(r, 0) = 0, \qquad 0 \leq r \leq 1, \tag{6}$$

where $f(r)$ describes the initial configuration of the membrane. For consistency we also require that $f(1) = 0$. Finally, we state explicitly the requirement that $u(r, t)$ is to be bounded for $0 \le r \le 1$.

Assuming that $u(r, t) = R(r)T(t)$, and substituting for $u(r, t)$ in Eq. (3), we obtain

$$\frac{R'' + (1/r)R'}{R} = \frac{1}{a^2}\frac{T''}{T} = -\lambda^2. \tag{7}$$

We have anticipated that the separation constant must be negative by writing it as $-\lambda^2$ with $\lambda > 0$.[9] Then Eq. (7) yields the following two ordinary differential equations:

$$r^2R'' + rR' + \lambda^2 r^2 R = 0, \tag{8}$$

$$T'' + \lambda^2 a^2 T = 0. \tag{9}$$

Thus, from Eq. (9),

$$T(t) = k_1 \sin \lambda at + k_2 \cos \lambda at. \tag{10}$$

Introducing the new independent variable $\xi = \lambda r$ into Eq. (8), we obtain

$$\xi^2 \frac{d^2R}{d\xi^2} + \xi \frac{dR}{d\xi} + \xi^2 R = 0, \tag{11}$$

which is Bessel's equation of order zero. Thus,

$$R = c_1 J_0(\xi) + c_2 Y_0(\xi), \tag{12}$$

where J_0 and Y_0 are Bessel functions of the first and second kinds, respectively, of order zero (see Section 11.4). In terms of r we have

$$R = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r). \tag{13}$$

The boundedness condition on $u(r, t)$ requires that R remain bounded as $r \to 0$. Since $Y_0(\lambda r) \to -\infty$ as $r \to 0$, we must choose $c_2 = 0$. The boundary condition (4) then requires that

$$J_0(\lambda) = 0. \tag{14}$$

Consequently, the allowable values of the separation constant are obtained from the roots of the transcendental equation (14). Recall from Section 11.4 that $J_0(\lambda)$ has an infinite set of discrete positive zeros, which we denote by $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots$, ordered in increasing magnitude. Further, the functions $J_0(\lambda_n r)$ are the eigenfunctions of a singular Sturm–Liouville problem, and can be used as the basis of a series expansion for the given function f. The fundamental solutions of this problem, satisfying the partial differential equation (3), the boundary condition (4), and boundedness condition, are

$$u_n(r, t) = J_0(\lambda_n r) \sin \lambda_n at, \qquad n = 1, 2, \ldots, \tag{15}$$

$$v_n(r, t) = J_0(\lambda_n r) \cos \lambda_n at, \qquad n = 1, 2, \ldots. \tag{16}$$

[9] By denoting the separation constant by $-\lambda^2$, rather than simply by $-\lambda$, we avoid the appearance of numerous radical signs in the following discussion.

Next we assume that $u(r, t)$ can be expressed as an infinite linear combination of the fundamental solutions (15), (16):

$$u(r,t) = \sum_{n=1}^{\infty} \left[k_n u_n(r,t) + c_n v_n(r,t)\right]$$
$$= \sum_{n=1}^{\infty} [k_n J_0(\lambda_n r) \sin \lambda_n a t + c_n J_0(\lambda_n r) \cos \lambda_n a t]. \tag{17}$$

The initial conditions require that

$$u(r,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r) \tag{18}$$

and

$$u_t(r,0) = \sum_{n=1}^{\infty} \lambda_n a k_n J_0(\lambda_n r) = 0. \tag{19}$$

From Eq. (26) of Section 11.4 we obtain

$$k_n = 0, \qquad c_n = \frac{\displaystyle\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\displaystyle\int_0^1 r[J_0(\lambda_n r)]^2\, dr}; \qquad n = 1, 2, \ldots. \tag{20}$$

Thus, the solution of the partial differential equation (3) satisfying the boundary condition (4) and the initial conditions (5) and (6) is given by

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \cos \lambda_n a t \tag{21}$$

with the coefficients c_n defined by Eq. (20).

PROBLEMS

1. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the parallelogram whose vertices are (0, 0), (2, 0), (3, 2), and (1, 2). Suppose that on the side $y = 2$ the boundary condition is $u(x, 2) = f(x)$ for $1 \le x \le 3$, and that on the other three sides $u = 0$ (see Figure 11.5.1).

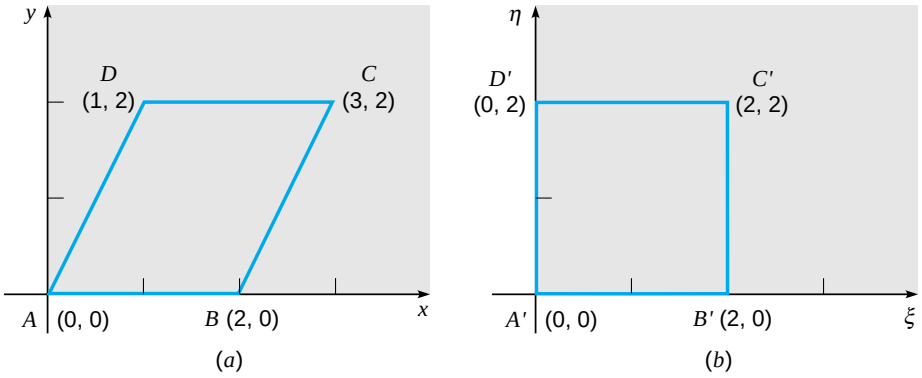


FIGURE 11.5.1 The region in Problem 1.

(a) Show that there are no nontrivial solutions of the partial differential equation of the form $u(x, y) = X(x)Y(y)$ that also satisfy the homogeneous boundary conditions.
(b) Let $\xi = x - \frac{1}{2}y$, $\eta = y$. Show that the given parallelogram in the xy-plane transforms into the square $0 \le \xi \le 2$, $0 \le \eta \le 2$ in the $\xi\eta$-plane. Show that the differential equation transforms into

$$\tfrac{5}{4}u_{\xi\xi} - u_{\xi\eta} + u_{\eta\eta} = 0.$$

How are the boundary conditions transformed?
(c) Show that in the $\xi\eta$-plane the differential equation possesses no solution of the form

$$u(\xi, \eta) = U(\xi)V(\eta).$$

Thus in the xy-plane the shape of the boundary precludes a solution by the method of the separation of variables, while in the $\xi\eta$-plane the region is acceptable but the variables in the differential equation can no longer be separated.

2. Find the displacement $u(r, t)$ in a vibrating circular elastic membrane of radius 1 that satisfies the boundary condition

$$u(1, t) = 0, \qquad t \ge 0,$$

and the initial conditions

$$u(r, 0) = 0, \qquad u_t(r, 0) = g(r), \qquad 0 \le r \le 1,$$

where $g(1) = 0$.
Hint: The differential equation to be satisfied is Eq. (3) of the text.

3. Find the displacement $u(r, t)$ in a vibrating circular elastic membrane of radius 1 that satisfies the boundary condition

$$u(1, t) = 0, \qquad t \ge 0,$$

and the initial conditions

$$u(r, 0) = f(r), \qquad u_t(r, 0) = g(r), \qquad 0 \le r \le 1,$$

where $f(1) = g(1) = 0$.

4. The wave equation in polar coordinates is

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = a^{-2}u_{tt}.$$

Show that if $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, then R, Θ, and T satisfy the ordinary differential equations

$$\begin{aligned} r^2R'' + rR' + (\lambda^2r^2 - n^2)R &= 0, \\ \Theta'' + n^2\Theta &= 0, \\ T'' + \lambda^2a^2T &= 0. \end{aligned}$$

5. In the circular cylindrical coordinates r, θ, z defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

Laplace's equation is

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} + u_{zz} = 0.$$

(a) Show that if $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, then R, Θ, and Z satisfy the ordinary differential equations

$$\begin{aligned} r^2R'' + rR' + (\lambda^2r^2 - n^2)R &= 0, \\ \Theta'' + n^2\Theta &= 0, \\ Z'' - \lambda^2Z &= 0. \end{aligned}$$

(b) Show that if $u(r, \theta, z)$ is independent of θ, then the first equation in part (a) becomes

$$r^2R'' + rR' + \lambda^2 r^2 R = 0,$$

the second is omitted altogether, and the third is unchanged.

6. Find the steady-state temperature in a semi-infinite rod $0 < z < \infty$, $0 \le r < 1$, if the temperature is independent of θ and approaches zero as $z \to \infty$. Assume that the temperature $u(r, z)$ satisfies the boundary conditions

$$\begin{aligned} u(1, z) &= 0, && z > 0, \\ u(r, 0) &= f(r), && 0 \le r \le 1. \end{aligned}$$

Hint: Refer to Problem 5.

7. The equation

$$v_{xx} + v_{yy} + k^2 v = 0$$

is a generalization of Laplace's equation, and is sometimes called the Helmholtz (1821–1894) equation.

(a) In polar coordinates the Helmholtz equation is

$$v_{rr} + (1/r)v_r + (1/r^2)v_{\theta\theta} + k^2 v = 0.$$

If $v(r, \theta) = R(r)\Theta(\theta)$, show that R and Θ satisfy the ordinary differential equations

$$r^2R'' + rR' + (k^2r^2 - \lambda^2)R = 0, \qquad \Theta'' + \lambda^2\Theta = 0.$$

(b) Consider the Helmholtz equation in the disk $r < c$. Find the solution that remains bounded at all points in the disk, that is periodic in θ with period 2π, and that satisfies the boundary condition $v(c, \theta) = f(\theta)$, where f is a given function on $0 \le \theta < 2\pi$.
Hint: The equation for R is a Bessel equation. See Problem 3 of Section 11.4.

8. Consider the flow of heat in an infinitely long cylinder of radius 1: $0 \le r < 1$, $0 \le \theta < 2\pi$, $-\infty < z < \infty$. Let the surface of the cylinder be held at temperature zero, and let the initial temperature distribution be a function of the radial variable r only. Then the temperature u is a function of r and t only, and satisfies the heat conduction equation

$$\alpha^2[u_{rr} + (1/r)u_r] = u_t, \qquad 0 < r < 1, \quad t > 0,$$

and the following initial and boundary conditions:

$$\begin{aligned} u(r, 0) &= f(r), && 0 \le r \le 1, \\ u(1, t) &= 0, && t > 0. \end{aligned}$$

Show that

$$u(r, t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\alpha^2 \lambda_n^2 t},$$

where $J_0(\lambda_n) = 0$. Find a formula for c_n.

9. In the spherical coordinates ρ, θ, ϕ ($\rho > 0$, $0 \le \theta < 2\pi$, $0 \le \phi \le \pi$) defined by the equations

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi,$$

Laplace's equation is

$$\rho^2 u_{\rho\rho} + 2\rho u_\rho + (\csc^2\phi)u_{\theta\theta} + u_{\phi\phi} + (\cot\phi)u_\phi = 0.$$

(a) Show that if $u(\rho, \theta, \phi) = \mathrm{P}(\rho)\Theta(\theta)\Phi(\phi)$, then P, Θ, and Φ satisfy ordinary differential equations of the form

$$\begin{aligned} \rho^2\mathrm{P}'' + 2\rho\mathrm{P}' - \mu^2\mathrm{P} &= 0, \\ \Theta'' + \lambda^2\Theta &= 0, \\ (\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi - \lambda^2)\Phi &= 0. \end{aligned}$$

The first of these equations is of the Euler type, while the third is related to Legendre's equation.

(b) Show that if $u(\rho, \theta, \phi)$ is independent of θ, then the first equation in part (a) is unchanged, the second is omitted, and the third becomes

$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi)\Phi = 0.$$

(c) Show that if a new independent variable is defined by $s = \cos\phi$, then the equation for Φ in part (b) becomes

$$(1-s^2)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} + \mu^2\Phi = 0, \qquad -1 \le s \le 1.$$

Note that this is Legendre's equation.

10. Find the steady-state temperature $u(\rho, \phi)$ in a sphere of unit radius if the temperature is independent of θ and satisfies the boundary condition

$$u(1, \phi) = f(\phi), \qquad 0 \le \phi \le \pi.$$

Hint: Refer to Problem 9 and to Problems 22 through 29 of Section 5.3. Use the fact that the only solutions of Legendre's equation that are finite at both ± 1 are the Legendre polynomials.

11.6 Series of Orthogonal Functions: Mean Convergence

In Section 11.2 we stated that under certain restrictions a given function f can be expanded in a series of eigenfunctions of a Sturm–Liouville boundary value problem, the series converging to $[f(x+) + f(x-)]/2$ at each point in the open interval. Under somewhat more restrictive conditions the series converges to $f(x)$ at each point in the closed interval. This type of convergence is referred to as pointwise convergence. In this section we describe a different kind of convergence that is especially useful for series of orthogonal functions, such as eigenfunctions.

Suppose that we are given the set of functions $\phi_1, \phi_2, \ldots, \phi_n$, which are continuous on the interval $0 \le x \le 1$ and satisfy the orthonormality condition

$$\int_0^1 r(x)\phi_i(x)\phi_j(x)\,dx = \begin{cases} 0, & i \ne j, \\ 1, & i = j, \end{cases} \tag{1}$$

where r is a nonnegative weight function. Suppose also that we wish to approximate a given function f, defined on $0 \le x \le 1$, by a linear combination of $\phi_1, \ldots, \phi_n$. That is, if

$$S_n(x) = \sum_{i=1}^{n} a_i\phi_i(x), \tag{2}$$

we wish to choose the coefficients $a_1, \ldots, a_n$ so that the function S_n will best approximate f on $0 \le x \le 1$. The first problem that we must face in doing this is to state precisely what we mean by "best approximate f on $0 \le x \le 1$." There are several reasonable meanings that can be attached to this phrase.

1. We can choose n points $x_1, \ldots, x_n$ in the interval $0 \le x \le 1$, and require that $S_n(x)$ have the same value as $f(x)$ at each of these points. The coefficients $a_1, \ldots, a_n$ are found by solving the set of linear algebraic equations

$$\sum_{i=1}^{n} a_i \phi_i(x_j) = f(x_j), \qquad j = 1, \ldots, n. \tag{3}$$

This procedure is known as the **method of collocation**. It has the advantage that it is very easy to write down Eqs. (3); one needs only to evaluate the functions involved at the points $x_1, \ldots, x_n$. If these points are well chosen, and if n is fairly large, then presumably $S_n(x)$ will not only be equal to $f(x)$ at the chosen points but will be reasonably close to it at other points as well. However, collocation has several deficiencies. One is that if one more base function ϕ_{n+1} is added, then one more point x_{n+1} is required, and *all* the coefficients must be recomputed. Thus, it is inconvenient to improve the accuracy of a collocation approximation by including additional terms. Further, the coefficients a_i depend on the location of the points $x_1, \ldots, x_n$, and it is not obvious how best to select these points.

2. Alternatively, we can consider the difference $|f(x) - S_n(x)|$ and try to make it as small as possible. The trouble here is that $|f(x) - S_n(x)|$ is a function of x as well as of the coefficients $a_1, \ldots, a_n$, and it is not clear how to calculate a_i. The choice of a_i that makes $|f(x) - S_n(x)|$ small at one point may make it large at another. One way to proceed is to consider instead the least upper bound[10] of $|f(x) - S_n(x)|$ for x in $0 \le x \le 1$, and then to choose $a_1, \ldots, a_n$ so as to make this quantity as small as possible. That is, if

$$E_n(a_1, \ldots, a_n) = \operatorname*{lub}_{0 \le x \le 1} |f(x) - S_n(x)|, \tag{4}$$

then choose $a_1, \ldots, a_n$ so as to minimize E_n. This approach is intuitively appealing, and is often used in theoretical calculations. However, in practice, it is usually very hard, if not impossible, to write down an explicit formula for $E_n(a_1, \ldots, a_n)$. Further, this procedure also shares one of the disadvantages of collocation; namely, on adding an additional term to $S_n(x)$, one must recompute all the preceding coefficients. Thus, it is not often useful in practical problems.

3. Another way to proceed is to consider

$$I_n(a_1, \ldots, a_n) = \int_0^1 r(x) |f(x) - S_n(x)| \, dx. \tag{5}$$

If $r(x) = 1$, then I_n is the area between the graphs of $y = f(x)$ and $y = S_n(x)$ (see Figure 11.6.1). We can then determine the coefficients a_i so as to minimize I_n. To avoid the complications resulting from calculations with absolute values, it is more convenient to consider instead

$$R_n(a_1, \ldots, a_n) = \int_0^1 r(x)[f(x) - S_n(x)]^2 \, dx \tag{6}$$

as our measure of the quality of approximation of the linear combination $S_n(x)$ to $f(x)$. While R_n is clearly similar in some ways to I_n, it lacks the simple geometric interpretation of the latter. Nevertheless, it is much easier mathematically to deal

[10] The least upper bound (lub) is an upper bound that is smaller than any other upper bound. The lub of a bounded function always exists, and is equal to its maximum if the function has one.

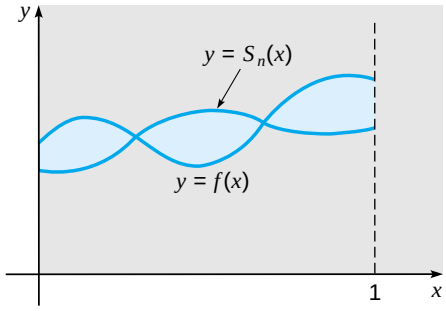


FIGURE 11.6.1 Approximation of $f(x)$ by $S_n(x)$.

with R_n than with I_n. The quantity R_n is called the **mean square error** of the approximation S_n to f. If $a_1, \ldots, a_n$ are chosen so as to minimize R_n, then S_n is said to approximate f in the mean square sense.

To choose $a_1, \ldots, a_n$ so as to minimize R_n we must satisfy the necessary conditions

$$\partial R_n/\partial a_i = 0, \qquad i = 1, \ldots, n. \tag{7}$$

Writing out Eq. (7), and noting that $\partial S_n(x; a_1, \ldots, a_n)/\partial a_i$ is equal to $\phi_i(x)$, we obtain

$$-\frac{\partial R_n}{\partial a_i} = 2\int_0^1 r(x)[f(x) - S_n(x)]\phi_i(x)\,dx = 0. \tag{8}$$

Substituting for $S_n(x)$ from Eq. (2) and making use of the orthogonality relation (1) lead to

$$a_i = \int_0^1 r(x)\,f(x)\phi_i(x)\,dx, \qquad i = 1, \ldots, n. \tag{9}$$

The coefficients defined by Eq. (9) are called the Fourier coefficients of f with respect to the orthonormal set $\phi_1, \phi_2, \ldots, \phi_n$ and the weight function r. Since the conditions (7) are only necessary and not sufficient for R_n to be a minimum, a separate argument is required to show that R_n is actually minimized if the a_i are chosen by Eq. (9). This argument is outlined in Problem 5.

Note that the coefficients (9) are the same as those in the eigenfunction series whose convergence, under certain conditions, was stated in Theorem 11.2.4. Thus, $S_n(x)$ is the nth partial sum in this series, and constitutes the best mean square approximation to $f(x)$ that is possible with the functions $\phi_1, \ldots, \phi_n$. We will assume hereafter that the coefficients a_i in $S_n(x)$ are given by Eq. (9).

Equation (9) is noteworthy in two other important respects. In the first place, it gives a formula for each a_i *separately*, rather than a set of linear algebraic equations for $a_1, \ldots, a_n$ as in the method of collocation, for example. This is due to the orthogonality of the base functions $\phi_1, \ldots, \phi_n$. Further, the formula for a_i is *independent* of n, the number of terms in $S_n(x)$. The practical significance of this is as follows. Suppose that, to obtain a better approximation to f, we desire to use an approximation with more terms, say, k terms, where $k > n$. It is then unnecessary to recompute the first n coefficients in $S_k(x)$. All that is required is to compute from Eq. (9) the coefficients $a_{n+1}, \ldots, a_k$ arising from the additional base functions $\phi_{n+1}, \ldots, \phi_k$. Of course, if

f, r, and the ϕ_n are complicated functions, it may be necessary to evaluate the integrals numerically.

Now let us suppose that there is an infinite sequence of functions $\phi_1, \ldots, \phi_n, \ldots$, which are continuous and orthonormal on the interval $0 \le x \le 1$. Suppose further that, as n increases without bound, the mean square error R_n approaches zero. In this event the infinite series

$$\sum_{i=1}^{\infty} a_i \phi_i(x)$$

is said to converge in the **mean square sense** (or, more simply, in the **mean**) to $f(x)$. Mean convergence is an essentially different type of convergence than the pointwise convergence considered up to now. A series may converge in the mean without converging at each point. This is plausible geometrically because the area between two curves, which behaves in the same way as the mean square error, may be zero even though the functions are not the same at every point. They may differ on any finite set of points, for example, without affecting the mean square error. It is less obvious, but also true, that even if an infinite series converges at every point, it may not converge in the mean. Indeed, the mean square error may even become unbounded. An example of this phenomenon is given in Problem 4.

Now suppose that we wish to know what class of functions, defined on $0 \le x \le 1$, can be represented as an infinite series of the orthonormal set ϕ_i, $i = 1, 2, \ldots$. The answer depends on what kind of convergence we require. We say that the set $\phi_1, \ldots, \phi_n, \ldots$ is **complete** with respect to mean square convergence for a set of functions $\mathcal{F}$, if for each function f in $\mathcal{F}$, the series

$$f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x), \tag{10}$$

with coefficients given by Eq. (9), converges in the mean. There is a similar definition for completeness with respect to pointwise convergence.

Theorems having to do with the convergence of series such as that in Eq. (10) can now be restated in terms of the idea of completeness. For example, Theorem 11.2.4 can be restated as follows: The eigenfunctions of the Sturm–Liouville problem

$$-[p(x)y']' + q(x)y = \lambda r(x)y, \qquad 0 < x < 1, \tag{11}$$

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0 \tag{12}$$

are complete with respect to ordinary pointwise convergence for the set of functions that are continuous on $0 \le x \le 1$, and have a piecewise continuous derivative there.

If pointwise convergence is replaced by mean convergence, Theorem 11.2.4 can be considerably generalized. Before we state such a companion theorem to Theorem 11.2.4, we first define what is meant by a square integrable function. A function f is said to be **square integrable** on the interval $0 \le x \le 1$ if both f and f^2 are integrable[11] on

[11] For the Riemann integral used in elementary calculus the hypotheses that f and f^2 are integrable are independent; that is, there are functions such that f is integrable but f^2 is not, and conversely (see Problem 6). A generalized integral, known as the Lebesgue integral, has the property (among others) that if f^2 is integrable, then f is also necessarily integrable. The term *square integrable* came into common use in connection with this type of integration.

that interval. The following theorem is similar to Theorem 11.2.4 except that it involves mean convergence.

Theorem 11.6.1 The eigenfunctions ϕ_i of the Sturm–Liouville problem (11), (12) are complete with respect to mean convergence for the set of functions that are square integrable on $0 \le x \le 1$. In other words, given any square integrable function f, the series (10), whose coefficients are given by Eq. (9), converges to $f(x)$ in the mean square sense.

It is significant that the class of functions specified in Theorem 11.6.1 is very large indeed. The class of square integrable functions contains some functions with many discontinuities, including some kinds of infinite discontinuities, as well as some functions that are not differentiable at any point. All these functions have mean convergent expansions in the eigenfunctions of the boundary value problem (11), (12). However, in many cases these series do not converge pointwise, at least not at every point. Thus mean convergence is more naturally associated with series of orthogonal functions, such as eigenfunctions, than ordinary pointwise convergence.

The theory of Fourier series discussed in Chapter 10 is just a special case of the general theory of Sturm–Liouville problems. For instance, the functions

$$\phi_n(x) = \sqrt{2}\sin n\pi x \tag{13}$$

are the normalized eigenfunctions of the Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0. \tag{14}$$

Thus, if f is a given square integrable function on $0 \le x \le 1$, then according to Theorem 11.6.1, the series

$$f(x) = \sum_{m=1}^{\infty} b_m \phi_m(x) = \sqrt{2}\sum_{m=1}^{\infty} b_m \sin m\pi x, \tag{15}$$

where

$$b_m = \int_0^1 f(x)\phi_m(x)\,dx = \sqrt{2}\int_0^1 f(x)\,\sin m\pi x\,dx, \tag{16}$$

converges in the mean. The series (15) is precisely the Fourier sine series discussed in Section 10.4. If f satisfies the further conditions stated in Theorem 11.2.4, then this series converges pointwise as well as in the mean. Similarly, a Fourier cosine series is associated with the Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(1) = 0. \tag{17}$$

EXAMPLE 1 Let $f(x) = 1$ for $0 < x < 1$. Expand $f(x)$ using the eigenfunctions (13) and discuss the pointwise and mean square convergence of the resulting series.

The series has the form (15) and its coefficients b_m are given by Eq. (16). Thus

$$b_m = \sqrt{2}\int_0^1 \sin m\pi x\,dx = \frac{\sqrt{2}}{m\pi}(1 - \cos m\pi) \tag{18}$$

and the nth partial sum of the series is

$$S_n(x) = 2\sum_{m=1}^{n} \frac{1-\cos m\pi}{m\pi} \sin m\pi x. \tag{19}$$

The mean square error is then

$$R_n = \int_0^1 [f(x) - S_n(x)]^2 \, dx. \tag{20}$$

By calculating R_n for several values of n and plotting the results, we obtain Figure 11.6.2. This figure indicates that R_n steadily decreases as n increases. Of course, Theorem 11.6.1 asserts that $R_n \to 0$ as $n \to \infty$. Pointwise, we know that $S_n(x) \to f(x) = 1$ as $n \to \infty$; further, $S_n(x)$ has the value zero for $x = 0$ or $x = 1$ for every n. Although the series converges pointwise for each value of x, the least upper bound of the error does not diminish as n increases. For each n there are points close to $x = 0$ and $x = 1$ where the error is arbitrarily close to 1.

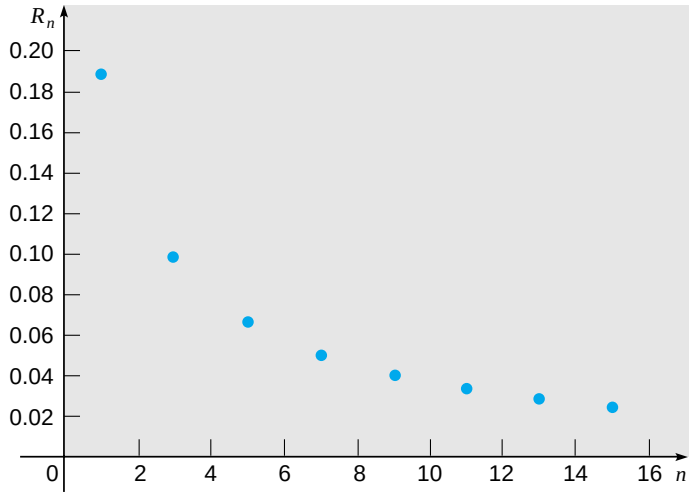


FIGURE 11.6.2 Dependence of the mean square error R_n on n in Example 1.

Theorem 11.6.1 can be extended to cover self-adjoint boundary value problems having periodic boundary conditions, such as the problem

$$y'' + \lambda y = 0, \tag{21}$$

$$y(-L) - y(L) = 0, \qquad y'(-L) - y'(L) = 0 \tag{22}$$

considered in Example 4 of Section 11.2. The eigenfunctions of the problem (21), (22) are $\phi_n(x) = \cos(n\pi x/L)$ for $n = 0, 1, 2, \ldots$ and $\psi_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, \ldots$. If f is a given square integrable function on $-L \le x \le L$, then its expansion in terms of the eigenfunctions ϕ_n and ψ_n is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{23}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots, \tag{24}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{25}$$

This expansion is exactly the Fourier series for f discussed in Sections 10.2 and 10.3. According to the generalization of Theorem 11.6.1, the series (23) converges in the mean for any square integrable function f, even though f may not satisfy the conditions of Theorem 10.3.1, which assure pointwise convergence.

PROBLEMS

1. Extend the results of Example 1 by finding the smallest value of n for which $R_n < 0.02$, where R_n is given by Eq. (20).
2. Let $f(x) = x$ for $0 < x < 1$ and let $\phi_m(x) = \sqrt{2}\sin m\pi x$.
   (a) Find the coefficients b_m in the expansion of $f(x)$ in terms of $\phi_1(x), \phi_2(x), \ldots$.
   (b) Calculate the mean square error R_n for several values of n and plot the results.
   (c) Find the smallest value of n for which $R_n < 0.01$.
3. Follow the instructions for Problem 2 using $f(x) = x(1-x)$ for $0 < x < 1$.
4. In this problem we show that pointwise convergence of a sequence $S_n(x)$ does not imply mean convergence, and conversely.
   (a) Let $S_n(x) = n\sqrt{x}e^{-nx^2/2}$, $0 \le x \le 1$. Show that $S_n(x) \to 0$ as $n \to \infty$ for each x in $0 \le x \le 1$. Show also that
   $$R_n = \int_0^1 [0 - S_n(x)]^2\,dx = \frac{n}{2}(1 - e^{-n}),$$
   and hence that $R_n \to \infty$ as $n \to \infty$. Thus pointwise convergence does not imply mean convergence.
   (b) Let $S_n(x) = x^n$ for $0 \le x \le 1$ and let $f(x) = 0$ for $0 \le x \le 1$. Show that
   $$R_n = \int_0^1 [f(x) - S_n(x)]^2\,dx = \frac{1}{2n+1},$$
   and hence $S_n(x)$ converges to $f(x)$ in the mean. Also show that $S_n(x)$ does not converge to $f(x)$ pointwise throughout $0 \le x \le 1$. Thus mean convergence does not imply pointwise convergence.
5. Suppose that the functions $\phi_1, \ldots, \phi_n$ satisfy the orthonormality relation (1), and that a given function f is to be approximated by $S_n(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x)$, where the coefficients c_i are not necessarily those of Eq. (9). Show that the mean square error R_n given by Eq. (6) may be written in the form
   $$R_n = \int_0^1 r(x)\,f^2(x)\,dx - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n}(c_i - a_i)^2,$$
   where the a_i are the Fourier coefficients given by Eq. (9). Show that R_n is minimized if $c_i = a_i$ for each i.
6. In this problem we show by examples that the (Riemann) integrability of f and f^2 are independent.

(a) Let $f(x) = \begin{cases} x^{-1/2}, & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$

Show that $\int_0^1 f(x)\,dx$ exists as an improper integral, but $\int_0^1 f^2(x)\,dx$ does not.

(b) Let $f(x) = \begin{cases} 1, & x \text{ rational}, \\ -1, & x \text{ irrational}. \end{cases}$

Show that $\int_0^1 f^2(x)\,dx$ exists, but $\int_0^1 f(x)\,dx$ does not.

7. Suppose that it is desired to construct a set of polynomials $f_0(x)$, $f_1(x)$, $f_2(x), \ldots$, $f_k(x), \ldots$, where $f_k(x)$ is of degree k, that are orthonormal on the interval $0 \le x \le 1$. That is, the set of polynomials must satisfy

$$(f_j, f_k) = \int_0^1 f_j(x) f_k(x)\,dx = \delta_{jk}.$$

(a) Find $f_0(x)$ by choosing the polynomial of degree zero such that $(f_0, f_0) = 1$.
(b) Find $f_1(x)$ by determining the polynomial of degree one such that $(f_0, f_1) = 0$ and $(f_1, f_1) = 1$.
(c) Find $f_2(x)$.
(d) The normalization condition $(f_k, f_k) = 1$ is somewhat awkward to apply. Let $g_0(x)$, $g_1(x), \ldots, g_k(x), \ldots$ be the sequence of polynomials that are orthogonal on $0 \le x \le 1$ and that are normalized by the condition $g_k(1) = 1$. Find $g_0(x)$, $g_1(x)$, and $g_2(x)$ and compare them with $f_0(x)$, $f_1(x)$, and $f_2(x)$.

8. Suppose that it is desired to construct a set of polynomials $P_0(x)$, $P_1(x), \ldots, P_k(x), \ldots$, where $P_k(x)$ is of degree k, that are orthogonal on the interval $-1 \le x \le 1$; see Problem 7. Suppose further that $P_k(x)$ is normalized by the condition $P_k(1) = 1$. Find $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$. Note that these are the first four Legendre polynomials (see Problem 24 of Section 5.3).

9. This problem develops some further results associated with mean convergence. Let $R_n(a_1, \ldots, a_n)$, $S_n(x)$, and a_i be defined by Eqs. (6), (2), and (9), respectively.
(a) Show that

$$R_n = \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{n} a_i^2.$$

Hint: Substitute for $S_n(x)$ in Eq. (6) and integrate, using the orthogonality relation (1).

(b) Show that $\sum_{i=1}^{n} a_i^2 \le \int_0^1 r(x) f^2(x)\,dx$. This result is known as Bessel's inequality.

(c) Show that $\sum_{i=1}^{\infty} a_i^2$ converges.

(d) Show that $\lim_{n\to\infty} R_n = \int_0^1 r(x) f^2(x)\,dx - \sum_{i=1}^{\infty} a_i^2$.

(e) Show that $\sum_{i=1}^{\infty} a_i \phi_i(x)$ converges to $f(x)$ in the mean if and only if

$$\int_0^1 r(x) f^2(x)\,dx = \sum_{i=1}^{\infty} a_i^2.$$

This result is known as Parseval's equation.

In Problems 10 through 12 let $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ be the normalized eigenfunctions of the Sturm–Liouville problem (11), (12).

10. Show that if a_n is the nth Fourier coefficient of a square integrable function f, then $\lim_{n\to\infty} a_n = 0$.

Hint: Use Bessel's inequality, Problem 9(b).

11. Show that the series

$$\phi_1(x) + \phi_2(x) + \cdots + \phi_n(x) + \cdots$$

cannot be the eigenfunction series for any square integrable function.
Hint: See Problem 10.

12. Show that the series

$$\phi_1(x) + \frac{\phi_2(x)}{\sqrt{2}} + \cdots + \frac{\phi_n(x)}{\sqrt{n}} + \cdots$$

is not the eigenfunction series for any square integrable function.
Hint: Use Bessel's inequality, Problem 9(b).

13. Show that Parseval's equation in Problem 9(e) is obtained formally by squaring the series (10) corresponding to f, multiplying by the weight function r, and integrating term by term.

REFERENCES

The following books were mentioned in the text in connection with certain theorems about Sturm–Liouville problems:

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Sagan, H., *Boundary and Eigenvalue Problems in Mathematical Physics* (New York: Wiley, 1961; New York: Dover, 1989).

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The following books are convenient sources of numerical and graphical data about Bessel and Legendre functions:

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The following books also contain much information about Sturm–Liouville problems:

Cole, R. H., *Theory of Ordinary Differential Equations* (New York: Irvington, 1968).

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Answers to Problems

CHAPTER 1

Section 1.1, page 8

See SSM for detailed solutions to 2, 4, 7, 11, 13, 15ab.

16, 21, 22

1. $y \to 3/2$ as $t \to \infty$
2. y diverges from $3/2$ as $t \to \infty$
3. y diverges from $-3/2$ as $t \to \infty$
4. $y \to -1/2$ as $t \to \infty$
5. y diverges from $-1/2$ as $t \to \infty$
6. y diverges from -2 as $t \to \infty$
7. $y' = 3 - y$
8. $y' = 2 - 3y$
9. $y' = y - 2$
10. $y' = 3y - 1$
11. $y = 0$ and $y = 4$ are equilibrium solutions; $y \to 4$ if initial value is positive; y diverges from 0 if initial value is negative.
12. $y = 0$ and $y = 5$ are equilibrium solutions; y diverges from 5 if initial value is greater than 5; $y \to 0$ if initial value is less than 5.
13. $y = 0$ is equilibrium solution; $y \to 0$ if initial value is negative; y diverges from 0 if initial value is positive.
14. $y = 0$ and $y = 2$ are equilibrium solutions; y diverges from 0 if initial value is negative; $y \to 2$ if initial value is between 0 and 2; y diverges from 2 if initial value is greater than 2.
15. (a) $dq/dt = 300(10^{-2} - q10^{-6})$ (b) $q \to 10^4$ g; no
16. $dV/dt = -kV^{2/3}$ for some $k > 0$.
17. (a) $dq/dt = 500 - 0.4q$ (b) $q \to 1250$ mg
18. (a) $mv' = mg - kv^2$ (b) $v \to \sqrt{mg/k}$ (c) $k = 2/49$
19. y is asymptotic to $t - 3$ as $t \to \infty$
20. $y \to 0$ as $t \to \infty$
21. $y \to \infty$, 0, or $-\infty$ depending on the initial value of y
22. $y \to \infty$ or $-\infty$ depending on the initial value of y
23. $y \to \infty$ or $-\infty$ or y oscillates depending on the initial value of y

See SSM for detailed solution for 24.

24. $y \to -\infty$ or is asymptotic to $\sqrt{2t-1}$ depending on the initial value of y
25. $y \to 0$ and then fails to exist after some $t_f \geq 0$
26. $y \to \infty$ or $-\infty$ depending on the initial value of y

Section 1.2, page 14

1b

1. (a) $y = 5 + (y_0 - 5)e^{-t}$ (b) $y = (5/2) + [y_0 - (5/2)]e^{-2t}$
   (c) $y = 5 + (y_0 - 5)e^{-2t}$
   Equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b); solution approaches equilibrium faster in (b) and (c) than in (a).
2. (a) $y = 5 + (y_0 - 5)e^{t}$ (b) $y = (5/2) + [y_0 - (5/2)]e^{2t}$
   (c) $y = 5 + (y_0 - 5)e^{2t}$
   Equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b); solution diverges from equilibrium faster in (b) and (c) than in (a).
3. (a) $y = ce^{-at} + (b/a)$
   (c) (i) Equilibrium is lower and is approached more rapidly. (ii) Equilibrium is higher. (iii) Equilibrium remains the same and is approached more rapidly.

4ab, 6b, 8ab,10a, 13abc

4. (a) $y_1(t) = ce^{at}$ (b) $y = ce^{at} + (b/a)$
5. $y = ce^{-at} + (b/a)$
6. (a) $T = 2\ln 18 \cong 5.78$ months (b) $T = 2\ln[900/(900 - p_0)]$ months
   (c) $p_0 = 900(1 - e^{-6}) \cong 897.8$
7. (a) $r = (\ln 2)/30$ day^{-1} (b) $r = (\ln 2)/N$ day^{-1}
8. (a) $T = 5\ln 50 \cong 19.56$ sec (b) 718.34 m
9. (a) $dv/dt = 9.8, \quad v(0) = 0$ (b) $T = \sqrt{300/4.9} \cong 7.82$ sec
   (c) $v \cong 76.68$ m/sec
10. (a) $r \cong 0.02828$ day^{-1} (b) $Q(t) = 100e^{-0.02828t}$ (c) $T \cong 24.5$ days
12. $1620\ln(4/3)/\ln 2 \cong 672.4$ years
13. (a) $Q(t) = CV(1 - e^{-t/RC})$ (b) $Q(t) \to CV = Q_L$
    (c) $Q(t) = CV\exp[-(t - t_1)/RC]$
14. (a) $Q' = 3(1 - 10^{-4}Q), \quad Q(0) = 0$
    (b) $Q(t) = 10^4(1 - e^{-3t/10^4})$, t in hrs; after 1 year $Q \cong 9277.77$ g
    (c) $Q' = -3Q/10^4, \quad Q(0) = 9277.77$
    (d) $Q(t) = 9277.77e^{-3t/10^4}$, t in hrs; after 1 year $Q \cong 670.07$ g
    (e) $T \cong 2.60$ years
15. (a) $q' = -q/300, \quad q(0) = 5000$ g (b) $q(t) = 5000e^{-t/300}$ (c) no
    (d) $T = 300\ln(25/6) \cong 7.136$ hr
    (e) $r = 250\ln(25/6) \cong 256.78$ gal/min

Section 1.3, page 22

2, 6, 8, 14, 16, 19

1. Second order, linear
2. Second order, nonlinear
3. Fourth order, linear
4. First order, nonlinear
5. Second order, nonlinear
6. Third order, linear

15. $r = -2$
16. $r = \pm 1$
17. $r = 2, -3$
18. $r = 0, 1, 2$
19. $r = -1, -2$
20. $r = 1, 4$

22, 26

21. Second order, linear
22. Second order, nonlinear
23. Fourth order, linear
24. Second order, nonlinear

CHAPTER 2

Section 2.1, page 38

1, 2, 3

1. (c) $y = ce^{-3t} + (t/3) - (1/9) + e^{-2t}$; y is asymptotic to $t/3 - 1/9$ as $t \to \infty$
2. (c) $y = ce^{2t} + t^3e^{2t}/3$; $y \to \infty$ as $t \to \infty$
3. (c) $y = ce^{-t} + 1 + t^2e^{-t}/2$; $y \to 1$ as $t \to \infty$

See SSM for detailed solutions to 4, 6, 7, 11, 13, 15, 18, 20.

4. (c) $y = (c/t) + (3\cos 2t)/4t + (3\sin 2t)/2$; y is asymptotic to $(3\sin 2t)/2$ as $t \to \infty$
5. (c) $y = ce^{2t} - 3e^{t}$; $y \to \infty$ or $-\infty$ as $t \to \infty$
6. (c) $y = (c - t\cos t + \sin t)/t^2$; $y \to 0$ as $t \to \infty$
7. (c) $y = t^2e^{-t^2} + ce^{-t^2}$; $y \to 0$ as $t \to \infty$
8. (c) $y = (\arctan t + c)/(1 + t^2)^2$; $y \to 0$ as $t \to \infty$
9. (c) $y = ce^{-t/2} + 3t - 6$; y is asymptotic to $3t - 6$ as $t \to \infty$
10. (c) $y = -te^{-t} + ct$; $y \to \infty, 0$, or $-\infty$ as $t \to \infty$
11. (c) $y = ce^{-t} + \sin 2t - 2\cos 2t$; y is asymptotic to $\sin 2t - 2\cos 2t$ as $t \to \infty$
12. (c) $y = ce^{-t/2} + 3t^2 - 12t + 24$; y is asymptotic to $3t^2 - 12t + 24$ as $t \to \infty$
13. $y = 3e^{t} + 2(t-1)e^{2t}$
14. $y = (t^2 - 1)e^{-2t}/2$
15. $y = (3t^4 - 4t^3 + 6t^2 + 1)/12t^2$
16. $y = (\sin t)/t^2$
17. $y = (t+2)e^{2t}$
18. $y = t^{-2}[(\pi^2/4) - 1 - t\cos t + \sin t]$
19. $y = -(1+t)e^{-t}/t^4, \quad t \neq 0$
20. $y = (t - 1 + 2e^{-t})/t, \quad t \neq 0$

21b, 24bc, 28

21. (b) $y = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + (a + \frac{4}{5})e^{t/2}$; $a_0 = -\frac{4}{5}$
    (c) y oscillates for $a = a_0$
22. (b) $y = -3e^{t/3} + (a+3)e^{t/2}$; $a_0 = -3$
    (c) $y \to -\infty$ for $a = a_0$
23. (b) $y = te^{-t} + (ea - 1)e^{-t}/t$; $a_0 = 1/e$
    (c) $y \to 0$ as $t \to 0$ for $a = a_0$
24. (b) $y = -\cos t/t^2 + \pi^2 a/4t^2$; $a_0 = 4/\pi^2$
    (c) $y \to \frac{1}{2}$ as $t \to 0$ for $a = a_0$
25. $(t, y) = (1.364312, 0.820082)$
26. $y_0 = -1.642876$
27. (a) $y = 12 + \frac{8}{65}\cos 2t + \frac{64}{65}\sin 2t - \frac{788}{65}e^{-t/4}$; y oscillates about 12 as $t \to \infty$
    (b) $t = 10.065778$
28. $y_0 = -5/2$

30, 32, 35, 36, 37

29. $y_0 = -16/3$; $y \to -\infty$ as $t \to \infty$ for $y_0 = -16/3$
36. See Problem 2.
37. See Problem 4.

Section 2.2, page 45

1, 4

1. $3y^2 - 2x^3 = c$; $\quad y \neq 0$
2. $3y^2 - 2\ln|1 + x^3| = c$; $\quad x \neq -1, y \neq 0$
3. $y^{-1} + \cos x = c$ if $y \neq 0$; $\quad$ also $y = 0$; $\quad$ everywhere
4. $3y + y^2 - x^3 + x = c$; $\quad y \neq -3/2$
5. $2\tan 2y - 2x - \sin 2x = c$ if $\cos 2y \neq 0$; $\quad$ also $y = \pm(2n+1)\pi/4$ for any integer n; everywhere

6, 10ac, 13, 15, 17a

6. $y = \sin[\ln|x| + c]$ if $x \neq 0$ and $|y| < 1$; $\quad$ also $y = \pm 1$
7. $y^2 - x^2 + 2(e^{y} - e^{-x}) = c$; $\quad y + e^{y} \neq 0$
8. $3y + y^3 - x^3 = c$; $\quad$ everywhere
9. (a) $y = 1/(x^2 - x - 6)$
   (c) $-2 < x < 3$
10. (a) $y = -\sqrt{2x - 2x^2 + 4}$
    (c) $-1 < x < 2$
11. (a) $y = [2(1-x)e^{x} - 1]^{1/2}$
    (c) $-1.68 < x < 0.77$ approximately
12. (a) $r = 2/(1 - 2\ln\theta)$
    (c) $0 < \theta < \sqrt{e}$
13. (a) $y = -[2\ln(1 + x^2) + 4]^{1/2}$
    (c) $-\infty < x < \infty$
14. (a) $y = \left[3 - 2\sqrt{1 + x^2}\right]^{-1/2}$
    (c) $|x| < \frac{1}{2}\sqrt{5}$
15. (a) $y = -\frac{1}{2} + \frac{1}{2}\sqrt{4x^2 - 15}$
    (c) $x > \frac{1}{2}\sqrt{15}$
16. (a) $y = -\sqrt{(x^2 + 1)/2}$
    (c) $-\infty < x < \infty$

17c, 19ac

17. (a) $y = 5/2 - \sqrt{x^3 - e^{x} + 13/4}$
    (c) $-1.4445 < x < 4.6297$ approximately
18. (a) $y = -\frac{3}{4} + \frac{1}{4}\sqrt{65 - 8e^{x} - 8e^{-x}}$
    (c) $|x| < 2.0794$ approximately
19. (a) $y = [\pi - \arcsin(3\cos^2 x)]/3$
    (c) $|x - \pi/2| < 0.6155$
20. (a) $y = \left[\frac{3}{2}(\arcsin x)^2\right]^{1/3}$
    (c) $-1 < x < 1$

See SSM for detailed solutions to 21, 23.

21. $y^3 - 3y^2 - x - x^3 + 2 = 0, \quad |x| < 1$
22. $y^3 - 4y - x^3 = -1, \quad |x^3 - 1| < 16/3\sqrt{3}$ or $-1.28 < x < 1.60$
23. $y = -1/(x^2/2 + 2x - 1); \quad x = -2$
24. $y = -3/2 + \sqrt{2x - e^x + 13/4}; \quad x = \ln 2$
25. $y = -3/2 + \sqrt{\sin 2x + 1/4}; \quad x = \pi/4$
26. $y = \tan(x^2 + 2x); \quad x = -1$

25, 27ab

27. (a) $y \to 4$ if $y_0 > 0$; $\quad y = 0$ if $y_0 = 0$; $\quad y \to -\infty$ if $y_0 < 0$
    (b) $T = 3.29527$
28. (a) $y \to 4$ as $t \to \infty$
    (b) $T = 2.84367$
    (c) $3.6622 < y_0 < 4.4042$

29, 30bd, 31, 33

29. $x = \dfrac{c}{a}y + \dfrac{ad - bc}{a^2}\ln|ay + b| + k; \quad a \neq 0, \ ay + b \neq 0$
30. (e) $|y + 2x|^3|y - 2x| = c$
31. (b) $\arctan(y/x) - \ln|x| = c$
32. (b) $x^2 + y^2 - cx^3 = 0$
33. (b) $|y - x| = c|y + 3x|^5$; $\quad$ also $y = -3x$
34. (b) $|y + x|\,|y + 4x|^2 = c$

35b

35. (b) $2x/(x + y) + \ln|x + y| = c$; $\quad$ also $y = -x$
36. (b) $x/(x + y) + \ln|x| = c$; $\quad$ also $y = -x$
37. (b) $|x|^3|x^2 - 5y^2| = c$
38. (b) $c|x|^3 = |y^2 - x^2|$

Section 2.3, page 57

1, 2, 3

1. $t = 100 \ln 100$ min $\cong 460.5$ min
2. $Q(t) = 120\gamma[1 - \exp(-t/60)]; \quad 120\gamma$
3. $Q = 50e^{-0.2}(1 - e^{-0.2})$ lb $\cong 7.42$ lb

4, 7abc, 9, 10, 12a, 16a

4. $Q(t) = 200 + t - [100(200)^2/(200 + t)^2]$ lb, $\quad t < 300; \quad c = 121/125$ lb/gal; $\lim_{t\to\infty} c = 1$ lb/gal
5. (a) $Q(t) = \frac{63{,}150}{2501}e^{-t/50} + 25 - \frac{625}{2501}\cos t + \frac{25}{5002}\sin t$
   (c) level $= 25$; amplitude $= 25\sqrt{2501}/5002 \cong 0.24995$
6. (a) $(\ln 2)/r$ years $\quad$ (b) 9.90 years $\quad$ (c) 8.66%
7. (a) $k(e^{rt} - 1)/r$ $\quad$ (b) $k \cong \$3930$ $\quad$ (c) 9.77%
8. (a) A: \$337,733.85; $\quad$ B: \$250,579.41
   (b) A: $2000e^{30r}(e^{10r} - 1)/r$; $\quad$ B: $2000(e^{30r} - 1)/r$
   (d) $r \cong 0.0609$
9. $k = \$3086.64$/year; $\quad \$1259.92$
10. (a) \$89,034.79 $\quad$ (b) \$102,965.21
11. (a) \$99,498.08 $\quad$ (b) \$188,501.92
12. (a) $t \cong 135.36$ months
    (b) \$152,698.56
13. (a) $(k/r) + [S_0 - (k/r)]e^{rt}$ $\quad$ (b) rS_0 $\quad$ (c) $(1/r)\ln[k/(k - k_0)]$ years
    (d) $T \cong 8.66$ years $\quad$ (e) $rS_0e^{rT}/(e^{rT} - 1)$ $\quad$ (f) \$119,716
14. (a) 0.00012097 year^{-1} $\quad$ (b) $Q_0 \exp(-0.00012097t)$, $\quad t$ in years
    (c) 13,305 years
15. $P = 201{,}977.31 - 1977.31e^{(\ln 2)t}, \quad 0 \le t \le t_f \cong 6.6745$ (weeks)
16. (a) $\tau \cong 2.9632$; no
    (b) $\tau = 10 \ln 2 \cong 6.9315$
    (c) $\tau = 6.3805$
17. (b) $y_c \cong 0.83$

18

18. $t = \ln\frac{13}{8} / \ln\frac{13}{12}$ min $\cong 6.07$ min

See SSM for detailed solutions to 19, 20ab, 21ab, 22, 24ab

19. (a) $x = 0.04[1 - \exp(-t/12{,}000)]$ (b) $\tau \cong 36$ min
$_0 - k - (P/r)]e^{-rt/V}$; $\lim_{t\to\infty} c = k + (P/r)$
(b) $T = (V \ln 2)/r$; $T = (V \ln 10)/r$
(c) Superior, $T = 431$ years; Michigan, $T = 71.4$ years; Erie, $T = 6.05$ years; Ontario, $T = 17.6$ years
21. (a) 50.408 m (b) 5.248 sec 22. (a) 45.783 m (b) 5.129 sec
23. (a) 48.562 m (b) 5.194 sec

24cd, 26ab, 27ab, 28

24. (a) 176.7 ft/sec (b) 1074.5 ft (c) 15 ft/sec (d) 256.6 sec
25. (a) $x_m = -\dfrac{m^2 g}{k^2}\ln\left(1 + \dfrac{kv_0}{mg}\right) + \dfrac{mv_0}{k}$; $t_m = \dfrac{m}{k}\ln\left(1 + \dfrac{kv_0}{mg}\right)$
26. (a) $v = -(mg/k) + [v_0 + (mg/k)]\exp(-kt/m)$ (b) $v = v_0 - gt$; yes
(c) $v = 0$ for $t > 0$
27. (a) $v_L = 2a^2 g(\rho - \rho')/9\mu$ (b) $e = 4\pi a^3 g(\rho - \rho')/3E$
28. (a) 11.58 m/sec (b) 13.45 m (c) $k \geq 0.2394$ kg/sec

29b, 30, 31bdef

29. (a) $v = R\sqrt{2g/(R + x)}$ (b) 50.6 hr
30. $v_e = \left(\dfrac{2gR}{1 + \xi}\right)^{1/2}$; altitude $\cong 1536$ miles
31. (b) $x = ut\cos A$, $y = -gt^2/2 + ut\sin A + h$
(d) $-16L^2/(u^2\cos^2 A) + L\tan A + 3 \geq H$
(e) 0.63 rad $\leq A \leq$ 0.96 rad
(f) $u = 106.89$ ft/sec, $A = 0.7954$ rad
32. (a) $v = (u\cos A)e^{-rt}$, $w = -g/r + (u\sin A + g/r)e^{-rt}$
(b) $x = u\cos A(1 - e^{-rt})/r$, $y = -gt/r + (u\sin A + g/r)(1 - e^{-rt})/r + h$
(d) $u = 145.3$ ft/sec, $A = 0.644$ rad
33. (d) $k = 2.193$

Section 2.4, page 72

1, 4, 8, 11, 13, 17

1. $0 < t < 3$ 2. $0 < t < 4$
3. $\pi/2 < t < 3\pi/2$ 4. $-\infty < t < -2$
5. $-2 < t < 2$ 6. $1 < t < \pi$
7. $2t + 5y > 0$ or $2t + 5y < 0$ 8. $t^2 + y^2 < 1$
9. $1 - t^2 + y^2 > 0$ or $1 - t^2 + y^2 < 0$, $t \neq 0$, $y \neq 0$
10. Everywhere 11. $y \neq 0$, $y \neq 3$
12. $t \neq n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$; $y \neq -1$ 13. $y = \pm\sqrt{y_0^2 - 4t^2}$ if $y_0 \neq 0$; $|t| < |y_0|/2$
14. $y = [(1/y_0) - t^2]^{-1}$ if $y_0 \neq 0$; $y = 0$ if $y_0 = 0$; interval is $|t| < 1/\sqrt{y_0}$ if $y_0 > 0$; $-\infty < t < \infty$ if $y_0 \leq 0$
15. $y = y_0/\sqrt{2ty_0^2 + 1}$ if $y_0 \neq 0$; $y = 0$ if $y_0 = 0$; interval is $-1/2y_0^2 < t < \infty$ if $y_0 \neq 0$; $-\infty < t < \infty$ if $y_0 = 0$
16. $y = \pm\sqrt{\frac{2}{3}\ln(1 + t^3) + y_0^2}$; $-[1 - \exp(-3y_0^2/2)]^{1/3} < t < \infty$
17. $y \to 3$ if $y_0 > 0$; $y = 0$ if $y_0 = 0$; $y \to -\infty$ if $y_0 < 0$
18. $y \to -\infty$ if $y_0 < 0$; $y \to 0$ if $y_0 \geq 0$ 19. $y \to 0$ if $y_0 \leq 9$; $y \to \infty$ if $y_0 > 9$
20. $y \to -\infty$ if $y_0 < y_c \approx -0.019$; otherwise y is asymptotic to $\sqrt{t - 1}$
21. (a) No (b) Yes; set $t_0 = 1/2$ in Eq. (19) in text.
(c) $|y| \leq (4/3)^{3/2} \cong 1.5396$

22abc

22. (a) $y_1(t)$ is a solution for $t \geq 2$; $y_2(t)$ is a solution for all t
(b) f_y is not continuous at $(2, -1)$

See SSM for detailed solutions to 23a, 24, 25, 27ab

26. (a) $y_1(t) = \dfrac{1}{\mu(t)}$; $y_2(t) = \dfrac{1}{\mu(t)} \displaystyle\int_{t_0}^{t} \mu(s)g(s)\,ds$

28. $y = \pm[5t/(2 + 5ct^5)]^{1/2}$ 29. $y = r/(k + cre^{-rt})$

30. $y = \pm[\epsilon/(\sigma + c\epsilon e^{-2\epsilon t})]^{1/2}$

28, 29, 32

31. $y = \pm\left\{\mu(t)\Big/\left[2\displaystyle\int_{t_0}^{t} \mu(s)\,ds + c\right]\right\}^{1/2}$, where $\mu(t) = \exp(2\Gamma \sin t + 2Tt)$

32. $y = \frac{1}{2}(1 - e^{-2t})$ for $0 \le t \le 1$; $y = \frac{1}{2}(e^2 - 1)e^{-2t}$ for $t > 1$

33. $y = e^{-2t}$ for $0 \le t \le 1$; $y = e^{-(t+1)}$ for $t > 1$

Section 2.5, page 84

1. $y = 0$ is unstable
2. $y = -a/b$ is asymptotically stable, $y = 0$ is unstable

3, 6, 7c, 9

3. $y = 1$ is asymptotically stable, $y = 0$ and $y = 2$ are unstable
4. $y = 0$ is unstable 5. $y = 0$ is asymptotically stable
6. $y = 0$ is asymptotically stable
7. (c) $y = [y_0 + (1 - y_0)kt]/[1 + (1 - y_0)kt]$
8. $y = 1$ is semistable
9. $y = -1$ is asymptotically stable, $y = 0$ is semistable, $y = 1$ is unstable
10. $y = -1$ and $y = 1$ are asymptotically stable, $y = 0$ is unstable

11, 14, 16bc, 17a, 18abc

11. $y = 0$ is asymptotically stable, $y = b^2/a^2$ is unstable
12. $y = 2$ is asymptotically stable, $y = 0$ is semistable, $y = -2$ is unstable
13. $y = 0$ and $y = 1$ are semistable
15. (a) $\tau = (1/r)\ln 4$; 55.452 years (b) $T = (1/r)\ln[\beta(1 - \alpha)/(1 - \beta)\alpha]$; 175.78 years
16. (a) $y = 0$ is unstable, $y = K$ is asymptotically stable
    (b) Concave up for $0 < y \le K/e$, concave down for $K/e \le y < K$
17. (a) $y = K\exp\{[\ln(y_0/K)]e^{-rt}\}$ (b) $y(2) \cong 0.7153K \cong 57.6 \times 10^6$ kg
    (c) $\tau \cong 2.215$ years
18. (b) $(h/a)\sqrt{k/\alpha\pi}$; yes 19. (b) $k^2/2g(\alpha a)^2$
    (c) $k/\alpha \le \pi a^2$

20bd, 21abd, 24abc, 26ab

20. (c) $Y = Ey_2 = KE[1 - (E/r)]$ (d) $Y_m = Kr/4$ for $E = r/2$
21. (a) $y_{1,2} = K[1 \mp \sqrt{1 - (4h/rK)}]/2$
22. (a) $y = 0$ is unstable, $y = 1$ is asymptotically stable
    (b) $y = y_0/[y_0 + (1 - y_0)e^{-\alpha t}]$
23. (a) $y = y_0 e^{-\beta t}$ (b) $x = x_0\exp[-\alpha y_0(1 - e^{-\beta t})/\beta]$ (c) $x_0\exp(-\alpha y_0/\beta)$
24. (b) $z = 1/[\nu + (1 - \nu)e^{\beta t}]$ (c) 0.0927
26. (a) $\displaystyle\lim_{t\to\infty} x(t) = \min(p, q)$; $x(t) = \dfrac{pq[e^{\alpha(q-p)t} - 1]}{qe^{\alpha(q-p)t} - p}$

    (b) $\displaystyle\lim_{t\to\infty} x(t) = p$; $x(t) = \dfrac{p^2\alpha t}{p\alpha t + 1}$

Section 2.6, page 95

1. $x^2 + 3x + y^2 - 2y = c$ 2. Not exact

3

3. $x^3 - x^2y + 2x + 2y^3 + 3y = c$ 4. $x^2y^2 + 2xy = c$

5, 7, 9, 12, 14

5. $ax^2 + 2bxy + cy^2 = k$ 6. Not exact
7. $e^x \sin y + 2y\cos x = c$; also $y = 0$ 8. Not exact
9. $e^{xy}\cos 2x + x^2 - 3y = c$ 10. $y\ln x + 3x^2 - 2y = c$
11. Not exact 12. $x^2 + y^2 = c$
13. $y = \left[x + \sqrt{28 - 3x^2}\right]/2$, $|x| < \sqrt{28/3}$
14. $y = [x - (24x^3 + x^2 - 8x - 16)^{1/2}]/4$, $x > 0.9846$

15

15. $b = 3$; $x^2y^2 + 2x^3y = c$ 16. $b = 1$; $e^{2xy} + x^2 = c$

17. $\int N(x,y)\,dy + \int \left[M(x,y) - \int N_x(x,y)\,dy \right] dx$

See SSM for detailed solutions to 19, 22, 23

19. $x^2 + 2\ln|y| - y^{-2} = c$; also $y = 0$
20. $e^x \sin y + 2y\cos x = c$
21. $xy^2 - (y^2 - 2y + 2)e^y = c$
22. $x^2 e^x \sin y = c$
24. $\mu(t) = \exp \int R(t)\,dt$, where $t = xy$

25, 26, 27, 29, 31

25. $\mu(x) = e^{3x}$; $(3x^2y + y^3)e^{3x} = c$
26. $\mu(x) = e^{-x}$; $y = ce^x + 1 + e^{2x}$
27. $\mu(y) = y$; $xy + y\cos y - \sin y = c$
28. $\mu(y) = e^{2y}/y$; $xe^{2y} - \ln|y| = c$; also $y = 0$
29. $\mu(y) = \sin y$; $e^x \sin y + y^2 = c$
30. $\mu(y) = y^2$; $x^4 + 3xy + y^4 = c$
31. $\mu(x,y) = xy$; $x^3y + 3x^2 + y^3 = c$

Section 2.7, page 103

1d, 3a

1. (a) 1.2, 1.39, 1.571, 1.7439
   (b) 1.1975, 1.38549, 1.56491, 1.73658
   (c) 1.19631, 1.38335, 1.56200, 1.73308
   (d) 1.19516, 1.38127, 1.55918, 1.72968
2. (a) 1.1, 1.22, 1.364, 1.5368
   (b) 1.105, 1.23205, 1.38578, 1.57179
   (c) 1.10775, 1.23873, 1.39793, 1.59144
   (d) 1.1107, 1.24591, 1.41106, 1.61277

3bcd, 4d, 6, 9

3. (a) 1.25, 1.54, 1.878, 2.2736
   (b) 1.26, 1.5641, 1.92156, 2.34359
   (c) 1.26551, 1.57746, 1.94586, 2.38287
   (d) 1.2714, 1.59182, 1.97212, 2.42554
4. (a) 0.3, 0.538501, 0.724821, 0.866458
   (b) 0.284813, 0.513339, 0.693451, 0.831571
   (c) 0.277920, 0.501813, 0.678949, 0.815302
   (d) 0.271428, 0.490897, 0.665142, 0.799729
5. Converge for $y \geq 0$; undefined for $y < 0$
6. Converge for $y \geq 0$; diverge for $y < 0$
7. Converge
8. Converge for $|y(0)| < 2.37$ (approximately); diverge otherwise
9. Diverge
10. Diverge
11. (a) 2.30800, 2.49006, 2.60023, 2.66773, 2.70939, 2.73521
    (b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209
    (c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053
    (d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959
12. (a) 1.70308, 3.06605, 2.44030, 1.77204, 1.37348, 1.11925
    (b) 1.79548, 3.06051, 2.43292, 1.77807, 1.37795, 1.12191
    (c) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328
    (d) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411

13a, 15ac, 16

13. (a) −1.48849, −0.412339, 1.04687, 1.43176, 1.54438, 1.51971
    (b) −1.46909, −0.287883, 1.05351, 1.42003, 1.53000, 1.50549
    (c) −1.45865, −0.217545, 1.05715, 1.41486, 1.52334, 1.49879
    (d) −1.45212, −0.173376, 1.05941, 1.41197, 1.51949, 1.49490
14. (a) 0.950517, 0.687550, 0.369188, 0.145990, 0.0421429, 0.00872877
    (b) 0.938298, 0.672145, 0.362640, 0.147659, 0.0454100, 0.0104931
    (c) 0.932253, 0.664778, 0.359567, 0.148416, 0.0469514, 0.0113722
    (d) 0.928649, 0.660463, 0.357783, 0.148848, 0.0478492, 0.0118978
15. (a) −0.166134, −0.410872, −0.804660, 4.15867
    (b) −0.174652, −0.434238, −0.889140, −3.09810
16. A reasonable estimate for y at $t = 0.8$ is between 5.5 and 6. No reliable estimate is possible at $t = 1$ from the specified data.

See SSM for detailed solutions to 18, 22.

17. A reasonable estimate for y at $t = 2.5$ is between 18 and 19. No reliable estimate is possible at $t = 3$ from the specified data.
18. (b) $2.37 < \alpha_0 < 2.38$
19. (b) $0.67 < \alpha_0 < 0.68$

Section 2.8, page 113

1, 4a

1. $dw/ds = (s+1)^2 + (w+2)^2, \quad w(0) = 0$
2. $dw/ds = 1 - (w+3)^3, \quad w(0) = 0$
3. (a) $\phi_n(t) = \sum_{k=1}^{n} \dfrac{2^k t^k}{k!}$ (c) $\lim_{n\to\infty} \phi_n(t) = e^{2t} - 1$

4c, 7

4. (a) $\phi_n(t) = \sum_{k=1}^{n} \dfrac{(-1)^k t^k}{k!}$ (c) $\lim_{n\to\infty} \phi_n(t) = e^{-t} - 1$
5. (a) $\phi_n(t) = \sum_{k=1}^{n} (-1)^{k+1} t^{k+1}/(k+1)!2^{k-1}$
   (c) $\lim_{n\to\infty} \phi_n(t) = 4e^{-t/2} + 2t - 4$
6. (a) $\phi_n(t) = t - \dfrac{t^{n+1}}{(n+1)!}$
   (c) $\lim_{n\to\infty} \phi_n(t) = t$
7. (a) $\phi_n(t) = \sum_{k=1}^{n} \dfrac{t^{2k-1}}{1\cdot 3\cdot 5\cdots(2k-1)}$
8. (a) $\phi_n(t) = -\sum_{k=1}^{n} \dfrac{t^{3k-1}}{2\cdot 5\cdot 8\cdots(3k-1)}$
9. (a) $\phi_1(t) = \dfrac{t^3}{3}; \quad \phi_2(t) = \dfrac{t^3}{3} + \dfrac{t^7}{7\cdot 9}; \quad \phi_3(t) = \dfrac{t^3}{3} + \dfrac{t^7}{7\cdot 9} + \dfrac{2t^{11}}{3\cdot 7\cdot 9\cdot 11} + \dfrac{t^{15}}{(7\cdot 9)^2\cdot 15}$
10. (a) $\phi_1(t) = t; \quad \phi_2(t) = t - \dfrac{t^4}{4}; \quad \phi_3(t) = t - \dfrac{t^4}{4} + \dfrac{3t^7}{4\cdot 7} - \dfrac{3t^{10}}{16\cdot 10} + \dfrac{t^{13}}{64\cdot 13}$

11

11. (a) $\phi_1(t) = t, \quad \phi_2(t) = t - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \dfrac{t^6}{6!} + O(t^7),$

$$\phi_3(t) = t - \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{7t^5}{5!} + \frac{14t^6}{6!} + O(t^7),$$

$$\phi_4(t) = t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{7t^5}{5!} + \frac{31t^6}{6!} + O(t^7)$$

12. (a) $\phi_1(t) = -t - t^2 - \dfrac{t^3}{2},$

$$\phi_2(t) = -t - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{24} + O(t^7),$$

$$\phi_3(t) = -t - \frac{t^2}{2} + \frac{t^4}{12} - \frac{3t^5}{20} + \frac{4t^6}{45} + O(t^7),$$

$$\phi_4(t) = -t - \frac{t^2}{2} + \frac{t^4}{8} - \frac{7t^5}{60} + \frac{t^6}{15} + O(t^7)$$

Section 2.9, page 124

2, 5

1. $y_n = (-1)^n (0.9)^n y_0; \quad y_n \to 0$ as $n \to \infty$
2. $y_n = y_0/(n+1); \quad y_n \to 0$ as $n \to \infty$
3. $y_n = y_0\sqrt{(n+2)(n+1)/2}; \quad y_n \to \infty$ as $n \to \infty$
4. $y_n = \begin{cases} y_0, & \text{if } n = 4k \text{ or } n = 4k-1; \\ -y_0, & \text{if } n = 4k-2 \text{ or } n = 4k-3; \end{cases}$ y_n has no limit as $n \to \infty$
5. $y_n = (0.5)^n (y_0 - 12) + 12; \quad y_n \to 12$ as $n \to \infty$
6. $y_n = (-1)^n (0.5)^n (y_0 - 4) + 4; \quad y_n \to 4$ as $n \to \infty$
7. 7.25%
8. \$2283.63
9. \$258.14

See SSM for detailed solutions to 10, 13, 14, 17, 17a

10. (a) \$804.62 (b) \$877.57 (c) \$1028.61
11. 30 years: \$804.62/month; \$289,663.20 total 20 years: \$899.73/month; \$215,935.20 total
12. \$103,624.62 13. 9.73%
16. (b) $u_n \to -\infty$ as $n \to \infty$

17b, 18

19. (a) 4.7263 (b) 1.223% (c) 3.5643 (e) 3.5699

Miscellaneous Problems, page 126

1 - 7

1. $y = (c/x^2) + (x^3/5)$
2. $\arctan(y/x) - \ln\sqrt{x^2+y^2} = c$
3. $x^2 + xy - 3y - y^3 = 0$
4. $x = ce^y + ye^y$
5. $x^2y + xy^2 + x = c$
6. $y = x^{-1}(1 - e^{1-x})$
7. $(x^2 + y^2 + 1)e^{-y^2} = c$
8. $y = (4 + \cos 2 - \cos x)/x^2$
9. $x^2y + x + y^2 = c$
10. $(y^2/x^3) + (y/x^2) = c$
11. $x^3/3 + xy + e^y = c$
12. $y = ce^{-x} + e^{-x}\ln(1 + e^x)$
13. $2(y/x)^{1/2} - \ln|x| = c$; also $y = 0$
14. $x^2 + 2xy + 2y^2 = 34$

8 - 32

15. $y = c/\cosh^2(x/2)$
16. $(2/\sqrt{3})\arctan[(2y - x)/\sqrt{3}x] - \ln|x| = c$
17. $y = ce^{3x} - e^{2x}$
18. $y = cx^{-2} - x$
19. $3y - 2xy^3 - 10x = 0$
20. $e^x + e^{-y} = c$
21. $e^{-y/x} + \ln|x| = c$
22. $y^3 + 3y - x^3 + 3x = 2$
23. $\dfrac{1}{y} = -x\displaystyle\int \frac{e^{2x}}{x^2}\,dx + cx$; also $y = 0$
24. $\sin^2 x \sin y = c$
25. $x^2/y + \arctan(y/x) = c$
26. $x^2 + 2x^2y - y^2 = c$
27. $\sin x \cos 2y - \frac{1}{2}\sin^2 x = c$
28. $2xy + xy^3 - x^3 = c$
29. $\arcsin(y/x) - \ln|x| = c$; also $y = x$ and $y = -x$
30. $xy^2 - \ln|y| = 0$
31. $x + \ln|x| + x^{-1} + y - 2\ln|y| = c$; also $y = 0$
32. $x^3y^2 + xy^3 = -4$

CHAPTER 3

Section 3.1, page 136

3, 5, 7, 10, 15

1. $y = c_1e^t + c_2e^{-3t}$
2. $y = c_1e^{-t} + c_2e^{-2t}$
3. $y = c_1e^{t/2} + c_2e^{-t/3}$
4. $y = c_1e^{t/2} + c_2e^t$
5. $y = c_1 + c_2e^{-5t}$
6. $y = c_1e^{3t/2} + c_2e^{-3t/2}$
7. $y = c_1\exp[(9 + 3\sqrt{5})t/2] + c_2\exp[(9 - 3\sqrt{5})t/2]$
8. $y = c_1\exp[(1 + \sqrt{3})t] + c_2\exp[(1 - \sqrt{3})t]$
9. $y = e^t$; $y \to \infty$ as $t \to \infty$
10. $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$; $y \to 0$ as $t \to \infty$
11. $y = 12e^{t/3} - 8e^{t/2}$; $y \to -\infty$ as $t \to \infty$
12. $y = -1 - e^{-3t}$; $y \to -1$ as $t \to \infty$
13. $y = \frac{1}{26}(13 + 5\sqrt{13})\exp[(-5 + \sqrt{13})t/2] + \frac{1}{26}(13 - 5\sqrt{13})\exp[(-5 - \sqrt{13})t/2]$; $y \to 0$ as $t \to \infty$
14. $y = (2/\sqrt{33})\exp[(-1 + \sqrt{33})t/4] - (2/\sqrt{33})\exp[(-1 - \sqrt{33})t/4]$; $y \to \infty$ as $t \to \infty$
15. $y = \frac{1}{10}e^{-9(t-1)} + \frac{9}{10}e^{t-1}$; $y \to \infty$ as $t \to \infty$
16. $y = -\frac{1}{2}e^{(t+2)/2} + \frac{3}{2}e^{-(t+2)/2}$; $y \to -\infty$ as $t \to \infty$

17, 19, 21

17. $y'' + y' - 6y = 0$
18. $2y'' + 5y' + 2y = 0$
19. $y = \frac{1}{4}e^t + e^{-t}$; minimum is $y = 1$ at $t = \ln 2$
20. $y = -e^t + 3e^{t/2}$; maximum is $y = \frac{9}{4}$ at $t = \ln(9/4)$, $y = 0$ at $t = \ln 9$
21. $\alpha = -2$
22. $\beta = -1$

See SSM for detailed solutions to 24, 25a.

25c, 27, 28, 30, 34

23. $y \to 0$ for $\alpha < 0$; y becomes unbounded for $\alpha > 1$
24. $y \to 0$ for $\alpha < 1$; there is no α for which all nonzero solutions become unbounded.
25. (a) $y = \frac{1}{5}(1+2\beta)e^{-2t} + \frac{1}{5}(4-2\beta)e^{t/2}$
    (b) $y \cong 0.71548$ when $t = \frac{2}{5}\ln 6 \cong 0.71670$ (c) $\beta = 2$
26. (a) $y = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$
    (b) $t_m = \ln[(12+3\beta)/(12+2\beta)]$, $y_m = \frac{4}{27}(6+\beta)^3/(4+\beta)^2$
    (c) $\beta = 6(1+\sqrt{3}) \cong 16.3923$ (d) $t_m \to \ln(3/2)$, $y_m \to \infty$
27. $y'' + 3y' + 2y = 0$ is one such equation.
28. $y = c_1 t^{-1} + c_2 + \ln t$
29. $y = c_1 \ln t + c_2 + t$
30. $y = (1/k)\ln|(k-t)/(k+t)| + c_2$ if $c_1 = k^2 > 0$; $y = (2/k)\arctan(t/k) + c_2$ if $c_1 = -k^2 < 0$; $y = -2t^{-1} + c_2$ if $c_1 = 0$; also $y = c$
31. $y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2$; also $y = c$ *Hint:* $\mu(v) = v^{-3}$ is an integrating factor.
32. $y = c_1 e^{-t} + c_2 - te^{-t}$
33. $c_1^2 y = c_1 t - \ln|1 + c_1 t| + c_2$ if $c_1 \neq 0$; $y = \frac{1}{2}t^2 + c_2$ if $c_1 = 0$; also $y = c$
34. $y^2 = c_1 t + c_2$
35. $y = c_1 \sin(t + c_2) = k_1 \sin t + k_2 \cos t$
36. $\frac{1}{3}y^3 - 2c_1 y + c_2 = 2t$; also $y = c$
37. $t + c_2 = \pm\frac{2}{3}(y - 2c_1)(y + c_1)^{1/2}$
38. $y\ln|y| - y + c_1 y + t = c_2$; also $y = c$
39. $e^y = (t + c_2)^2 + c_1$

39, 40, 43

40. $y = \frac{4}{3}(t+1)^{3/2} - \frac{1}{3}$
41. $y = 2(1-t)^{-2}$
42. $y = 3\ln t - \frac{3}{2}\ln(t^2+1) - 5\arctan t + 2 + \frac{3}{2}\ln 2 + \frac{5}{4}\pi$
43. $y = \frac{1}{2}t^2 + \frac{3}{2}$

Section 3.2, page 145

2, 4, 8

1. $-\frac{7}{2}e^{t/2}$
2. 1
3. e^{-4t}
4. $x^2 e^x$
5. $-e^{2t}$
6. 0
7. $0 < t < \infty$
8. $-\infty < t < 1$
9. $0 < t < 4$
10. $0 < t < \infty$

12, 14, 15, 18, 21

11. $0 < x < 3$
12. $2 < x < 3\pi/2$
14. The equation is nonlinear.
15. The equation is nonhomogeneous.
16. No
17. $3te^{2t} + ce^{2t}$
18. $te^t + ct$
19. $5W(f, g)$
20. $-4(t\cos t - \sin t)$
21. $y_1(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$, $y_2(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t$
22. $y_1(t) = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)}$, $y_2(t) = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}$
23. Yes
24. Yes

25, 27, 30, 32, 34, 36

25. Yes
26. Yes
28. Yes, $y = c_1 e^{-x^2/2}\int_{x_0}^{x} e^{t^2/2}\,dt + c_2 e^{-x^2/2}$
29. No
30. Yes, $y = \dfrac{1}{\mu(x)}\left[c_1\displaystyle\int_{x_0}^{x}\frac{\mu(t)}{t}\,dt + c_2\right]$, $\mu(x) = \exp\left[-\displaystyle\int\left(\frac{1}{x} + \frac{\cos x}{x}\right)dx\right]$
31. Yes, $y = c_1 x^{-1} + c_2 x$
33. $x^2\mu'' + 3x\mu' + (1 + x^2 - \nu^2)\mu = 0$
34. $(1-x^2)\mu'' - 2x\mu' + \alpha(\alpha+1)\mu = 0$
35. $\mu'' - x\mu = 0$

37

37. The Legendre and Airy equations are self-adjoint.

Section 3.3, page 152

2, 6, 7, 12, 15

1. Independent
2. Dependent
3. Independent
4. Dependent
5. Dependent
6. Independent
7. Independent if origin is interior to interval; otherwise dependent

8. Independent if origin is interior to interval; otherwise dependent
9. Independent; W is not always zero
10. Independent; W is not always zero
11. $W(c_1y_1, c_2y_2) = c_1c_2W(y_1, y_2) \neq 0$
12. $W(y_3, y_4) = -2W(y_1, y_2)$
13. $a_1b_2 - a_2b_1 \neq 0$
15. ct^2e^t
16. $c\cos t$
17. c/x
18. $c/(1-x^2)$
20. $2/25$
21. $3\sqrt{e} \cong 4.946$
22. $p(t) = 0$ for all t
26. If t_0 is an inflection point, and $y = \phi(t)$ is a solution, then from the differential equation $p(t_0)\phi'(t_0) + q(t_0)\phi(t_0) = 0$.

See SSM for detailed solutions to 20, 24, 26, 27

28

Section 3.4, page 158

1, 5, 7

1. $e\cos 2 + ie\sin 2 \cong -1.1312 + 2.4717i$
2. $e^2\cos 3 - ie^2\sin 3 \cong -7.3151 - 1.0427i$
3. -1
4. $e^2\cos(\pi/2) - ie^2\sin(\pi/2) = -e^2i \cong -7.3891i$
5. $2\cos(\ln 2) - 2i\sin(\ln 2) \cong 1.5385 - 1.2779i$
6. $\pi^{-1}\cos(2\ln\pi) + i\pi^{-1}\sin(2\ln\pi) \cong -0.20957 + 0.23959i$
7. $y = c_1e^t\cos t + c_2e^t\sin t$
8. $y = c_1e^t\cos\sqrt{5}t + c_2e^t\sin\sqrt{5}t$
9. $y = c_1e^{2t} + c_2e^{-4t}$
10. $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$

11, 14, 18, 22, 23a

11. $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$
12. $y = c_1\cos(3t/2) + c_2\sin(3t/2)$
13. $y = c_1e^{-t}\cos(t/2) + c_2e^{-t}\sin(t/2)$
14. $y = c_1e^{t/3} + c_2e^{-4t/3}$
15. $y = c_1e^{-t/2}\cos t + c_2e^{-t/2}\sin t$
16. $y = c_1e^{-2t}\cos(3t/2) + c_2e^{-2t}\sin(3t/2)$
17. $y = \frac{1}{2}\sin 2t$; steady oscillation
18. $y = e^{-2t}\cos t + 2e^{-2t}\sin t$; decaying oscillation
19. $y = -e^{t-\pi/2}\sin 2t$; growing oscillation
20. $y = (1+2\sqrt{3})\cos t - (2-\sqrt{3})\sin t$; steady oscillation
21. $y = 3e^{-t/2}\cos t + \frac{5}{2}e^{-t/2}\sin t$; decaying oscillation
22. $y = \sqrt{2}e^{-(t-\pi/4)}\cos t + \sqrt{2}\,e^{-(t-\pi/4)}\sin t$; decaying oscillation
23. (a) $u = 2e^{t/6}\cos(\sqrt{23}\,t/6) - (2/\sqrt{23})e^{t/6}\sin(\sqrt{23}\,t/6)$
    (b) $t = 10.7598$

23b, 25abcd, 31, 33

24. (a) $u = 2e^{-t/5}\cos(\sqrt{34}\,t/5) + (7/\sqrt{34})\,e^{-t/5}\sin(\sqrt{34}\,t/5)$
    (b) $T = 14.5115$
25. (a) $y = 2e^{-t}\cos\sqrt{5}t + [(\alpha+2)/\sqrt{5}]\,e^{-t}\sin\sqrt{5}t$ (b) $\alpha = 1.50878$
    (c) $t = \{\pi - \arctan[2\sqrt{5}/(2+\alpha)]\}/\sqrt{5}$ (d) $\pi/\sqrt{5}$
26. (a) $y = e^{-at}\cos t + ae^{-at}\sin t$ (b) $T = 1.8763$
    (c) $\alpha = \frac{1}{4}$, $T = 7.4284$; $\alpha = \frac{1}{2}$, $T = 4.3003$; $\alpha = 2$, $T = 1.5116$

35, 38, 39

35. Yes, $y = c_1\cos x + c_2\sin x$, $x = \int e^{-t^2/2}\,dt$
36. No
37. Yes, $y = c_1e^{-t^2/4}\cos(\sqrt{3}\,t^2/4) + c_2e^{-t^2/4}\sin(\sqrt{3}\,t^2/4)$
39. $y = c_1\cos(\ln t) + c_2\sin(\ln t)$
40. $y = c_1t^{-1} + c_2t^{-2}$
41. $y = c_1t^{-1}\cos(\frac{1}{2}\ln t) + c_2t^{-1}\sin(\frac{1}{2}\ln t)$
42. $y = c_1t^6 + c_2t^{-1}$

Section 3.5, page 166

1, 9, 12

1. $y = c_1e^t + c_2te^t$
2. $y = c_1e^{-t/3} + c_2te^{-t/3}$
3. $y = c_1e^{-t/2} + c_2e^{3t/2}$
4. $y = c_1e^{-3t/2} + c_2te^{-3t/2}$
5. $y = c_1e^t\cos 3t + c_2e^t\sin 3t$
6. $y = c_1e^{3t} + c_2te^{3t}$
7. $y = c_1e^{-t/4} + c_2e^{-4t}$
8. $y = c_1e^{-3t/4} + c_2te^{-3t/4}$
9. $y = c_1e^{2t/5} + c_2te^{2t/5}$
10. $y = e^{-t/2}\cos(t/2) + c_2e^{-t/2}\sin(t/2)$
11. $y = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$, $y \to -\infty$ as $t \to \infty$
12. $y = 2te^{3t}$, $y \to \infty$ as $t \to \infty$
13. $y = -e^{-t/3}\cos 3t + \frac{5}{9}e^{-t/3}\sin 3t$, $y \to 0$ as $t \to \infty$

See SSM for detailed solutions to 14, 17ab

14. $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}$, $y \to 0$ as $t \to \infty$
15. (a) $y = e^{-3t/2} - \frac{5}{2}te^{-3t/2}$ (b) $t = \frac{2}{5}$
    (c) $t_0 = 16/15$, $y_0 = -\frac{5}{3}e^{-8/5} \cong -0.33649$
    (d) $y = e^{-3t/2} + (b + \frac{3}{2})te^{-3t/2}$; $b = -\frac{3}{2}$
16. $y = 2e^{t/2} + (b-1)te^{t/2}$; $b = 1$
17. (a) $y = e^{-t/2} + \frac{5}{2}te^{-t/2}$ (b) $t_0 = \frac{8}{5}$, $y_0 = 5e^{-4/5} \cong 2.24664$

17cd, 19, 21, 25

(c) $y = e^{-t/2} + (b + \frac{1}{2})te^{-t/2}$
(d) $t_M = 4b/(1+2b) \to 2$ as $b \to \infty$; $y_M = (1+2b)\exp[-2b/(1+2b)] \to \infty$ as $b \to \infty$

18. (a) $y = ae^{-2t/3} + (\frac{2}{3}a - 1)te^{-2t/3}$ (b) $a = \frac{3}{2}$

23. $y_2(t) = t^3$
24. $y_2(t) = t^{-2}$
25. $y_2(t) = t^{-1}\ln t$
26. $y_2(t) = te^t$

27, 30, 31b

27. $y_2(x) = \cos x^2$
28. $y_2(x) = x$
29. $y_2(x) = x^{1/4}e^{-2\sqrt{x}}$
30. $y_2(x) = x^{-1/2}\cos x$
32. $y = c_1e^{-\delta x^2/2}\int_0^x e^{\delta s^2/2}\,ds + c_2e^{-\delta x^2/2}$

33, 35, 38

33. $y_2(t) = y_1(t)\int_{t_0}^{t} y_1^{-2}(s)\exp\left[-\int_{s_0}^{s} p(r)\,dr\right]ds$
34. $y_2(t) = t^{-1}\ln t$
35. $y_2(t) = \cos t^2$
36. $y_2(x) = x$
37. $y_2(x) = x^{-1/2}\cos x$
39. (b) $y_0 + (a/b)y_0'$
41. $y = c_1t^2 + c_2t^2\ln t$

42

42. $y = c_1t^{-1/2} + c_2t^{-1/2}\ln t$

Section 3.6, page 178

1, 4, 6, 8

1. $y = c_1e^{3t} + c_2e^{-t} - e^{2t}$
2. $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$
3. $y = c_1e^{3t} + c_2e^{-t} + \frac{3}{16}te^{-t} + \frac{3}{8}t^2e^{-t}$
4. $y = c_1 + c_2e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$
5. $y = c_1\cos 3t + c_2\sin 3t + \frac{1}{162}(9t^2 - 6t + 1)e^{3t} + \frac{2}{3}$
6. $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$
7. $y = c_1e^{-t} + c_2e^{-t/2} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$
8. $y = c_1\cos t + c_2\sin t - \frac{1}{3}t\cos 2t - \frac{5}{9}\sin 2t$
9. $u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + (\omega_0^2 - \omega^2)^{-1}\cos\omega t$
10. $u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + (1/2\omega_0)t\sin\omega_0 t$

11, 13, 16

11. $y = c_1e^{-t/2}\cos(\sqrt{15}\,t/2) + c_2e^{-t/2}\sin(\sqrt{15}\,t/2) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$
12. $y = c_1e^{-t} + c_2e^{2t} + \frac{1}{6}te^{2t} + \frac{1}{8}e^{-2t}$
13. $y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}$
14. $y = \frac{7}{10}\sin 2t - \frac{19}{40}\cos 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$
15. $y = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4$
16. $y = e^{3t} + \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$
17. $y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t$
18. $y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$

19a, 22a

19. (a) $Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + D\sin 3t + E\cos 3t$
    (b) $A_0 = 2/15$, $A_1 = -2/9$, $A_2 = 8/27$, $A_3 = -8/27$, $A_4 = 16/81$, $B_0 = -1/9$, $B_1 = -1/9$, $B_2 = -2/27$, $D = -1/18$, $E = -1/18$
20. (a) $Y(t) = A_0t + A_1 + t(B_0t + B_1)\sin t + t(D_0t + D_1)\cos t$
    (b) $A_0 = 1$, $A_1 = 0$, $B_0 = 0$, $B_1 = 1/4$, $D_0 = -1/4$, $D_1 = 0$
21. (a) $Y(t) = e^t(A\cos 2t + B\sin 2t) + (D_0t + D_1)e^{2t}\sin t + (E_0t + E_1)e^{2t}\cos t$
    (b) $A = -1/20$, $B = -3/20$, $D_0 = -3/2$, $D_1 = -5$, $E_0 = 3/2$, $E_1 = 1/2$
22. (a) $Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}\cos t + t(D_0t^2 + D_1t + D_2)e^{-t}\sin t$
    (b) $A = 3$, $B_0 = -2/3$, $B_1 = 0$, $B_2 = 1$, $D_0 = 0$, $D_1 = 1$, $D_2 = 1$

23. (a) $Y(t) = A_0t^2 + A_1t + A_2 + t^2(B_0t + B_1)e^{2t} + (D_0t + D_1)\sin 2t + (E_0t + E_1)\cos 2t$
(b) $A_0 = 1/2,\ A_1 = 1,\ A_2 = 3/4,\ B_0 = 2/3,\ B_1 = 0,\ D_0 = 0,\ D_1 = -1/16,$ $E_0 = 1/8,\ E_1 = 1/16$
24. (a) $Y(t) = t(A_0t^2 + A_1t + A_2)\sin 2t + t(B_0t^2 + B_1t + B_2)\cos 2t$
(b) $A_0 = 0,\ A_1 = 13/16,\ A_2 = 7/4,\ B_0 = -1/12,\ B_1 = 0,\ B_2 = 13/32$
25. (a) $Y(t) = (A_0t^2 + A_1t + A_2)e^t\sin 2t + (B_0t^2 + B_1t + B_2)e^t\cos 2t$ $+ e^{-t}(D\cos t + E\sin t) + Fe^t$
(b) $A_0 = 1/52,\ A_1 = 10/169,\ A_2 = -1233/35152,\ B_0 = -5/52,\ B_1 = 73/676,$ $B_2 = -4105/35152,\ D = -3/2,\ E = 3/2,\ F = 2/3$
26. (a) $Y(t) = t(A_0t + A_1)e^{-t}\cos 2t + t(B_0t + B_1)e^{-t}\sin 2t$ $+ (D_0t + D_1)e^{-2t}\cos t + (E_0t + E_1)e^{-2t}\sin t$
(b) $A_0 = 0,\ A_1 = 3/16,\ B_0 = 3/8,\ B_1 = 0,\ D_0 = -2/5,\ D_1 = -7/25,\ E_0 = 1/5,$ $E_1 = 1/25$
27. $y = c_1\cos\lambda t + c_2\sin\lambda t + \sum_{m=1}^{N}[a_m/(\lambda^2 - m^2\pi^2)]\sin m\pi t$

See SSM for detailed solutions to 28

28. $y = \begin{cases} t, & 0 \le t \le \pi \\ -(1+\pi/2)\sin t - (\pi/2)\cos t + (\pi/2)e^{\pi - t}, & t > \pi \end{cases}$

29. $y = \begin{cases} \frac{1}{5} - \frac{1}{10}e^{-t}\sin 2t - \frac{1}{5}e^{-t}\cos 2t, & 0 \le t \le \pi/2 \\ -\frac{1}{5}(1 + e^{\pi/2})e^{-t}\cos 2t - \frac{1}{10}(1 + e^{\pi/2})e^{-t}\sin 2t, & t > \pi/2 \end{cases}$

30, 33

30. No

33. $y = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}$

Section 3.7, page 183

2

1. $Y(t) = e^t$
2. $Y(t) = -\frac{2}{3}te^{-t}$
3. $Y(t) = \frac{3}{2}t^2e^{-t}$
4. $Y(t) = 2t^2e^{t/2}$

5, 11, 14

5. $y = c_1\cos t + c_2\sin t - (\cos t)\ln(\tan t + \sec t)$
6. $y = c_1\cos 3t + c_2\sin 3t + (\sin 3t)\ln(\tan 3t + \sec 3t) - 1$
7. $y = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln t$
8. $y = c_1\cos 2t + c_2\sin 2t + \frac{3}{4}(\sin 2t)\ln\sin 2t - \frac{3}{2}t\cos 2t$
9. $y = c_1\cos(t/2) + c_2\sin(t/2) + t\sin(t/2) + 2[\ln\cos(t/2)]\cos(t/2)$
10. $y = c_1e^t + c_2te^t - \frac{1}{2}e^t\ln(1 + t^2) + te^t\arctan t$
11. $y = c_1e^{2t} + c_2e^{3t} + \int[e^{3(t-s)} - e^{2(t-s)}]\,g(s)\,ds$
12. $y = c_1\cos 2t + c_2\sin 2t + \frac{1}{2}\int[\sin 2(t-s)]\,g(s)\,ds$
13. $Y(t) = \frac{1}{2} + t^2\ln t$
14. $Y(t) = -2t^2$
15. $Y(t) = \frac{1}{2}(t-1)e^{2t}$
16. $Y(t) = -\frac{1}{2}(2t-1)e^{-t}$
17. $Y(x) = \frac{1}{6}x^2(\ln x)^3$
18. $Y(x) = -\frac{3}{2}x^{1/2}\cos x$

18, 22

19. $Y(x) = \int\frac{xe^t - te^x}{(1-t)^2e^t}g(t)\,dt$
20. $Y(x) = x^{-1/2}\int t^{-3/2}\sin(x-t)g(t)\,dt$
23. (b) $y = y_0\cos t + y_0'\sin t + \int_{t_0}^{t}\sin(t-s)g(s)\,ds$
24. $y = (b-a)^{-1}\int_{t_0}^{t}[e^{b(t-s)} - e^{a(t-s)}]\,g(s)\,ds$

25, 29

25. $y = \mu^{-1}\int_{t_0}^{t}e^{\lambda(t-s)}\sin\mu(t-s)g(s)\,ds$
26. $y = \int_{t_0}^{t}(t-s)e^{a(t-s)}g(s)\,ds$
29. $y = c_1t + c_2t^2 + 4t^2\ln t$
30. $y = c_1t^{-1} + c_2t^{-5} + \frac{1}{12}t$
31. $y = c_1(1+t) + c_2e^t + \frac{1}{2}(t-1)e^{2t}$
32. $y = c_1e^t + c_2t - \frac{1}{2}(2t-1)e^{-t}$

Section 3.8, page 197

See SSM for detailed solutions to 2, 6

1. $u = 5\cos(2t - \delta), \quad \delta = \arctan(4/3) \cong 0.9273$
2. $u = 2\cos(t - 2\pi/3)$
3. $u = 2\sqrt{5}\cos(3t - \delta), \quad \delta = -\arctan(1/2) \cong -0.4636$
4. $u = \sqrt{13}\cos(\pi t - \delta), \quad \delta = \pi + \arctan(3/2) \cong 4.1244$
5. $u = \frac{1}{4}\cos 8t$ ft, t in sec; $\omega = 8$ rad/sec, $T = \pi/4$ sec, $R = 1/4$ ft
6. $u = \frac{5}{7}\sin 14t$ cm, t in sec; $t = \pi/14$ sec
7. $u = (1/4\sqrt{2})\sin(8\sqrt{2}\,t) - \frac{1}{12}\cos(8\sqrt{2}\,t)$ ft, t in sec; $\omega = 8\sqrt{2}$ rad/sec, $T = \pi/4\sqrt{2}$ sec, $R = \sqrt{11/288} \cong 0.1954$ ft, $\delta = \pi - \arctan(3/\sqrt{2}) \cong 2.0113$

8, 9, 12, 17

8. $Q = 10^{-6}\cos 2000t$ coulombs, t in sec
9. $u = e^{-10t}[2\cos(4\sqrt{6}\,t) + (5/\sqrt{6})\sin(4\sqrt{6}\,t)]$ cm, t in sec; $\mu = 4\sqrt{6}$ rad/sec, $T_d = \pi/2\sqrt{6}$ sec, $T_d/T = 7/2\sqrt{6} \cong 1.4289$, $\tau \cong 0.4045$ sec
10. $u = (1/8\sqrt{31})e^{-2t}\sin(2\sqrt{31}\,t)$ ft, t in sec; $t = \pi/2\sqrt{31}$ sec, $\tau \cong 1.5927$ sec
11. $u \cong 0.057198e^{-0.15t}\cos(3.87008\,t - 0.50709)$ m, t in sec; $\mu = 3.87008$ rad/sec, $\mu/\omega_0 = 3.87008/\sqrt{15} \cong 0.99925$
12. $Q = 10^{-6}(2e^{-500t} - e^{-1000t})$ coulombs; t in sec
13. $\gamma = \sqrt{20/9} \cong 1.4907$
16. $r = \sqrt{A^2 + B^2}$, $r\cos\theta = B$, $r\sin\theta = -A$; $R = r$; $\delta = \theta + (4n+1)\pi/2$, $n = 0, 1, 2, \ldots$

19, 20, 23, 24

17. $\gamma = 8$ lb-sec/ft
18. $R = 10^3$ ohms
20. $v_0 < -\gamma u_0/2m$
22. $2\pi/\sqrt{31}$
23. $\gamma = 5$ lb-sec/ft
24. $k = 6$, $v = \pm 2\sqrt{5}$
25. (a) $\tau \cong 41.715$ (d) $\gamma_0 \cong 1.73$, $\min \tau \cong 4.87$
    (e) $\tau = (2/\gamma)\ln(400/\sqrt{4 - \gamma^2})$
26. (a) $u(t) = e^{-\gamma t/2m}\left[u_0\sqrt{4km - \gamma^2}\cos\mu t + (2mv_0 + \gamma u_0)\sin\mu t\right]/\sqrt{4km - \gamma^2}$
    (b) $R^2 = 4m(ku_0^2 + \gamma u_0 v_0 + mv_0^2)/(4km - \gamma^2)$

27, 29ac, 30a

27. $\rho l u'' + \rho_0 g u = 0, \quad T = 2\pi\sqrt{\rho l/\rho_0 g}$
28. (a) $u = \sqrt{2}\sin\sqrt{2}\,t$ (c) clockwise
29. (a) $u = (16/\sqrt{127})e^{-t/8}\sin(\sqrt{127}\,t/8)$ (c) clockwise
30. (b) $u = a\cos(\sqrt{k/m}\,t) + b\sqrt{m/k}\sin(\sqrt{k/m}\,t)$
32. (b) $u = \sin t$, $A = 1$, $T = 2\pi$ (c) $A = 0.98$, $T = 6.07$
    (d) $\epsilon = 0.2$, $A = 0.96$, $T = 5.90$; $\epsilon = 0.3$, $A = 0.94$, $T = 5.74$
    (f) $\epsilon = -0.1$, $A = 1.03$, $T = 6.55$; $\epsilon = -0.2$, $A = 1.06$, $T = 6.90$; $\epsilon = -0.3$, $A = 1.11$, $T = 7.41$

Section 3.9, page 205

1, 5, 7ac, 10, 11a

1. $-2\sin 8t\sin t$
2. $2\sin(t/2)\cos(13t/2)$
3. $2\cos(3\pi t/2)\cos(\pi t/2)$
4. $2\sin(7t/2)\cos(t/2)$
5. $u'' + 256u = 16\cos 3t$, $u(0) = \frac{1}{6}$, $u'(0) = 0$, u in ft, t in sec
6. $u'' + 10u' + 98u = 2\sin(t/2)$, $u(0) = 0$, $u'(0) = 0.03$, u in m, t in sec
7. (a) $u = \frac{151}{1482}\cos 16t + \frac{16}{247}\cos 3t$ (c) $\omega = 16$ rad/sec
8. (a) $u = \frac{1}{153,281}[160e^{-5t}\cos(\sqrt{73}\,t) + \frac{383,443}{7300}e^{-5t}\sin(\sqrt{73}\,t) - 160\cos(t/2) + 3128\sin(t/2)]$
   (b) The first two terms are the transient. (d) $\omega = 4\sqrt{3}$ rad/sec
9. $u = \frac{64}{45}(\cos 7t - \cos 8t) = \frac{128}{45}\sin(t/2)\sin(15t/2)$ ft, t in sec
10. $u = (\cos 8t + \sin 8t - 8t\cos 8t)/4$ ft, t in sec; 1/8, $\pi/8$, $\pi/4$, $3\pi/8$ sec

11b

11. (a) $\frac{8}{901}(30\cos 2t + \sin 2t)$ ft, t in sec (b) $m = 4$ slugs
12. $u = (\sqrt{2}/6)\cos(3t - 3\pi/4)$ m, t in sec

See SSM for detailed solutions to 15 and 16

15. $u = \begin{cases} F_0(t - \sin t), & 0 \le t \le \pi \\ F_0[(2\pi - t) - 3\sin t], & \pi < t \le 2\pi \\ -4F_0 \sin t, & 2\pi < t < \infty \end{cases}$

16. $Q(t) = 10^{-6}(e^{-4000t} - 4e^{-1000t} + 3)$ coulombs, t in sec, $Q(0.001) \cong 1.5468 \times 10^{-6}$; $Q(0.01) \cong 2.9998 \times 10^{-6}$; $Q(t) \to 3 \times 10^{-6}$ as $t \to \infty$

17. (a) $u = [32(2 - \omega^2)\cos \omega t + 8\omega \sin \omega t]/(64 - 63\omega^2 + 16\omega^4)$

22 and 24

(b) $A = 8/\sqrt{64 - 63\omega^2 + 16\omega^4}$ (d) $\omega = 3\sqrt{14}/8 \cong 1.4031$, $A = 64/\sqrt{127} \cong 5.6791$

18. (a) $u = 3(\cos t - \cos \omega t)/(\omega^2 - 1)$

19. (a) $u = [(\omega^2 + 2)\cos t - 3\cos \omega t]/(\omega^2 - 1) + \sin t$

CHAPTER 4 Section 4.1, page 212

2, 8, 13, 17, 19c

1. $-\infty < t < \infty$
2. $t > 0$ or $t < 0$
3. $t > 1$, or $0 < t < 1$, or $t < 0$
4. $t > 0$
5. $\ldots, -3\pi/2 < x < -\pi/2, \quad -\pi/2 < x < 1, \quad 1 < x < \pi/2, \quad \pi/2 < x < 3\pi/2, \ldots$
6. $-\infty < x < -2, \quad -2 < x < 2, \quad 2 < x < \infty$
7. Linearly independent
8. Linearly dependent; $f_1(t) + 3f_2(t) - 2f_3(t) = 0$
9. Linearly dependent; $2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0$
10. Linearly independent
11. 1
12. 1
13. $-6e^{-2t}$
14. e^{-2t}
15. $6x$
16. $6/x$
17. $\sin^2 t = \frac{1}{10}(5) - \frac{1}{2}\cos 2t$
19. (a) $a_0[n(n-1)(n-2)\cdots 1] + a_1[n(n-1)\cdots 2]t + \cdots + a_n t^n$
    (b) $(a_0 r^n + a_1 r^{n-1} + \cdots + a_n)e^{rt}$
    (c) $e^t, e^{-t}, e^{2t}, e^{-2t}$; yes, $W(e^t, e^{-t}, e^{2t}, e^{-2t}) \ne 0, \ -\infty < t < \infty$

21, 27

21. $W = ce^{-2t}$
22. $W = c$
23. $W = c/t^2$
24. $W = c/t$
27. $y = c_1 e^t + c_2 t + c_3 t e^t$
28. $y = c_1 t^2 + c_2 t^3 + c_3(t + 1)$

Section 4.2, page 219

2, 8, 12

1. $\sqrt{2}\, e^{i[(\pi/4) + 2m\pi]}$
2. $2e^{i[(2\pi/3) + 2m\pi]}$
3. $3e^{i(\pi + 2m\pi)}$
4. $e^{i[(3\pi/2) + 2m\pi]}$
5. $2e^{i[(11\pi/6) + 2m\pi]}$
6. $\sqrt{2}e^{i[(5\pi/4) + 2m\pi]}$
7. $1, \ \frac{1}{2}(-1 + i\sqrt{3}), \ \frac{1}{2}(-1 - i\sqrt{3})$
8. $2^{1/4}e^{-\pi i/8}, \ 2^{1/4}e^{7\pi i/8}$
9. $1, \ i, \ -1, \ -i$
10. $(\sqrt{3} + i)/\sqrt{2}, \ -(\sqrt{3} + i)/\sqrt{2}$
11. $y = c_1 e^t + c_2 t e^t + c_3 e^{-t}$
12. $y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$
13. $y = c_1 e^t + c_2 e^{2t} + c_3 e^{-t}$
14. $y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$

15, 23

15. $y = c_1 \cos t + c_2 \sin t + e^{\sqrt{3}t/2}(c_3 \cos \frac{1}{2}t + c_4 \sin \frac{1}{2}t) + e^{-\sqrt{3}t/2}(c_5 \cos \frac{1}{2}t + c_6 \sin \frac{1}{2}t)$
16. $y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}$
17. $y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 e^{-t} + c_5 t e^{-t} + c_6 t^2 e^{-t}$
18. $y = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 \cos t + c_6 \sin t$
19. $y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 \cos t + c_5 \sin t$
20. $y = c_1 + c_2 e^{2t} + e^{-t}(c_3 \cos \sqrt{3}\, t + c_4 \sin \sqrt{3}\, t)$
21. $y = e^t[(c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t] + e^{-t}[(c_5 + c_6 t)\cos t + (c_7 + c_8 t)\sin t]$
22. $y = (c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t$
23. $y = c_1 e^t + c_2 e^{(2+\sqrt{5})t} + c_3 e^{(2-\sqrt{5})t}$
24. $y = c_1 e^{-t} + c_2 e^{(-2+\sqrt{2})t} + c_3 e^{(-2-\sqrt{2})t}$
25. $y = c_1 e^{-t/2} + c_2 e^{-t/3}\cos(t/\sqrt{3}) + c_3 e^{-t/3}\sin(t/\sqrt{3})$
26. $y = c_1 e^{3t} + c_2 e^{-2t} + c_3 e^{(3+\sqrt{3})t} + c_4 e^{(3-\sqrt{3})t}$

See SSM for detailed solutions to 27 ,29 ,30, 31

27. $y = c_1e^{-t/3} + c_2e^{-t/4} + c_3e^{-t}\cos 2t + c_4e^{-t}\sin 2t$
28. $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t + c_3e^{-2t}\cos(\sqrt{3}\,t) + c_4e^{-2t}\sin(\sqrt{3}\,t)$
29. $y = 2 - 2\cos t + \sin t$
30. $y = \frac{1}{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2}) - \frac{1}{2}e^{t/\sqrt{2}}\sin(t/\sqrt{2})$
31. $y = 2t - 3$
32. $y = 2\cos t - \sin t$

34, 37, 38a, 39ac

33. $y = -\frac{2}{3}e^{t} - \frac{1}{10}e^{2t} - \frac{1}{6}e^{-2t} - \frac{16}{15}e^{-t/2}$
34. $y = \frac{2}{13}e^{-t} + \frac{24}{13}e^{t/2}\cos t + \frac{3}{13}e^{t/2}\sin t$
35. $y = 8 - 18e^{-t/3} + 8e^{-t/2}$
36. $y = \frac{21}{13}e^{-t}\cos t - \frac{38}{13}e^{-t}\sin t - \frac{8}{13}e^{-2t}\cos(\sqrt{3}\,t) + \dfrac{17\sqrt{3}}{39}e^{-2t}\sin(\sqrt{3}\,t)$
37. $y = \frac{1}{2}(\cosh t - \cos t) + \frac{1}{2}(\sinh t - \sin t)$
38. (a) $W = c$, a constant (b) $W = -8$ (c) $W = 4$
39. (b) $u_1 = c_1\cos t + c_2\sin t + c_3\cos\sqrt{6}\,t + c_4\sin\sqrt{6}\,t$

Section 4.3, page 224

1

1. $y = c_1e^{t} + c_2te^{t} + c_3e^{-t} + \frac{1}{2}te^{-t} + 3$
2. $y = c_1e^{t} + c_2e^{-t} + c_3\cos t + c_4\sin t - 3t - \frac{1}{4}t\sin t$
3. $y = c_1e^{-t} + c_2\cos t + c_3\sin t + \frac{1}{2}te^{-t} + 4(t-1)$
4. $y = c_1 + c_2e^{t} + c_3e^{-t} + \cos t$

5, 9, 13

5. $y = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t} - \frac{1}{3}e^{t} - \frac{1}{48}t^4 - \frac{1}{16}t^2$
6. $y = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t + 3 + \frac{1}{9}\cos 2t$
7. $y = c_1 + c_2t + c_3t^2 + c_4e^{-t} + e^{t/2}[c_5\cos(\sqrt{3}\,t/2) + c_6\sin(\sqrt{3}\,t/2)] + \frac{1}{24}t^4$
8. $y = c_1 + c_2t + c_3t^2 + c_4e^{-t} + \frac{1}{20}\sin 2t + \frac{1}{40}\cos 2t$
9. $y = \frac{3}{16}(1 - \cos 2t) + \frac{1}{8}t^2$
10. $y = (t-4)\cos t - (\frac{3}{2}t + 4)\sin t + 3t + 4$
11. $y = 1 + \frac{1}{4}(t^2 + 3t) - te^{t}$
12. $y = -\frac{2}{5}\cos t - \frac{4}{5}\sin t + \frac{1}{20}e^{-t} + \frac{81}{40}e^{t} + \frac{73}{520}e^{-3t} + \frac{77}{65}\cos 2t - \frac{49}{130}\sin 2t$
13. $Y(t) = t(A_0t^3 + A_1t^2 + A_2t + A_3) + Bt^2e^{t}$
14. $Y(t) = t(A_0t + A_1)e^{-t} + B\cos t + C\sin t$

17, 20, 22a

15. $Y(t) = At^2e^{t} + B\cos t + C\sin t$
16. $Y(t) = At^2 + (B_0t + B_1)e^{t} + t(C\cos 2t + D\sin 2t)$
17. $Y(t) = t(A_0t^2 + A_1t + A_2) + (B_0t + B_1)\cos t + (C_0t + C_1)\sin t$
18. $Y(t) = Ae^{t} + (B_0t + B_1)e^{-t} + te^{-t}(C\cos t + D\sin t)$

22be

19. $k_0 = a_0, \quad k_n = a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n$

Section 4.4, page 229

1, 4, 5, 7

1. $y = c_1 + c_2\cos t + c_3\sin t - \ln\cos t - (\sin t)\ln(\sec t + \tan t)$
2. $y = c_1 + c_2e^{t} + c_3e^{-t} - \frac{1}{2}t^2$
3. $y = c_1e^{t} + c_2e^{-t} + c_3e^{2t} + \frac{1}{30}e^{4t}$
4. $y = c_1 + c_2\cos t + c_3\sin t + \ln(\sec t + \tan t) - t\cos t + (\sin t)\ln\cos t$
5. $y = c_1e^{t} + c_2\cos t + c_3\sin t - \frac{1}{5}e^{-t}\cos t$
6. $y = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t - \frac{1}{8}t^2\sin t$
7. $y = c_1e^{t} + c_2\cos t + c_3\sin t - \frac{1}{2}(\cos t)\ln\cos t + \frac{1}{2}(\sin t)\ln\cos t - \frac{1}{2}t\cos t - \frac{1}{2}t\sin t + \frac{1}{2}e^{t}\displaystyle\int_{t_0}^{t}\left(e^{-s}/\cos s\right)ds$

8. $y = c_1 + c_2e^t + c_3e^{-t} - \ln\sin t + \ln(\cos t + 1) + \frac{1}{2}e^t \int_{t_0}^{t} \left(e^{-s}/\sin s\right) ds$
$+ \frac{1}{2}e^{-t} \int_{t_0}^{t} \left(e^{s}/\sin s\right) ds$
9. $c_1 = 0,\ c_2 = 2,\ c_3 = 1$ in answer to Problem 4
10. $c_1 = 2,\ c_2 = \frac{7}{8},\ c_3 = -\frac{7}{8},\ c_4 = \frac{1}{2}$ in answer to Problem 6

See SSM for detailed solutions to 11, 14

11. $c_1 = \frac{3}{2},\ c_2 = \frac{1}{2},\ c_3 = -\frac{5}{2},\ t_0 = 0$ in answer to Problem 7
12. $c_1 = 3,\ c_2 = 0,\ c_3 = -e^{\pi/2},\ t_0 = \pi/2$ in answer to Problem 8
13. $Y(x) = x^4/15$
14. $Y(t) = \frac{1}{2}\int_{t_0}^{t} [e^{t-s} - \sin(t-s) - \cos(t-s)]g(s)\, ds$
15. $Y(t) = \frac{1}{2}\int_{t_0}^{t} [\sinh(t-s) - \sin(t-s)]g(s)\, ds$

16

16. $Y(t) = \frac{1}{2}\int_{t_0}^{t} e^{(t-s)}(t-s)^2 g(s)\, ds; \quad Y(t) = -te^t \ln|t|$
17. $Y(x) = \frac{1}{2}\int_{x_0}^{x} [(x/t^2) - 2(x^2/t^3) + (x^3/t^4)]g(t)\, dt$

CHAPTER 5 Section 5.1, page 237

2, 5, 9, 12, 13

1. $\rho = 1$
2. $\rho = 2$
3. $\rho = \infty$
4. $\rho = \frac{1}{2}$
5. $\rho = \frac{1}{2}$
6. $\rho = 1$
7. $\rho = 3$
8. $\rho = e$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \rho = \infty$
10. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \rho = \infty$
11. $1 + (x-1), \quad \rho = \infty$
12. $1 - 2(x+1) + (x+1)^2, \quad \rho = \infty$
13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad \rho = 1$
14. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n, \quad \rho = 1$
15. $\displaystyle\sum_{n=0}^{\infty} x^n, \quad \rho = 1$
16. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n, \quad \rho = 1$
17. $y' = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots + (n+1)^2x^n + \cdots$
$y'' = 2^2 + 3^2 \cdot 2x + 4^2 \cdot 3x^2 + 5^2 \cdot 4x^3 + \cdots + (n+2)^2(n+1)x^n + \cdots$

18, 19, 23, 25, 28

18. $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + (n+1)a_{n+1}x^n + \cdots$
$= \displaystyle\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$
$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots$
$= \displaystyle\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$
21. $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$
22. $\displaystyle\sum_{n=2}^{\infty} a_{n-2}x^n$
23. $\displaystyle\sum_{n=0}^{\infty} (n+1)a_nx^n$
24. $\displaystyle\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n$
25. $\displaystyle\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n$
26. $\displaystyle a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}]x^n$
27. $\displaystyle\sum_{n=0}^{\infty} [(n+1)na_{n+1} + a_n]x^n$
28. $a_n = (-2)^n a_0/n!, \quad n = 1, 2, \ldots; \quad a_0e^{-2x}$

Section 5.2, page 247

1. $a_{n+2} = a_n/(n+2)(n+1)$
$y_1(x) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$
$y_2(x) = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$

See SSM for detailed solutions to 2

2. $a_{n+2} = a_n/(n+2)$
$y_1(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{2\cdot 4} + \dfrac{x^6}{2\cdot 4\cdot 6} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$
$y_2(x) = x + \dfrac{x^3}{3} + \dfrac{x^5}{3\cdot 5} + \dfrac{x^7}{3\cdot 5\cdot 7} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}$

3

3. $(n+2)a_{n+2} - a_{n+1} - a_n = 0$
$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$
$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$

4. $a_{n+4} = -k^2 a_n/(n+4)(n+3); \quad a_2 = a_3 = 0$
$y_1(x) = 1 - \dfrac{k^2x^4}{3\cdot 4} + \dfrac{k^4x^8}{3\cdot 4\cdot 7\cdot 8} - \dfrac{k^6x^{12}}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} + \cdots$
$= 1 + \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{3\cdot 4\cdot 7\cdot 8\cdots(4m+3)(4m+4)}$
$y_2(x) = x - \dfrac{k^2x^5}{4\cdot 5} + \dfrac{k^4x^9}{4\cdot 5\cdot 8\cdot 9} - \dfrac{k^6x^{13}}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13} + \cdots$
$= x\left[1 + \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^{m+1}(k^2x^4)^{m+1}}{4\cdot 5\cdot 8\cdot 9\cdots(4m+4)(4m+5)}\right]$
Hint: Let $n = 4m$ in the recurrence relation, $m = 1, 2, 3, \ldots$.

5, 8

5. $(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n = 0, \quad n \geq 1; \quad a_2 = -\frac{1}{2}a_0$
$y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots, \quad y_2(x) = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \cdots$

6. $a_{n+2} = -(n^2 - 2n + 4)a_n/[2(n+1)(n+2)], \quad n \geq 2; \quad a_2 = -a_0, \quad a_3 = -\frac{1}{4}a_1$
$y_1(x) = 1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6 + \cdots, \quad y_2(x) = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 - \frac{19}{1920}x^7 + \cdots$

7. $a_{n+2} = -a_n/(n+1), \quad n = 0, 1, 2, \ldots$
$y_1(x) = 1 - \dfrac{x^2}{1} + \dfrac{x^4}{1\cdot 3} - \dfrac{x^6}{1\cdot 3\cdot 5} + \cdots = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1\cdot 3\cdot 5\cdots(2n-1)}$
$y_2(x) = x - \dfrac{x^3}{2} + \dfrac{x^5}{2\cdot 4} - \dfrac{x^7}{2\cdot 4\cdot 6} + \cdots = x + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2\cdot 4\cdot 6\cdots(2n)}$

8. $a_{n+2} = -[(n+1)^2 a_{n+1} + a_n + a_{n-1}]/(n+1)(n+2), \quad n = 1, 2, \ldots$
$a_2 = -(a_0 + a_1)/2$
$y_1(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots$
$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots$

9. $(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n = 0; \quad n = 0, 1, 2, \ldots$
$y_1(x) = 1 - 3x^2, \quad y_2(x) = x - x^3/3$

10. $4(n+2)a_{n+2} - (n-2)a_n = 0; \quad n = 0, 1, 2, \ldots$
$y_1(x) = 1 - \dfrac{x^2}{4}, \quad y_2(x) = x - \dfrac{x^3}{12} - \dfrac{x^5}{240} - \dfrac{x^7}{2240} - \cdots - \dfrac{x^{2n+1}}{4^n(2n-1)(2n+1)} - \cdots$

11. $3(n+2)a_{n+2} - (n+1)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5}{432}x^6 + \cdots + \frac{3 \cdot 5 \cdots (2n-1)}{3^n \cdot 2 \cdot 4 \cdots (2n)}x^{2n} + \cdots$$

$$y_2(x) = x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{945}x^7 + \cdots + \frac{2 \cdot 4 \cdots (2n)}{3^n \cdot 3 \cdot 5 \cdots (2n+1)}x^{2n+1} + \cdots$$

12. $(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots, \quad y_2(x) = x$$

13. $2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots + (-1)^n \frac{3 \cdot 5 \cdots (2n+1)}{2^n(2n)!}x^{2n} + \cdots$$

$$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{20} - \frac{x^7}{210} + \cdots + (-1)^n \frac{4 \cdot 6 \cdots (2n+2)}{2^n(2n+1)!}x^{2n+1} + \cdots$$

See SSM for detailed solutions to 14, 16a, 19

14. $2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n = 0; \quad n = 0, 1, 2, \ldots$

$$y_1(x) = 1 - \tfrac{3}{4}(x-2)^2 + \tfrac{3}{8}(x-2)^3 + \tfrac{1}{64}(x-2)^4 + \cdots$$

$$y_2(x) = (x-2) - \tfrac{3}{4}(x-2)^2 + \tfrac{1}{24}(x-2)^3 + \tfrac{9}{64}(x-2)^4 + \cdots$$

15. (a) $y = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$ (c) about $|x| < 0.7$

16. (a) $y = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \cdots$ (c) about $|x| < 0.7$

17. (a) $y = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 + \cdots$ (c) about $|x| < 0.5$

18. (a) $y = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 + \cdots$ (c) about $|x| < 0.9$

19. $y_1(x) = 1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + \cdots$

$$y_2(x) = (x-1) - \tfrac{1}{4}(x-1)^4 - \tfrac{1}{20}(x-1)^5 + \tfrac{1}{28}(x-1)^7 + \cdots$$

21. (a) $y_1(x) = 1 - \dfrac{\lambda}{2!}x^2 + \dfrac{\lambda(\lambda-4)}{4!}x^4 - \dfrac{\lambda(\lambda-4)(\lambda-8)}{6!}x^6 + \cdots$

$$y_2(x) = x - \frac{\lambda-2}{3!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!}x^7 + \cdots$$

(b) $1,\ x,\ 1-2x^2,\ x - \frac{2}{3}x^3,\ 1 - 4x^2 + \frac{4}{3}x^4,\ x - \frac{4}{3}x^3 + \frac{4}{15}x^5$

(c) $1,\ 2x,\ 4x^2 - 2,\ 8x^3 - 12x,\ 16x^4 - 48x^2 + 12,\ 32x^5 - 160x^3 + 120x$

22b, 23, 26

22. (b) $y = x - x^3/6 + \cdots$

Section 5.3, page 253

1, 6, 9afh

1. $\phi''(0) = -1, \quad \phi'''(0) = 0, \quad \phi^{\text{iv}}(0) = 3$
2. $\phi''(0) = 0, \quad \phi'''(0) = -2, \quad \phi^{\text{iv}}(0) = 0$
3. $\phi''(1) = 0, \quad \phi'''(1) = -6, \quad \phi^{\text{iv}}(1) = 42$
4. $\phi''(0) = 0, \quad \phi'''(0) = -a_0, \quad \phi^{\text{iv}}(0) = -4a_1$
5. $\rho = \infty, \quad \rho = \infty$
6. $\rho = 1, \quad \rho = 3, \quad \rho = 1$
7. $\rho = 1, \quad \rho = \sqrt{3}$
8. $\rho = 1$
9. (a) $\rho = \infty$ (b) $\rho = \infty$ (c) $\rho = \infty$ (d) $\rho = \infty$ (e) $\rho = 1$
   (f) $\rho = \sqrt{2}$ (g) $\rho = \infty$ (h) $\rho = 1$ (i) $\rho = 1$ (j) $\rho = 2$
   (k) $\rho = \sqrt{3}$ (l) $\rho = 1$ (m) $\rho = \infty$ (n) $\rho = \infty$

See SSM for detailed solutions to 10a

10. (a) $y_1(x) = 1 - \dfrac{\alpha^2}{2!}x^2 - \dfrac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \dfrac{(4^2-\alpha^2)(2^2-\alpha^2)\alpha^2}{6!}x^6 - \cdots$
$- \dfrac{[(2m-2)^2-\alpha^2]\cdots(2^2-\alpha^2)\alpha^2}{(2m)!}x^{2m} - \cdots$

$y_2(x) = x + \dfrac{1-\alpha^2}{3!}x^3 + \dfrac{(3^2-\alpha^2)(1-\alpha^2)}{5!}x^5 + \cdots$
$+ \dfrac{[(2m-1)^2-\alpha^2]\cdots(1-\alpha^2)}{(2m+1)!}x^{2m+1} + \cdots$

10b, 11

(b) $y_1(x)$ or $y_2(x)$ terminates with x^n as $\alpha = n$ is even or odd.

(c) $n = 0,\ y = 1;\quad n = 1,\ y = x;\quad n = 2,\ y = 1 - 2x^2;\quad n = 3,\ y = x - \frac{4}{3}x^3$

11. $y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \cdots,\quad y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots,$ $\rho = \infty$

12. $y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \cdots,\quad y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{60}x^6 + \cdots,$ $\rho = \infty$

13. $y_1(x) = 1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots,\quad y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \cdots,$ $\rho = \pi/2$

14. $y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{120}x^6 + \cdots,\quad y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{7}{120}x^5 + \cdots,$ $\rho = 1$

15. Cannot specify arbitrary initial conditions at $x = 0$; hence $x = 0$ is a singular point.

16. $y = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots = e^x$

17. $y = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{2\cdot 4} + \dfrac{x^6}{2\cdot 4\cdot 6} + \cdots + \dfrac{x^{2n}}{2^n\cdot n!} + \cdots$

18

18. $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots$

19. $y = 1 + x + x^2 + \cdots + x^n + \cdots = \dfrac{1}{1-x}$

20. $y = a_0\left(1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots\right) + 2\left(\dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^n}{n!} + \cdots\right)$

$= a_0e^x + 2\left(e^x - 1 - x - \dfrac{x^2}{2}\right) = ce^x - 2 - 2x - x^2$

20, 22

21. $y = a_0\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{2^2 2!} - \dfrac{x^6}{2^3 3!} + \cdots + \dfrac{(-1)^n x^{2n}}{2^n n!} + \cdots\right)$
$+ \left(x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{2\cdot 4} + \dfrac{x^5}{3\cdot 5} + \cdots\right)$

$= a_0e^{-x^2/2} + \left(x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{2\cdot 4} + \dfrac{x^5}{3\cdot 5} + \cdots\right)$

26, 28

23. $1,\ 1 - 3x^2,\ 1 - 10x^2 + \frac{35}{3}x^4;\quad x,\ x - \frac{5}{3}x^3,\ x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

24. (a) $1,\ x,\ (3x^2-1)/2,\ (5x^3-3x)/2,\ (35x^4-30x^2+3)/8,\ (63x^5-70x^3+15x)/8$
(c) $P_1,\ 0;\quad P_2,\ \pm 0.57735;\quad P_3,\ 0,\ \pm 0.77460;\quad P_4,\ \pm 0.33998,\ \pm 0.86114;$
$P_5,\ 0,\ \pm 0.53847, \pm 0.90618$

Section 5.4, page 259

See SSM for detailed solutions to 1, 5

1. $x = 0$, regular
2. $x = 0$, regular; $x = 1$, irregular
3. $x = 0$, irregular; $x = 1$, regular
4. $x = 0$, irregular; $x = \pm 1$, regular
5. $x = 1$, regular; $x = -1$, irregular
6. $x = 0$, regular
7. $x = -3$, regular
8. $x = 0, -1$, regular; $x = 1$, irregular
9. $x = 1$, regular; $x = -2$, irregular
10. $x = 0, 3$, regular

12, 17

11. $x = 1, -2$, regular
12. $x = 0$, regular
13. $x = 0$, irregular
14. $x = 0$, regular
15. $x = 0$, regular
16. $x = 0, \pm n\pi$, regular
17. $x = 0, \pm n\pi$, regular
18. $x = 0$, irregular; $x = \pm n\pi$, regular

19, 21, 23

19. $y = a_0\left(1 - \dfrac{x^2}{2\cdot 5} + \dfrac{x^4}{2\cdot 4\cdot 5\cdot 9} - \cdots\right)$

25

22. Irregular singular point
23. Regular singular point
24. Regular singular point
25. Irregular singular point
26. Irregular singular point
27. Irregular singular point

Section 5.5, page 265

2, 4, 9, 13, 16

1. $y = c_1x^{-1} + c_2x^{-2}$
2. $y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}$
3. $y = c_1x^2 + c_2x^2\ln|x|$
4. $y = c_1x^{-1}\cos(2\ln|x|) + c_2x^{-1}\sin(2\ln|x|)$
5. $y = c_1x + c_2x\ln|x|$
6. $y = c_1(x-1)^{-3} + c_2(x-1)^{-4}$
7. $y = c_1|x|^{(-5+\sqrt{29})/2} + c_2|x|^{(-5-\sqrt{29})/2}$
8. $y = c_1|x|^{3/2}\cos(\frac{1}{2}\sqrt{3}\ln|x|) + c_2|x|^{3/2}\sin(\frac{1}{2}\sqrt{3}\ln|x|)$
9. $y = c_1x^3 + c_2x^3\ln|x|$
10. $y = c_1(x-2)^{-2}\cos(2\ln|x-2|) + c_2(x-2)^{-2}\sin(2\ln|x-2|)$
11. $y = c_1|x|^{-1/2}\cos(\frac{1}{2}\sqrt{15}\ln|x|) + c_2|x|^{-1/2}\sin(\frac{1}{2}\sqrt{15}\ln|x|)$
12. $y = c_1x + c_2x^4$
13. $y = 2x^{3/2} - x^{-1}$
14. $y = 2x^{-1/2}\cos(2\ln x) - x^{-1/2}\sin(2\ln x)$
15. $y = 2x^2 - 7x^2\ln|x|$

17, 21, 21a, 22, 25, 31

16. $y = x^{-1}\cos(2\ln x)$
17. $\alpha < 1$
18. $\beta > 0$
19. $\gamma = 2$
20. $\alpha > 1$
21. (a) $\alpha < 1$ and $\beta > 0$
    (b) $\alpha < 1$ and $\beta \geq 0$, or $\alpha = 1$ and $\beta > 0$
    (c) $\alpha > 1$ and $\beta > 0$
    (d) $\alpha > 1$ and $\beta \geq 0$, or $\alpha = 1$ and $\beta > 0$
    (e) $\alpha = 1$ and $\beta > 0$
24. $y = c_1x^{-1} + c_2x^2$
25. $y = c_1x^2 + c_2x^2\ln x + \frac{1}{4}\ln x + \frac{1}{4}$
26. $y = c_1x^{-1} + c_2x^{-5} + \frac{1}{12}x$
27. $y = c_1x + c_2x^2 + 3x^2\ln x + \ln x + \frac{3}{2}$
28. $y = c_1\cos(2\ln x) + c_2\sin(2\ln x) + \frac{1}{3}\sin(\ln x)$
29. $y = c_1x^{-3/2}\cos(\frac{3}{2}\ln x) + c_2x^{-3/2}\sin(\frac{3}{2}\ln x)$
31. $x > 0$: $c_1 = k_1$, $c_2 = k_2$; $x < 0$: $c_1(-1)^{r_1} = k_1$, $c_2 = k_2$

Section 5.6, page 271

1. $r(2r-1)=0;\quad a_n=-\dfrac{a_{n-2}}{(n+r)[2(n+r)-1]};\quad r_1=\frac{1}{2},\ r_2=0$

$$y_1(x)=x^{1/2}\left[1-\frac{x^2}{2\cdot 5}+\frac{x^4}{2\cdot 4\cdot 5\cdot 9}-\frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}+\cdots+\frac{(-1)^n x^{2n}}{2^n n!5\cdot 9\cdot 13\cdots(4n+1)}+\cdots\right]$$

$$y_2(x)=1-\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 7}-\frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}+\cdots+\frac{(-1)^n x^{2n}}{2^n n!3\cdot 7\cdot 11\cdots(4n-1)}+\cdots$$

See SSM for detailed solutions to 2

2. $r^2-\frac{1}{9}=0;\quad a_n=-\dfrac{a_{n-2}}{(n+r)^2-\frac{1}{9}};\quad r_1=\frac{1}{3}, r_2=-\frac{1}{3}$

$$y_1(x)=x^{1/3}\left[1-\frac{1}{1!(1+\frac{1}{3})}\left(\frac{x}{2}\right)^2+\frac{1}{2!(1+\frac{1}{3})(2+\frac{1}{3})}\left(\frac{x}{2}\right)^4+\cdots+\frac{(-1)^m}{m!(1+\frac{1}{3})(2+\frac{1}{3})\cdots(m+\frac{1}{3})}\left(\frac{x}{2}\right)^{2m}+\cdots\right]$$

$$y_2(x)=x^{-1/3}\left[1-\frac{1}{1!(1-\frac{1}{3})}\left(\frac{x}{2}\right)^2+\frac{1}{2!(1-\frac{1}{3})(2-\frac{1}{3})}\left(\frac{x}{2}\right)^4+\cdots+\frac{(-1)^m}{m!(1-\frac{1}{3})(2-\frac{1}{3})\cdots(m-\frac{1}{3})}\left(\frac{x}{2}\right)^{2m}+\cdots\right]$$

Hint: Let $n=2m$ in the recurrence relation, $m=1,2,3,\ldots$.

3. $r(r-1)=0;\quad a_n=-\dfrac{a_{n-1}}{(n+r)(n+r-1)};\quad r_1=1,\ r_2=0$

$$y_1(x)=x\left[1-\frac{x}{1!2!}+\frac{x^2}{2!3!}+\cdots+\frac{(-1)^n}{n!(n+1)!}x^n+\cdots\right]$$

4

4. $r^2=0;\quad a_n=\dfrac{a_{n-1}}{(n+r)^2};\quad r_1=r_2=0$

$$y_1(x)=1+\frac{x}{(1!)^2}+\frac{x^2}{(2!)^2}+\cdots+\frac{x^n}{(n!)^2}+\cdots$$

5. $r(3r-1)=0;\quad a_n=-\dfrac{a_{n-2}}{(n+r)[3(n+r)-1]};\quad r_1=\frac{1}{3},\ r_2=0$

$$y_1(x)=x^{1/3}\left[1-\frac{1}{1!7}\left(\frac{x^2}{2}\right)+\frac{1}{2!7\cdot 13}\left(\frac{x^2}{2}\right)^2+\cdots+\frac{(-1)^m}{m!7\cdot 13\cdots(6m+1)}\left(\frac{x^2}{2}\right)^m+\cdots\right]$$

$$y_2(x)=1-\frac{1}{1!5}\left(\frac{x^2}{2}\right)+\frac{1}{2!5\cdot 11}\left(\frac{x^2}{2}\right)^2+\cdots+\frac{(-1)^m}{m!5\cdot 11\cdots(6m-1)}\left(\frac{x^2}{2}\right)^m+\cdots$$

Hint: Let $n=2m$ in the recurrence relation, $m=1,2,3,\ldots$.

6. $r^2-2=0;\quad a_n=-\dfrac{a_{n-1}}{(n+r)^2-2};\quad r_1=\sqrt{2},\ r_2=-\sqrt{2}$

$$y_1(x)=x^{\sqrt{2}}\left[1-\frac{x}{1(1+2\sqrt{2})}+\frac{x^2}{2!(1+2\sqrt{2})(2+2\sqrt{2})}+\cdots+\frac{(-1)^n}{n!(1+2\sqrt{2})(2+2\sqrt{2})\cdots(n+2\sqrt{2})}x^n+\cdots\right]$$

$$y_2(x)=x^{-\sqrt{2}}\left[1-\frac{x}{1(1-2\sqrt{2})}+\frac{x^2}{2!(1-2\sqrt{2})(2-2\sqrt{2})}+\cdots+\frac{(-1)^n}{n!(1-2\sqrt{2})(2-2\sqrt{2})\cdots(n-2\sqrt{2})}x^n+\cdots\right]$$

7. $r^2=0;\quad (n+r)a_n=a_{n-1};\quad r_1=r_2=0$

$$y_1(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots=e^x$$

8. $2r^2+r-1=0;\quad (2n+2r-1)(n+r+1)a_n+2a_{n-2}=0;\quad r_1=\frac{1}{2},\ r_2=-1$

$$y_1(x)=x^{1/2}\left(1-\frac{x^2}{7}+\frac{x^4}{2!7\cdot 11}-\cdots+\frac{(-1)^m x^{2m}}{m!7\cdot 11\cdots(4m+3)}+\cdots\right)$$

$$y_2(x)=x^{-1}\left(1-x^2+\frac{x^4}{2!5}-\cdots+\frac{(-1)^m x^{2m}}{m!5\cdot 9\cdots(4m-3)}+\cdots\right)$$

9. $r^2-4r+3=0;\quad (n+r-3)(n+r-1)a_n-(n+r-2)a_{n-1}=0;\quad r_1=3,\ r_2=1$

$$y_1(x)=x^3\left(1+\frac{2}{3}x+\frac{x^2}{4}+\cdots+\frac{2x^n}{n!(n+2)}+\cdots\right)$$

10. $r^2-r+\frac{1}{4}=0;\quad (n+r-\frac{1}{2})^2a_n+a_{n-2}=0;\quad r_1=r_2=1/2$

$$y_1(x)=x^{1/2}\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}-\cdots+\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}+\cdots\right)$$

See SSM for detailed solutions to 11

11. $r^2=0;\quad r_1=0,\ r_2=0$

$$y_1(x)=1+\frac{\alpha(\alpha+1)}{2\cdot 1^2}(x-1)-\frac{\alpha(\alpha+1)[1\cdot 2-\alpha(\alpha+1)]}{(2\cdot 1^2)(2\cdot 2^2)}(x-1)^2+\cdots+(-1)^{n+1}\frac{\alpha(\alpha+1)[1\cdot 2-\alpha(\alpha+1)]\cdots[n(n-1)-\alpha(\alpha+1)]}{2^n(n!)^2}(x-1)^n+\cdots$$

12. (a) $r_1=\frac{1}{2},\ r_2=0$ at both $x=\pm 1$

(b) $y_1(x)=|x-1|^{1/2}$

14 $\times\left[1+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(1+2\alpha)\cdots(2n-1+2\alpha)(1-2\alpha)\cdots(2n-1-2\alpha)}{2^n(2n+1)!}(x-1)^n\right]$

$$y_2(x)=1+\sum_{n=1}^{\infty}\frac{(-1)^n\alpha(1+\alpha)\cdots(n-1+\alpha)(-\alpha)(1-\alpha)\cdots(n-1-\alpha)}{n!1\cdot 3\cdot 5\cdots(2n-1)}(x-1)^n$$

13. $r^2=0;\quad r_1=0,\quad r_2=0;\quad a_n=\dfrac{(n-1-\lambda)a_{n-1}}{n^2}$

15 $y_1(x)=1+\dfrac{-\lambda}{(1!)^2}x+\dfrac{(-\lambda)(1-\lambda)}{(2!)^2}x^2+\cdots+\dfrac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}x^n+\cdots$

For $\lambda=n$, the coefficients of all terms past x^n are zero.

16ab 16. (b) $[(n-1)^2-1]b_n=-b_{n-2}$, and it is impossible to determine b_2.

Section 5.7, page 278

See SSM for detailed solutions to 1, 3, 9

1. $x = 0; \quad r(r-1) = 0; \quad r_1 = 1,\ r_2 = 0$
2. $x = 0; \quad r^2 - 3r + 2 = 0; \quad r_1 = 2,\ r_2 = 1$
3. $x = 0; \quad r(r-1) = 0; \quad r_1 = 1,\ r_2 = 0$
   $x = 1; \quad r(r+5) = 0; \quad r_1 = 0,\ r_2 = -5$
4. None
5. $x = 0; \quad r^2 + 2r - 2 = 0; \quad r_1 = -1 + \sqrt{3} \cong 0.732,\ r_2 = -1 - \sqrt{3} \cong -2.73$
6. $x = 0; \quad r(r - \frac{3}{4}) = 0; \quad r_1 = \frac{3}{4},\ r_2 = 0$
   $x = -2; \quad r(r - \frac{5}{4}) = 0; \quad r_1 = \frac{5}{4},\ r_2 = 0$
7. $x = 0; \quad r^2 + 1 = 0; \quad r_1 = i,\ r_2 = -i$
8. $x = -1; \quad r^2 - 7r + 3 = 0; \quad r_1 = (7 + \sqrt{37})/2 \cong 6.54,\ r_2 = (7 - \sqrt{37})/2 \cong 0.459$
9. $x = 1; \quad r^2 + r = 0; \quad r_1 = 0,\ r_2 = -1$
10. $x = -2; \quad r^2 - (5/4)r = 0; \quad r_1 = 5/4,\ r_2 = 0$
11. $x = 2; \quad r^2 - 2r = 0; \quad r_1 = 2,\ r_2 = 0$
    $x = -2; \quad r^2 - 2r = 0; \quad r_1 = 2,\ r_2 = 0$
12. $x = 0; \quad r^2 - (5/3)r = 0; \quad r_1 = 5/3,\ r_2 = 0$
    $x = -3; \quad r^2 - (r/3) - 1 = 0; \quad r_1 = (1 + \sqrt{37})/6 \cong 1.18,$
    $r_2 = (1 - \sqrt{37})/6 \cong -0.847$
13. (b) $r_1 = 0, \quad r_2 = 0$
    (c) $y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots$
    $y_2(x) = y_1(x)\ln x - 2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 + \cdots$
14. (b) $r_1 = 1,\ r_2 = 0$
    (c) $y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \cdots$
    $y_2(x) = -6y_1(x)\ln x + 1 - 33x^2 + \frac{449}{6}x^3 + \cdots$
15. (b) $r_1 = 1,\ r_2 = 0$
    (c) $y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \cdots$
    $y_2(x) = 3y_1(x)\ln x + 1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 + \cdots$
16. (b) $r_1 = 1,\ r_2 = 0$
    (c) $y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$
    $y_2(x) = -y_1(x)\ln x + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \cdots$

17abc

17. (b) $r_1 = 1,\ r_2 = -1$
    (c) $y_1(x) = x - \frac{1}{24}x^3 + \frac{1}{720}x^5 + \cdots$
    $y_2(x) = -\frac{1}{3}y_1(x)\ln x + x^{-1} - \frac{1}{90}x^3 + \cdots$

18

18. $r_1 = \frac{1}{2},\ r_2 = 0$
    $y_1(x) = (x-1)^{1/2}[1 - \frac{3}{4}(x-1) + \frac{53}{480}(x-1)^2 + \cdots], \quad \rho = 1$

20abc, 21bd

19. (c) *Hint:* $(n-1)(n-2) + (1+\alpha+\beta)(n-1) + \alpha\beta = (n-1+\alpha)(n-1+\beta)$
    (d) *Hint:* $(n-\gamma)(n-1-\gamma) + (1+\alpha+\beta)(n-\gamma) + \alpha\beta = (n-\gamma+\alpha)(n-\gamma+\beta)$

Section 5.8, page 289

1

1. $y_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)!}$
   $y_2(x) = -y_1(x)\ln x + \frac{1}{x}\left[1 - \displaystyle\sum_{n=1}^{\infty} \frac{H_n + H_{n-1}}{n!(n-1)!}(-1)^n x^n\right]$

2

2. $y_1(x) = \frac{1}{x}\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n!)^2}, \quad y_2(x) = y_1(x)\ln x - \frac{2}{x}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n!)^2}x^n$

See SSM for detailed solutions to 3, 4, 5

3. $y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n, \quad y_2(x) = y_1(x)\ln x - 2\sum_{n=1}^{\infty} \frac{(-1)^n 2^n H_n}{(n!)^2} x^n$

4. $y_1(x) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n$

$y_2(x) = -y_1(x)\ln x + \frac{1}{x^2}\left[1 - \sum_{n=1}^{\infty} \frac{H_n + H_{n-1}}{n!(n-1)!}(-1)^n x^n\right]$

5. $y_1(x) = x^{3/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\frac{3}{2})(2+\frac{3}{2})\cdots(m+\frac{3}{2})}\left(\frac{x}{2}\right)^{2m}\right]$

7, 12

$y_2(x) = x^{-3/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\frac{3}{2})(2-\frac{3}{2})\cdots(m-\frac{3}{2})}\left(\frac{x}{2}\right)^{2m}\right]$

13, 14

Hint: Let $n = 2m$ in the recurrence relation, $m = 1, 2, 3, \ldots$. For $r = -\frac{3}{2}$, $a_1 = 0$ and a_3 is arbitrary.

CHAPTER 6

Section 6.1, page 298

1, 2, 5b, 6, 9

1. Piecewise continuous
2. Neither
3. Continuous
4. Piecewise continuous
5. (a) $1/s^2, \quad s > 0$
   (b) $2/s^3, \quad s > 0$
   (c) $n!/s^{n+1}, \quad s > 0$
6. $s/(s^2 + a^2), \quad s > 0$

7. $\frac{s}{s^2 - b^2}, \quad s > |b|$
8. $\frac{b}{s^2 - b^2}, \quad s > |b|$
9. $\frac{s-a}{(s-a)^2 - b^2}, \quad s - a > |b|$
10. $\frac{b}{(s-a)^2 - b^2}, \quad s - a > |b|$
11. $\frac{b}{s^2 + b^2}, \quad s > 0$
12. $\frac{s}{s^2 + b^2}, \quad s > 0$

13, 16, 19, 21

13. $\frac{b}{(s-a)^2 + b^2}, \quad s > a$
14. $\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
15. $\frac{1}{(s-a)^2}, \quad s > a$
16. $\frac{2as}{(s^2 + a^2)^2}, \quad s > 0$
17. $\frac{s^2 + a^2}{(s-a)^2(s+a)^2}, \quad s > |a|$
18. $\frac{n!}{(s-a)^{n+1}}, \quad s > a$
19. $\frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}, \quad s > 0$
20. $\frac{2a(3s^2 + a^2)}{(s^2 - a^2)^3}, \quad s > |a|$
21. Converges
22. Converges
23. Diverges
24. Converges

25, 27abcd

26. (d) $\Gamma(3/2) = \sqrt{\pi}/2; \quad \Gamma(11/2) = 945\sqrt{\pi}/32$

Section 6.2, page 307

2, 4, 7

1. $\frac{3}{2}\sin 2t$
2. $2t^2 e^t$
3. $\frac{2}{5}e^t - \frac{2}{5}e^{-4t}$
4. $\frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}$
5. $2e^{-t}\cos 2t$
6. $2\cosh 2t - \frac{3}{2}\sinh 2t$
7. $2e^t\cos t + 3e^t \sin t$
8. $3 - 2\sin 2t + 5\cos 2t$
9. $-2e^{-2t}\cos t + 5e^{-2t}\sin t$
10. $2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$

11, 14, 15

11. $y = \frac{1}{5}(e^{3t} + 4e^{-2t})$
12. $y = 2e^{-t} - e^{-2t}$
13. $y = e^t \sin t$
14. $y = e^{2t} - te^{2t}$
15. $y = 2e^t\cosh\sqrt{3}\,t - (2/\sqrt{3})e^t\sinh\sqrt{3}\,t$
16. $y = 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$

See SSM for detailed solutions to 17, 20, 22, 24

17. $y = te^t - t^2e^t + \frac{2}{3}t^3e^t$
18. $y = \cosh t$
19. $y = \cos\sqrt{2}\,t$
20. $y = (\omega^2 - 4)^{-1}[(\omega^2 - 5)\cos\omega t + \cos 2t]$
21. $y = \frac{1}{5}(\cos t - 2\sin t + 4e^t\cos t - 2e^t\sin t)$
22. $y = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$
23. $y = 2e^{-t} + te^{-t} + 2t^2e^{-t}$
24. $Y(s) = \dfrac{s}{s^2+4} + \dfrac{1-e^{-\pi s}}{s(s^2+4)}$
25. $Y(s) = \dfrac{1}{s^2(s^2+1)} - \dfrac{e^{-s}(s+1)}{s^2(s^2+1)}$

27b, 30, 32

26. $Y(s) = (1-e^{-s})/s^2(s^2+4)$
29. $1/(s-a)^2$
30. $2b(3s^2 - b^2)/(s^2+b^2)^3$
31. $n!/s^{n+1}$
32. $n!/(s-a)^{n+1}$
33. $2b(s-a)/[(s-a)^2+b^2]^2$
34. $[(s-a)^2 - b^2]/[(s-a)^2+b^2]^2$

36a, 38ab

36. (a) $Y' + s^2Y = s$ (b) $s^2Y'' + 2sY' - [s^2 + \alpha(\alpha+1)]Y = -1$

Section 6.3, page 314

2, 4, 8

7. $F(s) = 2e^{-s}/s^3$
8. $F(s) = e^{-s}(s^2+2)/s^3$
9. $F(s) = \dfrac{e^{-\pi s}}{s^2} - \dfrac{e^{-2\pi s}}{s^2}(1+\pi s)$
10. $F(s) = \dfrac{1}{s}(e^{-s} + 2e^{-3s} - 6e^{-4s})$
11. $F(s) = s^{-2}[(1-s)e^{-2s} - (1+s)e^{-3s}]$
12. $F(s) = (1-e^{-s})/s^2$

14, 21, 22, 27

13. $f(t) = t^3e^{2t}$
14. $f(t) = \frac{1}{3}u_2(t)[e^{t-2} - e^{-2(t-2)}]$
15. $f(t) = 2u_2(t)e^{t-2}\cos(t-2)$
16. $f(t) = u_2(t)\sinh 2(t-2)$
17. $f(t) = u_1(t)e^{2(t-1)}\cosh(t-1)$
18. $f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$
20. $f(t) = 2(2t)^n$
21. $f(t) = \frac{1}{2}e^{-t/2}\cos t$
22. $f(t) = \frac{1}{6}e^{t/3}(e^{2t/3} - 1)$
23. $f(t) = \frac{1}{2}e^{t/2}u_2(t/2)$
24. $F(s) = s^{-1}(1-e^{-s}), \quad s > 0$
25. $F(s) = s^{-1}(1 - e^{-s} + e^{-2s} - e^{-3s}), \quad s > 0$
26. $F(s) = \dfrac{1}{s}[1 - e^{-s} + \cdots + e^{-2ns} - e^{-(2n+1)s}] = \dfrac{1-e^{-(2n+2)s}}{s(1+e^{-s})}, \quad s > 0$
27. $F(s) = \dfrac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-ns} = \dfrac{1/s}{1+e^{-s}}, \quad s > 0$

28, 30

29. $\mathcal{L}\{f(t)\} = \dfrac{1/s}{1+e^{-s}}, \quad s > 0$
30. $\mathcal{L}\{f(t)\} = \dfrac{1-e^{-s}}{s(1+e^{-s})}, \quad s > 0$
31. $\mathcal{L}\{f(t)\} = \dfrac{1-(1+s)e^{-s}}{s^2(1-e^{-s})}, \quad s > 0$
32. $\mathcal{L}\{f(t)\} = \dfrac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})}, \quad s > 0$
33. (a) $\mathcal{L}\{f(t)\} = s^{-1}(1-e^{-s}), \quad s > 0$
    (b) $\mathcal{L}\{g(t)\} = s^{-2}(1-e^{-s}), \quad s > 0$
    (c) $\mathcal{L}\{h(t)\} = s^{-2}(1-e^{-s})^2, \quad s > 0$
34. (b) $\mathcal{L}\{p(t)\} = \dfrac{1-e^{-s}}{s^2(1+e^{-s})}, \quad s > 0$

Section 6.4, page 321

1

1. $y = 1 - \cos t + \sin t - u_{\pi/2}(t)(1 - \sin t)$
2. $y = e^{-t}\sin t + \frac{1}{2}u_\pi(t)[1 + e^{-(t-\pi)}\cos t + e^{-(t-\pi)}\sin t]$
   $-\frac{1}{2}u_{2\pi}(t)[1 - e^{-(t-2\pi)}\cos t - e^{-(t-2\pi)}\sin t]$

3

3. $y = \frac{1}{6}[1 - u_{2\pi}(t)](2\sin t - \sin 2t)$
4. $y = \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}u_\pi(t)(2\sin t + \sin 2t)$
5. $y = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t)[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}]$

6. $y = e^{-t} - e^{-2t} + u_2(t)[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}]$
7. $y = \cos t + u_{3\pi}(t)[1 - \cos(t - 3\pi)]$

See SSM for detailed solutions to 8, 10, 16bc

8. $y = h(t) - u_{\pi/2}(t)h(t - \pi/2), \quad h(t) = \frac{4}{25}(-4 + 5t + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t)$
9. $y = \frac{1}{2}\sin t + \frac{1}{2}t - \frac{1}{2}u_6(t)[t - 6 - \sin(t - 6)]$
10. $y = h(t) + u_\pi(t)h(t - \pi), \quad h(t) = \frac{4}{17}[-4\cos t + \sin t + 4e^{-t/2}\cos t + e^{-t/2}\sin t]$
11. $y = u_\pi(t)[\frac{1}{4} - \frac{1}{4}\cos(2t - 2\pi)] - u_{3\pi}(t)[\frac{1}{4} - \frac{1}{4}\cos(2t - 6\pi)]$
12. $y = u_1(t)h(t - 1) - u_2(t)h(t - 2), \quad h(t) = -1 + (\cos t + \cosh t)/2$
13. $y = h(t) - u_\pi(t)h(t - \pi), \quad h(t) = (3 - 4\cos t + \cos 2t)/12$
14. $f(t) = [u_{t_0}(t)(t - t_0) - u_{t_0+k}(t)(t - t_0 - k)](h/k)$
15. $g(t) = [u_{t_0}(t)(t - t_0) - 2u_{t_0+k}(t)(t - t_0 - k) + u_{t_0+2k}(t)(t - t_0 - 2k)](h/k)$
16. (b) $u(t) = 4ku_{3/2}(t)h(t - \frac{3}{2}) - 4ku_{5/2}(t)h(t - \frac{5}{2})$,
    $h(t) = \frac{1}{4} - (\sqrt{7}/84)\,e^{-t/8}\sin(3\sqrt{7}\,t/8) - \frac{1}{4}e^{-t/8}\cos(3\sqrt{7}\,t/8)$
    (d) $k = 2.51$ (e) $\tau = 25.6773$
17. (a) $k = 5$
    (b) $y = [u_5(t)h(t - 5) - u_{5+k}(t)h(t - 5 - k)]/k, \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$
18. (b) $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$;
    $y = [u_{4-k}(t)h(t - 4 + k) - u_{4+k}(t)h(t - 4 - k)]/2k$,
    $h(t) = \frac{1}{4} - \frac{1}{4}e^{-t/6}\cos(\sqrt{143}\,t/6) - (\sqrt{143}/572)\,e^{-t/6}\sin(\sqrt{143}\,t/6)$

19abd

19. (b) $y = 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)[1 - \cos(t - k\pi)]$

20, 20abc

21. (b) $y = 1 - \cos t + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t)[1 - \cos(t - k\pi)]$
23. (a) $y = 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t)[1 - \cos(t - 11k/4)]$

Section 6.5, page 328

1, 3, 5, 7

1. $y = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)e^{-(t-\pi)}\sin(t - \pi)$
2. $y = \frac{1}{2}u_\pi(t)\sin 2(t - \pi) - \frac{1}{2}u_{2\pi}(t)\sin 2(t - 2\pi)$
3. $y = -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t} + u_5(t)[-e^{-2(t-5)} + e^{-(t-5)}] + u_{10}(t)[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}]$
4. $y = \cosh(t) - 20u_3(t)\sinh(t - 3)$
5. $y = \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t + (1/\sqrt{2})\,u_{3\pi}(t)e^{-(t-3\pi)}\sin\sqrt{2}(t - 3\pi)$
6. $y = \frac{1}{2}\cos 2t + \frac{1}{2}u_{4\pi}(t)\sin 2(t - 4\pi)$
7. $y = \sin t + u_{2\pi}(t)\sin(t - 2\pi)$
8. $y = u_{\pi/4}(t)\sin 2(t - \pi/4)$
9. $y = u_{\pi/2}(t)[1 - \cos(t - \pi/2)] + 3u_{3\pi/2}(t)\sin(t - 3\pi/2) - u_{2\pi}(t)[1 - \cos(t - 2\pi)]$

10

10. $y = (1/\sqrt{31})\,u_{\pi/6}(t)\exp[-\frac{1}{4}(t - \pi/6)]\sin(\sqrt{31}/4)(t - \pi/6)$
11. $y = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + u_{\pi/2}(t)e^{-(t-\pi/2)}\sin(t - \pi/2)$

13a, b

12. $y = u_1(t)[\sinh(t - 1) - \sin(t - 1)]/2$
13. (a) $-e^{-T/4}\delta(t - 5 - T), \quad T = 8\pi/\sqrt{15}$
14. (a) $y = (4/\sqrt{15})\,u_1(t)e^{-(t-1)/4}\sin(\sqrt{15}/4)(t - 1)$
    (b) $t_1 \cong 2.3613, \quad y_1 \cong 0.71153$
    (c) $y = (8\sqrt{7}/21)\,u_1(t)e^{-(t-1)/8}\sin(3\sqrt{7}/8)\,(t - 1); \quad t_1 \cong 2.4569, \quad y_1 \cong 0.83351$
    (d) $t_1 = 1 + \pi/2 \cong 2.5708, \quad y_1 = 1$
15. (a) $k_1 \cong 2.8108$ (b) $k_1 \cong 2.3995$ (c) $k_1 = 2$
16. (a) $\phi(t, k) = [u_{4-k}(t)h(t - 4 + k) - u_{4+k}(t)h(t - 4 - k)]/2k, \quad h(t) = 1 - \cos t$
    (b) $u_4(t)\sin(t - 4)$ (c) Yes

See SSM for detailed solutions to 17b

17. (b) $y = \sum_{k=1}^{20} u_{k\pi}(t)\sin(t-k\pi)$

18. (b) $y = \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi}(t)\sin(t-k\pi)$

19. (b) $y = \sum_{k=1}^{20} u_{k\pi/2}(t)\sin(t-k\pi/2)$

20. (b) $y = \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi/2}(t)\sin(t-k\pi/2)$

21b

21. (b) $y = \sum_{k=1}^{15} u_{(2k-1)\pi}(t)\sin[t-(2k-1)\pi]$

22. (b) $y = \sum_{k=1}^{40} (-1)^{k+1} u_{11k/4}(t)\sin(t-11k/4)$

25b

23. (b) $y = \frac{20}{\sqrt{399}} \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi}(t) e^{-(t-k\pi)/20}\sin[\sqrt{399}(t-k\pi)/20]$

24. (b) $y = \frac{20}{\sqrt{399}} \sum_{k=1}^{15} u_{(2k-1)\pi}(t) e^{-[t-(2k-1)\pi]/20}\sin\{\sqrt{399}[t-(2k-1)\pi]/20\}$

Section 6.6, page 335

1c

3. $\sin t * \sin t = \frac{1}{2}(\sin t - t\cos t)$ is negative when $t = 2\pi$, for example.

4

4. $F(s) = 2/s^2(s^2+4)$

5. $F(s) = 1/(s+1)(s^2+1)$

6. $F(s) = 1/s^2(s-1)$

7. $F(s) = s/(s^2+1)^2$

8, 13, 15, 17

8. $f(t) = \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau\, d\tau$

9. $f(t) = \int_0^t e^{-(t-\tau)}\cos 2\tau\, d\tau$

10. $f(t) = \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)}\sin 2\tau\, d\tau$

11. $f(t) = \int_0^t \sin(t-\tau)g(\tau)\, d\tau$

12. $y = \frac{1}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t \sin\omega(t-\tau)g(\tau)\, d\tau$

13. $y = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin\alpha\tau\, d\tau$

14. $y = \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\sin 2(t-\tau)g(\tau)\, d\tau$

15. $y = e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t + \int_0^t e^{-(t-\tau)/2}\sin(t-\tau)[1-u_\pi(\tau)]\, d\tau$

16. $y = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\, d\tau$

17. $y = 2e^{-t} - e^{-2t} + \int_0^t [e^{-(t-\tau)} - e^{-2(t-\tau)}]\cos\alpha\tau\, d\tau$

18. $y = \frac{1}{2}\int_0^t [\sinh(t-\tau) - \sin(t-\tau)]g(\tau)\, d\tau$

19. $y = \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{6}\int_0^t [2\sin(t-\tau) - \sin 2(t-\tau)]g(\tau)\, d\tau$

20

20. $\Phi(s) = \dfrac{F(s)}{1+K(s)}$

21. (c) $\phi(t) = \frac{1}{3}(4\sin 2t - 2\sin t)$ (d) $u(t) = \frac{1}{3}(2\sin t - \sin 2t)$

CHAPTER 7

Section 7.1, page 344

2, 4, 5

1. $x_1' = x_2, \quad x_2' = -2x_1 - 0.5x_2$
2. $x_1' = x_2, \quad x_2' = -2x_1 - 0.5x_2 + 3\sin t$
3. $x_1' = x_2, \quad x_2' = -(1-0.25t^{-2})x_1 - t^{-1}x_2$
4. $x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = x_1$
5. $x_1' = x_2, \quad x_2' = -q(t)x_1 - p(t)x_2 + g(t); \quad x_1(0) = u_0, \quad x_2(0) = u_0'$
6. $y_1' = y_2, \quad y_2' = -(k_1+k_2)y_1/m_1 + k_2y_3/m_1 + F_1(t)/m_1,$
   $y_3' = y_4, \quad y_4' = k_2y_1/m_2 - (k_2+k_3)y_3/m_2 + F_2(t)/m_2$

7. (a) $x_1 = c_1e^{-t} + c_2e^{-3t}, \quad x_2 = c_1e^{-t} - c_2e^{-3t}$
   (b) $c_1 = 5/2, \quad c_2 = -1/2$ in solution in (a)
   (c) Graph approaches origin in the first quadrant tangent to the line $x_1 = x_2$.

See SSM for detailed solutions to 8, 9

8. $x_1 = \frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}, \quad x_2 = \frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}$
   Graph is asymptotic to the line $x_1 = 2x_2$ in the first quadrant.
9. $x_1 = -\frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t}, \quad x_2 = \frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t}$
   Graph is asymptotic to the line $x_1 = x_2$ in the third quadrant.
10. $x_1 = -7e^{-t} + 6e^{-2t}, \quad x_2 = -7e^{-t} + 9e^{-2t}$
    Graph approaches the origin in the third quadrant tangent to the line $x_1 = x_2$.
11. $x_1 = 3\cos 2t + 4\sin 2t, \quad x_2 = -3\sin 2t + 4\cos 2t$
    Graph is a circle, center at origin, radius 5, traversed clockwise.

12, 14, 19

12. $x_1 = -2e^{-t/2}\cos 2t + 2e^{-t/2}\sin 2t, \quad x_2 = 2e^{-t/2}\cos 2t + 2e^{-t/2}\sin 2t$
    Graph is a clockwise spiral, approaching the origin.
13. $LRCI'' + LI' + RI = 0$

21abc

21. (a) $Q_1' = \frac{3}{2} - \frac{1}{10}Q_1 + \frac{3}{40}Q_2, \quad Q_1(0) = 25$
        $Q_2' = 3 + \frac{1}{10}Q_1 - \frac{1}{5}Q_2, \quad Q_2(0) = 15$
    (b) $Q_1^E = 42, \quad Q_2^E = 36$
    (c) $x_1' = -\frac{1}{10}x_1 + \frac{3}{40}x_2, \quad x_1(0) = -17$
        $x_2' = \frac{1}{10}x_1 - \frac{1}{5}x_2, \quad x_2(0) = -21$
22. (a) $Q_1' = 3q_1 - \frac{1}{15}Q_1 + \frac{1}{100}Q_2, \quad Q_1(0) = Q_1^0$
        $Q_2' = q_2 + \frac{1}{30}Q_1 - \frac{3}{100}Q_2, \quad Q_2(0) = Q_2^0$
    (b) $Q_1^E = 6(9q_1 + q_2), \quad Q_2^E = 20(3q_1 + 2q_2)$
    (c) No
    (d) $\frac{10}{9} \le Q_2^E/Q_1^E \le \frac{20}{3}$

Section 7.2, page 355

1ac

1. (a) $\begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} -15 & 6 & -12 \\ 7 & -18 & -1 \\ -26 & -3 & -5 \end{pmatrix}$
   (c) $\begin{pmatrix} 6 & -12 & 3 \\ 4 & 3 & 7 \\ 9 & 12 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} -8 & -9 & 11 \\ 14 & 12 & -5 \\ 5 & -8 & 5 \end{pmatrix}$
2. (a) $\begin{pmatrix} 1-i & -7+2i \\ -1+2i & 2+3i \end{pmatrix}$ (b) $\begin{pmatrix} 3+4i & 6i \\ 11+6i & 6-5i \end{pmatrix}$
   (c) $\begin{pmatrix} -3+5i & 7+5i \\ 2+i & 7+2i \end{pmatrix}$ (d) $\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -4 \end{pmatrix}$
3. (a) $\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{pmatrix}$ (c), (d) $\begin{pmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$
4. (a) $\begin{pmatrix} 3-2i & 2-i \\ 1+i & -2+3i \end{pmatrix}$ (b) $\begin{pmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{pmatrix}$ (c) $\begin{pmatrix} 3+2i & 2+i \\ 1-i & -2-3i \end{pmatrix}$
5. $\begin{pmatrix} 10 & 6 & -4 \\ 0 & 4 & 10 \\ 4 & 4 & 6 \end{pmatrix}$

See SSM for detailed solutions to 6, 10

6. (a) $\begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 0 & -1 \\ 2 & 7 & 4 \\ -1 & 1 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}$

8. (a) $4i$ (b) $12 - 8i$ (c) $2 + 2i$ (d) 16

10. $\begin{pmatrix} \frac{3}{11} & -\frac{4}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$

11. $\begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$

12

12. $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

13. $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$

14

14. Singular

15. $\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

16. $\begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{10} & \frac{4}{10} & -\frac{2}{10} \\ -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$

17. Singular

18. $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

19. $\begin{pmatrix} 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$

21. (a) $\begin{pmatrix} 7e^{t} & 5e^{-t} & 10e^{2t} \\ -e^{t} & 7e^{-t} & 2e^{2t} \\ 8e^{t} & 0 & -e^{2t} \end{pmatrix}$

(b) $\begin{pmatrix} 2e^{2t} - 2 + 3e^{3t} & 1 + 4e^{-2t} - e^{t} & 3e^{3t} + 2e^{t} - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t} & 2 + 2e^{-2t} + e^{t} & 6e^{3t} + e^{t} + e^{4t} \\ -2e^{2t} - 3 + 6e^{3t} & -1 + 6e^{-2t} - 2e^{t} & -3e^{3t} + 3e^{t} - 2e^{4t} \end{pmatrix}$

22, 25

(c) $\begin{pmatrix} e^{t} & -2e^{-t} & 2e^{2t} \\ 2e^{t} & -e^{-t} & -2e^{2t} \\ -e^{t} & -3e^{-t} & 4e^{2t} \end{pmatrix}$ (d) $(e-1)\begin{pmatrix} 1 & 2e^{-1} & \frac{1}{2}(e+1) \\ 2 & e^{-1} & -\frac{1}{2}(e+1) \\ -1 & 3e^{-1} & e+1 \end{pmatrix}$

Section 7.3, page 366

1

1. $x_1 = -\frac{1}{3}, \quad x_2 = \frac{7}{3}, \quad x_3 = -\frac{1}{3}$

2. No solution

2, 3

3. $x_1 = -c, \quad x_2 = c + 1, \quad x_3 = c$, where c is arbitrary

4. $x_1 = c, \quad x_2 = -c, \quad x_3 = -c$, where c is arbitrary

6, 8

5. $x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$

6. Linearly independent

7. $\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}$

8. $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} + 4\mathbf{x}^{(3)} - \mathbf{x}^{(4)} = \mathbf{0}$

9. Linearly independent

10. $\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}$

14

12. $3\mathbf{x}^{(1)}(t) - 6\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$

13. Linearly independent

15. $\lambda_1 = 2, \ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \quad \lambda_2 = 4, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

15

16. $\lambda_1 = 1 + 2i, \ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}; \quad \lambda_2 = 1 - 2i, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$

17. $\lambda_1 = -3, \ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \lambda_2 = -1, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

See SSM for detailed solutions to 18, 21

18. $\lambda_1 = 0,\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}; \quad \lambda_2 = 2,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

19. $\lambda_1 = 2,\ \mathbf{x}^{(1)} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}; \quad \lambda_2 = -2,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

20. $\lambda_1 = -1/2,\ \mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}; \quad \lambda_2 = -3/2,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

21. $\lambda_1 = 1,\ \mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}; \quad \lambda_2 = 1 + 2i,\ \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}; \quad \lambda_3 = 1 - 2i,\ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$

22. $\lambda_1 = 1,\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_2 = 2,\ \mathbf{x}^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; \quad \lambda_3 = 3,\ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

23. $\lambda_1 = 1,\ \mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}; \quad \lambda_2 = 2,\ \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; \quad \lambda_3 = -1,\ \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

24

27

24. $\lambda_1 = -1,\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}; \quad \lambda_2 = -1,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_3 = 8,\ \mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

Section 7.4, page 371

1, 2a

2. (c) $W(t) = c \exp \int [p_{11}(t) + p_{22}(t)]\, dt$

2bc, 6abcd

6. (a) $W(t) = t^2$
(b) $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent at each point except $t = 0$; they are linearly independent on every interval.
(c) At least one coefficient must be discontinuous at $t = 0$.
(d) $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix} \mathbf{x}$

7. (a) $W(t) = t(t-2)e^t$
(b) $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent at each point except $t = 0$ and $t = 2$; they are linearly independent on every interval.
(c) There must be at least one discontinuous coefficient at $t = 0$ and $t = 2$.
(d) $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \dfrac{2-2t}{t^2-2t} & \dfrac{t^2-2}{t^2-2t} \end{pmatrix} \mathbf{x}$

Section 7.5, page 381

1

1. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

2. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$

3. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

4. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

5

5. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

6. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

6, 7, 9

7. $\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

8. $\mathbf{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$

9. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t}$

10. $\mathbf{x} = c_1 \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-it}$

11. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}$

12. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$

13. $\mathbf{x} = c_1 \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

14. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$

15. $\mathbf{x} = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

16. $\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$

See SSM for detailed solutions to 16

17. $\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$

18. $\mathbf{x} = 6 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{t} + 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}$

20, 25

20. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$

21. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4$

22. $\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}$

23. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$

29. (a) $x_1' = x_2, \quad x_2' = -(c/a)x_1 - (b/a)x_2$

30. (a) $\mathbf{x} = -\frac{55}{8} \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20} + \frac{29}{8} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4}$

(c) $T \cong 74.39$

31. (a) $\mathbf{x} = c_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{(-2+\sqrt{2})t/2} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-2-\sqrt{2})t/2}$;

$r_{1,2} = (-2 \pm \sqrt{2})/2$; node

(b) $\mathbf{x} = c_1 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{(-1-\sqrt{2})t}$; $r_{1,2} = -1 \pm \sqrt{2}$; saddle point

31c

(c) $r_{1,2} = -1 \pm \sqrt{\alpha}$; $\alpha = 1$

32. (a) $\begin{pmatrix} I \\ V \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

33. (a) $\left(\frac{1}{CR_2} - \frac{R_1}{L} \right)^2 - \frac{4}{CL} > 0$

Section 7.6, page 390

1

1. $\mathbf{x} = c_1 e^{t} \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$

2. $\mathbf{x} = c_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}$

3. $\mathbf{x} = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$

4. $\mathbf{x} = c_1 e^{t/2}\begin{pmatrix} 5\cos\frac{3}{2}t \\ 3(\cos\frac{3}{2}t + \sin\frac{3}{2}t)\end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} 5\sin\frac{3}{2}t \\ 3(-\cos\frac{3}{2}t + \sin\frac{3}{2}t)\end{pmatrix}$

5. $\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t\end{pmatrix}$

6. $\mathbf{x} = c_1\begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t\end{pmatrix} + c_2\begin{pmatrix} -2\sin 3t \\ \sin 3t - 3\cos 3t\end{pmatrix}$

See SSM for detailed solutions to 7, 9

7. $\mathbf{x} = c_1\begin{pmatrix} 2 \\ -3 \\ 2\end{pmatrix}e^t + c_2 e^t\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t\end{pmatrix} + c_3 e^t\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t\end{pmatrix}$

8. $\mathbf{x} = c_1\begin{pmatrix} 2 \\ -2 \\ 1\end{pmatrix}e^{-2t} + c_2 e^{-t}\begin{pmatrix} -\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t\end{pmatrix}$
$+ c_3 e^{-t}\begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t\end{pmatrix}$

9. $\mathbf{x} = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t\end{pmatrix}$

10. $\mathbf{x} = e^{-2t}\begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t\end{pmatrix}$

11ad, 15ab

11. (a) $r = -\frac{1}{4} \pm i$

12. (a) $r = \frac{1}{5} \pm i$

13. (a) $r = \alpha \pm i$ (b) $\alpha = 0$

14. (a) $r = (\alpha \pm \sqrt{\alpha^2 - 20})/2$ (b) $\alpha = -\sqrt{20},\ 0,\ \sqrt{20}$

15. (a) $r = \pm\sqrt{4 - 5\alpha}$ (b) $\alpha = 4/5$

16abc, 18abc

16. (a) $r = \frac{5}{4} \pm \frac{1}{2}\sqrt{3\alpha}$ (b) $\alpha = 0,\ 25/12$

17. (a) $r = -1 \pm \sqrt{-\alpha}$ (b) $\alpha = -1,\ 0$

18. (a) $r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{49 - 24\alpha}$ (b) $\alpha = 2,\ 49/24$

19. (a) $r = \frac{1}{2}\alpha - 2 \pm \sqrt{\alpha^2 + 8\alpha - 24}$ (b) $\alpha = -4 - 2\sqrt{10},\ -4 + 2\sqrt{10},\ 5/2$

20. (a) $r = -1 \pm \sqrt{25 + 8\alpha}$ (b) $\alpha = -25/8,\ -3$

21, 23abc, 29a

21. $\mathbf{x} = c_1 t^{-1}\begin{pmatrix} \cos(\sqrt{2}\ln t) \\ \sqrt{2}\sin(\sqrt{2}\ln t)\end{pmatrix} + c_2 t^{-1}\begin{pmatrix} \sin(\sqrt{2}\ln t) \\ -\sqrt{2}\cos(\sqrt{2}\ln t)\end{pmatrix}$

22. $\mathbf{x} = c_1\begin{pmatrix} 5\cos(\ln t) \\ 2\cos(\ln t) + \sin(\ln t)\end{pmatrix} + c_2\begin{pmatrix} 5\sin(\ln t) \\ -\cos(\ln t) + 2\sin(\ln t)\end{pmatrix}$

23. (a) $r = -\frac{1}{4} \pm i,\ -\frac{1}{4}$

24. (a) $r = -\frac{1}{4} \pm i,\ \frac{1}{10}$

25. (b) $\begin{pmatrix} I \\ V\end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\sin(t/2)\end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\cos(t/2)\end{pmatrix}$
(c) Use $c_1 = 2$, $c_2 = -\frac{3}{4}$ in answer to part (b).
(d) $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} V(t) = 0$; no

26. (b) $\begin{pmatrix} I \\ V\end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ -\sin t + \cos t\end{pmatrix}$
(c) Use $c_1 = 2$ and $c_2 = 3$ in answer to part (b).
(d) $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} V(t) = 0$; no

28. (b) $r = \pm i\sqrt{k/m}$ (d) $|r|$ is the natural frequency.

29bce

29. (a) $y_1' = y_2,\ y_2' = -2y_1 + y_3,\ y_3' = y_4,\ y_4' = y_1 - 2y_3$
(b) $r = \pm i,\ \pm\sqrt{3}\,i$
(c) $y_1 = y_3 = \sin t + 2\cos t,\ y_2 = y_4 = -2\sin t + \cos t$
(d) $y_1 = -y_3 = \sin\sqrt{3}t + 2\cos\sqrt{3}t,\ y_2 = -y_4 = -2\sqrt{3}\sin\sqrt{3}t + \sqrt{3}\cos\sqrt{3}t$

Section 7.7, page 400

See SSM for detailed solutions to 2, 4

1. $\mathbf{\Phi}(t) = \begin{pmatrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} \end{pmatrix}$

2. $\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{-t/2} + \frac{1}{2}e^{-t} & e^{-t/2} - e^{-t} \\ \frac{1}{4}e^{-t/2} - \frac{1}{4}e^{-t} & \frac{1}{2}e^{-t/2} + \frac{1}{2}e^{-t} \end{pmatrix}$

3. $\mathbf{\Phi}(t) = \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{pmatrix}$

4. $\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} & -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t} \\ -\frac{4}{5}e^{-3t} + \frac{4}{5}e^{2t} & \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t} \end{pmatrix}$

5. $\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$

6, 10

6. $\mathbf{\Phi}(t) = \begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}$

7. $\mathbf{\Phi}(t) = \begin{pmatrix} -\frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \end{pmatrix}$

8. $\mathbf{\Phi}(t) = \begin{pmatrix} e^{-t}\cos t + 2e^{-t}\sin t & e^{-t}\sin t \\ 5e^{-t}\sin t & e^{-t}\cos t - 2e^{-t}\sin t \end{pmatrix}$

9. $\mathbf{\Phi}(t) = \begin{pmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{pmatrix}$

10. $\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{6}e^{t} + \frac{1}{3}e^{-2t} + \frac{1}{2}e^{3t} & -\frac{1}{3}e^{t} + \frac{1}{3}e^{-2t} & \frac{1}{2}e^{t} - e^{-2t} + \frac{1}{2}e^{3t} \\ -\frac{2}{3}e^{t} - \frac{1}{3}e^{-2t} + e^{3t} & \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t} & -2e^{t} + e^{-2t} + e^{3t} \\ -\frac{1}{6}e^{t} - \frac{1}{3}e^{-2t} + \frac{1}{2}e^{3t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} & -\frac{1}{2}e^{t} + e^{-2t} + \frac{1}{2}e^{3t} \end{pmatrix}$

11

11. $\mathbf{x} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}$

12. $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{-t}\cos 2t + \begin{pmatrix} -2 \\ 3/2 \end{pmatrix}e^{-t}\sin 2t$

Section 7.8, page 407

1

1. $\mathbf{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}\right]$

2. $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right]$

3, 5

3. $\mathbf{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}e^{-t}\right]$

4. $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t/2} + \begin{pmatrix} 0 \\ \frac{2}{5} \end{pmatrix}e^{-t/2}\right]$

5. $\mathbf{x} = c_1\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{2t} + c_3\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}\right]$

6. $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{-t}$

7. $\mathbf{x} = \begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix} e^{-3t}$

8. $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t}$

See SSM for detailed solutions to 9

9. $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{t/2} + \frac{3}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2}$

10. $\mathbf{x} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 14 \begin{pmatrix} 3 \\ -1 \end{pmatrix} t$

11, 12, 14

11. $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} e^{t} + 4 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$

12. $\mathbf{x} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + \frac{1}{3} \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix} e^{-7t/2}$

13. $\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right]$

14. $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3} \ln t - \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} t^{-3} \right]$

15, 17a

16. (b) $\begin{pmatrix} I \\ V \end{pmatrix} = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t/2} \right]$

17bcd

17. (b) $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

(c) $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$

(d) $\mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t}$

17ef

(e) $\mathbf{\Psi}(t) = e^{2t} \begin{pmatrix} 0 & 1 & t+2 \\ 1 & t+1 & (t^2/2)+t \\ -1 & -t & -(t^2/2)+3 \end{pmatrix}$

(f) $\mathbf{T} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix},$

$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

18. (a) $\mathbf{x}^{(1)}(t) = \begin{pmatrix}1\\0\\2\end{pmatrix} e^t, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix}0\\2\\-3\end{pmatrix} e^t$

(d) $\mathbf{x}^{(3)}(t) = \begin{pmatrix}2\\4\\-2\end{pmatrix} te^t + \begin{pmatrix}0\\0\\-1\end{pmatrix} e^t$

(e) $\boldsymbol{\Psi}(t) = e^t \begin{pmatrix}1 & 0 & 2t\\0 & 2 & 4t\\2 & -3 & -2t-1\end{pmatrix}$ or $e^t \begin{pmatrix}1 & 2 & 2t\\0 & 4 & 4t\\2 & -2 & -2t-1\end{pmatrix}$

(f) $\mathbf{T} = \begin{pmatrix}1 & 2 & 0\\0 & 4 & 0\\2 & -2 & -1\end{pmatrix}, \qquad \mathbf{T}^{-1} = \begin{pmatrix}1 & -1/2 & 0\\0 & 1/4 & 0\\2 & -3/2 & -1\end{pmatrix},$

$\mathbf{J} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$

See SSM for detailed solutions to 19abc

19. (a) $\mathbf{J}^2 = \begin{pmatrix}\lambda^2 & 2\lambda\\0 & \lambda^2\end{pmatrix}, \qquad \mathbf{J}^3 = \begin{pmatrix}\lambda^3 & 3\lambda^2\\0 & \lambda^3\end{pmatrix}, \qquad \mathbf{J}^4 = \begin{pmatrix}\lambda^4 & 4\lambda^3\\0 & \lambda^4\end{pmatrix}$

(c) $\exp(\mathbf{J}t) = e^{\lambda t}\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}$

19d

(d) $\mathbf{x} = \exp(\mathbf{J}t)\mathbf{x}^0$

20. (c) $\exp(\mathbf{J}t) = e^{\lambda t}\begin{pmatrix}1 & 0 & 0\\0 & 1 & t\\0 & 0 & 1\end{pmatrix}$

21. (c) $\exp(\mathbf{J}t) = e^{\lambda t}\begin{pmatrix}1 & t & t^2/2\\0 & 1 & t\\0 & 0 & 1\end{pmatrix}$

Section 7.9, page 417

1, 2

1. $\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t - \frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}e^t + \begin{pmatrix}1\\2\end{pmatrix}t - \begin{pmatrix}0\\1\end{pmatrix}$

2. $\mathbf{x} = c_1\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}e^{-2t} - \begin{pmatrix}2/3\\1/\sqrt{3}\end{pmatrix}e^t + \begin{pmatrix}-1\\2/\sqrt{3}\end{pmatrix}e^{-t}$

3

3. $\mathbf{x} = \frac{1}{5}(2t - \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t + c_1)\begin{pmatrix}5\cos t\\2\cos t + \sin t\end{pmatrix}$

$+\frac{1}{5}(-t - \frac{1}{2}\sin 2t + \frac{3}{2}\cos 2t + c_2)\begin{pmatrix}5\sin t\\-\cos t + 2\sin t\end{pmatrix}$

4

4. $\mathbf{x} = c_1\begin{pmatrix}1\\-4\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t} - \begin{pmatrix}0\\1\end{pmatrix}e^{-2t} + \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}e^t$

5. $\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\left[\begin{pmatrix}1\\2\end{pmatrix}t - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right] - 2\begin{pmatrix}1\\2\end{pmatrix}\ln t + \begin{pmatrix}2\\5\end{pmatrix}t^{-1} - \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}t^{-2}$

6. $\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-5t} + \begin{pmatrix}1\\2\end{pmatrix}\ln t + \frac{8}{5}\begin{pmatrix}1\\2\end{pmatrix}t + \frac{4}{25}\begin{pmatrix}-2\\1\end{pmatrix}$

7. $\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} + \frac{1}{4}\begin{pmatrix}1\\-8\end{pmatrix}e^t$

8. $\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + \begin{pmatrix}1\\0\end{pmatrix}e^{t} + 2\begin{pmatrix}1\\1\end{pmatrix}te^{t}$

9. $\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t/2} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t} + \begin{pmatrix}\frac{5}{2}\\\frac{3}{2}\end{pmatrix}t - \begin{pmatrix}\frac{17}{4}\\\frac{15}{4}\end{pmatrix} + \begin{pmatrix}\frac{1}{6}\\\frac{1}{2}\end{pmatrix}e^{t}$

10. $\mathbf{x} = c_1\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}e^{-t} + c_2\begin{pmatrix}-\sqrt{2}\\1\end{pmatrix}e^{-4t} - \frac{1}{3}\begin{pmatrix}\sqrt{2}-1\\2-\sqrt{2}\end{pmatrix}te^{-t} + \frac{1}{9}\begin{pmatrix}2+\sqrt{2}\\-1-\sqrt{2}\end{pmatrix}e^{-t}$

11. $\mathbf{x} = (\frac{1}{2}\sin^2 t + c_1)\begin{pmatrix}5\cos t\\2\cos t + \sin t\end{pmatrix} + (-\frac{1}{2}t - \frac{1}{2}\sin t\cos t + c_2)\begin{pmatrix}5\sin t\\-\cos t + 2\sin t\end{pmatrix}$

See SSM for detailed solutions to 12, 14

12. $\mathbf{x} = [\frac{1}{5}\ln(\sin t) - \ln(-\cos t) - \frac{2}{5}t + c_1]\begin{pmatrix}5\cos t\\2\cos t + \sin t\end{pmatrix}$
$+ [\frac{2}{5}\ln(\sin t) - \frac{4}{5}t + c_2]\begin{pmatrix}5\sin t\\-\cos t + 2\sin t\end{pmatrix}$

13. (a) $\mathbf{\Psi}(t) = \begin{pmatrix}e^{-t/2}\cos\frac{1}{2}t & e^{-t/2}\sin\frac{1}{2}t\\4e^{-t/2}\sin\frac{1}{2}t & -4e^{-t/2}\cos\frac{1}{2}t\end{pmatrix}$ (b) $\mathbf{x} = e^{-t/2}\begin{pmatrix}\sin\frac{1}{2}t\\4-4\cos\frac{1}{2}t\end{pmatrix}$

14. $\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}t + c_2\begin{pmatrix}1\\3\end{pmatrix}t^{-1} - \begin{pmatrix}2\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\3\end{pmatrix}t - \begin{pmatrix}1\\1\end{pmatrix}t\ln t - \frac{1}{3}\begin{pmatrix}4\\3\end{pmatrix}t^2$

15. $\mathbf{x} = c_1\begin{pmatrix}2\\1\end{pmatrix}t^2 + c_2\begin{pmatrix}1\\2\end{pmatrix}t^{-1} + \begin{pmatrix}3\\2\end{pmatrix}t + \frac{1}{10}\begin{pmatrix}-2\\1\end{pmatrix}t^4 - \frac{1}{2}\begin{pmatrix}2\\3\end{pmatrix}$

CHAPTER 8 Section 8.1, page 427

1ac, 5ac, 7ac

1. (a) 1.1975, 1.38549, 1.56491, 1.73658
   (b) 1.19631, 1.38335, 1.56200, 1.73308
   (c) 1.19297, 1.37730, 1.55378, 1.72316
   (d) 1.19405, 1.37925, 1.55644, 1.72638
2. (a) 1.59980, 1.29288, 1.07242, 0.930175
   (b) 1.61124, 1.31361, 1.10012, 0.962552
   (c) 1.64337, 1.37164, 1.17763, 1.05334
   (d) 1.63301, 1.35295, 1.15267, 1.02407
3. (a) 1.2025, 1.41603, 1.64289, 1.88590
   (b) 1.20388, 1.41936, 1.64896, 1.89572
   (c) 1.20864, 1.43104, 1.67042, 1.93076
   (d) 1.20693, 1.42683, 1.66265, 1.91802
4. (a) 1.10244, 1.21426, 1.33484, 1.46399
   (b) 1.10365, 1.21656, 1.33817, 1.46832
   (c) 1.10720, 1.22333, 1.34797, 1.48110
   (d) 1.10603, 1.22110, 1.34473, 1.47688
5. (a) 0.509239, 0.522187, 0.539023, 0.559936
   (b) 0.509701, 0.523155, 0.540550, 0.562089
   (c) 0.511127, 0.526155, 0.545306, 0.568822
   (d) 0.510645, 0.525138, 0.543690, 0.566529
6. (a) −0.920498, −0.857538, −0.808030, −0.770038
   (b) −0.922575, −0.860923, −0.812300, −0.774965
   (c) −0.928059, −0.870054, −0.824021, −0.788686
   (d) −0.926341, −0.867163, −0.820279, −0.784275
7. (a) 2.90330, 7.53999, 19.4292, 50.5614
   (b) 2.93506, 7.70957, 20.1081, 52.9779
   (c) 3.03951, 8.28137, 22.4562, 61.5496
   (d) 3.00306, 8.07933, 21.6163, 58.4462

8. (a) 0.891830, 1.25225, 2.37818, 4.07257
   (b) 0.908902, 1.26872, 2.39336, 4.08799
   (c) 0.958565, 1.31786, 2.43924, 4.13474
   (d) 0.942261, 1.30153, 2.42389, 4.11908
9. (a) 3.95713, 5.09853, 6.41548, 7.90174
   (b) 3.95965, 5.10371, 6.42343, 7.91255
   (c) 3.96727, 5.11932, 6.44737, 7.94512
   (d) 3.96473, 5.11411, 6.43937, 7.93424

See SSM for detailed solutions to 9c

10. (a) 1.60729, 2.46830, 3.72167, 5.45963
    (b) 1.60996, 2.47460, 3.73356, 5.47774
    (c) 1.61792, 2.49356, 3.76940, 5.53223
    (d) 1.61528, 2.48723, 3.75742, 5.51404
11. (a) −1.45865, −0.217545, 1.05715, 1.41487
    (b) −1.45322, −0.180813, 1.05903, 1.41244
    (c) −1.43600, −0.0681657, 1.06489, 1.40575
    (d) −1.44190, −0.105737, 1.06290, 1.40789
12. (a) 0.587987, 0.791589, 1.14743, 1.70973
    (b) 0.589440, 0.795758, 1.15693, 1.72955
    (c) 0.593901, 0.808716, 1.18687, 1.79291
    (d) 0.592396, 0.804319, 1.17664, 1.77111

15, 16

15. 1.595, 2.4636
16. $e_{n+1} = [2\phi(\bar{t}_n) - 1]h^2, \quad |e_{n+1}| \le \left[1 + 2\max_{0\le t\le 1}|\phi(t)|\right]h^2,$
    $e_{n+1} = e^{2\bar{t}_n}h^2, \quad |e_1| \le 0.012, \quad |e_4| \le 0.022$
17. $e_{n+1} = [2\phi(\bar{t}_n) - \bar{t}_n]h^2, \quad |e_{n+1}| \le \left[1 + 2\max_{0\le t\le 1}|\phi(t)|\right]h^2,$
    $e_{n+1} = 2e^{2\bar{t}_n}h^2, \quad |e_1| \le 0.024, \quad |e_4| \le 0.045$

19

18. $e_{n+1} = [\bar{t}_n + \bar{t}_n^2\phi(\bar{t}_n) + \phi^3(\bar{t}_n)]h^2$
19. $e_{n+1} = [19 - 15\bar{t}_n\phi^{-1/2}(\bar{t}_n)]h^2/4$
20. $e_{n+1} = \{1 + [\bar{t}_n + \phi(\bar{t}_n)]^{-1/2}\}h^2/4$
21. $e_{n+1} = \{2 - [\phi(\bar{t}_n) + 2\bar{t}_n^2]\exp[-\bar{t}_n\phi(\bar{t}_n)] - \bar{t}_n\exp[-2\bar{t}_n\phi(\bar{t}_n)]\}h^2/2$
22. (a) $\phi(t) = 1 + (1/5\pi)\sin 5\pi t$ (b) 1.2, 1.0, 1.2
    (c) 1.1, 1.1, 1.0, 1.0 (d) $h < 1/\sqrt{50\pi} \cong 0.08$

22d, 23abc, 24

24. $e_{n+1} = -\frac{1}{2}\phi''(\bar{t}_n)h^2$
25. (a) 1.55, 2.34, 3.46, 5.07
    (b) 1.20, 1.39, 1.57, 1.74

25b

    (c) 1.20, 1.42, 1.65, 1.90
26. (a) 0 (b) 60 (c) −92.16
27. $0.224 \ne 0.225$

Section 8.2, page 434

1a, 4

1. (a) 1.19512, 1.38120, 1.55909, 1.72956
   (b) 1.19515, 1.38125, 1.55916, 1.72965
   (c) 1.19516, 1.38126, 1.55918, 1.72967
2. (a) 1.62283, 1.33460, 1.12820, 0.995445
   (b) 1.62243, 1.33386, 1.12718, 0.994215
   (c) 1.62234, 1.33368, 1.12693, 0.993921
3. (a) 1.20526, 1.42273, 1.65511, 1.90570
   (b) 1.20533, 1.42290, 1.65542, 1.90621
   (c) 1.20534, 1.42294, 1.65550, 1.90634
4. (a) 1.10483, 1.21882, 1.34146, 1.47263
   (b) 1.10484, 1.21884, 1.34147, 1.47262
   (c) 1.10484, 1.21884, 1.34147, 1.47262

5. (a) 0.510164, 0.524126, 0.542083, 0.564251
   (b) 0.510168, 0.524135, 0.542100, 0.564277
   (c) 0.510169, 0.524137, 0.542104, 0.564284
6. (a) −0.924650, −0.864338, −0.816642, −0.780008
   (b) −0.924550, −0.864177, −0.816442, −0.779781
   (c) −0.924525, −0.864138, −0.816393, −0.779725
7. (a) 2.96719, 7.88313, 20.8114, 55.5106
   (b) 2.96800, 7.88755, 20.8294, 55.5758
8. (a) 0.926139, 1.28558, 2.40898, 4.10386
   (b) 0.925815, 1.28525, 2.40869, 4.10359
9. (a) 3.96217, 5.10887, 6.43134, 7.92332
   (b) 3.96218, 5.10889, 6.43138, 7.92337

See SSM for detailed solutions to 10, 11

10. (a) 1.61263, 2.48097, 3.74556, 5.49595
    (b) 1.61263, 2.48092, 3.74550, 5.49589
11. (a) −1.44768, −0.144478, 1.06004, 1.40960
    (b) −1.44765, −0.143690, 1.06072, 1.40999
12. (a) 0.590897, 0.799950, 1.16653, 1.74969
    (b) 0.590906, 0.799988, 1.16663, 1.74992

14abc, 15

15. $e_{n+1} = (38h^3/3)\exp(4\bar{t}_n)$, $|e_{n+1}| \le 37,758.8h^3$ on $0 \le t \le 2$, $|e_1| \le 0.00193389$
16. $e_{n+1} = (2h^3/3)\exp(2\bar{t}_n)$, $|e_{n+1}| \le 4.92604h^3$ on $0 \le t \le 1$, $|e_1| \le 0.000814269$
17. $e_{n+1} = (4h^3/3)\exp(2\bar{t}_n)$, $|e_{n+1}| \le 9.85207h^3$ on $0 \le t \le 1$, $|e_1| \le 0.00162854$

19, 24

18. $h \cong 0.071$
19. $h \cong 0.023$
20. $h \cong 0.081$
21. $h \cong 0.117$
23. 1.19512, 1.38120, 1.55909, 1.72956
24. 1.62268, 1.33435, 1.12789, 0.995130
25. 1.20526, 1.42273, 1.65511, 1.90570
26. 1.10485, 1.21886, 1.34149, 1.47264

Section 8.3, page 438

1. (a) 1.19516, 1.38127, 1.55918, 1.72968
   (b) 1.19516, 1.38127, 1.55918, 1.72968
2. (a) 1.62231, 1.33362, 1.12686, 0.993839
   (b) 1.62230, 1.33362, 1.12685, 0.993826
3. (a) 1.20535, 1.42295, 1.65553, 1.90638
   (b) 1.20535, 1.42296, 1.65553, 1.90638

4, 11

4. (a) 1.10484, 1.21884, 1.34147, 1.47262
   (b) 1.10484, 1.21884, 1.34147, 1.47262
5. (a) 0.510170, 0.524138, 0.542105, 0.564286
   (b) 0.520169, 0.524138, 0.542105, 0.564286
6. (a) −0.924517, −0.864125, −0.816377, −0.779706
   (b) −0.924517, −0.864125, −0.816377, −0.779706
7. (a) 2.96825, 7.88889, 20.8349, 55.5957
   (b) 2.96828, 7.88904, 20.8355, 55.5980
8. (a) 0.925725, 1.28516, 2.40860, 4.10350
   (b) 0.925711, 1.28515, 2.40860, 4.10350
9. (a) 3.96219, 5.10890, 6.43139, 7.92338
   (b) 3.96219, 5.10890, 6.43139, 7.92338
10. (a) 1.61262, 2.48091, 3.74548, 5.49587
    (b) 1.61262, 2.48091, 3.74548, 5.49587
11. (a) −1.44764, −0.143543, 1.06089, 1.41008
    (b) −1.44764, −0.143427, 1.06095, 1.41011
12. (a) 0.590909, 0.800000, 1.166667, 1.75000

14abc

    (b) 0.590909, 0.800000, 1.166667, 1.75000

Section 8.4, page 444

1. (a) 1.7296801, 1.8934697
   (b) 1.7296802, 1.8934698
   (c) 1.7296805, 1.8934711
2. (a) 0.993852, 0.925764
   (b) 0.993846, 0.925746
   (c) 0.993869, 0.925837

See SSM for detailed solutions to 4a

3. (a) 1.906382, 2.179567
   (b) 1.906391, 2.179582
   (c) 1.906395, 2.179611
4. (a) 1.4726173, 1.6126215
   (b) 1.4726189, 1.6126231
   (c) 1.4726199, 1.6126256

4bc, 7a

5. (a) 0.56428577, 0.59090918
   (b) 0.56428581, 0.59090923
   (c) 0.56428588, 0.59090952
6. (a) −0.779693, −0.753135
   (b) −0.779692, −0.753137
   (c) −0.779680, −0.753089

7bc

7. (a) 2.96828, 7.88907, 20.8356, 55.5984
   (b) 2.96829, 7.88909, 20.8357, 55.5986
   (c) 2.96831, 7.88926, 20.8364, 55.6015
8. (a) 0.9257133, 1.285148, 2.408595, 4.103495
   (b) 0.9257124, 1.285148, 2.408595, 4.103495
   (c) 0.9257248, 1.285158, 2.408594, 4.103493
9. (a) 3.962186, 5.108903, 6.431390, 7.923385
   (b) 3.962186, 5.108903, 6.431390, 7.923385
   (c) 3.962186, 5.108903, 6.431390, 7.923385
10. (a) 1.612622, 2.480909, 3.745479, 5.495872
    (b) 1.612622, 2.480909, 3.745479, 5.495873
    (c) 1.612623, 2.480905, 3.745473, 5.495869
11. (a) −1.447639, −0.1436281, 1.060946, 1.410122
    (b) −1.447638, −0.1436762, 1.060913, 1.410103
    (c) −1.447621, −0.1447219, 1.060717, 1.410027

16

12. (a) 0.5909091, 0.8000000, 1.166667, 1.750000
    (b) 0.5909091, 0.8000000, 1.166667, 1.750000
    (c) 0.5909092, 0.8000002, 1.166667, 1.750001

Section 8.5, page 454

2ab

1. (b) $\phi_2(t) - \phi_1(t) = 0.001e^t \to \infty$ as $t \to \infty$

2c, 4ab

2. (b) $\phi_1(t) = \ln[e^t/(2 - e^t)]$; $\phi_2(t) = \ln[1/(1 - t)]$
3. (a,b) $h = 0.00025$ is sufficient. (c) $h = 0.005$ is sufficient.

4cd, 5ac

4. (a) $y = 4e^{-10t} + (t^2/4)$ (c) Runge-Kutta is stable for $h = 0.25$ but unstable for $h = 0.3$. (d) $h = 5/13 \cong 0.384615$ is small enough.
5. (a) $y = t$
6. (a) $y = t^2$

Section 8.6, page 457

1. (a) 1.26, 0.76; 1.7714, 1.4824; 2.58991, 2.3703; 3.82374, 3.60413; 5.64246, 5.38885
   (b) 1.32493, 0.758933; 1.93679, 1.57919; 2.93414, 2.66099; 4.48318, 4.22639; 6.84236, 6.56452
   (c) 1.32489, 0.759516; 1.9369, 1.57999; 2.93459, 2.66201; 4.48422, 4.22784; 6.8444, 6.56684

2ab

2. (a) 1.451, 1.232; 2.16133, 1.65988; 3.29292, 2.55559; 5.16361, 4.7916; 8.54951, 12.0464
   (b) 1.51844, 1.28089; 2.37684, 1.87711; 3.85039, 3.44859; 6.6956, 9.50309; 15.0987, 64.074
   (c) 1.51855, 1.2809; 2.3773, 1.87729; 3.85247, 3.45126; 6.71282, 9.56846; 15.6384, 70.3792

3. (a) 0.582, 1.18; 0.117969, 1.27344; −0.336912, 1.27382; −0.730007, 1.18572; −1.02134, 1.02371
   (b) 0.568451, 1.15775; 0.109776, 1.22556; −0.32208, 1.20347; −0.681296, 1.10162; −0.937852, 0.937852
   (c) 0.56845, 1.15775; 0.109773, 1.22557; −0.322081, 1.20347; −0.681291, 1.10161; −0.937841, 0.93784
4. (a) −0.198, 0.618; −0.378796, 0.28329; −0.51932, −0.0321025; −0.594324, −0.326801; −0.588278, −0.57545
   (b) −0.196904, 0.630936; −0.372643, 0.298888; −0.501302, −0.0111429; −0.561270, −0.288943; −0.547053, −0.508303
   (c) −0.196935, 0.630939; −0.372687, 0.298866; −0.501345, −0.0112184; −0.561292, −0.28907; −0.547031, −0.508427
5. (a) 2.96225, 1.34538; 2.34119, 1.67121; 1.90236, 1.97158; 1.56602, 2.23895; 1.29768, 2.46732
   (b) 3.06339, 1.34858; 2.44497, 1.68638; 1.9911, 2.00036; 1.63818, 2.27981; 1.3555, 2.5175
   (c) 3.06314, 1.34899; 2.44465, 1.68699; 1.99075, 2.00107; 1.63781, 2.28057; 1.35514, 2.51827
6. (a) 1.42386, 2.18957; 1.82234, 2.36791; 2.21728, 2.53329; 2.61118, 2.68763; 2.9955, 2.83354
   (b) 1.41513, 2.18699; 1.81208, 2.36233; 2.20635, 2.5258; 2.59826, 2.6794; 2.97806, 2.82487
   (c) 1.41513, 2.18699; 1.81209, 2.36233; 2.20635, 2.52581; 2.59826, 2.67941; 2.97806, 2.82488

See SSM for detailed solutions to 7, 8

7. For $h = 0.05$ and 0.025: $x = 10.227$, $y = -4.9294$; these results agree with the exact solution to five digits.
8. 1.543, 0.0707503; 1.14743, −1.3885 9. 1.99521, −0.662442

CHAPTER 9 Section 9.1, page 468

1abd, 4ab

1. (a) $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = (1, 2)^T$; $r_2 = 2$, $\boldsymbol{\xi}^{(2)} = (2, 1)^T$ (b) saddle point, unstable
2. (a) $r_1 = 2$, $\boldsymbol{\xi}^{(1)} = (1, 3)^T$; $r_2 = 4$, $\boldsymbol{\xi}^{(2)} = (1, 1)^T$ (b) node, unstable
3. (a) $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = (1, 3)^T$; $r_2 = 1$, $\boldsymbol{\xi}^{(2)} = (1, 1)^T$ (b) saddle point, unstable

4d, 7abd, 10ab

4. (a) $r_1 = r_2 = -3$, $\boldsymbol{\xi}^{(1)} = (1, 1)^T$ (b) improper node, asymptotically stable
5. (a) $r_1, r_2 = -1 \pm i$; $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = (2 \pm i, 1)^T$ (b) spiral point, asymptotically stable
6. (a) $r_1, r_2 = \pm i$; $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = (2 \pm i, 1)^T$ (b) center, stable
7. (a) $r_1, r_2 = 1 \pm 2i$; $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = (1, 1 \mp i)^T$ (b) spiral point, unstable
8. (a) $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = (1, 0)^T$; $r_2 = -1/4$, $\boldsymbol{\xi}^{(2)} = (4, -3)^T$
   (b) node, asymptotically stable
9. (a) $r_1 = r_2 = 1$, $\boldsymbol{\xi}^{(1)} = (2, 1)^T$ (b) improper node, unstable

10d, 13, 17

10. (a) $r_1, r_2 = \pm 3i$; $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = (2, -1 \pm 3i)^T$ (b) center, stable
11. (a) $r_1 = r_2 = -1$; $\boldsymbol{\xi}^{(1)} = (1, 0)^T$, $\boldsymbol{\xi}^{(2)} = (0, 1)^T$ (b) proper node, asymptotically stable
12. (a) $r_1, r_2 = (1 \pm 3i)/2$; $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = (5, 3 \mp 3i)^T$ (b) spiral point, unstable
13. $x_0 = 1$, $y_0 = 1$; $r_1 = \sqrt{2}, r_2 = -\sqrt{2}$; saddle point, unstable
14. $x_0 = -1$, $y_0 = 0$; $r_1 = -1, r_2 = -3$; node, asymptotically stable

18ab, 19abc

15. $x_0 = -2$, $y_0 = 1$; $r_1, r_2 = -1 \pm \sqrt{2}\, i$; spiral point, asymptotically stable
16. $x_0 = \gamma/\delta$, $y_0 = \alpha/\beta$; $r_1, r_2 = \pm\sqrt{\beta\delta}\, i$; center, stable

20

17. $c^2 > 4km$, node, asymptotically stable; $c^2 = 4km$, improper node, asymptotically stable; $c^2 < 4km$, spiral point, asymptotically stable

Section 9.2, page 477

See SSM for detailed solutions to 1, 3

1. $x = 4e^{-t}$, $y = 2e^{-2t}$, $y = x^2/8$
2. $x = 4e^{-t}$, $y = 2e^{2t}$, $y = 32x^{-2}$; $x = 4e^{-t}$, $y = 0$
3. $x = 4\cos t$, $y = 4\sin t$, $x^2 + y^2 = 16$; $x = -4\sin t$, $y = 4\cos t$, $x^2 + y^2 = 16$
4. $x = \sqrt{a}\cos\sqrt{ab}\,t$, $y = -\sqrt{b}\sin\sqrt{ab}\,t$; $(x^2/a) + (y^2/b) = 1$
5. (a, c) $(-\frac{1}{2}, 1)$, saddle point, unstable; $(0, 0)$, (proper) node, unstable
6. (a, c) $(-\sqrt{3}/3, -\frac{1}{2})$, saddle point, unstable; $(\sqrt{3}/3, -\frac{1}{2})$, center, stable

7abc, 12a

7. (a, c) $(0, 0)$, node, unstable; $(0, 2)$, node, asymptotically stable; $(\frac{1}{2}, \frac{1}{2})$, saddle point, unstable; $(1, 0)$, node, asymptotically stable
8. (a, c) $(0, 0)$, saddle point, unstable; $(0, 1)$, spiral point, asymptotically stable; $(-2, -2)$, node, asymptotically stable; $(3, -2)$, node, unstable
9. (a, c) $(0, 0)$, spiral point, asymptotically stable; $(1 - \sqrt{2}, 1 + \sqrt{2})$, saddle point, unstable; $(1 + \sqrt{2}, 1 - \sqrt{2})$, saddle point, unstable
10. (a, c) $(0, 0)$, saddle point, unstable; $(2, 2)$, spiral point, asymptotically stable; $(-1, -1)$, spiral point, asymptotically stable; $(-2, 0)$, saddle point, unstable
11. (a, c) $(0, 0)$, saddle point, unstable; $(0, 1)$, saddle point, unstable; $(\frac{1}{2}, \frac{1}{2})$, center, stable; $(-\frac{1}{2}, \frac{1}{2})$, center, stable

12bc, 15ab, 19a, 21a

12. (a, c) $(0, 0)$, saddle point, unstable; $(\sqrt{6}, 0)$, spiral point, asymptotically stable; $(-\sqrt{6}, 0)$, spiral point, asymptotically stable
13. (a, c) $(0, 0)$, saddle point, unstable; $(-2, 2)$, node, unstable; $(4, 4)$, spiral point, asymptotically stable
14. (a, c) $(0, 0)$, spiral point, unstable
15. (a) $4x^2 - y^2 = c$
16. (a) $4x^2 + y^2 = c$
17. (a) $(y - 2x)^2(x + y) = c$
18. (a) $\arctan(y/x) - \ln\sqrt{x^2 + y^2} = c$
19. (a) $2x^2y - 2xy + y^2 = c$
20. (a) $x^2y^2 - 3x^2y - 2y^2 = c$
21. (a) $(y^2/2) - \cos x = c$

23, 24, 25, 26

22. (a) $x^2 + y^2 - (x^4/12) = c$

Section 9.3, page 487

1. linear and nonlinear: saddle point, unstable
2. linear and nonlinear: spiral point, asymptotically stable

3, 4

3. linear: center, stable; nonlinear: spiral point or center, indeterminate
4. linear: improper node, unstable; nonlinear: node or spiral point, unstable
5. (a, b, c) $(0, 0)$; $u' = -2u + 2v$, $v' = 4u + 4v$; $r = 1 \pm \sqrt{17}$; saddle point, unstable
   $(-2, 2)$; $u' = 4u$, $v' = 6u + 6v$; $r = 4, 6$; node, unstable
   $(4, 4)$; $u' = -6u + 6v$, $v' = -8u$; $r = -3 \pm \sqrt{39}\,i$; spiral point, asymptotically stable

6abc

6. (a, b, c) $(0, 0)$; $u' = u$, $v' = 3v$; $r = 1, 3$; node, unstable
   $(1, 0)$; $u' = -u - v$, $v' = 2v$; $r = -1, 2$; saddle point, unstable
   $(0, \frac{3}{2})$; $u' = -\frac{1}{2}u$, $v' = -\frac{3}{2}u - 3v$; $r = -\frac{1}{2}, -3$; node, asymptotically stable
   $(-1, 2)$; $u' = u + v$, $v' = -2u - 4v$; $r = (-3 \pm \sqrt{17})/2$; saddle point, unstable
7. (a, b, c) $(1, 1)$; $u' = -v$, $v' = 2u - 2v$; $r = -1 \pm i$; spiral point, asymptotically stable
   $(-1, 1)$; $u' = -v$, $v' = -2u - 2v$; $r = -1 \pm \sqrt{3}$; saddle point, unstable
8. (a, b, c) $(0, 0)$; $u' = u$, $v' = \frac{1}{2}v$; $r = 1, \frac{1}{2}$; node, unstable
   $(0, 2)$; $u' = -u$, $v' = -\frac{3}{2}u - \frac{1}{2}v$; $r = -1, -\frac{1}{2}$; node, asymptotically stable
   $(1, 0)$; $u' = -u - v$, $v' = -\frac{1}{4}v$; $r = -1, -\frac{1}{4}$; node, asymptotically stable
   $(\frac{1}{2}, \frac{1}{2})$; $u' = -\frac{1}{2}u - \frac{1}{2}v$, $v' = -\frac{3}{8}u - \frac{1}{8}v$; $r = (-5 \pm \sqrt{57})/16$; saddle point, unstable

9. (a, b, c) $(0, 0)$; $u' = -u + v$, $v' = 2u$; $r = -2, 1$; saddle point, unstable
$(0, 1)$; $u' = -u - v$, $v' = 3u$; $r = (-1 \pm \sqrt{11}\,i)/2$; spiral point, asymptotically stable
$(-2, -2)$; $u' = -5u + 5v$, $v' = -2v$; $r = -5, -2$; node, asymptotically stable
$(3, -2)$; $u' = 5u + 5v$, $v' = 3v$; $r = 5, 3$; node, unstable

See SSM for detailed solutions to 10ab

10. (a, b, c) $(0, 0)$; $u' = u$, $v' = v$; $r = 1, 1$; node or spiral point, unstable
$(-1, 0)$; $u' = -u$, $v' = 2v$; $r = -1, 2$; saddle point, unstable

10c

11. (a, b, c) $(0, 0)$; $u' = 2u + v$, $v' = u - 2v$; $r = \pm\sqrt{5}$; saddle point, unstable
$(-1.1935, -1.4797)$; $u' = -1.2399u - 6.8393v$, $v' = 2.4797u - 0.80655v$;
$r = -1.0232 \pm 4.1125i$; spiral point, asymptotically stable

12. (a, b, c) $(0, \pm 2n\pi)$, $n = 0, 1, 2, \ldots$; $u' = v$, $v' = -u$; $r = \pm i$; center or spiral point, indeterminate
$[2, \pm(2n-1)\pi]$, $n = 1, 2, 3, \ldots$; $u' = -3v$, $v' = -u$; $r = \pm\sqrt{3}$;
saddle point, unstable

13. (a, b, c) $(0, 0)$; $u' = u$, $v' = v$; $r = 1, 1$; node or spiral point, unstable
$(1, 1)$; $u' = u - 2v$, $v' = -2u + v$; $r = 3, -1$; saddle point, unstable

14. (a, b, c) $(1, 1)$; $u' = -u - v$, $v' = u - 3v$; $r = -2, -2$;
node or spiral point, asymptotically stable
$(-1, -1)$; $u' = u + v$, $v' = u - 3v$; $r = -1 \pm \sqrt{5}$; saddle point, unstable

15. (a, b, c) $(0, 0)$; $u' = -2u - v$, $v' = u - v$; $r = (-3 \pm \sqrt{3}\,i)/2$;
spiral point, asymptotically stable
$(-0.33076, 1.0924)$ and $(0.33076, -1.0924)$; $u' = -3.5216u - 0.27735v$,
$v' = 0.27735u + 2.6895v$; $r = -3.5092, 2.6771$; saddle point, unstable

18abc

16. (a, b, c) $(0, 0)$; $u' = u + v$, $v' = -u + v$; $r = 1 \pm i$; spiral point, unstable

19. (b, c) Refer to Table 9.3.1.

21. (a) $R = A$, $T \cong 3.17$ (b) $R = A$, $T \cong 3.20,\ 3.35,\ 3.63,\ 4.17$
(c) $T \to \pi$ as $A \to 0$ (d) $A = \pi$

22ab, 23a, 27a

22. (b) $v_c \cong 4.00$ 23. (b) $v_c \cong 4.51$

27b, 28ab

28. (a) $dx/dt = y$, $dy/dt = -g(x) - c(x)y$
(b) The linear system is $dx/dt = y$, $dy/dt = -g'(0)x - c(0)y$.
(c) The eigenvalues satisfy $r^2 + c(0)r + g'(0) = 0$.

Section 9.4, page 501

1. (b, c) $(0, 0)$; $u' = \frac{3}{2}u$, $v' = 2v$; $r = \frac{3}{2}, 2$; node, unstable
$(0, 2)$; $u' = \frac{1}{2}u$, $v' = -\frac{3}{2}u - 2v$; $r = \frac{1}{2}, -2$; saddle point, unstable
$(\frac{3}{2}, 0)$; $u' = -\frac{3}{2}u - \frac{3}{4}v$, $v' = \frac{7}{8}v$; $r = -\frac{3}{2}, \frac{7}{8}$; saddle point, unstable
$(\frac{4}{5}, \frac{7}{5})$; $u' = -\frac{4}{5}u - \frac{2}{5}v$, $v' = -\frac{21}{20}u - \frac{7}{5}v$; $r = (-22 \pm \sqrt{204})/20$;
node, asymptotically stable

2. (b, c) $(0, 0)$; $u' = \frac{3}{2}u$, $v' = 2v$; $r = \frac{3}{2}, 2$; node, unstable
$(0, 4)$; $u' = -\frac{1}{2}u$, $v' = -6u - 2v$; $r = -\frac{1}{2}, -2$; node, asymptotically stable
$(\frac{3}{2}, 0)$; $u' = -\frac{3}{2}u - \frac{3}{4}v$, $v' = -\frac{1}{4}v$; $r = -\frac{1}{4}, -\frac{3}{2}$; node, asymptotically stable
$(1, 1)$; $u' = -u - \frac{1}{2}v$, $v' = -\frac{3}{2}u - \frac{1}{2}v$; $r = (-3 \pm \sqrt{13})/4$;
saddle point, unstable

3. (b, c) $(0, 0)$; $u' = \frac{3}{2}u$, $v' = 2v$; $r = \frac{3}{2}, 2$; node, unstable

3bc

$(0, 2)$; $u' = -\frac{1}{2}u$, $v' = -\frac{9}{4}u - 2v$; $r = -\frac{1}{2}, -2$; node, asymptotically stable
$(3, 0)$; $u' = -\frac{3}{2}u - 3v$, $v' = -\frac{11}{8}v$; $r = -\frac{3}{2}, -\frac{11}{8}$; node, asymptotically stable

3e

$(\frac{4}{5}, \frac{11}{10})$; $u' = -\frac{2}{5}u - \frac{4}{5}v$, $v' = -\frac{99}{80}u - \frac{11}{10}v$; $r = -1.80475,\ 0.30475$;
saddle point, unstable

4. (b, c) $(0, 0)$; $u' = \frac{3}{2}u$, $v' = \frac{3}{4}v$; $r = \frac{3}{2}, \frac{3}{4}$; node, unstable
$(0, \frac{3}{4})$; $u' = \frac{3}{4}u$, $v' = -\frac{3}{4}v$; $r = \pm\frac{3}{4}$; saddle point, unstable
$(3, 0)$; $u' = -\frac{3}{2}u - 3v$, $v' = \frac{3}{8}v$; $r = -\frac{3}{2}, \frac{3}{8}$; saddle point, unstable
$(2, \frac{1}{2})$; $u' = -u - 2v$, $v' = -\frac{1}{16}u - \frac{1}{2}v$; $r = -1.18301$, -0.31699; node, asymptotically stable

See SSM for detailed solutions to 5bc, 6bc

5. (b, c) $(0, 0)$; $u' = u$, $v' = \frac{3}{2}v$; $r = 1, \frac{3}{2}$; node, unstable
$(0, \frac{3}{2})$; $u' = -\frac{1}{2}u$, $v' = -\frac{3}{2}u - \frac{3}{2}v$; $r = -\frac{1}{2}, -\frac{3}{2}$; node, asymptotically stable
$(1, 0)$; $u' = -u - v$, $v' = \frac{1}{2}v$; $r = -1, \frac{1}{2}$; saddle point, unstable

6d

6. (b, c) $(0, 0)$; $u' = u$, $v' = \frac{5}{2}v$; $r = 1, \frac{5}{2}$; node, unstable
$(0, \frac{5}{3})$; $u' = \frac{11}{6}u$, $v' = \frac{5}{12}u - \frac{5}{2}v$; $r = \frac{11}{6}$, $-\frac{5}{2}$; saddle point, unstable
$(1, 0)$; $u' = -u + \frac{1}{2}v$, $v' = \frac{11}{4}v$; $r = -1, \frac{11}{4}$; saddle point, unstable

6ef, 8a

$(2, 2)$; $u' = -2u + v$, $v' = \frac{1}{2}u - 3v$; $r = (-5 \pm \sqrt{3})/2$; node, asymptotically stable

8. (a) Critical points are $x = 0$, $y = 0$; $x = \epsilon_1/\sigma_1$, $y = 0$; $x = 0$, $y = \epsilon_2/\sigma_2$.
$x \to 0$, $y \to \epsilon_2/\sigma_2$ as $t \to \infty$; the redear survive.

8b, 9ab, 12a

(b) Same as part (a) except $x \to \epsilon_1/\sigma_1$, $y \to 0$ as $t \to \infty$; the bluegill survive.

9. (a) $X = (B - \gamma_1 R)/(1 - \gamma_1\gamma_2)$, $Y = (R - \gamma_2 B)/(1 - \gamma_1\gamma_2)$
(b) X is reduced, Y is increased; yes, if B becomes less than $\gamma_1 R$, then $x \to 0$ and $y \to R$ as $t \to \infty$.

10. (a) $\sigma_1\epsilon_2 - \alpha_2\epsilon_1 \neq 0$: $(0, 0)$, $(0, \epsilon_2/\sigma_2)$, $(\epsilon_1/\sigma_1, 0)$
$\sigma_1\epsilon_2 - \alpha_2\epsilon_1 = 0$: $(0, 0)$, and all points on the line $\sigma_1 x + \alpha_1 y = \epsilon_1$
(b) $\sigma_1\epsilon_2 - \alpha_2\epsilon_1 > 0$: $(0, 0)$ is unstable node; $(\epsilon_1/\sigma_1, 0)$ is saddle point; $(0, \epsilon_2/\sigma_2)$ is asymptotically stable node.
$\sigma_1\epsilon_2 - \alpha_2\epsilon_1 < 0$: $(0, 0)$ is unstable node; $(0, \epsilon_2/\sigma_2)$ is saddle point; $(\epsilon_1/\sigma_1, 0)$ is asymptotically stable node.
(c) $(0, 0)$ is unstable node; points on the line $\sigma_1 x + \alpha_1 y = \epsilon_1$ are stable, nonisolated critical points.

12bcd

12. (a) $(0, 0)$, $(0, 2 + 2\alpha)$, $(4, 0)$, $(2, 2)$
(b) $\alpha = 0.75$, asymptotically stable node; $\alpha = 1.25$, (unstable) saddle point
(c) $u' = -2u - 2v$, $v' = -2\alpha u - 2v$
(d) $r = -2 \pm 2\sqrt{\alpha}$; $\alpha_0 = 1$

13. (a) $(0, 0)$, saddle point; $(0.15, 0)$, spiral point if $\gamma^2 < 1.11$, node if $\gamma^2 \geq 1.11$; $(2, 0)$, saddle point
(c) $\gamma \cong 1.20$

Section 9.5, page 509

1. (b, c) $(0, 0)$; $u' = \frac{3}{2}u$, $v' = -\frac{1}{2}v$; $r = \frac{3}{2}, -\frac{1}{2}$; saddle point, unstable
$(\frac{1}{2}, 3)$; $u' = -\frac{1}{4}v$, $v' = 3u$; $r = \pm\sqrt{3}\,i/2$; center or spiral point, indeterminate

2. (b, c) $(0, 0)$; $u' = u$, $v' = -\frac{1}{4}v$; $r = 1, -\frac{1}{4}$; saddle point, unstable
$(\frac{1}{2}, 2)$; $u' = -\frac{1}{4}v$, $v' = u$; $r = \pm\frac{1}{2}i$; center or spiral point, indeterminate

3bc

3. (b, c) $(0, 0)$; $u' = u$, $v' = -\frac{1}{4}v$; $r = 1, -\frac{1}{4}$; saddle point, unstable
$(2, 0)$; $u' = -u - v$, $v' = \frac{3}{4}v$; $r = -1, \frac{3}{4}$; saddle point, unstable

3ef

$(\frac{1}{2}, \frac{3}{2})$; $u' = -\frac{1}{4}u - \frac{1}{4}v$, $v' = \frac{3}{4}u$; $r = (-1 \pm \sqrt{11}\,i)/8$; spiral point, asymptotically stable

4. (b, c) $(0, 0)$; $u' = \frac{9}{8}u$, $v' = -v$; $r = \frac{9}{8}, -1$; saddle point, unstable
$(\frac{9}{8}, 0)$; $u' = -\frac{9}{8}u - \frac{9}{16}v$, $v' = \frac{1}{8}v$; $r = -\frac{9}{8}, \frac{1}{8}$; saddle point, unstable
$(1, \frac{1}{4})$; $u' = -u - \frac{1}{2}v$, $v' = \frac{1}{4}u$; $r = (-1 \pm \sqrt{0.5})/2$; node, asymptotically stable

5. (b, c) $(0, 0)$; $u' = -u, v' = -\frac{3}{2}v$; $r = -1, -\frac{3}{2}$; node, asymptotically stable
   $(\frac{1}{2}, 0)$; $u' = \frac{3}{4}u - \frac{3}{20}v, v' = -v$; $r = -1, \frac{3}{4}$; saddle point, unstable
   $(2, 0)$; $u' = -3u - \frac{3}{5}v, v' = \frac{1}{2}v$; $r = -3, \frac{1}{2}$; saddle point, unstable
   $(\frac{3}{2}, \frac{5}{3})$; $u' = -\frac{3}{4}u - \frac{9}{20}v$, $v' = \frac{5}{3}u$; $r = (-3 \pm \sqrt{39}\,i)/8$; spiral point, asymptotically stable
6. $t = 0, T, 2T, \ldots$: H is a max., dP/dt is a max.
   $t = T/4, 5T/4, \ldots$: dH/dt is a min., P is a max.
   $t = T/2, 3T/2, \ldots$: H is a min., dP/dt is a min.
   $t = 3T/4, 7T/4, \ldots$: dH/dt is a max., P is a min.

See SSM for detailed solutions to 7abc, 11, 13

7. (a) $\sqrt{c}\,\alpha/\sqrt{a}\,\gamma$ (b) $\sqrt{3}$
   (d) The ratio of prey amplitude to predator amplitude increases very slowly as the initial point moves away from the equilibrium point.
8. (a) $4\pi/\sqrt{3} \cong 7.2552$
   (c) The period increases slowly as the initial point moves away from the equilibrium point.
9. (a) $T \cong 6.5$ (b) $T \cong 3.7$, $T \cong 11.5$ (c) $T \cong 3.8$, $T \cong 11.1$
11. Trap foxes in half-cycle when $dP/dt > 0$, trap rabbits in half-cycle when $dH/dt > 0$, trap rabbits and foxes in quarter-cycle when $dP/dt > 0$ and $dH/dt > 0$, trap neither in quarter-cycle when $dP/dt < 0$ and $dH/dt < 0$.
12. $dH/dt = aH - \alpha HP - \beta H, dP/dt = -cP + \gamma HP - \delta P$, where β and δ are constants of proportionality. New center is $H = (c + \delta)/\gamma > c/\gamma$ and $P = (a - \beta)/\alpha < a/\alpha$, so equilibrium value of prey is increased and equilibrium value of predator is decreased!
13. Let $A = a/\sigma - c/\gamma > 0$. The critical points are $(0, 0)$, $(a/\sigma, 0)$, and $(c/\gamma, \sigma A/\alpha)$, where $(0, 0)$ is a saddle point, $(a/\sigma, 0)$ is a saddle point, and $(c/\gamma, \sigma A/\alpha)$ is an asymptotically stable node if $(c\sigma/\gamma)^2 - 4c\sigma A \geq 0$ or an asymptotically stable spiral point if $(c\sigma/\gamma)^2 - 4c\sigma A < 0$. $(H, P) \to (c/\gamma, \sigma A/\alpha)$ as $t \to \infty$.

Section 9.7, page 530

1, 2, 4, 7, 8a

1. $r = 1,\ \theta = t + t_0$, stable limit cycle
2. $r = 1,\ \theta = -t + t_0$, semistable limit cycle
3. $r = 1,\ \theta = t + t_0$, stable limit cycle; $r = 3,\ \theta = t + t_0$, unstable periodic solution
4. $r = 1,\ \theta = -t + t_0$, unstable periodic solution; $r = 2,\ \theta = -t + t_0$, stable limit cycle
5. $r = 2n - 1,\ \theta = t + t_0,\ n = 1, 2, 3, \ldots$, stable limit cycle;
   $r = 2n,\ \theta = t + t_0,\ n = 1, 2, 3, \ldots$, unstable periodic solution
6. $r = 2,\ \theta = -t + t_0$, semistable limit cycle;
   $r = 3,\ \theta = -t + t_0$, unstable periodic solution
8. (a) Counterclockwise

8b, 9, 11, 13, 16a

   (b) $r = 1,\ \theta = t + t_0$, stable limit cycle; $r = 2,\ \theta = t + t_0$, semistable limit cycle;
   $r = 3,\ \theta = t + t_0$, unstable periodic solution
9. $r = \sqrt{2},\ \theta = -t + t_0$, unstable periodic solution
14. (a) $\mu = 0.2,\ T \cong 6.29$; $\mu = 1,\ T \cong 6.66$; $\mu = 5,\ T \cong 11.60$
15. (a) $x' = y, y' = -x + \mu y - \mu y^3/3$
   (b) $0 < \mu < 2$, unstable spiral point; $\mu \geq 2$, unstable node
   (c) $A \cong 2.16,\ T \cong 6.65$
   (d) $\mu = 0.2,\ A \cong 1.99,\ T \cong 6.31$; $\mu = 0.5,\ A \cong 2.03,\ T \cong 6.39$;
   $\mu = 2,\ A \cong 2.60,\ T \cong 7.65$; $\mu = 5,\ A \cong 4.36,\ T \cong 11.60$
16. (a) $k = 0,\ (1.1994,\ -0.62426)$; $k = 0.5,\ (0.80485,\ -0.13106)$
   (b) $k_0 \cong 0.3465$ (c) $T \cong 12.54$ (d) $k_1 \cong 0.3369$

Section 9.8, page 538

1ab

1. (b) $\lambda = \lambda_1,\ \boldsymbol{\xi}^{(1)} = (0, 0, 1)^T$; $\lambda = \lambda_2,\ \boldsymbol{\xi}^{(2)} = (20, 9 - \sqrt{81 + 40r}, 0)^T$;
   $\lambda = \lambda_3,\ \boldsymbol{\xi}^{(3)} = (20, 9 + \sqrt{81 + 40r}, 0)^T$

See SSM for detailed solutions to 1c, 2abc, 3b 3c, 4, 5ab

(c) $\lambda_1 \cong -2.6667$, $\boldsymbol{\xi}^{(1)} = (0, 0, 1)^T$; $\lambda_2 \cong -22.8277$, $\boldsymbol{\xi}^{(2)} \cong (20, -25.6554, 0)^T$; $\lambda_3 \cong 11.8277$, $\boldsymbol{\xi}^{(3)} \cong (20, 43.6554, 0)^T$

2. (c) $\lambda_1 \cong -13.8546$; $\lambda_2, \lambda_3 \cong 0.0939556 \pm 10.1945i$

5. (a) $dV/dt = -2\sigma[rx^2 + y^2 + b(z - r)^2 - br^2]$

CHAPTER 10

Section 10.1, page 547

2, 3, 7, 11, 15

1. $y = -\sin x$
2. $y = (\cot\sqrt{2}\pi \cos\sqrt{2}x + \sin\sqrt{2}x)/\sqrt{2}$
3. $y = 0$ for all L; $y = c_2 \sin x$ if $\sin L = 0$.
4. $y = -\tan L \cos x + \sin x$ if $\cos L \neq 0$; no solution if $\cos L = 0$.
5. No solution.
6. $y = (-\pi \sin\sqrt{2}x + x \sin\sqrt{2}\pi)/2 \sin\sqrt{2}\pi$
7. No solution.
8. $y = c_2 \sin 2x + \frac{1}{3}\sin x$
9. $y = c_1 \cos 2x + \frac{1}{3}\cos x$
10. $y = \frac{1}{2}\cos x$
11. $\lambda_n = [(2n-1)/2]^2$, $y_n(x) = \sin[(2n-1)x/2]$; $n = 1, 2, 3, \ldots$
12. $\lambda_n = [(2n-1)/2]^2$, $y_n(x) = \cos[(2n-1)x/2]$; $n = 1, 2, 3, \ldots$
13. $\lambda_0 = 0$, $y_0(x) = 1$; $\lambda_n = n^2$, $y_n(x) = \cos nx$; $n = 1, 2, 3, \ldots$
14. $\lambda_n = [(2n-1)\pi/2L]^2$, $y_n(x) = \cos[(2n-1)\pi x/2L]$; $n = 1, 2, 3, \ldots$
15. $\lambda_0 = 0$, $y_0(x) = 1$; $\lambda_n = (n\pi/L)^2$, $y_n(x) = \cos(n\pi x/L)$; $n = 1, 2, 3, \ldots$
16. $\lambda_n = -[(2n-1)\pi/2L]^2$, $y_n(x) = \sin[(2n-1)\pi x/2L]$; $n = 1, 2, 3, \ldots$

Section 10.2, page 555

3, 5, 7, 10

1. $T = 2\pi/5$
2. $T = 1$
3. Not periodic
4. $T = 2L$
5. $T = 1$
6. Not periodic
7. $T = 2$
8. $T = 4$
9. $f(x) = 2L - x$ in $L < x < 2L$; $f(x) = -2L - x$ in $-3L < x < -2L$
10. $f(x) = x - 1$ in $1 < x < 2$; $f(x) = x - 8$ in $8 < x < 9$
11. $f(x) = -L - x$ in $-L < x < 0$

13ab, 15ab

13. (b) $f(x) = \dfrac{2L}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L}$

14. (b) $f(x) = \dfrac{1}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x/L]}{2n-1}$

15a, 21abcd, 25

15. (b) $f(x) = -\dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{2\cos(2n-1)x}{\pi(2n-1)^2} + \frac{(-1)^{n+1}\sin nx}{n}\right]$

16. (b) $f(x) = \dfrac{1}{2} + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}$

17. (b) $f(x) = \dfrac{3L}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{2L\cos[(2n-1)\pi x/L]}{(2n-1)^2\pi^2} + \frac{(-1)^{n+1}L\sin(n\pi x/L)}{n\pi}\right]$

18. (b) $f(x) = \displaystyle\sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}$

19. (b) $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x/2]}{2n-1}$

20. (b) $f(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$

21. (b) $f(x) = \dfrac{2}{3} + \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos\dfrac{n\pi x}{2}$

22. (b) $f(x) = \dfrac{1}{2} + \dfrac{12}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos[(2n-1)\pi x/2]}{(2n-1)^2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin\dfrac{n\pi x}{2}$

23. (b) $f(x) = \dfrac{11}{12} + \dfrac{1}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n - 5}{n^2} \cos\dfrac{n\pi x}{2} + \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{4[1-(-1)^n]}{n^3\pi^3} - \dfrac{(-1)^n}{n\pi}\right] \sin\dfrac{n\pi x}{2}$

24. (b) $f(x) = \dfrac{9}{8} + \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{162[(-1)^n - 1]}{n^4\pi^4} - \dfrac{27(-1)^n}{n^2\pi^2}\right] \cos\dfrac{n\pi x}{3} - \displaystyle\sum_{n=1}^{\infty} \dfrac{108(-1)^n + 54}{n^3\pi^3} \sin\dfrac{n\pi x}{3}$

See SSM for detailed solutions to 27a

25. $m = 81$
26. $m = 27$
28. $\displaystyle\int_0^x f(t)\,dt$ may not be periodic; for example, let $f(t) = 1 + \cos t$.

Section 10.3, page 562

1. (a) $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)\pi x}{2n-1}$

2ab, 4a

2. (a) $f(x) = \dfrac{\pi}{4} - \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{2}{(2n-1)^2\pi} \cos(2n-1)x + \dfrac{(-1)^n}{n} \sin nx\right]$

3. (a) $f(x) = \dfrac{L}{2} + \dfrac{4L}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos[(2n-1)\pi x/L]}{(2n-1)^2}$

4b, 7abc

4. (a) $f(x) = \dfrac{2}{3} + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} \cos n\pi x$

5. (a) $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n-1} \cos(2n-1)x$

6. (a) $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$;

$a_0 = \dfrac{1}{3}, \quad a_n = \dfrac{2(-1)^n}{n^2\pi^2}, \quad b_n = \begin{cases} -1/n\pi, & n \text{ even} \\ 1/n\pi - 4/n^3\pi^3, & n \text{ odd} \end{cases}$

7. (a) $f(x) = -\dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{1-\cos n\pi}{\pi n^2} \cos nx - \dfrac{(-1)^n}{n} \sin nx\right]$

(b) $n = 10$; $\max|e| = 1.6025$ at $x = \pm\pi$
$n = 20$; $\max|e| = 1.5867$ at $x = \pm\pi$
$n = 40$; $\max|e| = 1.5788$ at $x = \pm\pi$

(c) Not possible

8. (a) $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{1-\cos n\pi}{n^2} \cos n\pi x$

(b) $n = 10$; $\max|e| = 0.02020$ at $x = 0, \pm 1$
$n = 20$; $\max|e| = 0.01012$ at $x = 0, \pm 1$
$n = 40$; $\max|e| = 0.005065$ at $x = 0, \pm 1$

(c) $n = 21$

9. (a) $f(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin n\pi x$

(b) $n = 10, 20, 40$; $\max|e| = 1$ at $x \pm 1$

(c) Not possible

10. (a) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{6(1-\cos n\pi)}{n^2\pi^2} \cos\frac{n\pi x}{2} + \frac{2\cos n\pi}{n\pi} \sin\frac{n\pi x}{2} \right]$

(b) $n = 10$; $\text{lub}|e| = 1.0606$ as $x \to 2$
$n = 20$; $\text{lub}|e| = 1.0304$ as $x \to 2$
$n = 40$; $\text{lub}|e| = 1.0152$ as $x \to 2$

(c) Not possible

11. (a) $f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \left[\frac{2\cos n\pi}{n^2\pi^2} \cos n\pi x - \frac{2 - 2\cos n\pi + n^2\pi^2 \cos n\pi}{n^3\pi^3} \sin n\pi x \right]$

(b) $n = 10$; $\text{lub}|e| = 0.5193$ as $x \to 1$
$n = 20$; $\text{lub}|e| = 0.5099$ as $x \to 1$
$n = 40$; $\text{lub}|e| = 0.5050$ as $x \to 1$

(c) Not possible

See SSM for detailed solutions to 12a

12b, 14

12. (a) $f(x) = -\frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x$

(b) $n = 10$; $\max|e| = 0.001345$ at $x = \pm 0.9735$
$n = 20$; $\max|e| = 0.0003534$ at $x = \pm 0.9864$
$n = 40$; $\max|e| = 0.00009058$ at $x = \pm 0.9931$

(c) $n = 4$

13. $y = (\omega \sin nt - n \sin \omega t)/\omega(\omega^2 - n^2), \quad \omega^2 \neq n^2$
$y = (\sin nt - nt \cos nt)/2n^2, \quad \omega^2 = n^2$

14. $y = \sum_{n=1}^{\infty} b_n (\omega \sin nt - n \sin \omega t)/\omega(\omega^2 - n^2), \quad \omega \neq 1, 2, 3, \ldots$

$y = \sum_{\substack{n=1 \\ n\neq m}}^{\infty} b_n (m \sin nt - n \sin mt)/m(m^2 - n^2) + b_m (\sin mt - mt \cos mt)/2m^2, \quad \omega = m$

15, 16, 18a

15. $y = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{\omega^2 - (2n-1)^2} \left[\frac{1}{2n-1} \sin(2n-1)t - \frac{1}{\omega} \sin \omega t \right]$

18b

16. $y = \cos \omega t + \frac{1}{2\omega^2}(1 - \cos \omega t) + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t - \cos \omega t}{(2n-1)^2[\omega^2 - (2n-1)^2]}$

Section 10.4, page 570

3, 6

1. Odd
2. Neither
3. Odd
4. Even
5. Even
6. Neither

7, 10, 13, 14

14. $f(x) = \frac{1}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi/2)}{n^2} \cos\frac{n\pi x}{2}$

$f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(n\pi/2) - \sin(n\pi/2)}{n^2} \sin\frac{n\pi x}{2}$

15, 18

15. $f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\frac{(2n-1)\pi x}{2}$

16. $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(-\cos n\pi + \frac{2}{n\pi} \sin\frac{n\pi}{2} \right) \sin\frac{n\pi x}{2}$

17. $f(x) = 1$

18. $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$

19. $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos\frac{n\pi}{3} + \cos\frac{2n\pi}{3} - 2\cos n\pi \right) \sin\frac{nx}{3}$

See SSM for detailed solutions to 20

20. $f(x) = \dfrac{1}{2} - \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin 2n\pi x}{n}$

21. $f(x) = \dfrac{L}{2} + \dfrac{4L}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos[(2n-1)\pi x/L]}{(2n-1)^2}$

22. $f(x) = \dfrac{2L}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin(n\pi x/L)}{n}$

23. (a) $f(x) = \dfrac{\pi}{4} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{2\pi}{n}\sin\dfrac{n\pi}{2} + \dfrac{4}{n^2}\left(\cos\dfrac{n\pi}{2} - 1\right)\right]\cos\dfrac{nx}{2}$

24. (a) $f(x) = 2\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}\sin nx$

25abc, 28b

25. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty}\left[\dfrac{4n^2\pi^2(1+\cos n\pi)}{n^3\pi^3} + \dfrac{16(1-\cos n\pi)}{n^3\pi^3}\right]\sin\dfrac{n\pi x}{2}$

26. (a) $f(x) = \dfrac{4}{3} + \dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{1+3\cos n\pi}{n^2}\cos\dfrac{n\pi x}{4}$

27. (b) $g(x) = \dfrac{3}{2} + \dfrac{6}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{1-\cos n\pi}{n^2}\cos\dfrac{n\pi x}{3}$

$h(x) = \dfrac{6}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{n\pi x}{3}$

28cd, 31, 32

28. (b) $g(x) = \dfrac{1}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4\cos(n\pi/2) + 2n\pi\sin(n\pi/2) - 4}{n^2\pi^2}\cos\dfrac{n\pi x}{2}$

$h(x) = \displaystyle\sum_{n=1}^{\infty} \dfrac{4\sin(n\pi/2) - 2n\pi\cos(n\pi/2)}{n^2\pi^2}\sin\dfrac{n\pi x}{2}$

29. (b) $g(x) = -\dfrac{5}{12} + \displaystyle\sum_{n=1}^{\infty} \dfrac{12\cos n\pi + 4}{n^2\pi^2}\cos\dfrac{n\pi x}{2}$

$h(x) = -\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty} \dfrac{n^2\pi^2(3+5\cos n\pi) + 32(1-\cos n\pi)}{n^3\pi^3}\sin\dfrac{n\pi x}{2}$

34, 35, 37ab

30. (b) $g(x) = \dfrac{1}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{6n^2\pi^2(2\cos n\pi - 5) + 324(1-\cos n\pi)}{n^4\pi^4}\cos\dfrac{n\pi x}{3}$

38, 39

$h(x) = \displaystyle\sum_{n=1}^{\infty}\left[\dfrac{4\cos n\pi + 2}{n\pi} + \dfrac{144\cos n\pi + 180}{n^3\pi^3}\right]\sin\dfrac{n\pi x}{3}$

40. Extend $f(x)$ antisymmetrically into $(L, 2L]$; that is, so that $f(2L - x) = -f(x)$ for $0 \le x < L$. Then extend this function as an even function into $(-2L, 0)$.

Section 10.5, 579

3, 5

1. $xX'' - \lambda X = 0, \quad T' + \lambda T = 0$
2. $X'' - \lambda x X = 0, \quad T' + \lambda t T = 0$
3. $X'' - \lambda(X' + X) = 0, \quad T' + \lambda T = 0$
4. $[p(x)X']' + \lambda r(x)X = 0, \quad T'' + \lambda T = 0$
5. Not separable
6. $X'' + (x + \lambda)X = 0, \quad Y'' - \lambda Y = 0$
7. $u(x,t) = e^{-400\pi^2 t}\sin 2\pi x - 2e^{-2500\pi^2 t}\sin 5\pi x$

8

8. $u(x,t) = 2e^{-\pi^2 t/16}\sin(\pi x/2) - e^{-\pi^2 t/4}\sin\pi x + 4e^{-\pi^2 t}\sin 2\pi x$
9. $u(x,t) = \dfrac{100}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1-\cos n\pi}{n}e^{-n^2\pi^2 t/1600}\sin\dfrac{n\pi x}{40}$

See SSM for detailed solutions to 10

10. $u(x,t) = \dfrac{160}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}e^{-n^2\pi^2t/1600}\sin\frac{n\pi x}{40}$

11. $u(x,t) = \dfrac{100}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi/4)-\cos(3n\pi/4)}{n}e^{-n^2\pi^2t/1600}\sin\frac{n\pi x}{40}$

12. $u(x,t) = \dfrac{80}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2t/1600}\sin\frac{n\pi x}{40}$

13. $t = 5,\ n = 16;\quad t = 20,\ n = 8;\quad t = 80,\ n = 4$

15abcd, 18a

14. (d) $t = 673.35$ 15. (d) $t = 451.60$ 16. (d) $t = 617.17$

17. (b) $t = 5,\ x = 33.20;\quad t = 10,\ x = 31.13;\quad t = 20,\ x = 28.62;\quad t = 40,\ x = 25.73;$ $t = 100,\ x = 21.95;\quad t = 200,\ x = 20.31$
(e) $t = 524.81$

18. $u(x,t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}e^{-n^2\pi^2\alpha^2t/400}\sin\frac{n\pi x}{20}$

18b, 19b, 20, 22

(a) 35.91°C (b) 67.23°C (c) 99.96°C

19. (a) 76.73 sec (b) 152.56 sec (c) 1093.36 sec

21. (a) $aw_{xx} - bw_t + (c - b\delta)w = 0$ (b) $\delta = c/b$ if $b \neq 0$

22. $X'' + \mu^2 X = 0,\quad Y'' + (\lambda^2 - \mu^2)Y = 0,\quad T' + \alpha^2\lambda^2 T = 0$

23. $r^2R'' + rR' + (\lambda^2r^2 - \mu^2)R = 0,\quad \Theta'' + \mu^2\Theta = 0,\quad T' + \alpha^2\lambda^2 T = 0$

Section 10.6, page 588

3, 7, 9abd

1. $u = 10 + \frac{3}{5}x$ 2. $u = 30 - \frac{5}{4}x$ 3. $u = 0$
4. $u = T$ 5. $u = 0$ 6. $u = T$
7. $u = T(1+x)/(1+L)$ 8. $u = T(1+L-x)/(1+L)$

9. (a) $u(x,t) = 3x + \displaystyle\sum_{n=1}^{\infty}\frac{70\cos n\pi + 50}{n\pi}e^{-0.86n^2\pi^2t/400}\sin\frac{n\pi x}{20}$ (d) 160.29 sec

10. (a) $f(x) = 2x,\ 0 \le x \le 50;\quad f(x) = 200 - 2x,\ 50 < x \le 100$

(b) $u(x,t) = 20 - \dfrac{x}{5} + \displaystyle\sum_{n=1}^{\infty}c_n e^{-1.14n^2\pi^2t/(100)^2}\sin\frac{n\pi x}{100},$

$c_n = \dfrac{800}{n^2\pi^2}\sin\dfrac{n\pi}{2} - \dfrac{40}{n\pi}$ (d) $u(50,t) \to 10$ as $t \to \infty$; 3754 sec

11. (a) $u(x,t) = 30 - x + \displaystyle\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2t/900}\sin\frac{n\pi x}{30},$

$c_n = \dfrac{60}{n^3\pi^3}[2(1-\cos n\pi) - n^2\pi^2(1+\cos n\pi)]$

12abcd

12. (a) $u(x,t) = \dfrac{2}{\pi} + \displaystyle\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2\alpha^2t/L^2}\cos\frac{n\pi x}{L},$

$c_n = \begin{cases} 0, & n \text{ odd;} \\ -4/(n^2-1)\pi, & n \text{ even} \end{cases}$

(b) $\lim\limits_{t\to\infty} u(x,t) = 2/\pi$

13. (a) $u(x,t) = \dfrac{200}{9} + \displaystyle\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2t/6400}\cos\frac{n\pi x}{40},$

$c_n = -\dfrac{160}{3n^2\pi^2}(3+\cos n\pi)$

(c) 200/9 (d) 1543 sec

14abc

14. (a) $u(x,t) = \dfrac{25}{6} + \displaystyle\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2t/900}\cos\frac{n\pi x}{30},$

$c_n = \dfrac{50}{n\pi}\left(\sin\dfrac{n\pi}{3} - \sin\dfrac{n\pi}{6}\right)$

See SSM for detailed solutions to 15a

15b, 19

15. (b) $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(2n-1)^2\pi^2\alpha^2 t/4L^2} \sin\dfrac{(2n-1)\pi x}{2L}$,

$c_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\dfrac{(2n-1)\pi x}{2L}\,dx$

16. (a) $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(2n-1)^2\pi^2 t/3600} \sin\dfrac{(2n-1)\pi x}{60}$,

$c_n = \dfrac{120}{(2n-1)^2\pi^2}[2\cos n\pi + (2n-1)\pi]$

(c) x_m increases from $x = 0$ and reaches $x = 30$ when $t = 104.4$.

17. (a) $u(x,t) = 40 + \sum_{n=1}^{\infty} c_n e^{-(2n-1)^2\pi^2 t/3600} \sin\dfrac{(2n-1)\pi x}{60}$,

$c_n = \dfrac{40}{(2n-1)^2\pi^2}[6\cos n\pi - (2n-1)\pi]$

19. $u(x) = \begin{cases} T\dfrac{x}{a}\left[\dfrac{\xi}{\xi + (L/a) - 1}\right], & 0 \le x \le a, \\ T\left[1 - \dfrac{L-x}{a}\dfrac{1}{\xi + (L/a) - 1}\right], & a \le x \le L, \end{cases}$ where $\xi = \kappa_2 A_2/\kappa_1 A_1$

20. (e) $u_n(x,t) = e^{-\lambda_n^2\alpha^2 t}\sin\lambda_n x$

Section 10.7, page 600

1a

1. (a) $u(x,t) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$

1bce, 6ab

2. (a) $u(x,t) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$

3. (a) $u(x,t) = \dfrac{32}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{2+\cos n\pi}{n^3}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$

4. (a) $u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)\sin(n\pi/L)}{n}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$

5. (a) $u(x,t) = \dfrac{8L}{a\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$

6. (a) $u(x,t) = \dfrac{8L}{a\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/4) + \sin(3n\pi/4)}{n^3}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$

6c, 9

7. (a) $u(x,t) = \dfrac{32L}{a\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{\cos n\pi + 2}{n^4}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$

8. (a) $u(x,t) = \dfrac{4L}{a\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)\sin(n\pi/L)}{n^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$

9. $u(x,t) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi at}{2L}$,

$c_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$

See SSM for detailed solutions to 10a

10bc, 13, 14

10. (a) $u(x,t) = \dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi}{4}\sin\frac{(2n-1)\pi}{2L}\sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi at}{2L}$

11. (a) $u(x,t) = \dfrac{512}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{(2n-1)\pi + 3\cos n\pi}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi at}{2L}$

14. $\phi(x+at)$ represents a wave moving in the negative x direction with speed $a > 0$.

15. (a) 248 ft/sec (b) $49.6\pi n$ rad/sec (c) Frequencies increase; modes are unchanged.

16, 17abc

16. $u(x,t) = \displaystyle\sum_{n=1}^{\infty} c_n \cos\beta_n t \sin(n\pi x/L), \quad \beta_n^2 = (n\pi a/L)^2 + \alpha^2,$

18abc, 24

$c_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(n\pi x/L)\,dx$

22. $r^2R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0, \quad \Theta'' + \mu^2\Theta = 0, \quad T'' + \lambda^2 a^2 T = 0$

Section 10.8, page 611

1abc

1. (a) $u(x,y) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}, \quad c_n = \frac{2/a}{\sinh(n\pi b/a)}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx$

1d

(b) $u(x,y) = \dfrac{4a}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\sin(n\pi/2)}{\sinh(n\pi b/a)}\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$

2, 3a, 4

2. $u(x,y) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}, \quad c_n = \frac{2/a}{\sinh(n\pi b/a)}\int_0^a h(x)\sin\frac{n\pi x}{a}\,dx$

3. (a) $u(x,y) = \displaystyle\sum_{n=1}^{\infty} c_n^{(1)}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} + \sum_{n=1}^{\infty} c_n^{(2)}\sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a},$

$c_n^{(1)} = \dfrac{2/b}{\sinh(n\pi a/b)}\displaystyle\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy, \quad c_n^{(2)} = \frac{2/a}{\sinh(n\pi b/a)}\int_0^a h(x)\sin\frac{n\pi x}{a}\,dx$

(b) $c_n^{(1)} = \dfrac{2}{n\pi\sinh(n\pi a/b)}, \quad c_n^{(2)} = -\dfrac{2}{n^3\pi^3}\dfrac{(n^2\pi^2 - 2)\cos n\pi + 2}{\sinh(n\pi b/a)}$

5, 7

5. $u(r,\theta) = \dfrac{c_0}{2} + \displaystyle\sum_{n=1}^{\infty} r^{-n}(c_n\cos n\theta + k_n\sin n\theta);$

$c_n = \dfrac{a^n}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta, \quad k_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$

6. (a) $u(r,\theta) = \displaystyle\sum_{n=1}^{\infty} c_n r^n \sin n\theta, \quad c_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta$

(b) $c_n = \dfrac{4}{\pi a^n}\dfrac{\cos n\pi + 1}{n^3}$

7. $u(r,\theta) = \displaystyle\sum_{n=1}^{\infty} c_n r^{n\pi/\alpha}\sin\frac{n\pi\theta}{\alpha}, \quad c_n = (2/\alpha)a^{-n\pi/\alpha}\int_0^{\alpha} f(\theta)\sin\frac{n\pi\theta}{\alpha}\,d\theta$

8abc

8. (a) $u(x,y) = \displaystyle\sum_{n=1}^{\infty} c_n e^{-n\pi y/a}\sin\frac{n\pi x}{a}, \quad c_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

(b) $c_n = \dfrac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$ (c) $y_0 \cong 6.6315$

10. (b) $u(x,y) = c_0 + \displaystyle\sum_{n=1}^{\infty} c_n \cosh\frac{n\pi x}{b}\cos\frac{n\pi y}{b}, \quad c_n = \frac{2/n\pi}{\sinh(n\pi a/b)}\int_0^b f(y)\cos\frac{n\pi y}{b}\,dy$

11. $u(r,\theta) = c_0 + \displaystyle\sum_{n=1}^{\infty} r^n(c_n\cos n\theta + k_n\sin n\theta),$

$c_n = \dfrac{1}{n\pi a^{n-1}}\displaystyle\int_0^{2\pi} g(\theta)\cos n\theta\,d\theta, \quad k_n = \frac{1}{n\pi a^{n-1}}\int_0^{2\pi} g(\theta)\sin n\theta\,d\theta;$

necessary condition is $\displaystyle\int_0^{2\pi} g(\theta)\,d\theta = 0.$

12. (a) $u(x,y)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}, \quad c_n=\frac{2/a}{\cosh(n\pi b/a)}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx$

    (b) $c_n=\dfrac{4a\sin(n\pi/2)}{n^2\pi^2\cosh(n\pi b/a)}$

See SSM for detailed solutions to 13a

13. (a) $u(x,y)=\sum_{n=1}^{\infty} c_n \sinh\frac{(2n-1)\pi x}{2b}\sin\frac{(2n-1)\pi y}{2b}$,

    $c_n=\dfrac{2/b}{\sinh[(2n-1)\pi a/2b]}\displaystyle\int_0^b f(y)\sin\frac{(2n-1)\pi y}{2b}\,dy$

    (b) $c_n=\dfrac{32b^2}{(2n-1)^3\pi^3\sinh[(2n-1)\pi a/2b]}$

14. (a) $u(x,y)=\dfrac{c_0 y}{2}+\sum_{n=1}^{\infty} c_n\cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$,

    $c_0=\dfrac{2}{ab}\displaystyle\int_0^a g(x)\,dx, \quad c_n=\frac{2/a}{\sinh(n\pi b/a)}\int_0^a g(x)\cos\frac{n\pi x}{a}\,dx$

    (b) $c_0=\dfrac{2}{b}\left(1+\dfrac{a^4}{30}\right), \quad c_n=-\dfrac{24a^4(1+\cos n\pi)}{n^4\pi^4\sinh(n\pi b/a)}$

CHAPTER 11 Section 11.1, page 626

2, 4, 5, 9

1. Homogeneous
2. Nonhomogeneous
3. Nonhomogeneous
4. Homogeneous
5. Nonhomogeneous
6. Homogeneous

7. $\phi_n(x)=\sin\sqrt{\lambda_n}\,x$, where $\sqrt{\lambda_n}$ satisfies $\sqrt{\lambda}=-\tan\sqrt{\lambda}\,\pi$;
   $\lambda_1\cong 0.6204, \quad \lambda_2\cong 2.7943, \quad \lambda_n\cong(2n-1)^2/4$ for large n
8. $\phi_n(x)=\cos\sqrt{\lambda_n}\,x$, where $\sqrt{\lambda_n}$ satisfies $\sqrt{\lambda}=\cot\sqrt{\lambda}$;
   $\lambda_1\cong 0.7402, \quad \lambda_2\cong 11.7349, \quad \lambda_n\cong(n-1)^2\pi^2$ for large n
9. $\phi_n(x)=\sin\sqrt{\lambda_n}\,x+\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x$, where $\sqrt{\lambda_n}$ satisfies
   $(\lambda-1)\sin\sqrt{\lambda}-2\sqrt{\lambda}\cos\sqrt{\lambda}=0$;
   $\lambda_1\cong 1.7071, \quad \lambda_2\cong 13.4924, \quad \lambda_n\cong(n-1)^2\pi^2$ for large n

10, 11ab

10. $\lambda_0=0; \quad \phi_0(x)=1-x$
    For $n=1,2,3,\ldots$, $\phi_n(x)=\sin\mu_n x-\mu_n\cos\mu_n x$ and $\lambda_n=-\mu_n^2$, where μ_n satisfies $\mu=\tan\mu$.
    $\lambda_1\cong -20.1907, \quad \lambda_2\cong -59.6795, \quad \lambda_n\cong -(2n+1)^2\pi^2/4$ for large n

13, 18a, 20

12. $\mu(x)=e^{-x^2}$
13. $\mu(x)=1/x$
14. $\mu(x)=e^{-x}$
15. $\mu(x)=(1-x^2)^{-1/2}$
16. $a^2X''+(\lambda-k)X=0, \quad T''+cT'+\lambda T=0$
17. (a) $s(x)=e^x$ (b) $\lambda_n=n^2\pi^2, \quad \phi_n(x)=e^x\sin n\pi x; \quad n=1,2,3,\ldots$
18. Positive eigenvalues $\lambda=\lambda_n$, where $\sqrt{\lambda_n}$ satisfies $\sqrt{\lambda}=\frac{2}{3}\tan 3\sqrt{\lambda}L$; corresponding eigenfunctions are $\phi_n(x)=e^{-2x}\sin 3\sqrt{\lambda_n}\,x$. If $L=\frac{1}{2}$, $\lambda_0=0$ is eigenvalue, $\phi_0(x)=xe^{-2x}$ is eigenfunction; if $L\neq\frac{1}{2}$, $\lambda=0$ is not eigenvalue. If $L\le\frac{1}{2}$, there are no negative eigenvalues; if $L>\frac{1}{2}$, there is one negative eigenvalue $\lambda=-\mu^2$, where μ is a root of $\mu=\frac{2}{3}\tanh 3\mu L$; corresponding eigenfunction is $\phi_{-1}(x)=e^{-2x}\sinh 3\mu x$.
19. No real eigenvalues.
20. Only eigenvalue is $\lambda=0$; eigenfunction is $\phi(x)=x-1$.
21. (a) $2\sin\sqrt{\lambda}-\sqrt{\lambda}\cos\sqrt{\lambda}=0$; $\lambda_1\cong 18.2738, \quad \lambda_2\cong 57.7075$

22abcd, 23

(b) $2\sinh\sqrt{\mu}-\sqrt{\mu}\cosh\sqrt{\mu}=0, \quad \mu=-\lambda$; $\lambda_{-1}\cong -3.6673$

See SSM for detailed solutions to 24b

24. (a) $\lambda_n = \mu_n^4$, where μ_n is a root of $\sin \mu L \sinh \mu L = 0$, hence $\lambda_n = (n\pi/L)^4$;
$\lambda_1 \cong 97.409/L^4$, $\phi_n(x) = \sin(n\pi x/L)$
(b) $\lambda_n = \mu_n^4$, where μ_n is a root of $\sin \mu L \cosh \mu L - \cos \mu L \sinh \mu L = 0$;
$\lambda_1 \cong 237.72/L^4$, $\phi_n = \dfrac{\sin \mu_n x \sinh \mu_n L - \sin \mu_n L \sinh \mu_n x}{\sinh \mu_n L}$
(c) $\lambda_n = \mu_n^4$, where μ_n is a root of $1 + \cosh \mu L \cos \mu L = 0$; $\lambda_1 \cong 12.362/L^4$,

$$\phi_n(x) = \frac{[(\sin \mu_n x - \sinh \mu_n x)(\cos \mu_n L + \cosh \mu_n L) + (\sin \mu_n L + \sinh \mu_n L)(\cosh \mu_n x - \cos \mu_n x)]}{\cos \mu_n L + \cosh \mu_n L}$$

25. (c) $\phi_n(x) = \sin \sqrt{\lambda_n}\, x$, where λ_n satisfies $\cos \sqrt{\lambda_n}\, L - \gamma \sqrt{\lambda_n}\, L \sin \sqrt{\lambda_n}\, L = 0$;
$\lambda_1 \cong 1.1597/L^2$, $\lambda_2 \cong 13.276/L^2$

Section 11.2, page 639

1, 3, 5

1. $\phi_n(x) = \sqrt{2} \sin(n - \frac{1}{2})\pi x$; $n = 1, 2, \ldots$
2. $\phi_n(x) = \sqrt{2} \cos(n - \frac{1}{2})\pi x$; $n = 1, 2, \ldots$
3. $\phi_0(x) = 1$, $\phi_n(x) = \sqrt{2} \cos n\pi x$; $n = 1, 2, \ldots$
4. $\phi_n(x) = \dfrac{\sqrt{2} \cos \sqrt{\lambda_n}\, x}{(1 + \sin^2 \sqrt{\lambda_n})^{1/2}}$, where λ_n satisfies $\cos \sqrt{\lambda_n} - \sqrt{\lambda_n} \sin \sqrt{\lambda_n} = 0$
5. $\phi_n(x) = \sqrt{2}\, e^x \sin n\pi x$; $n = 1, 2, \ldots$ 6. $a_n = \dfrac{2\sqrt{2}}{(2n-1)\pi}$; $n = 1, 2, \ldots$

7, 10, 14, 17, 21a

7. $a_n = \dfrac{4\sqrt{2}(-1)^{n-1}}{(2n-1)^2\pi^2}$; $n = 1, 2, \ldots$
8. $a_n = \dfrac{2\sqrt{2}}{(2n-1)\pi} \{1 - \cos[(2n-1)\pi/4]\}$; $n = 1, 2, \ldots$
9. $a_n = \dfrac{2\sqrt{2} \sin(n - \frac{1}{2})(\pi/2)}{(n - \frac{1}{2})^2\pi^2}$; $n = 1, 2, \ldots$

In Problems 10 through 13, $\alpha_n = (1 + \sin^2 \sqrt{\lambda_n})^{1/2}$ and $\cos \sqrt{\lambda_n} - \sqrt{\lambda_n} \sin \sqrt{\lambda_n} = 0$.

10. $a_n = \dfrac{\sqrt{2} \sin \sqrt{\lambda_n}}{\sqrt{\lambda_n}\alpha_n}$; $n = 1, 2, \ldots$ 11. $a_n = \dfrac{\sqrt{2}(2 \cos \sqrt{\lambda_n} - 1)}{\lambda_n \alpha_n}$; $n = 1, 2, \ldots$
12. $a_n = \dfrac{\sqrt{2}(1 - \cos \sqrt{\lambda_n})}{\lambda_n \alpha_n}$; $n = 1, 2, \ldots$ 13. $a_n = \dfrac{\sqrt{2} \sin(\sqrt{\lambda_n}/2)}{\sqrt{\lambda_n}\alpha_n}$; $n = 1, 2, \ldots$
14. Not self-adjoint 15. Self-adjoint
16. Not self-adjoint 17. Self-adjoint
18. Self-adjoint

21bc, 23abc, 24

21. (a) If $a_2 = 0$ or $b_2 = 0$, then the corresponding boundary term is missing.

25bc

25. (a) $\lambda_1 = \pi^2/L^2$; $\phi_1(x) = \sin(\pi x/L)$
(b) $\lambda_1 \cong (4.4934)^2/L^2$; $\phi_1(x) = \sin \sqrt{\lambda_1}\, x - \sqrt{\lambda_1}\, x \cos \sqrt{\lambda_1}\, L$
(c) $\lambda_1 = (2\pi)^2/L^2$; $\phi_1(x) = 1 - \cos(2\pi x/L)$
26. $\lambda_1 = \pi^2/4L^2$; $\phi_1(x) = 1 - \cos(\pi x/2L)$

Section 11.3, page 651

1

1. $y = 2 \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{(n^2\pi^2 - 2)n\pi}$

2

2. $y = 2 \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n - \frac{1}{2})\pi x}{[(n - \frac{1}{2})^2\pi^2 - 2](n - \frac{1}{2})^2\pi^2}$

See SSM for detailed solutions to 3, 5, 8

3. $y = -\dfrac{1}{4} - 4\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{[(2n-1)^2\pi^2 - 2](2n-1)^2\pi^2}$

4. $y = 2\displaystyle\sum_{n=1}^{\infty} \frac{(2\cos\sqrt{\lambda_n} - 1)\cos\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 2)(1 + \sin^2\sqrt{\lambda_n})}$ 5. $y = 8\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\sin n\pi x}{(n^2\pi^2 - 2)n^2\pi^2}$

6–9. For each problem the solution is

$$y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x), \qquad c_n = \int_0^1 f(x)\phi_n(x)\,dx, \qquad \mu \neq \lambda_n,$$

where $\phi_n(x)$ is given in Problems 1, 2, 3, and 4, respectively, in Section 11.2, and λ_n is the corresponding eigenvalue. In Problem 8 summation starts at $n = 0$.

10, 11, 12, 14, 18

10. $a = -\dfrac{1}{2}, \quad y = \dfrac{1}{2\pi^2}\cos\pi x + \dfrac{1}{\pi^2}\left(x - \dfrac{1}{2}\right) + c\sin\pi x$

11. No solution 12. a is arbitrary, $y = c\cos\pi x + a/\pi^2$

13. $a = 0, \quad y = c\sin\pi x - (x/2\pi)\sin\pi x$ 17. $v(x) = a + (b-a)x$

18. $v(x) = 1 - \frac{3}{2}x$

19, 22

19. $u(x,t) = \sqrt{2}\left[-\dfrac{4c_1}{\pi^2} + \left(\dfrac{4c_1}{\pi^2} + \dfrac{1}{\sqrt{2}}\right)e^{-\pi^2 t/4}\right]\sin\dfrac{\pi x}{2}$

$$- \sqrt{2}\sum_{n=2}^{\infty} \frac{4c_n}{(2n-1)^2\pi^2}[1 - e^{-(n-1/2)^2\pi^2 t}]\sin(n - \tfrac{1}{2})\pi x,$$

$$c_n = \frac{4\sqrt{2}(-1)^{n+1}}{(2n-1)^2\pi^2}, \quad n = 1, 2, \ldots$$

20. $u(x,t) = \sqrt{2}\displaystyle\sum_{n=1}^{\infty}\left[\frac{c_n}{\lambda_n - 1}(e^{-t} - e^{-\lambda_n t}) + \alpha_n e^{-\lambda_n t}\right]\frac{\cos\sqrt{\lambda_n}\,x}{(1 + \sin^2\sqrt{\lambda_n})^{1/2}},$

$$c_n = \frac{\sqrt{2}\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}(1 + \sin^2\sqrt{\lambda_n})^{1/2}}, \quad \alpha_n = \frac{\sqrt{2}(1 - \cos\sqrt{\lambda_n})}{\lambda_n(1 + \sin^2\sqrt{\lambda_n})^{1/2}},$$

and λ_n satisfies $\cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\sin\sqrt{\lambda_n} = 0$.

21. $u(x,t) = 8\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^4\pi^4}(1 - e^{-n^2\pi^2 t})\sin n\pi x$

22. $u(x,t) = \sqrt{2}\displaystyle\sum_{n=1}^{\infty}\frac{c_n(e^{-t} - e^{-(n-1/2)^2\pi^2 t})\sin(n - \frac{1}{2})\pi x}{(n - \frac{1}{2})^2\pi^2 - 1},$

$$c_n = \frac{2\sqrt{2}(2n-1)\pi + 4\sqrt{2}(-1)^n}{(2n-1)^2\pi^2}$$

23. (a) $r(x)w_t = [p(x)w_x]_x - q(x)w, \quad w(0,t) = 0, \quad w(1,t) = 0, \quad w(x,0) = f(x) - v(x)$

24, 28a

24. $u(x,t) = x^2 - 2x + 1 + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2 t}\sin(2n-1)\pi x}{2n-1}$

25. $u(x,t) = -\cos\pi x + e^{-9\pi^2 t/4}\cos(3\pi x/2)$

28bcd

30bcd

30e

31–34. In all cases solution is $y = \displaystyle\int_0^1 G(x,s)\,f(s)\,ds$, where $G(x,s)$ is given below.

31. $G(x,s) = \begin{cases} 1 - x, & 0 \le s \le x \\ 1 - s, & x \le s \le 1 \end{cases}$ 32. $G(x,s) = \begin{cases} s(2-x)/2, & 0 \le s \le x \\ x(2-s)/2, & x \le s \le 1 \end{cases}$

See SSM for detailed solutions to 33

33. $G(x,s) = \begin{cases} \cos s \sin(1-x)/\cos 1, & 0 \le s \le x \\ \sin(1-s)\cos x/\cos 1, & x \le s \le 1 \end{cases}$

34. $G(x,s) = \begin{cases} s, & 0 \le s \le x \\ x, & x \le s \le 1 \end{cases}$

Section 11.4, page 661

1. $y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n}\,x), \quad c_n = \int_0^1 f(x) J_0(\sqrt{\lambda_n}\,x)\,dx \Big/ \int_0^1 x J_0^2(\sqrt{\lambda_n}\,x)\,dx,$
$\sqrt{\lambda_n}$ satisfies $J_0(\sqrt{\lambda}) = 0$

2abc 2. (c) $y = -\frac{c_0}{\mu} + \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n}\,x);$
$c_0 = 2\int_0^1 f(x)\,dx; \quad c_n = \int_0^1 f(x) J_0(\sqrt{\lambda_n}\,x)\,dx \Big/ \int_0^1 x J_0^2(\sqrt{\lambda_n}\,x)\,dx, \ n = 1, 2, \ldots;$
$\sqrt{\lambda_n}$ satisfies $J_0'(\sqrt{\lambda}) = 0$

3. (d) $a_n = \int_0^1 x J_k(\sqrt{\lambda_n}\,x) f(x)\,dx \Big/ \int_0^1 x J_k^2(\sqrt{\lambda_n}\,x)\,dx$
(e) $y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_k(\sqrt{\lambda_n}\,x), \quad c_n = \int_0^1 f(x) J_k(\sqrt{\lambda_n}\,x)\,dx \Big/ \int_0^1 x J_k^2(\sqrt{\lambda_n}\,x)\,dx$

4ab 4. (b) $y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} P_{2n-1}(x), \quad c_n = \int_0^1 f(x) P_{2n-1}(x)\,dx \Big/ \int_0^1 P_{2n-1}^2(x)\,dx$

Section 11.5, page 666

1abc 1. (b) $u(\xi, 2) = f(\xi + 1), \quad u(\xi, 0) = 0, \quad 0 \le \xi \le 2$
$u(0, \eta) = u(2, \eta) = 0, \quad 0 \le \eta \le 2$

2 2. $u(r,t) = \sum_{n=1}^{\infty} k_n J_0(\lambda_n r) \sin \lambda_n a t, \quad k_n = \frac{1}{\lambda_n a} \int_0^1 r J_0(\lambda_n r) g(r)\,dr \Big/ \int_0^1 r J_0^2(\lambda_n r)\,dr$

4, 5ab, 6 3. Superpose the solution of Problem 2 and the solution [Eq. (21)] of the example in the text.

6. $u(r,z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} J_0(\lambda_n r), \quad c_n = \int_0^1 r J_0(\lambda_n r) f(r)\,dr \Big/ \int_0^1 r J_0^2(\lambda_n r)\,dr,$
and λ_n satisfies $J_0(\lambda) = 0$.

7b, 9a, 10 7. (b) $v(r,\theta) = \frac{1}{2} c_0 J_0(kr) + \sum_{m=1}^{\infty} J_m(kr)(b_m \sin m\theta + c_m \cos m\theta),$

$b_m = \frac{1}{\pi J_m(kc)} \int_0^{2\pi} f(\theta) \sin m\theta\,d\theta; \quad m = 1, 2, \ldots$

$c_m = \frac{1}{\pi J_m(kc)} \int_0^{2\pi} f(\theta) \cos m\theta\,d\theta; \quad m = 0, 1, 2, \ldots$

8. $c_n = \int_0^1 r f(r) J_0(\lambda_n r)\,dr \Big/ \int_0^1 r J_0^2(\lambda_n r)\,dr$

10. $u(\rho, s) = \sum_{n=0}^{\infty} c_n \rho^n P_n(s)$, where $c_n = \int_{-1}^1 f(\arccos s) P_n(s)\,ds \Big/ \int_{-1}^1 P_n^2(s)\,ds;$
P_n is the nth Legendre polynomial and $s = \cos\phi$.

Section 11.6, page 675

See SSM for detailed solutions to 2abc, 4a

5, 7bcd, 8

9abcde, 10, 12

1. $n = 21$
2. (a) $b_m = (-1)^{m+1}\sqrt{2}/m\pi$ (c) $n = 20$
3. (a) $b_m = 2\sqrt{2}(1 - \cos m\pi)/m^3\pi^3$ (c) $n = 1$
7. (a) $f_0(x) = 1$ (b) $f_1(x) = \sqrt{3}(1 - 2x)$ (c) $f_2(x) = \sqrt{5}(-1 + 6x - 6x^2)$
   (d) $g_0(x) = 1$, $g_1(x) = 2x - 1$, $g_2(x) = 6x^2 - 6x + 1$
8. $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$

INDEX

A
Abel, Niels Henrik, 149, 216
Abel's formula, 149, 167, 168, 213, 228
  for systems of equations, 370
Acceleration of convergence, 564
Adams, John Couch, 439
Adams–Bashforth formula, fourth order, 440
  second order, 440
Adams–Moulton formula, fourth order, 441
  second order, 441
Adaptive numerical method, 427, 433
Adjoint, differential equation, 146
  matrix, 348
Airy, George Biddell, 243
Airy equation, 147, 243–247, 252, 260, 291, 309
Almost linear systems, 479–491
Amplitude, of simple harmonic motion, 190
Amplitude modulation, 202
Analytic function, 235, 250
Angular momentum, principle of, 473–474
Annihilators, method of, 225–226
Asymptotic stability, *see* Stability
Augmented matrix, 358
Autonomous, equation, 74
  system, 471–472

B
Backward differentiation formulas, 443–444
Backward Euler formula, 422–424
Basin of attraction, 487, 498, 516–519
Beats, 201–202, 323–324
Bendixson, Ivar Otto, 525
Bernoulli, Daniel, 25, 87, 88, 573, 591
Bernoulli, Jakob, 24, 63, 73, 336
Bernoulli, Johann, 24, 63
Bernoulli equation, 73
Bessel, Friedrich Wilhelm, 280
Bessel equation of order:
  k, 662
  nu, 147, 152, 238, 256, 259, 260, 280, 290, 627, 657
  one, 272, 287–289, 290
  one-half, 285–286, 289
  zero, 272, 280–284, 290, 308, 613, 658, 665
Bessel functions, 25
  $J_0(x)$, 272, 281, 289, 308, 309, 658, 660–661, 662, 665, 668
    asymptotic approximation to, 284
    Laplace transform of, 308
    zeros of, 291, 658, 665
  $J_1(x)$, 272, 287, 289
  $J_{1/2}(x)$, 285
  $J_{-1/2}(x)$, 286
  orthogonality of, 291, 661, 662
  $Y_0(x)$, 283, 658, 665
    asymptotic approximation to, 284
  $Y_1(x)$, 289
Bessel inequality, 676
Bessel series expansion, 661, 666
Bifurcation (points), 88, 89, 122, 383, 480, 502, 533
Boundary conditions: 542
  for elastic string, 591, 600
  for heat equation, 574, 582, 584, 616
  for Laplace's equation, 605
  nonhomogeneous, 582–584, 650
  periodic, 638, 674
  separated, 630
  time dependent, 650
Boundary layer, 450
Boundary value problems:
  heat conduction equation, 573–590, 614–617, 646–649

homogeneous, 542–544, 629–641
Laplace's equation, 604–613
nonhomogeneous, 542–543, 641–645
self-adjoint, 637–639, 660
singular, 656–663
Sturm-Liouville, 629–630
two-point, 541–547, 623
wave equation, 591–604, 617–619, 653, 664–666
see also Homogeneous boundary value problems; Nonhomogeneous boundary value problems
Brachistochrone, 24, 63
Buckling of elastic column, 640–641

C
Capacitance, 195, 196
Cardano, Girolamo, 216
Cayley, Arthur, 348
Center, 387, 467, 479, 490
Change of independent variable, 159–160, 291
for Euler equation, 160, 264, 266
Chaotic solution, of logistic difference equation, 122, 126
of Lorenz equations, 536
Characteristic equation, 132, 214, 303
complex roots, 153, 217
real and equal roots, 161, 218
real and unequal roots, 132, 215
Characteristic polynomial, 214, 303
Chebyshev, Pafnuty L., 253, 511
Chebyshev equation, 253, 271, 627, 663
Chebyshev polynomials, 253, 663
Chemical reactions, 89
Collocation, method of, 670
Competing species, 491–503
Complementary solution, 170
Complete set of functions, 672
Complex exponentials, 153–155, 219
Compound interest, 51–54
Computer use in differential equations, 21
Conjugate matrix, 348
Continuous spectrum, 660
Convergence:
of an improper integral, 294
of a numerical approximation, 424
of a power series, 232
Convergence in mean, 672
Converging solutions, 4, 12, 102
Convolution integral, 185, 330–337
Laplace transform of, 330–333
Cosine series, 566
Critical amplitude, 82
Critical damping, 193
Critical point:
approached by trajectory, 467
center of linear system, 387, 467, 479, 490
definition of, 460, 472
for first order equation, 77
improper node, 404, 462–464
isolated, 481
node of linear system, 378, 390, 460–461
nonisolated, 470
proper node of linear system, 462
saddle point of linear system, 375, 389, 461–462
spiral point of linear system, 386, 390, 464–466
stability of, *see* Stability of critical point
Cycloid, 64, 337

D
D'Alembert, Jean, 161, 573, 591, 602
Dal Ferro, Scipione, 216
Damping force, 187–188, 473
Decay, radioactive, 16, 59
Degenerate node, *see* Improper node
Diagonalization, of matrices, 397–398
of homogeneous systems, 398–400
of nonhomogeneous systems, 411–413
Difference equation, 115–126
equilibrium solution of, 116
first order, 115–126
initial condition for, 116
iteration of, 116
linear, 115, 116–118
logistic, 118–126
chaotic solutions of, 122, 126
nonlinear, 115, 118–126
solution of, 115
Differential operator, 138, 630
Diffusion equation, *see* Heat conduction equation
Dimensionless variables, 580, 601
Dirac, Paul A. M., 326
Dirac delta function, 326, 656
Laplace transform of, 326–327
Direction field, for first order equations, 3, 5
for systems, 373
Dirichlet, Peter Gustav Lejeune, 605
Dirichlet problem, 605
for circle, 609–611, 612
for rectangle, 606–608, 611–612
for sector, 612
for semicircle, 612
for semi-infinite strip, 612
Discontinuous forcing function, 74, 317–324
Diverging solutions, 6, 10, 102
Drag, 2, 8
Duffing equation, 478

E
Eigenfunctions, 545, 623
normalized, 633
orthogonality of, 632, 660
series of, 635–636, 642, 660–661, 672

Eigenvalues of linear homogeneous boundary value problems, 545, 623
of Sturm–Liouville problem, existence, 631
real, 547, 631
simple, 633
when positive, 640
Eigenvalues of matrix, 362–366, 401–402
multiplicity, 364
real, 365
simple, 364
Eigenvectors of matrix, 362–366, 401–402
generalized, 405, 409
linear independence of, 364, 365
normalized, 364
orthogonality of, 366
Elastic bar, vibrations of, 628–629
Elastic membrane, vibrations of, 604, 664–666
Elastic string:
boundary value problems for, 591–604
derivation of wave equation, 617–619
free at one end, 600
general problem, 598–600
infinite length, 602–603
justification of solution, 596–598, 603–604
natural frequencies of, 594
natural modes of, 594
propagation of discontinuities in initial conditions, 598
wavelength of, 594
Electrical networks, 16, 195–196, 306–307, 340, 345–346
Elliptic integral, 491
Environmental carrying capacity, 78
Epidemics, 87–88
Equilibrium solution, 4, 6, 12, 76, 116, 460
see also Critical point
Error:
for Adams–Bashforth formula, 440
for Adams–Moulton formula, 441
effect of step size, 426, 445–447
for Euler method, 425–427, 428
for Fourier series, 554–555, 561–562
global truncation, 424, 426, 445–447
for improved Euler method, 431, 434–435
local truncation, 424
mean square, 671
for modified Euler method, 435
round-off, 424, 429, 445–447
for Runge–Kutta method, 436
Escape velocity, 56–57
Euler, Leonhard, 24, 25, 97, 571, 573, 591, 604
Euler equation, 160, 169, 260–267, 382, 391, 408, 610
change of independent variable, 160, 264, 266
Euler formula for $\exp(it)$, 154
Euler–Fourier formulas, 550–551
Euler–Mascheroni constant, 283
Euler method, 96–105, 419–429, 455–457
convergence of, 104–105
global truncation error, 426
local truncation error, 425–427
Even functions, 564
Even periodic extension, 568
Exact equations, 89–93, 146
necessary and sufficient condition for existence of solution, 25, 91
for second order equations, 146
Exchange of stability, 120
Existence and uniqueness theorems, 20, 21
for first order equations, 66, 106
for first order linear equations, 65
for *n*th order linear equations, 210
proof of, for $y' = f(t, y)$, 106–113
for second order linear equations, 138
for series solution of second order linear equations, 250, 277–278
for systems of first order equations, 343
Exponential growth, 75
Exponential matrix, 396–397, 400
Exponential order, functions of, 296
Exponents at the singularity, 269, 274

F
Falling object problem, 2, 3, 12–14
Feigenbaum number, 126
Ferrari, Ludovico, 216
First order ordinary differential equations:
applications of, 47–64, 74–89
Bernoulli, 73
direction field, 3, 5
exact, 89–93
existence and uniqueness theorems, 65, 66, 106
proof of, 106–113
general solution of, 11, 30, 36, 70, 71
graphical construction of integral curves, 71
homogeneous, 24, 46–47
implicit solutions, 70–71
integrating factor for, 30–31, 35–36, 93–95, 96
interval of definition, 37, 43, 69–70
linear, 24, 29–39
nonlinear, 64–72
numerical solution of, *see* Numerical methods

separable, 40–46
series solution of, 254
systems of, *see* Systems
Fitzhugh–Nagumo equations, 531
Fourier, Joseph, 547, 573
Fourier coefficients, 671
Fourier series, 547–572
acceleration of convergence, 564
choice of series, 568–569
convergence of, 559, 564
convergence of partial sums, 554, 561
cosine series, 566, 635
error, 554–555, 561–562
Euler–Fourier formulas, 550–551
Gibbs phenomenon, 561, 567
integration of, 550
orthogonality of sines and cosines, 549–550
Parseval equation, 564, 571, 604, 676
periodicity of sines and cosines, 548–549
for sawtooth wave, 567–568, 571
sine series, 566-567, 635
specialized kinds, 571–572
for square wave, 560–562, 571
for triangular wave, 551–552, 553–555, 571
Fredholm, Erik Ivar, 644
Fredholm alternative theorem, 644
Frequency, natural, 190
of simple harmonic motion, 190
of vibrating string, 594
Frobenius, Ferdinand Georg, 268, 348, 459
Frobenius, method of, 268
Fuchs, Immanuel Lazarus, 251, 459
Fundamental matrix, 393–401, 405–406
Fundamental set of solutions, 142, 211, 370
Fundamental theorem of algebra, 215

G

Galois, Evariste, 216
Gamma function, 298–299
Gauss, Carl Friedrich, 215
Gaussian elimination, 352
Generalized function, 326
General solution of linear equations:
first order, 11, 30, 36, 70, 71
*n*th order, 211
second order, 133, 142, 143, 155, 163, 170
systems of first order equations, 370
Gibbs, Josiah Willard, 561
Gibbs phenomenon, 561, 567
Global asymptotic stability, 487
Gompertz equation, 85
Graphical construction of integral curves, 71
Gravity, 2, 56
Green, George, 653
Green's function, 653–656

H

Half-life, 16, 59
Harvesting a renewable resource, 86–87
Heat conduction equation, 17, 574, 615
bar with insulated ends, 584–587
boundary conditions, 574, 582, 584, 616
derivation of, 613–617
fundamental solutions of, 576, 586
nonhomogeneous boundary conditions, 582–584
nonhomogeneous source term, 617, 646–649
in polar coordinates, 581
smoothing of discontinuities in initial conditions, 582
solution of fundamental problem, 575–577
steady state solution, 582
transient solution, 583
Heaviside, Oliver, 294
Heaviside function, 310
Helmholtz equation, 668
Hereditary systems, 331
Hermite, Charles, 248, 348
Hermite equation, 248, 260, 627
Hermite polynomials, 248
Hermitian matrix, 248, 365–366, 379, 397, 398, 402, 637, 653
Heun formula, 431
History of differential equations, 23–26
Hodgkin, Alan L., 503
Hodgkin–Huxley equations, 503
Homogeneous algebraic equations, 357
Homogeneous boundary value problems, 542–544, 623–626
eigenfunctions, 545, 623
eigenvalues, 545, 623
singular Sturm–Liouville, 656–663
Sturm–Liouville, 629–641
Homogeneous differential equations, with constant coefficients, 25, 129–137, 153–159, 160–169, 214–219
definition of, 130, 343
general theory of, 137–153, 210–211, 368–372
systems, 373–410
Homogeneous first order differential equations, 24, 46–47
Hooke, Robert, 187
Hooke's law, 187
Huxley, Andrew F., 503
Huygens, Christian, 336
Hypergeometric equation, 279

I

Identity matrix, 351
Implicit numerical method, 423, 441, 453

Implicit solution, 70–71
Improper integral, 294
  comparison theorem for, 296
Improper node, 404, 462–464
Improved Euler method, 430–435
Impulse functions, 324–330
Impulse response, 335
Indicial equation, 269, 273, 274, 278
Inductance, 195, 196
Initial conditions, 11, 72, 116, 130, 210, 342
  propagation of discontinuities for wave equation, 598
  smoothing of discontinuities for heat equation, 582
Initial value problem, 11, 65, 66, 105, 130, 210, 342
Inner product:
  for functions, 549, 631
  for vectors, 351
Instability:
  of critical point, 79, 117, 119, 468, 471, 472, 475, 514
  of numerical method, 448–453
  of periodic orbit, 524
  see also Diverging solutions
Integral curves, 11, 71
Integral equation, 106
  Volterra, 336
Integral transform, 293
Integrating factor, 25, 30–31, 35–36, 93–95, 96, 146
Interval of convergence, 233
Inverse Laplace transform, 303
Inverse matrix, 352–354
Invertible matrix, 352
Irregular singular point, 258, 279–280
Iteration, method of, 106
  see also Successive approximations
  of difference equation, 116

J

Jordan, Camille, 348, 406
Jordan form of matrix, 406–407, 409, 410
Jump discontinuity, 295, 559, 598

K

Kernel, of convolution, 185
  of integral transform, 293
Kirchhoff, Gustav R., 196
Kirchhoff's laws, 196, 345
Kutta, M. Wilhelm, 436

L

Lagrange, Joseph-Louis, 25, 180, 547, 573, 591
Lagrange's identity, 630, 659–660
Laguerre, Edmond Nicolas, 272
Laguerre equation, 272, 627
Landau, Lev D., 89
Landau equation, 89
Laplace, Pierre Simon de, 25, 294, 604
Laplace's equation, 25, 604–613, 666
  boundary conditions, 605
  in cylindrical coordinates, 613, 667
  Dirichlet problem, 605
    for circle, 609–611, 612
    for rectangle, 606–608, 611–612
    for sector, 612
    for semicircle, 612
    for semi-infinite strip, 612
  mixed problem, 613
  Neumann problem, 605
    for circle, 613
    for rectangle, 612
  in polar coordinates, 609
  in spherical coordinates, 668
Laplace transform, 293–338
  of convolution, 330-333
  definition of, 294
  of derivatives, 300–301
  of Dirac delta function, 326–327
  existence of, 296
  of integral equations, 336
  inverse of, 303
  of periodic functions, 315
  of rectified sine wave, 316
  of sawtooth wave, 316
  of square wave, 315, 316
  of step function, 311
  table of, 304
  translation formula, 311, 313
Legendre, Adrien Marie, 149, 254
Legendre equation of order alpha, 147, 152, 238, 252, 254–255, 256, 258, 271, 309, 657, 662, 669
Legendre polynomials, 254–255, 662, 676
Leibniz, Gottfried Wilhelm, 24, 63, 73, 336, 571
Length, of a vector, 351
L'Hospital, Marquis de, 63
Liapunov, Aleksandr M., 511
Liapunov function, 514
Liapunov's second method, 511–521
Libby, Willard F., 59
Liénard equation, 491, 520
Limit cycle, 524
Linear dependence and independence, of functions, 147–153, 211
  of vector functions, 362
  of vectors, 360
Linear operator, 138, 226, 298, 303
Linear ordinary differential equations:
  adjoint equation, 146
  change of independent variable, 159–160, 264, 266, 291
  characteristic equation, 132, 214, 303
    complex roots, 153, 217
    real and equal roots, 161, 218
    real and unequal roots, 132, 215
  complementary solution, 170
  definition of, 19
  Euler equation, 160, 169, 260–267
  exact, 146

existence and uniqueness theorems, 65, 138, 210, 250, 277–278, 343
first order, 24, 29–39
fundamental set of solutions, 142, 211, 370
general solution of, 11, 25, 30, 36, 70, 71, 133, 142, 143, 155, 163, 170, 211, 370
homogeneous equation with constant coefficients, 25, 129–137, 153–159, 160–169, 214–219
integrating factor, 25, 30–31, 35–36, 146
nonhomogeneous equation, 130, 169–185, 211–212, 222–230, 411–418
ordinary point, 238, 250, 254, 260
particular solution, 170, 212
reduction of order, 165–166, 212
self-adjoint, 147
series solution of, *see* Series solution
singular point, 238, 250, 255–260
systems of, *see* Systems
undetermined coefficients, 171–179, 222–226
variation of parameters, 25, 39, 179–185, 226–230
Liouville, Joseph, 106, 629
Logarithmic decrement, 198
Logistic equation, 76, 118, 511
Logistic growth, 75–81, 83–84
Lorenz, Edward N., 532
Lorenz equations, 532–539
Lotka, Alfred James, 504
Lotka–Volterra equations, 18, 340–341, 503–511

M

Magnitude, of a vector, 351
Malthus, Thomas, 75
Mathematical model, 2, 47–49
analysis of, 48–49
comparison with experiment, 49
construction of, 7, 14, 48
Matrices, 348–367
addition of, 349
adjoint, 348
augmented, 358
conjugate, 348
diagonalizable, 397–398
eigenvalues of, 362–366, 544
eigenvectors of, 362–366, 544
equality of, 349
exponential, 396–397, 400
fundamental, 393–401, 405–406
Gaussian elimination, 352
Hermitian, 248, 365–366, 379, 397, 402, 637, 653
identity, 351
inverse, 352–354
invertible, 352
Jordan form of, 406–407, 409, 410
multiplication of, 349–350
multiplication by numbers, 349
noninvertible, 352
nonsingular, 352
row reduction of, 352
self-adjoint, 365
similar, 398
singular, 352
subtraction of, 349
transpose, 348
zero, 349
Matrix functions, 354–355
Mean convergence, 672
Mean square error, 671
Millikan, Robert A., 62
Mixing problems, 49–51, 54–55
Mode, natural (of vibrating string), 594
Modified Euler formula, 435
Moulton, Forest Ray, 440
Multiplicity of eigenvalue, 364
Multistep method, 439–445

N

Negative definite function, 513
Negative semidefinite function, 513
Neumann, Karl Gottfried, 605
Neumann problem, 605
for circle, 613
for rectangle, 612
Newton, Isaac, 24, 63
Newton's law:
of cooling, 60
of motion, 2, 187, 345, 618
Node, 378, 390, 461
See also Improper node; Proper node
Nonhomogeneous algebraic equations, 358
Nonhomogeneous boundary value problems, 542–543, 641–656
Fredholm alternative, 644
solution, by eigenfunction expansion, 641–653
by Green's function, 653–656
Nonhomogeneous linear differential equations, 130, 169–185, 211–212, 222–230, 334–335, 343, 411–418
Noninvertible matrix, 352
Nonlinear ordinary differential equations:
autonomous systems, 471–539
definition of, 19, 130
existence and uniqueness theorems, 66, 106, 343
first order, 64–72
methods of solving, 40–47, 89–96, 137
periodic solutions of, 503–511, 521–531
$y'' = f(t, y')$, 137
$y'' = f(y, y')$, 137
Nonsingular matrix, 352
Normalization condition, 633
Nullcline, 499
Numerical dependence, 452
Numerical methods, 419–458
Adams–Bashforth formula, 440

Adams–Moulton formula, 441
adaptive, 427, 433
backward differentiation formulas, 443–444
backward Euler formula, 422–424
comparison of, 444
convergence of, 424
effect of step size, 426, 445–447
error *see* Error
Euler, 96–105, 419–429, 455–457
Heun, 431
improved Euler, 430–435
modified Euler, 435
multistep, 439–445
one-step, 439
predictor–corrector, 441, 458
Runge–Kutta, 435–439, 456
stability of, 448–453
for stiff equations, 450–451
for systems of first order equations, 455–458
vertical asymptotes, 447–448
Numerical stability, 448–453

O
Odd function, 565
Odd periodic extension, 568
One-step method, 439
Orbital stability, 524
Order of differential equation, 18
Ordinary differential equation, definition of, 17
Ordinary point, 238, 250, 254
at infinity, 260
Orthogonality, of Bessel functions, 291, 661, 662
of Chebyshev polynomials, 663
of eigenfunctions of Sturm–Liouville problems, 632, 660
of functions, 549
of Legendre polynomials, 255, 662, 676
of sine and cosine functions, 549–550
of vectors, 351
Orthonormal set, 633, 669
Overdamped motion, 193

P
Parseval equation, 564, 571, 604, 676
Partial differential equation, definition of, 17
See also Heat conduction equation, Laplace's equation, Wave equation
Partial fraction expansion, 302, 305, 306, 309
Particular solution, 170, 212
Pendulum equation:
generalized nonlinear undamped, 519
linear undamped, 19
nonlinear damped, 473–475, 481–482, 483, 484–487, 489, 519–520
nonlinear undamped, 19, 478, 488–489, 490–491, 511–513, 516
period, 490–491
Period, of nonlinear undamped pendulum, 490–491
of simple harmonic motion, 190
Periodic boundary conditions, 638, 674
Periodic forcing terms, 563
Periodic functions, 548
cosine and sine, 548–549
derivative of, 557
fundamental period, 548
integral of, 556–557
Laplace transform of, 315
linear combination of, 549
product of, 549
Periodic solutions of autonomous systems, 503–511, 521–531
Phase, of simple harmonic motion, 190
Phase plane, 373, 460
Phase plot, 207
Phase portrait, 373, 460
Picard, Charles Émile, 106
Picard, method of, 106
Piecewise continuous functions, 295, 558
Poincaré, Henri, 459
Poincaré-Bendixson theorem, 525
Population, growth of, 74–84
Positive definite function, 513
Positive semidefinite function, 513
Potential equation, 604–613
See also Laplace's equation
Power series, properties of, 232–235
Predator–prey equations, 18, 340–341, 503–511
Predictor–corrector method, 441, 458
Proper node, 462

Q
Quasi frequency, 192
Quasi period, 193

R
Radioactive decay, 16, 59
Radiocarbon dating, 59
Radius of convergence, 232, 251
Ramp loading, 320
Rate function, 5
Rate of growth (decline), 5, 75
intrinsic, 76
Rayleigh equation, 531
Rectified sine wave, 316
Recurrence relation, 240, 268, 269, 274
Reduction of order, 165–166, 212
Reduction to systems of equations, 341–342
Region of asymptotic stability, 487, 498, 516–519
Regular singular point, 255–260
at infinity, 260
Resistance, electric, 195, 196
see also Damping force
Resonance, 202–203, 323–324
Rodrigues' formula, 255

Round-off error, 424, 429, 445–447
Row reduction, 352
Runge, Carl D., 436
Runge–Kutta method, 436–439, 456

S
Saddle point, 375, 389, 461–462
Saturation level, 78
Sawtooth wave, 316, 567–568, 571
Scalar product, *see* Inner product
Schaefer model for fish population, 86
Schrödinger, Erwin, 326
Self-adjoint:
  boundary value problem, 637–639, 660
  equation, 147
  matrix, 365
Separable equations, 24, 40–46
Separated boundary conditions, 630
Separation constant, 575, 585, 593, 606, 623, 665
Separation of variables, 573
  further remarks, 664–669
  for heat conduction equation, 575, 585
  for Laplace's equation, 606
    in polar coordinates, 609
  for wave equation, 593
    in polar coordinates, 667
Separatrix, 487, 498
Series of eigenfunctions, 635–636, 642, 660–661, 672
Series solution:
  existence theorem for, 250, 277–278
  first order equations, 254
  indicial equation, 269, 273, 274, 278
  near an ordinary point, 238–255
  near a regular singular point, 267–280
  recurrence relation, 240, 268, 269, 274
  when roots of indicial equation:
    are equal, 270, 276–277, 278, 280–284
    differ by integer, 270, 277, 278, 285–289
Shift of index of summation, 235–236
Similarity transformation, 397–398
Similar matrices, 398
Simple eigenvalues, 364
Simple harmonic motion, 190
Simply connected region, 91
Simpson's rule, 437
Sine series, 567
Singular matrix, 352
Singular point, 238, 250, 255–260
  irregular, 258, 279–280
  regular, 255–260
    at infinity, 260
Sink, nodal, 461
  spiral, 465
Slope field, *see* Direction field
Solution of ordinary differential equation, 20
  general solution of linear equations, 11, 30, 36, 70, 71, 133, 142, 143, 155, 163, 170, 211
    systems, 370
  implicit, 70–71
  of systems of equations, 342
Source, nodal, 461
  spiral, 465
Spiral point, 386, 390, 464–466, 479–480
Spring–mass system, 186–195, 200–207, 306, 392–393
  two degrees of freedom, 220, 345, 393
Square integrable function, 672
Square wave, 315, 316, 560–562, 571
Stability:
  for almost linear systems, 483–484
  asymptotic, 79, 117, 119, 468, 471, 473, 475, 514–515, 524
    see also Converging solutions
  basin of attraction, 487, 498, 516–519
  of critical point, 468, 471, 472–473, 475, 479, 514
  definition of, 472
  exchange of, 120
  globally asymptotically stable, 487
  Liapunov theorems, 514
  for linear systems, 468, 479–480
  of numerical method, 448–453
  orbital, 524
  region of asymptotic stability, 487, 498, 516–519
  semistable, 84, 524
  stable, 468, 471, 472, 475, 514, 524
  unstable, 79, 117, 119, 468, 471, 472, 475, 514, 524
    see also Diverging solutions
Stairstep diagram, 120
Star point, 462
Steady state solution, 203, 582
Step functions, 310
Stiff equations, 450–451
Stokes, George Gabriel, 62
Stokes' law, 62
Strange attractor, 536
Sturm, Charles, 629
Sturm–Liouville boundary value problems, 629–641
  eigenfunctions orthogonal, 632
  eigenvalues, real, 631
    simple, 633
  nonhomogeneous, 641–645
  self-adjointness of, 637–639, 660
  singular, 656–663
    continuous spectrum, 660
Successive approximations, method of, 106, 401

Superposition principle, 140, 344, 368, 600, 663
Sylvester, James, 348
Systems:
  of algebraic equations, 357–360, 542–543
  of differential equations, 18
    autonomous, 471–472
    existence and uniqueness theorem, 343
    initial conditions, 342
    linear, 343
    nonlinear, 343
    numerical solution of, 455–458
    reduction to, 341–342
    solution of, 342
  of linear first order equations, 368–418
    definition of, 343
    diagonalization, 398–400, 411–413
    Euler, 382, 391, 408
    existence and uniqueness theorem, 343
    fundamental matrix, 393–401, 405–406
    fundamental set of solutions, 370
    general solution of, 370
    homogeneous, 343, 368–372
    homogeneous with constant coefficients, 373–410
      complex eigenvalues, 384–393, 464–467
      real and unequal eigenvalues, 373–383, 460–462
      repeated eigenvalues, 401–410, 462–464
    nonhomogeneous, 343, 411–418
    superposition of solutions, 368
    undetermined coefficients, 413–414
    variation of parameters, 414–416

T

Tangent line method, *see* Euler method
Tautochrone, 336–337
Taylor, Brook, 234
Taylor series, 154, 234
  for functions of two variables, 434
Telegraph equation, 619, 627, 653
Thermal diffusivity, 574, 616
Threshold models, 81–84
Trajectories, 342, 373, 460, 467–468, 475–477
  almost linear systems, 483–484
  linear systems, 460–467
Transfer function, 335
Transient solution, 203, 583
Translation of function, 311
Transpose of matrix, 348
Triangular wave, 551–552, 553–555, 571
Truncation error, global, 424, 426, 445–447
  local, 424

U

Undetermined coefficients, 171–178, 222–226
  for systems of equations, 413–414
Uniqueness theorems, *see* Existence and uniqueness theorems
Unit step function, 310
  Laplace transform of, 311

V

Van der Pol equation, 478, 526–530
Variation of parameters, 25, 39, 179–185, 226–230
  for systems of equations, 414–416
Vector space, 151
Vectors, 151, 349
  inner product, 351
  length, 351
  linear dependence and independence of, 360–362
  magnitude, 351
  multiplication of, 351
  orthogonality, 351
Verhulst, Pierre F., 76
Verhulst equation, 76
Vibrations:
  of elastic bar, 628–629
  of elastic membrane, 604, 664–666
  of elastic string, 591–604, 617–619
  of spring–mass system, 186–195, 200–207, 306, 392
  two degrees of freedom, 220, 345, 393
Volterra, Vito, 504
Volterra integral equation, 336

W

Wave equation, 17, 591, 617–619
  in polar coordinates, 664, 667
  solution of, 602
  see also Elastic string, derivation of wave equation
Wavelength, of vibrating string, 594
Wave velocity, 591, 601, 619
Wronski, Józef, 141
Wronskian, 141, 211
  Abel's identity for, 149, 213
  for systems of equations, 369
    Abel's identity, 370

Y

Yield, maximum sustainable, 87

Student Solutions Manual

Charles W. Haines
Rochester Institute of Technology

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to accompany

Elementary Differential Equations, 7e and Elementary Differential Equations and Boundary Value Problems, 7e

William E. Boyce
Richard C. DiPrima

Rensselaer Polytechnic Institute



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PREFACE

This supplement has been prepared for use in conjunction with the seventh editions of ELEMENTARY DIFFERNTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS and ELEMENTARY DIFFERENTIAL EQUATIONS, both by W.E. Boyce and R.C. DiPrima. The supplement contains a sampling of the problems from each section of the text. In most cases the complete details in determining the solutions are given while in the remainder of the problems helpful hints are provided. The problems chosen in each section represent, wherever possible, the variety of applications and types of examples that are covered in the written material of the text, thereby providing the student a complete set of examples from which to learn.

Students should be aware that following these solutions is very different from designing and constructing one's own solution.Using this supplemental resource appropriately for learning differential equations is outlined as follows:

1. Make an honest attempt to solve the problem without using the guide.
2. If needed, glance at the beginning of the solution in the guide and then try again to generate the complete solution.Continue using the guide for hints when you reach an impasse.
3. Compare your final solution with the one provided to see whether yours is more or less efficient than the guide, since there is frequently more than one correct way to solve a problem.
4. Ask yourself why that particular problem was assigned.

The use of a symbolic computational software package, in many cases, would greatly simplify finding the solution to a given problem, but the details given in this solutions manual are important for the student's understanding of the underlying mathematical principles and applications. In other cases, these software packages are essential for completing the given problem, as the calculations would be overwhelming using analytical techniques. In these cases, some steps or hints are given and then reference made to the use of an appropriate software package.

In order to simplify the text, the following abbreviations have been used:

D.E.	differential equation(s)
O.D.E.	ordinary differential equation(s)
P.D.E.	partial differential equation(s)
I.C.	initial condition(s)
I.V.P.	initial value problem(s)
B.C.	boundary condition(s)
B.V.P.	boundary value problem(s)

I wish to express my appreciation to Mrs. Susan A. Hickey for her excellent typing and proofreading of all stages of the manuscript. Dr. Josef Torok has also provided invaluable assistance with the preparation of the figures as well as assistance with many of the solutions involving the use of symbolic computational software.

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June 2000

CONTENTS

CHAPTER 1 1

CHAPTER 2 6

CHAPTER 3 35

CHAPTER 4 64

CHAPTER 5 73

CHAPTER 6 103

CHAPTER 7 121

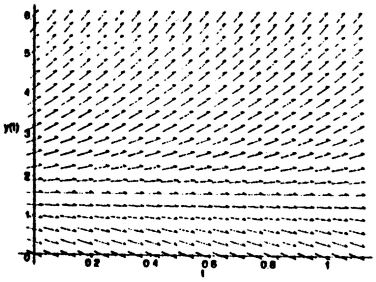
CHAPTER 8 160

CHAPTER 9 174

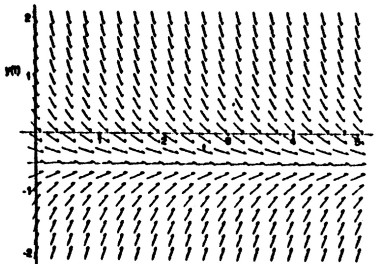
CHAPTER 10 203

CHAPTER 11 235

Section 1.1, Page 8



2. For $y > 3/2$ we see that $y' > 0$ and thus $y(t)$ is increasing there. For $y < 3/2$ we have $y' < 0$ and thus $y(t)$ is decreasing there. Hence $y(t)$ diverges from 3/2 as $t \to \infty$.



4. Observing the direction field, we see that for $y > -1/2$ we have $y' < 0$, so the solution is decreasing here. Likewise, for $y < -1/2$ we have $y' > 0$ and thus $y(t)$ is increasing here. Since the slopes get closer to zero as y gets closer to $-1/2$, we conclude that $y \to -1/2$ as $t \to \infty$.

7. If all solutions approach 3, then 3 is the equilibrium solution and we want $\frac{dy}{dt} < 0$ for $y > 3$ and $\frac{dy}{dt} > 0$ for $y < 3$. Thus $\frac{dy}{dt} = 3-y$.

11. For $y = 0$ and $y = 4$ we have $y' = 0$ and thus $y = 0$ and $y = 4$ are equilibrium solutions. For $y > 4$, $y' < 0$ so if $y(0) > 4$ the solution approaches $y = 4$ from above. If $0 < y(0) < 4$, then $y' > 0$ and the solutions "grow" to $y = 4$ as $t \to \infty$. For $y(0) < 0$ we see that $y' < 0$ and the solutions diverge from 0.

13. Since $y' = y^2$, $y = 0$ is the equilibrium solution and $y' > 0$ for all y. Thus if $y(0) > 0$, solutions will diverge from 0 and if $y(0) < 0$, solutions will aproach $y = 0$ as $t \to \infty$.

15a. Let $q(t)$ be the number of grams of the substance in the water at any time. Then $\frac{dq}{dt} = 300(.01) - \frac{300q}{1{,}000{,}000} = 300(10^{-2} - 10^{-6}q)$.

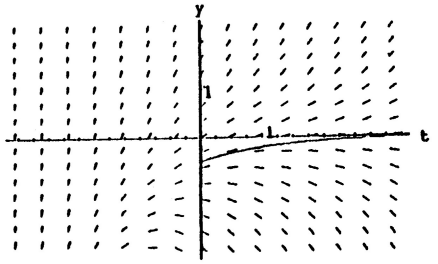
15b. The equilibrium solution occurs when $q' = 0$, or $c = 10^4$gm, independent of the amount present at $t = 0$ (all solutions approach the equilibrium solution).

16. The D.E. expressing the evaporation is $\frac{dV}{dt} = -aS$, $a > 0$.

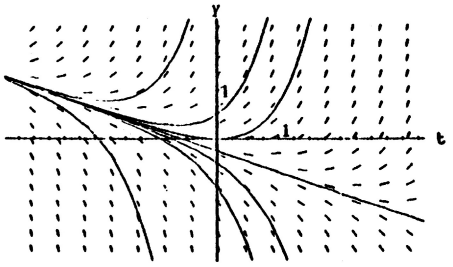
Now $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, so $S = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}$.

Thus $\frac{dv}{dt} = -kV^{2/3}$, for $k > 0$.

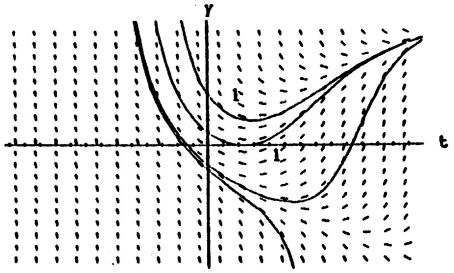
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22.



24.



Section 1.2 Page 14

1b. dy/dt = -2y+5 can be rewritten as $\frac{dy}{y-5/2} = -2dt$. Thus $\ln|y-5/2| = -2t+c_1$, or $y-5/2 = ce^{-2t}$. $y(0) = y_0$ yields $c = y_0 - 5/2$, so $y = 5/2 + (y_0-5/2)e^{-2t}$.
If $y_0 > 5/2$, the solution starts above the equilibrium solution and decreases exponentially and approaches 5/2 as $t\to\infty$. Conversely, if $y < 5/2$, the solution starts below 5/2 and grows exponentially and approaches 5/2 from

below as $t \to \infty$.

4a. Rewrite Eq.(ii) as $\frac{dy/dt}{y} = a$ and thus $\ln|y| = at + c_1$; or $y = ce^{at}$.

4b. If $y = y_1(t) + k$, then $\frac{dy}{dt} = \frac{dy_1}{dt}$. Substituting both these into Eq.(i) we get $\frac{dy_1}{dt} = a(y_1+k) - b$. Since $\frac{dy_1}{dt} = ay_1$, this leaves $ak - b = 0$ and thus $k = b/a$.
Hence $y = y_1(t) + b/a$ is the solution to Eq(i).

6b. From Eq.(11) we have $p = 900 + ce^{t/2}$. If $p(0) = p_0$, then $c = p_0 - 900$ and thus $p = 900 + (p_0 - 900)e^{t/2}$. If $p_0 < 900$, this decreases, so if we set $p = 0$ and solve for T (the time of extinction) we obtain $e^{T/2} = 900/(900-p_0)$, or $T = 2\ln[900/(900-p_0)]$.

8a. Use Eq.(26).

8b. Use Eq.(29).

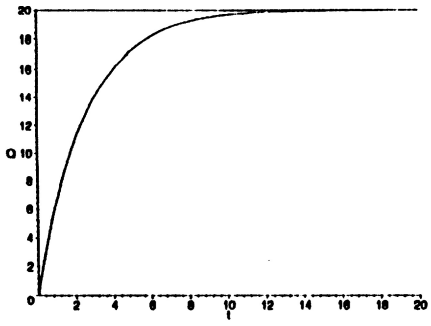
10a. $\frac{dQ}{dt} = -rQ$ yields $\frac{dQ/dt}{Q} = -r$, or $\ln|Q| = -rt + c_1$. Thus $Q = ce^{-rt}$ and $Q(0) = 100$ yields $c = 100$. Hence $Q = 100e^{-rt}$. Setting $t = 1$, we have $82.04 = 100e^{-r}$, which yields $r = .1980$/wk or $r = .02828$/day.

13a. Rewrite the D.E. as $\frac{dQ/dt}{Q-CV} = \frac{-1}{CR}$, thus upon integrating and simplifying we get $Q = De^{-t/CR} + CV$. $Q(0) = 0 \Rightarrow D = -CV$ and thus $Q(t) = CV(1 - e^{-t/CR})$.

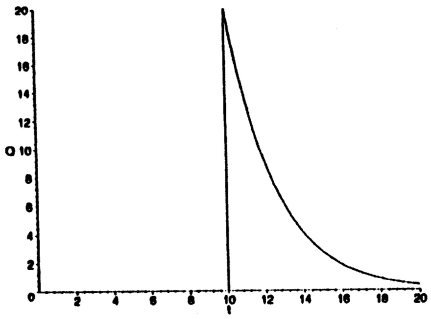
13b. $\lim_{t\to\infty} Q(t) = CV$ since $\lim_{t\to\infty} e^{-t/CR} = 0$.

13c. In this case $R\frac{dQ}{dt} + \frac{Q}{C} = 0$, $Q(t_1) = CV$. The solution of this D.E. is $Q(t) = Ee^{-t/CR}$, so $Q(t_1) = Ee^{-t_1/CR} = CV$, or $E = CVe^{t_1/CR}$. Thus $Q(t) = CVe^{t_1/CR}e^{-t/CR} = CVe^{-(t-t_1)/CR}$.

13a. CV = 20,CR = 2.5



13c. CV = 20, CR = 2.5 $t_1 = 10$



Section 1.3, Page 22

2. The D.E. is second order since there is a second derivative of y appearing in the equation. The equation is nonlinear due to the y^2 term (as well as due to the y^2 term multiplying the y″ term).

6. This is a third order D.E. since the highest derivative is y‴ and it is linear since y and all its derivatives appear to the first power only. The terms t^2 and $\cos^2 t$ do not affect the linearity of the D.E.

8. For $y_1(t) = e^{-3t}$ we have $y_1'(t) = -3e^{-3t}$ and $y_1''(t) = 9e^{-3t}$. Substitution of these into the D.E. yields $9e^{-3t} + 2(-3e^{-3t}) - 3(e^{-3t}) = (9-6-3)e^{-3t} = 0$.

14. Recall that if $u(t) = \int_0^t f(s)ds$, then $u'(t) = f(t)$.

16. Differentiating e^{rt} twice and substituting into the D.E. yields $r^2e^{rt} - e^{rt} = (r^2-1)e^{rt}$. If $y = e^{rt}$ is to be a solution of the D.E. then the last quantity must be zero for all t. Thus $r^2-1 = 0$ since e^{rt} is never zero.

19. Differentiating t^r twice and substituting into the D.E. yields $t^2[r(r-1)t^{r-2}] + 4t[rt^{r-1}] + 2t^r = [r^2+3r+2]t^r$. If $y = t^r$ is to be a solution of the D.E., then the last term must be zero for all t and thus $r^2 + 3r + 2 = 0$.

22. The D.E. is second order since there are second partial derivatives of u(x,y) appearing. The D.E. is nonlinear due to the product of u(x,y) times u_x(or u_y). 5

26. Since $\frac{\partial u_1}{\partial t} = -\alpha^2 e^{-\alpha^2 t}\sin x$ and $\frac{\partial^2 u_1}{\partial x^2} = -e^{-\alpha^2 t}\sin x$ we have

$\alpha^2[-e^{-\alpha^2 t}\sin x] = -\alpha^2 e^{-\alpha^2 t}\sin x$, which is true for all t and x.

CHAPTER 2

Section 2.1, Page 38

1. $\mu(t) = \exp(\int 3dt) = e^{3t}$. Thus $e^{3t}(y'+3y) = e^{3t}(t+e^{-2t})$ or $\frac{d}{dt}(ye^{3t}) = te^{3t} + e^{t}$. Integration of both sides yields $ye^{3t} = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^{t} + c$, and division by e^{3t} gives the general solution. Note that $\int te^{3t}dt$ is evaluated by integration by parts, with u = t and $dv = e^{3t}dt$.

2. $\mu(t) = e^{-2t}$.

3. $\mu(t) = e^{t}$.

4. $\mu(t) = \exp(\int \frac{dt}{t}) = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution.

6. The equation must be divided by t so that it is in the form of Eq.(3): $y' + (2/t)y = (\sin t)/t$. Thus $\mu(t) = \exp(\int \frac{2dt}{t} = t^2$, and $(t^2y)' = t\sin t$. Integration then yields $t^2y = -t\cos t + \sin t + c$.

7. $\mu(t) = e^{t^2}$.

8. $\mu(t) = \exp(\int \frac{4tdt}{1+t^2}) = (1+t^2)^2$.

11. $\mu(t) = e^{t}$ so $(e^{t}y)' = 5e^{t}\sin 2t$. To integrate the right side you can integrate by parts (twice), use an integral table, or use a symbolic computational software program to find $e^{t}y = e^{t}(\sin 2t - 2\cos 2t) + c$.

13. $\mu(t) = e^{-t}$ and $y = 2(t-1)e^{2t} + ce^{t}$. To find the value for c, set t = 0 in y and equate to 1, the initial value of y. Thus -2+c = 1 and c = 3, which yields the solution of the given initial value problem.

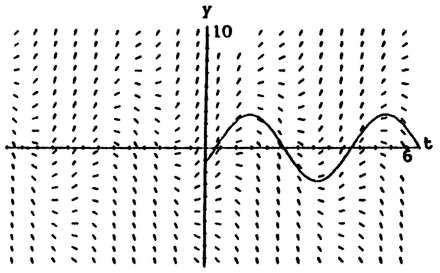
15. $\mu(t) = \exp(\int \frac{2dt}{t}) = t^2$ and $y = t^2/4 - t/3 + 1/2 + c/t^2$. Setting t = 1 and y = 1/2 we have c = 1/12.

18. $\mu(t) = t^2$. Thus $(t^2y)' = t\sin t$ and $t^2y = -t\cos t + \sin t + c$. Setting $t = \pi/2$ and y = 1 yields $c = \pi^2/4 - 1$.

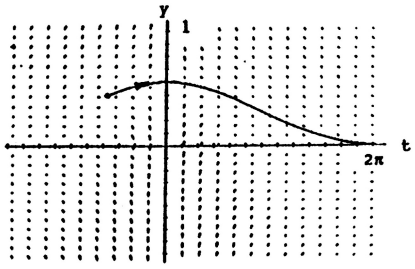
20. $\mu(t) = te^{t}$.

21b. $\mu(t) = e^{-t/2}$ so $(e^{-t/2}y)' = 2e^{-t/2}\cos t$. Integrating (see comments in #11) and dividing by $e^{-t/2}$ yields $y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + ce^{t/2}$. Thus $y(0) = -\frac{4}{5} + c = a$, or $c = a + \frac{4}{5}$ and $y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + (a + \frac{4}{5})e^{t/2}$. If $(a + \frac{4}{5}) = 0$, then the solution is oscillatory for all t, while if $(a + \frac{4}{5}) \neq 0$, the solution is unbounded as $t \to \infty$. Thus $a_0 = -\frac{4}{5}$.

21a.



24a.



24b. $\mu(t) = \exp\int \frac{2dt}{t} = t^2$, so $(t^2y)' = \sin t$ and $y(t) = \frac{-\cos t}{t^2} + \frac{c}{t^2}$. Setting $t = -\frac{\pi}{2}$ yields $\frac{4c}{\pi^2} = a$ or $c = \frac{a\pi^2}{4}$ and hence $y(t) = \frac{a\pi^2/4 - \cos t}{t^2}$, which is unbounded as $t \to 0$ unless $a\pi^2/4 = 1$ or $a_0 = 4/\pi^2$.

24c. For $a = 4/\pi^2$ $y(t) = \frac{1 - \cos t}{t^2}$. To find the limit as $t \to 0$ L'Hopital's Rule must be used: $\lim_{t\to 0} y(t) = \lim_{t\to 0} \frac{\sin t}{2t} = \lim_{t\to 0} \frac{\cos t}{2} = \frac{1}{2}$.

28. $(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$ so $e^{-t}y = -e^{-t} - 3e^{-t}(\frac{\sin t+\cos t}{2}) + c$ or $y(t) = -1 - (\frac{3}{2})e^{-t}(\sin t+\cos t) + ce^{t}$. If $y(t)$ is to remain bounded, we must have $c = 0$. Thus $y(0) = -1 - \frac{3}{2} + c = y_0$ or $c = y_0 + \frac{5}{2} = 0$ and $y_0 = -\frac{5}{2}$.

30. $\mu(t) = e^{at}$ so the D.E. can be written as $(e^{at}y)' = be^{at}e^{-\lambda t} = be^{(a-\lambda)t}$. If $a \neq \lambda$, then integration and solution for y yields $y = [b/(a-\lambda)]e^{-\lambda t} + ce^{-at}$. Then $\lim_{x\to\infty} y$ is zero since both λ and a are positive numbers.
If $a = \lambda$, then the D.E. becomes $(e^{at}y)' = b$, which yields $y = (bt+c)/e^{\lambda t}$ as the solution. L'Hopital's Rule gives $\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{(bt+c)}{e^{\lambda t}} = \lim_{t\to\infty} \frac{b}{\lambda e^{\lambda t}} = 0$.

32. There is no unique answer for this situation. One possible response is to assume $y(t) = ce^{-2t} + 3 - t$, then $y'(t) = -2ce^{-2t} - 1$ and thus $y' + 2y = 5 - 2t$.

35. This problem demonstrates the central idea of the method of variation of parameters for the simplest case. The solution (ii) of the homogeneous D.E. is extended to the corresponding nonhomogeneous D.E. by replacing the constant A by a function A(t), as shown in (iii).

36. Assume $y(t) = A(t)\exp(-\int(-2)dt) = A(t)e^{2t}$. Differentiating y(t) and substituting into the D.E. yields $A'(t) = t^2$ since the terms involving A(t) add to zero. Thus $A(t) = t^3/3 + c$, which substituted into y(t) yields the solution.

37. $y(t) = A(t)\exp(-\int \frac{dt}{t}) = A(t)/t$.

Section 2.2, Page 45

Problems 1 through 20 follow the pattern of the examples worked in this section. The first eight problems, however, do not have I.C. so the integration constant, c, cannot be found.

1. Write the equation in the form $ydy = x^2dx$. Integrating the left side with respect to y and the right side with respect to x yields
$\frac{y^2}{2} = \frac{x^3}{3} + C$, or $3y^2 - 2x^3 = c$.

4. For $y \neq -3/2$ multiply both sides of the equation by $3 + 2y$ to get the separated equation $(3+2y)dy = (3x^2-1)dx$. Integration then yields

$3y + y^2 = x^3 - x + c.$

6. We need $x \neq 0$ and $|y| < 1$ for this problem to be defined. Separating the variables we get $(1-y^2)^{-1/2}dy = x^{-1}dx$. Integrating each side yields $\arcsin y = \ln|x|+c$, so $y = \sin[\ln|x|+c]$, $x \neq 0$ (note that $|y| < 1$). Also, $y = \pm 1$ satisfy the D.E., since both sides are zero.

10a. Separating the variables we get $ydy = (1-2x)dx$, so $\frac{y^2}{2} = x - x^2 + c$. Setting $x = 0$ and $y = -2$ we have $2 = c$ and thus $y^2 = 2x - 2x^2 +$ or $y = -\sqrt{2x - 2x^2 + 4}$. The negative square root must be used since $y(0) = -2$.

10c. Rewriting $y(x)$ as $-\sqrt{2(2-x)(x+1)}$, we see that y is defined for $-1 \leq x \leq 2$, However, since y' does not exist for $x = -1$ or $x = 2$, the solution is valid only for the open interval $-1 < x < 2$.

13. Separate variables by factoring the denominator of the right side to get $ydy = \frac{2x}{1+x^2}dx$. Integration yields $y^2/2 = \ln(1+x^2)+c$ and use of the I.C. gives $c = 2$. Thus $y = \pm [2\ln(1+x^2)+4]^{1/2}$, but we must discard the plus square root because of the I.C. Since $1 + x^2 > 0$, the solution is valid for all x.

15. Separating variables and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y^2 + y = x^2-4$. To solve for y complete the square on the left side by adding $1/4$ to both sides. This yields $y^2 + y + \frac{1}{4} = x^2 - 4 + \frac{1}{4}$ or $(y + \frac{1}{2})^2 = x^2 - 15/4$. Taking the square root of both sides yields $y + \frac{1}{2} = \pm\sqrt{x^2 - 15/4}$, where the positive square root must be taken in order to satisfy the I.C. Thus $y = -\frac{1}{2} + \sqrt{x^2 - 15/4}$, which is defined for $x^2 \geq 15/4$ or $x \geq \sqrt{15}/2$. The possibility that $x < -\sqrt{15}/2$ is discarded due to the I.C.

17a. Separating variables gives $(2y-5)dy = (3x^2-e^x)dx$ and

integration then gives $y^2 - 5y = x^3 - e^x + c$. Setting $x = 0$ and $y = 1$ we have $1 - 5 = 0 - 1 + c$, or $c = -3$. Thus $y^2 - 5y - (x^3 - e^x - 3) = 0$ and using the quadratic formula then gives $y(x) = \frac{5 \pm \sqrt{25+4(x^3-e^x-3)}}{2} = \frac{5}{2} - \sqrt{\frac{13}{4} + x^3 - e^x}$. The negative square root is chosen due to the I.C.

17c. The interval of definition for y must be found numerically. Approximate values can be found by plotting $y_1(x) = \frac{13}{4} + x^3$ and $y_2(x) = e^x$ and noting the values of x where the two curves cross.

19a. As above we start with $\cos 3y\,dy = -\sin 2x\,dx$ and integrate to get $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + c$. Setting $y = \pi/3$ when $x = \pi/2$ (from the I.C.) we find that $0 = -\frac{1}{2} + c$ or $c = \frac{1}{2}$, so that $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$ (using the appropriate trigonometric identity). To solve for y we must choose the branch that passes through the point $(\pi/2, \pi/3)$ and thus $3y = \pi - \arcsin(3\cos^2 x)$, or $y = \frac{\pi}{3} - \frac{1}{3}\arcsin(3\cos^2 x)$.

19c. The solution in part a is defined only for $0 \le 3\cos^2 x \le 1$, or $-\sqrt{1/3} \le \cos x \le \sqrt{1/3}$. Taking the indicated square roots and then finding the inverse cosine of each side yields $.9553 \le x \le 2.1863$, or $|x - \pi/2| \le 0.6155$, as the approximate interval.

21. We have $(3y^2 - 6y)dy = (1 + 3x^2)dx$ so that $y^3 - 3y^2 = x + x^3 - 2$, once the I.C. are used. From the D.E., the integral curve will have a vertical tangent when $3y^2 - 6y = 0$, or $y = 0, 2$. For $y = 0$ we have $x^3 + x - 2 = 0$, which is satisfied for $x = 1$, which is the only zero of the function $w = x^3 + x - 2$. Likewise, for $y = 2$, $x = -1$.

23. Separating variables gives $y^{-2}dy = (2 + x)dx$, so

$-y^{-1} = 2x + \frac{x^2}{2} + c$. $y(0) = 1$ yields $c = -1$ and thus $y = \frac{-1}{\frac{x^2}{2} + 2x - 1} = \frac{2}{2 - 4x - x^2}$. This gives $\frac{dy}{dx} = \frac{8 + 4x}{(2-4x-x^2)^2}$, so the minimum value is attained at $x = -2$. Note that the solution is defined for $-2 - \sqrt{6} < x < -2 + \sqrt{6}$ (by finding the zeros of the denominator) and has vertical asymptotes at the end points of the interval.

25. Separating variables and integrating yields $3y + y^2 = \sin 2x + c$. $y(0) = -1$ gives $c = -2$ so that $y^2 + 3y + (2 - \sin 2x) = 0$. The quadratic formula then gives $y = -\frac{3}{2} + \sqrt{\sin 2x + 1/4}$, which is defined for $-.126 < x < 1.697$ (found by solving $\sin 2x = -.25$ for x and noting $x = 0$ is the initial point). Thus we have $\frac{dy}{dx} = \frac{\cos 2x}{(\sin 2x + \frac{1}{4})}$, which yields $x = \pi/4$ as the only critical point in the above interval. Using the second derivative test or graphing the solution clearly indicates the critical point is a maximum.

27a. By sketching the direction field and using the D.E.we note that $y' < 0$ for $y > 4$ and y' approaches zero as y approaches 4. For $0 < y < 4$, $y' > 0$ and again approaches zero as y approaches 4. Thus $\lim_{t\to\infty} y = 4$ if $y_0 > 0$. For $y_0 < 0$, $y' < 0$ for all y and hence y becomes negatively unbounded ($-\infty$) as t increases. If $y_0 = 0$, then $y' = 0$ for all t, so $y = 0$ for all t.

27b. Separating variables and using a partial fraction expansion we have $(\frac{1}{y} - \frac{1}{y-4})dy = \frac{4}{3}t\,dt$. Hence $\ln\left|\frac{y}{y-4}\right| = \frac{2}{3}t^2 + c_1$ and thus $\left|\frac{y}{y-4}\right| = e^{c_1}e^{2t^2/3} = ce^{2t^2/3}$, where c is positive. For $y_0 = .5$ this becomes $\frac{y}{4 - y} = ce^{2t^2/3}$ and thus $c = \frac{.5}{3.5} = \frac{1}{7}$. Using this value

for c and solving for y yields $y(t) = \frac{4}{1 + 7e^{-2t^2/3}}$.
Setting this equal to 3.98 and solving for t yields
t = 3.29527.

29. Separating variables yields $\frac{cy+d}{ay+b}\,dy = dx$. If $a \neq 0$ and $ay+b \neq 0$ then $dx = \left(\frac{c}{a} + \frac{ad-bc}{a(ay+b)}\right)dy$. Integration then yields the desired answer.

30c. If v = y/x then y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and thus the D.E. becomes $v + x\frac{dv}{dx} = \frac{v-4}{1-v}$. Subtracting v from both sides yields $x\frac{dv}{dx} = \frac{v^2-4}{1-v}$.

30d. The last equation in (c) separates into $\frac{1-v}{v^2-4}dv = \frac{1}{x}dx$. To integrate the left side use partial fractions to write $\frac{1-v}{v-4} = \frac{A}{v-2} + \frac{B}{v+2}$, which yields A = -1/4 and B = -3/4. Integration then gives $-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| - k$, or $\ln|x^4||v-2||v+2|^3 = 4k$ after manipulations using properties of the ln function.

31a. Simplifying the right side of the D.E. gives $dy/dx = 1 + (y/x) + (y/x)^2$ so the equation is homogeneous.

31b. The substitution y = vx leads to $v + x\frac{dv}{dx} = 1 + v + v^2$ or $\frac{dv}{1 + v^2} = \frac{dx}{x}$. Solving, we get $\arctan v = \ln|x| + c$. Substituting for v we obtain $\arctan(y/x) - \ln|x| = c$.

33b. Dividing the numerator and denominator of the right side by x and substituting y = vx we get $v + x\frac{dv}{dx} = \frac{4v - 3}{2-v}$ which can be rewritten as $x\frac{dv}{dx} = \frac{v^2 + 2v - 3}{2 - v}$. Note that $v = -3$ and $v = 1$ are solutions of this equation. For $v \neq 1, -3$ separating variables gives

$\frac{2 - v}{(v+3)(v-1)}\,dv = \frac{1}{x}dx$. Applying a partial fraction decomposition to the left side we obtain $[\frac{1}{4}\frac{1}{v-1} - \frac{5}{4}\frac{1}{v+3}]dv = \frac{dx}{x}$, and upon integrating both sides we find that $\frac{1}{4}\ln|v-1| - \frac{5}{4}\ln|v+3| = \ln|x| + c$.
Substituting for v and performing some algebraic manipulations we get the solution in the implicit form $|y-x| = c|y+3x|^5$. $v = 1$ and $v = -3$ yield $y = x$ and $y = -3x$, respectively, as solutions also.

35b. As in Prob.33, substituting $y = vx$ into the D.E. we get $v + x\frac{dv}{dx} = \frac{1+3v}{1-v}$, or $x\frac{dv}{dx} = \frac{(v+1)^2}{1-v}$. Note that $v = -1$ (or $y = -x$) satisfies this D.E. Separating variables yields $\frac{1-v}{(v+1)^2}dv = \frac{dx}{x}$. Integrating the left side by parts we obtain $\frac{v-1}{v+1} - \ln|v+1| = \ln|x| + c$. Letting $v = \frac{y}{x}$ then yields $\frac{y-x}{y+x} - \ln|\frac{y+x}{x}| = \ln|x| + c$, or $\frac{y-x}{y+x} - \ln|y+x| = c$.
This answer differs from that in the text. The answer in the text can be obtained by integrating the left side, above, using partial fractions. By differentiating both answers, it can be verified that indeed both forms satisfy the D.E.

Section 2.3, Page 57

1. Note that q(0) = 200 gm, where q(t) represents the amount of dye present at any time.

2. Let S(t) be the amount of salt that is present at any time t, then S(0) = 0 is the original amount of salt in the tank, 2 is the amount of salt entering per minute, and 2(S/120) is the amount of salt leaving per minute (all amounts measured in grams). Thus
dS/dt = 2 - 2S/120, S(0) = 0.

3. We must first find the amount of salt that is present after 10 minutes. For the first 10 minutes (if we let Q(t) be the amount of salt in the tank):
= (2) - 2, Q(0) = 0. This I.V.P. has the solution: Q(10) = 50(1-) 9.063 lbs. of salt in the tank after the first 10

minutes. At this point no more salt is allowed to enter, so the new I.V.P. (letting P(t) be the amount of salt in the tank) is:
= (0)(2) - 2, P(0) = Q(10) = 9.063. The solution of this problem is P(t) = 9.063, which yields P(10) = 7.42 lbs.

4. Salt flows into the tank at the rate of (1)(3) lb/min. and it flows out of the tank at the rate of (2) lb/min. since the volume of water in the tank at any time t is 200 + (1)(t) gallons (due to the fact that water flows into the tank faster than it flows out). Thus the I.V.P. is dQ/dt = 3 - Q(t), Q(0) = 100.

7a. Set = 0 in Eq.(16) (or solve Eq.(15) with S(0) = 0).

7b. Set r = .075, t = 40 and S(t) = $1,000,000 in the answer to (a) and then solve for k.

7c. Set k = $2,000, t = 40 and S(t) = $1,000,000 in the answer to (a) and then solve numerically for r.

9. The rate of accumulation due to interest is .1S and the rate of decrease is k dollars per year and thus dS/dt = .1S - k, S(0) = $8,000. Solving this for S(t) yields S(t) = 8000 - 10k(-1). Setting S = 0 and substitution of t = 3 gives k = $3,086.64 per year. For 3 years this totals $9,259.92, so $1,259.92 has been paid in interest.

10. Since we are assuming continuity, either convert the monthly payment into an annual payment or convert the yearly interest rate into a monthly interest rate for 240 months. Then proceed as in Prob. 9.

12a. Using Eq. (15) we have - S = 800(1) or
S (800t), S(0) 100,000. Using an integrating factor and integration by parts (or using a D.E. solver) we get S(t) t c. Using the I.C. yields c . Substituting this value into S, setting S(t) 0, and solving numerically for t yields t 135.363 months.

16a. This problem can be done numerically using appropriate D.E. solver software. Analytically we have
(.1.2sint)dt by separating variables and thus
y(t) cexp(.1t.2cost). y(0) 1 gives c , so
y(t) exp(.2.1t.2cost). Setting y 2 yields
ln2 .2 .1 .2cos, which can be solved numerically to give
2.9632. If y(0) , then as above,
y(t) exp(.2.1t.2cost). Thus if we set y 2 we get the

same numerical equation for and hence the doubling time has not changed.

18. From Eq.(26) we have 19 = 7 + (20-7) or k = -ln = ln(13/12). Hence if (T) = 15 we get: 15 = 7 + 13. Solving for T yields T = ln(8/13)/-ln(13/12) = ln(13/8)/ln(13/12) min.

19. Hint: let Q(t) be the quantity of carbon monoxide in the room at any time t. Then the concentration is given by x(t) = Q(t)/1200.

20a. The required I.V.P. is dQ/dt = kr + P - r, Q(0) = V. Since c = Q(t)/V, the I.V.P. may be rewritten Vc(t) = kr + P - rc, c(0) = , which has the solution c(t) = k + + (- k -).

20b. Set k = 0, P = 0, t = T and c(T) = .5 in the solution found in (a).

21a. If we measure x positively upward from the ground, then Eq.(4) of Section 1.1 becomes m = -mg, since there is no air resistance. Thus the I.V.P. for v(t) is dv/dt = -g, v(0) = 20. Hence = v(t) = 20 - gt and x(t) = 20t - (g/2) + c. Since x(0) = 30, c = 30 and x(t) = 20t - $(g/2)t^2$ + 30. At the maximum height $v(t_m) = 0$ and thus t_m = 20/9.8 = 2.04 sec., which when substituted in the equation for x(t) yields the maximum height.

21b. At the ground $x(t_g) = 0$ and thus $20t_g - 4.9t_g^2 + 30 = 0$.

22. The I.V.P. in this case is $m\frac{dv}{dt} = -\frac{1}{30}v - mg$, v(0) = 20, where the positive direction is measured upward.

24a. The I.V.P. is $m\frac{dv}{dt}$ = mg - .75v, v(0) = 0 and v is measured positively downward. Since m = 180/32, the D.E. becomes $\frac{dv}{dt} = 32 - \frac{2}{15}v$ and thus $v(t) = 240(1-e^{-2t/15})$ so that v(10) = 176.7 ft/sec.

24b. Integration of v(t) as found in (a) yields $x(t) = 240t + 1800(e^{-2t/15}-1)$ where x is measured positively down from the altitude of 5000 feet. Set

t = 10 to find the distance traveled when the parachute opens.

24c. After the parachute opens the I.V.P. is $m\frac{dv}{dt} = mg-12v$, $v(0) = 176.7$, which has the solution $v(t) = 161.7e^{-32t/15} + 15$ and where $t = 0$ now represents the time the parachute opens. Letting $t\to\infty$ yields the limiting velocity.

24d. Integrate v(t) as found in (c) to find $x(t) = 15t - 75.8e^{-32t/15} + C_2$. $C_2 = 75.8$ since $x(0) = 0$, x now being measured from the point where the parachute opens. Setting x = 3925.5 will then yield the length of time the skydiver is in the air after the parachute opens.

26a. Again, if x is measured positively upward, then Eq.(4) of Sect.1.1 becomes $m\frac{dv}{dt} = -mg - kv$.

26b. From part (a) $v(t) = -\frac{mg}{k} + [v_0 + \frac{mg}{k}]e^{-kt/m}$. As $k \to 0$ this has the indeterminant form of $-\infty + \infty$. Thus rewrite v(t) as $v(t) = [-mg + (v_0k + mg)e^{-kt/m}]/k$ which has the indeterminant form of 0/0, as $k \to 0$ and hence L'Hopital's Rule may be applied with k as the variable.

27a. The equation of motion is m(dv/dt) = w-R-B which, in this problem, is $\frac{4}{3}\pi a^3\rho(dv/dt) = \frac{4}{3}\pi a^3\rho g - 6\pi\mu av - \frac{4}{3}\pi a^3\rho' g$. The limiting velocity occurs when dv/dt = 0.

27b. Since the droplet is motionless, v = dv/dt = 0, we have the equation of motion $0 = (\frac{4}{3})\pi a^3\rho g - Ee - (\frac{4}{3})\pi a^3\rho' g$, where ρ is the density of the oil and ρ' is the density of air. Solving for e yields the answer.

28. All three parts can be answered from one solution if k represents the resistance and if the method of solution of Example 4 is used. Thus we have $m\frac{dv}{dt} = mv\frac{dv}{dx} = mg - kv$, $v(0) = 0$, where we have assumed the velocity is a function of x. The solution of this

I.V.P. involves a logarithmic term, and thus the answers to parts (a) and (c) must be found using a numerical procedure.

29b. Note that 32 ft/sec^2 = 78,545 m/hr^2.

30. This problem is the same as Example 4 through Eq.(29). In this case the I.C. is $v(\xi R) = v_o$, so $c = \frac{v_0^2}{2} - \frac{gR}{1+\xi}$. The escape velocity, v_e, is found by noting that $v_o^2 \geq \frac{2gR}{1+\xi}$ in order for v^2 to always be positive. From Example 4, the escape velocity for a surface launch is $v_e(0) = \sqrt{2gR}$. We want the escape velocity of $x_o = \xi R$ to have the relation $v_e(\xi R) = .85v_e(0)$, which yields $\xi = (0.85)^{-2} - 1 \cong 0.384$. If R = 4000 miles then $x_o = \xi R = 1536$ miles.

31b. From part a) $\frac{dx}{dt} = v = u\cos A$ and hence $x(t) = (u\cos A)t + d_1$. Since $x(0) = 0$, we have $d_1 = 0$ and $x(t) = (u\cos A)t$. Likewise $\frac{dy}{dt} = -gt + u\sin A$ and therefore $y(t) = -gt^2/2 + (u\sin A)t + d_2$. Since $y(0) = h$ we have $d_2 = h$ and $y(t) = -gt^2/2 + (u\sin A)t + h$.

31d. Let t_w be the time the ball reaches the wall. Then $x(t_w) = L = (u\cos A)t_w$ and thus $t_w = \frac{L}{u\cos A}$. For the ball to clear the wall $y(t_w) \geq H$ and thus (setting $t_w = \frac{L}{u\cos A}$, g = 32 and h = 3 in y) we get $\frac{-16L^2}{u^2\cos^2 A} + L\tan A + 3 \geq H$.

31e. Setting L = 350 and H = 10 we get $\frac{-161.98}{\cos^2 A} + 350\frac{\sin A}{\cos A} \geq 7$ or $7\cos^2 A - 350\cos A\sin A + 161.98 \leq 0$. This can be solved numerically or by plotting the left side as a function of A and finding where the zero crossings are.

31f. Setting L = 350, and H = 10 in the answer to part d

yields $\frac{-16(350)^2}{u^2\cos^2 A} + 350\tan A = 7$, where we have chosen the equality sign since we want to just clear the wall. Solving for u^2 we get $u^2 = \frac{1{,}960{,}000}{175\sin 2A - 7\cos^2 A}$. Now u will have a minimum when the denominator has a maximum. Thus $350\cos 2A + 7\sin 2A = 0$, or $\tan 2A = -50$, which yields $A = .7954$ rad. and $u = 106.89$ ft./sec.

Section 2.4, Page 72

1. If the equation is written in the form of Eq.(1), then $p(t) = (\ln t)/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0,3)$ and $(3,\infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

4. $p(t) = 2t/(2-t)(2+t)$ and $g(t) = 3t^2/(2-t)(2+t)$.

8. Theorem 2.4.2 guarantees a unique solution to the D.E. through any point (t_0, y_0) such that $t_0^2 + y_0^2 < 1$ since $\frac{\partial f}{\partial y} = -y(1-t^2-y^2)^{1/2}$ is defined and continuous only for $1-t^2-y^2 > 0$. Note also that $f = (1-t^2-y^2)^{1/2}$ is defined and continuous in this region as well as on the boundary $t^2+y^2 = 1$. The boundary can't be included in the final region due to the discontinuity of $\frac{\partial f}{\partial y}$ there.

11. In this case $f = \frac{1+t^2}{y(3-y)}$ and $\frac{\partial f}{\partial y} = \frac{1+t^2}{y(3-y)^2} - \frac{1+t^2}{y^2(3-y)}$, which are both continuous everywhere except for $y = 0$ and $y = 3$.

13. The D.E. may be written as $y\,dy = -4t\,dt$ so that $\frac{y^2}{2} = -2t^2 + c$, or $y^2 = c - 4t^2$. The I.C. then yields $y_0^2 = c$, so that $y^2 = y_0^2 - 4t^2$ or $y = \pm\sqrt{y_0^2 - 4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

17. From the direction field and the given D.E. it is noted that for $t > 0$ and $y < 0$ that $y' < 0$, so $y \to -\infty$ for $y_0 < 0$. Likewise, for $0 < y_0 < 3$, $y' > 0$ and $y' \to 0$ as

$y \to 3$, so $y \to 3$ for $0 < y_0 < 3$ and for $y_0 > 3$, $y' < 0$ and again $y' \to 0$ as $y \to 3$, so $y \to 3$ for $y_0 > 3$. For $y_0 = 3$, $y' = 0$ and $y = 3$ for all t and for $y_0 = 0$, $y' = 0$ and $y = 0$ for all t.

22a. For $y_1 = 1-t$, $y_1' = -1 = \dfrac{-t+[t^2+4(1-t)]^{1/2}}{2}$

$$= \frac{-t+[(t-2)^2]^{1/2}}{2}$$

$$= \frac{-t+|t-2|}{2} = -1 \text{ if}$$

$(t-2) \geq 0$, by the definition of absolute value. Setting $t = 2$ in y_1 we get $y_1(2) = -1$, as required.

22b. By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t,y) = \dfrac{-t+(t^2+4y)^{1/2}}{2}$ and $f_y(t,y) = (t^2+4y)^{-1/2}$ are continuous. In this case the initial point (2,-1) lies in the region $t^2 + 4y \leq 0$, in which case $\dfrac{\partial f}{\partial y}$ is not continuous and hence the theorem is not applicable and there is no contradiction.

22c. If $y = y_2(t)$ then we must have $ct + c^2 = -t^2/4$, which is not possible since c must be a constant.

23a. To show that $\phi(t) = e^{2t}$ is a solution of the D.E., take its derivative and substitute into the D.E.

24. $[c\,\phi(t)]' + p(t)[c\phi(t)] = c[\phi'(t) + p(t)\phi(t)] = 0$ since $\phi(t)$ satisfies the given D.E.

25. $[y_1(t) + y_2(t)]' + p(t)[y_1(t) + y_2(t)] =$
$y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = 0 + g(t)$.

27a. For n = 0,1, the D.E. is linear and Eqs.(3) and (4) apply.

27b. Let $v = y^{1-n}$ then $\dfrac{dv}{dt} = (1-n)y^{-n}\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = \dfrac{1}{1-n}y^n\dfrac{dv}{dt}$, which makes sense when $n \neq 0,1$. Substituting into the D.E. yields $\dfrac{y^n}{1-n}\dfrac{dv}{dt} + p(t)y = q(t)y^n$ or

$v' + (1-n)p(t)y^{1-n} = (1-n)q(t)$. Setting $v = y^{1-n}$ then yields a linear D.E. for v.

28. $n = 3$ so $v = y^{-2}$ and $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ or $\frac{dy}{dt} = -\frac{1}{2}y^3\frac{dv}{dt}$. Substituting this into the D.E. gives $-\frac{1}{2}y^3\frac{dv}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3$. Simplifying and setting $y^{-2} = v$ then gives the linear D.E. $v' - \frac{4}{t}v = -\frac{2}{t^2}$, where $\mu(t) = \frac{1}{t^4}$ and $v(t) = ct^4 + \frac{2}{5t} = \frac{2+5ct^5}{5t}$. Thus $y = \pm[5t/(2+5ct^5)]^{1/2}$.

29. $n = 2$ so $v = y^{-1}$ and $\frac{dv}{dt} = -y^2\frac{dv}{dt}$. Thus the D.E. becomes $-y^2\frac{dv}{dt} - ry = -ky^2$ or $\frac{dv}{dt} + rv = k$. Hence $\mu(t) = e^{rt}$ and $v = k/r + ce^{-rt}$. Setting $v = 1/y$ then yields the solution.

32. Since g(t) is continuous on the interval $0 \le t \le 1$ we may solve the I.V.P. $y_1' + 2y_1 = 1$, $y_1(0) = 0$ on that interval to obtain $y_1 = 1/2 - (1/2)e^{-2t}$, $0 \le t \le 1$. g(t) is also continuous for $1 < t$; and hence we may solve $y_2' + 2y_2 = 0$ to obtain $y_2 = ce^{-2t}$, $1 < t$. The solution y of the original I.V.P. must be continuous (since its derivative must exist) and hence we need c in y_2 so that y_2 at 1 has the same value as y_1 at 1. Thus $ce^{-2} = 1/2 - e^{-2}/2$ or $c = (1/2)(e^2-1)$ and we obtain

$$y = \begin{cases} 1/2 - (1/2)e^{-2t} & 0 \le t \le 1 \\ 1/2(e^2-1)e^{-2t} & 1 \le t \end{cases} \quad \text{and}$$

$$y' = \begin{cases} e^{-2t} & 0 \le t \le 1 \\ (1-e^2)e^{-2t} & 1 < t. \end{cases}$$

Evaluating the two parts of y' at $t_0 = 1$ we see that they are different, and hence y' is not continuous at $t_0 = 1$.

Section 2.5, Page 84

Problems 1 through 13 follow the pattern illustrated in Fig.2.5.3 and the discussion following Eq.(11).

3. The critical points are found by setting $\frac{dy}{dt}$ equal to zero. Thus y = 0,1,2 are the critical points. The graph of y(y-1)(y-2) is positive for O < y < 1 and 2 < y and negative for 1 < y < 2. Thus y(t) is increasing ($\frac{dy}{dt} > 0$) for 0 < y < 1 and 2 < y and decreasing ($\frac{dy}{dt} < 0$) for 1 < y < 2. Therefore 0 and 2 are unstable critical points while 1 is an asymptotically stable critical point.

6. $\frac{dy}{dt}$ is zero only when arctany is zero (ie, y = 0). $\frac{dy}{dt} > 0$ for y < 0 and $\frac{dy}{dt} < 0$ for y > 0. Thus y = 0 is an asymptotically stable critical point.

7c. Separate variables to get $\frac{dy}{(1-y)^2} = kt$. Integration yields $\frac{1}{1-y} = kt + c$, or $y = 1 - \frac{1}{kt + c} = \frac{kt + c - 1}{kt + c}$. Setting t = 0 and y(0) = y_0 yields $y_0 = \frac{c-1}{c}$ or $c = \frac{1}{1-y_0}$. Hence $y(t) = \frac{(1-y_0)kt + y_0}{(1-y_0)kt + 1}$. Note that for $y_0 < 1$ $y \to (1-y_0)k/(1-y_0)k = 1$ as $t \to \infty$. For $y_0 > 1$ notice that the denominator will have a zero for some value of t, depending on the values chosen for y_0 and k. Thus the solution has a discontinuity at that point.

9. Setting $\frac{dy}{dt} = 0$ we find y = 0, ± 1 are the critical points. Since $\frac{dy}{dt} > 0$ for y < -1 and y > 1 while $\frac{dy}{dt} < 0$ for -1 < y < 1 we may conclude that y = -1 is asymptotically stable, y = 0 is semistable, and y = 1 is unstable.

11. $y = b^2/a^2$ and $y = 0$ are the only critical points. For $0 < y < b^2/a^2$, $\frac{dy}{dt} < 0$ and thus $y = 0$ is asymptotically stable. For $y > b^2/a^2$, $dy/dt > 0$ and thus $y = b^2/a^2$ is unstable.

14. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asysmtotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.

16b. By taking the derivative of $y \ln(K/y)$ it can be shown that the graph of $\frac{dy}{dt}$ vs y has a maximum point at $y = K/e$. Thus $\frac{dy}{dt}$ is positive and increasing for $0 < y < K/e$ and thus y(t) is concave up for that interval. Similarly $\frac{dy}{dt}$ is positive and decreasing for $K/e < y < K$ and thus y(t) is concave down for that interval.

16c. $\ln(K/y)$ is very large for small values of y and thus $(ry)\ln(K/y) > ry(1 - y/K)$ for small y. Since $\ln(K/y)$ and $(1 - y/K)$ are both strictly decreasing functions of y and since $\ln(K/y) = (1 - y/K)$ only for $y = K$, we may conclude that $\frac{dy}{dt} = (ry)\ln(K/y)$ is never less than $\frac{dy}{dt} = ry(1 - y/K)$.

17a. If $u = \ln(y/K)$ then $y = Ke^u$ and $\frac{dy}{dt} = Ke^u\frac{du}{dt}$ so that the D.E. becomes $du/dt = -ru$.

18a. The D.E. is $dV/dt = k - \alpha\pi r^2$. The volume of a cone of height L and radius r is given by $V = \pi r^2 L/3$ where $L = hr/a$ from symmetry. Solving for r yields the desired solution.

18b. Equilibrium is given by $k - \alpha\pi r^2 = 0$.

18c. The equilibrium height must be less than h.

20b. Use the results of Problem 14.

20d. Differentiate Y with respect to E.

21a. Set $\frac{dy}{dt} = 0$ and solve for y using the quadratic formula.

21b. Use the results of Problem 14.

21d. If h > rK/4 there are no critical points (see part a) and $\frac{dy}{dt} < 0$ for all t.

24a. If z = x/n then $dz/dt = \frac{1}{n}\frac{dx}{dt} - \frac{x}{n^2}\frac{dn}{dt}$. Use of Equations (i) and (ii) then gives the I.V.P. (iii).

24b. Separate variables to get $\frac{\nu dz}{z(1-\nu z)} = -\beta dt$. Using partial fractions this becomes $\frac{dz}{z} + \frac{dz}{1-z} = -\beta dt$. Integration and solving for z yields the answer.

24c. Find z(20).

26a. Plot dx/dt vs x and observe that x = p and x = q are critical points. Also note that dx/dt > 0 for x < min(p,q) and x > max(p,q) while dx/dt < 0 for x between min(p,q) and max(p,q). Thus x = min(p,q) is an asymptotically stable point while x = max(p,q) is unstable. To solve the D.E., separate variables and use partial fractions to obtain $\frac{1}{q-p}[\frac{dx}{q-x} - \frac{dx}{p-x}] = \alpha dt$. Integration and solving for x yields the solution.

26b. x = p is a semistable critical point and since $\frac{dx}{dt} > 0$, x(t) is an increasing function. Thus for x(0) = 0, x(t) approaches p as $t \to \infty$. To solve the D.E., separate variables and integrate.

Section 2.6, Page 95

3. $M(x,y) = 3x^2-2xy+2$ and $N(x,y) = 6y^2-x^2+3$, so $M_y = -2x = N_x$ and thus the D.E. is exact. Integrating M(x,y) with

respect to x we get $\psi(x,y) = x^3 - x^2y + 2x + H(y)$. Taking the partial derivative of this with respect to y and setting it equal to N(x,y) yields $-x^2+h'(y) = 6y^2-x^2+3$, so that $h'(y) = 6y^2 + 3$ and $h(y) = 2y^3 + 3y$. Substitute this h(y) into $\psi(x,y)$ and recall that the equation which defines y(x) implicitly is $\psi(x,y) = c$. Thus $x^3 - x^2y + 2x + 2y^3 + 3y = c$ is the equation that yields the solution.

5. Writing the equation in the form $M(x,y)dx + N(x,y)dy = 0$ gives $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Now $M_y = b = N_x$ and the equation is exact. Integrating M(x,y) with respect to x yields $\psi(x,y) = (a/2)x^2 + bxy + h(y)$. Differentiating ψ with respect to y (x constant) and setting $\psi_y(x,y) = N(x,y)$ we find that $h'(y) = cy$ and thus $h(y) = (c/2)y^2$. Hence the solution is given by $(a/2)x^2 + bxy + (c/2)y^2 = k$.

7. $M_y(x,y) = e^x\cos y - 2\sin x = N_x(x,y)$ and thus the D.E. is exact. Integrating M(x,y) with respect to x gives $\psi(x,y) = e^x\sin y + 2y\cos x + h(y)$. Finding $\psi_y(x,y)$ from this and setting that equal to N(x,y) yields $h'(y) = 0$ and thus h(y) is a constant. Hence an implicit solution of the D.E. is $e^x\sin y + 2y\cos x = c$. The solution $y = 0$ is also valid since it satisfies the D.E. for all x.

9. If you try to find $\psi(x,y)$ by integrating M(x,y) with respect to x you must integrate by parts. Instead find $\psi(x,y)$ by integrating N(x,y) with respect to y to obtain $\psi(x,y) = e^{xy}\cos 2x - 3y + g(x)$. Now find g(x) by differentiating $\psi(x,y)$ with respect to x and set that equal to M(x,y), which yields $g'(x) = 2x$ or $g(x) = x^2$.

12. As long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2 + y^2)^{3/2}$. This gives the exact equation $xdx + ydy = 0$. The solution to this equation is given implicitly by $x^2 + y^2 = c$. If you apply Theorem 2.6.1 and its construction without the simplification, you get $(x^2 + y^2)^{-1/2} = C$ which can be written as $x^2 + y^2 = c$ under the same assumption required for the simplification.

14. $M_y = 1$ and $N_x = 1$, so the D.E. is exact. Integrating

$M(x,y)$ with respect to x yields $\psi(x,y) = 3x^3 + xy - x + h(y)$. Differentiating this with respect to y and setting $\psi_y(x,y) = N(x,y)$ yields $h'(y) = -4y$ or $h(y) = -2y^2$. Thus the implicit solution is $3x^3 + xy - x - 2y^2 = c$. Setting $x = 1$ and $y = 0$ gives $c = 2$ so that $2y^2 - xy + (2+x-3x^3) = 0$ is the implicit solution satisfying the given I.C. Use the quadratic formula to find $y(x)$, where the negative square root is used in order to satisfy the I.C. The solution will be valid for $24x^3 + x^2 - 8x - 16 > 0$.

15. We want $M_y(x,y) = 2xy + bx^2$ to be equal to $N_x(x,y) = 3x^2 + 2xy$. Thus we must have $b = 3$. This gives $\psi(x,y) = \frac{1}{2}x^2y^2 + x^3y + h(y)$ and consequently $h'(y) = 0$. After multiplying through by 2, the solution is given implicitly by $x^2y^2 + 2x^3y = c$.

19. $M_y(x,y) = 3x^2y^2$ and $N_x(x,y) = 1 + y^2$ so the equation is not exact by Theorem 2.6.1. Multiplying by the integrating factor $\mu(x,y) = 1/xy^3$ we get $x + \frac{(1+y^2)}{y^3}y' = 0$, which is an exact equation since $M_y = N_x = 0$ (it is also separable). In this case $\psi = \frac{1}{2}x^2 + h(y)$ and $h'(y) = y^{-3} + y^{-1}$ so that $x^2 - y^{-2} + 2\ln|y| = c$ gives the solution implicitly.

22. Multiplication of the given D.E. (which is not exact) by $\mu(x,y) = xe^x$ yields $(x^2 + 2x)e^x\sin y\,dx + x^2e^x\cos y\,dy$, which is exact since $M_y(x,y) = N_x(x,y) = (x^2+2x)e^x\cos y$. To solve this exact equation it's easiest to integrate $N(x,y) = x^2e^x\cos y$ with respect to y to get $\psi(x,y) = x^2e^x\sin y + g(x)$. Solving for $g(x)$ yields the implicit solution.

23. This problem is similar to the derivation leading up to Eq.(26). Assuming that μ depends only on y, we find from Eq.(25) that $\mu' = Q\mu$, where $Q = (N_x - M_y)/M$ must depend on y alone. Solving this last D.E. yields $\mu(y)$ as given. This method provides an alternative approach to Problems 27 through 30.

25. The equation is not exact so we must attempt to find an integrating factor. Since $\frac{1}{N}(M_y - N_x) = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$ is a function of x alone there is an integrating factor depending only on x, as shown in Eq.(26). Then $d\mu/dx = 3\mu$, and the integrating factor is $\mu(x) = e^{3x}$. Hence the equation can be solved as in Example 4.

26. An integrating factor can be found which is a function of x only, yielding $\mu(x) = e^{-x}$. Alternatively, you might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

27. Using the results of Problem 23, it can be shown that $\mu(y) = y$ is an integrating factor. Thus multiplying the D.E. by y gives $ydx + (x - ysiny)dy = 0$, which can be identified as an exact equation. Alternatively, one can rewrite the last equation as $(ydx + xdy) - ysiny\, dy = 0$. The first term is d(xy) and the last can be integrated by parts. Thus we have $xy + ycosy - siny = c$.

29. Multiplying by siny we obtain $e^x siny\, dx + e^x cosy\, dy + 2y\, dy = 0$, and the first two terms are just $d(e^x siny)$. Thus, $e^x siny + y^2 = c$.

31. Using the results of Problem 24, it can be shown that $\mu(xy) = xy$ is an integrating factor. Thus, multiplying by xy we have $(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0$, which can be identified as an exact equation. Alternatively, we can observe that the above equation can be written as $d(x^3y) + d(3x^2) + d(y^3) = 0$, so that $x^3y + 3x^2 + y^3 = c$.

Section 2.7, Page 103

1d. The exact solution to this I.V.P. is $y = \phi(t) = t + 2 - e^{-t}$.

3a. The Euler formula is $y_{n+1} = y_n + h(2y_n - t_n + 1/2)$ for $n = 0,1,2,3$ and with $t_0 = 0$ and $y_0 = 1$. Thus
$y_1 = y_0 + .1(2y_0 - t_0 + 1/2) = 1.25$,
$y_2 = 1.25 + .1[2(1.25) - (.1) + 1/2] = 1.54$,
$y_3 = 1.54 + .1[2(1.54) - (.2) + 1/2] = 1.878$, and
$y_4 = 1.878 + .1[2(1.878) - (.3) + 1/2] = 2.2736$.

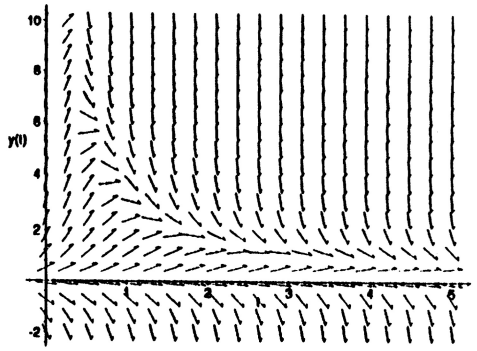
3b. Use the same formula as in Problem 3a, except now h = .05 and n = 0,1...7. Notice that only results for n = 1,3,5 and 7 are needed to compare with part a.

3c. Again, use the same formula as above with h = .025 and n = 0,1...15. Notice that only results for n = 3,7,11 and 15 are needed to compare with parts a and b.

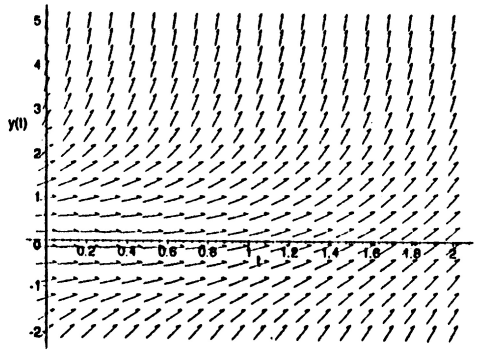
3d. $y' = 1/2 - t + 2y$ is a first order linear D.E. Rewrite the equation in the form $y' - 2y = 1/2 - t$ and multiply both sides by the integrating factor e^{-2t} to obtain $(e^{-2t}y)' = (1/2 - t)e^{-2t}$. Integrating the right side by parts and multiplying by e^{2t} we obtain $y = ce^{-2t} + t/2$. The I.C. $y(0) = 1 \rightarrow c = 1$ and hence the solution of the I.V.P. is $y = \phi(x) = e^{2t} + t/2$. Thus $\phi(0.1) = 1.2714$, $\phi(0.2) = 1.59182$, $\phi(0.3) = 1.97212$, and $\phi(0.4) = 2.42554$.

4d. The exact solution to this I.V.P. is $y = \phi(t) = (6\cos t + 3\sin t - 6e^{-2t})/5$.

6. For $y(0) > 0$ the solutions appear to converge. For $y(0)<0$ the solutions diverge.



9. All solutions seem to diverge.



13a. The Euler formula is

$y_{n+1} = y_n + h(\frac{4-t_n y_n}{1 + y_n^2})$, where $t_0 = 0$ and $y_0 = y(0) = -2$. Thus, for h = .1, we get

$y_1 = -2 + .1(4/5) = -1.92$

$y_2 = -1.92 + .1(\frac{4-.1(-1.92)}{1 + (1.92)^2}) = -1.83055$

$y_3 = -1.83055 + .1(\frac{4-.2(-1.83055)}{1+(1.83055)^2}) = -1.7302$

$y_4 = -1.7302 + .1(\frac{4-.3(-1.7302)}{1 + (1.7302)^2}) = -1.617043$

$y_5 = -1.617043 + .1(\frac{4 - .4(-1.617043)}{1 + (1.617043)^2}) = -1.488494.$

Thus, $y(.5) \cong -1.488494$.

15a. The Euler formula is

$y_{n+1} = y_n + .1\ (\frac{3t_n^2}{3y_n^2 - 4})$, where $t_0 = 1$ and $y_0 = 0$. Thus

$y_1 = 0 + .1\ (\frac{3}{-4}) = -.075$ and

$y_2 = -.075 + .1\ (\frac{3(1.1)^2}{3(.075)^2 - 4}) = .166134.$

15c. There are two factors that explain the large differences. From the limit, the slope of y, y′, becomes very “large” for values of y near -1.155. Also, the slope changes sign at y = -1.155. Thus for part a, $y(1.7) \cong y_7 = -1.178$, which is close to -1.155 and the slope y′ here is large and positive, creating the large change in $y_8 \cong y(1.8)$. For part b, $y(1.65) \cong -1.125$, resulting in a large negative slope, which yields $y(1.70) \cong -3.133$. The slope at this point is now positive and the remainder of the solutions “grow” to -3.098 for the approcimation to y(1.8).

16. For the four step sizes given, the approximte values for y(.8) are 3.5078, 4.2013, 4.8004 and 5.3428. Thus, since these changes are still rather “large”, it is hard to give an estimate other than y(.8) is at least 5.3428. By using h = .005, .0025 and .001, we find further approximate values of y(.8) to be 5.576, 5.707 and 5.790. Thus a better estimate now is for y(.8) to be between 5.8 and 6. No reliable estimate is obtainable for y(1), which is consistent with the direction field of Prob.9.

18. It is helpful, in understanding this problem, to also calculate $y'(t_n) = y_n(.1\,y_n^2 - t_n)$. For $\alpha = 2.38$ this term remains positive and grows very large for $t_n > 2$. On the other hand, for $\alpha = 2.37$ this term decreases and eventually becomes negative for $t_n \cong 1.6$ (for $h = .01$). For $\alpha = 2.37$ and $h = .1$, .05 and .01, $y(2.00)$ has the approximations of 4.48, 4.01 and 3.50 respectively. A small step size must be used, due to the sensitivety of the slope field, given by $y_n\,(.1y_n^2 - t_n)$.

22. Using Eq.(8) we have $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$. Setting $n + 1 = k$ (and hence $n = k-1$) this becomes $y_k = (1 + 2h)y_{k-1} - h$, for $k = 1,2,\ldots$. Since $y_0 = 1$, we have $y_1 = 1 + 2h - h = 1 + h = (1 + 2h)/2 + 1/2$, and hence $y_2 = (1 + 2h)y_1 - h = (1 + 2h)^2/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^2/2 + 1/2$;
$y_3 = (1 + 2h)y_2 - h = (1 + 2h)^3/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^3/2 + 1/2$. Continuing in this fashion (or using induction) we obtain $y_k = (1 + 2h)^k/2 + 1/2$. For fixed $x > 0$ choose $h = x/k$. Then substitute for h in the last formula to obtain $y_k = (1 + 2x/k)^k/2 + 1/2$. Letting $k \to \infty$ we find (See hint for Problem 20d.)
$y(x) = y_k \to e^{2x}/2 + 1/2$, which is the exact solution.

Section 2.8, Page 113

1. Let $s = t-1$ and $w(s) = y(t(s)) - 2$, then when $t = 1$ and $y = 2$ we have $s = 0$ and $w(0) = 0$. Also,
$\frac{dw}{ds} = \frac{dw}{dt}\cdot\frac{dt}{ds} = \frac{d}{dt}(y-2)\frac{dt}{ds} = \frac{dy}{dt}$ and hence
$\frac{dw}{ds} = (s+1)^2 + (w+2)^2$, upon substitution into the given D.E.

4a. Following Ex. 1 of the text, from Eq.(7) we have $\phi_{n+1}(t) = \int_0^t f(s,\phi(s))ds$, where $f(t,\phi) = -1 - \phi$ for this problem. Thus if $\phi_0(t) = 0$, then $\phi_1(t) = -\int_0^t ds = -t$;
$\phi_2(t) = -\int_0^t (1-s)ds = -t + \frac{t^2}{2}$;

$\phi_3(t) = -\int_0^t (1-s + \frac{s^2}{2})ds = -t + \frac{t^2}{2} - \frac{t^3}{2\cdot 3}$;

$\phi_4(t) = -\int_0^t (1 - s + \frac{s^2}{2} - \frac{s^3}{3!})ds = -t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!}$.

Based upon these we hypothesize that: $\phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^k t^k}{k!}$ and use mathematical induction to verify this form for $\phi_n(t)$. Using Eq.(7) again we have:

$\phi_{n+1}(t) = -\int_0^t [1 + \phi_n(s)]\, ds = -t - \sum_{k=1}^{n} \frac{(-1)^k t^{k+1}}{(k+1)!}$

$= \sum_{k=0}^{n} \frac{(-1)^{k+1} t^{k+1}}{(k+1)!} = \sum_{i=1}^{n+1} \frac{(-1)^i t^i}{i!}$, where i = k+1. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

4c. From part a, let $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{k!}$

$= -t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots$.

Since this is a power series, recall from calculus that: $e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \ldots$. If we let a = -1, then we have $e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots = 1 + \phi(t)$.

Hence $\phi(t) = e^{-t} - 1$.

7. As in Prob.4,

$\phi_1(t) = \int_0^t (s\phi_0(s) + 1)ds = s\big|_0^t = t$

$\phi_2(t) = \int_0^t (s^2+1)ds = (\frac{s^3}{3} + s)\big|_0^t = t + \frac{t^3}{3}$

$\phi_3(t) = \int_0^t (s^2 + \frac{s^4}{3} + 1)ds = (\frac{s^3}{3} + \frac{s^5}{3\cdot 5} + s)\big|_0^t = t + \frac{t^3}{3} + \frac{t^5}{3\cdot 5}$.

Based upon these we hypothesize that:

$\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k-1}}{1\cdot3\cdot5\cdots(2k-1)}$ and use mathematical induction to verify this form for $\phi_n(t)$. Using Eq.(7) again we have:

$$\phi_{n+1}(t) = \int_0^t \left(\sum_{k=1}^{n} \frac{s^{2k}}{1\cdot3\cdot5\cdots(2k-1)} + 1 \right) ds$$

$$= \sum_{k=1}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)} + t$$

$$= \sum_{k=0}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)}$$

$$= \sum_{i=1}^{n+1} \frac{t^{2i-1}}{1\cdot3\cdot5\cdots(2i-1)},$$ where i = k+1. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

11. Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)$. Thus, for $\phi_2(t) = t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + O(t^7)$ we have

$$\sin[\phi_2(t)] = \left(t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) - \frac{\left(t - \frac{t^2}{2!}\right)^3}{3!} + \frac{t^5}{5!} + O(t^7).$$

Section 2.9, Page 124

2. Using the given difference equation we have for n=0, $y_1 = y_0/2$; for n=1, $y_2 = 2y_1/3 = y_0/3$; and for n=2, $y_3 = 3y_2/4 = y_0/4$. Thus we guess that $y_n = y_0/(n+1)$, and the given equation then gives $y_{n+1} = \frac{n+1}{n+2} y_n = y_0/(n+2)$, which, by mathematical induction, verifies $y_n = y_0/(n+1)$ as the solution for all n.

5. From the given equation we have $y_1 = .5y_0+6$.

$y_2 = .5y_1 + 6 = (.5)^2 y_0 + 6(1 + \frac{1}{2})$ and

$y_3 = .5y_2 + 6 = (.5)^3 y_0 + 6(1 + \frac{1}{2} + \frac{1}{4})$. In general, then

$y_n = (.5)^n y_0 + 6(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}})$

$= (.5)^n y_0 + 6(\frac{1 - (1/2)^n}{1 - 1/2})$

$= (.5)^n y_0 + 12 - (.5)^n 12$

$= (.5)^n (y_0 - 12) + 12$. Mathematical induction can now be used to prove that this is the correct solution.

10. The governing equation is $y_{n+1} = \rho y_n - b$, which has the solution $y_n = \rho^n y_0 - \frac{1-\rho^n}{1-\rho} b$ (Eq.(14) with a negative b). Setting $y_{360} = 0$ and solving for b we obtain

$b = \frac{(1-\rho)\rho^{360} y_0}{1-\rho^{360}}$, where $\rho = 1.0075$ for part a.

13. You must solve Eq.(14) numerically for ρ when $n = 240$, $y_{240} = 0$, $b = -\$900$ and $y_0 = \$95{,}000$.

14. Substituting $u_n = \frac{\rho-1}{\rho} + v_n$ into Eq.(21) we get

$\frac{\rho-1}{\rho} + v_{n+1} = \rho(\frac{\rho-1}{\rho} + v_n)(1 - \frac{\rho-1}{\rho} - v_n)$ or

$v_{n+1} = -\frac{\rho-1}{\rho} + (\rho - 1 + \rho v_n)(\frac{1}{\rho} - v_n)$

$= \frac{1-\rho}{\rho} + \frac{\rho-1}{\rho} - (\rho-1)v_n + v_n - \rho v_n^2 = (2-\rho)v_n - \rho v_n^2$.

15a. For $u_0 = .2$ we have $u_1 = 3.2u_0(1-u_0) = .512$ and $u_2 = 3.2u_1(1-u_1) = .7995392$. Likewise $u_3 = .51288406$, $u_4 = .7994688$, $u_5 = .51301899$, $u_6 = .7994576$ and $u_7 = .5130404$. Continuing in this fashion, $u_{14} = u_{16} = .79945549$ and $u_{15} = u_{17} = .51304451$.

17. For both parts of this problem a computer spreadsheet was used and an initial value of $u_0 = .2$ was chosen. Different initial values or different computer programs may need a slightly different number of iterations to reach the limiting value.

17a. The limiting value of .65517 (to 5 decimal places) is reached after approximately 100 iterations for $\rho = 2.9$. The limiting value of .66102 (to 5 decimal places) is reached after approximately 200 iterations for $\rho = 2.95$. The limiting value of .66555 (to 5 decimal places) is reached after approximately 910 iterations for $\rho = 2.99$.

17b. The solution oscillates between .63285 and .69938 after approximately 400 iterations for $\rho = 3.01$. The solution oscillates between.59016 and .73770 after approximately 130 iterations for $\rho = 3.05$. The solution oscillates between .55801 and .76457 after approximately 30 iterations for $\rho = 3.1$. For each of these cases additional iterations verified the oscillations were correct to five decimal places.

18. For an initial value of .2 and $\rho = 3.448$ we have the solution oscillating between .4403086 and .8497146. After approximately 3570 iterations the eighth decimal place is still not fixed, though. For the same initial value and $\rho = 3.45$ the solution oscillates between the four values: .43399155, .84746795, .44596778 and .85242779 after 3700 iterations.. For $\rho = 3.449$, the solution is still varying in the fourth decimal place after 3570 iterations, but there appear to be four values.

Miscellaneous Problems, Page 126

Before trying to find the solution of a D.E. it is necessary to know its type. The student should first classify the D.E. before reading this section, which indentifies the type of each equation in Problems 1 through 32.

1. Linear
2. Homogeneous
3. Exact
4. Linear equation in x(y)
5. Exact
6. Linear
7. Letting $u = x^2$ yields $\frac{dy}{dx} = 2x\frac{dy}{du}$ and thus $\frac{du}{dy} - 2yu = 2y^3$ which is linear in u(y).

8. Linear
9. Exact
10. Integrating factor depends on x only
11. Exact
12. Linear
13. Homogeneous
14. Exact or homogeneous
15. Separable
16. Homogeneous
17. Linear
18. Linear or homogeneous
19. Integrating factor depends on x only
20. Separable
21. Homogeneous
22. Separable
23. Bernoulli equation
24. Separable
25. Exact
26. Integrating factor depends on x only
27. Integrating factor depends on x only
28. Exact
29. Homogeneous
30. Linear equation in x(y)
31. Separable
32. Integrating factor depends on y only.

CHAPTER 3

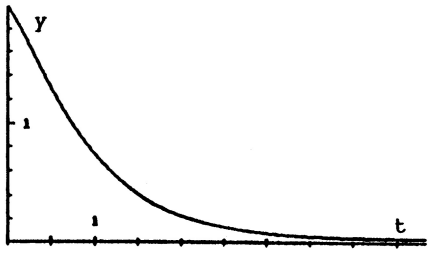
Section 3.1, Page 136

3. Assume $y = e^{rt}$, which gives $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substitution into the D.E. yields $(6r^2-r-1)e^{rt} = 0$. Since $e^{rt} \neq 0$, we have the characteristic equation $6r^2-r-1 = 0$, or $(3r+1)(2r-1) = 0$. Thus $r = -1/3, 1/2$ and $y = c_1e^{t/2} + c_2e^{-t/3}$.

5. The characteristic equation is $r^2 + 5r = 0$, so the roots are $r_1 = 0$, and $r_2 = -5$. Thus $y = c_1e^{0t} + c_2e^{-5t} = c_1 + c_2e^{-5t}$.

7. The characteristic equation is $r^2 - 9r + 9 = 0$ so that $r = (9 \pm \sqrt{81-36})/2 = (9 \pm 3\sqrt{5})/2$ using the quadratic formula. Hence $y = c_1\exp[(9+3\sqrt{5})t/2] + c_2\exp[(9-3\sqrt{5})t/2]$.

10. Substituting $y = e^{rt}$ in the D.E. we obtain the characteristic equation $r^2 + 4r + 3 = 0$, which has the roots $r_1 = -1$, $r_2 = -3$. Thus $y = c_1e^{-t} + c_2e^{-3t}$ and $y' = -c_1e^{-t} - 3c_2e^{-3t}$. Substituting $t = 0$ we then have $c_1 + c_2 = 2$ and $-c_1 - 3c_2 = -1$, yielding $c_1 = 5/2$ and $c_2 = -1/2$. Thus $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$ and hence $y \to 0$ as $t \to \infty$.



15. The characteristic equation is $r^2 + 8r - 9 = 0$, so that $r_1 = 1$ and $r_2 = -9$ and the general solution is $y = c_1e^{t} + c_2e^{-9t}$. Since the I.C. are given at $t = 1$, it is convenient to write the general solution in the form $y = k_1e^{(t-1)} + k_2e^{-9(t-1)}$. Note that $c_1 = k_1e^{-1}$ and $c_2 = k_2e^{9}$. The advantage of the latter form of the general solution becomes clear when we apply

the I.C. $y(1) = 1$ and $y'(1) = 0$. This latter form of y gives $y' = k_1 e^{(t-1)} - 9k_2 e^{-9(t-1)}$ and thus setting $t = 1$ in y and y′ yields the equations $k_1 + k_2 = 1$ and $k_1 - 9k_2 = 0$. Solving for k_1 and k_2 we find that $y = (9e^{(t-1)} + e^{-9(t-1)})/10$. Since $e^{(t-1)}$ has a positive exponent for $t > 1$, $y \to \infty$ as $t \to \infty$.

17. Comparing the given solution to Eq(17), we see that $r_1 = 2$ and $r_2 = -3$ are the two roots of the characteristic equation. Thus we have $(r-2)(r+3) = 0$, or $r^2 + r - 6 = 0$ as the characteristic equation. Hence the given solution is for the D.E. $y'' + y' - 6y = 0$.

19. The roots of the characteristic equation are $r = 1, -1$ and thus the general solution is $y(t) = c_1 e^{t} + c_2 e^{-t}$. $y(0) = c_1 + c_2 = \frac{5}{4}$ and $y'(0) = c_1 - c_2 = -\frac{3}{4}$, yielding $y(t) = \frac{1}{4}e^{t} + e^{-t}$. From this $y'(t) = \frac{1}{4}e^{t} - e^{-t} = 0$ or $e^{2t} = 4$ or $t = \ln 2$. The second derivative test or a graph of the solution indicates this is a minimum point.

21. The general solution is $y = c_1 e^{-t} + c_2 e^{2t}$. Using the I.C. we obtain $c_1 + c_2 = \alpha$ and $-c_1 + 2c_2 = 2$, so adding the two equations we find $3c_2 = \alpha + 2$. If y is to approach zero as $t \to \infty$, c_2 must be zero. Thus $\alpha = -2$.

24. The roots of the characteristic equation are given by $r = -2, \alpha - 1$ and thus $y(t) = c_1 e^{-2t} + c_2 e^{(\alpha-1)t}$. Hence, for $\alpha < 1$, all solutions tend to zero as $t \to \infty$. For $\alpha > 1$, the second term becomes unbounded, but not the first, so there are no values of α for which all solutions become unbounded.

25a. The characteristic equation is $2r^2 + 3r - 2 = 0$, so $r_1 = -2$ and $r_2 = 1/2$ and $y = c_1 e^{-2t} + c_2 e^{t/2}$. The I.C. yield $c_1 + c_2 = 2$ and $-2c_1 + \frac{1}{2}c_2 = -\beta$ so that $c_1 = (1 + 2\beta)/5$ and $c_2 = (4-2\beta)/5$.

25c. From part (a), if $\beta = 2$ then $y(t) = e^{-2t}$ and the solution simply decays to zero. For $\beta > 2$, the solution becomes unbounded negatively, and again there is no minimum point.

27. The second solution must decay faster than e^{-t}, so choose e^{-2t} or e^{-3t} etc. as the second solution. Then proceed as in Problem 17.

28. Let $v = y'$, then $v' = y''$ and thus the D.E. becomes $t^2v' + 2tv - 1 = 0$ or $t^2v' + 2tv = 1$. The left side is recognized as $(t^2v)'$ and thus we may integrate to obtain $t^2v = t + c$ (otherwise, divide both sides of the D.E. by t^2 and find the integrating factor, which is just t^2 in this case). Solving for $v = dy/dt$ we find $dy/dt = 1/t + c/t^2$ so that $y = \ln t + c_1/t + c_2$.

30. Set $v = y'$, then $v' = y''$ and thus the D.E. becomes $v' + tv^2 = 0$. This equation is separable and has the solution $-v^{-1} + t^2/2 = c$ or $v = y' = -2/(c_1 - t^2)$ where $c_1 = 2c$. We must consider separately the cases $c_1 = 0$, $c_1 > 0$ and $c_1 < 0$. If $c_1 = 0$, then $y' = 2/t^2$ or $y = -2/t + c_2$. If $c_1 > 0$, let $c_1 = k^2$. Then $y' = -2/(k^2-t^2) = -(1/k)[1/(k-t) + 1/(k+t)]$, so that $y = (1/k)\ln|(k-t)/(k+t)|+c_2$. If $c_1 < 0$, let $c_1 = -k^2$. Then $y' = 2/(k^2+ t^2)$ so that $y = (2/k)\tan^{-1}(t/k) + c_2$. Finally, we note that $y =$ constant is also a solution of the D.E.

34. Following the procedure outlined, let $v = dy/dt$ and $y'' = dv/dt = v\,dv/dy$. Thus the D.E. becomes $yvdv/dy + v^2 = 0$, which is a separable equation with the solution $v = c_1/y$. Next let $v = dy/dt = c/y$, which again separates to give the solution $y^2 = c_1t + c_2$.

37. Again let $v = y'$ and $v' = vdv/dy$ to obtain $2y^2v\,dv/dy + 2yv^2 = 1$. This is an exact equation with solution $v = \pm y^{-1}(y + c_1)^{1/2}$. To solve this equation,

we write it in the form $\pm\, ydy/(y+c_1)^{1/2} = dt$. On observing that the left side of the equation can be written as $\pm[(y+c_1) - c_1]dy/(y+c_1)^{1/2}$ we integrate and find $\pm\,(2/3)(y-2c_1)(y+c_1)^{1/2} = t + c_2$.

39. If $v = y'$, then $v' = vdv/dy$ and the D.E. becomes $v\,dv/dy + v^2 = 2e^{-y}$. Dividing by v we obtain $dv/dy + v = 2v^{-1}e^{-y}$, which is a Bernoulli equation (see Prob.27, Section 2.4). Let $w(y) = v^2$, then $dw/dy = 2v\,dv/dy$ and the D.E. then becomes $dw/dy + 2w = 4e^{-y}$, which is linear in w. Its solution is $w = v^2 = ce^{-2y} + 4e^{-y}$. Setting $v = dy/dt$, we obtain a separable equation in y and t, which is solved to yield the solution.

40. Since both t and y are missing, either approach used above will work. In this case it's easier to use the approach of Problems 28-33, so let $v = y'$ and thus $v' = y''$ and the D.E. becomes $vdv/dt = 2$.

43. The variable y is missing. Let $v = y'$, then $v' = y''$ and the D.E. becomes $vv' - t = 0$. The solution of the separable equation is $v^2 = t^2 + c_1$. Substituting $v = y'$ and applying the I.C. $y'(1) = 1$, we obtain $y' = t$. The positive square root was chosen because $y' > 0$ at $t = 1$. Solving this last equation and applying the I.C. $y(1) = 2$, we obtain $y = t^2/2 + 3/2$.

Section 3.2, Page 145

2. $W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$

4. $W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x.$

8. Dividing by (t-1) we have $p(t) = -3t/(t-1)$, $q(t) = 4/(t-1)$ and $g(t) = \sin t/(t-1)$, so the only point of discontinuity is $t = 1$. By Theorem 3.2.1, the largest interval is $-\infty < t < 1$, since the initial point is $t_0 = -2$.

12. $p(x) = 1/(x-2)$ and $q(x) = \tan x$, so $x = \pi/2, 2, 3\pi/2...$ are points of discontinuity. Since $t_0 = 3$, the interval specified by Theorem 3.2.1 is $2 < x < 3\pi/2$.

14. For $y = t^{1/2}$, $y' = \frac{1}{2}t^{-1/2}$ and $y'' = -\frac{1}{4}t^{-3/2}$. Thus $yy'' + (y')^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$. Similarly $y = 1$ is also a solution. If $y = c_1(1) + c_2t^{1/2}$ is substituted in the D.E. you will get $-c_1c_2/4t^{3/2}$, which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus the linear combination of two solutions is not, in general, a solution. Theorem 3.2.2 is not contradicted however, since the D.E. is not linear.

15. $y = \phi(t)$ is a solution of the D.E. so $L[\phi](t) = g(t)$. Since L is a linear operator, $L[c\phi](t) = cL[\phi](t) = cg(t)$. But, since $g(t) \neq 0$, $cg(t) = g(t)$ if and only if $c = 1$. This is not a contradiction of Theorem 3.2.2 since the linear D.E. is not homogeneous.

18. $W(f,g) = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g = t^2e^t$, or $g' - \frac{1}{t}g = te^t$. This has an integrating factor of $\frac{1}{t}$ and thus $\frac{1}{t}g' - \frac{1}{t^2}g = e^t$ or $(\frac{1}{t}g)' = e^t$. Integrating and multiplying by t we obtain $g(t) = te^t + ct$.

21. From Section 3.1, e^t and e^{-2t} are two solutions, and since $W(e^t,e^{-2t}) \neq 0$ they form a fundamental set of solutions. To find the fundamental set specified by Theorem 3.2.5, let $y(t) = c_1e^t + c_2e^{-2t}$, where c_1 and c_2 satisfy
$c_1 + c_2 = 1$ and $c_1 - 2c_2 = 0$ for y_1. Solving, we find $y_1 = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$. Likewise, c_1 and c_2 satisfy
$c_1 + c_2 = 0$ and $c_1 - 2c_2 = 1$ for y_2, so that
$y_2 = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$.

25. For $y_1 = x$, we have $x^2(0) - x(x+2)(1) + (x+2)(x) = 0$ and for $y_2 = xe^x$ we have $x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)xe^x = 0$. From Problem 4, $W(x, xe^x) = x^2e^x \neq 0$ for $x > 0$, so y_1 and y_2 form a fundamental set of solutions.

27. Suppose that $P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]'$. On expanding the right side and equating coefficients, we find $f'(x) = R(x)$ and $P'(x) + f(x) = Q(x)$. These two conditions on f can be satisfied if $R(x) = Q'(x) - P''(x)$ which gives the necessary condition $P''(x) - Q'(x) + R(x) = 0$.

30. We have $P(x) = x$, $Q(x) = -\cos x$, and $R(x) = \sin x$ and the condition for exactness is satisfied. Also, from Problem 27, $f(x) = Q(x) - P'(x) = -\cos x - 1$, so the D.E. becomes $(xy')' - [(1 + \cos x)y]' = 0$. Hence $xy' - (1 + \cos x)y = c_1$. This is a first order linear D.E. and the integrating factor (after dividing by x) is $\mu(x) = \exp[-\int x^{-1}(1 + \cos x)dx]$. The general solution is $y = [\mu(x)]^{-1}[c_1\int_{x_0}^{x} t^{-1}\,\mu(t)dt + c_2]$.

32. We want to choose $\mu(x)$ and $f(x)$ so that $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [f(x)y]'$. Expand the right side and equate coefficients of y'', y' and y. This gives $\mu'(x)P(x) + \mu(x)P'(x) + f(x) = \mu(x)Q(x)$ and $f'(x) = \mu(x)R(x)$. Differentiate the first equation and then eliminate $f'(x)$ to obtain the adjoint equation $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$.

34. $P = 1-x^2$, $Q = -2x$ and $R = \alpha(\alpha+1)$. Thus $2P' - Q = -4x + 2x = -2x = Q$ and $P'' - Q' + R = -2 + 2 + \alpha(\alpha+1) = \alpha(\alpha+1) = R$.

36. Write the adjoint D.E. given in Problem 32 as $\hat{P}\mu'' + \hat{Q}\mu' + \hat{R}\mu = 0$ where $\hat{P} = P$, $\hat{Q} = 2P' - Q$, and $\hat{R} = P'' - Q' + R$. The adjoint of this equation, namely the adjoint of the adjoint, is $\hat{P}y'' + (2\hat{P}' - \hat{Q})y' + (\hat{P}'' - \hat{Q}' + \hat{R})y = 0$. After substituting for $\hat{P}$, $\hat{Q}$, and $\hat{R}$ and simplifying, we obtain $Py'' + Qy' + Ry = 0$. This is the same as the original equation.

37. From Problem 32 the adjoint of $Py'' + Qy' + Ry = 0$ is $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$. The two equations are the same if $2P' - Q = Q$ and $P'' - Q' + R = R$. This will be true if $P' = Q$. Hence the original D.E. is self-adjoint if $P' = Q$. For Problem 33, $P(x) = x^2$ so $P'(x) = 2x$ and $Q(x) = x$. Hence the Bessel equation of order ν is not self-adjoint. In a similar manner we find that Problems 34 and 35 are self-adjoint.

Section 3.3, Page 152

2. Since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ we have $\cos 3\theta - (4\cos^3\theta - 3\cos\theta) = 0$ for all θ. From Eq.(1) we have $k_1 = 1$ and $k_2 = -1$ and thus $\cos 3\theta$ and $4\cos^3\theta - 3\cos\theta$ are linearly dependent.

6. $W(t, t^{-1}) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -2/t \neq 0.$

7. For $t>0$ $g(t) = t$ and hence $f(t) - 3g(t) = 0$ for all t. Therefore f and g are linearly dependent on $0<t$. For $t<0$ $g(t) = -t$ and $f(t) + 3g(t) = 0$, so again f and g are linearly dependent on $t<0$. For any interval that includes the origin, such as $-1<t<2$, there is no c for which $f(t) + cg(t) = 0$ for all t, and hence f and g are linearly independent on this interval.

12. The D.E. is linear and homogeneous. Hence, if y_1 and y_2 are solutions, then $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are solutions. $W(y_3, y_4) = y_3y_4' - y_3'y_4 = (y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2) = -2(y_1y_2' - y_1'y_2) = -2W(y_1, y_2)$, is not zero since y_1 and y_2 are linearly independent solutions. Hence y_3 and y_4 form a fundamental set of solutions. Conversely, solving the first two equations for y_1 and y_2, we have $y_1 = (y_3+y_4)/2$ and $y_2 = (y_3-y_4)/2$, so y_1 and y_2 are solutions. Finally, from above we have $W(y_1y_2) = -W(y_3, y_4)/2$.

15. Writing the D.E. in the form of Eq.(7), we have $p(t) = -(t+2)/t$. Thus Eq.(8) yields

$W(t) = c\exp[-\int \frac{-(t+2)}{t}\,dt] = ct^2e^t.$

20. From Eq.(8) we have $W(y_1,y_2) = c\exp[-\int p(t)dt]$, where $p(t) = 2/t$ from the D.E. Thus $W(y_1,y_2) = c/t^2$. Since $W(y_1,y_2)(1) = 2$ we find $c = 2$ and thus $W(y_1,y_2)(5) = 2/25$.

24. Let c be the point in I at which both y_1 and y_2 vanish. Then $W(y_1,y_2)(c) = y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0$. Hence, by Theorem 3.3.3 the functions y_1 and y_2 cannot form a fundamental set.

26. Suppose that y_1 and y_2 have a point of inflection at t_0 and either $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Since $y_1''(t_0) = 0$ and $y_2''(t_0) = 0$ it follows from the D.E. that $p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0$ and $p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0$. If $p(t_0) = 0$ and $q(t_0) \neq 0$ then $y_1(t_0) = y_2(t_0) = 0$, and $W(y_1,y_2)(t_0) = 0$ so the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$ then $y_1'(t_0) = y_2'(t_0) = 0$ and $W(y_1,y_2)(t_0) = 0$, so again the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$ then $y_1'(t_0) = q(t_0)y_1(t_0)/p(t_0)$ and $y_2'(t_0) = q(t_0)y_2(t_0)/p(t_0)$ and thus

$$\begin{aligned} W(y_1,y_2)(t_0) &= y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \\ &= y_1(t_0)[q(t_0)y_2(t_0)/p(t_0)] - \\ &\quad [q(t_0)y_1(t_0)/p(t_0)]y_2(t_0) \\ &= 0. \end{aligned}$$

27. Let $-1 < t_0, t_1 < 1$ and $t_0 \neq t_1$. If $y_1 = t$ and $y_2 = t^2$ are linearly dependent then $c_1t_1 + c_2t_1^2 = 0$ and $c_1t_0 + c_2t_0^2 = 0$ have a solution for c_1 and c_2 such that c_1 and c_2 are not both zero. But this system of equations has a non-zero solution only if $t_1 = 0$ or $t_0 = 0$ or $t_1 = t_0$. Hence, the only set c_1 and c_2 that satisfies the system for every choice of t_0 and t_1 in $-1 < t < 1$ is $c_1 = c_2 = 0$. Therefore t and t^2 are linearly independent

on $-1 < t < 1$. Next, $W(t,t^2) = t^2$ clearly vanishes at $t = 0$. Since $W(t,t^2)$ vanishes at $t = 0$, but t and t^2 are linearly independent on $-1 < t < 1$, it follows that t and t^2 cannot be solutions of Eq.(7) on $-1 < t < 1$. To show that the functions $y_1 = t$ and $y_2 = t^2$ are solutions of $t^2y'' - 2ty' + 2y = 0$, substitute each of them in the equation. Clearly, they are solutions. There is no contradiction to Theorem 3.3.3 since $p(t) = -2/t$ and $q(t) = 2/t^2$ are discontinuous at $t = 0$, and hence the theorem does not apply on the interval $-1 < t < 1$.

28. On $0 < t < 1$, $f(t) = t^3$ and $g(t) = t^3$. Hence there are nonzero constants, $c_1 = 1$ and $c_2 = -1$, such that $c_1f(t) + c_2g(t) = 0$ for each t in $(0,1)$. On $-1 < t < 0$, $f(t) = -t^3$ and $g(t) = t^3$; thus $c_1 = c_2 = 1$ defines constants such that $c_1f(t) + c_2g(t) = 0$ for each t in $(-1,0)$. Thus f and g are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$. We will show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$ by demonstrating that it is impossible to find constants c_1 and c_2, not both zero, such that $c_1f(t) + c_2g(t) = 0$ for all t in $(-1,1)$. Assume that there are two such nonzero constants and choose two points t_0 and t_1 in $-1 < t < 1$ such that $t_0 < 0$ and $t_1 > 0$. Then $-c_1t_0^3 + c_2t_0^3 = 0$ and $c_1t_1^3 + c_2t_1^3 = 0$. These equations have a nontrivial solution for c_1 and c_2 only if the determinant of coefficients is zero. But the determinant of coefficients is $-2t_0^3t_1^3 \neq 0$ for t_0 and t_1 as specified. Hence $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.

Section 3.4, Page 158

1. $\exp(1+2i) = e^{1+2i} = ee^{2i} = e(\cos 2 + i\sin 2)$.

5. Recall that $2^{1-i} = e^{\ln(2^{1-i})} = e^{(1-i)\ln 2}$

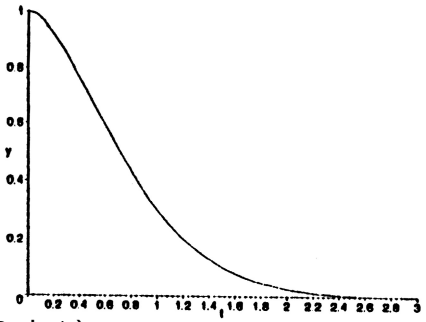
7. As in Section 3.1, we seek solutions of the form $y = e^{rt}$. Substituting this into the D.E. yields the characteristic equation $r^2 - 2r + 2 = 0$, which has the roots $r_1 = 1 + i$

and $r_2 = 1 - i$, using the quadratic formula. Thus $\lambda = 1$ and $\mu = 1$ and from Eq.(17) the general solution is $y = c_1e^t\cos t + c_2e^t\sin t$.

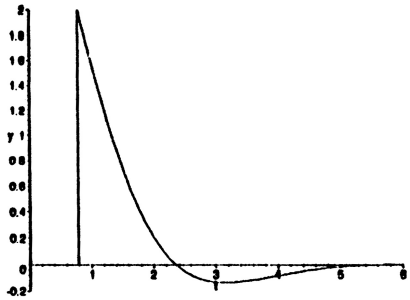
11. The characteristic equation is $r^2 + 6r + 13 = 0$, which has the roots $r = \frac{-6\pm\sqrt{-16}}{2} = -3\pm 2i$. Thus $\lambda = -3$ and $\mu = 2$, so Eq.(17) becomes $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$.

14. The characteristic equation is $9r^2 + 9r - 4$, which has the real roots -4/3 and 1/3. Thus the solution has the same form as in Section 3.1, $y(t) = c_1e^{t/3} + c_2e^{-4t/3}$.

18. The characteristic equation is $r^2 + 4r + 5 = 0$, which has the roots $r_1, r_2 = -2 \pm i$. Thus $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$ and $y' = (-2c_1+c_2)e^{-2t}\cos t + (-c_1-2c_2)e^{-2t}\sin t$, so that $y(0)= c_1= 1$ and $y'(0)= -2c_1+c_2= 0$, or $c_2 = 2$. Hence $y = e^{-2t}(\cos t + 2\sin t)$.



22. The characteristic equation is $r^2 + 2r + 2 = 0$, so $r_1, r_2 = -1 \pm i$. Since the I.C. are given at $\pi/4$ we want to alter Eq.(17) by letting $c_1 = e^{\pi/4}d_1$ and $c_2 = e^{\pi/4}d_2$. Thus, for $\lambda = -1$ and $\mu = 1$ we have $y = e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t)$; so $y' = -e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t) + e^{-(t-\pi/4)}(-d_1\sin t + d_2\cos t)$. Thus $\sqrt{2}\,d_1/2 + \sqrt{2}\,d_2/2 = 2$ and $-\sqrt{2}\,d_1 = -2$ and hence $y = \sqrt{2}\,e^{-(t-\pi/4)}(\cos t + \sin t)$.



23a. The characteristic equation is $3r^2 - r + 2 = 0$, which has the roots $r_1, r_2 = \frac{1}{6} \pm \frac{\sqrt{23}}{6}i$. Thus $u(t) = e^{t/6}(c_1\cos\frac{\sqrt{23}}{6}t + c_2\sin\frac{\sqrt{23}}{6}t)$ and we obtain $u(0) = c_1 = 2$

and $u'(0) = \frac{1}{6}c_1 + \frac{\sqrt{23}}{6}c_2 = 0$. Solving for c_2 we find

$u(t) = e^{t/6}(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t)$.

23b. To estimate the first time that $|u(t)| = 10$ plot the graph of u(t) as found in part (a). Use this estimate in an appropriate computer software program to find t = 10.7598.

25a. The characteristic equation is $r^2 + 2r + 6 = 0$, so $r_1, r_2 = -1 \pm \sqrt{5}\,i$ and $y(t) = e^{-t}(c_1\cos\sqrt{5}\,t + c_2\sin\sqrt{5}\,t)$. Thus $y(0) = c_1 = 2$ and $y'(0) = -c_1 + \sqrt{5}\,c_2 = \alpha$ and hence

$y(t) = e^{-t}(2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t)$.

25b. $y(1) = e^{-1}(2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}) = 0$ and hence

$\alpha = -2 - \frac{2\sqrt{5}}{\tan\sqrt{5}} = 1.50878$.

25c. For $y(t) = 0$ we must have $2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t = 0$ or

$\tan\sqrt{5}\,t = \frac{-2\sqrt{5}}{\alpha+2}$. For the given α (actually, for $\alpha > -2$) this yields $\sqrt{5}\,t = \pi - \arctan\frac{2\sqrt{5}}{\alpha+2}$ since $\arctan x < 0$ when $x < 0$.

25d. From part (c) $\arctan\frac{2\sqrt{5}}{\alpha+2} \to 0$ as $\alpha \to \infty$, so $t \to \pi/\sqrt{5}$.

31. $\frac{d}{dt}[e^{\lambda t}(\cos\mu t + i\sin\mu t)] = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t)$
$+ e^{\lambda t}(-\mu\sin\mu t + i\mu\cos\mu t) = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t)$
$+ i\mu e^{\lambda t}(i\sin\mu t + \cos\mu t) = e^{\lambda t}(\lambda+i\mu)(\cos\mu t + i\sin\mu t)$.
Setting $r = \lambda+i\mu$ we then have $\frac{d}{dt}e^{rt} = re^{rt}$.

33. Suppose that t = a and t = b (b>a) are consecutive zeros of y_1. We must show that y_2 vanishes once and only once in the interval $a < t < b$. Assume that it does not

vanish. Then we can form the quotient y_1/y_2 on the interval $a \le t \le b$. Note $y_2(a) \neq 0$ and $y_2(b) \neq 0$, otherwise y_1 and y_2 would not be linearly independent solutions. Next, y_1/y_2 vanishes at $t = a$ and $t = b$ and has a derivative in $a < t < b$. By Rolles theorem, the derivative must vanish at an interior point. But $(\frac{y_1}{y_2})' = \frac{y_1'y_2 - y_2'y_1}{y_2^2} = \frac{-W(y_1,y_2)}{y_2^2}$, which cannot be zero since y_1 and y_2 are linearly independent solutions. Hence we have a contradiction, and we conclude that y_2 must vanish at a point between a and b. Finally, we show that it can vanish at only one point between a and b. Suppose that it vanishes at two points c and d between a and b. By the argument we have just given we can show that y_1 must vanish between c and d. But this contradicts the hypothesis that a and b are consecutive zeros of y_1.

35. We use the result of Problem 34. Note that $q(t) = e^{-t^2} > 0$ for $-\infty < t < \infty$. Next, we find that $(q' + 2pq)/q^{3/2} = 0$. Hence the D.E. can be transformed into an equation with constant coefficients by letting $x = u(t) = \int e^{-t^2/2}dt$. Substituting $x = u(t)$ in the differential equation found in part (b) of Problem 34 we obtain, after dividing by the coefficient of d^2y/dx^2, the D.E. $d^2y/dx^2 - y = 0$. Hence the general solution of the original D.E. is $y = c_1\cos x + c_2\sin x$, $x = \int e^{-t^2/2}dt$.

38. Rewrite the D.E. as $y'' + (\alpha/t)y' + (\beta/t^2)y = 0$ so that $p = \alpha/t$ and $q = \beta/t^2$, which satisfy the conditions of parts (c) and (d) of Problem 34. Thus $x = \int (1/t^2)^{1/2}dt = \ln t$ will transform the D.E. into $dy^2/dx^2 + (\alpha-1)dy/dx + \beta y = 0$. Note that since β is constant, it can be neglected in defining x.

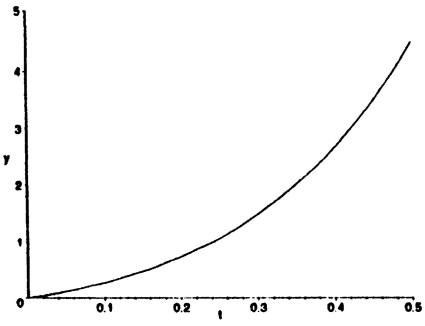
39. By direct substitution, or from Problem 38, $x = \ln t$ will transform the D.E. into $d^2y/dx^2 + y = 0$, since $\alpha = 1$ and $\beta = 1$. Thus $y = c_1\cos x + c_2\sin x$, with $x = \ln t$, $t > 0$.

Section 3.5, Page 166

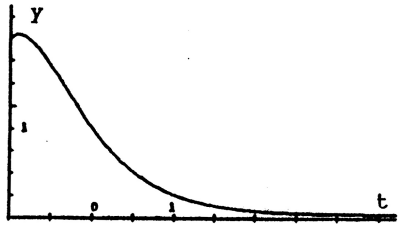
1. Substituting $y = e^{rt}$ into the D.E., we find that $r^2 - 2r + 1 = 0$, which gives $r_1 = 1$ and $r_2 = 1$. Since the roots are equal, the second linearly independent solution is te^t and thus the general solution is $y = c_1e^t + c_2te^t$.

9. The characteristic equation is $25r^2 - 20r + 4 = 0$, which may be written as $(5r-2)^2 = 0$ and hence the roots are $r_1, r_2 = 2/5$. Thus $y = c_1e^{2t/5} + c_2\, te^{2t/5}$.

12. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, which has the repeated root $r = 3$. Thus $y = c_1e^{3t} + c_2te^{3t}$, which gives $y(0) = c_1 = 0$, $y'(t) = c_2(e^{3t}+3te^{3t})$ and $y'(0) = c_2 = 2$. Hence $y(t) = 2te^{3t}$.



14. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$, which has the repeated root $r = -2$. Since the I.C. are given at $t = -1$, write the general solution as $y = c_1e^{-2(t+1)} + c_2te^{-2(t+1)}$. Then $y' = -2c_1e^{-2(t+1)} + c_2e^{-2(t+1)} - 2c_2te^{-2(t+1)}$ and hence $c_1-c_2 = 2$ and $-2c_1+3c_2 = 1$ which yield $c_1 = 7$ and $c_2 = 5$. Thus $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}$, a decaying exponential as shown in the graph.



17a. The characteristic equation is $4r^2 + 4r + 1 = (2r+1)^2 = 0$, so we have $y(t) = (c_1+c_2t)e^{-t/2}$. Thus $y(0) = c_1 = 1$ and $y'(0) = -c_1/2 + c_2 = 2$ and hence $c_2 = 5/2$ and $y(t) = (1 + 5t/2)e^{-t/2}$.

17b. From part (a), $y'(t) = -\frac{1}{2}(1 + 5t/2)e^{-t/2} + \frac{5}{2}e^{-t/2} = 0$, when $-\frac{1}{2} - \frac{5t}{4} + \frac{5}{2} = 0$, or $t_0 = \frac{8}{5}$ and $y_0 = 5e^{-4/5}$.

17c. From part (a), $-\frac{1}{2} + c_2 = b$ or $c_2 = b + \frac{1}{2}$ and
$y(t) = [1 + (b + \frac{1}{2})t]e^{-t/2}$.

17d. From part (c), $y'(t) = -\frac{1}{2}[1 + (b+\frac{1}{2})t]e^{-t/2} + (b+\frac{1}{2})e^{-t/2} = 0$
which yields $t_M = \frac{4b}{2b+1}$ and
$y_M = (1 + \frac{2b+1}{2} \cdot \frac{4b}{2b+1})e^{-2b/(2b+1)} = (1 + 2b)e^{-2b/(2b+1)}$.

19. If $r_1 = r_2$ then $y(t) = (c_1 + c_2t)e^{r_1t}$. Since the exponential is never zero, $y(t)$ can be zero only if $c_1 + c_2t = 0$, which yields at most one positive value of t if c_1 and c_2 differ in sign. If $r_2 > r_1$ then
$y(t) = c_1e^{r_1t} + c_2e^{r_2t} = e^{r_1t}(c_1 + c_2e^{(r_2-r_1)t})$. Again, this is zero only if c_1 and c_2 differ in sign, in which case
$t = \frac{\ln(-c_1/c_2)}{(r_2-r_1)}$.

21. If $r_2 \neq r_1$ then $\phi(t;r_1,r_2) = (e^{r_2t} - e^{r_1t})/(r_2 - r_1)$ is defined for all t. Note that ϕ is a liner combination of two solutions, e^{r_1t} and e^{r_2t}, of the D.E. Hence, ϕ is a solution of the differential equation. Think of r_1 as fixed and let $r_2 \to r_1$. The limit of ϕ as $r_2 \to r_1$ is indeterminate. If we use L'Hopital's rule, we find
$\lim_{r_2 \to r_1} \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} \cdot = \lim_{r_2 \to r_1} \frac{te^{r_2t}}{1} = te^{r_1t}$. Hence, the solution $\phi(t;r_1,r_2) \to te^{r_1t}$ as $r_2 \to r_1$.

25. Let $y_2 = v/t$. Then $y_2' = v'/t - v/t^2$ and
$y_2'' = v''/t - 2v'/t^2 + 2v/t^3$. Substituting in the D.E. we obtain
$t^2(v''/t - 2v'/t^2 + 2v/t^3) + 3t(v'/t - v/t^2) + v/t = 0$.
Simplifying the left side we get $tv'' + v' = 0$, which yields $v' = c_1/t$. Thus $v = c_1\ln t + c_2$. Hence a second solution is $y_2(t) = (c_1\ln t + c_2)/t$. However, we may set $c_2 = 0$ and $c_1 = 1$ without loss of generality and thus we have $y_2(t) = (\ln t)/t$ as a second solution. Note that in

the form we actually calculated, $y_2(t)$ is a linear combination of 1/t and lnt/t, and hence is the general solution.

27. In this case the calculations are somewhat easier if we do not use the explicit form for $y_1(x) = \sin x^2$ at the beginning but simply set $y_2(x) = y_1v$. Substituting this form for y_2 in the D.E. gives $x(y_1v)''-(y_1v)'+4x^3(y_1v) = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $xy_1v'' + (2xy_1' - y_1)v' = 0$. This is a first order linear equation for v′, which has the solution $v' = cx/(\sin x^2)^2$. Setting $u = x^2$ allows integration of this to get $v = c_1 \cot x^2 + c_2$. Setting $c_1 = 1$, $c_2 = 0$ and multiplying by $y_1 = \sin x^2$ we obtain $y_2(x) = \cos x^2$ as the second solution of the D.E.

30. Substituting $y_2(x) = y_1(x)v(x)$ in the D.E. gives $x^2(y_1v)'' + x(y_1v)' + (x^2 - \frac{1}{4})y_1v = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $x^2y_1v'' + (2x^2y_1' + xy_1)v' = 0$. This is a first order linear equation for v′, $v'' + (2y_1'/y_1 + 1/x)v' = 0$, with solution

$$v'(x) = c\exp[-\int(2\frac{y_1'}{y_1} + \frac{1}{x})dx] = c\exp[-2\ln y_1 - \ln x]$$

$$= c\frac{1}{xy_1^2} = \frac{c}{x(x^{-1}\sin^2 x)} = c\csc^2 x,$$

where c is an arbitrary constant, which we will take to be one. Then $v(x) = \int \csc^2 x\,dx = -\cot x + k$ where again k is an arbitrary constant which can be taken equal to zero. Thus $y_2(x) = y_1(x)v(x) = (x^{-1/2}\sin x)(-\cot x) = -x^{-1/2}\cos x$. The second solution is usually taken to be $x^{-1/2}\cos x$. Note that c = -1 would have given this solution.

31b. Let $y_2(x) = e^xv(x)$, then $y_2' = e^xv' + e^xv$, and $y_2'' = e^xv'' + 2e^xv' + e^xv$. Substituting in the D.E. we obtain $xe^xv'' + (xe^x-Ne^x)v' = 0$, or $v'' + (1-N/x)v' = 0$. This is a first order linear D.E. for v′ with integrating

factor $\mu(x) = \exp[\int(1-N/x)dx] = x^{-N}e^{x}$. Hence $(x^{-N}e^{x}v')' = 0$, and $v' = cx^{N}e^{-x}$ which gives $v(x) = c\int x^{N}e^{-x}dx + k$. On taking $k = 0$ we obtain as the second solution $y_2(x) = ce^{x}\int x^{N}e^{-x}dx$. The integral can be evaluated by using the method of integration by parts. At each stage let $u = x^{N}$ or x^{N-1}, or whatever the power of x that remains, and let $dv = e^{-x}$. Note that this dv is not related to the $v(x)$ in $y_2(x)$. For $N = 2$ we have

$$y_2(x) = ce^{x}\int x^{2}e^{-x}dx = ce^{x}[x^{2}\frac{e^{-x}}{-1} - \int 2x\frac{e^{-x}}{-1}dx]$$

$$= -cx^{2} + ce^{x}[2x\frac{e^{-x}}{-1} - \int 2\frac{e^{-x}}{-1}dx]$$

$$= c(-x^{2} - 2x - 2) = -2c(1 + x + x^{2}/2!).$$

Choosing $c = -1/2!$ gives the desired result. For the general case $c = -1/N!$

33. $(y_2/y_1)' = (y_1y_2' - y_1'y_2)/y_1^{2} = W(y_1,y_2)/y_1^{2}$. Abel's identity is $W(y_1,y_2) = c\exp[-\int_{t_0}^{t}p(r)dr]$. Hence $(y_2/y_1)' = cy_1^{-2}\exp[-\int_{t_0}^{t}p(r)dr]$. Integrating and setting $c = 1$ (since a solution y_2 can be multiplied by any constant) and taking the constant of integration to be zero we obtain

$$y_2(t) = y_1(t)\int_{t_0}^{t}\frac{\exp[-\int_{s_0}^{s}p(r)dr]}{[y_1(s)]^{2}}\,ds.$$

35. From Problem 33 and Abel's formula we have $(\frac{y_2}{y_1})' = \frac{\exp[\int(1/t)dt]}{\sin^{2}(t^{2})} = \frac{e^{\ln t}}{\sin^{2}(t^{2})} = t\csc^{2}(t^{2})$. Thus $y_2/y_1 = -(1/2)\cot(t^{2})$ and hence we can choose $y_2 = \cos(t^{2})$ since $y_1 = \sin^{2}(t^{2})$.

38. The general solution of the D.E. is $y = c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1,r_2 = (-b \pm \sqrt{b^{2}-4ac})/2a$ provided $b^{2} - 4ac \neq 0$. In this case there are two possibilities. If $b^{2} - 4ac > 0$ then $(b^{2} - 4ac)^{1/2} < b$ and r_1 and r_2 are real and <u>negative</u>. Consequently $e^{r_1t} \to 0$ and $e^{r_2t} \to 0$; and hence

$y \to 0$, as $t \to \infty$. If $b^2 - 4ac < 0$ then r_1 and r_2 are complex conjugates with <u>negative</u> real part. Again $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$; and hence $y \to 0$, as $t \to \infty$. Finally, if $b^2 - 4ac = 0$, then $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$ where $r_1 = -b/2a < 0$. Hence, again $y \to 0$ as $t \to \infty$. This conclusion does not hold if either $b = 0$ (since $y(t) = c_1 \cos\omega t + c_2 \sin\omega t$) or $c = 0$ (since $y_1(t) = c_1$).

42. Substituting $z = \ln t$ into the D.E. gives $\frac{d^2y}{dz^2} + \frac{dy}{dz} + 0.25y = 0$, which has the solution $y(z) = c_1 e^{-z/2} + c_2 z e^{-z/2}$ so that $y(t) = c_1 t^{-1/2} + c_2 t^{-1/2} \ln t$.

<u>Section 3.6, Page 178</u>

1. First we find the solution of the homogeneous D.E., which has the characteristic equation $r^2 - 2r - 3 = (r-3)(r+1) = 0$. Hence $y_c = c_1 e^{3t} + c_2 e^{-t}$ and we can assume $Y = Ae^{2t}$ for the particular solution. Thus $Y' = 2Ae^{2t}$ and $Y'' = 4Ae^{2t}$ and substituting into the D.E. yields $4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$. Thus $-3A = 3$ and $A = -1$, yielding $y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$.

4. Initially we assume $Y = A + B\sin 2t + C\cos 2t$. However, since a constant is a solution of the related homogeneous D.E. we must modify Y by multiplying the constant A by t and thus the correct form is $Y = At + B\sin 2t + C\cos 2t$.

6. Since $y_c = c_1 e^{-t} + c_2 t e^{-t}$ we must assume $Y = At^2 e^{-t}$, so that $Y' = 2Ate^{-t} - At^2 e^{-t}$ and $y'' = 2Ae^{-t} - 4Ate^{-t} + At^2 e^{-t}$. Substituting in the D.E. gives $(At^2 - 4At + 2A)e^{-t} + 2(-At^2 + 2At)e^{-t} + At^2 e^{-t} = 2e^{-t}$. Notice that all terms on the left involving t^2 and t add to zero and we are left with $2A = 2$, or $A = 1$. Hence $y = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}$.

8. The assumed form is $Y = (At + B)\sin 2t + (Ct + D)\cos 2t$, which is appropriate for both terms appearing on the right side of the D.E. Since none of the terms appearing

in Y are solutions of the homogeneous equation, we do not need to modify Y.

11. First solve the homogeneous D.E. Substituting $y = e^{rt}$ gives $r^2 + r + 4 = 0$. Hence $y_c = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)]$. We replace sinht by $(e^t - e^{-t})/2$ and then assume $Y(t) = Ae^t + Be^{-t}$. Since neither e^t nor e^{-t} are solutions of the homogeneous equation, there is no need to modify our assumption for Y. Substituting in the D.E., we obtain $6Ae^t + 4Be^{-t} = e^t - e^{-t}$. Hence, A = 1/6 and B = -1/4. The general solution is $y = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)] + e^t/6 - e^{-t}/4$. [For this problem we could also have found a particular solution as a linear combination of sinht and cosht: Y(t) = Acosht + Bsinht. Substituting this in the D.E. gives (5A + B)cosht + (A + 5B)sinht = 2sinht. The solution is A = -1/12 and B = 5/12. A simple calculation shows that $-(1/12)\cosh t + (5/12)\sinh t = e^t/6 - e^{-t}/4$.]

13. $y_c = c_1e^{-2t} + c_2e^t$ so for the particular solution we assume Y = At + B. Since neither At or B are solutions of the homogeneous equation it is not necessary to modify the original assumption. Substituting Y in the D.E. we obtain 0 + A -2(At+B) = 2t or -2A = 2 and A-2B = 0. Solving for A and B we obtain $y = c_1e^{-2t} + c_2e^t - t - 1/2$ as the general solution. $y(0) = 0 \rightarrow c_1 + c_2 - 1/2 = 0$ and $y'(0) = 1 \rightarrow -2c_1 + c_2 - 1 = 1$, which yield $c_1 = -1/2$ and $c_2 = 1$. Thus $y = e^t - (1/2)e^{-2t} - t - 1/2$.

16. Since the nonhomogeneous term is the product of a linear polynomial and an exponential, assume Y of the same form: $Y = (At+B)e^{2t}$. Thus $Y' = Ae^{2t} + 2(At+B)e^{2t}$ and $Y'' = 4Ae^{2t}+4(At+B)e^{2t}$. Substituting into the D.E. we find -3At = 3t and 2A - 3B = 0, yielding A = -1 and B = -2/3. Since the characteristic equation is $r^2 - 2r - 3 = 0$, the general solution is $y = c_1e^{3t} + c_2e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$.

19a. The solution of the homogeneous D.E. is $y_c = c_1e^{-3t} + c_2$. After inspection of the nonhomogeneous term, for $2t^4$ we must assume a fourth order polynominial, for t^2e^{-3t} we must assume a quadratic polynomial times the exponential, and for sin3t we must assume Csin3t + Dcos3t. Thus $Y(t) = (A_0t^4+A_1t^3+A_2t^2+A_3t+A_4) + (B_0t^2+B_1t+B_2)e^{-3t}+Csin3t+Dcos3t$. However, since e^{-3t} and a constant are solutions of the homogeneous D.E., we must multiply the coefficient of e^{-3t} and the polynomial by t. The correct form is

$$Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + Csin3t + Dcos3t.$$

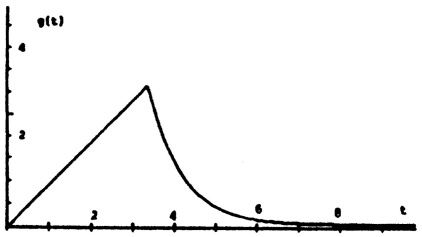
22a. The solution of the homgeneous D.E. is $y_c = e^{-t}[c_1cost + c_2sint]$. After inspection of the nonhomogeneous term, we assume $Y(t) = Ae^{-t} + (B_0t^2 + B_1t + B_2)e^{-t}cost + (C_0t^2 + C_1t + C_2)e^{-t}sint$. Since $e^{-t}cost$ and $e^{-t}sint$ are solutions of the homogeneous D.E., it is necessary to multiply both the last two terms by t. Hence the correct form is

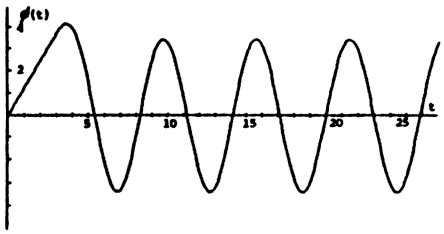
$$Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}cost + t(C_0t^2 + C_1t + C_2)e^{-t}sint.$$

28. First solve the I.V.P. $y'' + y = t$, $y(0) = 0$, $y'(0) = 1$ for $0 \le t \le \pi$. The solution of the homogeneous D.E. is $y_c(t) = c_1cost + c_2sint$. The correct form for Y(t) is $y(t) = A_0t + A_1$. Substituting in the D.E. we find $A_0 = 1$ and $A_1 = 0$. Hence, $y = c_1cost + c_2sint + t$. Applying the I.C., we obtain $y = t$. For $t > \pi$ we have $y'' + y = \pi e^{\pi-t}$ so the form for Y(t) is $Y(t) = Ee^{\pi-t}$. Substituting Y(t) in the D.E., we obtain $Ee^{\pi-t} + Ee^{\pi-t} = \pi e^{\pi-t}$ so $E = \pi/2$. Hence the general solution for $t > \pi$ is $Y = D_1cost + D_2sint + (\pi/2)e^{\pi-t}$. If y and y′ are to be continuous at $t = \pi$, then the solutions and their derivatives for $t \le \pi$ and $t > \pi$ must have the same value at $t = \pi$. These conditions require $\pi = -D_1 + \pi/2$ and $1 = -D_2 - \pi/2$. Hence $D_1 = -\pi/2$, $D_2 = -(1 + \pi/2)$, and

$$y = \phi(t) = \begin{cases} t, & 0 \le t \le \pi \\ -(\pi/2)\cos t - (1 + \pi/2)\sin t + (\pi/2)e^{\pi - t}, & t > \pi. \end{cases}$$

The graphs of the nonhomogeneous term and ϕ follow.





30. According to Theorem 3.6.1, the difference of any two solutions of the linear second order nonhomogeneous D.E. is a solution of the corresponding homogeneous D.E. Hence $Y_1 - Y_2$ is a solution of $ay'' + by' + cy = 0$. In Problem 38 of Section 3.5 we showed that if $a > 0$, $b > 0$, and $c > 0$ then every solution of this D.E. goes to zero as $t \to \infty$. If $b = 0$, then y_c involves only sines and cosines, so $Y_1 - Y_2$ does not approach zero as $t \to \infty$.

33. From Problem 32 we write the D.E. as $(D-4)(D+1)y = 3e^{2t}$. Thus let $(D+1)y = u$ and then $(D-4)u = 3e^{2t}$. This last equation is the same as $du/dt - 4u = 3e^{2t}$, which may be solved by multiplying both sides by e^{-4t} and integrating (see section 2.1). This yields $u = (-3/2)e^{2t} + Ce^{4t}$. Substituting this form of u into $(D+1)y = u$ we obtain $dy/dt + y = (-3/2)e^{2t} + Ce^{4t}$. Again, multiplying by e^{t} and integrating gives $y = (-1/2)e^{2t} + C_1e^{4t} + C_2e^{-t}$, where $C_1 = C/5$.

Section 3.7, Page 183

2. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$. Assume $Y = u_1(t)e^{2t} + u_2(t)e^{-t}$, then $Y'(t) = [2u_1(t)e^{2t} - u_2(t)e^{-t}] + [u_1'(t)e^{2t}$

$+ u_2'(t)e^{-t}]$. We set $u_1'(t)e^{2t} + u_2'(t)e^{-t} = 0$. Then $Y'' = 4u_1e^{2t} + u_2e^{-t} + 2u_1'e^{2t} - u_2'e^{-t}$ and substituting in the D.E. gives $2u_1'(t)e^{2t} - u_2'(t)e^{-t} = 2e^{-t}$. Thus we have two algebraic equations for $u_1'(t)$ and $u_2'(t)$ with the solution $u_1'(t) = 2e^{-3t}/3$ and $u_2'(t) = -2/3$. Hence $u_1(t) = -2e^{-3t}/9$ and $u_2(t) = -2t/3$. Substituting in the formula for $Y(t)$ we obtain $Y(t) = (-2e^{-3t}/9)e^{2t} + (-2t/3)e^{-t} = (-2e^{-t}/9) - (2te^{-t}/3)$. Since e^{-t} is a solution of the homogeneous D.E., we can choose $Y(t) = -2te^{-t}/3$.

5. Since cost and sint are solutions of the homogeneous D.E., we assume $Y = u_1(t)\cos t + u_2(t)\sin t$. Thus $Y' = -u_1(t)\sin t + u_2(t)\cos t$, after setting $u_1'(t)\cos t + u_2'(t)\sin t = 0$. Finding Y'' and substituting into the D.E. then yields $-u_1'(t)\sin t + u_2'(t)\cos t = \tan t$. The two equations for $u_1'(t)$ and $u_2'(t)$ have the solution: $u_1'(t) = -\sin^2 t/\cos t = -\sec t + \cos t$ and $u_2'(t) = \sin t$. Thus $u_1(t) = \sin t - \ln(\tan t + \sec t)$ and $u_2(t) = -\cos t$, which when substituted into the assumed form for Y, simplified, and added to the homogeneous solution yields
$y = c_1\cos t + c_2\sin t - (\cos t)\ln(\tan t + \sec t)$.

11. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Applying Theorem 3.7.1 with $W(y_1,y_2)(t) = -e^{5t}$, we obtain

$$Y(t) = -e^{3t}\int \frac{e^{2s}g(s)}{-e^{5s}}\,ds + e^{2t}\int \frac{e^{3s}g(s)}{-e^{5s}}\,ds$$
$$= \int [e^{3(t-s)} - e^{2(t-s)}]g(s)\,ds.$$

The complete solution is then obtained by adding $c_1e^{3t} + c_2e^{2t}$ to $Y(t)$.

14. That t and te^t are solutions of the homogeneous D.E. can be verified by direction substitution. Thus we assume $Y = tu_1(t) + te^tu_2(t)$. Following the pattern of earlier problems we find $tu_1'(t) + te^tu_2'(t) = 0$ and

$u_1'(t) + (t+1)e^t u_2' = 2t$. [Note that $g(t) = 2t$, since the D.E. must be put into the form of Eq.(16)]. The solution of these equations gives $u_1'(t) = -2$ and $u_2'(t) = 2e^{-t}$. Hence, $u_1(t) = -2t$ and $u_2(t) = -2e^{-t}$, and $Y(t) = t(-2t) + te^t(-2e^{-t}) = -2t^2 - 2t$. However, since t is a solution of the homogeneous D.E. we can choose as our particular solution $Y(t) = -2t^2$.

18. For this problem, and for many others, it is probably easier to rederive Eqs.(26) without using the explicit form for $y_1(x)$ and $y_2(x)$ and then to substitute for $y_1(x)$ and $y_2(x)$ in Eqs.(26). In this case if we take $y_1 = x^{-1/2}\sin x$ and $y_2 = x^{-1/2}\cos x$, then $W(y_1,y_2) = -1/x$. If the D.E. is put in the form of Eq.(16), then $g(x) = 3x^{-1/2}\sin x$ and thus $u_1'(x) = 3\sin x\cos x$ and $u_2'(x) = -3\sin^2 x = 3(-1 + \cos 2x)/2$. Hence $u_1(x) = (3\sin^2 x)/2$ and $u_2(x) = -3x/2 + 3(\sin 2x)/4$, and

$$Y(x) = \frac{3\sin^2 x}{2}\,\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin 2x}{4}\right)\frac{\cos x}{\sqrt{x}}$$

$$= \frac{3\sin^2 x}{2}\,\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin x\cos x}{2}\right)\frac{\cos x}{\sqrt{x}}$$

$$= \frac{3\sin x}{2\sqrt{x}} - \frac{3\sqrt{x}\cos x}{2}.$$

The first term is a multiple of $y_1(x)$ and thus can be neglected for Y(x).

22. Putting limits on the integrals of Eq.(28) and changing the integration variable to s yields

$$Y(t) = -y_1(t)\int_{t_0}^{t}\frac{y_2(s)g(s)ds}{W(y_1,y_2)(s)} + y_2(t)\int_{t_0}^{t}\frac{y_1(s)g(s)ds}{W(y_1,y_2)(s)}$$

$$= \int_{t_0}^{t}\frac{-y_1(t)y_2(s)g(s)ds}{W(y_1,y_2)(s)} + \int_{t_0}^{t}\frac{y_2(t)y_1(s)g(s)ds}{W(y_1,y_2)(s)}$$

$$= \int_{t_0}^{t}\frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]g(s)ds}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}.$$

To show that Y(t) satisfies $L[y] = g(t)$ we must take the derivative of Y using Leibnitz's rule, which says that if $Y(t) = \int_{t_0}^{t} G(t,s)ds$, then $Y'(t) = G(t,t) + \int_{t_0}^{t}\frac{\partial G}{\partial t}(t,s)ds$. Letting G(t,s) be the above integrand, then $G(t,t) = 0$

and $\frac{\partial G}{\partial t} = \frac{y_1(s)y_2'(t) - y_1'(t)y_2(s)}{W(y_1,y_2)(s)}\, g(s)$. Likewise

$$Y'' = \frac{\partial G(t,t)}{\partial t} + \int_{t_0}^{t} \frac{\partial^2 G}{\partial t^2}(t,s)\,ds$$

$$= g(t) + \int_{t_0}^{t} \frac{y_1(s)y_2''(t) - y_1''(t)y_2(s)}{W(y_1,y_2)(s)}\,ds.$$

Since y_1 amd y_2 are solutions of $L[y] = 0$, we have $L[Y] = g(t)$ since all the terms involving the integral will add to zero. Clearly $Y(t_0) = 0$ and $Y'(t_0) = 0$.

25. Note that $y_1 = e^{\lambda t}\cos\mu t$ and $y_2 = e^{\lambda t}\sin\mu t$ and thus $W(y_1,y_2) = \mu e^{2\lambda t}$. From Problem 22 we then have:

$$Y(t) = \int_{t_0}^{t} \frac{e^{\lambda s}\cos\mu s e^{\lambda t}\sin\mu t - e^{\lambda t}\cos\mu t e^{\lambda s}\sin\mu s}{\mu e^{2\lambda s}}\, g(s)\,ds$$

$$= \mu^{-1}\int_{t_0}^{t} e^{\lambda(t-s)}[\cos\mu s \sin\mu t - \cos\mu t \sin\mu s]g(s)\,ds$$

$$= \mu^{-1}\int_{t_0}^{t} e^{\lambda(t-s)}[\sin\mu(t-s)]g(s)\,ds.$$

29. First, we put the D.E. in standard form by dividing by t^2: $y'' - 2y'/t + 2y/t^2 = 4$. Assuming that $y = tv(t)$ and substituting in the D.E. we obtain $tv'' = 4$. Hence $v'(t) = 4\ln t + c_2$ and $v(t) = 4\int \ln t\,dt + c_2 t = 4(t\ln t - t) + c_2 t$. The general solution is $c_1 y_1(t) + tv(t) = c_1 t + 4(t^2\ln t - t^2) + c_2 t^2$. Since $-4t^2$ is a multiple of $y_2 = c_2 t^2$ we can write $y = c_1 t + c_2 t^2 + 4t^2\ln t$.

Section 3.8, Page 197

2. From Eq.(15) we have $R\cos\delta = -1$, and $R\sin\delta = \sqrt{13}$. Thus $R = \sqrt{1+3} = 2$ and $\delta = \tan^{-1}(-\sqrt{3}) + \pi = 2\pi/3 \cong 2.09440$. Note that we have to "add" π to the inverse tangent value since δ must be a second quadrant angle. Thus $u = 2\cos(t-2\pi/3)$.

6. The motion is an undamped free vibration. The units are in the CGS system. The spring constant $k = (100\text{ gm})(980\text{cm/sec}^2)/5\text{cm}$. Hence the D.E. for the motion is $100u'' + [(100 \cdot 980)/5]u = 0$ where u is measured in cm and time in sec. We obtain $u'' + 196u = 0$

so
$u = A\cos 14t + B\sin 14t$. The I.C. are $u(0) = 0 \rightarrow A = 0$ and $u'(0) = 10$ cm/sec $\rightarrow B = 10/14 = 5/7$. Hence $u(t) = (5/7)\sin 14t$, which first reaches equilibrium when $14t = \pi$, or $t = \pi/14$.

8. We use Eq.(33) without R and E(t) (there is no resistor or impressed voltage) and with L = 1 henry and $1/C = 4\text{x}10^6$ since $C = .25\text{x}10^{-6}$ farads. Thus the I.V.P. is $Q'' + 4\text{x}10^6\, Q = 0$, $Q(0) = 10^{-6}$ coulombs and $Q'(0) = 0$.

9. The spring constant is $k = (20)(980)/5 = 3920$ dyne/cm. The I.V.P. for the motion is $20u'' + 400u' + 3920u = 0$ or $u'' + 20u' + 196u = 0$ and $u(0) = 2$, $u'(0) = 0$. Here u is measured in cm and t in sec. The general solution of the D.E. is $u = Ae^{-10t}\cos 4\sqrt{6}\,t + Be^{-10t}\sin 4\sqrt{6}\,t$. The I.C. $u(0) = 2 \rightarrow A = 2$ and $u'(0) = 0 \rightarrow -10A + 4\sqrt{6}\,B = 0$. The solution is $u = e^{-10t}[2\cos 4\sqrt{6}\,t + 5(\sin 4\sqrt{6}\,t)/\sqrt{6}\,]$cm. The quasi frequency is $\mu = 4\sqrt{6}$, the quasi period is $T_d = 2\pi\mu = \pi/2\sqrt{6}$ and $T_d/T = 7/2\sqrt{6}$ since $T = 2\pi/14 = \pi/7$. To find an upper bound for τ, write u in the form of Eq.(30): $u(t) = \sqrt{4+25/6}\,e^{-10t}\cos(4\sqrt{6}\,t-\delta)$. Now, since $|\cos(4\sqrt{6}\,t-\delta)| \le 1$, we have $|u(t)| < .05 \Rightarrow \sqrt{4+25/6}\,e^{-10t} < .05$, which yields $\tau = .4046$. A more precise answer can be obtained with a computer algebra system, which in this case yields $\tau = .4045$. The original estimate was unusually close for this problem since $\cos(4\sqrt{6}\,t-\delta) = -0.9996$ for $t = .4046$.

12. Substituting the given values for L, C and R in Eq.(33), we obtain the D.E. $.2Q'' + 3\text{x}10^2\, Q' + 10^5\, Q = 0$. The I.C. are $Q(0) = 10^{-6}$ and $Q'(0) = I(0) = 0$. Assuming $Q = e^{rt}$, we obtain the roots of the characteristic equation as $r_1 = -500$ and $r_2 = -1000$. Thus $Q = c_1e^{-500t} + c_2e^{-1000t}$ and hence $Q(0) = 10^{-6} \rightarrow c_1 + c_2 = 10^{-6}$ and $Q'(0) = 0 \rightarrow -500c_1 - 1000c_2 = 0$. Solving for c_1 and c_2 yields the solution.

17. The mass is 8/32 lb-sec^2/ft, and the spring constant is $8/(1/8) = 64$ lb/ft. Hence $(1/4)u'' + \gamma u' + 64u = 0$ or $u'' + 4\gamma u' + 256u = 0$, where u is measured in ft, t in sec and the units of γ are lb-sec/ft. We look for solutions of the D.E. of the form $u = e^{rt}$ and find $r^2 + 4\gamma r + 256 = 0$, so $r_1, r_2 = [-4\gamma \pm \sqrt{16\gamma^2 - 1024}\,]/2$. The system will be overdamped, critically damped or

underdamped as $(16\gamma^2 - 1024)$ is > 0, =0, or < 0, respectively. Thus the system is critically damped when $\gamma = 8$ lb-sec/ft.

19. The general solution of the D.E. is $u = Ae^{r_1 t} + Be^{r_2 t}$ where $r_1, r_2 = [-\gamma \pm (\gamma^2 - 4km)^{1/2}]/2m$ provided $\gamma^2 - 4km \neq 0$, and where A and B are determined by the I.C. When the motion is overdamped, $\gamma^2 - 4km > 0$ and $r_1 > r_2$. Setting $u = 0$, we obtain $Ae^{r_1 t} = -Be^{r_2 t}$ or $e^{(r_1 - r_2)t} = -B/A$. Since the exponential function is a monotone function, there is at most one value of t (when $B/A < 0$) for which this equation can be satisfied. Hence u can vanish at most once. If the system is critically damped, the general solution is $u(t) = (A + Bt)e^{-\gamma t/2m}$. The exponential function is never zero; hence u can vanish only if $A + Bt = 0$. If $B = 0$ then u never vanishes; if $B \neq 0$ then u vanishes once at $t = -A/B$ provided $A/B < 0$.

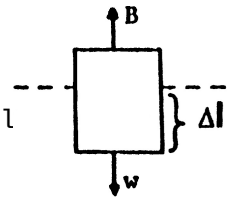
20. The general solution of Eq.(21) for the case of critical damping is $u = (A + Bt)e^{-\gamma t/2m}$. The I.C. $u(0) = u_0 \rightarrow A = u_0$ and $u'(0) = v_0 \rightarrow A(-\gamma/2m) + B = v_0$. Hence $u = [u_0 + (v_o + \gamma u_0/2m)t]e^{-\gamma t/2m}$. If $v_0 = 0$, then $u = u_0(1 + \gamma t/2m)e^{-\gamma t/2m}$, which is never zero since γ and m are postive. By L'Hopital's Rule $u \rightarrow 0$ as $t \rightarrow \infty$. Finally for $u_0 > 0$, we want the condition which will insure that $v = 0$ at least once. Since the exponential function is never zero we require $u_0 + (v_0 + \gamma u_0/2m)t = 0$ at a positive value of t. This requires that $v_0 + \gamma u_0/2m \neq 0$ and that $t = -u_0(v_0 + u_0\gamma/2m)^{-1} > 0$. We know that $u_0 > 0$ so we must have $v_0 + \gamma u_0/2m < 0$ or $v_0 < -\gamma u_0/2m$.

23. From Problem 21: $\Delta = \dfrac{2\pi\gamma}{\mu(2m)} = T_d\gamma/2m$. Substituting the known values we find $\gamma = \dfrac{(1/2)\ (3)}{.3} = 5$ lb sec/ft.

24. From Eq.(13) $\omega_0^2 = \dfrac{2k}{3}$ so $P = 2\pi/\sqrt{2k/3} = \pi \rightarrow k = 6$. Thus $u(t) = c_1\cos 2t + c_2 \sin 2t$ and $u(0) = 2 \rightarrow c_1 = 2$ and $u'(0) = v \rightarrow c_2 = v/2$. Hence $u(t) = 2\cos 2t + \dfrac{v}{2}\sin 2t =$

$\sqrt{4+\frac{v^2}{4}}\cos(2t-\gamma)$. Thus $\sqrt{4+\frac{v^2}{4}} = 3$ and $v = \pm 2\sqrt{5}$.

27. First, consider the static case. Let Δl denote the length of the block below the surface of the water. The weight of the block, which is a downward force, is $w = \rho l^3 l^3 g$. This is balanced by an equal and opposite buoyancy force B, which is equal to the weight of the displaced water. Thus $B = (\rho_0 l^2 \Delta l)g = \rho \Delta l^3 g$ so $\rho_0 \Delta l = \rho l$. Now let x be the displacement of the block from its equilibrium position. We take downward as the positive direction. In a displaced position the forces acting on the block are its weight, which acts downward and is unchanged, and the buoyancy force which is now $\rho_0 l^2(\Delta l + x)g$ and acts upward. The resultant force must be equal to the mass of the block times the acceleration, namely $\rho l^3 x''$. Hence $\rho l^3 g - \rho_0 l^2(\Delta l + x)g = \rho l^3 x''$. The D.E. for the motion of the block is $\rho l^3 x'' + \rho_0 l^2 g x = 0$. This gives a simple harmonic motion with frequency $(\rho_0 g/\rho l)^{1/2}$ and natural period $2\pi(\rho l/\rho_0 g)^{1/2}$.



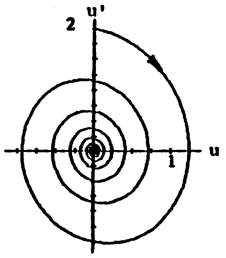
29a. The characteristic equation is $4r^2 + r + 8 = 0$, so $r = (-1 \pm \sqrt{127})/8$ and hence

$u(t) = e^{-t/8}\left(c_1 \cos\frac{\sqrt{127}}{8}t + c_2 \sin\frac{\sqrt{127}}{8}t\right)$. $u(0) = 0 \rightarrow c_1 = 0$

and $u'(0) = 2 \rightarrow \frac{\sqrt{127}}{8}c_2 = 2$. Thus

$u(t) = \frac{16}{\sqrt{127}}e^{-t/8}\sin\frac{\sqrt{127}}{8}t$.

29c. The phase plot is the spiral shown and the direction of motion is clockwise since the graph starts at (0,2) and u increases initially.



30c. Using u(t) as found in part(b), show that $ku^2/2 + m(u')^2/2 = (ka^2 + mb^2)/2$ for all t.

Section 3.9, Page 205

1. We use the trigonometric identities $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ to obtain $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$. If we choose $A + B = 9t$ and $A - B = 7t$, then $A = 8t$ and $B = t$. Substituting in the formula just derived, we obtain $\cos 9t - \cos 7t = -2\sin 8t \sin t$.

5. The mass $m = 4/32 = 1/8$ lb-sec^2/ft and the spring constant $k = 4/(1/8) = 32$ lb/ft. Since there is no damping, the I.V.P. is $(1/8)u'' + 32u = 2\cos 3t$, $u(0) = 1/6$, $u'(0) = 0$ where u is measured in ft and t in sec.

7a. From the solution to Problem 5, we have $m = 1/8$, $F_0 = 2$, $\omega_0^2 = 256$, and $\omega^2 = 9$, so Eq.(3) becomes $u = c_1\cos 16t + c_2\sin 16t + \frac{16}{247}\cos 3t$. The I.C. $u(0) = 1/6 \rightarrow c_1 + 16/247 = 1/6$ and $u'(0) = 0 \rightarrow 16c_2 = 0$, so the solution is $u = (151/1482)\cos 16t + (16/247)\cos 3t$ ft.

7c. Resonance occurs when the frequency ω of the forcing function $4\sin\omega t$ is the same as the natural frequency ω_0 of the system. Since $\omega_0 = 16$, the system will resonate when $\omega = 16$ rad/sec.

10. The I.V.P. is $.25u'' + 16u = 8\sin 8t$, $u(0) = 3$ and $u'(0) = 0$. Thus, the particular solution has the form $t(A\cos 8t + B\sin 8t)$ and resonance occurs.

11a. For this problem the mass $m = 8/32$ lb-sec^2/ft and the spring constant $k = 8/(1/2) = 16$ lb/ft, so the D.E. is $0.25u'' + 0.25u' + 16u = 4\cos 2t$ where u is measured in ft and t in sec. To determine the steady state response we need only compute a particular solution of the nonhomogeneous D.E. since the solutions of the homogeneous D.E. decay to zero as $t \rightarrow \infty$. We assume $u(t) = A\cos 2t + B\sin 2t$, and substitute in the D.E.: $-A\cos 2t - B\sin 2t + (1/2)(-A\sin 2t + B\cos 2t) + 16(A\cos 2t + B\sin 2t) = 4\cos 2t$. Hence $15A + (1/2)B = 4$ and $-(1/2)A + 15B = 0$, from which we obtain $A = 240/901$ and $B = 8/901$. The steady state response is $u(t) = (240\cos 2t + 8\sin 2t)/901$.

11b. In order to determine the value of m that maximizes the steady state response, we note that the present problem has exactly the form of the problem considered in the text. Referring to Eqs.(8) and (9), the response is a maximum when Δ is a minimum since F_0 is constant. Δ, as given in Eq.(10), will be a minimum when $f(m) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$, where $\omega_0^2 = k/m$, is a minimum. We calculate df/dm and set this quantity equal to zero to obtain $m = k/\omega^2$. We verify that this value of m gives a minimum of f(m) by the second derivative test. For this problem k = 16 lb/ft and ω = 2 rad/sec so the value of m that maximizes the response of the system is m = 4 slugs.

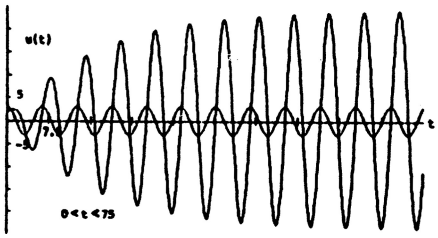
15. We must solve the three I.V.P.: (1)$u_1'' + u_1 = F_0 t$, $0 < t < \pi$, $u_1(0) = u_1'(0) = 0$; (2) $u_2'' + u_2 = F_0(2\pi - t)$, $\pi < t < 2\pi$, $u_2(\pi) = u_1(\pi)$, $u_2'(\pi) = u_1'(\pi)$; and (3) $u_3'' + u_3 = 0$, $2\pi < t$, $u_3(2\pi) = u_2(2\pi)$, $u_3'(2\pi) = u_2'(2\pi)$. The conditions at π and 2π insure the continuity of u and u′ at those points. The general solutions of the D.E. are $u_1 = b_1\cos t + b_2\sin t + F_0 t$, $u_2 = c_1\cos t + c_2\sin t + F_0(2\pi - t)$, and $u_3 = d_1\cos t + d_2\sin t$. The I.C. and matching conditions, in order, give $b_1 = 0$, $b_2 + F_0 = 0$, $-b_1 + \pi F_0 = -c_1 + \pi F_0$, $-b_2 + F_0 = -c_2 - F_0$, $c_1 = d_1$, and $c_2 - F_0 = d_2$. Solving these equations we obtain

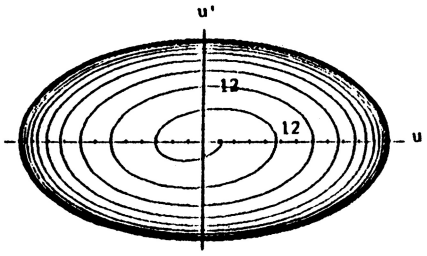
$$u = F_0 \begin{cases} t - \sin t & , \ 0 \le t \le \pi \\ (2\pi - t) - 3\sin t & , \ \pi < t \le 2\pi \\ -4\sin t & , \ 2\pi < t. \end{cases}$$

16. The I.V.P. is $Q'' + 5\text{x}10^3 Q' + 4\text{x}10^6 Q = 12$, $Q(0) = 0$, and $Q'(0) = 0$. The particular solution is of the form Q = A, so that upon substitution into the D.E. we obtain $4\text{x}10^6 A = 12$ or $A = 3\text{x}10^{-6}$. The general solution of the D.E. is $Q = c_1 e^{r_1 t} + c_2 e^{r_2 t} + 3\text{x}10^{-6}$, where r_1 and r_2 satisfy $r^2 + 5\text{x}10^3 r + 4\text{x}10^6 = 0$ and thus are $r_1 = -1000$ and $r_2 = -4000$. The I.C. yield $c_1 = -4\text{x}10^{-6}$ and $c_2 = 10^{-6}$ and thus $Q = 10^{-6}(e^{-4000t} - 4e^{-1000t} + 3)$ coulombs. Substituting t = .001 sec we obtain $Q(.001) = 10^{-6}(e^{-4} - 4e^{-1} + 3) = 1.5468 \text{ x } 10^{-6}$ coulombs. Since exponentials are to a negative power $Q(t) \to 3\text{x}10^{-6}$ coulombs as $t \to \infty$, which is the steady state charge.

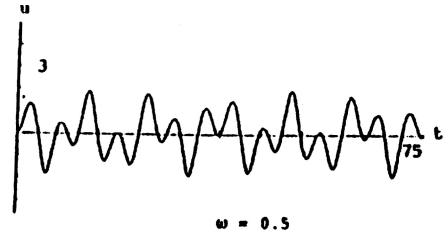
22. The amplitude of the steady state response is seven or eight times the amplitude (3) of the forcing term. This large an increase is due to the fact that the forcing function has the same frequency as the natural frequency, ω_0, of the system.

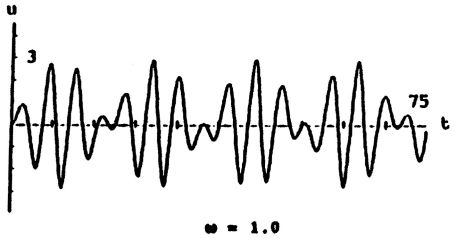
There also appears to be a phase lag of approximately 1/4 of a period. That is, the maximum of the response occurs 1/4 of a period after the maximum of the forcing function. Both these results are substantially different than those of either Problems 21 or 23.

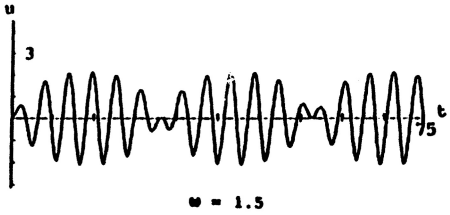


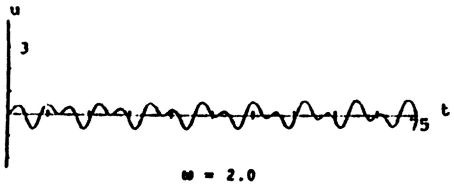


24.









From viewing the above graphs, it appears that the system exhibits a beat near $\omega = 1.5$, while the pattern for $\omega = 1.0$ is more irregular. However, the system exhibits the resonance characteristic of the linear system for ω near 1, as the amplitude of the response is the largest here.

CHAPTER 4

Section 4.1, Page 212

2. Writing the equation in standard form, we obtain $y''' + [(\sin t)/t]y'' + (3/t)y = \cos t/t$. The functions $p_1(t) = \sin t/t$, $p_3(t) = 3/t$ and $g(t) = \cos t/t$ have discontinuities at $t = 0$. Hence Theorem 4.1.1 guarantees that a solution exists for $t < 0$ and for $t > 0$.

8. We have $W(f_1, f_2, f_3) = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix} = 0$ for all t.

Thus by the extension of Theorem 3.3.1 the given functions are linearly dependent. Thus $c_1(2t-3) + c_2(2t^2+1) + c_3(3t^2+t) = (2c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+c_2) = 0$ when $(2c_2 + 3c_3) = 0$, $2c_1 + c_3 = 0$ and $-3c_1 + c_2 = 0$. Thus $c_1 = 1$ $c_2 = 3$ and $c_3 = -2$.

13. That e^t, e^{-t}, and e^{-2t} are solutions can be verified by direct substitution. Computing the Wronskian we obtain,

$$W(e^t,e^{-t},e^{-2t}) = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = e^{-2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{-2t}$$

17. To show that the given Wronskian is zero, it is helpful, in evaluating the Wronskian, to note that $(\sin^2 t)' = 2\sin t\cos t = \sin 2t$. This result can be obtained directly since $\sin^2 t = (1 - \cos 2t)/2 = \frac{1}{10}(5) + (-1/2)\cos 2t$ and hence $\sin^2 t$ is a linear combination of 5 and $\cos 2t$. Thus the functions are linearly dependent and their Wronskian is zero.

19c. If we let $L[y] = y^{iv} - 5y'' + 4y$ and if we use the result of Problem 19b, we have $L[e^{rt}] = (r^4 - 5r^2 + 4)e^{rt}$. Thus e^{rt} will be a solution of the D.E. provided $(r^2-4)(r^2-1) = 0$. Solving for r, we obtain the four solutions e^t, e^{-t}, e^{2t} and e^{-2t}. Since

$W(e^t, e^{-t}, e^{2t}, e^{-2t}) \neq 0$, the four functions form a fundamental set of solutions.

21. Comparing this D.E. to that of Problem 20 we see that $p_1(t) = 2$ and thus from the results of Problem 20 we have $W = ce^{-\int 2dt} = ce^{-2t}$.

27. As in Problem 26, let $y = v(t)e^t$. Differentiating three times and substituting into the D.E. yields $(2-t)e^t v''' + (3-t)e^t v'' = 0$. Dividing by $(2-t)e^t$ and letting $w = v''$ we obtain the first order separable equation $w' = -\frac{t-3}{t-2}w = (-1 + \frac{1}{t-2})w$. Separating t and w, integrating, and then solving for w yields $w = v'' = c_1(t-2)e^{-t}$. Integrating this twice then gives $v = c_1te^{-t} + c_2t + c_3$ so that $y = ve^t = c_1t + c_2te^t + c_3e^t$, which is the complete solution, since it contains the given $y_1(t)$ and three constants.

Section 4.2, Page 219

2. If $-1 + i\sqrt{3} = Re^{i\theta}$, then $R = [(-1)^2 + (\sqrt{3})^2]^{1/2} = 2$. The angle θ is given by $R\cos\theta = 2\cos\theta = -1$ and $R\sin\theta = 2\sin\theta = \sqrt{3}$. Hence $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$ which has the solution $\theta = 2\pi/3$. The angle θ is only determined up to an additive integer multiple of $\pm 2\pi$.

8. Writing (1-i) in the form $Re^{i\theta}$, we obtain $(1-i) = \sqrt{2}\,e^{i(-\pi/4+2m\pi)}$ where m is any integer. Hence, $(1-i)^{1/2} = [2^{1/2}e^{i(-\pi/4+2m\pi)}]^{1/2} = 2^{1/4}e^{i(-\pi/8+m\pi)}$. We obtain the two square roots by setting m = 0,1. They are $2^{1/4}e^{-i\pi/8}$ and $2^{1/4}e^{i7\pi/8}$. Note that any other integer value of m gives one of these two values. Also note that 1-i could be written as $1-i = \sqrt{2}\,e^{i(7\pi/4 + 2m\pi)}$.

12. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E., we obtain the characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$ which has roots r = 1,1,1. Since the roots are repeated, the general solution is $y = c_1e^t + c_2te^t + c_3t^2e^t$.

15. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E. we obtain the characteristic equation $r^6 + 1 = 0$. The six roots of -1 are obtained by setting $m = 0,1,2,3,4,5$ in $(-1)^{1/6} = e^{i(\pi+2m\pi)/6}$. They are $e^{i\pi/6} = (\sqrt{3} + i)/2$, $e^{i\pi/2} = i$, $e^{i5\pi/6} = (-\sqrt{3} + i)/2$, $e^{i7\pi/6} = (-\sqrt{3} - i)/2$, $e^{i3\pi/2} = -i$, and $e^{i11\pi/6} = (\sqrt{3} - i)/2$. Note that there are three pairs of conjugate roots. The general solution is
$y = e^{\sqrt{3}t/2}[c_1\cos(t/2) + c_2\sin(t/2)]$
$+ e^{-\sqrt{3}t/2}[c_3\cos(t/2) + c_4\sin(t/2)] + c_5\cos t + c_6\sin t.$

23. The characteristic equation is $r^3 - 5r^2 + 3r + 1 = 0$. Using the procedure suggested following Eq. (12) we try $r = 1$ as a root and find that indeed it is. Factoring out $(r-1)$ we are then left with $r^2 - 4r - 1 = 0$, which has the roots $2 \pm \sqrt{5}$.

27. The characteristic equation in this case is $12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$. Using an equation solver we find $r = -\frac{1}{4}, -\frac{1}{3}, -1 \pm 2i$. Thus
$y = c_1e^{-t/4} + c_2e^{-t/3} + e^{-t}(c_3\cos 2t + c_4\sin 2t)$. As in Problem 23, it is possible to find the first two of these roots without using an equation solver.

29. The characteristic equation is $r^3 + r = 0$ and hence $r = 0, +i, -i$ are the roots and the general solution is $y(t) = c_1 + c_2\cos t + c_3\sin t$. $y(0) = 0$ implies $c_1 + c_2 = 0$, $y'(0) = 1$ implies $c_3 = 1$ and $y''(0) = 2$ implies $-c_2 = 2$. Use this last equation in the first to find $c_1 = 2$ and thus $y(t) = 2 - 2\cos t + \sin t$, which continues to oscillate as $t \to \infty$.

30. The general solution is given by Eq. (21).

31. The general solution would normally be written $y(t) = c_1 + c_2t + c_3e^{2t} + c_4te^{2t}$. However, in order to evaluate the c's when the initial conditions are given at $t = 1$, it is advantageous to rewrite $y(t)$ as $y(t) = c_1 + c_2t + c_5e^{2(t-1)} + c_6(t-1)e^{2(t-1)}$.

34. The characteristic equation is $4r^3 + r + 5 = 0$, which has roots $-1, \frac{1}{2} \pm i$. Thus
$y(t) = c_1e^{-t} + e^{t/2}(c_2\cos t + c_3\sin t)$,
$y'(t) = -c_1e^{-t} + e^{t/2}[(c_2/2 + c_3)\cos t + (-c_2 + c_3/2)\sin t]$
and
$y''(t) = c_1e^{-t} + e^{t/2}[(-3c_2/4 + c_3)\cos t + (-c_2 - 3c_3/4)\sin t]$.
The I.C. then yield $c_1 + c_2 = 2$, $-c_1 + c_2/2 + c_3 = 1$ and $c_1 - 3c_2/4 + c_3 = -1$. Solving these last three equations give $c_1 = 2/13$, $c_2 = 24/13$ and $c_3 = 3/13$.

37. The approach developed in this section for solving the D.E. would normally yield $y(t) = c_1\cos t + c_2\sin t + c_5e^{t} + c_6e^{-t}$ as the solution. Now use the definition of coshx and sinht to yield the desired result. It is convenient to use cosht and sinht rather than e^{t} and e^{-t} because the I.C. are given at t = 0. Since cosht and sinht and all of their derivatives are either 0 or 1 at t = 0, the algebra in satisfying the I.C. is simplified.

38a. Since $p_1(t) = 0$, $W = ce^{-\int 0dt} = c$.

39a. As in Section 3.8, the force that the spring designated by k_1 exerts on mass m_1 is $-3u_1$. By an analysis similar to that shown in Section 3.8, the middle spring exerts a force of $-2(u_1-u_2)$ on mass m_1 and a force of $-2(u_2-u_1)$ on mass m_2. In all cases the positive direction is taken in the direction shown in Figure 4.2.4.

39c. From Eq.(i) we have $u_1''(0) = 2u_2(0) - 5u_1(0) = -1$ and $u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 0$. From Prob.39b we have $u_1 = c_1\cos t + c_2\sin t + c_3\cos\sqrt{6}\,t + c_4\sin\sqrt{6}\,t$. Thus $c_1+c_3 = 1$, $c_2+\sqrt{6}\,c_4 = 0$, $-c_1-6c_3 = -1$ and $-c_2-6\sqrt{6}\,c_4 = 0$, which yield $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$, so that $u_1 = \cos t$. The first of Eqs.(i) then gives u_2.

Section 4.3, Page 224

1. First solve the homogeneous D.E. The characteristic

equation is $r^3 - r^2 - r + 1 = 0$, and the roots are r = -1, 1, 1; hence $y_c(t) = c_1e^{-t} + c_2e^t + c_3te^t$. Using the superposition principle, we can write a particular solution as the sum of particular solutions corresponding to the D.E. $y'''-y''-y'+y = 2e^{-t}$ and $y'''-y''-y'+y = 3$. Our initial choice for a particular solution, Y_1, of the first equation is Ae^{-t}; but e^{-t} is a solution of the homogeneous equation so we multiply by t. Thus, $Y_1(t) = Ate^{-t}$. For the second equation we choose $Y_2(t) = B$, and there is no need to modify this choice. The constants are determined by substituting into the individual equations. We obtain A = 1/2, B = 3. Thus, the general solution is $y = c_1e^{-t} + c_2e^t + c_3te^t + 3 + (te^{-t})/2$.

5. The characteristic equation is $r^4 - 4r^2 = r^2(r^2-4) = 0$, so $y_c(t) = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t}$. For the particular solution correspnding to t^2 we assume $Y_1 = t^2(At^2 + Bt + C)$ and for the particular solution corresponding to e^t we assume $Y_2 = De^t$. Substituting Y_1, in the D.E. yields -48A = 1, B = 0 and 24A-8C = 0 and substituting Y_2 yields -3D = 1. Solving for A, B, C and D gives the desired solution.

9. The characteristic equation for the related homogeneous D.E. is $r^3 + 4r = 0$ with roots r = 0, +2i, -2i. Hence $y_c(t) = c_1 + c_2\cos 2t + c_3\sin 2t$. The initial choice for Y(t) is At + B, but since B is a solution of the homogeneous equation we must multiply by t and assume Y(t) = t(At+B). A and B are found by substituting in the D.E., which gives A = 1/8, B = 0, and thus the general solution is $y(t) = c_1 + c_2\cos 2t + c_3\sin 2t + (1/8)t^2$. Applying the I.C. we have $y(0) = 0 \rightarrow c_1 + c_2 = 0$, $y'(0) = 0 \rightarrow 2c_3 = 0$, and $y''(0) = 1 \rightarrow -4c_2 + 1/4 = 1$, which have the solution $c_1 = 3/16$, $c_2 = -3/16$, $c_3 = 0$. For small t the graph will approximate $3(1-\cos 2t)/16$ and for large t it will be approximated by $t^2/8$.

13. The characteristic equation for the homogeneous D.E. is $r^3 - 2r^2 + r = 0$ with roots r = 0,1,1. Hence the complementary solution is $y_c(t) = c_1 + c_2e^t + c_3te^t$. We

consider the differential equations $y''' - 2y'' + y' = t^3$ and $y''' - 2y'' + y' = 2e^t$ separately. Our initial choice for a particular solution, Y_1, of the first equation is $A_0t^3 + A_1t^2 + A_2t + A_3$; but since a constant is a solution of the homogeneous equation we must multiply by t. Thus $Y_1(t) = t(A_0t^3 + A_1t^2 + A_2t + A_3)$. For the second equation we first choose $Y_2(t) = Be^t$, but since both e^t and te^t are solutions of the homogeneous equation, we multiply by t^2 to obtain $Y_2(t) = Bt^2e^t$. Then $Y(t) = Y_1(t) + Y_2(t)$ by the superposition principle and $y(t) = y_c(t) + Y(t)$.

17. The complementary solution is $y_c(t) = c_1 + c_2e^{-t} + c_3e^t + c_4te^t$. The superposition principle allows us to consider separately the D.E. $y^{iv} - y''' - y'' + y' = t^2 + 4$ and $y^{iv} - y''' - y'' + y' = tsint$. For the first equation our initial choice is $Y_1(t) = A_0t^2 + A_1t + A_2$; but this must be multiplied by t since a constant is a solution of the homogeneous D.E. Hence $Y_1(t) = t(A_0t^2 + A_1t + A_2)$. For the second equation our initial choice that $Y_2 = (B_0t + B_1)cost + + (C_0t + C_1)sint$ does not need to be modified. Hence
$Y(t) = t(A_0t^2 + A_1t + A_2) + (B_0t + B_1)cost + (C_0t + C_1)sint.$

20. $(D-a)(D-b)f = (D-a)(Df-bf) = D^2f - (a+b)Df + abf$ and $(D-b)(D-a)f = (D-b)(Df-af) = D^2f - (b+a)Df + baf$. Since $a+b = b+a$ and $ab = ba$, we find the given equation holds for any function f.

22a. The D.E. of Problem 13 can be written as $D(D-1)^2y = t^3 + 2e^t$. Since D^4 annihilates t^3 and $(D-1)$ annihilates $2e^t$, we have $D^5(D-1)^3y = 0$, which corresponds to Eq.(ii) of Problem 21. The solution of this equation is $y(x) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + A_5 + (B_1t^2 + B_2t + B_3)e^{-t}$. Since $A_5 + (B_2t + B_3)e^{-t}$ are solutions of the homogeneous equation related to the original D.E., they may be deleted and thus $Y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + B_1t^2e^{-t}$.

22b. $(D+1)^2(D^2+1)$ annihilates the right side of the D.E. of Problem 14.

22e. $D^3(D^2+1)^2$ annihilates the right side of the D.E. of Problem 17.

Section 4.4, Page 229

1. The complementary solution is $y_c = c_1 + c_2\cos t + c_3\sin t$ and thus we assume a particular solution of the form $Y = u_1(t) + u_2(t)\cos t + u_3(t)\sin t$. Differentiating and assuming Eq.(5), we obtain $Y' = -u_2\sin t + u_3\cos t$ and

$$u_1' + u_2'\cos t + u_3'\sin t = 0 \quad \text{(a)}.$$

Continuing this process we obtain $Y'' = -u_2\cos t - u_3\sin t$, $Y''' = u_2\sin t - u_3\cos t - u_2'\cos t - u_3'\sin t$ and

$$-u_2'\sin t + u_3'\cos t = 0 \quad \text{(b)}.$$

Substituting Y and its derivatives, as given above, into the D.E. we obtain the third equation:

$$-u_2'\cos t - u_3'\sin t = \tan t \quad \text{(c)}.$$

Equations (a), (b) and (c) constitute Eqs.(10) of the text for this problem and may be solved to give $u_1' = \tan t$, $u_2' = -\sin t$, and $u_3' = -\sin^2 t/\cos t$. Thus $u_1 = -\ln\cos t$, $u_2 = \cos t$ and $u_3 = \sin t - \ln(\sec t + \tan t)$ and $Y = -\ln\cos t + 1 - (\sin t)\ln(\sec t + \tan t)$. Note that the constant 1 can be absorbed in c_1.

4. Replace tant in Eq. (c) of Prob. 1 by sect and use Eqs. (a) and (b) as in Prob. 1 to obtain $u_1' = \sec t$, $u_2' = -1$ and $u_3' = -\sin t/\cos t$.

5. Replace sect in Problem 7 with $e^{-t}\sin t$.

7. Since e^t, cost and sint are solutions of the related homogenous equation we have $Y(t) = u_1e^t + u_2\cos t + u_3\sin t$. Eqs. (10) then are

$u_1'e^t + u_2'\cos t + u_3'\sin t = 0$

$u_1'e^t - u_2'\sin t + u_3'\cos t = 0$

$u_1'e^t - u_2'\cos t - u_3'\sin t = \sec t$.

Using Abel's identity, $W(t) = c\exp(-\int p_1(t)dt) = ce^t$.

Using the above equations, $W(0) = 2$, so $c = 2$ and $W(t) = 2e^t$. From Eq.(11), we have $u'_1(t) = \dfrac{\text{sect } W_1(t)}{2e^t}$, where $W_1 = \begin{vmatrix} 0 & \text{cost} & \text{sint} \\ 0 & -\text{sint} & \text{cost} \\ 1 & -\text{cost} & -\text{sint} \end{vmatrix} = 1$ and thus

$u'_1(t) = \frac{1}{2}e^{-t}/\text{cost}$. Likewise

$u'_2 = \dfrac{\text{sect } W_2(t)}{2e^t} = -\frac{1}{2}\text{sect}(\text{cost} - \text{sint})$ and

$u'_3 = \dfrac{\text{sect } W_3(t)}{2e^t} = -\frac{1}{2}\text{sect}(\text{sint} + \text{cost})$. Thus

$u_1 = \frac{1}{2}\int_{t_0}^{t} \dfrac{e^{-s}ds}{\text{coss}}$, $u_2 = -\frac{1}{2}t - \frac{1}{2}\ln(\text{cost})$ and $u_3 = -\frac{1}{2}t + \frac{1}{2}\ln(\text{cost})$ which, when substituted into the assumed form for Y, yields the desired solution.

11. Since the D.E. is the same as in Problem 7, we may use the complete solution from that, with $t_0 = 0$. Thus

$y(0) = c_1 + c_2 = 2$, $y'(0) = c_1 + c_3 - \frac{1}{2} + \frac{1}{2} = -1$ and

$y''(0) = c_1 - c_2 + \frac{1}{2} - 1 + \frac{1}{2} = 1$. Again, a computer algebra system may be used to yield the respective derivatives.

14. Since a fundamental set of solutions of the homogeneous D.E. is $y_1 = e^t$, $y_2 = \text{cost}$, $y_3 = \text{sint}$, a particular solution is of the form $Y(t) = e^t u_1(t) + (\text{cost})u_2(t) + (\text{sint})u_3(t)$. Differentiating and making the same assumptions that lead to Eqs.(10), we obtain

$$u'_1e^t + u'_2\text{cost} + u'_3\text{sint} = 0$$
$$u'_1e^t - u'_2\text{sint} + u'_3\text{cost} = 0$$
$$u'_1e^t - u'_2\text{cost} - u'_3\text{sint} = g(t)$$

Solving these equations using either determinants or by elimination, we obtain $u'_1 = (1/2)e^{-t}g(t)$, $u'_2 = (1/2)(\text{sint} - \text{cost})g(t)$, $u'_3 = -(1/2)(\text{sint} + \text{cost})g(t)$. Integrating these and substituting into Y yields

$$Y(t) = \frac{1}{2}\{e^t\int_{t_0}^{t} e^{-s}g(s)ds + \cos t \int_{t_0}^{t}(\sin s - \cos s)g(s)ds$$
$$-\sin t\int_{t_0}^{t}(\sin s + \cos s)g(s)ds\}.$$

This can be written in the form

$$Y(t) = (1/2)\int_{t_0}^{t}(e^{t-s} + \cos t\sin s - \cos t\cos s$$
$$-\sin t\sin s - \sin t\cos s)g(s)ds.$$

If we use the trigonometric identities $\sin(A-B) = \sin A\cos B - \cos A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$, we obtain the desired result. Note: Eqs.(11) and (12) of this section give the same result, but it is not recommended to memorize these equations.

16. The particular solution has the form $Y = e^t u_1(t) + te^t u_2(t) + t^2e^t u_3(t)$. Differentiating, making the same assumptions as in the earlier problems, and solving the three linear equations for u_1', u_2', and u_3' yields $u_1' = (1/2)t^2e^{-t}g(t)$, $u_2' = -te^{-t}g(t)$ and $u_3' = (1/2)e^{-t}g(t)$. Integrating and substituting into Y yields the desired solution. For instance

$$te^t u_2 = -te^t\int_{t_0}^{t} se^{-s}g(s)ds = -\frac{1}{2}\int_{t_0}^{t} 2tse^{(t-s)}g(s)ds, \text{ and}$$

likewise for u_1 and u_3. If $g(t) = t^{-2}e^t$ then $g(s) = e^s/s^2$ and the integration is accomplished using the power rule. Note that terms involving t_0 become part of the complimentary solution.

CHAPTER 5

Section 5.1, Page 237

2. Use the ratio test:
$$\lim_{n\to\infty} \frac{|(n+1)x^{n+1}/2^{n+1}|}{|nx^n/2^n|} = \lim_{n\to\infty} \frac{n+1}{n}\,\frac{1}{2}\,|x| = \frac{|x|}{2}.$$
Therefore the series converges absolutely for $|x| < 2$. For $x = 2$ and $x = -2$ the n^{th} term does not approach zero as $n \to \infty$ so the series diverge. Hence the radius of convergence is $\rho = 2$.

5. Use the ratio test:
$$\lim_{n\to\infty} \frac{|(2x+1)^{n+1}/(n+1)^2|}{|(2x+1)^n/n^2|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}|2x+1| = |2x+1|.$$
Therefore the series converges absolutely for $|2x+1| < 1$, or $|x+1/2| < 1/2$. At $x = 0$ and $x = -1$ the series also converge absolutely. However, for $|x+1/2| > 1/2$ the series diverges by the ratio test. The radius of convergence is $\rho = 1/2$.

9. For this problem $f(x) = \sin x$. Hence $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \ldots$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1, \ldots$. The even terms in the series will vanish and the odd terms will alternate in sign. We obtain $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$. From the ratio test it follows that $\rho = \infty$.

12. For this problem $f(x) = x^2$. Hence $f'(x) = 2x$, $f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Then $f(-1) = 1$, $f'(-1) = -2$, $f''(-1) = 2$ and $x^2 = 1 - 2(x+1) + 2(x+1)^2/2! = 1 - 2(x+1) + (x+1)^2$. Since the series terminates after a finite number of terms, it converges for all x. Thus $\rho = \infty$.

13. For this problem $f(x) = \ln x$. Hence $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 1\cdot 2/x^3, \ldots$, and $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 1\cdot 2, \ldots$, $f^{(n)}(1) = (-1)^{n+1}(n-1)!$ The Taylor series is $\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n/n$. It follows from the ratio test that

the series converges absolutely for $|x-1| < 1$. However, the series diverges at $x = 0$ so $\rho = 1$.

18. Writing the individual terms of y, we have
$y = a_0 + a_1x + a_2x^2 + ... + a_ax^n + ...$, so
$y' = a_1 + 2a_2x + 3a_3x^2 + ... + (n+1)a_{n+1}x^n + ...$, and
$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + ... + (n+2)(n+1)a_{n+2}x^n + ...$.
If $y'' = y$, we then equate coefficients of like powers of x to obtain $2a_2 = a_0$, $3\cdot 2a_3 = a_1$, $4\cdot 3a_4 = a_2$, ... $(n+2)(n+1)a_{n+2} = a_n$, which yields the desired result for n = 0,1,2,3... .

19. Set m = n-1 on the right hand side of the equation. Then n = m+1 and when n = 1, m = 0. Thus the right hand side becomes $\sum_{m=0}^{\infty} a_m(x-1)^{m+1}$, which is the same as the left hand side when m is replaced by n.

23. Multiplying each term of the first series by x yields
$x\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=1}^{\infty} na_nx^n = \sum_{n=0}^{\infty} na_nx^n$, where the last equality can be verified by writing out the first few terms. Changing the index from k to n (n=k) in the second series then yields
$\sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} (n+1)a_nx^n$.

25. $\sum_{m=2}^{\infty} m(m-1)a_mx^{m-2} + x\sum_{k=1}^{\infty} ka_kx^{k-1} =$
$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} ka_kx^k =$
$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n$. In the first case we have let n = m − 2 in the first summation and multiplied each term of the second summation by x. In the second case we have let n = k and noted that for n = 0, $na_n = 0$.

28. If we shift the index of summation in the first sum by

letting m = n-1, we have $\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m$. Substituting this into the given equation and letting m = n again, we obtain:

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0, \text{ or}$$

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n]x^n = 0.$$

Hence $a_{n+1} = -2a_n/(n+1)$ for n = 0,1,2,3,... . Thus $a_1 = -2a_0$, $a_2 = -2a_1/2 = 2^2a_0/2$, $a_3 = -2a_2/3 = -2^3a_0/2\cdot 3 = -2^3a_0/3!$... and $a_n = (-1)^n 2^n a_0/n!$. Notice that for n = 0 this formula reduces to a_0 so we can write

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n 2^n\, a_0 x^n/n! = a_0\sum_{n=0}^{\infty} (-2x)^n/n! = a_0 e^{-2x}.$$

Section 5.2, Page 247

2. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and since we must multiply y′ by x in the D.E. we do not shift the index; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Substituting in the D.E., we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In order to have the starting point the same in all three summations, we let n = 0 in the first and third terms to obtain the following

$$(2\cdot 1\ a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0.$$

Thus $a_{n+2} = a_n/(n+2)$ for n = 1,2,3,... . Note that the recurrence relation is also correct for n = 0. We show how to calculate the odd a's:

$a_3 = a_1/3$, $a_5 = a_3/5 = a_1/5\cdot3$, $a_7 = a_5/7 = a_1/7\cdot5\cdot3, \ldots$.
Now notice that $a_3 = 2a_1/(2\cdot3) = 2a_1/3!$, that
$a_5 = 2\cdot4a_1/(2\cdot3\cdot4\cdot5) = 2^2\cdot2a_1/5!$, and that
$a_7 = 2\cdot4\cdot6a_1/(2\cdot3\cdot4\cdot5\cdot6\cdot7) = 2^3\cdot3!\ a_1/7!$. Likewise
$a_9 = a_7/9 = 2^3\cdot3!\ a_1/(7!)9 = 2^3\cdot3!\ 8a_1/9! = 2^4\cdot4!\ a_1/9!$.
Continuing we have $a_{2m+1} = 2^m m!\ a_1/(2m+1)!$. In the same way we find that the even a's are given by $a_{2m} = a_0/2^m\ m!$. Thus

$$y = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} \frac{2^m m!\ x^{2m+1}}{(2m+1)!}.$$

3. $y = \sum_{n=0}^{\infty} a_n(x-1)^n$; $y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n$,
and
$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n$.
Substituting in the D.E. and setting $x = 1 + (x-1)$ we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0,$$

where the third term comes from:

$$-(x-1)y' = \sum_{n=1}^{\infty} (n+1)a_{n+1}(x-1)^{n+1} = -\sum_{n=1}^{\infty} na_n(x-1)^n.$$

Letting n = 0 in the first, second, and the fourth sums, we obtain

$$(2\cdot1\cdot a_2 - 1\cdot a_1 - a_0)(x-1)^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n](x-1)^n = 0.$$

Thus $(n+2)a_{n+2} - a_{n+1} - a_n = 0$ for $n = 0,1,2, \ldots$. This recurrance relation can be used to solve for a_2 in terms of a_0 and a_1, then for a_3 in terms of a_0 and a_1, etc. In many cases it is easier to first take $a_0 = 0$ and generate

one solution and then take $a_1 = 0$ and generate the second linearly independent solution. Thus, choosing $a_0 = 0$ we find that $a_2 = a_1/2$, $a_3 = (a_2+a_1)/3 = a_1/2$, $a_4 = (a_3+a_2)/4 = a_1/4$, $a_5 = (a_4+a_3)/5 = 3a_1/20, \ldots$. This yields the solution $y_2(x) = a_1[(x-1) + (x-1)^2/2 + (x-1)^3/2 + (x-1)^4/4 + 3(x-1)^5/20 + \ldots]$. The second independent solution may be obtained by choosing $a_1 = 0$. Then $a_2 = a_0/2$, $a_3 = (a_2+a_1)/3 = a_0/6$, $a_4 = (a_3+a_2)/4 = a_0/6$, $a_5 = (a_4+a_3)/5 = a_0/15, \ldots$. This yields the solution $y_1(x) = a_0[1+(x-1)^2/2+(x-1)^3/6+(x-1)^4/6+(x-1)^5/15+\ldots]$.

5. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substituting in the D.E. and shifting the index in both summations for y'' gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$(2\cdot 1\cdot a_2 + a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n]x^n = 0.$$

Thus $a_2 = -a_0/2$ and $a_{n+2} = na_{n+1}/(n+2) - a_n/(n+2)(n+1)$, $n = 1,2, \ldots$. Choosing $a_0 = 0$ yields $a_2 = 0$, $a_3 = -a_1/6$, $a_4 = 2a_3/4 = -a_1/12, \ldots$ which gives one solution as $y_2(x) = a_1(x - x^3/6 - x^4/12 + \ldots)$. A second linearly independent solution is obtained by choosing $a_1 = 0$. Then $a_2 = -a_0/2$, $a_3 = a_2/3 = -a_0/6$, $a_4 = 2a_3/4 - a_2/12 = -a_0/24, \ldots$ which gives $y_1(x) = a_0(1 - x^2/2 - x^3/6 - x^4/24 + \ldots)$.

8. If $y = \sum_{n=1}^{\infty} a_n(x-1)^n$ then

$$xy = [1+(x-1)]y = \sum_{n=1}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_n(x-1)^{n+1},$$

$$y' = \sum_{n=1}^{\infty} n a_n(x-1)^{n-1}, \text{ and}$$

$$xy'' = [1+(x-1)]y''$$

$$= \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-1}.$$

14. You will need to rewrite x+1 as 3 + (x-2) in order to multiply x+1 times y′ as a power series about $x_0 = 2$.

16a. From Problem 6 we have

$y(x) = c_1(1 - x^2 + \frac{1}{6}x^4+...) + c_2(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + ...)$.

Now $y(0) = c_1 = -1$ and $y'(0) = c_2 = 3$ and thus

$y(x) = -1+x^2-\frac{1}{6}x^4+3x-\frac{3}{4}x^3 = -1+3x+x^2-\frac{3}{4}x^3-\frac{1}{6}x^4 + ...$.

16c. By plotting $f = -1 + 3x + x^2 - 3x^3/4$ and $g = f - x^4/6$ between −1 and 1 it appears that f is a reasonable approximation for $|x| < 0.7$.

19. The D.E. transforms into $u''(t) + t^2u'(t) + (t^2+2t)u(t) = 0$.

Assuming that $u(t) = \sum_{n=0}^{\infty} a_nt^n$, we have $u'(t) = \sum_{n=1}^{\infty} na_nt^{n-1}$ and

$u''(t) = \sum_{n=2}^{\infty} n(n-1)a_nt^{n-2}$. Substituting in the D.E. and shifting indices yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0,$$

$$2\cdot1\cdot a_2t^0 + (3\cdot2\cdot a_3 + 2\cdot a_0)t^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}]t^n = 0.$$

It follows that $a_2 = 0$, $a_3 = -a_0/3$ and $a_{n+2} = -a_{n-1}/(n+2) - a_{n-2}/[(n+2)(n+1)]$, $n = 2,3,4...$. We obtain one solution by choosing $a_1 = 0$. Then $a_4 = -a_0/12$, $a_5 = -a_2/5 - a_1/20 = 0$, $a_6 = -a_3/6 - a_2/30 = a_0/18,...$. Thus one solution is $u_1(t) = a_0(1 - t^3/3 - t^4/12 + t^6/18 + ...)$ so

$y_1(x) = u_1(x-1) = a_0[1 - (x-1)^3/3 - (x-1)^4/12 + (x-1)^6/18 + ...]$.

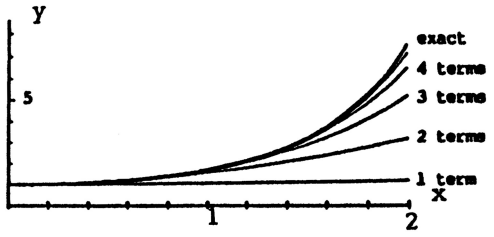
We obtain a second solution by choosing $a_0 = 0$. Then $a_4 = -a_1/4$, $a_5 = -a_2/5 - a_1/20 = -a_1/20$, $a_6 = -a_3/6 - a_2/30 = 0$, $a_7 = -a_4/7 - a_3/42 = a_1/28, \ldots$. Thus a second linearly independent solution is $u_2(t) = a_1[t - t^4/4 - t^5/20 + t^7/28 + \ldots]$ or $y_2(x) = u_2(x-1)$

$= a_1[(x-1) - (x-1)^4/4 - (x-1)^5/20 + (x-1)^7/28 + \ldots]$.

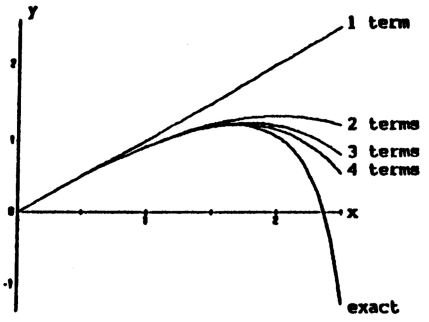
The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = (x-1) + 1$ so $x^2 = (x-1)^2 + 2(x-1) + 1$ and $x^2 - 1 = (x-1)^2 + 2(x-1)$. The D.E. now appears as $y'' + (x-1)^2y' + [(x-1)^2 + 2(x-1)]y = 0$ which is identical to the transformed equation with $t = x - 1$.

22b. $y = a_0 + a_1x + a_2x^2 + \ldots$, $y^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + \ldots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \ldots$, and $(y')^2 = a_1^2 + 4a_1a_2x + (6a_1a_3 + 4a_2^2)x^2 + \ldots$. Substituting these into $(y')^2 = 1 - y^2$ and collecting coefficients of like powers of x yields $(a_1^2 + a_0^2 - 1) + (4a_1a_2 + 2a_0a_1)x + (6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2)x^2 + \ldots = 0$. As in the earlier problems, each coefficient must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_0^2 - 1 = 0$ gives $a_1^2 = 1$. However, the D.E.indicates that y' is always positive, so $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1a_2 + 2a_0a_1 = 0$ implies that $a_2 = 0$; and $6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2 = 6a_1a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus $y = x - x^3/3! + \ldots$.

23.



26.



26. We have $y(x) = a_0y_1 + a_1y_2$, where y_1 and y_2 are found in

Problem 10. Now $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$. Thus $y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240}$.

Section 5.3, Page 253

1. The D.E. can be solved for y'' to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then $\phi''(x) = -x\phi'(x) - \phi(x)$ and thus setting $x = 0$ we obtain $\phi''(0) = - 0 - 1 = -1$. Differentiating the equation for y'' yields $y''' = -xy'' - 2y'$ and hence setting $y = \phi(x)$ again yields $\phi'''(0) = -0 - 0 = 0$. In a similar fashion $y^{iv} = -xy''' - 3y''$ and thus $\phi^{iv}(0) = - 0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

6. The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is 1,3, and 1, respectively. Thus a lower bound for the radius of convergence for series solutions in powers of $(x-4)$, $(x+4)$, and x is $\rho = 1$, $\rho = 3$, and $\rho = 1$, respectively.

9a. Since $P(x) = 1$ has no zeros, the radius of convergence for $x_0 = 0$ is $\rho = \infty$.

9f. Since $P(x) = x^2 + 2$ has zeros at $x = \pm\sqrt{2}\ i$, the lower bound for the radius of convergence of the series solution about $x_0 = 0$ is $\rho = \sqrt{2}$.

9h. Since $x_0 = 1$ and $P(x) = x$ has a zero at $x = 0$, $\rho = 1$.

10a. If we assume that $y = \sum_{n=2}^{\infty} a_n x^n$, then $y' = \sum_{n=2}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Substituting in the D.E., shifting indices of summation, and collecting coefficients of like powers of x yields the equation $(2\cdot 1\cdot a_2 + \alpha^2 a_0)x^0 + [3\cdot 2\cdot a_3 + (\alpha^2 - 1)a_1]x^1$

$$+ \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (\alpha^2-n^2)a_n]x^n = 0.$$

Hence the recurrence relation is $a_{n+2} = (n^2-\alpha^2)a_n/(n+2)(n+1)$, $n = 0, 1, 2, \ldots$. For the first solution we choose $a_1 = 0$. We find that $a_2 = -\alpha^2 a_0/2\cdot 1$, $a_3 = 0$, $a_4 = (2^2-\alpha^2)a_2/4\cdot 3 = -(2^2-\alpha^2)\alpha^2 a_0/4!$ $\ldots$, $a_{2m} = -[(2m-2)^2 - \alpha^2]\ldots (2^2-\alpha^2)\alpha^2 a_0/(2m)!$, and $a_{2m+1} = 0$, so $y_1(x) = 1 - \frac{\alpha^2}{2!}x^2 - \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \ldots$

$$-\frac{[(2m-2)^2-\alpha^2]\ldots(2^2-\alpha^2)\alpha^2}{(2m)!}x^{2m} - \ldots,$$

where we have set $a_0 = 1$. For the second solution we take $a_0 = 0$ and $a_1 = 1$ in the recurrence relation to obtain the desired solution.

10b. If α is an even integer $2k$ then $(2m-2)^2 - \alpha^2 = (2m-2)^2 - 4k^2 = 0$. Thus when $m = k+1$ all terms in the series for $y_1(x)$ are zero after the x^{2k} term. A similar argument shows that if $\alpha = 2k+1$ then all terms in $y_2(x)$ are zero after the x^{2k+1}.

11. The Taylor series about $x = 0$ for $\sin x$ is $\sin x = x - x^3/3! + x^5/5! - \ldots$. Assuming that $y = \sum_{n=2}^{\infty} a_n x^n$ we find $y'' + (\sin x)y = 2a_2 + 6a_3 x + 12a_4 x^2$

$+ 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots$

$+ (x-x^3/3!+x^5/5!-\ldots)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$

$= 2a_2 + (6a_3+a_0)x + (12a_4+a_1)x^2 + (20a_5+a_2-a_0/6)x^3 +$

$(30a_6+ a_3-a_1/6)x^4 + (42a_7+a_4+a_0/5!)x^5 + \ldots = 0$. Hence $a_2 = 0$, $a_3 = -a_0/6$, $a_4 = -a_1/12$, $a_5 = a_0/120$, $a_6 = (a_1+a_0)/180$, $a_7 = -a_0/7! + a_1/504$, $\ldots$. We set $a_0 = 1$ and $a_1 = 0$ and obtain $y_1(x) = (1 - x^3/6 + x^5/120 + x^6/180 + \ldots)$. Next we set $a_0 = 0$ and $a_1 = 1$ and obtain $y_2(x) = (x - x^4/12 + x^6/180 + x^7/504 + \ldots)$. Since

$p(x) = 1$ and $q(x) = \sin x$ both have $\rho = \infty$, the solution in this case converges for all x, that is, $\rho = \infty$

18. We know that $e^x = 1 + x + x^2/2! + X^3/3! + \ldots$, and therefore $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + \ldots$. Hence, if $y = \Sigma a_n x^n$, we have

$$a_1 + 2a_2x + 3a_3x^2 + \ldots = (1+x^2 + x^4/2+\ldots)(a_0+a_1x+a_2x^2+\ldots)$$
$$= a_0 + a_1x + (a_0+a_2)x^2 + \ldots:$$

Thus, $a_1 = a_0$ $2a_2 = a_1$ and $3a_3 = a_0 + a_2$, which yield the desired solution.

20. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the D.E. we obtain

$\sum_{n=2}^{\infty} n a_n x^{n-1} - \sum_{n=2}^{\infty} a_n x^n = x^2$. Shifting indices in the summation yields $\sum_{n=2}^{\infty} [(n+1)a_{n+1} - a_n]x^n = x^2$. Equating coefficients of both sides then gives: $a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 1$ and $(n+1)a_{n+1} = a_n$ for $n = 3,4,\ldots$. Thus $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = 1/3 + a_2/3 = 1/3 + a_0/2\cdot 3$, $a_4 = a_3/4 = 1/3\cdot 4 + a_0/2\cdot 3\cdot 4$, ..., $a_n = a_{n-1}/n = 2/n! + a_0/n!$ and hence

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\ldots\right) + 2\left(\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+\ldots\right).$$

Using the power series for e^x, the first and second sums can be rewritten as $a_0e^x + 2(e^x - 1 - x - x^2/2)$.

22. Substituting $y = \sum_{n=2}^{\infty} a_n x^n$ into the Legendre equation, shifting indices, and collecting coefficients of like powers of x yields

$[2\cdot 1\cdot a_2 + \alpha(\alpha+1)a_0]x^0 + \{3\cdot 2\cdot a_3 - [2\cdot 1 - \alpha(\alpha+1)]a_1\}x^1 +$

$\sum_{n=2}^{\infty} \{(n+2)(n+1)a_{n+2} - [n(n+1) - \alpha(\alpha+1)]a_n\}x^n = 0$. Thus

$a_2 = -\alpha(\alpha+1)a_0/2!$, $a_3 = [2\cdot 1 - \alpha(\alpha+1)]a_1/3! =$ $-(\alpha-1)(\alpha+2)a_1/3!$ and the recurrence relation is

$(n+2)(n+1)a_{n+2} = -[\alpha(\alpha+1) - n(n+1)]a_n = -(\alpha-n)(\alpha+n+1)a_n$, $n = 2,3,\ldots$. Setting $a_1 = 0$, $a_0 = 1$ yields a solution with $a_3 = a_5 = a_7 = \ldots = 0$ and

$a_4 = \alpha(\alpha-2)(\alpha+1)(\alpha+3)/4!,\ldots,$ $a_{2m} = (-1)^m\alpha(\alpha-2)(\alpha-4)\ldots(\alpha-2m+2)(\alpha+1)(\alpha+3)\ldots(\alpha+2m-1)/(2m)!,\ldots$. The second linearly independent solution is obtained by setting $a_0 = 0$ and $a_1 = 1$. The coefficients are $a_2 = a_4 = a_6 = \ldots = 0$ and $a_3 = -(\alpha-1)(\alpha+2)/3!$, $a_5 = -(\alpha-3)(\alpha+4)a_3/5\cdot4 = (\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)/5!,\ldots$.

26. Using the chain rule we have:

$$\frac{dF(\phi)}{d\phi} = \frac{dF[\phi(x)]}{dx}\,\frac{dx}{d\phi} = -f'(x)\sin\phi(x) = -f'(x)\sqrt{1-x^2},$$

$$\frac{d^2F(\phi)}{d\phi^2} = \frac{d}{dx}[-f'(x)\sqrt{1-x^2}]\,\frac{dx}{d\phi} = (1-x^2)f''(x) - xf'(x),$$

which when substituted into the D.E. yields the desired result.

28. Since $[(1-x^2)y']' = (1-x^2)y'' - 2xy'$, the Legendre Equation, from Problem 22, can be written as shown. Thus, carrying out the steps indicated yields the two equations:

$$P_m[(1-x^2)P_n']' = -n(n+1)P_nP_m$$

$$P_n[(1-x^2)P_m']' = -m(m+1)P_nP_m.$$

As long as $n \neq m$ the second equation can be subtracted from the first and the result integrated from -1 to 1 to obtain

$$\int_{-1}^{1}\{P_m[(1-x^2)P_n']'-P_n[(1-x^2)P_m']'\}dx = [m(m+1)-n(n+1)]\int_{-1}^{1}P_nP_m dx$$

The left side may be integrated by parts to yield

$$[P_m(1-x^2)P_n' - P_n(1-x^2)P_m']_{-1}^{1} + \int_{-1}^{1}[P_m'(1-x^2)P_n' - P_n'(1-x^2)P_m']dx,$$

which is zero. Thus $\int_{-1}^{1}P_n(x)P_m(x)dx = 0$ for $n \neq m$.

Section 5.4, Page 259

1. Since the coefficients of y, y′ and y″ have no common factors and since P(x) vanishes only at x = 0 we conclude that x = 0 is a singular point. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we obtain $p(x) = (1-x)/x$ and $q(x) = 1$. Thus for the singular point we have

$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} 1-x = 1$, $\lim_{x\to 0} x^2 q(x) = 0$ and thus $x = 0$ is a regular singular point.

5. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we find $p(x) = x/(1-x)(1+x)^2$ and $q(x) = 1/(1-x^2)(1+x)$. Therefore $x = \pm 1$ are singular points. Since $\lim_{x\to 1} (x-1)p(x)$ and $\lim_{x\to 1} (x-1)^2 q(x)$ both exist, we conclude $x = 1$ is a regular singular point. Finally, since $\lim_{x\to -1} (x+1)p(x)$ does not exist, we find that $x = -1$ is an irregular singular point.

12. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = e^x/x$ and $q(x) = (3\cos x)/x$. Thus $x = 0$ is a singular point. Since $xp(x) = e^x$ is analytic at $x = 0$ and $x^2 q(x) = 3x\cos x$ is analytic at $x = 0$ the point $x = 0$ is a regular singular point.

17. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = \dfrac{x}{\sin x}$ and $q(x) = \dfrac{4}{\sin x}$. Since $\lim_{x\to 0} q(x)$ does not exist, the point $x_0 = 0$ is a singular point and since neither $\lim_{x\to \pm n\pi} p(x)$ nor $\lim_{x\to \pm n\pi} q(x)$ exist either the points $x_0 = \pm n\pi$ are also singular points. To determine whether the singular points are regular or irregular we must use Eq.(8) and the result #7 of multiplication and division of power series from Section 5.1. For $x_0 = 0$, we have

$$xp(x) = \frac{x^2}{\sin x} = \frac{x^2}{x - \frac{x^3}{6} + \cdots} = x\left[1 + \frac{x^2}{6} + \cdots\right]$$

$$= x + \frac{x^3}{6} + \cdots,$$

which converges about $x_0 = 0$ and thus $xp(x)$ is analytic at $x_0 = 0$. $x^2 q(x)$, by similar steps, is also analytic at $x_0 = 0$ and thus $x_0 = 0$ is a regular singular point. For $x_0 = n\pi$, we have

$$(x-n\pi)p(x) = \frac{(x-n\pi)x}{\sin x} = \frac{(x-n\pi)[(x-n\pi) + n\pi]}{\pm(x-n\pi) \mp \frac{(x-n\pi)^3}{6} \pm \cdots}$$

$= [(x-n\pi)+n\pi][\pm 1 \pm \frac{(x-n\pi)^2}{6} \pm \ldots]$, which converges about $x_0 = n\pi$ and thus $(x-n\pi)p(x)$ is analytic at $x= n\pi$. Similarly $(x+n\pi)p(x)$ and $(x\pm n\pi)^2 q(x)$ are analytic and thus $x_0 = \pm n\pi$ are regular singular points.

19. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the D.E. yields $2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 3\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$. The last sum becomes $\sum_{n=2}^{\infty} a_{n-2} x^{n-1}$ by replacing n+1 by n-1, the first term of the middle sum is $3a_1$, and thus we have $3a_1 + \sum_{n=2}^{\infty} \{[2n(n-1)+3n]a_n + a_{n-2}\}x^{n-1} = 0$. Hence $a_1 = 0$ and $a_n = \frac{-a_{n-2}}{n(2n+1)}$, which is the desired recurrance relation. Thus all even coefficients are found in terms of a_0 and all odd coefficients are zero, thereby yielding only one solution of the desired form.

21. If $\xi = 1/x$ then

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -\frac{1}{x^2}\frac{dy}{d\xi} = -\xi^2\frac{dy}{d\xi},$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\xi}(-\xi^2\frac{dy}{d\xi})\frac{d\xi}{dx} = (-2\xi\frac{dy}{d\xi} - \xi^2\frac{d^2y}{d\xi^2})(-\frac{1}{x^2})$$

$$= \xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}.$$

Substituting in the D.E. we have

$$P(1/\xi)[\xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}] + Q(1/\xi)[-\xi^2\frac{dy}{d\xi}] + R(1/\xi)y = 0,$$

$$\xi^4 P(1/\xi)\frac{d^2y}{d\xi^2} + [2\xi^3 P(1/\xi) - \xi^2 Q(1/\xi)]\frac{dy}{d\xi} + R(1/\xi)y = 0.$$

The result then follows from the theory of singular points at $\xi = 0$.

23. Since $P(x) = x^2$, $Q(x) = x$ and $R(x) = -4$ we have

$f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4 P(1/\xi) = -4/\xi^2$. Thus the point at infinity is a singular point. Since both $\xi f(\xi)$ and $\xi^2 g(\xi)$ are analytic at $\xi = 0$, the point at infinity is a regular singular point.

25. Since $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \upsilon^2$, $f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4 P(1/\xi) = (1/\xi^2 - \upsilon^2)/\xi^2 = 1/\xi^4 - \upsilon^2/\xi^2$. Thus the point at infinity is a singular point. Although $\xi f(\xi) = 1$ is analytic at $\xi = 0$, $\xi^2 g(\xi) = 1/\xi^2 - \upsilon^2$ is not, so the point at infinity is an irregular singular point.

Section 5.5, Page 265

2. Comparing the D.E. to Eq.(27), we seek solutions of the form $y = (x+1)^r$ for $x + 1 > 0$. Substitution of y into the D.E. yields $[r(r-1) + 3r + 3/4](x+1)^r = 0$. Thus $r^2 + 2r + 3/4 = 0$, which yields $r = -3/2, -1/2$. The general solution of the D.E. is then $y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}$, $x \neq -1$.

4. If $y = x^r$ then $r(r-1) + 3r + 5 = 0$. So $r^2 + 2r + 5 = 0$ and $r = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$. Thus the general solution of the D.E. is $y = c_1 x^{-1}\cos(2\ln|x|) + c_2 x^{-1}\sin(2\ln|x|)$, $x \neq 0$.

9. Again let $y = x^r$ to obtain $r(r-1) - 5r + 9 = 0$, or $(r-3)^2 = 0$. Thus the roots are $x = 3,3$ and $y = c_1 x^3 + c_2 x^3 \ln|x|$, $x \neq 0$, is the solution of the D.E.

13. If $y = x^r$, then $F(r) = 2r(r-1) + r - 3 = 2r^2 - r - 3 = (2r-3)(r+1) = 0$, so $y = c_1 x^{3/2} + c_2 x^{-1}$ and $y' = \frac{3}{2} c_1 x^{1/2} - c_2 x^{-2}$. Setting $x = 1$ in y and y′ we obtain $c_1 + c_2 = 1$ and $\frac{3}{2} c_1 - c_2 = 4$, which yield $c_1 = 2$ and $c_2 = -1$. Hence $y = 2x^{3/2} - x^{-1}$. As $x \to 0^+$ we have $y \to -\infty$ due to the second term.

16. We have $F(r) = r(r-1) + 3r + 5 = r^2 + 2r + 5 = 0$. Thus

$r_1, r_2 = -1 \pm 2i$ and $y = x^{-1}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$. Then $y(1) = c_1 = 1$ and $y' = -x^{-2}[\cos(2\ln x) + c_2\sin(2\ln x)] + x^{-1}[-\sin(2\ln x)2/x + c_2\cos(2\ln x)2/x]$ so that $y'(1) = -1-2c_2 = -1$, or $c_2 = 0$.

17. Substituting $y = x^r$, we find that $r(r-1) + \alpha r + 5/2 = 0$ or $r^2 + (\alpha-1)r + 5/2 = 0$. Thus $r_1, r_2 = [-(\alpha-1) \pm \sqrt{(\alpha-1)^2-10}]/2$. In order for solutions to approach zero as $x \to 0$ it is necessary that the real parts of r_1 and r_2 be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of r_1 and r_2 will be negative. Suppose that $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose that $\alpha < 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $|\alpha-1| = 1 - \alpha$; hence the real parts of r_1 and r_2 will be positive. Thus if $\alpha < 1$ the solutions of the D.E. will approach zero as $x \to 0$.

21. In all cases the roots of $F(r) = 0$ are given by Eq.(5) and the forms of the solution are given in Theorem 5.5.1.

21a. The real part of the root must be positive so, from Eq.(5), $\alpha < 0$. Also $\beta > 0$, since the $\sqrt{(\alpha-1)^2-4\beta}$ term must be less than $|\alpha-1|$.

22. Assume that $y = v(x)x^{r_1}$. Then $y' = v(x)r_1x^{r_1-1} + v'(x)x^{r_1}$ and $y'' = v(x)r_1(r_1-1)x^{r_1-2} + 2v'(x)r_1x^{r_1-1} + v''(x)x^{r_1}$. Substituting in the D.E. and collecting terms yields $x^{r_1+2}\, v'' + (\alpha + 2r_1)x^{r_1+1}\, v' + [r_1(r_1-1) + \alpha r_1 + \beta]x^{r_1}\, v = 0$. Now we make use of the fact that r_1 is a double root of $f(r) = r(r-1) + \alpha r + \beta$. This means that $f(r_1) = 0$ and $f'(r_1) = 2r_1 - 1 + \alpha = 0$. Hence the D.E. for v reduces to $x^{r_1+2}\, v'' + x^{r_1+1}\, v'$. Since $x > 0$ we may divide by x^{r_1+1} to obtain $xv'' + v' = 0$. Thus $v(x) = \ln x$ and a second solution is $y = x^{r_1}\ln x$.

25. The change of variable $x = e^z$ transforms the D.E. into $u'' - 4u' + 4u = z$, which has the solution

$u(z) = c_1e^{2z} + c_2ze^{2z} + (1/4)z + 1/4$. Hence
$y(x) = c_1x^2 + c_2x^2\ln x + (1/4)\ln x + 1/4$.

31. If $x > 0$, then $|x| = x$ and $|x|^{r_1} = x^{r_1}$ so we can choose $c_1 = k_1$. If $x < 0$, then $|x| = -x$ and $|x|^{r_1} = (-x)^{r_1} = (-1)^{r_1}x^{r_1}$ and we can choose $c_1 = (-1)^{r_1}k_1$, or $k_1 = (-1)^{-r_1}c_1 = (-1)^{r_1}c_1$. In both cases we have $c_2 = k_2$.

Section 5.6, Page 271

2. If the D.E. is put in the standard form $y'' + p(x)y + q(x)y = 0$, then $p(x) = x^{-1}$ and $q(x) = 1 - 1/9x^2$. Thus $x = 0$ is a singular point. Since $xp(x) \to 1$ and $x^2q(x) \to -1/9$ as $x \to 0$ it follows that $x = 0$ is a regular singular point. In determining a series solution of the D.E. it is more convenient to leave the equation in the form given rather than divide by the x^2, the coefficient of y''. If we substitute $y = \sum_{n=0}^{\infty} a_nx^{n+r}$, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_nx^{n+r} + \sum_{n=0}^{\infty} (n+r)a_nx^{n+r} + (x^2 - \frac{1}{9})\sum_{n=0}^{\infty} a_nx^{n+r} = 0.$$

Note that $x^2\sum_{n=0}^{\infty} a_nx^{n+r} = \sum_{n=0}^{\infty} a_nx^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$. Thus we have $[r(r-1) + r - \frac{1}{9}]a_0x^r + [(r+1)r + (r+1) - \frac{1}{9}]a_1x^{r+1} +$

$$\sum_{n=2}^{\infty} \{[(n+r)(n+r-1) + (n+r) - \frac{1}{9}]a_n + a_{n-2}\} x^{n+r} = 0.$$

From the first term, the indicial equation is $r^2 - 1/9 = 0$ with roots $r_1 = 1/3$ and $r_2 = -1/3$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2 - 1/9]a_n = -a_{n-2}$. For $r = 1/3$ we have

$$a_n = \frac{-a_{n-2}}{(n + \frac{1}{3})^2-(\frac{1}{3})^2} = -\frac{a_{n-2}}{(n + \frac{2}{3})n}, \quad n = 2,3,4,\ldots .$$

Since $a_1 = 0$ it follows from the recurrence relation that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients it is convenient to let $n = 2m$, $m = 1,2,3,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m(m + \frac{1}{3})$. The first few coefficients are given by

$$a_2 = \frac{(-1)a_0}{2^2(1 + \frac{1}{3})1}, \quad a_4 = \frac{(-1)a_2}{2^2(2 + \frac{1}{3})2} = \frac{a_0}{2^4(1 + \frac{1}{3})(2 + \frac{1}{3})2!}$$

$$a_6 = \frac{(-1)a_4}{2^2(3 + \frac{1}{3})3} = \frac{(-1)a_0}{2^6(1 + \frac{1}{3})(2 + \frac{1}{3})(3 + \frac{1}{3})3!}, \text{ and the}$$

coefficent of x^{2m} for $m = 1, 2,\ldots$ is

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!(1 + \frac{1}{3})(2 + \frac{1}{3}) \ldots (m + \frac{1}{3})}. \text{ Thus one}$$

solution (on setting $a_0 = 1$) is

$$y_1(x) = x^{1/3}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(1 + \frac{1}{3})(2 + \frac{1}{3})\ldots(m + \frac{1}{3})}(\frac{x}{2})^{2m}].$$

Since $r_2 = -1/3 \neq r_1$ and $r_1 - r_2 = 2/3$ is not an integer, we can calculate a second series solution corresponding to $r = -1/3$. The recurrence relation is $n(n-2/3)a_n = -a_{n-2}$, which yields the desired solution following the steps just outlined. Note that $a_1 = 0$, as in the first solution, and thus all the odd coefficients are zero.

4. Putting the D.E. in standard form $y''+p(x)y'+q(x)y = 0$, we see that $p(x) = 1/x$ and $q(x) = -1/x$. Thus $x = 0$ is a singular point, and since $xp(x) \to 1$ and $x^2q(x) \to 0$, as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in $xy'' + y' - y = 0$ and shifting indices we obtain

$$\sum_{n=-1}^{\infty} a_{n+1}(r+n+1)(r+n)x^{n+r} + \sum_{n=-1}^{\infty} a_{n+1}(r+n+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

$[r(r-1) + r]a_0x^{-1+r} + \sum_{n=0}^{\infty} [(r+n+1)^2a_{n+1} - a_n]x^{n+r} = 0$. The indicial equation is $r^2 = 0$ so $r = 0$ is a double root. Thus we will obtain only one series of the form $y = x^r \sum_{n=0}^{\infty} a_nx^n$. The recurrence relation is $(n+1)^2a_{n+1} = a_n$, $n = 0,1,2,\ldots$. The coefficients are $a_1 = a_0$, $a_2 = a_1/2^2 = a_0/2^2$, $a_3 = a_2/3^2 = a_0/3^2\cdot2^2$, $a_4 = a_3/4^2 = a_0/4^2\cdot3^2\cdot2^2, \ldots$ and $a_n = a_0/(n!)^2$. Thus one solution (on setting $a_0 = 1$) is $y = \sum_{n=0}^{\infty} x^n/(n!)^2$.

11. If we make the change of variable $t = x-1$ and let $y = u(t)$, then the Legendre equation transforms to $(t^2 + 2t)u''(t) + 2(t+1)u'(t) - \alpha(\alpha+1)u(t) = 0$. Since $x = 1$ is a regular singular point of the original equation, we know that $t = 0$ is a regular singular point of the transformed equation. Substituting $u = \sum_{n=0}^{\infty} a_nt^{n+r}$ in the transformed equation and shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_nt^{n+r} + 2\sum_{n=-1}^{\infty} (n+r+1)(n+r)a_{n+1}t^{n+r}$$
$$+ 2\sum_{n=0}^{\infty} (n+r)a_nt^{n+r} + 2\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}t^{n+r}$$
$$- \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nt^{n+r} = 0, \text{ or}$$
$$[2r(r-1) + 2r]a_0\, t^{r-1} + \sum_{n=0}^{\infty} \{2(n+r+1)^2a_{n+1}$$
$$+ [(n+r)(n+r+1) - \alpha(\alpha+1)]a_n\}t^{n+r} = 0.$$

The indicial equation is $2r^2 = 0$ so $r = 0$ is a double root. Thus there will be only one series solution of the

form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$. The recurrence relation is $2(n+1)^2 a_{n+1} = [\alpha(\alpha+1) - n(n+1)]a_n, n = 0,1,2,\ldots$. We have $a_1 = [\alpha(\alpha+1)]a_0/2\cdot 1^2$, $a_2 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2]a_0/2^2\cdot 2^2\cdot 1^2$, $a_3 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2][\alpha(\alpha+1) - 2\cdot 3]a_0/2^3\cdot 3^2\cdot 2^2\cdot 1^2, \ldots$, and $a_n = [\alpha(\alpha+1)][\alpha(\alpha+1)-1\cdot 2]..[\alpha(\alpha+1)-(n-1)n]a_0/2^n(n!)^2$. Reverting to the variable x it follows that one solution of the Legendre equation in powers of x-1 is

$$y_1(x) = \sum_{n=0}^{\infty} [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2] \ldots$$

$[\alpha(\alpha+1) - (n-1)n](x-1)^n/2^n(n!)^2$ where we have set $a_0 = 1$, which is equivalent to the answer in the text if a (-1) is taken out of each square bracket.

14. The standard form is $y'' + p(x)y' + q(x)y = 0$, with $p(x) = 1/x$ and $q(x) = 1$. Thus $x = 0$ is a singular point; and since $xp(x) \to 1$ and $x^2q(x) \to 0$ as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + x^2y = 0$ and shifting indices appropriately, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

or

$$[r(r-1)+r]a_0x^r + [(1+r)r+1+r]a_1x^{r+1} + \sum_{n=2}^{\infty} [(n+r)^2a_n + a_{n-2}]x^{n+r} = 0.$$

The indicial equation is $r^2 = 0$ so $r = 0$ is a double root. It is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation in $n^2a_n = -a_{n-2}$, $n = 2,3,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients we let $n = 2m$, $m = 1,2,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m^2$ so $a_2 = -a_0/2^2\cdot 1^2$, $a_4 = a_0/2^2\cdot 2^2\cdot 1^2\cdot 2^2, \ldots$, and $a_{2m} = (-1)^m a_0/2^{2m}(m!)^2$. Thus one solution of the Bessel

equation of order zero is $J_0(x) = 1 + \sum_{m=1}^{\infty} (-1)^m x^{2m}/2^{2m}(m!)^2$ where we have set $a_0 = 1$. Using the ratio test it can be shown that the series converges for all x. Also note that $J_0(x) \to 1$ as $x \to 0$.

15. In order to determine the form of the integral for x near zero we must study the integrand for x small. Using the above series for J_0, we have

$$\frac{1}{x[J_0(x)]^2} = \frac{1}{x[1 - x^2/2 + \ldots]^2} = \frac{1}{x[1 - x^2 + \ldots]} =$$

$\frac{1}{x}[1 + x^2 + \ldots]$ for x small. Thus

$$y_2(x) = J_0(x)\int \frac{dx}{x[J_0(x)]^2} = J_0(x)\int [\frac{1}{x} + x + \ldots]dx$$

$= J_0(x)[\ln x + \frac{x^2}{x} + \ldots]$, and it is clear that $y_2(x)$ will contain a logarithmic term.

16a. Putting the D.E. in the standard form $y'' + p(x)y' + q(x)y = 0$ we see that $p(x) = 1/x$ and $q(x) = (x^2-1)/x^2$. Thus $x = 0$ is a singular point and since $xp(x) \to 1$ and $x^2q(x) \to -1$ as $x \to 0$, $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + (x^2-1)y = 0$, shifting indices appropriately, and collecting coefficients of common powers of x we obtain $[r(r-1) + r - 1]a_0x^r + [(1+r)r + 1 + r - 1]a_1x^{r+1}$

$$+ \sum_{n=2}^{\infty} \{[(n+r)^2 - 1]a_n + a_{n-2}\}x^{n+r} = 0.$$

The indicial equation is $r^2-1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,4\ldots$. For $r = 1$ we have $a_n = -a_{n-2}/[n(n+2)]$, $n = 2,3,4,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. Let $n = 2m$. Then $a_{2m} = -a_{2m-2}/2^2m(m+1)$, $m =$

$1,2,\ldots$, so $a_2 = -a_0/2^2 \cdot 1 \cdot 2$, $a_4 = -a_2/2^2 \cdot 1 \cdot 2 \cdot 3 = a_0/2^2 \cdot 2^2 \cdot 1 \cdot 2 \cdot 2 \cdot 3, \ldots$, and $a_{2m} = (-1)^m a_0/2^{2m} m!(m+1)!$. Thus one solution (set $a_0 = 1/2$) of the Bessel equation of order one is $J_1(x) = (x/2) \sum_{n=0}^{\infty} (-1)^n x^{2n}/(n+1)!n!2^{2n}$. The ratio test shows that the series converges for all x. Also note that $J_1(x) \to 0$ as $x \to 0$.

16b. For $r = -1$ the recurrence relation is $[(n-1)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,\ldots$. Substituting $n = 2$ into the relation yields $[(2-1)^2 - 1]a_2 = 0$ $a_2 = -a_0$. Hence it is impossible to determine a_2 and consequently impossible to find a series solution of the form $x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

Secton 5.7, Page 278

1. The D.E. has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x$, $Q(x) = 2x$, and $R(x) = 6e^x$. From this we find $p(x) = Q(x)/P(x) = 2$ and $q(x) = R(x)/P(x) = 6e^x/x$ and thus $x = 0$ is a singular point. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are analytic at $x = 0$ we conclude that $x = 0$ is a regular singular point. Next, we have $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$ and thus the indicial equation is $r(r-1) + 0 \cdot r + 0 = r^2 - r = 0$, which has the roots $r_1 = 1$ and $r_2 = 0$.

3. The equation has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1)$, $Q(x) = 6x^2$ and $R(x) = 3$. Since $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors and $P(0) = 0$ and $P(1) = 0$, we conclude that $x = 0$ and $x = 1$ are singular points. The first point, $x = 0$, can be shown to be a regular singular point using steps similar to those to shown in Problem 1. For $x = 1$, we must put the D.E. in a form similar to Eq.(1) for this case. To do this, divide the D.E. by x and multiply by $(x-1)$ to obtain $(x-1)^2y'' + 6x(x-1)y + \frac{3}{x}(x-1)y = 0$. Comparing this to Eq.(1) we find that $(x-1)p(x) = 6x$ and $(x-1)^2q(x) = 3(x-1)/x$ which are both analytic at

$x = 1$ and hence $x = 1$ is a regular singular point. These last two expressions approach $p_0 = 6$ and $q_0 = 0$ respectively as $x \to 1$, and thus the indicial equation is $r(r-1) + 6r + 0 = r(r+5) = 0$.

9. For this D.E., $p(x) = \dfrac{-(1+x)}{x^2(1-x)}$ and $q(x) = \dfrac{2}{x(1-x)}$ and thus $x = 0, -1$ are singular points. Since $xp(x)$ is not analytic at $x = 0$, $x = 0$ is not a regular singular point. Looking at $(x-1)p(x) = \dfrac{1+x}{x^2}$ and $(x-1)^2 q(x) = \dfrac{2(1-x)}{x}$ we see that $x = 1$ is a regular singular point and that $p_0 = 2$ and $q_0 = 0$.

17a. We have $p(x) = \dfrac{\sin x}{x^2}$ and $q(x) = -\dfrac{\cos x}{x^2}$, so that $x = 0$ is a singular point. Note that $xp(x) = (\sin x)/x \to 1 = p_0$ as $x \to 0$ and $x^2 q(x) = -\cos x \to -1 = q_0$ as $x \to 0$. In order to assert that $x = 0$ is a regular singular point we must demonstrate that $xp(x)$ and $x^2 q(x)$, with $xp(x) = 1$ at $x = 0$ and $x^2 q(x) = -1$ at $x = 0$, have convergent power series (are analytic) about $x = 0$. We know that $\cos x$ is analytic so we need only consider $(\sin x)/x$. Now

$$\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)! \text{ for } -\infty < x < \infty \text{ so}$$

$$(\sin x)/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)! \text{ and hence is analytic.}$$

Thus we may conclude that $x = 0$ is a regular singular point.

17b. From part a) it follows that the indicial equation is $r(r-1) + r - 1 = r^2 - 1 = 0$ and the roots are $r_1 = 1$, $r_2 = -1$.

17c. To find the first few terms of the solution corresponding to $r_1 = 1$, assume that

$y = x(a_0 + a_1 x + a_2 x^2 + \ldots) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots$.

Substituting this series for y in the D.E. and expanding $\sin x$ and $\cos x$ about $x = 0$ yields

$x^2(2a_1 + 6a_2 x + 12a_3 x^2 + 20a_4 x^3 + \ldots) +$

$(x - x^3/3! + x^5/5! - \ldots)(a_0 + 2a_1 x + 3a_2 x^2 + 4a_3 x^3 + 5a_4 x^4 +$

...) - (1 - $x^2/2!$ + $x^4/4!$ - ...)($a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5$ + ...) = 0. Collecting terms, $(2a_1 + 2a_1 - a_1)x^2$ + $(6a_2 + 3a_2 - a_0/6 - a_2 + a_0/2)x^3$ + $(12a_3 + 4a_3 - 2a_1/6 - a_3 + a_1/2)x^4$ + $(20a_4 + 5a_4 - 3a_2/6 + a_0/120 - a_4 + a_2/2 - a_0/24)x^5$ + ... = 0. Simplifying, $3a_1x^2 + (8a_2 + a_0/3)x^3$ + $(15a_3 + a_1/6)x^4 + (24a_4 - a_0/30)x^5$ + ... = 0. Thus, $a_1 = 0$, $a_2 = -a_0/4!$, $a_3 = 0$, $a_4 = a_0/6!$,... . Hence $y_1(x) = x - x^3/4! + x^5/6!$ + ... where we have set $a_0 = 1$. From Eq. (24) the second solution has the form

$$y_2(x) = ay_1(x)\ln x + x^{-1}\left(1+\sum_{n=1}^{\infty}c_nx^n\right)$$

$$= ay_1(x)\ln x + \frac{1}{x} + c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots, \text{ so}$$

$y_2' = ay_1'\ln x + ay_1x^{-1} - x^{-2} + c_2 + 2c_3x + 3c_4x^2 + \ldots$, and

$y_2'' = ay_1''\ln x + 2ay_1'x^{-1} - ay_1x^{-2} + 2x^{-3} + 2c_3 + 3c_4x + \ldots$.

When these are substituted in the given D.E. the terms including lnx will appear as

$a[x^2y_1'' + (\sin x)y_1' - (\cos x)y_1]$, which is zero since y_1 is a solution. For the remainder of the terms, use $y_1 = x - x^3/24 + x^5/720$ and the cosx and sinx series as shown earlier to obtain

$-c_1 + (2/3+2a)x + (3c_3+c_1/2)x^2 + (4/45+c_2/3+8c_4)x^3 + \ldots = 0$.

These yield $c_1 = 0$, $a = -1/3$, $c_3 = 0$, and $c_4 = -c_2/24 - 1/90$. We may take $c_2 = 0$, since this term will simply generate $y_1(x)$ over again. Thus

$y_2(x) = -\frac{1}{3}y_1(x)\ln x + x^{-1} - \frac{1}{90}x^3$. If a computer algebra system is used, then additional terms in each series may be obtained without much additional effort. The next terms, in each case, are shown here:

$$y_1(x) = x - \frac{x^3}{24} + \frac{x^5}{720} - \frac{43x^7}{1451520} + \ldots \text{ and}$$

$$y_2(x) = -\frac{1}{3}y_1(x)\ln x + \frac{1}{x}\left[1 - \frac{x^4}{90} + \frac{41x^6}{120960} - \ldots\right].$$

18. We first write the D.E. in the standard form as given for Theorem 5.7.1 except that we are expanding in powers of (x-1) rather than powers of x:

$(x-1)^2y'' + (x-1)[(x-1)/2\ln x]y' + [(x-1)^2/\ln x]y = 0$. since $\ln 1 = 0$, $x = 1$ is a singular point. To show it is a regular singular point of this D.E. we must show that $(x-1)/\ln x$ is analytic at $x = 1$; it will then follow that $(x-1)^2/\ln x = (x-1)[(x-1)/\ln x]$ is also analytic at $x = 1$. If we expand $\ln x$ in a Taylor series about $x = 1$ we find that $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$.

Thus

$$(x-1)/\ln x = [1 - \tfrac{1}{2}(x-1) + \tfrac{1}{3}(x-1)^2 - \ldots]^{-1} = 1 + \tfrac{1}{2}(x-1) + \ldots$$

has a power series expansion about $x = 1$, and hence is analytic. We can use the above result to obtain the indicial equation at $x = 1$. We have

$(x-1)^2y'' + (x-1)[\frac{1}{2} + \frac{1}{4}(x-1) + \ldots]y' + [(x-1) +$

$\frac{1}{2}(x-1)^2 + \ldots]y = 0$. Thus $p_0 = 1/2$, $q_0 = 0$ and the indicial equation is $r(r-1) + r/2 = 0$. Hence $r = 1/2$ and $r = 0$. In order to find the first three non-zero terms in a series solution corresponding to $r = 1/2$, it is better to keep the differential equation in its original form and to substitute the above power series for $\ln x$:

$$[(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \ldots]y'' + \tfrac{1}{2}y' + y = 0.$$

Next we substitute $y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + \ldots$ and collect coefficients of like powers of $(x-1)$ which are then set equal to zero. This requires some algebra before we find that $6a_1/4 + 9a_0/8 = 0$ and $5a_2 + 5a_1/8 - a_0/12 = 0$. These equations yield $a_1 = -3a_0/4$ and $a_2 = 53a_0/480$. With $a_0 = 1$ we obtain the solution

$y_1(x) = (x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + \ldots$. Since the radius of convergence of the series for $\ln x$ is 1, we would expect $\rho = 1$.

20a. If we write the D.E. in the standard form as given in Theorem 5.7.1 we obtain $x^2y'' + x[\alpha/x]y' + [\beta/x]y = 0$ where $xp(x) = \alpha/x$ and $x^2q(x) = \beta/x$. Neither of these terms are analytic at $x = 0$ so $x = 0$ is an irregular singular point.

20b. Substituting $y = x^r\sum_{n=0}^{\infty} a_n x^n$ in $x^3y'' + \alpha xy' + \beta y = 0$ gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \beta \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Shifting the index in the first series and collecting coefficients of common powers of x we obtain $(\alpha r + \beta)a_0 x^r$ $+ \sum_{n=1}^{\infty} (n+r-1)(n+r-2)a_{n-1} + [\alpha(n+r) + \beta]a_n x^{n+r} = 0$. Thus the indicial equation is $\alpha r + \beta = 0$ with the single root $r = -\beta/\alpha$.

20c. From part b, the recurrence relation is

$$a_n = \frac{(n+r-1)(n+r-2)a_{n-1}}{\alpha(n+r) + \beta}, \quad n = 1,2,...$$

$$= \frac{(n - \frac{\beta}{\alpha} - 1)(n - \frac{\beta}{\alpha} - 2)a_{n-1}}{\alpha n}, \text{ for } r = -\beta/\alpha.$$

For $\frac{\beta}{\alpha} = -1$, then, $a_n = \frac{n(n-1)a_{n-1}}{\alpha n}$, which is zero for $n = 1$ and thus $y(x) = x$ is the solution. Similarly for $\frac{\beta}{\alpha} = 0$, $a_n = \frac{(n-1)(n-2)}{\alpha n}$ and again for $n = 1$ $a_1 = 0$ and $y(x) = 1$ is the solution. Continuing in this fashion, we see that the series solution will terminate for β/α any positive integer as well as 0 and -1. For other values of β/α, we have $\frac{a_n}{a_{n-1}} = \frac{(n-\frac{\beta}{2}-1)(n-\frac{\beta}{\alpha}-2)}{\alpha n}$, which approaches ∞ as $n \to \infty$ and thus the ratio test yields a zero radius of convergence.

21b. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E. in standard form gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \alpha \sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1-s}$$

$$+ \beta \sum_{n=0}^{\infty} a_n x^{n+r+2-t} = 0.$$

If $s = 2$ and $t = 2$ the first term in each of the three series is $r(r-1)a_0x^r$, $\alpha r a_0 x^{r-1}$, and $\beta a_0 x^r$, respectively. Thus we must have $\alpha r a_0 = 0$ which requires $r = 0$. Hence there is at most one solution of the assumed form.

21d. In order for the indicial equation to be quadratic in r it is necessary that the first term in the first series contribute to the indicial equation. This means that the first term in the second and the third series cannot appear before the first term of the first series. The first terms are $r(r-1)a_0x^r$, $\alpha r a_0 x^{r+1-s}$, and $\beta a_0 x^{r+2-t}$, respectively. Thus if $s \leq 1$ and $t \leq 2$ the quadratic term will appear in the indicial equation.

Section 5.8, Page 289

1. It is clear that $x = 0$ is a singular point. The D.E. is in the standard form given in Theorem 5.7.1 with $xp(x) = 2$ and $x^2q(x) = x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 2r]a_0x^r + \sum_{n=1}^{\infty} [(r+n)(r+n+1)a_n + a_{n-1}]x^{r+n} = 0.$$

The indicial equation is $F(r) = r(r+1) = 0$ with roots $r_1 = 0$, $r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$, $n = 1,2,...$ if $F(r+n) \neq 0$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2)$,..., and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)...F(r+n)$, provided $F(r+n) \neq 0$ for $n = 1,2,...$. For the case $r_1 = 0$, we have $a_n(0) = (-1)^n a_0/F(1)F(2) \ldots F(n) = (-1)^n a_0/n!(n+1)!$ so one solution is $y_1(x) = \sum_{n=0}^{\infty} (-1)^n x^n/n!(n+1)!$ where we have set $a_0 = 1$.

If we try to use the above recurrence relation for

the case $r_2 = -1$ we find that $a_n(-1) = -a_{n-1}/n(n-1)$, which is undefined for n = 1. Thus we must follow the procedure described at the end of Section 5.7 to calculate a second solution of the form given in Eq.(24). Specifically, we use Eqs.(19) and (20) of that section to calculate a and $c_n(r_2)$, where $r_2 = -1$. Since $r_1 - r_2 = 1 = N$, we have $a_N(r) = a_1(r) = -1/F(r+1)$, with $a_0 = 1$. Hence

$$a = \lim_{r \to -1} [(r+1)(-1)/F(r+1)] = \lim_{r \to -1} [-(r+1)/(r+1)(r+2)] = -1.$$

Next

$$c_n(-1) = \frac{d}{dr}[(r+1)a_n(r)]\Big|_{r=-1} = (-1)^n \frac{d}{dr}\left[\frac{(r+1)}{F(r+1)\ \ldots\ F(r+n)}\right]\Bigg|_{r=-1},$$

where we again have set $a_0 = 1$. Observe that $(r+1)/F(r+1)\ldots F(r+n) = 1/[(r+2)^2(r+3)^2\ldots(r+n)^2(r+n+1)] = 1/G_n(r)$. Hence $c_n(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. Notice that $G_n(-1) = 1^2\cdot 2^2\cdot 3^2\ldots(n-1)^2 n = (n-1)!n!$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \ldots + 1/(n-1)] + 1/n = H_n + H_{n-1}$. Thus $c_n(-1) = (-1)^{n+1}(H_n + H_{n-1})/(n-1)!n!$. From Eq.(24) of Section 5.7 we obtain the second solution

$$y_2(x) = -y_1(x)\ln x + x^{-1}\left[1 - \sum_{n=1}^{\infty}(-1)^n(H_n + H_{n-1})x^n/n!(n-1)!\right].$$

2. It is clear that x = 0 is a singular point. The D.E. is in the standard form given in Theorem 5.7.1 with $xp(x) = 3$ and $x^2q(x) = 1+x$. Both are analytic at x = 0, so x = 0 is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 3r + 1]a_0 x^r + \sum_{n=1}^{\infty}\{[(r+n)(r+n+2) + 1]a_n + a_{n-1}\}x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 + 2r + 1 = (r+1)^2 = 0$ with the double root $r_1 = r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$,

$n = 1,2,\ldots$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2),\ldots$, and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\ldots F(r+n)$. Setting $r = -1$ we find that $a_n(-1)=(-1)^n a_0/(n!)^2$, $n = 1,2,\ldots$. Hence one solution is $y_1(x) = x^{-1}\sum_{n=0}^{\infty}(-1)^n x^n/(n!)^2$ where we have set $a_0 = 1$. To find a second solution we follow the procedure described in Section 5.7 for the case when the roots of the indicial equation are equal. Specifically, the second solution will have the form given in Eq.(17) of that section. We must calculate $a_n'(-1)$. If we let $G_n(r) = F(r+1)\ldots F(r+n) = (r+2)^2(r+3)^2\ldots (r+n+1)^2$ and take $a_0 = 1$, then $a_n'(-1) = (-1)^n[1/G_n(r)]'$ evaluated $r = -1$. Hence $a_n'(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. But $G_n(-1) = (n!)^2$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \ldots + 1/n] = 2H_n$. Thus a second solution is

$$y_2(x) = y_1(x)\ln x - 2x^{-1}\sum_{n=1}^{\infty}(-1)^n H_n x^n/(n!)^2.$$

3. The roots of the indicial equation are r_1 and $r_2 = 0$ and thus the analysis is similar to that for Problem 2.

4. The roots of the indicial equation are $r_1 = -1$ and $r_2 = -2$ and thus the analysis is similar to that for Problem 1.

5. Since $x = 0$ is a regular singular point, substitute $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shift indices appropriately, and collect coefficients of like powers of x to obtain

$$[r^2 - 9/4]a_0 x^r + [(r+1)^2 - 9/4]a_1 x^{r+1} + \sum_{n=2}^{\infty}\{[(r+n)^2 - 9/4]a_n + a_{n-2}\}\, x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 - 9/4 = 0$ with roots $r_1 = 3/2$, $r_2 = -3/2$. Treating a_n as a function of r we see that $a_n(r)= -a_{n-2}(r)/F(r+n)$, $n = 2,3,\ldots$ if $F(r+n) \neq$

0. For the case $r_1 = 3/2$, $F(r_1+1)$, which is the coefficient of x^{r_1+1} is $\neq 0$ so we must set $a_1 = 0$. It follows that $a_3 = a_5 = \ldots = 0$. For the even coefficients, set $n = 2m$ so $a_{2m}(3/2) = -a_{2m-2}(3/2)/F(3/2 + 2m) = -a_{2m-2}/2^2m(m+3/2)$, $m = 1,2\ldots$. Thus $a_2(3/2) = - a_0/2^2\cdot 1(1 + 3/2)$, $a_4(3/2) = a_0/2^4\cdot 2!(1 + 3/2)(2 + 3/2),\ldots,$ and $a_{2m}(3/2) = (-1)^m/2^{2m}m!\cdot(1 + 3/2)\ldots(m + 3/2)$. Hence one solution is

$$y_1(x) = x^{3/2}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1 + 3/2)(2 + 3/2)\ldots(m + 3/2)}(\frac{x}{2})^{2m}],$$

where we have set $a_0 = 1$. For this problem, the roots r_1 and r_2 of the indicial equation differ by an integer: $r_1 - r_2 = 3/2 - (-3/2) = 3$. Hence we can anticipate that there may be difficulty in calculating a second solution corresponding to $r = r_2$. This difficulty will occur in calculating $a_3(r) = - a_1(r)/F(r+3)$ since when $r = r_2 = -3/2$ we have $F(r_2+3) = F(r_1) = 0$. However, in this problem we are fortunate because $a_1 = 0$ and it will not be necessary to use the theory described at the end of Section 5.7. Notice for $r = r_2 = -3/2$ that the coefficient of x^{r_2+1} is $[(r_2+1)^2 - 9/4]a_1$, which does not vanish unless $a_1 = 0$. Thus the recurrence relation for the odd coefficients yields $a_5 = -a_3/F(7/2)$, $a_7 = -a_5/F(11/2) = a_3/F(11/2)F(7/2)$ and so forth. Substituting these terms into the assumed form we see that a multiple of $y_1(x)$ has been obtained and thus we may take $a_3 = 0$ without loss of generality. Hence $a_3 = a_5 = a_7 = \ldots = 0$. The even coefficients are given by $a_{2m}(-3/2) = -a_{2m-2}(-3/2)/F(2m - 3/2)$, $m = 1,2\ldots$. Thus $a_2(-3/2) = -a_0/2^2\cdot 1\cdot(1 - 3/2)$, $a_4(-3/2) = a_0/2^4\cdot 2!(1 - 3/2)(2 - 3/2),\ldots,$ and $a_{2m}(-3/2) = (-1)^m a_0/2^{2m}m!(1 - 3/2)(2 - 3/2) \ldots (m - 3/2)$. Thus a second solution is

$$y_2(x) = x^{-3/2}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1 - 3/2)(2 - 3/2) \ldots (m - 3/2)}(\frac{x}{2})^{2m}].$$

7. Apply the ratio test:

$$\lim_{m\to\infty}\frac{|(-1)^{m+1}\,x^{2m+2}/2^{2m+2}[(m+1)!]^2|}{|(-1)^m\,x^{2m}/2^{2m}(m!)^2|} = |x^2|\lim_{m\to\infty}\frac{1}{2^2(m+1)^2} = 0$$

for every x. Thus the series for $J_0(x)$ converges absolutely for all x.

12. If $\xi = \alpha x^\beta$, then $dy/dx = \frac{1}{2}x^{-1/2}f + x^{1/2}f'\alpha\beta x^{\beta-1}$ where f' denotes $df/d\xi$. Find d^2y/dx^2 in a similar fashion and use algebra to show that f satisfies the D.E. $\xi^2 f'' + \xi f' + [\xi^2 - \upsilon^2]f = 0$.

13. To compare $y'' - xy = 0$ with the D.E. of Problem 12, we must multiply by x^2 to get $x^2y'' - x^3y = 0$. Thus $2\beta = 3$, $\alpha^2\beta^2 = -1$ and $1/4 - \upsilon^2\beta^2 = 0$. Hence $\beta = 3/2$, $\alpha = 2i/3$ and $\upsilon^2 = 1/9$ which yields the desired result.

14. First we verify that $J_0(\lambda_j x)$ satisfies the D.E. We know that $J_0(t)$ is a solution of the Bessel equation of order zero:

$$t^2J_0''(t) + tJ_0'(t) + t^2J_0(t) = 0 \text{ or}$$

$$J_0''(t) + t^{-1}J_0'(t) + J_0(t) = 0.$$

Let $t = \lambda_j x$. Then

$$\frac{d}{dx}\,J_0(\lambda_j x) = \frac{d}{dt}\,J_0(t)\frac{dt}{dx} = \lambda_j J_0'(t)$$

$$\frac{d^2}{dx^2}\,J_0(\lambda_j x) = \lambda_j\frac{d}{dt}[J_0'(t)]\frac{dt}{dx} = \lambda_j^2 J_0''(t).$$

Substituting $y = J_0(\lambda_j x)$ in the given D.E. and making use of these results, we have

$$\lambda_j^2J_0''(t) + (\lambda_j/t)\,\lambda_j J_0'(t) + \lambda_j^2J_0(t) =$$

$$\lambda_j^2[J_0''(t) + t^{-1}J_0'(t) + J_0(t)] = 0.$$

Thus $y = J_0(\lambda_j x)$ is a solution of the given D.E. For the second part of the problem we follow the hint. First, rewrite the D.E. by multiplying by x to yield $xy'' + y' + \lambda_j^2 xy = 0$, which can be written as $(xy')' = -\lambda_j^2 xy$. Now let $y_i(x) = J_0(\lambda_i x)$ and $y_j(x) = J_0(\lambda_j x)$ and we have, respectively: $(xy_i')' = -\lambda_i^2 xy_i$

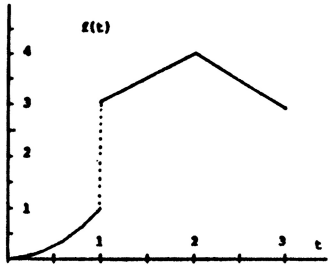
$$(xy_j')' = -\lambda_j^2 xy_j.$$

Now multiply the first equation by y_j, the second by y_i,

CHAPTER 6

Section 6.1, Page 298

1. The graph of f(t) is shown. Since the function is continuous on each interval, but has a jump discontinuity at t = 1, f(t) is piecewise continuous.



2. Note that $\lim_{t\to 1^+} (t-1)^{-1} = \infty$.

5b. Since t^2 is continuous for $0 \le t \le A$ for any positive A and since $t^2 \le e^{at}$ for any $a > 0$ and for t sufficiently large, it follows from Theorem 6.1.2 that $\mathcal{L}\{t^2\}$ exists for $s > 0$. $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}t^2dt = \lim_{M\to\infty}\int_0^M e^{-st}t^2dt$

$$= \lim_{M\to\infty}\left[\frac{-t^2}{s}e^{-st}\Big|_0^M + \frac{2}{s}\int_0^M e^{-st}tdt\right]$$
$$= \frac{2}{s}\lim_{M\to\infty}\left[-\frac{1}{s}te^{-st}\Big|_0^M + \frac{1}{s}\int_0^M e^{-st}dt\right]$$
$$= \frac{2}{s^2}\lim_{M\to\infty} -\frac{1}{s}e^{-st}\Big|_0^M = \frac{2}{s^3}.$$

6. That f(t) = cosat satisfies the hypotheses of Theorem 6.1.2 can be verified by recalling that $|\cos at| \le 1$ for all t. To determine $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at dt$ we must integrate by parts twice to get $\int_0^\infty e^{-st}\cos at dt = \lim_{M\to\infty}[(-s^{-1}e^{-st}\cos at + as^{-2}e^{-st}\sin at)|_0^M$ $- (a^2/s^2)\int_0^M e^{-st}\cos at\,dt]$. Evaluating the first two terms, letting $M \to \infty$, and adding the third term to both sides, we obtain $[1 + a^2/s^2]\int_0^\infty e^{-st}\cos at\,dt = 1/s$, $s > 0$. Division by $[1 + a^2/s^2]$ and simplification yields the desired solution.

9. From the definition for coshbt we have $\mathcal{L}\{e^{at}\cosh bt\} = \mathcal{L}\{\frac{1}{2}[e^{(a+b)t} + e^{(a-b)t}]\}$. Using the linearity

property of £, Eq.(5), the right side becomes $\frac{1}{2}£\{e^{(a+b)t}\} + \frac{1}{2}£\{e^{(a-b)t}\}$ which can be evaluated using the result of Example 5 and thus

$$£\{e^{at}\cosh bt\} = \frac{1/2}{s-(a+b)} + \frac{1/2}{s-(a-b)}$$
$$= \frac{s-a}{(s-a)^2-b^2}, \text{ for } s-a > |b|.$$

13. We write $\sinh at = (e^{iat} - e^{-iat})/2i$, then the linearity of the Laplace transform operator allows us to write $£\{e^{at}\sinh bt\} = (1/2i)£\{e^{(a+ib)t}\}-(1/2i)£\{e^{(a-ib)t}\}$. Each of these two terms can be evaluated by using the result of Example 5, where we now have to require s to be greater than the real part of the complex numbers $a \pm ib$ in order for the integrals to converge. Complex algebra then gives the desired result. An alternate method of evaluation would be to use integration on the integral appearing in the definition of $£\{e^{at}\sinh bt\}$, but that method requires integration by parts twice.

16. As in Problem 13, $£\{t\sinh at\} = (1/2i)£\{te^{iat}\} - (1/2i)£\{te^{-iat}\}$. Using the result of Problem 15 we obtain $£\{t\sinh at\} = (1/2i)[(s-b)^{-2} - (s+b)^{-2}]$ where $b = ia$ and $s > 0$. Hence $£\{t\sinh at\} = 2as/(s^2+a^2)^2$, $s > 0$.

19. Use the approach shown in Problem 16 with the result of Problem 18, for $n = 2$. A computer algebra system may also be used.

21. The integral $\int_0^A (t^2 + 1)^{-1}dt$ can be evaluated in terms of the arctan function and then Eq. (3) can be used. To illustrate Theorem 6.1.1, however, consider that $\frac{1}{t^2+1} < \frac{1}{t^2}$ for $t \geq 1$ and, from Example 3, $\int_1^\infty t^{-2}dt$ converges and hence $\int_1^\infty (t^2 + 1)^{-1}dt$ also converges. $\int_0^1 (t^2 + 1)^{-1}dt$ is finite and hence does not affect the convergence of $\int_0^\infty (t^2 + 1)^{-1}dt$ at infinity.

25. If we let $u = f$ and $dv = e^{-st}dt$ then $F(s) = \int_0^\infty e^{-st}f(t)dt$ $= \lim_{M \to \infty} -\frac{1}{s}e^{-st}f(t)|_0^M + \frac{1}{s}\int_0^\infty e^{-st}f'(t)dt$. Now use an argument similar to that given to establish Theorem 6.1.2.

27a. Make a transformation of variables with $x = st$ and $dx = sdt$. Then use the definition of $\Gamma(P+1)$ from Problem 26.

27b. From part a, $\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n dx = \frac{n}{s^{n+1}}\int_0^\infty e^{-x}x^{n-1}dx$ $= \frac{n!}{s^{n+1}}\int_0^\infty e^{-x}dx$, using integration by parts successively. Evaluation of the last integral yields the desired answer.

27c. From part a, $\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}dx$. Let $x = y^2$, then $2dy = x^{-1/2}dx$ and thus $\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-y^2}dy$.

27d. Use the definition of $\mathcal{L}\{t^{1/2}\}$ and integrate by parts once to get $\mathcal{L}\{t^{1/2}\} = (1/2s)\mathcal{L}\{t^{-1/2}\}$. The result follows from part c.

Section 6.2, Page 307

Problems 1 through 10 are solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.

2. We have $\frac{4}{(s-1)^3} = 2\frac{2!}{(s-1)^{2+1}}$ and thus the inverse Laplace transform is $2t^2e^t$, using line 11.

4. We have $\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{9/5}{s-3} + \frac{6/5}{s+2}$ using partial fractions. Thus $(9/5)e^{3t} + (6/5)e^{-2t}$ is the inverse transform, from line 2.

7. We have $\frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$, where

we first used the concept of completing the square (in the denominator) and then added and subtracted appropriately to put the numerator in the desired form. Lines 9 and 10 may now be used to find the desired result.

In each of the Problems 11 through 23 it is assumed that the I.V.P. has a solution $y = \phi(t)$ which, with its first two derivatives, satisfies the conditions of the Corollary to Theorem 6.2.1.

11. Take the Laplace transform of the D.E., using Eq.(1) and Eq.(2), to get
$s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = 0$.
Using the I.C. and solving for Y(s) we obtain
$Y(s) = \frac{s-2}{s^2-s-6}$. Following the pattern of Eq.(12) we have
$\frac{s-2}{s^2-s-6} = \frac{a}{s+2} + \frac{b}{s-3} = \frac{a(s-3)+b(s+2)}{(s+2)(s-3)}$. Equating like powers in the numerators we find $a+b = 1$ and $-3a + 2b = -2$. Thus $a = 4/5$ and $b = 1/5$ and
$Y(s) = \frac{4+5}{s+2} + \frac{1/5}{s-3}$, which yields the desired solution using Table 6.2.1.

14. Taking the Laplace transform we have $s^2Y(s) - sy(0) - y'(0) - 4[sY(s)-y(0)] + 4Y(s) = 0$. Using the I.C. and solving for Y(s) we find $Y(s) = \frac{s-3}{s^2-4s+4}$. Since the denominator is a perfect square, the partial fraction form is $\frac{s-3}{s^2-4s+4} = \frac{a}{(s-2)^2} + \frac{b}{s-2}$. Solving for a and b, as shown in examples of this section or in Problem 11, we find $a = -1$ and $b = 1$.
Thus $Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$, from which we find
$y(t) = e^{2t} - te^{2t}$ (lines 2 and 11 in Table 6.2.1).

15. Note that $Y(s) = \frac{2s-4}{s^2-2s-2} = \frac{2s-4}{(s-1)^2-3} = \frac{2(s-1)}{(s-1)^2-3} - \frac{2}{(s-1)^2-3}$.
Three formulas in Table 6.2.1 are now needed: F(s-c) in line 14 in conjunction with the ones for coshat and sinhat, lines 7 and 8.

17. The Laplace transform of the D.E. is $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4[s^3Y(s)-s^2y(0) -sy'(0) - y''(0)] + 6[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] +Y(s) = 0$. Using the I.C. and solving for Y(s) we find

$Y(s) = \dfrac{s^2 - 4s + 7}{s^4-4s^3+6s^2-4s+1}$. The correct partial fraction form for this is $\dfrac{a}{(s-1)^4} + \dfrac{b}{(s-1)^3} + \dfrac{c}{(s-1)^2} + \dfrac{d}{s-1}$. Setting this equal to Y(s) above and equating the numerators we have $s^2-4s+7 = a + b(s-1) + c(s-1)^2 + d(s-1)^3$. Solving for a,b,c, and d and use of Table 6.2.1 yields the desired solution.

20. The Laplace transform of the D.E. is $s^2Y(s) - sy(0) - y'(0) + \omega^2Y(s) = s/(s^2+4)$. Applying the I.C. and solving for Y(s) we get $Y(s) = s/[(s^2+4)(s^2+\omega^2)] + s/(s^2+\omega^2)$. Decomposing the first term by partial fractions we have

$$Y(s) = \frac{s}{(\omega^2-4)(s^2+4)} - \frac{s}{(\omega^2-4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2}$$

$$= (\omega^2-4)^{-1}\left[\frac{(\omega^2-5)s}{s^2+\omega^2} + \frac{s}{s^2+4}\right].$$

Then, using Table 6.1.2, we have $y = (\omega^2-4)^{-1}[(\omega^2-5)\cos\omega t + \cos 2t]$.

22. Solving for Y(s) we find $Y(s) = 1/[(s-1)^2 + 1] + 1/(s+1)[(s-1)^2 + 1]$. Using partial fractions on the second term we obtain

$Y(s) = 1/[(s-1)^2 + 1] + \{1/(s+1) - (s-3)/[(s-1)^2 + 1]\}/5$
$= (1/5)\{(s+1)^{-1}-(s-1)[(s-1)^2 + 1]^{-1} + 7[(s-1)^2 + 1]^{-1}\}$.
Hence, $y = (1/5)(e^{-t} - e^t\cos t + 7e^t\sin t)$.

24. Under the standard assumptions, the Lapace transform of the left side of the D.E. is $s^2Y(s) - sy(0) - y'(0) + 4Y(s)$. To transform the right side we must revert to the definition of the Laplace trasnform to determine $\int_0^\infty e^{-st}f(t)dt$. Since f(t) is piecewise continuous we are able to calculate $\mathcal{L}\{f(t)\}$ by

$$\int_0^\infty e^{-st}f(t)dt = \int_0^\pi e^{-st}\,dt + \lim_{M\to\infty}\int_\pi^M (e^{-st})(0)dt$$

$$= \int_0^\pi e^{-st}dt = (1 - e^{-\pi s})/s.$$

Hence, the Laplace transform Y(s) of the solution is given by $Y(s) = s/(s^2+4) + (1 - e^{-\pi s})/s(s^2+4)$.

27b. The Taylor series for f about t = 0 is

$f(t) = \sum_{n=0}^{\infty}(-1)^n\, t^{2n}/(2n+1)!$, which is obtained from part(a) by dividing each term of the sine series by t. Also, f is continuous for t > 0 since $\lim_{t\to 0+}(\sin t)/t = 1$. Assuming that we can compute the Laplace transform of f term by term, we obtain $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n+1)!\}$

$$= \sum_{n=0}^{\infty}[(-1)^n/(2n+1)!\mathcal{L}\{t^{2n}\}$$

$$= \sum_{n=0}^{\infty}[(-1)^n(2n)!/(2n+1)!]s^{-(2n+1)}$$

$$= \sum_{n=0}^{\infty}[(-1)^n/(2n+1)]s^{-(2n+1)}, \text{ which converges for } s > 1.$$

The Taylor series for arctan x is given by $\sum_{n=0}^{\infty}(-1)^n\, x^{2n+1}/(2n+1)$, for $|x| < 1$. Comparing $\mathcal{L}\{f(t)\}$ with the Taylor series for arctanx, we conclude that $\mathcal{L}\{f(t)\} = \arctan(1/s)$, $s > 1$.

30. Setting n = 2 in Problem 28b, we have

$\mathcal{L}\{t^2\sin bt\} = \frac{d^2}{ds^2}[b/(s^2+b^2)] = \frac{d}{ds}[-2bs/(s^2+b^2)^2] =$

$-2b/(s^2+b^2)^2 + 8bs^2/(s^2+b^2)^3 = 2b(3s^2-b^2)/(s^2+b^2)^3$.

32. Using the result of Problem 28a. we have

$\mathcal{L}\{te^{at}\} = -\frac{d}{ds}(s-a)^{-1} = (s-a)^{-2}$

$\mathcal{L}\{t^2e^{at}\} = -\frac{d}{ds}(s-a)^{-2} = 2(s-a)^{-3}$.

$\mathcal{L}\{t^3e^{at}\} = -\frac{d}{ds}2(s-a)^{-3} = 3!(s-a)^{-4}$. Continuing in this

fashion, or using induction, we obtain the desired result.

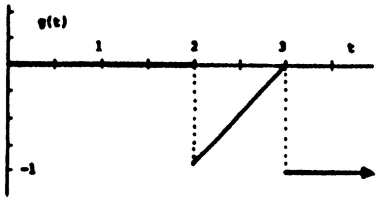
36a. Taking the Laplace transform of the D.E. we obtain
$\mathcal{L}\{y''\} - \mathcal{L}\{ty\} = \mathcal{L}\{y''\} + \mathcal{L}\{-ty\}$
$= s^2Y(s) - sy(0) - y'(0) + Y'(s) = 0.$
Hence, Y satisfies $Y' + s^2Y = s$.

38a. From Eq(i) we have $A_k = \lim_{s\to r_k} (s-r_k)\,\dfrac{P(r_k)}{Q(r_k)}$, since Q has distinct zeros. Thus $A_k = P(r_k) \lim_{S\to r_k} \dfrac{s-r_k}{Q(r_k)} = \dfrac{P(r_k)}{Q'(r_k)}$, by L'Hopital's Rule.

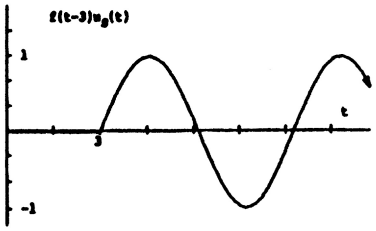
38b. Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s-r_k}\right\} = e^{r_k t}$, the result follows.

Section 6.3, Page 314

2. From the definition of $u_c(t)$ we have:
$g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
$$= \begin{cases} 0 - 0 = 0, & 0\le t<2 \\ (t-3) - 0 = t-3, & 2\le t<3. \\ (t-3) - (t-2) = -1, & 3 \le t \end{cases}$$



4. As indicated in the discussion following Example 1, the unit step function can be used to translate a given function f, with domain $t\ge 0$, a distance c to the right by the multiplication $u_c(t)f(t-c)$. Hence the required graph of $y = u_3(t)f(t-3)$ for $f(t) = \sin t$ is shown.



8. In order to use Theorem 6.3.1 we must write f(t) in terms of $u_c(t)$. Since $t^2 - 2t + 2 = (t-1)^2 + 1$ (by completing the square), we can thus write $f(t) = u_1(t)g(t-1)$, where $g(t) = t^2+1$. Now applying Theorem 6.3.1 we have $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)g(t-1)\} = e^{-s}\,\mathcal{L}\{g(t)\} = e^{-s}(2/s^3 + 1/s)$.

14. Use partial fractions to write $F(s) = e^{-2s}[(s-1)^{-1} - (s+2)^{-1}]/3$. For ease in calculations let us define $G(s) = (s-1)^{-1}$ and $H(s) = (s+2)^{-1}$. Then $F(s) = [e^{-2s} G(s) - e^{-2s} H(s)]/3$. Using the fact that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$ and applying Theorem 6.3.1, we have $F(s) = [e^{-2s} \mathcal{L}\{e^{t}\} - e^{-2s} \mathcal{L}\{e^{-2t}\}]/3$. Thus $F(s) = [\mathcal{L}\{u_2(t)e^{(t-2)}\} - \mathcal{L}\{u_2(t)e^{-2(t-2)}\}]/3$. Using the linearity of the Laplace transform, we have $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t)[e^{t-2} - e^{-2(t-2)}]/3\}$. Hence, $f(t) = [u_2(t)(e^{t-2} - e^{-2(t-2)})]/3$. An alternate method is to complete the square in the denominator: $F(s) = \dfrac{e^{-2s}}{(s+1/2)^2 - 9/4}$. This gives $f(t) = (2/3)u_2(t)e^{-(t-2)/2} \sinh\frac{3}{2}(t-2)$, which can be shown to be the same as that found above.

21. By completing the square in the denominator of F we can write $F(s) = (2s+1)/[(2s+1)^2 + 4]$. This has the form $G(2s+1)$ where $G(u) = u/(u^2+4)$. We must find $\mathcal{L}^{-1}\{G(2s+1)\}$. Applying the results of Problem 19(c), we have $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-t/2} \cos(\frac{2t}{2})$.

22. If the approach of Problem 21 is used we find $f(t) = (1/3)e^{2t/3} \sinh(t/3)$, which is equivalent to the given answer using the definition of sinht.

27. Assuming that term-by-term integration of the infinite series is permissible and recalling that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$ for $s > 0$, we have $\mathcal{L}\{f(t)\} = (1/s) + \sum_{k=1}^{\infty}(-1)^k \mathcal{L}\{u_k(t)\}$ $= (1/s) + \sum_{k=1}^{\infty}(-1)^k e^{-ks}/s = [\sum_{k=0}^{\infty}(-e^{-s})^k]/s$. We recognize the last infinite series as the geometric series, $\sum_{k=0}^{\infty}ar^k$, with $a = 1$ and $r = -e^{-s}$. This series converges to $[1/(1+e^{-s})]$ if $|r| < 1$. Hence, $\mathcal{L}\{f(t)\} = (1/s)[1/(1+e^{-s})]$, $s > 0$.

28. Using the definition of the Laplace transform we have $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$. Since f is periodic with period T, we have $f(t+T) = f(t)$. This suggests that we rewrite the improper integral as $\int_0^\infty e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt$. The periodicity of f also suggests that we make the change of variable $t = r + nT$. Hence, $F(s) = \sum_{n=0}^{\infty} \int_0^T e^{-s(r+nT)} f(r+nT)dr = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-rs} f(r)dr$, where we have used the fact that $f(r+nT) = f(r+(n-1)T) = \ldots = f(r+T) = f(r)$ from the definition that f is periodic. We recognize this last series as the geometric series, $\sum_{n=0}^{\infty} au^n$, with $a = \int_0^T e^{-rs} f(r)dr$ and $u = e^{-sT}$. The geometric series converges to $a/(1-u)$ for $|u| < 1$ and consequently we obtain
$F(s) = (1 - e^{-sT})^{-1} \int_0^T e^{-rs} f(r)dr$, $s > 0$.

30. The function f is periodic with period 2. The result of Problem 28 gives us $\mathcal{L}\{f(t)\} = \int_0^2 e^{-st} f(t)dt/(1-e^{-2s})$. Calculating the integral we have
$\int_0^2 e^{-st} f(t)dt = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt$
$= (1-e^{-s})/s + (e^{-2s}-e^{-s})/s$
$= (e^{-2s}-2e^{-s}+1)/s$
$= (1-e^{-s})^2/s$. Since the denominator of $\mathcal{L}\{f(t)\}$, $1 - e^{-2s}$, may be written as $(1-e^{-s})(1+e^{-s})$ we obtain the desired answer.

Section 6.4, Page 321

1. $f(t)$ can be written in the form $f(t) = 1 - u_{\pi/2}(t)$ and thus the Laplace transforms of the D.E. is $(s^2+1)Y(s) - sy(0) - y'(0) = (1/s) - e^{-\pi s/2}/s$. Introducing the I.C. and solving for Y(s), we obtain $Y(s) = (s^2+1)^{-1} + [s(s^2+1)]^{-1} - e^{-\pi s/2}/s(s^2+1)$. Using partial fractions on the second and third terms we find

$Y(s)=(1/s) + (s^2+1)^{-1} - s/(s^2+1) - e^{-\pi s/2}/s + e^{-\pi s/2}s/(s^2+1)$. The inverse transform of the first three terms can be obtained directly from Table 6.2.1. Using Theorem 6.3.1 to find the inverse transform of the last two terms we have $\mathcal{L}^{-1}\{e^{-\pi s/2}/s\} = u_{\pi/2}(t)g(t - \pi/2)$ where

$g(t) = \mathcal{L}^{-1}\{1/s\} = 1$ and
$\mathcal{L}^{-1}\{e^{-\pi s/2}s/(s^2+1)\} = u_{\pi/2}(t)h(t - \pi/2)$ where

$h(t) = \mathcal{L}^{-1}\{s/(s^2+1)\} = \cos t$. Hence,
$y = 1 + \sin t - \cos t + u_{\pi/2}(t)[\cos(t - \pi/2) - 1]$
$= 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \sin t]$. The graph of the forcing function is a unit pulse for $0 \le t < \pi/2$ and 0 thereafter. The graph of the solution will be composed of two segments. The first, for $0 \le t < \pi/2$, is a sinusoid oscillating about 1, which represents the system response to a unit forcing function and the given initial conditions. For $t \ge \pi/2$, the forcing function, f(t), is zero and the "initial" conditions are
$y(\pi/2) = \lim_{t\to\pi^-/2} 1 + \sin t - \cos t = 2$ and
$y'(\pi/2) = \lim_{t\to\pi^-/2} \cos t + \sin t = 1$. In this case the system response is $y(t) = 2\sin t - \cos t$, which is a sinusoid oscillating about zero.

3. According to Theorem 6.3.1,
$\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} = e^{-2\pi s}\,\mathcal{L}\{\sin t\} = e^{-2\pi s}/(s^2+1)$.
Transforming the D.E., we have
$(s^2+4)Y(s) - sy(0) - y'(0) = 1/(s^2+1) - e^{-2\pi s}/(s^2+1)$.
Introducing the I.C. and solving for Y(s), we obtain
$Y(s) = (1-e^{-2\pi s})/(s^2+1)(s^2+4)$. We apply partial fractions to write
$Y(s) = [(s^2+1)^{-1} - (s^2+4)^{-1} - e^{-2\pi s}(s^2+1)^{-1} + e^{-2\pi s}(s^2+4)^{-1}]/3$.
We compute the inverse transform of the first two terms directly from Table 6.2.1 after noting that
$(s^2+4)^{-1} = (1/2)[2/(s^2+4)]$. We apply Theorem 6.3.1 to the last two terms to obtain the solution,
$y = (1/3)\{\sin t - (1/2)\sin 2t - u_{2\pi}(t)[\sin(t-2\pi) - (1/2)\sin 2(t-2\pi)]\}$.
This may be simplified, using trigonometric identities, to $y = [(2\sin t - \sin 2t)(1-u_{2\pi}(t))]/6$. Note that the forcing function is $\sin t - \sin(t-2\pi) = 0$ for $t \ge 2\pi$. The solution is $y(t) = 2\sin t - \sin 2t$ for $0 \le t < 2\pi$. Thus $y(2\pi^-) = 0$ and $y'(2\pi^-) = 2\cos 2\pi - 2\cos 4\pi = 0$. Hence the "initial" value problem for $t \ge 2\pi$ is $y'' + 4y = 0$, $y(2\pi) = 0$, $y'(2\pi) = 0$, which has the trivial solution $y \equiv 0$.

8. Taking the Laplace transform, applying the I.C. and using Theorem 6.3.1 we have $(s^2+s+5/4)Y(s) = (1-e^{-\pi s/2})/s^2$. Thus

$$Y(s) = \frac{1-e^{-s/2}}{s^2(s^2+s+5/4)}$$

$$= (1-e^{-\pi s/2})\left\{\frac{4/5}{s^2} - \frac{16/25}{s} + \frac{(16/25)s-4/25}{(s+1/2)^2+1}\right\}$$

$= (1-e^{-\pi s/2})H(s)$, where we have used partial fractions and completed the square in the denominator of the last term. Since the numerator of the last term of H can be written as $\frac{16}{25}[(s+1/2) - 3/4]$, we see that $\mathcal{L}^{-1}\{H(s)\} = (4/25)(5t - 4 + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t)$, which yields the desired solution. The graph of the forcing function is a ramp ($f(t) = t$) for $0 \le t < \pi/2$ and a constant ($f(t) = \pi/2$) for $t \ge \pi/2$. The solution will be a damped sinusoid oscillating about the "ramp" $(20t-16)/25$ for $0 \le t < \pi/2$ and oscillating about $2\pi/5$ for $t \ge \pi/2$.

10. Note that $g(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$. Proceeding as in Problem 8 we find $Y(s) = (1+e^{-\pi s})\frac{1}{(s^2+1)(s^2+s+5/4)}$. The correct partial fraction expansion of the quotient is $\frac{as+b}{s^2+1} + \frac{cs+d}{s^2+s+5/4}$, where
$a+c = 0$, $a+b+d = 0$, $(5/4)a+b+c = 0$ and $(5/4)b+d = 1$ by equating coefficients. Solving for the constants yields the desired solution.

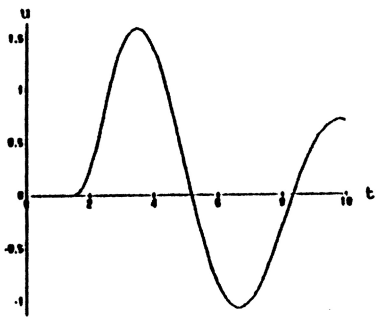
16b. Taking the Laplace transform of the D.E. we obtain $U(s^2+s/4+1) = k(e^{-3s/2}-e^{-5s/2})/s$, since the I.C. are zero. Solving for U and using partial fractions yields $U(s) = k(e^{-3s/2}-e^{-5s/2})(\frac{1}{s}-\frac{s+1/4}{s^2+s/4+1})$. Thus, if $H(s) = (\frac{1}{s}-\frac{s+1/4}{s^2+s/4+1})$, then

$h(t) = 1 - e^{-t/8}(\cos\frac{3\sqrt{7}}{8}t+\frac{\sqrt{7}}{21}\sin\frac{3\sqrt{7}}{8}t)$ and

$u(t) = ku_{3/2}(t)h(t-3/2) - ku_{5/2}(t)h(t-5/2)$.

16c. In all cases the plot will be zero for $0 \le t < 3/2$. For $3/2 \le t < 5/2$ the plot will be the system response

(damped sinusoid) to a step input of magnitude k. For $t \geq 5/2$, the plot will be the system response to the I.C. $u(5^-/2)$, $u'(5^-/2)$ with no forcing function. The graph shown is for k = 2. Varying k will just affect the amplitude. Note that the amplitude never reaches 2, which would be the steady state response for the step input $2u_{3/2}(t)$. Note also that the solution and its derivative are continuous at t = 5/2.



19a. The graph on $0 \leq t < 6\pi$ will depend on how large n is. For instance, if n = 2 then $f(t) = \begin{cases} 1 & 0 \leq t < \pi,\ 2\pi \leq t < 6\pi \\ -1 & -\pi \leq t < 2\pi \end{cases}$. For

$n \geq 6$, $f(t) = \begin{cases} 1 & 0 \leq t < \pi,\ 2\pi \leq t < 3\pi,\ 4\pi \leq t < 5\pi \\ -1 & \pi \leq t < 2\pi,\ 3\pi \leq t < 4\pi,\ 5\pi \ \ t < 6\pi \end{cases}$.

19b. Taking the Laplace transform of the D.E. and using the I.C. we have $Y(s) = \dfrac{1}{s(s^2+1)}[1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi k s}]$, since $\mathcal{L}\{u_{\pi k}(t)\} = \dfrac{e^{-\pi k s}}{s}$. Since $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, we then obtain

$y(t) = 1 - \cos t + 2\sum_{k=1}^{\infty}(-1)^{-k} u_{\pi k}(t)[1 - \cos(t-\pi k)]$, using line 13 in Table 6.2.1.

19d. Since $\cos(t-\pi k) = (-1)^k \cos t$, the solution in part b can be written as

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{\infty}(-1)^k u_{\pi k}(t) - 2\sum_{k=1}^{\infty}(-1)^{2k}\cos t$$

$$= 1 - \cos t - 2n\cos t + 2\sum_{k=1}^{\infty}(-1)^k u_{\pi k}(t) \text{ which diverges for } n \to \infty.$$

20. In this case

$Y(s) = \dfrac{1}{s(s^2+.1s+1)}\ [1 + 2\sum_{n=1}^{\infty}(-1)^k e^{-\pi k s}]$. Using partial fractions we have

$H(s) = \dfrac{1}{s(s^2+.1s+1)} = \dfrac{1}{s} - \dfrac{s+.1}{s^2+.1s+1}$

$= \dfrac{1}{s} - \dfrac{s+.05}{(s+.05)^2+b^2} - \dfrac{.05}{(s+.05)^2+b^2}$, where

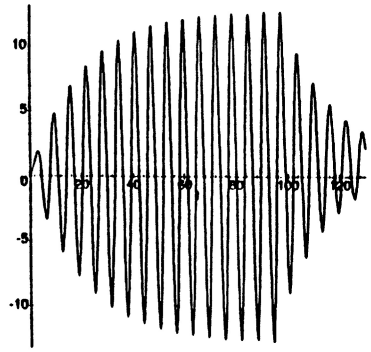
$b^2 = [1-(.05)^2] = .9975$. Now let

$h(t) = \pounds^{-1}\{H(s)\} = 1 - e^{-.05t}\cos bt - \dfrac{.05}{b}e^{-.05t}\sin bt$. Hence,

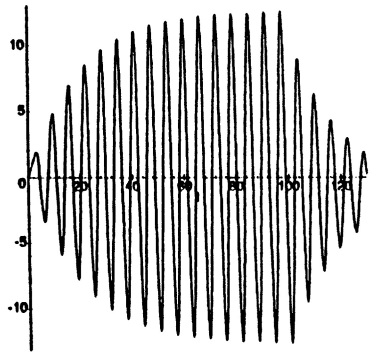
$y(t) = h(t) + 2\sum_{n=1}^{\infty}(-1)^k u_{\pi k}(t)\ h(t-\pi k)$, and thus, for $t > n$ the solution will be approximated by

$\pm 1 - Ae^{-.05(t-n\pi)}\cos[b(t-n\pi) + \delta]$, and therefore converges as $t \to \infty$.

20a. y(t) for n = 30

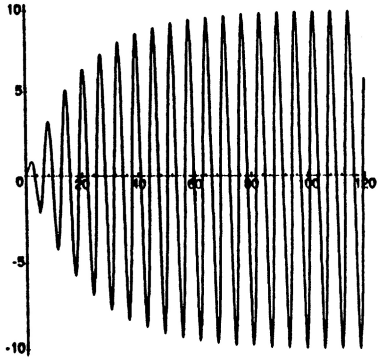


y(t) for n = 31



20b. From the graph of part a, $A \cong 12.5$ and the frequency is 2π.

20c. From the graph (or analytically) A = 10 and the frequency is 2π.



Section 6.5, Page 328

1. Proceeding as in Example 1, we take the Laplace transform of the D.E. and apply the I.C.: $(s^2 + 2s + 2)Y(s) = s + 2 + e^{-\pi s}$. Thus, $Y(s) = (s+2)/[(s+1)^2 + 1] + e^{-\pi s}/[(s+1)^2 + 1]$. We write the first term as $(s+1)/[(s+1)^2 + 1] + 1/[(s+1)^2 + 1]$. Applying Theorem 6.3.1 and using Table 6.2.1, we obtain the solution, $y = e^{-t}\cos t + e^{-t}\sin t - u_\pi(t)e^{-(t-\pi)}\sin t$. Note that $\sin(t-\pi) = -\sin t$.

3. Taking the Laplace transform and using the I.C. we have $(s^2+ 3s+2)Y(s) = \frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}$. Thus

$$Y(s) = \frac{1/2}{s^2+3s+2} + \frac{e^{-5s}}{s^2+3s+2} + e^{-10s}\left(\frac{1/2}{s}+\frac{1/2}{s+2}-\frac{1}{s+1}\right)$$ and hence

$$y(t) = \frac{1}{2}h(t) + u_5(t)h(t-5) + u_{10}(t)\left[\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}\right]$$

where $h(t) = e^{-t} - e^{-2t}$.

5. The Laplace transform of the D.E. is $(s^2+2s+3)Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$, so

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}\left[\frac{1}{s^2+2s+3}\right].$$ Using partial fractions or a computer algebra system we obtain

$$y(t) = \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_{3\pi}(t)h(t-3\pi),$$

where $h(t) = e^{-t}\sin\sqrt{2}\,t$.

7. Taking the Laplace transform of the D.E. yields $(s^2+1)Y(s) - y'(0) = \int_0^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$. Since $\delta(t-2\pi) = 0$ for $t \neq 2\pi$ the integral on the right is equal to $\int_{-\infty}^{\infty} e^{-st}\delta(t-2\pi)\cos t\,dt$ which equals $e^{-2\pi s}\cos 2\pi$ from Eq.(16). Substituting for $y'(0)$ and solving for $Y(s)$ gives $Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$ and hence

$$y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}$$

10. See the solution for Problem 7.

13a. From Eq. (22) y(t) will complete one cycle when $\sqrt{15}\,(t-5)/4 = 2\pi$ or $T = t - 5 = 8\pi/\sqrt{15}$, which is consistent with the plot in Fig. 6.5.3. Since an impulse causes a discontinuity in the first derivative, we need to find the value of y′ at t = 5 and t = 5 + T. From Eq. (22) we have, for $t \ge 5$,
$y' = e^{-(t-5)/4}\left[\frac{-1}{2\sqrt{15}}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{1}{2}\cos\frac{\sqrt{15}}{4}(t-5)\right]$. Thus
$y'(5) = \frac{1}{2}$ and $y'(5+T) = \frac{1}{2}e^{-T/4}$. Since the original impulse, $\delta(t-5)$, caused a discontinuity in y′ of 1/2, we must choose the impulse at t = 5 + T to be $-e^{-T/4}$, which is equal and opposite to y′ at 5 + T.

13b. Now consider $2y'' + y' + 2y = \delta(t-5) + k\delta(t-5-T)$ with $y(0) = 0$, $y'(0) = 0$. Using the results of Example 1 we have

$$y(t) = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{2k}{\sqrt{15}}u_{5+T}(t)e^{-(t-5-T)/4}\sin\frac{\sqrt{15}}{4}(t-5-T)$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t)\sin\frac{\sqrt{15}}{4}(t-5) + ku_{5+T}(t)e^{T/4}\sin\frac{\sqrt{15}}{4}(t-5-T)\right]$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t) + ke^{T/4}u_{5+T}(t)\right]\sin\frac{\sqrt{15}}{4}(t-5).$$

If $y(t) \equiv 0$ for $t \ge 5 + T$, then $1 + ke^{T/4} = 0$, or $k = -e^{-T/4}$, as found in part (a).

17b. We have $(s^2+1)Y(s) = \sum_{k=1}^{20} e^{-k\pi s}$ so that $Y(s) = \sum_{k=1}^{20} \frac{e^{-ks}}{s^2+1}$ and hence $y(t) = \sum_{k=1}^{20} u_{k\pi}(t)\sin(t-k\pi)$

$= u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) + \dots + u_{10\pi}\sin(t-10\pi)$.

For $0 \le t < \pi$, $y(t) \equiv 0$. For $\pi \le t < 2\pi$, $y(t) = \sin(t-\pi) = -\sin t$. For $2\pi \le t < 3\pi$, $y(t) = \sin(t-\pi) + \sin(t-2\pi) = -\sin t + \sin t \equiv 0$. Due to the periodicity of sint, the solution will exhibit this behavior in alternate intervals for $0 \le t < 20\pi$. After $t = 20\pi$ the solution remains at zero.

21b. Taking the transform and using the I.C. we have

$(s^2+1)Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi}$ so that $Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi}}{s^2+1}$.

Thus $y(t) = \sum_{k=1}^{15} u_{(2k-1)\pi}(t)\sin[t-(2k-1)\pi]$

$= \sin(t-\pi) + \sin(t-3\pi) \ldots + \sin(t-29\pi)$

$= -\sin t - \sin t \ldots -\sin t$

$= -15\sin t.$

25b. Substituting for f(t) we have $y = \int_0^t e^{-(t-\tau)}\delta(\tau-\pi)\sin(t-\tau)d\tau$. We know that the integration variable is always less than t (the upper limit) and thus for $t < \pi$ we have $\tau < \pi$ and thus $\delta(\tau-\pi) = 0$. Hence $y = 0$ for $t < \pi$. For $t > \pi$ utilize Eq.(16).

Section 6.6, Page 335

1c. Using the format of Eqs.(2) and (3) we have

$f*(g*h) = \int_0^t f(t-\tau)(g*h)(\tau)d\tau$

$= \int_0^t f(t-\tau)[\int_0^\tau g(\tau-\eta)h(\eta)d\eta]d\tau$

$= \int_0^t [\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau]h(\eta)(d\eta).$

The last double integral is obtained from the previous line by interchanging the order of the η and τ integrations. Making the change of variable $\omega = \tau - \eta$ on the inside integral yields

$f*(g*h) = \int_0^t [\int_0^{t-\eta} f(t-\eta-\omega)g(\omega)d\omega]h(\eta)d\eta$

$= \int_0^t (f*g)(t-\eta)h(\eta)d\eta = (f*g)*h.$

4. It is possible to determine f(t) explicitly by using integration by parts and then find its transform F(s). However, it is much more convenient to apply Theorem 6.6.1. Let us define $g(t) = t^2$ and $h(t) = \cos 2t$. Then, $f(t) = \int_0^t g(t-\tau)h(\tau)d\tau$. Using Table 6.2.1, we have $G(s) = \mathcal{L}\{g(t)\} = 2/s^3$ and $H(s) = \mathcal{L}\{h(t)\} = s/(s^2+4)$. Hence, by Theorem 6.6.1, $\mathcal{L}\{f(t)\} = F(s) = G(s)H(s) = 2/s^2(s^2+4)$.

8. As was done in Example 1 think of F(s) as the product of s^{-4} and $(s^2+1)^{-1}$ which, according to Table 6.2.1, are the transforms of $t^3/6$ and sint, respectively. Hence, by Theorem 6.6.1, the inverse transform of F(s) is $f(t) = (1/6)\int_0^t (t-\tau)^3 \sin\tau d\tau$.

13. We take the Laplace transform of the D.E. and apply the I.C.: $(s^2 + 2s + 2)Y(s) = \alpha/(s^2 + \alpha^2)$. Solving for Y(s), we have $Y(s) = [\alpha/(s^2+\alpha^2)][(s+1)^2 + 1]^{-1}$, where the second factor has been written in a convenient way by completing the square. Thus Y(s) is seen to be the product of the transforms of $\sin\alpha t$ and $e^{-t}\sin t$ respectively. Hence, according to Theorem 6.6.1, $y = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin\alpha\tau d\tau$.

15. Proceeding as in Problem 13 we obtain

$$Y(s) = \frac{s}{s^2+s+5/4} + \frac{1-e^{-s}}{s(s^2+s+5/4)}$$

$$= \frac{(s+1/2) - 1/2}{(s+1/2)^2+1} + \frac{1-e^{-s}}{s} \cdot \frac{1}{(s+1/2)^2+1},$$

where the first term is obtained by completing the square in the denominator and the second term is written as the product of two terms whose inverse transforms are known, so that Theorem 6.6.1 can be used. Note that $\mathcal{L}^{-1}\{(1-e^{-s})/s\} = 1 - u_\pi(t)$. Also note that a different form of the same solution would be obtained by writing the second term as $(1-e^{-\pi s})\left(\frac{a}{s} + \frac{bs + c}{(s+1/2)^2+1}\right)$ and solving for a, b and c. In this case $\mathcal{L}^{-1}\{1-e^{-s}\} = \delta(t) - \delta(t-\pi)$ from Section 6.5.

17. Taking the Laplace transform, using the I.C. and solving, we have $Y(s) = (s+3)/(s+1)(s+2) + s/(s^2+\alpha^2)(s+1)(s+2)$. As in Problem 15, there are several correct ways the second term can be treated in order to use the convolution integral. In order to obtain the desired answer, write the second term as $\frac{s}{s^2+\alpha^2}\left(\frac{a}{s+1} + \frac{b}{s+2}\right)$ and solve for a and b.

20. To find $\Phi(s)$ you must recognize the integral that

appears in the equation as a convolution integral. Taking the transform of both sides then yields

$$\Phi(s) + K(s)\Phi(s) = F(s), \text{ or } \Phi(s) = \frac{F(s)}{1+K(s)}.$$

CHAPTER 7

Section 7.1, Page 344

2. As in Example 1, let $x_1 = u$ and $x_2 = u'$, then $x_1' = x_2$ and $x_2' = u'' = 3\sin t - .5x_2 - 2x_1$.

4. In this case let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, and $x_4 = u'''$.

5. Let $x_1 = u$ and $x_2 = u'$; then $x_1' = x_2$ is the first of the desired pair of equations. The second equation is obtained by substituting $u'' = x_2'$, $u' = x_2$, and $u = x_1$ in the given D.E. The I.C. become $x_1(0) = u_0$, $x_2(0) = u_0'$.

8. Follow the steps outlined in Problem 7. Solve the first D.E. for x_2 to obtain $x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'$. Substitute this into the second D.E. to obtain $x_1'' - x_1' - 2x_1 = 0$, which has the solution $x_1 = c_1e^{2t} + c_2e^{-t}$. Differentiating this and substituting into the above equation for x_2 yields $x_2 = \frac{1}{2}c_1e^{2t} + 2c_2e^{-t}$. The I.C. then give

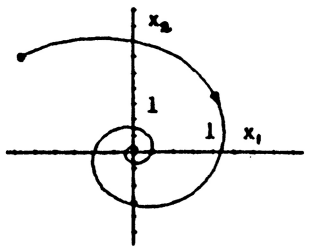
   $c_1 + c_2 = 3$ and $\frac{1}{2}c_1 + 2c_2 = \frac{1}{2}$, which yield

   $c_1 = \frac{11}{3}$, $c_2 = -\frac{2}{3}$. Thus $x_1 = \frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}$ and

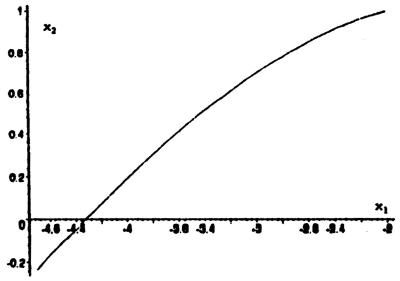
   $x_2 = \frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}$. Note that for large t, the second term in each solution vanishes and we have $x_1 \cong \frac{11}{3}e^{2t}$ and $x_2 \cong \frac{11}{6}e^{2t}$, so that $x_1 \cong 2x_2$. This says that the graph will be asymptotic to the line $x_1 = 2x_2$ for large t.

9. Solving the first D.E. for x_2 gives $x_2 = \frac{4}{3}x_1' - \frac{5}{3}x_1$, which substituted into the second D.E. yields $x_1'' - 2.5x_1' + x_1 = 0$. Thus $x_1 = c_1e^{t/2} + c_2e^{2t}$ and $x_2 = -c_1e^{t/2} + c_2e^{2t}$. Using the I.C. yields $c_1 = -3/2$ and $c_2 = -1/3$. For large t, $x_1 \cong (-1/2)e^{2t}$ and $x_2 \cong (-1/2)e^{2t}$ and thus the graph is asymptotic to $x_1 = x_2$ in the third quadrant. The graph is shown on the right.

12.



9.



12. Solving the first D.E. for x_2 gives $x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$ and substitution into the second D.E. gives $x_1'' + x_1' + \frac{17}{4}x_1 = 0$. Thus $x_1 = e^{-t/2}(c_1\cos 2t + c_2\sin 2t)$ and $x_2 = e^{-t/2}(c_2\cos 2t - c_1\sin 2t)$. The I.C. yields $c_1 = -2$ and $c_2 = 2$.

14. If $a_{12} \neq 0$, then solve the first equation for x_2, obtaining $x_2 = [x_1' - a_{11}x_1 - g_1(t)]/a_{12}$. Upon substituting this expression into the second equation, we have a second order linear O.D.E. for x_1. One I.C. is $x_1(0) = x_1^0$. The second I.C. is $x_2(0) = [x_1'(0) - a_{11}x_1(0) - g_1(0)]/a_{12} = x_2^0$. Solving for $x_1'(0)$ gives $x_1'(0) = a_{12}x_2^0 + a_{11}x_1^0 + g_1(0)$. These results hold when $a_{11}, \ldots, a_{22}$ are functions of t as long as the derivatives exist and $a_{12}(t)$ and $a_{21}(t)$ are not both zero on the interval. The initial conditions will involve $a_{11}(0)$ and $a_{12}(0)$.

19. Let us number the nodes 1,2, and 3 clockwise beginning with the top right node in Figure 7.1.4. Also let I_1, I_2, I_3, and I_4 denote the currents through the resistor R = 1, the inductor L = 1, the capacitor $C = \frac{1}{2}$, and the resistor R = 2, respectively. Let V_1, V_2, V_3, and V_4 be the corresponding voltage drops. Kirchhoff's first law applied to nodes 1 and 2, respectively, gives (i) $I_1 - I_2 = 0$ and (ii) $I_2 - I_3 - I_4 = 0$. Kirchhoff's second law applied to each loop gives (iii) $V_1 + V_2 + V_3 = 0$ and (iv) $V_3 - V_4 = 0$. The current-

voltage relation through each circuit element yields four more equations: (v) $V_1 = I_1$, (vi) $I_2' = v_2$, (vii) $(1/2)V_3' = I_3$ and (viii) $V_4 = 2I_4$. We thus have a system of eight equations in eight unknowns, and we wish to eliminate all of the variables except I_2 and V_3 from this system of equations. For example, we can use Eqs.(i) and (iv) to eliminate I_1 and V_4 in Eqs.(v) and (viii). Then use the new Eqs.(v) and (viii) to eliminate V_1 and I_4 in Eqs.(ii) and (iii). Finally, use the new Eqs. (ii) and (iii) in Eqs.(vi) and (vii) to obtain $I_2' = -I_2 - V_3$, $V_3' = 2I_2 - V_3$. These equations are identical (when subscripts on the remaining variables are dropped) to the equations given in the text.

21a. Note that the amount of water in each tank remains constant. Thus $Q_1(t)/30$ and $Q_2(t)/20$ represent oz./gal of salt in each tank. As in Example 1 of Section 2.3, we assume the mixture in each tank is well stirred. Then, for the first tank we have $\frac{dQ_1}{dt} = 1.5 - 3\frac{Q_1(t)}{30} + 1.5\frac{Q_2(t)}{20}$, where the first term on the right represents the amount of salt per minute entering the mixture from an external source, the second term represents the loss of salt per minute going to Tank 2 and the third term represents the gain of salt per minute entering from Tank 2. Similarly, we have $\frac{dQ_2}{dt} = 3 + 3\frac{Q_1(t)}{30} - 4\frac{Q_2(t)}{20}$ for Tank 2.

21b. Solve the second equation for $Q_1(t)$ to obtain $Q_1(t) = 10Q_2' + 2Q_2 - 30$. Substitution into the first equation then yields $10Q_2'' + 3Q_2' + \frac{1}{8}Q_2 = \frac{9}{2}$. The steady state solution for this is $Q_2^E = 8(9/2) = 36$. Substituting this value into the equation for Q_1 yields $Q_1^E = 72 - 30 = 42$.

21c. Substitute $Q_1 = x_1 + 42$ and $Q_2 = x_2 + 36$ into the equations found in part a.

Section 7.2, Page 355

1a. $2\mathbf{A} = \begin{pmatrix} 2 & -4 & 0 \\ 6 & 4 & -2 \\ -4 & 2 & 6 \end{pmatrix}$ so that

$$2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -4-2 & 0+3 \\ 6-1 & 4+5 & -2+0 \\ -4+6 & 2+1 & 6+2 \end{pmatrix} = \begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}$$

1c. Using Eq.(9) and following Example 1 we have

$$\mathbf{AB} = \begin{pmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{pmatrix},$$

which yields the correct answer.

6. $\mathbf{AB} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix}$ so that

$$\mathbf{(AB)C} = \mathbf{A(BC)} = \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}.$$

In problems 10 through 19 the method of row reduction, as illustrated in Example 2, can be used to find the inverse matrix or else to show that none exists. We start with the original matrix augmented by the indentity matrix, describe a suitable sequence of elementary row operations, and show the result of applying these operations.

10. Start with the given matrix augmented by the identity matrix. $\left(\begin{array}{ccc|ccc} 1 & 4 & . & 1 & 0 \\ & & . & & \\ -2 & 3 & . & 0 & 1 \end{array}\right)$

Add 2 times the first row to the second row.

$$\left(\begin{array}{ccc|ccc} 1 & 4 & . & 1 & 0 \\ & & . & & \\ 0 & 11 & . & 2 & 1 \end{array}\right)$$

Multiply the second row by (1/11).

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Add (-4) times the second row to the first row.

$$\left(\begin{array}{cc|cc} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Since we have performed the same operation on the given matrix and the identity matrix, the 2 x 2 matric appearing on the right side of this augmented matrix is the desired inverse matrix. The answer can be checked by multiplying it by the given matrix; the result should be the indentity matrix.

12. The augmented matrix in this case is:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right)$$

Add (-2) times the first row to the second row and (-3) times the first row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)$$

Multiply the second and third rows by (-1) and interchange them.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

Add (-3) times the third row to the first and second

rows.
$$\left(\begin{array}{ccc:ccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

Add (-2) times the second row to the first row.

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

The desired answer appears on the right side of this augmented matrix.

14. Again, start with the given matrix augmented by the identity matrix.
$$\left(\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right)$$

Add (2) times the first row to the second row and add(-1) times the first row to the third row.

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right)$$

Add (4/5) times the second row to the third row.

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3/5 & 4/5 & 0 \end{array}\right)$$

Since the third row of the left matrix is all zeros, no further reduction can be performed, and the given matrix is singular.

22. $\mathbf{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$; and

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

25. $\Psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}.$

Section 7.3, Page 366

1. Form the augmented matrix, as in Example 1, and use row reduction.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right)$$

Add (-3) times the first row to the second and add the first row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Add (-1) times the second row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

The third row is equivalent to $-3x_3 = 1$ or $x_3 = -1/3$. Likewise the second row is equivalent to $x_2 + 4x_3 = 1$, so $x_2 = 7/3$. Finally, from the first row, $x_1 - x_3 = 0$, so $x_1 = -1/3$. The answer can be checked by substituting into the original equations.

2. The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array}\right)$. Row reduction then yields $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, and there is no choice of x_1, x_2, and x_3 that satisfies this equation. Hence the given system of equations has no solution.

3. Form the augmented matrix and use row reduction.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right)$$

Add (-2) times the first row to the second and add (-1) times the first row to the third.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right)$$

Add (-1) times the second row to the third row and then multiply the second row by (-1/3).

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the last row has only zero entries, it may be dropped. The second row corresponds to the equation $x_2 - x_3 = 1$. We can assign an arbitrary value to either x_2 or x_3 and use this equation to solve for the other. For example, let $x_3 = c$, where c is arbitrary. Then $x_2 = 1 + c$. The first row corresponds to the equation $x_1 + 2x_2 - x_3 = 2$, so $x_1 = 2 - 2x_2 + x_3 = 2 - 2(1+c)+c = -c$.

6. To determine whether the given set of vectors is linearly independent we must solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$ for c_1, c_2, and c_3. Writing this in scalar form, we have

$$\begin{aligned} c_1 \qquad + c_3 &= 0 \\ c_1 + c_2 \qquad &= 0, \text{ so the} \\ c_2 + c_3 &= 0 \end{aligned}$$

augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$

Row reduction yields $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$.

From the third row we have $c_3 = 0$. Then from the second row, $c_2 - c_3 = 0$, so $c_2 = 0$. Finally from the first row $c_1 + c_3 = 0$, so $c_1 = 0$. Since $c_1 = c_2 = c_3 = 0$, we conclude that the given vectors are linearly independent.

8. As in Problem 6 we wish to solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$ for c_1, c_2, c_3, and c_4. Form the augmented matrix and use row reduction.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 2 & 0 & -1 & 0 & 0 \\ 2 & 3 & 1 & -1 & 0 \\ 3 & 1 & 0 & 3 & 0 \end{array}\right)$$

Add (-2) times the first row to the second, add (-2) times the first row to the third, and add (-3) times the first row to the fourth.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 4 & 6 & 12 & 0 \end{array}\right)$$

Multiply the second row by (1/2) and then add (-5) times the second row to the third and add (-4) times the second row to the fourth.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 1 & 3/2 & 3 & 0 \\ 0 & 0 & -5/2 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The third row is equivalent to the equation $c_3 + 4c_4 = 0$. One way to satisfy this equation is by choosing $c_4 = -1$; then $c_3 = 4$. From the second row we then have $c_2 = -(3/2)c_3 - 3c_4 = -6 + 3 = -3$. Then, from the first row, $c_1 = c_2 + 2c_3 + 3c_4 = -3 + 8 - 3 = 2$. Hence the given vectors are linearly dependent, and satisfy $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} + 4\mathbf{x}^{(3)} - \mathbf{x}^{(4)} = \mathbf{0}$.

14. Let $t = t_0$ be a fixed value of t in the interval $0 \le t \le 1$. To determine whether $\mathbf{x}^{(1)}(t_0)$ and $\mathbf{x}^{(2)}(t_0)$ are

linearly dependent we must solve $c_1\mathbf{x}^{(1)}(t_0)+c_2\mathbf{x}^{(2)}(t_0)=\mathbf{0}$. We have the augmented matrix

$$\begin{pmatrix} e^{t_0} & 1 & . & 0 \\ t_0e^{t_0} & t_0 & . & 0 \end{pmatrix}.$$

Multiply the first row by $(-t_0)$ and add to the second row to obtain $\begin{pmatrix} e^{t_0} & 1 & . & 0 \\ 0 & 0 & . & 0 \end{pmatrix}$.

Thus, for example, we can choose $c_1 = 1$ and $c_2 = -e^{t_0}$, and hence the given vectors are linearly dependent at t_0. Since t_0 is arbitrary the vectors are linearly dependent at each point in the interval. However, there is no linear relation between $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ that is valid throughout the interval $0 \le t \le 1$. For example, if $t_1 \ne t_0$, and if c_1 and c_2 are chosen as above, then

$$c_1\mathbf{x}^{(1)}(t_1) + c_2\mathbf{x}^{(2)}(t_1) = \begin{pmatrix} e^{t_1} \\ t_1e^{t_1} \end{pmatrix} + -e^{t_0}\begin{pmatrix} 1 \\ t_1 \end{pmatrix} = \begin{pmatrix} e^{t_1} - e^{t_0} \\ t_1e^{t_1} - t_1e^{t_0} \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence the given vectors must be linearly independent on $0 \le t \le 1$. In fact, the same argument applies to any interval.

15. To find the eigenvalues and eigenvectors of the given matrix we must solve $\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(5-\lambda)(1-\lambda) - (-1)(3) = 0$, or $\lambda^2 - 6\lambda + 8 = 0$. Hence $\lambda_1 = 2$ and $\lambda_2 = 4$ are the eigenvalues. The eigenvector corresponding to λ_1 must satisfy $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $3x_1 - x_2 = 0$. If we let $x_1 = 1$, then $x_2 = 3$ and the eigenvector is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, or any constant multiple of this vector. Similarly, the eigenvector corresponding to λ_2 must satisfy

$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 - x_2 = 0$. Hence $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, or a multiple thereof.

18. Since $\bar{a}_{12} = a_{21}$, the given matrix is Hermitian and we know in advance that its eigenvalues are real. To find the eigenvalues and eigenvectors we must solve $\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)^2 - i(-i) = \lambda^2 - 2\lambda$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$; observe that they are indeed real even though the given matrix has imaginary entries. The eigenvector corresponding to λ_1 must satisfy $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 + ix_2 = 0$. Note that the second equation $-ix_1 + x_2 = 0$ is a multiple of the first. If $x_1 = 1$, then $x_2 = i$, and the eigenvector is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. In a similar way we find that the eigenvector associated with λ_2 is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

21. The eigenvalues and eigenvectors satisfy $\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)[(1-\lambda)^2 + 4] = 0$, which has roots $\lambda = 1, 1 \pm 2i$. For $\lambda = 1$, we then have $2x_1 - 2x_3 = 0$ and $3x_1 + 2x_2 = 0$. Choosing $x_1 = 2$ then yields $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ as the eigenvector corresponding to $\lambda = 1$. For $\lambda = 1 + 2i$ we have $-2ix_1 = 0$, $2x_1 - 2ix_2 - 2x_3 = 0$ and $3x_1 + 2x_2 - 2ix_3 = 0$, yielding $x_1 = 0$ and $x_3 = -ix_2$. Thus $\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 + 2i$. A similar calculation shows that

$\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 - 2i$.

24. Since the given matrix is real and symmetric, we know that the eigenvalues are real. Further, even if there are repeated eigenvalues, there will be a full set of three linearly independent eigenvectors. To find the eigenvalues and eigenvectors we must solve

$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(3-\lambda)[-\lambda(3-\lambda)-4] - 2[2(3-\lambda) -8] + 4[4+4\lambda] = -\lambda^3 + 6\lambda^2 + 15\lambda + 8$. Setting this equal to zero and solving we find $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 8$. The eigenvectors corresponding to λ_1 and λ_2 must satisfy

$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; hence there is only the single relation $2x_1 + x_2 + 2x_3 = 0$ to be satisfied. Consequently, two of the variables can be selected arbitrarily and the third is then determined by this equation. For example, if $x_1 = 1$ and $x_3 = 1$, then $x_2 = -4$, and we obtain the eigenvector $x^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$. Similarly, if $x_1 = 1$ and $x_2 = 0$, then $x_3 = -1$, and we have the eigenvector $x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, which is linearly independent of $x^{(1)}$. There are many other choices that could have been made; however, by Eq.(38) there can be no more than two linearly independent eigenvectors corresponding to the eigenvalue -1. To find the eigenvector corresponding to λ_3 we must solve

$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Interchange the first and second rows and use row reduction to obtain the equivalent system $x_1 - 4x_2 + x_3 = 0$, $2x_2 - x_3 = 0$. Since there are two equations to satisfy only one variable can be assigned an arbitrary value. If we let $x_2 = 1$, then $x_3 = 2$ and $x_1 = 2$, so we find that $\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

27. We are given that $\mathbf{Ax} = \mathbf{b}$ has solutions and thus we have $(\mathbf{Ax},\mathbf{y}) = (\mathbf{b},\mathbf{y})$. From Problem 26, though, $(\mathbf{Ax},\mathbf{y}) = (\mathbf{x}, \mathbf{A^*y}) = 0$. Thus $(\mathbf{b},\mathbf{y}) = \mathbf{0}$. For Example 2, $\mathbf{A^*} = \bar{\mathbf{A}}^{\mathbf{T}} = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix}$ and, using row reduction, the augmented matrix for $\mathbf{A^*y} = \mathbf{0}$ becomes $\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus $y = c\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and hence $(\mathbf{b},\mathbf{y}) = b_1 + 3b_2 + b_3 = 0$.

Section 7.4, Page 371

1. Use Mathematical Induction. It has already been proven that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions, then so is $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$. Assume that if $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, …, $\mathbf{x}^{(k)}$ are solutions, then $\mathbf{x} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$ is a solution. Then use Theorem 7.4.1 to conclude that $\mathbf{x} + c_{k+1}\mathbf{x}^{(k+1)}$ is also a solution and thus $c_1\mathbf{x}^{(1)} + \cdots + c_{k+1}\mathbf{x}^{(k+1)}$ is a solution if $\mathbf{x}^{(1)}$, …, $\mathbf{x}^{(k+1)}$ are solutions.

2a. From Eq.(10) we have $W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = x_1^{(1)}\, x_2^{(2)} - x_2^{(1)}\, x_1^{(2)}$. Taking the derivative of these two products yields four terms which may be written as

$$\frac{dW}{dt} = \left[\frac{dx_1^{(1)}}{dt}x_2^{(2)} - x_2^{(1)}\frac{dx_1^{(2)}}{dt}\right] + \left[x_1^{(1)}\frac{dx_2^{(2)}}{dt} - \frac{dx_2^{(1)}}{dt}x_1^{(2)}\right].$$

The terms in the square brackets can now be recognized as the respective determinants appearing in the desired solution. A similar result was mentioned in Problem 20 of Section 4.1.

2b. If $\mathbf{x}^{(1)}$ is substituted into Eq.(3) we have

$$\frac{dx_1^{(1)}}{dt} = p_{11}\,x_1^{(1)} + p_{12}\,x_2^{(1)}$$

$$\frac{dx_2^{(1)}}{dt} = p_{21}\,x_1^{(1)} + p_{22}\,x_2^{(1)}.$$

Substituting the first equation above and its counterpart for $\mathbf{x}^{(2)}$ into the first determinant appearing in dW/dt and evaluating the result yields $p_{11}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = p_{11}W$.

Similarly, the second determinant in dW/dt is evaluated as $p_{22}W$, yielding the desired result.

2c. From prt b we have $\frac{dW}{W} = [p_{11}(t) + p_{22}(t)]dt$ which gives

$$W(t) = c\exp\int[p_{11}(t) + p_{22}(t)]dt.$$

6a. $W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$

6b. Pick $t = t_0$, then $c_1\mathbf{x^{(1)}}(t_0) + c_2\mathbf{x^{(2)}}(t_0) = \mathbf{0}$ implies

$c_1\begin{pmatrix} t_0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t_0^2 \\ 2t_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a non-zero solution for c_1 and c_2 if and only if $\begin{vmatrix} t_0 & t_0^2 \\ 1 & 2t_0 \end{vmatrix} = 2t_0^2 - t_0^2 = t_0^2 = \mathbf{0}$.

Thus $\mathbf{x^{(1)}}(t)$ and $\mathbf{x^{(2)}}(t)$ are linearly independent at each point except $t = 0$. Thus they are linearly independent on every interval.

6c. From part a we see that the Wronskian vanishes at $t = 0$, but not at any other point. By Theorem 7.4.3, if p(t), from Eq.(3), is continuous, then the Wronskian is either identically zero or else never vanishes. Hence, we conclude that the D.E. satisfied by $\mathbf{x^{(1)}}(t)$ and $\mathbf{x^{(2)}}(t)$ must have at least one discontinuous coefficient at $t = 0$.

6d. To obtain the system satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ we

consider

$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$, or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

Taking the derivative we obtain $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 2 \end{pmatrix}$.

Solving this last system for c_1 and c_2 we find $c_1 = x_1' - tx_2'$ and $c_2 = x_2'/2$. Thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1' - tx_2') \begin{pmatrix} t \\ 1 \end{pmatrix} + \frac{x_2'}{2} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$, which yields $x_1 = tx_1' - \frac{t^2}{2} x_2'$ and $x_2 = x_1'$. Writing this system in matrix form we have $\mathbf{x} = \begin{pmatrix} t - t^2/2 & \\ 1 & 0 \end{pmatrix}\mathbf{x}'$. Finding the inverse of the matrix multiplying $\mathbf{x}'$ yields the desired solution.

Section 7.5, Page 381

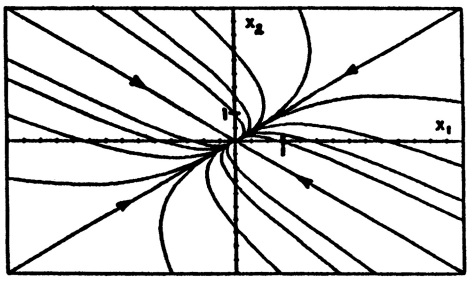
1. Assuming that there are solutions of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, we substitute into the D.E. to find $r\boldsymbol{\xi}e^{rt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi}e^{rt}$. Since $\boldsymbol{\xi} = I\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi}$, we can write this equation as $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$ and thus we must solve $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for r, ξ_1, ξ_2. The determinant of the coefficients is $(3-r)(-2-r) + 4 = r^2 - r - 2$, so the eigenvalues are $r = -1, 2$. The eigenvector corresponding to $r = -1$ satisfies $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which yields $2\xi_1 - \xi_2 = 0$. Thus $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{-t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$, where we have set $\xi_1 = 1$. (Any other non zero choice would also work). In a similar fashion, for $r = 2$, we have $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $\xi_1 - 2\xi_2 = 0$. Hence $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}^{(2)}e^{2t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ by

setting $\xi_2 = 1$. The general solution is then $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$. To sketch the trajectories we follow the steps illustrated in Examples 1 and 2.

Setting $c_2 = 0$ we have $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ or $x_1 = c_1e^{-t}$ and $x_2 = 2c_1e^{-t}$ and thus one asymptote is given by $x_2 = 2x_1$. In a similar fashion $c_1 = 0$ gives $x_2 = (1/2)x_1$ as a second asymptote. Since the roots differ in sign, the trajectories for this problem are similar in nature to those in Example 1. For $c_2 \neq 0$, all solutions will be asymptotic to $x_2 = (1/2)x_1$ as $t \to \infty$. For $c_2 = 0$, the solution approaches the origin along the line $x_2 = 2x_1$.



5. Proceeding as in Problem 1 we assume a solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, where r, ξ_1, ξ_2 must now satisfy $\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Evaluating the determinant of the coefficients set equal to zero yields $r = -1, -3$ as the eigenvalues. For $r = -1$ we find $\xi_1 = \xi_2$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $r = -3$ we find $\xi_2 = -\xi_1$ and hence $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution is then $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$. Since there are two negative eigenvalues, we would expect the trajectories to be similar to those of Example 2. Setting $c_2 = 0$ and eliminating t (as in Problem 1) we find that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ approaches the origin along the line $x_2 = x_1$. Similarly $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$ approaches the origin along the line



$x_2 = -x_1$. As long as $c_1 \neq 0$ (since e^{-t} is the dominant term as $t \to 0$), all trajectories approach the origin asymptotic to $x_2 = x_1$. For $c_1 = 0$, the trajectory approaches the origin along $x_2 = -x_1$, as shown in the graph.

6. The characteristic equation is $(5/4 - r)^2 - 9/16 = 0$, so $r = 2, 1/2$. Since the roots are of the same size, the behavior of the solutions is similar to Problem 5, except the trajectories are reversed since the roots are positive.

7. Again assuming $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ we find that r, ξ_1, ξ_2 must satisfy $\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of the coefficients set equal to zero yields $r = 0, -2$. For $r = 0$ we find $4\xi_1 = 3\xi_2$. Choosing $\xi_2 = 4$ we find $\xi_1 = 3$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Similarly for $r = -2$ we have $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and thus $\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$. To sketch the trajectories, note that the general solution is equivalent to the simultaneous equations $x_1 = 3c_1 + c_2 e^{-2t}$ and $x_2 = 4c_1 + 2c_2 e^{-2t}$. Solving the first equation for $c_2 e^{-2t}$ and substituting into the second yields $x_2 = 2x_1 - 2c_1$ and thus the trajectories are parallel straight lines.

9. The eigvalues are given by $\begin{vmatrix} 1-r & i \\ -i & 1-r \end{vmatrix} = (1-r)^2 + i^2 = r(r-2) = 0$. For $r = 0$ we have $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$ or $-i\xi_1 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is one eigenvector. Similarly $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigenvector for $r = 2$.

14. The eigenvalues and eigenvectors of the coefficient

matrix satisfy $\begin{pmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients set equal to zero reduces to $r^3 - 2r^2 - 5r + 6 = 0$, so the eigenvalues are $r_1 = 1$, $r_2 = -2$, and $r_3 = 3$. The eigenvector corresponding to r_1 must satisfy $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Using row reduction we obtain the equivalent system $\xi_1 + \xi_3 = 0$, $\xi_2 - 4\xi_3 = 0$. Letting $\xi_1 = 1$, it follows that

$\xi_3 = -1$ and $\xi_2 = -4$, so $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$. In a similar way the eigenvectors corresponding to r_2 and r_3 are found to be $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{\xi}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, respectively. Thus the general solution of the given D.E. is

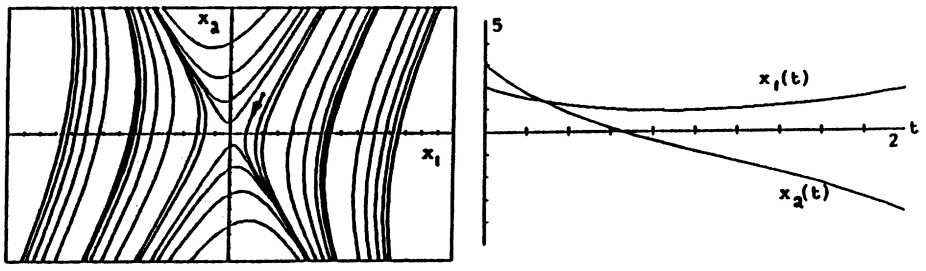
$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. Notice that the "trajectories" of this solution would lie in the $x_1\ x_2\ x_3$ three dimensional space.

16. The eigenvalues and eigenvectors of the coefficient matrix are found to be $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = 3$, $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Thus the general solution of the given D.E. is $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$. The I.C. yields the system of equations $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The augmented

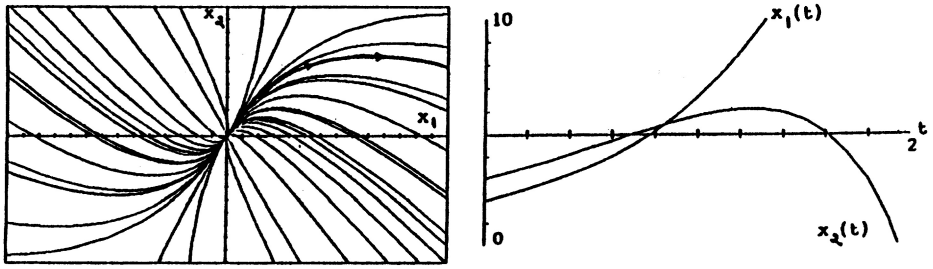
matrix of this system is $\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$ and by row reduction we obtain $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus $c_2 = 1/2$ and $c_1 = 1/2$. Substituting these values in the general solution gives the solution of the I.V.P. As $t \to \infty$, the solution becomes asymptotic to $\mathbf{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$, or $x_2 = 5x_1$.

20. Substituting $\mathbf{x} = \boldsymbol{\xi} t^r$ into the D.E. we obtain $r\boldsymbol{\xi} t^r = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{\xi} t^r$. For $t \neq 0$ this equation can be written as $\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The eigenvalues and eigenvectors are $r_1 = 1$, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = -1$, $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Substituting these in the assumed form we obtain the general solution $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$.

25.



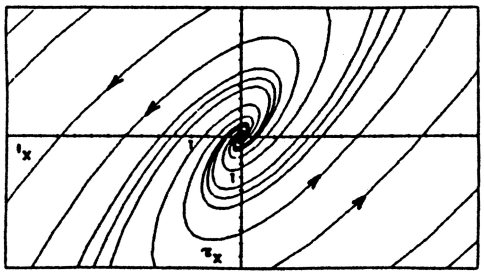
27.



31c. The eigevalues are given by
$\begin{vmatrix} -1-r & -1 \\ -\alpha & -1-r \end{vmatrix} = r^2 + 2r + 1 - \alpha = 0$. Thus $r_{1,2} = -1 \pm \sqrt{\alpha}$.
Note that in Part (a) the eigenvalues are both negative while in Part (b) they differ in sign. Thus, in this part, if we choose $\alpha = 1$, then one eigenvalue is zero, which is the transition of the one root from negative to positive. This is the desired bifurcation point.

Section 7.6, Page 390

1. We assume a solution of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ thus r and $\boldsymbol{\xi}$ are solutions of $\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(r^2-2r-3) + 8 = r^2 - 2r + 5$, so the eigenvalues are $r = 1 \pm 2i$. The eigenvector corresponding to $1 + 2i$ satisfies $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $(2-2i)\xi_1 - 2\xi_2 = 0$. If $\xi_1 = 1$, then $\xi_2 = 1-i$ and $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ and thus one complex-valued solution of the D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t}$.



To find real-valued solutions (see Eqs.8 and 9) we take the real and imaginary parts, respectively of $\mathbf{x}^{(1)}(t)$. Thus $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t(\cos 2t + i \sin 2t)$

$$= e^t \begin{pmatrix} \cos 2t + i \sin 2t \\ \cos 2t + \sin 2t + i(\sin 2t - \cos 2t) \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

Hence the general solution of the D.E. is $\mathbf{x} = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$. The solutions spiral to ∞ as $t \to \infty$ due to the e^t terms.

7. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients reduces to $(1-r)(r^2 - 2r + 5)$ so the eigenvalues are $r_1 = 1$, $r_2 = 1 + 2i$, and $r_3 = 1 - 2i$. The eigenvector corresponding to r_1 satisfies $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; hence $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we let $\xi_2 = -3$ then $\xi_1 = 2$ and $\xi_3 = 2$, so one solution of the D.E. is $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t$. The eigenvector corresponding to r_2 satisfies $\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence $\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -i$. Thus a complex-valued solution is $\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t(\cos 2t + i \sin 2t)$. Taking the real and imaginary parts, see prob. 1, we obtain $\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t$ and $\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix} e^t$, respectively. Thus the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t + c_2e^t\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3e^t\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix},$$ which spirals to ∞ about the x_1 axis in the $x_1x_2x_3$ space as $t \to \infty$.

9. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $r^2 + 2r + 2$ so that the eigenvalues are $r = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$

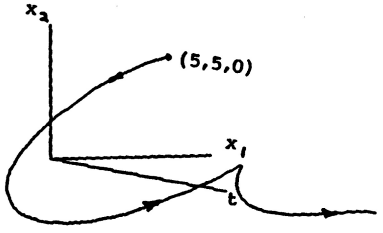
is given by $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathbf{0}$ so that $\xi_1 = (2+i)\xi_2$ and thus one complex-valued solution is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$. Finding the real and complex parts of $\mathbf{x}^{(1)}$ leads to the general solution $\mathbf{x} = c_1e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$. Setting $t = 0$ we find $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is equivalent to the system $\begin{matrix} 2c_1 + c_2 = 1 \\ c_1 + 0 = 1 \end{matrix}$. Thus $c_1 = 1$ and $c_2 = -1$ and

$$\mathbf{x}(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}, \text{ which spirals to zero as}$$

$t \to \infty$ due to the e^{-t} term.

11a. The eigenvalues are given by $\begin{vmatrix} 3/4-r & -2 \\ 1 & -5/4-r \end{vmatrix} = r^2 + r/2 + 17/16 = 0.$

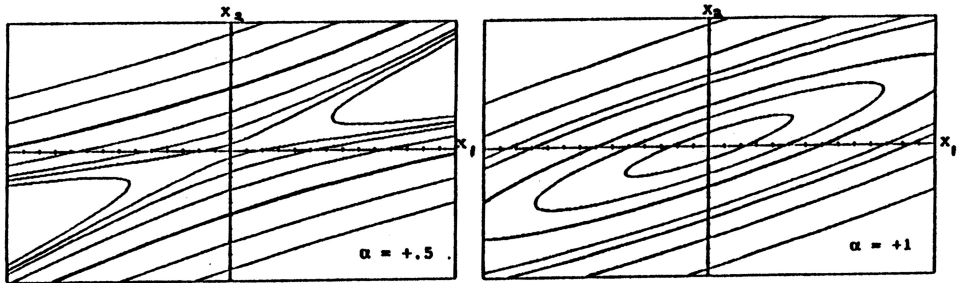
11d. The trajectory starts at (5,5) in the x_1x_2 plane and spirals around and converges to the t axis as $t \to \infty$.



15a. The eigenvalues satisfy $\begin{vmatrix} 2-r & -5 \\ \alpha & -2-r \end{vmatrix} = r^2 - 4 + 5\alpha = 0$, so $r_1, r_2 = \pm\sqrt{4-5\alpha}$.

15b. The critical value of α yields $r_1 = r_2 = 0$, or $\alpha = 4/5$.

15c.

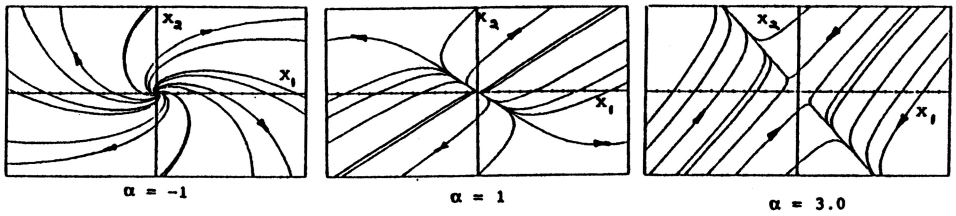


16a. $\begin{vmatrix} 5/4-r & 3/4 \\ \alpha & 5/4-r \end{vmatrix} = r^2 - 5r/2 + (25/16 - 3\alpha/4) = 0$, so

$r_{1,2} = 5/4 \pm \sqrt{3\alpha}\,/2$.

16b. There are two critical values of α. For $\alpha < 0$ the eigenvalues are complex, while for $\alpha > 0$ they are real. There will be a second critical value of α when $r_2 = 0$, or $\alpha = 25/12$. In this case the second real eigenvalue goes from positive to negative.

16c.

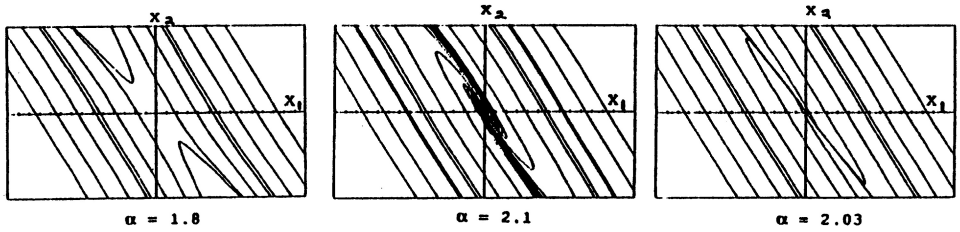


18a. We have $\begin{vmatrix} 3-r & \alpha \\ -6 & -4-r \end{vmatrix} = r^2 + r - 12 + 6\alpha = 0$, so

$r_1, r_2 = -1/2 \pm \sqrt{49-24\alpha}\,/2$.

18b. The critical values occur when $49 - 24\alpha = 1$ (in which case $r_2 = 0$) and when $49 - 24\alpha = 0$, in which case $r_1 = r_2 = -1/2$. Thus $\alpha = 2$ and $\alpha = 49/24 \cong 2.04$.

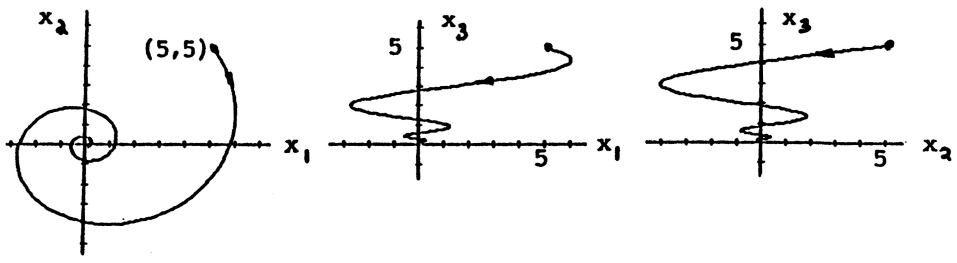
18c.



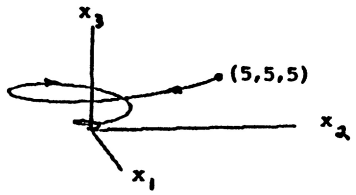
21. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} t^r$, then r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix. Thus r and $\boldsymbol{\xi}$ satisfy $\begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(-1-r)^2 + 2 = r^2 + 2r + 3$, so the eigenvalues are $r = -1 \pm \sqrt{2}\,i$. The eigenvector corresponding to $-1 + \sqrt{2}\,i$ satisfies $\begin{pmatrix} -\sqrt{2}\,i & -1 \\ 2 & -\sqrt{2}\,i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\sqrt{2}\,i\xi_1 + \xi_2 = 0$. If we let $\xi_1 = 1$, then $\xi_2 = -\sqrt{2}\,i$, and $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix}$. Thus a complex-valued solution of the given D.E. is $\begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} t^{-1+\sqrt{2}\,i}$. From Eq. (15) of Section 5.5 we have (since $t^{\sqrt{2}\,i} = e^{\ln t^{\sqrt{2}\,i}} = e^{\sqrt{2}\,i \ln t}$) $t^{-1+\sqrt{2}\,i} = t^{-1}[\cos(\sqrt{2}\ln t) + i\sin(\sqrt{2}\ln t)]$ for $t > 0$. Separating the complex valued solution into real and imaginary parts, we obtain the two real-valued solutions $\mathbf{u} = t^{-1}\begin{pmatrix} \cos(\sqrt{2}\ln t) \\ \sqrt{2}\,\sin(\sqrt{2}\ln t) \end{pmatrix}$ and $\mathbf{v} = t^{-1}\begin{pmatrix} \sin(\sqrt{2}\ln t) \\ -\sqrt{2}\cos(\sqrt{2}\ln t) \end{pmatrix}$.

23a. The eigenvalues are given by $(r+1/4)[(r+1/4)^2 + 1] = 0$.

23b.



23c. Graph starts in the first octant and spirals around the x_3 axis, converging to zero.



29a. We have $y_1' = x_1' = y_2$, $y_3' = x_2' = y_4$, $y_2' = -2y_1 + y_3$, and $y_4' = y_1 - 2y_3$. Thus

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \mathbf{y}.$$

29b. The eigenvalues are given by $r^4 + 4r^2 + 3 = 0$, which yields $r^2 = -1, -\sqrt{3}$, so $r = \pm i, \pm\sqrt{3}\,i$.

29c. For $r = \pm i$ the eigenvectors are given by $\begin{pmatrix} -i & 1 & 0 & 0 \\ -2 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & -2 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = 0$. Choosing $\xi_1 = 1$ yields $\xi_2 = i$ and choosing $\xi_3 = 1$ yields $\xi_4 = i$, so $(1, i, 1, i)^T(\cos t + i\sin t)$ is a solution. Finding the real and imaginary parts yields
$\mathbf{w}_1 = (\cos t, -\sin t, \cos t, -\sin t)^T$ and
$\mathbf{w}_2 = (\sin t, \cos t, \sin t, \cos t)^T$ as two real solutions.
In a similar fashion, for $r = \pm\sqrt{3}\,i$, we obtain
$\boldsymbol{\xi} = (1, \sqrt{3}\,i, -1, -\sqrt{3}\,i)$ and
$\mathbf{w}_3 = (\cos\sqrt{3}\,t, -\sqrt{3}\sin\sqrt{3}\,t, -\cos\sqrt{3}\,t, \sqrt{3}\sin\sqrt{3}\,t)^T$ and
$\mathbf{w}_4 = (\sin\sqrt{3}\,t, \sqrt{3}\cos\sqrt{3}\,t, -\sin\sqrt{3}\,t, -\sqrt{3}\cos\sqrt{3}\,t)^T$.
Thus $\mathbf{y} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + c_4\mathbf{w}_4$, so $\mathbf{y}^T(0) = (2, 1, 2, 1)$ yields $c_1 + c_3 = 2$, $c_2 + \sqrt{3}\,c_4 = 1$, $c_1 - c_3 = 2$, and $c_2 - \sqrt{3}\,c_4 = 1$, which yields $c_1 = 2$, $c_2 = 1$, and $c_3 = c_4 = 0$. Hence $\mathbf{y} = \begin{pmatrix} 2\cos t + \sin t \\ -2\sin t + \cos t \\ 2\cos t + \sin t \\ -2\sin t + \cos t \end{pmatrix}$.

29e. The natural frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$, which are the absolute value of the eigenvalues. For any other choice of I.C., both frequencies will be present, and thus another mode of oscillation with a different frequency (depending on the I.C.) will be present.

Section 7.7, Page 400

Each of the Problems 1 through 10, except 2 and 8, has been solved in one of the previous sections. Thus a fundamental matrix for the given systems can be readily written down. The fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(0) = \mathbf{I}$ can then be

found, as shown in the following problems.

2. The characteristic equation is given by $\begin{vmatrix} -3/4-r & 1/2 \\ 1/8 & -3/4-r \end{vmatrix} = r^2 + 3r/2 + 1/2 = 0$, so $r = 1, 1/2$. For $r = 1$ we have $\begin{pmatrix} 1/4 & 1/2 \\ 1/8 & 1/4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Likewise $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and thus $\mathbf{x}^{(1)}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t/2}$. To find the first column of $\boldsymbol{\Phi}$ we choose c_1 and c_2 so that $c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which yields $-2c_1 + 2c_2 = 1$ and $c_1 + c_2 = 0$. Thus $c_1 = -1/4$ and $c_2 = 1/4$ and the first column of $\boldsymbol{\Phi}$ is $\begin{pmatrix} 1/2e^{-t/2} + 1/2e^{t} \\ 1/4e^{-t/2} - 1/4e^{-t/2} \end{pmatrix}$. The second colunm of $\boldsymbol{\Phi}$ is determined by $d_1\mathbf{x}^{(1)}(0) + d_2\mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which yields $d_1 = d_2 = 1/2$ and thus the second column of $\boldsymbol{\Phi}$ is $\begin{pmatrix} e^{-t/2} - e^{-t} \\ 1/2e^{-t/2} + 1/2e^{-t} \end{pmatrix}$.

4. From Problem 4 of Section 7.5 we have the two linearly independent solutions $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$. Hence a fundamental matrix $\boldsymbol{\Psi}$ is given by $\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$. To find the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying the I.C. $\boldsymbol{\Phi}(0) = \mathbf{I}$ we can proceed in either of two ways. One way is to find $\boldsymbol{\Psi}(0)$, invert it to obtain $\boldsymbol{\Psi}^{-1}(0)$, and then to form the product $\boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$, which is $\boldsymbol{\Phi}(t)$. Alternatively, we can find the first column of $\boldsymbol{\Phi}$ by determining the linear combination $c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This requires that $c_1 + c_2 = 1$, $-4c_1 + c_2 = 0$, so we obtain $c_1 = 1/5$ and $c_2 = 4/5$. Thus the first column of $\boldsymbol{\Phi}(t)$ is

$\begin{pmatrix} (1/5)e^{-3t} + (4/5)e^{2t} \\ -(4/5)e^{-3t} + (4/5)e^{2t} \end{pmatrix}$. Similarly, the second column of $\mathbf{\Phi}$ is that linear combination of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus we must have

$c_1 + c_2 = 0$, $-4c_1 + c_2 = 1$; therefore $c_1 = -1/5$ and $c_2 = 1/5$. Hence the second column of $\mathbf{\Phi}(t)$ is $\begin{pmatrix} -(1/5)e^{-3t} + (1/5)e^{2t} \\ (4/5)e^{-3t} + (1/5)e^{2t} \end{pmatrix}$.

6. Two linearly independent real-valued solutions of the given D.E. were found in Problem 2 of Section 7.6. Using the result of that problem, we have $\mathbf{\Psi}(t) = \begin{pmatrix} -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \\ e^{-t}\cos 2t & e^{-t}\sin 2t \end{pmatrix}$. To find $\mathbf{\Phi}(t)$ we determine the linear combinations of the columns of $\mathbf{\Psi}(t)$ that satisfy the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively.

In the first case c_1 and c_2 satisfy $0c_1 + 2c_2 = 1$ and $c_1 + 0c_2 = 0$. Thus $c_1 = 0$ and $c_2 = 1/2$. In the second case we have $0c_1 + 2c_2 = 0$ and $c_1 + 0c_2 = 1$, so $c_1 = 1$ and $c_2 = 0$. Using these values of c_1 and c_2 to form the first and second columns of $\mathbf{\Phi}(t)$ respectively, we obtain $\mathbf{\Phi}(t) = \begin{pmatrix} e^{t}\cos 2t & -2e^{-t}\sin 2t \\ (1/2)e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}$.

10. From Problem 14 Section 7.5 we have $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t}$, $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-2t}$ and $\mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. For the first column of $\mathbf{\Phi}$ we want to choose c_1, c_2, c_3 such that $c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) + c_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Thus $c_1 + c_2 + c_3 = 1$, $-4c_1 - c_2 + 2c_3 = 0$ and $-c_1 - c_2 + c_3 = 0$, which yield $c_1 = 1/6$, $c_2 = 1/3$ and $c_3 = 1/2$. The first column of $\mathbf{\Phi}$ is then $(1/6e^{t} + 1/3e^{-2t} + 1/2e^{3t}, -2/3e^{t} - 1/3e^{-2t} + e^{3t}, -1/6e^{t} - 1/3e^{-2t} + 1/2e^{3t})^{T}$. Likewise, for the second

column we have $d_1\mathbf{x}^{(1)}(0) + d_2\mathbf{x}^{(2)}(0) + d_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which yields $d_1 = -1/3$, $d_2 = 1/3$ and $d_3 = 0$ and thus $(-1/3e^{t} + 1/3e^{-2t},\ 4/3e^{t} - 1/3e^{-2t},\ 1/3e^{t} - 1/3e^{-2t})^{T}$ is the second column of $\mathbf{\Phi}(t)$. Finally, for the third column we have $e_1\mathbf{x}^{(1)}(0) + e_2\mathbf{x}^{(2)}(0) + e_3\mathbf{x}^{(3)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which gives $e_1 = 1/2$, $e_2 = -1$ and $e_3 = 1/2$ and hence $(1/2e^{t} - e^{-2t} + 1/2e^{3t},\ -2e^{t} + e^{-2t} + e^{3t},\ -1/2e^{t} + e^{-2t} + 1/2e^{3t})^{T}$ is the third column of $\mathbf{\Phi}(t)$.

11. From Eq. (14) the solution is given by $\mathbf{\Phi}(t)\mathbf{x}^{0}$. Thus

$$\mathbf{x} = \begin{pmatrix} 3/2e^{t}-1/2e^{-t} & -1/2e^{t}+1/2e^{-t} \\ 3/2e^{t}-3/2e^{-t} & -1/2e^{t}+3/2e^{-t} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7/2e^{t}-3/2e^{t} \\ 7/2e^{t}-9/2e^{-t} \end{pmatrix} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

Section 7.8, Page 407

1. The eigenvalues and eigenvectors of the given coefficient matrix satisfy $\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(3-r)(-1-r) + 4 = r^2 - 2r + 1 = (r-1)^2$ so $r_1 = 1$ and $r_2 = 1$. The eigenvectors corresponding to this double eigenvalue satisfy $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $\xi_1 - 2\xi_2 = 0$. Thus the only eigenvectors are multiples of $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. One solution of the given D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$, but there is no second solution of this form. To find a second solution we assume, as in Eq. (13), that $\mathbf{x} = \boldsymbol{\xi}te^{t} + \boldsymbol{\eta}e^{t}$ and substitute this expression into the D.E. As in Example 2 we find that $\boldsymbol{\xi}$ is an eigenvector, so we choose $\boldsymbol{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then $\boldsymbol{\eta}$ must

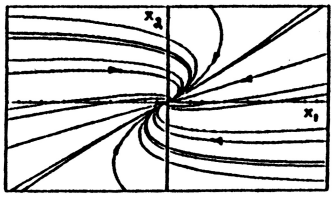
satisfy $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which verifies Eq.(16).

Solving these equations yields $\eta_1 - 2\eta_2 = 1$. If $\eta_2 = k$, where k is an arbitrary constant, then $\eta_1 = 1 + 2k$. Hence the second solution that we obtain is

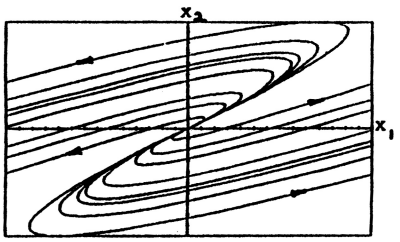
$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 + 2k \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + k\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

The last term is a multiple of the first solution $\mathbf{x}^{(1)}(t)$ and may be neglected, that is, we may set $k = 0$. Thus $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ and the general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$. All solutions diverge to infinity as $t \to \infty$. The graph is shown on the right.

3. The origin is attracting



1.



5. Substituting $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ into the given system, we find that the eigenvalues and eigenvectors satisfy

$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $-r^3 + 3r^2 - 4$ and thus $r_1 = -1$, $r_2 = 2$ and $r_3 = 2$. The eigenvector corresponding to r_1 satisfies

$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which yields $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and

$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}$. The eigenvectors corresponding to the double eigenvalue must satsify $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

which yields the single eigenvector $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and hence $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields $\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \boldsymbol{\eta} e^{2t}$.

Substituting this into the given system, or using Eq.(16), we find that $\boldsymbol{\eta}$ satisfies $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Using row reduction we find that $\eta_1 = 1$ and $\eta_2 + \eta_3 = 1$, where either η_2 or η_3 is arbitrary. If we choose $\eta_2 = 0$, then $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and thus $\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$. The general solution is then $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}$.

9. We have $\begin{vmatrix} 2-r & 3/2 \\ -3/2 & -1-r \end{vmatrix} = (r-1/2)^2 = 0$. For $r = 1/2$, the eigenvector is given by $\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$, so $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$ is one solution. For the second solution we have $\mathbf{x} = \boldsymbol{\xi} te^{t/2} + \boldsymbol{\eta} e^{t/2}$, where $(\mathbf{A} - \frac{1}{2}\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$, $\mathbf{A}$ being the coefficient matrix for this problem. This last equation reduces to $3\boldsymbol{\eta}_1/2 + 3\boldsymbol{\eta}_2/2 = 1$ and $-3\boldsymbol{\eta}_1/2 - 3\boldsymbol{\eta}_2/2 = -1$. Choosing $\boldsymbol{\eta}_2 = 0$ yields $\boldsymbol{\eta}_1 = 2/3$ and hence

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2}.$$

$\mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ gives $c_1 + 2c_2/3 = 3$ and $-c_1 = -2$, and hence $c_1 = 2$, $c_2 = 3/2$. Substituting these into the above $\mathbf{x}$ yields the solution.

11. The eigenvalues are r = 1,1,2. For r = 2, we have $\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields $\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so one solution is $\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$. For r = 1, we have $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields the second solution $\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^{t}$. The third solution is of the form $\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + \eta e^{t}$, where $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$ and thus $\boldsymbol{\eta_1} = -1/4$ and $6\boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 = -21/4$. Choosing $\boldsymbol{\eta}_2 = 0$ gives $\boldsymbol{\eta}_3 = -21/4$ and hence

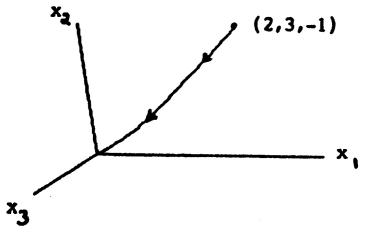
$$\mathbf{x}(t) = c_1\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^{t} + c_2\left[\begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t}\right] + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

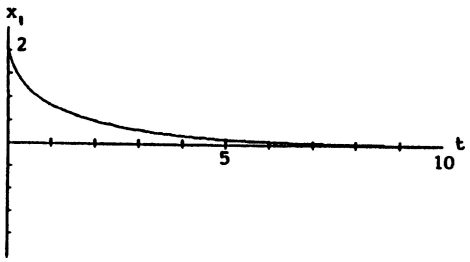
The I.C. then yield $c_1 = 2$, $c_2 = 4$ and $c_3 = 3$ and hence

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} e^{t} + 4\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^{t} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t},$$

which become unbounded as $t \to \infty$.

12.





14. Assuming $\mathbf{x} = \boldsymbol{\xi} t^{r}$ and substituting into the given system, we find r and $\boldsymbol{\xi}$ must satisfy $\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has the double eigenvalue r = -3 and single eigenvector

$\begin{pmatrix}1\\1\end{pmatrix}$. Hence one solution of the given D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix}1\\1\end{pmatrix} t^{-3}$. By analogy with the scalar case considered in Section 5.5 and Example 2 of this section, we seek a second solution of the form $\mathbf{x} = \boldsymbol{\eta} t^{-3}\ln t + \boldsymbol{\zeta} t^{-3}$. Substituting this expression into the D.E. we find that $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations $(\mathbf{A} + 3\mathbf{I})\boldsymbol{\eta} = \mathbf{0}$ and $(\mathbf{A} + 3\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$, where $\mathbf{A} = \begin{pmatrix}1 & -4\\ 4 & -7\end{pmatrix}$ and $\mathbf{I}$ is the identity matrix. Thus $\boldsymbol{\eta} = \begin{pmatrix}1\\1\end{pmatrix}$, from above, and $\boldsymbol{\zeta}$ is found to be $\begin{pmatrix}0\\-1/4\end{pmatrix}$. Thus a second solution is $\mathbf{x}^{(2)}(t) = \begin{pmatrix}1\\1\end{pmatrix} t^{-3}\ln t + \begin{pmatrix}1\\-1/4\end{pmatrix} t^{-3}$.

15. All solutions of the given system approach zero as $t \to \infty$ if and only if the eigenvalues of the coefficient matrix either are real and negative or else are complex with negative real part. Write down the determinantal equation satisfied by the eigenvalues and determine when the eigenvalues are as stated.

17a. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix}1-r & 1 & 1\\ 2 & 1-r & -1\\ -3 & 2 & 4-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. The determinant of coefficients is $8 - 12r + 6r^2 - r^3 = (2-r)^3$, so the eigenvalues are $r_1 = r_2 = r_3 = 2$. The eigenvectors corresponding to this triple eigenvalue satisfy $\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Using row reduction we can reduce this to the equivalent system $\xi_1 - \xi_2 - \xi_3 = 0$, and $\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -1$ and $\xi_1 = 0$, so the only eigenvectors are multiples of $\boldsymbol{\xi} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$.

17b. From part a, one solution of the given D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$, but there are no other linearly independent solutions of this form.

17c. We now seek a second solution of the form $\mathbf{x} = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t}$. Thus $\mathbf{Ax} = \mathbf{A}\boldsymbol{\xi} te^{2t} + \mathbf{A}\boldsymbol{\eta} e^{2t}$ and $\mathbf{x}' = 2\boldsymbol{\xi} te^{2t} + \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\eta} e^{2t}$. Equating like terms, we then have $(\mathbf{A}-2\mathbf{I})\boldsymbol{\xi} = 0$ and $(\mathbf{A}-2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$. Thus $\boldsymbol{\xi}$ is as in part a and the second equation yields $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. By row reduction this is equivalent to the system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If we choose $\eta_3 = 0$, then $\eta_2 = 1$ and $\eta_1 = 1$, so $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence a second solution of the D.E. is

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

17d. Assuming $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t}$, we have $\mathbf{Ax} = \mathbf{A}\boldsymbol{\xi}(t^2/2)e^{2t} + \mathbf{A}\boldsymbol{\eta} te^{2t} + \mathbf{A}\boldsymbol{\xi} e^{2t}$ and $\mathbf{x}' = \boldsymbol{\xi} te^{2t} + 2\boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} e^{2t} + 2\boldsymbol{\eta} te^{2t} + 2\boldsymbol{\xi} e^{2t}$ and thus $(\mathbf{A}-2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, $(\mathbf{A}-2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ and $(\mathbf{A}-2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$. Again, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are as found previously and the last equation is equivalent to $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. By row reduction we find the equivalent system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$. If we let

$\zeta_2 = 0$, then $\zeta_3 = 3$ and $\zeta_1 = 2$, so $\boldsymbol{\zeta} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and

$$\mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t}.$$

17e. $\boldsymbol{\Psi}$ is the matrix with $\mathbf{x}^{(1)}$ as the first column, $\mathbf{x}^{(2)}$ as the second column and $\mathbf{x}^{(3)}$ as the third column.

17f. $\mathbf{T} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and using row operations on $\mathbf{T}$ and $\mathbf{I}$, or a computer algebra system, $\mathbf{T}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$ and thus

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{J}.$$

19a. $\mathbf{J}^2 = \mathbf{J}\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$

$$\mathbf{J}^3 = \mathbf{J}\mathbf{J}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

19b. Based upon the results of part a, assume $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$, then

$$\mathbf{J}^{n+1} = \mathbf{J}\mathbf{J}^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}, \text{ which is the same as } \mathbf{J}^n \text{ with } n$$

replaced by n+1. Thus, by mathematical induction, $\mathbf{J}^n$ has the desired form.

19c. From Eq.(23), Section 7.7, we have

$$\exp(\mathbf{J}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{J}^n t^n}{n!}$$

$$= \mathbf{I} + \sum_{n=1}^{\infty} \begin{pmatrix} \frac{\lambda^n t^n}{n!} & \frac{n\lambda^{n-1} t^n}{n!} \\ 0 & \frac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} & \sum_{n=1}^{\infty} \frac{\lambda^{n-1} t^n}{(n-1)!} \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}, \text{ since}$$

$$\sum_{n=1}^{\infty} \frac{\lambda^{n-1} t^n}{(n-1)!} = t\left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}\right) = te^{\lambda t}.$$

19d. From Eq. (28), Section 7.7, we have

$$\mathbf{x} = \exp(\mathbf{J}t)\mathbf{x}^0 = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} x_1^0 e^{\lambda t} + x_2^0 te^{\lambda t} \\ x_2^0 e^{\lambda t} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} e^{\lambda t} + \begin{pmatrix} x_2^0 \\ 0 \end{pmatrix} te^{\lambda t}.$$

Section 7.9, Page 417

1. From Section 7.5 Problem 3 we have $\mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. Note that $\mathbf{g}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$ and that $r = 1$ is an eigenvalue of the coefficient matrix. Thus if the method of undetermined coefficients is used, the assumed form is given by Eq.(18).

2. Using methods of previous sections, we find that the eigenvalues are $r_1 = 2$ and $r_2 = -2$, with corresponding eigenvectors $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$. Thus

$\mathbf{x}^{(c)} = c_1\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\ -\sqrt{3}\end{pmatrix}e^{-2t}$. Writing the nonhomogeneous term as $\begin{pmatrix}1\\ 0\end{pmatrix}e^{t} + \begin{pmatrix}0\\ \sqrt{3}\end{pmatrix}e^{-t}$ we see that we can assume $\mathbf{x}^{(p)} = \mathbf{a}e^{t} + \mathbf{b}e^{-t}$. Substituting this in the D.E., we obtain

$\mathbf{a}e^{t} - \mathbf{b}e^{-t} = \mathbf{A}\mathbf{a}e^{t} + \mathbf{A}\mathbf{b}e^{-t} + \begin{pmatrix}1\\ 0\end{pmatrix}e^{t} + \begin{pmatrix}0\\ \sqrt{3}\end{pmatrix}e^{-t}$, where $\mathbf{A}$ is the given coefficient matrix. All the terms involving e^{t} must add to zero and thus we have $\mathbf{A}\mathbf{a} - \mathbf{a} + \begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. This is equivalent to the system $\sqrt{3}\,a_2 = -1$ and $\sqrt{3}\,a_1 - 2a_2 = 0$, or $a_1 = -2/3$ and $a_2 = -1/\sqrt{3}$. Likewise the terms involving e^{-t} must add to zero, which yields $\mathbf{A}\mathbf{b} + \mathbf{b} + \begin{pmatrix}0\\ \sqrt{3}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. The solution of this system is $b_1 = -1$ and $b_2 = 2/\sqrt{3}$. Substituting these values for $\mathbf{a}$ and $\mathbf{b}$ into $\mathbf{x}^{(p)}$ and adding $\mathbf{x}^{(p)}$ to $\mathbf{x}^{(c)}$ yields the desired solution.

3. The method of undetermined coefficients is not straight forward since the assumed form of $\mathbf{x}^{(p)} = \mathbf{a}\cos t + \mathbf{b}\sin t$ leads to singular equations for $\mathbf{a}$ and $\mathbf{b}$. From Problem 3 of Section 7.6 we find that a fundamental matrix is

$\Psi(t) = \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t + \sin t & -\cos t + 2\sin t\end{pmatrix}$. The inverse matrix is

$\Psi^{-1}(t) = \begin{pmatrix}\dfrac{\cos t - 2\sin t}{5} & \sin t\\ \dfrac{2\cos t + \sin t}{5} & -\cos t\end{pmatrix}$, which may be found as

in Section 7.2 or by using a computer algebra system. Thus we may use the method of variation of parameters where $\mathbf{x} = \Psi(t)\mathbf{u}(t)$ and $\mathbf{u}(t)$ is given by $\mathbf{u}'(t) = \Psi^{-1}(t)g(t)$ from Eq.(27). For this problem $\mathbf{g}(t) = \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$ and thus

$$\mathbf{u}'(t) = \begin{pmatrix} \dfrac{\cos t - 2\sin t}{5} & \sin t \\ \dfrac{2\cos t + \sin t}{5} & -\cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 - 3\cos 2t + \sin 2t \\ -1 - \cos 2t - 3\sin 2t \end{pmatrix},$$

after multiplying and using appropriate trigonometric identities. Integration and multiplication by $\mathbf{\Psi}$ yields the desired solution.

4. In this problem we use the method illustrated in Example 1. From Problem 4 of Section 7.5 we have the transformation matrix $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. Inverting $\mathbf{T}$ we find that $\mathbf{T}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$. If we let $\mathbf{x} = \mathbf{T}\mathbf{y}$ and substitute into the D.E., we obtain

$$\mathbf{y}' = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{y} + \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} + \frac{1}{5} \begin{pmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{pmatrix}.$$ This corresponds to the two scalar equations

$$y_1' + 3y_1 = (1/5)e^{-2t} + (2/5)e^{t},$$
$$y_2' - 2y_2 = (4/5)e^{-2t} - (2/5)e^{t},$$

which may be solved by the methods of Section 2.1. For the first equation the integrating factor is e^{3t} and we obtain $(e^{3t}y_1)' = (1/5)e^{t} + (2/5)e^{4t}$, so $e^{3t}y_1 = (1/5)e^{t} + (1/10)e^{4t} + c_1$. For the second equation the integrating factor is e^{-2t}, so $(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}$. Hence $e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2$. Thus

$$\mathbf{y} = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/10 \\ 2/5 \end{pmatrix} e^{t} + \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{pmatrix}.$$ Finally, multiplying by $\mathbf{T}$, we obtain

$$\mathbf{x} = \mathbf{T}\mathbf{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t} + c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, while the first two terms constitute a particular solution of the nonhomogeneous system.

12. Since the coefficient matrix is the same as that of Problem 3, use the same procedure as done in that problem, including the $\mathbf{\Psi}^{-1}$ found there. In the interval $\pi/2 < t < \pi$ sint > 0 and cost < 0; hence |sint| = sint, but |cost| = -cost.

14. To verify that the given vector is the general solution of the corresponding system, it is sufficient to substitute it into the D.E. Note also that the two terms in $\mathbf{x}^{(c)}$ are linearly independent. If we seek a solution of the form $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ then we find that the equation corresponding to Eq.(26) is $t\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, where $\mathbf{\Psi}(t) = \begin{pmatrix} t & 1/t \\ t & 3/t \end{pmatrix}$ and $\mathbf{g}(t) = \begin{pmatrix} 1-t^2 \\ 2t \end{pmatrix}$. Thus $\mathbf{u}' = (1/t)\mathbf{\Psi}^{-1}(t)\mathbf{g}(t)$. Using a computer algebra system or row operations on $\mathbf{\Psi}$ and $\mathbf{I}$, we find that $\mathbf{\Psi}^{-1} = \begin{pmatrix} 3/2t & -1/2t \\ -t/2 & t/2 \end{pmatrix}$ and hence $u_1' = \frac{3}{2t^2} - \frac{3}{2} - \frac{1}{t}$ and $u_2' = \frac{-1}{2} + \frac{t^2}{2} + t$, which yields $u_1 = \frac{-3}{2t} - \frac{3t}{2} - \ln t + c_1$ and $u_2 = -\frac{1}{2}t + \frac{t^3}{6} + \frac{t^2}{2} + c_2$. Multiplication of $\mathbf{u}$ by $\mathbf{\Psi}(t)$ yields the desired solution.

CHAPTER 8

Section 8.1, Page 427

In the following problems that ask for a large number of numerical calculations the first few steps are shown. It is then necessary to use these samples as a model to format a computer program or calculator to find the remaining values.

1a. The Euler formulas is $y_{n+1} = y_n + h(3 + t_n - y_n)$ for $n = 0,1,2,3...$ and with $t_0 = 0$ and $y_0 = 1$. Thus
$y_1 = 1 + .05(3 + 0 - 1) = 1.1$
$y_2 = 1.1 + .05(3 + .05 - 1.1) = 1.1975 \cong y(.1)$
$y_3 = 1.1975 + .05(3 + .1 - 1.1975) = 1.29263$
$y_4 = 1.29263 + .05(3 + .15 - 1.29263) = 1.38549 \cong y(.2)$.

1c. The backward Euler formula is $y_{n+1} = y_n + h(3 + t_{n+1} - y_{n+1})$. Solving this for y_{n+1} we find $y_{n+1} = [y_n + h(3 + t_{n+1})]/(1+n)$.
Thus $y_1 = \dfrac{1 + .05(3.05)}{1.05} = 1.097619$ and
$y_2 = \dfrac{1.097619 + .05(3.1)}{1.05} = 1.192971$.

5a. $y_1 = y_0 + h\dfrac{y_0^2 + 2t_0y_0}{3 + t_0^2} = .5 + .05\dfrac{(.5)^2 + 0}{3 + 0} = .504167$

$y_2 = .504167 + .05\dfrac{(.504167)^2 + 2(.05)(.504167)}{3 + (.05)^2} = .509239$

5c. $y_1 = .5 + .05\dfrac{y_1^2 + 2(.05)y_1}{3 + (.05)^2}$, which is a quadratic equation in y_1. Using the quadratic formula, or an equation solver, we obtain $y_1 = .5050895$. Thus
$y_2 = .5050895 + .05\dfrac{y_2^2 + 2(.1)y_2}{3 + (.1)^2}$ which is again quadratic in y_2, yielding $y_2 = .5111273$.

7a. For part a eighty steps must be taken, that is, $n = 0,1,...79$ and for part b 160 steps must taken with $n = 0,1,...159$. Thus use of a programmable calculator or a computer is required.

7c. We have $y_{n+1} = y_n + h(.5-t_{n+1} + 2y_{n+1})$, which is linear in y_{n+1} and thus we have $y_{n+1} = \dfrac{y_n + .5h - ht_{n+1}}{1 - 2h}$. Again, 80

steps are needed here and 160 steps in part d. In This case a spreadsheet is very useful. The first few, the middle three and last two lines are shown for h = .025:

n	y_n	t_n	y_{n+1}
0	1	0	1.06513
1	1.06513	.025	1.13303
2	1.13303	.050	1.20381
⋮			
38	7.49768	.950	7.87980
39	7.87980	.975	8.28137
40	8.28137	1.000	8.70341
⋮			
78	55.62105	1.950	58.50966
79	58.50966	1.975	61.54964
80	61.54964	2.000	

At least eight decimal places were used in all calculations.

9c. The backward Euler formula gives $y_{n+1} = y_n + h\sqrt{t_{n+1} + y_{n+1}}$. Subtracting y_n from both sides, squaring both sides, and solving for y_{n+1} yields $y_{n+1} = y_n + \frac{h^2}{2} + h\sqrt{y_n + t_{n+1} + h^2/4}$. Alternately, an equation solver can be used to solve $y_{n+1} = y_n + h\sqrt{t_{n+1} + y_{n+1}}$ for y_{n+1}. The first few values, for h = 0.25, are $y_1 = 3.043795$, $y_2 = 2.088082$, $y_3 = 3.132858$ and $y_4 = 3.178122 \cong y(.1)$.

15. If $y' = 1 - t + 4y$ then $y'' = -1 + 4y' = -1 + 4(1-t+4y) = 3 - 4t + 16y$. In Eq.(12) we let y_n, y_n' and y_n'' denote the approximate values of $\phi(t_n)$, $\phi'(t_n)$, and $\phi''(t_n)$, respectively. Keeping the first three terms in the Taylor series we have

$y_{n+1} = y_n + y_n'h + y_n'' h^2/2$

$= y_n + (1 - t_n + 4y_n)h + (3 - 4t_n + 16y_n)h^2/2$. For n = 0, $t_0 = 0$ and $y_0 = 1$ we have

$$y_1 = 1 + (1 - 0 + 4)(.1) + (3 - 0 + 16)\frac{(.1)^2}{2} = 1.595.$$

16. If $y = \phi(t)$ is the exact solution of the I.V.P., then $\phi'(t) = 2\phi(t) - 1$ and $\phi''(t) = 2\phi'(t) = 4\phi(t) - 2$. From Eq.(21), $e_{n+1} = [2\phi(\bar{t}_n) - 1]h^2$, $t_n < \bar{t}_n < t_n + h$. Thus a

bound for e_{n+1} is $|e_{n+1}| \le [1 + 2\max_{0\le t\le 1}|\phi(t)|]h^2$. Since the exact solution is $y = \phi(t) = [1 + \exp(2t)]/2$, $e_{n+1} = h^2\exp(2\bar{t}_n)$. Therefore $|e_1| \le (0.1)^2\exp(0.2) = 0.012$ and $|e_4| \le (0.1)^2\exp(0.8) = 0.022$, since the maximum value of $\exp(2\bar{t}_n)$ occurs at $t = .1$ and $t = .4$ respectively. From Problem 2 of Section 2.7, the actual error in the first step is .0107.

19. The local truncation error is $e_{n+1} = \phi''(\bar{t}_n)h^2/2$. For this problem $\phi'(t) = 5t - 3\phi^{1/2}(t)$ and thus $\phi''(t) = 5 - (3/2)\phi^{-1/2}\phi' = 19/2 - (15/2)t\phi^{-1/2}$. Substituting this last expression into e_{n+1} yields the desired answer.

22d. Since $y'' = -5\pi\sin 5\pi t$, Eq.(21) gives $e_{n+1} = -(5\pi/2)\sin(5\pi\bar{t}_n)h^2$. Thus $|e_{n+1}| < \frac{5\pi}{2}h^2 < .05$, or $h < \frac{1}{\sqrt{50\pi}} \cong .08$.

23a. From Eq.(14) we have $E_n = \phi(t_n) - y_n$. Using this in Eq.(20) we obtain

$E_{n+1} = E_n + h\{f[t_n,\phi(t_n)] - f(t_n,y_n)\} + \phi''(\bar{t}_n)h^2/2$. Using the given inequality involving L we have $|f[t_n,\phi(t_n)] - f(t_n,y_n)| \le L\,|\phi(t_n) - y_n| = L|E_n|$ and thus $|E_{n+1}| \le |E_n| + hL|E_n| + \max_{t_0\le t\le t_n}|\phi''(t)|h^2/2 = \alpha|E_n| + \beta h^2$.

23b. Since $\alpha = 1 + hL$, $\alpha - 1 = hL$. Hence $\beta h^2(\alpha^n-1)/(\alpha-1) = \beta h^2[(1+hL)^n - 1]/hL = \beta h[(1+hL)^n - 1]/L$.

23c. $(1+hL)^n \le \exp(nhL)$ follows from the observation that $\exp(nhL) = [\exp(nL)]^n = (1 + hL + h^2L^2/2! + \ldots)^n$. Noting that $nh = t_n - t_0$, the rest follows from Eq.(ii).

24. The Taylor series for $\phi(t)$ about $t = t_{n+1}$ is

$$\phi(t) = \phi(t_{n+1}) + \phi'(t_{n+1})(t-t_{n+1}) + \phi''(t_{n+1})\frac{(t-t_{n+1})^2}{2} + \ldots.$$

Letting $\phi'(t) = f(t,\phi(t))$, $t = t_n$ and $h = t_{n+1} - t_n$ we have $\phi(t_n) = \phi(t_{n+1}) - f(t_{n+1},\phi(t_{n+1}))h + \phi''(\bar{t}_n)h^2/2$, where $t_n < \bar{t}_n < t_{n+1}$. Thus $\phi(t_{n+1}) = \phi(t_n) + f(t_{n+1},\phi(t_{n+1}))h - \phi''(\bar{t}_n)h^2/2$. Comparing this to Eq. 13 we then have $e_{n+1} = -\phi''(\bar{t}_n)h^2/2$.

25b. From Problem 1 we have $y_{n+1} = y_n + h(3 + t_n - y_n)$, so
$y_1 = 1 + .05(3 + 0 - 1) = 1.1$
$y_2 = 1.1 + .05(3 + .05 - 1.1) = 1.20 \cong y(.1)$
$y_3 = 1.20 + .05(3 + .1 - 1.20) = 1.30$
$y_4 = 1.30 + .05(3 + .15 - 1.30) = 1.39 \cong y(.2)$.

Section 8.2, Page 434

1a. The improved Euler formula is
$y_{n+1} = y_n + [y_n' + f(t_n + h, y_n + hy_n')]h/2$ where
$y' = f(t,y) = 3 + t - y$. Hence $y_n' = 3 + t_n - y_n$ and
$f(t_n + h, y_n + hy_n') = 3 + t_{n+1} - (y_n + hy_n')$. Thus we obtain

$$y_{n+1} = y_n + (3 + t_n - y_n)h + \frac{h^2}{2}(1 - y_n')$$

$$= y_n + (3 + t_n - y_n)h + \frac{h^2}{2}(-2 - t_n + y_n).$$ Thus

$$y_1 = 1 + (3-1)(.05) + \frac{(.05)^2}{2}(-2+1) = 1.098750 \text{ and}$$

$$y_2 = y_1 + (3 +.05 - y_1)(.05) + \frac{(.05)^2}{2}(-2 -.05 + y_1) = 1.19512$$

are the first two steps. In this case, the equation specifying y_{n+1} is somewhat more complicated when $y_n' = 3 + t_n - y_n$ was substituted. When designing the steps to calculate y_{n+1} on a computer, y_n' can be calculated first and thus the simpler formula for y_{n+1} can be used. The exact solution is $y(t) = 2 + t - e^{-t}$, so $y(.1) = 1.19516$, $y(.2) = 1.38127$, $y(.3) = 1.55918$ and $y(.04) = 1.72968$, so the approximations using $h = .0125$ are quite accurate to five decimal places.

4. In this case $y_n' = 2t_n + e^{-t_n}y_n$ and thus the improved Euler formula is

$$y_{n+1} = y_n + \frac{[(2t_n + e^{-t_n}y_n) + 2t_{n+1} + e^{-t_{n+1}(y_n + hy_n')}]h}{2}.$$ For

$n = 0, 1, 2$ we get $y_1 = 1.05122$, $y_2 = 1.10483$ and $y_3 = 1.16072$ for $h = .05$.

10. See Problem 4.

11. The improved Euler formula is

$$y_{n+1} = y_n + \frac{f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))}{2} h.$$ As suggested in the text, it's best to perform the following steps when

implementing this formula: let $k_1 = (4 - t_n y_n)/(1 + y_n^2)$, $k_2 = y_n + hk_1$ and $k_3 = (4 - t_{n+1}k_2)/(1 + k_2^2)$. Then $y_{n+1} = y_n + (k_1 + k_3)h/2$.

14a. Since $\phi(t_n + h) = \phi(t_{n+1})$ we have, using the first part of Eq.(5) and the given equation,
$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = [\phi(t_n) - y_n] + [\phi'(t_n) - \frac{y_n' + f(t_n+h,\ y_n+hy_n')}{2}]h + \phi''(t_n)h^2/2! + \phi'''(\bar{t}_n)h^3/3!$.
Since $y_n = \phi(t_n)$ and $y_n' = \phi'(t_n) = f(t_n,y_n)$ this reduces to
$e_{n+1} = \phi''(t_n)h^2/2! - \{f[t_n+h,\ y_n + hf(t_n,y_n)] - f(t_n,y_n)\}h/2! + \phi'''(\bar{t}_n)h^3/3!$,
which can be written in the form of Eq.(i).

14b. First observe that $y' = f(t,y)$ and $y'' = f_t(t,y) + f_y(t,y)y'$. Hence $\phi''(t_n) = f_t(t_n,y_n) + f_y(t_n,y_n)f(t_n,y_n)$. Using the given Taylor series, with $a = t_n$, $h = h$, $b = y_n$ and $k = hf(t_n,y_n)$ we have

$$f[t_n+h,y_n+hf(t_n,y_n)] = f(t_n,y_n)+f_t(t_n,y_n)h+f_y(t_n,y_n)hf(t_n,y_n) + [f_{tt}(\xi,\eta)h^2+2f_{ty}(\xi,\eta)h^2f(t_n,y_n)+f_{yy}(\xi,\eta)h^2f^2(t_n,y_n)]/2!$$

where $t_n < \xi < t_n + h$ and $|\eta - y_n| < h|f(t_n,y_n)|$. Substituting this in Eq.(i) and using the earlier expression for $\phi''(t_n)$ we find that the first term on the right side of Eq.(i) reduces to
$-[f_{tt}(\xi,\eta) + 2f_{ty}(\xi,\eta)f(t_n,y_n) + f_{yy}(\xi,\eta)f^2(t_n,y_n)]h^3/4$,
which is proportional to h^3 plus, possibly, higher order terms. The reason that there may be higher order terms is because ξ and η will, in general, depend upon h.

14c. If f(t,y) is linear in t and y, then $f_{tt} = f_{ty} = f_{yy} = 0$ and the terms appearing in the last formula of part (b) are all zero.

15. Since $\phi(t) = [4t - 3 + 19\exp(4t)]/16$ we have $\phi'''(t) = 76\exp(4t)$ and thus from Problem 14c, since f is linear in t and y, we find
$e_{n+1} = 38[\exp(4\bar{t}_n)]h^3/3$. Thus
$|e_{n+1}| \le (38h^3/3)\exp(8) = 37{,}758.8h^3$ on $0 \le t \le 2$. For n = 1, we have $|e_1| = |\phi(t_1) - y_1| \le (38/3)\exp(0.2)(.05)^3 = .001934$, which is approximately 1/15 of the error indicated in Eq.(27) of the previous section.

19. The Euler method gives
$y_1 = y_0 + h(5t_0 - 3\sqrt{y_0}) = 2 + .1(-3\sqrt{2}) = 1.57574$ and the improved Euler method gives

$$y_1 = y_0 + \frac{f(t_0,y_0) + f(t_1,y_1)}{2}\, h$$
$$= 2 + [-3\sqrt{2} + (.5 - 3\sqrt{1.57574})].05 = 1.62458.$$

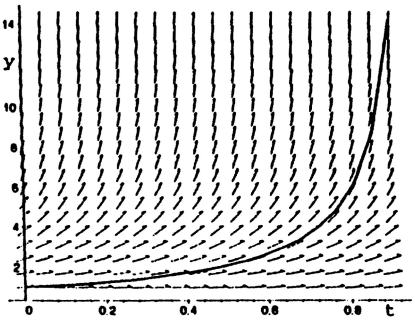
Thus, the estimated error in using the Euler method is 1.62458 - 1.57574 = .04884. Since we want our error tolerance to be no greater than .0025 we need to adjust the step size downward by a factor of $\sqrt{.0025/.04884} \cong .226$. Thus a step size of h = (.1)(.23) = .023 would be needed for the required local truncation error bound of .0025.

24. The modified Euler formula is
$y_{n+1} = y_n + hf[t_n + h/2,\ y_n + (h/2)f(t_n,y_n)]$ where
$f(t,y) = 5t - 3\sqrt{y}$. Thus
$y_1 = 2 + .05[5(t_0 + .025) - 3\text{sqrt}(2 + .025(5t_0 - 3\sqrt{2}))]$
$= 1.79982$ for $t_0 = 0$. The values obtained here are between the values for h = .05 and for h = .025 using the Euler method in Problem 2.

Section 8.3, Page 438

4. The Runge-Kutta formula is
$y_{n+1} = y_n + h(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})/6$ where k_{n1}, k_{n2} etc. are given by Eqs.(3). Thus for
$f(t,y) = 2t + e^{-ty}$, $(t_0,y_0) = (0,1)$ and h = .1 we have
$k_{01} = 0 + e^0 = 1$
$k_{02} = 2(0 + .05) + e^{-(0+.05)(1+.05k_{01})} = 1.048854$
$k_{03} = 2(.05) + e^{-(.05)(1+.05k_{02})} = 1.048738$
$k_{04} = 2(.1) + e^{-(.1)(1+.1k_{03})} = 1.095398$ and hence
$y(.1) \cong y_1 = 1 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = 1.104843.$

11. We have $f(t_n,y_n) = (4 - t_ny_n)/(1 + y_n^2)$. Thus for $t_0 = 0$, $y_0 = -2$ and h = .1 we have
$k_{01} = f(0,-2) = .8$
$k_{02} = f(.05, -2 + .05(.8)) = f(.05, -1.96) = .846414,$
$k_{03} = f(.05, -2 + .05k_{02}) = f(.05, -1.957679) = .847983,$
$k_{04} = f(.1, -2 + .1k_{03}) = f(.1, -1.915202) = .897927,$ and
$y_1 = -2 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = -1.915221.$ For comparison, see Problem 11 in Sections 8.1 and 8.2.

14a. 

14b. We have $f(t_n,y_n) = t_n^2 + y_n^2$, $t_0 = 0$, $y_0 = 1$ and $h = .1$ so
$k_{01} = 0^2 + 1^2 = 1$
$k_{02} = (.05)^2 + (1 + .05)^2 = 1.105$
$k_{03} = (.05)^2 + [1 + .05(1.105)]^2 = 1.11605$
$k_{04} = (.1)^2 + [1 + .1(1.11605)]^2 = 1.245666$ and thus
$y_1 = 1 + .1(k_{01} + 2k_{02} + 2k_{03} + k_{04})/6 = 1.11146$. Using these steps in a computer program, we obtain the following values for y:

t	h =.1	h = .05	h = .025	h = 0.125
.8	5.842	5.8481	5.8483	5.8486
.9	14.0218	14.2712	14.3021	14.3046
.95		46.578	49.757	50.3935

14c. No accurate solution can be obtained for y(1), as the values at t = .975 for h = .025 and h = .0125 are 1218 and 23,279 respectively. These are caused by the slope field becoming vertical as $t \to 1$.

Section 8.4, Page 444

4a. The predictor formula is
$y_{n+1} = y_n + h(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})/24$
and the corrector formula is
$y_{n+1} = y_n + h(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})/24$, where
$f_n = 2t_n + \exp(-t_n y_n)$. Using the Runge-Kutta method, from Section 8.3, Problem 4a, we have for $t_0 = 0$ and $y_0 = 1$, $y_1 = 1.1048431$, $y_2 = 1.2188411$ and $y_3 = 1.3414680$. Thus the predictor formula gives $y_4 = 1.4725974$, so $f_4 = 1.3548603$ and the corrector formula then gives $y_4 = 1.4726173$, which is the desired value. These results, and the next step, are summarized in the following table:

n	y_n	f_n	y_{n+1}	f_{n+1}	y_{n+1} Corrected
0	1	1			
1	1.1048431	1.0954004			
2	1.2188411	1.1836692			
3	1.3414680	1.2686862	1.4725974	1.3548603	1.4726173
4	1.4726173	1.3548559	1.6126246	1.4465016	1.6126215
5	1.6126215				

where f_n is given above, y_{n+1} is given by the predictor formula, and the corrected y_{n+1} is given by the corrector formula. Note that the value for f_4 on the line for $n = 4$ uses the corrected value for y_4, and differs slightly from the f_4 on the line for $n = 3$, which uses the predicted value for y_4.

4b. The fourth order Adams-Moulton method is given by Eq. (10): $y_{n+1} = y_n + (h/24)(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$. Substituting $h = .1$ we obtain $y_{n+1} = y_n + (.0375)(19f_n - 5f_{n-1} + f_{n-2}) + .0375f_{n+1}$. For $n = 2$ we then have

$y_3 = y_2 + (.0375)(19f_2 - 5f_1 + f_0) + .0375f_3$
$= 1.293894103 + .0375(.6 + e^{-.3y_3})$, using values for y_2, f_0, f_1, f_2 from part a. An equation solver then yields $y_3 = 1.341469821$. Likewise

$y_4 = y_3 + (.0375)(19f_3 - 5f_2 + f_1) + .0375f_4$
$= 1.421811841 + .0375(.8 + e^{-.4y_4})$, where f_3 is calculated using the y_3 found above. This last equation yields $y_4 = 1.472618922$. Finally

$y_5 = y_4 + (.0375)(19f_4 - 5f_3 + f_2) + .0375f_5$
$= 1.558379316 + .0375(1.0 + e^{-.5y_5})$, which gives $y_5 = 1.612623138$.

4c. We use Eq. (16): $y_{n+1} = (1/25)(48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1})$. Thus $y_4 = .04(48y_3 - 36y_2 + 16y_1 - 3y_0) + .048f_4$
$= 1.40758686 + .048(.8 + e^{-.4y_4})$, using values for y_0, y_1. y_2, y_3 from part a. An equation solver then yields $y_4 = 1.472619913$. Likewise
$y_5 = .04(48y_4 - 36y_3 + 16y_2 - 3y_1) + .048f_5$
$= 1.54319349 + .048(1 + e^{-.5y_5})$, which gives $y_5 = 1.612625556$.

7a. Using the predictor and corrector formulas (Eqs.6 and 10) with $f_n = .5 - t_n + 2y_n$ and using the Runge-Kutta method to calculate y_1,y_2 and y_3, we obtain the following table for $h = .05$, $t_0 = 0$, $y_0 = 1$:

n	y_n	f_n	y_{n+1}	f_{n+1}	y_{n+1} corrected
0	1	2.5			
1	1.130171	2.710342			
2	1.271403	2.9420805			
3	1.424858	3.199717	1.591820	3.483640	1.591825
4	1.591825	3.483649	1.773716	3.797433	1.773721
5	1.773721	3.797443	1.972114	4.144227	1.972119
6	1.972119	4.144238	2.188747	4.527495	2.188753
7	2.188753	4.527507	2.425535	4.951070	2.425542
8	2.425542	4.951084	2.684597	5.419194	2.684604
9	2.684604	5.419209	2.968276	5.936551	2.968284
10	2.968284				

7b. From Eq.(10) we have

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$

$$= y_n + \frac{h}{24}[9(.5 - t_{n+1} + 2y_{n+1}) + 19f_n - 5f_{n-1} + f_{n-2}].$$

Solving for y_{n+1} we obtain

$$y_{n+1} = [y_n + \frac{h}{24}(19f_n - 5f_{n-1} + f_{n-2} + 4.5 - 9t_{n+1})]/(1-.75h).$$

For h = .05, $t_0 = 0$, $y_0 = 1$ and using y_1 and y_2 as calculated using the Runge-Kutta formula, we obtain the following table:

n	y_n	f_n	y_{n+1}
0	1	2.5	
1	1.130171	2.710342	
2	1.271403	2.942805	1.424859
3	1.424859	3.199718	1.591825
4	1.591825	3.483650	1.773722
5	1.773722	3.797444	1.972120
6	1.972120	4.144241	2.188755
7	2.188755	4.527510	2.425544
8	2.425544	4.951088	2.684607
9	2.684607	5.419214	2.968287
10	2.968287		

7c. From Eq (16) we have

$$y_{n+1} = (48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1})/25$$

$$= [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h(.5-t_{n+1})]/25+(24/25)hy_{n+1}.$$

Solving for y_{n+1} we have

$$y_{n+1} = [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h(.5-t_{n+1})]/(25-24h).$$

Again, using Runge-Kutta to find y_1 and y_2, we then obtain the following table:

n	y_n	y_{n+1}
0	1	
1	1.130170833	
2	1.271402571	
3	1.424858497	1.591825573
4	1.591825573	1.773724801
5	1.773724801	1.972125968
6	1.972125968	2.188764173
7	2.188764173	2.425557376
8	2.425557376	2.684625416
9	2.684625416	2.968311063
10	2.968311063	

The exact solution is $y(t) = e^t + t/2$ so $y(.5) = 2.9682818$ and $y(2) = 55.59815$, so we see that the predictor-corrector method in part a is accurate through three decimal places.

16. Let $P_2(t) = At^2 + Bt + C$. As in Eqs. (12) and (13) let $P_2(t_{n-1}) = y_{n-1}$, $P_2(t_n) = y_n$, $P_2(t_{n+1}) = y_{n+1}$ and $P_2'(t_{n+1}) = f(t_{n+1},y_{n+1}) = f_{n+1}$. Recall that $t_{n-1} = t_n - h$ and $t_{n+1} = t_n + h$ and thus we have the four equations:

$$A(t_n-h)^2 + B(t_n-h) + C = y_{n-1} \quad \text{(i)}$$
$$At_n^2 + Bt_n + C = y_n \quad \text{(ii)}$$
$$A(t_n+h)^2 + B(t_n+h) + C = y_{n+1} \quad \text{(iii)}$$
$$2A(t_n+h) + B = f_{n+1} \quad \text{(iv)}$$

Subtracting Eq. (i) from Eq. (ii) to get Eq. (v) (not shown) and subtracting Eq. (ii) from Eq. (iii) to get Eq. (vi) (not shown), then subtracting Eq. (v) from Eq. (vi) yields $y_{n+1} - 2y_n + y_{n-1} = 2Ah^2$, which can be solved for A. Thus $B = f_{n+1} - 2A(t_n+h)$ [from Eq. (iv)] and $C = y_n - t_nf_{n+1} + At_n^2 + 2At_nh$ [from Eq. (ii)]. Using these values for A, B and C in Eq. (iv) yields $y_{n+1} = (1/3)(4y_n-y_{n-1}+2hf_{n+1})$, which is Eq. (15).

Section 8.5, Page 454

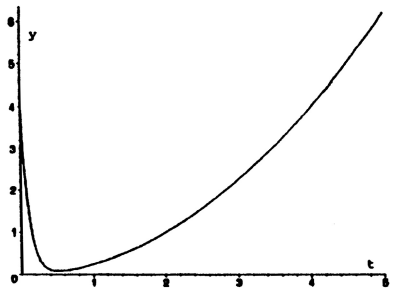
2a. If $0 \le t \le 1$ then we know $0 \le t^2 \le 1$ and hence $e^y \le t^2 + e^y \le 1 + e^y$. Since each of these terms represents a slope, we may conclude that the solution of Eq.(i) is bounded above by the solution of Eq.(iii) and is bounded below by the solution of Eq.(iv).

2b. $\phi_1(t)$ and $\phi_2(t)$ can each be found by separation of

variables. For $\phi_1(t)$ we have $\frac{1}{1+e^y}dy = dt$, or $\frac{e^{-y}}{e^{-y}+1}dy = dt$. Integrating both sides yields $-\ln(e^{-y}+1) = t + c$. Solving for y we find $y = \ln[1/(c_1e^{-t}-1)]$. Setting $t = 0$ and $y = 0$, we obtain $c_1 = 2$ and thus $\phi_1(t) = \ln[e^t/(2-e^t)]$. As $t \to \ln 2$, we see that $\phi_1(t) \to \infty$. A similar analysis shows that $\phi_2(t) = \ln[1/(c_2-t)]$, where $c_2 = 1$ when the I.C. are used. Thus $\phi_2(t) \to \infty$ as $t \to 1$ and thus we conclude that $\phi(t) \to \infty$ for some t such that $\ln 2 \le t \le 1$.

2c. From Part b: $\phi_1(.9) = \ln[1/(c_1e^{-.9}-1)] = 3.4298$ yields $c_1 = 2.5393$ and thus $\phi_1(t) \to \infty$ when $t \cong .9319$. Similarly for $\phi_2(t)$ we have $c_2 = .9324$ and thus $\phi_2(t) \to \infty$ when $t \cong .932$.

4a. The D.E. is $y' + 10y = 2.5t^2 + .5t$. So $y_h = ce^{-10t}$ is the solution of the related homogeneous equation and the particular solution, using undetermined coefficients, is $y_p = At^2 + Bt + C$. Substituting this into the D.E. yields $A = 1/4$, $B = C = 0$. To satisfy the I.C., $c = 4$, so $y(t) = 4e^{-10t} + (t^2/4)$, which is shown in the graph.



4b. From the discussion following Eq (15), we see that h must be less than $\frac{2}{|r|}$ for the Euler method to be stable. Thus, for $r = 10$, $h < .2$. For $h = .2$ we obtain the following values:

t =	4	4.2	4.4	4.6	4.8	5.0
y =	8	.4	8.84	1.28	9.76	2.24

and for $h = .18$ we obtain:

t =	4.14	4.32	4.50	4.68	4.86	5.04
y =	4.26	4.68	5.04	5.48	5.89	6.35.

Clearly the second set of values is stable, although far from accurate.

4c. For a step size of .25 we find

t =	4	4.25	4.75	5.00
y =	4.018	4.533	5.656	6.205,

for a step size of .28 we find

t =	4.2	4.48	4.76	5.00
y =	10.14	10.89	11.68	12.51,

and for a step size of .3 we find

t =	4.2	4.5	4.8	5.1
y =	353	484	664	912.

Thus instability appears to occur about h = .28 and certainly by h = .3. Note that the exact solution for t = 5 is y = 6.2500, so for h = .25 we do obtain a good approximation.

4d. For h = .5 the error at t = 5 is .013, while for h = .385, the error at t = 5.005 is .01.

5a. The general solution of the D.E. is $y(t) = t + ce^{\lambda t}$, where $y(0) = 0 \rightarrow c = 0$ and thus $y(t) = t$, which is independent of λ.

5c. Your result in Part b will depend upon the particular computer system and software that you use. If there is sufficient accuracy, you will obtain the solution y = t for t on $0 \leq t \leq 1$ for each value of λ that is given, since there is no discretization error. If there is not sufficient accuracy, then round-off error will affect your calculations. For the larger values of λ, the numerical solution will quickly diverge from the exact solution, y = t, to the general solution $y = t + ce^{\lambda t}$, where the value of c depends upon the round-off error. If the latter case does not occur, you may simulate it by computing the numerical solution to the I.V.P. $y' - \lambda y = 1 - \lambda t$, $y(.1) = .10000001$. Here we have assumed that the numerical solution is exact up to the point t = .09 [i.e. y(.09) = .09] and that at t = .1 round-off error has occurred as indicated by the slight error in the I.C. It has also been found that a larger step size (h = .05 or h = .1) may also lead to round-off error.

Section 8.6, Page 457

2a. The Euler formula is $\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n$, where $\mathbf{f}_n = \begin{pmatrix} 2x_n+t_ny_n \\ x_ny_n \end{pmatrix}$, $x_0 = 1$ and $y_0 = 1$. Thus $f_0 = \begin{pmatrix} 2-0 \\ (1)(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{x}_1 = \begin{pmatrix} 1+.1(2) \\ 1+.1(1) \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.1 \end{pmatrix}$,

$f_1 = \begin{pmatrix} 2.4+.1(1.1) \\ (1.2)(1.1) \end{pmatrix} = \begin{pmatrix} 2.51 \\ 1.32 \end{pmatrix}$

and $\mathbf{x}_2 = \begin{pmatrix} 1.2+.1(2.51) \\ 1.1+.1(1.32) \end{pmatrix} = \begin{pmatrix} 1.451 \\ 1.232 \end{pmatrix} \cong \begin{pmatrix} \phi(.2) \\ \psi(.2) \end{pmatrix}$

2b. Eqs. (7) give:

$\mathbf{k}_{01} = \begin{pmatrix} f(0,1,1) \\ g(0,1,1) \end{pmatrix} = \begin{pmatrix} 2+0 \\ (1)(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\mathbf{k}_{02} = \begin{pmatrix} 2.4+.1(1.1) \\ (1.2)(1.1) \end{pmatrix} = \begin{pmatrix} 2.51 \\ 1.32 \end{pmatrix}$

$\mathbf{k}_{03} = \begin{pmatrix} 2.502+.1(1.132) \\ (1.251)(1.132) \end{pmatrix} = \begin{pmatrix} 2.6152 \\ 1.41613 \end{pmatrix}$

$\mathbf{k}_{04} = \begin{pmatrix} 3.04608+.2(1.28323) \\ (1.52304)(1.28323) \end{pmatrix} = \begin{pmatrix} 3.30273 \\ 1.95441 \end{pmatrix}$

Using Eq. (6) in scalar form, we then have
$x_1 = 1+(.2/6)[2+2(2.51)+2(2.6152)+3.30273] = 1.51844$
$y_1 = 1+(.2/6)[1+2(1.32)+2(1.41613)+1.95440] = 1.28089.$

7. Write a computer program to do this problem as there are twenty steps or more for $h \le .05$.

8. If we let $y = x'$, then $y' = x''$ and thus we obtain the system $x' = y$ and $y' = t-3x-t^2y$, with $x(0) = 1$ and $y(0) = x'(0) = 2$. Thus $f(t,x,y) = y$, $g(t,x,y) = t - 3x - t^2y$, $t_0 = 0$, $x_0 = 1$ and $y_0 = 2$. If a program has been written for an earlier problem, then its best to use that. Otherwise, the first two steps are as follows:

$\mathbf{k}_{01} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$$\mathbf{k}_{02} = \begin{pmatrix} 2+(-.15) \\ .05-3(1.1)-(.05)^2(1.85) \end{pmatrix} = \begin{pmatrix} 1.85 \\ -3.25463 \end{pmatrix}$$

$$\mathbf{k}_{03} = \begin{pmatrix} 2+(-.16273) \\ .05-3(1.0925)-(.05)^2(1.83727) \end{pmatrix} = \begin{pmatrix} 1.83727 \\ -3.23209 \end{pmatrix}$$

$$\mathbf{k}_{04} = \begin{pmatrix} 2+(-.32321) \\ .1-3(1.18373)-(.1)^2(1.67679) \end{pmatrix} = \begin{pmatrix} 1.67679 \\ -3.46796 \end{pmatrix}$$

and thus

$x_1 = 1+(.1/6)[2 + 2(1.85)+2(1.83727)+(1.67679)]=1.18419,$
$y_1 = 2+(.1/6)[-3-2(3.25463)-2(3.23209)-3.46796]=1.67598,$

which are approximations to $x(.1)$ and $y(.1) = x'(.1)$.
In a similar fashion we find

$$\mathbf{k}_{11} = \begin{pmatrix} 1.67598 \\ -3.46933 \end{pmatrix} \qquad \mathbf{k}_{12} = \begin{pmatrix} 1.50251 \\ -3.68777 \end{pmatrix}$$

$$\mathbf{k}_{13} = \begin{pmatrix} 1.49159 \\ -3.66151 \end{pmatrix} \qquad \mathbf{k}_{14} = \begin{pmatrix} 1.30983 \\ -3.85244 \end{pmatrix}$$

and thus

$x_2=x_1+(.1/6)[1.67598+2(1.50251)+2(1.49159)+1.30983]=1.33376$
$y_2=y_1-(.1/6)[3.46933+2(3.68777)+2(3.66151)+3.85244]=1.30897.$

Three more steps must be taken in order to approximate $x(.5)$ and $y(.5) = x'(.5)$. The intermediate steps yield $x(.3) \cong 1.44489$, $y(.3) \cong .9093062$ and $x(.4) \cong 1.51499$, $y(.4) \cong .4908795$.

CHAPTER 9

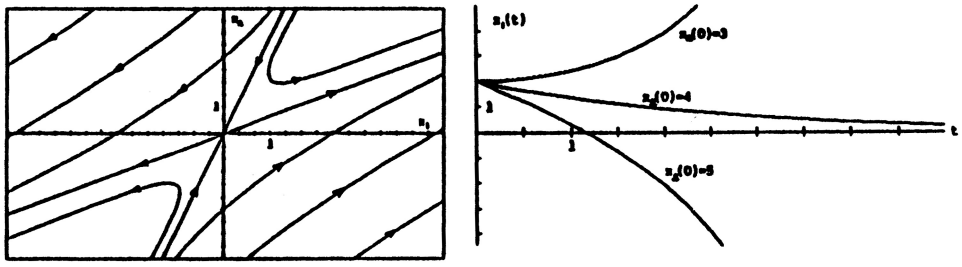
Section 9.1, Page 468

For Problems 1 through 16, once the eigenvalues have been found, Table 9.1.1 will, for the most part, quickly yield the type of critical point and the stability. In all cases it can be easily verified that **A** is nonsingular.

1a. The eigenvalues are found from the equation $\det(\mathbf{A}-r\mathbf{I})=0$. Substituting the values for **A** we have $\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} =$ $r^2 - r - 2 = 0$ and thus the eigenvalues are $r_1 = -1$ and $r_2 = 2$. For $r_1 = -1$, we have $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and for r_2 we have $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

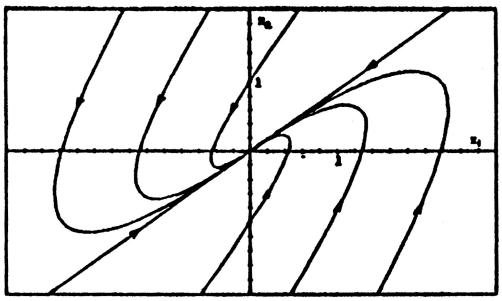
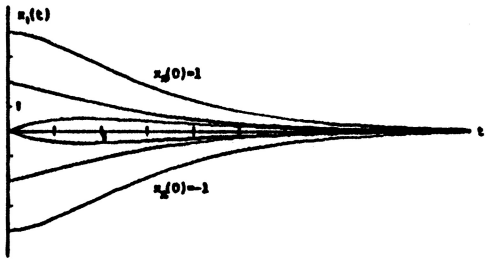
1b. Since the eigenvalues differ in sign, the critical point is a saddle point and is unstable.

1d.



4a. Again the eigenvalues are given by $\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} =$ $r^2 + 6r + 9 = 0$ and thus $r_1 = r_2 = -3$. The eigenvectors are solutions of $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and hence there is just one eigenvector $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4b. Since the eigenvalues are negative, (0,0) is an improper node which is asymptotically stable. If we had found that there were two independent eigenvectors then (0,0) would have been a proper node, as indicated in Case 3a.

4d.  

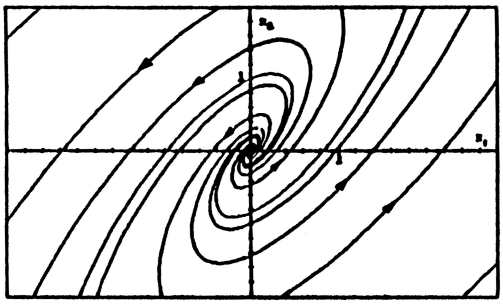
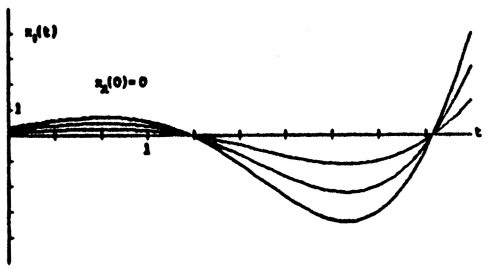
7a. In this case $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 5$ and thus the eigenvalues are $r_{1,2} = 1 \pm 2i$. For $r_1 = 1 + 2i$ we have

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 8-8i & -8 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ and thus

$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Similarly for $r_2 = 1-2i$ we have

$$\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ and hence $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$.

7b. Since the eigenvalues are complex with positive real part, we conclude that the critical point is a spiral point and is unstable.

7d.  

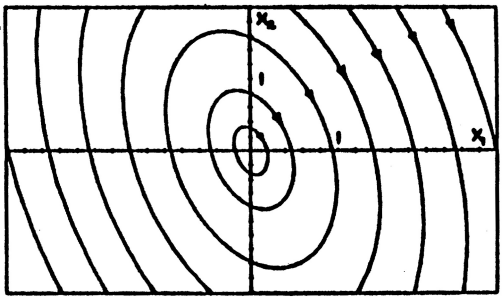
10a. Again, $\det(\mathbf{A}-r\mathbf{I}) = r^2 + 9$ and thus we have $r_{1,2} = \pm 3i$.

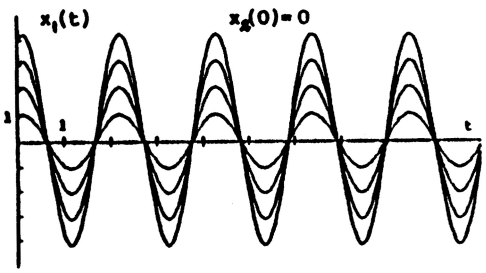
For $r_1 = 3i$ we have $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus

$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$. Likewise for $r_2 = -3i$,

$\begin{pmatrix} 1+3i & 2 \\ -5 & -1+3i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so that $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$.

10b. Since the eigenvalues are pure imaginary the critical point is a center, which is stable.

10d. 



13. If we let $\mathbf{x} = \mathbf{x^0} + \mathbf{u}$ then $\mathbf{x}' = \mathbf{u}'$ and thus the system becomes $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x^0} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ which will be in the form of Eq.(2) if $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x^0} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Using row operations, this last set of equations is equivalent to $\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x^0} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and thus $x_1^0 = 1$ and $x_2^0 = 1$. Since $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u}$ has (0,0) as the critical point, we conclude that (1,1) is the critical point of the original system. As in the earlier problems, the eigenvalues are given by $\begin{vmatrix} 1-r & 1 \\ 1 & -1-r \end{vmatrix} = r^2 - 2 = 0$ and thus $r_{1,2} = \pm\sqrt{2}$. Hence the critical point (1,1) is an unstable saddle point.

17. The equivalent system is dx/dt = y, dy/dt = -(k/m)x-(c/m)y which is written in the form of Eq.(2) as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The point (0,0) is clearly a critical point, and since **A** is nonsingular, it is the only one. The characteristic equation is $r^2+(c/m)r+k/m=0$ so $r_1, r_2 = [-c \pm (c^2 - 4km)^{1/2}]/2m$. In the underdamped case $c^2 - 4km < 0$, the characteristic roots are complex with negative real parts and thus the critical point (0,0) is an asymptotically stable spiral point. In the overdamped case $c^2 - 4km > 0$, the characteristic roots are real, unequal, and negative and hence the critical point (0,0) is asymptotically stable node. In the critically damped case $c^2 - 4km = 0$, the characteristic roots are equal and negative. As indicated in the solution to Problem 4, to determine whether this is an improper or proper node we must determine whether there

are one or two linearly independent eigenvectors. The eigenvectors satisfy the equation $\begin{pmatrix} c/2m & 1 \\ -k/m & -c/2m \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has just one solution if $c^2 - 4km = 0$. Thus the critical point (0,0) is an asymptotically stable improper node.

18a. If **A** has one zero eigenvalue then for $r = 0$ we have $\det(\mathbf{A}-r\mathbf{I}) = \det\mathbf{A} = 0$. Hence **A** is singular which means $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions and consequently there are infinitely many critical points.

18b. From Chapter 7, the solution is $\mathbf{x}(t) = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}e^{r_2t}$, which can be written in scalar form as $x_1 = c_1\xi_1^{(1)} + c_2\xi_1^{(2)}e^{r_2t}$ and $x_2 = c_1\xi_2^{(1)} + c_2\xi_2^{(2)}e^{r_2t}$. Assuming $\xi_1^{(2)} \neq 0$, the first equation can be solved for $c_2e^{r_2t}$, which is then substituted into the second equation to yield $x_2 = c_1\xi_2^{(1)} + [\xi_2^{(2)}/\xi_1^{(2)}][x_1-c_1\xi_1^{(1)}]$. These are straight lines parallel to the vector $\boldsymbol{\xi}^{(2)}$. Note that the family of lines is independent of c_2. If $\xi_1^{(2)} = 0$, then the lines are vertical. If $r_2 > 0$, the direction of motion will be in the same direction as indicated for $\boldsymbol{\xi}^{(2)}$. If $r_2 < 0$, then it will be in the opposite direction.

19a. $\det(A-rI) = r^2 - (a_{11}+a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$. If $a_{11} + a_{22} = 0$, then $r^2 = -(a_{11}a_{22} - a_{21}a_{12}) < 0$ if $a_{11}a_{22} - a_{21}a_{12} > 0$.

19b. Eq.(i) can be written in scalar form as $dx/dt = a_{11}x + a_{12}y$ and $dy/dt = a_{21}x + a_{22}y$, which then yields Eq.(iii). Ignoring the middle quotient in Eq.(iii), we can rewrite that equation as $(a_{21}x + a_{22}y)dx - (a_{11}x + a_{12}y)dy = 0$, which is exact since $a_{22} = -a_{11}$ from Eq.(ii)..

19c. Integrating $\phi_x = a_{21}x + a_{22}y$ we obtain $\phi = a_{21}x^2/2 + a_{22}xy + g(y)$ and thus $a_{22}x + g' = -a_{11}x - a_{12}y$ or $g' = -a_{12}y$ using Eq.(ii). Hence $a_{21}x^2/2 + a_{22}xy - a_{12}y^2/2 = k/2$ is the solution to Eq.(iii). The quadratic equation $Ax^2 + Bxy + Cy^2 = D$ is an ellipse provided $B^2 - 4AC < 0$. Hence for our problem if

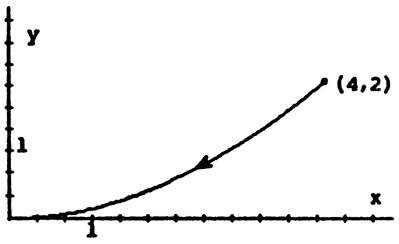
$a_{22}^2 + a_{21}a_{12} < 0$ then Eq.(iv) is an ellipse. Using $a_{11} + a_{22} = 0$ we have $a_{22}^2 = -a_{11}a_{22}$ and hence $-a_{11}a_{22} + a_{21}a_{12} < 0$ or $a_{11}a_{22} - a_{21}a_{12} > 0$, which is true by Eqs.(ii). Thus Eq.(iv) is an ellipse under the conditions of Eqs.(ii).

20. The given system can be written as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Thus the eigenvalues are given by $r^2-(a_{11}+a_{22})r + a_{11}a_{22}-a_{12}a_{21} = 0$ and using the given definitions we rewrite this as $r^2 - pr + q = 0$ and thus $r_{1,2} = (p \pm \sqrt{p^2-4q}\,)/2 = (p \pm \sqrt{\Delta}\,)/2$. The results are now obtained using Table 9.1.1.

Section 9.2, Page 477

1. Solutions of the D.E. for x are y are $x = Ae^{-t}$ and $y = Be^{-2t}$ respectively. $x(0) = 4$ and $y(0) = 2$ yield $A = 4$ and $B = 2$, so $x = 4e^{-t}$ and $y = 2e^{-2t}$. Solving the first equation for e^{-t} and then substituting into the second yields $y = 2[x/4]^2 = x^2/8$, which is a parabola. From the original D.E., or from the parametric solutions, we find that $0 < x \le 4$ and $0 < y \le 2$ for $t \ge 0$ and thus only the portion of the parabola shown is the trajectory, with the direction of motion indicated.

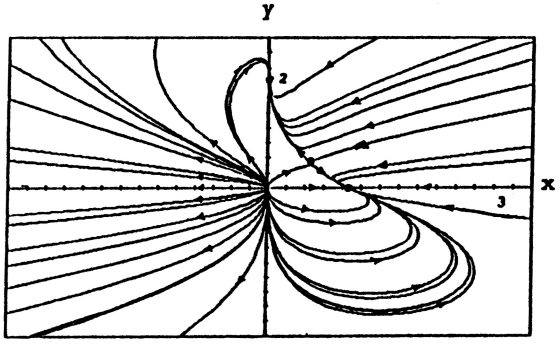


3. Utilizing the approach indicated in Eq.(14), we have $dy/dx = -x/y$, which separates into $xdx + ydy = 0$. Integration then yields the circle $x^2 + y^2 = c^2$, where $c^2 = 16$ for both sets of I.C. The direction of motion can be found from the original D.E. and is counterclockwise for both I.C. To obtain the parametric equations, we write the system in the form $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, which has the characteristic equation $\begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0$, or $r = \pm i$. Following

the procedures of Section 7.6, we find that one solution of the above system is $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i \sin t \\ \sin t - i \cos t \end{pmatrix}$ and thus two real solutions are $\mathbf{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ and $\mathbf{v}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$. The general solution of the system is then $\begin{pmatrix} x \\ y \end{pmatrix} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$ and hence the first I.C. yields $c_1 = 4$, $c_2 = 0$, or $x = 4\cos t$, $y = 4\sin t$. The second I.C. yields $c_1 = 0$, $c_2 = -4$, or $x = -4\sin t$, $y = 4\cos t$. Note that both these parametric representations satsify the form of the trajectories found in the first part of this problem.

7a. The critical points are given by the solutions of $x(1-x-y) = 0$ and $y(1/2 - y/4 - 3x/4) = 0$. The solutions corresponding to either $x = 0$ or $y = 0$ are seen to be $x = 0$, $y = 0$; $x = 0$, $y = 2$; $x = 1$, $y = 0$. In addition, there is a solution corresponding to the intersection of the lines $1 - x - y = 0$ and $1/2 - y/4 - 3x/4 = 0$ which is the point $x = 1/2$, $y = 1/2$. Thus the critical points are (0,0), (0,2), (1,0), and (1/2,1/2).

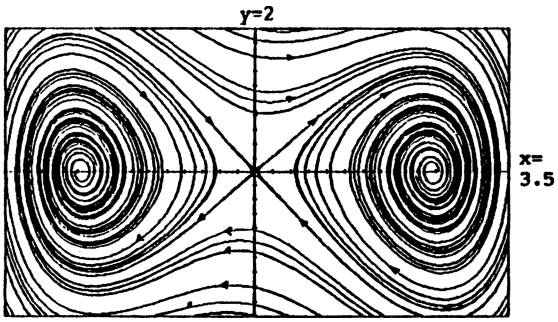
7b.



7c. For (0,0) since all trajectories leave this point, this is an unstable node. For (0,2) and (1,0) since the trajectories tend to these points, respectively, they are asymptotically stable nodes. For (1/2,1/2), one trajectory tends to (1/2,1/2) while all others tend to infinity, so this is an unstable saddle point.

12a. The critical points are given by $y = 0$ and $x(1 - x^2/6 - y/5) = 0$, so (0,0), $(\sqrt{6},0)$ and $(-\sqrt{6},0)$ are the only critical points.

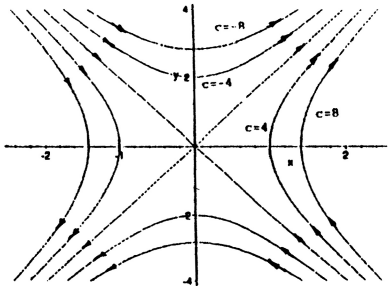
12b.



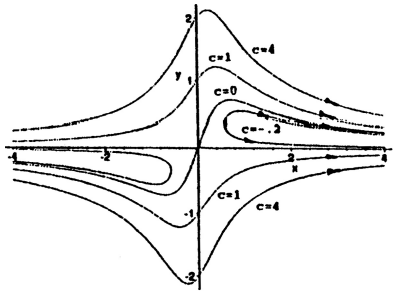
12c. Clearly $(\sqrt{6},0)$ and $(-\sqrt{6},0)$ are spiral points, and are asymptotically stable since the trajectories tend to each point, respectively. (0,0) is a saddle point, which is unstable, since the trajectories behave like the ones for (1/2,1/2) in Problem 7.

15a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8x}{2y}$, so $4xdx - ydy = 0$ and thus $4x^2 - y^2 = c$, which are hyperbolas for $c \neq 0$ and straight lines $y = \pm 2x$ for $c = 0$.

15b.



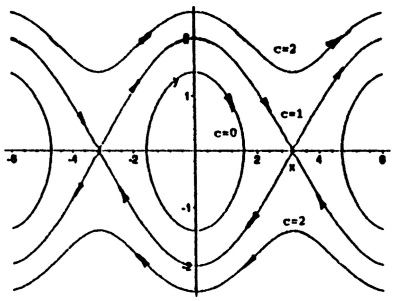
19b.



19a. $\frac{dy}{dx} = \frac{y-2xy}{-x+y+x^2}$, so $(y-2xy)dx + (x-y-x^2)dy = 0$, which is an exact D.E. Therefore $\phi(x,y) = xy - x^2y + g(y)$ and hence $\frac{\partial\phi}{\partial y} = x - x^2 + g'(y) = x - y - x^2$, so $g'(y) = -y$ and $g(y) = -y^2/2$. Thus $2x^2y - 2xy + y^2 = c$ (after multiplying by -2) is the desired solution.

21a. $\frac{dy}{dx} = \frac{-\sin x}{y}$, so $ydy + \sin x dx = 0$ and thus $y^2/2 - \cos x = c$.

21b.



23. We know that $\phi'(t) = F[\phi(t), \psi(t)]$ and $\psi'(t) = G[\phi(t), \psi(t)]$ for $\alpha < t < \beta$. By direct substitution we have
$\Phi'(t) = \phi'(t-s) = F[\phi(t-s), \psi(t-s)] = F[\Phi(t), \Psi(t)]$ and
$\Psi'(t) = \psi'(t-s) = G[\phi(t-s), \psi(t-s)] = G[\Phi(t), \Psi(t)]$ for
$\alpha < t-s < \beta$ or $\alpha+s < t < \beta+s$.

24. Suppose that $t_1 > t_0$. Let $s = t_1 - t_0$. Since the system is autonomous, the result of Problem 23, with s replaced by -s shows that $x = \phi_1(t+s)$ and $y = \psi_1(t+s)$ generates the same trajectory (C_1) as $x = \phi_1(t)$ and $y = \psi_1(t)$. But at $t = t_0$ we have $x = \phi_1(t_0+s) = \phi_1(t_1) = x_0$ and $y = \psi_1(t_0+s) = \psi_1(t_1) = y_0$. Thus the solution $x = \phi_1(t+s)$, $y = \psi_1(t+s)$ satisfies <u>exactly</u> the same initial conditions as the solution $x = \phi_0(t)$, $y = \psi_0(t)$ which generates the trajectory C_0. Hence C_0 and C_1 are the same.

25. From the existence and uniqueness theorem we know that if the two solutions $x = \phi(t)$, $y = \psi(t)$ and $x = x_0$, $y = y_0$ satisfy $\phi(a) = x_0$, $\psi(a) = y_0$ and $x = x_0$, $y = y_0$ at $t = a$, then these solutions are identical. Hence $\phi(t) = x_0$ and $\psi(t) = y_0$ for all t contradicting the fact that the trajectory generated by $[\phi(t), \psi(t)]$ started at a noncritical point.

26. By direct substitution
$\Phi'(t) = \phi'(t+T) = F[\phi(t+T), \psi(t+T)] = F[\Phi(t), \Psi(t)]$ and
$\Psi'(t) = \psi'(t+T) = G[\phi(t+T), \psi(t+T)]$, $G[\Phi(t), \Psi(t)]$.
Furthermore $\Phi(t_0) = x_0$ and $\Psi(t_0) = y_0$. Thus by the existence and uniqueness theorem $\Phi(t) = \phi(t)$ and $\Psi(t) = \psi(t)$ for all t.

<u>Section 9.3, Page 487</u>

In Problems 1 through 4, write the system in the form of

Eq.(4). Then if $\mathbf{g(0)} = \mathbf{0}$ we may conclude that (0,0) is a critical point. In addition, if $\mathbf{g}$ satisfies Eq.(5) or Eq.(6), then the system is almost linear. In this case the linear system, Eq.(1), will determine, in most cases, the type and stability of the critical point (0,0) of the almost linear system. These results are summarized in Table 9.3.1.

3. In this case the system can be written as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1 - \cos y \end{pmatrix}$. However, the coefficient matrix is singular and $g_1(x,y) = (1+x)\sin y$ does not satisfy Eq.(6). However, if we consider the Taylor series for siny, we see that $(1+x)\sin y - y = \sin y - y + x\sin y = -y^3/3! + y^5/5! + \cdots + x(y - y^3/3! + \cdots)$, which does satisfy Eq.(6), using $x = r\cos\theta$, $y = r\sin\theta$. Thus the first equation now becomes $\frac{dx}{dt} = y + [(1+x)\sin y - y]$ and hence $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y - y \\ 1-\cos y \end{pmatrix}$, where the coefficient matrix is now nonsingular and $\mathbf{g}(x,y) = \begin{pmatrix} (1+x)\sin y - y \\ 1-\cos y \end{pmatrix}$ satisfies Eq.(6).

4. In this case the system can be written as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$ and thus $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$. Since $\mathbf{g(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we conclude that (0,0) is a critical point. Following the procedure of Example 1, we let $x = r\cos\theta$ and $y = r\sin\theta$ and thus $g_1(x,y)/r = \frac{r^2\sin^2\theta}{r} \to 0$ as $r \to 0$ and thus the system is almost linear. Since $\det(\mathbf{A}-r\mathbf{I}) = (r-1)^2$, we find that the eigenvalues are $r_1 = r_2 = 1$. Since the roots are equal, we must determine whether there are one or two eigenvectors to classify the type of critical point. The eigenvectors are determined by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and hence

there is only one eigenvector $\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus the critical point for the linear system is an unstable improper node. From Table 9.3.1 we then conclude that the given system, which is almost linear, has a critical point near (0,0) which is either a node or spiral point (depending on how the roots bifurcate) which is unstable.

6a. The critical points are the solutions of $x(1-x-y) = 0$ and $y(3-x-2y) = 0$. Solutions are $x = 0$, $y = 0$; $x = 0$, $3 - 2y = 0$ which gives $y = 3/2$; $y = 0$ and $1 - x = 0$ which give $x = 1$; and $1 - x - y = 0$, $3 - x - 2y = 0$ which give $x = -1$, $y = 2$. Thus the critical points are (0,0), (0,3/2), (1,0) and (-1,2).

6b, 6c. For the critical point (0,0) the D.E. is already in the form of an almost linear system; and the corresponding linear system is $du/dt = u$, $dv/dt = 3v$ which has the eigenvalues $r_1 = 1$ and $r_2 = 3$. Thus the critical point (0,0) is an unstable node. Each of the other three critical points is dealt with in the same manner; we consider only the critical point (-1,2). In order to translate this critical point to the origin we set $x(t) = -1 + u(t)$, $y(t) = 2 + v(t)$ and substitute in the D.E. to obtain

$du/dt = -1 + u - (-1+u)^2 - (-1+u)(2+v) = u + v - u^2 - uv$ and

$dv/dt = 3(2+v) - (-1+u)(2+v) - 2(2+v)^2 = -2u - 4v - uv - 2v^2$.

Writing this in the form of Eq.(4) we find that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}$ and $\mathbf{g} = -\begin{pmatrix} u^2 + uv \\ uv + 2v^2 \end{pmatrix}$ which is an almost linear system. The eigenvalues of the corresponding linear system are $r = (-3 \pm \sqrt{9 + 8})/2$ and hence the critical point (-1,2), of the original system, is an unstable saddle point.

10a. The critical points are solutions of $x + x^2 + y^2 = 0$ and $y(1-x) = 0$, which yield (0,0) and (-1,0).

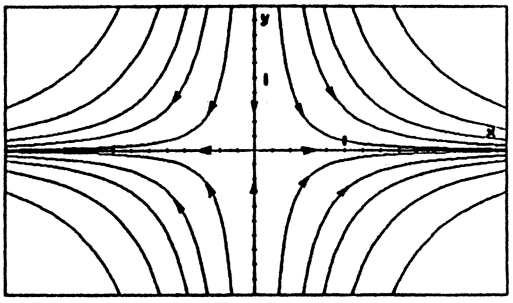
10b. For (0,0) the D.E. is already in the form of an almost linear system and thus $du/dt = u$ and $dv/dt = v$. For (-1,0) we let $u = x+1$, $v = y$ so that substituting $x = u-1$ and $y = v$ into the D.E. we obtain $\dfrac{du}{dt} = -u + u^2 + v^2$ and

$\frac{dv}{dt} = 2v - uv$. Thus the corresponding linear system is $u' = -u$ and $v' = -2v$.

10c. For (0,0) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which has $r_1 = r_2 = 1$, so that (0,0), for the nonlinear system, will be either a node or spiral point, depending on how the roots bifurcate. In any case, since r_1 and r_2 are positive, the system will be unstable. For (-1,0) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ and thus $r_1 = -1$ and $r_2 = 2$, and hence the nonlinear system, from Table 9.3.1, has an unstable saddle point at (-1,0).

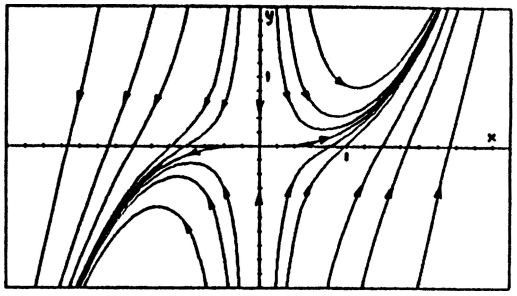
18a. The system is $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ x^3 \end{pmatrix}$ and thus is almost linear using the procedures outlined in the earlier problems. The corresponding linear system has the eigenvalues $r_1 = 1$, $r_2 = -2$ and thus (0,0) is an unstable saddle point for both the linear and almost linear systems.

18b. The trajectories of the linear system are the solutions of $dx/dt = x$ and $dy/dt = -2y$ and thus $x(t) = c_1e^t$ and $y(t) = c_2e^{-2t}$. To sketch these, solve the first equation for e^t and substitute into the second to obtain $y = c_1^2c_2/x^2$, $c_1 \neq 0$. Several trajectories are shown in the figure. Since $x(t) = c_1e^t$, we must pick $c_1 = 0$ for $x \to 0$ and $t \to \infty$. Thus $x = 0$, $y = c_2e^{-2t}$ (the vertical axis) is the only trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$.

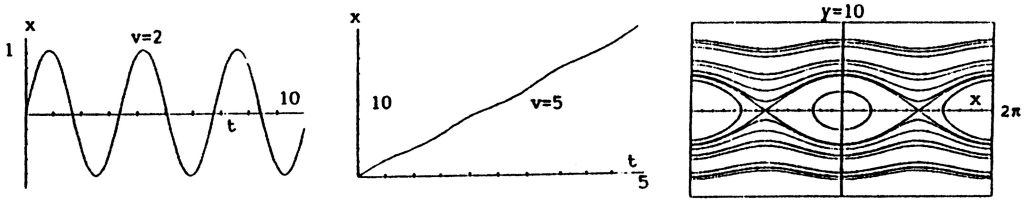


18c. For $x \neq 0$ we have $dy/dx = (dy/dt)/(dx/dt) = (-2y+x^3)/x$. This is a linear equation, and the general solution is $y = x^3/5 + k/x^2$, where k is an arbitrary constant. In addition the system of equations has the solution $x = 0$, $y = Be^{-2t}$. Any solution with its initial point on the y-axis (x=0) is given by the latter solution. The trajectories corresponding to these solutions approach

the origin as $t \to \infty$. The trajectory that passes through the origin and divides the family of curves is given by $k = 0$, namely $y = x^3/5$. This trajectory corresponds to the trajectory $y = 0$ for the linear problem. Several trajectories are sketched in the figure.

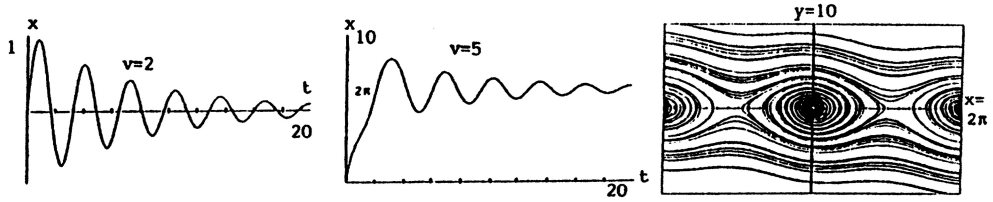


22a.



22b. From the graphs in part a, we see that v_c is between $v = 2$ and $v = 5$. Using several values for v, we estimate $v_c \cong 4.00$.

23a.



For $v = 2$, the motion is damped oscillatory about $x = 0$. For $v = 5$, the pendulum swings all the way around once and then is a damped oscillation about $x = 2\pi$ (after one full rotation). For problem 22, this later case is not damped, so x continues to increase, as shown earlier.

27a. Setting $c = 0$ in Eq.(10) of Section 9.2 we obtain $mL^2d^2\theta/dt^2 + mgL\sin\theta = 0$. Considering $d\theta/dt$ as a function of θ and using the chain rule we have $\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{1}{2}\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)^2$. Thus $(1/2)mL^2d[(d\theta/dt)^2]/d\theta = -mgL\sin\theta$. Now integrate both sides from α to θ where $d\theta/dt = 0$ at $\theta = \alpha$: $(1/2)mL^2(d\theta/dt)^2 = mgL(\cos\theta - \cos\alpha)$. Thus $(d\theta/dt)^2 = (2g/L)(\cos\theta - \cos\alpha)$. Since we are releasing the pendulum with zero velocity from a positive angle α,

the angle θ will initially be decreasing so $d\theta/dt < 0$. If we restrict attention to the range of θ from $\theta = \alpha$ to $\theta = 0$, we can assert $d\theta/dt = -\sqrt{2g/L}\ \sqrt{\cos\theta - \cos\alpha}$. Solving for dt gives $dt = -\sqrt{L/2g}\, d\theta/\sqrt{\cos\theta - \cos\alpha}$.

27b. Since there is no damping, the pendulum will swing from its initial angle α through 0 to $-\alpha$, then back through 0 again to the angle α in one period. It follows that $\theta(T/4) = 0$. Integrating the last equation and noting that as t goes from 0 to T/4, θ goes from α to 0 yields $T/4 = -\sqrt{L/2g}\ \int_{\alpha}^{0} (1/\sqrt{\cos\theta - \cos\alpha})d\theta$.

28a. If $\frac{dx}{dt} = y$, then $\frac{d^2x}{dt^2} = \frac{dy}{dt} = -g(x) - c(x)y$.

28b. Under the given assumptions we have $g(x) = g(0) + g'(0)x + g''(\xi_1)x^2/2$ and $c(x) = c(0) + c'(\xi_2)x$, where $0 < \xi_1, \xi_2 < x$ and $g(0) = 0$. Hence

$$\frac{dy}{dt} = (-g(0) - g'(0)x) - c(0)y - [g''(\xi_1)x^2/2 - c'(\xi_2)xy]$$

and thus the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -g''(\xi_1)x^2/2 - c'(\xi_2)xy \end{pmatrix},$$

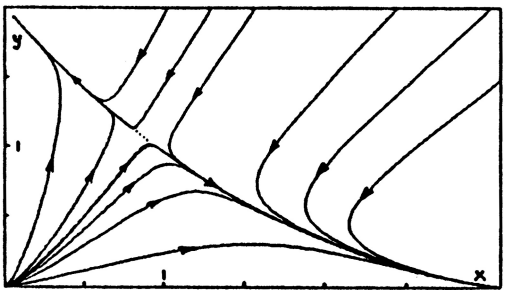
from which the results follow.

Section 9.4, Page 501

3b. $x(1.5 - .5x - y) = 0$ and $y(2 - y - 1.125x) = 0$ yield (0,0), (0,2) and (3,0) very easily. The fourth critical point is the intersection of $.5x + y = 1.5$ and $1.125x + y = 2$, which is (.8,1.1).

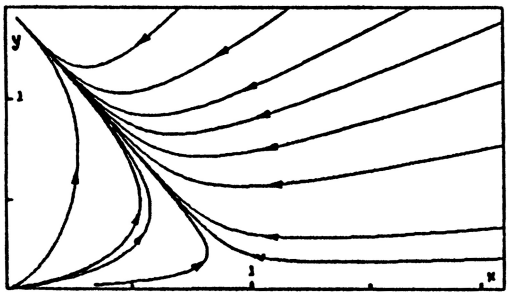
3c. From Eq (5) we get $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1.5-x_0-y_0 & -x_0 \\ -1.125y_0 & 2-2y_0-1.125x_0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$. For (0,0) we get $u' = 1.5u$ and $v' = 2v$, so $r = 3/2$ and $r = 2$, and thus (0,0) is an unstable node. For (0,2) we have $u' = -.5u$ and $v' = -2.25u-2v$, so $r = -.5, -2$ and thus (0,2) is an asymptotically stable node. For (3,0) we get $u' = -1.5u-3v$ and $v' = -1.375v$, so $r = -1.5, -1.375$ and hence (3,0) is an symptotically stable node. For (.8,1.1) we have $u' = -.4u - .8v$ and $v' = -1.2375u - 1.1v$ which give $r = -1.80475, .30475$ and thus (.8,1.1) is an unstable saddle point.

3e.



5b. The critical points are found by setting dx/dt = 0 and dy/dt = 0 and thus we need to solve x(1 - x - y) = 0 and y(1.5 - y - x) = 0. The first yields x = 0 or y = 1 - x and the second yields y = 0 or y = 1.5 - x. Thus (0,0), (0,3/2) and (1,0) are the only critical points since the two straight lines do not intersect in the first quadrant (or anywhere in this case). This is an example of one of the cases shown in Figure 9.4.5 a or b.

5e.



6b. The critical points are found by setting dx/dt = 0 and dy/dt = 0 and thus we need to solve x(1-x + y/2) = 0 and y(5/2 - 3y/2 + x/4) = 0. The first yields x = 0 or y = 2x - 2 and the second yields y = 0 or y = x/6 + 5/3. Thus we find the critical points (0,0), (1,0), (0,5/3) and (2,2). The last point is the intersection of the two straight lines, which will be used again in part d.

6c. For (0,0) the linearized system is $x' = x$ and $y' = 5y/2$, which has the eigenvalues $r_1 = 1$ and $r_2 = 5/2$. Thus the origin is an unstable node. For (2,2) we let $x = u + 2$ and $y = v + 2$ in the given system to find (since $x' = u'$ and $y' = v'$) that
$du/dt = (u+2)[1 - (u+2) + (v+2)/2] = (u+2)(-u+v/2)$ and
$dv/dt = (v+2)[5/2 - 3(v+2)/2 + (u+2)/4] = (v+2)(u/4 - 3v/2)$.
Hence the linearized equations are $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 1/2 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$
which has the eigenvalues $r_{1,2} = (-5 \pm \sqrt{3})/2$. Since these are both negative we conclude that (2,2) is an asymptotically stable node. In a similar fashion for

(1,0) we let $x = u + 1$ and $y = v$ to obtain the linearized system $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -1 & 1/2 \\ 0 & 11/4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. This has $r_1 = -1$ and $r_2 = 11/4$ as eigenvalues and thus (1,0) is an unstable saddle point. Likewise, for (0,5/3) we let $x = u$, $y = v + 5/3$ to find $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 11/6 & 0 \\ 5/12 & -5/2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ as the corresponding linear system. Thus $r_1 = 11/6$ and $r_2 = -5/2$ and thus (0,5/3) is an unstable saddle point.

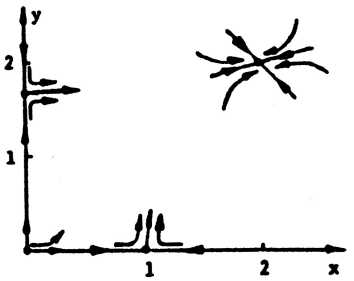
6d. To sketch the required trajectories, we must find the eigenvectors for each of the linearized systems and then analyze the behavior of the linear solution near the critical point. Using this approach we find that the solution near (0,0) has the form $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{5t/2}$ and thus the origin is approached only for large negative values of t. In this case e^t dominates $e^{5t/2}$ and hence in the neighborhood of the origin all trajectories are tangent to the x-axis except for one pair ($c_1 = 0$) that lies along the y-axis.
For (2,2) we find the eigenvector corresponding to $r = (-5 + \sqrt{3})/2 = -1.63$ is given by $(1-\sqrt{3})\xi_1/2 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ (\sqrt{3}-1)/2 \end{pmatrix} = \begin{pmatrix} 1 \\ .37 \end{pmatrix}$ is one eigenvector. For $r = (-5 - \sqrt{3})/2 = -3.37$ we have $(1 + \sqrt{3})\xi_1/2 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ -(\sqrt{3}+1)/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.37 \end{pmatrix}$ is the second eigenvector.
Hence the linearized solution is $\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ .37 \end{pmatrix} e^{-1.63t} + c_2 \begin{pmatrix} 1 \\ -1.37 \end{pmatrix} e^{-3.37t}$. For large positive values of t the first term is the dominant one and thus we conclude that all trajectories but two approach (2,2) tangent to the straight line with slope .37. If $c_1 = 0$, we see that there are exactly two ($c_2 > 0$ and $c_2 < 0$) trajectories that lie on the straight line with slope -1.37.
In similar fashion, we find the linearized solutions near (1,0) and (0,5/3) to be, respectively,

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 15/2 \end{pmatrix} e^{11t/4}$$

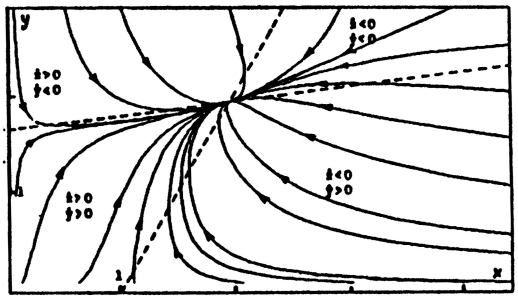
and

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t/2} + c_2 \begin{pmatrix} 1 \\ 5/52 \end{pmatrix} e^{11t/6},$$

which, along with the above analysis, yields the sketch shown.



6e. From the above sketch, it appears that $(x,y) \to (2,2)$ as
6f. $t \to \infty$ as long as (x,y) starts in the first quadrant. To ascertain this, we need to prove that x and y cannot become unbounded as $t \to \infty$. From the given system, we can observe that, since $x > 0$ and $y > 0$, that dx/dt and dy/dt have the same sign as the quantities $1 - x + y/2$ and $5/2 - 3y/2 + x/4$ respectively. If we set these quantities equal to zero we get the straight lines $y = 2x - 2$ and $y = x/6 + 5/3$, which divide the first quadrant into the four sectors shown. The signs of x' and y' are indicated, from which it can be concluded that x and y must remain bounded [and in fact approach (2,2)] as $t \to \infty$. The discussion leading up to Fig.9.4.4 is also useful here.



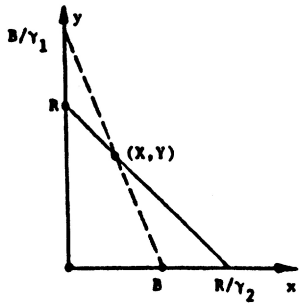
8a. Setting the right sides of the equations equal to zero gives the critical points (0,0), $(0, \varepsilon_2/\sigma_2)$, $(\varepsilon_1/\sigma_1, 0)$, and possibly $([\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1]/[\sigma_1\sigma_2 - \alpha_1\alpha_2], [\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2]/[\sigma_1\sigma_2 - \alpha_1\alpha_2])$. (The last point can be obtained from Eq.(36) also). The conditions $\varepsilon_2/\alpha_2 > \varepsilon_1/\sigma_1$ and $\varepsilon_2/\sigma_2 > \varepsilon_1/\alpha_1$ imply that $\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2 > 0$ and $\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1 < 0$. Thus either the x coordinate or the y coordinate of the last critical point is negative so a mixed state is not possible. The linearized system for (0,0) is $x' = \varepsilon_1 x$ and $y' = \varepsilon_2 y$ and thus (0,0) is an unstable equilibrium point. Similarly, it can be shown [by linearizing the given system or by using Eq.(35)] that $(0, \varepsilon_2/\sigma_2)$ is an asymptotically stable critical point and that $(\varepsilon_1\sigma_1, 0)$ is an unstable critical point. Thus the fish represented by y(redear) survive.

8b. The conditions $\varepsilon_1/\sigma_1 > \varepsilon_2/\alpha_2$ and $\varepsilon_1/\alpha_1 > \varepsilon_2/\sigma_2$ imply that $\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2 < 0$ and $\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1 > 0$ so again one of the coordinates of the fourth point in 8a. is negative and hence a mixed state is not possible. An analysis similar to that in part(a) shows that (0,0) and $(0,\varepsilon_2/\sigma_2)$ are unstable while $(\varepsilon_1/\sigma_1,0)$ is stable. Hence the bluegill (represented by x) survive in this case.

9a. $x' = \varepsilon_1 x(1 - \frac{\sigma_1}{\varepsilon_1}x - \frac{\alpha_1}{\varepsilon_1}y) = \varepsilon_1 x(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y)$

$y' = \varepsilon_2 y(1 - \frac{\sigma_2}{\varepsilon_2}y - \frac{\alpha_2}{\varepsilon_2}x) = \varepsilon_2 y(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x)$. The coexistence equilibrium point is given by $\frac{1}{B}x + \frac{\gamma_1}{B}y = 1$ and $\frac{\gamma_2}{R}x + \frac{1}{R}y = 1$. Solving these (using determinants) yields $X = (B - \gamma_1 R)/(1 - \gamma_1\gamma_2)$ and $Y = (R - \gamma_2 B)/(1-\gamma_1\gamma_2)$.

9b. If B is reduced, it is clear from the answer to part(a) that X is reduced and Y is increased. To determine whether the bluegill will die out, we give an intuitive argument which can be confirmed by doing the analysis. Note that $B/\gamma_1 = \varepsilon_1/\alpha_1 > \varepsilon_2/\sigma_2 = R$ and $R/\gamma_2 = \varepsilon_2/\alpha_2 > \varepsilon_1/\sigma_1 = B$ so that the graph of the lines $1 - x/B - \gamma_1 y/B = 0$ and $1 - y/R - \gamma_2 x/R = 0$ must appear as indicated in the figure, where critical points are inidcated by heavy dots. As B is decreased, X decreases, Y increases (as indicated above) and the point of intersection moves closer to (0,R). If $B/\gamma_1 < R$ coexistence is not possible, and the only critical points are (0,0),(0,R)and (B,0). It can be shown that (0,0) and (B,0) are unstable and (0,R) is asymptotically stable. Hence we conlcude, when coexistence is no longer possible, that $x \to 0$ and $y \to R$ and thus the bluegill population will die out.



12a. Setting each equation equal to zero, we obtain $x = 0$ or $(4 - x - y) = 0$ and $y = 0$ or $(2 + 2\alpha - y - \alpha x) = 0$. Thus we have (0,0), (4,0), $(0,2 + 2\alpha)$, and the intersection of $x + y = 4$ and $\alpha x + y = 2 + 2\alpha$. If $\alpha \neq 1$, this yields (2,2) as the fourth critical point.

12b. For $\alpha = .75$ the linear system is $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ -1.5 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, which has the characteristic equation $r^2 + 4r + 1 = 0$ so that $r = -2 \pm \sqrt{3}$. Thus the critical point is an asymptotically stable node. For $\alpha = 1.25$, we have $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ -2.5 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, so $r^2 + 4r - 1 = 0$ and $r = -2 \pm \sqrt{5}$. Thus (2,2) is an unstable saddle point.

12c. Letting $x = u+2$ and $y = v+2$ yields
$u' = (u+2)(4 - u-2 - v-2) = -2u - 2v - u^2 - uv$ and
$v' = (v+2)(2 + 2\alpha - v-2 - \alpha u - 2\alpha) = -2\alpha u - 2v - v^2 - \alpha uv$.
Thus the approximate linear system is $u' = -2u - 2v$ and $v' = -2\alpha u - 2v$.

12d. The eigenvalues are given by
$\begin{vmatrix} -2-r & -2 \\ -2\alpha & -2-r \end{vmatrix} = r^2 + 4r + 4 - 4\alpha = 0$, or $r = -2 \pm 2\sqrt{\alpha}$.
Thus for $0 < \alpha < 1$ there are 2 negative real roots (asymptotially stable node) and for $\alpha > 1$ the real roots differ in sign, yielding an unstable saddle point. $\alpha = 1$ is the bifurcation point.

Section 9.5, Page 509

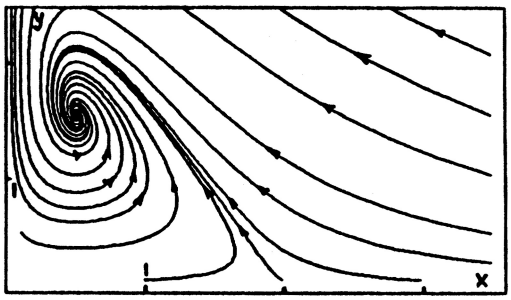
3b. We have $x = 0$ or $(1 - .5x - .5y) = 0$ and $y = 0$ or $(-.25 + .5x) = 0$ and thus we have three critical points: (0,0), (2,0) and (1/2,3/2).

3c. For (0,0) the linear system is $dx/dt = x$ and $dy/dt = -.25y$ and hence $A = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}$ which has eigenvalues $r_1 = 1$ and $r_2 = -1/4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus (0,0) is an unstable saddle point.
For (2,0), we let $x = 2 + u$ and $y = v$ in the given equations and obtain $\frac{du}{dt} = -(u+v) - \frac{1}{2}u(u+v)$ and $\frac{dv}{dt} = \frac{3}{4}v + \frac{1}{2}uv$. The linear portion of this has matrix

$A = \begin{pmatrix} -1 & -1 \\ 0 & 3/4 \end{pmatrix}$, which has the eigenvalues $r_1 = -1$, $r_2 = 3/4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$. Thus (2,0) is also an unstable saddle point.

For $\left(\frac{1}{2}, \frac{3}{2}\right)$ we let $x = 1/2 + u$ and $y = 3/2 + v$ in the given equations, which yields $\frac{du}{dt} = -\frac{1}{4}u - \frac{1}{4}v$, $\frac{dv}{dt} = \frac{3}{4}u$ as the linear portion. Thus $A = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{pmatrix}$, which has eigenvalues $r_{1,2} = (-1 \pm \sqrt{11}\,i)/8$. Thus $\left(\frac{1}{2}, \frac{3}{2}\right)$ is an asymptotically stable spiral point since the eigenvalues are complex with negative real part. Using $r_1 = (-1 + \sqrt{11}\,i)/8$ we find that one eigenvector is $\begin{pmatrix} -2 \\ 1 + \sqrt{11}\,i \end{pmatrix}$ and by Section 7.6 the second eigenvector is $\begin{pmatrix} -2 \\ 1 - \sqrt{11}\,i \end{pmatrix}$.

3e.



3f. For (x,y) above the line $x + y = 2$ we see that $x' < 0$ and thus x must remain bounded. For (x,y) to the right of $x = 1/2$, $y' > 0$ so it appears that y could grow large asymptotic to x = constant. However, this implies a contradiction (x = constant implies $x' = 0$, but as y gets larger, x' gets increasingly negative) and hence we conclude y must remain bounded and hence $(x,y) \to (1/2,3/2)$ as $t \to \infty$, again assuming they start in the first quadrant.

7a. The amplitude ratio is $(cK/\gamma)/(\sqrt{ac}\,K/\alpha) = \alpha\sqrt{c}\,/\gamma\sqrt{a}$.

7b. From Eq (2) $\alpha = .5$, $a = 1$, $\gamma = .25$ and $c = .75$, so the ratio is $.5\sqrt{.75}\,/.25\sqrt{1} = 2\sqrt{.75} = \sqrt{3} \tilde{=} 1.732$.

7c. A rough measurement of the amplitudes is (6.1 - 1)/2 = 2.55 and (3.8 -. 9)/2 = 1.45 and thus the ratio is approximately 1.76. In this case the linear approximation is a good predictor.

11. The presence of a trapping company actually would require a modification of the equations, either by altering the coefficients or by including nonhomogeneous terms on the right sides of the D.E. The effects of indiscreminate trapping could decrease the populations of both rabbits and fox significantly or decrease the fox population which could possibly lead to a large increase in the rabbit population. Over the long run it makes sense for a trapping company to operate in such a way that a consistent supply of pelts is available and to disturb the predator-prey system as little as possible. Thus, the company should trap fox only when their population is increasing, trap rabbits only when their population is increasing, trap rabbits and fox only during the time when both their populations are increasing, and trap neither during the time both their populations are decreasing. In this way the trapping company can have a moderating effect on the population fluctuations, keeping the trajectory close to the center.

13. The critical points of the system are the solutions of the algebraic equations $x(a - \sigma x - \alpha y) = 0$, and $y(-c + \gamma x) = 0$. the critical points are $x = 0$, $y = 0$; $x = a/\sigma$, $y = 0$; and $x = c/\gamma$, $y = a/\alpha - c\sigma/\alpha\gamma = \sigma A/\alpha$ where $A = a/\sigma - c/\gamma > 0$.
To study the critical point (0,0) we discard the nonlinear terms in the system of D.E. to obtain the corresponding linear system $dx/dt = ax$, $dy/dt = -cy$. The characteristic equation is $r^2 - (a+c)r - ac = 0$ so $r_1 = a$, $r_2 = -c$. Thus the critical point (0,0) is an unstable saddle point.
To study the critical point $(a/\sigma,0)$ we let $x = (a/\sigma) + u$, $y = 0 + v$ and substitute in the D.E. to obtain the almost linear system $du/dt = -au - (a\alpha/\sigma)v - \sigma u^2 - \alpha uv$, $dv/dt = \gamma Av + \gamma uv$. The corresponding linear system is $du/dt = -au - (a\alpha/\sigma)v$, $dv/dt = \gamma Av$. The characteristic equation is $r^2 + (a - \gamma A)r - a\gamma A = 0$ so $r_1 = -a$, $r_2 = \gamma A$.

Thus the critical point $(a/\sigma,0)$ is an unstable saddle point.
To study the critical point $(c/\gamma, \sigma A/\alpha)$ we let $x = (c/\gamma) + u$, $y = (\sigma A/\alpha) + v$ and substitute in the D.E. to obtain the almost linear system

$$\frac{du}{dt} = -(c\sigma/\gamma)u - (ac/\gamma)v - \sigma u^2 - \alpha uv$$

$$\frac{dv}{dt} = (\sigma A\gamma/\alpha)u + \gamma uv$$

The corresponding linear system is $du/dt = -(c\sigma/\gamma)u - (\alpha c/\gamma)v$, $dv/dt = (\sigma A\gamma/\alpha)u$. The characteristic equation is $r^2 + (c\sigma/\gamma)r + c\sigma A = 0$, so $r_1, r_2 = [-(c\sigma/\gamma) \pm \sqrt{(c\sigma/\gamma)^2 - 4c\sigma A}\,]/2$. Thus, depending on the sign of the discriminant we have that $(c/\gamma, \sigma A/\alpha)$ is either an asymptotically stable spiral point or an asymptotically stable node. Thus for nonzero initial data $(x,y) \to (c/\gamma, \sigma A/\alpha)$ as $t \to \infty$.

Section 9.6, Page 519

1. Assuming that $V(x,y) = ax^2 + cy^2$ we find $V_x(x,y) = 2ax$, $V_y = 2cy$ and thus Eq.(7) yields $\dot{V}(x,y) = 2ax(-x^3 + xy^2) + 2cy(-2x^2y - y^3) = -[2ax^4 + 2(2c-a)x^2y^2 + 2cy^4]$. If we choose a and c to be any positive real numbers with $2c > a$, then $\dot{V}$ is a negative definite. Also, V is positive definite by Theorem 9.6.4. Thus by Theorem 9.6.1 the origin is an asymptotically stable critical point.

3. Assuming the same form for V(x,y) as in Problem 1, we have
$\dot{V}(x,y) = 2ax(-x^3 + 2y^3) + 2cy(-2xy^2) = -2ax^4 + 4(a-c)xy^3$.
If we choose $a = c > 0$, then $\dot{V}(x,y) = -2ax^4 \le 0$ in any neighborhood containing the origin and thus $\dot{V}$ is negative semidefinite and V is positive definite. Theorem 9.6.1 then concludes that the origin is a stable critical point. Note that the origin may still be asymptotically stable, however, the V(x,y) used here is not sufficient to prove that.

6a. The correct system is $dx/dt = y$ and $dy/dt = -g(x)$. Since $g(0) = 0$, we conclude that (0,0) is a critical point.

6b. From the given conditions, the graph of g must be positive for $0 < x < k$ and negative for $-k < x < 0$. Thus

if $0 < x < k$ then $\int_0^x g(s)ds > 0$,

if $-k < x < 0$ then $\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0$.

Since $V(0,0) = 0$ it follows that $V(x,y) = y^2/2 + \int_0^x g(s)ds$ is positive definite for $-k < x < k$, $-\infty < y < \infty$. Next, we have $\dot{V}(x,y) = V_x\frac{dx}{dt} + V_y\frac{dy}{dt} = g(x)y + y[-g(x)] = 0$.

Since $\dot{V}(x,y)$ is never positive, we may conclude that it is negative semidefinite and hence by Theorem 9.6.1 (0,0) is at least a stable critical point.

7b. V is positive definite by Theorem 9.6.4. Since $V_x(x,y) = 2x$, $V_y(x,y) = 2y$, we obtain

$\dot{V}(x,y) = 2xy - 2y^2 - 2y\sin x = 2y[-y + (x - \sin x)]$. If $x < 0$, then $\dot{V}(x,y) < 0$ for all $y > 0$. If $x > 0$, choose y so that $0 < y < x - \sin x$. Then $\dot{V}(x,y) > 0$. Hence V is not a Liapunov function.

7c. Since $V(0,0) = 0$, $1 - \cos x > 0$ for $0 < |x| < 2\pi$ and $y^2 > 0$ for $y \neq 0$, it follows that $V(x,y)$ is positive definite in a neighborhood of the origin. Next $V_x(x,y) = \sin x$, $V_y(x,y) = y$, so

$\dot{V}(x,y) = (\sin x)(y) + y(-y - \sin x) = -y^2$. Hence $\dot{V}$ is negative semidefinite and (0,0) is a stable critical point by Theorem 9.6.1.

7d. $V(x,y) = (x+y)^2/2 + x^2 + y^2/2 = 3x^2/2 + xy + y^2$ is positive definite by Theorem 9.6.4. Next $V_x(x,y) = 3x + y$, $V_y(x,y) = x + 2y$ so

$$\begin{aligned}\dot{V}(x,y) &= (3x+y)y - (x+2y)(y+\sin x)\\ &= 2xy - y^2 - (x+2y)\sin x\\ &= 2xy - y^2 - (x+2y)(x - \alpha x^3/6)\\ &= -x^2 - y^2 + \alpha(x+2y)x^3/6\\ &= -r^2 + \alpha r^4(\cos\theta + 2\sin\theta)(\cos^3\theta)/6 < -r^2 + r^4/2\\ &= -r^2(1-r^2/2).\end{aligned}$$

Thus $\dot{V}$ is negative definite for $r < \sqrt{2}$. From Theorem 9.6.1 it follows that the origin is an asymptotically stable critical point.

8. Let $x = u$ and $y = du/dt$ to obtain the system $dx/dt = y$ and $dy/dt = -c(x)y - g(x)$. Now consider $V(x,y) = y^2/2 + \int_0^x g(s)ds$, which yields

$$\dot{V} = g(x)y + y[-c(x)y - g(x)] = -y^2c(x).$$

10b. Since $V_x(x,y) = 2Ax + By$, $V_y(x,y) = Bx + 2Cy$, we have

$$\dot{V}(x,y) = (2Ax + By)(a_{11}x + a_{12}y) + (Bx + 2Cy)(a_{21}x + a_{22}y) = (2Aa_{11} + Ba_{21})x^2 + [2(Aa_{12} + Ca_{21}) + B(a_{11}+a_{22})]xy + (2Ca_{22} + Ba_{12})y^2.$$

We choose A, B, and C so that $2Aa_{11} + Ba_{21} = -1$, $2(Aa_{12} + Ca_{21}) + B(a_{11}+a_{22}) = 0$, and $2Ca_{22} + Ba_{12} = -1$. The first and third equations give us A and C in terms of B, respectively. We substitute in the second equation to find B and then calculate A and C. The result is given in the text.

10c. Since $a_{11}a_{22} - a_{12}a_{21} > 0$ and $a_{11} + a_{22} < 0$, we see that $\Delta < 0$ and so $A > 0$. Using the expressions for A, B, and C found in part (b) we obtain

$$\begin{aligned}(4AC-B^2)\Delta^2 &= [a_{21}^2+a_{22}^2 + (a_{11}a_{22}-a_{12}a_{21})][a_{11}^2+a_{12}^2 + (a_{11}a_{22}-a_{12}a_{21})] \\ &\quad - (a_{12}a_{22}+a_{11}a_{21})^2 \\ &= (a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)(a_{11}a_{22}-a_{12}a_{21}) + (a_{11}^2+a_{12}^2)(a_{21}^2+a_{22}^2) \\ &\quad + (a_{11}a_{22}-a_{12}a_{21})^2 - (a_{12}a_{22}+a_{11}a_{21})^2 \\ &= (a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)(a_{11}a_{22}-a_{12}a_{21}) + 2(a_{11}a_{22}-a_{12}a_{21})^2.\end{aligned}$$

Since $a_{11}a_{22} - a_{12}a_{21} > 0$ it follows that $4AC - B^2 > 0$.

11a. For $V(x,y) = Ax^2 + Bxy + Cy^2$ we have

$$\begin{aligned}\dot{V} &= (2Ax + By)(a_{11}x+a_{12}y + F_1(x,y)) + (Bx+2Cy)(a_{21}x+a_{22}y + G_1(x,y)) \\ &= (2Ax+By)(a_{11}x+a_{12}y)+(Bx+2Cy)(a_{21}x+a_{22}y) \\ &\quad + (2Ax+By)F_1(x,y) + (Bx+2Cy)G_1(x,y) \\ &= -x^2-y^2 + (2Ax+By)F_1(x,y) + (Bx+2Cy)G_1(x,y),\end{aligned}$$

if A,B and C are chosen as in Problem 10.

11b. Substituting $x = r\cos\theta$, $y = r\sin\theta$ we find that

$$\dot{V}[x(r,\theta), y(r,\theta)] = -r^2+r(2A\cos\theta+B\sin\theta)F_1[x(r,\theta), y(r,\theta)] + r(B\cos\theta + 2C\sin\theta)G_1[x(r,\theta), y(r,\theta)].$$

Now we make use of the facts that (1) there exists an M such that $|2A| \le M$, $|B| \le M$, and $|2C| \le M$; and (2) given any $\varepsilon > 0$ there exists a circle $r = R$ such that $|F_1(x,y)| < \varepsilon r$ and $|G_1(x,y)| < \varepsilon r$ for $0 \le r < R$. We have $|2A\cos\theta + B\sin\theta| \le 2M$ and $|B\cos\theta + 2C\sin\theta| \le 2M$. Hence

$$\dot{V}[x(r,\theta), y(r,\theta)] \le -r^2 + 2Mr(\varepsilon r)+2Mr(\varepsilon r) = -r^2(1 - 4M\varepsilon).$$

If we choose $\varepsilon = M/8$ we obtain $\dot{V}[x(r,\theta), y(r,\theta)] \le -r^2/2$ for $0 \le r < R$. Hence $\dot{V}$ is negative definite in $0 \le r < R$ and from Problem 10c V is positive definite and thus V is a Liapunov function for the almost linear system.

Section 9.7, Page 530

1. Note that $r = 1$, $\theta = t + t_0$ satisfy the two equations for all t and is thus a periodic solution. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $r > 1$, then the direction of motion is inward. It follows that the periodic solution $r = 1$, $\theta = t + t_0$ is a stable limit cycle.

2. $r = 1$, $\theta = -t + t_0$ is a periodic solution. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $r > 1$, the $dr/dt > 0$, and the direction of motion is still outward. It follows that the solution $r = 1$, $\theta = -t + t_0$ is a semistable limit cycle.

4. $r = 1$, $\theta = -t + t_0$ and $r = 2$, $\theta = -t + t_0$ are periodic solutions. If $r < 1$, then $dr/dt < 0$, and the direction of motion on a trajectory is inward. If $1 < r < 2$, then $dr/dt > 0$, and the direction of motion is outward. Similarly, if $r > 2$, the direction of motion is inward. It follows that the periodic solution $r = 1$, $\theta = -t + t_0$ is unstable and the periodic solution $r = 2$, $\theta = -t + t_0$ is a stable limit cycle.

7. Differentiating x and y with respect to t we find that $dx/dt = (dr/dt)\cos\theta - (r\sin\theta)d\theta/dt$ and $dy/dt = (dr/dt)\sin\theta + (r\cos\theta)d\theta/dt$. Hence

$$\begin{aligned} y\,dx/dt - x\,dy/dt &= (r\sin\theta\cos\theta)dr/dt - (r^2\sin^2\theta)d\theta/dt - \\ &\quad (r\cos\theta\sin\theta)dr/dt - (r^2\cos\theta)d\theta/dt \\ &= -r^2 d\theta/dt. \end{aligned}$$

8a. Multiplying the first equation by x and the second by y and adding yields $x\,dx/dt + y\,dy/dt = (x^2+y^2)f(r)/r$, or $r\,dr/dt = rf(r)$, as in the derivation of Eq.(8), and thus $dr/dt = f(r)$. To obtain an equation for θ multiply the first equation by y, the second by x and substract to obtain $y\,dx/dt - x\,dy/dt = -x^2-y^2$, or $-r^2 d\theta/dt = -r^2$, using the results of Problem 7. Thus $d\theta/dt = 1$. It follows that periodic solutions are given by $r = c$, $\theta = t + t_0$ where $f(c) = 0$. Since $\theta = t + t_0$, the motion is counterclockwise.

8b. First note that $f(r) = r(r-2)^2(r-3)(r-1)$. Thus $r = 1$, $\theta = t + t_0$; $r = 2$, $\theta = t + t_0$; and $r = 3$, $\theta = t + t_0$ are periodic solutions. If $r < 1$, then $dr/dt > 0$, and the direction of motion on a trajectory is outward. If $1 < r < 2$, then $dr/dt < 0$ and the direction of motion is inward. Thus the periodic solution $r = 1$, $\theta = t + t_0$ is a stable limit cycle. If $2 < r < 3$, then $dr/dt < 0$, and the direction of motion is inward. Thus the periodic solution $r = 2$, $\theta = t + t_0$ is a semistable limit cycle. If $r > 3$, then $dr/dt > 0$, and the direction of motion is outward. Thus the periodic solution $r = 3$, $\theta = t + t_0$ is unstable.

9. Setting $x = r\cos\theta$, $y = r\sin\theta$ and using the techniques of Problem 8 the equations transform to $dr/dt = r^2 - 2$, $d\theta/dt = -1$. This system has a periodic solution $r = \sqrt{2}$, $\theta = -t + t_0$. If $r < \sqrt{2}$, then $dr/dt < 0$, and the direction of motion along a trajectory is inward. If $r > \sqrt{2}$, then $dr/dt > 0$, and the direction of motion is outward. Thus the periodic solution $r = \sqrt{2}$, $\theta = -t + t_0$ is unstable.

11. If $F(x,y) = x+y+x^3-y^2$, $G(x,y) = -x+2y+x^2y+y^3/3$, then $F_x(x,y) + G_y(x,y) = 1+3x^2+2+x^2+y^2 = 3+4x^2+y^2$. Since the conditions of Theorem 9.7.2 are satisfied for all x and y, and since $F_x + G_y > 0$ for all x and y, it follows that the system has no periodic nonconstant solution.

13. Since $x = \phi(t)$, $y = \psi(t)$ is a solution of Eqs.(15), we have $d\phi/dt = F[\phi(t),\psi(t)]$, $d\psi/dt = G[\phi(t),\psi(t)]$. Hence on the curve C,
$F(x,y)dy - G(x,y)dx = \phi'(t)\psi'(t)dt - \psi'(t)\phi'(t)dt = 0$. It follows that the line integral around C is zero. However, if $F_x + G_y$ has the same sign throughout D, then the double integral cannot be zero. This gives a contradiction. Thus either the solution of Eqs.(15) is not periodic or if it is, it cannot lie entirely in D.

16a. Setting $x' = 0$ and solving for y yields $y = x^3/3 - x + k$. Substituting this into $y' = 0$ then gives
$x + .8(x^3/3 - x + k) - .7 = 0$ Using an equation solver we obtain $x = 1.1994$, $y = -.62426$ for $k = 0$ and $x = .80485$, $y = -.13106$ for $k = .5$. To determine the type of critical points these are, we use Eq.(13) of Section 9.3 to find the

linear coefficient matrix to be $\mathbf{A} = \begin{pmatrix} 3(1-x_c^2) & 3 \\ -1/3 & -.8/3 \end{pmatrix}$, where x_c is the critical point. For $x_c = 1.1994$ we obtain complex conjugate eigenvalues with a negative real part, and therefore k = 0 yields an asymptotically stable spiral point. For $x_c = .80485$ the eigenvalues are also complex conjugates, but with positive real parts, so k = .5 yields an unstable spiral point.

16b. Letting k = .1, .2, .3, .4 in the cubic equation of part (a) and finding the corresponding eigenvalues from the matrix in part (a), we find that the real part of the eigenvalues change sign between k = .3 and k = .4. Continuing to iterate in this fashion we find that for k = .3464 that the real part of the eigenvalue is -.0002 while for k = .3465 the real part is .00005, which indicates $k_0 = .3465$ is the critical point for which the system changes from stable to unstable.

16d. You must plot the solution for values of k slightly less than k_0, found in part (c), to determine whether a limit cycle exists.

Section 9.8, Page 538

1a. From Eq (6), $\lambda = -8/3$ is clearly one eigenvalue and the other two may be found from $\lambda^2 + 11\lambda - 10(r-1) = 0$ using the quadratic formula.

1b. For $\lambda = \lambda_1$ we have

$$\begin{pmatrix} -10+8/3 & -10 & 0 \\ r & -1+8/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\xi}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ which requires } \boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = 0$$

and $\boldsymbol{\xi}_3$ arbitrary and thus $\boldsymbol{\xi}^{(1)} = (0,0,1)^T$.

For $\lambda = \lambda_3 = (-11 + \alpha)/2$, where $\alpha = \sqrt{81+40r}$, we have

$$\begin{pmatrix} -10+(11-\alpha)/2 & 10 & 0 \\ r & -1+(11-\alpha)/2 & 0 \\ 0 & 0 & -8/3+(11-\alpha)/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The last line implies $\xi_3 = 0$ and multiplying the first line by

$(-9+\alpha)/2$ we obtain $\begin{pmatrix} (81-\alpha^2)/4 & 10(-9+\alpha)/2 \\ r & (9-\alpha)/2 \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.
Substituting $\alpha^2 = 81+40r$ we have
$\begin{pmatrix} -10r & -10(9-\alpha)/2 \\ r & (9-\alpha)/2 \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Thus $\xi^{(3)} = \begin{pmatrix} 9 - \sqrt{81+40r} \\ -2r \\ 0 \end{pmatrix}$,
which is proportional to the answer given in the text.
Similar calculations give $\boldsymbol{\xi}^{(2)}$.

1c. Simply substitute r = 28 into the answers in parts (a) and (b).

2a. The calculations are somewhat simplified if you let $x = \beta + u$, $y = \beta + v$, and $z = (r-1)+w$, where $\beta = \sqrt{8(r-1)/3}$. An alternate approach is to extend Eq.(13) of Section 9.3, which is:
$$\begin{pmatrix}u\\v\\w\end{pmatrix}' = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix}_{(x_0,y_0,z_0)} \begin{pmatrix}u\\v\\w\end{pmatrix}.$$
In this example $F = -10x + 10y$, $G = rx - y - xz$ and $H = -8z/3 + xy$ and thus
$\begin{pmatrix}u\\v\\w\end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & -x_0 \\ y_0 & x_0 & -8/3 \end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}$, which is Eq.(8) for P_2 since
$x_0 = y_0 = \sqrt{8(r-1)/3}$.

2b. Eq.(9) is found by evaluating $\begin{vmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & -\beta \\ \beta & \beta & -8/3-\lambda \end{vmatrix} = 0.$

2c. If r = 28, then Eq.(9) is $3\lambda^3 + 41\lambda^2 + 304\lambda + 4320 = 0$, which has the roots -13.8546 and .093956 ± 10.1945i.

3b. If r_1, r_2, r_3 are the three roots of a cubic polynomial, then the polynomial can be factored as $(x-r_1)(x-r_2)(x-r_3)$. Expanding this and equating to the given polynomial we have $A = -(r_1+r_2+r_3)$, $B = r_1r_2 + r_1r_3 + r_2r_3$ and $C = -r_1r_2r_3$. We are interested in the case when the real part of the complex conjugate roots changes sign. Thus let $r_2 = \alpha+i\beta$ and $r_3 = \alpha-i\beta$, which yields
$A = -(r_1+2\alpha)$, $B = 2\alpha r_1 + \alpha^2 + \beta^2$ and $C = -r_1(\alpha^2+\beta^2)$.
Hence, if AB = C, we have

$-(r_1+2\alpha)(2\alpha r_1+\alpha^2+\beta^2) = -r_1(\alpha^2+\beta^2)$ or
$-2\alpha[r_1^2 + 2\alpha r_1 + (\alpha^2+\beta^2)] = 0$ or $-2\alpha[(r_1+\alpha)^2+\beta^2] = 0$.
Since the square bracket term is positive, we conclude that if AB = C, then $\alpha = 0$. That is, the conjugate complex roots are pure imaginary. Note that the converse is also true. That is, if the conjugate complex roots are pure imaginary then AB = C.

3c. Comparing Eq.(9) to that of part b, we have A = 41/3, B = 8(r+10)/3 and C = 160(r-1)/3. Thus AB = C yields r = 470/19.

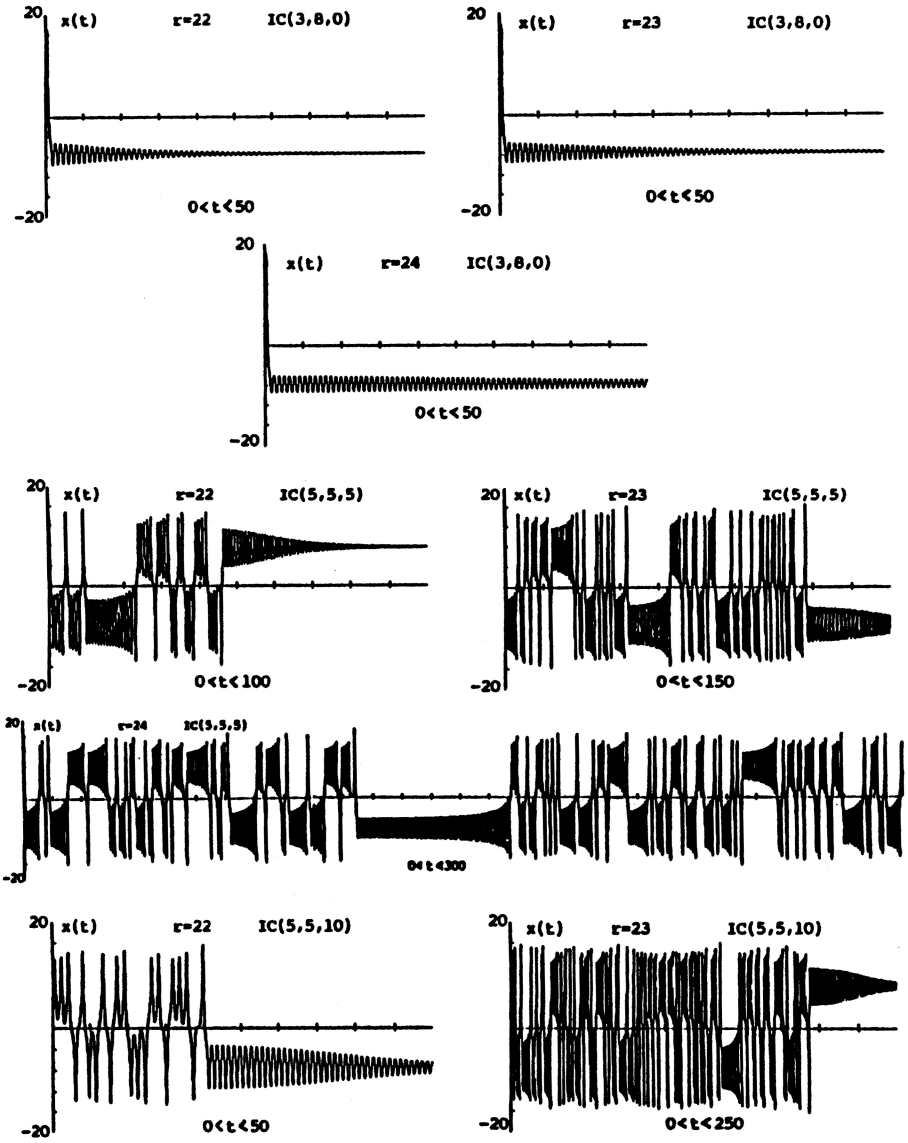
4. We have $\dot{V} = 2x[\sigma(-x+y)] + 2\sigma y[rx-y-xz] + 2\sigma z[-bz+xy]$
$= -2\sigma x^2 + 2\sigma xy + 2\sigma rxy - 2\sigma y^2 - 2\sigma bz^2$
$= 2\sigma\{-[x^2-(r+1)xy+y^2]-bz^2\}$. For $r < 1$, the term in the square brackets remains positive for all values of x and y, by Theorem 9.6.4, and thus $\dot{V}$ is negative definite. Thus, by the extension of Theorem 9.6.1 to three equations, we conclude that the origin is an asymptotically stable critical point.

5a. $V = rx^2 + \sigma y^2 + \sigma(z-2r)^2 = c > 0$ yields $\frac{dv}{dt} = 2rx[\sigma(-x+y)] + 2\sigma y(rx-y-xz) + 2\sigma(z-2r)(-bz+xy)$. Thus
$\dot{V} = -2\sigma[rx^2+y^2 + b(z^2 - 2rz)] = -2\sigma[rx^2 + y^2 + b(z-r)^2 - br^2]$.

5b. From the proof of Theorem 9.6.1, we find that we need to show that $\dot{V}$, as found in part a, is always negative as it crosses V(x,y,z) = c. (Actually, we need to use the extension of Theorem 9.6.1 to three equations, but the proof is very similar using the vector calculus approach.) From part a we see that
$\dot{V} < 0$ if $rx^2 + y^2 + b(z-r)^2 > br^2$, which holds if (x,y,z) lies outside the ellipsoid $\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} = 1$. (i)
Thus we need to choose c such that V = c lies outside Eq.(i). Writing V = c in the form of Eq.(i) we obtain the ellipsoid $\frac{x^2}{c/r} + \frac{y^2}{c/\sigma} + \frac{(z-2r)^2}{c/\sigma} = 1$. (ii) Now let $M = \max(\sqrt{br}, r\sqrt{b}, r)$, then the ellipsoid (i) is contained inside the sphere S1: $\frac{x^2}{M^2} + \frac{y^2}{M^2} + \frac{(z-r)^2}{M^2} = 1$.
Let S2 be a sphere centered at (0,0,2r) with radius

M+r: $\frac{x^2}{(M+r)^2} + \frac{y^2}{(M+r)^2} + \frac{(z-2r)}{(M+r^2)} = 1$, then S1 is contained in S2. Thus, if we choose c, in Eq.(ii), such that $\frac{c}{r} > (M+r)^2$ and $\frac{c}{\sigma} > (M+r)^2$, then $\dot{V} < 0$ as the trajectory crosses $V(x,y,z) = c$. Note that this is a sufficient condition and there may be many other "better" choices using different techniques.

8b. Several cases are shown. Results may vary, particularly for r = 24, due to the closeness of r to $r_3 \cong 24.06$.



CHAPTER 10

Section 10.1 Page 547

2. $y(x) = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$ is the general solution of the D.E. Thus $y'(x) = -\sqrt{2}\,c_1\sin\sqrt{2}x + \sqrt{2}\,c_2\cos\sqrt{2}x$ and hence $y'(0) = \sqrt{2}\,c_2 = 1$, which gives $c_2 = 1/\sqrt{2}$. Now, $y'(\pi) = -\sqrt{2}\,c_1\sin\sqrt{2}\pi + \cos\sqrt{2}\pi = 0$ then yields $c_1 = \dfrac{\cos\sqrt{2}\pi}{\sqrt{2}\sin\sqrt{2}\pi} = \cot\sqrt{2}\pi/\sqrt{2}$. Thus the desired solution is $y = (\cot\sqrt{2}\pi\cos\sqrt{2}x + \sin\sqrt{2}x)/\sqrt{2}$.

3. We have $y(x) = c_1\cos x + c_2\sin x$ as the general solution and hence $y(0) = c_1 = 0$ and $y(L) = c_2\sin L = 0$. If $\sin L \neq 0$, then $c_2 = 0$ and $y(x) = 0$ is the only solution. If $\sin L = 0$, then $y(x) = c_2\sin x$ is a solution for arbitrary c_2.

7. $y(x) = c_1\cos 2x + c_2\sin 2x$ is the solution of the related homogeneous equation and $y_p(x) = \frac{1}{3}\cos x$ is a particular solution, yielding $y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\cos x$ as the general solution of the D.E. Thus $y(0) = c_1 + \frac{1}{3} = 0$ and $y(\pi) = c_1 - \frac{1}{3} = 0$ and hence there is no solution since there is no value of c_1 that will satisfy both boundary conditions.

11. If $\lambda < 0$, the general solution of the D.E. is $y = c_1\sinh\sqrt{\mu}x + c_2\cosh\sqrt{\mu}x$ where $-\lambda = \mu$. The two B.C. require that $c_2 = 0$ and $c_1 = 0$ so $\lambda < 0$ is not an eigenvalue. If $\lambda = 0$, the general solution of the D.E. is $y = c_1 + c_2x$. The B.C. require that $c_1 = 0$, $c_2 = 0$ so again $\lambda = 0$ is not an eigenvalue. If $\lambda > 0$, the general solution of the D.E. is $y = c_1\sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x$. The B.C. require that $c_2 = 0$ and $\sqrt{\lambda}\,c_1\cos\sqrt{\lambda}\pi = 0$. The second condition is satisfied for $\lambda \neq 0$ and $c_1 \neq 0$ if $\sqrt{\lambda}\pi = (2_{n-1})\pi/2$, $n = 1,2,\ldots$. Thus the eigenvalues are $\lambda_n = (2n-1)^2/4$, $n = 1,2,3\ldots$ with the corresponding eigenfunctions $y_n(x) = \sin[(2n-1)x/2]$, $n = 1,2,3\ldots$.

15. For $\lambda < 0$ there are no eigenvalues, as shown in Problem 11. For $\lambda = 0$ we have $y(x) = c_1 + c_2x$, so $y'(0) = c_2 = 0$ and

$y'(\pi) = c_2 = 0$, and thus $\lambda = 0$ is an eigenvalue, with $y_0(x) = 1$ as the eigenfunction. For $\lambda > 0$ we again have $y(x) = c_1\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$, so $y'(0) = \sqrt{\lambda}\,c_1 = 0$ and $y'(L) = -c_2\sqrt{\lambda}\sin\sqrt{\lambda}\,L = 0$. We know $\lambda > 0$, in this case, so the eigenvalues are given by $\sin\sqrt{\lambda}\,L = 0$ or $\sqrt{\lambda}\,L = n\pi$. Thus $\lambda_n = (n\pi/L)^2$ and $y_n(x) = \cos(n\pi x/L)$ for $n = 1,2,3\ldots$.

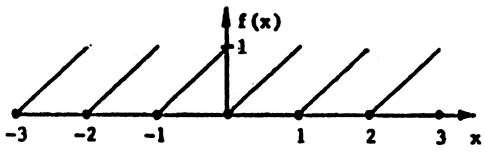
Section 10.2, Page 555

3. We look for values of T for which $\sinh 2(x+T) = \sinh 2x$ for all x. Expanding the left side of this equation gives $\sinh 2x\cosh 2T + \cosh 2x\sinh 2T = \sinh 2x$, which will be satisfied for all x if we can choose T so that $\cosh 2T = 1$ and $\sinh 2T = 0$. The only value of T satisfying these two constraints is $T = 0$. Since T is not positive we conclude that the function $\sinh 2x$ is not periodic.

5. We look for values of T for which $\tan\pi(x+T) = \tan\pi x$. Expanding the left side gives $\tan\pi(x+T) = (\tan\pi x + \tan\pi T)/(1-\tan\pi x\tan\pi T)$ which is equal to $\tan\pi x$ only for $\tan\pi T = 0$. The only positive solutions of this last equation are $T = 1,2,3\ldots$ and hence $\tan\pi x$ is periodic with fundamental period $T = 1$.

7. To start, let $n = 0$, then $f(x) = \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}$; for $n = 1$, $f(x) = \begin{cases} 0 & 1 \le x < 2 \\ 1 & 2 \le x < 3 \end{cases}$; and for $n = 2$, $f(x) = \begin{cases} 0 & 3 \le x < 4 \\ 1 & 4 \le x < 5 \end{cases}$. By continuing in this fashion, and drawing a graph, it can be seen that $T = 2$.

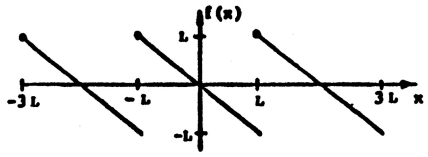
10. The graph of f(x) is:



We note that f(x) is a straight line with a slope of one in any interval. Thus f(x) has the form x+b in any interval for the correct value of b. Since f(x+2) = f(x), we may set x = -1/2 to obtain f(3/2) = f(-1/2). Noting that 3/2 is on the interval $1 < x < 2$[f(3/2) = 3/2 + b] and that -1/2 is on the interval $-1 < x < 0$[f(-1/2) = -1/2 + 1], we conclude that 3/2 + b = -1/2 + 1, or b = -1 for the interval $1 < x < 2$. For the interval $8 < x < 9$ we have f(x+8) = f(x+6) = ... = f(x) by successive applications of the periodicity condition. Thus for x = 1/2 we have f(17/2) = f(1/2) or 17/2 + b = 1/2 so b = -8 on the interval $8 < x < 9$.

In Problems 13 through 18 it is often necessary to use integration by parts to evaluate the coefficients, although all the details will not be shown here.

13a. The function represents a sawtooth wave. It is periodic with period 2L.



13b. The Fourier series is of the form

$f(x) = a_0/2 + \sum_{m=1}^{\infty} (a_m \cos m\pi x/L + b_m \sin m\pi x/L)$, where the coefficients are computed from Eqs. (13) and (14). Substituting for f(x) in these equations yields $a_0 = (1/L)\int_{-L}^{L}(-x)dx = 0$ and $a_m = (1/L)\int_{-L}^{L}(-x)\cos(m\pi x/L)dx = 0$, m = 1,2... (these can be shown by direct integration, or using the fact that $\int_{-a}^{a} g(x)dx = 0$ when g(x) is an odd function). Finally,

$b_m = (1/L)\int_{-L}^{L}(-x)\sin(m\pi x/L)dx$

$= (x/m\pi)\cos(m\pi x/L)\Big|_{-L}^{L} - (1/m\pi)\int_{-L}^{L}\cos(m\pi x/L)dx$

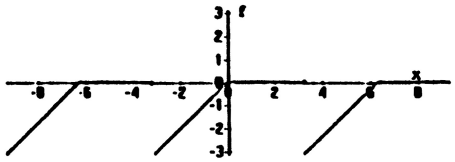
$= (2L\cos m\pi)/m\pi - (L/m^2\pi^2)\sin(m\pi x/L)\Big|_{-L}^{L} = 2L(-1)^m/m\pi$

Substituting these terms in the above Fourier series for f(x) yields the desired answer.

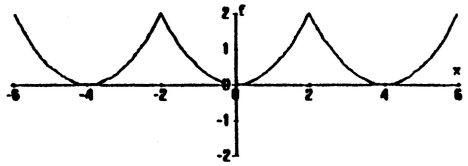
15a. See the next page.

15b. In this case f(x) is periodic of period 2π and thus $L = \pi$ in Eqs. (9), (13,) and (14). The constant a_0 is found to be $a_0 = (1/\pi)\int_{-\pi}^{0} x dx = -\pi/2$ since f(x) is zero on the interval $[0,\pi]$. Likewise $a_n = (1/\pi)\int_{-\pi}^{0} x\cos nx dx = [1 - (-1)^n]/n^2\pi$, using integration by parts and recalling that $\cos n\pi = (-1)^n$. Thus $a_n = 0$ for n even and $a_n = 2/n^2\pi$ for n odd, which may be written as $a_{2n-1} = 2/(2n-1)^2\pi$ since 2n-1 is always an odd number. In a similar fashion $b_n = (1/\pi)\int_{-\pi}^{0} x\sin nx dx = (-1)^{n+1}/n$ and thus the desired solution is obtained. Notice that in this case both cosine and sine terms appear in the Fourier series for the given f(x).

15a.



21a.



21b. $a_0 = \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2}dx = \frac{1}{12}x^3 \Big|_{-2}^{2} = \frac{4}{3}$, so $\frac{a_0}{2} = \frac{2}{3}$ and

$$a_n = \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2}\cos\frac{n\pi x}{2}dx$$

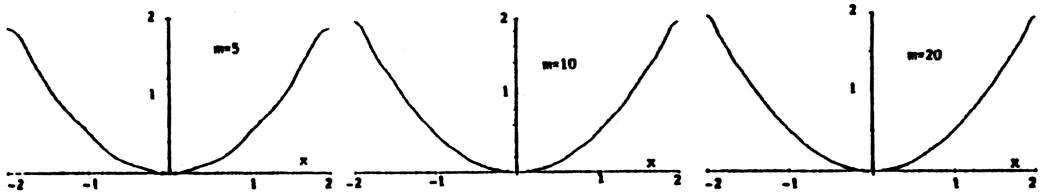
$$= \frac{1}{4}\left[\frac{2x^2}{n\pi}\sin\frac{n\pi x}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]\Big|_{-2}^{2}$$

$$= (8/n^2\pi^2)\cos(n\pi) = (-1)^n 8/n^2\pi^2$$

where the second line for a_n is found by integration by parts or a computer algebra system. Similarly,

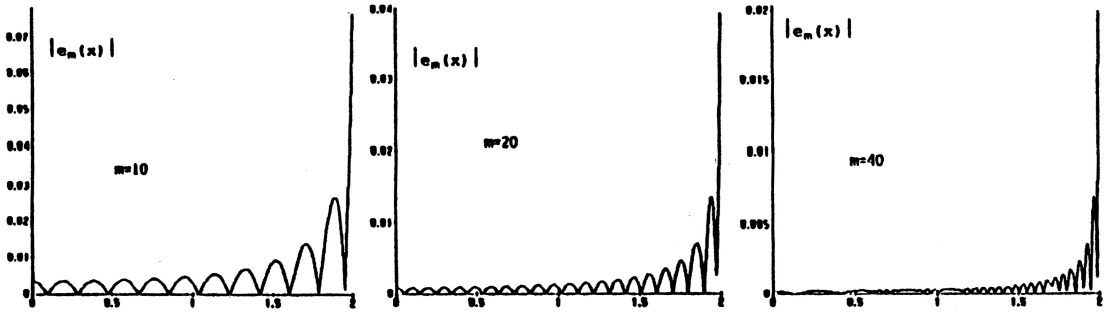
$b_n = \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2}\sin\frac{n\pi x}{2}dx = 0$, since $x^2\sin\frac{n\pi x}{2}$ is an odd function. Thus $f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$.

21c. As in Eq. (27), we have $s_m(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{m} \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$



21d. Observing the graphs we see that the Fourier series converges quite rapidly, except, at x = -2 and x = 2, where there is a sharp “point” in the periodic function.

25.



27a. First we have $\int_T^{a+T} g(x)dx = \int_0^a g(s)ds$ by letting $x = s + T$ in the left integral. Now, if $0 \le a \le T$, then from elementary calculus we know that $\int_a^{a+T} g(x)dx = \int_a^T g(x)dx + \int_T^{a+T} g(x)dx = \int_a^T g(x)dx + \int_0^a g(x)dx$ using the equality derived above. This last sum is $\int_0^T g(x)dx$ and thus we have the desired result.

Section 10.3, Page 562

2a. Substituting for f(x) in Eqs.(2) and (3) with $L = \pi$ yields $a_0 = (1/\pi)\int_0^\pi x dx = \pi/2$;

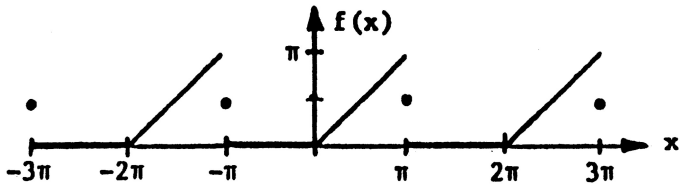
$a_m = (1/\pi)\int_0^\pi x\cos mx dx = (\cos m\pi - 1)/\pi m^2 = 0$ for m even and

$= -2/\pi m^2$ for m odd; and

$b_m = (1/\pi)\int_0^\pi x\sin mx dx = -(\pi\cos m\pi)/m\pi = (-1)^{m+1}/m$,

$m = 1,2\ldots$. Substituting these values into Eq.(1) with $L = \pi$ yields the desired solution.

2b. The function to which the series converges is indicated in the figure and is periodic with period 2π. Note that



the Fourier series converges to $\pi/2$ at $x = -\pi, \pi$, etc., even though the function is defined to be zero there. This value $(\pi/2)$ represents the mean value of the left and right hand limits at those points. In $(-\pi, 0)$, $f(x) = 0$ and $f'(x) = 0$ so both f and f′ are continuous and have finite limits as $x \to -\pi$ from the right and as $x \to 0$ from the left. In $(0, \pi)$, $f(x) = x$ and $f'(x) = 1$ and again both f and f′ are continuous and have limits as $x \to 0$ from the right and as $x \to \pi$ from the left. Since f and f′ are piecewise continuous on $[-\pi, \pi]$ the conditions of the Fourier theorem are satisfied.

4a. Substituting for f(x) in Eqs.(2) and (3), with $L = 1$

yields $a_0 = \int_{-1}^1 (1-x^2)dx = 4/3$;

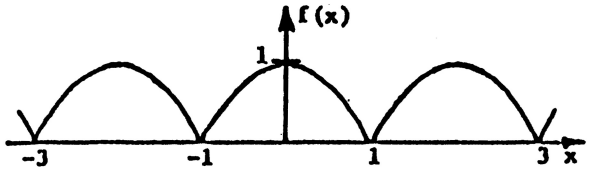
$a_n = \int_{-1}^{1}(1-x^2)\cos n\pi x\,dx = (2/n\pi)\int_{-1}^{1} x\sin n\pi x\,dx$

$= (-2/n^2\pi^2)[x\cos n\pi x\Big|_{-1}^{1} - \int_{-1}^{1}\cos n\pi x\,dx]$

$= 4(-1)^{n+1}/n^2\pi^2$; and

$b_n = \int_{-1}^{1}(1-x^2)\sin n\pi x\,dx = 0$. Substituting these values into Eq.(1) gives the desired series.

4b. The function to which the series converges is shown in the figure and is periodic of fundamental period 2. In $[-1,1]$ $f(x)$ and $f'(x) = -2x$ are both continuous and have finite limits as the endpoints of the interval are approached from within the interval.



7a. As in Problem 15, Section 10.2, we have

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2\cos(2n-1)x}{\pi(2n-1)^2} + \frac{(-1)^{n+1}\sin nx}{n}\right].$$

7b. $$e_n(x) = f(x) + \frac{\pi}{4} - \sum_{k=1}^{n}\left[\frac{2\cos(2k-1)x}{\pi(2k-1)^2} + \frac{(-1)^{k+1}\sin kx}{k}\right].$$

Using a computer algebra system, we find that for $n = 5$, 10 and 20 the maximum error occurs at $x = -\pi$ in each case and is 1.6025, 1.5867 and 1.5787 respectively. Note that the author's n values are 10, 20 and 40, since he has included the zero cosine coefficient terms and the sine terms are all zero at $x = -\pi$.

7c. It's not possible in this case, due to Gibb's phenomenon, to satisfy $|e_n(x)| \le 0.01$ for all x.

12a. $a_0 = \int_{-1}^{1}(x-x^3)\,dx = 0$ and $a_n = \int_{-1}^{1}(x-x^3)\cos n\pi x\,dx = 0$ since $(x-x^3)$ and $(x-x^3)\cos n\pi x$ are odd functions.

$$b_n = \int_{-1}^{1}(x-x^3)\sin n\pi x\,dx$$

$$= \left[\frac{x^3}{n\pi}\cos n\pi x - \frac{3x^2}{n^2\pi^2}\sin n\pi x - \frac{(n^2\pi^2+6)}{n^3\pi^3}x\cos n\pi x + \frac{(n^2\pi^2+6)}{n^4\pi^4}\sin n\pi x\right]_{-1}^{1}$$

$$= \frac{-12}{n^3\pi^3}\cos n\pi, \text{ so } f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x.$$

12b. $e_n(x) = f(x) + \frac{12}{\pi^3}\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\sin k\pi x$. These errors will be much smaller than in the earlier problems due to the n^3 factor in the denominator. Convergence is much more rapid in this case.

14. The solution to the corresponding homogeneous equation is found by methods presented in Section 3.4 and is $y(t) = c_1\cos\omega t + c_2\sin\omega t$. For the nonhomogeneous terms we use the method of superposition and consider the sequence of equations $y_n'' + \omega^2 y_n = b_n \sin nt$ for $n = 1,2,3\ldots$. If $\omega > 0$ is not equal to an integer, then the particular solution to this last equation has the form $Y_n = a_n\cos nt + d_n\sin nt$, as previously discussed in Section 3.6. Substituting this form for Y_n into the equation and solving, we find $a_n = 0$ and $d_n = b_n/(\omega^2-n^2)$. Thus the formal general solution of the original nonhomogeneous D.E. is

$y(t) = c_1\cos\omega t + c_2\sin\omega t + \sum_{n=1}^{\infty} b_n(\sin nt)/(\omega^2-n^2)$, where we have superimposed all the Y_n terms to obtain the infinite sum. To evalute c_1 and c_2 we set $t = 0$ to obtain $y(0) = c_1 = 0$ and

$y'(0) = \omega c_2 + \sum_{n=1}^{\infty} n b_n/(\omega^2-n^2) = 0$ where we have formally differentiated the infinite series term by term and evaluated it at $t = 0$. (Differentiation of a Fourier Series has not been justified yet and thus we can only consider this method a formal solution at this point).

Thus $c_2 = -(1/\omega)\sum_{n=1}^{\infty} n b_n/(\omega^2-n^2)$, which when substituted into the above series yields the desired solution.

If $\omega = m$, a positive integer, then the particular solution of $y_m'' + m^2 y_m = b_m \sin mt$ has the form $Y_m = t(a_m\cos mt + d_m\sin mt)$ since $\sin mt$ is a solution of the related homogeneous D.E. Substituting Y_m into the D.E. yields $a_m = -b_m/2m$ and $d_m = 0$ and thus the general solution of the D.E. (when $\omega = m$) is now $y(t) = c_1\cos mt$

$$+ c_2\sin mt - b_m t(\cos mt)/2m + \sum_{n=1,\, n\neq m}^{\infty} b_n(\sin nt)/(m^2-n^2).$$

To evaluate c_1 and c_2 we set $y(0) = 0 = c_1$ and

$y'(0) = c_2 m - b_m/2m + \sum_{n=1, n\neq m}^{\infty} b_n n/(m^2-n^2) = 0$. Thus

$c_2 = b_m/2m^2 - \sum_{n=1, n\neq m}^{\infty} b_n n/m(m^2-n^2)$, which when substituted into the equation for y(t) yields the desired solution.

15. In order to use the results of Problem 14, we must first find the Fourier series for the given f(t). Using Eqs.(2) and (3) with $L = \pi$, we find that

$a_0 = (1/\pi)\int_0^{\pi} dx - (1/\pi)\int_{\pi}^{2\pi} dx = 0;$

$a_n = (1/\pi)\int_0^{\pi} \cos nx\,dx - (1/\pi)\int_{\pi}^{2\pi} \cos nx\,dx = 0;$ and

$b_n = (1/\pi)\int_0^{\pi} \sin nx\,dx - (1/\pi)\int_{\pi}^{2\pi} \sin nx\,dx = 0$ for n even and $= 4/n\pi$ for n odd. Thus

$f(t) = (4/\pi)\sum_{n=1}^{\infty} \sin(2n-1)t/(2n-1)$. Comparing this to the forcing function of Problem 14 we see that b_n of Problem 14 has the specific values $b_{2n} = 0$ and $b_{2n-1} = (4/\pi)/(2n-1)$ in this example. Substituting these into the answer to Problem 14 yields the desired solution. Note that we have asumed ω is not a positive integer. Note also, that if the solution to Problem 14 is not available, the procedure for solving this problem would be exactly the same as shown in Problem 14.

16. From Problem 8, the Fourier series for f(t) is given by

$f(t) = 1/2 + (4/\pi^2)\sum_{n=1}^{\infty} \cos(2n-1)\pi t/(2n-1)^2$ and thus we may not use the form of the answer in Problem 14. The procedure outlined there, however, is applicable and will yield the desired solution.

18a. We will assume f(x) is continuous for this part. For the case where f(x) has jump discontinuities, a more detailed proof can be developed, as shown in part b. From Eq.(3) we have $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$. If we let u = f(x) and $dv = \sin\frac{n\pi x}{L}dx$, then du = f'(x)dx and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$. Thus

$$b_n = \frac{1}{L}\left[\frac{-L}{n\pi} f(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx\right]$$

$$= -\frac{1}{n\pi}[f(L)\cos n\pi - f(-L)\cos(-n\pi)] + \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx$$

$$= \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx, \text{ since } f(L) = f(-L) \text{ and}$$

$\cos(-n\pi) = \cos n\pi$. Hence $nb_n = \frac{1}{\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}dx$, which exists for all n since $f'(x)$ is piecewise continuous. Thus nb_n is bounded as $n \to \infty$. Likewise, for a_n, we obtain $na_n = -\frac{1}{\pi}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}dx$ and hence na_n is also bounded as $n \to \infty$.

18b. Note that f and f′ are continuous at all points where f″ is continuous. Let $x_1, \ldots, x_m$ be the points in (-L,L) where f″ is not continuous. By splitting up the interval of integration at these points, and integrating Eq.(3) by parts twice, we obtain

$$n^2 b_n = \frac{n}{\pi}\sum_{i=1}^{m}[f(x_i+)-f(x_i-)]\cos\frac{n\pi x_i}{L} - \frac{n}{\pi}[f(L-)-f(-L+)]\cos n\pi$$

$$-\frac{L}{\pi^2}\sum_{i=1}^{m}[f'(x_i+)-f'(x_1-)]\sin\frac{n\pi x_i}{L} - \frac{L}{\pi^2}\int_{-L}^{L} f''(x)\sin\frac{n\pi x}{L}dx, \text{ where}$$

we have used the fact that cosine is continuous. We want the first two terms on the right side to be zero, for otherwise they grow in magnitude with n. Hence f must be continuous throughout the closed interval [-L,L]. The last two terms are bounded, by the hypotheses on f′ and f″. Hence $n^2 b_n$ is bounded; similarly $n^2 a_n$ is bounded. Convergence of the Fourier series then follows by comparison with $\sum_{n=1}^{\infty} n^{-2}$.

Section 10.4, Page 570

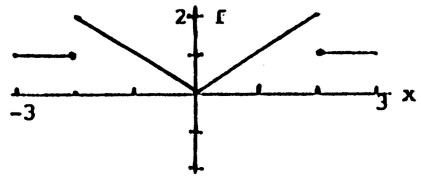
3. Let $f(x) = \tan 2x$, then

$$f(-x) = \tan(-2x) = \frac{\sin(-2x)}{\cos(-2x)} = \frac{-\sin(2x)}{\cos(2x)} = -\tan 2x = -f(x)$$

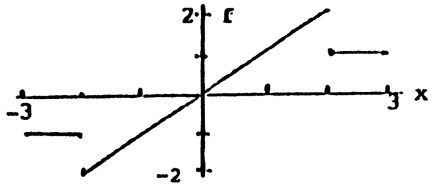
and thus tan2x is an odd function.

6. Let $f(x) = e^{-x}$, then $f(-x) = e^{x}$ so that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ and thus e^{-x} is neither even nor odd.

7.

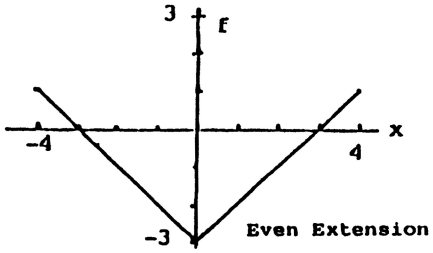


Even Extension

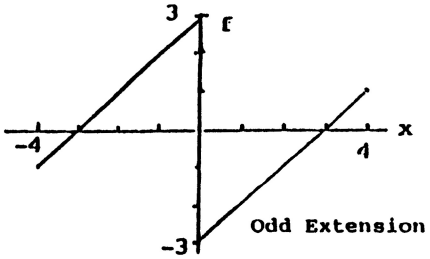


Odd Extension

10.



Even Extension



Odd Extension

13. By the hint $f(-x) = g(-x) + h(-x) = g(x) - h(x)$, since g is an even function and f is an odd function. Thus $f(x) + f(-x) = 2g(x)$ and hence $g(x) = [f(x) + f(-x)]/2$ defines g(x). Likewise $f(x)-f(-x) = g(x)-g(-x) + h(x)-h(-x) = 2h(x)$ and thus $h(x) = [f(x) - f(-x)]/2$.

All functions and their derivatives in Problems 14 through 30 are piecewise continuous on the given intervals and their extensions. Thus the Fourier Theorem applies in all cases.

14. For the cosine series we use the even extension of the function given in Eq.(13) and hence

$$f(x) = \begin{cases} 0 & -2 \le x < -1 \\ 1+x & -1 \le x < 0 \end{cases} \text{ on the interval } -2 \le x < 0.$$

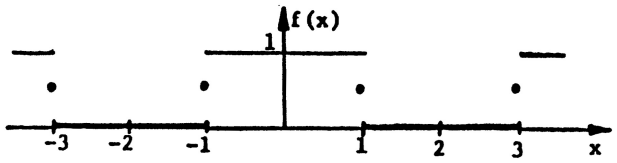
However, we don't really need this, as the coefficients in this case are given by Eqs.(7), which just use the original values for f(x) on $0 < x \le 2$. Applying Eqs.(7) we have L = 2 and thus $a_0 = (2/2)\int_0^1 (1-x)dx + (2/2)\int_1^2 0dx = 1/2$. Similarly,

$a_n = (2/2)\int_0^1 (1-x)\cos(n\pi x/2)dx = 4[1-\cos(n\pi/2)]/n^2\pi^2$ and $b_n = 0$. Substituting these values in the Fourier series yields the desired results.

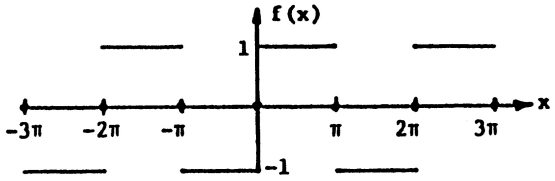
For the sine series, we use Eqs.(8) with L = 2. Thus $a_n = 0$ and

$b_n = (2/2)\int_0^1 (1-x)\sin(n\pi x/2)dx = 4[n\pi/2 - \sin(n\pi/2)]/n^2\pi^2$.

15. The graph of the function to which the series converges is shown in the figure. Using Eqs.(7) with L = 2 we have $a_0 = \int_0^1 dx = 1$ and $a_n = \int_0^1 \cos(n\pi x/2)dx = 2\sin(n\pi/2)/n\pi$. Thus $a_n = 0$ for n even, $a_n = 2/n\pi$ for n = 1,5,9,...and $a_n = -2/n\pi$ for n = 3,7,11,... . Hence we may write $a_{2n} = 0$ and $a_{2n-1} = 2(-1)^{n+1}/(2n-1)\pi$, which when substituted into the series gives the desired answer.

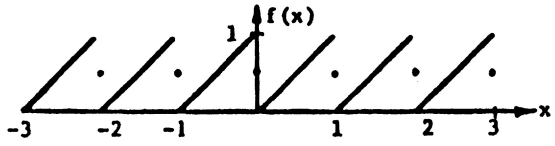


18. The graph of the function to which the series converges is indicated in the figure.



Since we want a sine series, we use Eqs.(8) to find, with L = π, that $b_n = (2/\pi)\int_0^\pi \sin nx dx = 2[1-(-1)^n]/n\pi$ and thus $b_n = 0$ for n even and $b_n = 4/n\pi$ for n odd.

20. The graph of the function to which the series converges is shown in the figure.



We note that f(x) is specified over its entire fundamental period (T = 1) and hence we cannot extend f to make it either an odd or an even function. Using Eqs.(2) and (3) from Section 10.3 we have (L = 1/2) $a_0 = 2\int_0^1 x dx = 1$, $a_n = 2\int_0^1 x\cos(2n\pi x)dx = 0$ and

$b_n = 2\int_0^1 x\sin(2n\pi x)dx = -1/n\pi$. [Note: We have used the

results of Problem 27 of Section 10.2 in writing these integrals. That is, if f(x) is periodic of period T, then every integral of f over an interval of length T has the same value. Thus we integrate from 0 to 1 here, rather than −1/2 to 1/2.] Substituting the above values into Eq.(1) of Section 10.3 yields the desired solution. It can also be observed from the above graph that g(x) = f(x) - 1/2 is an odd function. If Eqs.(8) are used with g(x), then it is found that

g(x) = (-1/π) $\sum_{n=1}^{\infty}$ sin(2nπx)/n and thus we obtain the same series for f(x) as found above.

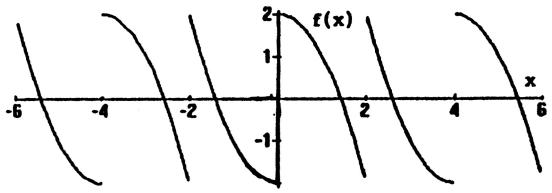
25a. $b_n = \frac{2}{2}\int_0^2 (2-x^2)\sin\frac{n\pi x}{2}\,dx$

$= \left[\frac{-2}{n\pi}(2-x^2)\cos\frac{n\pi x}{2} - \frac{8x}{n^2\pi^2}\sin\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\cos\frac{n\pi x}{2}\right]\Big|_0^2$

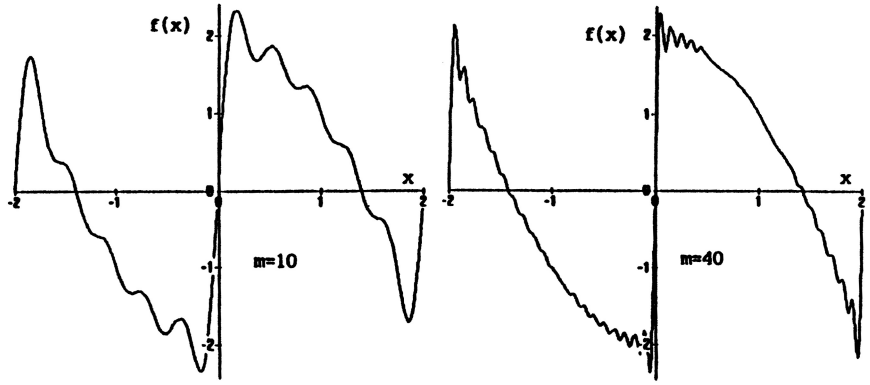
$= \frac{4}{n\pi}(1+\cos n\pi) + \frac{16}{n^3\pi^3}(1-\cos n\pi)$ and thus

$f(x) = \sum_{n=1}^{\infty}\left(\frac{4n^2\pi^2(1+\cos n\pi) + 16(1-\cos n\pi)}{n^3\pi^3}\right)\sin\frac{n\pi x}{2}$

25b.



25c.



28b. For the cosine series(even extension) we have

$a_0 = \frac{2}{2}\int_0^1 x\,dx = \frac{1}{2}$

$$a_n = \frac{2}{2}\int_0^1 x\cos\frac{n\pi x}{2}\,dx = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Big|_0^1$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}, \text{ so}$$

$$g(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4\cos(n\pi/2) + 2n\pi\sin(n\pi/2) - 4}{n^2\pi^2}\cos\frac{n\pi x}{2}.$$

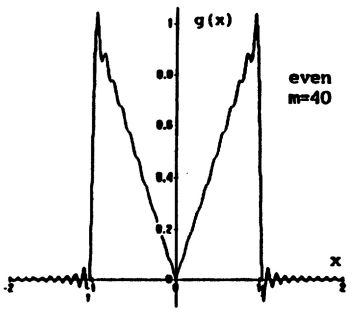
For the sine series (odd extension) we have

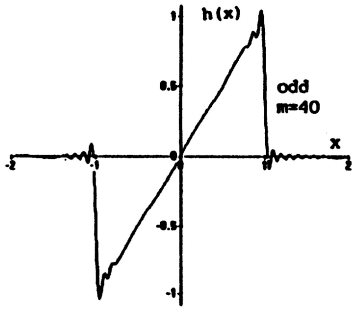
$$b_n = \frac{2}{2}\int_0^1 x\sin\frac{n\pi x}{2}\,dx = \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Big|_0^1$$

$$= \frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}, \text{ so}$$

$$h(x) = \sum_{n=1}^{\infty}\frac{4\sin(n\pi/2) - 2n\pi\cos(n\pi/2)}{n^2\pi^2}\sin\frac{n\pi x}{2}.$$

28c.





28d. The maximum error does not approach zero in either case, due to Gibb's phenomenon. Note that the coefficients in both series behave like 1/n as $n \to \infty$ since there is an n in the numerator.

31. We have $\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$. Now, if we let $x = -y$ in the first integral on the right, then $\int_{-L}^{0} f(x)dx = \int_{L}^{0} f(-y)(-dy) = \int_{0}^{L} f(-y)dy = -\int_{0}^{L} f(y)dy$. Thus $\int_{-L}^{L} f(x)dx = -\int_{0}^{L} f(y)dy + \int_{0}^{L} f(x)dx = 0$.

32. To prove property 2 let f_1 and f_2 be odd functions and let $f(x) = f_1(x) \pm f_2(x)$. Then $f(-x) = f_1(-x) \pm f_2(-x) = -f_1(x) \pm [-f_2(x)] = -f_1(x) \mp f_2(x) = -f(x)$, so $f(x)$ is odd. Now let $g(x)= f_1(x)f_2(x)$, then $g(-x) = f_1(-x)f_2(-x) = [-f_1(x)][-f_2(x)] = f_1(x)f_2(x) = g(x)$ and thus $g(x)$ is even. Finally, let $h(x) = f_1(x)/f_2(x)$ and hence $h(-x) = f_1(-x)/f_2(-x) = [-f_1(x)]/[-f_2(x)] = f_1(x)/f_2(x) = h(x)$, which says $h(x)$ is also even. Property 3 is proven in a similar manner.

34. Since $F(x) = \int_0^x f(t)dt$ we have $F(-x) = \int_0^{-x} f(t)dt = -\int_0^x f(-s)ds$ by letting $t = -s$. If f is an even function, $f(-s) = f(s)$, we then have $F(-x) = -\int_0^x f(s)ds = -F(x)$ from the original definition of F. Thus $F(x)$ is an odd function. The argument is similar if f is odd.

35. Set $x = L/2$ in Eq.(6) of Section 10.3. Since we know f is continuous at $L/2$, we may conclude, by the Fourier theorem, that the series will converge to $f(L/2) = L$ at this point. Thus we have

$L = L/2 + (2L/\pi) \sum_{n=1}^{\infty} (-1)^{n+1}/(2n-1)$, since $\sin[(2n-1)\pi/2] = (-1)^{n+1}$. Dividing by L and simplifying yields $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

37a. Multiplying both sides of the equation by $f(x)$ and integrating from 0 to L gives

$$\int_0^L [f(x)]^2 dx = \int_0^L [f(x) \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)]dx$$

$$= \sum_{n=1}^{\infty} b_n \int_0^L f(x)\sin(n\pi x/L)dx = (L/2) \sum_{n=1}^{\infty} b_n^2, \text{ by Eq.(8).}$$

This result is identical to that of Problem 17 of Section 10.3 if we set $a_n = 0$, $n = 0,1,2,\ldots$, since $\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{2}{L}\int_0^L [f(x)]^2 dx$. In a similar manner, it can be shown that

$$(2/L)\int_0^L [f(x)]^2 dx = a_0^2/2 + \sum_{n=1}^{\infty} a_n^2.$$

37b. Since $f(x) = x$ and $b_n = 2L(-1)^{n+1}/n\pi$ (from Eq.(9)), we obtain

$$(2/L)\int_0^L [f(x)]^2 dx = (2/L)\int_0^L x^2 dx = 2L^2/3 = \sum_{n=1}^{\infty} b_n^2 =$$

$$\sum_{n=1}^{\infty} 4L^2/n^2\pi^2 = 4L^2/\pi^2 \sum_{n=1}^{\infty} (1/n^2) \text{ or } \pi^2/6 = \sum_{n=1}^{\infty} (1/n^2).$$

38. We assume that the extensions of f and f′ are piecewise continuous on [-2L,2L]. Since f is an odd periodic function of fundamental period 4L it follows from properties 2 and 3 that f(x)cos(nπx/2L) is odd and f(x)sin(nπx/2L) is even. Thus the Fourier coefficients of f are given by Eqs.(8) with L replaced by 2L; that is $a_n = 0$, n = 0,1,2,... and $b_n = (2/2L)\int_0^{2L} f(x)\sin(n\pi x/2L)dx$, n = 1,2,... . The Fourier sine series for f is $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2L)$.

39. From Problem 38 we have $b_n = (1/L)\int_0^{2L} f(x)\sin(n\pi x/2L)dx$
$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx + (1/L)\int_L^{2L} f(2L-x)\sin(n\pi x/2L)dx$
$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx - (1/L)\int_L^{0} f(s)\sin[n\pi(2L-s)/2L]ds$
$= (1/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx - (1/L)\int_0^{L} f(s)\cos(n\pi)\sin(n\pi s/2L)ds$
and thus $b_n = 0$ for n even and $b_n = (2/L)\int_0^{L} f(x)\sin(n\pi x/2L)dx$ for n odd. The Fourier series for f is given in Problem 38, where the b_n are given above.

Section 10.5, Page 579

3. We seek solutions of the form u(x,t) = X(x)T(t). Substituting into the P.D.E. yields X″T + X′T′ + XT′ = X″T + (X′ + X)T′ = 0. Formally dividing by the quantity (X′ + X)T gives the equation X″/(X′ + X) = -T′/T in which the variables are separated. In order for this equation to be valid on the domain of u it is necessary that both sides be equal to the same constant λ. Hence X″/(X′ + X) = -T′/T = λ or equivalently, X″ - λ(X′ + X) = 0 and T′ + λT = 0.

5. We seek solutions of the form u(x,y) = X(x)Y(y). Substituting into the P.D.E. yields X″Y + (x+y)XY″ = X″Y + xXY″ + yXY″ = 0. Formally dividing by XY yields X″/X + xY″/Y + yY″/Y = 0. From this equation we see that the presence of the independent variable x multiplying the term u_{yy} in the original equation leads to the term xY″/Y when we attempt to separate the variables. It follows that the argument for a separation constant does not carry through and we cannot replace the P.D.E. by two O.D.E.

8. Following the procedures of Eqs.(5) through (8), we set $u(x,y) = X(x)T(t)$ in the P.D.E. to obtain $X''T = 4XT'$, or $X''X = 4T'/T$, which must be a constant. As stated in the text this separation constant must be $-\lambda^2$ (we choose $-\lambda^2$ so that when a square root is used later, the symbols are simpler) and thus $X'' + \lambda^2 X = 0$ and $T' + (\lambda^2/4)T = 0$. Now $u(0,t) = X(0)T(t) = 0$, for all $t > 0$, yields $X(0) = 0$, as discussed after Eq.(11) and similarly $u(2,t) = X(2)T(t) = 0$, for all $t > 0$, implies $X(2) = 0$. The D.E. for X has the solution $X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ and $X(0) = 0$ yields $C_1 = 0$. Setting $x = 2$ in the remaining form of X yields $X(2) = C_2 \sin 2\lambda = 0$, which has the solutions $2\lambda = n\pi$ or $\lambda = n\pi/2$, $n = 1,2,\ldots$. Note that we exclude $n = 0$ since then $\lambda = 0$ would yield $X(x) = 0$, which is unacceptable. Hence $X(x) = \sin(n\pi x/2)$, $n = 1,2,\ldots$. Finally, the solution of the D.E. for T yields $T(t) = \exp(-\lambda^2 t/4) = \exp(-n^2\pi^2 t/16)$. Thus we have found $u_n(x,t) = \exp(-n^2\pi^2 t/16)\sin(n\pi x/2)$. Setting $t = 0$ in this last expression indicates that $u_n(x,0)$ has, for the correct choices of n, the same form as the terms in $u(x,0)$, the initial condition. Using the principle of superposition we know that $u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t) + c_4 u_4(x,t)$ satisfies the P.D.E. and the B.C. and hence we let $t = 0$ to obtain $u(x,0) = c_1 u_1(x,0) + c_2 u_2(x,0) + c_4 u_4(x,0) = c_1 \sin\pi x/2 + c_2 \sin\pi x + c_4 \sin 2\pi x$. If we choose $c_1 = 2$, $c_2 = -1$ and $c_4 = 4$ then $u(x,0)$ here will match the given initial condition, and hence substituting these values in $u(x,t)$ above then gives the desired solution.

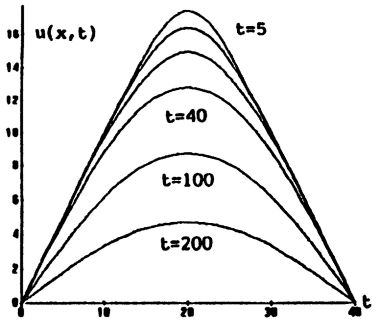
10. Since the B.C. for this heat conduction problem are $u(0,t) = u(40,t) = 0$, $t > 0$, the solution $u(x,t)$ is given by Eq.(19) with $\alpha^2 = 1$ cm^2/sec, $L = 40$ cm, and the coefficients c_n determined by Eq.(21) with the I.C. $u(x,0) = f(x) = x$, $0 \le x \le 20$; $= 40 - x$, $20 \le x \le 40$.

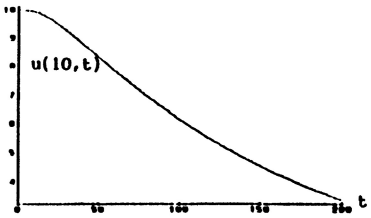
$$\text{Thus } c_n = \frac{1}{20}\left[\int_0^{20} x \sin\frac{n\pi x}{40}\,dx + \int_{20}^{40} (40-x)\sin\frac{n\pi x}{40}\,dx\right]$$

$$= \frac{160}{n^2\pi^2}\sin\frac{n\pi}{2}. \quad \text{It follows that}$$

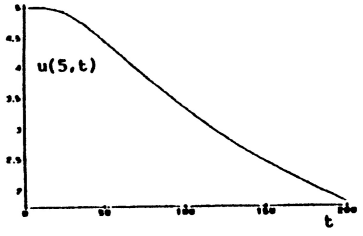
$$u(x,t) = \frac{160}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}e^{-n^2\pi^2 t/1600}\sin\frac{n\pi x}{40}.$$

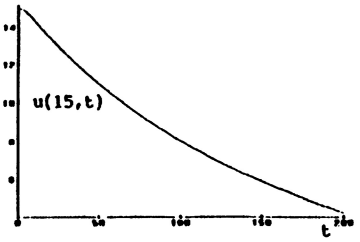
15a.



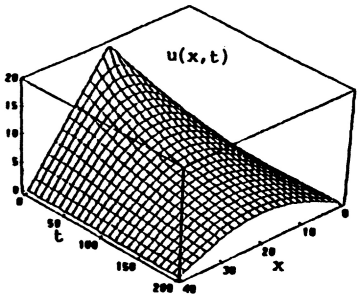


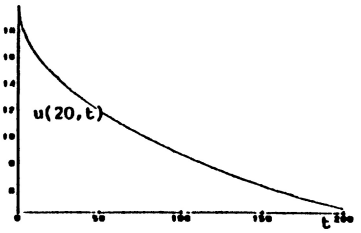
15b.





15c.





15d. As in Example 1, the maximum temperature will be at the midpoint, x = 20, and we use just the first term, since the others will be negligible for this temperature, since t is so large. Thus $u(20,t) = 1 = \frac{160}{\pi^2}\sin(\pi/2)e^{-\pi^2 t/1600}\sin(20\pi/40)$. Solving for t, we obtain $e^{-\pi^2 t/1600} = \pi^2/160$, or $t = \frac{1600}{\pi^2}\ln\frac{160}{\pi^2} = 451.60$ sec.

18a. Since the B.C. for this heat conduction problem are $u(0,t) = u(20,t) = 0$, $t > 0$, the solution $u(x,t)$ is given by Eq.(19) with L = 20 cm, and the coefficients c_n determined by Eq.(21) with the I.C. $u(x,0) = f(x) = 100^{o}C$. Thus $c_n = (1/10)\int_0^{20}(100)\sin(n\pi x/20)dx = -200[(-1)^n-1]/n\pi$ and hence

$c_{2n} = 0$ and $c_{2n-1} = 400/(2n-1)\pi$. Substituting these values into Eq.(19) yields

$$u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2\alpha^2 t/400}}{2n-1}\sin\frac{(2n-1)\pi x}{20}$$

18b. For aluminum, we have $\alpha^2 = .86$ cm^2/sec (from Table 10.5.1) and thus the first two terms give
$u(10,30) = \frac{400}{\pi}\{e^{-\pi^2(.86)30/400} - \frac{1}{3}e^{-9\pi^2(.86)30/400}\}$
$= 67.228^{o}$C. If an additional term is used, the temperature is increased by
$\frac{80}{\pi}e^{-25\pi^2(.86)30/400} = 3 \times 10^{-6}$ degrees C.

19b. Using only one term in the series for u(x,t), we must solve the equation $5 = (400/\pi)\exp[-\pi^2(.86)t/400]$ for t. Taking the logarithm of both sides and solving for t yields $t \cong 400\ln(80/\pi)/\pi^2(.86) = 152.56$ sec.

20. Applying the chain rule to partial differentiation of u with respect to x we see that $u_x = u_\xi\xi_x = u_\xi(l/L)$ and $u_{xx} = u_{\xi\xi}(l/L)^2$. Substituting $u_{\xi\xi}/L^2$ for u_{xx} in the heat equation gives $\alpha^2 u_{\xi\xi}/L^2 = u_t$ or $u_{\xi\xi} = (L^2/\alpha^2)u_t$. In a similar manner, $u_t = u_\tau\tau_t = u_\tau(\alpha^2/L^2)$ and hence $\frac{L^2}{\alpha^2}u_t = u_\tau$ and thus $u_{\xi\xi} = u_\tau$.

22. Substituting $u(x,y,t) = X(x)Y(y)T(t)$ in the P.D.E. yields $\alpha^2(X''YT + XY''T) = XYT'$, which is equivalent to $\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T}$. By keeping the independent variables x and y fixed and varying t we see that T'/α^2T must equal some constant σ_1 since the left side of the equation is fixed. Hence, $X''/X + Y''/Y = T'/\alpha^2T = \sigma_1$, or $X''/X = \sigma_1 - Y''/Y$ and $T' - \sigma_1\alpha^2T = 0$. By keeping x fixed and varying y in the equation involving X and Y we see that $\sigma_1 - Y''/Y$ must equal some constant σ_2 since the left side of the equation is fixed. Hence, $X''/X = \sigma_1 - Y''/Y = \sigma_2$ so $X'' - \sigma_2X = 0$ and $Y'' - (\sigma_1 - \sigma_2)Y = 0$. For $T' - \sigma_1\alpha^2T = 0$ to have solutions that remain bounded as $t \to \infty$ we must have $\sigma_1 < 0$. Thus, setting $\sigma_1 = -\lambda^2$, we have $T' + \alpha^2\lambda^2T = 0$. For $X'' - \sigma_2X = 0$ and homogeneous B.C., we conclude, as in Sect. 10.1, that $\sigma_2 < 0$ and, if we let

$\sigma_2 = -\mu^2$, then $X'' + \mu^2 X = 0$. With these choices for σ_1 and σ_2 we then have $Y'' + (\lambda^2 - \mu^2)Y = 0$.

Section 10.6, Page 588

3. The steady-state temperature distribution $v(x)$ must satisfy Eq.(9) and also satsify the B.C. $v_x(0) = 0$, $v(L) = 0$. The general solution of $v'' = 0$ is $v(x) = Ax + B$. The B.C. $v_x(0) = 0$ implies $A = 0$ and then $v(L) = 0$ implies $B = 0$, so the steady state solution is $v(x) = 0$.

7. Again, $v(x)$ must satisfy $v'' = 0$, $v'(0) - v(0) = 0$ and $v(L) = T$. The general solution of $v'' = 0$ is $v(x) = ax + b$, so $v(0) = b$, $v'(0) = a$ and $v(L) = T$. Thus $a - b = 0$ and $aL + b = T$, which give $a = b = T/(1+L)$. Hence $v(x) = T(x+1)/(L+1)$.

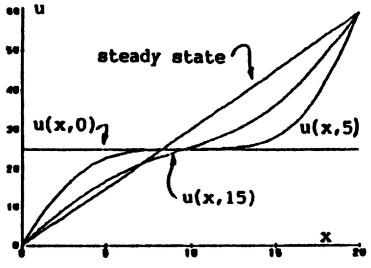
9a. Since the B.C. are not homogeneous, we must first find the steady state solution. Using Eqs.(9) and (10) we have $v'' = 0$ with $v(0) = 0$ and $v(20) = 60$, which has the solution $v(x) = 3x$. Thus the transient solution $w(x,t)$ satisfies the equations $\alpha^2 w_{xx} = w_t$, $w(0,t) = 0$, $w(20,t) = 0$ and $w(x,0) = 25 - 3x$, which are obtained from Eqs.(13) to (15). The solution of this problem is given by Eq.(4) with the c_n given by Eq.(6):

$c_n = \frac{1}{10}\int_0^{20}(25-3x)\sin\frac{n\pi x}{20}dx = (70\cos n\pi + 50)/n\pi$, and thus

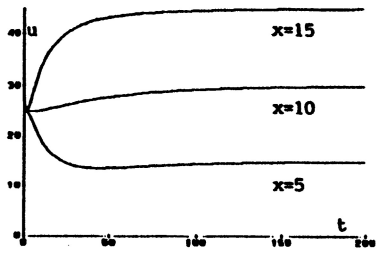
$$u(x,t) = 3x + \sum_{n=1}^{\infty}\frac{70\cos n\pi + 50}{n\pi}e^{-0.86n^2\pi^2 t/400}\sin\frac{n\pi x}{20}$$ since

$\alpha^2 = .86$ for aluminum.

9b.



9c.



9d. Using just the first term of the sum, we have $u(5,t) = 15 - \frac{20}{\pi}e^{-0.86\pi^2 t/400}\sin\frac{\pi}{4} = 15 \pm .15$. Thus

$\frac{20}{\pi}e^{-0.86\pi^2 t/400}\sin\frac{\pi}{4}$ = .15, which yields t = 160.30 sec.

To obtain the answer in the text, the first two terms of the sum must be used, which requires an equation solver to solve for t. Note that this reduces t by only .01 seconds.

12a. Since the B.C. are $u_x(0,t) = u_x(L,t) = 0$, $t > 0$, the solution u(x,t) is given by Eq.(35) with the coefficients c_n determined by Eq.(37). Substituting the I.C. $u(x,0) = f(x) = \sin(\pi x/L)$ into Eq.(37) yields

$c_0 = (2/L)\int_0^L \sin(\pi x/L)dx = 4/\pi$ and

$c_n = (2/L)\int_0^L \sin(\pi x/L)\cos(n\pi x/L)dx$

$= (1/L)\int_0^L \{\sin[(n+1)\pi x/L] - \sin[(n-1)\pi x/L]\}dx$

$= (1/\pi)\{[1 - \cos(n+1)\pi]/(n+1) - [1 - \cos(n-1)\pi]/(n-1)\}$

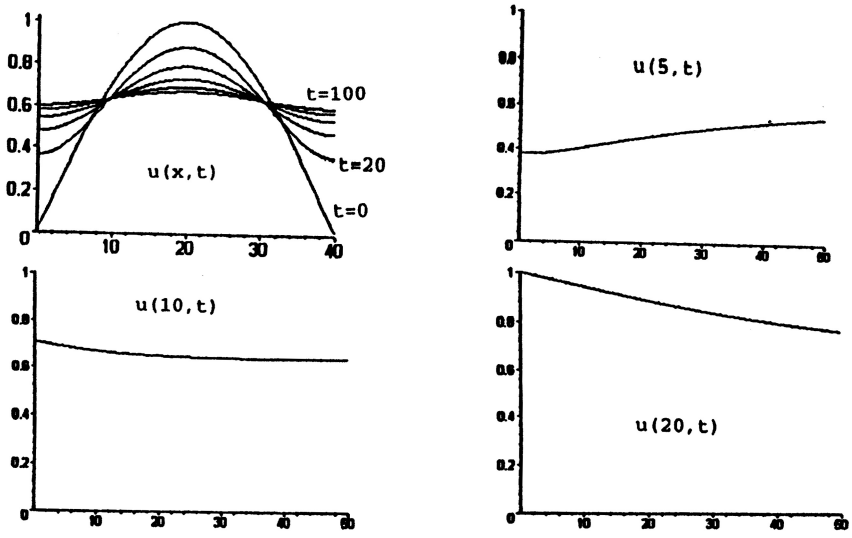
$= 0$, n odd; $= -4/(n^2-1)\pi$, n even. Thus

$$u(x,t) = 2/\pi - (4/\pi)\sum_{n=1}^{\infty} \exp[-4n^2\pi^2\alpha^2 t/L^2]\cos(2n\pi x/L)/(4n^2-1)$$

where we are now summing over even terms by setting n = 2n.

12b. As $t \to \infty$ we see that all terms in the series decay to zero except the constant term, $2/\pi$. Hence $\lim_{t\to\infty} u(x,t) = 2/\pi$.

12c.



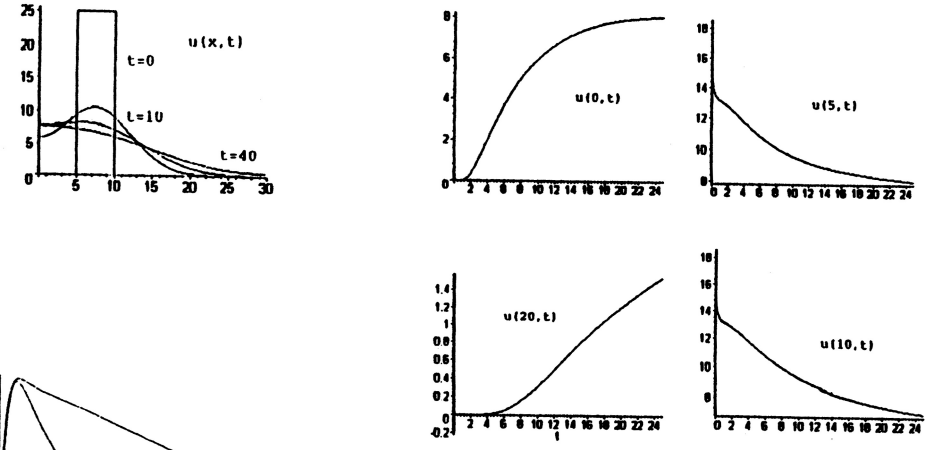
12d. The original heat in the rod is redistributed to give the final temperature distribution, since no heat is lost.

14a. Since the ends are insulated, the solution to this problem is given by Eq.(35), with $\alpha^2 = 1$ and L = 30, and Eq.(37). Thus $u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t/900)\cos(n\pi x/30)$, where

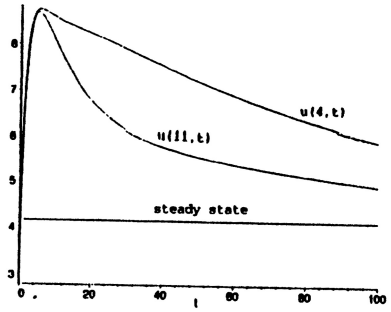
$c_0 = \frac{2}{30}\int_0^{30} f(x)dx = \frac{1}{15}\int_5^{10} 25dx = \frac{25}{3}$ and

$c_n = \frac{2}{30}\int_0^{3} f(x)\cos\frac{n\pi x}{30}dx = \frac{1}{15}\int_5^{10} 25\cos\frac{n\pi x}{30}dx = \frac{50}{n\pi}[\sin\frac{n\pi}{3} - \sin\frac{n\pi}{6}]$.

14b.



14c.



Although x = 4 and x = 1 are symmetrical to the initial temperature pulse, they are not symmetrical to the insulated end points.

15a. Substituting u(x,t) = X(x)T(t) into Eq.(1) leads to the two O.D.E. $X'' - \sigma X = 0$ and $T' - \alpha^2\sigma T = 0$. An argument similar to the one in the text implies that we must have X(0) = 0 and X′(L) = 0. Also, by assuming σ is real and considering the three cases σ < 0, σ = 0, and σ > 0 we can show that only the case σ < 0 leads to nontrivial solutions of $X'' - \sigma X = 0$ with X(0) = 0 and X′(L) = 0. Setting $\sigma = -\lambda^2$, we obtain $X(x) = k_1\sin\lambda x + k_2\cos\lambda x$. Now, $X(0) = 0 \rightarrow k_2 = 0$ and thus $X(x) = k_1\sin\lambda x$.

Differentiating and setting $x = L$ yields $\lambda k_1 \cos\lambda L = 0$. Since $\lambda = 0$ and $k_1 = 0$ lead to $u(x,t) = 0$, we must choose λ so that $\cos\lambda L = 0$, or $\lambda = (2n-1)\pi/2L$, $n = 1,2,3,\ldots$. These values for λ imply that $\sigma = -(2n-1)^2\pi^2/4L^2$ so the solutions $T(t)$ of $T' - \alpha^2\sigma T = 0$ are proportional to $\exp[-(2n-1)^2\pi^2\alpha^2 t/4L^2]$. Combining the above results leads to the desired set of fundamental solutions.

15b. In order to satisfy the I.C. $u(x,0) = f(x)$ we assume that $u(x,t)$ has the form $u(x,t) = \sum_{n=1}^{\infty} c_n \exp[-(2n-1)^2\pi^2\alpha^2 t/4L^2]\sin[(2n-1)\pi x/2L]$. The coefficients c_n are determined by the requirement that $u(x,0) = \sum_{n=1}^{\infty} c_n \sin[(2n-1)\pi x/2L] = f(x)$. Referring to Problem 39 of Section 10.4 reveals that such a representation for $f(x)$ is possible if we choose the coefficients $c_n = (2/L)\int_0^L f(x)\sin[(2n-1)\pi x/2L]dx$.

19. We must solve $v_1''(x) = 0$, $0 \le x \le a$ and $v_2''(x) = 0$, $a \le x \le L$ subject to the B.C. $v_1(0) = 0$, $v_2(L) = T$ and the continuity conditions at $x = a$. For the temperature to be continuous at $x = a$ we must have $v_1(a) = v_2(a)$ and for the rate of heat flow to be continuous we must have $-\kappa_1 A_1 v_1'(a) = -\kappa_2 A_2 v_2'(a)$, from Eq.(2) of Appendix A. The general solutions to the two O.D.E. are $v_1(x) = C_1 x + D_1$ and $v_2(x) = C_2 x + D_2$. By applying the boundary and continuity conditions we may solve for C_1, D_1, and C_2 and D_2 to obtain the desired solution.

Section 10.7, Page 600

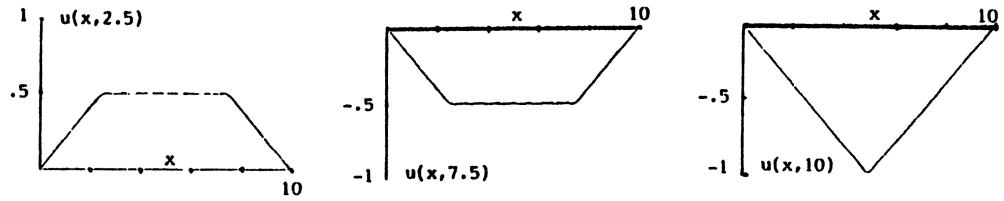
1a. Since the initial velocity is zero, the solution is given by Eq.(20) with the coefficients c_n given by Eq.(22). Substituting $f(x)$ into Eq.(22) yields

$$c_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2x}{L}\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L} \frac{2(L-x)}{L}\sin\frac{n\pi x}{L}dx\right]$$

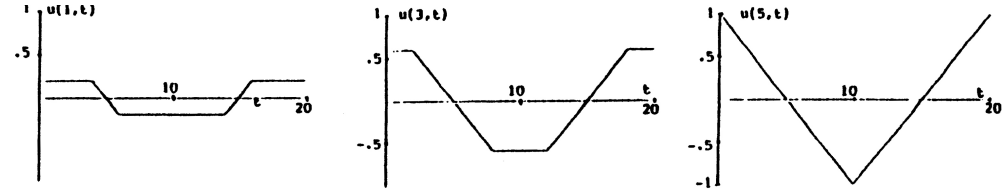
$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}.$$ Thus Eq. (20) becomes

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}.$$

1b.



1c.



1e. The graphs in part b can best be understood using Eq.(28) (or the results of Problems 13 and 14). The original triangular shape is composed of two similar triangles of 1/2 the height, one of which moves to the right, h(x-at), and the other to the left, h(x+at). Recalling that the series are periodic then gives the results shown. The graphs in part c can then be visualized from those in part b.

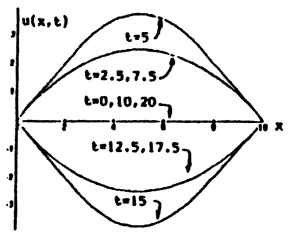
6a. The motion is governed by Eqs.(1), (3) and (31), and thus the solution is given by Eq.(34) where the k_n are given by Eq.(36):

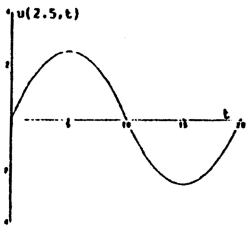
$$k_n = \frac{2}{n\pi a}\left[\int_0^{L/4}\frac{4x}{L}\sin\frac{n\pi x}{L}dx + \int_{L/4}^{3L/4}\sin\frac{n\pi x}{L}dx + \int_{3L}^{L}\frac{4(L-x)}{L}\sin\frac{n\pi x}{L}dx\right]$$

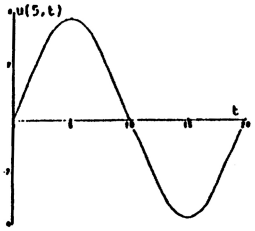
$$= \frac{8L}{n^3\pi^3 a}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right).$$ Substituting this in Eq.(34) yields

$$u(x,t) = \frac{8L}{a\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}}{n^3}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}.$$

6b.



6c. 



9. Assuming that $u(x,t) = X(x)T(t)$ and substituting for u in Eq.(1) leads to the pair of O.D.E. $X'' + \sigma X = 0$, $T'' + a^2\sigma T = 0$. Applying the B.C. $u(0,t) = 0$ and $u_x(L,t) = 0$ to $u(x,t)$ we see that we must have $X(0) = 0$ and $X'(L) = 0$. By considering the three cases $\sigma < 0$, $\sigma = 0$, and $\sigma > 0$ it can be shown that nontrivial solutions of the problem $X'' + \sigma X = 0$, $X(0) = 0$, $X'(L) = 0$ are possible if and only if $\sigma = (2n-1)^2\pi^2/4L^2$, $n = 1,2,\ldots$ and the corresponding solutions for $X(x)$ are proportional to $\sin[(2n-1)\pi x/2L]$. Using these values for σ we find that $T(t)$ is a linear combination of $\sin[(2n-1)\pi at/2L]$ and $\cos[(2n-1)\pi at/2L]$. Now, the I.C. $u_t(x,0)$ implies that $T'(0) = 0$ and thus functions of the form $u_n(x,t) = \sin[(2n-1)\pi x/2L]\cos[(2n-1)\pi at/2L]$, $n = 1,2,\ldots$ satsify the P.D.E. (1), the B.C. $u(0,t) = 0$, $u_x(L,t) = 0$, and the I.C. $u_t(x,0) = 0$. We now seek a superposition of these fundamental solutions u_n that also satisfies the I.C. $u(x,0) = f(x)$. Thus we assume that $u(x,t) = \sum_{n=1}^{\infty} c_n \sin[(2n-1)\pi x/2L]\cos[(2n-1)\pi at/2L]$. The I.C. now implies that we must have $f(x) = \sum_{n=1}^{\infty} c_n \sin[(2n-1)\pi x/2L]$. From Problem 39 of Section 10.4 we see that $f(x)$ can be represented by such a series and that $c_n = (2/L)\int_0^L f(x)\sin[(2n-1)\pi x/2L]dx$, $n = 1,2,\ldots$. Substituting these values into the above series for $u(x,t)$ yields the desired solution.

10a. From Problem 9 we have

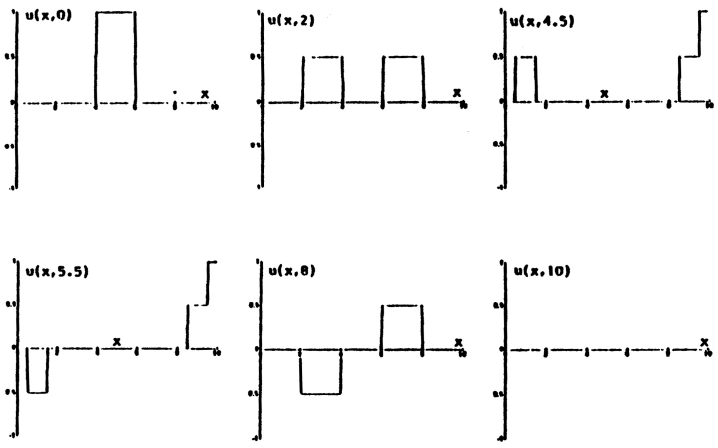
$$c_n = \frac{2}{L}\int_{(L-2)/2}^{(L+2)/2} \sin\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-4}{(2n-1)\pi}\left[\cos\left(\frac{(2n-1)\pi(L+2)}{4L}\right) - \cos\left(\frac{(2n-1)\pi(L-2)}{4L}\right)\right]$$

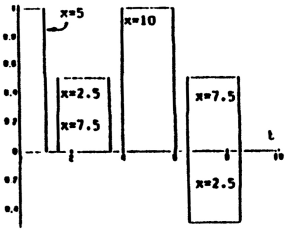
$$= \frac{8}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{4}\sin\frac{(2n-1)\pi}{2L}$$ using the

trigonometric relations for $\cos(A \pm B)$. Substituting this value of c_n into $u(x,t)$ in Problem 9 yields the desired solution.

10b.

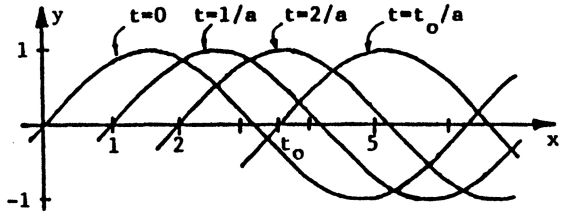


10c.



13. Using the chain rule we obtain $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$ since $\xi_x = \eta_x = 1$. Differentiating a second time gives $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$. In a similar way we obtain $u_t = u_\xi \xi_t + u_\eta \eta_t = -au_\xi + au_\eta$, since $\xi_t = -a$, $\eta_t = a$. Thus $u_{tt} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$. Substituting for u_{xx} and u_{tt} in the wave equation, we obtain $u_{\xi\eta} = 0$. Integrating both sides of $u_{\xi\eta} = 0$ with respect to η yields $u_\xi(\xi,\eta) = \gamma(\xi)$ where γ is an arbitrary function of ξ. Integrating both sides of $u_\xi(\xi,\eta) = \gamma(\xi)$ with respect to ξ yields $u(\xi,\eta) = \int\gamma(\xi)d\xi + \psi(\eta) = \phi(\xi) + \psi(\eta)$ where $\psi(\eta)$ is an arbitrary function of η and $\int\gamma(\xi)d\xi$ is some function of ξ denoted by $\phi(\xi)$. Thus $u(x,t) = u(\xi(x,t),\eta(x,t)) = \phi(x - at) + \psi(x + at)$.

14. The graph of $y = \sin(x-at)$ for the various values of t is indicated in the figure on the next page. Note that the graph of $y = \sin x$ is displaced to the right by the distance "at" for each value of t.



Similarly, the graph of $y = \phi(x + at)$ would be displaced to the left by a distance "at" for each t. Thus $\phi(x + at)$ represents a wave moving to the left.

16. Write the equation as $a^2u_{xx} = u_{tt} + \alpha^2u$ and assume $u(x,t) = X(x)T(t)$. This gives $a^2X''T = XT'' + \alpha^2XT$, or $\frac{X''}{X} = \frac{1}{a^2}(\frac{T''}{T} + \alpha^2) = \sigma$. The separation constant σ is $-\lambda^2$ using the same arguments as in the text and earlier problems. Thus $X'' + \lambda^2X = 0$, $X(0) = 0$, $X(L) = 0$ and $T'' + (\alpha^2 + z^2\lambda^2)T = 0$, $T'(0) = 0$. If we let $\beta_n^2 = \lambda_n^2a^2+\alpha^2$, we then have $u_n(x,t) = \cos\beta_n t \sin\frac{n\pi x}{L}$, where $\lambda_n = \frac{n\pi x}{L}$.

Using superposition we obtain $u(x,t) = \sum_{n=1}^{\infty} c_n \cos\beta_n t \sin\frac{n\pi x}{L}$

and thus $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x)$. Hence c_n are given by Eq. (22).

17a. We have $u(x,t) = \phi(x-at) + \psi(x+at)$ and thus $u_t(x,t) = -a\phi'(x-at) + a\chi'(x+at)$. Hence $u(x,0) = \phi(x) + \psi(x) = f(x)$ and $u_t(x,0) = -a\phi'(x) + a\psi'(x) = 0$. Dividing the last equation by a yields the desired result.

17b. Using the hint and the first equation obtained in part (a) leads to $\phi(x) + \psi(x) = 2\phi(x) + c = f(x)$ so $\phi(x) = (1/2)f(x) - c/2$ and $\psi(x) = (1/2)f(x) + c/2$. Hence $u(x,t) = \phi(x - at) + \psi(x + at) = (1/2)[f(x - at) - c] + (1/2)[f(x + at) + c] = (1/2)[f(x - at) + f(x + at)]$.

17c. Substituting $x + at$ for x in $f(x)$ yields

$$f(x + at) = \begin{cases} 2 & -1 < x + at < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Subtracting "at" from both sides of the inequality then yields

$$f(x + at) = \begin{cases} 2 & -1 - at < x < 1 - at \\ 0 & \text{otherwise} \end{cases}.$$

18a. As in Problem 17a, we have $u(x,0) = \phi(x) + \psi(x) = 0$ and $u_t(x,0) = -a\phi'(x) + a\psi'(x) = g(x)$.

18b. From part (a) we have $\psi(x) = -\phi(x)$ which yields $-2a\phi(x) = g(x)$ from the second equation in part a. Integration then yields $\phi(x) - \phi(x_0) = \frac{-1}{2a}\int_{x_0}^{x} g(\xi)d\xi$ and hence
$\psi(x) = (1/2a)\int_{x_0}^{x} g(\xi)d\xi - \phi(x_0)$.

18c. $u(x,t) = \phi(x-at) + \psi(x+at)$
$= -(1/2a)\int_{x_0}^{x-at} g(\xi)d\xi + \phi(x_0) + (1/2a)\int_{x_0}^{x+at} g(\xi)d\xi - \phi(x_0)$
$= (1/2a)[\int_{x_0}^{x+at} g(\xi)d\xi - \int_{x_0}^{x-at} g(\xi)d\xi]$
$= (1/2a)[\int_{x_0}^{x+at} g(\xi)d\xi + \int_{x-at}^{x_0} g(\xi)d\xi]$
$= (1/2a\int_{x-at}^{x+at} g(\xi)d\xi.$

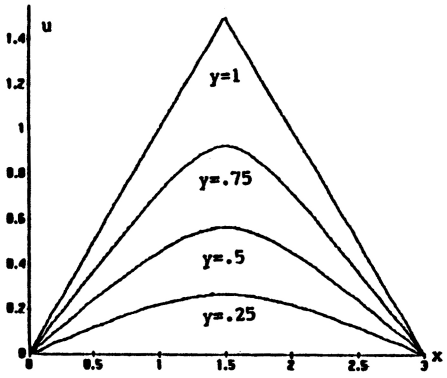
24. Substituting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ into the P.D.E. yields $R''\Theta T + R'\Theta T/r + R\Theta''T/r^2 = R\Theta T''/a^2$ or equivalently $R''/R + R'/rR + \Theta''/\Theta r^2 = T''/a^2T$. In order for this equation to be valid for $0 < r < r_0$, $0 \le \theta \le 2\pi$, $t > 0$, it is necessary that both sides of the equation be equal to the same constant $-\sigma$. Otherwise, by keeping r and θ fixed and varying t, one side would remain unchanged while the other side varied. Thus we arrive at the two equations $T'' + \sigma a^2 T = 0$ and $r^2R''/R + rR'/R + \sigma r^2 = -\Theta''/\Theta$. By an argument similar to the one above we conclude that both sides of the last equation must be equal to the same constant δ. This leads to the two equations $r^2R'' + rR' + (\sigma r^2 - \delta)R = 0$ and $\Theta'' + \delta\Theta = 0$. Since the circular membrane is continuous, we must have $\Theta(2\pi) = \Theta(0)$, which requires $\delta = \mu^2$, μ a non-negative integer. The condition $\Theta(2\pi) = \Theta(0)$ is also known as the periodicity condition. Since we also desire solutions which vary periodically in time, it is clear that the separation constant σ should be positive, $\sigma = \lambda^2$. Thus we arrive at the three equations $r^2R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0$, $\Theta'' + \mu^2\Theta = 0$, and $T'' + \lambda^2 a^2 T = 0$.

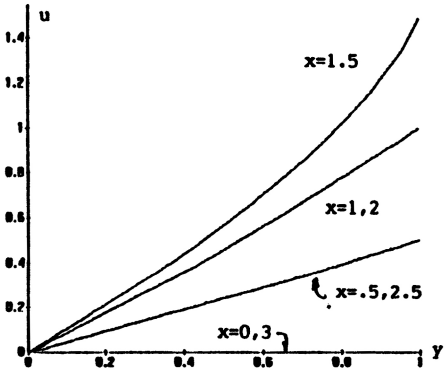
Section 10.8, Page 611

1a. Assuming that $u(x,y) = X(x)Y(y)$ leads to the two O.D.E. $X'' - \sigma X = 0$, $Y'' + \sigma Y = 0$. The B.C. $u(0,y) = 0$, $u(a,y) = 0$ imply that $X(0) = 0$ and $X(a) = 0$. Thus nontrivial solutions to $X'' - \sigma X = 0$ which satisfy these boundary conditions are possible only if $\sigma = -(n\pi/a)^2$, $n = 1,2...$; the corresponding solutions for $X(x)$ are proportional to $\sin(n\pi x/a)$. The B.C. $u(x,0) = 0$ implies that $Y(0) = 0$. Solving $Y'' - (n\pi/a)^2 Y = 0$ subject to this condition we find that $Y(y)$ must be proportional to $\sinh(n\pi y/a)$. The fundamental solutions are then $u_n(x,y) = \sin(n\pi x/a)\sinh(n\pi y/a)$, $n = 1,2,...$, which satisfy Laplace's equation and the homogeneous B.C. We assume that $u(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/a)\sinh(n\pi y/a)$, where the coefficients c_n are determined from the B.C. $u(x,b) = g(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/a)\sinh(n\pi b/a)$. It follows that
$c_n \sinh(n\pi b/a) = (2/a)\int_0^a g(x)\sin(n\pi x/a)dx$, $n = 1,2,...$.

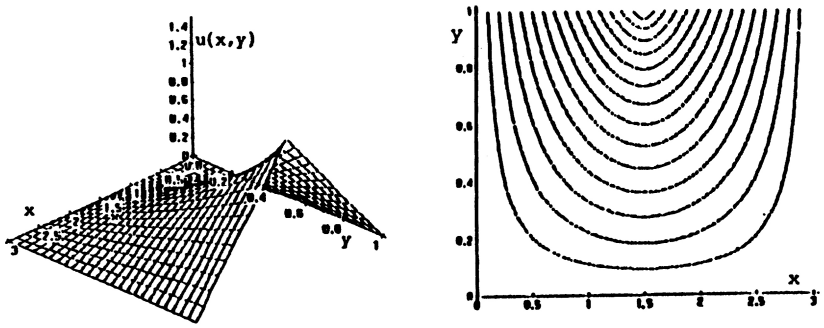
1b. Substituting for $g(x)$ in the equation for c_n we have $c_n \sinh(n\pi b/a) = (2/a)[\int_0^{a/2} x\sin(n\pi x/a)dx + \int_{a/2}^a (a-x)\sin(n\pi x/a)dx] = [4a \sin(n\pi/2)]/n^2\pi^2$, $n = 1,2,...$, so $c_n = [4a \sin(n\pi/2)]/[n^2\pi^2 \sinh(n\pi b/a)]$. Substituting these values for c_n in the above series yields the desired solution.

1c.





1d.



2. In solving the D.E. $Y'' - \lambda^2 Y = 0$, one normally writes $Y(y) = c_1\sinh\lambda y + c_2\cosh\lambda y$. However, since we have $Y(b) = 0$, it is advantageous to rewrite Y as $Y(y) = d_1\sinh\lambda(b-y) + d_2\cosh\lambda(b-y)$, where d_1, d_2 are also arbitrary constants and can be related to c_1, c_2 using the appropriate hyperbolic trigonometric identities. The important thing, however, is that the second form also satisfies the D.E. and thus $Y(y) = d_1\sinh\lambda(b-y)$ satisfies the D.E. and the homogeneous B.C. $Y(b) = 0$. The rest of the problem follows the pattern of Problem 1.

3a. Let $u(x,y) = v(x,y) + w(x,y)$, where u, v and w all satisfy Laplace's Eq., v(x,y) satisfies the conditions in Eq. (4) and w(x,y) satisfies the conditions of Problem 2.

4. Following the pattern of Problem 3, one could consider adding the solutions of four problems, each with only one non-homogeneous B.C. It is also possible to consider adding the solutions of only two problems, each with only two non-homogeneous B.C., as long as they involve the same variable. For instance, one such problem would be $u_{xx} + u_{yy} = 0$, $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = k(y)$, $u(a,y) = f(y)$, which has the fundamental solutions $u_n(x,y) = [c_n\sinh(n\pi x/b) + d_n\cosh(n\pi x/b)]\sin(n\pi y/b)$.

Assuming $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$ and using the B.C. $u(0,y) = k(y)$ we obtain $d_n = (2/b)\int_0^b k(y)\sin(n\pi y/b)dy$.

Using the B.C. $u(a,y) = f(y)$ we obtain $c_n\sinh(n\pi a/b) + d_n\cosh(n\pi a/b) = (2/b)\int_0^b f(y)\sin(n\pi y/b)dy$, which can be solved for c_n, since d_n is already known. The second problem, in this approach, would be $u_{xx} + u_{yy} = 0$, $u(x,0) = h(x)$, $u(x,b) = g(x)$, $u(0,y) = 0$ and $u(a,y) = 0$. This has the fundamental solutions

$u_n(x,y) = [a_n\sinh(n\pi y/a) + b_n\cosh(n\pi y/a)]\sin(n\pi x/a$, so that $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$. Thus $u(x,0) = h(x)$ gives $b_n = (2/a)\int_0^a h(x)\sin(n\pi x/a)dx$ and $u(x,b) = g(x)$ gives $a_n\sinh(n\pi b/a) + b_n\cosh(n\pi b/a) = (2/a)\int_0^a g(x)\sin(n\pi x/a)dx$, which can be solved for a_n since b_n is known.

5. Using Eq.(20) and following the same arguments as presented in the text, we find that $R(r) = k_1r^n + k_2r^{-n}$ and $\Theta(\theta) = c_1\sin n\theta + c_2\cos n\theta$, for n a positive integer, and $u_0(r,\theta) = 1$ for $n = 0$. Since we require that $u(r,\theta)$ be bounded as $r \to \infty$, we conclude that $k_1 = 0$. The fundamental solutions are therefore $u_n(r,\theta) = r^{-n}\cos n\theta$, $v_n(r,\theta) = r^{-n}\sin n\theta$, $n = 1,2,...$ together with $u_0(r,\theta) = 1$. Assuming that u can be expressed as a linear combination of the fundamental solutions we have $u(r,\theta) = c_0/2 + \sum_{n=1}^{\infty} r^{-n}(c_n\cos n\theta + k_n\sin n\theta)$. The B.C. requires that $u(a,\theta) = c_0/2 + \sum_{n=1}^{\infty} a^{-n}(c_n\cos n\theta + k_n\sin n\theta) = f(\theta)$ for $0 \le \theta < 2\pi$. This is precisely the Fourier series representation for $f(\theta)$ of period 2π and thus $a^{-n}c_n = (1/\pi)\int_0^2 f(\theta)\cos n\theta d\theta$, $n = 0,1,2,...$ and

$a^{-n}k_n = (1/\pi)\int_0^2 f(\theta)\sin n\theta d\theta$, $n = 1,2... .$

7. Again we let $u(r,\theta) = R(r)\Theta(\theta)$ and thus we have $r^2R'' + rR' - \sigma R = 0$ and $\Theta'' + \sigma\Theta = 0$, with $R(0)$ bounded and the B.C. $\Theta(0) = \Theta(\alpha) = 0$. For $\sigma \le 0$ we find that $\Theta(0) \equiv 0$, so we let $\sigma = \lambda^2$ (λ^2 real) and thus $\Theta(\theta) = c_1\cos\lambda\theta + c_2\sin\lambda\theta$. The B.C. $\Theta(0) = 0 \to c_1 = 0$ and the B.C. $\Theta(\alpha) = 0 \to \lambda = n\pi/\alpha$, $n = 1,2,... .$ Substituting these values into Eq.(31) we obtain $R(r) = k_1r^{n\pi/\alpha} + k_2r^{-n\pi/\alpha}$. However $k_2 = 0$ since $R(0)$ must be bounded, and thus the fundamental solutions are $u_n(r,\theta) = r^{n\pi/\alpha}\sin(n\pi\theta/\alpha)$. The desired solution may now be formed using previously discussed procedures.

8a. Separating variables, as before, we obtain $X'' + \lambda^2 X = 0$, $X(0) = 0$, $X(a) = 0$ and $Y'' - \lambda^2 Y = 0$, $Y(y)$ bounded as $y \to \infty$. Thus $X(x) = \sin(n\pi x/a)$, and $\lambda^2 = (n\pi/a)^2$. However, since neither sinhy nor coshy are bounded as $y \to \infty$, we must write the solution to $Y'' - (n\pi/a)^2 Y = 0$ as $Y(y) = c_1 \exp[n\pi y/a] + c_2 \exp[-n\pi y/a]$. Thus we must choose $c_1 = 0$ so that $u(x,y) = X(x)Y(y) \to 0$ as $y \to \infty$. The fundamental solutions are then $u_n(x,t) = e^{-n\pi y/a} \sin(n\pi x/a)$.

$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y)$ then gives

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/a) = f(x)$ so that $c_n = \frac{2}{a}\int_0^1 f(x)\sin\frac{n\pi x}{a}\,dx$.

8b. $c_n = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx = \frac{4a^2}{n^3\pi^3}(1-\cos n\pi)$

8c. Using just the first term and letting $a = 5$, we have $u(x,y) = \frac{200}{\pi^3} e^{-\pi y/5} \sin\frac{\pi x}{5}$, which, for a fixed y, has a maximum at $x = 5/2$ and thus we need to find y such that $u(5/2,y) = \frac{200}{\pi^3} e^{-\pi y/5} = .1$. Taking the logarithm of both sides and solving for y yields $y_0 = 6.6315$. With an equation solver, more terms can be used. However, to four decimal places, three terms yield the same result as above.

13a. Assuming that $u(x,y) = X(x)Y(y)$ and substituting into Eq.(1) leads to the two O.D.E. $X'' - \sigma X = 0$, $Y'' + \sigma Y = 0$. The B.C. $u(x,0) = 0$, $u_y(x,b) = 0$ imply that $Y(0) = 0$ and $Y'(b) = 0$. For nontrivial solutions to exist for $Y'' + \sigma Y = 0$ with these B.C. we find that σ must take the values $(2n-1)^2\pi^2/4b^2$, $n = 1,2,\ldots$; the corresponding solutions for $Y(y)$ are proportional to $\sin[(2n-1)\pi y/2b]$. Solutions to $X'' - [(2n-1)^2\pi^2/4b^2]X = 0$ are of the form $X(x) = A\sinh[(2n-1)\pi x/2b] + B\cosh[(2n-1)\pi x/2b]$. However, the boundary condition $u(0,y) = 0$ implies that $X(0) = B = 0$. It follows that the fundamental solutions are $u_n(x,y) = c_n \sinh[(2n-1)\pi x/2b]\sin[(2n-1)\pi y/2b]$, $n = 1,2,\ldots$. To satisfy the remaining B.C. at $x = a$ we assume that we can represent the solution $u(x,y)$ in the form $u(x,y) = \sum_{n=1}^{\infty} c_n \sinh[(2n-1)\pi x/2b]\sin[(2n-1)\pi y/2b]$.

The coefficients c_n are determined by the B.C.

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh[(2n-1)\pi a/2b]\sin[(2n-1)\pi y/2b] = f(y).$$

By properly extending f as a periodic function of period 4b as in Problem 39, Section 10.4, we find that the coefficients c_n are given by

$$c_n \sinh[(2n-1)\pi a/2b] = (2/b)\int_0^b f(y)\sin[(2n-1)\pi y/2b]dy,$$

$n = 1,2,\ldots$.

CHAPTER 11

Section 11.1, Page 626

2. Since the B.C. at $x = 1$ is nonhomogeneous, the B.V.P. is nonhomogeneous.

4. The D.E. may be written $y'' + (\lambda - x^2)y = 0$ and is thus homogeneous, as are both B.C.

5. Since the D.E. contains the nonhomogeneous term 1, the B.V.P. is nonhomogeneous.

9. If $\lambda = 0$, then $y(x) = c_1x + c_2$ and thus $y(0) = c_2$, $y(1) = c_1 + c_2$, $y'(0) = c_1$ and $y'(1) = c_1$. Hence the B.C. yield the two equations $c_2 - c_1 = 0$ and $c_1 + c_2 + c_1 = 0$ which give $c_1 = c_2 = 0$ and thus $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$, the general solution of the D.E. is $y = c_1\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$. The B.C. require that $c_2 - \sqrt{\lambda}\,c_1 = 0$, and $(\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda})c_1 + (\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda})c_2 = 0$. In order to have nontrivial solutions λ must satisfy $(\lambda-1)\sin\sqrt{\lambda} - 2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$. In this case $c_2 = \sqrt{\lambda}\,c_1$ and thus $\phi_n = \sin\sqrt{\lambda_n}\,x + \sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x$. If $\lambda \neq 1$, the eigenvalue equation is equivalent to $\tan\sqrt{\lambda} = 2\sqrt{\lambda}/(\lambda-1)$ and thus by graphing $f(\sqrt{\lambda}) = \tan\sqrt{\lambda}$ and $g(\sqrt{\lambda}) = 2\sqrt{\lambda}/(\lambda-1)$ we can estimate the eigenvalues. Since $g(\sqrt{\lambda})$ has a vertical asymptote at $\lambda = 1$ and $f(\sqrt{\lambda})$ has a vertical asymptote at $\sqrt{\lambda} = \pi/2$, we see that $1 < \sqrt{\lambda_1} < \pi/2$. By interating numerically, we find $\sqrt{\lambda_1} \cong 1.30655$ and thus $\lambda_1 \cong 1.7071$. The second eigenvalue will lie to the right of π, the second zero of $\tan\sqrt{\lambda}$. Again iterating numerically, we find $\sqrt{\lambda_2} \cong 3.6732$ and thus $\lambda_2 \cong 13.4924$. For large values of n, we see from the graph that $\sqrt{\lambda_n} \cong (n-1)\pi$, which are the zeros of $\tan\sqrt{\lambda}$. Thus $\lambda_n \cong (n-1)^2\pi^2$ for large n.

For $\lambda < 0$, the discussion follows the pattern of Example 1 yielding $y(x) = c_1\sinh\sqrt{\mu}\,x + c_2\cosh\sqrt{\mu}\,x$. The B.C. then yield $c_2 - \sqrt{\mu}\,c_1 = 0$ and $(\sinh\sqrt{\mu} + \sqrt{\mu}\cosh\sqrt{\mu})c_1 + (\cosh\sqrt{\mu} + \sqrt{\mu}\sinh\sqrt{\mu})c_2 = 0$, which

have a non-zero solution if and only if $(\mu+1)\sinh\sqrt{\mu} + 2\sqrt{\mu}\cosh\sqrt{\mu} = 0$. By plotting $y = \tanh\sqrt{\mu}$ and $y = -2\sqrt{\mu}/(\mu+1)$ we see that they intersect only at $\mu = 0$, and thus there are no negative eigenvalues.

10. If $\lambda = 0$, the general solution of the D.E. is $y = c_1 + c_2x$. The B.C. $y(0) + y'(0) = 0$ requires $c_1 + c_2 = 0$ and the B.C. $y(1) = 0$ requires $c_1 + c_2 = 0$ and thus $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $\phi_0(x) = 1-x$.

If $\lambda < 0$, set $-\lambda = \mu^2$ to obtain $y = c_1\cos\mu x + c_2\sin\mu x$. In this case the B.C. require $c_1 + \mu c_2 = 0$ and $c_1\cos\mu + c_2\sin\mu = 0$ which yields nontrivial solutions for c_1 and c_2 (i.e., $c_1 = -\mu c_2$) if and only if $\tan\mu = \mu$. By plotting on the same graph $f(\mu) = \mu$ and $g(\mu) = \tan\mu$, we see that they intersect at $\mu_0 = 0$ ($\mu = 0 \rightarrow \lambda = 0$, which has already been discussed), $\mu_1 \cong 4.49341$ (which is just to the left of the vertical asymptote of $\tan\mu$ at $\mu = 3\pi/2$, $\mu_2 \cong 7.72525$ (which is just to the left of the vertical asymptote of $\tan\mu$ at $\mu = 5\pi/2$) and for larger values $\mu_n \cong (2n+1)\pi/2$. Since $\lambda_n = -\mu_n^2$, we have $\lambda_1 \cong -20.1907$, $\lambda_2 = -59.6795$, $\lambda_n \cong -(2n+1)^2\pi^2/4$ and $\phi_n = \sin\mu_n x - \mu_n\cos\mu_n x$.

If $\lambda > 0$, the general solution of the D.E. is $y(x) = c_1\cosh\sqrt{\lambda}x + c_2\sinh\sqrt{\lambda}x$. The B.C. respectively require that $c_1 + \sqrt{\lambda}c_2 = 0$ and $c_1\cosh\sqrt{\lambda} + c_2\sinh\sqrt{\lambda} = 0$ and thus λ must satisfy $\tanh\sqrt{\lambda} = \sqrt{\lambda}$ in order to have nontrivial solutions. The only solution of this equation is $\lambda = 0$ and thus there are no positive eigenvalues.

11a. From Eq.(i) the coefficient of y' is μQ and from Eq.(ii) the coefficient of y' is $(\mu P)'$. Thus $(\mu P)' = \mu Q$, which gives Eq.(iii).

11b. Eq.(iii) is both linear and separable. Using the latter approach we have $d\mu/\mu = [(Q-P')/P]dx$ and thus $\ln\mu = \int_{x_0}^{x}[Q(s)/P(s)]ds - \ln P$. Taking the exponential of both sides yields Eq.(iv). The choice of x_0 simply alters the constant of integration, which is immaterial here.

13. Since $P(x) = x^2$ and $Q(x) = x$, we find that $\mu(x) = (1/x^2)\exp[\int_{x_0}^{x}(s/s^2)ds] = k/x$, where k is an arbitrary constant which may be set equal to 1. It follows that Bessel's equation takes the form $(xy')' + (x^2-\upsilon^2/x)y = 0$.

18a. Assuming $y = s(x)u$, we have $y' = s'u + su'$ and $y'' = s''u + 2s'u' + su''$ and thus the D.E. becomes $su'' + (2s'+4s)u' + [s'' + 4s' + (4+9\lambda)s]u = 0$. Setting $2s' + 4s = 0$ we find $s(x) = e^{-2x}$ and the D.E. becomes $u'' + 9\lambda u = 0$. The B.C. $y(0) = 0$ yields $s(0)u(0) = 0$, or $u(0) = 0$ since $s(0) \neq 0$. The B.C. at L is $y'(L) = s'(L)u(L) + s(L)u'(L) = e^{-2L}(-2u(L) + u'(L)) = 0$ and thus $u'(L) - 2u(L) = 0$. Thus the B.V.P. satisfied by $u(x)$ is $u'' + 9\lambda u = 0$, $u(0) = 0$, $u'(L) - 2u(L) = 0$.

If $\lambda < 0$, the general solution of the D.E. $u'' + 9\lambda u = 0$ is $u = c_1\sinh 3\mu x + c_2\cosh 3\mu x$ where $-\lambda = \mu^2$. The B.C. require that $c_2 = 0$, $c_1(3\mu\cosh 3\mu L - 2\sinh 3\mu L) = 0$. In order to have nontrivial solutions μ must satisfy the equation $3\mu/2 = \tanh 3\mu L$. A graphical analysis reveals that for $L \leq 1/2$ this equation has no solutions for $\mu \neq 0$ so there are no negative eigenvalues for $L \leq 1/2$. If $L > 1/2$ there is one solution and hence one negative eigenvalue with eigenfuction $\phi_{-1}(x) = e^{-2x}\sinh 3\mu x$.

If $\lambda = 0$, the general solution of the D.E. $u'' + 9\lambda u = 0$ is $u = c_1 + c_2x$. The B.C. require that $c_1 = 0$, $c_2(1-2L) = 0$ so nontrivial solutions are possible only if $L = 1/2$. In this case the eigenfuction is $\phi_0(x) = xe^{-2x}$.

If $\lambda > 0$, the general solution of the D.E. $u'' + 9\lambda u = 0$ is $u = c_1\sin 3\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$. The B.C. require that $c_2 = 0$, $c_1(3\sqrt{\lambda}\cos 3\sqrt{\lambda}\,L - 2\sin 3\sqrt{\lambda}\,L) = 0$. In order to have nontrivial solutions λ must satisfy the equation $\sqrt{\lambda} = (2/3)\tan 3\sqrt{\lambda}\,L$. A graphical analysis reveals that there is an infinite number of solutions to this eigenvalue equation. Thus the eigenfunctions are $\phi_n(x) = e^{-2x}\sin 3\sqrt{\lambda_n}\,x$ where the eigenvalues λ_n satisfy $\sqrt{\lambda_n} = (2/3)\tan 3\sqrt{\lambda_n}\,L$.

20. This is an Euler equation whose characteristic equation has roots $r_1 = \lambda$ and $r_2 = 1$. If $\lambda = 1$ the general solution of the D.E. is $y = c_1x + c_2x\ln x$ and the B.C.

require that $c_1 = c_2 = 0$ and thus $\lambda = 1$ is not an eigenvalue. If $\lambda \neq 1$, $y = c_1x + c_2x^{\lambda}$ is the general solution and the B.C. require that $c_1 + c_2 = 0$ and $2c_1 + c_2 2^{\lambda} - (c_1 + \lambda c_2 2^{\lambda-1}) = 0$. Thus nontrivial solutions exist if and only if $\lambda = 2(1-2^{-\lambda})$. The graphs of $f(\lambda) = \lambda$ and $g(\lambda) = 2(1-2^{-\lambda})$ intersect only at $\lambda = 1$ (which has already been discussed) and $\lambda = 0$. Thus the only eigenvalue is $\lambda = 0$ with corresponding eigenfunction $\phi(x) = x - 1$ (since $c_1 = -c_2$).

22a. For positive λ, the general solution of the D.E. is $y = c_1\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}\,x$. The B.C. require that $\sqrt{\lambda}\,c_1 + \alpha c_2 = 0$, $c_1\sin\sqrt{\lambda} + c_2\cos\sqrt{\lambda} = 0$. Nontrivial solutions exist if and only if $\sqrt{\lambda}\cos\sqrt{\lambda} - \alpha\sin\sqrt{\lambda} = 0$. If $\alpha = 0$ this equation is satisfied by the sequence $\lambda_n = [(2n-1)\pi/2]^2$, $n = 1,2,\ldots$. If $\alpha \neq 0$, λ must satisfy the equation $\sqrt{\lambda}/\alpha = \tan\sqrt{\lambda}$. A plot of the graphs of $f(\sqrt{\lambda}) = \sqrt{\lambda}/\alpha$ and $g(\sqrt{\lambda}) = \tan\sqrt{\lambda}$ reveals that there is an infinite sequence of postive eigenvalues for $\alpha < 0$ and $\alpha > 0$.

22b. By procedures shown previously, the cases $\lambda < 0$ and $\lambda = 0$, when $\alpha < 1$, lead to only the trivial solution and thus by part a all real eigenvalues are positive. For $0 < \alpha < 1$, the graphs of $f(\sqrt{\lambda})$ and $g(\sqrt{\lambda})$ (see part a) intersect once on $0 < \sqrt{\lambda} < \pi/2$. As α approaches 1 from below, the slope of $f(\sqrt{\lambda})$ decreases and thus the intersection point approaches zero.

22c. If $\lambda = 0$, then $y(x) = c_1 + c_2x$ and the B.C. yield $\alpha c_1 + c_2 = 0$ and $c_1 + c_2 = 0$, which have a non-zero solution if and only if $\alpha = 1$.

22d. Let $-\lambda = \mu^2$, then $y(x) = c_1\cosh\mu x + c_2\sinh\mu x$ and thus the B.C. yield $\alpha c_1 + \mu c_2 = 0$ and $(\cosh\mu)c_1 + (\sinh\mu)c_2 = 0$, which have non-zero solutions if and only if $\tanh\mu = \mu/\alpha$. For $\alpha>1$, the straight line $y = \mu/\alpha$ intersects the curve $y = \tanh\mu$ in one point, which increases as α increases. Thus $\lambda = -\mu^2$ decreases as α increases.

23. Using the D.E. ϕ_m and following the hint yields: $\lim\int_0^L \phi_m''\phi_n dx + \lambda_m\int_0^L \phi_m\phi_n dx = 0$. Integrating the first term by

parts yields: $\phi_m'\phi_n\big|_0^L - \int_0^L \phi_m'\phi_n' dx = -\lambda_m \int_0^L \phi_m\phi_n dx$. Upon utilizing he B.C. the first term on the left vanishes and thus $\int_0^L \phi_n'\phi_m' dx = \lambda_m \int_0^L \phi_m\phi_n dx$. Similarly, the D.E. for ϕ_n yields $\int_0^L \phi_m'\phi_n' dx = \lambda_n \int_0^L \phi_n\phi_m dx$ and thus $(\lambda_n - \lambda_m) \int_0^L \phi_m\phi_n dx = 0$. If $\lambda_n \neq \lambda_m$ the desired result follows.

24b. The general solution of the D.E. is $y = c_1\sin\mu x + c_2\cos\mu x + c_3\sinh\mu x + c_4\cosh\mu x$ where $\lambda = \mu^4$. The B.C. require that $c_2 + c_4 = 0$, $-c_2 + c_4 = 0$, $c_1\sin\mu L + c_2\cos\mu L + c_3\sinh\mu L + c_4\cosh\mu L = 0$, and $c_1\cos\mu L - c_2\sin\mu L + c_3\cosh\mu L + c_4\sinh\mu L = 0$. The first two equations yield $c_2 = c_4 = 0$, and the last two have nontrivial solutions if and only if $\sin\mu L\cosh\mu L - \cos\mu L\sinh\mu L = 0$. In this case the third equation yields $c_3 = -c_1\sin\mu L/\sinh\mu L$ and thus the desired eigenfunctions are obtained. The quantity μL can be approximated by finding the intersetion of $f(x) = \tan x$ and $g(x) = \tanh x$, where $x = \mu L$. The first intersection is at $x \cong 3.9266$, which gives $\lambda_1 \cong 237.72/L^4$ and the second intersection is at $x \cong 7.0686$, which gives $\lambda_2 \cong 2{,}496.5/L^4$.

Section 11.2, Page 639

1. We have $y(x) = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$ and thus $y(0) = 0$ yields $c_1 = 0$ and $y'(1) = 0$ yields $\sqrt{\lambda}\,c_2\cos\sqrt{\lambda} = 0$. $\lambda = 0$ gives $y(x) = 0$, so is not an eigenvalue. Otherwise $\lambda = (2n-1)^2\pi^2/4$ and the eigenfuctions are $\sin[(2n-1)\pi x/2]$, $n = 1,2,\ldots$. Thus, by Eq.(20), we must choose k_n so that $\int_0^1 \{k_n\sin[(2n-1)\pi x/2]\}^2 dx = 1$, since the weight function $r(x) = 1$ (by comparing the D.E. to Eq.(1)). Evaluating the integral yields $k_n^2/2 = 1$ and thus $k_n = \sqrt{2}$ and the desired normalized eigenfuctions are obtained.

3. Note here that $\phi_0(x) = 1$ satisifes Eq.(20) and hence it is already normalized.

5. From Problem 17 of Section 11.1 we have $e^x\sin n\pi x$, $n = 1,2,\ldots$ as the eigenfunctions and thus k_n must be chosen so that $\int_0^1 r(x)k_n^2 e^{2x}\sin^2 n\pi x dx = 1$. To determine $r(x)$, we must write the D.E. in the form of Eq.(1). That is, we multiply

the D.E. by r(x) to obtain $ry'' - 2ry' + ry + \lambda ry = 0$. Now choose r so that $(ry')' = ry''-2ry'$, which yields $r' = -2r$ or $r(x) = e^{-2x}$. Thus the above integral becomes $\int_0^1 k_n^2 \sin^2 n\pi x dx = 1$ and $k_n = \sqrt{2}$. Hence $\phi_n(x) = \sqrt{2}\, e^x \sin n\pi x$ are the normalized eigenfunctions.

7. Using Eq.(34) with r(x) = 1, we find that the coefficients of the series (32) are determined by
$a_n = (f,\phi_n) = \sqrt{2}\int_0^1 x\sin[(2n-1)\pi x/2]dx$
$= (4\sqrt{2}/(2n-1)^2\pi^2)\sin(2n-1)\pi/2$. Thus Eq.(32) yields
$$f(x) = \frac{4\sqrt{2}}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sqrt{2}\sin[(2n-1)\pi x/2],\ 0 \le x \le 1,$$
which agrees with the expansion using the approach developed in Problem 39 of Section 10.4.

10. In this case $\phi_n(x) = (\sqrt{2}/\alpha_n)\cos\sqrt{\lambda_n}\, x$, where $\alpha_n = (1 + \sin^2\sqrt{\lambda_n})^{1/2}$. Thus Eq.(34) yields
$a_n = (\sqrt{2}/\alpha_n)\int_0^1 \cos\sqrt{\lambda_n}\, x dx = \sqrt{2}\sin\sqrt{\lambda_n}/\alpha_n\sqrt{\lambda_n}$.

14. In this case $L[y] = y'' + y' + 2y$ is not of the form shown in Eq.(3) and thus the B.V.P. is not self adjoint.

17. In this case $L[y] = [(1+x^2)y']' + y$ and thus the D.E. has the form shown in Eq.(3). However, the B.C. are not separated and thus we must determine by integration whether Eq.(8) is satisfied. Therefore, for u and v satisfying the B.C., integration by parts yields the following:
$$(L[u],v) = \int_0^1\{[(1+x^2)u']'+u\}v dx = vu'(1+x^2)\Big|_0^1 - \int_0^1\{(1+x^2)v'u'+uv\}dx$$
$$= vu'(1+x^2)\Big|_0^1 - uv'(1+x^2)\Big|_0^1 + \int_0^1\{[(1+x^2)v']'+v\}u dx$$
$$= (u,L[v])$$
since the integrated terms add to zero with the given B.C. Thus the B.V.P. is self-adjoint.

21a. Substituting ϕ for y in the D.E., multiplying both sides by ϕ, and integrating form 0 to 1 yields
$\lambda\int_0^1 r\phi^2 dx = \int_0^1\{-[p(x)\phi']'\phi + q(x)\phi^2\}dx$. Integrating the first term on the right side once by parts, we obtain

$\lambda\int_0^1 r\phi^2 dx = -p(1)\phi'(1)\phi(1) + p(0)\phi'(0)\phi(0) + \int_0^1 (p\phi'^2+q\phi^2)dx$. If $a_2 \neq 0$, $b_2 \neq 0$, then $\phi'(1) = -b_1\phi(1)/b_2$ and $\phi'(0) = -a_1\phi(0)/a_2$ and the result follows. If $a_2 = 0$, then $\phi(0) = 0$ and the boundary term at 0 will be missing. A similar result is obtained if $b_2 = 0$.

21b. From the text, $p(x) > 0$ and $r(x) > 0$ for $0 \leq x \leq 1$ (following Eq.(4)). If $q(x) \geq 0$ and if b_1/b_2 and $-a_1/a_2$ are non-negative, then all terms in the final equation of part a are non-negative and thus λ must be non-negative. Note that λ could be zero if $q(x) = 0$, $b_1 = 0$, $a_1 = 0$ and $\phi(x) = 1$.

21c. If either $q(x) \neq 0$ or $a_1 \neq 0$ or $b_1 \neq 0$, there is at least one positive term on the right and thus λ must be positive.

23a. Using $\phi(x) = U(x) + iV(x)$ in Eq.(4) we have $L[\phi] = L[U(x) + iV(x)] = \lambda r(x)[U(x)+iV(x)]$. Using the linearity of L and the fact that λ and $r(x)$ are real we have $L[U(x)] + iL[V(x)] = \lambda r(x)U(x) + i\lambda r(x)V(x)$. Equating the real and imaginary parts shows that both U and V satisfy Eq.(1). The B.C. Eq.(2) are also satisfied by both U and V, using the same arguments, and thus both U and V are eigenfunctions.

23b. By Theorem 11.2.3 each eigenvalue λ has only one linearly independent eigenfuction. By part a we have U and V being eigenfunctions corresponding to λ and thus U and V must be linearly dependent.

23c. From part b we have $V(x) = cU(x)$ and thus $\phi(x) = U(x) + icuU(x) = (1+ic)U(x)$.

24. This is an Euler equation, so for $y = x^r$ we have $r^2 - (\lambda+1)r + \lambda = 0$ or $(r-1)(r-\lambda) = 0$. If $\lambda = 1$, the general solution to the D.E. is $y = c_1x + c_2x\ln x$. The B.C. require that $c_1 = 0$, $2c_1 + 2(\ln 2)c_2 = 0$ so $c_1 = c_2 = 0$ and $\lambda = 1$ is not an eigenvalue. If $\lambda \neq 1$, the general solution to the D.E. is $y = c_1x + c_2x^\lambda$. The B.C. require that $c_1 + c_2 = 0$ and $2c_1 + 2^\lambda c_2 = 0$. Nontrivial solutions exist if and only if $2^\lambda - 2 = 0$. If λ is real, this equation has no solution (other than $\lambda = 1$) and again $y = 0$ is the only solution to the boundary value problem. Suppose that $\lambda = a + bi$ with $b \neq 0$. Then $2^\lambda = 2^{a+bi} = 2^a 2^{bi} = 2^a\exp(ib\ln 2)$, which upon substitution into $2^\lambda = 2$ yields the equation

$\exp(ib\ln 2) = 2^{1-a}$. Since $e^{ix} = \cos x + i\sin x$, it follows that $\cos(b\ln 2) = 2^{1-a}$ and $\sin(b\ln 2) = 0$, which yield $a = 1$ and $b(\ln 2) = 2n\pi$ or $b = 2n\pi/\ln 2$, $n = \pm 1, \pm 2, \ldots$. Thus the only eigenvalues of the problem are $\lambda_n = 1 + i(2n\pi/\ln 2)$, $n = \pm 1, \pm 2, \ldots$.

25b. For $\lambda \le 0$, there are no eigenfuctions. For $\lambda > 0$ the general solution of the D.E. is $y = c_1 + c_2 x + c_3 \sin\sqrt{\lambda}\, x + c_4 \cos\sqrt{\lambda}\, x$. The B.C. require that $c_1 + c_4 = 0$, $c_4 = 0$, $c_1 + c_2 L + c_3 \sin\sqrt{\lambda}\, L + c_4 \cos\sqrt{\lambda}\, L = 0$, and $c_2 + \sqrt{\lambda}\, c_3 \cos\sqrt{\lambda}\, L - \sqrt{\lambda}\, c_4 \sin\sqrt{\lambda}\, L = 0$. Thus $c_1 = c_4 = 0$ and for nontrivial solutions to exist λ must satisfy the equation $\sqrt{\lambda}\, L\cos\sqrt{\lambda}\, L - \sin\sqrt{\lambda}\, L = 0$. In this case $c_2 = (-\sqrt{\lambda}\cos\sqrt{\lambda}\, L)(c_3)$ and it follows that the eigenfunction ϕ_1 is given by $\phi_1(x) = \sin(\sqrt{\lambda_1}\, x) - \sqrt{\lambda_1}\, x\cos\sqrt{\lambda_1}\, L$ where λ_1 is the smallest positive solution of the equation $\sqrt{\lambda}\, L = \tan\sqrt{\lambda}\, L$. A graphical or numerical estimate of λ_1 reveals that $\lambda_1 \cong (4.4934)^2/L^2$.

25c. Assuming $\lambda \ne 0$ the eigenvalue equation is $2(1-\cos x) = x\sin x$, where $x = \sqrt{\lambda}\, L$. Graphing $f(x) = 2(1-\cos x)$ and $g(x) = x\sin x$ we see that there is an intersection for $6 < x < 7$. Since both $f(x)$ and $g(x)$ are zero for $x = 2\pi$, this then is the precise root and thus $\lambda_1 = (2\pi)^2/L^2$. In addition, it appears there might be an intersection for $0 < x < 1$. Using a Taylor series representation for $f(x)$ and $g(x)$ about $x = 0$, however, shows there is no intersection for $0 < x < 1$. Of course $x = 0$ is also an intersection, which yields $\lambda = 0$, which gives the trivial solution and hence $\lambda = 0$ is not an eigenvalue.

Section 11.3, Page 651

1. We must first find the eigenvalues and normalized eigenfunctions of the associated homogeneous problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) = 0$. This problem has the solutions $\phi_n(x) = k_n \sin n\pi x$, for $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$.

Choosing k_n so that $\int_0^1 \phi_n^2 dx = 1$ we find $k_n = \sqrt{2}$. Hence the solution of the original nonhomogeneous problem is given by $y = \sum_{n=1}^{\infty} b_n \phi_n(x)$, where the coefficients b_n are found from Eq.(12), $b_n = c_n/(\lambda_n - 2)$ where c_n is given by $c_n = \sqrt{2}\int_0^1 x \sin n\pi x dx$ (Eq.9). [Note that the original problem can be written as $-y'' = 2y + x$ and therefore comparison with Eq.(1) yields $r(x) = 1$ and $f(x) = x$]. Integrating the expression for c_n by parts yields $c_n = \sqrt{2}(-1)^{n+1}/n\pi$ and thus $y = \sum_{n=1}^{\infty} \frac{\sqrt{2}(-1)^{n+1}}{(n^2\pi^2 - 2)n\pi} \sqrt{2} \sin n\pi x$.

2. From Problem 1 of Section 11.2 we have $\phi_n = \sqrt{2}\sin[(2n-1)\pi x/2]$ for $\lambda_n = (2n-1)^2\pi^2/4$ and from Problem 7 of that section we have $c_n = 4\sqrt{2}(-1)^{n+1}/(2n-1)^2\pi^2$. Substituting these values into $b_n = c_n/(\lambda_n - 2)$ and $y = \sum_{n=1}^{\infty} b_n \phi_n$ yields the desired result.

3. Referring to Problem 3 of Section 11.2 we have $y = b_0 + \sum_{n=1}^{\infty} b_n(\sqrt{2}\cos n\pi x)$, where $b_n = c_n/(\lambda_n - 2)$ for $n = 0,1,2,\ldots$. The rest of the calculations follow those of Problem 1.

5. Note that the associated eigenvalue problem is the same as for Problem 1 and that $|1-2x| = 1-2x$ for $0 \le x \le 1/2$ while $|1-2x| = 2x-1$ for $1/2 \le x \le 1$.

8. Writing the D.E. in the form Eq.(1), we have $-y'' = \mu y + f(x)$, so $r(x) = 1$. The associated eigenvalue problem is $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$ which has the eigenvalues $\lambda_n = n^2\pi^2$, $n = 0,1,2\ldots$ and the normalized eigenfunctions $\phi_0 = 1$, $\phi_n(x) = \sqrt{2}\cos n\pi x$, $n = 1,2\ldots$, as found in Problem 3, Section 11.2. Now, we assume $y(x) = b_0 + \sum_{n=1}^{\infty} b_n\phi_n(x) = \sum_{n=0}^{\infty} b_n\phi_n(x)$, Eq.(5), and thus $b_n = \frac{c_n}{\lambda_n - \mu}$, where $c_n = \int_0^1 f(x)\phi_n(x)dx$,

$n = 0,1,2...$, Eq.(9). Thus

$y(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x) = -c_0/\mu + \sqrt{2}\sum_{n=1}^{\infty} \frac{c_n \cos n\pi x}{\lambda_n - \mu}$, where we have assumed $\mu \neq \lambda_n$, $n = 0,1,2...$.

10. Since $\mu = \pi^2$ is an eigenvalue of the corresponding homogeneous equation, Theorem 11.3.1 tells us that a solution will exist only if $-(a+x)$ is orthogonal to the corresponding eigenfunction $\sqrt{2}\sin\pi x$. Thus we require $\int_0^1 (a+x)\sin\pi x\,dx = 0$, which yields $a = -1/2$. With $a = -1/2$, we find that $Y = (x-1/2)/\pi^2$ and $y_c = c\sin\pi x + d\cos\pi x$ by methods of Chapter 3. Setting $y = y_c + Y$ and choosing d to satisfy the B.C. we obtain the desired family of solutions.

11. Note that in this case $\mu = 4\pi^2$ and $\phi_2 = \sqrt{2}\sin 2\pi x$ are the eigenvalue and eigenfunction respectively of the corresponding homogeneous equation. However, there is no value of a for which $-\sqrt{2}\int_0^1 (a+x)\sin 2\pi x\,dx = 0$, and thus there is no solution.

12. In this case a solution will exist only if $-a$ is orthogonal to $\sqrt{2}\cos\pi x.$, that is if $\int_0^1 a\cos\pi x\,dx = 0$. Since this condition is valid for all a, a family of solutions exists.

14. Since $\sum_{n=1}^{\infty} c_n\phi_n(x)$ converges to zero we have $\sum_{n=1}^{\infty} c_n\phi_n(x) = 0$. Multiplying and integrating as suggested yields

$\int_0^1 [\sum_{n=1}^{\infty} c_n\phi_n(x)]r(x)\phi_m(x)\,dx = 0$ or

$\sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_n(x)\phi_m(x)\,dx = 0$. The integral that multiplies c_n is just δ_{nm}[Eq.(22) of Section 11.2]. Thus the infinite sum becomes c_m and the last equation yields $c_m = 0$.

18. A twice differentiable function v satisfying the boundary conditions can be found by assuming that $v = ax + b$. Thus $v(0) = b = 1$ and $v(1) = a + 1$ while $v'(1) = a$. Hence $2a + 1 = -2$ or $a = -3/2$ and $v(x) = 1 - 3x/2$.

Assuming $y = u + v$ we have $(u+v)'' + 2(u+v) = u'' + 2u + 2(1-3x/2) = 2 - 4x$ or $u'' + 2u = -x$, $u(0) = 0$, $u(1) + u'(1) = 0$ which is the same as Example 1 of the text.

19. From Eq.(30) we assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$, where the ϕ_n are the eigenfunctions of the related eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$ and the $b_n(t)$ are given by Eq.(42). From Problem 2, we have $\phi_n = \sqrt{2}\sin[(2n-1)\pi x/2]$ and $\lambda_n = (2n-1)^2\pi^2/4$. To evaluate Eq.(42) we need to calculate $\alpha_n = \int_0^1 \sin(\pi x/2)\sqrt{2}\sin[(2n-1)\pi x/2dx$ [Eq.(41) with $r(x) = 1$ and $f(x) = \sin(\pi x/2)$], which is zero except for $n = 1$ in which case $\alpha_1 = \sqrt{2}/2$, and $\gamma_n = \int_0^1 (-x)\sqrt{2}\sin[(2n-1)\pi x/2]dx$ [Eq.(35) with $F(x,t) = -x$]. This integral is the negative of the c_n in Problem 2 and thus $\gamma_n = -4\sqrt{2}(-1)^{n+1}/(2n-1)^2\pi^2 = -c_n$, $n = 1,2,\ldots$. Setting $\gamma_n = -c_n$ in Eq.(42) we then have

$$b_1 = \frac{\sqrt{2}}{2}e^{-\pi^2 t/4} - c_1\int_0^t e^{-\pi^2(t-s)/4}ds$$

$$= \frac{\sqrt{2}}{2}e^{-\pi^2 t/4} - \frac{4c_1}{\pi^2}e^{-\pi^2(t-s)/4}\Big|_0^t$$

$$= \frac{\sqrt{2}}{2}e^{-\pi^2 t/4} - \frac{4c_1}{\pi^2} + \frac{4c_1}{\pi^2}e^{-\pi^2 t/4} \text{ and}$$

similarly $b_n = -c_n\int_0^t e^{-\lambda_n(t-s)}ds = -(c_n/\lambda_n)e^{-\lambda_n(t-s)}\Big|_0^t$

$= -(c_n/\lambda_n)(1-e^{-\lambda_n t})$, where $\lambda_n = (2n-1)^2\pi^2/4$, $n = 2,3,\ldots$. Substituting these values for b_n along with $\phi_n = \sqrt{2}\sin[(2n-1)\pi x/2]$ into the series for $u(x,t)$ yields the solution to the given problem.

22. In this case $\alpha_n = 0$ for all n and γ_n is given by

$\gamma_n = \int_0^1 e^{-t}(1-x)\sqrt{2}\sin[(2n-1)\pi x/2]dx$

$= e^{-t}\int_0^1 (1-x)\sqrt{2}\sin[(2n-1)\pi x/2]dx$. This last integral can be written as the sum of two integrals, each of which has been evaluated in either Problem 6 or 7 of Section

11.2. Letting c_n denote the value obtained, we then have $\gamma_n = c_n e^{-t}$ and thus $b_n = c_n\int_0^t e^{-\lambda_n(t-s)}e^{-s}ds =$ $c_n e^{-\lambda_n t}\int_0^t e^{(\lambda_n-1)s}ds = [c_n/(\lambda_n-1)](e^{-t}-e^{-\lambda_n t})$, where $\lambda_n = (2n-1)^2\pi^2/4$. Substituting these values into Eq.(30) yields the desired solution.

24. Using the approach of Problem 23 we find that v(x) satisfies $v'' = 2$, $v(0) = 1$, $v(1) = 0$. Thus $v(x) = x^2 + c_1x + c_2$ and the B.C. yield $v(0) = c_2 = 1$ and $v(1) = 1 + c_1 + 1 = 0$ or $c_1 = -2$. Hence $v(x) = x^2 - 2x + 1$ and $w(x,t) = u(x,t) - v(x)$ where, from Problem 23, we have $w_t = w_{xx}$, $w(0,t) = 0$, $w(1,t) = 0$ and $w(x,0) = x^2 - 2x + 2 - v(x) = 1$. This last problem can be solved by methods of this section or by methods of Chapter 10. Using the approach of this section we have

$$w(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) \text{ where } \phi_n(x) = \sqrt{2}\sin n\pi x$$

[which are the normalized eigenfunctions of the associated eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) = 0$] and the b_n are given by Eq.(42). Since the P.D.E. for w(x,t) is homogeneous Eq.(42) reduces to $b_n = \alpha_n e^{-\lambda_n t}$ ($\lambda_n = n^2\pi^2$ from the above eigenvalue problem), where $\alpha_n = \int_0^1 1\cdot\sqrt{2}\sin n\pi x dx = \sqrt{2}\,[1-(-1)^n]/n\pi$. Thus

$$u(x,t) = x^2-2x + 1 + \sum_{n=1}^{\infty}\frac{\sqrt{2}\,[1-(-1)^n]}{n\pi}\, e^{-n^2\pi^2 t}\sqrt{2}\sin n\pi x,$$

which simplifies to the desired solution.

28a. Since $y_c = c_1+c_2x$, we assume that $Y(x) = u_1(x) + xu_2(x)$. Then $Y' = u_2$ since we require $u_1' + xu_2' = 0$. Differentiating again yields $Y'' = u_2'$ and thus $u_2' = -f(x)$ by substitution into the D.E. Hence $u_2(x) = -\int_0^x f(s)ds$, $u_1' = xf(x)$, and $u_1(x) = \int_0^x sf(s)ds$. Therefore $Y = \int_0^x sf(s)ds - x\int_0^x f(s)ds = -\int_0^x (x-s)f(s)ds$ and $\phi(x) = c_1 + c_2x - \int_0^x (x-s)f(s)ds$.

28b. From part a we have $y(0) = c_1 = 0$. Thus $y(x) = c_2x - \int_0^x (x-s)f(s)ds$ and hence $y(1) = c_2 - \int_0^1 (1-s)f(s)ds = 0$, which yields the desired value of c_2.

28c. From parts a and b we have

$$\phi(x) = x\int_0^1 (1-s)f(s)ds - \int_0^x (x-s)f(s)ds$$

$$= \int_0^x x(1-s)f(s)ds + \int_x^1 x(1-s)f(s)ds - \int_0^x (x-s)f(s)ds$$

$$= \int_0^x (x-xs-x+s)f(s)ds + \int_x^1 x(1-s)f(s)ds$$

$$= \int_0^x s(1-x)f(s)ds + \int_x^1 x(1-s)f(s)ds.$$

28d. We have

$$\phi(x) = \int_0^1 G(x,s)f(s)ds$$

$$= \int_0^x G(x,s)f(s)ds + \int_x^1 G(x,s)f(s)ds$$

$$= \int_0^x s(1-x)ds + \int_x^1 x(1-s)ds,$$ which is the same as found in part c.

30b. In this case $y_1(x) = \sin x$ and $y_2(x) = \sin(1-x)$ [assume $y_2(x) = c_1\cos x + c_2\sin x$, let $x = 1$, solve for c_2 in terms of c_1 using $y(1) = 0$ and then let $c_1 = \sin 1$]. Using these functions for y_1 and y_2 we find $W(y_1,y_2) = -\sin 1$ and thus $G(x,s) = -\sin s \sin(1-x)/(-\sin 1)$, since $p(x) = 1$, for $0 \le s \le x$. Interchanging the x and s verifies $G(x,s)$ for $x \le s \le 1$.

30c. Since $W(y_1,y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ we find that

$$[p(x)W(y_1,y_2)(x)]' = p'(x)[y_1(x)y_2'(x) - y_2(x)y_1'(x)]$$

$$+ p(x)[y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_2'(x)y_1'(x) - y_2(x)y_1''(x)]$$

$$= y_1[py_2']' - y_2[py_1']' = y_1[q(x)y_2] - y_2[q(x)y_1] = 0.$$

30d. Let $c = p(x)W(y_1,y_2)(x)$. If $0 \le s \le x$, then $G(x,s) = -y_1(s)y_2(x)/c$. Since the first argument in $G(s,x)$ is less than the second argument, the bottom expression of formula (iv) must be used to determine $G(s,x)$. Thus, $G(s,x) = -y_1(s)y_2(x)/c$. A similar argument holds if $x \le s \le 1$.

30e. We have $\phi(x) = \int_0^1 G(x,s)f(s)ds$

$= -\int_0^x \frac{y_1(s)y_2(x)f(s)}{c}ds - \int_x^1 \frac{y_1(x)y_2(s)f(s)}{c}ds$

(where $c = p(x)W(y_1,y_2)$ and thus, by Leibnitz's rule,

$c\phi'(x) = -y_1(x)y_2(x)f(x) - \int_0^x y_1(s)y_2'(x)f(s)ds + y_1(x)y_2(x)f(x)$

$-\int_x^1 y_1'(x)y_2(s)f(s)ds$. From this we obtain

$-c(p\phi')' = (py_2')'\int_0^x y_1(s)f(s)ds + py_2'y_1f(x)$

$+ (py_1')'\int_x^1 y_2(s)f(s)ds - py_1'y_2f(x)$. Dividing by c and adding $q(x)\phi(x)$ we get

$(-p\phi')'+q\phi = \frac{(py_2')'}{c}\int_0^x y_1(s)f(s)ds - \frac{qy_2}{c}\int_0^x y_1(s)f(s)ds$

$+ \frac{(py_1')'}{c}\int_x^1 y_2(s)f(s)ds - \frac{qy_1}{c}\int_x^1 y_2(s)f(s)ds + f(x)$

$= \frac{(py_2')'-qy_2}{c}\int_0^x y_1(s)f(s)ds + \frac{(py_1')'-qy_1}{c}\int_x^1 y_2(s)f(s)ds + f(x)$

$= f(x)$,

since y_1 and y_2 satisfy $L[y] = 0$. Using $\phi(x)$ and $\phi'(x)$ as found above, the B.C. are both satisfied since $y_1(x)$ satisfies one B.C. and $y_2(x)$ satisfies the other B.C.

33. In general $y(x) = c_1\cos x + c_2\sin x$. For $y'(0) = 0$ we must choose $c_2 = 0$ and thus $y_1(x) = \cos x$. For $y(1) = 0$ we have $c_1\cos 1 + c_2\sin 1 = 0$, which yields $c_2 = -c_1(\cos 1)/\sin 1$ and thus $y_2(x) = c_1\cos x - c_1(\cos 1)\sin x/\sin 1$

$= c_1(\sin 1\cos x - \cos 1\sin x)/\sin 1$

$= \sin(1-x)$ [by setting $c_1 = \sin 1$].

Furthermore, $W(y_1,y_2) = -\cos 1$ and thus

$$G(x,s) = \begin{cases} \frac{\cos s \sin(1-x)}{\cos 1} & 0 \le s \le x \\ \frac{\cos x \sin(1-s)}{\cos 1} & x \le s \le 1 \end{cases},$$

and hence

$\phi(x) = \int_0^x [\cos s \sin(1-x)f(s)/\cos 1]ds$

$+ \int_x^1 [\cos x \sin(1-s)f(s)/\cos 1]ds$ is the solution of the given B.V.P.

Section 11.4, Page 661

2a. The D.E. is the same as Eq.(6) and thus, from Eq.(9), the general solution of the D.E. is $y = c_1 J_0(\sqrt{\lambda}\,x) + c_2 Y_0(\sqrt{\lambda}\,x)$. The B.C. at $x = 0$ requires that $c_2 = 0$, and the B.C. at $x = 1$ requires $c_1\sqrt{\lambda}\,J_0'(\sqrt{\lambda}\,) = 0$. For $\lambda = 0$ we have $\phi_0(x) = J_0(0) = 1$ and if λ_n is the n^{th} positive root of $J_0'(\sqrt{\lambda}\,) = 0$ then $\phi_n(x) = J_0(\sqrt{\lambda_n}\,x)$. Note that for $\lambda = 0$ the D.E. becomes $(xy')' = 0$, which has the general solution $y = c_1 \ln x + c_2$. To satisfy the bounded conditions at $x = 0$ we must choose $c_1 = 0$, thus obtaining the same solution as above.

2b. For $n \neq 0$, set $y = J_0(\sqrt{\lambda_n}\,x)$ in the D.E. and integrate from 0 to 1 to obtain $-\int_0^1 (xJ_0')'dx = \lambda_n \int_0^1 xJ_0(\sqrt{\lambda_n}\,x)dx$. Integrating the left side of this equation yields $\int_0^1 (xJ_0')'dx = xJ_0'(\sqrt{\lambda_n}\,x)\Big|_0^1 = J_0'(\sqrt{\lambda_n}\,) - 0 = 0$ since the λ_n are eigenvalues from part a. Thus $\int_0^1 xJ_0(\sqrt{\lambda_n}\,x)dx = 0$. For other n and m, we let $L[y] = -(xy')'$. Then $L[J_0(\sqrt{\lambda_n}\,x)] = \lambda_n xJ_0(\sqrt{\lambda_n}\,x)$ and $L[J_0(\sqrt{\lambda_m}\,x)] = \lambda_m xJ_0(\sqrt{\lambda_m}\,x)$. Multiply the first equation by $J_0(\sqrt{\lambda_m}\,x)$, the second by $J_0(\sqrt{\lambda_n}\,x)$, subtract the second from the first, and integrate from 0 to 1 to obtain $\int_0^1 \{J_0(\sqrt{\lambda_m}\,x)L[J_0(\sqrt{\lambda_n}\,x)] - J_0(\sqrt{\lambda_n}\,x)L[J_0(\sqrt{\lambda_m}\,x)]\}dx = (\lambda_n - \lambda_m)\int_0^1 xJ_0(\sqrt{\lambda_n}\,x)J_0(\sqrt{\lambda_m}\,x)dx$. Again the left side is zero after each term is integrated by parts once, as was done above. If $\lambda_n \neq \lambda_m$, the result follows with $\phi_n(x) = J_0(\sqrt{\lambda_n}\,x)$.

2c. Since $\lambda = 0$ is an eigenvalue we assume that $y = b_0 + \sum_{n=1}^{\infty} b_n J_0(\sqrt{\lambda_n}\,x)$. Since $-[xJ_0'(\sqrt{\lambda_n}\,x)]' = \lambda_n xJ_0(\sqrt{\lambda_n}\,x)$, $n = 0,1,\ldots$, we find that

$-(xy')' = x\sum_{n=1}^{\infty}\lambda_n b_n J_0(\sqrt{\lambda_n}\,x)$ [note that $\lambda_0 = 0$ and b_0 are missing on the right]. Now assume $f(x)/x = c_0 + \sum_{n=1}^{\infty}c_n J_0(\sqrt{\lambda_n}\,x)$. Multiplying both sides by $xJ_0(\sqrt{\lambda_m}\,x)$, integrating from 0 to 1 and using the orthogonality relations of part b, we find $c_n = \int_0^1 f(x)J_0(\sqrt{\lambda_n}\,x)dx / \int_0^1 xJ_0^2(\sqrt{\lambda_n}\,x)dx$, n = 0,1,2,... . [Note that $c_0 = 2\int_0^1 f(x)dx$ since the denominator can be integrated.] Substituting the series for y and f(x)/x into the D.E., using the above result for $-(xy')'$, and simplifying we find that $(\mu b_0 + c_0) + \sum_{n=1}^{\infty}[c_n - b_n(\lambda_n - \mu)]J_0(\sqrt{\lambda_n}\,x) = 0$. Thus $b_0 = -c_0/\mu$ and $b_n = c_n/(\lambda_n - \mu)$, n = 1,2,..., where $\sqrt{\lambda_n}$ are obtained from $J_0'(\sqrt{\lambda_n}) = 0$.

4a. Let $L[y] = -[(1-x^2)y']'$. Then $L[\phi_n] = \lambda_n\phi_n$ and $L[\phi_m] = \lambda_m\phi_m$. Multiply the first equation by ϕ_m, the second by ϕ_n, subtract the second from the first, and integrate from 0 to 1 to obtain $\int_0^1(\phi_m L[\phi_n] - \phi_n L[\phi_m])dx = (\lambda_n - \lambda_m)\int_0^1 \phi_n\phi_m dx$. The integral on the left side can be shown to be 0 by integrating each term once by parts. Since $\lambda_n \neq \lambda_m$ if $m \neq n$, the result follows. Note that the result may also be written as $\int_0^1 P_{2m-1}(x)P_{2n-1}(x)dx = 0$, $m \neq n$.

4b. First let $f(x) = \sum_{n=1}^{\infty}c_n\phi_n(x)$, multiply both sides by $\phi_m(x)$, and integrate term by term from x = 0 to x = 1. The orthogonality condition yields

$c_n = \int_0^1 f(x)\phi_n(x)dx / \int_0^1 \phi_n^2(x)dx$, n = 1,2,... where it is understood that $\phi_n(x) = P_{2n-1}(x)$. Now assume

$y = \sum_{n=1}^{\infty}b_n\phi_n(x)$. As in Problem 2 and in the text

$-[(1-x^2)y']' = \sum_{n=1}^{\infty}\lambda_n b_n \phi_n$ since the ϕ_n are eigenfunctions. Thus, substitution of the series for y and f into the D.E. and simplification yields $\sum_{n=1}^{\infty}[b_n(\lambda_n-\mu) - c_n]\phi_n(x) = 0$. Hence $b_n = c_n/(\lambda_n - \mu)$, n = 1,2,... and the desired solution is obtained [after setting $\phi_n(x) = P_{2n-1}(x)$].

Section 11.5, Page 666

1a. Since u(x,0) = 0 we have Y(0) = 0. However, since the other two boundaries are given by y = 2x and y = 2(x-2) we cannot separate x and y dependence and thus neither X nor Y satisfy homogeneous B.C. at both end points.

1b. The line y = 2x is transformed into $\xi = 0$ and y = 2(x-2) is transformed into $\xi = 2$. The lines y = 0 and y = 2 are transformed into $\eta = 0$ and $\eta = 2$ respectively, so the parallelogram is transformed into a square of side 2. From the given equations, we have $x = \xi + \eta/2$ and $y = \eta$. Thus
$u_\xi = u_x x_\xi + u_y y_\xi = u_x$ and
$u_\eta = u_x x_\eta + u_y y_\eta = 1/2u_x + u_y$. Likewise
$u_{\xi\xi} = u_{xx} - x_\xi + u_{xy} y_\xi = u_{xx}$
$u_\eta = u_{xx} x_\eta + u_{xy} y_\eta = 1/2u_{xx} + u_{xy}$ and
$u_{\eta\eta} = 1/2u_{xx}x_\eta + 1/2u_{xy}y_\eta + u_{yx}x_\eta + u_{yy}y_\eta$
$= 1/4u_{xx} + u_{xy} + u_{yy}$. Therefore,
$5/4u_{\xi\xi} - u_{\xi\eta} + u_{\eta\eta} = u_{xx} + u_{yy} = 0$.

1c. Substituting $u(\xi,\eta) = U(\xi)V(\eta)$ into the equation of part b yields $5/4U''V - U'V' + UV'' = 0$ or upon dividing by UV $\frac{5}{4}\frac{U''}{U} + \frac{V''}{V} = \frac{U'V'}{UV}$, which is not separable. The B.C. become $U(\xi,0) = 0$, $U(\xi,2) = f(\xi+1)$ (since $x = \xi + \eta/2$), $U(0,\eta) = 0$, and $U(2,\eta) = 0$.

2. This problem is very similar to the example worked in the text. The fundamental solutions satisfying the P.D.E.(3), the B.C. $u(1,t) = 0$, $t \geq 0$ and the finiteness condition are given by Eqs.(15) and (16). Thus assume u(r,t) is of the form given by Eq.(17). The I.C. require that $u(r,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = 0$ and

$u_t(r,0) = \sum_{n=1}^{\infty} \lambda_n a k_n J_0(\lambda_n r) = g(r)$. From Eq.(26) of Section 11.4 we obtain $c_n = 0$ and $\lambda_n k_n a = \int_0^1 r g(r) J_0(\lambda_n r)dr / \int_0^1 r J_0^2(\lambda_n r)dr$, $n = 1,2,\ldots$.

4. This problem is the same as Problem 22 of Section 10.7. The periodicity condition requires that μ of that problem be an integer and thus substituting $\mu^2 = n^2$ into the previous results yields the given equations.

5a. Substituting $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ into Laplace's equation yields $R''\Theta Z + R'\Theta Z/r + R\Theta''Z/r^2 + R\Theta Z'' = 0$ or equivalently $R''/R + R'/rR + \Theta''/r^2\Theta = - Z''/Z = \sigma$. In order to satisfy arbitrary B.C. it can be shown that σ must be negative, so assume $\sigma = -\lambda^2$, and thus $Z'' - \lambda^2 Z = 0$ and, after some algebra, it follows that $r^2R''/R + rR'/R + \lambda^2 r^2 = -\Theta''/\Theta = \alpha$. The periodicity condition $\Theta(0) = \Theta(2\pi)$ requires that $\sqrt{\alpha}$ be an integer n so $\alpha = n^2$. Thus $r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$, $\Theta'' + n^2\Theta = 0$, and $Z'' - \lambda^2 Z = 0$.

5b. If $u(r,\theta,z)$ is independent of θ, then the $\Theta''/r^2\Theta$ term does not appear in the second equation of part a and thus $R''/R + R'/rR = - Z''/Z = -\lambda^2$, from which the desired result follows.

6. Assuming that $u(r,z) = R(r)Z(z)$ it follows from Problem 5 that $R = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$, from Eq.(13), and $Z = k_1 e^{-\lambda z} + k_2 e^{\lambda z}$. Since $u(r,z)$ is bounded as $r \to 0$ and approaches zero as $z \to \infty$ we require that $c_2 = 0$, $k_2 = 0$. The B.C. $u(1,z) = 0$ requires that $J_0(\lambda) = 0$ leading to an infinite set of discrete positive eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n \ldots$. The fundamental solutions of the problem are then $u_n(r,z) = J_0(\lambda_n r)e^{-\lambda_n z}$, $n = 1,2,\ldots$. Thus assume $u(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)e^{-\lambda_n z}$. The B.C. $u(r,0) = f(r)$, $0 \le r \le 1$ requires that $u(r,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r)$ so $c_n = \int_0^1 r f(r) J_0(\lambda_n r)dr / \int_0^1 r J_0^2(\lambda_n r)dr$, $n = 1,2,\ldots$.

7b. Again, Θ periodic of period 2π implies $\lambda^2 = n^2$. Thus the solutions to the D.E. are $R(r) = c_1 J_n(kr) + c_2 Y_n(kr)$ (note that λ and k here are the reverse of Problem 3 of Section 11.4) and $\Theta(\theta) = d_1 \cos n\theta + d_2 \sin n\theta$, $n = 0,1,2\ldots$. For the solution to remain bounded, $c_2 = 0$ and thus

$$v(r,\theta) = (1/2)c_0 J_0(kr) + \sum_{m=1}^{\infty} J_m(kr)(b_m \sin m\theta + c_m \cos m\theta).$$

Hence $v(c,\theta)$ is then a Fourier Series of period 2π and the coefficients are found as in Section 10.2, Eqs.(13), (14) and Problem 27.

9a. Substituting $u(\rho,\theta,\phi) = P(\rho)\Theta(\theta)\Phi(\phi)$ into Laplace's equation leads to
$\rho^2 P''/P + 2\rho P'/P = -(\csc^2\phi)\Theta''/\Theta - \Phi''/\Phi - (\cot\phi)\Phi'/\Phi = \sigma$.
In order to satisfy arbitrary B.C. it can be shown that σ must be positive, so assume $\sigma = \mu^2$.
Thus $\rho^2 P'' + 2\rho P' - \mu^2 P = 0$. Then we have
$(\sin^2\phi)\Phi''/\Phi + (\sin\phi\cos\phi)\Phi'/\Phi + \mu^2\sin^2\phi = -\Theta''/\Theta = \alpha$.
The periodicity condition $\Theta(0) = \Theta(2\pi)$ requires that $\sqrt{\alpha}$ be an integer λ so $\alpha = \lambda^2$. Hence $\Theta'' + \lambda^2\Theta = 0$ and
$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi - \lambda^2)\Phi = 0$.

10. Since u is independent of θ, only the first and third of the Eqs. in 9a hold. The general solution to the Euler equation is

$P = c_1\rho^{r_1} + c_2\rho^{r_2}$ where $r_1 = (-1+\sqrt{1+4\mu^2})/2 > 0$ and $r_2 = (-1-\sqrt{1+4\mu^2})/2 < 0$. Since we want u to be bounded as $\rho \to 0$, we set $c_2 = 0$. As found in Problem 22 of Section 5.3, the solutions of Legendre's equation, Problem 9c, are either singular at 1, at -1, or at both unless $\mu^2 = n(n+1)$, where n is an integer. In this case, one of the two linearly independent solutions is a polynomial denoted by P_n (Problems 23 and 24 of Section 5.3). Since $r_1 = (-1 + \sqrt{1+4n(n+1)})/2 = n$, the fundamental solutions of this problem satisfying the finiteness condition are $u_n(\rho,\phi) = \rho^n P_n(s) = \rho^n P_n(\cos\phi)$, $n = 1,2,\ldots$. It can be shown that an arbitrary piecewise continuous function on $[-1,1]$ can be expressed as a linear combination of Legendre polynomials. Hence we assume that

$u(\rho,\phi) = \sum_{n=1}^{\infty} c_n \rho^n P_n(\cos\phi)$. The B.C. $u(1,\phi) = f(\phi)$ requires that $u(1,\phi) = \sum_{n=1}^{\infty} c_n P_n(\cos\phi) = f(\phi)$, $0 \le \phi \le \pi$. From Problem 28 of Section 5.3 we know that $P_n(x)$ are orthogonal. However here we have $P_n(\cos\phi)$ and thus we must rewrite the equation in Problem 9b to find $-[(\sin\phi)\Phi']' = \mu^2(\sin\phi)\Phi$. Thus $P_n(\cos\phi)$ and $P_m(\cos\phi)$ are orthogonal with weight function $\sin\phi$. Thus we must multiply the series expansion for $f(\phi)$ by $\sin\phi P_m(\cos\phi)$ and integrate from 0 to π to obtain $c_m = \int_0^{\pi} f(\phi)\sin\phi P_m(\cos\phi)d\phi / \int_0^{\pi} \sin\phi P_m^2(\cos\phi)d\phi$. To obtain the answer as given in the text let $s = \cos\phi$.

Section 11.6, Page 675

2a. $b_m = \sqrt{2}\int_0^1 x\sin m\pi x dx = \sqrt{2}\,(-1)^{m+1)}/m\pi$ and thus

$$S_n = \frac{2}{\pi}\sum_{m=1}^{n} \frac{(-1)^{m+1}\sin m\pi x}{m}.$$

2b. We have $R_n = \int_0^1 [x - S_n(x)]^2 dx$, where $S_n(x)$ is given in part a. Using appropriate computer software we find $R_1 = .1307$, $R_2 = .0800$, $R_5 = .0367$, $R_{10} = .0193$, $R_{15} = .0131$ and $R_{19} = .0104$.

2c. As in part b, we find $R_{20} = .0099$, and thus n = 20 will insure a mean square error less than .01.

4a. Write $S_n(x)$ as $n\sqrt{x}/e^{nx^2/2}$ and use L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{n\sqrt{x}}{e^{nx^2/2}} = \lim_{n\to\infty} \frac{\sqrt{x}}{\frac{x^2}{2}e^{nx^2/2}} = 0 \text{ for } x \ne 0. \text{ For } x = 0,$$

$S_n(0) = 0$ and thus $\lim_{n\to\infty} S_n(x) = 0$ for all x in [0,1].

$R_n > \int_0^1 [0 - S_n(x)]^2 dx = n^2\int_0^1 x e^{-nx^2} dx$

$= -\frac{n}{2}e^{nx^2}\Big|_0^1 = \frac{n}{2}(1-e^{-n})$. Since $e^{-n} \to 0$ as $n \to \infty$, we have that $R_n \to \infty$ as $n \to \infty$.

5. Expanding the integrand we get

$$R_n = \int_0^1 r(x)[f(x) - S_n(x)]^2 dx$$

$$= \int_0^1 r(x)f^2(x)dx - 2\sum_{i=1}^{n} c_i \int_0^1 r(x)f(x)\phi_i(x)dx + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \int_0^1 r(x)\phi_i(x)\phi_j(x)dx,$$

where the last term is obtained by calculating $S_n^2(x)$. Using Eqs.(1) and (9) this becomes

$$R_n = \int_0^1 r(x)f^2(x)dx - 2\sum_{i=1}^{n} c_i a_i + \sum_{i=1}^{n} c_i^2$$

$$= \int_0^1 r(x)f^2(x)dx - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} (c_i - a_i)^2,$$ by completing the square. Since all terms involve a real quantity squared (and $r(x) > 0$) we may conclude R_n is minimized by choosing $c_i = a_i$. This can also be shown by calculating $\partial R_n/\partial c_i = 2(c_i - a_i)$ and setting equal to zero.

7b. From part a we have $f_0(x) = 1$ and thus $f_1(x) = c_1 + c_2 x$ must satisfy $(f_0, f_1) = \int_0^1 (c_1 + c_2 x)dx = 0$ and $(f_1, f_1) = \int_0^1 (c_1 + c_2 x)^2 dx = 1$. Evaluating the integrals yields $c_1 + c_2/2 = 0$ and $c_1^2 + c_1 c_2 + c_2^2/3 = 1$, which have the solution $c_1 = \sqrt{3}$, $c_2 = -2\sqrt{3}$ and thus $f_1(x) = \sqrt{3}(1-2x)$.

7c. $f_2(x) = c_1 + c_2 x + c_3 x^2$ must satisfy $(f_0, f_2) = 0$, $(f_1, f_2) = 0$ and $(f_2, f_2) = 1$.

7d. For $g_2(x) = c_1 + c_2 x + c_3 x^2$ we have $(g_0, g_2) = 0$ and $(g_1, g_2) = 0$, which yield the same ratio of coefficients as found in 7c. Thus $g_2(x) = c f_2(x)$, where c may now be found from $g_2(1) = 1$.

8. This problem follows the pattern of Problem 7 except now the limits on the orthogonality integral are from -1 to 1. That is $(P_i, P_j) = \int_{-1}^{1} P_i(x)P_j(x)dx = 0$, $i \neq j$. For $i = 0$ we have $P_0(x) = 1$ and for $i = 0$ and $j = 1$ we

have$(P_0,P_1) = \int_{-1}^{1}(c_1+c_2x)dx = (c_1x+c_2x^2/2)\Big|_{-1}^{1} = 2c_1 = 0$ and thus $P(1) = 1$ yields $P_1(x) = x$. The others follow in a similar fashion.

9a. This part has essentially been worked in Problem 5 by setting $c_i = a_i$.

9b. Eq.(6) shows that $R_n \geq 0$ since $r(x) \geq 0$ and thus $\int_0^1 r(x)f^2(x)dx - \sum_{i=1}^{n} a_i^2 \geq 0$. The result follows.

9c. Since f is square integrable, $\int_0^1 r(x)f^2(x)dx = M < \infty$ and therefore the monotone increasing sequence of partial sums $T_n = \sum_{i=1}^{n} a_i^2$ is bounded above. Thus $\lim_{n\to\infty} T_n$ exists, which proves the convergence of the given sum.

9d. This result follows from part a and part c.

9e. By definition if $\sum_{i=1}^{\infty} a_i\phi_i(x)$ converges to f(x) in the mean, then $R_n \to 0$ as $n \to \infty$. Hence $\int_0^1 r(x)f^2(x)dx = \sum_{i=1}^{\infty} a_i^2$. Conversely, if $\int_0^1 r(x)f^2(x)dx = \sum_{i=1}^{\infty} a_i^2$, $\lim_{n\to\infty} R_n = 0$ and $\sum_{i=1}^{\infty} a_i\phi_i(x)$ converges to f(x) in the mean.

10. Bessel's inequality implies that $\sum_{i=1}^{\infty} a_i^2$ converges and thus the n^{th} term $a_n \to 0$ as $n \to \infty$.

12. If the series were the eigenfunction series for a square integrable function, the series $\sum_{i=1}^{\infty} a_i^2$ would have to converge. But $a_0 = 1$, $a_1 = 1/\sqrt{2}, \ldots, a_n = 1/\sqrt{n}, \ldots,$ and $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} 1/n$ is the well-known harmonic series which does not converge.

ODE ARCHITECT

Companion



ODE ARCHITECT

Companion

C·ODE·E

(Consortium for ODE Experiments)



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PREFACE

This workbook was designed to accompany the software package ODE Architect, and that's why we call it a *Companion*. Each of the 13 Companion chapters corresponds to a multimedia module in the Architect and provides background and opportunities for you to extend the ideas contained in the module. Each chapter ends with several problem sets, called Explorations, related to the chapter and module topics. The Exploration pages can be photocopied so that you can write in answers and derivations, and hand them in along with printouts of graphs produced by the Architect. There is also a notepad facility in the Architect which, with the cut and paste features, makes it possible to write reports.

ODE Architect

ODE Architect provides a highly interactive environment for constructing and exploring your own mathematical models of real-world phenomena, whether they lead to linear or nonlinear systems of ODEs. The Architect's multimedia front end guides you through experiments to build and explore your own ODEs. The software has numerical solvers, 2- and 3-D graphics, and the ability to build physical representations of systems such as pendulums and spring-mass systems as well as the ability to animate them. Together with its library of ODEs, the ODE Architect brings a wealth of opportunities to gain insights about solutions to ODEs.

The overall guiding feature is for the software to be easy to use. Navigational paths are clearly marked and simple to follow. When starting the software, you are presented with a title screen followed by a main menu allowing selection of a specific module. You may prefer to go directly to the Architect Tool to run your own experiments. At any place in the software, you will be able to call up the contents menus and access the material in any order. We expect that most will work through the multimedia modules. Let's look at each of the three principal parts of the ODE Architect in more detail: The Multimedia ODE Architect, The ODE Architect Tool, and the ODE Library.

Multimedia ODE Architect

C·ODE·E members and colleagues have authored the multimedia modules, each with its own theme. The modeling process is detailed, supported by highly interactive simulations. You will explore the problem-solving process via "what-if" scenarios and exercises. You are guided to build your own ODEs and solve them numerically and graphically, and compare the predicted results to empirical data when appropriate.

Each module has up to four submodules, and they range from the straightforward to the advanced. The animations are often funny, the voice-overs and text informal, but the modeling and the mathematics are the real thing. Most submodules go through a model-building process and several experiment screens, and then end with some questions (Things-to-Think-About, or TTAs). These questions extend the topics of the submodule and take you to the solver tool to produce solution curves and orbits, or write a report connecting the mathematics, the models, and the pictures. When you open the Tool using a TTA link, the pertinent equations and parameter settings will automatically be entered into the equation quadrant of the Tool. You are then poised to think about, and without constraint, explore the model introduced in the submodule.

ODE Architect Tool

The ODE Architect Tool is a first-rate, research-quality numerical ODE solver and graphics package. The ODE Architect Tool employs a graphical user interface to enter and edit equations, control solver settings and features, and to create and edit a wide variety of graphics. A second mode of operation, the Expert Mode, provides access to more advanced features.

The Tool is the heart of the software, and it is a workspace where you can:

- Construct, solve, and explore ODEs
- Input data tables
- Graph and animate solution curves, phase plane graphs, 3D graphs, Poincaré sections, discrete maps, direction fields, etc.
- Build, analyze, and animate physical representations of dynamical systems.

The robust Tool will solve systems of up to 10 first-order ODEs which can be entered using a simple, natural scripting language. Auxiliary functions involving the state variables can be defined. A solver/grapher feature for discrete dynamical systems is also available from the Tool. A variety of engineering functions such as square waves, sawtooth waves, and step functions are included in the Tool function library. Two- and three-dimensional graphics are supported, as well as time and parameter animations of solution

data. Initial conditions can be entered by clicking in a graph window or via the keyboard. Graph scales can be set automatically or manually. Numerical values of solutions can be viewed in tabular form. Parameter-sensitive analysis is made easy with a built-in parameter-sweep tool. You can do parameter and initial-value sweeps to see the effects of data changes on orbits and solution curves. Graphs are editable and you can scale and label axes, mark equidistant-in-time orbital points, color the graphs, change line styles, overlay graphs of functions and solution curves for different ODEs—all with no programming or special commands to remember.

The solvers in the ODE Architect are state-of-the-art numerical solvers based on those developed by Dr. L.F. Shampine and Dr. I. Gladwell at Southern Methodist University. For a delightfully readable account on using numerical ODE solvers in teaching ODEs, please refer to their paper:

> Shampine, L.F., and Gladwell, I., "Teaching Numerical Methods in ODE Courses"

in the book *Revolutions in Differential Equations*, edited by Michael J. Kallaher in the MAA Notes series.

Module 1, "Modeling with the ODE Architect", is an on-line tutorial for many of the features of the Tool. The Architect also has help facilities and the multimedia side is self-documenting.

ODE Library

The ODE Library has dozens of pre-programmed, editable, and interactive ODE files covering a wide range of topics from mathematics, physics, chemistry, population biology, and epidemiology. There are also many ODEs to illustrate points such as data compression, ODEs with singular coefficients, bifurcations, limit cycles, and so on. Each Library file has explanatory text along with the equations and includes an illustrative graph or graphs. The Library files are organized into folders by topic and they have descriptive titles to facilitate browsing. These files also provide a marvelous way to learn how to use the tool.

ACKNOWLEDGMENTS

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INFORMATION ABOUT MODULES/CHAPTERS

Overview

Modules/Chapters 1–3 are all introductory modules for first-order ODEs and simple systems of ODEs. Any of these modules/chapters can be used at the beginning of an ODE course, or at appropriate places in elementary calculus courses.

Modules/Chapters 4–9 involve higher-order ODEs and systems and their applications. Once students understand how to deal with two-dimensional systems graphically, any of these modules/chapters is easily accessible.

Modules/Chapters 10–12 apply two-dimensional systems to models that illustrate more advanced techniques and theory; the multimedia approach makes them nevertheless quite accessible. The modules are intended to enable students to get much further with the technical aspects explained in the chapters than would be otherwise possible.

Module/Chapter 13 treats discrete dynamical systems in an introductory fashion that could be used in a course in ODEs, calculus, or even a non-calculus course.

A Multimedia appendix on numerical methods gives insight into the ways in which numerical solutions are constructed.

Description/Prerequisites for Individual Modules/Chapters

We list below for each Module/Chapter its prerequisites and some comments on its level and goals. In general, each module progresses from easier to harder submodules, but the first section of nearly every module is at an introductory level.

The modules can be accessed in different orders. It is not expected that they will be assigned in numerical order. Consequently, we have tried to explain each concept wherever it appears, or to indicate where an explanation is provided. For example, Newton's second law, $F = ma$, is described every time it is invoked.

There is far more material in ODE Architect than could possibly fit into a single course.

Module/Chapter 1: Modeling with the ODE Architect

Assumed concepts: Precalculus; derivative as a rate of change

This module is unlike all the others in that it is not divided into submodules, and it provides a tutorial for learning how to navigate ODE Architect. It carries that tutorial process along in tandem with an introduction to modeling that assumes very little background.

Module/Chapter 2: Introduction to ODEs

Assumed concepts: Derivatives; slopes; slope fields

The module begins with some simple first-order ODEs and their solutions and continues with slope fields (and a slope field game).

The Juggler and the Sky Diver submodules use second-order differential equations, but both the chapter and the module explain the transformation to systems of two first-order differential equations.

Module/Chapter 3: Some Cool ODEs

Assumed concepts: Basic concepts of first-order ODEs, solutions, and solution curves

Newton's law of cooling, and solving the resulting ODEs by separation of variables or as linear equations with integrating factors, are presented thoroughly enough that there need be no prerequisites.

The submodule for Cooling a House extends Newton's law of cooling to real world cases that are easily handled by ODE Architect (and not so easily by traditional methods). This section makes the point that rate equations and numerical solutions are often a much smarter way to go than to trudge toward a solution formula.

Module/Chapter 4: Second-Order Linear Equations

Assumed concepts: Euler's formula for complex exponentials

The module and chapter treat only constant coefficient ODEs. The chapter begins by demonstrating how to treat a second-order ODE as a system of first-order ODEs which can be entered in ODE Architect. Both the first submodule and the chapter explain from scratch all the traditional details of an oscillating system such as amplitude, period, frequency, damping, forcing, and beats.

The Seismograph submodule is a real world application. The derivation of the equation of motion is not simple, but the multimedia module gives insight into the workings of a seismograph, and it is not necessary to understand the details of the derivation to use and explore the modeling ODE.

Module/Chapter 5: Models of Motion

Assumed concepts: Newton's second law of motion

This module's collection of models of motion in one and two dimensions is supported by a chapter that gives background on vectors, forces, Newton's laws, and the details of the specific submodules; so it stands on its own without further prerequisites.

Module/Chapter 6: First-Order Linear Systems

Assumed concepts: Basic matrix notation and operations (multiplication, determinants); complex numbers; Euler's formula

This unit introduces all of the basic notions, both algebraic (emphasized in the chapter) and geometric (emphasized in the module), for linear systems. The central roles of eigenvalues and eigenvectors are explained. The Tool can be used to calculate eigenvalues and eigenvectors.

The Explorations bring in coupled tank problems (Chapter 8 introduces compartment models) and small motions of a double pendulum (which are extended in Chapter 7).

Module/Chapter 7: Nonlinear Systems

Assumed concepts: Equilibrium points; phase plane and component plots; matrices; eigenvalues and eigenvectors

The goal is to use graphical solutions to make handling nonlinear systems as easy (almost) as linear systems. Linearization of a nonlinear ODE is introduced as a basic concept, and the chapter goes on to elaborate perturbations and bifurcations. The Tool can be used to find equilibrium points, and calculate the Jacobian matrix and its eigenvalues/eigenvectors at each equilibrium point. The predator-prey and saxophone reed models are introduced and explained in the module while the spinning bodies and double pendulum models are treated in the chapter and also in the Library with an animated model linked to the ODE.

Module/Chapter 8: Compartment Models

Assumed concepts: Systems of ODEs

Both the module and the chapter use 1D, 2D, 3D, and 4D applications (in sequence) to illustrate principles of the Balance Law and interpretations of solutions. The final submodule introduces Hopf bifurcations and the interesting behavior of chemical reactions in an autocatalator. Three of the models are linear; the last is nonlinear.

Module/Chapter 9: Population Models

Assumed concepts: Systems of ODEs

The module and chapter introduce simple 1D, 2D, and 3D nonlinear models, and give a discussion of the biology behind the models.

Module/Chapter 10: The Pendulum and Its Friends

Assumed concepts: Systems of ODEs; the first submodule of Module 4; the arctangent function; parametric curves on a surface

The pendulum submodule explores all the traditional aspects of a pendulum, using integrals of motion. Child on a Swing and Geodesics on a Torus give new extensions of pendulum analysis; supporting detail is given in the chapter. The approach to modeling is a little different in this chapter—for example, how to invent functions that behave as needed (Child on a Swing), or how to exploit part of an ODE that looks familiar (Geodesics on a Torus).

Module/Chapter 11: Applications of Series Solutions

Assumed concepts: Systems of ODEs; acquaintance with infinite series and convergence; the first submodule of Module 4

The module introduces the techniques and limitations of series solutions of second-order linear ODEs. The Robot and Egg provides motivation for the subject and Aging Springs illustrates Bessel functions. The chapter contains information about the mathematics of series solutions.

Module/Chapter 12: Chaos and Control

Assumed concepts: The pendulum ODEs of Module 10; systems of ODEs; experience with Poincaré sections and/or discrete dynamical systems (Chapter 13) is helpful

The three submodules of this unit tell a story, and in the process illustrate a theorem from current research. This module uses sensitivity to initial conditions and the Poincaré section to assist with the analysis. Sinks, saddles, basins, and stability are described. Finally, the elusive boundaries of the Tangled Basin provide a mechanism for control of the chaotically wandering pendulum. The module ends in a fascinating control game that is both fun to play and illuminates the theorem mentioned above.

Module/Chapter 13: Discrete Dynamical Systems

Assumed concepts: Acquaintance with complex numbers and the ideas of equilibrium and stability are helpful

The module provides a gentle introduction to an increasingly important subject. The chapter fills in the technical and mathematical background.

This module could be used successfully in a liberal arts course for students with no calculus.

Level-of-Difficulty of Modules

The chart below is a handy reference for the levels of the submodules.

Elementary		Intermediate		Advanced
1				
2.1, 2.2	2.3, 2.4			
3.1	3.3	3.2		
4.1				4.2
5.1	5.2	5.3		
	6.1	6.2	6.3	
	7.1		7.2	7.3
8.1	8.2	8.3	8.4	
9.1	9.2	9.3		
		10.1	10.2	10.3
		11.1	11.2	
		12.1		12.2, 12.3
13.1	13.2		13.3	

In constructing this chart we have used the following criteria:

Elementary: Straightforward, self-contained, can be used as a unit in any introductory calculus or ODE course.

Intermediate: Builds on some prior experience, including earlier submodules and chapters.

Advanced: More challenging models or mathematics, especially suitable for term or group projects.

User's Guide

The *User's Guide* is the basic reference for the features of the ODE Architect Tool. The Guide is included on the CD-ROM for ODE Architect.

CONTENTS

1 Modeling with the ODE Architect 1

Douglas Campbell & Wade Ellis, West Valley Community College

Building a Model of the Pacific Sardine Population 2
The Logistic Equation 10
Introducing Harvesting via Landing Data 12
How to Model in Eight Steps 15
Explorations 17

2 Introduction to ODEs 25

Margie Hale & Michael Branton, Stetson University

Differential Equations 26
Solutions to Differential Equations 26
Solving a Differential Equation 27
Slope Fields 27
Initial Values 28
Finding a Solution Formula 28
Modeling 30
The Juggler 30
The Sky Diver 31
Explorations 35

3 Some Cool ODEs 43

Margie Hale, Stetson University & Douglas Quinney, University of Keele

Newton's Law of Cooling 44
Cooling an Egg 44
Finding a General Solution 44
Time-Dependent Outside Temperature 46
Air Conditioning a Room 47
The Case of the Melting Snowman 49
Explorations 51

4 Second-Order Linear Equations 57
William Boyce & William Siegmann, Rensselaer Polytechnic Institute

Second-Order ODEs and the Architect 58
Undamped Oscillations 58
The Effect of Damping 60
Forced Oscillations 61
Beats 63
Electrical Oscillations: An Analogy 64
Seismographs 64
Explorations 69

5 Models of Motion 77
Robert Borrelli & Courtney Coleman, Harvey Mudd College

Vectors 78
Forces and Newton's Laws 79
Dunk Tank 80
Longer to Rise or to Fall? 81
Indiana Newton 82
Ski Jumping 84
Explorations 85

6 First-Order Linear Systems 93
William Boyce & William Siegmann, Rensselaer Polytechnic Institute

Background 94
Examples of Systems: Pizza and Video, Coupled Springs 94
Linear Systems with Constant Coefficients 95
Solution Formulas: Eigenvalues and Eigenvectors 97
Calculating Eigenvalues and Eigenvectors 98
Phase Portraits 99
Using ODE Architect to Find Eigenvalues and Eigenvectors 102
Separatrices 103
Parameter Movies 103
Explorations 105

7 Nonlinear Systems 115
Michael Branton, Stetson University

Linear vs. Nonlinear 116
The Geometry of Nonlinear Systems 116
Linearization 117
Separatrices and Saddle Points 120

Behavior of Solutions Away from Equilibrium Points 121
Bifurcation to a Limit Cycle 122
Higher Dimensions 123
Spinning Bodies: Stability of Steady Rotations 123
The Planar Double Pendulum 126
Explorations 129

8 Compartment Models 135

Courtney Coleman & Michael Moody, Harvey Mudd College

Lake Pollution 136
Allergy Relief 137
Lead in the Body 139
Equilibrium 141
The Autocatalator and a Hopf Bifurcation 142
Explorations 147

9 Population Models 155

Michael Moody, Harvey Mudd College

Modeling Population Growth 156
The Logistic Model 156
Two-Species Population Models 158
Predator and Prey 159
Species Competition 160
Mathematical Epidemiology: The SIR Model 161
Explorations 163

10 The Pendulum and Its Friends 173

John Hubbard & Beverly West, Cornell University

Modeling Pendulum Motion 174
Conservative Systems: Integrals of Motion 176
The Effect of Damping 177
Separatrices 180
Pumping a Swing 182
Writing the Equations of Motion for Pumping a Swing 182
Geodesics 185
Geodesics on a Surface of Revolution 186
Geodesics on a Torus 188
Explorations 193

11 Applications of Series Solutions 203
Anne Noonburg & Ben Pollina, University of Hartford

Infinite Series 204
Recurrence Formulas 204
Ordinary Points 206
Regular Singular Points 207
Bessel Functions 209
Transforming Bessel's Equation to the Aging Spring Equation 210
Explorations 213

12 Chaos and Control 221
John Hubbard & Beverly West, Cornell University

Introduction 222
Solutions as Functions of Time 222
Poincaré Sections 223
Periodic Points 224
The Unforced Pendulum 224
The Damped Forced Pendulum 226
Tangled Basins, the Wada Property 226
Gaining Control 228
Explorations 231

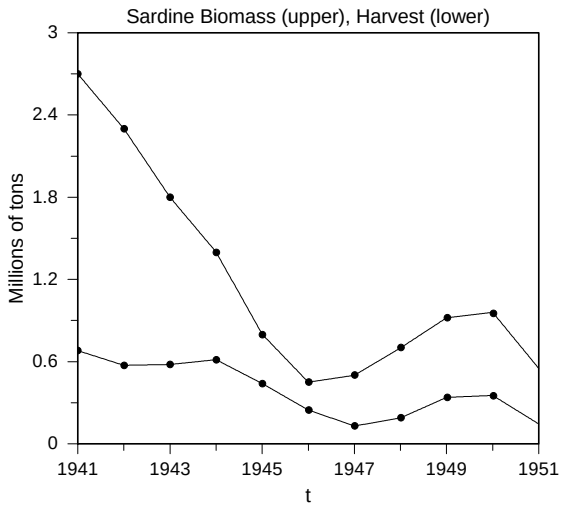
13 Discrete Dynamical Systems 233
Thomas LoFaro, Washington State University

Equilibrium States 235
Linear versus Nonlinear Dynamics 236
Stability of a Discrete Dynamical System 237
Bifurcations 238
Periodic and Chaotic Dynamics 240
What is Chaos? 241
Complex Numbers and Functions 242
Iterating a Complex Function 243
Julia Sets, the Mandelbrot Set, and Cantor Dust 244
Explorations 249

Glossary 257

CHAPTER 1

Modeling with the ODE Architect



Pacific sardine population and harvest.

Overview In the two decades from 1932 to 1951, the Pacific sardine fishery completely collapsed. In this chapter you will learn to use the ODE Architect to construct a mathematical model which describes this event rather well. This will have two purposes: it will familiarize you with the menus and features of the ODE Architect, and it will acquaint you with the principles of mathematical modeling.

First we'll construct a model for the Pacific sardine population during the years 1930–1950 as if it were unharvested. Then we will focus on the harvesting that actually took place and see how it contributed greatly to the destruction of the sardine population.

Key words Modeling; Pacific sardine; population model; initial conditions; exponential growth; carrying capacity; logistic equation; harvesting

See also Chapter 9 for more on population models.

◆ Building a Model of the Pacific Sardine Population

Step 1: State the problem and its context

☞ In this chapter we will build a model for the sardine fishery in California and also introduce features of the ODE Architect Tool for solving differential equations. Consult the *User's Guide* for a full description of all features of the tool.

The Pacific sardine (*Sardinops sagax caerulea*) has historically experienced long-range cycles of abundance and depletion off the West Coast of California. It was during one of the abundant periods, 1920 through 1951, that a huge sardine fishing and canning industry developed. The total catch for the California coastline reached a peak of 726,124 tons during the 1936–37 season (June through the following May). The Pacific sardine population then began a serious decline during the 1940s until, as one estimate has it, by 1959 the sardine biomass was 5% (0.2 million tons) of the 1934 level (4 million tons). (The *biomass* is the amount of a particular organism in its habitat.) There is general agreement that heavy harvesting played a role in the decimation of the Pacific sardine during that period. The fishing industry had a serious decline after the 1950–51 season: increasing numbers of fishermen went bankrupt or moved to other fisheries. Undoubtedly the canneries were also affected.

After 50 years of fishing for the Pacific sardine, a moratorium was imposed by the California legislature in 1967. The Pacific sardine seems to be making a comeback as of the mid-1980s, though the numbers are not yet near the abundant levels of the 1930s.

Here are the goals of your model:

☞ Problem statement.

1. Determine the extent to which the precipitous decline of the Pacific sardine population was due to over-harvesting from 1941 to 1951.
2. Ascertain an optimal harvest rate that would stabilize and sustain the sardine population during that time period.

Step 2: Identify and assign variables

Assigning the variables in a mathematical model is a skill that requires some practice. Doing some background reading and studying the context of the problem and the problem statement helps to clarify which are the most important features of the system you wish to model.

It turns out that there tend to be long-range cycles of Pacific sardine abundance and scarcity. These cycles are not yet completely understood, but it is certain that factors such as ocean temperature, nutrient upwelling from deep waters, currents that aid fish migration, predator populations of larger fish and of sea lions, and, of course, fishing, play a vital role in the cycles. In this model we will focus on a period when the harvesting of the sardine was very heavy. Due to the large magnitude of the harvesting, its effect was dominant for the period of time we will model, 1941–1951, so we will neglect the other factors. That the other factors still operate on the population is evidenced by the difficulty in getting the model to match the data perfectly. Nevertheless, you'll see how modeling, while not always explaining every aspect, provides insight into the dynamics of an otherwise very complex biological relationship.

Given the information we have at this point we need the following variables and parameters in our model:

☞ Sardine biomass is a state variable, the other quantities are parameters.

1. Sardine biomass (in units of million tons)
2. Growth rate for the Pacific sardine (in units of million tons/year)
3. Maximum biomass, or *carrying capacity* (in units of million tons)
4. Sardine harvesting (in units of million tons/year)

Note that we opted to define sardine biomass in million tons, rather than numbers of fish, to be consistent with the data and estimates used.

It's good practice to introduce as few parameters as necessary into a model at first. Additional parameters can be added if they are needed to improve the accuracy of the model. The model may be refined until the desired level of accuracy is achieved.

With the variables and parameters identified, the next step is to construct an equation for the rate of change of the state variable in terms of the state variable itself, the model's parameters, and possibly also time. This equation is known as a differential equation (abbreviated ODE). When an ODE is entered into ODE Architect along with an initial value of the state variable, the Architect Tool displays a graphical representation of the solution.

☞ Graphical representation of the problem.

Figure 1.1 represents the estimated Pacific sardine biomass and harvest during the period 1941–1951. Note again the use of sardine biomass in million tons, rather than numbers of fish.

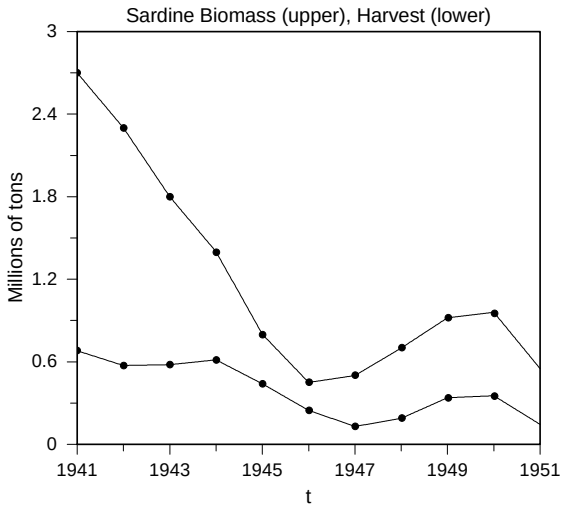


Figure 1.1: Sardine biomass and harvest.

Our first task is to use ODE Architect to build a mathematical model to simulate the growth and decline of the Pacific sardine biomass without harvesting. Then we can explore the impact of harvesting on that biomass.

☞ Click on the spinning orb to go directly to the Architect Tool.

Place the ODE Architect CD-ROM in your computer and start the ODE Architect Tool. Four quadrants will be displayed on the screen (see Figure 1.2). The upper left quadrant is the equation quadrant; it should be empty now. The two right-hand quadrants should be empty. These are plot quadrants that will display 2- or 3-dimensional plots when you solve differential equations. The lower left quadrant currently shows the initial conditions (**IC**) display. Notice that it is selectable using the four tabs (**IC**, **Sweep**, **Solver**, **Equilibrium**) on the lower edge of the quadrant. For now leave **IC** selected.

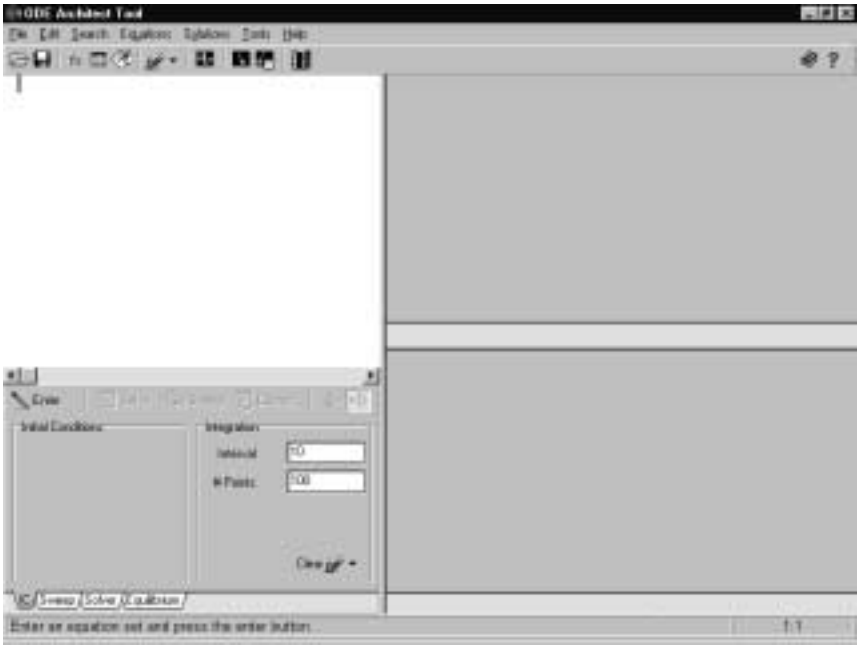


Figure 1.2: ODE Architect tool screen

Step 3: State the relationships that govern the variables

We begin by simulating the unchecked growth (no harvesting) of the Pacific sardine population, which we will designate as the state variable *sardine.* Basic biology suggests that it is reasonable to assume that the rate of biomass growth (i.e., the derivative $d(sardine)/dt$) at a given time t is *proportional* to the quantity of sardines (the size of the biomass) present at that time t.

Step 4: Translate the laws into equations

Since $sardine'$ is a common notation for the derivative (rate of change) of *sardine*, we can write

$$sardine' = r * sardine$$

where r is a proportionality factor that we will refer to as the *growth rate factor*.

Fishery and biomass data collected over the period 1932–1958 indicate that the Pacific sardine population has had a volatile history. The Pacific sardine biomass, if not manipulated or constrained, can grow at a rate of between 10% and 40% per year. We will assume a moderate position and set the growth rate factor at $r = 0.20$. A modeler often has to make assumptions and guess parameter values to get a model started; you can refine the assumptions later.

Step 5: Solve the resulting differential equations

☞ Entering differential equations.

Point and click the cursor in the equation quadrant and type in

$$sardine' = r * sardine \tag{1}$$

using an apostrophe for the prime, and an asterisk for multiplication; hit **Return** (or **Enter**) and assign the value 0.20 to the parameter r by typing in

$$r = 0.20$$

Now click the cursor on the box marked **Enter** just below the equation quadrant. Notice that this causes scales to appear in the two plot quadrants.

☞ Setting initial conditions.

Now go to the lower left quadrant to set the initial conditions. Double click in the appropriate box to select a variable; then type in the new value. Set **t** (time) to start at 1930 and set **sardine** to be 1 (unit of million tons). We'll go back later and put in a more realistic estimate for *sardine.* In the **Integration** panel, set the solve time to 20 by inserting the number 20 in the **Interval** box. Leave the default value of 100 in the **# Points** box.

☞ Setting the time interval.

Click the **Solve** icon and notice that the right arrow is automatically selected. Your screen will look something like this (Figure 1.3):

☞ Starting the solution.

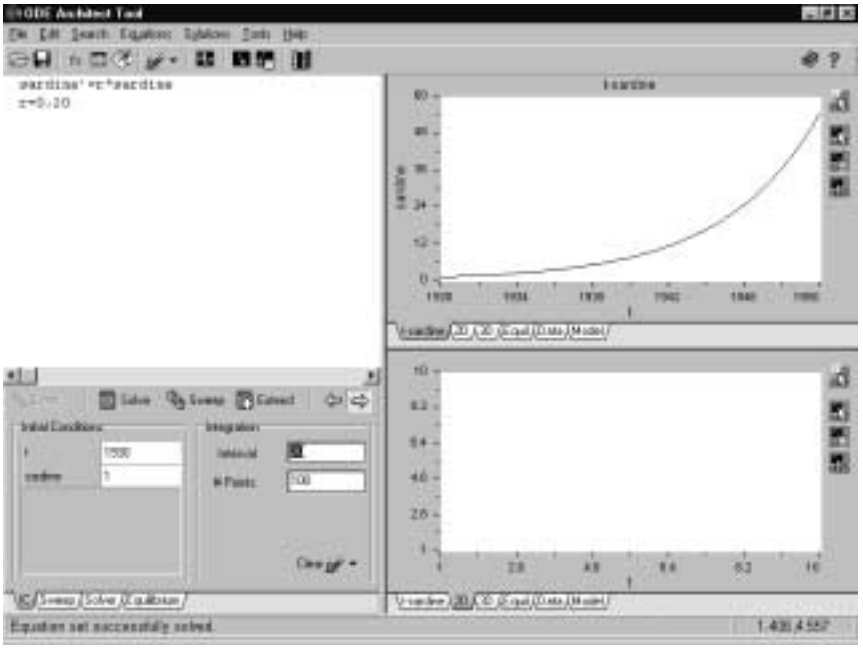


Figure 1.3: Exponentially growing sardine population.

Step 6: Interpret and test the solutions in context

☞ Unbounded growth.

There is now a classical exponential growth curve in the upper right quadrant. This implies that the sardine biomass grows without bound, which can't be true as there is not enough room on the planet! The exponential growth must be limited by factors like available food supplies, disease, predators, and so on; therefore we have to modify our model to reflect this fact. We learned earlier that the sardine biomass has been as large as 4 million tons, but we don't truly know the maximum sustainable biomass (*carrying capacity*), so to start let's assume a carrying capacity of 6 million tons. We can refine this guess later if we have trouble fitting the model to actual data. As we said before, it's not uncommon to have to make informed guesses for values that are not known or available. Then the values can perhaps be deduced by "fine tuning" (refining) the model in subsequent iterations to conform to reality.

Step 7: Refine the model to predict the empirical data

☞ Introduce a carrying capacity.

The following differential equation is sometimes used to exhibit maximum carrying capacity behavior in a population:

$$sardine' = r(capacity - sardine)$$

This equation says that the growth rate at any time is proportional to the "room to grow" factor $capacity - sardine$. Now click the cursor in the equations region. Using our assumed growth rate constant of $r = 0.20$ per year and a carrying capacity of 6 million tons, we modify the sardine growth to be

$$sardine' = r * (6 - sardine) \tag{2}$$

Before entering the new sardine ODE, clear the graphics screens by clicking on **Clear** at the lower left and choosing **Clear All Runs**. A "confirmation" window will pop up; click on **Yes**. Now click in the equation region, make the corrections to your equation, and then click on the box marked **Enter**. Click on the **Solve** icon and notice in the plot window that the graph of the sardine biomass climbs and levels out at the assumed carrying capacity of 6 million tons.

✓ "Check" your understanding by comparing this curve with the earlier one and notice some significant differences: (1.) The first curve was concave up; this one is concave down. Why is that significant? (2.) The first curve grew without bound and had no asymptotes; the second curve has a horizontal asymptote. Explain why.

Examine the two graphs carefully at early values of t, say the first five years. Recall that the slope of a line tangent to the solution curve is the growth rate of the biomass at that time. How do the two curves differ in this regard? When is the rate of change of the biomass the greatest? Is it realistic for a biomass to exhibit its greatest rate of increase when the population is smallest? The answer to these questions is not as simple as you might think. For

many biological populations, rate of change is proportional to the size of the population. The solution of ODE (1) exhibits this proportionality but it is unconstrained and so it's not useful over its whole domain. The solution curve of ODE (2) doesn't exhibit the proportional growth property. Which of the two is most appropriate for the Pacific sardine? We'll come back to that question after a little exploration with ODE (2).

☞ Exceed carrying capacity.

Now let's see if ODE (2) will allow us to exceed the carrying capacity for any length of time. Change the initial biomass to 12 million tons of sardines in the IC window. Click on the **Solve** icon. Notice that the vertical scale in the graph changes to accommodate the revised values and that both the old (lower) and the new (upper) curves are displayed on the graph. Observe what happened to the "overstocked" sardine population. How does this compare to what happened when the initial sardine biomass was 1 million tons? If you examine the two plots closely, you'll see that both plots stabilize at a level of about 6 million tons (see Figure 1.4).

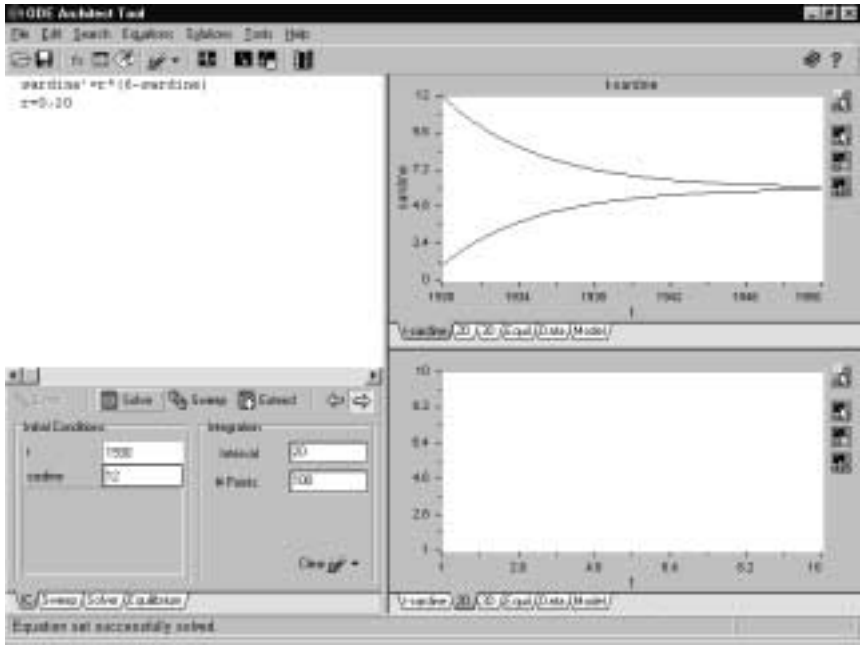


Figure 1.4: Sardine populations approach carrying capacity.

☞ To set scaling of axes.

Sometimes it's advantageous to change the scales of the axes to make graphs easier to read and interpolate, so we'd like to show you how to reset the vertical and horizontal scales. (The default setting for scales for ODE Architect is **Auto Scale**.) Select the upper right graph by placing the cursor arrow on the graph and clicking the *right-most* mouse button (or click on the icon at the upper right corner of the graph). You will see various plot window options presented. Select **Scales** from the resulting dropdown menu using the *left-most* mouse button. Click on **Auto Scale** to toggle it off. (The check in the

box will disappear.) To change other values, double click in the box to select the value and just type to make a change. On the **X-Scale** menu set **Minimum** = 1930; **Maximum** = 1950; **Number of Ticks** = 10; and **Label every** = 2. (Adjust the number of ticks by clicking on the down arrow and selecting, or by double clicking the box and typing in the new value.) Make sure **Linear** is selected (not **Log**). Now select the **Y-Scale** menu (at the top): click the **Auto Scale** to toggle it off; set **Minimum** = 0; **Maximum** = 12; **Number of Ticks** = 10; and **Label every** = 2; and check that the **Linear** button is selected. Your screen should have a window that looks like Figure 1.5:

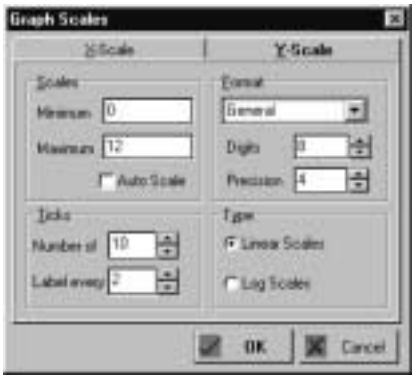


Figure 1.5: Plot scales window.

☞ You must click the **OK** button to enter changes.

Click on **OK** to cause the graph to be rescaled. (In this particular case, it turns out that the scale did not change from the automatically selected value.)

Now click on the **Clear** box in the **Integration** panel and choose **Clear All Runs**. What has changed? Next click on the **Solve** icon again. Notice that you got only the most recent curve (carrying capacity exceeded); you cleared the previous solution.

☞ Sweeping a variable.

We can extend the ease of making comparisons by sweeping through several possible initial values for *sardine* and displaying them all on one graph. Click on the **Clear** box and choose **Clear All Runs**. Now notice the **Sweep** tab beneath the **Initial Conditions** panel; click on **Sweep**. Click on **Single** for type of sweep. We will choose **Sweep 1** to be a sweep (multi-plot) of various initial values of *sardine*. In the **Sweep 1** box, click on the down arrow and select **sardine**. Set **Start** = 1; **Stop** = 7; and **# Points** = 3. Now click on the **Sweep** box next to the **Solve** icon (*not* the **Sweep** tab) (Figure 1.6).

☞ If the **# Points** is set to *n*, you'll get *n* overlayed graphs.

Notice that ODE Architect makes several runs. Notice also that the initial value for *sardine* located in the IC window was ignored and the values we entered in the sweep conditions were used instead.

To better see these results, let's rescale the vertical axis (**Y-Scale**) to **Minimum** = 1 and **Maximum** = 7. Look back at page 7 if you do not recall how to do this. Figure 1.7 shows that multiple runs are easily comparable in this format. Which initial value for *sardine* created the most stable or flattest curve? Does the population always stabilize around the same biomass?

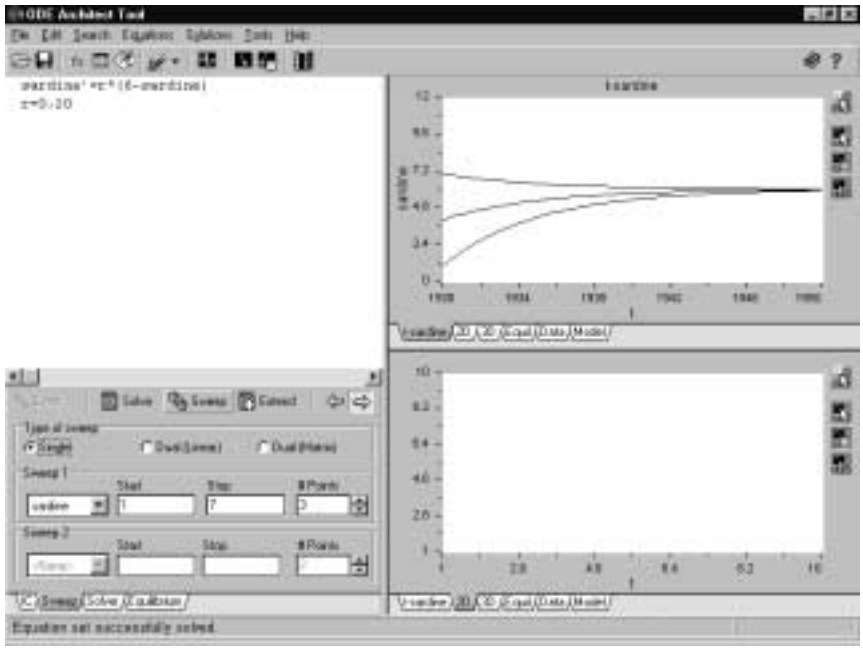


Figure 1.6: Setting up the sweep for *sardine* with three initial values 1, 4, 7.

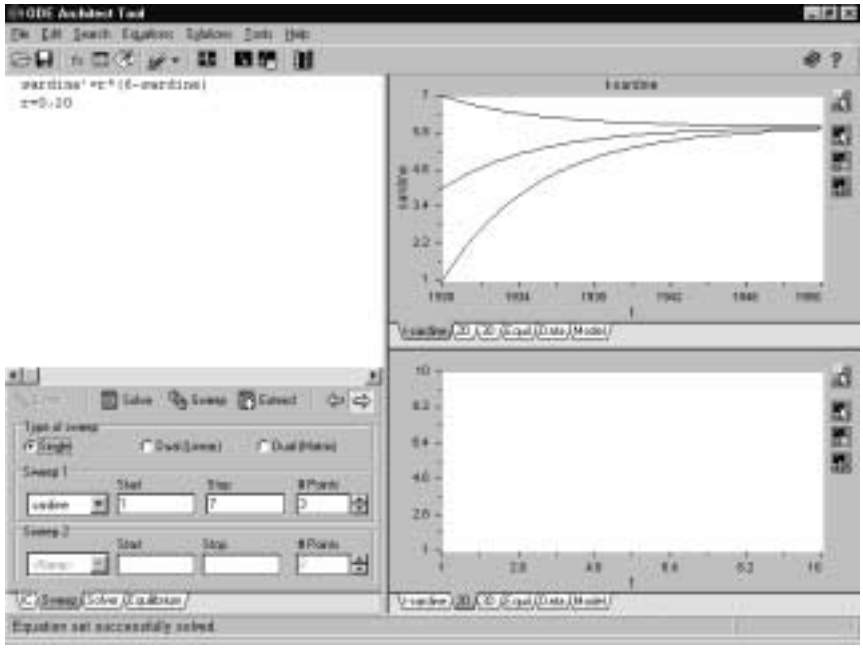


Figure 1.7: The sardine curves with a rescaled vertical axis.

Do the resulting curves accurately represent the growth you'd expect over the whole range of values for t? Growth is usually proportional to population size when well below the carrying capacity. However, when you look at your graph notice that for small populations of sardines, the growth rate is rather steep. As the sardine population approaches the carrying capacity the biomass should level off, which the preceding curves do reflect.

Now we'll examine the properties of the model we created in ODE (2). The growth rate is proportional to $(capacity - sardine)$ and so for small sardine biomass, the biomass grows at a nearly constant rate. Near the carrying capacity, the factor $(capacity - sardine)$ causes a leveling off (see Figure 1.7): the factor forces growth to be proportional to the distance from capacity.

◆ The Logistic Equation

Combining the elements of the proportional growth model given by ODE (1) and the restricted growth model given by ODE (2) leads to what is called the *logistic equation for growth* (or the *Verhulst equation*, after the nineteenth century Belgian mathematician and biologist P. F. Verhulst):

$$sardine' = r\ sardine\frac{6 - sardine}{6} \tag{3}$$

Notice that for values of *sardine* very near zero, the factor $r * sardine$ dominates the computation, causing approximate exponential growth behavior. This is because the factor $(6 - sardine)/6$ has a value very near 1. For values of *sardine* near 6 (the carrying capacity), the factor $(6 - sardine)/6$ is near zero, and so growth slows to approach zero. Therefore we can expect exponential growth for small biomass with growth tapering off as the biomass approaches carrying capacity. Let's see if this refinement improves the model.

Click on the **IC** tab to clear the graph and enter a new equation. After clicking on the **Clear** box, and choosing **Clear All Runs**, click in the equations quadrant and modify the growth ODE to read:

☞ Changing the equation.

$$sardine' = r * sardine * (6 - sardine)/6 \tag{4}$$

Don't forget to click the box labeled **Enter**. Reset the initial sardine biomass to **1**. Finally, click on the **Solve** icon. Your screen should look something like Figure 1.8.

Notice that the graph now displays a mathematical representation more like what we expect of the sardine biomass over the long term. It is an elongated S-shaped curve with slow growth for small biomass, maximum growth near the midrange, and slow growth near the carrying capacity.

☞ Try various initial values. As before, use a **Single** sweep.

Use the sweep feature now to see how the logistic growth curve responds for various initial conditions for the variable *sardine*. **Sweep 1** *sardine*; **Start** = 1; **Stop** = 7; **# points** = 4. Click on **Sweep**. Figure 1.9 shows the four solution curves.

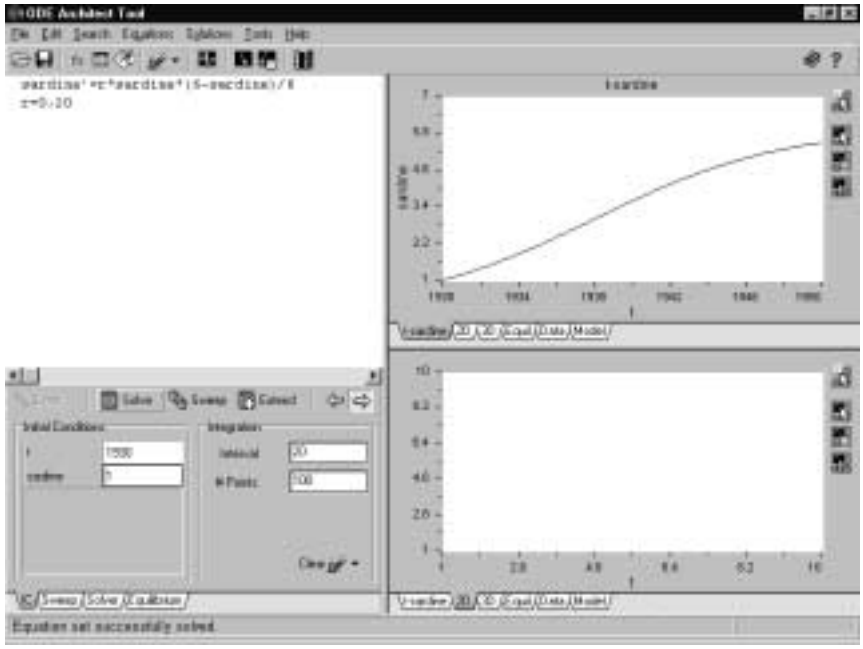


Figure 1.8: A logistic growth curve.

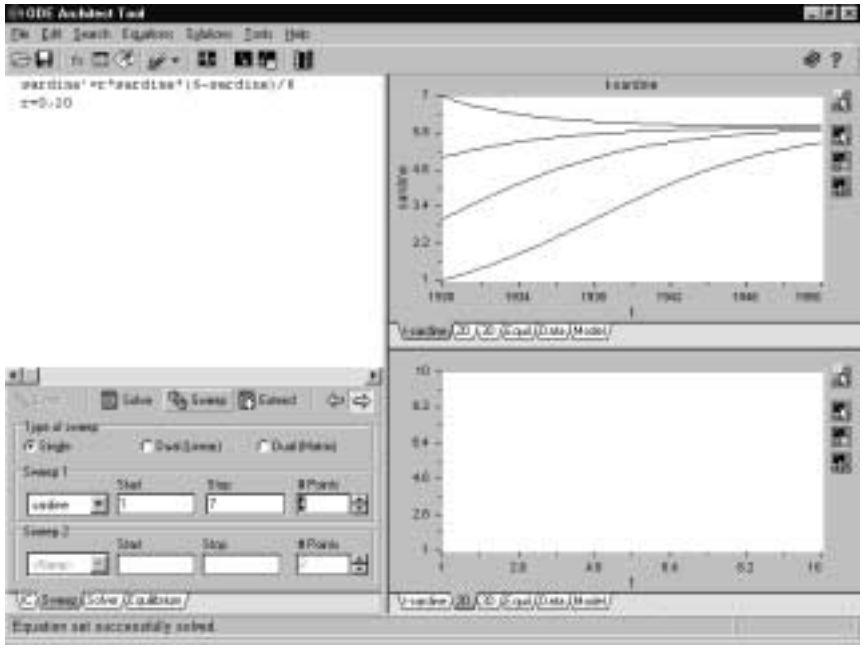


Figure 1.9: Four logistic solution curves.

✓ Does the model respond to your initial conditions in a reasonable manner? Do you think that this is a good population model to use for modeling the biomass of the Pacific sardine?

Let's now use the model given in equation (3) to explore the harvesting that took place in the years 1941–1951.

◆ Introducing Harvesting via Landing Data

In Figure 1.1 you saw a graph of the Pacific sardine harvest and the resulting biomass decline during the years 1941–51. We have not yet taken into account this harvest (or *landing*) data in our model; so our model does not yet reflect the collapse of the sardine fishery that occurred. We'll now incorporate the landing data into our model in the form of a lookup table.

The tutorial in Module 1 provides you with landing data for the Pacific sardine over the time period 1941–1951 in the form of a table with 11 rows and 2 columns (tutorial steps 13 and 14). This data can be entered as a *lookup table* named HTABLE by following these directions (which also appear in the tutorial):

- Start by clicking on the **Equations** entry on the menu bar and choosing **Lookup Tables** to display the lookup table manager window.
- Double click on <**Create New Table**> to display the new table window. Enter the name HTABLE, and specify 11 rows and 2 columns in the appropriate boxes. Then click the **OK** button. An array of empty cells will appear with 11 rows and 2 columns.
- Enter the data (provided in the tutorial window) in the array by clicking on cell $[1, 1]$ to start. When all of the data is entered, click the **OK** button.
- Close the lookup table manager window.

Now you have a lookup table called HTABLE.

☞ Defining the harvest.

Go to the equation quadrant and, on a new line, add the following

$$harvest = \text{lookupval}(\text{HTABLE}, 1, t, 2)$$

Be sure to click the **Enter** box. The value returned by lookupval is the data in column 2 of HTABLE corresponding to the t-value of the data in column 1. (If t is not an integer, the value returned is computed by linear interpolation.)

☞ Using the **2D** custom tab.

Let's now look at this harvest data. Since *harvest* is not an ODE state variable, the Architect does not automatically generate a plot tab; we will have to make it by hand. Click on the **2D** tab at the lower right to select what we want to plot on each of the two axes. Place the cursor on the lower right plot quadrant, and after clicking the *right-most* mouse button, select **Edit** with the *left-most* mouse button. Leave the **X-Axis** variable set to **t**. For the **Y-Axis**

click on the down arrow after **1:** <**None**> and select **harvest**. Now click on **Titles** at the top of the edit window and type in the **Graph Title** box: Harvest. In the **X-axis Title** box type: Year; and in the **Y-axis Title** box: Harvest. See Figure 1.10. Click **OK**. Using the *right-most* mouse button again on the lower graph, select **Scales**. Set the **X-Scale** as follows: deselect **Auto Scale**; set **Minimum** = 1940; **Maximum** = 1955; **Number of Ticks** = 3; **Label every** = 2. Select the **Y-Scale**, deselect **Auto Scale**, and set: **Minimum** = 0; **Maximum** = 1; **Number of Ticks** = 5; and **Label every** = 1. Click **OK**.

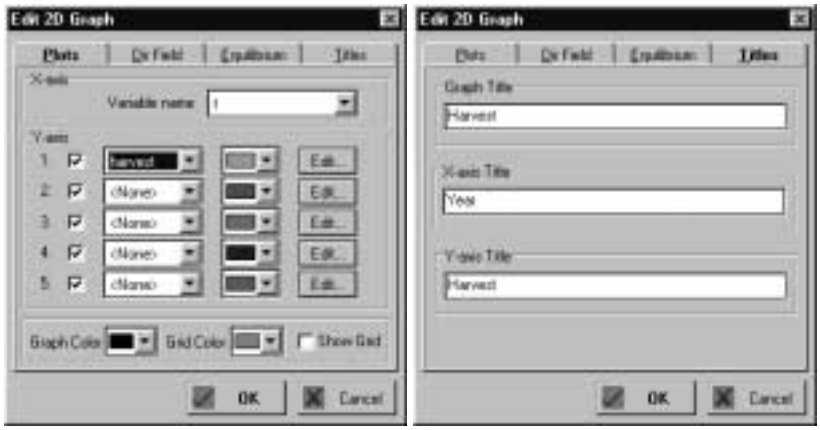


Figure 1.10: **Plots** and **Titles** panels for **2D** tab.

☞ See the lower right graph in Figure 1.11.

Click on the **Solve** icon. Notice that a graphical representation of the Pacific sardine landings appears in the lower graph but the upper graph has not been affected. That's because we have not included harvest (landings) in the sardine model yet. (Note: the two graphs have different vertical axis scales.)

☞ Including *harvest* in the sardine model.

After clicking on **Clear** and choosing **Clear All Runs**, go to the equation quadrant and modify the *sardine* ODE as follows:

$$sardine' = r * sardine * (6 - sardine)/6 - harvest \qquad (5)$$

You may have to scroll the equations quadrant (on some computers) in order to see the whole equation. (You can also move the dividing line between the right and left quadrants, at the slight expense of the graphing resolution.) Click the **Enter** box. Before we run the model, we must change the initial conditions to reflect the reality of the Pacific sardine population at that time. In the literature, the most reliable data for the Pacific sardine biomass starts in 1941. Thus set the **IC** for *t* to 1941 and the **IC** for *sardine* to 2.71. Reset **Interval** to 10. Now click the **Solve** icon and note the results. For best viewing of the top right graph window choose the **X-Scale**; deselect **Auto Scale**; set

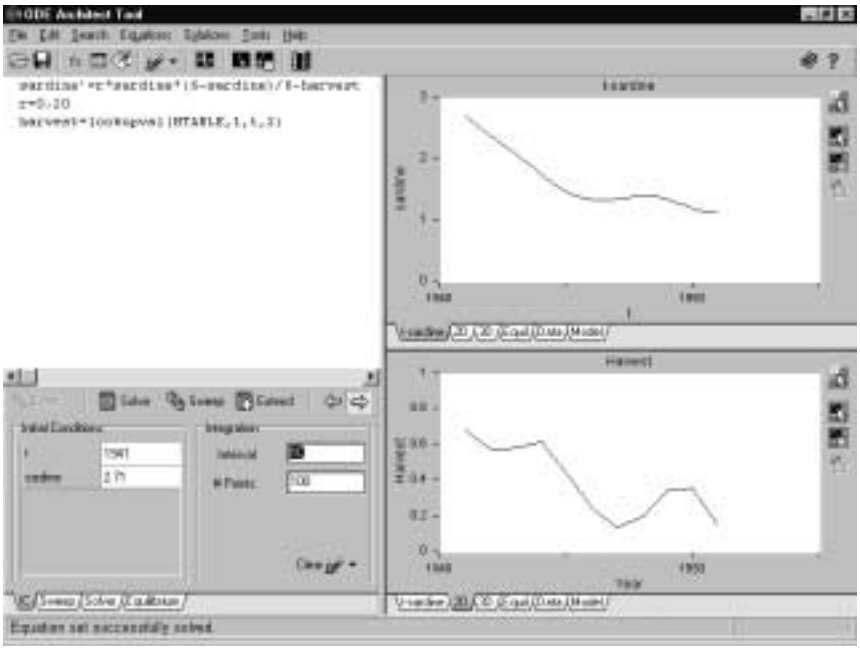


Figure 1.11: Model sardine biomass (upper), harvested biomass (lower).

Minimum = 1940; **Maximum** = 1955; **Number of Ticks** = 3; **Label every** = 2. Rescale the **Y-Scale** axis to: **Minimum** = 0; **Maximum** = 3; **Number of Ticks** = 3; **Label every** = 1. See Figure 1.11 for the graphs.

☞ Fishery collapse.

How do these model results compare with the expected behavior at the beginning of the chapter? While the overall behavior is captured in general terms by the model, it is unusual to have a model match the estimated data exactly.

✓ What are your thoughts about the model as it relates to historical behavior? Explain any discrepancies.

Step 8: Interpret the implications of the model

☞ Analysis.

It is now clear that while over-exploitation of the sardine landings was not the sole factor, it played a very large role in the collapse of the California fishery in the early 1950s. Since we now have a functioning model of that ten-year period in time, you have the amazing power to use your computer to revise history and attempt to save the fishing industry. What limit on the landings would have allowed a sizable sardine harvest[1] but not a collapse of the fishery?

[1]Historical note: A limit to the total catch of sardines at between 200,000 and 300,000 tons was recommended as early as 1929, and repeatedly over the next several years.

Now it is your turn to examine some options and try some alternate scenarios in the Explorations that follow.

◆ How to Model in Eight Steps

Modeling a situation mathematically involves many ideas and activities, but modeling is not always straightforward. There are many times when you may be puzzled, confused, and frustrated and you must retrace or rethink the steps involved. We summarize the steps in an order that allows for easy reference, but keep in mind the need to retreat, reassess, and redefine your thinking.

1. State the problem and its context.
2. Identify and assign variables.
3. State the laws that govern the relationships between the variables.
4. Translate the laws into equations.
5. Solve the resulting equations.
6. Interpret and test the solutions in the context of the natural environment.
7. Refine the model until it predicts the empirical data.
8. Interpret the implications of the model.

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Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 1.1. Constant Harvesting of a Biomass

1. *No harvesting.*
Let's examine the rate of growth (the derivative) of the sardine biomass using the logistic model of ODE (3). To do this we'll look at the values of $sardine'$ as a function of sardine biomass size. Go to the equations quadrant and type in the ODE

$$sardine' = r * sardine * (6 - sardine)/6$$
$$r = 0.20$$

Click the **Enter** box.

To create a plot of *sardine* vs. $sardine'$, select the **2D** tab (if necessary), place the cursor over the lower right graph, press the *right-most* mouse button, and select **Edit**. For the **X-Axis** use the down arrow to select *sardine*. For **Y-Axis 1**, select $sardine'$. Click now on the **Titles** tab at the top of the edit window. Type Rate of Growth vs. Biomass as the Graph Title, Sardine as the X-axis label and Sardine$'$ as the Y-axis label. Click **OK**. Place the cursor over the lower right graph again, press the *right-most* mouse button and select **Auto Scales: Both** (if necessary). Next set the **IC** for t to 1941 and the **IC** for *sardine* to 1, and set the **Interval** to 20. Click **Clear** and select **Clear All Runs** (if necessary), then click the **Solve** icon.

The top graph shows (by default) *sardine* vs. time. Notice in the lower graph that the sardine growth rate, $sardine'$, is maximized somewhere near a midsized sardine population of about 3 million tons. Rescale the Y-axis of the top graph (if necessary) to **Minimum** = 0; **Maximum** = 6; **Number of Ticks** = 6. Verify that the sardine biomass grows at the rate of approximately 10% to 40% per year, depending upon the size of the biomass.

2. *Constant harvesting.*
Let's analyze the effect of constant harvesting on the logistic sardine population of Problem 1. Since the sardine biomass was 2.71 million tons in 1941, reset the *sardine* IC to 2.71 and keep the *sardine* vertical scale set on the range 0 to 6 and re-solve to observe the relative stabilization of the population.

Now insert a constant harvesting term in the model by modifying the ODE in the equation quadrant to read

$$sardine' = r * sardine * (6 - sardine)/6 - harvest$$

Try a harvest value that is slightly less than the biomass growth amount for 2.71 million tons by setting a constant harvest in the equation quadrant. For example you could try $harvest = 0.28$ (280,000 tons per year) and solve the model. (Be sure to click on the **Enter** box first.)

☞ Clearing a 2D custom graph. The same procedure clears a 3D custom graph.

Now click on the **2D** tab in the lower graph quadrant. Clear the graph in that quadrant by setting all axes to <**None**> in the **Plots** tab of the **Edit** box, then going to the **Titles** tab and deleting all titles. For the upper right graph you can set up and run a sweep of *harvest* over the values 0.1 (100,000 tons/year) to 0.7 (700,000 tons/year) using 7 points in the sweep. Describe the biomass behavior for harvest levels of 0.1; 0.3; 0.5; 0.7. From your exploration, determine what constant harvest amount provides a large harvest yet does not jeopardize the long-term viability of the Pacific sardine population. Explain what you mean by "large harvest" and "long-term viability." Did the harvest levels suggested by fishery researchers stand up?

3. *How does the IC affect the optimum harvest level?*
Is the optimum harvest level that you determined in Problem 2 affected by the initial biomass of the sardine in 1941? Try some different values for the IC and explain what you learn about the relationship between initial biomass and the optimum constant harvest amount.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 1.2. Constant Effort Harvesting

1. *Using a constant effort harvesting function.*
Another model for harvesting is to land a certain percentage of the existing biomass each year. This is called *constant effort* (or *proportional*) *harvesting*. Introduce constant effort harvesting into ODE (5) by setting

$$harvest = 0.25 * sardine$$

to harvest 25% of the sardine population each year. Try a run. What happens?

Go back and revise the harvest function to

$$harvest = k * sardine$$

and sweep through several values of your choosing for the harvest coefficient k. Summarize your results. What is the optimum harvest coefficient? Explain what you mean by "optimum" harvest coefficient.

2. *How does the IC affect the optimum harvest percentage?*
Run some experiments to determine if the optimum harvest percentage you select in Problem 1 is sensitive to the initial biomass of the sardine in 1941. Explain your results. How do your results compare to the results of Problem 3 in Exploration 1.1?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 1.3. Investigating a Harvesting Function

1. *A unifying harvest strategy.*
We can combine the strategies used in Explorations 1.1 (Problem 2) and 1.2 (Problem 1) by using a function that approximates each strategy at the appropriate time: proportional harvest for small sardine biomass and constant harvest for sufficiently large sardine biomass. A function suitable to this purpose is

$$harvest = \frac{\alpha * sardine}{\beta + sardine}$$

Use some algebra to demonstrate that the function does behave as claimed. Approximately what is the proportional harvest coefficient? Approximately what is the constant harvest level?

2. *Testing the function.*
Determine values for α and β suitable for the Pacific sardine based on what you learned from Explorations 1.1 (Problem 2) and 1.2 (Problem 1). Is the optimal choice of α and β dependent on the initial biomass of the sardine?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 1.4. The Ricker Growth Rate Model

☞ Compare the Ricker with the logistic function: $R = rP(1 - P/K)$ for positive constants r and K.

Biologists commonly use the *Ricker function* to model fish population reproduction. The Ricker function is $R = \alpha P e^{(P_r - P)/P_m}$, where R is the reproduction rate, α is a constant, P is the parental or spawning stock population, P_r is the stock size at which $R = P$, and P_m is the stock size that yields maximum reproduction in the absolute sense. Calibrated for the Pacific sardine during the time period 1941 through 1951, this function is: $R = 0.15 P e^{(2.4-P)/1.7}$.

1. *The Ricker population model.*
Replace the logistic term in ODE (5) with the Ricker function to obtain

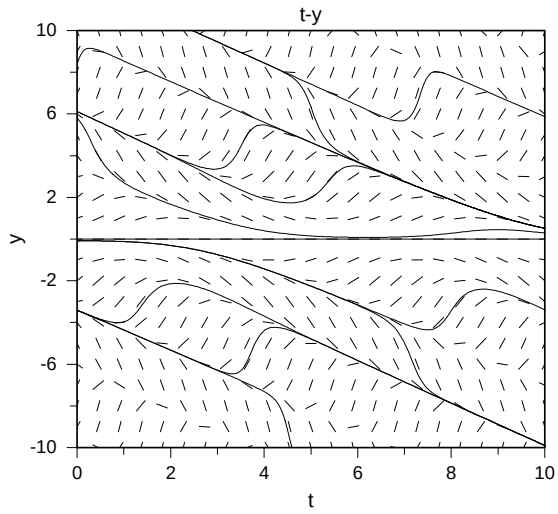
$$sardine' = 0.15 * (sardine) * \exp((2.4 - sardine)/1.7)$$

This function exhibits "compensatory behavior" that biologists know many fish populations exhibit. Plot two sardine populations vs. time on the same set of axes for comparison: $sardine1'$ as per the Ricker function above and $sardine2'$ as per the logistic growth model used earlier. You have to select the **2D** tab on the graphics window when defining the graph to get both populations on the same graph. To compare their respective growth patterns, plot the two sardine populations from 1920 to 1960 (**# Points** = 40) with **IC** set to 1 on both plots and with no harvesting. Based on this comparison, speculate what "compensatory" behavior is as envisioned by the biologists and reflected by the Ricker function.

2. *Repeat the harvest experiments.*
Repeat Exploration 1.1, Problem 2, using the Ricker function in the *sardine* ODE. What harvest level would provide a stable sustainable Pacific sardine population? Test whether the optimal harvest rate depends on the population IC. Are the results significantly different than when you used the logistic function?

CHAPTER 2

Introduction to ODEs



A slope field and some solution curves for $y' = y\sin(t + y)$.

Overview Ordinary differential equations (ODEs) model many natural processes, so solutions of ODEs can be used to predict the behavior of those processes.

This chapter will investigate ODEs and initial value problems, their solutions, and their solution curves, along with some methods for finding solution formulas. Slope fields are introduced and used as guides to the behavior of solution curves. The path of a juggler's ball and the descent of a sky diver are modeled by ODEs.

Key words Differential equation; solution; integration; separation of variables; initial values; modeling; slope field; direction field; juggling; sky diving; free fall; parachute; gravity; Newton's second law

See also Chapter 1 for more on modeling, and Chapter 5 for more on models of motion.

◆ Differential Equations

Differential equations were first used in the seventeenth century to describe physical phenomena, such as the motion of orbiting planets or swinging pendulums. Since then they have been applied to processes, such as the growth of biological populations, the management of investment portfolios, and many other dynamical systems.

An *ordinary differential equation* is an equation involving an unknown function of one variable and one or more of its derivatives. For example, the ODE

$$\frac{dy}{dt} = y\cos t$$

is a statement about an unknown function y (the dependent variable) whose independent variable is t. To *solve* the ODE we need to find all the functions $y(t)$ that satisfy the ODE (we will discuss what we mean by a solution in the next section).

✓ "Check" your understanding by identifying the independent and dependent variables and the *order* of each ODE (i.e., the highest-order derivative that appears):

$$\frac{dy}{dx} = 2y + 2x$$

$$3\frac{d^2z}{dt^2} - 4\frac{dz}{dt} + 7z = 4\sin(2t)$$

◆ Solutions to Differential Equations

A function is a solution of an ODE if it yields a true statement when substituted into the equation. For example, $y = 2t^2$ is a solution of the equation

$$\frac{dy}{dt} = 4t \tag{1}$$

✓ Can you find another solution of ODE (1)?

☞ Most ODEs have infinitely many solutions.

Actually, ODE (1) has infinitely many solutions. A single solution is called a *particular solution*. The set of all solutions is called the *general solution*. For example, the general solution of ODE (1) is $y = 2t^2 + C$, where C is any constant, while $y = 2t^2 + 3$ is a particular solution.

◆ Solving a Differential Equation

Solving a differential equation involves finding a function, just as solving an algebraic equation involves finding a number.

An ODE such as $dy/dt = 2ty$ gives us information about an unknown function y in terms of its derivative(s). In your differential equations class, you'll learn some methods for finding solutions of ODEs. The section "Finding a Solution Formula" later in this chapter also describes two techniques.

◆ Slope Fields

☞ Slopes for $y' = y\cos t$:

Point	Slope
$(0,0)$	0
$(0,1)$	1
$(0,2)$	2
$(0,-1)$	-1
$(0,-2)$	-2
$(\frac{\pi}{2},y)$	0

One useful way to get information about solutions of an ODE is to graph them; graphs of solutions are called *solution curves*. For first-order ODEs, you can actually get a good idea of what solution curves look like without solving the equation! Notice that for the ODE $y' = y\cos t$ the slope of the solution curve passing through the point (t, y) is given by $y\cos t$. Every first-order ODE gives you direct information about the slope of the solution curve through a point, so you can visualize solution curves by drawing small line segments with the correct slopes on a grid of fixed points. With patience (or a computer), you can draw many such line segments (as in the chapter cover figure). This is called a *slope field*. (Some books call it a *direction field*.) With practice you'll be able to imagine some of the line segments running together to make a graph. This approximates the graph of a solution to the ODE, that is, a solution curve. Figure 2.1 shows a slope field with several solution curves.

☞ Each segment of a slope field is tangent at its midpoint to the solution curve through that midpoint.

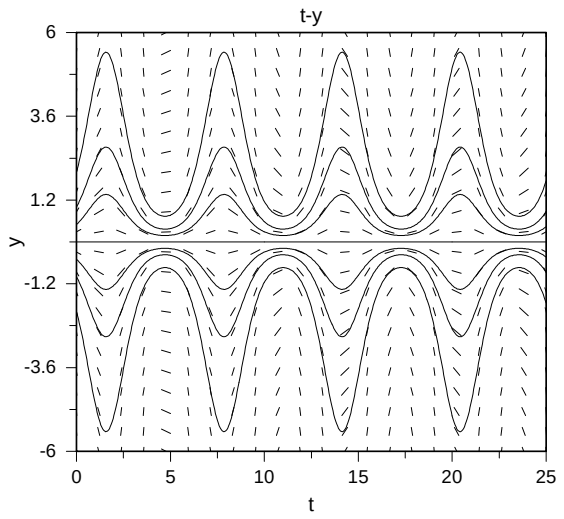


Figure 2.1: Slope field and seven solution curves for $y' = y\cos t$.

◆ Initial Values

We have seen that an ODE can have many solutions. In fact, the general solution formula involves an arbitrary constant. What happens if we specify that the solution must satisfy another property, such as passing through a given point? For example, all functions $y = 2t^2 + C$ are solutions of the ODE $dy/dt = 4t$, but only the *specific* solution $y = 2t^2 + 3$ satisfies the condition that $y = 5$ when $t = 1$. So, if we graph solution curves in the ty-plane, only the graph of the solution $y = 2t^2 + 3$ goes through the point $(1, 5)$.

The requirement that $y(1) = 5$ is an example of an *initial condition*, and the combination of the ODE and an initial condition

$$\frac{dy}{dt} = 4t, \quad y(1) = 5 \tag{2}$$

is called an *initial value problem* (IVP). Its solution is $y = 2t^2 + 3$.

✓ Replace the condition $y(1) = 5$ in IVP (2) by $y(2) = 3$ and find the solution of this new initial value problem. How many solutions are there?

◆ Finding a Solution Formula

An ODE usually has many solutions. How can you find a solution, and how can you describe it? A solution formula provides a useful description, but graphs and tables generated by ODE Architect are also useful, especially in the all-too-frequent case where no formula can be found. Two techniques to find solution formulas are summarized here, and others are in your textbook.

Integration

If $f(t)$ is a continuous function, then the general solution of the ODE

$$\frac{dy}{dt} = f(t)$$

☞ A table of integrals comes in handy here.

is $y(t) = F(t) + C$, where $F(t)$ is an antiderivative of f. For example, the general solution of $dy/dt = \sin t$ is $y = -\cos t + C$.

Separation of Variables

If you can write a differential equation in the form

$$\frac{dy}{dt} = f(t)g(y)$$

then wherever $g(y) \neq 0$ you can rewrite it as

$$\frac{1}{g(y)}\frac{dy}{dt} = f(t)$$

so that

$$\int \frac{1}{g(y)}\,dy = \int f(t)\,dt$$

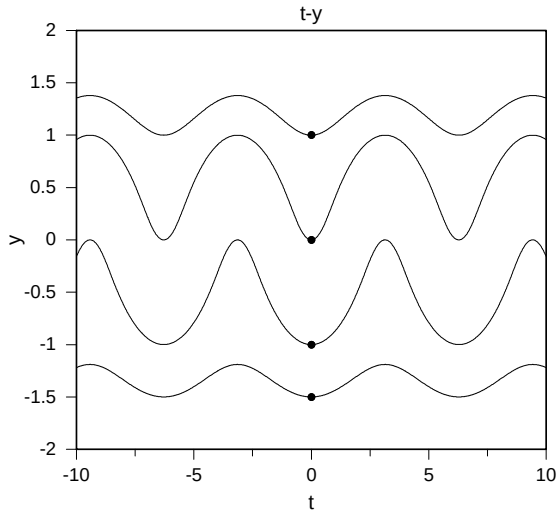


Figure 2.2: Four solution curves of $dy/dt = \sin t/(3y^2+1)$ through the marked initial points.

☞ Keep that table of integrals handy!

If $H(y)$ is an antiderivative of $1/g(y)$ and $F(t)$ is an antiderivative of $f(t)$, then a solution $y(t)$ of the ODE solves the equation

$$H(y(t)) = F(t) + C$$

for some constant C.

Here's an example of a separable ODE:

$$\frac{dy}{dt} = \frac{\sin t}{3y^2+1} \tag{3}$$

Separating the variables and finding the antiderivatives, we see that

$$(3y^2+1)\frac{dy}{dt} = \sin t$$
$$y^3 + y = -\cos t + C \tag{4}$$

We won't attempt to express a solution $y(t)$ directly in terms of t (and C), but we can check that formula (4) is correct by differentiating each side with respect to t. This gives

$$3y^2\frac{dy}{dt} + \frac{dy}{dt} = \sin t$$

which has the form of ODE (3) if we divide each side by $3y^2+1$. Figure 2.2 shows solution curves of ODE (3) through the initial points $(0,-1.5)$, $(0,-1)$, $(0,0)$, $(0,1)$. The curves were plotted by using ODE Architect to solve ODE (3) with the given initial data.

Solution formulas are useful, but they exist only for a small number of ODEs of special forms. That's where numerical solvers like ODE Architect come in—they don't need solution formulas.

◆ Modeling

A *mathematical model* is a system of mathematical equations relating specific variables that represent some aspect of a natural process. Modeling involves several steps:

☞ The eight steps are described in Chapter 1.

1. State the problem and its context.
2. Identify and assign variables.
3. State the laws that govern the relationships between the variables.
4. Translate the laws into equations.
5. Solve the resulting equations.
6. Interpret and test the solutions in the context of the natural environment.
7. Refine the model until it predicts the empirical data.
8. Interpret the implications of the model.

The models we consider all involve ODEs.

◆ The Juggler

You can observe the modeling process in the following juggler problem.

1. Find an ODE that describes the height of a ball between the time it leaves the juggler's hand, moving vertically upward, and the time it falls back into the hand.
2. Let $t =$ time (in seconds), $h =$ height of the ball above the floor (in feet), $v =$ velocity (in ft/sec), and $a =$ acceleration (in ft/sec^2).
3. Apply Newton's second law of motion to the ball: the mass m of a body times its acceleration is equal to the sum of all of the forces acting on the body. We treat the ball as a point mass encountering negligible air resistance (drag) so the only force acting on the ball is that due to gravity, which acts downward.
4. By Newton's second law, we have that $ma = -mg$, where $g = 32$ ft/sec^2 is the acceleration due to gravity near the surface of the earth, and the minus sign indicates the downward direction of the gravitational force. Since the ball's acceleration is $a = v'$ where v is its velocity, and $v = h'$, we can model the ball's motion by $h'' = -32$. The initial height h_0 of the ball is that of the juggler's hand above the floor when the ball is launched upward, and that is easy to measure. The initial velocity v_0 is harder to measure directly; it is simpler to solve the model first and then experiment to deduce a reasonable value for v_0.

☞ So the juggler's ODE for vertical motion is $h'' = -32$.

5–8. Solving and testing are up to you. See Figure 2.3 for graphs of $h(t)$ corresponding to $h_0 = 4$ ft and five values of v_0.

☞ In the multimedia module $h_0 = 4.5$ ft.

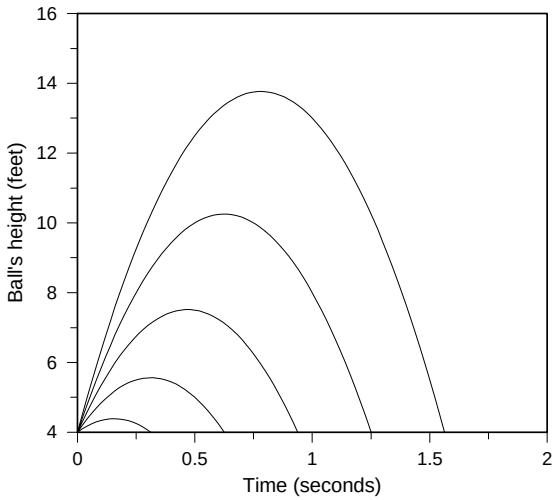


Figure 2.3: Five tosses of the juggler's ball: initial velocities v_0 range from 5 to 25 ft/sec. Which time-height curve corresponds to $v_0 = 25$?

✓ How must you revise the process when the ball is thrown to the juggler's *other* hand? (The result appears on Screen 3.4 of Module 2.)

◆ The Sky Diver

You might think that the path of a sky diver in free fall looks like the downward path of the ball in the simplest juggler problem of vertical motion. However, as the sky diver's velocity becomes large the effects of air resistance (or *drag*) become noticeable and must be included in the model. A revised model (starting with Step 3) follows:

☞ This kind of air resistance is called *viscous* damping.

3. In this case, Newton's second law says that mass times acceleration is equal to the force due to gravity plus that due to air resistance. Experience has shown that the force of air resistance can be modeled fairly well by a term that is proportional to velocity and opposite in direction.
4. We have $mh'' = mv' = -mg - kv$, where k is a constant coefficient of air resistance. The initial velocity of the sky diver is $v_0 = 0$ ft/sec; the initial height when the sky diver jumps from the plane is h_0 ft.
5. We solve the second-order ODE for h in two steps, first for v (by separating the variables) and then for h (by integrating the expression we

find for v, since $v = h'$). Here are the steps:

$$v' = -g - \frac{k}{m}v, \quad v(0) = 0$$

$$\frac{1}{g + kv/m}\frac{dv}{dt} = -1$$

$$\int \frac{1}{g + kv/m}\,dv = -\int dt$$

☞ C is an arbitrary constant.

$$(m/k)\ln(g + kv/m) = -t + C$$

$$\ln(g + kv/m) = (k/m)(-t + C)$$

Exponentiating and setting $K = \exp(kC/m)$ we obtain

$$g + kv/m = Ke^{-(k/m)t}$$

Since $v = 0$ when $t = 0$, we find that $K = g$. Solving for v we obtain

$$v = \frac{-mg}{k} + \frac{mg}{k}e^{-(k/m)t}$$

That means that $h(t)$ solves the IVP

$$h' = v = \frac{-mg}{k} + \frac{mg}{k}e^{-(k/m)t}, \quad h(0) = h_0$$

We find the formula for $h(t)$ by integration and the fact that $h = h_0$ at $t = 0$:

$$h = \frac{-mg}{k}t - \frac{m^2 g}{k^2}e^{-(k/m)t} + \frac{m^2 g}{k^2} + h_0$$

In our example of free fall (Screen 4.3), these equations become

☞ So the sky diver's free fall ODE is $h'' = -32 - (k/5)v$.

$$h'' = v' = -32 - \frac{k}{5}v \quad \text{if } m = 5 \text{ slugs}$$

$$h' = v = \frac{-160}{k} + \frac{160}{k}e^{-(k/5)t}$$

$$h = \frac{-160}{k}t - \frac{800}{k^2}e^{-(k/5)t} + \frac{800}{k^2} + 13500 \qquad (5)$$

See Figure 2.4 for some time-height curves.

Since the mass m of the sky diver doesn't drop out of the ODE when damping is added, we have to use appropriate units for the mass. In English units (which the English have been wise enough to discard) we have

$$\text{mass} = \frac{\text{force}}{\text{acceleration}} = \frac{\text{weight}}{\text{gravity}} = \frac{\text{lbs}}{\text{ft/sec}^2} = \text{slugs}$$

Opening the Parachute

If we wish to model what happens when the parachute opens, we'll need to alter the model slightly to account for the sudden change in drag—that is, for how the value of k suddenly changes.

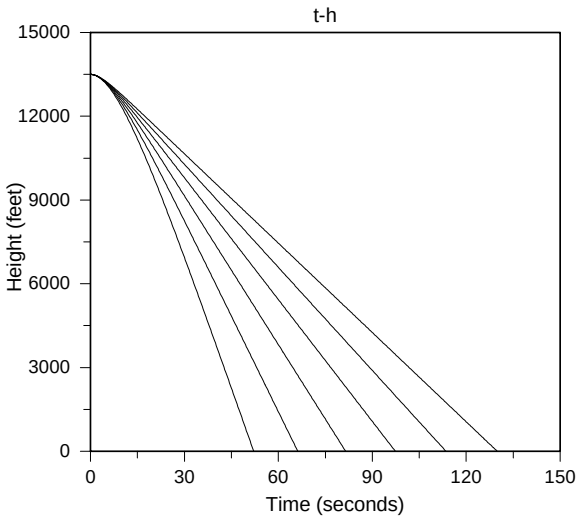


Figure 2.4: Six sky divers in free fall from 13,500 ft: viscous damping constants range from 0.5 to 1.5 slug/sec. Which sky diver has the smallest damping constant?

☞ Opening the chute changes k from k_{ff} to k_p.

4. We can use experimental values for the drag coefficients: in free fall $k_{ff} = 0.86$, and, after the parachute opens, $k_p = 6.71$, both in slugs/sec. The parachute opens at time t_p, when h is 2500 feet. It's hard to calculate t_p from formula (5), so we can approximate it by reading the graph of h vs. t (use the Explore feature on graphs of Screen 4.4).

We noticed on Screen 4.5 of Module 2 that an instantaneous opening of the parachute would exert an enormous force on the sky diver, so the model was further revised to allow the chute to open over a few seconds (a more realistic model), and we let k grow gradually, in a linear way, as it goes from k_{ff} to k_p. Take a look at Exploration 2.4, Problem 3.

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Borrelli, R. L., and Coleman, C. S., *Differential Equations: A Modeling Perspective*, (1998: John Wiley & Sons, Inc.)

Boyce, W. E., and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed. (1997: John Wiley & Sons, Inc.)

Hale, M., and Skidmore, A., *A Guided Tour of Differential Equations*, (1997: Prentice-Hall)

C·ODE·E Newsletter, http://www.math.hmc.edu/codee, for articles on modeling with ODEs

IDEA (Internet Differential Equations Activities), created by Thomas LoFaro and Kevin Cooper, offers an interactive virtual lab book with models. http://www.sci.wsu.edu/idea

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 2.1. ODEs and Their Solutions

1. *Where is that constant?*
Solution formulas for first-order ODEs often involve an arbitrary constant C, and it can show up in all sorts of strange places in the formulas. Solve each of the following ODEs for y in terms of t and C.

(a) $y' = 1 + \sin t$ **(b)** $y' = -y/3$ **(c)** $y' = t/y$ **(d)** $y' = 2t^2 y/\ln y$

2. *Let's check out the ODE Architect.*
You can see how good the ODE Architect solver is by creating initial value problems for the ODEs of Problem 1 and using the Architect to solve them and graph the solutions. Then compare the solver graphs with those obtained using the solution formula. For example, use ODE Architect to solve and plot the solution of the IVP $y' = -y/3,\ \ y(0) = 1$. Then graph the solution $y = e^{-t/3}$ and compare. To do this, enter the following two equations on the editor screen:

$$y' = -y/3$$
$$u = e^{-t/3}$$

Next enter the initial condition for the ODE, then solve and plot the solution on one of the graphics screens. Use the custom 2D plot tab to overlay the graph of u. Do the graphs match? Repeat with your own initial data for each of the other three ODEs in Problem 1.

3. *How many*[1] *solutions does this IVP have?*
Find formulas for two different solutions for the IVP $y' = y^{1/3}$, $y(0) = 0$. Which solution does ODE Architect give? Repeat with $y' = y^{2/3}$, $y(0) = 0$. [*Hint:* Is $y(t) = 0$ for all t a solution?]

4. *The effect of a singularity in the differential equation.*
The ODE $y' = y/t$ has a singularity at the point $(0, 0)$ because at that point, $y/t = 0/0$, which is undefined. Find a formula for all solutions of the ODE. Does the IVP $y' = y/t$, $y(0) = 0$, have any solutions? Use ODE Architect for $y' = y/t$, $y(1) = a$, for various positive values of a and then solve backward in time to see what happens as t gets near zero. Explain.

[1]Usually an IVP has a single solution, but in this Exploration you will see some exceptions. You can find out why by reading about "existence" and "uniqueness" in your text.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 2.2. Slope Fields

1. *What happens in the long term?*
The following ODEs are given in Screen 2.2 (Experiment 1). Using ODE Architect, describe what the solutions do as t gets very large. Include sketches or printouts of your solution curves and their slope fields.

(a) $y' = y - 1$ **(b)** $y' = t/4$ **(c)** $y' = (y - t)/10$

2. *More long-term behavior.*
Repeat Problem 1 with the following ODEs.

(a) $y' = ty$ **(b)** $y' = (y^2 - 4)/10$ **(c)** $y' = (y - 3)/5$

3. *Still more long-term behavior.*
Using ODE Architect, describe the long-term behavior of the solutions of $y' = y\sin(t + y)$.

4. *Strange solutions.*
Make up your own ODEs, especially ones whose solution curves or slope fields form strange patterns. Use ODE Architect to display your results. Describe the long-term behavior of solution curves. Attach printouts of your graphs.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 2.3. The Juggler

Second-order ODEs of the form $y'' = f(t, y, y')$ are to be solved in Explorations 2.3 and 2.4. Since ODE Architect only accepts first-order ODEs, we will replace $y'' = f$ by an equivalent pair of first-order ODEs. We do this by introducing $v = y'$ as another dependent variable:

$$y' = v$$
$$v' = f(t, y, v)$$

1. *What goes up must come down.*
Use ODE Architect to find the position of the ball at several different times t for several different initial velocities. Assume no air resistance and that the ball moves in a vertical line. What is the name for the shapes of the solution curves in the ty-plane? Does it take longer for the ball to rise or to fall? Show and explain the difference (if there is one!).

2. *Hand-to-hand motion of the ball.*
For a given initial speed v_0, find the range of values of the angle θ_0 so that the ball goes from one hand to the other. Now increase the initial speed. What happens to the range of successful values of θ_0? Explain. [Suggestion: First take a look at Screen 3.5 (Experiment 2 in Module 2); then enter the equations in ODE Architect and vary θ_0 with fixed v_0 to find the ranges. You may also want to take a look at Screens 1.2 and 1.3 in Module 5.]

3. *Raise your hand!*
Suppose the juggler raises his catching hand one foot higher. Repeat Problem 2 in this setting.

4. *Juggling two balls.*
Construct model ODEs for tossing two balls, one after the other, from one hand to the other. Use ODE Architect to find the positions of both balls at time t.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 2.4. The Sky Diver

Second-order ODEs of the form $y'' = f(t, y, y')$ are to be solved in Explorations 2.3 and 2.4. Since ODE Architect only accepts first-order ODEs, we will replace $y'' = f$ by an equivalent pair of first-order ODEs. We do this by introducing $v = y'$ as another dependent variable:

$$y' = v$$
$$v' = f(t, y, v)$$

1. *Terminal speed of a falling body.*
Use ODE Architect and determine the sky diver's terminal speeds for several different values of the viscous damping coefficient (use $m = 5$ slugs and $g = 32$ ft/sec^2). Is there any difference if the sky diver jumps at 25,000 feet instead of 13,500 feet? [*Suggestion:* After entering the ODE and solving, click on a Data tab in either of the two graphics windows and use approximate data you find there.]

2. *Slow down!*
If a sky diver can survive a free-fall jump only if she hits the ground at no more than 30 ft/sec, what values of the viscous drag coefficient k make this possible? Are these k-values realistic? (Use $m = 5$ slugs and $g = 32$ ft/sec^2.)

3. *A Modeling Challenge!*

Let's construct a model for a parachute that opens over a 3 second time span. The ODEs for this model are given on Screen 4.5 (Experiment 2), but we have to define $k(t)$. Assume that the sky diver has a mass of 5 slugs and that she jumps from 13,500 ft. The parachute starts to open after 65 seconds of free fall and the damping coefficient changes linearly from $k_{ff} = 0.86$ slugs/ft to $k_p = 6.71$ slugs/ft as the chute opens. In other words,

$$k(t) = \begin{cases} k_{ff}, & t < 65 \\ k_{ff} + \frac{k_p - k_{ff}}{3}(t - 65), & 65 \le t \le 68 \\ k_p, & t > 68 \end{cases}$$

☞ A step function is one of the engineering functions. You can find them by going to ODE Architect and clicking on Help, Topic Search, and Engineering Functions.

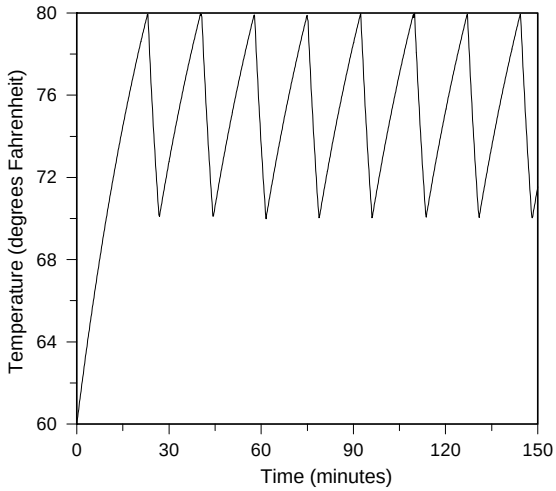
(a) Write an expression for $k(t)$ using the properties of step functions. *Hint*:

$$\text{Step}(t, 65) - \text{Step}(t, 68) = \begin{cases} 1, & 65 \le t \le 68 \\ 0, & \text{otherwise} \end{cases}$$

(b) Use ODE Architect to plot the sky diver's acceleration, velocity, and height vs. time, using your expression for $k(t)$.

CHAPTER 3

Some Cool ODEs



A room heats up in the morning, and the air conditioner in the room starts its on-off cycles.

Overview In this chapter, we'll use Newton's law of cooling to build mathematical models of a number of situations that involve the variation of temperature in a body with time. Some of our models involve ODEs that can be solved analytically; others will be solved numerically by ODE Architect. We'll compare the analytical solutions and the numerical results and see how both can be used to verify predictions made by the models.

Key words Modeling; Newton's law of cooling (and warming); initial conditions; general solution; separation of variables; integrating factor; heat energy; air conditioning; melting

See also Chapter 1 for more on modeling and Chapter 2 for the technique of separation of variables.

◆ Newton's Law of Cooling

Have you ever gotten an order of piping hot French fries, only to find them ice cold in what seems like a matter of moments? Whenever an object (or substance) is warmer than its surroundings, it cools because it loses heat energy. The greater the temperature difference between the object and its surroundings, the faster the object cools. The temperature of a body rises if its surroundings are at a higher temperature than it is. What happens to the ice cream in a cone on a hot day?

Although it is an oversimplification, we will assume that the temperature is uniform at all points in the objects we wish to model, but the temperature may change with time. Let's assume that the rate of change of the object's temperature is proportional to the difference between its temperature and that of its surroundings. Stated mathematically, we have:

☞ This becomes a "law of warming" if the surroundings are hotter than the object.

Newton's law of cooling. If $T(t)$ is the temperature of an object at time t and $T_{out}(t)$ is the temperature of its surroundings, then

$$\frac{dT}{dt} = k(T_{out} - T) \tag{1}$$

where k is a positive constant called the *cooling coefficient*.

◆ Cooling an Egg

What happens to the temperature of a hard-boiled egg when you take it out of a pot of boiling water? At first, the egg is the same temperature as the boiling water. Once you take it out of the water the egg begins to cool, rapidly at first and then more slowly. The temperature of the egg, $T(t)$, drops at a rate proportional to the difference between the temperature of the air, T_{out}, and $T(t)$. Notice from ODE (1) that if $T_{out} < T(t)$, the rate of change of temperature, dT/dt, is negative, so $T(t)$ decreases and your egg cools.

✓ "Check" your understanding by answering this question: What happens to the temperature of an egg if it is boiled at 212°F and then transferred to an oven at 400°F?

◆ Finding a General Solution

Equation (1) is a first-order ODE and its general solution contains one arbitrary constant. We can see this as follows: If T_{out} is a constant, then ODE (1) is separable, and separating the variables we have

☞ See Chapter 2 for how to solve a separable ODE.

$$\int \frac{dT}{T_{out} - T} = \int k\,dt$$

Finding an antiderivative for each side we obtain

☞ Why are the absolute value signs needed?

$$-\ln|T_{out} - T(t)| = kt + K$$

where K is an arbitrary constant. Multiplying through by -1 and exponentiating gives us

$$|T_{out} - T(t)| = e^{-K}e^{-kt}$$

or, after dropping the absolute value signs, we have that

$$T(t) = T_{out} + Ce^{-kt} \tag{2}$$

where $C = \pm e^{-K}$ is now the arbitrary constant. The solution formula (2) is called the *general solution* of ODE (1).

✓ How does the temperature $T(t)$ in (2) behave as $t \to +\infty$? Why can the constant C be positive or negative?

Given an initial condition, we can determine C uniquely and identify a single solution from the general solution (2). If we take $T(0) = T_0$, then since $T(0) = T_{out} + C$ we see that $C = T_0 - T_{out}$ and we get the unique solution

$$T(t) = T_{out} + (T_0 - T_{out})e^{-kt} \tag{3}$$

The constant of proportionality, k, in ODE (1) determines the rate at which the body cools. It can be evaluated in a number of ways, for example, by measuring the body's temperature at two different times and using formula (3) to solve for T_0 and k. Figure 3.1 shows temperature curves corresponding to $T_{out} = 70^\circ$F, $T_0 = 212^\circ$F, and five values of k.

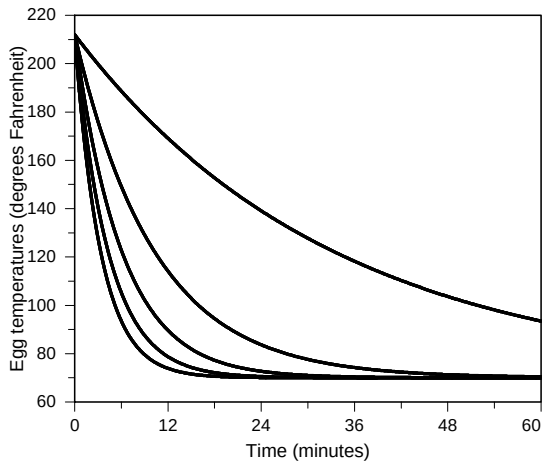


Figure 3.1: The cooling coefficient k ranges from 0.03 to 0.3 min^{-1} for eggs of different sizes. Which is the $k = 0.03$ egg?

✓ An object is initially at 212°F and cools to 190°F after 5 minutes in a room that is at 72°F. Find the coefficient of cooling, k, and determine how long it will take to cool to 100°F.

Finding the general solution formula (2) for ODE (1) was straightforward. However, the vast majority of ODEs are not so simple to solve and we have to use numerical methods. To demonstrate the accuracy of such methods, you can compare the numerical solutions from ODE Architect with a known solution formula.

☞ ODE Architect helps out again.

✓ How long will it take for a 212°F egg to cool to 190°F in a 72°F room if $k = 0.03419\ \text{min}^{-1}$? Use ODE Architect and formula (3) and compare the results.

◆ Time-Dependent Outside Temperature

When considering the cooling of an egg, ODE (1) is separable because T_{out} is constant in this instance. Let's consider what happens when the outside temperature changes with time.

We can still use Newton's law of cooling, so that if $T(t)$ is the egg's temperature and $T_{out}(t)$ is the room's temperature, then

$$\frac{dT}{dt} = k(T_{out}(t) - T) \tag{4}$$

Note that ODE (4) is not separable (because T_{out} varies with time) but it is linear, so we can find its general solution as follows. Rearrange the terms to give the linear ODE in standard form:

$$\frac{dT}{dt} + kT = kT_{out}(t)$$

Multiply both sides by e^{kt}, so that

$$e^{kt}\left(\frac{dT}{dt} + kT\right) = kT_{out}(t)e^{kt} \tag{5}$$

Since the left-hand side of ODE (5) is $(d/dt)(e^{kt}T(t))$, it can be rewritten:

$$\frac{d}{dt}\left(e^{kt}T\right) = kT_{out}(t)e^{kt} \tag{6}$$

Integrating both sides of ODE (6) we have that

$$e^{kt}T = \int kT_{out}(t)e^{kt}dt + C$$

where C is an arbitrary constant. The magic factor $\mu(t) = e^{kt}$ that enabled us to do this is called an *integrating factor*. So ODE (4) has the general solution

$$T(t) = e^{-kt}\left(\int kT_{out}(t)e^{kt}dt + C\right)$$

☞ Every ODE text discusses integrating factors and first-order linear ODEs.

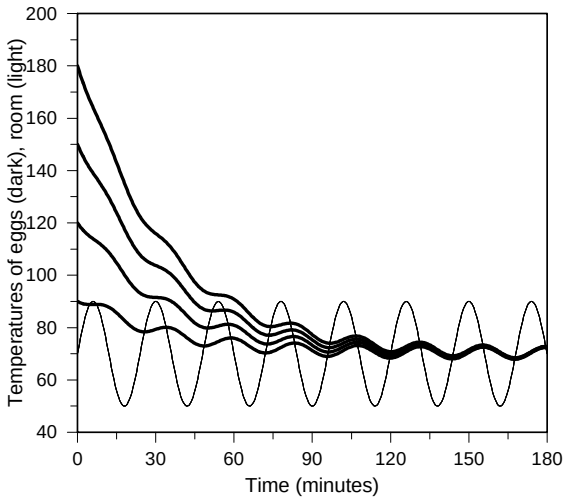


Figure 3.2: Eggs at initial temperatures of 180, 150, 120, and 90°F cool in a room whose temperature oscillates sinusoidally about 70°F for $k = 0.03$ min^{-1}. Do the initial temperatures matter in the long term?

Finally, letting $T(0) = T_0$ and integrating from 0 to t, we get the solution

$$T(t) = e^{-kt}\left(\int_0^t kT_{out}(s)e^{ks}\,ds + T_0\right) \tag{7}$$

It may be possible to evaluate the integral (7) analytically, but it is easier to use ODE Architect right from the start. See Figure 3.2 for egg temperatures in a room whose temperature oscillates between hot and cold.

✓ Show that if T_{out} is a constant, then formula (7) reduces to formula (3).

☞ You may find a computer algebra system or a table of integrals helpful!

✓ Use equation (7) to find a formula for $T(t)$ if

$$T_{out}(t) = 82 - 10\sin\left(\frac{2\pi(t+3)}{24}\right)$$

Use a table of integrals to carry out the integration.

◆ Air Conditioning a Room

Now let's build a model that describes a room cooled by an air conditioner. Without air conditioning, we can model the change in temperature using ODE (1). When the air conditioner is running, its coils remove heat energy at a rate proportional to the difference between T_r, the room temperature, and the

temperature T_{ac} of the coils. So, using Newton's law of cooling for the temperature change due to both the air outside the room and the air conditioner coils, our model ODE is

☞ Newton's law of cooling (twice)!

$$\frac{dT_r}{dt} = k(T_{out} - T_r) + k_{ac}(T_{ac} - T_r)$$

where T_{out} is the temperature of the outside air and k and k_{ac} are the appropriate cooling coefficients. If the unit is turned off, then $k_{ac} = 0$ and this equation reduces to ODE (1).

Let's assume that the initial temperature of the room is 60°F and the outside temperature is a constant 100°F. The air conditioner operates with a coil temperature of 40°F, switches on when the room reaches 80°F, and switches off at 70°F. Initially, the unit is off and the change in the room temperature is modeled by

☞ Time t is measured in minutes.

$$\frac{dT_r}{dt} = 0.03(100 - T_r), \quad T_r(0) = 60 \tag{8}$$

where we have taken the cooling coefficient $k = 0.03\ \text{min}^{-1}$. As we expect, the temperature in the room will rise as time passes. At some time t_{on} the room's temperature will reach 80°F and the air conditioner will switch on. If $k_{ac} = 0.1\ \text{min}^{-1}$, then for $t > t_{on}$ the temperature is modeled by the IVP

$$\frac{dT_r}{dt} = 0.03(100 - T_r) + 0.1(40 - T_r), \quad T_r(t_{on}) = 80 \tag{9}$$

which is valid until the room cools to 70°F at some time t_{off}. Then for $t > t_{off}$ the room temperature satisfies the IVP (8) but with the new initial condition $T_r(t_{off}) = 70$. Each time the unit turns on or off the ODE alternates between the two forms given in (8) and (9).

☞ The modeling here is more advanced than you have seen up to this point. You may want to just use the equations and skim the modeling.

Solving the problem by hand in the manner just described is very tedious. However, we can use ODE Architect to change the ODE automatically and without having to find t_{on} and t_{off}. The key is to define k_{ac} to be a function of temperature by using a step function; here's how we do it. In the equation quadrant of ODE Architect write the ODE as

☞ The step function is one of the engineering functions. You can find them by going to ODE Architect and clicking on Help, Topic Search, and Engineering Functions.

$$Tr' = 0.03 * (100 - Tr) + kac * (40 - Tr)$$

Now define k_{ac} as follows:

$$kac = 0.1 * \text{Step}(Tr, Tc)$$

where

$$Tc = 75 + 5 * B$$

Here T_c is the control temperature and

$$B = 2 * \text{Step}(Tr', 0) - 1$$

Note that $B = +1$ when $T_r' > 0$ (the room is warming) and $B = -1$ when $T_r' < 0$ (the room is cooling). This causes T_c to change from 80°F to 70°F (or

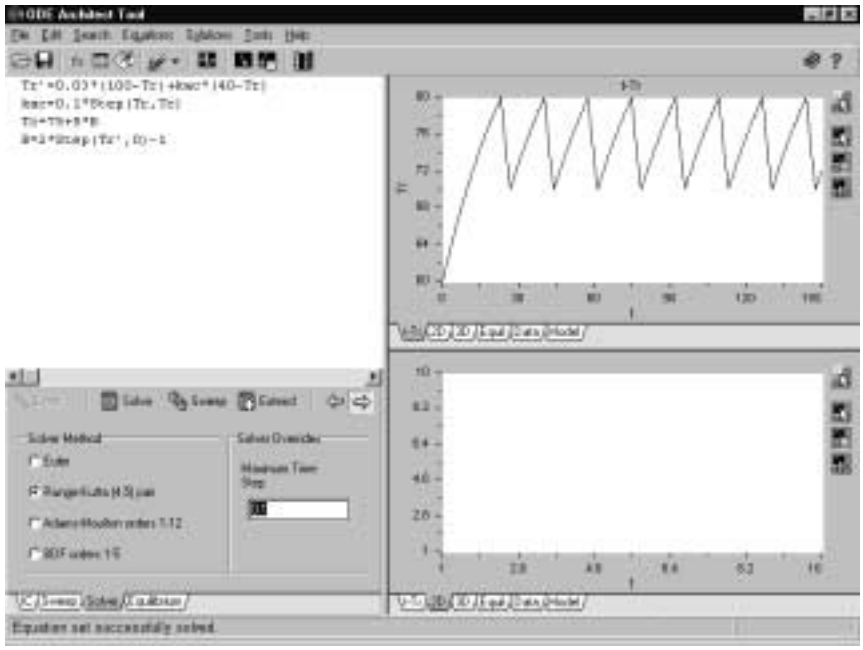


Figure 3.3: Air conditioning keeps the room temperature in the comfort zone, $70^\circ\text{F} \le T_r \le 80^\circ\text{F}$.

the reverse) depending on whether the room is warming or cooling. Finally, k_{ac} is zero (the air conditioner is off) when $T_r < T_c$, and $k_{ac} = 0.1$ (the air conditioner is on) when $T_r > T_c$.

The overall effect is that the air conditioner switches on only if the room temperature is above 80°F, then it runs until the room is cooled to 70°F, and then it switches off. The room temperature rises again to 80°F, and the process repeats. The temperature-vs.-time plot is shown in Figure 3.3. The accompanying screen image shows that we have set the maximum time step to 0.1 (under the Solver tab). If the internal time steps are not kept small, the Architect will not correctly notice when the step functions turn on and off.

◆ The Case of the Melting Snowman

It is difficult to model the melting of a snowman because of its complicated geometry: a large roundish ball of snow with another smaller mound on top. So let's simplify the model by treating the snowman as a single spherical ball of snow. The rate at which the snowman melts is proportional to the rate at which it gains thermal energy from the surrounding air, and it is given by

$$\frac{dV}{dt} = -h\frac{dE}{dt} \tag{10}$$

where V is the ball's volume, E is thermal energy, and h is a positive constant.

Our snowman will gain thermal energy only at its surface, where it is exposed to the warm air. So, it is reasonable to assume that the energy gain is proportional to both the surface area of the snowman and the temperature difference between the air and the snow:

☞ Remember that the snowman's temperature is always 32°F.

$$\frac{dE}{dt} = \kappa A(V)(T_{out} - 32) \tag{11}$$

where κ is a positive constant, and $A(V)$ is the surface area of a sphere of volume V.

☞ This is the snowman's law of melting.

If we combine equations (10) and (11) and take $k = \kappa h$, we obtain

$$\frac{dV}{dt} = -kA(V)(T_{out} - 32) \tag{12}$$

✓ The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$. Eliminate r between these two formulas to express A as a function of V. (You will need this soon.)

Note that ODE (12) is separable even when the outside temperature T_{out} is a function of time. Separating the variables and integrating we find the formula

$$\int \frac{1}{A(V)}\, dV = -\int k(T_{out}(t) - 32)\, dt + C \tag{13}$$

which defines V implicitly as a function of t. We can find the constant of integration C from the volume of the snowman at a specific time. However, expressions for the integrals in formula (13) may be hard to find. Once again ODE Architect comes to the rescue and solves ODE (12) numerically, given formulas for $A(V)$, $T_{out}(t)$, and the initial volume.

✓ If $k = 0.1451$ ft/(hr °F), the original volume of the snowman is 10 ft^3, and the outside temperature is 40°F, how many hours does it take the snowman's volume to shrink to 5 ft^3?

References

Nagle, R.K., and Saff, E.B., *Fundamentals of Differential Equations*, 3rd ed. (1993: Addison-Wesley)

Farlow, S.J., *An Introduction to Differential Equations and their Applications*, (1994: McGraw-Hill)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 3.1. Cooling Bodies

1. *Too hot to handle.*
When eating an egg, you don't want it to be too hot! If an egg with an initial temperature of 15°C is boiled and reaches 95°C after 5 minutes, how long will you have to wait until it cools to 70°C?

2. *A dead body, methinks.*
In forensic science, it is important to be able to estimate the time of death if the circumstances are suspicious. Assume that a corpse cools according to Newton's law of cooling. Suppose the victim has a temperature of 72°F when it is found in a 40°F walk-in refrigerator. However, it has cooled to 66.8°F two hours later when the forensic pathologist arrives. Estimate the time of death.[1]

[1]From "Estimating the Time of Death" by T.K. Marshall and F.E. Hoare, *Journal of Forensic Sciences*, Jan. 1962.

3. *In hot water.*
Heat a pan of water to 120°F and measure its temperature at five-minute intervals as it cools. Plot a graph of temperature vs. time. For various values of the constant k in Newton's law of cooling, use ODE Architect to solve the rate equation for the water temperature. What value of k gives you a graph that most closely fits your experimental data?

4. *More hot water.*
In Problem 3 you may have found it difficult to find a suitable value of k. Here is the preferred way to determine k. The solution to ODE (1) is

$$T(t) = T_{out} + (T_0 - T_{out})e^{-kt}$$

where in this context T_{out} is the room temperature. We can measure T_{out} and the initial temperature, T_0. Rearranging and taking the natural logarithm of both sides gives

$$\ln|T(t) - T_{out}| = \ln|T_0 - T_{out}| - kt$$

Using the data of Problem 3, plot $\ln|T(t) - T_{out}|$ against t. What would you expect the graph to look like? Use your graph to estimate k, then use ODE Architect to check your results.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 3.2. Keeping Your Cool

1. *On again, off again.*
When a room is cooled by an air conditioner, the unit switches on and off periodically, causing the temperature in the room to oscillate. How does the period of oscillation depend on the following factors?

- The upper and lower settings of the control temperature
- The outside temperature
- The coil temperature, T_{ac}

2. *Keeping your cool for less.*
The cost of operating an air conditioner depends on how much it runs. Which is the most economical way of cooling a room over a given time period?

- Set a small difference between the control temperatures, so that the temperature is always close to the average.
- Allow a large difference between the control temperatures so that the unit switches on and off less frequently.

Make sure the average of the control temperatures is the same in all your tests.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 3.3. The Return of the Melting Snowman

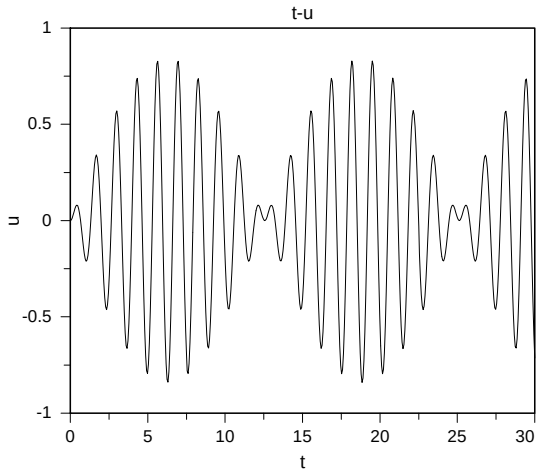
1. *The half-life of a snowman.*
Use ODE Architect to plot volume vs. time for several different initial snowman volumes between 5 and 25 ft^3, assuming that $k = 0.1451$ ft/(hr $^\circ$F) and $T_{out} = 40^\circ$F. For each initial volume use the Explore feature of ODE Architect to find the time it takes the snowman to melt to half of its original size, and make a plot of this "half-life" vs. initial volume. Any conclusions? [To access the Explore feature, click on Solutions on the menu bar and choose Explore. This will bring up a dialog box and a pair of crosshairs in the graphics window. Move the crosshairs to the appropriate point on the solution curve and read the coordinates of that point from the dialog box. Note that the Index entry gives the corresponding line in the Data table.]

2. *Sensitivity to outside temperature.*
Now fix the snowman's initial volume at 10 ft^3 and use ODE Architect to plot a graph of volume vs. time for several different outside temperatures between 35°F and 45°F, with $k = 0.1451$ ft/(hr $^\circ$F). Find the time it takes the snowman to melt to 5 ft^3 for each outside temperature used and plot that time against temperature. Describe the shape of the graph.

3. *Other snowmen.*
In developing our snowman model, we assumed that the snowman could be modeled as a sphere. Sometimes snowmen are built by rolling the snow in a way that makes the body cylindrical. How would you model a cylindrical snowman? Which type of snowman melts faster, given the same initial volume and air temperature?

CHAPTER 4

Second-Order Linear Equations



The phenomenon of beats.

Overview Second-order linear differential equations, especially those with constant coefficients, have a host of important applications. In this chapter we explore some phenomena involving mechanical and electrical oscillations. The first part deals with some basic features common to oscillations of all sorts. The second part applies some of these results to seismographs, which are instruments used for recording earthquake data.

Key words Newton's second law; oscillation; period; frequency; amplitude; phase; simple harmonic motion; viscous damping; underdamping; overdamping; critical damping; transient; steady-state solution; forced oscillation; seismograph; Kirchhoff's laws

See also Chapter 5 for more on vectors and damping, Chapters 6 and 10 for more on oscillations, and Chapter 12 for more on forced oscillations.

◆ Second-Order ODEs and the Architect

ODE Architect will accept only first-order ODEs, so how can we use it to solve a second-order ODE? There is a neat trick that does the job, and an example will show how. Suppose we want to use ODE Architect to study the behavior of the *initial value problem* (or IVP):

$$u'' + 3u' + 10u = 5\cos(2t), \quad u(0) = 1, \quad u'(0) = 0 \tag{1}$$

Let's write $v = u'$, then

$$v' = \frac{d}{dt}(v) = \frac{d}{dt}(u') = u''$$
$$u'' = -10u - 3u' + 5\cos(2t)$$

so IVP (1) becomes

$$\begin{aligned} u' &= v, \quad u(0) = 1 \\ v' &= -10u - 3v + 5\cos(2t), \quad v(0) = 0 \end{aligned} \tag{2}$$

☞ ODE Architect only accepts ODEs in *normal form*; for example, write $2x' - x = 6$ as $x' = x/2 + 3$ with the x' term alone on the left.

ODE Architect won't accept IVP (1), but it will accept the equivalent IVP (2). The components u and v give the solution of IVP (1) and its first derivative $u' = v$. If we use ODE Architect to solve and plot the component curve $u(t)$ of system (2), we are simultaneously plotting the solution $u(t)$ of IVP (1).

✓ "Check" your understanding by converting the IVP

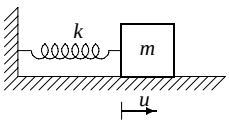
$$2u'' - 2u' + 3u = -\sin(4t), \quad u(0) = -1, \quad u'(0) = 2$$

to an equivalent IVP involving a system of two normalized first-order ODEs.

◆ Undamped Oscillations

Second-order differential equations arise naturally in physical situations; for example, the motion of an object is described by Newton's second law, $F = ma$. Here, F is the sum of the forces acting on the object of mass m and a is the acceleration, which is the second derivative of the object's position. Many of these differential equations lead to *oscillations* or *vibrations*. Many oscillating systems can be modeled by a system consisting of a mass attached to a spring where the motion takes place in a horizontal direction on a table. This simplifies the derivation of the equation of motion, but the same equation also describes the up-and-down motion of a mass suspended by a vertical spring.

Let's assume an ideal situation: there is no friction between the mass and the table, there is no air resistance, and there is no dissipation of energy in the spring or anywhere else in the system. The differential equation describing the motion of the mass is



$$m\frac{d^2u}{dt^2} = -ku \tag{3}$$

where $u(t)$ is the position of the mass m relative to its equilibrium and k is the spring constant. The natural tendency of the spring to return to its equilibrium position is represented by the restoring force $-ku$. Two initial conditions,

☞ This is also called *Hooke's law restoring force.*

$$u(0) = u_0, \qquad u'(0) = v_0 \tag{4}$$

where u_0 and v_0 are the initial position and velocity of the mass, respectively, determine the position of the mass uniquely. ODE (3) together with the initial conditions (4) constitute a well-formulated initial value problem whose solution predicts the position of the mass at any future time.

☞ See the first two references for derivations of formula (5).

The general solution of ODE (3) is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \tag{5}$$

where C_1 and C_2 are arbitrary constants and $\omega_0^2 = k/m$. Applying the initial conditions (4), we find that $C_1 = u_0$ and $C_2 = v_0/\omega_0$. Thus the solution of the IVP (3), (4) is

$$u(t) = u_0 \cos(\omega_0 t) + (v_0/\omega_0) \sin(\omega_0 t) \tag{6}$$

☞ The term "circular frequency" is only used with trigonometric functions.

The corresponding motion of the mass is *periodic*, which means that it repeats itself after the passage of a time interval T called the *period*. If we measure time in seconds, then the quantity ω_0 is the *natural (circular) frequency* in radians/sec, and T is given by

$$T = 2\pi/\omega_0 \tag{7}$$

The reciprocal of T, or $\omega_0/2\pi$, is the *frequency* of the oscillations measured in cycles per second, or *hertz*. Notice that since $\omega_0 = \sqrt{k/m}$, the frequency and the period depend only on the mass and the spring constant and not on the initial data u_0 and v_0.

By using a trigonometric identity, the solution (6) can be rewritten in the *amplitude-phase form* as a single cosine term:

☞ This motion is called *simple harmonic motion.* See Screen 1.3 of Module 4 for graphs.

$$u(t) = A \cos(\omega_0 t - \delta) \tag{8}$$

where A and δ are expressed in terms of u_0 and v_0/ω_0 by the equations

$$A = \sqrt{u_0^2 + (v_0/\omega_0)^2}, \quad \tan\delta = \frac{v_0}{u_0\omega_0} \tag{9}$$

The quantity A determines the magnitude or *amplitude* of the oscillation (8); δ is called the *phase* (or *phase shift*) because δ/ω_0 measures the time translation from a standard cosine curve.

✓ Show that (8) is equivalent to (6) when A and δ are defined by (9).

◆ The Effect of Damping

Equation (8) predicts that the periodic oscillation will continue indefinitely. A more realistic model of an oscillating spring must include damping. A simple, useful model results if we represent the damping force by a single term that is proportional to the velocity of the mass. This model is known as the *viscous damping* model; it leads to the differential equation

☞ The viscous damping force is $-c\,du/dt$.

$$m\frac{d^2u}{dt^2} = -ku - c\frac{du}{dt} \tag{10}$$

where the positive constant c is the viscous damping coefficient.

The behavior of the solutions of ODE (10) is determined by the roots r_1 and r_2 of the *characteristic polynomial equation*,

$$mr^2 + cr + k = 0$$

Using the quadratic formula, we find that the *characteristic roots* r_1 and r_2 are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \quad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \tag{11}$$

The nature of the solutions of ODE (10) depends on the sign of the *discriminant* $c^2 - 4mk$. If $c^2 \neq 4mk$, then $r_1 \neq r_2$ and the general solution of ODE (10) is

☞ Check that this equation gives a solution of ODE (10).

$$u = C_1e^{r_1t} + C_2e^{r_2t} \tag{12}$$

where C_1 and C_2 are arbitrary constants.

The most important case is *underdamping* and occurs when $c^2 - 4mk < 0$, which means that the damping is relatively small. In the underdamped case, the characteristic roots r_1 and r_2 in formula (11) are the complex numbers

$$r_1 = -\frac{c}{2m} + i\mu, \quad r_2 = -\frac{c}{2m} - i\mu, \quad \text{where } \mu = \frac{\sqrt{4mk - c^2}}{2m} \neq 0 \tag{13}$$

Euler's formula implies that

$$e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

for any real numbers α and β, so

$$e^{r_1t} = e^{-ct/2m}(\cos\mu t + i\sin\mu t), \quad e^{r_2t} = e^{-ct/2m}(\cos\mu t - i\sin\mu t) \tag{14}$$

Now, using the initial conditions together with equations (12) and (14), we find that the solution of the IVP

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0, \quad u(0) = u_0, \quad u'(0) = v_0 \tag{15}$$

is given by

☞ Solutions of an underdamped ODE oscillate with circular frequency μ and an exponentially decaying amplitude.

$$u = e^{-ct/2m}\left\{u_0\cos(\mu t) + \left[\frac{v_0}{\mu} + \frac{cu_0}{2m\mu}\right]\sin(\mu t)\right\} \tag{16}$$

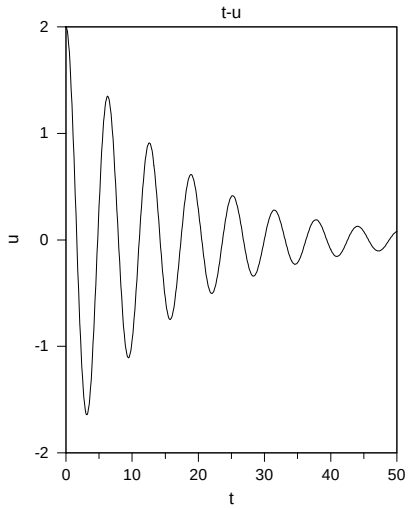


Figure 4.1: A solution curve of the underdamped spring-mass ODE, $u'' + 0.125u' + u = 0$.

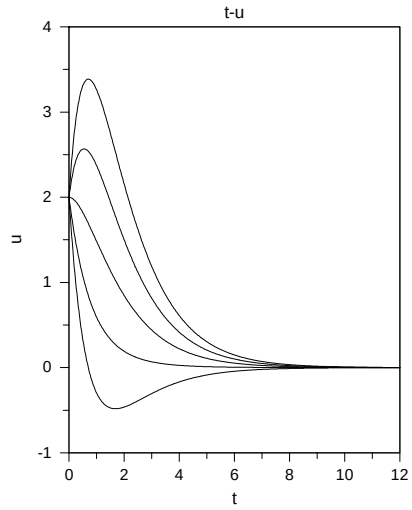


Figure 4.2: Solution curves of the overdamped spring-mass ODE, $u'' + 2.1u' + u = 0$.

✓ Verify that $u(t)$ defined in formula (16) is a solution of IVP (15).

The solution (16) represents an oscillation with circular frequency μ and an exponentially decaying amplitude (see Figure 4.1). From the formula in (13) we see that $\mu < \omega_0$, where $\omega_0 = \sqrt{k/m}$, but the difference is small for small c.

☞ Take a look at Screen 1.6 of Module 4.

If the damping is large enough so that $c^2 - 4mk > 0$, then we have *overdamping* and the solution of IVP (15) decays exponentially to the equilibrium position but does not oscillate (see Figure 4.2). The transition from oscillatory to nonoscillatory motion occurs when $c^2 - 4mk = 0$. The corresponding value of c, given by $c_0 = 2\sqrt{mk}$, is called *critical damping*.

◆ Forced Oscillations

☞ $F(t)$ is also called the *input*, or *driving term*; solutions $u(t)$ are the *responses* to the input and the initial data.

Now let's see what happens when an external force is applied to the oscillating mass described by ODE (10). If $F(t)$ represents the external force, then ODE (10) becomes

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F(t) \tag{17}$$

Some interesting things happen if $F(t)$ is periodic, so we will look at the ODE

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F_0\cos(\omega t) \tag{18}$$

where F_0 and ω are the amplitude and circular frequency, respectively, of the external force F. Then, in the underdamped case, the general solution of ODE (18) has the form

☞ Check that this formula gives solutions of ODE (18).

$$u(t) = e^{-ct/2m}[C_1 \cos(\mu t) + C_2 \sin(\mu t)] + a\cos(\omega t) + b\sin(\omega t) \qquad (19)$$

where a and b are constants determined so that $a\cos(\omega t) + b\sin(\omega t)$ is a solution of ODE (18). The constants a and b depend on m, c, k, F_0, and ω of ODE (18), but not on the initial data. The constants C_1 and C_2 can be chosen so that $u(t)$ given by formula (19) satisfies given initial conditions.

The first term on the right side of the solution (19) approaches zero as $t \to +\infty$; this is called the *transient* term. The remaining two terms do not diminish as t increases, and their sum is called the *steady-state solution* (or the *forced oscillation*), here denoted by $u_s(t)$. Since the steady-state solution persists forever with constant amplitude, it is frequently the most interesting solution. Notice that it oscillates with the circular frequency ω of the driving force F. It can be written in the amplitude-phase form (8) as

$$u_s(t) = A\cos(\omega t - \delta) \qquad (20)$$

where A and δ are now given by

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \quad \tan\delta = \frac{c\omega}{m(\omega_0^2 - \omega^2)} \qquad (21)$$

Figure 4.3 shows a graphical example of solutions that tend to a forced oscillation.

For an underdamped system with fixed c, k, and m, the amplitude A of the steady-state solution depends upon the frequency of the driving force. It

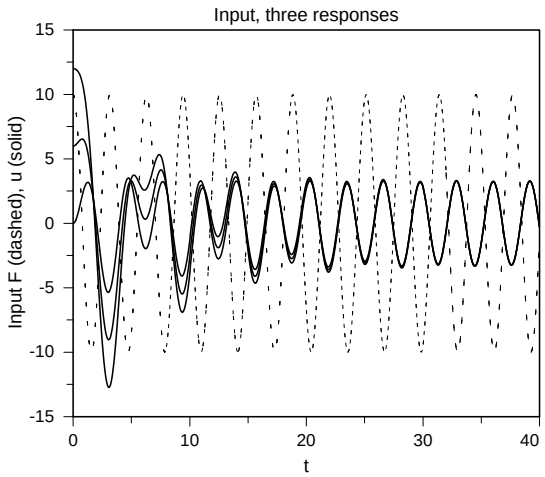


Figure 4.3: Solutions of $u'' + 0.3u' + u = 10\cos 2t$ approach a unique forced oscillation with the circular frequency 2 of the input.

is important to know whether there is a value $\omega = \omega_r$ for which the amplitude is maximized. If so, then driving the system at the circular frequency ω_r produces the greatest response. Using methods of calculus, it can be shown that if $c^2 < 2mk$ then ω_r is given by

☞ Recall that the *natural circular frequency* ω_0 is the value $\omega_0 = \sqrt{k/m}$.

$$\omega_r^2 = \omega_0^2 \left(1 - \frac{c^2}{2mk}\right) \tag{22}$$

The corresponding maximum value A_r of the amplitude when $\omega = \omega_r$ is

$$A_r = \frac{F_0}{c\omega_0\sqrt{1 - (c^2/4mk)}} \tag{23}$$

✓ Does A have a maximum value when $2mk < c^2 < 4mk$?

✓ Find the forced oscillation for the ODE of Figure 4.3.

◆ Beats

Let's polish the table and streamline the mass so that damping is negligible. Then we apply a forcing function whose frequency is close to the natural frequency of the spring-mass system, and watch the response. We can model this by the IVP

$$u'' + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0 \tag{24}$$

where $|\omega_0 - \omega|$ is small (but not zero). The solution is

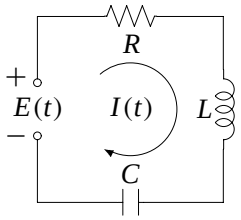
$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)}[\cos(\omega t) - \cos(\omega_0 t)] \\ &= \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\right]\sin\left(\frac{\omega_0 + \omega}{2}t\right) \end{aligned} \tag{25}$$

where trigonometric identities have been used to get from the first form of the solution to the second. The term in square brackets in formula (25) can be viewed as a varying amplitude for the sinusoid term $\sin[(\omega_0 + \omega)/2]t$. Since $|\omega_0 - \omega|$ is small, the circular frequency $(\omega_0 + \omega)/2$ is much higher than the low circular frequency $(\omega_0 - \omega)/2$ of the varying amplitude. Therefore we have a rapid oscillation with a slowly varying amplitude. This is the *beat phenomenon* illustrated on the chapter cover figure for the IVP

$$u'' + 25u = 2\cos(4.5t), \quad u(0) = 0, \quad u'(0) = 0$$

If you try this out with a driven mass on a spring you will see rapid oscillations whose amplitude slowly grows and then diminishes in a repeating pattern. This phenomenon can actually be heard when a pair of tuning forks which have nearly equal frequencies are struck simultaneously. We hear the "beats" as each acts as a driving force for the other.

◆ Electrical Oscillations: An Analogy



Linear differential equations with constant coefficients are important because they arise in so many different physical contexts. For example, an ODE similar to ODE (17) can be used to model charge oscillations in an electrical circuit. Suppose an electrical circuit contains a resistor, an inductor, and a capacitor connected in series. The current I in the circuit and the charge Q on the capacitor are functions of time t. Let's assume we know the resistance R, the inductance L, and the capacitance C. By Kirchhoff's voltage law for a closed circuit, the applied voltage $E(t)$ is equal to the sum of the voltage drops through the various elements of the circuit. Observations of circuits suggest that these voltage drops are as follows:

☞ See also the electrical circuit files in the Physical Systems and Model Based Animation folders in the Library.

- The voltage drop through the resistor is RI (Ohm's law);
- The voltage drop through the inductor is $L(dI/dt)$ (Faraday's law);
- The voltage drop through the capacitor is Q/C (Coulomb's law).

Thus, by Kirchhoff's law, we obtain the differential equation

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \tag{26}$$

Since $I = dQ/dt$, we can write ODE (26) entirely in terms of Q,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \tag{27}$$

ODE (27) models the charge $Q(t)$ on the capacitor of what is called the *simple RLC circuit* with voltage source $E(t)$. ODE (27) is equivalent to ODE (17), except for the symbols and their interpretations. Therefore we can also apply conclusions about our spring-mass system to electrical circuits. For example, we can interpret the ODE $u'' + 0.3u + u = 10\cos 2t$, whose solutions are graphed in Figure 4.3, as a model for either the oscillations of a damped and driven spring-mass system, or the charge on the capacitor of a driven RLC circuit. We see that a mathematical model can have many interpretations, and any mathematical conclusions about the model apply to every interpretation.

✓ What substitutions of parameters and variables would you have to make in ODE (27) to transform it to ODE (17)?

◆ Seismographs

☞ Look at "Earthquakes and the Richter Scale" in Module 4.

Seismographs are instruments that record the displacement of the ground as a function of time, and a seismometer is the part of a seismograph that responds to the motion. Seismographs come in two generic types. Matt's friend Seismo is a horizontal-component seismograph, which records one of the horizontal

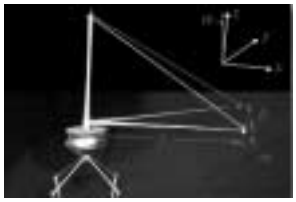
components of the earth's local motion. Of course, two horizontal components are required to specify fully horizontal motion, usually by means of north-south and east-west components. The other type of seismograph records the vertical component of motion. Both of these instruments are based on pendulums that respond to the motion of the ground relative to the seismograph.

Since Seismo is an animation of a horizontal-component seismograph, we'll outline the derivation of the ODEs that govern the motion of his arm. The starting point is the angular form of Newton's second law of motion, also known as the *angular momentum law*:

☞ If you're queasy about cross products or approximating functions (used in formulas (28), (31), (32)) you may prefer to skip directly to ODE (33) or ODE (34).

$$\frac{d}{dt}\mathbf{L} = \mathbf{R} \times \mathbf{F} \tag{28}$$

where $\mathbf{L}$ is the angular momentum of a mass (Seismo's arm and hand) about a fixed axis, $\mathbf{F}$ is the force acting on the mass, $\mathbf{R}$ is the position vector from the center of mass of Seismo's arm and hand to the axis, and $\times$ is the vector cross product.



We'll apply this law using an orthogonal xyz-coordinate system which is illustrated on Screen 2.2 of Module 4. In this system the y-axis is horizontal. Seismo's body is parallel to the z-axis and the rest position of Seismo's arm is parallel to the x-axis. The z-axis is not parallel to the local vertical, but instead is the axis which results from rotating the local vertical through a small angle α about the y-axis. Because of this small tilt, the x-axis points slightly downward and the arm is in a stable equilibrium position when it is parallel to the x-axis. The seismic disturbance is assumed to be in the direction of the y-axis. The xz-plane is called Seismo's *rest plane*.

Seismo's hand writes on the paper in the xy-plane, and the angle θ measures the angular displacement of his arm from its rest position. Consider an axis pointing in the z-direction and through the center of mass of Seismo's arm and hand, and let m represent the mass of the arm and hand. The z-component of the angular momentum about that axis is $mr^2(d\theta/dt)$ where r is the radius of gyration of the arm.

To compute the right-hand side of ODE (28), we need to know $\mathbf{R}$, the position vector from the center of mass of Seismo's arm and hand to the origin. Note that

$$\mathbf{R} = -l\cos\theta\hat{\mathbf{x}} - l\sin\theta\hat{\mathbf{y}}$$

where l is the distance from the center of mass to Seismo's body, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors along the positive x- and y-axes. For small θ, we have the approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$, so

$$\mathbf{R} = -l\hat{\mathbf{x}} - l\theta\hat{\mathbf{y}} \tag{29}$$

Using equation (29) in ODE (28) and computing the cross product, we obtain

$$mr^2\frac{d^2\theta}{dt^2} = -lF^{(y)} + l\theta F^{(x)} \tag{30}$$

indicating by superscripts the components of the net force **F** exerted on the arm and hand.

Now we need expressions for the two components of **F** in ODE (30). If the x-component of friction is assumed negligible then the two force components acting in the x-direction are the x-component of the gravitational force and the x-component of the force due to the seismic disturbance. Because the arm displacement angle θ and the body inclination angle α are both assumed small, the x-component of the force due to the seismic disturbance can be shown to be negligible also. Therefore the x-component of the net force, $F^{(x)}$, is given by the simple form

$$F^{(x)} \approx mg\alpha \tag{31}$$

The right side of equation (31) is the gravity component $mg\sin\alpha$ approximated by $mg\alpha$.

In the y-direction, the forces acting are the force due to the seismic disturbance and to friction, the latter assumed to be proportional to the angular velocity $d\theta/dt$. The force due to the seismic disturbance can be computed as follows: Let h be a small ground displacement in the y-direction. Then the y-coordinate of the center of mass is approximately $h + l\theta$. Therefore the force due to the earthquake is approximated by

☞ This is *viscous friction*.

$$m\frac{d^2}{dt^2}(h + l\theta) = m\frac{d^2h}{dt^2} + ml\frac{d^2\theta}{dt^2}$$

and the net force in the y-direction is

$$F^{(y)} \approx m\frac{d^2h}{dt^2} + ml\frac{d^2\theta}{dt^2} - k\frac{d\theta}{dt} \tag{32}$$

where k is a positive constant characterizing the effect of friction.

Combining ODE (30) with formulas (31) and (32), we find that the motions of Seismo's arm are governed by the ODE

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \omega_0^2\theta = -\frac{1}{L}\frac{d^2h}{dt^2} \tag{33}$$

In ODE (33) the quantities ω_0^2, L, and c are given by $\omega_0^2 = g\alpha/L$, where $L = (r^2 + l^2)/l$, and $c = k/(mL)$. We can interpret the terms in (33) as follows. The first term on the left arises from the inertia of Seismo's hand and arm. The second term models the frictional force due to the angular motion of the arm. The third term, arising from gravity and the tilt of the arm, is the restoring force for the oscillations of the arm and hand. Finally, the term on the right arises from the effective force of the seismic displacement.

To simplify ODE (33) a little more, we let $h(t) = Hf(t)$, where H is the maximum ground displacement, which means that the maximum value of the dimensionless ground displacement $f(t)$ is one. Then ODE (33) becomes the following equation for the dimensionless arm displacement $y(t) = L\theta(t)/H$:

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2f}{dt^2} \tag{34}$$

This is the ODE used in Screen 2.3 of Module 4.

☞ A given ODE can model a variety of phenomena.

It's an important result that ODE (34) is the same type as ODE (18), the only differences being in the definitions of the parameters that multiply the individual terms, and in the choice of variables. A second striking result is that this same ODE (34) applies as well to the motions of a vertical component seismograph. All that is necessary are other modifications in the meanings of the parameters and functions. Details are available in the book by Bullen and Bolt in the references.

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Borrelli, R. L., and Coleman, C. S., *Differential Equations: A Modeling Perspective*, (1998: John Wiley & Sons, Inc.)

Boyce, W. E., and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed. (1997: John Wiley & Sons, Inc.)

Bullen, K. E., and Bolt, B. A., *An Introduction to the Theory of Seismology*, 4th ed. (1985: Cambridge University Press)

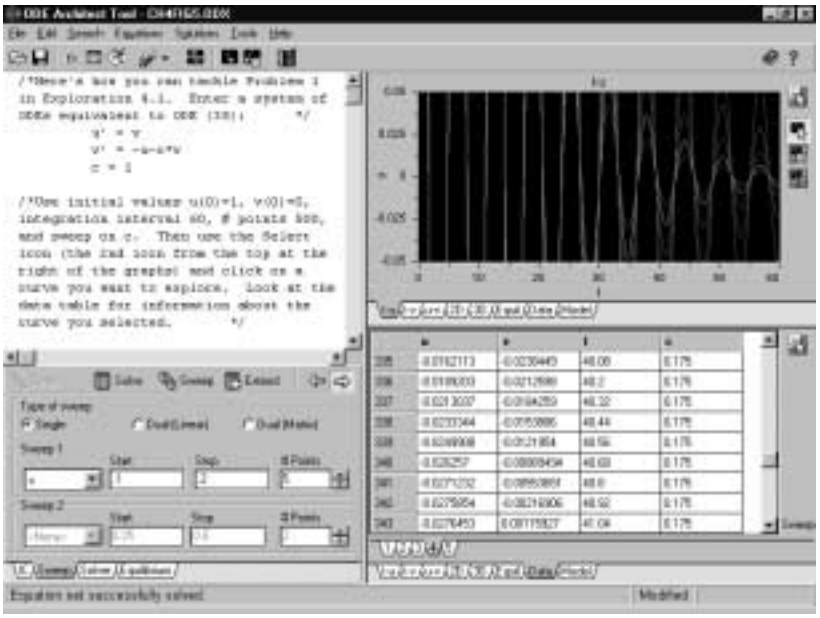


Figure 4.4: The sweep on c generated five solution curves. The selected curve is highlighted, and the corresponding solution $u(t)$ satisfies the condition $|u(t)| < 0.05$ for $t \geq 40$. The data tells us that $c = 0.175$ for the curve.

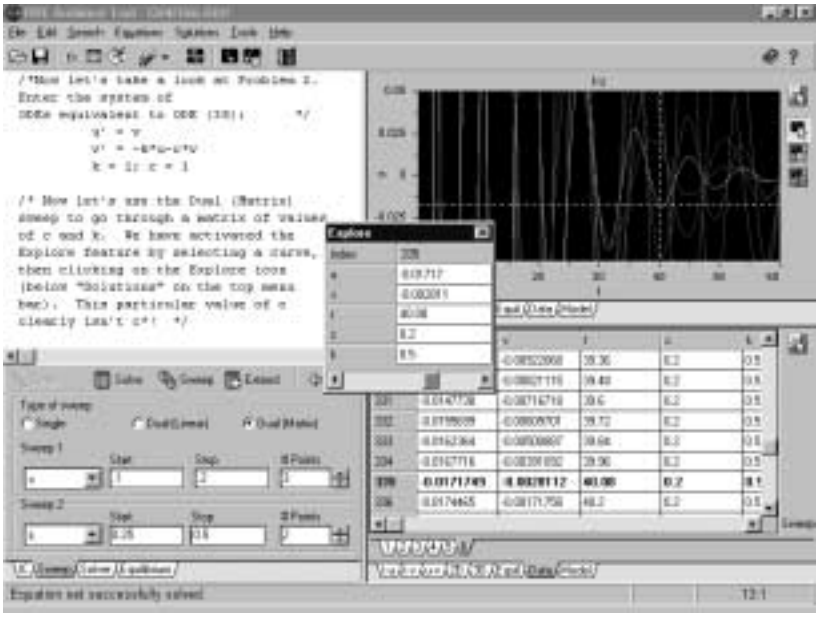


Figure 4.5: The Dual (Matrix) feature produces six solutions for various values of c and k. We have selected one of them (the highlighted curve) and used the Explore option to get additional information.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 4.1. The Damping Coefficient

Assume that Dogmatic's oscillations satisfy the IVP

$$u'' + cu' + ku = 0, \quad u(0) = 1, \quad u'(0) = 0 \tag{35}$$

1. Let $k = 1$ and use ODE Architect to estimate the smallest value c^* of the damping coefficient c so that $|u(t)| \leq 0.05$ for all $t \geq 40$. [*Suggestion:* Figure 4.4 illustrates one way to estimate c^* by using the Select feature and the Data table.]

2. Repeat Problem 1 for other values of k, including $k = \frac{1}{4}$, $\frac{1}{2}$, 2, and 4. How does c^* change as k changes? [*Suggestion:* Figure 4.5 shows the outcome of using a Dual (Matrix) sweep on the values of c and k, and then using the Explore feature.]

3. Let $k = 10$ in IVP (35).

(a) Find the value of c for which the ratio of successive maxima in the graph of u vs. t is 0.75.

(b) Why is the ratio between successive maxima always the same?
Note: Since the values of the maxima can be observed experimentally, this provides a practical way to determine the value of the damping coefficient c, which may be difficult to measure directly.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 4.2. Response to the Forcing Frequency

1. Suppose that Dogmatic's oscillations satisfy the differential equation

$$2u'' + u' + 4u = 2\cos(\omega t)$$

Let $\omega = 1$. Select your own initial conditions and use ODE Architect to plot the solution over a long enough time interval that the transient part of the solution becomes negligible. From the graph, determine the amplitude A_s of Dogmatic's steady-state solution.

2. Repeat Problem 1 for other values of ω. Plot the corresponding pairs ω, A_s and sketch the graph of A_s vs. ω. Estimate the value of ω for which A_s is a maximum. *Note*: You may want to use the Lookup Table feature of ODE Architect (see Module 1 and Chapter 1 for details).

3. In Problems 1 and 2, the value of the damping coefficient c is 1. Repeat your calculations for $c = \frac{1}{2}$ and $c = \frac{1}{4}$. How does the maximum value of A_s change as the value of c changes? Compare your results with the predictions of formula (23).

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 4.3. Low- and High-Frequency Quakes

In experiments with Seismo, you used ODE (34) to find the response of his arm to different ground displacements of sinusoidal type, $f(t) = \cos \omega t$, when $1 \leq \omega \leq 5$. In this exploration you'll investigate what happens for ground displacements with frequencies that are lower or higher than these values.

1. Choose $c = 2$ and $\omega_0 = 3$ in ODE (34), and set the initial conditions $y(0)$ and $y'(0)$ to zero. Use $f(t) = \cos \omega t$ with $\omega = 0.5$ for the ground displacement. Use ODE Architect to plot the displacement $y(t)$ determined from ODE (34); also plot $f(t)$ on the same graph. How do the features of $y(t)$ compare with those of $f(t)$?

2. Repeat Problem 1 for values of ω smaller than 0.5. Be sure to plot for a long enough time interval to see the relevant time variations. What do you think is Seismo's arm response as ω approaches zero? How does this compare with the corresponding response of a mass on a spring from ODE (18)?

3. Repeat Problem 1 for values of ω larger than 5, such as $\omega = 10$ and $\omega = 20$. What do you think is Seismo's arm response as ω becomes very large?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 4.4. Different Ground Displacements

In explorations with Seismo, we assumed that the dimensionless ground displacements $f(t)$ are sinusoidal, with a single frequency. Real earthquakes however, are not so simple: you'll investigate other possibilities in the following problems. The ODE for Seismo's dimensionless arm displacement $y(t)$ is

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2 f}{dt^2} \tag{36}$$

1. Suppose the ground displacement can be modeled by the function

$$f(t) = \begin{cases} (t/T)^2, & 0 \le t \le T \\ 1, & t > T \end{cases}$$

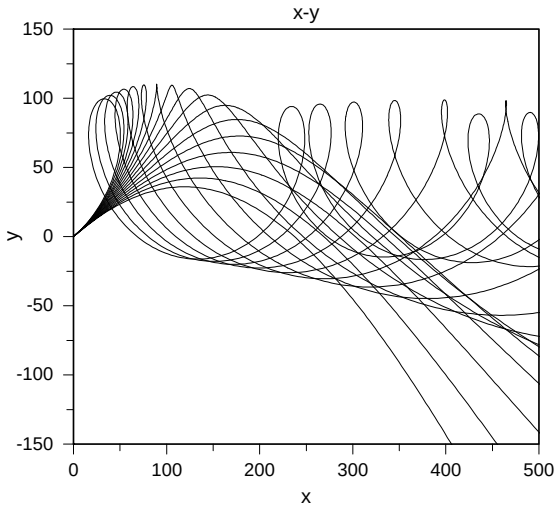
How do you interpret this motion? Choose $c = 2$ and $\omega_0 = 3$, and set $y(0) = y'(0) = 0$. Use ODE Architect to find $y(t)$ from ODE (36) for the case $T = 2$, and display both $y(t)$ and $f(t)$. *Note*: d^2f/dt^2 can be written using a step function. How do the features of $y(t)$ compare with those of $f(t)$? For example, what is the maximum magnitude of $y(t)$, and when does it occur?

2. Now suppose that the ground motion is given by the function $f = e^{-at}\sin(\pi t)$. Choose some values of a in the range $0 < a \leq 0.5$ and study how Seismo's arm displacements change with the parameter a.

3. How do you think the results of Problem 2 would change if the period of the sinusoidal oscillation were different from 2? Try a few cases to check your predictions.

CHAPTER 5

Models of Motion



Seventeen ski jumpers take off from an upward-tilted ski-jump.

Overview How would you model the motion of a baseball thrown at a target, or the rise and fall of a whiffle ball, or the trajectory of a ski jumper? You need modeling principles to explain the effects of the surroundings on the motion of a body.

Building on the work of Galileo, Newton formulated the fundamental laws of motion that describe the forces acting on a body in terms of the body's acceleration and mass. Newton's second law of motion, for example, relates the mass and the acceleration of a moving body to the forces acting on it and ultimately leads to differential equations for the motion.

Bodies moving through the air near the surface of the earth (e.g., a whiffle ball, Indiana Newton jumping onto a boxcar, or a ski jumper) are subject to the forces of gravity and air resistance, so these forces will affect their motion.

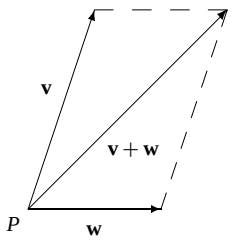
Key words Vectors; force; gravity; Newton's laws; acceleration; trajectory; air resistance; viscous drag; Newtonian drag; lift

See also Chapter 1 for more on modeling, and Chapter 2 for "The Juggler" and "The Sky Diver".

◆ Vectors

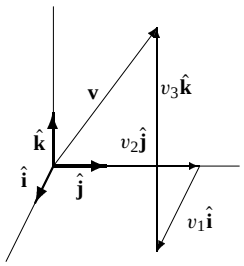
A *vector* is a directed line segment and can be represented by an arrow with a head and a tail. We use boldface letters to denote vectors.

Some terminology:



- The *length* of a vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$.
- Two vectors $\mathbf{v}$ and $\mathbf{w}$ are *equivalent* if they can be made to coincide by translations. (Translations preserve length and direction of vectors.) So parallel vectors of equal length and pointing in the same direction are equivalent.
- The *sum* $\mathbf{v}+\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ is defined by the parallelogram law as follows: $\mathbf{v}+\mathbf{w}$ is the diagonal vector of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ as shown in the margin figure.
- If r is any real number, then the *product* $r\mathbf{v}$ is the vector of length $|r|\,|\mathbf{v}|$ that points in the direction of $\mathbf{v}$ if $r > 0$ and in the direction opposite to $\mathbf{v}$ if $r < 0$.
- If a vector $\mathbf{u} = \mathbf{u}(t)$ depends on a variable t, then the *derivative* $d\mathbf{u}/dt$ [or $\mathbf{u}'(t)$] is defined as the limit of a difference quotient:

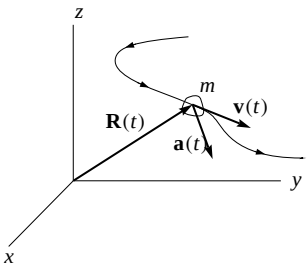
$$\mathbf{u}'(t) = \frac{d\mathbf{u}}{dt} = \lim_{h\to 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h}$$



- A *coordinate frame* is a triple of vectors, denoted by $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, that are mutually orthogonal and all of unit length. Every vector can be uniquely written as the sum of vectors parallel to $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. So for each vector $\mathbf{v}$ there is a unique set of real numbers v_1, v_2, and v_3 such that $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Here v_1,v_2, and v_3 are called the *coordinates* (or *components*) of $\mathbf{v}$ in the frame $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$.

Let's see how to use vectors in a real-life situation. Suppose a particle of mass m moves in a manner described by the *position vector*

$$\mathbf{R} = \mathbf{R}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$



If $\mathbf{R}$ is differentiable, then

$$\mathbf{R}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}$$

The vector $\mathbf{R}'(t) = \mathbf{v}(t)$ is the *velocity vector* of the particle at time t, and $\mathbf{v}(t)$ is tangent to the path of the particle's motion at the point $\mathbf{R}(t)$. If $\mathbf{R}'(t)$ is differentiable, then

$$\mathbf{R}''(t) = \mathbf{v}'(t) = x''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}} + z''(t)\hat{\mathbf{k}}$$

The vector $\mathbf{R}''(t) = \mathbf{a}(t)$ is the *acceleration vector* for the particle.

✓ "Check" your understanding by answering this question: If a particle moves at a constant speed around a circle, does the acceleration vector from the particle point to the inside of the circle or to the outside of the circle?

✓ If a particle's acceleration vector is always tangent to its path, what is the path?

Next, let's use vectors to express Newton's laws of motion.

◆ Forces and Newton's Laws

☞ Deceleration is just negative acceleration.

Our environment creates forces that act on bodies in a way that causes the bodies to accelerate or decelerate. Forces have magnitudes and directions and so can be represented by vectors. Newton formulated two laws that describe how the forces on a body relate to its motion.

Newton's First Law. A body remains in a state of rest, or in a state of uniform motion in a straight line if there is no net external force acting on it.

But the more interesting situation is when there *is* a net external force acting on the body.

Newton's Second Law. For a body with acceleration $\mathbf{a}$ and constant mass m,

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the sum of all external forces acting on the body.

Sometimes it's easier to visualize Newton's second law in terms of the x-, y-, and z-components of the position vector $\mathbf{R}$ of the moving body. If we project the acceleration vector $\mathbf{a} = \mathbf{R}''$ and the forces onto the x-, y-, and z-axes, then for a body of mass m,

$$mx'' = \text{the sum of the forces in the } x\text{-direction}$$
$$my'' = \text{the sum of the forces in the } y\text{-direction}$$
$$mz'' = \text{the sum of the forces in the } z\text{-direction}$$

We'll look at motion in a plane with x measuring the horizontal distance and y measuring the vertical distance up from the ground. We don't need the z-axis for our examples because the motion is entirely along a line or in a plane.

◆ Dunk Tank

Imagine your favorite professor seated over a dunk tank. Let's construct a model that will help you find the secret to hitting the target and giving your teacher a swim!

You hurl a ball at the target from a height of 6 ft with speed v_0 ft/sec and with a launch angle of θ_0 degrees from the horizontal[1]. The target is centered 10 ft above the ground and 20 ft away. Let's suppose that air resistance doesn't have much effect on the ball over its short path, so that gravity, acting downwards, is the only force acting on the ball.

Newton's second law says that

$$m\mathbf{R}'' = -mg\hat{\mathbf{j}}$$

where m is the ball's mass, $\mathbf{R}(t)$ is the position of the ball at time t relative to your hand (which is 6 ft above the ground at the instant $t = 0$ of launch), and $g = 32$ ft/sec^2 is the acceleration due to gravity. In coordinate terms,

$$\begin{aligned} mx'' &= 0 \\ my'' &= -mg \end{aligned}$$

Since $x'(0) = v_0 \cos\theta_0$ and $y'(0) = v_0 \sin\theta_0$, one integration of these second-order ODEs gives us

$$\begin{aligned} x'(t) &= v_0 \cos\theta_0 \\ y'(t) &= v_0 \sin\theta_0 - gt \end{aligned} \tag{1}$$

☞ What is the ball doing if $\theta_0 = 90°$?

Then because $x(0) = 0$ and $y(0) = 6$, a second integration yields

$$\begin{aligned} x(t) &= (v_0 \cos\theta_0)t \\ y(t) &= 6 + (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \end{aligned} \tag{2}$$

To hit the target at some time T we want $x(T) = 20$ and $y(T) = 10$. So values of $T > 0$, θ_0, and v_0 such that

$$\begin{aligned} x(T) = 20 &= (v_0 \cos\theta_0)T \\ y(T) = 10 &= 6 + (v_0 \sin\theta_0)T - \frac{1}{2}gT^2 \end{aligned} \tag{3}$$

lead to hitting the target right in the bull's eye and dunking your professor.

☞ Figure 5.1 was done in the Tool, where θ_0 is in radians.

You can try to use system (3), or you can just adjust your launch angle and pitching speed by intuition and experience. The screen shot in Figure 5.1 shows you how to get started with the latter approach. If you play the dunking game on Screen 1.3 you'll find that you can dunk without hitting the target head-on, but that a little up or a little down from the center works fine.

[1]The sin, cos and other trig functions in the ODE Architect Tool expect angles to be measured in radians. Note that $\theta_0 = 1$ radian corresponds to $360/2\pi \approx 57.3$ degrees. The multimedia modules, however, will accept angles measured in degrees.

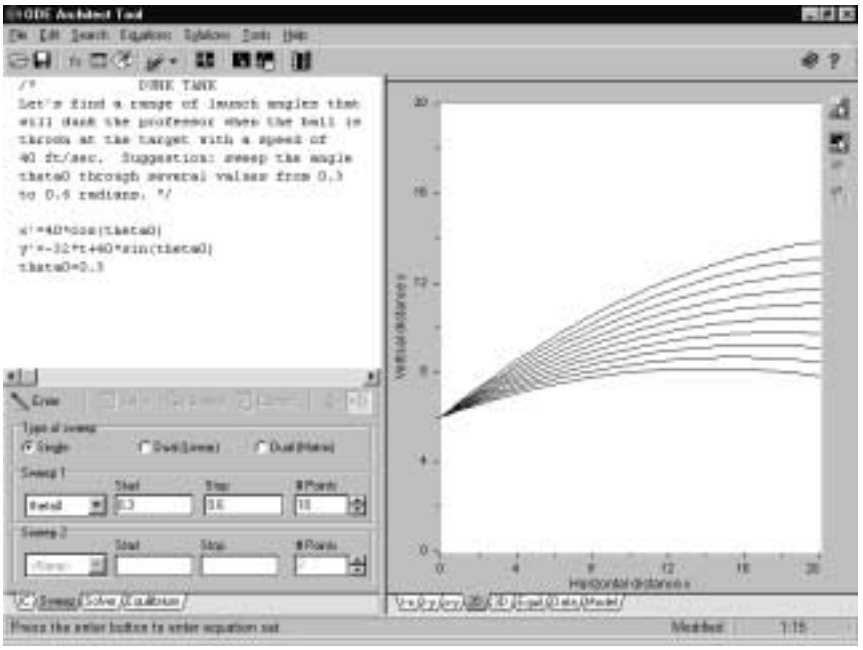


Figure 5.1: This ODE Architect screen shows paths of a ball thrown at ten different angles θ_0. Which paths lead to dunking the prof?

◆ Longer to Rise or to Fall?

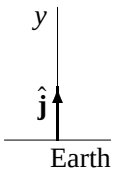
Throw a ball straight up in the air and ask observers whether the ball takes longer to rise or to fall. You'll get four answers:

1. Longer to rise
2. Longer to fall
3. Rise-time and fall-time are the same
4. It all depends . . .

What's your answer?

A mathematical model and ODE Architect suggest the answer. The forces acting on the ball of mass m are gravity and air resistance, so Newton's second law states that

$$m\mathbf{R}''(t) = -mg\hat{\mathbf{j}} + \mathbf{F}$$



where $\mathbf{R}$ is the position vector of the ball, and $\mathbf{F}$ is the drag on the ball caused by air resistance. In this case $\mathbf{R}(t) = y(t)\hat{\mathbf{j}}$ where $\hat{\mathbf{j}}$ is a unit vector pointing upward (the positive y direction). If the drag is negligible, we can set $\mathbf{F} = 0$. For a light ball with an extended surface, like a whiffle ball, the drag, called *viscous drag*, exerts a force approximately proportional to the ball's velocity but opposite in direction:

$$\mathbf{F}(\mathbf{v}) = -k\mathbf{v} = -ky'\hat{\mathbf{j}}$$

☞ Drag forces are usually determined by observation. They differ widely from one body to another.

If the ball is solid and dense, like a baseball or a bowling ball, then we have *Newtonian drag*, which acts opposite to the velocity with magnitude proportional to the square of the speed:

$$\mathbf{F}(\mathbf{v}) = -k|\mathbf{v}|\mathbf{v} = -k|y'|y'\hat{\mathbf{j}}$$

Summarizing, we have the models

$$y'' = -g - \begin{cases} 0 & \text{no drag} \\ \dfrac{k}{m}y' & \text{viscous} \\ \dfrac{k}{m}|y'|y' & \text{Newtonian} \end{cases}$$

or, in system form,

$$y' = v$$

$$v' = -g - \begin{cases} 0 & \text{no drag} \\ \dfrac{k}{m}v & \text{viscous} \\ \dfrac{k}{m}|v|v & \text{Newtonian} \end{cases}$$

☞ ODE Architect only accepts first-order ODEs, so that's why we use the first-order system form.

To observe different rise times and fall times, you can set $y(0) = 0,\ v(0) = v_0$ and see what happens for various positive values of v_0. See Figure 5.2 for graphs of $y(t)$ with viscous damping, four different initial velocities, $k/m = 2\ \text{sec}^{-1}$, and $g = 32\ \text{ft/sec}^2$. In this setting v is the rate of change of y, so v is positive as the ball rises and negative as it falls.

◆ Indiana Newton

You notice that Indiana Newton is about to jump from a ledge onto a boxcar of a speeding train. His timing has to be perfect. He also gets to choose his drag: none, viscous, or Newtonian. If you knew the train's position at all times, and how long it takes Indy to drop from the ledge to the top of the boxcar, then you could give him good advice about which drag to choose.

The initial value problem that models Indy's situation is

$$\begin{aligned} y' &= v & y(0) &= h \\ v' &= -g - F(v)/m & v(0) &= 0 \end{aligned}$$

where m is his mass, $F(v)$ is a drag function, g is the acceleration due to gravity, and h is the height of the ledge above the boxcar. His life is in your hands! Figure 5.3 shows Indy's free-fall solution curves $y(t)$ from a height of 100 ft with three different drag functions.

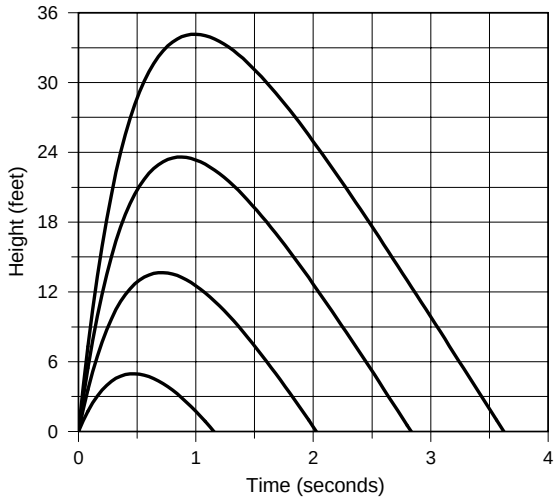


Figure 5.2: Height vs. time of a whiffle ball thrown straight up four times with viscous damping and different initial velocities. Does the ball take longer to rise or to fall?

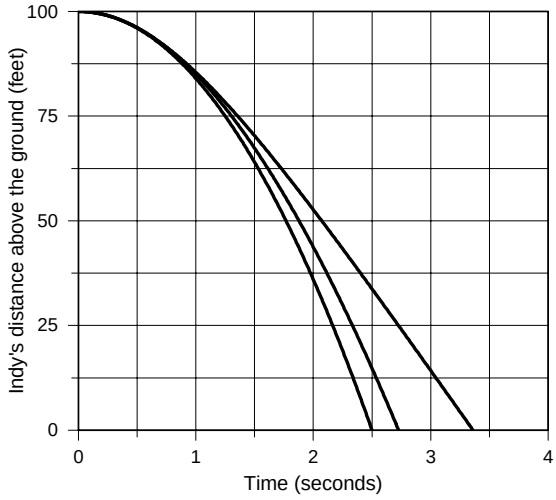


Figure 5.3: Indiana jumps with no drag (left curve), viscous drag $-0.2y'$ (middle), Newtonian drag $-0.02|y'|y'$ (right).

◆ Ski Jumping

When a ski jumper is aloft she is subject to gravitational, drag, and lift forces. She can diminish the drag and increase the lift by her posture, ski angle, and choice of clothing. *Drag* acts opposite to velocity and its magnitude is usually taken to be proportional to the skier's velocity $\mathbf{R}'$:

$$\text{Drag force} = -\delta\mathbf{R}' = -\delta x'\hat{\mathbf{i}} - \delta y'\hat{\mathbf{j}}$$

The lift force is what makes ski jumping fun. The *lift force* is that force which acts perpendicular to the velocity and enables the jumper to soar. Its magnitude is usually taken to be proportional to the speed, so

☞ Check that this force is perpendicular to velocity.

$$\text{Lift force} = -\lambda y'\hat{\mathbf{i}} + \lambda x'\hat{\mathbf{j}}$$

Newton's second law in the $\hat{\mathbf{i}}$- and $\hat{\mathbf{j}}$-directions gives us

☞ The origin of the xy-plane is at the edge of the ski jump (x-horizontal, y-vertical). The edge is horizontal so $x'(0) = v_0 > 0$, but $y'(0) = 0$.

$$\begin{aligned} mx'' &= -\delta x' - \lambda y' & x'(0) &= v_0, & x(0) &= 0 \\ my'' &= -mg + \lambda x' - \delta y' & y'(0) &= 0, & y(0) &= 0 \end{aligned}$$

where m is the skier's mass and δ, λ, and v_0 are positive constants. Integration of each of these ODEs yields

$$\begin{aligned} mx' - mv_0 &= -\delta x - \lambda y \\ my' &= -mgt + \lambda x - \delta y \end{aligned}$$

Divide by the mass to get the system IVP

$$\begin{aligned} x' &= -ax - by + v_0 & x(0) &= 0 \\ y' &= -gt + bx - ay & y(0) &= 0 \end{aligned}$$

where $a = \delta/m$ and $b = \lambda/m$ are the drag and lift coefficients, respectively.

When Newtonian drag and lift occur, δ and λ are not constants, so we can no longer integrate once to get x' and y', and we must treat the original second-order ODE differently:

$$\begin{aligned} x' &= v & x(0) &= 0 \\ v' &= -\delta v/m - \lambda w/m & v(0) &= v_0 \\ y' &= w & y(0) &= 0 \\ w' &= -g + \lambda v/m - \delta w/m & w(0) &= 0 \end{aligned}$$

where v and w are the velocities in the $\hat{\mathbf{i}}$- and $\hat{\mathbf{j}}$-directions, respectively.

We have assumed that the bottom edge of the ski jump is horizontal, but everything can be modified to accommodate a tilt in the launch angle (see the chapter cover figure and Exploration 5.4, Problem 1).

References

Halliday, D., and Resnick, R., *Physics*, (1994: John Wiley & Sons, Inc.)

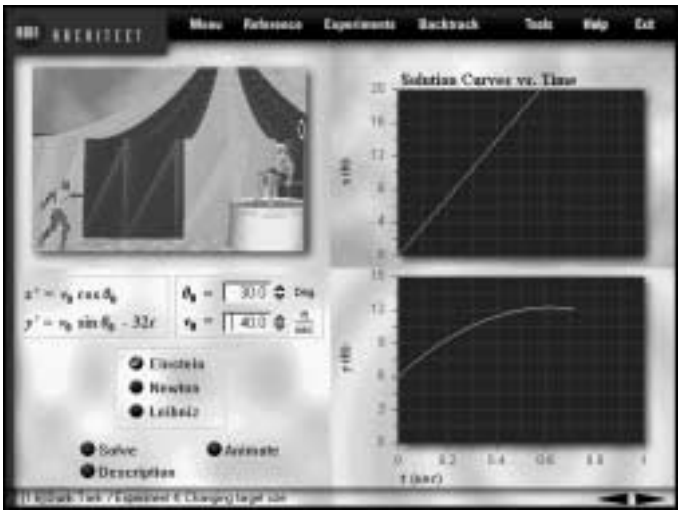
True, Ernest, "The flight of a ski jumper" in *C·ODE·E*, Spring 1993, pp. 5–8, http://www.math.hmc.edu/codee

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 5.1. Dunk Tank



1. *How big is the target?*
Play the dunk tank game on Screen 1.6 of Module 5 and use various launch angles and speeds to help you determine the heights and diameters of the Einstein, Leibniz, and Newton targets, given that the ball has a 4-inch diameter.

2. *One speed, two angle ranges for success.*
Use the ODE Architect tool to find two quite different launch angles that will dunk Einstein if the launch speed is 40 ft/sec. Repeat for Leibniz and Newton.

3. *Launch angles and speeds that dunk Einstein.*
Find the region in the $\theta_0 v_0$ plane for which the ball hits the target and dunks Einstein. *Hint*: Start with $v_0 = 40$ ft/sec and determine the ranges for θ_0 using ODE Architect by playing the dunk tank game. Then repeat for other values of v_0.

4. *Solution formulas for the dunk tank model.*
The position and velocity of the ball at time t is given by formula (2). Find a formula that relates the launch angle to the initial speed and the time T needed to hit the bull's eye. If you had to choose between using your formula and using ODE Architect computer simulations to find winning combinations of speed and launch angle, which would you choose? Why?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 5.2. Longer to Rise or to Fall?

1. *Throw a ball up in the air.*
Do just that, and determine as best you can the time it takes to rise and to fall. You can use a whiffle ball for slower motion. Explain your results. (No computers here, and no math, either!)

2. *Longer to rise or to fall in a vacuum?*
What if there were no air to slow the ball down? Use ODE Architect to determine whether it takes the ball longer to rise or to fall. Try various initial speeds between 5 and 60 ft/sec. [*Suggestion:* Use the Sweep feature.]

3. *Longer to rise or to fall with viscous drag?*
Suppose that air exerts a viscous drag force on a whiffle ball (a reasonable assumption). For various initial speeds, use the ODE Architect to determine whether it takes longer to rise or to fall. Does your answer depend on the initial speed? What physical explanation can you give for your results?

4. *Longer to rise or to fall with your own drag?*
Repeat Problem 3, but make up several of your own formulas for the drag force. Include Newtonian drag as one case. This isn't as outlandish an idea as it may seem, since the drag force depends very much on the nature of the moving body, e.g., rough or smooth surface, holes through the body, and so on. Discuss your results.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 5.3. Indiana Newton

1. *Indiana Newton lands on the boxcar (no drag).*
Indiana Newton jumps from a height h of 100 ft and intends to land on the boxcar of a train moving at a speed of 30 ft/sec. Assuming that there is no air resistance, use Screen 2.6 of Module 5 to find the time window of opportunity for jumping from the ledge.

2. *Indiana Newton lands on the boxcar (Newtonian drag).*
Repeat Problem 1 but with Newtonian drag (coefficient $k/m = 0.05$ ft^{-1}). Compare fall-times with the no-drag and also with the viscous-drag ($k/m = 0.05$ sec^{-1}) cases. Find nonzero values of the coefficients so that Indiana Newton hits the train sooner with Newtonian drag than with viscous drag. How do the fall-times change as Indy's jump height h varies?

3. *How long is the boxcar?*
Use computer simulations of Indiana Newton jumping onto the boxcar and estimate the length of the boxcar.

4. *Indiana Newton floats down.*
First, explore Indiana's fall-times in the viscous- and Newtonian-drag cases where the coefficient k/m has magnitudes ranging from 0 to 0.5. The units for the coefficient are sec^{-1} (viscous damping) and ft^{-1} (Newtonian damping). Then find a formula in terms of h, k/m, and t for Indiana's position after he jumps: first in the viscous-drag case, then in the Newtonian-drag case. *Suggestion*: In the viscous case, first solve $v' = -32 - kv/m, \quad v(0) = 0$, and then integrate and use $y(0) = h$ to get $y(t)$. In the Newtonian-drag case proceed similarly but with $v' = -32 - kv^2$. (This one is hard!) Choose values of the parameters in each case, and compare the graphs of the height function $y(t)$ from your formula with the graphs obtained by the ODE Architect. Any differences?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 5.4. Ski Jumping

1. *Tilt the edge of the ski jump upward.*
Use the first Things-to-Think-About on Screen 3.7 of Module 5 to see what happens to the ski jumper's path if the edge of the ski jump structure is tilted upward at 0.524 radians (about 30°). Set $a = 0.01$ sec^{-1}, and $v_0 = 85$ ft/sec, and sweep on the lift coefficient b from 0 to 1.0 in 20 steps. Compare your graphs of the jumper's path with the chapter cover figure. Then animate your graphs. Now fix b at the value 0.02 sec^{-1} and sweep on θ (in radians) to see the effect of the tilt angle on the jumper's path. Explain your results.

2. *Loop-the-loop.*
The second Things-to-Think-About on Screen 3.7 of Module 5 asks you to use the ODE Architect to estimate the smallest value of the viscous damping coefficient b that will allow the ski jumper to loop-the-loop. If $a = 0.01$ sec^{-1} and $v_0 = 85$ ft/sec, estimate that value. Then increase b by increments from that value upward all the way to the unrealistic value of 5.0 sec^{-1} and describe what you see.

3. *Complex eigenvalues and loop-the-loops.*

☞ You need to know about matrices and eigenvalues to complete this part. See also Chapter 6.

The system matrix of the viscous drag/lift model for ski jumping is

$$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}$$

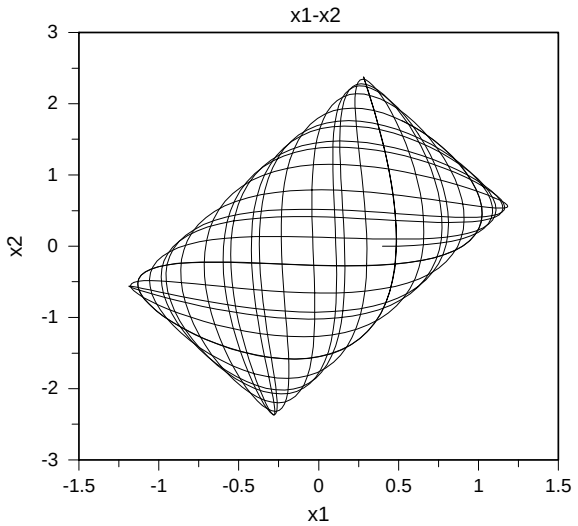
Explain why the eigenvalues of this matrix are complex conjugates with negative real parts if a and b are any positive real numbers. Explain why you are more likely to see loop-the-loops if a is small and b is large. Do some simulations with the ODE Architect for various values of a and b that support your explanation. If you plot a loop-the-loop path over a long enough time interval, you will see no loops at all near the end of the interval. Any explanation?

4. *Newton on skis.*

The fourth Things-to-Think-About on Screen 3.7 of Module 5 puts Indiana Newton on skis with Newtonian drag (of course!). This situation takes you to the expert solver in the ODE Architect, where you are asked to explore every scenario you can think of and to explain what you see in the graphs.

CHAPTER 6

First-Order Linear Systems



Oscillating displacements $x_1(t)$ and $x_2(t)$ of two coupled springs play off against each other.

Overview This chapter outlines some of the main facts concerning systems of first-order linear ODEs, especially those with constant coefficients. You'll have the opportunity to work with physical problems that have two or more dependent variables. Such problems can be modeled using systems of differential equations, which can always be written as systems of first-order equations, as can higher-order differential equations. The eigenvalues and eigenvectors of a matrix of coefficients help us understand the behavior of solutions of these systems.

Key words Linear systems; pizza and video; coupled springs; connected tanks; linearized double pendulum; matrix; system matrix; Jacobian matrix; component; component plot; phase space; phase plane; phase portrait; equilibrium point; eigenvalue; eigenvector; saddle point; node; spiral; center; source; sink

See also Chapter 5 for definitions of vector mathematics.

◆ Background

Many applications involve a single independent variable (usually time) and two or more dependent variables. Some examples of dependent variables are:

- the concentrations of a chemical in organs of the body
- the voltage drops across the elements of an electrical network
- the populations of several interacting species
- the profits of businesses in a mall

Applications with more than one dependent variable lead naturally to *systems* of ordinary differential equations. Such systems, as well as higher-order ODEs, can be rewritten as systems of first-order ODEs.

☞ How to convert a second-order ODE to a system of first-order ODEs.

Here's how to reduce a second-order ODE to a system of first-order ODEs (see also Chapter 4). Let's look at the the second-order ODE

$$y'' = f(t, y, y') \tag{1}$$

Introduce the variables $x_1 = y$ and $x_2 = y'$. Then we get the first-order system

$$x_1' = x_2 \tag{2}$$

$$x_2' = f(t, x_1, x_2) \tag{3}$$

ODE (2) follows from the definition of x_1 and x_2, and ODE (3) is ODE (1) rewritten in terms of x_1 and x_2.

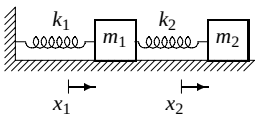
✓ "Check" your understanding now by reducing the second-order ODE $y'' + 5y' + 4y = 0$ to a system of first-order ODEs.

◆ Examples of Systems: Pizza and Video, Coupled Springs

Module 6 shows how to model the profits $x(t)$ and $y(t)$ of a pizza parlor and a video store by a system that looks like this:

$$x' = ax + by + c$$
$$y' = fx + gy + h$$

where a, b, c, f, g, and h are constants. Take another look at Screens 1.1–1.4 in Module 6 to see how ODE Architect handles these systems.



Module 6 also presents a model system of second-order ODEs for oscillating springs and masses. A pair of coupled springs with spring constants k_1 and k_2 are connected to masses m_1 and m_2 that glide back forth on a table. As shown in the "Coupled Springs" submodule, if damping is negligible then the second-order linear ODEs that model the displacements of the masses from equilibria are

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 - k_2 x_2$$

☞ A system of first-order ODEs is *autonomous* if the terms on the right-hand sides of the equations do not explicitly depend on time.

Let's set $m_1 = 4,\ m_2 = 1,\ k_1 = 3$, and $k_2 = 1$. Then, setting $x_1' = v_1,\ x_2' = v_2$, the corresponding autonomous system of four first-order ODEs is

$$\begin{aligned} x_1' &= v_1 \\ v_1' &= -x_1 + \frac{1}{4}x_2 \\ x_2' &= v_2 \\ v_2' &= x_1 - x_2 \end{aligned}$$

☞ Trajectories of an autonomous system can't intersect because to do so would violate the uniqueness property that only one trajectory can pass through a given point.

The cover figure of this chapter shows how x_1 and x_2 play off against each other when $x_1(0) = 0.4,\ v_1(0) = 1,\ x_2(0) = 0$, and $v_2(0) = 0$. The trajectories for this IVP are defined in the 4-dimensional $x_1v_1x_2v_2$-space and cannot intersect themselves. However, the projections of the trajectories onto any plane *can* intersect, as we see in the cover figure.

◆ Linear Systems with Constant Coefficients

The model first-order systems of ODEs for pizza and video and for coupled springs have the special form of linear systems with constant coefficients. Now we shall see just what linearity means and how it allows us (sometimes) to construct solution formulas for linear systems.

Let t (time) be the independent variable and let $x_1, x_2, \ldots, x_n$ denote the dependent variables. Then a general system of first-order linear *homogeneous* ODEs with constant coefficients has the form

☞ Dependent variables are also called *state variables*.

☞ *Homogeneous* means that there are no free terms, that is, terms that don't involve any x_i.

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{4}$$

where $a_{11}, a_{12}, \ldots, a_{nn}$ are given constants. To find a unique solution, we need a set of initial conditions, one for each dependent variable:

$$x_1(t_0) = \alpha_1, \quad \ldots, \quad x_n(t_0) = \alpha_n \tag{5}$$

where t_0 is a specific time and $\alpha_1, \ldots, \alpha_n$ are given constants. The system (4) and the initial conditions (5) together constitute an *initial value problem* (IVP) for $x_1, \ldots, x_n$ as functions of t. Note that $x_1 = \cdots = x_n = 0$ is an equilibrium point of system (4).

☞ An equilibrium point of an autonomous system of ODEs is a point where all the rates are zero; it corresponds to a constant solution.

The model on Screen 1.4 of Module 6 for the profits of the pizza and video stores is the system

$$\begin{aligned} x' &= 0.06x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05y - 0.013 \end{aligned} \tag{6}$$

☞ If $n = 2$, we often use x and y for the dependent variables.

with the initial conditions

$$x(0) = 0.30, \quad y(0) = 0.20 \tag{7}$$

The ODEs (6) are nonhomogeneous due to the presence of the free term -0.013 in each equation. The coordinates of an equilibrium point of a system are values of the dependent variables for which all of the derivatives $x_1', \ldots, x_n'$ are zero. For the system (6) the only equilibrium point is $(0.2, 0.1)$. The translation $X = x - 0.2,\ Y = y - 0.1$ transforms the system (6) into the system

☞ A change of variables puts the equilibrium point at the origin.

$$\begin{aligned} X' &= 0.06X + 0.01Y \\ Y' &= 0.04X + 0.05Y \end{aligned} \tag{8}$$

which is homogeneous and has the same coefficients as the system (6). In terms of X and Y, the initial conditions (7) become

$$X(0) = 0.1, \quad Y(0) = 0.1 \tag{9}$$

Although we have converted a nonhomogeneous system to a homogeneous system in this particular case, it isn't always possible to do so.

☞ Vectors and matrices appear as bold letters.

☞ **A** is called the *linear system matrix*, or the *Jacobian matrix* (often denoted by J).

It is useful here to introduce matrix notation: it saves space and it expresses system (4) in the form of a single equation. Let $\mathbf{x}$ be the vector with components $x_1, x_2, \ldots, x_n$ and let $\mathbf{A}$ be the matrix of the coefficients, where a_{ij} is the element in the ith row and jth column of $\mathbf{A}$. The derivative of the vector $\mathbf{x}$, written $d\mathbf{x}/dt$ or $\mathbf{x}'$, is defined to be the vector with the components $dx_1/dt, \ldots, dx_n/dt$. Therefore we can write the system (4) in the compact form

☞ The vector $\mathbf{x}(t)$ is called the *state* of system (10) at time t.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{10}$$

In vector notation, the initial conditions (5) become

$$\mathbf{x}(t_0) = \boldsymbol{\alpha} \tag{11}$$

where $\boldsymbol{\alpha}$ is the vector with components $\alpha_1, \ldots, \alpha_n$.

✓ Find the linear system matrix for system (8).

A solution of the initial value problem (10) and (11) is a set of functions

$$\begin{aligned} x_1 &= x_1(t) \\ &\vdots \\ x_n &= x_n(t) \end{aligned} \tag{12}$$

that satisfy the differential equations and initial conditions. Using our new notation, if $\mathbf{x}(t)$ is the vector whose components are $x_1(t), \ldots, x_n(t)$, then $\mathbf{x} = \mathbf{x}(t)$ is a solution of the corresponding vector IVP, (10) and (11). The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is homogeneous, while a nonhomogeneous system would have the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{F}$, where $\mathbf{F}$ is a vector function of t or else a constant vector.

◆ Solution Formulas: Eigenvalues and Eigenvectors

To find a solution formula for system (10) let's look for an exponential solution of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t} \tag{13}$$

where λ is a constant and $\mathbf{v}$ is a constant vector to be determined. Substituting $\mathbf{x}$ as given by (13) into the ODE (10), we find that $\mathbf{v}$ and λ must satisfy the algebraic equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{14}$$

Equation (14) can also be written in the form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \tag{15}$$

where $\mathbf{I}$ is the *identity matrix* and $\mathbf{0}$ is the *zero vector* with zero for each component. Equation (15) has nonzero solutions if and only if λ is a root of the nth-degree polynomial equation

☞ The determinant of a matrix is denoted by det.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{16}$$

called the *characteristic equation* for the system (10). Such a root is called an *eigenvalue* of the matrix $\mathbf{A}$. We will denote the eigenvalues by $\lambda_1, \ldots, \lambda_n$. For each eigenvalue λ_i there is a corresponding nonzero solution $\mathbf{v}^{(i)}$, called an *eigenvector*. The eigenvectors are not determined uniquely but only up to an arbitrary multiplicative constant.

☞ The keys to finding a solution formula for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are the eigenvalues and eigenvectors of $\mathbf{A}$.

For each eigenvalue-eigenvector pair $(\lambda_i, \mathbf{v}^{(i)})$ there is a corresponding vector solution $\mathbf{v}^{(i)}e^{\lambda_i t}$ of the ODE (10). If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different, then there are n such solutions,

$$\mathbf{v}^{(1)}e^{\lambda_1 t}, \ldots, \mathbf{v}^{(n)}e^{\lambda_n t}$$

In this case the *general solution* of system (10) is the linear combination

$$\mathbf{x} = C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + \cdots + C_n\mathbf{v}^{(n)}e^{\lambda_n t} \tag{17}$$

☞ Formula (17) is called the general solution formula of system (10) because every solution has the form of (17) for some choice of the constants C_j. The other way around, every choice of the constants yields a solution of system (10).

The arbitrary constants $C_1, \ldots, C_n$ can always be chosen to satisfy the initial conditions (11). If the eigenvalues are not distinct, then the general solution takes on a slightly different (but similar) form. The texts listed in the references give the formulas for this case. If some of the eigenvalues are complex, then the solution given by formula (17) is complex-valued. However, if all of the coefficients a_{ij} are real, then the complex eigenvalues and eigenvectors occur in complex conjugate pairs, and it is always possible to express the solution formula (17) in terms of real-valued functions. Look ahead to formulas (20) and (21) for a way to accomplish this feat.

◆ Calculating Eigenvalues and Eigenvectors

Here's how to find the eigenvalues and eigenvectors of a 2×2 real matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

First define the *trace* of $\mathbf{A}$ (denoted by $\operatorname{tr}\mathbf{A}$) to be the sum $a+d$ of the diagonal entries, and the *determinant* of $\mathbf{A}$ (denoted by $\det\mathbf{A}$) to be the number $ad-bc$. Then the characteristic equation for $\mathbf{A}$ is

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} \\ &= 0 \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots λ_1 and λ_2 of this quadratic equation. We assume $\lambda_1 \neq \lambda_2$. For the eigenvalue λ_1 we can find a corresponding eigenvector $\mathbf{v}^{(1)}$ by solving the vector equation

$$\mathbf{A}\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)}$$

for $\mathbf{v}^{(1)}$. In a similar fashion we can find an eigenvector $\mathbf{v}^{(2)}$ corresponding to the eigenvalue λ_2.

Example: Take a look at the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

Since

$$\operatorname{tr}\mathbf{A} = 0+3 = 3 \quad \text{and} \quad \det\mathbf{A} = 0\cdot 3 - 1\cdot(-2) = 2$$

the characteristic equation is

$$\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. To find an eigenvector $\mathbf{v}^{(1)}$ for λ_1, let's solve

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)} = \mathbf{v}^{(1)}$$

for $\mathbf{v}^{(1)}$. Denoting the components of $\mathbf{v}^{(1)}$ by α and β, we have

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ -2\alpha+3\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This gives two equations for α and β:

$$\beta = \alpha, \quad -2\alpha + 3\beta = \beta$$

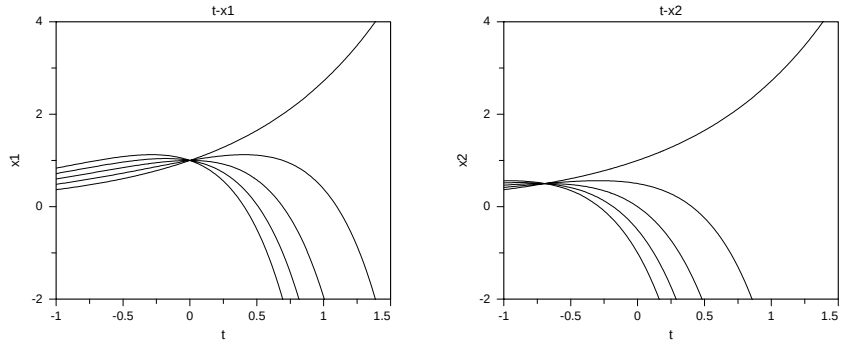


Figure 6.1: Graphs of five solutions $x_1(t)$ (left), $x_2(t)$ (right) of system (18).

The second equation is equivalent to the first, so we may as well set $\alpha = \beta = 1$, which gives us an eigenvector $\mathbf{v}^{(1)}$. In a similar way for the eigenvalue λ_2, we can find an eigenvector $\mathbf{v}^{(2)}$ with components $\alpha = 1,\ \beta = 2$. So the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in this case is

$$\begin{aligned}\mathbf{x} &= C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + C_2\mathbf{v}^{(2)}e^{\lambda_2 t} \\ &= C_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} + C_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}\end{aligned}$$

or in component form

$$\begin{aligned}x_1 &= C_1e^{t} + C_2e^{2t} \\ x_2 &= C_1e^{t} + 2C_2e^{2t}\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

✓ Find a formula for the solution of system (18) if $x_1(0) = 1,\ x_2(0) = -1$. Figure 6.1 shows graphs of $x_1(t)$ and $x_2(t)$ where $x_1(0) = 1,\ x_2(0) = 0, \pm 0.5, \pm 1$. Which graphs correspond to $x_1(0) = 1,\ x_2(0) = -1$? What happens as $t \to +\infty$? As $t \to -\infty$?

◆ Phase Portraits

We can view solutions graphically in several ways. For example, we can draw plots of $x_1(t)$ vs. t, $x_2(t)$ vs. t, and so on. These plots are called *component plots* (see Figure 6.1). Alternatively, we can interpret equations (12) as a set of parametric equations with t as the parameter. Then each specific value of t corresponds to a set of values for $x_1, \ldots, x_n$. We can view this set of values as coordinates of a point in $x_1x_2\cdots x_n$-space, called the *phase space*. (If $n = 2$ it's called the *phase plane*.) For an interval of t-values, the corresponding points form a curve in phase space. This curve is called a *phase plot*, a *trajectory*, or an *orbit*.

☞ Another term for phase space is *state space*.

Phase plots are particularly useful if $n = 2$. In this case it is often worthwhile to draw several trajectories starting at different initial points on the same set of axes. This produces a *phase portrait*, which gives us the best possible overall view of the behavior of solutions. Whatever the value of n, the trajectories of system (10) can never intersect because system (10) is autonomous.

If $\mathbf{A}$ in system (10) is a 2×2 matrix, then it is useful to examine and classify the various cases that can arise. There aren't many cases when $n = 2$, but even so these cases give important information about higher-dimensional linear systems, as well as nonlinear systems (see Chapter 7). We won't consider here the cases where the two eigenvalues are equal, or where one or both of them are zero.

A *direction field* (or *vector field*) for an autonomous system when $n = 2$ is a field of line segments. The slope of the segment at the point (x_1, x_2) is x_2'/x_1'. The trajectory through (x_1, x_2) is tangent to the segment. An arrowhead on the segment shows the direction of the flow. See Figures 6.2–6.5 for examples.

Real Eigenvalues

If the eigenvalues λ_1 and λ_2 are real, the general solution is

$$\mathbf{x} = C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + C_2\mathbf{v}^{(2)}e^{\lambda_2 t} \tag{19}$$

where C_1 and C_2 are arbitrary real constants.

Let's first look at the case where λ_1 and λ_2 have opposite signs, with $\lambda_1 > 0$ and $\lambda_2 < 0$. The term in formula (19) involving λ_1 dominates as $t \to +\infty$, and the term involving λ_2 dominates as $t \to -\infty$. Thus as $t \to +\infty$ the trajectories approach the line that goes through the origin and has the same slope as $\mathbf{v}^{(1)}$, and as $t \to -\infty$, they approach the line that goes through the origin and has the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case is shown in Figure 6.2. The origin is called a *saddle point*, and it is *unstable*, since most solutions move away from the point.

☞ Trajectories starting on either line at $t = 0$ stay on the line.

☞ Eigenvalues of opposite signs imply a *saddle*.

Now suppose that λ_1 and λ_2 are both negative, with $\lambda_2 < \lambda_1 < 0$. The solution is again given by formula (19), but in this case both terms approach

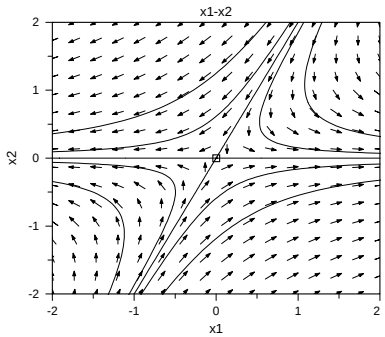


Figure 6.2: Phase portrait of a saddle: $x_1' = x_1 - x_2,\ x_2' = -x_2$.

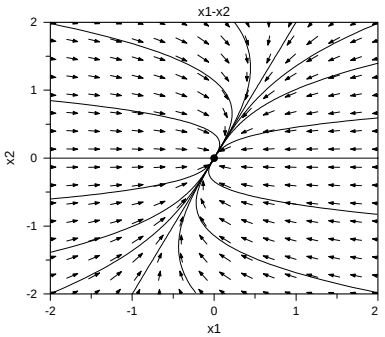


Figure 6.3: Phase portrait of a nodal sink: $x_1 = -3x_1 + x_2,\ x_2' = -x_2$.

zero as $t \to +\infty$. However, for large positive t, the factor $e^{\lambda_2 t}$ is much smaller than $e^{\lambda_1 t}$, so for $C_1 \neq 0$ the trajectories approach the origin tangent to the line with the same slope as $\mathbf{v}^{(1)}$, and if $C_1 = 0$ the trajectory lies on the line with the same slope as $\mathbf{v}^{(2)}$. For large negative t, the term involving λ_2 is the dominant one and the trajectories approach asymptotes that have the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case is shown in Figure 6.3. The origin attracts all solutions and is called an *asymptotically stable node*. It is also called a *sink* because all nearby orbits get pulled in as $t \to +\infty$.

☞ Both eigenvalues negative imply a *nodal sink*.

If both eigenvalues are positive, the situation is similar to when both eigenvalues are negative, but in this case the direction of motion on the trajectories is reversed. For example, suppose that $0 < \lambda_1 < \lambda_2$: then the trajectories are unbounded as $t \to +\infty$ and asymptotic to lines parallel to $\mathbf{v}^{(2)}$. As $t \to -\infty$ the trajectories approach the origin either tangent to the line through the origin with the same slope as $\mathbf{v}^{(1)}$ or lying on the line through the origin with the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case looks like Figure 6.3 but with the arrows reversed. The origin is an *unstable node*. It is also called a *source* because all orbits (except $\mathbf{x} = \mathbf{0}$ itself) flow out and away from the origin as t increases from $-\infty$.

☞ Both eigenvalues positive imply a *nodal source*.

✓ Find the eigenvalues and eigenvectors of the systems of Figures 6.2 and 6.3 and interpret them in terms of the phase plane portraits.

Complex Eigenvalues

Now suppose that the eigenvalues are complex conjugates $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The exponential form (13) of a solution remains valid, but usually it is preferable to use Euler's formula:

$$e^{i\beta t} = \cos(\beta t) + i \sin(\beta t) \tag{20}$$

This allows us to write the solution in terms of real-valued functions. The result is

$$\mathbf{x} = C_1 e^{\alpha t}[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)] + C_2 e^{\alpha t}[\mathbf{b}\cos(\beta t) + \mathbf{a}\sin(\beta t)] \tag{21}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the eigenvector $\mathbf{v}^{(1)}$ associated with λ_1, and C_1 and C_2 are constants. The trajectories are spirals about the origin. If $\alpha > 0$, then the spirals grow in magnitude and the origin is called a *spiral source* or an *unstable spiral point*. A typical phase portrait in this case looks like Figure 6.4. If $\alpha < 0$, then the spirals approach the origin as $t \to +\infty$, and the origin is called a *spiral sink* or an *asymptotically stable spiral point*. In both cases the spirals encircle the origin and may be directed in either the clockwise or counterclockwise direction (but not both directions in the same system).

☞ Complex eigenvalues with nonzero real parts imply a *spiral sink* or a *spiral source*.

Finally, consider the case $\lambda = \pm i\beta$, where β is real and positive. Now the exponential factors in solution formula (21) are absent so the trajectory is bounded as $t \to \pm\infty$, but it does not approach the origin. In fact, the

trajectories are ellipses centered on the origin (see Figure 6.5), and the origin is called a *center*. It is *stable*, but not asymptotically stable.

☞ Pure imaginary eigenvalues imply a *center*.

✓ Find the eigenvalues of the systems of Figures 6.4 and 6.5, and interpret them in terms of the phase plane portraits. Why can't you "see" the eigenvectors in these portraits?

There is one other graphing technique that is often useful. If $n = 2$, ODE Architect can draw a plot of the solution in tx_1x_2-space. If we project this curve onto each of the coordinate planes, we obtain the two component plots and the phase plot (Figure 6.6).

◆ Using ODE Architect to Find Eigenvalues and Eigenvectors

ODE Architect will find equilibrium points of a system and the eigenvalues and eigenvectors of the Jacobian matrix of an autonomous system at an equilibrium point. Here are the steps:

- Enter an autonomous system of first-order ODEs.
- Click on the lower left Equilibrium tab; enter a guess for the coordinates of an equilibrium point.
- The Equil. tab at the lower right will bring up a window with calculated coordinates of an equilibrium point close to your guess.

☞ Use this Architect feature to calculate the eigenvalues, eigenvectors.

- Double click anywhere on the boxed coordinates of an equilibrium in the window (or click on the window's editing icon) to see the eigenvalues, eigenvectors, and the Jacobian matrix.

If you complete these steps for a system of two first-order, autonomous ODEs, ODE Architect will insert a symbol at the equilibrium point in the phase plane: An open square for a saddle, a solid dot for a sink, an open dot for a source, and a plus sign for a center (Figures 6.2–6.5). The symbols can be edited using the Equilibrium tab on the edit window.

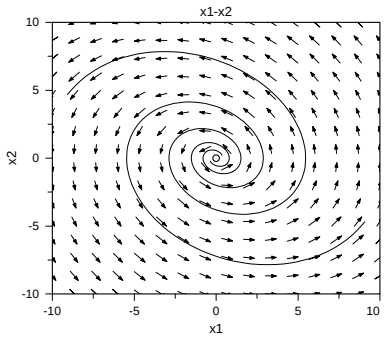


Figure 6.4: Phase portrait of a spiral source: $x_1' = x_2,\ x_2' = x_1 + 0.4x_2$.

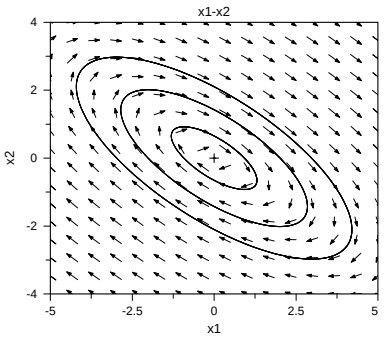


Figure 6.5: Phase portrait of a center: $x_1' = x_1 + 2x_2,\ x_2' = -x_1 - x_2$.

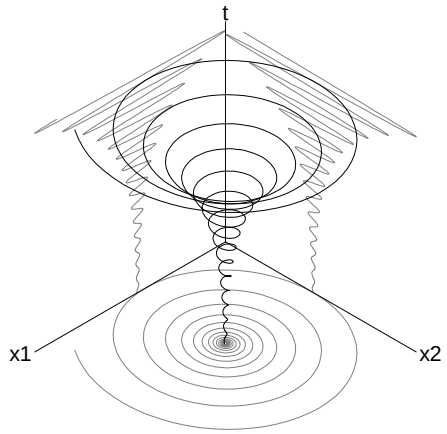


Figure 6.6: Solution curve of $x_1' = x_2,\ x_2' = -100.25x_1 + x_2,\ x_1(0) = 1,\ x_2(0) = 1$, the two component curves, and the trajectory in the x_1x_2-phase plane.

✓ Use ODE Architect to find the eigenvalues and eigenvectors of the systems in Figures 6.2–6.5.

◆ Separatrices

A trajectory Γ of a planar autonomous system is a *separatrix* if the long-term behavior of trajectories on one side of Γ is quite different from the behavior of those on the other side. Take a look at the four *saddle separatrices* in Figure 6.2, each of which is parallel to an eigenvector of the system matrix. The two separatrices that approach the saddle point as t increases are the *stable separatrices*, and the two that leave are the *unstable separatrices*.

◆ Parameter Movies

The eigenvalues of a 2×2 matrix $\mathbf{A}$ depend on the values of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$, and the behavior of the trajectories of $\mathbf{x}' = \mathbf{Ax}$ depends very much on the eigenvalues. So it makes sense to see what happens to trajectories as we vary the values of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$. When we do this varying, we can make the eigenvalues change sign, or move into the complex plane, or become equal. As the changes occur the behavior of the trajectories has to change as well. Take a look at the "Parameter Movies" part of Module 6 for some surprising views of the changing phase plane portraits as we follow along a path in the parameter plane of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$.

References

Borrelli, R.L., and Coleman, C.S., *Differential Equations: A Modeling Perspective* (1998: John Wiley & Sons, Inc.)

Boyce, W.E., and DiPrima, R.C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed., (1997: John Wiley & Sons, Inc.)

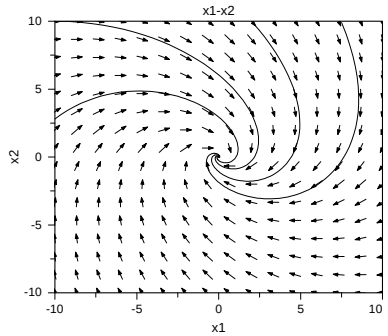
Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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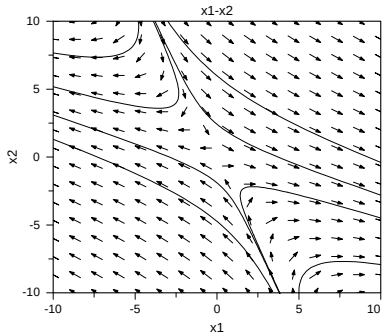
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Exploration 6.1. Eigenvalues, Eigenvectors, and Graphs

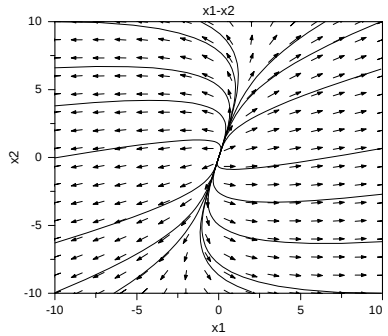
1. Each of the phase portraits in the graphs below is associated with a planar autonomous linear system with equilibrium point at the origin. What can you say about the eigenvalues of the system matrix **A** (e.g., are they real, complex, positive)? Sketch by hand any straight line trajectories. What can you say about the eigenvectors of **A**?



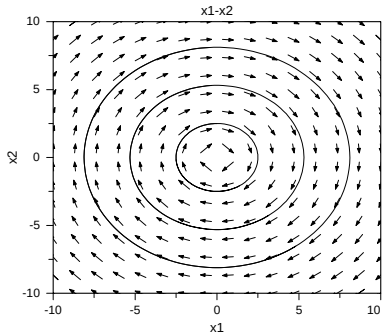
(a)



(b)



(c)



(d)

2. What does the phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ look like if $\mathbf{A}$ is a 2×2 matrix with one eigenvalue zero and the other nonzero? How many equilibrium points are there? Include portraits of specific examples.

3. Using Figure 6.6 as a guide, make your own gallery of 2D and 3D graphs to illustrate solution curves, component curves, trajectories, and phase-plane portraits of the systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix of constants. List eigenvalues and eigenvectors of $\mathbf{A}$. Include examples of the following types of equilibrium points:

- Saddle
- Nodal sink
- Nodal source
- Spiral sink
- Spiral source
- Center
- Eigenvalues of $\mathbf{A}$ are equal and negative

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 6.2. Pizza and Video

Sometimes business enterprises are strongly affected by periodic (e.g., seasonal) influences. We can illustrate this in the case of Diffey and Cue.

The model describing Diffey's and Cue's profits on Screen 1.4 in Module 6 is

$$\begin{aligned} x' &= 0.06x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05y - 0.013 \end{aligned} \tag{22}$$

Let's introduce a periodic fluctuation in the coefficient of x in the first ODE and in the coefficient of y in the second ODE.

Sine and cosine functions are often used to model periodic phenomena. We'll use $\sin(2\pi t)$ so that the fluctuations have a period of one time unit. We will also include a variable amplitude parameter a so that the intensity of the fluctuations can be easily controlled. We have the modified system

$$\begin{aligned} x' &= 0.06\left(1 + \frac{1}{2}a\sin(2\pi t)\right)x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05\left(1 + \frac{3}{10}a\sin(2\pi t)\right)y - 0.013 \end{aligned} \tag{23}$$

Note that if $a = 0$, we recover system (22), and that as a increases the amplitude of the fluctuations in the coefficients also increases.

1. Interpret the terms involving $\sin(2\pi t)$ in the context of Diffey's and Cue's businesses. Use ODE Architect to solve the system (23) subject to the initial conditions $x(0) = 0.3,\ y(0) = 0.2$ for $a = 1$. Use the time interval $0 \le t \le 10$, or an even longer interval. Plot x vs. t, y vs. t, and y vs. x. Compare the plots with the corresponding plots for the system (22). What is the effect of the fluctuating coefficients on the solution? Repeat with the same initial data, but sweeping a from 0 to 5 in 11 steps. What is the effect of increasing a on the solution?

2. Use ODE Architect to solve the system (23) subject to the initial conditions $x(0) = 0.25,\ y(0) = 0$ for $a = 3$. Draw a plot of y vs. x only. Be sure to use a sufficiently large t-interval to make clear the ultimate behavior of the solution. Repeat using the initial conditions $x(0) = 0.2,\ y(0) = -0.2$. Explain what you see.

3. For the two initial conditions in Problem 2 you should have found solutions that behave quite differently. Consider initial points on the line joining $(0.25, 0)$ and $(0.2, -0.2)$. For $a = 3$, estimate the coordinates of the point where the solution changes from one type of behavior to the other.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 6.3. Control of Interconnected Water Tanks

☞ Take a look at Chapter 8 for a way to diagram this "compartment" model.

Consider two interconnected tanks containing salt water. Initially Tank 1 contains 5 gal of water and 3 oz of salt while Tank 2 contains 4 gal of water and 5 oz of salt.

Water containing p_1 oz of salt per gal flows into Tank 1 at a rate of 2 gal/min. The mixture in Tank 1 flows out at a rate of 6 gal/min, of which half goes into Tank 2 and half leaves the system.

Water containing p_2 oz of salt per gal flows into Tank 2 at a rate of 3 gal/min. The mixture in Tank 2 flows out at a rate of 6 gal/min: 4 gal/min goes to Tank 1, and the rest leaves the system.

1. Draw a diagram showing the tank system. Does the amount of water in each tank remain the same during this flow process? Explain. If $q_1(t)$ and $q_2(t)$ are the amounts of salt (in oz) in the respective tanks at time t, show that they satisfy the system of differential equations:

$$q_1' = 2p_1 - \tfrac{6}{5}q_1 + q_2$$
$$q_2' = 3p_2 + \tfrac{3}{5}q_1 - \tfrac{3}{2}q_2$$

What are the initial conditions associated with this system of ODEs?

2. Suppose that $p_1 = 1$ oz/gal and $p_2 = 1$ oz/gal. Solve the IVP, plot $q_1(t)$ vs. t, and estimate the limiting value q_1^* that $q_1(t)$ approaches after a long time. In a similar way estimate the limiting value q_2^* for $q_2(t)$. Repeat for your own initial conditions, but remember that $q_1(0)$ and $q_2(0)$ must be nonnegative. How are q_1^* and q_2^* affected by changes in the initial conditions? Now use ODE Architect to find q_1^* and q_2^*. [*Hint:* Use the Equilibrium tab.] Is the equilibrium point a source or a sink? A node, saddle, spiral, or center?

3. The operator of this system (you) can control it by adjusting the input parameters p_1 and p_2. Note that q_1^* and q_2^* depend on p_1 and p_2. Find values of p_1 and p_2 so that $q_1^* = q_2^*$. Can you find values of p_1 and p_2 so that $q_1^* = 1.5q_2^*$? So that $q_2^* = 1.5q_1^*$?

4. Let c_1^* and c_2^* be the limiting concentrations of salt in each tank. Express c_1^* and c_2^* in terms of q_1^* and q_2^*, respectively. Find p_1 and p_2, if possible, so as to achieve each of the following results:

(a) $c_1^* = c_2^*$ **(b)** $c_1^* = 1.5c_2^*$ **(c)** $c_2^* = 1.5c_1^*$

Finally, consider all possible (nonnegative) values of p_1 and p_2. Describe the set of limiting concentrations c_1^* and c_2^* that can be obtained by adjusting p_1 and p_2.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 6.4. Three Interconnected Tanks

☞ Take a look at Chapter 8 for a way to diagram this "compartment" model.

Consider three interconnected tanks containing salt water. Initially Tanks 1 and 2 contain 10 gal of water while Tank 3 contains 15 gal. Each tank initially contains 6 oz of salt.

Water containing 2 oz of salt per gal flows into Tank 1 at a rate of 1 gal/min. The mixture in Tank 1 flows into Tank 2 at a rate of r gal/min. Furthermore, the mixture in Tank 1 is discharged into the drain at a rate of 2 gal/min. Water containing 1 oz of salt per gal flows into Tank 2 at a rate of 2 gal/min. The mixture in Tank 2 flows into Tank 3 at a rate of $r+1$ gal/min and also flows back into Tank 1 at a rate of 1 gal/min. The mixture in Tank 3 flows into Tank 1 at a rate of r gal/min, and down the drain at a rate of 1 gal/min.

1. Draw a diagram that depicts the tank system. Does the amount of water in each tank remain constant during the process? Show that the flow process is modeled by the following system of equations, where $q_1(t)$, $q_2(t)$, and $q_3(t)$ are the amounts of salt (in oz) in the respective tanks at time t:

$$q_1' = 2 - \frac{r+2}{10}q_1 + \frac{1}{10}q_2 + \frac{r}{15}q_3$$
$$q_2' = 2 + \frac{r}{10}q_1 - \frac{r+2}{10}q_2$$
$$q_3' = \frac{r+1}{10}q_2 - \frac{r+1}{15}q_3$$

What are the corresponding initial conditions?

2. Let $r = 1$, and use ODE Architect to plot q_1 vs. t, q_2 vs. t, and q_3 vs. t for the IVP in Problem 1. Estimate the limiting value of the amount of salt in each tank after a long time. Now suppose that the flow rate r is increased to 4 gal/min. What effect do you think this will have on the limiting values for q_1, q_2, and q_3? Check your intuition with ODE Architect. What do you think will happen to the limiting values if r is increased further? For each value of r use ODE Architect to find the limiting values for q_1, q_2, and q_3.

3. Although the two sets of graphs in Problem 2 may look similar, they're actually slightly different. Calculate the eigenvalues of the coefficient matrix when $r = 1$ and when $r = 4$. There is a certain "critical" value $r = r_0$ between 1 and 4 where complex eigenvalues first occur. Determine r_0 to two decimal places.

☞ Use ODE Architect to find the eigenvalues.

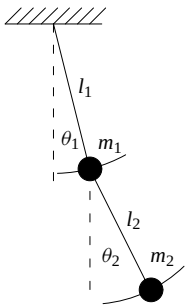
4. Complex eigenvalues lead to sinusoidal solutions. Explain why the oscillatory behavior characteristic of the sine and cosine functions is not apparent in your graphs from Problem 2 for $r = 4$. Devise a plan that will enable you to construct plots showing the oscillatory part of the solution for $r = 4$. Then execute your plan to make sure that it is effective.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 6.5. Small Motions of a Double Pendulum; Coupled Springs



Another physical system with two degrees of freedom is the planar double pendulum. This consists of two rods of length l_1 and l_2 and two masses m_1 and m_2, all attached together so that motions are confined to a vertical plane. Here we'll investigate motions for which the pendulum system doesn't move too far from its stable equilibrium position in which both rods are hanging vertically downward. We'll assume the damping in this system is negligible.

A sketch of the double pendulum system is shown in the margin. A derivation of the nonlinear equations in terms of the angles $\theta_1(t)$ and $\theta_2(t)$ that govern the oscillations of the system is given in Chapter 7 (beginning on page 126). The equations of interest here are the linearized ODEs in θ_1 and θ_2 where both of these angles are required to be small:

$$l_1\theta_1'' + \frac{m_2}{m_1 + m_2} l_2\theta_2'' + g\theta_1 = 0$$

$$l_2\theta_2'' + l_1\theta_1'' + g\theta_2 = 0$$

For small values of θ_1, θ_1', θ_2, and θ_2' these ODEs are obtained by linearizing ODEs (19) and (20) on page 127.

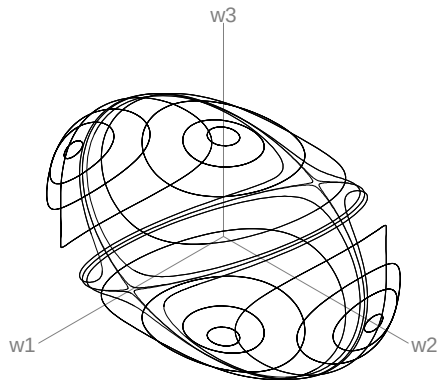
1. Consider the special case where $m_1 = m_2 = m$ and $l_1 = l_2 = l$, and define $g/l = \omega_0^2$. Write the equations above as a system of four first-order equations. Use ODE Architect to generate motions for different values of ω_0. Experiment with different initial conditions. Try to visualize the motions of the pendulum system that correspond to your solutions. Then use the model-based animation tool in ODE Architect and watch the animated double pendulums gyrate as your initial value problems are solved.

2. Assume $\omega_0^2 = 10$ in Problem 1. Can you find in-phase and out-of-phase oscillations that are analogous to those of the coupled mass-spring system? Determine the relationships between the initial conditions $\theta_1(0)$ and $\theta_2(0)$ that are needed to produce these motions. Plot θ_2 against θ_1 for these motions. Then change $\theta_1(0)$ or $\theta_2(0)$ to get a motion which is neither in-phase nor out-of-phase. Overlay this graph on the first plot. Explain what you see. Use the model-based animation feature in ODE Architect to help you "see" the in-phase and out-of-phase motions, and those that are neither. Describe what you see.

3. Show that the linearized equations for the double pendulum in Problem 2 are equivalent to those for a particular coupled mass-spring system. Find the corresponding values of (or constraints on) the mass-spring parameters m_1, m_2, k_1, and k_2. Does this connection extend to other double-pendulum parameter values besides those in Problems 1 and 2? If so, find the relationships between the parameters of the corresponding systems. Use the model-based animation feature in ODE Architect and watch the springs vibrate and the double pendulum gyrate. Describe what you see.

CHAPTER 7

Nonlinear Systems



Angular velocity: twenty-four ways to spin a book.

Overview While many natural processes can be modeled by linear systems of ODEs, others require *nonlinear systems*. Fortunately, some of the ideas used to understand linear systems can be modified to apply to nonlinear systems. In particular, state (or phase) spaces and equilibrium solutions (as well as eigenvalues and eigenvectors) continue to play a key role in understanding the long-term behavior of solutions. You will also see some new phenomena that occur only in nonlinear systems. We restrict our attention to *autonomous* equations, that is, equations in which time does not explicitly appear in the rate functions.

Key words Nonlinear systems of differential equations; linearization; direction fields; state (phase) space; equilibrium points; Jacobian matrices; eigenvalues; separatrices; Hopf bifurcations; limit cycles; predator-prey; van der Pol system; saxophone; spinning bodies; conservative systems; integrals of motion; angular velocity; Newton's second law; nonlinear double pendulum

See also Chapter 6 for background on linear systems and Chapters 8–10 and 12 for more examples of nonlinear systems.

◆ Linear vs. Nonlinear

In modeling a dynamical process with ODEs we aim for a model that is both reasonably accurate and solvable. By the latter we mean that there are either explict solution formulas that reveal how solutions behave, or reliable numerical solvers for approximating solutions. Constant-coefficient linear ODEs and linear systems have explicit solution formulas (see Chapters 4 and 6), and that is one reason linearity is widely assumed in modeling. However, nonlinearity is an essential feature of many dynamical processes, but explicit solution formulas for nonlinear ODEs are rare. So for nonlinear systems we turn to the alternative approaches, and that's what this chapter is about.

◆ The Geometry of Nonlinear Systems

Let's start with the *linear* system of ODEs that models the motion of a certain viscously damped spring-mass system that obeys Hooke's Law for the displacement x of a unit mass from equilibrium:

$$x' = y, \quad y' = -x - 0.1y \tag{1}$$

In Chapter 4 we saw that the equivalent linear second-order ODE, $x'' + 0.1x' + x = 0$, has an explicit solution formula, which we can use to determine the behavior of solutions and of trajectories in the xy-phase plane.

Now let's suppose that the Hooke's-law spring is replaced by a stiffening spring, which can be modeled by replacing the Hooke's-law restoring force $-x$ in system (1) with the nonlinear restoring force $-x - x^3$. We obtain the system

$$x' = y, \quad y' = -x - x^3 - 0.1y \tag{2}$$

As in the linear system (1), the nonlinear system (2) defines a *vector* (or *direction*) *field* in the *xy-state* (or *phase*) *plane*. The field lines are tangent to the *trajectories* (or *orbits*) and point in the direction of increasing time.

There are no solution formulas for system (2), so we turn to direction fields and ODE Architect for visual clues to solution behavior. As you can see from Figure 7.1, the graphs generated by ODE Architect tell us that the trajectories of both systems spiral into the *equilibrium point* at the origin as $t \to +\infty$, even though the shapes of the trajectories differ. The origin corresponds to the constant solution $x = 0,\ y = 0$, which is called a *spiral sink* for each system because of the spiraling nature of the trajectories and because the trajectories, like water in a draining sink, are "pulled" into the origin with the advance of time. This is an indication of *long-term* or *asymptotic* behavior. Note that in this case the nonlinearity does not affect long-term behavior, but clearly does affect short-term behavior.

☞ The equilibrium points of a system correspond to the constant solutions, that is, to the points where all the rate functions of the system are zero.

✓ "Check" your understanding by answering these questions: Do the systems (1) and (2) have any equilibrium points other than the origin? How do the corresponding springs and masses behave as time increases? Why does the $-x^3$ term seem to push orbits toward the y-axis if $|x| \geq 1$, but not have much effect if $|x|$ is close to zero?

◆ Linearization

If we start with a nonlinear system such as (2), we can often use *linear approximations* to help us understand some features of its solutions. Our approximations will give us a corresponding linear system and we can apply what we know about that linear system to try to understand the nonlinear system. In particular, we will be able to verify our earlier conclusions about the long-term behavior of the nonlinear spring-mass system (2).

The nonlinearity of system (2) comes from the $-x^3$ term in the rate function $g(x, y) = -x - x^3 - 0.1y$. In calculus you may have seen the following formula for the linear approximation of the function $g(x, y)$ near the point (x_0, y_0):

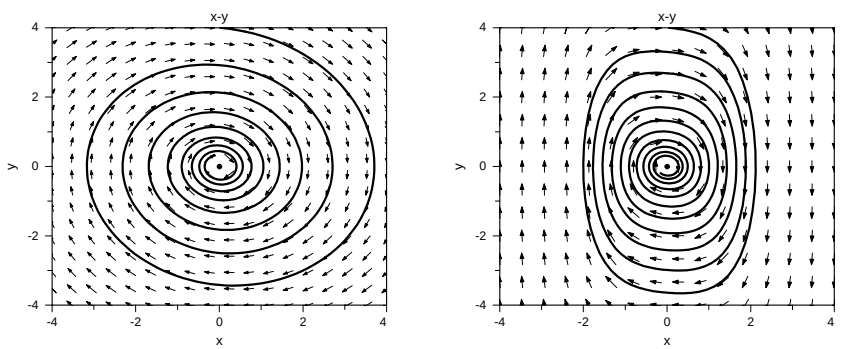
☞ This is a *finite Taylor series* approximation to $g(x, y)$.

$$g(x, y) \approx g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (3)$$

However, $g(x_0, y_0)$ will always be zero at an equilibrium point (do you see why?), so formula (3) simplifies in this case to

$$g(x, y) \approx \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (4)$$

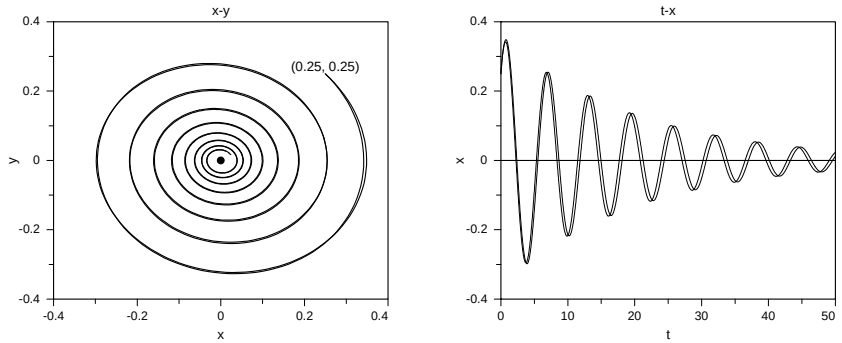
Since we're interested in long-term behavior and the trajectories of system (2) seem to be heading toward the origin, we want to use the equilibrium point



Linear System
$x' = y,\ y' = -x - 0.1y$

Nonlinear System
$x' = y,\ y' = -x - x^3 - 0.1y$

Figure 7.1: Trajectories of both systems have the same long-term, spiral-sink behavior, but behavior differs in the short-term.



Trajectories *t*-*x* curves

Figure 7.2: Near the equilibrium at the origin trajectories and tx-component curves of nonlinear system (2) and its linearization (1) are nearly look-alikes.

$(x_0, y_0) = (0, 0)$ in formula (4). Near the origin, the rate function for our nonlinear spring can be approximated by

$$g(x, y) \approx -x - 0.1y$$

since $\partial g/\partial x = -1$ and $\partial g/\partial y = -0.1$ at $x_0 = 0,\ y_0 = 0$. Therefore the nonlinear system (2) reduces to the *linearized system* (1). You can see the approximation when the phase portraits are overlaid. The trajectories and tx-component curves of both systems, issuing from a common initial point close to the origin, are shown in Figure 7.2. The linear approximation is pretty good because the nonlinearity $-x^3$ is small near $x = 0$. Take another look at Figure 7.1; the linear approximation is *not* very good when $|x| > 1$.

☞ Linear and nonlinear look-alikes.

✓ How good an approximation to system (2) is the linearized system (1) if the initial point of a trajectory is far away from the origin? Explain what you mean by "good" and "far away."

In matrix notation, linear system (1) takes the form

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{5}$$

so the characteristic equation of the system matrix is $\lambda^2 + 0.1\lambda + 1 = 0$. The matrix has eigenvalues $\lambda = (-0.1 \pm i\sqrt{3.99})/2$, making $(0, 0)$ a spiral sink (due to the negative real part of both eigenvalues). This supports our earlier conclusion that was based on the computer-generated pictures in Figure 7.2. The addition of a nonlinear term to a linear system (in this example, a cubic nonlinearity) does not change the stability of the equilibrium point (a sink in this case) or the spiraling nature of the trajectories (suggested by the complex eigenvalues).

☞ Look back at Chapter 6 for more on complex eigenvalues and spiral sinks.

The linear and nonlinear trajectories and the *tx*-components shown in Figure 7.2 look pretty much alike. This is often the case for a system

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \tag{6}$$

☞ The point $\mathbf{x}_0$ is an equilibrium point of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ if $\mathbf{F}(\mathbf{x}_0) = 0$.

and its *linearization*

$$\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) \tag{7}$$

at an equilibrium point $\mathbf{x}_0$. Let's assume that the dependent vector variable $\mathbf{x}$ has n components $x_1, \ldots x_n$, that $F_1(\mathbf{x}), \ldots, F_n(\mathbf{x})$ are the components of $\mathbf{F}(\mathbf{x})$, and that these components are at least twice continuously differentiable functions. Then the $n \times n$ constant matrix $\mathbf{A}$ in system (7) is the matrix of the first partial derivatives of the components of $\mathbf{F}(\mathbf{x})$ with respect to the components of $\mathbf{x}$, all evaluated at $\mathbf{x}_0$:

$$\mathbf{A} = \left[\begin{array}{ccc} \dfrac{\partial \mathbf{F}_1}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{F}_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \mathbf{F}_n}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{F}_n}{\partial x_n} \end{array}\right]_{\mathbf{x}=\mathbf{x}_0}$$

$\mathbf{A}$ is called the *Jacobian matrix* of $\mathbf{F}$ at $\mathbf{x}_0$, and is often denoted by $\mathbf{J}$ or $\mathbf{J}(\mathbf{x}_0)$. As an example, look back at system (1) and its linearization, system (2) or system (5).

☞ Here's why linearization is so widely used.

It is known that if none of the eigenvalues of the Jacobian matrix at an equilibrium point is zero or pure imaginary, then close to the equilibrium point the trajectories and component curves of systems (6) and (7) look alike. We can use ODE Architect to find equilibrium points, calculate Jacobian matrices and their eigenvalues, and so, check out whether the eigenvalues meet the conditions just stated. If $n = 2$, we can apply the vocabulary of planar linear systems from Chapter 6 to nonlinear systems. We can talk about a *spiral sink*, a *nodal source*, a *saddle point*, etc. ODE Architect uses a solid dot for a sink, an open dot for a source, a plus sign for a center, and an open square for a saddle.

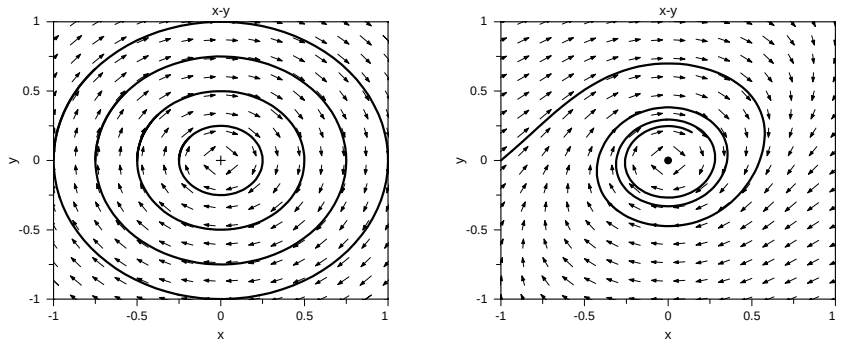
What happens when, say, the matrix $\mathbf{A}$ *does* have pure imaginary eigenvalues? Then all bets are off, as the following example shows.

Start with the linear system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

The system matrix has the pure imaginary eigenvalues $\pm i$, making the origin a center. Now give the system a nonlinear perturbation to get

$$\begin{aligned} x' &= y - x^3 \\ y' &= -x \end{aligned}$$



Linear system Perturbed system

Figure 7.3: Nonlinear terms convert a linear center to a nonlinear sink.

☞ The behavior of the linearized system does not always accurately predict the behavior of the nonlinear system.

By picturing the direction field defined by this system, we can see that each vector has been nudged slightly inward, toward the origin. This causes solutions to spiral inward, making $(0, 0)$ a spiral sink. Figure 7.3 shows trajectories from the original linear system on the left, and a trajectory of the nonlinear system on the right, spiraling inward. Now it should be clear why we had to exclude pure imaginary eigenvalues!

✓ What happens if you perturb the linear system by adding the x^3 term, instead of subtracting? What about the system $x' = y - x^3$, $y' = -x + y^3$?

◆ Separatrices and Saddle Points

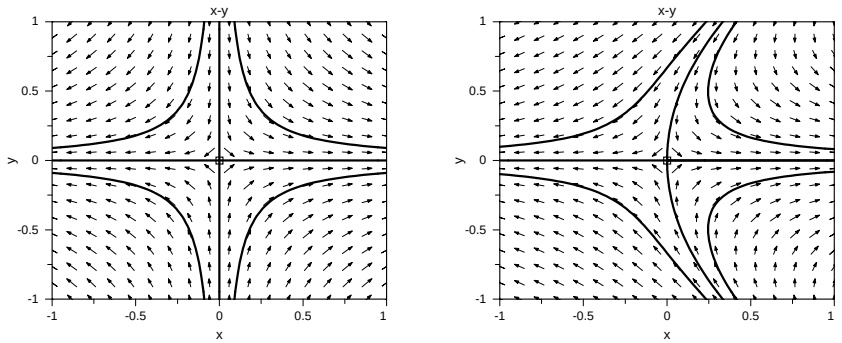
A linear saddle point has two trajectories that leave the point (as time increases from $-\infty$) along a straight line in the direction of an eigenvector. Another two trajectories approach the point (as time increases to $+\infty$) along a straight line in the direction of an eigenvector. These four trajectories are called *saddle separatrices* because they divide the neighborhood of the saddle point into regions of quite different long-term trajectory behavior. The left plot in Figure 7.4 shows the four separatrices along the x- and y- axes for the linear system

$$x' = x, \quad y' = -y \tag{8}$$

☞ See Chapters 10 and 12 for more on separatrices.

with a saddle point at the origin. The two that leave the origin as t increases are the *unstable separatrices*, and the two that enter the origin are the *stable separatrices*.

If we add some higher-order nonlinear terms to a linear saddle-point system, the separatrices persist but their shapes may change. They still divide a neighborhood of the equilibrium point into regions of differing long-term behavior. And, most importantly, they still leave or approach the equilibrium



(a) Trajectories of System (8) (b) Trajectories of System (9)

Figure 7.4: Saddle separatrices lie along the axes in (a); two of the separatrices are bent to the right by a nonlinearity in (b).

point tangent to eigenvectors of the linearized system. The right plot in Figure 7.4 suggests all this for the system (8) with a nonlinear term tacked on:

$$x' = x - y^2, \quad y' = -y \tag{9}$$

Note how the nonlinearity bends two of the separatrices.

◆ Behavior of Solutions Away from Equilibrium Points

While we can use linearization in most cases to determine the long-term behavior of solutions near an equilibrium point, it may not be a good method for studying the behavior of solutions "far away" from the equilibrium point. Consider, for example, the spider-fly system of Module 7:

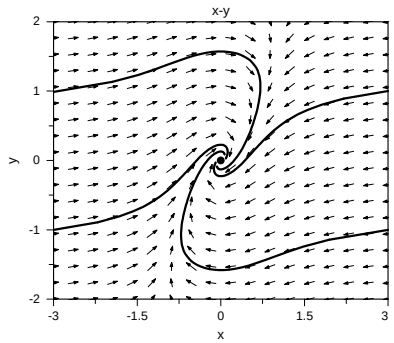
$$S' = -4S + 2SF, \quad F' = 3\left(1 - \frac{F}{5}\right)F - 2SF$$

where S is a population of spiders preying on F, a population of flies (all measured in thousands). This nonlinear system has several equilibrium points, one of which is at $p^* = (0.9, 2)$.

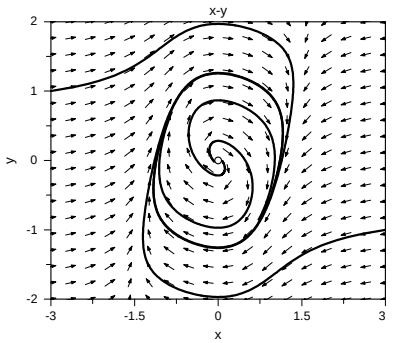
Take a look at the graphics windows in Experiment 2 of "The Spider and Fly" (Screen 1.4). The trajectories of the linearized system that are close to p^* approximate well those of the nonlinear system. However, trajectories of the linearized system that are not near the equilibrium point diverge substantially from those of the nonlinear system, and may even venture into a region of the state space where the population of spiders is negative!

☞ The nonlinear terms are critical in determining trajectory behavior far away from an equilibrium point.

✓ Look at the Library file "Mutualism: Symbiotic Interactions" in the "Population Models" folder and investigate the long-term behavior of solution curves by using linear approximations near equilibrium points.



An attracting spiral sink ($a = -1$).



A repelling spiral source and attracting limit cycle ($a = 1$).

Figure 7.5: The system, $x' = y + ax - x^3$, $y' = -x$, undergoes a Hopf bifurcation to an attracting limit cycle as the parameter transits the value $a = 0$.

◆ Bifurcation to a Limit Cycle

☞ If you are not "into" nonlinear electrical circuits, ignore the modeling here and just consider ODE (10) as a particular nonlinear system.

The model equations for an electrical circuit (the *van der Pol circuit*) containing a nonlinear resistor, an inductor, and a capacitor, all in series, are

$$x' = y + ax - x^3, \quad y' = -x \tag{10}$$

☞ Current, voltages and time are scaled to dimensionless quantities in system (10).

where x is the current in the circuit and y is the voltage drop across the capacitor. The voltage drop across the nonlinear resistor is $ax - x^3$, where a is a parameter. The characteristics of the resistor, and thus the performance of the circuit, change when we change the value of this parameter. Let's look at the phase portrait and the corresponding eigenvalues of the linearization of this system at the equilibrium point $(0, 0)$ for different values of a.

As a increases from -1 to 1, the eigenvalues of the Jacobian matrix of system (10) at the origin change from complex numbers with negative real parts to complex numbers with positive real parts, but at $a = 0$ they are pure imaginary. The circuit's behavior changes as a increases, and it changes in a qualitative way at $a = 0$. The phase portrait shows a spiral sink at $(0, 0)$ for $a \leq 0$, then a spiral source for $a > 0$. Further, the trajectories near the source spiral out to a closed curve that is itself a trajectory. Our electrical circuit has gone from one where current and voltage die out to one that achieves a continuing oscillation described by a periodic steady state. A change like this in the behavior of a model at a particular value of a parameter is called a *Hopf bifurcation*. Figure 7.5 shows the changes in a trajectory of system (10) due to the bifurcation that occurs when a is increased through zero.

✓ Find the Jacobian matrix of system (10) at the origin and calculate its eigenvalues in terms of the parameter a. Write out the linearized version of system (10). Check your work by using ODE Architect's equilibrium, Jacobian, and eigenvalue capabilities.

☞ A limit cycle is exclusively a nonlinear phenomenon. Any cycle in a linear autonomous system is always part of a family of cycles, none of which are limit cycles.

The closed solution curve in Figure 7.5 that represents a periodic steady state is called an *attracting limit cycle* because all nearby trajectories spiral into it as time increases. As the parameter value changes in a Hopf bifurcation, you can observe an equilibrium point that is a spiral sink changing into a source with nearby orbits spiralling onto the limit cycle. You'll investigate this kind of phenomenon when you use ODE Architect to investigate the model system in the "Saxophone" submodule of Module 7.

◆ Higher Dimensions

So far we have looked at systems of nonlinear ODEs involving only two state variables. However it is not uncommon for a model to have a system with more than two state variables. Fortunately our ideas extend in a natural way to cover these situations. Analysis by linear approximation may still work in these cases, and ODE Architect can always be used to find equilibrium points, Jacobian matrices, and eigenvalues in any dimension. See for example Problem 3 in Exploration 7.3.

The chapter cover figure shows trajectories of a system with three state variables; this system describes the angular velocity of a spinning body. The "Spinning Bodies" submodule of Module 7 and Problem 1 in Exploration 7.3 model the rotational motion of an object thrown into space; this model is described below.

✓ How could you visualize the trajectories of a system of four equations?

◆ Spinning Bodies: Stability of Steady Rotations

☞ As we shall see, not all axes **L** will support steady rotations.

Suppose that a rigid body is undergoing a steady rotation about an axis **L** through its center of mass. In a plane perpendicular to **L** let θ be the angle swept out by a point in the body, but not on the axis. Steady rotations about **L** are characterized by the fact that $\theta' = d\theta/dt =$ constant, for all time. In mechanics, it is useful to describe such steady rotations by a vector $\boldsymbol{\omega}$ parallel to **L** whose magnitude $|\boldsymbol{\omega}| = d\theta/dt$ is constant. Notice that $-\boldsymbol{\omega}$ in this case also corresponds to a steady spin about **L**, but in the opposite direction. The vector $\boldsymbol{\omega}$ is called the *angular velocity*, and for steady rotations we see that $\boldsymbol{\omega}$ is a constant vector. The angular velocity vector $\boldsymbol{\omega}$ can also be defined for an unsteady rotation of the body, but in this case $\boldsymbol{\omega}(t)$ is not a constant vector.

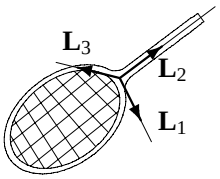
It turns out that in a uniform force field (such as the gravitational field near the earth's surface), the differential equations for the rotational motion of the body about its center of mass decouple from the ODEs for the translational motion of the center of mass. How shall we track the rotational motion of the body? For each rigid body there is a natural triple of orthogonal axes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ (called *body axes*) which, as it turns out, makes it relatively easy to

model the rotational motion by a system of ODEs. To define the body axes we need the *inertia tensor* I of the body. Given a triple of orthogonal axes through the body's center of mass, put an orientation on each axis and label them to form a *right-handed frame* (i.e., it follow the right-hand rule). In that frame, I is represented by a 3×3 *positive definite matrix*. Body axes are just the frame for which the representation of I is a diagonal matrix with the positive entries I_1, I_2, and I_3 along the diagonal. These values I_1, I_2, and I_3 are called the *principal moments of inertia* of the body. Note that I_k is the moment of inertia about the principal axis $\mathbf{L}_k$, for $k = 1, 2, 3$. If a body has uniform density and an axis $\mathbf{L}$ such that turning the body 180° about that axis brings the body into coincidence with itself again, then that axis $\mathbf{L}$ is a principal axis.

☞ A matrix A is positive definite if it is symmetric (i.e., $A^T = A$) and all of its eigenvalues are positive.

Let's say that a book has uniform density (not quite true, but nearly so). Then the three axes of rotational symmetry through the center of mass are the principal axes: $\mathbf{L}_3$, the *short axis* through the center of the book's front and back covers; $\mathbf{L}_2$, the *long axis* parallel to the book's spine; and $\mathbf{L}_1$, the *intermediate axis* which is perpendicular to $\mathbf{L}_2$ and $\mathbf{L}_3$. For a tennis racket, the body axis $\mathbf{L}_2$ is obvious on geometrical grounds. The other axes $\mathbf{L}_1$ and $\mathbf{L}_3$ are a bit more difficult to discern, but they are given in the margin sketch.

☞ Described graphically in Module 7.



Throw a tennis racket up into the air and watch its gyrations. Wrap a rubber band around a book, toss it into the air, and look at its spinning behavior. Now try to get the racket or the book to spin steadily about each of three perpendicular body axes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$. Not so hard to do about two of the axes—but nearly impossible about the third. Why is that? Let's construct a model for the rotation of the body and answer this question.

☞ A complete derivation of the model ODEs can be found in the first of the listed references.

Let's confine our attention to the body's angular motion while aloft, not its vertical motion. Let's ignore air resistance. The key parameters that influence the angular motion are the principal inertias I_1, I_2, I_3 about the respective body axes $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$. Let ω_1, ω_2, and ω_3 be the components of the vector $\boldsymbol{\omega}$ along the body axes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$. There is an analogue of Newton's second law applied to the body which involves the angular velocity vector $\boldsymbol{\omega}$. The components of the rotational equation of motion in the body axes frame are given by $I_1\omega_1' = (I_2 - I_3)\omega_2\omega_3$, $I_2\omega_2' = (I_3 - I_1)\omega_1\omega_3$, $I_3\omega_3' = (I_1 - I_2)\omega_1\omega_2$.

Dividing by the principal inertias, we have the nonlinear system

$$\begin{aligned} \omega_1' &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \omega_2' &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\ \omega_3' &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \end{aligned} \tag{11}$$

Let's measure angles in radians and time in seconds, so that each ω_i has units of radians per second.

☞ Pure steady rotations are possible about any body axis.

First, we note that for any constant $\alpha \neq 0$, the equilibrium point $\boldsymbol{\omega} = (\alpha, 0, 0)$ of system (11) represents a pure steady rotation (or spinning motion)

about the first body axis $\mathbf{L}_1$ with angular velocity α. The equilibrium point $(-\alpha, 0, 0)$ represents steady rotation about $\mathbf{L}_1$ in the opposite direction. Similar statements are true for the equilibrium points $\boldsymbol{\omega} = (0, \alpha, 0)$ and $(0, 0, \alpha)$.

Now the *kinetic energy of angular rotation* is given by

$$KE(\omega_1, \omega_2, \omega_3) = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

The value of KE stays fixed on an orbit of system (11) since

$$\begin{aligned}\frac{d(KE)}{dt} &= I_1\omega_1\omega_1' + I_2\omega_2\omega_2' + I_3\omega_3\omega_3' \\ &= (I_2 - I_3)\omega_1\omega_2\omega_3 + (I_3 - I_1)\omega_1\omega_2\omega_3 + (I_1 - I_2)\omega_1\omega_2\omega_3 = 0\end{aligned}$$

☞ A system of autonomous ODEs is *conservative* if there is a function F of the dependent variables whose value is constant along each orbit (i.e., trajectory), but varies from one orbit to another. F is said to be an *integral of motion* of the system.

So system (11) is *conservative* and KE is an *integral*. The ellipsoidal integral surface $KE = C$, where C is a positive constant, is called an *inertial ellipsoid* for system (11). Note that any orbit of (11) that starts on one of the ellipsoids stays on the ellipsoid, and orbits on that ellipsoid share the same value of KE.

✓ Show that the functions

$$K = \frac{I_3 - I_1}{I_2}\omega_1^2 - \frac{I_2 - I_3}{I_1}\omega_2^2$$

and

$$M = \frac{I_1 - I_2}{I_3}\omega_1^2 - \frac{I_2 - I_3}{I_1}\omega_3^2$$

are also integrals for system (11). Describe the surfaces $K = \text{const.}$, $M = \text{const.}$

Let's put in some numbers for I_1, I_2, and I_3 and see what happens. Set $I_1 = 2$, $I_2 = 1$, $I_3 = 3$. Then system (11) becomes

$$\begin{aligned}\omega_1' &= -\omega_2\omega_3 \\ \omega_2' &= \omega_1\omega_3 \\ \omega_3' &= \frac{1}{3}\omega_1\omega_2\end{aligned} \tag{12}$$

With the given values for I_1, I_2, I_3 we have the integral

$$KE = \frac{1}{2}\left(2\omega_1^2 + \omega_2^2 + 3\omega_3^2\right) \tag{13}$$

☞ Body axis $\mathbf{L}_1$ is parallel to the ω_1-axis, $\mathbf{L}_2$ to the ω_2-axis, and $\mathbf{L}_3$ to the ω_3-axis in Figure 7.6.

The left graph in Figure 7.6, which is also the chapter cover figure, shows the inertial ellipsoid $KE = 12$ and twenty-four orbits on the surface. The geometry of the orbits indicates that if the body is started spinning about an axis very near the body axes $\mathbf{L}_2$ or $\mathbf{L}_3$, then the body continues to spin almost steadily about those body axes. Attempting to spin the body about the intermediate body axis $\mathbf{L}_1$ is another matter. Any attempt to spin the body about the $\mathbf{L}_1$ body axis leads to strange gyrations. Note in Figure 7.6 that each of the four trajectories that starts near the equilibrium point $(\sqrt{12}, 0, 0)$ where the

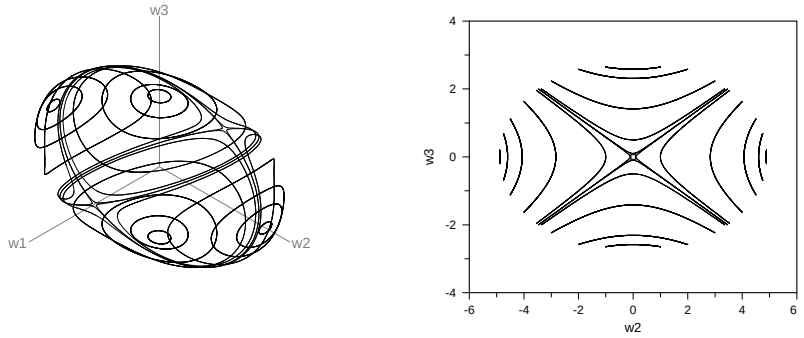


Figure 7.6: Twenty-four trajectories on the inertial ellipsoid $KE = 12$ (left); head-on view from the ω_1-axis (right) shows a saddle point on the ellipsoid.

ω_1-axis pierces the ellipsoid goes back near the antipodal point (and reverses its direction of rotation) then returns in an endlessly repeating periodic path. This corresponds to unstable gyrations near the ω_1-axis.

✓ Match up the trajectories in Figure 7.6 with actual book rotations. Put a rubber band around a book, flip the book into the air, and check out the rotations. Do the projected trajectories in the right graph of Figure 7.6 really terminate, or is something else going on?

◆ The Planar Double Pendulum

☞ This is pretty advanced material here, so feel free to skip the text and go directly to the "Double Pendulum Movies". Just click on the ODE Architect library, open the "Physical Models" folder and the "The Nonlinear Double Pendulum" file, and create chaos!

The planar double pendulum is an interesting physical system with two degrees of freedom. It consists of two rods of lengths l_1 and l_2, and two masses, specified by m_1 and m_2, attached together so that the rods are constrained to oscillate in a vertical plane. We'll neglect effects of damping in this system.

The governing equations are most conveniently written in terms of the angles $\theta_1(t)$ and $\theta_2(t)$ shown in Figure 7.7. One way to obtain the equations of motion is by applying Newton's second law to the motions of the masses. First we'll consider mass m_2 and the component in the direction shown by the unit vector $\mathbf{u}_3$ in Figure 7.7. Define a coordinate system centered at mass m_1 and rotating with angular velocity $\Omega = (d\theta_1/dt)\mathbf{k}$, where $\mathbf{k}$ is the unit vector normal to the plane of motion. If $\hat{\mathbf{a}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{r}}$ denote the acceleration, velocity, and position of m_2 with respect to the rotating coordinate system, then the acceleration $\mathbf{a}$ with respect to a coordinate system at rest is known to be

☞ The "$\times$" here denotes the vector cross product.

$$\mathbf{a} = \hat{\mathbf{a}} + \frac{d\Omega}{dt} \times \hat{\mathbf{r}} + 2\Omega \times \hat{\mathbf{v}} + \Omega \times (\Omega \times \hat{\mathbf{r}}) \tag{14}$$

For our configuration it follows that $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}} = -[l_1 \sin(\theta_2 - \theta_1)]\mathbf{u}_3 + [l_2 + l_1 \cos(\theta_2 - \theta_1)]\mathbf{u}_4 \tag{15}$$

with the unit vector $\mathbf{u}_4$ in the direction shown in Figure 7.7. Since the only forces acting are gravity and the tensile forces in the rods, the $\mathbf{u}_3$-component of $\mathbf{F} = m_2\mathbf{a}$ in combination with Eqs. (14) and (15) gives

☞ The Library file, "The Nonlinear Double Pendulum" uses different symbols for θ_1, θ_1', θ_2, and θ_2'.

$$m_2 l_2(\theta_2 - \theta_1)'' + m_2[l_2 + l_1\cos(\theta_2 - \theta_1)]\theta_1'' + m_2 l_1(\theta_1')^2\sin(\theta_2 - \theta_1) = -m_2 g\sin\theta_2 \quad (16)$$

Similarly, the component of Newton's law in the direction of the unit vector $\mathbf{u}_1$ is given by

$$m_2 l_1\theta_1'' + m_2 l_2\cos(\theta_2 - \theta_1)\theta_2'' - m_2 l_2(\theta_2')^2\sin(\theta_2 - \theta_1) = -m_2 g\sin\theta_1 - f_2\sin(\theta_2 - \theta_1) \quad (17)$$

where f_2 is the magnitude of the tensile force in the rod l_2. Equations (16) and (17) will provide the system governing the motion, once the quantity f_2 is determined. An equation for f_2 is found from the $\mathbf{u}_1$-component of Newton's law applied to the mass m_1:

$$m_1 l_1\theta_1'' = -m_1 g\sin\theta_1 + f_2\sin(\theta_2 - \theta_1) \quad (18)$$

Eliminating f_2 between Eqs. (17) and (18) and simplifying Eq. (16) slightly, we obtain the governing nonlinear system of second-order ODEs for the undamped planar double pendulum:

☞ Before equations (19) and (20) can be solved by the Tool, they need to be converted to a normalized system of first order ODEs.

$$(m_1 + m_2)l_1\theta_1'' + m_2 l_2\cos(\theta_2 - \theta_1)\theta_2'' - m_2 l_2(\theta_2')^2\sin(\theta_2 - \theta_1) + (m_1 + m_2)g\sin\theta_1 = 0 \quad (19)$$

$$m_2 l_2\theta_2'' + m_2 l_1\cos(\theta_2 - \theta_1)\theta_1'' + m_2 l_1(\theta_1')^2\sin(\theta_2 - \theta_1) + m_2 g\sin\theta_2 = 0 \quad (20)$$



Figure 7.7: Geometry and unit vectors for the double pendulum.

Another way to derive the equations of motion of the double pendulum system is to use Lagrange's equations. These are

$$\frac{d}{dt}\left[\frac{\partial}{\partial\theta_1'}(T-V)\right]-\frac{\partial}{\partial\theta_1}(T-V)=0 \tag{21}$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial\theta_2'}(T-V)\right]-\frac{\partial}{\partial\theta_2}(T-V)=0 \tag{22}$$

where T is the kinetic energy of the system and V is its potential energy. The respective kinetic energies of the masses m_1 and m_2 are

$$T_1=\frac{1}{2}m_1l_1^2(\theta_1')^2$$

$$T_2=\frac{1}{2}m_2(l_1\theta_1'\sin\theta_1+l_2\theta_2'\sin\theta_2)^2+\frac{1}{2}m_2(l_1\theta_1'\cos\theta_1+l_2\theta_2'\cos\theta_2)^2$$

The corresponding potential energies of m_1 and m_2 are

$$V_1=m_1gl_1(1-\cos\theta_1)$$

$$V_2=m_2gl_1(1-\cos\theta_1)+m_2gl_2(1-\cos\theta_2)$$

Then we have $T=T_1+T_2$ and $V=V_1+V_2$. Inserting the expressions for T and V into Eqs. (21) and (22), we find the equations of motion of the double pendulum. These equations are equivalent to the ones obtained previously using Newton's law. The formalism of Lagrange pays the dividend of producing the equations with "relatively" shorter calculations.

References

Borrelli, R.L., Coleman, C.S., and Boyce, W.E., *Differential Equations Laboratory Workbook*, (1992: John Wiley & Sons, Inc.), Experiment 6.9

Hirsch, M., and Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra* (1974: Academic Press), Chapter 10

Hubbard, J., and West, B., *Differential Equations: A Dynamical Systems Approach*, Vol. 18 of Texts in Appl. Math. (1995: Springer-Verlag), Sections 8.2, 8.3

Marion, J.B., and Thornton, S.T., *Classical Dynamics of Particles and Systems*, 3rd ed. (1988: Harcourt Brace Jovanovich)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 7.1. Predator and Prey: Linearization and Stability

1. Let F represent the number of flies and S the number of spiders (both in 1000s). Assume that the model for their interaction is given by:

$$S' = -4S + 2SF, \quad F' = 3F - 2SF \tag{23}$$

where the SF-term is a measure of the interaction between the two species.

☞ Take a look at the "Spider and Fly" submodule of Module 7.

(a) Why is the SF-term negative in the first ODE and positive in the second when (S, F) is inside the population quadrant?

(b) Show that the system has an equilibrium point at $(1.5, 2)$.

(c) Show that the system matrix of the linearization of system (23) about $(1.5, 2)$ has pure imaginary eigenvalues.

☞ This makes the point $(1.5, 2)$ a center for the linearized system.

(d) Now plot phase portraits for system (23) and for its linearization about $(1.5, 2)$. What do you see?

2. Suppose that an insecticide reduces the spider population at a rate proportional to the size of the population.

(a) Modify the predator-prey model of system (23) to account for this.

(b) Model how insecticide can be made more or less effective.

(c) Use the model to predict the long-term behavior of the populations.

3. In a predator-prey system that models spider-fly interaction

$$S' = -4S + 2SF, \quad F' = 3\left(1 - \frac{F}{N}\right)F - 2SF$$

the number N represents the maximum fly population (in 1000s). Investigate the effect of changing the value of N. What's the largest the spider population can get? The fly population?

4. Suppose the spider-fly model is modified so that there are two predators, spiders and lizards, competing to eat the flies. One model for just the two predator populations is

$$S' = 4\left(1 - \frac{S}{5}\right)S - SL, \quad L' = 3\left(1 - \frac{L}{2}\right)L - SL$$

(a) What do the numbers 2, 3, 4, and 5 represent?

(b) What does the term SL represent? Why is it negative?

(c) What will become of the predator populations in the long run?

5. Take a look at the library file "A Predator-Prey System with Resource Limitation" in the "Biological Models" folder. Compare and contrast the system you see in that file with that given in Problem 2. Create a system where both the predator and the prey are subject to resource limitations, and analyze the behavior of the trajectories.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 7.2. Bifurcations and Limit Cycles

1. Alter the model in the "Saxophone" submodule of Module 10 by adding a parameter c:

$$u' = v, \quad v' = -su + cv - \frac{1}{b}v^3$$

(a) What part of the model does this affect?

(b) How do solutions behave for values of c between 0 and 2, taking $s = b = 1$?

(c) As c increases, what happens to the pitch and amplitude?

2. Suppose the model for a *simple harmonic oscillator* (a linear model),

$$x' = y, \quad y' = -x$$

is modified by adding a parameter c:

$$x' = cx + y, \quad y' = -x + cy$$

(a) What happens to the equilibrium point as c goes from -1 to 1?

(b) What happens to the eigenvalues of the matrix of coefficients as c changes from -1 to 1?

3. Suppose we further modify the system of Problem 2:

$$x' = cx + y - x(x^2 + y^2), \qquad y' = -x + cy - y(x^2 + y^2)$$

where $-1 \leq c \leq 1$. Analyze the behavior of the equilibrium point at $(0, 0)$ as c increases from -1 to 1. How does it compare with the behavior you observed in Problem 2?

4. You can modify the system for a simple, undamped nonlinear pendulum (see Chapter 10) to produce a *torqued pendulum*:

$$x' = y, \qquad y' = -\sin(x) + a$$

☞ Use the model-based pendulum animation in ODE Architect and watch the pendulum gyrate.

Here a represents a torque applied about the axis of rotation of the pendulum arm. Investigate the behavior of this torqued pendulum for the values of a between 0 and 2 by building the model and animating the phase space as a increases. Explain what kind of behavior the pendulum exhibits as a increases; explain the behavior of any equilibrium points you see.

5. The motion of a thin, flexible steel beam, affixed to a rigid support over two magnets, can be modeled by *Duffing's equation*:

$$x' = y, \quad y' = ax - x^3$$

☞ Sweep on the parameter a, and then animate. To animate a graph with multiple trajectories corresponding to different values of a, click on the animate icon below the word "Tools" at the top left of the tools screen.

where x represents the horizontal displacement of the beam from the rest position and a is a parameter that is related to the strength of the magnets. Investigate the behavior of this model for $-1 \leq a \leq 1$. In particular:

(a) Find all equilibrium points and classify them as to type (e.g., center, saddle point), verifying your phase plots with eigenvalue calculations (use ODE Architect for the eigenvalue calculations). Some of your answers will depend on a.

(b) Give a physical interpretation of your answers to Question (a).

(c) What happens to the equilibrium points as the magnets change from weak ($a \leq 0$) to strong ($a > 0$)?

(d) What happens if you add a linear damping term to the model? (Say, $y' = ax - x^3 - \nu y$.)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 7.3. Higher Dimensions

1. *Spinning Bodies.*
Use ODE Architect to draw several distinct trajectories on the ellipsoid of inertia, $0.5(2\omega_1^2 + \omega_2^2 + 3\omega_3^2) = 6$, for system (12).

☞ For example, $\omega_1 = 0$, $\omega_2 = 3$, $\omega_3 = 1$ can be taken as an initial point and so can $\omega_1 = 1$, $\omega_2 = 0$, $\omega_3 = (10/3)^{0.5}$.

Choose initial data on the ellipsoid so that the trajectories become the "visible skeleton" of the invisible ellipsoid. What do the trajectories look like? What kind of motion does each represent? You should be able to get a picture that resembles the chapter cover figure and Figure 7.6. Project your 3D graphs onto the $\omega_1\omega_2$-, $\omega_2\omega_3$-, and $\omega_1\omega_3$-planes, and describe what you see. Now apply the equilibrium/eigenvalue/eigenvector calculations from ODE Architect to equilibrium points on each of the ω_1-, ω_2-, and ω_3- axes. Describe the results and their correlation with what you saw on the coordinate planes. Now go to the Library file "A Conservative System: The Momentum Ellipsoid" in the folder "Physical Models" and explain what you see in terms of the previous questions in this problem.

2. Exploration 7.1 (Problem 4) gives a predator-prey model where two species, spiders and lizards, prey on flies. Construct a system of three differential equations that includes the prey in the model. You'll need to represent growth rates and interactions, and you may want to limit population sizes. Make some reasonable assumptions about these parameters. What long-term behavior does your model predict?

3. Take another look at the ODEs of the coupled springs model in Module 6. Use ODE Architect for the system of four ODEs given in Experiment 1 of that section (use $c = 0.05$). Make 3D plots of any three of the five variables x_1, x_1', x_2, x_2', and t. What do the plots tell you about the corresponding motions of the springs? Now use the Tool to find the eigenvalues of the Jacobian matrix of the system at the equilibrium point. What do the eigenvalues tell you about the motions?

4. Modify the coupled springs model from Module 6 (where coupled linear springs move on a frictionless horizontal surface) by making one of the springs hard or soft: add a term like $\pm x^3$ to the restoring force. Does this change the long-term behavior of the system? Make and interpret graphs as in Problem 3.

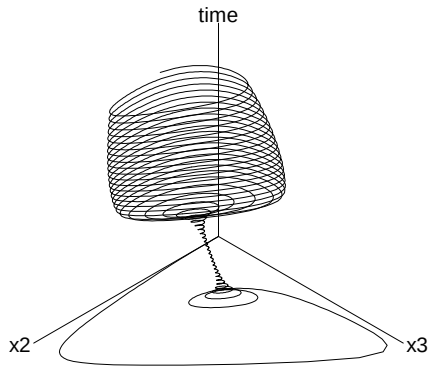
5. *Chaos in three dimensions.*
Some nonlinear 3D systems seem to behave chaotically. Orbits stay bounded as time advances, but the slightest change in the initial data leads to an orbit that eventually seems to be completely uncorrelated with the original orbit. This is thought to be one feature of chaotic dynamics. Choose one of the following three Library files located in the folder "Higher Dimensional Systems":

- "The Scroll Circuit: Organized Chaos"
- "The Lorenz System: Chaos and Sensitivity"
- "The Roessler System: A Strange Attractor"

Change parameters until you see an example of this kind of chaos. You may want to look at Chapter 12 for additional insight into the meaning of chaos.

CHAPTER 8

Compartment Models



Oscillating chemical reactions on a wineglass?

Overview A salt solution is pumped through a series of tanks. We'll use the *balance law* to model the rate of change of the amount of salt in each tank:

$$\left\{\begin{array}{c}\text{Net rate of change of}\\ \text{amount of salt in tank}\end{array}\right\} = \left\{\begin{array}{c}\text{Rate into}\\ \text{tank}\end{array}\right\} - \left\{\begin{array}{c}\text{Rate out of}\\ \text{tank}\end{array}\right\}$$

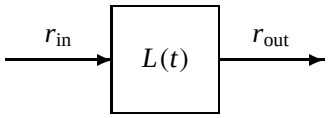
If we know the initial amount of salt and the inflow and outflow rates of the solution in each tank, then we can set up an IVP that models the physical system. We'll use this "balance law" approach to model the pollution level in a lake; the flow of a medication; the movement of lead among the blood, tissues, and bones of a body; and an autocatalytic chemical reaction.

Key words Compartment model; balance law; lake pollution; pharmacokinetics; chemical reactions; chemical law of mass action; autocatalysis; Hopf bifurcation

See also Chapter 9 for the SIR compartment model, and Chapter 6 for linear systems and flow through interconnected tanks.

◆ Lake Pollution

Modeling how pollutants move through an environment is important in the prediction of harmful effects, as well as the formulation of environmental policies and regulations. The simplest situation has a single source of pollution that contaminates a well-defined habitat, such as a lake. To build a model of this system, we picture the lake as a *compartment*; pollutants in the water flow into and out of the compartment. The rates of flow determine the amount of build-up or dissipation of pollutants. It is useful to represent this conceptual model with a *compartment diagram*, where a box represents a compartment and an arrow represents a flow rate. Here is a compartment diagram for a simple model of lake pollution:



The amount of pollutant in the lake at time t is $L(t)$, while r_{in} is the rate of flow of pollutant into the lake and r_{out} is the rate of flow of pollutant out of the lake. To obtain the equation for the rate of change of the amount of pollutant in the lake, we apply the *balance law*: the net rate of change of the amount of a substance in a compartment is the difference between the rate of flow into the compartment and the rate of flow out of the compartment:

$$\frac{dL}{dt} = r_{\text{in}} - r_{\text{out}}$$

This ODE is sufficient when we know the rates r_{in} and r_{out}, but these rates are usually not constant: they depend on the rate of flow of water into the lake, the rate of flow of water out of the lake, and the pollutant concentration in the inflowing water. Let s_{in} and s_{out} represent the volume rates of flow of water into and out of the lake, V the volume of water in the lake, and p_{in} the concentration of pollutant in the incoming water. Now we can calculate the rates shown in the compartment diagram:

☞ You can get the volume $V(t)$ of water in the lake by solving the IVP $V' = s_{in} - s_{out}$, $V(0) = V_0$.

$$r_{\text{in}} = p_{\text{in}} s_{\text{in}}, \quad r_{\text{out}} = \frac{L}{V} s_{\text{out}}$$

The ODE for the amount of pollutant in the lake is now

$$\frac{dL}{dt} = p_{\text{in}} s_{\text{in}} - \frac{L}{V} s_{\text{out}} \tag{1}$$

☞ So we need to know $V(0)$, $L(0)$, p_{in}, s_{in}, and s_{out} in order to determine $L(t)$ and $V(t)$.

To obtain an IVP, we need to specify $L(0)$, the initial amount of pollutant in the lake. The solution to this IVP will reveal how the level of pollution varies in time. Figure 8.1 shows a solution to the ODE (1) for the pollution level in the lake if the inflow is contaminated for the first six months of every year and is clean for the last six months (so $p_{in}(t)$ is a square wave function).

☞ Take a look at Screen 1.4 in Module 8 for on-off inflow concentrations.

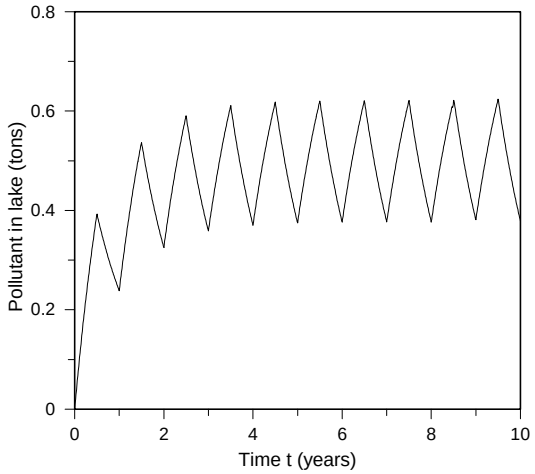


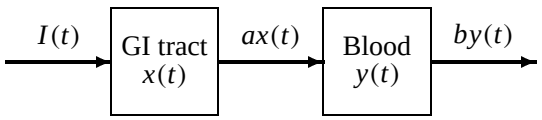
Figure 8.1: Pollutant level in a lake (on-off inflow rates).

✓ "Check" your understanding by finding the volume $V(t)$ of water in the lake at time t if $V(0) = 10$, $s_{in} = 3$, and $s_{out} = 1$, 3, or 5 (all quantities in suitable units). Does the lake dry up, overflow, or stay at a constant volume?

◆ Allergy Relief

Medications that relieve the symptoms of hay fever often contain an antihistamine and a decongestant bundled into a single capsule. The capsule dissolves in the gastrointestinal (or GI) tract and the contents move through the intestinal walls and into the bloodstream at rates proportional to the amounts of each medication in the tract. The kidneys clear medications from the bloodstream at rates proportional to the amounts in the blood.

Here is a compartment diagram for this system:



The symbols in this diagram have the following meanings:

- $I(t)$: The rate at which the dissolving capsule releases a medication (for example, a decongestant) into the GI tract
- $x(t)$: The amount of medication in the GI tract at time t
- $ax(t)$: The clearance rate of the medication from the GI tract, which equals the entrance rate into the blood (a is a positive rate constant)
- $y(t)$: The amount of medication in the blood at time t
- $by(t)$: The clearance rate of the medication from the blood (b is a positive rate constant)

Applying the balance law to each compartment, we have a system of first-order linear ODEs:

$$\begin{aligned} x' &= I - ax \\ y' &= ax - by \end{aligned} \tag{2}$$

If we know $I(t)$, the rate constants a and b, and the initial amounts $x(0)$ and $y(0)$ of medication in the GI tract and the bloodstream, we can use ODE Architect to track the flow of the medication through the body. From a pharmacological point of view, the goal is to get the medication levels in the blood into the effective (but safe) zone as quickly as possible and then to keep them there until the patient recovers.

☞ Medication levels in the blood (easily measured) indicate the levels in the tissues (hard to measure), where the medication does its good work.

There are two kinds of medication-release mechanisms: continuous and on-off. In the first kind, the medication is released continuously at an approximately constant rate, so $I(t)$ is a positive constant. In the on-off case, each capsule releases the medication at a constant rate over a brief span of time and then the process repeats when the next capsule is taken. In this case we model $I(t)$ by a square wave:

$$I(t) = A \ \text{SqWave}(t, T_{per}, T_{on})$$

☞ Screen 2.4 in Module 8 shows what happens if $a = 0.6931$ hr^{-1}, $T_{on} = 1$ hr, and b and A are adjustable parameters.

which has amplitude A, period T_{per}, and "on" time T_{on}. For example, if the capsule releases 6 units of medication over a half hour and the dosage is one capsule every six hours, then

$$I(t) = 12 \ \text{SqWave}(t, 6, 0.5) \tag{3}$$

Note that $A \cdot T_{on} = 12$ (units/hr)$\times 0.5$ (hr) $= 6$ units.

Compartment models described by equations such as (2) are called *cascades*. They can be solved explicitly, one equation at a time, by solving the first ODE, inserting the solution into the second ODE, solving it, and so on down the cascade. Although this approach theoretically yields explicit solution formulas, in practice the formulas farther along in the cascade of solutions get so complicated that they are difficult to interpret. That's one reason why it pays to use a numerical solver, like the ODE Architect. Figure 8.2 shows how the amounts of decongestant in the body change when administered by the on-off method [equation (3)].

☞ ODE Architect to the rescue!

✓ By inspecting Figure 8.2 decide which of the clearance coefficients a or b in system (2) is larger.

References

Borrelli, R.L., and Coleman, C.S., *Differential Equations: A Modeling Perspective*, (1998: John Wiley & Sons, Inc.)

Spitznagel, E., "Two-Compartment Phamacokinetic Models" in *C·ODE·E*, Fall 1992, pp. 2–4, http://www.math.hmc.edu/codee

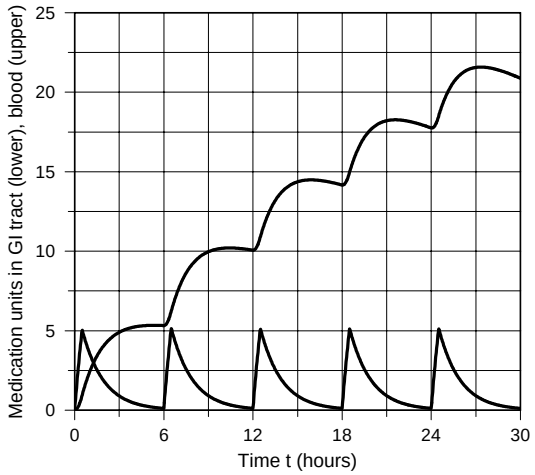
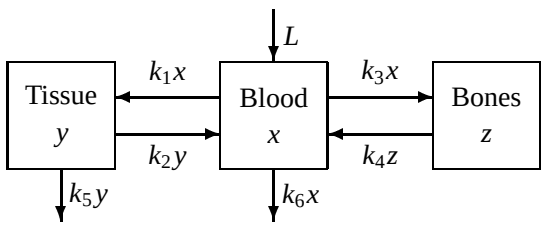


Figure 8.2: Decongestant levels in the GI tract and in the blood.

◆ Lead in the Body

☞ In ancient times lead was used to sweeten wine.

Lead gets into the digestive and respiratory systems of the body via contaminated food, air, and water, as well as lead-based paint, glaze, and crystalware. Lead moves into the bloodstream, which then distributes it to the tissues and bones. From those two body compartments it leaks back into the blood. Lead does the most damage to the brain and nervous system (treated here as tissues). Hair, nails, and perspiration help to clear lead from the tissues, and the kidneys clear lead from the blood. The rate at which lead leaves one compartment and enters another has been experimentally observed to be proportional to the amount that leaves the first compartment. Here is the compartment diagram that illustrates the flow of lead through the body.



In the diagram, L is the inflow rate of lead into the bloodstream (from the lungs and GI tract); x, y, and z are the respective amounts of lead in the blood, tissues, and bones; and $k_1, \ldots, k_6$ are experimentally determined positive rate constants. The amount of lead is measured in micrograms (1 microgram $= 10^{-6}$ gram), and time (t) is measured in days.

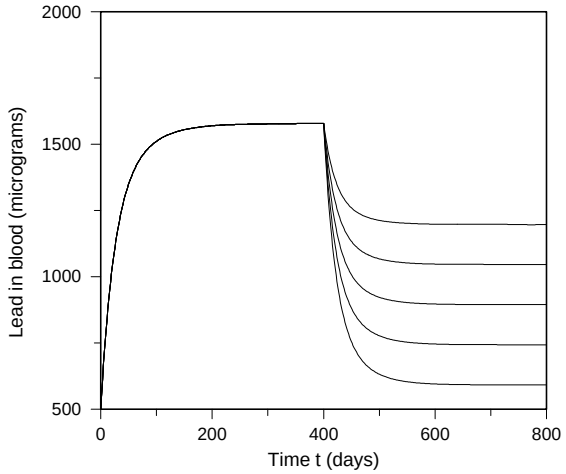


Figure 8.3: Five environmental clean-up scenarios for $t > 400$ days result in five different steady-state lead levels in the blood.

Applying the balance law to each compartment, we have the linear system of ODEs that models the flow of lead through the body compartments:

☞ System (4) is a driven linear system with constant coefficients, so eigenvalue/eigenvector techniques can be used to find solution formulas if L is a constant (see Chapter 6).

$$\begin{aligned} x' &= (L + k_2 y + k_4 z) - (k_1 + k_3 + k_6)x \\ y' &= k_1 x - (k_2 + k_5)y \\ z' &= k_3 x - k_4 z \end{aligned} \tag{4}$$

Unlike the allergy relief system (2), system (4) is not a cascade. Lead moves back and forth between compartments, so the system cannot be solved one ODE at a time. ODE Architect can be used to find $x(t)$, $y(t)$, and $z(t)$ if $x(0)$, $y(0)$, $z(0)$, $L(t)$, and $k_1, \ldots, k_6$ are known.

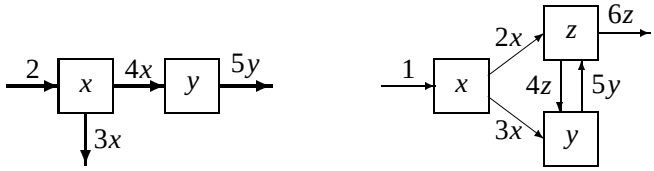
If the goal is to reduce the amount of lead in the blood (and therefore in the tissues and bones), we can clean up the environment (which reduces the inflow rate) or administer a medication that increases the clearance coefficient k_6. However, such medication carries its own risks, so most efforts today are aimed at removing lead from the environment. A major step in this direction was made in the 1970's and 80's when oil companies stopped adding lead to gasoline and paint manufacturers began to use other spreading agents in place of lead. Figure 8.3 shows the effects of changing the lead intake rate L.

☞ It was in the 1970's and 80's that most of the environmental protection laws were enacted.

The Food and Drug Administration and the National Institutes of Health have led the fight against lead pollution in the environment. They base their efforts on data acquired from several controlled studies of lead flow, where the study groups were made up of human volunteers in urban areas. The numbers we use in Submodule 3 of Module 8 and in this chapter come from one of those studies. Some references on the lead problem are listed below.

☞ See Screen 3.3 in Module 8 for the rate constants and the inflow rate L.

✓ Write down the systems of ODEs for the two compartment diagrams:



References

Batschelet, E., Brand, L., and Steiner, A., "On the kinetics of lead in the human body," *J. Math. Bio.*, **8** (1979), pp. 15–23

Kessel, I., and O'Conner, J., *Getting the Lead Out* (1997: Plenum)

Rabinowitz, M., Wetherill, G., and Kopple, J., "Lead metabolism in the normal human: Stable isotope studies," *Science*, **182** (1973), pp. 725–727.

◆ Equilibrium

In many compartment models, if the inflow rates from outside the system are constant, then the substance levels in each compartment tend to an equilibrium value as time goes on. Mathematically, we can find the equilibrium values by setting each rate equal to zero and solving the resulting system of equations simultaneously. For example, the equilibrium for the system

$$\begin{aligned} x' &= 1 - 2x \\ y' &= 2x - 3y \end{aligned} \tag{5}$$

is $x = 1/2$, $y = 1/3$, which is the solution to the algebraic system $1 - 2x = 0$ and $2x - 3y = 0$. If the system is complicated, you can use ODE Architect to find the equilibrium values. Just use the Equilibrium tabs in the lower left quadrant and in one of the right quadrants, and you will get approximate values for the equilibrium levels.

☞ The Equilibrium tabs in ODE Architect work for systems such as (5), where the rate functions don't depend explicitly on time (i.e., the systems are autonomous).

✓ Go to Things-to-Think-About 2 on Screen 3.5 of Module 8 for the lead flow model with constant values for L and the coefficients k_j. Use the Equilibrium tabs in the tool screen to estimate the equilibrium lead levels in the blood, tissues, and bones for the given data.

☞ You need to know about matrices to tackle this one.

✓ Suppose that x is a column vector with n entries, b is a column vector of n constants, and A is an $n \times n$ invertible matrix of real constants. Can you explain why the linear system $x' = Ax - b$ has a constant equilibrium x^*? Find a formula for x^* in terms of A^{-1} and b.

◆ The Autocatalator and a Hopf Bifurcation

So far all the compartments in our models have represented physical spaces through which substances move. However, there are other ways to think about compartments. For example, they can represent substances that transform into one another, such as uranium 238 and all of its seventeen radioactive decay products, ending with stable lead 206. Or think of a chemical reactor in which chemicals react with one another and produce other chemicals. The *autocatalator* is a mathematical model for one of these chemical reactions.

In an *autocatalytic reaction*, a chemical promotes its own production. For example, suppose that one unit of chemical X reacts with two units of chemical Y to produce three units of Y, a net gain of one unit of Y:

$$X + 2Y \xrightarrow{k} 3Y$$

where k is a positive rate constant. This is an example of *autocatalysis*. We'll come back to autocatalysis, but first we need to make a quick survey of how chemical reactions are modeled by ODEs.

Most chemical reactions are *first-order* in the sense that the rate of decay of each chemical in the reaction is directly proportional to its own concentration:

$$\frac{dz}{dt} = -kz \tag{6}$$

where $z(t)$ is the concentration of chemical Z at time t in the reactor and k is a positive rate constant.

While a first-order reaction is modeled by a *linear* ODE, such as (6), autocatalytic reactions are higher-order and the corresponding rate equations are *nonlinear*. In order to build models of higher-order chemical reactions, we will use a basic principle called the *Chemical Law of Mass Action*:

The Chemical Law of Mass Action. If molecules $X_1, \ldots, X_n$ react to produce molecules $Y_1, \ldots, Y_m$ in one step of the chemical reaction

$$X_1 + \cdots + X_n \xrightarrow{k} Y_1 + \cdots + Y_m$$

that occurs with rate constant k, then

$$x_i' = -kx_1x_2\cdots x_n,\ 1 \le i \le n$$
$$y_j' = kx_1x_2\cdots x_n,\ 1 \le j \le m$$

where x_i and y_j are, respectively, the concentrations of X_i and Y_j. The chemical species $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ need not be distinct from each other: more than one molecule of a given type may be involved in the reaction.

For example, the chemical law of mass action applied to the reaction

$$X + Y \xrightarrow{k} Z$$

gives

$$x' = -kxy, \quad y' = -kxy, \quad \text{and} \quad z' = kxy$$

where k is a positive rate constant and $x,\ y,\ z$ denote the respective concentrations of the chemicals $X,\ Y,\ Z$ in the reactor. The autocatalytic reaction

$$X + 2Y \xrightarrow{k} 3Y$$

is modeled by

$$\begin{aligned} x' &= -kxy^2 \\ y' &= -2kxy^2 + 3kxy^2 = kxy^2 \end{aligned}$$

because the rate of decrease of the reactant concentration x is kxy^2 (think of $X + 2Y$ as $X + Y + Y$), the rate of decrease of the reactant concentration y is $2kxy^2$ (because two units of Y are involved), and the rate of increase in the product concentration y is $3kxy^2$ (think of $3Y$ as $Y + Y + Y$).

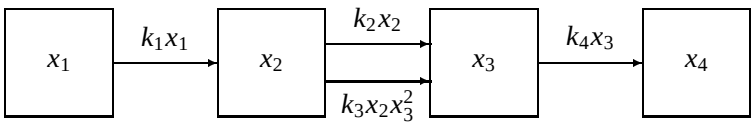
✓ If you want to speed up the reaction should you increase the rate constant k, or lower it? Any guesses about what would happen if you heat up the reactor? Put the reactor on ice?

With this background, we can model a sequence of reactions that has been studied in recent years:

$$X_1 \xrightarrow{k_1} X_2, \qquad X_2 \xrightarrow{k_2} X_3, \qquad X_2 + 2X_3 \xrightarrow{k_3} 3X_3, \qquad X_3 \xrightarrow{k_4} X_4$$

☞ In Figure 8.5, the multimedia module, and the Library file, different symbols are used for the rate constants.

Note the nonlinear autocatalytic step in the midst of the first-order reactions. A compartment diagram for this reaction is



where x_1, x_2, x_3, and x_4 denote the respective concentrations of the chemicals X_1, X_2, X_3, and X_4. The corresponding ODEs are:

$$\begin{aligned} x_1' &= -k_1x_1 \\ x_2' &= k_1x_1 - (k_2x_2 + k_3x_2x_3^2) \\ x_3' &= (k_2x_2 + k_3x_2x_3^2) - k_4x_3 \\ x_4' &= k_4x_3 \end{aligned} \tag{7}$$

☞ The term $k_3x_2x_3^2$ makes this system nonlinear.

In a reaction like this, we call X_1 the reactant, X_2 and X_3 intermediates, and X_4 the final product of the reaction. For certain ranges of values for the rate constants $k_1,\ k_2,\ k_3,\ k_4$ and for the initial reactant concentration $x_1(0)$, the

☞ See Screen 4.3 of Module 8 for values of the rate constants.

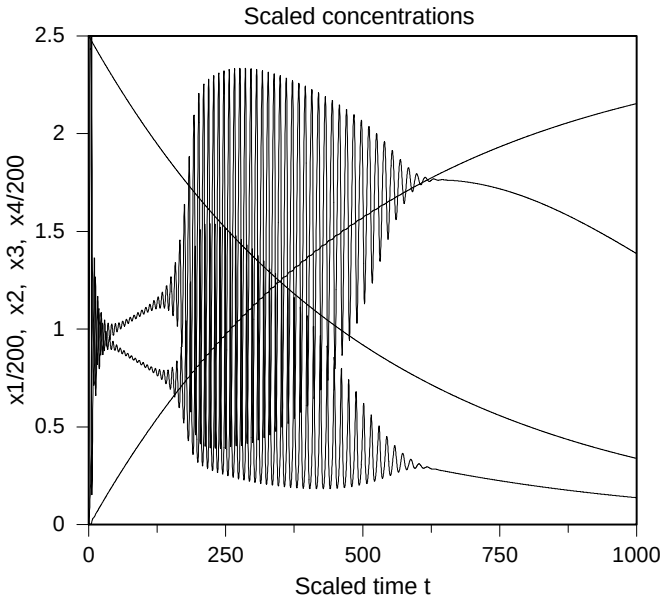


Figure 8.4: As the reactant falls into the Hopf bifurcation zone, the oscillations of the intermediates turn on as the product rises. Later the oscillations turn off and the reaction approaches completion.

☞ The chapter cover figure shows how the intermediate concentrations $x_2(t)$ and $x_3(t)$ play off against each other as time increases.

intermediate concentrations $x_2(t)$ and $x_3(t)$ will suddenly begin to oscillate. These oscillations eventually stop and the intermediates decay into the final reaction product. See Figure 8.4.

The onset of these oscillations is a kind of a *Hopf bifurcation* for $x_2(t)$ and $x_3(t)$. In this context, if we keep the value of x_1 fixed at, say x_1^*, the rate term $k_1 x_1^*$ in system (7) can be viewed as a parameter c. Then the middle two rate equations can be decoupled from the other two:

$$\begin{aligned} x_2' &= c - k_2 x_2 - k_3 x_2 x_3^2 \\ x_3' &= k_2 x_2 + k_3 x_2 x_3^2 - k_4 x_3 \end{aligned} \tag{8}$$

Now let's fix k_2, k_3, and k_4 and use the parameter c to turn the oscillations in $x_2(t)$ and $x_3(t)$ on and off. This is the setting for a *Hopf bifurcation*, so let's take a detour and explain what that is.

As a parameter transits a bifurcation value the behavior of the state variables suddenly changes. A Hopf bifurcation is a particular example of this kind of behavioral change. Suppose that we have a system that involves a parameter c,

$$\begin{aligned} x' &= f(x, y, c) \\ y' &= g(x, y, c) \end{aligned} \tag{9}$$

and that has an equilibrium point P at $x = a,\ y = b$ [so that $f(a, b, c) = 0$

and $g(a, b, c) = 0$]. Suppose that the matrix of partial derivatives

☞ This is the Jacobian matrix of system (9). See Chapter 7.

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}_{x=a,\,y=b}$$

has the complex conjugate eigenvalues $\alpha(c) \pm i\,\beta(c)$. The Dutch mathematician Eberhard Hopf showed that *if*:

(a) the functions f and g are twice differentiable,
(b) P is a stable, attracting sink for some value c_0 of the parameter c,
(c) $\alpha(c_0) = 0$,
(d) $[d\alpha/dc]_{c=c_0} \neq 0$,
(e) $\beta(c_0) \neq 0$,

☞ The Hopf conditions.

then as the parameter c varies through the bifurcation value c_0, the attracting equilibrium point P destabilizes and an attracting limit cycle appears (i.e., an attracting periodic orbit in the xy-phase plane) that grows in amplitude as c changes beyond the value c_0.

☞ Since $\alpha'(c_0) \neq 0$, $\alpha(c)$ changes sign as c goes through c_0; this means that P goes from a sink to a source, or the other way around.

It isn't always a simple matter to check the conditions for a Hopf bifurcation (especially condition (b)). It is often easier just to apply the Architect to the system and watch what happens to solution curves and trajectories when a parameter is swept over a range of values. For instance, for system (8) with values $k_2 = 0.08$ and $k_3 = k_4 = 1$ for the rate constants, we can sweep the parameter c and observe the results. In particular, we want to find values of c for which an attracting limit cycle is either spawned by P, or absorbed by P. At or near the special c values we can use the Equilibrium feature of the ODE Architect tool to locate the equilibrium point, calculate the Jacobian matrix, and find its eigenvalues. We expect the eigenvalues to be complex conjugates and the real part to change sign at the bifurcation value of c.

Figure 8.5 shows a sweep of twenty-one trajectories of system (8) with c sweeping down from 1.1 to 0.1 and the values of k_1, k_2, and k_3 as indicated in the figure. See also Problem 3, Exploration 8.4.

✓ (This is the first part of Problem 3 of Exploration 8.4.) Use ODE Architect to duplicate Figure 8.5. Animate (the right icon under Tools on the top menu bar) so that you can see how the trajectories change as c moves downward from 1.1. Then use the Explore feature to determine which values of c spawn or absorb a limit cycle. For what range of values of c does an attracting limit cycle exist?

This behavior of the system (8) carries over to the autocatalator system (7). Notice that the first equation in (7) is $x_1' = -k_1 x_1$, which is easily solved to give $x_1(t) = x_1(0)e^{-k_1 t}$. If k_1 is very small, say $k_1 = 0.002$, the exponential decay of x_1 is very slow, so that if $x_1(0) = 500$, the term $k_1 x_1(t)$,

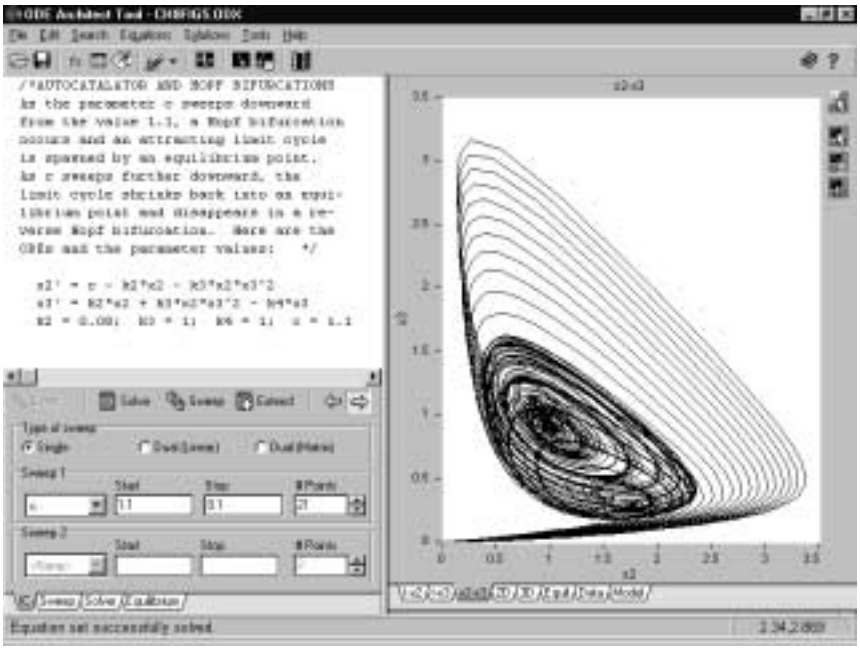


Figure 8.5: Twenty-one trajectories of system (8) for twenty-one values of c; initial data is $x_2(0) = x_3(0) = 0$, time interval is 100 with 1000 points.

though not constant, has values between 1 and 0.01 for a long time interval. The behavior of the autocatalator will be similar to that of system (8).

The section "Bifurcations to a Limit Cycle" in Chapter 7 gives another instance of a Hopf bifurcation. For more on bifurcations, see the references.

References

Gray, P., and Scott, S.K., *Chemical Oscillations and Instabilities* (1990: Oxford Univ. Press)

Hubbard, J.H., and West, B.H., *Differential Equations: A Dynamical Systems Approach, Part II: Higher Dimensional Systems*, (1995: Springer-Verlag)

Scott, S.K., *Chemical Chaos* (1991: Oxford Univ. Press)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 8.1. Tracking Pollution in a Lake

1. Suppose that the water flow rates into and out of a lake are $s_{\text{in}} = s_{\text{out}} = 10^9$ m^3/yr. The (constant) lake volume is $V = 10^{10}$ m^3, and the concentration of pollutant in the water flowing into the lake is $p_{\text{in}} = 0.0003$ lb/m^3. Solve the IVP with $L(0) = 0$ (no initial pollution) and describe in words how pollution builds up in the lake. Estimate the steady-state amount of pollution, and estimate the amount of time for the pollution level to increase to half of the asymptotic level.

2. Suppose that the lake in Problem 1 reaches its steady-state level of pollution, and then the source of pollution is removed. Build a new IVP for this situation, and estimate how much time it will take for the lake to clear out 50% of the pollution. How does this time compare to the time you estimated in Problem 1 for the build-up of pollutant?

3. What would be more effective in controlling pollution in the lake: (**i**) reducing the concentration of pollutant in the inflow stream by 50%, (**ii**) reducing the rate of flow of polluted water into the lake by 50%, or (**iii**) increasing the outflow rate from the lake by 50%?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 8.2. What Happens When You Take a Medication?

1. Go to the Library in ODE Architect and check out the file "Cold Pills I: A Model for the Flow of a Single Dose of Medication in the Body" in the folder "Biological Models." This model tracks a unit dose of medication as it moves from the GI tract into the blood and is then cleared from the blood. Read the file and carry out the explorations suggested there. Record your results below.

2. Go to the Library in ODE Architect and check out "Cold Pills II: A Model for the Flow of Medication with Periodic Dosage" in the folder "Biological Models." Carry out the suggested explorations.

3. Suppose you take a decongestant pill every four hours to relieve the symptoms of a cold. Each pill dissolves slowly and completely over the four-hour period between doses, releasing 16 units of decongestant at a constant rate. The decongestant diffuses from the GI tract into the bloodstream at a rate proportional to the amount in the GI tract (rate constant is $a = 0.5/$ hr) and is cleared from the bloodstream at a rate proportional to the amount in the blood (rate constant is $b = 0.1/$ hr). Assume that initially there is no decongestant in the body. Write a report in which you address the following points. Be sure to attach graphs.

(a) Write out ODEs for the amounts $x(t)$ and $y(t)$ in the GI tract and the blood, respectively, at time t.

(b) Find explicit formulas for $x(t)$ and $y(t)$ in terms of $x(0)$ and $y(0)$.

(c) Use ODE Architect to plot $x(t)$ and $y(t)$ for $0 \leq t \leq 100$ hr. What are the equilibrium levels of decongestant in the GI tract and in the blood (assuming that you continue to follow the same dosage regimen)?

(d) Graph $x(t)$ and $y(t)$ as given by the formulas you found in part (b) and overlay these graphs on those produced by ODE Architect. What are the differences?

(e) Imagine that you are an experimental pharmacologist for Get Well Pharmaceuticals. Set lower and upper bounds for decongestant in the bloodstream, bounds that will assure both effectiveness and safety. How long does it take from the time a patient starts taking the medication before the decongestant is effective? How long if you double the initial dosage (the "loading dose")? How about a triple loading dose?

(f) For the old or the chronically ill, the clearance rate constant from the blood may be much lower than the average rate for a random sample of people (because the kidneys don't function as well). Explore this situation and make a recommendation about lowering the dosage.

4. Repeat all of Problem 3 but assume the capsule is rapidly dissolving: it delivers the decongestant at a constant rate to the GI tract in just half an hour, then the dosage is repeated four hours later.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 8.3. Get the Lead Out

1. Check out the ODE Architect Library file "A Model for Lead in the Body" in the "Biological Models" folder and carry out the explorations suggested there. (The notation for the rate constants in the library file differs from the notation used in this chapter.)

2. Use the following rate constants: $k_1 = 0.0039,\ k_2 = 0.0111,\ k_3 = 0.0124,\ k_4 = 0.0162,\ k_5 = 0.000035,\ k_6 = 0.0211$, and put $L = 49.3\ \mu$g/day in the lead system (4). These values were derived directly from experiments with volunteer human subjects living in Los Angeles in the early 1970's. Using the data for the lead flow model, describe what happens if the lead inflow rate L is doubled, halved, or multiplied by a constant α. Illustrate your conclusions by using the ODE Architect to graph the lead levels in each of the three body compartments as functions of t. Do the long-term lead levels (i.e., the equilibrium levels) depend on the initial values? On L? Find the equilibrium levels for each of your values of L using ODE Architect. Find the eigenvalues of the Jacobian matrix for each of your values of L. With the names given in Chapter 6 to equilibrium points in mind, would you call the equilibrium lead levels sinks or sources? Nodes, spirals, centers, or saddles?

3. The bones act as a lead storage system, as you can see from the graphs in Submodule 3 of Module 8. What happens if the exit rate constant k_4 from the bones back into the blood is increased from 0.000035 to 0.00035? To 0.0035? Why might an increase in k_4 be harmful? See Problem 2 for the values of L and the rate constants k_i.

4. The medication now in use for acute lead poisoning works by improving the efficiency of the kidneys in clearing lead from the blood (i.e., it increases the value of the rate constant k_6). What if a medication were developed that increased the clearance coefficient k_5 from the tissues? Explore this possibility. See Problem 2 for the values of L and the rate constants k_i.

5. In the 1970's and 80's, special efforts were made to decrease the amount of lead in the environment because of newly enacted laws. Do you think this was a good decision, or do you think it would have been better to direct the efforts toward the development of a better antilead medication for cases of lead poisoning? Why? What factors are involved in making such a decision?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 8.4. Chemical Reactions: the Autocatalator

1. Check out "The Autocatalator Reaction" in the "Chemical Models" folder in the ODE Architect Library and graph the concentrations suggested. Describe how the concentrations of the various chemical species change in time.

2. Here are schematics for chemical reactions. Draw a compartment diagram for each reaction. Then write out the corresponding sets of ODEs for the individual chemical concentrations. [Use lower case letters for the concentrations (e.g., $x(t)$ for the concentration of chemical X at time t).]

(a) $X + Y \xrightarrow{k} Z$

(b) $X + Y \xrightarrow{k_1} Z \xrightarrow{k_2} W$

(c) $X + 2Y \xrightarrow{k} Z$

(d) $X + 2Y \xrightarrow{k} 3Y + Z$

3. Explore the behavior of $x_2(t)$ and $x_3(t)$ as governed by system (8). Start with $c = 1.1,\ k_2 = 0.08,\ k_3 = k_4 = 1$, and $x(0) = y(0) = 0$. Then sweep c from 1.1 down to 0.1 and describe what happens to orbits in the x_2x_3-plane. Find the range of values of c between 1.1 and 0.1 for which attracting limit cycles are visible. These are Hopf cycles. Fix c at a value that you think is interesting and sweep one of the parameters k_2, k_3, or k_4. Describe what you observe. [*Suggestion:* Take a look at Figure 8.5, and use the Animate feature of ODE Architect to scroll through the sweep trajectories. Then use the Explore option to get a data table with information about any of the trajectories you have selected.]

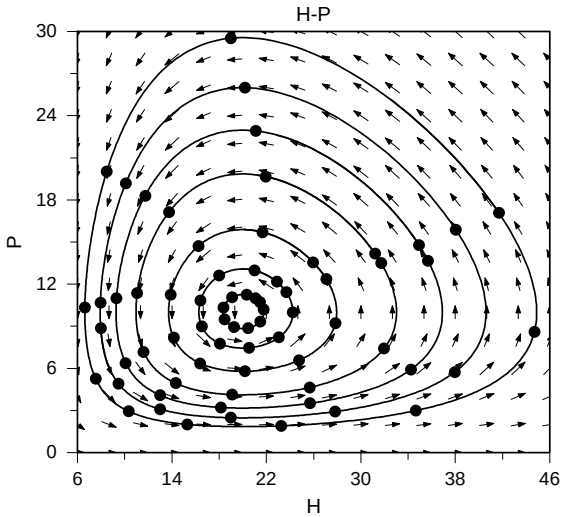
4. Look at the autocatalator system (7) with $x_1(0) = 500,\ \ x_2(0) = x_3(0) = x_4(0) = 0$ and $k_1 = 0.002,\ \ k_2 = 0.08,\ \ k_3 = k_4 = 1$. Graph $x_2(t)$ and $x_3(t)$ over various time ranges and estimate the times when sustained oscillations begin and when they finally stop. What are the time intervals between successive maxima of the oscillations in x_2? Plot a 3D tx_2x_3-graph over various time intervals ranging from $t = 100$ to $t = 1000$. Describe what you see. [*Suggestion:* Look at the chapter cover figure and Figure 8.4.]

☞ Use the Explore feature to estimate the starting and stopping times of the oscillator.

Now sweep the values of $x_1(0)$ downward from 500. What is the minimal value that generates sustained oscillations? Then fix $x_1(0)$ at 500 and try to turn the oscillations off by changing one or more of the rate constants k_1, k_2, k_3, k_4—this corresponds to heating or chilling the reactor. Describe your results.

CHAPTER 9

Population Models



Predator-prey cycles with a direction field. Markers are equally spaced in time.

Overview Population biology is the study of how communities of organisms change. The structure of a population can be quite intricate, such as species interactions in a tropical rain forest. Other communities may involve only a few species and are simpler to describe. There are many aspects of population biology, including ecology, demography, population genetics, and epidemiology. In each of these areas, mathematics plays an important role in modeling how populations change in time and how the interaction between the environment and the community affects that change. We'll explore mathematical models in ecology and epidemiology.

Key words Logistic model; growth rate; carrying capacity; equilibrium; steady state; competition; coexistence; exclusion; predator-prey; epidemiology; harvesting

See also Chapter 1 for more on the logistic equation, Chapter 7 for predator-prey models, and Chapter 8 for chemical mass action, Library folder "Population Models."

◆ Modeling Population Growth

The increasing awareness of environmental issues is an important development in modern society. This awareness ranges from concern about conserving important natural resources to concern about habitat destruction and the endangerment of species. Human population pressures are ever-increasing, and this growth has led to intense exploitation of the environment. To reduce the negative effects of this exploitation, scientists are seeking to understand the ecology and biology of natural populations. This understanding can be used to design management strategies and environmental policies.

The simplest ecological models describe the growth of a single species living in an *environment* or *habitat*. Characteristics of the habitat—moisture, temperature, availability of food—will affect how well the species survives and reproduces. Intrinsic biological characteristics of the species, such as the basic reproductive rate, also affect the growth of the species. However, a mathematical model that incorporates *all* possible effects on the growth of the population would be complicated and difficult to interpret.

✓ "Check" your understanding by answering this question: What are some other characteristics of a species and its environment that can affect the productivity of the species?

The most common procedure for modeling population growth is first to build elementary models with only a few biological or environmental features. Once the simple models are understood, more complicated models can be developed. In the next section we'll start with the *logistic model* for the growth of a single species—a model that is both simple and fundamental.

◆ The Logistic Model

The ecological situation that we want to model is that of a single species growing in an environment with limited resources. Examples of this situation abound in nature: the fish population of a mountain lake (the limited resource is food), a population of ferns on a forest floor (the limited resource is light), or the lichen population on a field of arctic rocks (the limited resource is space). We won't attempt to describe the biology or ecology of our population in detail: we want to keep the mathematical model simple. Instead, we'll summarize a number of such effects using two *parameters*. The first parameter is called the *intrinsic growth rate* of the population. It is often symbolized using the letter r, and it represents the average number of offspring, per unit time, that each individual contributes to the growth of the population. The second parameter is called the *carrying capacity of the environment*. Symbolized by K, the carrying capacity is the largest number of individuals that the environment can support in a steady state. If there are more individuals in the population than the carrying capacity, the population declines because there

are too few resources to support them. When there are fewer individuals than K, the environment has not been overexploited and the population grows.

Our mathematical model must capture these essential characteristics of the population's growth pattern. To begin, we define a variable that represents the size of the population as a function of time; call this size $N(t)$ at time t. Next, we specify how the size $N(t)$ changes in time. Creating a specific rule for the rate of change of population size is the first step in *building a mathematical model*. In general, a model for changing population size has the form

$$\frac{dN}{dt} = f(t, N), \quad N(0) = N_0$$

for some function f and initial population N_0. Once the details of f are given, based on the biological and ecological assumptions, we have a concrete mathematical model for the growth of the population.

To complete our description of the logistic model, we need to find a reasonable function f that captures the essential properties described above. We are looking for a simple function that gives rise to

- population growth when the population is below the carrying capacity (that is, $N'(t) > 0$ if $N(t) < K$);
- population decline if the population exceeds the carrying capacity (that is, $N'(t) < 0$ if $N(t) > K$).

One such function is $f(t, N) = rN(1 - N/K)$. The *logistic model* is the IVP

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

where r, K and N_0 are positive constants. Figure 9.1 shows some typical solution curves.

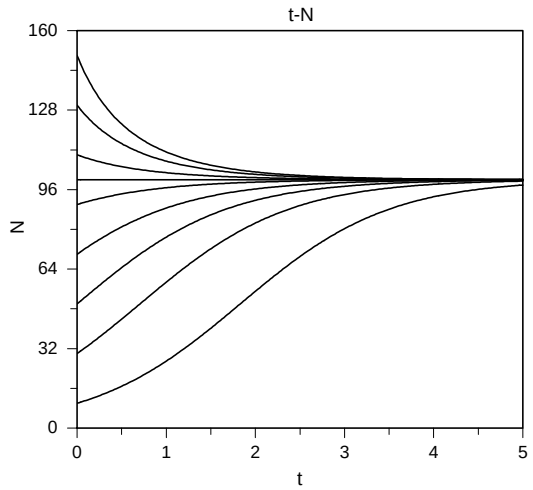


Figure 9.1: Some solution curves for the logistic equation.

Let's observe a few important features of our model. First, the algebraic sign of $N'(t)$ follows the desired relationship to the population size N. Second, if either $N = 0$ or $N = K$, there is no change in the population size: $N'(t) = 0$. Thus, $N = 0$ and $N = K$ are *equilibria* or *steady states*. The first steady state corresponds to the extinction of the species in its environment, and the second corresponds to a population in perfect balance, living at the carrying capacity of the habitat.

Notice the effect of the parameters r and K. As the carrying capacity K increases, the environment supports more individuals at equilibrium. As the growth rate r increases, the population attains its steady state in a faster time. An important part of understanding a mathematical model is to discover how changing the parameters affects the behavior of the system that is being modeled. This knowledge can lead to predictions about the system, and to a much deeper understanding of population processes. In Exploration 9.1 you will study the logistic model and variations of it. See also Chapter 1.

✓ Do you think the intrinsic annual growth rate r of the earth's human population is closer to 0.01, 0.03, or 0.05? It's anyone's guess as to the carrying capacity. What is your estimate, given that the current population is about 6 billion?

◆ Two-Species Population Models

The logistic model applies to a single species. Most habitats support a variety of species; interactions can be both *intraspecific* (between individuals of the same species) or *interspecific* (between individuals of different species). These interactions can take many forms. For example, competition between individuals of the same species for food resources, nesting sites, mates, and so on, are intraspecific interactions that lead to regulated population growth. Important interspecific interactions include predation, competition for food or other resources, and symbiotic relationships that are mutually beneficial. Such interactions can be very complex and can involve a large number of species. Again, the first step in modeling complicated ecologies is to build and analyze simple models. We'll present two such models here (involving only two species) and consider others in the explorations.

✓ Can you think of a mutually beneficial interaction between humans and another species?

◆ Predator and Prey

As we noted, an important interaction between species is that of predator and prey. Such interactions are very common: animals must eat to thrive, and for every eater there is the eaten! Spiders prey on flies, cows prey on grass, mosquitoes prey on humans, and humans prey on shiitake mushrooms, truffles, salmon, redwood trees, and just about everything else. We'll now build a simple model to describe such interactions.

Consider two species, the prey species (H, because they're "harvested" or "hunted") and the predator species (P), but for the moment, imagine that they don't interact. In the absence of the predator, we assume that the prey grows according to the logistic law, with carrying capacity K and intrinsic growth rate $a > 0$. The model for the prey under these conditions is

$$H' = aH(1 - H/K)$$

Now suppose that in the absence of its food source (the prey), the predator dies out; the model for the predator is

$$P' = -bP$$

where $b > 0$. If this situation persists, the prey will grow to fill the habitat and the predator will become extinct.

Now suppose that the predator does feed upon the prey, and that each predator consumes, on the average, a fraction c of the prey population, per unit time. The growth rate of the prey will then be decreased (since they're being eaten) by the amount cHP. The predators benefit from having consumed the prey, so their growth rate will increase. But because a given predator may have to consume a lot of prey to survive, not all prey produce new predators in a one-for-one way. Therefore the increase in the growth rate of the predators in this case is dHP, where d is a constant which may be different than c. Putting this all together, we obtain our model for the predator-prey system:

☞ When two species interact at a rate proportional to the product of the two populations, it's called *population mass action*.

$$\begin{aligned} \frac{dH}{dt} &= aH\left(1 - \frac{H}{K}\right) - cHP \\ \frac{dP}{dt} &= -bP + dHP \end{aligned} \tag{1}$$

Analyzing this model gives insight into a number of important ecological issues, such as the nature of coexistence of predator and prey, and the understanding of population cycles. Figure 9.2 on the next page shows a phase plot for the predator-prey system described by ODE (1). Exploration 9.3 examines this predator-prey model.

✓ What is the long-term future of the prey species in Figure 9.2? The predator species?

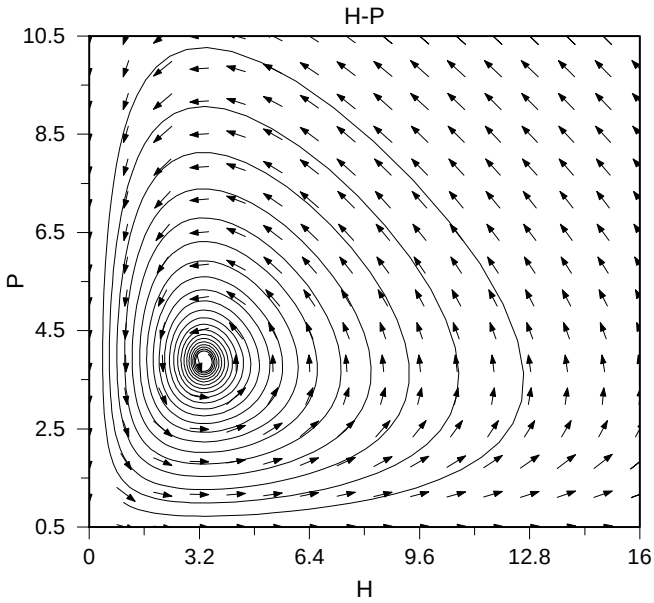


Figure 9.2: An orbit of a predator-prey model with logistic prey growth.

◆ Species Competition

Another common interaction between species is *competition*. Species can compete for space, food, light, or for other resources. In the absence of its competitor, each species grows logistically to its carrying capacity. However, the presence of the competitor changes the situation, and the growth rate of each species is diminished by the presence of the other. Let N_1 and N_2 represent the numbers of the two species. We model the competition between these species with the following equations:

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} - \alpha_{12} N_2\right) \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2} - \alpha_{21} N_1\right) \end{aligned} \tag{2}$$

The parameter α_{12} measures the effect of Species 2 on Species 1, and α_{21} measures the effect of Species 1 on Species 2. If $\alpha_{12} > \alpha_{21}$, then Species 2 dominates Species 1, because Species 2 reduces the growth rate of Species 1 more *per capita* than the reverse. The analysis of this model gives insight into how species maintain their diversity in the ecology (*coexistence*) or how such diversity might be lost (*competitive exclusion*). A phase plot for the competitive system is shown in Figure 9.3. Exploration 9.4 examines a related model for so-called mutualistic interactions.

☞ Could the competition be so fierce that both species become extinct?

✓ Does Figure 9.3 show coexistence or competitive exclusion?

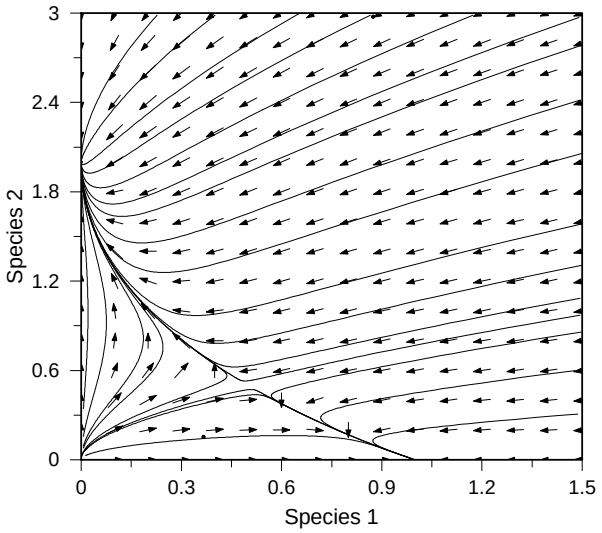


Figure 9.3: Orbits for a model of two competing species.

◆ Mathematical Epidemiology: The SIR Model

An important use of mathematical models is to describe how infectious diseases spread through populations. This field is called epidemiology. Quantitative models can predict the time course of a disease or the effectiveness of control strategies, such as immunization. Again, the development proceeds from the simplest model to more complex ones.

The most elementary model for an epidemic is the so-called SIR model (presented in Module 9): Consider a population of individuals divided into three groups—those susceptible (S) to a certain disease, those infected (I) with the disease , and those who have recovered (R) and are immune to reinfection, or who otherwise leave the population. The SIR model describes how the proportions of these groups change in time.

The susceptible population changes size as individuals become infected. Let's think of this process as "converting" susceptibles to infecteds. If we assume that each infected individual can infect a proportion a of the susceptible population per unit time, we obtain the rate equation

☞ Another instance of population mass action.

$$\frac{dS}{dt} = -aSI \tag{3}$$

The infected population is increased by conversion of susceptibles and is decreased when infected individuals recover. If b represents the proportion of infecteds that recover per unit time, then the rate of change of the infected population satisfies

$$\frac{dI}{dt} = aSI - bI \tag{4}$$

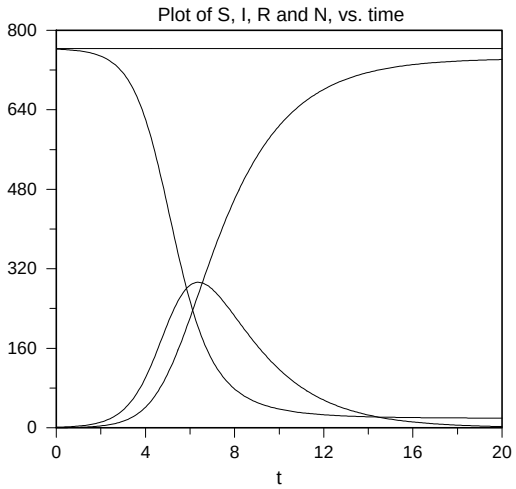


Figure 9.4: A plot of susceptibles (falling curve), infecteds (the "bump"), recovereds (rising curve), and their sum (top line).

Lastly, as infecteds recover they augment the recovered population, so that

$$\frac{dR}{dt} = bI \tag{5}$$

The ODEs (3)–(5) together with the initial values $S(0)$, $I(0)$, and $R(0)$ define the SIR model. A component plot of S, I, R, and $N = S + I + R$ appears in Figure 9.4.

✓ Can you explain why $N(t)$ stays constant as time changes?

We can learn many important things from this model about the spread of diseases. For example, analysis of the model can reveal how the rate of spread of the disease through the population is related to the infectiousness of the disease. Our common experience suggests that not all diseases become epidemic: sometimes a few people are afflicted and the disease dies out. Analysis of the SIR model can also give insight into the conditions under which a disease will become epidemic. This is an important phenomenon! These and other matters will be examined in Exploration 9.5.

✓ What factors can you think of that might influence the spread of a disease in a human population?

References

Bailey, N.T.J., *The Mathematical Theory of Infectious Diseases and its Applications*, 2nd ed., (1975: Hafner Press)

Edelstein-Keshet, L., *Mathematical Models in Biology*, (1988: McGraw-Hill)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 9.1. The Logistic Model

In this exploration you will consider a population that grows according to the logistic law: $N' = rN(1 - N/K)$, where r is the intrinsic growth rate and K is the carrying capacity of the environment.

☞ Using the ODE Architect Library.

1. Open the ODE Architect Library. In the folder "Population Models," open the file "Logistic Model of Population Growth." The logistic equation will be automatically entered into the Architect. The graphs show several solution curves. Set the initial condition for the population size to $N_0 = 25$ and set $K = 100$. Plot eight solutions by sweeping the growth rate constant from $r = -0.5$ to $r = 2$; print your graph. Describe the effect of r on the solutions of the logistic equation. Your description should address the following questions: How does the growth rate constant affect the long-term behavior of the population? How does the rate constant affect the dynamics of the system?

2. Set the IC to $N_0 = 25$ and $r = 1.2$. Plot eight solution curves by sweeping the carrying capacity K from 70 to 150; print your graph. Describe the effect of the parameter K on the solutions of the logistic equation. Your description should address the following questions: How does the carrying capacity affect the long-term behavior of the population? How does the carrying capacity affect the dynamics of the system?

3. Study the graphs that you produced for Problems 1 and 2. Notice that sometimes the rate of change of population size is increasing (i.e., $N'(t)$ is increasing and the graph of $N(t)$ is concave up) and sometimes it is decreasing ($N'(t)$ is decreasing and the graph of $N(t)$ is concave down). By analyzing your graphs, try to predict a relationship between r, K, and N that distinguishes between these two situations. Use ODE Architect to test your prediction by graphing more solution curves. Lastly, try to confirm your prediction by exact analysis of N'' using the logistic ODE.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 9.2. Harvesting a Natural Resource

Human societies use resources from their environments. We harvest animals and plants for food, construction, fuel, and many other uses. The harvesting of a biological resource must be done carefully, because overexploitation of the population can cause severe harm, or even extinction, to the resource. As a society we have become much more sensitive about the need to balance the benefits of resource consumption against the impact of that consumption on the exploited populations and their environment.

Resource management is an important tool for minimizing the negative effects of harvesting. Mathematical models are tools for understanding the impact of harvesting on a population, so that we can then design management policies, such as quotas on the annual harvest.

In this exploration, you will analyze a simple model for harvesting a single species. To be specific, suppose that the habitat is a forest and the resource is a species of pine tree. The number of trees grows logistically with an intrinsic growth rate r, and the forest will support at most K trees (measured in millions of board feet). You are a consulting ecologist, asked to model the effect of a lumber company's harvesting strategy on the pine forest. The company harvests the trees *proportionally*: in a unit of time (a year, for example), the company removes a fixed fraction h of the trees. Harvesting reduces the net rate of growth of the forest; this leads you to propose the following model for the effect of harvesting:

$$N' = rN\left(1 - \frac{N}{K}\right) - hN, \quad N(0) = N_0 \tag{6}$$

The last term, $-hN$, is the *harvesting term*. Notice that when $h = 0$ (i.e., no trees are harvested), the model reduces to the logistic equation.

1. Open ODE Architect and enter the ODE for the harvesting model given by equation (6). Set the growth rate to $r = 0.1$ year^{-1}, the carrying capacity to $K = 1000$ million board feet, and the population size IC to $N_0 = 100$ at $t = 0$. Describe the growth of the forest when there is no harvesting ($h = 0$). You'll have to choose a good time interval to best display your results.

2. Keep r and K fixed and plot solution curves for various (positive) values of the harvesting coefficient h. You can do this exploration most efficiently by sweeping the parameter h. After you have studied a variety of harvest rates, explain how harvesting affects the pine population. Your explanation should address the following questions: How does the growth of the pine population with harvesting compare to its growth without harvesting? What is the long-term effect of harvesting? How are the time dynamics of the forest growth affected by harvesting?

3. The annual yield Y of the harvest is the amount of lumber removed per year. This is just $Y = hN$ when there are N units of lumber in the forest. The yield will vary through time as the amount of lumber (trees) in the forest varies in time. If the harvest rate is too high, the long-term yield will tend to zero ($Y \to 0$) and the forest will become overexploited. If the harvest rate is very low, the yield will also be very low. As the consultant to the company, you are asked: What should the harvest rate be to give the largest *sustainable yield* of lumber? That is to say, what optimal harvest rate will maximize $\lim_{t\to\infty} Y(t)$? Attack the problem graphically using ODE Architect to plot graphs of the yield function for various values of h. Assume that $r = 0.1$, $K = 1000$, and $N_0 = 100$. If you can, provide an analytic solution to the question, and check your results using the Architect. Suppose that the company follows your recommendation and harvests pine at the optimal rate. When the size of the forest reaches equilibrium, how much lumber (trees) will there be, and how does this amount compare to the size of the forest without harvesting?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 9.3. Predator and Prey

Predator-prey interactions are very common in natural populations. These interactions can be modeled by a system of nonlinear equations:

$$H' = aH\left(1 - \frac{H}{K}\right) - cHP, \quad P' = -bP + dHP$$

where H and P are the prey and predator population sizes, respectively.

1. Give a biological interpretation of the parameters a, b, c, d, K of the predator-prey model.

2. Open ODE Architect and enter the modeling equations for the predator-prey system above. Assign the following values to the parameters: $a = 1.0$, $b = 0.25$, $c = 1.0$, $d = 0.15$, $K = 100$. After you have entered the equations and parameters, set the solve interval to 60, and the number of points plotted to 500. Solve the system forward in time using the initial conditions $H(0) = 1$, $P(0) = 1$. Plot graphs of the orbits in the HP-phase plane, and plot the individual component graphs for predator and prey. Experiment with other initial conditions. Describe the nature of the solutions and locate all equilibrium solutions.

3. Fix $a = 1.0$, $b = 0.25$, $c = 1.0$, and $d = 0.15$ as in Problem 2. Plot several solutions from the fixed initial conditions $H(0) = 1$, $P(0) = 1$, for varying values of K. For example, let K range over several values between 100 to 10,000. How does changing the carrying capacity of the prey affect the behavior of the system? Make a conjecture about the limiting behavior of the system as $K \to \infty$.

4. Test the conjecture that you made in Problem 3 in two steps:

(a) Take the limit as $K \to \infty$ in the predator-prey equations and obtain a new system of equations that describes a predator-prey system where there is no resource limitation for the prey.

(b) Explore this system using ODE Architect; this new system is often called the *Lotka–Volterra* model. Plot several orbits using markers that are equally spaced in time. Do the cycles have a common period? How do the time markers help you answer that question? Compare your graphs with the chapter cover figure. Also plot graphs of H against t and P against t for various values of $H(0)$ and $P(0)$. What do these graphs tell you about the periods?

☞ Finally, you get a chance to figure out what is going on in the chapter cover figure.

How does the behavior of the Lotka–Volterra model differ from the model you explored in Problems 1–3?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 9.4. Mutualism: Symbiotic Species Interactions

For both predator-prey and species competition, the growth rate of at least one of the species is reduced by the interaction. Though eating the prey helps the predator, it certainly harms the prey; for competitors the reduction in growth rate is reciprocal. Not all species interactions must be negative: there are many examples where the species cooperate or otherwise *mutually enhance* their respective growth rates. A famous example is the yucca plant–yucca moth system: the yucca plant can be pollinated only by the yucca moth, and the yucca moth is adapted to eat nectar only from the yucca plant. Each species benefits the other, and their interaction is positive for both. Such interactions are called *mutualistic* or *symbiotic* by ecologists. In this exploration we will present and analyze a simple model for mutualism.

Our model will be very similar to the competition model studied in Module 9. To obtain a model for mutualism, we just change the signs of the interaction terms so that they are always positive: each species enhances the growth rate of the other. We then obtain the following equations:

$$\frac{dN_1}{dt} = N_1\left(r_1 - e_1 N_1 + \alpha_{12} N_2\right), \quad \frac{dN_2}{dt} = N_2\left(r_2 - e_2 N_2 + \alpha_{21} N_1\right) \tag{7}$$

The parameters r_1, r_2, α_{12}, and α_{21} retain their meanings from ODE (2) in the competition model. However, the interaction terms $\alpha_{12}N_1N_2$ and $\alpha_{21}N_1N_2$ have *positive* sign and thus enhance the respective growth rates.

Notice that in the absence of interaction, the carrying capacities of the two species are $K_1 = r_1/e_1$ and $K_2 = r_2/e_2$ in this version of the model.

1. Open the ODE Architect Library, go to the "Population Models" folder, and open the file "Mutualism: Symbiotic Interactions." This file loads the equations that model a mutualistic interaction. Fix $r_1 = 1$, $r_2 = 0.5$, $e_1 = 1$, $e_2 = 0.75$. Vary the values of each of the interaction coefficients from 0 to 2. For each combination of values for α_{12} and α_{21} that you try, draw a phase portrait of the system (7) in the first quadrant. Describe every possible kind of behavior of the system; try enough combinations of the parameters to feel confident that you have covered all the possibilities. Answer the following questions: Is species coexistence possible? Can competitive exclusion occur? Will the populations of both species remain bounded as time increases?

☞ You may want to use the Dual (Matrix) sweep feature here.

2. Using pencil and paper, deduce the conditions under which a two-species equilibrium will be present. Check your conditions using the Architect to solve the model. When a two-species equilibrium is present, does it necessarily have to be stable? Compare two-species equilibria to single-species equilibria (the carrying capacities): does mutualism increase or decrease the abundance of the species at equilibrium?

3. Do you think that a mutualistic interaction is always beneficial to an ecosystem? Under what conditions might it be deleterious? Compare the behavior of mutually interacting species to that of competing species. How are the two behaviors similar? How are they different?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 9.5. Analyzing the SIR model for an Epidemic

We will now explore the SIR model for the spread of an epidemic. Recall the ODEs for this model: $S' = -aSI,\ I' = aSI - bI,\ R' = bI$. The parameter $a > 0$ is the infection rate constant and $b > 0$ is the removal (recovery) rate constant of infecteds. Notice that $S' + I' + R' = 0$, i.e., the total number of individuals N is constant and equals $S(0) + I(0) + R(0)$. The ODE Architect Library has an equation file for the SIR model in the "Population Models" folder. In this file you will find values of a, b, and N that correspond to an actual epidemic.

1. Set the IC to $I(0) = 20$ and $R(0) = 0$. Set the solve interval to 24 time units, and make ten plots by sweeping the initial number of susceptibles from $S(0) = 100$ to $S(0) = 500$. Now examine the graph panel for I vs. t. Which of the curves corresponds to $S(0) = 100$ and which to $S(0) = 500$? By definition, an epidemic occurs if $I(t)$ increases from its initial value $I(0)$. For which of the curves that you plotted did an epidemic occur?

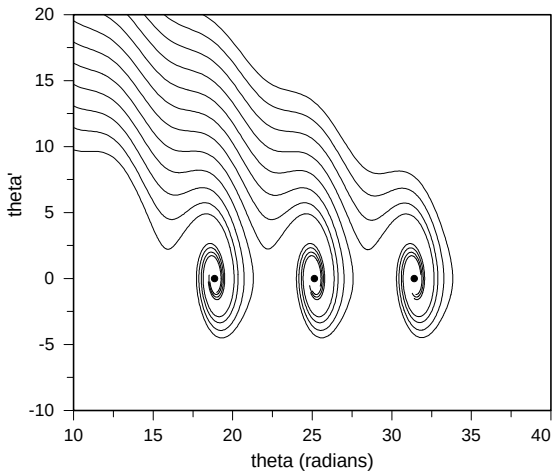
2. The behavior that you studied in Problem 1 is called a *threshold effect*. If the initial number of susceptible individuals is below a threshold value, there will be no epidemic. If the number of susceptibles exceeds this value, there will be an epidemic. Use ODE Architect to empirically determine the threshold value for $S(0)$; use the values of a and b in the Library file. Now analyze the equation for dI/dt and determine a sufficient condition for $I(t)$ to initially increase. Interpret your answer as a threshold effect. Use the values of the infection and removal rates that appear in the Library file to compare your analytic calculation of the threshold with that obtained from your empirical study.

3. Clear your previous results from the Architect but keep the same values for a and b. Set the initial conditions for I, R, and S to 10, 0, and 200, respectively. Solve the equations for a time interval of 24 units. Notice from the plot of $I(t)$ that the number of infecteds steadily diminishes from 10 to nearly zero. Also notice that over this same period of time, the number of susceptibles declines by almost 50, and the number of recovered individuals increases from zero to nearly 50. Explain this seemingly contradictory observation.

4. A disease is said to be *endemic* in a population if at equilibrium the number of infecteds is positive. Is it possible in the SIR model for the disease to be endemic at equilibrium? In other words, can $\lim_{t\to\infty} I(t) > 0$? Explain your answer.

CHAPTER 10

The Pendulum and Its Friends



High-energy trajectories of a damped pendulum ODE swing over the top and then settle into decaying oscillations about rest points.

Overview The whole range of fixed-length pendulum models—linear, nonlinear, damped, and forced—are presented in this chapter, and their behaviors are compared using insights provided by integrals. After discussing fixed-length pendulum ODEs, the effects of damping, and separatrices, we turn to a variable-length model. A child pumping a swing alters the length of its associated pendulum as the swing moves. We present a nontraditional autonomous model and show that phase-plane analysis leads to a successful description of the effects of the pumping action. Finding geodesics (the paths of minimum length between points) on a torus leads to an ODE with a resemblance to the pendulum ODE.

Key words Pendulum; damping; energy; equilibrium; separatrices; pumping (a swing); conservation laws; integrals of motion; torus; geodesics; limit cycle; bifurcation

See also Chapter 4 for a spring-mass system which has the same ODE as the linear pendulum; Chapter 11 for a study of damping effects in the Robot and Egg submodule, and a lengthening pendulum in Exploration 11.4; Chapters 6 and 7 for separatrices and integrals of motion; and Chapter 12 for elaboration on the forced, damped pendulum resulting in chaos (and control).

◆ Modeling Pendulum Motion

☞ The volumes by Halliday and Resnick (refs.) are good general references for physical models (including the pendulum).

Let's find the ODE that models the motion of a pendulum. For a pendulum bob of mass m at the end of a rod of negligible weight and fixed length L at an angle θ to the vertical, Newton's second law gives

$$\text{mass} \cdot \text{acceleration} = \text{sum of forces acting on the bob}$$

The bob moves along an arc of a circle of radius L. The tangential component of the bob's velocity and acceleration at time t are given by $L\theta'(t)$ and $L\theta''(t)$, respectively. The tangential component, $-mg\sin\theta$, of the gravitational force acts to restore the pendulum to its downward equilibrium. The viscous damping force, $-bL\theta'$, is proportional to the velocity and acts in a direction tangential to the motion, but oppositely directed. Other forces such as repeated pushes on the bob may also have components $F(t)$ in the tangential direction.

☞ Since the tensile force in the rod and the radial component of the gravitational force are equal and opposite, the radial acceleration is zero and the pendulum moves along a circular arc.

Equating the product of the mass and the tangential acceleration to the sum of the tangential forces, we obtain the pendulum ODE

$$mL\theta'' = -mg\sin\theta - bL\theta' + F(t) \tag{1}$$

The equivalent pendulum system is

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{g}{L}\sin\theta - \frac{b}{m}y + \frac{1}{mL}F(t) \end{aligned} \tag{2}$$

The angle θ is positive if measured counterclockwise from the downward vertical, and is negative otherwise; θ is measured in radians (1 radian is $360/2\pi$ or about 57°). We allow θ to increase or decrease without bound because we want to keep track of the number of times that the pendulum swings over the pivot, and in which direction. For example, if $\theta = -5$ radians then the pendulum has swung clockwise (the minus sign) once over the top from $\theta = 0$ because the angle -5 is between $-\pi$ (at the top clockwise from 0) and -3π (reaching the top a second time going clockwise).

We will work with the *undriven* pendulum ODE ($F = 0$) in this chapter. Since $\sin\theta \approx \theta$ if $|\theta|$ is small, we will on occasion replace $\sin\theta$ by θ to obtain a linear ODE. We treat both undamped ($b = 0$) and damped ($b > 0$) pendulum ODEs:

☞ The first two ODEs in this list have the form of the mass-spring ODEs of Chapter 4.

$$\theta'' + \frac{g}{L}\theta = 0 \quad \text{(undamped, linear)} \tag{3a}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\theta = 0 \quad \text{(damped, linear)} \tag{3b}$$

$$\theta'' + \frac{g}{L}\sin\theta = 0 \quad \text{(undamped, nonlinear)} \tag{3c}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\sin\theta = 0 \quad \text{(damped, nonlinear)} \tag{3d}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\sin\theta = \frac{1}{mL}F(t) \quad \text{(damped, nonlinear, forced)} \tag{3e}$$

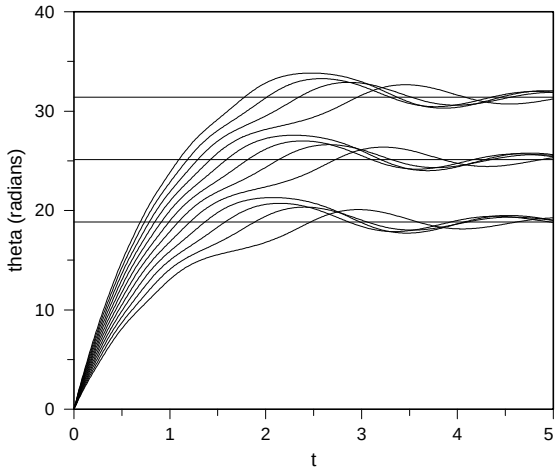


Figure 10.1: Solution curves of a damped pendulum system. What is the meaning of the horizontal solution curves?

Although the two linear ODEs are only good models of actual pendulum motions when $|\theta|$ is small, these ODEs have the advantages that their solutions have explicit formulas (see Chapter 4). The nonlinear ODEs model pendulum motions for all values of θ, but there are no explicit solution formulas. Figure 10.1 and the chapter cover figure, respectively, show some solution curves and trajectories of the damped, nonlinear pendulum ODE, $\theta'' + \theta' + 10\sin\theta = 0$.

Now fire up your computer, go to Screen 1.2 of Module 10, and visually explore the behavior of solution curves and trajectories of linear, nonlinear, damped, and undamped pendulum ODEs. Pay particular attention to the behavior of the animated pendulum at the upper left, and relate its motions to the trajectories and to the solution curves, and to what you think a real pendulum would do. Explore all the options in order to understand the differences.

✓ "Check" your understanding by matching solution curves of Figure 10.1 with the corresponding trajectories in the chapter cover figure. Describe the long-term behavior of the pendulum represented by each curve.

☞ This is also what Problem 1 of Exploration 10.1 is about.

✓ Go to Screen 1.2 of Module 10 and explore what happens to solutions of the undamped, linearized ODE, $\theta'' + \theta = 0$, if θ_0 is 0 and θ_0' is large. The motion of the animated pendulum is crazy, even though it accurately portrays the behavior of the solutions $\theta(t) = \theta_0' \sin t$. Explain what is going on. Is the linearized ODE a good model here? Repeat with the undamped, nonlinear ODE, $\theta'' + \sin\theta = 0$, and the same initial data as above. Is this a better model?

There is another way to look at pendulum motion, an approach based on integrals of motion. This approach goes beyond pendulum motion and applies to any physical system which can be modeled by a second-order ODE of a particular type.

◆ Conservative Systems: Integrals of Motion

In this section we will study solutions of the differential equation

☞ V is often called a potential function.

$$q'' = -\frac{dV}{dq} \tag{4}$$

for a generic variable q where $V(q)$ is a given function.

Example 1: The undamped, nonlinear pendulum ODE is the special case where $q = \theta$:

$$\theta'' = -\frac{g}{L}\sin\theta, \qquad V(\theta) = -\frac{g}{L}\cos\theta$$

Example 2: You will see later in this chapter that geodesics on a surface of revolution lead to the differential equation

$$u'' = M^2\frac{f'}{f^3}, \qquad V(u) = M^2\frac{1}{2f^2}$$

where the generic variable q is u in this case, M is a constant, and f is a function of u.

ODE (4) is autonomous and equivalent to the system

$$\begin{aligned} q' &= y \\ y' &= -\frac{dV}{dq} \end{aligned} \tag{5}$$

A solution to system (5) is a pair of functions, $q = q(t)$, $y = y(t)$. One way to analyze the behavior of these solutions is by a conservation law. A function $K(q, y)$ that remains constant on each solution [i.e., $K(q(t), y(t))$ is a constant for all t], but varies from one solution to another, is said to be a *conserved quantity*, or an *integral of motion* and the system is said to be *conservative*. For system (5) one conserved quantity is

$$K(q, y) = \frac{1}{2}y^2 + V(q) \tag{6}$$

Here's how to prove that $K(q(t), y(t))$ stays constant on a solution—use the chain rule and system (5) to show that dK/dt is zero:

$$\frac{dK}{dt} = y\frac{dy}{dt} + \frac{dV}{dq}\frac{dq}{dt} = y\left(-\frac{dV}{dq}\right) + \frac{dV}{dq}y = 0$$

Incidentally, if K is any conserved quantity, so also is $\alpha K + \beta$ where α and β are constants and $\alpha \neq 0$.

Example 3: Here's an example where we use $\alpha K + \beta$, rather than K, as the integral to show how integrals sometimes correspond to physical quantities. Look back at the function $V(\theta) = -(g/L)\cos\theta$ for the undamped, nonlinear pendulum of Example 1. Using formula (6), $E(\theta, y)$ is an integral, where

$$\begin{aligned} E(\theta, y) &= mL^2 K(\theta, y) + mgL \\ &= mL^2\left(\frac{1}{2}y^2 - \frac{g}{L}\cos\theta\right) + mgL \\ &= \frac{1}{2}m(Ly)^2 + mgL(1-\cos\theta) \\ &= \text{kinetic energy} + \text{potential energy} \end{aligned}$$

This integral is called the *total mechanical energy* of the pendulum. The constant mgL is inserted so that the potential energy is zero when the pendulum bob is at its lowest point.

✓ Find the conserved quantity E for the undamped, linear pendulum ODE $\theta'' + \theta = 0$. Draw level curves $E(\theta, y) = E_0$, where $y = \theta'$, in the θy-plane, and identify the curves (e.g., ellipses, parabolas, hyperbolas).

☞ So we can draw trajectories of system (5) by drawing level sets of an integral.

Drawing the level curves of a conserved quantity K in the qy-plane for system (5) gives phase plane trajectories of the system and so serves to describe the motions. This may be much easier than finding solution formulas, but even so, we can take some steps toward obtaining formulas. To see this, we have from equation (6) that if K has the value K_0 on a trajectory of system (5), then

$$\frac{1}{2}y^2 + V(q) = K_0, \quad \text{i.e.,} \quad y = q' = \pm\sqrt{2K_0 - 2V(q)}$$

This is a separable first-order differential equation (as discussed in Chapter 2) that can be solved by separating the variables and integrating:

$$\int \frac{dq}{\sqrt{K_0 - V(q)}} = \sqrt{2}t + C$$

It is hard to obtain explicit solution formulas because the integral cannot usually be expressed in terms of elementary functions.

◆ The Effect of Damping

Mechanical systems are usually damped by friction, and it is important to understand the effect of friction on the motions. Friction is not well described by the fundamental laws of physics, and any formula we write for it will be more or less *ad-hoc*. The system will now be modeled by a differential equation of the form

$$q'' + f(q, q') + \frac{dV}{dq} = 0$$

or, rewritten as a system of first-order ODEs,

$$\begin{aligned} q' &= y \\ y' &= -f(q,y) - dV/dq \end{aligned} \tag{7}$$

where $-f(q,y)$ represents the frictional force; the function $f(q,y)$ always has the sign of y.

At low velocities, $f(q,y) = by$ is a reasonably good approximation of the friction due to air, but higher powers of y are necessary at higher velocities. This latter fact is why reducing the speed limit actually helps reduce gasoline usage—there is less drag at lower speeds. If friction were only a linear function of velocity, the effects of a higher speed would be cancelled by the distance being covered in a shorter time, and the system would expend the same amount of energy in either case. But if friction depends on the cube of velocity, for instance, you gain a lot by going more slowly. We will examine more elaborate friction laws when we study the pumping of a swing, but for now we will use viscous damping with $f = by$.

Example 4: Let's model the motion of a linearized pendulum with and without damping:

$$\begin{aligned} \theta' &= y \\ y' &= -10\theta - by \end{aligned} \tag{8}$$

where $b = 0$ (no damping), or $b = 1$ (viscous damping). If there is no damping, then one conserved quantity is

$$K = \frac{1}{2}y^2 + 5\theta^2 \tag{9}$$

The left graph in Figure 10.2 displays the integral surface defined by formula (9). The surface is a bowl whose cross-sections $K = K_0$ are ellipses. Projecting the ellipses downward onto the θy-plane gives the trajectories of system (8) with $b = 0$.

Once damping is turned on, the function K in formula (9) no longer is constant on a trajectory of system (8). But the integral concept still gives a good geometric picture of the behavior of a system under damping, because the value of the function K in equation (6) decreases along trajectories. This fact follows from the following computation (using system (7)):

$$\begin{aligned} \frac{d}{dt}\left(\frac{y^2}{2} + V(q)\right) = y\frac{dy}{dt} + \frac{dV}{dq}\frac{dq}{dt} &= y\left(-f(q,y) - \frac{dV}{dq}\right) + \frac{dV}{dq}y \\ &= -yf(q,y) \le 0 \end{aligned}$$

where the final inequality follows from the fact that $f(q,y)$ has the sign of y. So if $b > 0$ in system (8) the value of K along a solution decreases, and will either tend to a finite limit, which can only happen if the solution tends to an equilibrium of the system, or the value of K will tend to $-\infty$. If V is bounded from below (as in all our examples), the latter does not happen.

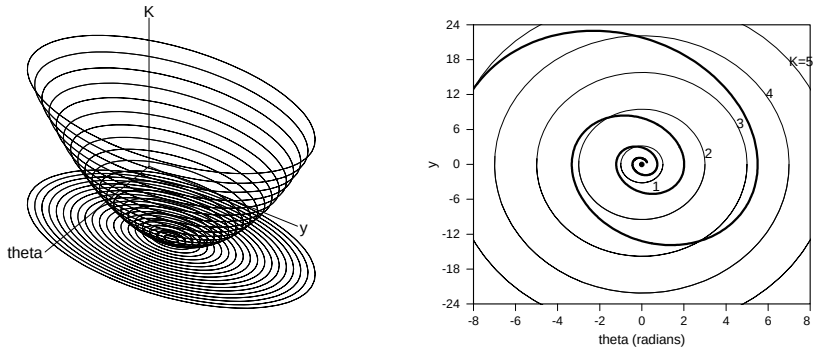


Figure 10.2: The left graph shows the integral surface $K = y^2/2 + 5\theta^2$ for the undamped, linearized system, $\theta' = y, \quad y' = -10\theta$, and the projections of the level curves $K = K_0$. The right graph shows a trajectory of the damped, linearized system, $\theta' = y, \quad y' = -10\theta - y$; as the trajectory cuts across the level curves of K, the value of K decreases.

Example 5: Let's turn on viscous damping (take $b = 1$ in system (8)) and see what happens. The right side of Figure 10.2 shows a trajectory of the damped, linear pendulum system as it cuts across the level curves of the integral function $K = y^2/2 + 5\theta^2$. K decreases as the trajectory approaches the spiral sink at $\theta = 0, \quad y = 0$. [The level curves of K are drawn by ODE Architect as trajectories of the undamped system (8) with $b = 0$.]

Now let's turn to the more realistic nonlinear pendulum and see how damping affects its motions.

Example 6: The nonlinear system is

$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta - by \end{aligned} \tag{10}$$

where $b = 0$ corresponds to no damping, and $b = 1$ gives viscous damping. In the no-damping case we can take the conserved quantity K to be

$$K = \frac{1}{2}y^2 - 10(\cos\theta - 1) \tag{11}$$

The left side of Figure 10.3 shows part of the surface defined by equation (11).

Example 7: With damping turned on (set $b = 1$ in system (10)) a trajectory with a high initial K-value may "swing over the pivot" several times before settling into a tightening spiral terminating at a sink, $\theta = 2n\pi, \quad y = 0$, for some value of n. The right side of Figure 10.3 shows one of these trajectories as it swings over the pivot once, and then heads toward the point, $\theta = 2\pi$, $y = 0$, where $K = 0$.

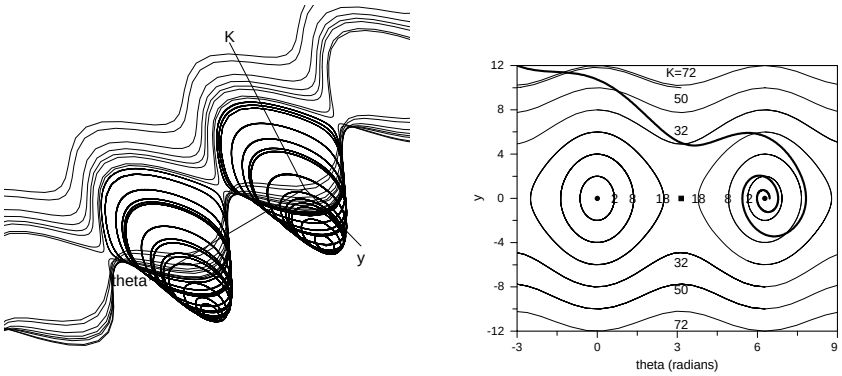


Figure 10.3: The left graph shows the surface $K = y^2/2 - 10(\cos\theta - 1)$ with two of its bowl-like projections that touch the θy-plane at equilibrium points of minimal K-value. The nonlinear pendulum system is $\theta' = y,\ y' = -10\sin\theta - by$ with $b = 0$. Turn on damping ($b = 1$) and watch a trajectory cut across the level sets $K = K_0$, with ever smaller values of K_0 (right graph).

✓ Would you increase or decrease b to cause the trajectory starting at $\theta = -3,\ y = 12$ to approach $\theta = 0,\ y = 0$? How about $\theta = 10\pi,\ y = 0$? What would you need to do to get the trajectory to approach $\theta = -2\pi,\ y = 0$, or is this even possible?

◆ Separatrices

A trajectory is a *separatrix* if the long-term behavior of trajectories on one side is quite different from the behavior on the other side. As we saw in Chapters 6 and 7 each saddle comes equipped with four separatrices: two approach the saddle with increasing time (the *stable* separatrices for that saddle point) and two approach as time decreases (the *unstable* separatrices). These separatrices are of the utmost importance in understanding how solutions behave in the long term.

Example 8: The undamped system

$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta \end{aligned} \tag{12}$$

☞ Use the Tool to find eigenvalues and eigenvectors of the Jacobian matrix of system (12) at $\theta = n\pi,\ y = 0$ for $n = 1, 2, 3$.

has equilibrium points at $\theta = n\pi,\ y = 0$. These points are (nonlinear) centers if n is even, and saddles if n is odd. Each separatrix at a saddle enters (or leaves) the saddle tangent to an eigenvector of the Jacobian matrix evaluated at the point. ODE Architect gives us these eigenvectors after it has located the saddle.

Example 9: (*Plotting a Separatrix:*) To find a point approximately on a saddle separatrix, just take a point close to a saddle and on an eigenvector. Then solve

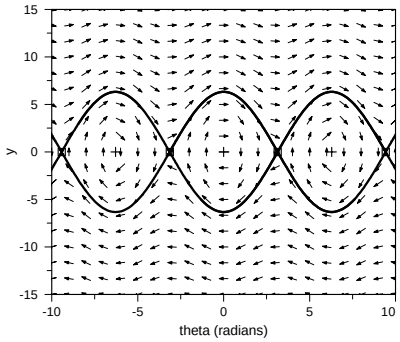


Figure 10.4: Saddle separatrices for the undamped, nonlinear pendulum system enclose centers.

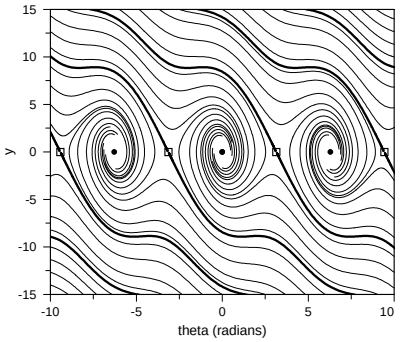


Figure 10.5: Basins of attraction of spiral sinks are bounded by stable saddle separatrices.

forward and backward to obtain a reasonable approximation to a separatrix. For example, at $\theta = \pi,\ \ y = 0$, ODE Architect tells us that $(0.3015, 0.9535)$ is an eigenvector corresponding to the eigenvalue 3.162, and so the saddle separatrix is unstable. To graph the corresponding separatrix we choose as the initial point $\theta_0 = \pi + 0.003015,\ \ y = 0.009535$ which is in the direction of the eigenvector and very close to the saddle point. Figure 10.4 shows several separatrices of system (12). The squares indicate saddle points, and the plus signs inside the regions bounded by separatrices indicate centers.

✓ Describe the motions that correspond to trajectories inside the regions bounded by separatrices. Repeat with the region above the separatrices. Can a separatrix be both stable and unstable when considered as an entire trajectory?

Example 10: Add in viscous damping and the picture completely changes: Figure 10.5 shows the stable separatrices at the saddle points for the system

$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta - y \end{aligned} \tag{13}$$

The equilibrium points at $\theta = 2n\pi,\ \ y = 0$ are no longer centers, but attracting spiral points (the solid dots). The *basin of attraction* of each sink (i.e., the points on the trajectories attracted to the sink) is bounded by four stable saddle separatrices.

✓ With a fine-tipped pen, draw the *unstable* separatrices at each saddle in Figure 10.5.

That's all we have to say about the motions of a constant-length pendulum for now. More (much more) is discussed in Chapter 12, where we add a driving term $F(t)$ to the pendulum equations.

◆ Pumping a Swing

Recall that in an *autonomous* differential equation, the time variable t does not appear explicitly. The central thing to realize is that *the ODE that models pumping a swing must be autonomous:* a child pumping the swing does not consult a watch when deciding how to lean back or sit up; the movements depend only on the position of the swing and its velocity. The swinger may change pumping strategies, deciding to go higher or to slow down, but the modeling differential equation for any particular strategy should be autonomous, depending on various parameters which describe the pumping strategy.

☞ If you use a different pumping strategy, make up a differential equation of your own!

If you observe a child pumping a swing, or do it yourself, you will find that one strategy is to lean back on the first half of the forward swing and to sit up the rest of the time. If you stand on the seat, the strategy is the same: you crouch during the forward down-swing, and stand up straight the rest of the time. The work is done when you bring yourself back upright during the forward up-swing, either by pulling on the ropes (if sitting), or simply by standing.

The pumping action effectively changes the length of the swing, which complicates the ODE considerably, for two reasons. Newton's second law must be stated differently, as will be shown below, and we must find an appropriate equation to model the changing length.

The question of friction is more subtle. Of course, the air creates a drag, but that is not the most important component of friction. We believe that things are quite different for a swing attached to the axle by something flexible, than if it were attached by rigid rods. Circus acrobats often drive swings right over the top; they always have rigid swings. We believe that a swing attached flexibly to the axle cannot be pumped to go over the top. Suppose the swing were to go beyond the horizontal—then at the end of the forward motion, the swinger would go into free-fall instead of swinging back; the jolt (heard as "ka-chunk") when the rope becomes tight again will drastically slow down the motion. If you get on a swing, you will find that this effect is felt before the amplitude of the swing reaches $\pi/2$; the ropes become loose near the top of the forward swing, and you slow down abruptly when they draw tight again.

☞ Do you agree with this claim about the behavior of flexibly supported swings?

We will now turn this description into a differential equation.

◆ Writing the Equations of Motion for Pumping a Swing

Modeling the pendulum with changing length requires a more careful look at Newton's second law of motion. The equation $F = ma = mq''$ is not correct when the mass is changing (as when you use a leaky bucket as the bob of a pendulum), or when the distance variable is changing with respect to position and velocity (as for the child on the swing). In cases such as this, force is the

rate of change of the *momentum* mq':

$$\text{Force} = (mq')' \tag{14}$$

When the mass and pendulum length are constant, equation (14) indeed reduces to the more familiar $F = ma$.

The analog in rotational mechanics about a pivot, where $q = L\theta$, is that the *torque* equals the rate of change of angular momentum:

$$\text{Torque} = (I\theta')'$$

where I is the moment of inertia (the rotational analog of the mass). If a force $\mathbf{F}$ is applied at a point p, then the torque about the pivot is the vector product $\mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the position vector from the pivot to p. For the undamped and nonlinear pendulum, the gravitational torque can be treated as the scalar $-mgL\sin\theta$, and the moment of inertia is $I = mL^2$. Then Newton's second law becomes

$$(mL^2\theta')' = -mgL\sin\theta \tag{15}$$

When L and m are constant, equation (15) is precisely the ODE of the undamped, nonlinear pendulum. In the case of the child pumping a swing, the mass m remains constant (and can be divided out of the equation), but L is a function of θ and θ', so we must differentiate $L^2\theta'$ in equation (15) using the chain rule to get

$$2L\left(\frac{\partial L}{\partial \theta}\theta' + \frac{\partial L}{\partial \theta'}\theta''\right)\theta' + L^2\theta'' = -gL\sin\theta$$

or, in system form

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{2y^2\partial L/\partial\theta + g\sin\theta}{2y\partial L/\partial y + L} \end{aligned} \tag{16}$$

The person pumping the swing is changing L as a function of θ and y. For the reasons given in Screen 2.3 of Module 10 we will use the following formula for L:

$$L = L_0 + \frac{\Delta L}{\pi^2}\left(\frac{\pi}{2} - \arctan 10\theta\right)\left(\frac{\pi}{2} + \arctan 10y\right) \tag{17}$$

where L_0 is the distance from the axle to the center of gravity of the swinger when upright, and ΔL is the amount by which leaning back (or crouching) increases this distance. Note that

$$\frac{1}{\pi}\left(\frac{\pi}{2} - \arctan 10\theta\right)$$

is a smoothed-out step function: roughly 1 when $\theta < 0$ and 0 when $\theta > 0$. The jump from one value to the other is fairly rapid because of the factor 10; other values would be appropriate if you were to sit (or stand) up more or less suddenly. A similar analysis applies to the second arctan factor in formula (17).

As for friction with the swing, we will use

$$f(\theta, y) = \varepsilon y + \left(\frac{\theta}{1.4}\right)^6 y$$

The first term corresponds to some small viscous air resistance. Admittedly, the second term is quite ad-hoc, but it serves to describe some sort of insurmountable "brick wall," which somewhat suddenly takes effect when $\theta > 3/2 \sim \pi/2$. So it does seem to reflect our qualitative description.

Writing the differential equation as an autonomous system is now routine—an unpleasant routine since we need to differentiate L, which leads to pretty horrific formulas. But with this summary, we have tried to make the structure clear. Now let's get real and insert friction into modeling system (16):

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{2y^2 \partial L/\partial\theta + g\sin\theta + \text{friction term}}{2y\partial L/\partial y + L} \end{aligned} \tag{18}$$

where L is given by formula (17) and

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -10\Delta L \frac{\frac{\pi}{2} + \arctan(10y)}{\pi^2(1+100\theta^2)} \\ \frac{\partial L}{\partial y} &= 10\Delta L \frac{\frac{\pi}{2} - \arctan(10\theta)}{\pi^2(1+100y^2)} \\ \text{friction term} &= \varepsilon y + \left(\frac{\theta}{1.4}\right)^6 y \end{aligned} \tag{19}$$

Example 11: Now set $g = 32$, $L_0 = 4$, $\Delta L = 1$, and $\varepsilon = 0$ (no viscous damping), and use ODE Architect to solve system (18). Figure 10.6 shows that you can pump up a swing from rest at an initial angle of 0.25 radian (about 14°) within a reasonable time, but not from the tiny angle of 0.01 radian. Do you see the approach to a stable, periodic, high-amplitude oscillation? This corresponds to an attracting limit cycle in the θy-plane.

What happens if we put viscous damping back in? See for yourself by going to Screen 2.4 of Module 10 and clicking on several initial points in the θy-screen. You should see *two* limit cycles now:

- a large attracting limit cycle representing an oscillation of amplitude close to $\pi/2$, due to the "brick wall" friction term, and (for $\varepsilon > 0$)
- a small repelling limit cycle near the downward equilibrium, due to friction and viscous air resistance.

In order to get going, the child must move the swing outside the small limit cycle, either by cajoling someone into pushing her, or backing up with her feet on the ground. Once outside the small limit cycle, the pumping will

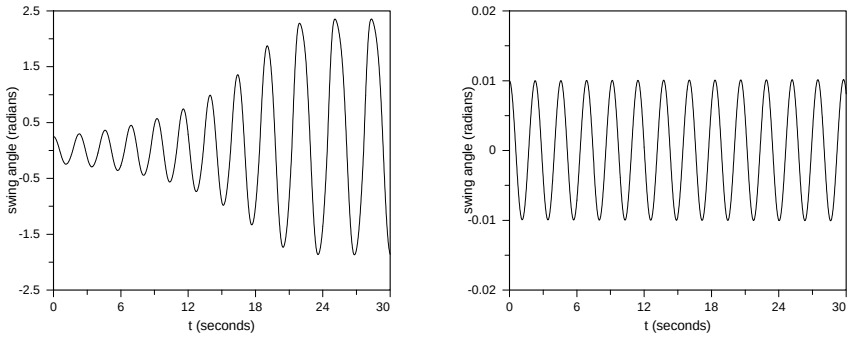


Figure 10.6: Successful pumping (left graph) starts at a moderately high angle ($\theta_0 = 0.25$ radian). If θ_0 is small (e.g., $\theta_0 = 0.01$ rad), then pumping doesn't help much (right graph).

push the trajectory to the attracting limit cycle, where it will stay until the child decides to slow down.

Please note that this structure of the phase plane, with two limit cycles, is necessary in order to account for the observed behavior: the origin must be a sink because of air resistance, and you cannot have an attracting limit cycle surrounding a sink without another limit cycle in between.

✓ Does the system without viscous damping have a small repelling limit cycle?

◆ Geodesics

Geodesics on a surface are curves that minimize length between sufficiently close points on the surface; they may, but need not, minimize length between distant points.

Example 12: Straight lines are geodesics on planes, and they minimize the distance between arbitrary points. Great circles are geodesics on the unit sphere, but they only minimize length between pairs of points if you travel in the right direction. If you travel along a great circle your path will be shortest until you get half-way around the world; but further along, you would have done better to go the other way.

☞ Arc length is denoted by t, rather than s, in Submodule 10.3.

To look for geodesics, we use the fact that parametrization of a curve γ by its arc length s results in traversing a curve at constant speed 1, that is, $|d\gamma/ds|$ is always 1.

On a surface in three-dimensional space, a geodesic γ is a curve for which the vector $d^2\gamma/ds^2$ is perpendicular to the surface at the point $\gamma(s)$. For now, let's assume that all curves are parametrized by arc length, so γ' means $d\gamma/ds$.

If any curve γ (not necessarily a geodesic) on the surface is parametrized at constant speed, we are guaranteed that γ'' is perpendicular to γ', but not necessarily to the surface. To see this, observe that $\gamma' \cdot \gamma' = 1$, where γ' is the velocity vector for the curve γ and the "dot" indicates the *dot* (or *scalar*) *product* of two vectors. Differentiating the dot product equation, we have $\gamma'' \cdot \gamma' + \gamma' \cdot \gamma'' = 0$, so γ'' is perpendicular to γ' (or else is the zero vector).

The statement that γ'' is perpendicular to the *surface* says that γ is going as "straight" as it can in the surface, and that the surface is exerting no force which would make the curve bend away from its path. Such a curve is a geodesic. See the book by Do Carmo for a full explanation of why geodesics defined as above minimize the distance between nearby points.

Example 13: On a sphere, the parallels of latitude $\gamma(s)$ yield vectors $\gamma''(s)$ in the plane of the parallel and perpendicular to the parallel (but not in general perpendicular to the surface), whereas any great circle yields vectors $\gamma''(s)$ pointing toward the center of the sphere and hence perpendicular to both the great circle and to the surface. The great circles are geodesics, but the parallels (except for the equator) are not.

◆ Geodesics on a Surface of Revolution

Suppose that

$$x = f(u), \quad z = g(u)$$

is a parametrization by arc length u of a curve in the xz-plane. One consequence of this parametrization is that $(f'(u))^2 + (g'(u))^2 = 1$. Let's rotate the curve by an angle θ around the z-axis, to find the surface parametrized by

$$P(u, \theta) = \begin{bmatrix} f(u)\cos\theta \\ f(u)\sin\theta \\ g(u) \end{bmatrix}$$

Let's suppose that curves γ on the surface are parametrized by arc length s and, hence, these curves have the parametric equation

$$\gamma(s) = \begin{bmatrix} f(u(s))\cos\theta(s) \\ f(u(s))\sin\theta(s) \\ g(u(s)) \end{bmatrix}$$

and we need to differentiate this twice to find

$$\gamma'(s) = \begin{bmatrix} f'(u(s))u'(s)\cos\theta(s) - f(u(s))\sin\theta(s)\theta'(s) \\ f'(u(s))u'(s)\sin\theta(s) + f(u(s))\cos\theta(s)\theta'(s) \\ g'(u(s))u'(s) \end{bmatrix}$$

$$\gamma''(s) = u'' \begin{bmatrix} f'(u)\cos\theta \\ f'(u)\sin\theta \\ g'(u) \end{bmatrix} + (u')^2 \begin{bmatrix} f''(u)\cos\theta \\ f''(u)\sin\theta \\ g''(u) \end{bmatrix} + 2u'\theta' \begin{bmatrix} -f'(u)\sin\theta \\ f'(u)\cos\theta \\ 0 \end{bmatrix} - (\theta')^2 \begin{bmatrix} f(u)\cos\theta \\ f(u)\sin\theta \\ 0 \end{bmatrix} + \theta'' \begin{bmatrix} -f(u)\sin\theta \\ f(u)\cos\theta \\ 0 \end{bmatrix} \tag{20}$$

This array is pretty terrifying, but the two equations

$$\gamma'' \cdot \frac{\partial P}{\partial u} = 0, \quad \text{and} \quad \gamma'' \cdot \frac{\partial P}{\partial \theta} = 0 \tag{21}$$

which express the fact that γ'' is perpendicular to the surface, give

$$u'' - (\theta')^2 f(u) f'(u) = 0 \quad \text{and} \quad 2u' f(u) f'(u)\theta' + \theta''(f(u))^2 = 0 \tag{22}$$

That γ'' is perpendicular to the surface if formulas (21) hold follows because the vectors $\partial P/\partial u$ and $\partial P/\partial \theta$ span the tangent plane to the surface at the point (θ, u). Formulas (22) follow from formulas (21), from the fact that $(f'(u))^2 + (g'(u))^2 = 1$, and from the formulas

$$\frac{\partial P}{\partial u} = \begin{bmatrix} f'(u)\cos\theta \\ f'(u)\sin\theta \\ g'(u) \end{bmatrix}, \qquad \frac{\partial P}{\partial \theta} = \begin{bmatrix} -f(u)\sin\theta \\ f(u)\cos\theta \\ 0 \end{bmatrix}$$

The quantity

$$M = (f(u))^2\theta' \tag{23}$$

is conserved along a trajectory of system (22) since

$$\frac{d}{ds}[(f(u))^2\theta'] = (f(u))^2\theta'' + 2u'\theta' f' f = 0$$

The integral M behaves like *angular momentum* (see Exploration 10.5 for the central force context that first gave rise to this notion).

Substituting $\theta' = M/(f(u))^2$ into the first ODE of equations (22) gives

$$u'' - \frac{M^2 f'(u)}{(f(u))^3} = 0 \tag{24}$$

Using equations (23) and (24), we obtain a system of first-order ODEs for the geodesics on a surface of revolution:

$$\begin{aligned} \theta' &= \frac{M}{(f(u))^2} \\ u' &= w \\ w' &= \frac{M^2 f'(u)}{(f(u))^3} \end{aligned} \tag{25}$$

We recognize that ODE (24) is of the form of ODE (4), so

$$u'' = -\frac{d}{du}\frac{M^2}{2(f(u))^2}$$

and we can analyze this ODE by the phase plane and conservation methods used earlier. Let us now specialize to the torus.

◆ Geodesics on a Torus

Rotate a circle of radius r about a line lying in the plane of the circle to obtain a torus. If R is the distance from the line to the center of the circle, then in the equations on page 186 for a parameterized curve we can set

$$\begin{aligned} x &= f(u) = R + r\cos u \\ z &= g(u) = r\sin u \end{aligned}$$

If we set $r = 1$, then we have $(f')^2 + (g')^2 = 1$ as required in the derivation of the geodesic ODEs. The system of geodesic ODEs (25) becomes

$$\begin{aligned} \theta' &= \frac{M}{(R+\cos u)^2} \\ u' &= w \\ w' &= -\frac{M^2 \sin u}{(R+\cos u)^3} \end{aligned} \tag{26}$$

where M is a constant. The variable u measures the angle up from the outer equator of the torus, and θ measures the angle around the outer equator from some fixed point. Figure 10.7 shows seventeen geodesics through the point

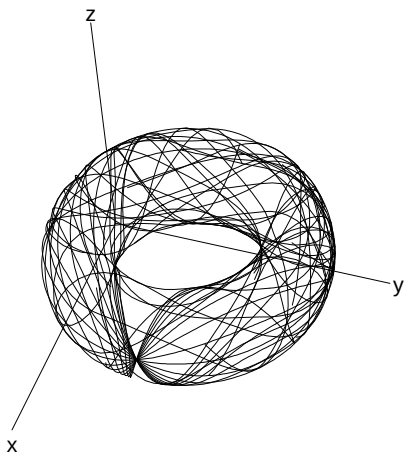


Figure 10.7: Seventeen geodesics through a point on the outer equator.

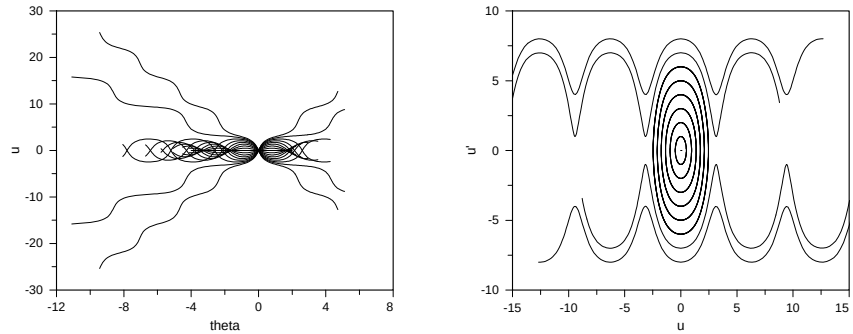


Figure 10.8: The seventeen geodesics of Figure 10.7 drawn in the θu-plane (left) and in the uu'-plane (right).

$\theta_0 = 0,\ u_0 = 0$ with w_0 sweeping from -8 to 8. In Figure 10.7 and subsequent figures we take $R = 3$ and $M = 16$. Figure 10.8 shows the geodesic curves in the θu-plane (left graph) and in the uu'-plane (right). Note the four outlying geodesics that coil around the torus, repeatedly cutting both the outer [$u = 2n\pi$] and the inner [$u = (2n+1)\pi$] equators and periodically going through the hole of the donut. Twelve geodesics oscillate about the geodesic along the outer equator.

Figure 10.9 shows the outer and inner equatorial geodesics (the horizontal lines) in the θu-plane, as well as three curving geodesics starting at $\theta_0 = 0$, $u_0 = 0$. One oscillates about the outer equator six times in one revolution

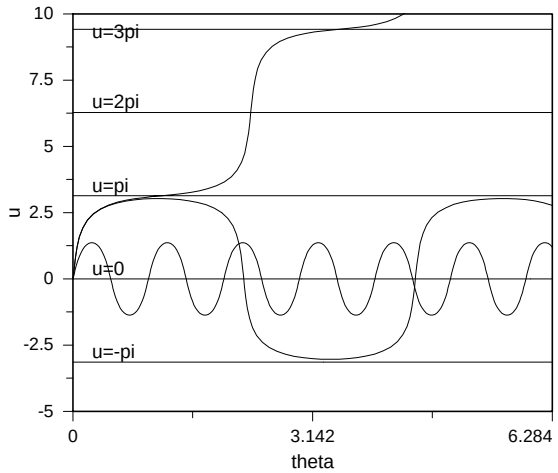


Figure 10.9: Equatorial geodesics (lines), a geodesic that rapidly oscillates around the outer equator, another that oscillates slowly around the outer equator, and a third that slowly coils around the torus.

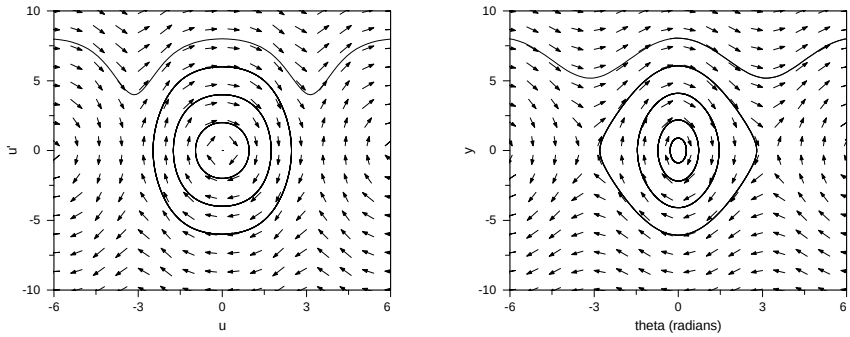


Figure 10.10: The graphs of the toroidal geodesics in the uu'-plane (left) look like the trajectories of an undamped, nonlinear pendulum (right).

(i.e., as θ increases from 0 to 2π). The other two start with values of u_0' that take them up over the torus and near the inner equator. One of these geodesics turns back and slowly oscillates about the outer equator. The other starts with a slightly larger value of u_0', cuts across the inner equator, and slowly coils around the torus. This suggests that the inner equator ($u = \pi$) is a *separatrix* geodesic, dividing the geodesics into those that oscillate about the outer equator from those that coil around the torus. This separatrix is unstable in the sense that if you start a geodesic near the inner equator (say at $\theta_0 = 3.14,\ u_0' = 0$) and solve the system (26), then the geodesic moves away from the separatrix.

☞ Try this and see how hard it is to stay on the inner equator.

Why do we call this geodesic model a "friend of the pendulum"? Take a look at the u' and w' ODEs in system (26). Note that if we delete the term "$\cos u$" from the denominator of the w' equation, then we obtain the system

$$\begin{aligned} u' &= w \\ w' &= -\frac{M^2}{R^3}\sin u \end{aligned} \tag{27}$$

which is precisely the system for an undamped, nonlinear pendulum with $g/L = M^2/R^3$. This fact suggests that geodesics of (26) plotted in the uu'-plane will look like trajectories of the pendulum system (27). Figure 10.10 compares the two sets of trajectories and shows how much alike they are. This illustrates a general principle (which, like most principles, has its exceptions): *If two systems of ODEs resemble one another, so will their trajectories.*

References

Arnold, V.I., *Ordinary Differential Equations* (1973: M.I.T.)

Note: Arnold's book is the classical text. Much of the considerable literature on modeling swings has been influenced by his description, which uses a nonautonomous length function, $L(t)$, instead of $L(\theta, \theta')$. In Sections 27.1 and 27.6, Arnold shows that if the child makes pumping motions at the right frequency at the bottom position of the swing, the motion eventually destabilizes, and the swing will start swinging without any

push. This is correct (we have seen it done, though it requires pumping at an uncomfortably high frequency), but appears to us to be completely unrelated to how swings are actually pumped. Since the usual swing-pumping is done without reference to a clock, a proper model must certainly give an autonomous equation.

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Halliday, D., Resnick, R., *Physics*, (v. I and II), 3rd ed. (1977-78: John Wiley & Sons, Inc.)

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Stewart, I., *Does God Play Dice?* The Mathematics of Chaos (1989: Blackwell Publishers). See Ch. 5, "The One Way Pendulum."

Werkes, S., Rand, R., Ruina, A., "Modeling the Pumping of a Swing" in *C·ODE·E*, Winter/Spring 1997, http://www.math.hmc.edu/codee.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 10.1. Explorations of Basic Pendulum Equation

1. If the nonlinear pendulum ODE (3c) is approximated by the linear ODE (3a), how closely do the trajectories and the component curves of the two ODEs match up? Screen 1.2 in Module 10 will be a big help here.

2. What would motions of the system, $x' = y$, $y' = -V(x)$, look like under different potential functions, such as $V(x) = x^4 - x^2$? What happens if a viscous damping term $-y$ is added to the second ODE of the system? Use graphical images like those in Figures 10.2 and 10.3 to guide your analysis. Use ODE Architect to draw trajectories in the xy-plane for both the undamped and damped case. Identify the equilibrium points in each case as saddles, centers, sinks, or sources. Plot the stable and the unstable saddle separatrices (if there are any) and identify the basin of attraction of each sink. [*Suggestion:* Use the Equilibrium feature of ODE Architect to locate the equilibrium points, calculate Jacobian matrices, find eigenvalues and eigenvectors, and so help to determine the nature of those points.]

3. Find all solutions of the undamped and linearized pendulum ODE,

$$\theta'' + (g/L)\theta = 0$$

Show that all solutions are periodic of period $2\pi\sqrt{L/g}$. Are all solutions of the corresponding nonlinear pendulum ODE, $\theta'' + (g/L)\sin\theta = 0$, periodic? If the latter ODE has periodic solutions, compare the periods with those of solutions of the linearized ODE that have the same initial conditions.

4. Use the sweep and the animate features of ODE Architect to make "movies" of the solution curves and the trajectories of the nonlinear pendulum ODE, $\theta'' + b\theta + \sin\theta = 0$, where $\theta_0 = 0$, $\theta_0' = 10$, and b is a nonnegative parameter. Interpret what you see in terms of the motions of a pendulum. In this regard, you may want to use the model-based pendulum animation feature of ODE Architect.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 10.2. Physical Variations for Child on a Swing

1. Module 10 and the text of this chapter describe a swing-pumping strategy where the swinger changes position only on the first half of the forward swing (i.e., where θ is negative but θ' is positive). Is this the strategy you would use to pump a swing? Try pumping a swing and then describe in words your most successful strategy.

2. Rebuild the model for the length function $L(\theta, \theta')$ of the "swing pendulum" to model your own pumping scenario. [*Suggestion:* Change the arguments of the arctan function used in Module 10 and the text of this chapter.] Use the ODE Architect to solve your set of ODEs. From plots of $t\theta$-curves and of $\theta\theta'$-trajectories, what do you conclude about the success of your modeling and your pumping strategy?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 10.3. Bifurcations

In these problems you will study the bifurcations in the swing-pumping model of Module 10 and this chapter as the viscous damping constant ε or the incremental pendulum length ΔL is changed.

1. There is a Hopf bifurcation for the small-amplitude repelling limit cycle at $\varepsilon = 0$ for the swing-pumping system (18) and (19). Plot lots of trajectories near the origin $\theta = 0, \quad y = 0$ for values of ε above and below $\varepsilon = 0$ and describe what you see. What does the ODE Architect equilibrium feature tell you about the nature of the equilibrium point at the origin if $\varepsilon < 0$? If $\varepsilon = 0$? If $\varepsilon > 0$?

2. Now sweep ΔL through a series of values and watch what happens to the large-amplitude attracting limit cycle. At a certain value of ΔL you will see a sudden change (called a *homoclinic, saddle-connection bifurcation*). What is this value of ΔL? Plot lots of trajectories for various values of ΔL and describe what you see.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 10.4. Geodesics on a Torus

The basic initial value problem for a geodesic starting on the outer equator of a torus is

$$\begin{aligned} \theta' &= \frac{M}{(R+\cos u)^2} \\ u'' &= -\frac{M^2 \sin u}{(R+\cos u)^3} \\ u(0) = 0, &\quad u'(0) = \alpha, \quad \theta(0) = 0 \end{aligned} \tag{28}$$

where M is a constant.

1. Make up your own "pretty pictures" of geodesic sprays on the surface of the torus by varying $u'(0)$. Explain what each geodesic is doing on the torus. If two geodesics through $u_0 = 0,\ \theta_0 = 0$ intersect at another point, which provides the shortest path between two points? Is every "meridian", $\theta =$ const., a geodesic? Is every "parallel", $u =$ const., a geodesic?

2. Repeat Problem 1 at other initial points on the torus, including a point on the inner equator.

3. Explore different values for R (between 2 and 5) for the torus—what does it mean for the solutions of the ODEs for the geodesics? To what extent does the ugly denominator in the ODEs mess up the similarity to the nonlinear pendulum equation?

Answer by discussing effects on θu-phase portraits.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 10.5. The Central Force and Kepler's Laws

An object at position $\mathbf{r}(t)$ (relative to a fixed coordinate frame) is moving under a central force if the force points toward or away from the origin, with a magnitude which depends only on the distance r from the origin. This is modeled by the differential equation $\mathbf{r}'' = f(r)\,\mathbf{r}$, where we will take $\mathbf{r}(t)$ to be a vector moving in a fixed plane.

Example 14: (*Newton's law of gravitation*) This, as applied to a planet and the sun, is perhaps the most famous differential equation of all of science. Newton's law describes the position of the planet by the differential equation

$$\mathbf{r}'' = -\frac{AG}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the vector from the center of gravity of the two bodies (located, for all practical purposes, at the sun) to the planet, G is the universal gravitational constant, and $A = M^3/(m+M)^2$, where m is the mass of the planet (so for all practical purposes, A is the mass of the sun).

1. Newton's law of gravitation is often called the "inverse square law," not the "inverse cube law." Explain.

2. The way to analyze a central force problem is to write it in polar coordinates, where

$$\mathbf{r} = r[\cos\theta, \sin\theta]$$
$$\mathbf{r}' = r'[\cos\theta, \sin\theta] + r\theta'[-\sin\theta, \cos\theta]$$
$$\mathbf{r}'' = (r'' - r(\theta')^2)[\cos\theta, \sin\theta] + (2r'\theta' + r\theta'')[-\sin\theta, \cos\theta]$$

☞ Another way to write the vector $\mathbf{r}$ is $\mathbf{r} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the positive x- and y-axes, respectively.

Show that the central force equation $\mathbf{r}'' = f(r)\mathbf{r}$ yields

$$2r'\theta' + r\theta'' = 0 \tag{29}$$

and

$$r'' - r(\theta')^2 = rf(r) \tag{30}$$

3. Show that the quantity $M = r^2\theta'$ is constant as a function of time during a motion in a central force system, using equation (29).

The quantity M (now called the angular momentum of the motion) was singled out centuries ago as a quantity of interest precisely because of the derivation above. You should see that the constancy of M is equivalent to *Kepler's second law*: the vector $\mathbf{r}$ sweeps out equal areas in equal times.

4. Substitute $\theta' = M/r^2$ into equation (30) and show that, for each value of the particular central force $f(r)$ and each angular momentum M, the resulting differential equation is of the expected form.

5. Specialize to Newton's inverse square law with $k = AG$ and show that the resulting system becomes

$$r'' = -\frac{k}{r^2} + \frac{M^2}{r^3}$$

or the system

$$r' = y$$
$$y' = -\frac{k}{r^2} + \frac{M^2}{r^3}$$

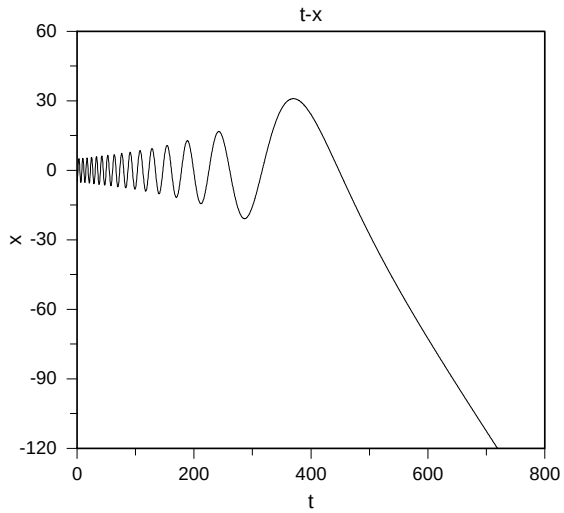
Make a drawing of the phase plane for this system, and analyze this drawing using the conserved quantity K, where

$$K(r, y) = \frac{y^2}{2} + \frac{k}{r} - \frac{M^2}{2r^2}$$

K is evidently defined only for $r > 0$, and K has a unique minimum, so the level curves of K are simple closed curves for $K \ll 0$, corresponding to the elliptic orbits of *Kepler's first law*, an unbounded level curve when $K = 0$, corresponding to a parabolic orbit, and other unbounded curves for $K > 0$ which correspond to hyperbolic orbits. (For discussion of these three cases and their relation to conic sections, see Hubbard and West, Part II, Section 6.7 pp. 43–47.)

CHAPTER 11

Applications of Series Solutions



An aging spring stretches.

Overview Many phenomena, especially those explained by Newton's second law, can be modeled by second-order linear ODEs with variable coefficients, for example:

1. Robot arms, which are modeled by a spring-mass equation with a time-varying damping coefficient; and
2. Aging springs, which are modeled by a spring-mass equation with a time-varying spring constant.

These two applications illustrate very different ways in which series solutions can be used to solve linear ODEs with nonconstant coefficients.

Key words Power series; series solutions; recurrence formula; ordinary point; singular point; regular singular point; Bessel's equations; Bessel functions; aging spring; lengthening pendulum

See also Chapter 4 for second-order linear ODEs with constant coefficients (i.e., without the time-dependence).

◆ Infinite Series

Certain second-order linear ODEs with nonconstant coefficients have been studied extensively, so their properties are well-known. We will look at some of these ODEs in the chapter.

If the general linear homogeneous (undriven) second-order ODE

$$x'' + p(t)x' + q(t)x = 0 \tag{1}$$

has coefficients p and q that are not both constants, the methods of Chapter 4 don't work. However, sometimes we can write a solution $x(t)$ as a power series:

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{2}$$

where we use ODE (1) to determine the coefficients a_n. Much useful information can be deduced about an ODE when its solutions can be expressed as power series.

☞ Look in your calculus book for Taylor series. The term "analytic" is frequently used for functions with convergent Taylor series. We assume that $p(t)$ and $q(t)$ are analytic on a common interval about t_0.

If a function $x(t)$ has a convergent Taylor series $x(t) = \sum a_n (t - t_0)^n$ in some interval about $t = t_0$, then $x(t)$ is said to be *analytic* at t_0. Since all derivatives of analytic functions exist, the derivatives x' and x'' of x can be obtained by differentiating that series term by term, producing series with the same radius of convergence as the series for x. If we substitute these series into ODE (1), we can determine the coefficients a_n. To begin with, a_0 and a_1 are equal to the initial values $x(t_0)$ and $x'(t_0)$, respectively.

✓ "Check" your understanding by evaluating the series (2) at $t = t_0$ to show that $a_0 = x(t_0)$. Now differentiate series (2) term by term to obtain a series for $x'(t)$; evaluate this series at t_0 to find that $a_1 = x'(t_0)$. Does a_2 equal $x''(t_0)$?

◆ Recurrence Formulas

A *recurrence formula* for the coefficients a_n is a formula that defines each a_n in terms of the coefficients $a_0, a_1, \ldots, a_{n-1}$. To find such a formula, we have to express each of the terms in ODE (1) [i.e., x'', $p(t)x'$, and $q(t)x$] as power series about $t = t_0$, which is the point at which the initial conditions are given. Then we combine these series to obtain a single power series which, according to ODE (1), must sum to zero for all t near t_0. This implies that the coefficient of each power of $t - t_0$ must be equal to zero, which yields an equation for each a_n in terms of the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$.

Example: Finding a recurrence formula

☞ We chose a first-order ODE for simplicity.

Let's solve the first-order IVP $x' + tx = 0,\ x(0) = 1$. First we write $x(t)$ in the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

where we have chosen $t_0 = 0$. The derivative of $x(t)$ is then

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting this into the given ODE, we get

$$x' + tx = \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

To make the power of t the same in both sums, replace n by $n-2$ in the second sum to obtain

☞ Notice in the second summation that n starts at 2, rather than 0. Do you see why?

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=2}^{\infty} a_{n-2} t^{n-1} = a_1 + \sum_{n=2}^{\infty} [n a_n + a_{n-2}] t^{n-1} = 0$$

The last equality is true if and only if $a_1 = 0$ and, if for every $n \geq 2$, we have that $n a_n + a_{n-2} = 0$. Therefore, the desired recurrence formula is

$$a_n = \frac{-a_{n-2}}{n}, \quad n = 2, 3, \ldots \tag{3}$$

Since $a_1 = 0$, formula (3) shows that the coefficients $a_3, a_5, \ldots, a_{2k+1}, \ldots$ must all be zero; and $a_2 = -a_0/2,\ a_4 = -a_2/4 = a_0/(2 \cdot 4), \ldots$. With a little algebra you can show that the series for $x(t)$ is

$$x(t) = a_0 - \frac{a_0}{2} t^2 + \frac{a_0}{2 \cdot 4} t^4 - \frac{a_0}{2 \cdot 4 \cdot 6} t^6 + \cdots$$

which can be simplified to

$$x(t) = a_0 \left(1 - \frac{t^2}{2} + \frac{1}{2!} \left(\frac{t^2}{2} \right)^2 - \frac{1}{3!} \left(\frac{t^2}{2} \right)^3 + \cdots \right)$$

If the initial condition $a_0 = x(0) = 1$ is used, this becomes the Taylor series for $e^{-t^2/2}$ about $t_0 = 0$. Although the series solution to the IVP, $x' + x = 0$, $x(0) = 1$, can be written in the form of a familiar function, for most IVPs that is rarely possible and usually the only form we can obtain is the series form of the solution.

✓ Check that $x(t) = e^{-t^2/2}$ is a solution of the IVP $x' + tx = 0,\ x(0) = 1$.

◆ Ordinary Points

If $p(t)$ and $q(t)$ are both analytic at t_0, then t_0 is called an *ordinary point* for the differential equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$. At an ordinary point, the method illustrated in the preceding example always produces solutions written in series form. The following theorem states this more precisely.

Ordinary Points Theorem. If t_0 is an ordinary point of the second-order differential equation

$$x'' + p(t)x' + q(t)x = 0 \tag{4}$$

that is, if $p(t)$ and $q(t)$ are both analytic at t_0, then the general solution of ODE (4) is given by the series

$$x(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0x_1(t) + a_1x_2(t) \tag{5}$$

where a_0 and a_1 are arbitrary and, for each $n \geq 2$, a_n can be written in terms of a_0 and a_1. When this is done, we get the right-hand term in formula (5), where $x_1(t)$ and $x_2(t)$ are linearly independent solutions of ODE (4) that are analytic at t_0. Further, the radius of convergence for each of the series solutions $x_1(t)$ and $x_2(t)$ is at least as large as the smaller of the two radii of convergence for the series for $p(t)$ and $q(t)$.

One goal of Module 11 is to give you a feeling for the interplay between infinite series and the functions they represent. In the first submodule, the position $x(t)$ of a robot arm is modeled by the second-order linear ODE

☞ Note that $p(t) = Ct$ and $q(t) = k$ are analytic for all t.

$$x'' + Ctx' + kx = 0 \tag{6}$$

where C and k are positive constants. Using the methods of the earlier example, we can derive a series solution (with $t_0 = 0$)

$$x(t) = 1 - \frac{kt^2}{2!} + \frac{k(2C + k)t^4}{4!} - \frac{k(2C + k)(4C + k)t^6}{6!} + \cdots \tag{7}$$

that satisfies $x(0) = 1$, $x'(0) = 0$. We then have to to determine how quickly the arm can be driven from the position $x = 1$ to $x = 0.005$ without letting x go below zero. The value of k is fixed at 9, so that only C is free to vary. When $C = k$, it turns out that series (7) is the Taylor series for $e^{-kt^2/2}$ about $t = 0$. It can then be demonstrated numerically, using ODE Architect, that $C = 9$ produces a solution that stays positive and is an optimal solution in the sense of requiring the least time for the value of x to drop from 1 to 0.005.

☞ Historically, new functions in engineering, science, and mathematics have often been introduced in the form of series solutions of ODEs.

In the majority of cases, however, it is *not* possible to recognize the series solution as one of the standard functions of calculus. Then the only way to approximate $x(t)$ at a given value of t is by summing a large number of terms

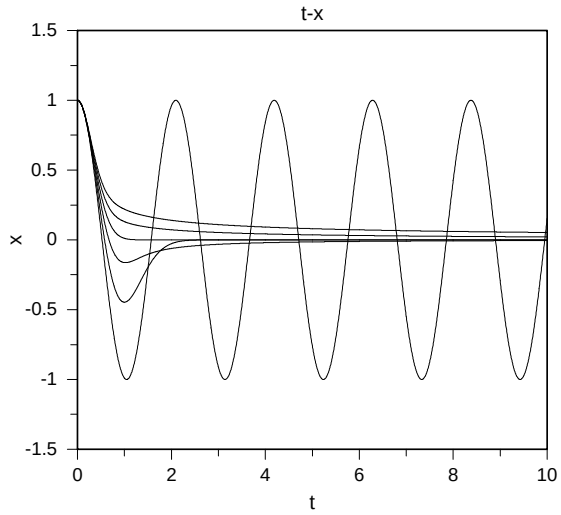


Figure 11.1: Solutions of ODE (6) for $k = 9$, $x(0) = 1$, $x'(0) = 0$, and $C = 0$, 3, 6, 9, 12, 15. Which is the $C = 15$ curve?.

in the series, or by using a numerical solver to solve the corresponding IVP. ODE Architect was used to graph solutions of ODE (6) for several values of C (Figure 11.1).

What if t_0 is *not* an ordinary point for the ODE, $x'' + p(t)x' + q(t)x = 0$, that is, what if $p(t)$ or $q(t)$ is not analytic at t_0? For example, in the ODE $x'' + x/(t-1) = 0$, $q(t)$ is not analytic at $t_0 = 1$. Such a point is said to be a *singular point* of the ODE. Thus, $t_0 = 1$ is a singular point for the ODE $x'' + x/(t-1) = 0$. Next we show how to deal with ODEs with certain kinds of singular points.

✓ Is $t = 0$ an ordinary point or a singular point of $x'' + t^2 x = 0$? What about $x'' + (\sin t)x = 0$ and $x'' + x/t = 0$?

◆ Regular Singular Points

A singular point of the ODE $x''(t) + p(t)x'(t) + q(t)x(t) = 0$ is a *regular singular point* if both $(t - t_0)\,p(t)$ and $(t - t_0)^2 q(t)$ are analytic at t_0. In this case we'll have to modify the method to find a series solution to the ODE.

✓ Is $t = 0$ a regular singular point of $x'' + x'/t + x = 0$? What about $x'' + x' + x/t^2 = 0$ and $x'' + x' + x/t^3 = 0$?

Since $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are analytic at t_0, they have power series expansions centered at t_0:

$$(t-t_0)p(t) = P_0 + P_1(t-t_0) + P_2(t-t_0)^2 + \cdots$$
$$(t-t_0)^2q(t) = Q_0 + Q_1(t-t_0) + Q_2(t-t_0)^2 + \cdots$$

As we shall soon see, the constant coefficients, P_0 and Q_0, in these two series are particularly important. The roots of the quadratic equation (called the *indicial equation*)

$$r(r-1) + P_0 r + Q_0 = 0 \tag{8}$$

☞ Assume that the roots of the indicial equation are real numbers.

are used in solution formula (9) below.

A theorem due to Frobenius tells us how to modify our original method of constructing power series solutions so that we can obtain series solutions near regular singular points.

Frobenius' Theorem. If t_0 is a regular singular point of the second-order differential equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$, then there is a series solution at t_0 of the form

$$x_1(t) = (t-t_0)^{r_1}\sum_{n=0}^{\infty} a_n(t-t_0)^n = \sum_{n=0}^{\infty} a_n(t-t_0)^{n+r_1} \tag{9}$$

where r_1 is the larger of the two roots r_1 and r_2 of the indicial equation.

☞ The second summation is called the *Frobenius series*.

☞ Consult the references for detailed instructions on how to find the coefficients a_n.

The coefficients a_n can be determined in the same way as in the earlier example: differentiate twice, substitute the series for qx_1, px_1', and x_1'' into the given differential equation, and then find a recurrence formula.

Here are a few things to keep in mind when finding a Frobenius series.

1. The roots of the indicial equation may not be integers, in which case the series representation of the solution would not be a power series, but is still a valid series.
2. If $r_1 - r_2$ is not an integer, then the smaller root r_2 of the indicial equation generates a second solution of the form
$$x_2(t) = (t-t_0)^{r_2}\sum_{n=0}^{\infty} b_n(t-t_0)^n$$
which is linearly independent of the first solution $x_1(t)$.
3. When $r_1 - r_2$ is an integer, a second solution of the form
$$x_2(t) = Cx_1(t)\ln(t-t_0) + \sum_{n=0}^{\infty} b_n(t-t_0)^{n+r_2}$$
exists, where the values of the coefficients b_n are determined by finding a recurrence formula, and C is a constant. The solution $x_2(t)$ is linearly independent of $x_1(t)$.

◆ Bessel Functions

For any nonnegative constant p, the differential equation

$$t^2x''(t) + tx'(t) + (t^2 - p^2)x(t) = 0$$

is known as *Bessel's equation of order* p, and its solutions are the *Bessel functions of order* p. In normalized form, Bessel's equation becomes

☞ If t is very large, Bessel's equation looks like the harmonic oscillator equation, $x'' + x = 0$.

$$x''(t) + \frac{1}{t}x'(t) + \left(\frac{t^2 - p^2}{t^2}\right)x(t) = 0$$

From this we can see that $tp(t) = 1$ and $t^2q(t) = t^2 - p^2$, so that $tp(t)$ and $t^2q(t)$ are analytic at $t_0 = 0$. Therefore zero is a regular singular point and, using equation (8), we find that the indicial equation (with $P_0 = 1$, $Q_0 = -p^2$) is

☞ The roots of the indicial equation are p and $-p$.

$$r(r-1) + r - p^2 = r^2 - p^2 = 0$$

Application of Frobenius' Theorem yields a solution J_p given by the formula

☞ Consult the references for the derivation of the formula for $J_p(t)$.

$$J_p(t) = t^p \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!(p+1)(p+2)\cdots(p+n)}t^{2n}$$

The function $J_p(t)$ is called the *Bessel function of order* p *of the first kind*. The series converges and is bounded for all t. If p is not an integer, it can be shown that a second solution of Bessel's equation is $J_{-p}(t)$ and that the general solution of Bessel's equation is a linear combination of $J_p(t)$ and $J_{-p}(t)$.

For the special case $p = 0$, we get the function $J_0(t)$ used in the aging spring model in the second submodule of Module 11:

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

Note that even though $t = 0$ is a singular point of the Bessel equation of order zero, the value of $J_0(0)$ is finite [$J_0(0) = 1$]. See Figure 11.2.

✓ Check that $J_0(t)$ is a solution of Bessel's equation of order 0.

When p is an integer we have to work much harder to get a second solution that is linearly independent of $J_p(t)$. The result is a function $Y_p(t)$ called the *Bessel function of order* p *of the second kind*. The general formula for $Y_p(t)$ is extremely complicated. We show only the special case $Y_0(t)$, used in the aging spring model:

$$Y_0(t) = \frac{2}{\pi}\left[\left(\gamma + \ln\left|\frac{t}{2}\right|\right)J_0(t) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_n}{(n!)^2}\left(\frac{t}{2}\right)^{2n}\right]$$

☞ Actually γ is an unending decimal and non-repeating (or so most mathematicians believe), and 0.5772 gives the first four digits.

where $H_n = 1 + (1/2) + (1/3) + \cdots + (1/n)$ and γ is *Euler's constant*: $\gamma = \lim_{n\to\infty}(H_n - \ln n) \approx 0.5772$.

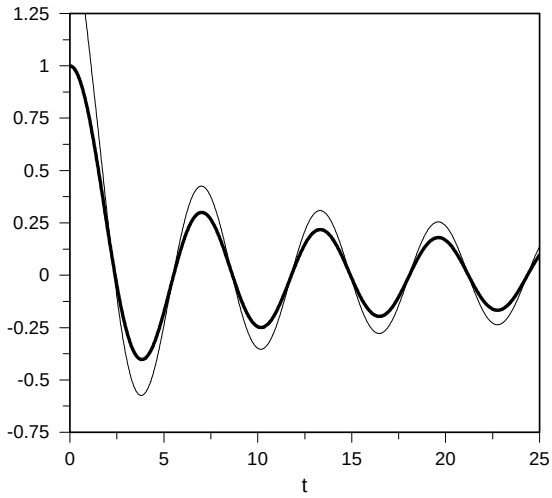


Figure 11.2: The graph of $J_0(t)$ [dark] looks like the graph of the decaying sinusoid $\sqrt{2/\pi t}\cos(t-\pi/4)$ [light].

The general solution of Bessel's equation of integer order p is

$$x(t) = c_1 J_p(t) + c_2 Y_p(t) \tag{10}$$

for arbitrary constants c_1 and c_2. An important thing to note here is that the value of $Y_p(t)$ at $t = 0$ *does* reflect the singularity at $t = 0$; in fact, $Y_p(t) \to -\infty$ as $t \to 0^+$, so that a solution having the form given in equation (10) is bounded only if $c_2 = 0$.

Bessel functions appear frequently in applications involving cylindrical geometry and have been extensively studied. In fact, except for the functions you studied in calculus, Bessel functions are the most widely used functions in science and engineering.

◆ Transforming Bessel's Equation to the Aging Spring Equation

☞ See "Aging Springs" in Module 11.

Bessel's equation of order zero can be transformed into the aging spring equation $x'' + e^{-at}x = 0$. To do this, we take

$$t = (2/a)\ln(2/as) \tag{11}$$

where the new independent variable s is assumed to be positive. Then we can use the chain rule to find the first two derivatives of the displacement x of the

aging spring with respect to s:

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{dx}{dt}\left(-\frac{2}{as}\right)$$

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left[\frac{dx}{dt}\right]\left(-\frac{2}{as}\right) + \frac{dx}{dt}\frac{d}{ds}\left(-\frac{2}{as}\right)$$

$$= \frac{d^2x}{dt^2}\frac{dt}{ds}\left(-\frac{2}{as}\right) + \frac{dx}{dt}\frac{2}{as^2}$$

$$= \frac{d^2x}{dt^2}\left(-\frac{2}{as}\right)\left(-\frac{2}{as}\right) + \frac{dx}{dt}\frac{2}{as^2}$$

$$= \frac{d^2x}{dt^2}\frac{4}{(as)^2} + \frac{dx}{dt}\frac{2}{as^2}$$

☞ We use w in place of x in the aging spring section of Module 11.

Bessel's equation of order $p = 0$ is given by:

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0$$

and when we substitute in the derivatives we just found, we obtain

$$s^2\left(\frac{d^2x}{dt^2}\frac{4}{(as)^2} + \frac{dx}{dt}\frac{2}{as^2}\right) + s\frac{dx}{dt}\left(-\frac{2}{as}\right) + s^2x = 0$$

Using the fact that

$$s = (2/a)e^{-at/2} \tag{12}$$

(found by solving equation (11) for s) in the last term, when we simplify this monster equation it collapses down to a nice simple one:

$$\frac{d^2x}{dt^2}\frac{4}{a^2} + \frac{4}{a^2}e^{-at}x = 0$$

Finally, if we divide through by $4/a^2$, we get the aging spring equation, $x'' + e^{-at}x = 0$.

The other way around works as well, that is, a change of variables will convert the aging spring equation to Bessel's equation of order zero. That means that solutions of the aging spring equation can be expressed in terms of Bessel functions. This can be accomplished by using $x = c_1J_0(s) + c_2Y_0(s)$ as the general solution of Bessel's equation of order zero, and then using formula (12) to replace s. Take another look at Experiments 3 and 4 on Screens 2.5 and 2.6 of Module 11. That will give you a graphical sense about the connection between aging springs and a Bessel's equation.

References

Borrelli, R. L., and Coleman, C. S., *Differential Equations: A Modeling Perspective*, (1998: John Wiley & Sons, Inc.)

Boyce, W. E., and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed., (1997: John Wiley & Sons, Inc.)

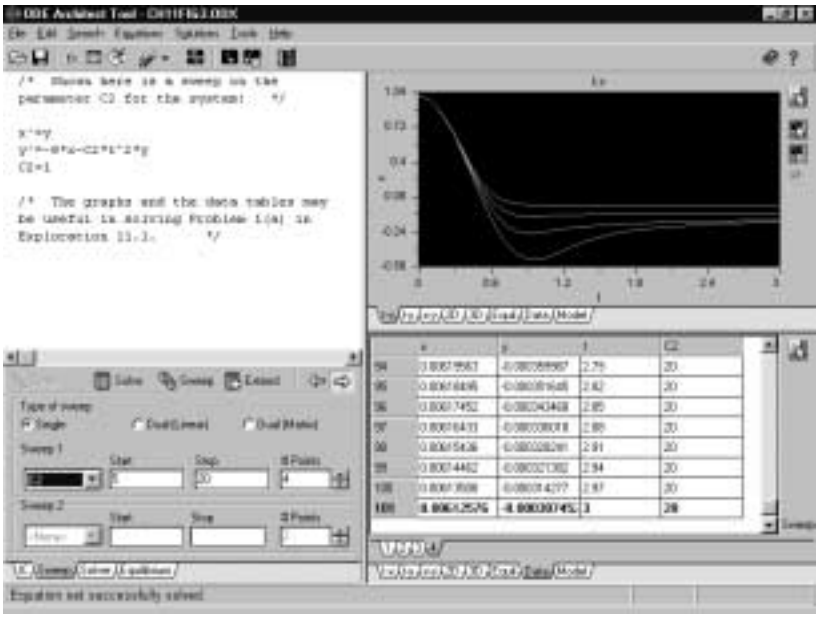


Figure 11.3: Here are some typical graphs for the solution of $x'' + C_2 t^2 x' + 9x = 0$ for various values of C_2. The graphs and the data tables are useful in Problem 1 of Exploration 11.1.

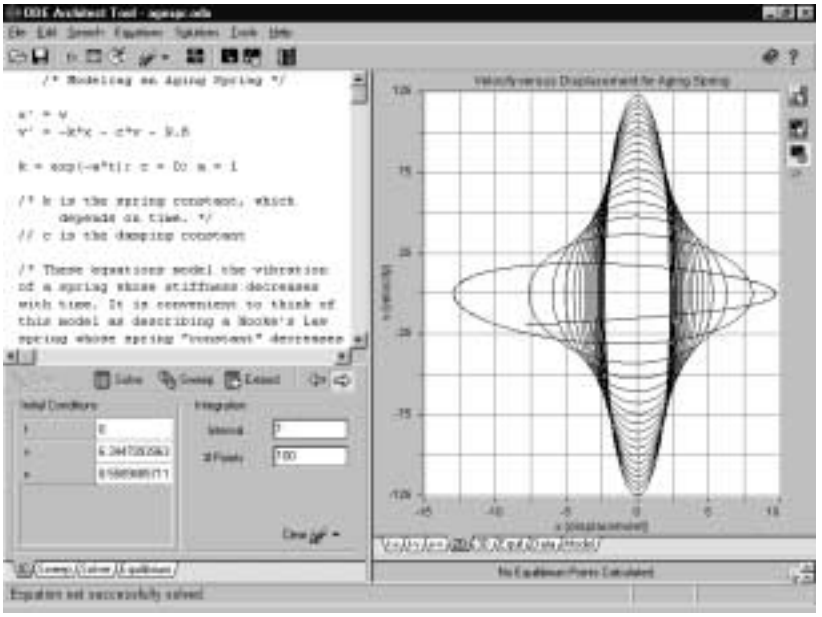


Figure 11.4: Here is a phase-plane portrait for an aging spring ODE, $x'' + e^{-t}x = -9.8$. See Problem 1 in Exploration 11.3.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 11.1. Damping a Robot Arm

In each of the following problems it is assumed that the displacement x of a robot arm satisfies an IVP of the form

$$x'' + b(t)x' + 9x = 0, \quad x(0) = 1, \quad x'(0) = 0$$

An optimal damping function $b(t)$ is one for which the solution $x(t)$ reaches 0.005 in minimal time t^* without ever going below zero.

1. Consider damping functions of the form $b(t) = C_k t^k$. For a positive integer k, let C_k^* be the value of C_k that gives the optimal solution, and denote the corresponding minimal time by t_k^*. In Module 11, Screen 1.4 and TTA 3 on Screen 1.7 you found that the optimal solution for $k = 1$ is $x(t) = e^{-9t^2/2}$, with $C_1^* = 9$ and $t_1^* \approx 1.0897$.

(a) Use ODE Architect to find an approximate optimal solution and values of C_k^* and t_k^* when $k = 2$. [*Suggestion:* Look at Figure 11.3.]

(b) Repeat with $k = 3$.

(c) Compare the optimal damping functions for $k = 1, 2, 3$, in the context of the given physical process.

2. For *quadratic damping*, $b(t) = C_2 t^2$, derive a power series solution $x(t) = \sum_{n=0}^{\infty} a_n t^n$. Show that the recurrence formula for the coefficients is

$$a_{n+2} = \frac{-[9a_n + C_2(n-1)a_{n-1}]}{(n+1)(n+2)}, \quad n \geq 1$$

and $a_2 = -9a_0/2$. Recall that $a_0 = x(0)$ and $a_1 = x'(0)$.

3. Let $P_6(t)$ be the Taylor polynomial $\sum_{n=0}^{6} a_n t^n$, where the a_n are given by the recurrence formula in Problem 2.

(a) Write out $P_6(t)$ with C_2 as a parameter; briefly describe how the graph of $P_6(t)$ changes as C_2 increases.

☞ You will need results from Problem 1(a) here.

(b) Graph the apparently optimal solution from Problem 1(a) over the interval $0 \le t \le t_2^*$ and compare it to the graph of $P_6(t)$ with $C_2 = C_2^*$.

4. If the robot arm is totally undamped, its position at time t is $x(t) = \cos 3t$; therefore the arm cannot reach $x = 0$ for all t, $0 \le t \le \pi/6$. In this situation the undamped arm can't remain above $x = 0$. The optimal damping functions $C_k^* t^k$ found in Problem 1 look more like step functions as the degree k increases. Try to improve the time t^* by using a step function for damping.

Assume the robot arm is allowed to fall without damping until just before it reaches $x = 0$, at which time a constant damping force is applied. This situation can be modeled by defining

$$b(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{\pi}{6} - \varepsilon \\ B_\varepsilon & \text{for } t \ge \frac{\pi}{6} - \varepsilon \end{cases}$$

for $\varepsilon = 0.2$, 0.1, and 0.05. Use ODE Architect to find values of B_ε that give an approximate optimal solution. Include a graph showing your best solution for each ε and give your best value of t^* in each case. What happens to the "optimal" B_ε as $\varepsilon \to 0$?

5. Find a formula for the solution for the situation in Problem 4. The value of ε should be treated as a parameter. Assume that $x(t) = \cos 3t$ for $t < (\pi/6) - \varepsilon$. Then the IVP to be solved is

$$\begin{aligned} &x'' + B_\varepsilon x' + 9x = 0 \\ &x(\pi/6 - \varepsilon) = \cos[3(\tfrac{\pi}{6} - \varepsilon)] = \sin 3\varepsilon \\ &x'(\pi/6 - \varepsilon) = -3\sin[3(\tfrac{\pi}{6} - \varepsilon)] = -3\cos 3\varepsilon \end{aligned}$$

The solution will be of the form $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $r_1 < r_2 < 0$, but the optimal solution requires that $c_2 = 0$. Why? For a fixed ε, find the value of B_ε so that $x(t)$ remains positive and reaches 0.005 in minimum time.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 11.2. Bessel Functions

1. Bessel functions resemble decaying sinusoids. Let's compare the graph of $J_0(t)$ with that of one of these sinusoids.

(a) On the same set of axes, graph the Bessel function $J_0(t)$ and the function

$$\sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi}{4}\right)$$

over the interval $0 \le t \le 10$.

(b) Now graph these same two functions over the interval $0 \le t \le 50$.

(c) Describe what you see.

[*Suggestion:* You can use ODE Architect to plot a good approximation of $J_0(t)$ by solving an IVP involving Bessel's equation in system form:

$$x' = y, \quad y' = -x - y/t, \qquad x(t_0) = 1, \quad x'(t_0) = 0$$

with $t_0 = 0.0001$. Actually, $J_0(0) = 1$ and $J_0'(0) = 0$, but $t_0 = 0$ is a singular point of the system so we must move slightly away from zero. You can plot the decaying sinusoid on the same axes as $J_0(t)$ by entering $a = \sqrt{\frac{2}{\pi t}}\cos(t - \frac{\pi}{4})$ in the same equation window as the IVP, selecting a custom 2D plot, and plotting both a and x vs. t.]

2. Repeat Problem 1 for the functions $Y_0(t)$ and $\sqrt{\frac{2}{\pi t}}\sin\left(t-\frac{\pi}{4}\right)$. To graph a good approximation of $Y_0(t)$, solve the system equivalent of Bessel's equation of order zero (from Problem 1) with initial data $t_0 = 0.89357$, $x(t_0) = 0$, $x'(t_0) = 0.87942$. As in Problem 1, we have to avoid the singularity at $t_0 = 0$, especially here because $Y_0(0) = -\infty$. The given initial data are taken from published values of Bessel functions and their derivatives.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 11.3. Aging Spring Models

1. Check out the Library file "Modeling an Aging Spring" in the "Physical Models" folder (see Figure 11.4). The ODE in the file models the motion of a vertically suspended damped and aging spring that is subject to gravity. Carry out the suggested explorations.

2. Show that

$$x(t) = \sqrt{\frac{t+1}{3}} \sin\left(\frac{\sqrt{3}}{2}\ln(t+1)\right) - \sqrt{t+1}\cos\left(\frac{\sqrt{3}}{2}\ln(t+1)\right)$$

is an analytic solution of the initial value problem

$$x''(t) + \frac{x(t)}{(t+1)^2} = 0, \quad x(0) = -1, \quad x'(0) = 0$$

Explain why this IVP provides another model for the motion of an aging spring that is sliding back and forth (without damping) on a support table. [*Suggestion:* let $s = t+1$, $u(s) = x(t)$. Then $u(s)$ satisifes the Euler equation $s^2u''(s) + u(s) = 0$. See the references on page 211 for solution formulas for Euler ODEs.]

3. Graph the solution $x(t)$ from Problem 2 over the interval $0 \leq t \leq 300$ and compare the graph to the one for the ODE, $x'' + e^{-at}x = 0,\ x = 0,\ x'(0) = 0$, $a = 0.04$ (see Screen 2.6 in Module 11). Repeat for other values of a.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 11.4. The Incredible Lengthening Pendulum

☞ The ODE for a pendulum of varying length is derived in Chapter 10 (see equation (15)).

Suppose that we have an undamped pendulum whose length $L = a + bt$ increases linearly over time. Then the ODE that models the motion of this pendulum is

$$(a + bt)\theta''(t) + 2b\theta'(t) + g\theta(t) = 0 \tag{13}$$

where θ is small enough that $\sin\theta \approx \theta$, the mass of the pendulum bob is 1, and the value of the acceleration due to gravity is $g = 32$.[1]

1. With $a = b = 1$ and initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$, use ODE Architect to solve ODE (13) numerically. What happens to $\theta(t)$ as $t \to +\infty$?

2. Under the same conditions, what happens to the oscillation time of the pendulum as $t \to +\infty$? (The oscillation time is the time between successive maxima of $\theta(t)$.)

[1] See the article "Poe's Pendulum" by Borrelli, Coleman, and Hobson in *Mathematics Magazine*, Vol. 58 (1985) No. 2, pp. 78–83. See also "Child on a Swing" in Module 10.

3. Show that the change of variables

$$s = (2/b)\sqrt{(a+bt)g}, \quad x = \theta\sqrt{a+bt}$$

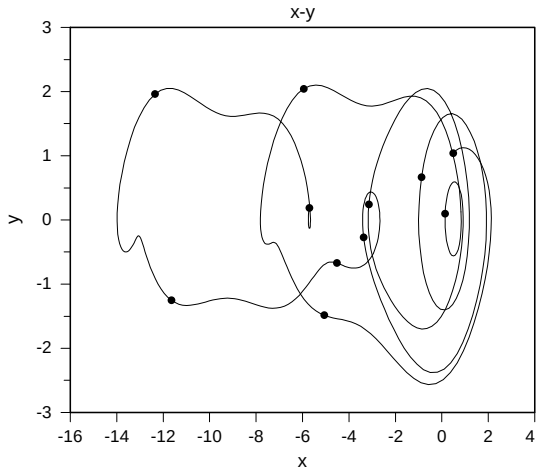
transforms Bessel's equation of order 1

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + (s^2-1)x = 0$$

into ODE (13) for the lengthening pendulum. [*Suggestion:* Take a look at the section "Transforming Bessel's Equation to the Aging Spring Equation" in this chapter to help you get started. Use the change of variables given above to express the solution of the IVP in Problem 1 using Bessel functions.]

12 Chaos and Control

CHAPTER



Poincaré map of a forced damped pendulum superimposed on a trajectory.

Overview In this chapter we'll look at solutions of a forced damped pendulum ODE. In the linear approximation of small oscillations, this ODE becomes the standard constant-coefficient ODE $x'' + cx' + kx = F(t)$, which can be solved explicitly in all cases. Without the linear approximation, the pendulum ODE contains the term $k \sin x$ instead of kx. Now the study becomes much more complicated. We'll focus on the special case of the nonlinear pendulum ODE

$$x'' + cx' + \sin x = A \cos t \tag{1}$$

but our results leave a world of further things to be discovered. We'll show that appropriate initial conditions will send the pendulum on any desired sequence of gyrations, and hint at how to control the chaos by finding such an initial condition.

Key words Forced damped pendulum; sensitivity to initial conditions; chaos; control; Poincaré sections; discrete dynamical systems; Lakes of Wada; control

See also Chapter 10 for background on the pendulum. Chapter 13 for more on discrete dynamical systems and other instances of chaos and sensitivity to initial conditions.

◆ Introduction

☞ See the glossary for one definition of chaos.

How might chaos and control possibly be related? These concepts appear at first to be opposites, but in fact they are two faces of the same coin!

A good way to start discussing this apparent paradox is to think about learning to ski. The beginning skier tries to be as stable as possible, with feet firmly planted far enough apart to give confidence that she or he will not topple over. If you try to ski in such a position, you cannot turn, and the only way to stop, short of running into a tree, is to fall down. Learning to ski is largely a matter of giving up on "stability," bringing your feet together so as to acquire controllability! You need to allow chaos in order to gain control.

Another example of the relation between chaos and control is the early aircraft available at the beginning of World War I, carefully designed for greatest stability. The result was that their course was highly predictable, an easy target for anti-aircraft fire. Very soon the airplane manufacturers started to build in enough instability to allow maneuverability!

◆ Solutions as Functions of Time

The methods of analysis we will give can be used for many other differential equations, such as Duffing's equation

$$x'' + cx' + x - x^3 = A\cos\omega t, \tag{2}$$

or the differential equation

$$x'' + cx' + x - x^2 = A\cos\omega t, \tag{3}$$

which arises when studying the stability of ships. The explorations at the end of this chapter suggest some strategies for these problems.

Let's begin to study ODE (1) with $c = 0.1$:

$$x'' + 0.1x' + \sin x = A\cos t \tag{4}$$

Let's compute some solutions, starting at $t = 0$ with $A = 1$ and various values of $x(0)$ and $x'(0)$, and observe the motion out to $t = 100$, or perhaps longer (see Figure 12.1). We see that most solutions eventually settle down to an oscillation with period 2π (the same period as the driving force). This xt-plot actually shows oscillations about x-values which differ by multiples of 2π.

This settling down of behaviors at various levels is definitely a feature of the parameter values chosen: for the amplitude $A = 2.5$ in ODE (4), for instance, there appears to be no steady-state oscillation at all.

Looking at such pictures is quite frustrating: it is very hard to see the pattern for which initial conditions settle down to which stable oscillations, and which will not settle down at all.

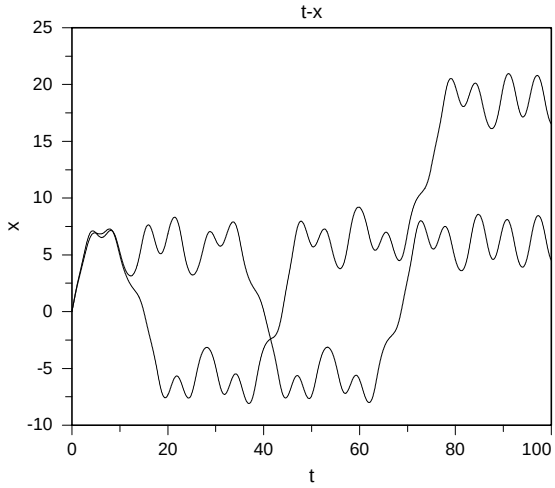


Figure 12.1: Solution curves of ODE (4) with $x(0) = 0,\ x'(0) = 2$, 2.1.

◆ Poincaré Sections

Poincaré found a way to understand and visualize the behavior of our differential equation: he sampled solutions of ODE (4) at multiples of the period 2π of the driving function:

$$0, 2\pi, 4\pi, \ldots, 2k\pi, \ldots$$

This is much like taking pictures with a strobe light.

An equivalent way of saying this is to say that we will iterate[1] the mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$ which takes a point (a, b) in $\mathbb{R}^2$, computes the solution $x(t)$ with $x(0) = a,\ x'(0) = b$, and sets

☞ Note that the clock starts at $t_0 = 0$ when generating Poincaré plots.

$$P(a, b) = (x(2\pi), x'(2\pi)) \tag{5}$$

This mapping P is called a *Poincaré mapping*. If you apply the operator P to (a, b) k times in succession, the result is $P^k(a, b)$ and we see that

$$P^k(a, b) = (x(2k\pi), x'(2k\pi))$$

☞ When the xx'-plane is used to chart the evolution of the points $P^k(a, b), k = 1, 2, \ldots$, it is called the Poincaré plane.

In a sense, the Poincaré section is simply a crutch: every statement about Poincaré sections corresponds to a statement about the original ODE, and vice versa. But this crutch is invaluable since the orbits of a nonautonomous ODE such as ODE (4) often intersect each other and themselves in a hopelessly tangled way.

[1] Chapter 13 discusses iterating maps $f : \mathbb{R} \to \mathbb{R}$; there you will find that the map $f(x) = \lambda x(1 - x)$ is filled with surprises. Before trying to understand the iteration of P, which is quite complicated indeed, the reader should experiment with several easier examples, like linear maps of $\mathbb{R}^2 \to \mathbb{R}^2$.

◆ Periodic Points

A good way to start investigating the Poincaré mapping P (or for that matter, the iteration of any map) is to ask: what periodic points does it have? Setting $x' = y$, a *periodic point* is a point (x, y) in $\mathbb{R}^2$ such that for some integer k we have $P^k(x, y) = (x, y)$. *Fixed* points are periodic points with $k = 1$, and are particularly important.

Periodic points of period k for P are associated with periodic solutions of ODE (4) of period $2k\pi$. In particular, if $x(t)$ is a solution which is periodic of period 2π, then

$$(x(0), x'(0)) = (x(2\pi), x'(2\pi))$$

is a fixed point of P. If you observe this solution with a strobe which flashes every 2π, you will always see the solution in the same place.

◆ The Unforced Pendulum

If there is no forcing term in ODE (4), then we have an autonomous ODE like those treated in Chapter 10.

Example: The ODE

$$x'' + x' + 10 \sin x = 0$$

models a damped pendulum without forcing. A phase plane portrait is shown in Figure 12.2 ($\theta = x$ and $y = x'$ in the figure). Note that the equilibrium points (of the equivalent system) at $x = 2n\pi,\ x' = 0$ are spiral sinks, but the equilibrium points at $x = (2n + 1)\pi,\ x' = 0$ are saddles. Note also that the phase plane is divided into slanting regions, each of which has the property that its points are attracted to the equilibrium point inside the region. These regions are called *basins of attraction*. If a forcing term is supplied, these basins become all tangled up (Figure 12.4 on page 227).

There is a Poincaré mapping P for the unforced damped pendulum, which is fairly easy to understand, and which you should become familiar with before tackling the forced pendulum. In this case, two solutions of ODE (4) with $A = 0$ stand out: the equilibria $x(t) = 0$ and $x(t) = \pi$ for all t. Certainly if the pendulum is at one of these equilibria and you illuminate it with a strobe which flashes every T seconds, where T is a positive number, you will always see the pendulum in the same place. Thus these points are fixed points of the corresponding Poincaré mapping P. In the xx'-plane, the same thing happens at the other equilibrium points, that is, at the points $\ldots, (-2\pi, 0), (0, 0)(2\pi, 0), \ldots$ for the "downward" stable equilibria, and at the points $\ldots, (-3\pi, 0), (-\pi, 0), (\pi, 0), \ldots$ for the unstable equilibria.

The analysis in Module 10 using an integral of motion should convince you that for the unforced damped pendulum, these are the only periodic points: if the pendulum is not at an equilibrium, the value of the integral decreases with time, and the system cannot return to where it was.

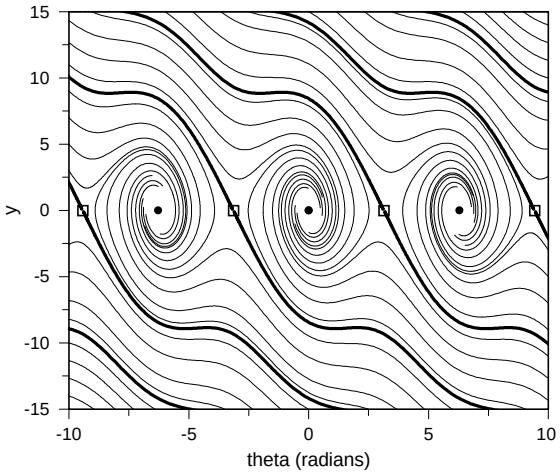


Figure 12.2: Basins of attraction of the stable downward equilibrium positions of the unforced damped pendulum are bounded by separatrices.

If you start the pendulum with both $x(0)$ and $x'(0)$ small, the damping will simply kill off the motion, and the pendulum will be attracted to the downward equilibrium. The point $(0, 0)$ in state space is called a *sink*.

The behavior is more interesting near an unstable equilibrium. Imagine imparting an initial velocity to the bob by kicking it. For a small kick, it will swing back. Now kick it a little harder: it will rise higher, and still swing back. Kick it harder still, and it will make it over the top, and hit you in the back if you aren't careful. Dividing the kicks which don't make it over from those that do is a very special kick, where the pendulum rises forever, more and more slowly, tending to the unstable equilibrium. Thus there are initial conditions which generate solutions that tend to the unstable equilibrium; in the Poincaré plane these solutions form two curves which meet end to end at the fixed point corresponding to the unstable equilibrium. Together they form the *stable separatrix* of the fixed point. There are also curves of initial conditions which come from the unstable equilibrium; together they form the *unstable separatrix* of the equilibrium. See Figure 12.3 (where $y = x'$).

As stated earlier, a good first thing to do when iterating a map is to search for the periodic points; a good second thing to do is to find the periodic points which correspond to unstable equilibria (*saddles*, in the case of the pendulum) and find their separatrices.

For the unforced damped pendulum, the equilibria of the differential equation and the fixed points of any Poincaré map coincide; so, too, do the separatrices of the unstable equilibria (in the phase plane) and the separatrices of the corresponding saddle fixed points in the Poincaré plane. These separatrices separate the trajectories which approach a given sink from the trajectories that approach a different sink.

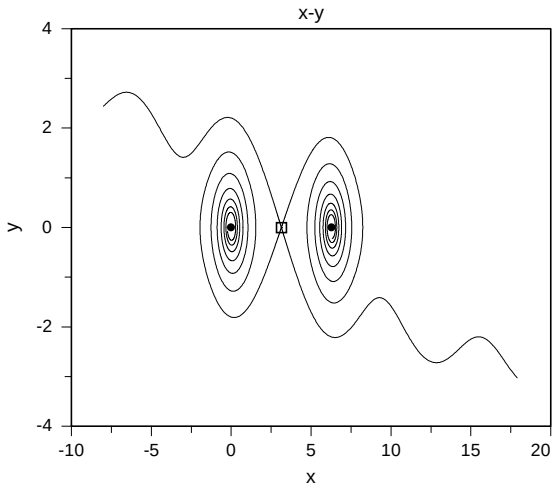


Figure 12.3: Stable and unstable separatrices at a saddle for an unforced damped pendulum. Which are the stable separatrices?

✓ "Check" your understanding by reproducing the plot in Figure 12.3.

◆ The Damped Forced Pendulum

We described above the Poincaré plane for the unforced pendulum. The same description holds for the forced pendulum. A figure showing a Poincaré map for a forced pendulum appears as the chapter cover figure. Thus, in the Poincaré plane, we expect to see a collection of fixed points corresponding to the oscillations to which the pendulum "settles down", and each has a basin: the set of initial conditions which will settle down to it. The basins appear to be extraordinarily tangled and complicated, and they are. The reader should put up the picture of the basins (Screen 2.6 in Module 12), and practice superimposing iterations on the figure, checking that if you start in the blue basin, the entire orbit remains in the blue basin, perhaps taking a complicated path to get near the sink, but making it in the end.

◆ Tangled Basins, the Wada Property

In the tangled basins Screen 3.3 of Module 12, each basin appears to be made of a central piece, and four canals which go off and meander around the plane. The meandering appears to be completely random and chaotic, and the only thing the authors really know about the shapes of the basins of our undamped pendulum is the following fact: The basins have the *Wada property*: every

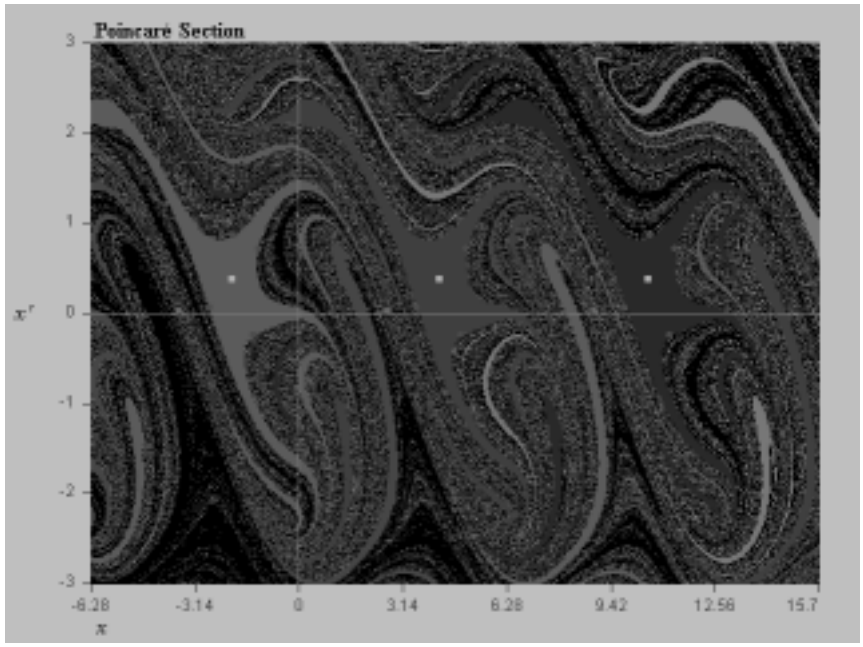


Figure 12.4: Tangled basins for a forced damped pendulum.

point of the boundary[2] of any basin is in the boundary of all the others. Thus if you start at a boundary point of any basin, and perturb the initial condition an arbitrarily small amount, you can land in any of the infinitely many basins.

A careful look at Figure 12.4 should convince you that this stands a good chance of being true: Each region of a canal boundary point includes pieces on many curves. It isn't clear, of course, that there are canals of *all* the basins between any two canals.

It is one thing to think that the Wada property is likely true, and quite another to prove it. It isn't clear how you would prove anything whatsoever about the basins: they do not appear to be amenable to precise study.

To get a grip on these basins, the first step is to understand why they appear to be bounded by smooth curves, and to figure out what these smooth curves are. For each sink (solid white squares in Figure 12.4), there are in fact four periodic points, each of period two, which are saddles, such that for each saddle one of its two unstable separatrices is entirely contained in the corresponding basin.

[2]The boundary ∂U of an open set $U \subset \mathbb{R}^2$ is a point $\mathbf{x} \in \mathbb{R}^2$ which is not in U, but such that there exists a sequence of points $\mathbf{x}_n \in U$ which converges to $\mathbf{x}$. Later we will encounter the notion of *accessible boundary*: the points $\mathbf{x} \in \partial U$ such that there exists a parametrized curve $\gamma : (0, 1] \to U$ such that $\lim_{t\to 0} \gamma(t) = \mathbf{x}$. For simple open sets, the boundary and the accessible boundary coincide, but not for our basins.

The next step is to show that the *accessible boundary* of the basin is made up of the stable separatrices of these saddles. This uses the technique of *basin cells*, as pioneered by Kennedy, Nusse, and Yorke[3]. To see a fleshed-out sketch, see the *C·ODE·E* article referenced at the end of this chapter.

◆ Gaining Control

The statement about the basins having the Wada property is, in some sense, a negative statement, saying that there is maximum possible disorder. Is there some positive statement one can make about the forced pendulum (for these parameter values)? It turns out that there is. The precise statement is as follows.

During one period of the forcing term, say during

$$t \text{ in the interval } I_k = [2k\pi, 2(k+1)\pi]$$

the pendulum will do one of the following four things:

- It will cross the bottom position exactly once moving clockwise (count this possibility as -1);
- It will cross the bottom position exactly once moving counterclockwise (count this possibility as $+1$);
- It will not cross the bottom position at all (count this possibility as 0);
- It will do something else (possibility NA).

Note that most solutions appear to be attracted to sinks, and that the stable oscillation corresponding to a sink crosses the bottom position twice during each I_k, and hence these oscillations (and most oscillations after they have settled down) belong to the NA category.

The essential control statement we can make about the pendulum is the following:

> For any biinfinite sequence $\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots$ of symbols ε_i selected from the set $\{-1, 0, 1\}$, there exist values $x(0), x'(0)$ such that the solution with this initial condition will do ε_k during the time interval I_k.

The chaos game in Module 12 suggests why this might be true; the techniques involved in the proof were originally developed by Smale[4].

[3]Judy Kennedy, Helena Nusse, and James Yorke are mathematicians at the Universities of Delaware, Utrecht, and Maryland, respectively.

[4]Stephen Smale is a contemporary mathematician who was awarded a Fields medal (the mathematical equivalent of a Nobel prize) in the early 1960's. See Devaney in the references at the end of this chapter.

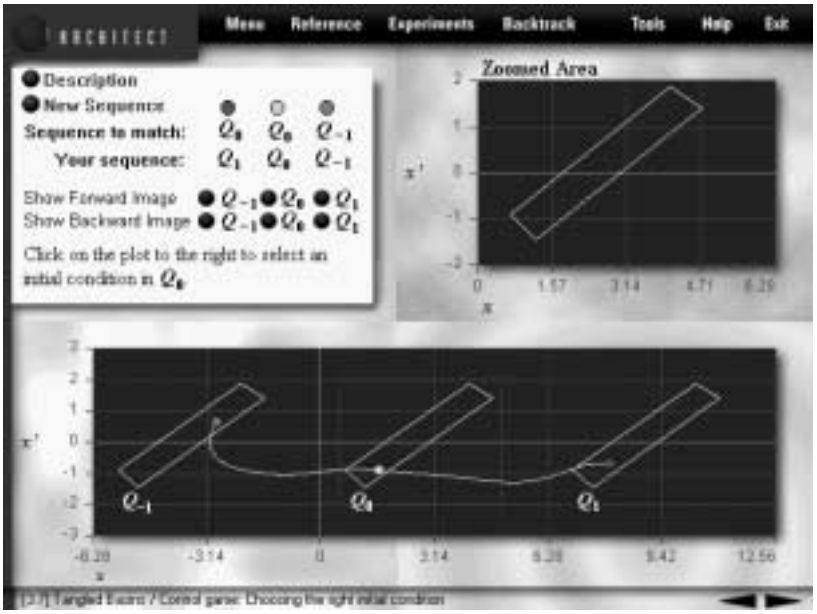


Figure 12.5: Start in quadrilateral Q_0 and reach forward into Q_1 and backward into Q_{-1}.

We start by drawing quadrilaterals Q_k around the kth saddle, long in the unstable direction and short in the stable direction, such that they cross a good part of the tangle. We can now translate our symbols ε_i, which refer to the differential equation, into the Poincaré mapping language.

If at time $t = 2k\pi$ the pendulum is in Q_k and at time $2(k+1)\pi$ it is in $Q_{k+\varepsilon_k}$, then during I_k the pendulum does ε_k. So it is the same thing to require that a trajectory of the pendulum realize a particular symbol sequence, and to require that an orbit of the Poincaré map visit a particular sequence of quadrangles, just so long as successive quadrangles be neighbors or identical.

Draw the forward image of that quadrilateral, and observe that it grows much longer in the unstable direction and shrinks in the stable direction; we will refer to $P(Q_k)$ as the kth *snake*, S_k. The entire proof comes down to understanding how S_k intersects Q_{k-1}, Q_k, and Q_{k+1}.

The thing to be checked is that S_k intersects all three quadrilaterals in elongated regions going from top to bottom, and that the top and bottom of Q_k map to parts of the boundary of S_k which are outside $Q_{k-1} \cup Q_k \cup Q_{k+1}$. See Figure 12.5 for an example of a winning strategy for three adjacent quadrilaterals.

Once you have convinced yourself that this is true, you will see that every symbolic sequence describing a history of the pendulum is realized by an intersection of thinner and thinner nested subquadrangles.

A similar argument shows that a symbolic sequence describing a future of the pendulum corresponds to a sequence of thinner and thinner subquadrangles going from left to right. The details are in the *C·ODE·E* paper by

J.H. Hubbard in the references.

We have shown how to gain control of the motions of the driven pendulum. In particular, if we want the pendulum "robot" to execute a prescribed set of rotations, all we have to do is put it in the right initial state and switch on the driving force. Although everything has been phrased in terms of a pendulum, the approach extends to almost any kind of chaotic motion. Engineers, scientists, and mathematicians are now designing prototypes of chaotic control systems based on these ideas. One of the most intriguing applications uses a chaotic sequence to encode a digital message. The sequence is "added" to the message, and the intended recipient then subtracts the chaos to read the message. Chaos is often an undesirable aspect of physical motions. Devices have recently been built that force a chaotic system to stay focused on a desired response. All of this is new, so we can't say just how the applications will evolve. See the books by Kapitaniak, Nayfeh, and Ott for more on chaotic controls and controlling chaos.

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Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 12.1.

In each problem describe what you see and explain what the figures tell you about the behavior of the pendulum.

1. Choose a value for $c \neq 0.1$, take $A = 1$ in ODE (1), and produce graphs like those in the chapter cover figure and Figure 12.1.

2. Choose a value for $A \neq 1$ and $c = 0.1$ in ODE (1) and produce graphs like those in the chapter cover figure and Figure 12.1.

3. Choose a value for $\omega \neq 1$ in the ODE

$$x'' + 0.1x' + \sin x = \cos \omega t$$

and produce graphs like those in the chapter cover figure and Figure 12.1.

4. Repeat Problems 1 and 2, but for the Duffing ODE,

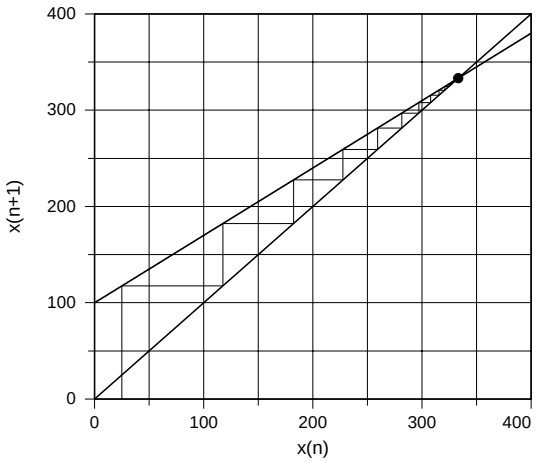
$$x'' + cx' + x - x^3 = A\cos t$$

5. Repeat Problems 1 and 2, but for the ODE with a quadratic nonlinearity,

$$x'' + cx' + x - x^2 = A\cos t$$

CHAPTER 13

Discrete Dynamical Systems



Supply and demand converge to a stable equilibrium.

Overview Processes such as population dynamics that evolve in discrete time steps are best modeled using discrete dynamical systems. These take the form $x_{n+1} = f(x_n)$, where the variable x_n is the state of the system at "time" n and x_{n+1} is the state of the system at time $n+1$. Discrete dynamical systems are widely used in ecology, economics, physics and many other disciplines. In this section we present the basic techniques and phenomena associated with discrete dynamical systems.

Key words Iteration; fixed point; periodic point; cobweb and stairstep diagrams; stability; sinks; sources; bifurcation diagrams; logisitic maps; chaos; sensitive dependence on initial conditions; Julia sets; Mandelbrot sets

See also Chapter 6 for more on sinks and sources in differential equations, and Chapter 12 for Poincaré sections and chaotic pendulum motion.

Discrete dynamical systems arise in a large variety of applications. For example, the population of a species that reproduces on an annual basis is best modeled using discrete systems. Discrete systems also play an important role in understanding many *continuous* dynamical systems. For example, points calculated by a numerical ODE solver form a discrete dynamical system that approximates the solution of an initial value problem for an ODE. The Poincaré section described in Chapter 12 is another example of a discrete dynamical system that gives information about a system of ODEs.

A *discrete* dynamical system is defined by the *iteration* of a function f, and takes the form

$$x_{n+1} = f(x_n), \quad n \geq 0, \quad x_0 \text{ given} \tag{1}$$

Here are two examples. In population dynamics, some populations are modeled using a *proportional growth* model

☞ The function $f(x) = \lambda x$ is denoted L_λ, and so $L_\lambda(x) = \lambda x$.

$$x_{n+1} = L_\lambda(x_n) = \lambda x_n, \quad n \geq 0, \quad x_0 \text{ given} \tag{2}$$

where x_n is the population density at generation n and λ is a positive number that measures population growth from generation to generation. Another common model is the *logistic growth* model:

☞ The function $\lambda x(1-x)$ is denoted by $g_\lambda(x)$.

$$x_{n+1} = g_\lambda(x_n) = \lambda x_n(1 - x_n), \quad n \geq 0, \quad x_0 \text{ given}$$

Let's return to the general discrete system (1). Starting with an initial condition x_0, we can generate a *sequence* using this rule for iteration: Given x_0, we get $x_1 = f(x_0)$ by evaluating the function f at x_0. We then compute $x_2 = f(x_1)$, $x_3 = f(x_2)$, and so on, generating a sequence of points $\{x_n\}$. Each x_n is the *n-fold composition* of f at x_0 since

☞ The superscript $^\circ$ reminds us that this is just the composition of f with itself; f is *not* being raised to a power.

$$\begin{aligned} x_2 &= f(f(x_0)) = f^{\circ 2}(x_0) \\ x_3 &= f(f(f(x_0))) = f^{\circ 3}(x_0) \\ &\vdots \\ x_n &= f^{\circ n}(x_0) \end{aligned}$$

(Some authors omit the superscript $^\circ$.)

The infinite sequence of iterates $O(x_0) = \{x_n\}_{n=0}^{\infty}$ is called the *orbit of* x_0 *under* f, and the function f is often referred to as a *map*. For example, if we take $\lambda = 1/2$ and the initial condition $x_0 = 1$ in the proportional growth model (2), we get the orbit for the map L:

$$x_0 = 1, \quad x_1 = 1/2, \quad x_2 = 1/4, \ldots$$

Refer to Screen 1.2 of Module 13 for four representations of the orbit of an iteration: as a *sequence* $\{x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots\}$; a *numerical list* whose columns are labeled n, x_n, $f(x_n)$; a *time series* where x_n is plotted against "time" n; and a *stairstep/cobweb diagram* for graphical analysis.

The chapter cover figure shows a stairstep diagram for the model $x_{n+1} = 0.7x_n + 100$. Figures 13.1 and 13.2 show cobweb diagrams for the logistic

model $x_{n+1} = \lambda x_n(1 - x_n)$, with $\lambda = 3.51$ and 3.9, respectively. In all of these figures, the diagonal line $x_{n+1} = x_n$ is also plotted. The stairstep and cobweb diagrams are constructed by selecting a value for x_0 on the horizontal axis, moving up to the graph of the iterated function to obtain x_1, horizontally over to the diagonal then up (or down) to the graph of the function to obtain x_2, and so on. These diagrams are used to guide the eye in moving from x_n to x_{n+1}.

◆ Equilibrium States

As with autonomous ODEs, it is useful to determine the equilibrium states for a discrete dynamical system. First we need some definitions:

☞ A fixed point of a discrete dynamical system is the analogue of an equilibrium point for a system of ODEs.

- A point x^* is a *fixed point* of f if $f(x^*) = x^*$. A fixed point is easy to spot in a stairstep or cobweb diagram even before the steps and webs are plotted: the fixed points of f are where the graph of f intersects the diagonal.
- A point x^* is a *periodic point of period n* of f if $f^{\circ n}(x^*) = x^*$ and $f^{\circ k}(x^*) \neq x^*$ for $k < n$. A fixed point is a periodic point of period 1.

Both the proportional and logistic growth models have the fixed point $x = 0$. For certain values of λ the logistic model has periodic points; Figure 13.1 suggests that the model has a period-4 orbit if $\lambda = 3.51$.

✓ "Check" your understanding by showing that the logistic model has a second fixed point $x^* = (\lambda - 1)/\lambda$. Does the proportional growth model for $\lambda > 0$ have any periodic points that are not fixed?

☞ This use of the words "stable" and "unstable" for points and orbits of a discrete system differs from the way the words are used for equilibrium points of an ODE. For example, a saddle point of an ODE is unstable, but a saddle point of a discrete system is neither stable nor unstable.

A fixed point x^* of f is said to be *stable* (or a *sink*, or an *attractor*) if every point p in some neighborhood of x^* approaches x^* under iteration by f, that is, if $f^{\circ n}(p) \to x^*$ as $n \to +\infty$. The set of *all* points such that $f^{\circ n}(p) \to x^*$ as $n \to +\infty$ is the *basin of attraction* of p. A fixed point x^* is *unstable* (or a *source* or *repeller*) if every point in some neighborhood of x^* moves out of the neighborhood under iteration by f. If x^* is a period-n point of f, then the orbit of x^* is said to be *stable* if x^* is stable as a fixed point of the map $f^{\circ n}$. The orbit is *unstable* if x^* is unstable as a fixed point of $f^{\circ n}$. Stability is determined by the *derivative* of the map f, as the following tests show:

- A fixed point x^* is stable if $|f'(x^*)| < 1$, and unstable if $|f'(x^*)| > 1$.
- A period-n point x^* (and its orbit) is stable if $|(f^{\circ n})'(x^*)| < 1$, and unstable if $|(f^{\circ n})'(x^*)| > 1$.

Stable periodic orbits are *attracting* because nearby orbits approach them, while unstable periodic orbits are *repelling* because nearby orbits move away from them.

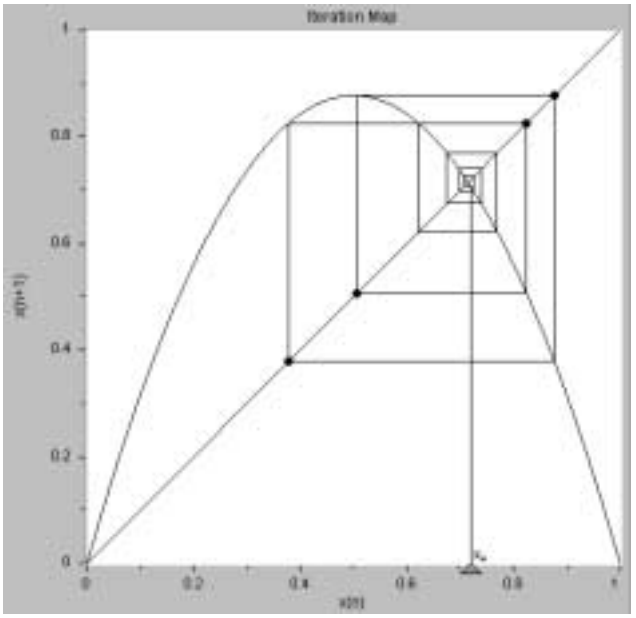


Figure 13.1: The cobweb diagram for the logistic map, $x_{n+1} = 3.51x_n(1 - x_n)$, suggests that iterates of $x_0 = 0.72$ approach a stable orbit of period 4.

✓ Is the fixed point in the chapter cover figure stable? Is the period-4 orbit in Figure 13.1 stable? How about the two fixed points in that figure? [*Suggestion:* Use the Discrete Tool as an aid in answering these questions.]

◆ Linear versus Nonlinear Dynamics

The solutions of linear and of nonlinear ODEs are compared and contrasted in Chapter/Modules 6 and 7. Now we will do the same comparison for linear and nonlinear maps of the real line into itself.

☞ Refer to the first submodule of Module 13 for examples.

Let's look at the iteration of linear functions such as the proportional growth model (2), $x_{n+1} = L_\lambda(x_n)$, which has a fixed point at $x^* = 0$. This fixed point is stable if $|\lambda| < 1$, so the orbit of every initial population tends to 0 as $n \to \infty$. If $\lambda = 1$, then $x_{n+1} = x_n$, and hence every point is a fixed point. The fixed point at $x^* = 0$ is unstable if $|\lambda| > 1$, and all initial populations tend to ∞ as $n \to \infty$. If $\lambda = -1$ then $x^* = 0$ is the only fixed point and every other point is of period 2 since $x_{n+2} = -x_{n+1} = -(-x_n) = x_n$.

The iteration of any linear function $f(x) = ax + b$ (with slope $a \neq 1$) behaves much like the proportional growth model. Fixed points are found by solving $ax^* + b = x^*$, and their stability is governed by the magnitude of a.

The iteration of nonlinear functions can be much more complex than that of linear functions. In particular, nonlinear functions can exhibit chaotic behavior, as well as periodic behavior. To illustrate the types of behavior typical

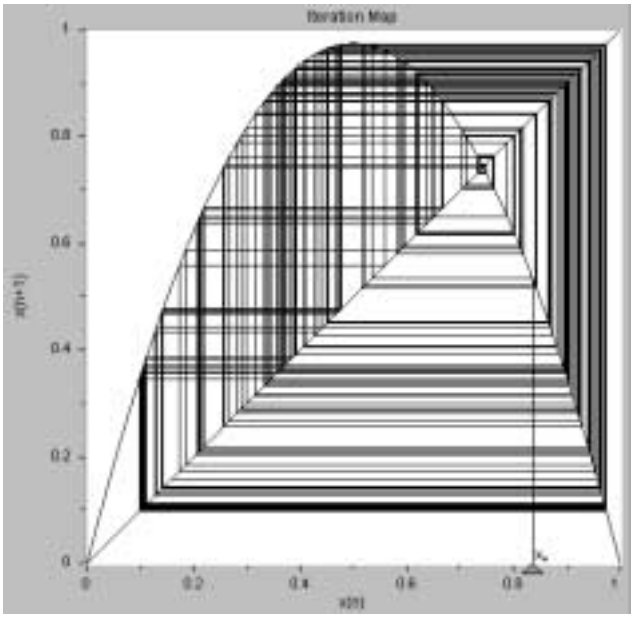


Figure 13.2: The cobweb diagram of the logistic map $x_{n+1} = 3.9x_n(1 - x_n)$ suggests that iterates of $x_0 = 0.8361$ either approach an attracting periodic orbit of very high period, or else wander chaotically.

to nonlinear functions we consider, in the second submodule of Module 13, the one-parameter family of *logistic* functions

$$g_\lambda(x) = \lambda x(1 - x)$$

Figure 13.2 shows how complex an orbit of a logistic map may be for certain values of λ.

◆ Stability of a Discrete Dynamical System

Now we turn our attention to the stability of an entire dynamical *system* rather than just that of a fixed point. One of the most important ideas of dynamical systems (discrete or continuous) is that of *hyperbolicity*. Hyperbolic points are stable to small changes in the parameters of a dynamical system. This does not mean that a perturbation (a small change) of the function leaves a fixed or periodic point unchanged. It simply means that the perturbed function will also have a fixed point or periodic point "nearby," and that this point has the stability properties of the corresponding point of the unperturbed function. For example, at $\lambda = 2$ the function $g_2(x)$ has an attracting fixed point $x^* = 0.5$. For values of λ near 2, the function $g_\lambda(x)$ also has an attracting fixed point $x^* = (\lambda - 1)/\lambda$. For example, if $\lambda = 2.1$ then the attracting fixed point is $x^* = 0.524$. Even though the fixed point moved a little as λ increased, the

☞ Recall that $g_2(x) = 2x(1 - x)$.

fixed point still exists and it is still attracting. The following theorem provides a way of determining whether fixed points and periodic orbits are hyperbolic.

THEOREM 13.1 Given a discrete dynamical system $x_{n+1} = f_\lambda(x_n)$, a fixed point x^* of $f_\lambda(x)$ is hyperbolic if $|f'_\lambda(x^*)| \neq 1$. Similarly, a periodic point x^* of period n (and its orbit) is hyperbolic if $|(f_\lambda^{\circ n})'(x^*)| \neq 1$.

Because the number and type of periodic points do not change at parameter values where $f_\lambda(x)$ has hyperbolic points, we say that the qualitative structure of the dynamical system remains unchanged. On the other hand, this theorem also implies that changes in the qualitative structure of a family of discrete dynamical systems can occur only when a fixed or periodic point is *not* hyperbolic. We see this in the proportional growth model $x_{n+1} = \lambda x_n$ when $\lambda = 1$ and $\lambda = -1$. For $\lambda = 1 - \varepsilon$ [and hence $L'_{1-\varepsilon}(1) = \lambda = 1 - \varepsilon$] the fixed point $x = 0$ is attracting. But for $\lambda = 1 + \varepsilon$ the fixed point is repelling. Thus, as λ passes through the value 1, the stability of the fixed point changes from attracting to repelling and the qualitative structure of the dynamical system changes.

◆ Bifurcations

A change in the qualitative structure of a discrete dynamical system, such as a change in the stability of a fixed point, is known as a *bifurcation*. If f_λ is nonlinear, several types of bifurcations can occur; here are two common types.

The first, known as a *saddle-node* bifurcation, occurs when x^* is a periodic point of period n and $(f_\lambda^{\circ n})'(x^*) = 1$. In a saddle-node bifurcation, the periodic point x^* splits into a pair of periodic points, both of period n, one of which is attracting and the other repelling. A saddle-node bifurcation occurs in the logistic growth family $g_\lambda(x)$ when $\lambda = 1$. At this value the fixed point $x^* = 0$ (which is attracting for $\lambda < 1$) splits into a pair of fixed points, $x^* = 0$ (repelling for $\lambda > 1$), and $x^* = (\lambda - 1)/\lambda$ (attracting for $\lambda > 1$). This type of bifurcation is sometimes called an *exchange of stability* bifurcation.

☞ See Chapter 7 for bifurcations of systems of ODEs.

The second important type of bifurcation is called *period-doubling* and occurs when x^* is a periodic point of period-n and $(f_\lambda^{\circ n})'(x^*) = -1$. In this bifurcation the attracting period-n point becomes repelling and an attracting period-$2n$ orbit is spawned. (Note that the stability can be reversed.) This occurs in the logistic family $g_\lambda(x)$ when $\lambda = 3$. At this parameter value, the attracting fixed point $x^* = (\lambda - 1)/\lambda$ becomes repelling and a stable period-2 orbit emerges with one point on each side of $x^* = (\lambda - 1)/\lambda$. Since the logistic equations model population growth, this says that the population converges to an equilibrium for growth rate constants λ less than 3. However, for values of λ greater than 3, the population oscillates through a sequence of values.

The bifurcations that occur in a one-parameter family of discrete dynamical systems can be summarized in a *bifurcation diagram*. For each value of the parameter (on the horizontal axis) the diagram shows the long-term behavior under iteration of a "typical" initial point. For example, if you see

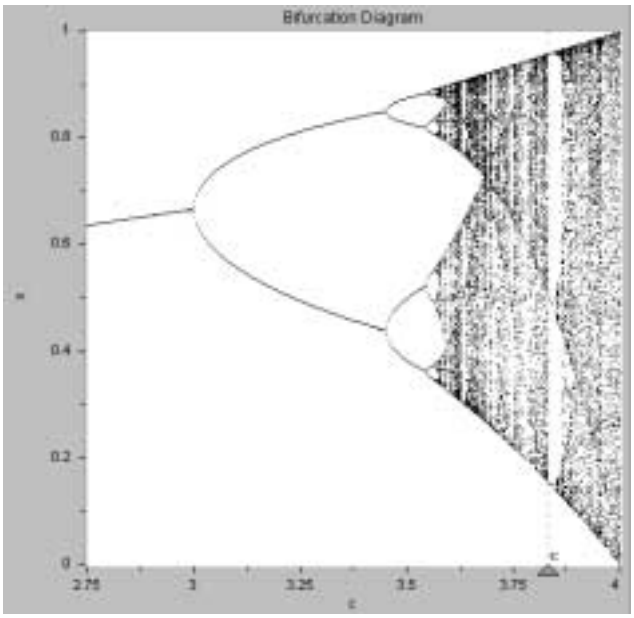


Figure 13.3: Part of the bifurcation diagram for the logistic map.

a single point in the diagram above a particular parameter value, that point corresponds to an attracting fixed point. The spot in the diagram where you see an arc of attracting fixed points split into two arcs corresponds to a bifurcation from an attracting fixed point to an attracting orbit of period 2 (i.e., period doubling). If the diagram shows a multitude of points above a given parameter value, then either you are seeing an attracting periodic orbit of a very high period, or else you are seeing chaotic wandering. It should be noted that when constructing the bifurcation diagram for each parameter value and initial point, the first 50 or so iterates are omitted so that only the long-term behavior is visible in the diagram. See Figure 13.3 and Screen 2.4 in Module 13 for the bifurcation diagram of the logistic map.

The stable arcs in these diagrams are usually straightforward to generate numerically. We constructed a bifurcation diagram on an interval $[\lambda_{\min}, \lambda_{\max}]$ for the logistic population model $x_{n+1} = g_\lambda(x_n)$ using the following procedure.

1. Fix $\lambda_{\min}$, $\lambda_{\max}$, λ_{inc}, $n_{\min}$, $n_{\max}$. Here λ_{inc} is the step size between successive values of λ while $n_{\min}$ and $n_{\max}$ are bounds on the number of iterates used to construct the diagram; they control the accuracy of the diagram. Typical values are $n_{\min} = 50$ and $n_{\max} = 150$.
2. Let $\lambda = \lambda_{\min}$.
3. Taking $x_0 = 0.5$ for example, compute the first $n_{\min}$ iterates of g_λ without plotting anything. This eliminates transient behavior.

4. As n increases from $n_{\min}$ to $n_{\max}$, plot the points $(\lambda, g_\lambda^{\circ n}(0.5))$. If the orbit of 0.5 converges to a periodic orbit, only points near this orbit are plotted. If the orbit of 0.5 isn't periodic, then the points above λ seem to be almost randomly distributed.
5. Let $\lambda = \lambda + \lambda_{\text{inc}}$.
6. If $\lambda < \lambda_{\max}$, go back to Step 3 and repeat the process.

✓ Go to the One-dimensional tab of the Discrete Tool. Use the default values, but set the value of c at 1 (c in the tool plays the role of λ in Chapter/Module 13). Turn on the bifurcation diagram. Keep your finger on the up-arrow for c and describe what is happening. Any attracting periodic orbits? For what values of c do these orbits occur? What are the periods?

◆ Periodic and Chaotic Dynamics

One of the most celebrated theorems of discrete dynamical systems is often paraphrased "Period 3 Implies Chaos." This theorem, originally proven by Šarkovskii and independently discovered by Li and Yorke[1], is a remarkable result in that it requires little information about the dynamical system and yet it returns a treasure trove of information.

THEOREM 13.2 If f is a continuous function on the real line and if there exists a point of period 3, then there exist points of every period.

For the logistic population model there exists an attracting period-3 orbit at $\lambda = \sqrt{8} + 1 \approx 3.83$, and most initial conditions in the unit interval converge to this orbit (see Figure 13.4). In terms of our model, most populations tend to oscillate between the three different values of the period-3 orbit. Theorem 13.2 states that even more is going on at $\lambda = \sqrt{8} + 1$ than meets the eye. If we pick any positive integer n, there exists a point p such that n is the smallest positive integer allowing $g_\lambda^{\circ n}(p) = p$. Thus, for example, there exists a point that returns to itself in 963 iterates. The reason we don't "see" this periodic orbit (or, indeed, any periodic orbit, except that of period 3) is that it is unstable, so no iterate can approach it. But orbits of every period are indeed present if $\lambda = \sqrt{8} + 1$.

[1] James Yorke and T.Y. Li are contemporary mathematicians who published their result in 1975 (see References). They were the first to apply the word "chaos" to the strange behavior of the iterates of functions such as g_λ. A.N. Šarkovskii published a stronger result in 1964, in Russian, in the *Ukrainian Mathematical Journal*, but it remained unknown in the West until after the paper by Yorke and Li had appeared.

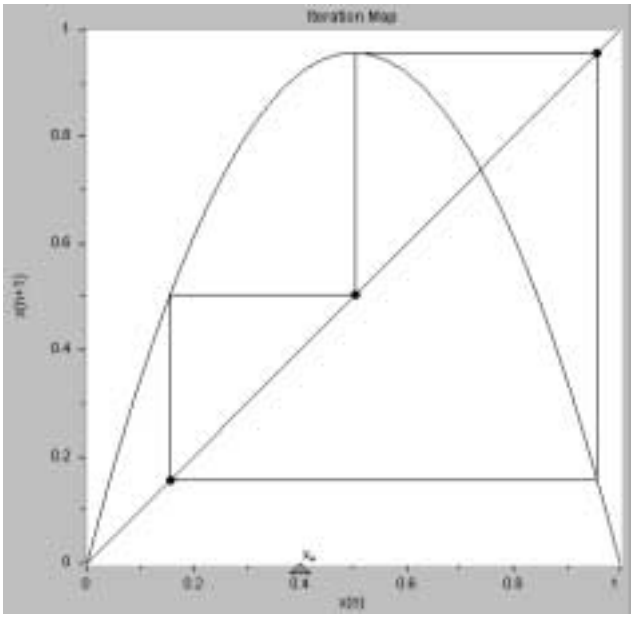


Figure 13.4: At $\lambda = \sqrt{8}+1 \approx 3.83$ the logistic map $g_\lambda = \lambda x(1-x)$ has an attracting orbit of period 3; the points $x_0, x_1, \ldots, x_{49}$ have been suppressed in this graph.

◆ What is Chaos?

So, you're probably asking, what is chaos? The definition of chaos is a bit slippery. In fact, mathematicians are still arguing about a proper definition. But to get the idea across we'll use one due to Devaney.

Let S be a set such that if x lies in S, then $f^{\circ n}(x)$ belongs to S for all positive integers n. The set S is called *invariant*. If you start in an invariant set, you can't get out! Now let's define what we mean by chaos in an invariant set S.

A map $f : S \to S$ is *chaotic* if:

1. periodic points are dense in S;
2. f displays sensitive dependence on initial conditions in S; and
3. f is topologically transitive in S.

☞ A set U of real numbers is *open* if every point p of U has the property that all points in some interval $(p-a, p+a)$ are also in U.

The first condition of this definition is explained like this: A set A is *dense* in another set B if for every point x in B and every open set U containing x there exists points of A that are also in U. Therefore, condition 1 says that periodic points are almost everywhere in S. This means that S contains many periodic points; Theorem 13.2 gives a condition guaranteeing infinitely many of these points.

☞ Module 13 has examples.

In the second condition, *sensitive dependence on initial conditions* means that points that are initially close to one another eventually get moved far apart under iteration by f.

Finally, f is *topologically transitive* (or *mixing*) if given any pair of open sets U and V in S, some iterate of f takes one or more points of U into V. This means that points of open sets get spread throughout the set S.

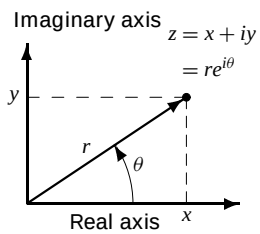
The most significant item on this list for applied problems is sensitive dependence on initial conditions. Let's consider the logistic growth model at a parameter value where the dynamics are chaotic. Sensitive dependence implies that no matter how close two populations may be today, there will be a time in the future when the populations differ significantly. So environmental disturbances that cause small population changes will eventually lead to large changes, if chaotic dynamics exist.

Chaotic dynamics occur in a wide range of models. Although the definitions above are given in terms of a single scalar dynamical system, everything extends to higher dimensions, and many of the applications are two- or three-dimensional. In addition to models of population dynamics, chaos has been observed in models of the weather, electrical circuits, fluid dynamics, planetary motion, and many other phenomena. The relatively recent understanding of chaos has shed new light on the complexity and beauty of the world we inhabit.

◆ Complex Numbers and Functions

Probably the most popular type of discrete dynamical system is a *complex dynamical system* where the variables are complex numbers instead of real numbers. The intricate fractal structures common to images generated using complex dynamics have appeared everywhere from calendars to art shows and have inspired both artists and scientists alike. Many of the fundamental ideas of complex dynamics are identical to those of real dynamics and have been discussed in previous sections. In what follows, we will highlight both the similarities and differences between real and complex dynamics.

Recall that complex numbers arise when factoring quadratic polynomials with negative discriminant. Because the discriminant is negative we must take the square root of a negative real number, which we do by defining i to be $\sqrt{-1}$. We then write the *complex number* as $z = x + iy$. We say that x is the *real part of* z and y is the *imaginary part of* z. The complex number z is represented graphically on the complex plane by the point having coordinates (x, y). It is often useful to represent complex numbers in polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$ so that



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

The remarkable relationship $\cos\theta + i\sin\theta = e^{i\theta}$ between polar coordinates and exponential functions is known as *Euler's formula*. The number $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point z in the complex plane

☞ Euler's formula is also used in Chapter 4.

and is sometimes called the *modulus* of z, and it is denoted by $r = |z|$. The angle θ is called the *argument* of z. Note that the usual properties of exponential functions hold in the complex plane. Thus, given two complex numbers $z = re^{i\theta}$ and $w = se^{i\phi}$, their product is

$$zw = rse^{i(\theta+\phi)}$$

A *complex function* $f(z)$ takes a complex number z as its argument and returns a complex number $w = f(z)$. Differentiation proceeds as in the real case; for example, $(z^3)' = 3z^2$. Unlike functions of one real variable, we cannot graph a complex function since both the domain and range are two-dimensional.

◆ Iterating a Complex Function

Iteration of a complex function is identical to the iteration of a real function. Given an initial z-value z_0, iteration generates a sequence of complex numbers $z_1 = f(z_0)$, $z_2 = f(z_1)$, etc. Fixed and periodic points are defined in the same way as for real functions, as are stability and instability. Here are the previous criteria for stability, but now applied to complex functions.

- A fixed point z^* is stable if $|f'(z^*)| < 1$, and unstable if $|f'(z^*)| > 1$.
- A period-n point z^* (and its orbit) is stable if $|(f^{\circ n})'(z^*)| < 1$, and unstable if $|(f^{\circ n})'(z^*)| > 1$.

Let's consider a simple example to illustrate these ideas. Let $f(z) = z^2$. Then $z^* = 0$ is an attracting fixed point since $f(0) = 0$ and $|f'(0)| = 0$. If z is any point such that $|z| < 1$, then the sequence $\{f^{\circ n}(z)\}_{n=0}^{\infty}$ converges to 0 as $n \to \infty$. On the other hand, if $|z| > 1$, then the sequence $\{f^{\circ n}(z)\}_{n=0}^{\infty}$ goes to infinity as $n \to \infty$. To see what happens to values of z having modulus equal to 1, let's write $z = e^{i\theta}$. Then $f(z) = e^{2i\theta}$, which also has modulus 1. Thus all iterates of points on the unit circle $|z| = 1$ stay on the unit circle. The point $z^* = 1$ is a repelling fixed point since $f(1) = 1$ and $|f'(1)| = 2$. The period-2 points are found by solving $f(f(z)) = z^4 = z$. We can rewrite this equation as

$$z(z^3 - 1) = 0$$

One solution to this equation is $z^* = 0$, corresponding to the attracting fixed point, and another solution is $z^* = 1$, corresponding to the repelling fixed point. Notice that the fixed points of $f(z)$ remain fixed points of $f(f(z))$, or equivalently, are also period-2 points of $f(z)$. To find the other two solutions, we write $z = e^{i\theta}$ in the equation $z^3 = 1$ to get the equation

☞ Any period-n point is also a periodic point of all periods which are positive integer multiples of n.

$$e^{3i\theta} = 1$$

which we need to solve for θ. Since we are working in polar coordinates, we note that $1 = e^{i2n\pi}$ where n is an integer. This implies that $3\theta = 2n\pi$ and from this we find a second pair of period-2 points at $z = e^{2\pi i/3}$ and $z = e^{4\pi i/3}$. Both of these are repelling.

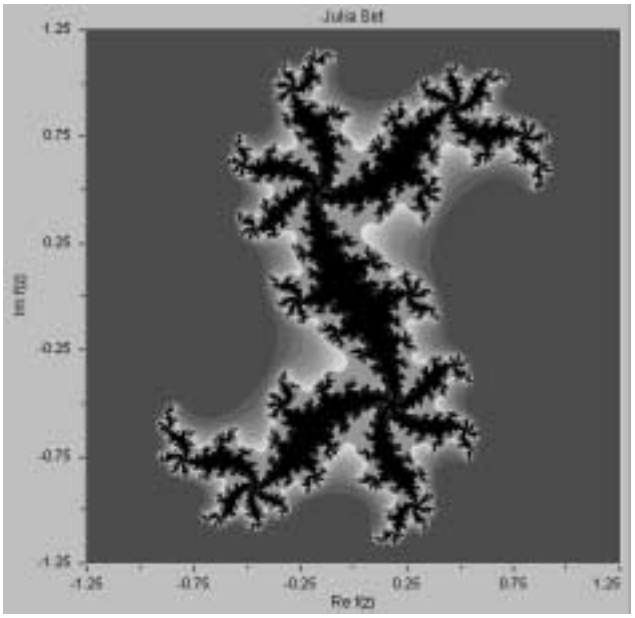


Figure 13.5: The filled Julia set for $f(z) = z^2 + c$, where $c = 0.4012 - 0.3245i$

✓ Show that $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are repelling period-2 points of $f = z^2$. Show that $f^{\circ n}(z) \to 0$ as $n \to \infty$ if $|z| < 1$, and that $|f^{\circ n}(z)| \to \infty$ if $|z| > 1$. What is the "basin of attraction" of the fixed point $z = 0$?

◆ Julia Sets, the Mandelbrot Set, and Cantor Dust

The set of repelling periodic points of the function $f = z^2$ is dense on the unit circle, although we don't show that here. This leads us to the definition of the Julia set.

DEFINITION The *filled Julia set* K of a complex-valued function f is the set of all points whose iterates remain bounded. The *Julia set* J of f is the closure of the set of repelling periodic points.

☞ The *closure* of a set A consists of the points of A together with all points that are limits of sequences of points of A.

For $f = z^2$, the filled Julia set K of f is the set of all complex numbers z with $|z| \leq 1$, while the Julia set J of f is the unit circle $|z| = 1$. This is a very simple example of a Julia set. In general, Julia sets are highly complicated objects having a very intricate fractal structure. For example, see Figure 13.5 and Screens 3.3 and 3.4 of Module 13.

In the above example, the Julia set J divides those points that iterate to infinity (points outside the unit circle) and those that converge to the attracting fixed point (points inside the unit circle). This division of the domain by the Julia set is often the case in complex dynamics and provides a way of numerically computing the filled Julia set of a given function f. Assign a

complex number to each screen pixel. Then use each pixel (i.e., complex number) as an initial condition and iterate to determine whether the orbit of that point exceeds some predetermined bound (for example $|z| = 50$). If it does, we say the orbit diverges and we color the point black. If not, we color the point red to indicate it is in the filled Julia set.

☞ In the Discrete Tool of ODE Architect, the coloring is reversed. Points in the filled Julia set are colored black and points whose orbits diverge past the predetermined bound are colored with various colors according to their divergence rates (e.g., red is the fastest, dark blue the slowest).

Earlier in this chapter we saw the importance of attracting periodic orbits in building a bifurcation diagram for a real map f. Although we didn't mention it then, we can home in on an attracting periodic orbit of f (if there is one) by starting at $x_0 = \tilde{x}$ if $f'(x)$ is zero at $\tilde{x}$ and nowhere else. Complex functions $f(z)$ for which $f'(\tilde{z}) = 0$ at exactly one point $\tilde{z}$ have the same property, as the following theorem shows.

☞ A complex function f is *analytic* if its derivatives of every order exist. A point $\tilde{z}$ is a *critical point* of f if $f'(\tilde{z}) = 0$.

THEOREM 13.3 Let f be an analytic complex-valued function with a unique critical point $\tilde{z}$. If f has an attracting periodic orbit, then the forward orbit of $\tilde{z}$ converges to this orbit.

Let's look at some of the implications of this theorem with the family of functions $f_c(z) = z^2 + c$ where $c = a + ib$ is a complex parameter. For each value of c the only critical point is $\tilde{z} = 0$. To find an attracting periodic orbit for a given value of c we need to compute the orbit

$$\{0, c, c^2 + c, \dots\}$$

and see if the orbit converges or not. If it does, we find the attracting periodic orbit; if not, there doesn't exist one. Let's see what happens when we set $c = 1$ to give the function $f_1(z) = z^2 + 1$. The orbit of the critical point is $\{0, 1, 2, 5, 26, \dots\}$, which goes to infinity. Thus, f_1 has *no* attracting periodic orbit and the Julia set does not divide points that converge to a periodic orbit from points that iterate to infinity. In fact, it can be shown that this Julia set is totally disconnected; it is sometimes referred to as *Cantor dust*. Click on outlying points on the edge of the Mandelbrot set (defined below) in Screen 3.5 of Module 13 and you will generate Cantor dust in the upper graphics screen.

This leads to another question. If some functions in the family f_c have connected Julia sets (such as $f_0 = z^2$) and other functions in the family have totally disconnected Julia sets (such as f_1), what set of points in the c plane separates these distinctive features? This set is the *boundary* of the Mandelbrot set. The *Mandelbrot set* M of the function $f_c(z) = z^2 + c$ is defined as the set of all complex numbers c such that the orbit $\{f_c^{\circ n}(0)_{n=1}^{\infty}\}$ remains bounded, that is, $|f_c^{\circ n}(0)| \leq K$ for some positive number K and all integers $n \geq 0$.

This definition leads us to an algorithm for computing the Mandelbrot set M. Assign to each pixel a complex number c. Choose a maximum number of iterations N and determine whether $|f_c^{\circ n}(0)| < 2$ for all $n \leq N$ (it can be proven that if $|f_c^{\circ n}(0)| > 2$ for some n, then the orbit goes to infinity). If so, then color this point green to indicate that it is (probably) in the Mandelbrot set. Otherwise, color this point black. It is this computation that gives the wonderfully intricate Mandelbrot set; see Figure 13.6 and Screens 3.4 and 3.5 of Module 13.

☞ The colors of the Mandelbrot set example in the Discrete Tool are different.

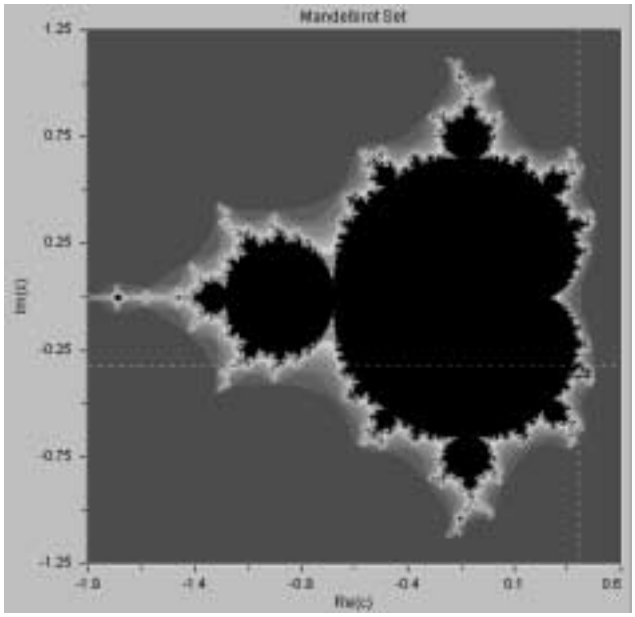


Figure 13.6: The Mandelbrot set; the cross-hairs are set on the point $c = 0.4012 - 0.3245i$, which gives the Julia set shown in Figure 13.5.

The Mandelbrot set actually contains much more information than is described here. It is, in fact, the bifurcation diagram for the family of functions $f_c(z) = z^2 + c$. Each "blob" of the set corresponds to an attracting periodic orbit of a particular period. Values of c in the big cardioid shown on Screen 3.4 of Module 13 give attracting period-1 orbits for f_c. Values of c in the circle immediately to the left of this cardioid give attracting period-2 orbits for f_c. Other "blobs" give other attracting periodic behaviors.

Although we have only defined the Mandelbrot set for the specific family $f_c = z^2 + c$, it can be defined in an analogous way for other families of complex functions (see the Discrete Tool). One final note on Julia sets and the Mandelbrot set. You've probably seen intricately colored versions of these objects on posters or elsewhere. The coloring is usually determined by how "fast" orbits tend to infinity. The color scheme is, of course, up to the programmer.

Module 13 introduces and lets you play with three important discrete dynamical systems—linear, logistic, and a third that uses complex numbers. Explorations 13.1–13.4 extend these ideas and introduce other maps with curious behavior under iteration.

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Doebeli, M. and Ruxton, G. "Controlling Spatial Chaos in Metapopulations with Long-Range Dispersal" in *Bulletin of Mathematical Biology*, Vol. 59 (1997) pp. 497–515. Emphasizes the frequent use of the linear map and the logistic map in population dynamics.

Guimez, J. and Matias, M.A. "Control of Chaos in Unidimensional Maps" in *Physics Letters A* Vol. 181 (September 1993) pp. 29–32

Peitgen, H.-O. and Richter, P.H., *The Beauty of Fractals*, (1986: Springer-Verlag)

Saha, P. and Strogatz, S.H. "The Birth of Period 3" in *Mathematics Magazine* Vol. 68 (1995) pp. 42–47

Strogatz, S., *Nonlinear Dynamics and Chaos*, (1994: Addison-Wesley). A very readable text, mostly on ODEs, but Chapter 10 on One-Dimensional Maps is particularly relevant to this chapter/module of ODE Architect.

Yorke, J. and Li, T.Y., "Period 3 Implies Chaos" in the *American Mathematical Monthly* 82 (1975) pp. 985–992

Here are three sources on chaos that are less technical and more accessible, but still give an accurate description of what chaos is all about.

Peak, D. and Frame, M., *Chaos under Control*, (1994: W.H. Freeman)

Stewart, I., *Does God Play Dice?*, (1990, Blackwell)

Stoppard, T., the play "Arcadia", (1993: Faber and Faber). Yes! A play about chaos.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 13.1. One-Dimensional Maps and the Discrete Tool

1. Go to the Discrete Tool and enter the proportional growth model $x_{n+1} = cx_n$, where c is the parameter. For the range $0 \leq n \leq 30$ and the initial condition $x_0 = 0.5$, explore and describe what happens to the iteration map, time series, and bifurcation diagram as the parameter is increased from -2 to 2. For what values of c is there a sudden change in the behavior of the iterates (the bifurcation values of c)? For what values of c are there 1, 2, or infinitely many fixed or periodic points? Which of these points are attractors? Repellers?

2. Go to the Discrete Tool and explore and describe what happens to the iteration map, the time series, and the bifurcation diagram as the parameter c for the logistic map $g_c(x) = cx(1 - x)$ is incremented from 1 to 4. Use the range $50 \leq n \leq 150$ to avoid any initial wandering, and the initial condition $x_0 = 0.5$. Describe what all three graphs on the tool screen look like at values of c where there is a periodic orbit. What is the period? Go as far forward as you can with the period-doubling sequence of values of c: 3, 3.434, ... What are the corresponding periods? [*Suggestion:* Zoom on the bifurcation diagram.] Repeat with the sequence 3.83, 3.842, ...

3. In the Discrete Tool enter the tent map T_c on the interval $0 \le x \le 1$:

$$T_c(x) = c(1 - 2\,\text{abs}(x - 0.5)) = \begin{cases} 2cx, & 0 \le x \le 0.5 \\ 2c(1-x), & 0.5 \le x \le 1 \end{cases}$$

where the parameter c is allowed to range from 0 to 1. Describe and explain what you see as c is incremented from 0 to 1. [*Suggestion:* use the Edit option in the Menu box for the bifurcation diagram and set $200 \le n \le 300$ in order to suppress the initial transients.] Any orbits of period 2? Period 3?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 13.2. Circle Maps

Another common type of discrete dynamical system is a *circle map*, which maps the perimeter of the unit circle onto itself. These functions arise when modeling coupled oscillators, such as pendulums or neurons. The simplest types of circle maps are rotations that take the form

$$R_\omega(\theta) = (\theta + \omega) \bmod 2\pi$$

where $0 \le \theta \le 2\pi$ and ω is a constant.

1. Show that if $\omega = (p/q)\pi$ with p and q positive integers and p/q in lowest terms, then every point has period q.

2. Show that if $\omega = a\pi$ with a an irrational number, then no point on the circle is periodic.

3. What is the long-term behavior of the orbit of a point on the circle if $\omega = a\pi$, where a is an irrational number?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 13.3. Two-Dimensional Maps and the Discrete Tool

A two-dimensional discrete dynamical system looks like this:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n, c) \\ y_{n+1} &= g(x_n, y_n, c) \end{aligned} \tag{3}$$

where f and g are given functions and c is a "place holder" for parameters. For given values of c, x_0, and y_0, system (3) defines an orbit of points

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \ldots$$

in the xy-plane. The two-dimensional tab in the Discrete Tool allows you to explore discrete systems of the form of system (3).

1. Open the Discrete Tool and explore the default system (a version of what is known as the *Hènon Map*):

$$\begin{aligned} x_{n+1} &= 1 + y_n - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned} \tag{4}$$

where a and b are parameters. For fixed values of the parameters a and b find the fixed points. Are they sinks, sources, or neither? How sensitive is the long-term behavior of an orbit to small changes in the initial point (x_0, y_0)? What happens if you increment a through a range of values? If you increment b? Any period-doubling sequences? In your judgment, is there any long-term chaotic wandering? [*Suggestion:* Keep the values of a and b within small ranges of their default values to avoid instabilities.]

2. Repeat Problem 1 with the following version of the Hènon map:

$$x_{n+1} = a - x_n^2 + by_n$$
$$y_{n+1} = x_n$$

Start with $a = 1.28,\ b = -0.3,\ x_0 = 0,\ y_0 = 0.$

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date _____

Course/Section _____

Exploration 13.4. Julia and Mandelbrot Sets and the Discrete Tool

Note that the color schemes for the Julia and Mandelbrot sets in Module 13 differ from those in the discrete tool.

1. Use the Discrete Tool to explore the Mandelbrot set and Julia sets for the complex family $f_c = z^2 + c$. What happens to the filled Julia sets as you move c from inside the Mandelbrot set up toward the boundary, then across the boundary and out beyond the Mandelbrot set? Describe how the Julia sets change as you "walk" along the edge of the Mandelbrot set.

2. Repeat Problem 1 for the complex family $g_c = c \sin z$.

3. Repeat Problem 1 for the family $h_c = ce^z$.

GLOSSARY

Acceleration The acceleration of a moving body whose position at time t is $u(t)$ is given by

$$\frac{d^2u}{dt^2}$$

Air resistance A body moving through air (or some other medium) is slowed down by a resistive force (also called a drag or damping force) that acts opposite to the body's velocity. See also "Viscous damping" and "Newtonian damping."

Amplitude The amplitude of a periodic oscillating function $u(t)$ is half the difference between its maximum and minimum values.

Angular momentum The angular momentum vector of a body rotating about an axis is its moment of inertia about the axis times its angular velocity vector.

This is the analog in rotational mechanics of momentum (mass times velocity) in linear mechanics.

Angular velocity An angular velocity vector, $\omega(t)$, is the key to the relation between rotating body axes and a fixed coordinate system of the observer. The component ω_j of the vector $\omega(t)$ along the jth body axis describes the spin rate of the body about that axis.

Autocatalator This is a chemical reaction of several steps, at least one of which is autocatalytic.

Autocatalytic reaction In an autocatalytic reaction, a chemical species stimulates more of its own production than is destroyed in the process.

Autonomous ODE An autonomous ODE has no explicit mention of the independent variable (usually t) in the rate equations. For example, $x' = x^2$ is autonomous, but $x' = x^2 + t$ is not.

Balance law The balance law states that the net rate of change of the amount of a substance in a compartment equals the net rate of flow in minus the net rate of flow out.

Beats When two sinusoids of nearly equal frequencies are added the result appears to be a high frequency sinusoid modulated by a low frequency sinusoid called a beat. A simple example is given by the function $(\sin t)(\sin 10t)$, where the first sine produces an "amplitude modulation" of the second.

Bessel functions of the 1st kind The Bessel function of the first kind of order zero,

$$J_0(s) = 1 - \frac{1}{4}s^2 + \cdots + (-1)^n \frac{s^{2n}}{n!^2 2^{2n}} + \cdots$$

is a solution of Bessel's equation of order zero, and is bounded and convergent for all s.

Bessel functions of the 2nd kind The Bessel function of the second kind of order zero, $Y_0(s)$, is another solution of Bessel's equation of order zero. It is much more complicated than $J_0(s)$, and

$$Y_0(s) \to -\infty \quad \text{as} \quad s \to 0+$$

See Chapter 11 for a complete formula for $Y_0(s)$ that involves a logarithmic term, $J_0(s)$, and a complicated (but convergent) infinite series.

Bessel's equation Bessel's equation of order $p \geq 0$ is

$$s^2 w''(s) + s w'(s) + (s^2 - p^2) w = 0$$

where p is a nonnegative constant. Module 11 considers only $p = 0$. See Chapter 11 for $p > 0$.

Bessel's equation, general solution of Bessel's equation of order zero is second-order and linear. The general solution is the set of all linear combinations of $J_0(s)$ and $Y_0(s)$.

Bifurcation diagram A bifurcation diagram describes how the behavior of a dynamical system changes as a parameter varies. It can appear in studies of iteration or of differential equations.

In the case of a single real parameter, a bifurcation diagram plots a parameter versus something indicative of the behavior, such as the variable being iterated (as in Module 13, Nonlinear Behavior) or a single variable marking location and stability of equilibrium points for a differential equation.

In iteration of a function of a complex variable, two dimensions are needed just to show the parameter, but different colors can be used to show different behaviors (as in Module 13, Complex Dynamics).

Cantor set, Cantor dust A Cantor set was first detailed by Henry Smith in 1875, but was named in honor of Georg Cantor, the founder of set theory, after he used this bizarre construction in 1883. Now Cantor sets are found in many guises in discrete dynamical systems.

A Cantor set is a closed, bounded, and totally disconnected set, with uncountably many points. A typical construction is to delete a band across the middle of a set, then to delete the middle of both pieces that are left, and then to repeat this process indefinitely.

Julia sets (see glossary) for parameter values outside the Mandelbrot set (see glossary) are Cantor dusts, constructed by a similar algorithm. See Chapter 13 for References.

Carrying capacity The carrying capacity K of an environment is the maximum number of individuals that the environment can support at steady state. If there are fewer individuals than the carrying capacity in the environment, the population will grow; if there are more individuals, the population will decline.

A widely used model for population dynamics involving a carrying capacity is the logisitc ODE

$$\frac{dN}{dt} = rN(1 - N/K)$$

where r is the intrinsic growth rate constant.

Cascade A cascade is a compartment model where the "flow" through the compartments is all one direction.

Center A center is an equilibrium point of an autonomous planar linear system for which the eigenvalues are conjugate imaginaries $\pm i\beta$, $\beta \neq 0$. All nonconstant orbits of an autonomous planar linear system with a center are simple closed curves enclosing the equilibrium.

Centering an equilibrium If p^* is an equilibrium point of the system $x' = f(x)$ (so $f(p^*) = 0$), then the change of coordinates $x = y + p^*$ moves p^* to the origin in the y-coordinate system.

Chain rule The chain rule for differentiating a function $L(\theta(t), y(t))$ with respect to t is

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} \\ &= L_\theta \theta' + L_y y'\end{aligned}$$

Chaos Mathematical chaos is a technical term that describes certain nonperiodic behavior of a discrete dynamical system (Module 13) or solutions to a differential equation (Module 12). A discrete system is said to be chaotic on an interval if all of the following are true.

- It exhibits sensitive dependence on initial conditions.
- Periodic unstable orbits occur almost everywhere.
- Iterates of intervals get "mixed up."

Chaotic behavior never repeats, revisits every neighborhood infinitely often, but is not random. Each step is completely determined by the previous step.

An equivalent list of requirements appears in Module 12, Screen 1.4. Further discussion appears in Chapter 13.

Characteristic equation The characteristic equation of a square matrix A is $\det(\lambda I - A) = |\lambda I - A| = 0$. For a 2×2 matrix, this reduces to $\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$ whose solutions, called eigenvalues of A, are

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{\operatorname{tr}^2 A - 4 \det A}}{2}$$

Chemical law of mass action The rate of a reaction step is proportional to the product of the concentrations of the reactants.

Example: If one unit of species X produces one unit of product Y in a reaction step, the rate of the step is kx, where k is a positive constant. Thus, we have

$$x' = -kx, \qquad y' = kx$$

Example (Autocatalysis): If one unit of species X reacts with two units of Y and produces three units of Y in an autocatalytic step, the reaction rate is

$$axyy = axy^2$$

where a is a positive constant. Thus, we have

$$x' = -axy^2, \qquad y' = 3axy^2 - 2axy^2 = axy^2$$

because one unit of X is destroyed, while three units of Y are created, and two are consumed.

Combustion model The changing concentration $y(t)$ of a reactant in a combustion process is modeled by the IVP

$$y' = y^2(1 - y), \qquad y(0) = a, \quad 0 \leq t \leq 2/a$$

where a is a small positive number that represents a disturbance from the pre-ignition state $y = 0$. R. E. O'Malley studied the problem in his book, *Singular Perturbation Methods for Ordinary Differential Equations*, (1991: Springer).

Compartment model A compartment model is a set of boxes (the compartments) and arrows that show the flow of a substance into and out of the different boxes.

Component graphs A component graph of a solution of a differential system is a graph of one of the dependent variables as a function of t.

Example: For the ODE system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

the component graphs are the plots of a solution $x = x(t)$ and $y = y(t)$ in the respective tx- and ty-planes.

Concentration The concentration of a substance is the amount of the substance dissolved per unit volume of solution.

Connected set A connected set is a set with no islands. In the early 1980's Adrien Douady (Université Paris XI, Orsay and Ecole Normale Supérieure) and John Hubbard (Cornell University) proved that the Mandelbrot set was connected. They did this by showing that its exterior could be put in a one-to-one correspondence with the exterior of a disk. They found in the process that all the angles one might note while walking around the boundary of the disk have special analogs on the Mandelbrot set. Halfway around the disk from the rightmost point corresponds to being at the tip of the Mandelbrot set, while one third or two thirds the way around the disk corresponds to the "neck" where the biggest ball attaches to the cardioid.

Conserved quantity A function $E(q, y)$ is conserved along a trajectory $q = q(t)$, $y = y(t)$, of a system $q' = f(q, y)$, $y' = g(q, y)$, if $dE(q(t), y(t))/dt = 0$.

As time changes, the value of E stays constant on each trajectory, although the value will vary from one trajectory to another. The graph of each trajectory in the qy-phase plane lies on one of the level sets $E =$ constant.

This idea of a conserved quantity can be extended to any autonomous system of ODEs. An autonomous system is conservative if there is a function E that stays constant along each trajectory, but is nonconstant on every region (i.e., varies from trajectory to trajectory).

Cycle In a discrete dynamical system, including a Poincaré section, a cycle is a sequence of iterates that repeats. The number of iterates in a cycle is its period.

For an autonomous differential system, a cycle is a nonconstant solution $x(t)$ such that $x(t + T) = x(t)$, for all t, where T is a positive constant. The smallest value of T for a cycle is its period.

For a cycle in a system of 2 ODEs, see Limit cycle.

Damped pendulum A real pendulum of length L is affected by friction or air resistance that is a function of L, θ, and θ', and acts opposite to the direction of motion.

Throughout the Linear and Nonlinear Pendulums submodule of Module 10, we assume that, if there is any damping, it is viscous; i.e., the damping force is given by $-bL\theta'$. The minus sign tells us that damping acts opposite to the velocity.

Module 4 makes a more detailed study of the effects of damping on a linear oscillator, as does Module 11 for the spring in the Robot and Egg.

Damping Damping can arise from several sources, including air resistance and friction. The most common model of damping is viscous damping—the damping force is assumed to be proportional to the velocity and acts opposite to the direction of motion. See also Newtonian damping.

Dense orbit An orbit $x(t)$ of a system of ODEs $x' = f(t, x)$ is dense in a region R of x-space if the orbit gets arbitrarily close to every point of R as time goes on.

That is, if x_1 is any point in R, and ε is any positive number, then, at some time t_1, the distance between $x(t_1)$ and x_1 is less than ε.

Determinant The determinant of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\det A = ad - bc$.

Deterministic A system of ODEs is said to be deterministic if the state of the system at time t is uniquely determined by the state of the system at the initial time.

For example, the single first-order ODE $x' = f(t, x)$ is deterministic if for each set of initial data (t_0, x_0) there is exactly one solution $x(t)$.

Thus, if you were to choose the same initial data a second time and watch the solution curve trace out in time again, you would see exactly the same curve.

Dimensionless variables Suppose that a variable x is measured in units of kilograms and that x varies from 10 to 500 kilograms. If we set $y =$ (x kilograms)/(100 kilograms), y is dimensionless, and $0.1 \le y \le 5$. The smaller range of values is useful for computing. The fact that y has no units is useful because it no longer matters if the units are kilograms, grams, or some other units.

When variables are scaled to dimensionless quantities, they are typically divided by a constant somewhere around the middle of the expected range of values. For example, by dividing a chemical concentration by a "typical" concentration, we obtain a dimensionless

concentration variable. Similarly, dimensionless time is obtained by dividing ordinary time by a "standard" time.

Direction field A direction field is a collection of line segments in the xy-plane which shows the slope of the trajectories for an autonomous ODE system

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

at a representative grid of points. An arrowhead on a segment shows the direction of motion.

Disconnected Julia set A disconnected Julia set is actually a Cantor dust. It is composed entirely of totally disconnected points, which means that it is almost never possible to land on a point in the Julia set by clicking on a pixel. You will probably find that every click you can make starts an iteration that goes to infinity, only because you cannot actually land on an exact enough value to show a stable iteration.

Discrete dynamical system A discrete dynamical system takes the form $u_{n+1} = f(u_n)$, where the variable u_n gives the state of the system at "time" n, and u_{n+1} is the state of the system at time $n+1$. See Module 13.

Eigenvalues The eigenvalues of a matrix A are the numbers λ for which

$$Av = \lambda v$$

for some nonzero vector v (the vector v is called an eigenvector). The eigenvalues λ of a 2×2 matrix A are the solutions to the characteristic equation of A:

$$\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$$

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}$$

where $\operatorname{tr} A$ is the trace of A, and $\det A$ is the determinant of A. If a linear or linearized system of ODEs is $z' = A(z - p^*)$, and if the real parts of the eigenvalues of A are positive, then trajectories flow away from the equilibrium point, p^*. If the real parts are negative, then trajectories flow toward p^*.

Eigenvector An eigenvector of a matrix A is a nonzero vector, v, that satisfies $Av = \lambda v$ for some eigenvalue λ. The ODE Architect Tool calculates eigenvalues and eigenvectors of Jacobian matrices at any equilibrium point of an autonomous system (linear or nonlinear). Eigenvectors play a strong role in the local geometry of phase portraits at an equilibrium point.

Energy In physics and engineering, energy is defined by

$$E = \text{kinetic energy} + \text{potential energy}$$

where kinetic energy is interpreted to be the energy of motion, and the potential energy is the energy due to some external force, such as gravity, or (in electricity) a battery, or a magnet. If energy is conserved, i.e., stays at a constant level, then the system is said to be conservative.

If we are dealing with the autonomous differential system

$$x' = y, \qquad y' = -v(x) \tag{5}$$

we can define an "energy function" by

$$E = \frac{1}{2}y^2 + V(x)$$

where $dV/dx = v(x)$. Note that E is constant along each trajectory, because $dE/dt = y\,dy/dt + (dV/dx)(dx/dt) = y(-v(x)) + v(x)(y) = 0$, where the ODEs in system (5) have been used. The term $(1/2)y^2$ is the "kinetic energy". $V(x)$ is the "potential energy" in this context. See Chapter 10 for more on these ideas.

Epidemic An epidemic occurs in an epidemiological model if the number of infectives, $I(t)$, increases above its initial value, I_0. Thus, an epidemic occurs if $I'(0) > 0$.

Equilibrium point An equilibrium point p^* in phase (or state) space of an autonomous ODE is a point at which all derivatives of the state variables are zero; also known as a stationary point or a steady-state value of the state variables. For example, for the autonomous system,

$$x' = F(x, y), \qquad y' = G(x, y)$$

if $F(x^*, y^*) = 0$, $G(x^*, y^*) = 0$, then $p^* = (x^*, y^*)$ is an equilibrium point, and $x = x^*$, $y = y^*$ (for all t) is a constant solution.

For a discrete dynamical system, an equilibrium point p^* is one for which $f(p^*) = p^*$, so that $p^*_{n+1} = p^*_n$, for all n; p^* is also called a fixed point of the system.

Estimated error For the solution $u(t)$ of the IVP $y' = f(t, y)$, $y(t_0) = y_0$, the local error at the nth step of the Euler approximation is given by

$$e_n = \text{Taylor series of } u(t) - \text{Euler approximation}$$
$$= \frac{1}{2}h^2u''(t_n) + h^3u'''(t_n) + \cdots$$

If the true solution, $u(t)$, is not known, we can approximate e_n for small h by

$$e_n \approx \text{Taylor approx.} - \text{Euler approx.} \approx \frac{1}{2}h^2u''(t_n)$$

Euler's method Look at the IVP $y' = f(t, y),\ y(t_0) = y_0$. Euler's method approximates the solution $y(t)$ at discrete t values. For step size h , put $t_{n+1} = t_n + h$ for $n = 0, 1, 2, \ldots$. Euler's method approximates

$$y(t_1), y(t_2), \ldots$$

by the values

$$y_1, y_2, \ldots$$

where

$$y_{n+1} = y_n + hf(t_n, y_n), \quad \text{for } n = 0, 1, 2, \ldots$$

Existence and uniqueness A basic uniqueness and existence theorem says that, for the IVP,

$$x' = F(x, y, t), \qquad y' = G(x, y, t),$$
$$x(t_0) = x_0, \qquad y(t_0) = y_0$$

a unique solution $x(t)$, $y(t)$ exists if F, G, $\partial F/\partial x$, $\partial F/\partial y$, $\partial G/\partial x$, and $\partial G/\partial y$ are all continuous in some region containing (x_0, y_0).

Fixed point A fixed point, p^*, of a discrete dynamical system is a point for which $x_{n+1} = f(x_n) = x_n$. That is, iteration of such a point simply gives the same point.

A fixed point can also be called an equilibrium or a steady state. A fixed point may be a sink, a source, or a saddle, depending on the character of the eigenvalues of the associated linearization matrix of the iterating function.

Forced damped pendulum A forced, viscously damped pendulum has the modeling equation

$$mx'' + bx' + ksinx = F(t)$$

The beginning of Module 10 explains the terms and parameters of this equation using θ instead of x. Module 12 examines a case where chaos can result, with $b = 0.1,\ m = 1,\ k = 1$, and $F(t) = \cos t$. All three submodules of Module 12 are involved in explaining the behaviors, and the introduction to the Tangled Basins submodule shows a movie of what happens when b is varied from 0 to 0.5.

Forced pendulum Some of the most complex and curious behavior occurs when the pendulum is driven by an external force. In Module 10, The Pendulum and Its Friends, you can experiment with three kinds of forces in the Linear and Nonlinear Pendulums submodule, and an internal pumping force in the Child on a Swing submodule. But, for truly strange behavior, take a look at Module 12, Chaos and Control.

Fractal dimension Benoit Mandelbrot in the early 1980's coined the word "fractal" to apply to objects with dimensions between integers. The boundary of the Mandelbrot set (see glossary) is so complicated that its dimension is surely greater than one (the dimension of any "ordinary" curve). Just how much greater remained an open question until 1992 when the Japanese mathematician Mitsuhiro Shishikura proved it is actually dimension two!

Frequency The frequency of a function of period T is $1/T$. Another widely used term is "circular frequency", which is defined to be $2\pi/T$. For example, the periodic function $\sin(3t)$ has period $T = 2\pi/3$, frequency $3/(2\pi)$, and circular frequency 3.

General solution Consider the linear system $x' = Ax + u$ [where x has 2 components, A is a 2×2 matrix of constants, and u is a constant vector or a function only of t]. Let A have distinct eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2. All solutions of the system are given by the so-called general solution:

$$x(t) = C_1e^{\lambda_1 t}v_1 + C_2e^{\lambda_2 t}v_2 + \tilde{x}$$

where $\tilde{x}$ is any one particular solution of the system and C_1 and C_2 are arbitrary constants.. If u is a constant vector, then $\tilde{x} = p^*$, the equilibrium of the system. If x has more than two dimensions, terms of the same form are added until all dimensions are covered. Note that, if $u = 0,\ p^* = 0$ is an equilibrium.

Geodesic Any smooth curve can be reparametrized to a unit speed curve $x(t)$, where $|x'(t)| = 1$. Unit-speed curves $x(t)$ on a surface are geodesics if the acceleration vector $x''(t)$ is perpendicular to the surface at each point $x(t)$.

It can be shown that a geodesic is locally length-minimizing, so, between any two points sufficiently close, the geodesic curve is the shortest path.

GI tract The gastro-intestinal (GI) tract consists of the stomach and the intestines.

Gravitational force The gravitational force is the force on a body due to gravity. If the body is near the earth's surface, the force has magnitude mg, where m is the body's mass, and the force acts downward. The value of acceleration due to gravity, g, is 32 ft/sec^2 (English units), 9.8 meters/sec^2 (metric units).

Great circle A great circle on a sphere is an example of a geodesic. You can test this with a ball and string. Hold one end of the string fixed on a ball. Choose another point some distance away, and find the geodesic or shortest path by pulling the string tight between the two points. You will find that it always is along a circle centered at the center of the ball, which is the definition of a great circle.

Hooke's law Robert Hooke, an English physicist in the seventeenth century, stated the law that a spring exerts a force, on an attached mass, which is proportional to the displacement of the mass from the equilibrium position and points back toward that position.

Initial condition An initial condition specifies the value of a state variable at some particular time, usually at $t = 0$.

Initial value problem An initial value problem (IVP) consists of a differential equation or a system of ODEs and an initial condition specifying the value of the state variables at some particular time, usually at $t = 0$.

Integral surfaces The surface S defined by $F(x, y, z) = C$, where C is a constant, is an integral surface of the autonomous system

$$x' = f(x, y, z), \quad y' = g(x, y, z), \quad z' = h(x, y, z)$$

if

$$\begin{aligned}\frac{d}{dt}F(x, y, z) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial F}{\partial x}f + \frac{\partial F}{\partial y}g + \frac{\partial F}{\partial z}h = 0\end{aligned}$$

for all x, y, z. We get a family of integral surfaces by varying the constant C. An orbit of the system that touches an integral surface stays on it. The function F is called an integral of the system.

For example, the family of spheres

$$F = x^2 + y^2 + z^2 = \text{constant}$$

is a family of integral surfaces for the system

$$x' = y, \qquad y' = z - x, \qquad z' = -y$$

because

$$2xx' + 2yy' + 2zz' = 2xy + 2y(z - x) + 2z(-y) = 0$$

Each orbit lies on a sphere, and each sphere is covered with orbits.

Intermediate An intermediate is a chemical produced in the course of a reaction which then disappears as the reaction comes to an end.

Intrinsic growth rate At low population sizes, the net rate of growth is essentially proportional to population size, so that $N' = rN$. The constant r is called the intrinsic growth rate constant. It gives information about how fast the population is changing before resources become limited and reduce the growth rate.

Iteration Iteration generates a sequence of numbers by using a given number x_0 and the rule $x_{n+1} = f(x_n)$, where $f(x)$ is a given function. Sometimes, x_n is written as $x(n)$.

IVP See initial value problem.

Jacobian matrix The system $x' = F(x, y)$, $y' = G(x, y)$, has the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix at an equilibrium point p^* help determine the local geometry of the phase portrait.

Jacobian matrices J can be defined for autonomous systems of ODEs with any number of state variables.

The ODE Architect Tool will find eigenvalues and eigenvectors of J at any equilibrium point.

Julia Set In complex dynamics, a Julia set for a given function $f(z)$ separates those points that iterate to infinity from those that do not. See the third submodule of Module 13, Dynamical Systems. Julia sets were discovered about 1910 by two French mathematicians, Pierre Fatou and Gaston Julia. But, without computer graphics, they were unable to see the details of ragged structure that today display Cantor sets, self-similarity and fractal properties.

Kinetic energy of rotation The kinetic energy of rotation of a gyrating body is

$$E = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

where I_j and ω_j are, respectively, the moment of inertia and the angular velocity about the body axis, j, for $j = 1, 2, 3$.

Lift The lift force on a body moving through air is a force that acts in a direction orthogonal to the motion. Its magnitude may be modeled by a term which is proportional to the speed or to the square of the speed.

Limit cycle A cycle is a closed curve orbit of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

A cycle is the orbit of a periodic solution.

An attracting limit cycle is a cycle that attracts all nearby orbits as time increases; a repelling limit cycle repels all nearby orbits as time increases.

Linearization For a nonlinear ODE, a linearization (or linear approximation) can be made about an equilibrium, $p^* = (x^*, y^*)$, as follows:

For $x' = F(x, y),\ y' = G(x, y)$, the linearized system is $z' = J(z - p^*)$, where J is the Jacobian matrix evaluated at p^*, i.e.,

$$z' = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}_{p^*} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} = J(z - p^*)$$

The eigenvalues and eigenvectors of the Jacobian matrix, J, at an equilibrium point, p^*, determine the geometry of the phase portrait close to the equilibrium point p^*. These ideas can be extended to any autonomous system of ODEs. A parallel definition applies to a discrete dynamical system.

Linear pendulum Pendulum motion can be modeled by a nonlinear ODE, but there is an approximating linear ODE that works well for small angles θ, where $\sin\theta \approx \theta$. In that case, the mathematics is the same as that discussed for the mass on a spring in Module 4.

Linear system A linear system of first-order ODEs has only terms that are linear in the state variables. The coefficients can be constants or functions (even nonlinear) of t.

Example: Here is a linear system with state variables x and y, and constant coefficients $a, b, \ldots, h$:

$$x' = ax + by + c$$
$$y' = fx + gy + h$$

This can be written in matrix/vector form as:

$$z' = Az + k$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ f & g \end{bmatrix}, \qquad k = \begin{bmatrix} c \\ h \end{bmatrix}$$

The example can be extended to n state variables and an $n \times n$ matrix A. If $z = p^*$ is an equilibrium point of a linear system, then $k = -Ap^*$ and the system may be written as

$$z' = A(z - p^*)$$

Linear algebra can be applied to find the general solution.

Lissajous figures Jules Antoine Lissajous was a 19th-century French physicist who devised ingenious ways to visualize wave motion that involves more than one frequency. For example, try plotting the parametric curve $x_1 = \sin 2t,\ \ x_2 = \sin 3t$ in the x_1x_2-plane with $0 \le t \le 320$.

The graph of a solution $x_1 = x_1(t),\ x_2 = x_2(t)$ of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{for } B \text{ a } 2 \times 2 \text{ constant matrix}$$

in the x_1x_2-plane is a Lissajous figure if the vector $(x_1(0), x_2(0))$ is not an eigenvector of B.

See also "Normal modes and frequencies."

Local IVP One-step methods for approximating solutions to the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

generate the $(n+1)$st approximation, y_{n+1}, from the nth, y_n, by solving the local IVP

$$u' = f(t, u), \qquad u(t_n) = y_n$$

This is exactly the same ODE, but the initial condition is different at each step.

Logistic model The logistic equation is the fundamental model for population growth in an environment with limited resources. Many advanced models in ecology are based on the logistic equation.

For continuous models, the logistic ODE is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r is the intrinsic growth rate constant, and K is the carrying capacity.

For discrete models, the logistic map is

$$f_\lambda(x) = \lambda x\left(1 - \frac{x}{K}\right)$$

where λ is the intrinsic growth rate constant, and K is again the carrying capacity (see Module 13).

Mandelbrot Set In complex dynamics, for $f_c(z) = z^2 + c$, the Mandelbrot set is a bifurcation diagram in the complex c-plane, computed by coloring all c-values for which z_0 (usualy $z_0 = 0$) does not iterate to infinity.

It acts as a catalog of all the Julia sets for individual values of c.

The boundary of the Mandelbrot set is even more complicated than the boundary of a given Julia set. More detail appears at every level of zoom, but no two regions are exactly self-similar.

Two mathematicians at UCLA, R. Brooks and J. P. Matelsky, published the first picture in 1978. It is now called the Mandelbrot set, because Benoit Mandelbrot of the Thomas J. Watson IBM Research Center made it famous in the early 1980's.

You can experiment with the Mandelbrot set in Module 13 and the Discrete Tool.

Matrix An $n \times n$ square matrix A of constants, where n is a positive integer, is an array of numbers arranged into n rows and n columns. The entry where the ith row meets the jth column is denoted by a_{ij}.

In ODEs we most often see matrices A as the array of coefficients of a linear system. For example, here is a planar linear system with a 2×2 coefficient matrix A:

$$\begin{aligned} x' &= 2x - 3y \\ y' &= 7x + 4y \end{aligned} \qquad A = \begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix}$$

Mixing A function $f : R \to R$ is "mixing" if given any two intervals I and J there exists an $n > 0$ such that the nth iterate of I intersects J.

Modeling A mathematical model is a collection of variables and equations representing some aspect of a physical system. In our case, the equations are differential equations. Steps involved in the modeling process are:

1. State the problem.
2. Identify the quantities to which variables are to be assigned; choose units.
3. State laws which govern the relationships and behaviors of the variables.
4. Translate the laws and other data into mathematical notation.
5. Solve the resulting equations.
6. Apply the mathematical solution to the physical system.
7. Test to see whether the solution is reasonable.
8. Revise the model and/or restate the problem, if necessary.

Moment of inertia The moment of inertia, I, of a body B about an axis is given by

$$I = \iiint_B r^2 \rho(x, y, z)\, dV(x, y, z)$$

where r is the distance from a general point in the body to the axis and ρ is the density function for B. Each moment of inertia plays the same role as mass does in nonrotational motion, but, now, the shape of the body and the position of the axis play a role.

Newtonian damping A body moving through air (or some other medium) is slowed down by a resistive force that acts opposite to the body's velocity, v. In Newtonian damping (or Newtonian drag), the magnitude of the force is proportional to the square of the magnitude of the velocity, i.e., to the square of the speed:

$$\text{force} = -k|v|v \quad \text{for some positive constant } k$$

Newton's law of cooling The temperature, T , of a warm body immersed in a cooler outside medium of temperature T_{out} changes at a rate proportional to the temperature difference,

$$\frac{dT}{dt} = k(T_{out} - T)$$

where T_{out} is assumed to be unaffected by T (unless stated otherwise). The same ODE works if T_{out} is larger than T (Newton's law of warming).

Newton's second law Newton's second law states that, for a body of constant mass,

$$\text{mass} \cdot \text{acceleration} = \text{sum of forces acting on body}$$

This is a differential equation, because acceleration is the rate of change of velocity, and velocity is the rate of change of position.

Nodal equilibrium The behavior of the trajectories of an autonomous system of ODEs is nodal at an equilibrium point if all nearby trajectories approach the equilibrium point with definite tangents as $t \to +\infty$ (nodal sink), or as $t \to -\infty$ (nodal source).

If the system is linear with the matrix of coefficients A, then the equilibrium is a nodal sink if all eigenvalues of A are negative, and a nodal source if all eigenvalues are positive. This also holds at an equilibrium point of any nonlinear autonomous system, where A is the Jacobian matrix at the equilibrium point.

Nonautonomous ODE A system of ODEs with t occurring explicitly in the expressions for the rates is nonautonomous; e.g., $x' = tx$ is nonautonomous.

Nonlinear center point An equilibrium point of a nonlinear system, $x' = F(x, y)$, $y' = G(x, y)$, is a center if all nearby orbits are simple closed curves enclosing the equilibrium point.

Nonlinear ODE A nonlinear ODE or system has at least some dependent variables appearing in nonlinear terms (e.g., xy, $\sin x$, $\sqrt{x}$). Thus, linear algebra cannot be applied to the system overall. But, near an equilibrium (of which there are usually more than one for a nonlinear system of ODEs), a linearization is (usually) a good approximation, and allows analysis with the important roles of the eigenvalues and eigenvectors.

Nonlinear pendulum Newton's laws of motion give us

$$\text{force} = \text{mass} \times \text{acceleration}$$

In the circular motion of a pendulum of fixed length, L, at angle θ, acceleration is given by $L\theta''$. The only forces acting on the undamped pendulum are those due to tension in the rod and gravity. The component of force in the direction of the pendulum bob's motion is:

$$F = mL\theta'' = -mg\sin\theta$$

where m is the mass of the pendulum bob, and g is the acceleration due to gravity. The mass of the rigid support rod is assumed to be negligible.

Normal modes and frequencies The normal modes of a second-order system $z'' = Bz$ (where B is a 2×2 matrix with negative eigenvalues μ_1, μ_2) are eigenvectors v_1, v_2 of B. The general solution is all linear combinations of the periodic oscillations z_1, z_2, z_3, z_4 along the normal modes:

$$z_1 = v_1 \cos\omega_1 t, \quad z_2 = v_1 \sin\omega_1 t,$$
$$z_3 = v_2 \cos\omega_2 t, \quad z_4 = v_2 \sin\omega_2 t$$

where $\omega_1 = \sqrt{-\mu_1}$, $\omega_2 = \sqrt{-\mu_2}$ are the normal frequencies.

See also "Second-order systems."

Normalized ODE In a normalized differential equation, the the highest-order derivative appears alone in a separate term and has a coefficient equal to one.

ODE See "Ordinary differential equation."

On-off function See "Square wave."

Orbit See "Trajectory."

Order of the method A method of numerical approximation to a solution of an IVP is order p, if there exists a constant C such that

$$\max(|\text{global error}|) < Ch^p$$

as $h \to 0$.

Ordinary differential equation An ordinary differential equation (ODE) is an equation involving an unknown function and one or more of its derivatives. The order of the ODE is the order of the highest derivative in the ODE. Examples:

$$\frac{dy}{dt} = 2t, \qquad \text{(first-order, unknown } y(t)\text{)}$$
$$y^2\frac{dy}{dt} = 2y + t, \qquad \text{(first-order, unknown } y(t)\text{)}$$
$$x'' - 4x' + 7x = 4\sin 2t, \qquad \text{(second-order, unknown } x(t)\text{)}$$

Oscillation times Oscillation times of a solution curve $x(t)$ of an ODE that oscillates around $x = 0$ are the times between successive crossings of $x = 0$ in the same direction. If the solution is periodic, the oscillation times all equal the period.

Oscillations A scalar function $x(t)$ oscillates if $x(t)$ alternately increases and decreases as time increases. The oscillation is periodic of period T if $x(t + T) = x(t)$ for all t and if T is the smallest positive number for which this is true.

Parametrization Each coordinate of a point in space may sometimes be given in terms of other variable(s) or parameter(s). A single parameter suffices to describe a curve in space. Two parameters are required to describe a two-dimensional surface.

Period The period of a periodic function $u(t)$ is the smallest time interval after which the graph of u versus t repeats itself. It can be found by estimating the time interval between any two corresponding points, e.g., successive absolute maxima.

The period of a cycle in a discrete dynamical system is the minimal number of iterations after which the entire cycle repeats.

Periodic phase plane The periodic xx'-phase plane for the pendulum ODE $x'' + 0.1x' + \sin x = \cos t$ is plotted periodically in x. An orbit leaving the screen on the right comes back on the left. In other words, the horizontal axis represents x mod 2π. This view ignores how many times the pendulum bob has gone over the top. See Module 12, Screen 1.4.

Phase angle The phase angle, δ, of the oscillatory function $u(t) = A\cos(\omega_0 t + \delta)$ shifts the the graph of $u(t)$ from the position of a standard cosine graph $u = \cos\omega_0 t$ by the amount δ/ω_0 . The phase angle may have either sign and must lie in the interval $-\pi/\omega_0 < \delta < \pi/\omega_0$.

Phase plane The phase plane, or state plane, is the xy-plane for the dependent variables x and y of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

The trajectory, or orbit, of a solution

$$x = x(t), \qquad y = y(t)$$

of the system is drawn in this plane with t as a parameter. A graph of trajectories is called a phase portrait for the system.

The higher-dimensional analog is called phase space, or state space.

Pitch The pitch (frequency) of an oscillating function $u(t)$ is the number of oscillations per unit of time t.

Poincaré Henri Poincaré (1854–1912) was one of the last mathematicians to have a universal grasp of all branches of the subject. He was also a great popular writer on mathematics. Poincaré's books sold over a million copies.

Poincaré section A Poincaré section of a second-order ODE $x'' = f(x, x', t)$, where f has period T in t, is a strobe picture of the xx'-phase plane that plots only the points of an orbit that occur at intervals separated by a period of T time units, i.e., the sequence of points

$$\begin{aligned} P_0 &= (x(0), x'(0)) \\ P_1 &= (x(T), x'(T)) \\ &\vdots \\ P_n &= (x(nT), x'(nT)) \\ &\vdots \end{aligned}$$

For further detail, see the second submodule of Module 12, Chaos and Control.

A Poincaré section is a two-dimensional discrete dynamical system. Another example of such a system is discussed in some detail in the third submodule of Module 13. These ideas can be extended to any system of ODEs and T can be any positive number.

Population quadrant In a two-species population model, the population quadrant of the phase plane is the one where both dependent variables are non-negative.

Post-image In a discrete dynamical system, a post-image of a set S_0 is the set of points, S_1 , where the iterates of S_0 land in one step.

For a Poincaré section of an ODE, S_1 would be the set of points arriving at S_1 when the ODE is solved from S_0 over one time period of the Poincaré section. See submodule 3 of Module 12.

Pre-image In a discrete dynamical system, a pre-image of a set S_0 is the set of points, S_{-1}, that iterate to S_0 in one step.

For a Poincaré section of an ODE, S_{-1} would be the set of points arriving at S_0 when the ODE is solved from S_{-1} over one time period of the Poincaré section. See submodule 3 of Module 12.

Products The products of a chemical reaction are the species produced by a reaction step. The end products are the species that remain after all of the reaction steps have ended.

Proportional Two quantities are proportional if their ratio is constant. Thus, the circumference, c, of a circle is proportional to the diameter, because $c/d = \pi$.

The basic linear differential equation

$$\frac{dy}{dt} = ky, \quad k \text{ a constant}$$

represents a quantity y whose derivative is proportional to its value.

Random Random motion is the opposite of deterministic motion. In random motion, there is no way to predict the future state of a system from knowledge of the initial state. For example, if you get heads on the first toss of a coin, you cannot predict the outcome of the fifth toss.

Rate constant Example: The constant coefficients a, b, and c in the rate equation

$$x'(t) = ax(t) - by(t) - cx^2(t)$$

are often called rate constants.

Rates of chemical reactions The rate of a reaction step is the speed at which a product species is created or (equivalently) at which a reactant species is destroyed in the step.

Reactant A chemical reactant produces other chemicals in a reaction.

Resonance This phenomenon occurs when the amplitude of a solution of a forced second-order ODE becomes either unbounded (in an undamped ODE) or relatively large (in a damped ODE) after long enough times.

Rotation system As Lagrange discovered in the 18th century, the equations of motion governing a gyrating body are

$$\begin{aligned} \omega_1' &= \frac{(I_2 - I_3)\omega_2\omega_3}{I_1} \\ \omega_2' &= \frac{(I_3 - I_1)\omega_1\omega_3}{I_2} \\ \omega_3' &= \frac{(I_1 - I_2)\omega_1\omega_2}{I_3} \end{aligned}$$

where I_j is the principal moment of inertia, and ω_j is the component of angular velocity about the jth body axis.

Saddle An equilibrium point of a planar autonomous ODE, or a fixed point of a discrete two-dimensional dynamical system, with the property that, in one direction (the unstable one), trajectories move away from it, while, in another direction (the stable one), trajectories move toward it.

At a saddle of an ODE, one eigenvalue of the associated linearization matrix must be real and positive, and at least one eigenvalue must be real and negative.

Scaling Before computing or plotting, variables are often scaled for convenience.

See also "Dimensionless variables."

Second-order systems Second-order systems of the form $z'' = Bz$ often arise in modeling mechanical structures with no damping, (and hence, no loss of energy). Here, z is an n-vector state variable, z'' denotes d^2z/dt^2, and B is an $n \times n$ matrix of real constants.

Although numerical solvers usually require that we introduce $v = z'$ and enter the system of $2n$ first-order ODEs, $z' = v$, $v' = Bz$, we can learn a lot about solutions directly from the eigenvalues and eigenvectors of the matrix B.

See also "Normal modes and frequencies" and Screen 3.4 in Module 6.

Sensitivity An ODE model contains elements, such as initial data, environmental parameters, and functions, whose exact values are experimentally determined. The effect on the solution of the model ODEs when these factors are changed is called sensitivity.

Sensitivity to initial conditions A dynamical system has sensitive dependence on initial conditions if every pair of nearby points eventually gets mapped to points far apart.

Separatrix Separatrices are trajectories of a planar autonomous system that enter or leave an equilibrium point p with definite tangents as $t \to \pm\infty$, and divide a neighborhood of p into distinct regions of quite different long-term trajectory behavior as t increases or decreases.

For more on separatrices see "Separatrices and Saddle Points" in Chapter 7.

Sink A sink is an equilibrium point of a system of ODEs, or a fixed point of a discrete dynamical system, with the property that all trajectories move toward the equilibrium.

If all eigenvalues of the associated linearization matrix at an equilibrium of a system of ODEs have negative real parts, then the equilibrium is a sink.

Slope The slope of a line segment in the xy-plane is given by the formula

$$m = \frac{\text{change in } y}{\text{change in } x}$$

The slope of a function $y = f(x)$ at a point is the value of the derivative of the function at that point.

Slope field A slope field for the first-order ODE $y' = f(t, y)$ is a collection of line segments whose slopes are determined by the value of $f(t, y)$ on a grid of points in the ty-plane.

Solution A solution to a differential equation is any function which gives a true statement when plugged into the equation. Such a function is called a particular solution. Thus,

$$y = t^2 - 2$$

is a particular solution to the equation

$$\frac{dy}{dt} = 2t$$

The set of all possible solutions to a differential equation is called the general solution. Thus,

$$y = t^2 + C$$

is the general solution to the equation

$$\frac{dy}{dt} = 2t$$

Source A source is an equilibrium of a system of ODEs, or a fixed point of a discrete dynamical system, with the property that all trajectories move away from the equilibrium.

If all eigenvalues of the associated linearization matrix at an equilibrium of a system of ODEs have positive real parts, then the equilibrium is a source.

Spiral equilibrium An equilibrium point of a planar autonomous system of ODEs is a spiral point if all nearby orbits spiral toward it (or away from it) as time increases.

If the system is linear with the matrix of coefficients A, then the equilibrium is a spiral sink if the eigenvalues of A are complex conjugates with negative real part, a spiral source if the real part is positive. This also holds at an equilibrium point of any nonlinear planar autonomous system, where A is the Jacobian matrix at the equilibrium.

Spring A Hooke's law restoring force (proportional to displacement, x, from equilibrium) and a viscous damping force (proportional to velocity, but oppositely directed) act on a body of mass m at the end of spring. By Newton's second law,

$$mx'' = -kx - bx'$$

where k and b are the constants of proportionality.

Spring force The spring force is often assumed to obey Hooke's law—the magnitude of the force in the spring is proportional to the magnitude of its displacement from equilibrium, and the force acts in the direction opposite to the displacement.

The proportionality constant, k, is called the spring constant. A large value of k corresponds to a stiff spring.

Square wave An on-off function (also called a square wave) is a periodic function which has a constant nonzero value for a fraction of each period; otherwise, it has a value of 0. For example, $y = A\text{SqWave}(t, 6, 2)$ is a square wave of amplitude A and period 6, which is "on" for the first 2 units of its period of 6 units, then is off the next 4 time units.

Stable An equilibrium point p^* of an autonomous system of ODEs is stable if trajectories that start near p^* stay near p^*, as time advances. The equilibrium point $p^* = 0$ of the linear system $z' = Az$, where A is a matrix of real constants, is stable if all eigenvalues of A are negative or have negative real parts.

State space The phase plane, or state plane, is the xy-plane for the dependent variables x and y of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

The trajectory, or orbit, of a solution

$$x = x(t), \qquad y = y(t)$$

of the system is drawn in this plane with t as a parameter. A graph of trajectories is called a phase portrait for the system.

The higher dimensional analog is called phase space, or state space.

State variables These are dependent variables whose values at a given time can be used with the modeling ODEs to determine the state of the system at any other time.

Steady state A steady state of a system of ODEs is an equilibrium position where no state variable changes with time. Sometimes only an attracting equilibrium point (or periodic solution) is called a steady state.

Surface A surface of a three-dimensional object is just its two-dimensional "skin," and does not include the space or volume enclosed by the surface.

Taylor remainder For an $(n+1)$-times differentiable function $u(t)$, the difference (or Taylor remainder)

$$u(t) - [u(t_0) + hu'(t_0) + \cdots + \frac{1}{n!}h^n u^{(n)}(t_0)]$$

can be written as

$$\frac{1}{(n+1)!}h^{(n+1)}u^{(n+1)}(c)$$

for some c in the interval $[t_0, t_0 + h]$, a fact which gives useful estimates.

Taylor series expansion For an infinitely differentiable function $u(t)$, the Taylor series expansion at t_0 for $u(t_0 + h)$ is

$$u(t_0) + hu'(t_0) + \frac{1}{2}h^2u''(t_0) + \cdots + \frac{1}{n!}h^n u^{(n)}(t_0) + \cdots$$

Taylor series method Look at the IVP $y' = f(t, y)$, $y(t_0) = y_0$. For a step size h, the three-term Taylor series method approximates the solution $y(t)$ at $t_{n+1} = t_n + h$, for $n = 0, 1, 2, \ldots$, using the algorithm

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{1}{2}h^2 f_t(t_n, y_n)$$

Trace The trace of a square matrix is the sum of its diagonal entries. So

$$\text{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$$

Trace-determinant parabola The eigenvalues λ_1, λ_2 of a 2×2 matrix A are given by

$$\lambda_1, \lambda_2 = \frac{\text{tr}\, A \pm \sqrt{\text{tr}^2\, A - 4 \det A}}{2}$$

The trace-determinant parabola, $4 \det A = \text{tr}^2\, A$, divides the $\text{tr}\, A - \det A$ plane into the upper region where A's eigenvalues are complex conjugates and the lower region where they are real. The two eigenvalues are real and equal on the parabola.

Trajectory A trajectory (or orbit, or path) is the parametric curve drawn in the xy-plane, called the phase plane or state plane, by $x = x(t)$ and $y = y(t)$ as t changes,

where $x(t)$, $y(t)$ is a solution of

$$x' = F(x, y, t)$$
$$y' = G(x, y, t)$$

The trajectory shows how $x(t)$ and $y(t)$ play off against each other as time changes.

For a higher-dimensional system, the definition extends to parametric curves in higher-dimensional phase space or state space.

Unstable An equilibrium point p^* of an autonomous system of ODEs is unstable if it is not stable. That means there is a neighborhood N of p^* with the property that, starting inside each neighborhood M of p^*, there is at least one trajectory that goes outside N as time advances.

Vector A vector is a directed quantity with length. In two dimensions, a vector can be written in terms of unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, directed along the positive x- and y-axes.

Viscous damping A body moving through air (or some other medium) is slowed down by a resistive force that acts opposite to the body's velocity, v. In viscous damping (or viscous drag), the force is proportional to the velocity:

$$\text{force} = -kv$$

for some positive constant k.

Wada property The Wada property, as described and illustrated on Screen 3.2 of Module 12 is the fact that:

> Any point on the boundary of any one of the areas described on Screen 3.2 is also on the boundary of all the others.

The geometry/topology example constructed by Wada was the first to have this property; we can now show that the basins of attraction for our forced, damped pendulum ODE have the same property. See Module 12 and Chapter 12.

All we know about Wada is that a Japanese manuscript asserts that someone by that name is responsible for constructing this example, showing that for three areas in a plane, they can become so utterly tangled that every boundary point touches all three areas!