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q-Fractional Calculus and Equations

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Dedicated to the Martyrs of the Great Egyptian 2011 Revolution

Foreword

When I was a graduate student in the early 1970s I was interested in q-series, special functions and I had some interest in fractional calculus. The prevailing wisdom at the time was that none of these subjects will lead anywhere. It is amazing how much things have changed since then and here I am, 40 years later, writing a Preface to a wonderful monograph on q-fractional calculus and q-difference equations. I worked with Waleed Al-Salam who was enthusiastic about the subject. Earlier he introduced the operators of q-fractional calculus and in the early 1970s he worked with Arun Verma on q-fractional Leibniz rule. They found q-analogues of part of a series of papers by Tom Osler on fractional calculus, whose first paper appeared in 1970.

In my opinion the recent developments in q-difference equations and q-fractional calculus naturally followed the developments in the theory of q-series and orthogonal polynomials by a quite a few mathematicians under the leadership of Richard Askey and George Andrews. By the late 1980s the importance of difference and q-difference equations, as well as the Askey–Wilson operator equations, became apparent and Mahmoud Annaby started studying the old papers by Adams, Birkhoff, and their collaborators. He started working with students and other mathematicians from Cairo on related problems and they were naturally led to work on operators of fractional and q-fractional calculus as well as difference and q-difference equations.

Fractional calculus has a long history and has recently gone through a period of rapid development. In writing any book one has to pick and choose from a wide range of topics. The present monograph covers a selection of topics in a rigorous way. It starts with elementary calculus of q-differences and q-integration. Then presents a study of q-difference equations. The existence and uniqueness theorems are derived using successive approximations, leading to systems of equations with retarded arguments. Regular q-Sturm–Liouville theory is also introduced; Green's function is constructed and the eigenfunction expansion theorem is established. The Abel integral equation is the prime example of a Volterra integral equation which can be solved via the Riemann–Liouville operators of fractional calculus. The authors study fractional q-calculi of many types including Riemann–Liouville; Grünwald–Letnikov; Caputo; Erdélyi–Kober and Weyl, in some detail. One aspect I liked is the rigorous treatment of fractional q-Leibniz rule and its application. This puts on solid ground many classical results starting from those early papers of F. H. Jackson down to the recent papers of Al-Salam and Verma. It is nice to see these results written and the domains of their validity precisely specified.

In special functions the tradition is that identities are proved but in many cases one side gives analytic continuation of the other side. A case in point is the Pfaff–Kummer transformation

$$ {}_2F_1(a,b,c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c,z/(z-1)). $$

This is an identity if $|z| < 1$ and $|z| < |1-z|$. On the other hand, the right-hand side gives an analytic continuation of the left-hand side if $|z| > 1$ but $|z| < |1-z|$. Many identities in this work can be considered as analytic continuation formulas to domains wider than their domains of validity.

Some integral equations of Volterra and Abel type work as an introductory material for the study of fractional q-calculi.

The investigation of q-fractional difference equations leads to families of q-Mittag-Leffler functions which are defined and their properties are investigated, especially the distribution, asymptotic and reality of their zeros, establishing q-counterparts of Wiman's results. Fractional q-difference equations are studied; existence and uniqueness theorems are given and classes of Cauchy-type problems are completely solved in terms of families of q-Mittag-Leffler functions. Among many q-analogues of classical results and concepts, q-Laplace and Fourier transforms are studied and their applications are investigated

As I said earlier this monograph is too brief to be encyclopedic, so the authors had to restrict their coverage of topics within the subject. For example dual integral and series equations are not covered but the reader can find this in Sneddon [276]. Another topic of interest is that operators of fractional calculus can be used to construct reproducing kernels with known eigenvalues and eigenfunctions. Another important missing topic is the characterization of the ranges of various fractional integral operators. This is related to the theory of multipliers for the Mellin transform. Some references are [259, 260]. Zeinab Mansour kindly informed me of her joint work in progress on q-analogues of these results. I look forward to seeing this work completed.

I am sure the publication of this book will stimulate further research in this area.

Orlando, FL Mourad Ismail

Preface

This book is a rigorous study of q-fractional calculus and q-fractional difference equations. Our study is developed starting from the work of Agarwal [17], Al-Salam [18, 19], and Al-Salam and Verma [20]. In [17], the q-fractional Riemann–Liouville calculus is defined formally and many properties are given as well. In comparison with the celebrated monographs on fractional calculus, for example, the first book on fractional calculus [227] written by Oldham and Spanier, the books of Samko et al. [269], Kilbas et al. [169], Podlubny [234], and Diethelm [82], we can see that the q-fractional theory is far from being a well-established q-counterpart of the existing fractional theory. However, we hope that the present work would be a takeoff point to establish a more comprehensive q-fractional theory. We would like to mention that our q-study is based on a q-difference operator and its associated right inverse. This q-difference operator goes back to Euler and may go back to Heine and is reintroduced by Jackson in [158]. Sometimes it is called Euler–Jackson q-difference operator or simply Jackson q-difference operator as we do through the entire book. The main objective of this book is to provide such an overview of the basic theory of fractional q-difference equations, methods of their solutions and applications, taking into account the audience of this book, namely the applied scientists interested in developing the q-theory, investigating and exploring its applications. Applications of the classical fractional calculus appeared in many publications. For example, Sneddon's book [276] on mixed boundary value problems from the mid-1960s included a survey of fractional calculus and how to use it to solve integral equation arising in elasticity. In addition, applications of fractional calculus in mathematical physics, probability, and modeling were introduced in the mid-1970s in [261]. Recently, more applications in classical mechanics, particle physics, diffusion systems, viscoelastic and disordered modern electrical systems, modeling and control are in [267]. Perhaps Leibniz [179] did not expect this number of applications when he sent a letter in 1695 to L'Hôpital asking about the meaning of the derivative of order half. From this point of view, we expect that in the long run, many applications of the fractional q-calculus will appear.

This book consists of nine chapters. Chapter 1 provides some basic definitions and properties of q-analysis as the q-difference operator, the q-integral operator, q-special functions, and q-integral transforms. Chapter 2 is a study of the existence and uniqueness of the solutions of first-order systems of q-difference equations and linear q-difference equations. This chapter also includes some results on zeros of q-trigonometric functions and q-Bessel functions. Chapter 3 includes the basic Sturm–Liouville problem formulated and studied in [30]. It also includes the reformulation introduced in [207] of the q^2-Fourier transform introduced by Rubin in [265, 266]. In Chap. 4, we survey the developments in the fractional q-theory since Al-Salam and Agarwal introduced their generalization to Jackson q-integral and derivatives to fractional orders. It contains the fractional q-calculus associated with Al-Salam and Agarwal fractional q-analogue of the Riemann–Liouville fractional derivatives. Chapter 5 is devoted to other approaches of extending the notion of q-integrals and q-derivatives to fractional orders like q-Caputo fractional derivatives and q-Weyl fractional derivatives. In this chapter we show that a generalization to Grünwald–Letnikov fractional derivatives in the q-settings leads to Al-Salam–Agarwal fractional q-derivatives. We outline the generalization of the Askey–Wilson q-difference operator to fractional orders introduced by Ismail and Rahman in [147]. We conclude this chapter with a generalization of the q-difference operator introduced by Rubin in [265, 266] to fractional orders. In Chap. 6, we give a rigorous proof of Al-Salam–Verma fractional q-Leibniz rule [20] and a generalization of the fractional q-Leibniz rule introduced by Agarwal in [15]. We also introduce a fractional q-Leibniz rule associated with Weyl fractional q-operator. This result is a generalization of the result introduced by Purohit in [248]. At the last section of this chapter, we derive some q-identities using the fractional q-Leibniz formulae represented in this chapter. Chapter 7 is fully devoted to q-Mittag-Leffler functions and their major properties. We explore the Mellin–Barnes contour representations and Hankel contour representations of the two q-analogues of the Mittag-Leffler functions considered in this book. Chapter 8 includes fundamental existence and uniqueness theorems for linear and nonlinear fractional q-difference equations as well as first-order systems of fractional q-difference equations, where the q-derivative is either the Riemann–Liouville fractional q-derivative or Caputo fractional q-derivative. Most of the results of this chapter are a generalization of the results mentioned in Chap. 2. In Chap. 9, the last chapter, we investigate the applications of the q-Laplace, q-Mellin, and q^2-Fourier integral transforms to constructing explicit solutions of certain classes of linear fractional q-difference equations. In the appendix, we include tables of fractional q-derivatives of q-special functions and generalized Rodrigues-type formulae for some q-special functions. The bibliography consists of 302 books and articles, including some recent preprints submitted for publications, up to 2011. However, it cannot be considered as a complete bibliography since this discipline is a fast-growing area. But, on the other hand, we believe that the references of the bibliography and references mentioned therein are enough to get a complete overview of the developments occurred in this subject up to the year 2011.

The authors would like to express their gratitude to their colleagues in the Mathematical Analysis seminar of the department of Mathematics, Faculty of Science, Cairo University for the wonderful environment they created. The authors are also grateful to Alexander von Humboldt foundation for supporting their visit to Mathematisches Seminar, A.C.U Zu Kiel, fall 2006 under the grant 3.4-AGY/1039259, where the first seeds of this book were planted. In this respect, they wish to thank professor Walter Bergweiler for hospitality. Mahmoud Annaby is so grateful to Professor Paul Butzer, RWTH-Aachen for the fruitful discussions held during his visits to Lehrstuhl A für Mathematik, RWTH-Aachen, Germany. Again, Annaby thanks Alexander von Humboldt for supporting his visit, 3.4-8131-KAT/1039259, to the Institute of Mathematics, University of Lübeck, where he started the final revision of the book. Thanks also to Professor Jürgen Prestin. The authors would like to express their gratitude to Professor Mourad Ismail for his constant inspiration, encouragement, and support throughout the whole period of preparing this book project. Particularly, for the fruitful discussions he held with Zeinab Mansour during her visit as a Fulbright Scholar in the academic year 2008/2009 at the university of Central Florida, Florida, USA, and at City University of Hong Kong, Hong Kong, July 2011. Moreover, Zeinab Mansour is greatly thankful and grateful to her husband, Abdallah, for his constant support, encouragement, and understanding; with him, a lot of her dreams came true. She is also thankful to her colleague, Dr. Ahmad El-Guindy, for his constant support and fruitful discussions and to Ms. Zeinab Yussuf for her support in the manuscript revision. Finally, Zeinab Mansour would like to thankfully acknowledge the financial support awarded by King Saud University, Riyadh through grant DSFP/MATH 01 and by NPST Program of King Saud University; project number 10-MAT1293-02.

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Contents

Acronyms

Here, we collect a list of symbols that have been used in this book.
Sets

$\mathbb{N}$ $= \{1, 2, 3, \ldots\}$, the sets of natural numbers.
$\mathbb{N}_0$ $= \mathbb{N} \cup \{0\}$.
$\mathbb{Z}^-$ $= \{0, -1, -2, -3, \ldots\}$, the set of all non positive integers.
$\mathbb{R}^+$ $= \{x \in \mathbb{R} \,:\, x > 0\}$.
$\mathbb{R}_{q,+}$ $= \{q^k \,:\, k \in \mathbb{Z}\}$.
$\mathbb{R}_q$ $= \{\pm q^k \,:\, k \in \mathbb{Z}\}$.
$\mathbb{R}$ The set of all real numbers.

Function Spaces

$L^p_{q,\eta}(0, a)$; $\eta \in \mathbb{R}$, $p \geq 1$ The space of all functions f defined on $(0, a]$ that satisfy $\int_0^a t^\eta |f(t)|^p \, d_q t < \infty$.

$\mathscr{L}^p_{q,\eta}(0, a)$; $\eta \in \mathbb{R}$, $p \geq 1$ The space of all functions f defined on $(0, a]$ that satisfies $t^\eta f^p(t) \in L^p_{q,\eta}(0, x)$, $0 < x \leq a$. See Definition 1.4.1

$\mathscr{A}C_q[0, a]$ The space of all q-absolutely continuous functions defined on $[0, a]$, see Definition 4.3.1.

$L^2_q\big((0, a) \times (0, a)\big)$ See Definition 1.4.3.

$\mathscr{A}C^{(n)}_q[0, a]$ See Definition 4.3.2.

$\mathscr{S}_{q,\mu}$; $\mu > 0$ The space of all functions f defined on a q^{-1}-geometric set A such that

$$|f(xq^{-n})| = O(q^{\mu n(n+\nu)}) \quad \text{as} \quad n \to \infty.$$

$L^p(\mathbb{R}_{q,+})$ The space of all functions f defined on $\mathbb{R}_{q,+}$ that satisfy

$$\int_0^\infty |f(t)|^p \, d_q t < \infty.$$

$L^p(\mathbb{R}_q)$ The space of all functions f defined on $\mathbb{R}_q$ that satisfy

$$\int_{-\infty}^{\infty} |f(t)|^p \, d_q t < \infty.$$

$\mathscr{L}^p(\Omega)$ where $\Omega \subseteq \mathbb{C}$ is a neighborhood or a deleted neighborhood of zero is the space of all functions defined on Ω such that

$$\int_0^{|z|} |f(t)|^p \, d_q t < \infty \quad \text{for all} \quad z \in \Omega.$$

$C_q^n[a,b]$ The space of all continues functions with continuous q-derivatives up to order $n-1$ on the interval $[a,b]$. See Definition 1.4.4.

Functions

$\lceil \cdot \rceil$ Ceiling function, $\lceil x \rceil := \min\{n \in \mathbb{N}_0 : x \leq n\}$.

$\operatorname{Re} z, \operatorname{Im} z$ Real and imaginary part of the complex number z.

$\Gamma_q(z)$ The q-analogue of the Euler's gamma function.

$B_q(\alpha, \beta)$ The q-analogue of the Euler's beta function.

${}_r\phi_s$ The basic hypergeometric function, see (1.9).

$J_\nu^{(1)}(z;q)$ First Jackson q-Bessel function.

$J_\nu^{(2)}(z;q)$ Second Jackson q-Bessel function.

$J_\nu^{(3)}(z;q)$ Third Jackson q-Bessel function.

$e_q(z),\ E_q(z)$ q-analogues of the exponential function.

$\sin_q z,\ \cos_q z,\ \sinh_q z,\ \cosh_q z$ The q-analogues of the sine, cosine, hyperbolic sine, and hyperbolic cosine associated with $e_q(z)$.

$\operatorname{Sin}_q(z), \operatorname{Cos}_q(z), \operatorname{Sinh}_q(z), \operatorname{Cosh}_q(z)$ The q-analogues of the sine, cosine, hyperbolic sine, and hyperbolic cosine associated with $E_q(z)$.

$\sin(z;q), \cos(z;q)$ q-analogues of the sine and cosine functions, see (2.75)–(2.76).

$\operatorname{Sin}(z;q^2), \operatorname{Cos}(z;q^2)$ q^2-analogues of the sine and cosine functions, see (3.70).

$e(z;q^2)$ $= \operatorname{Cos}(-iz;q^2) + i \operatorname{Sin}(-iz;q^2),\ z \in \mathbb{C}$

$e_{\nu,\mu}(z;q),\ E_{\nu,\mu}(z;q)$ q-analogues of the two parameter Mittag-Leffler functions, see (7.3).

${}_r\psi_s(z;q)$ q-analogue of the Fox-Wright function, see Definition 1.8.1.

$\operatorname{Erf}(z;q),\ \operatorname{Erf}_q(z),\ \operatorname{erf}(z;q)$ q-analogues of the error function, see (1.73), (1.75), (1.77).

$\gamma_q(s,x),\ \Gamma_q(s,x)$ q-analogues of the incomplete gamma function, see (1.78).

Operators

D_q Jackson q- divided difference operator.

∂_q The q-divided difference operator introduced by Rubin, see (3.77).

W_q is a q-analogue of the Wronskian determinant, see Definition 2.7.1.

$\mathscr{D}_q$ The Askey–Wilson operator.

${}_qL_s$, ${}_q\mathscr{L}_s$ The q-analogues of the Laplace integral transforms.

$\mathscr{M}_q$ A q type Mellin integral transform.

$\mathscr{F}_q$ A q^2 type Fourier transform.

D_q^α The Riemann–Liouville fractional q-derivative of order α.

${}^cD_q^\alpha$ The q-Caputo fractional q-derivative of order α.

D_{*q} The fractional Caputo q-difference operator of order α.

$\mathscr{D}_q^{k\alpha}$ The sequential Riemann–Liouville fractional q-derivative of order $k\alpha$, $\alpha > 0$, see (8.74).

${}^c\mathscr{D}_q^{k\alpha}$ The sequential Caputo fractional q-derivative of order $k\alpha$, $0\alpha > 0$.

Other Symbols

$W_{q,\alpha}$, $|W_{q,\alpha}|$ The q,α Wronskian matrix and the q,α Wronskian determinant associated with the Riemann–Liouville fractional q-derivative, respectively, see Definition 8.6.2.

$W^C_{q,\alpha}$, $|W^C_{q,\alpha}|$ The q,α Wronskian matrix and the q,α Wronskian determinant associated with the Caputo fractional q-derivative, respectively, see Definition 8.7.2.

$w_{\alpha,\beta}(z;q)$, $W_{\alpha,\beta}(z;q)$ q-analogues of the Write functions, see (1.83) and (1.84), respectively.

o Sometimes called little-o or Landau symbol. For example, $f = o(g)$ as $x \to a$ means that $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

O Sometimes called big-O or Landau symbol. For example. $f = O(g)$ as $x \to a$ means $f(x)| \le A|g(x)|$ for some constant A and for all x in a neighborhood of the point a.

$\sim$ For example, $f \sim g$ as $x \to a$ means that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.

Chapter 1
Preliminaries

Abstract This chapter includes definitions and properties of Jackson q-difference and q-integral operators, q-gamma and q-beta functions and finally q-analogues of Laplace and Mellin integral transforms.

1.1 Some Classical Results

In this section, we collect results from complex analysis which we shall use in this book. We will also introduce the definitions and terminology used in the text. Let f, g be entire functions and $a \in \mathbb{C}$, we say that

$$f(z) = O\big(g(z)\big), \quad \text{as} \quad z \to a,$$

if $f(z)\big/g(z)$ is bounded in a neighborhood of a. We write

$$f(z) \sim g(z), \text{ as } z \to a, \text{ if } \lim_{z\to a} \frac{f(z)}{g(z)} = 1.$$

If $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is an entire function, then the maximum modulus is defined for $r > 0$ by

$$M(r; f) := \sup\{|f(z)| : |z| = r\}. \tag{1.1}$$

The order of f, $\rho(f)$, is, cf. [57, 128, 181],

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \limsup_{n\to\infty} \frac{n \log n}{\log |a|_n^{-1}}. \tag{1.2}$$

Theorem 1.1. [57, 122]. *If f is entire and $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros, or it is a polynomial.*

The following two results of Pólya, cf. [237, 238], concerning the zeros of cosine and sine transforms, are essential in our investigations.

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Theorem 1.2. *Let $f(t)$ be a real-valued twice continuously differentiable function on the interval* $[0, 1]$. *If* $|f(1)| > |f(0)|$, *the entire functions*

$$U(z) = \int_0^1 f(t)\cos(zt)\,dt, \quad V(z) = \int_0^1 f(t)\sin(zt)\,dt$$

have infinitely many real zeros and a finite number of complex zeros.

Theorem 1.3. [238]. *If the function $f(t) \in L_1(0,1)$ is positive and increasing, then the zeros of the entire functions of exponential type*

$$U(z) = \int_0^1 f(t)\cos(zt)\,dt, \quad V(z) = \int_0^1 f(t)\sin(zt)\,dt$$

are real, infinite and simple. The even function $U(z)$ has no zeros in $[0, \frac{\pi}{2})$, *and its positive zeros are situated in the intervals* $(\pi k - \pi/2, \pi k + \pi/2)$, $1 \leqslant k < \infty$, *one zero in each interval. The odd function $V(z)$ has only one zero $z = 0$ in* $[0, \pi)$, *and its positive zeros are situated in the intervals* $(\pi k, \pi(k+1))$, $1 \leqslant k < \infty$, *one zero in each interval.*

The following version of Hurwitz–Biehler theorem for entire functions of order zero, cf. [181, Chap. 7] will be used in the sequel.

Theorem 1.4. *Let $F(z)$ be an entire function of order zero and assume that*

$$F(z) = P(z) + iQ(z),$$

where $P(z)$ and $Q(z)$ are entire functions with real coefficients. All roots of $F(z)$ lie in the upper half plane, $\Im z > 0$, if and only if $P(z)$ and $Q(z)$ have real, simple and interlacing zeros.

Katkova and Vishnyakova[166] derived the following interesting result.

Theorem 1.5. *Assume that $F(z) := \sum_{k=0}^{\infty} a_k z^k$ is an entire function, $a_k > 0$ for all $k \in \mathbb{N}_0$, and x_0 is the unique positive root of the polynomial $x^3 - x^2 - 2x - 1$, $x_0 \approx 2.1479$. Then x_0 is smallest possible constant such that if*

$$a_k a_{k+1} \geqslant x_0 a_{k-1} a_{k+2} \quad (k \in \mathbb{N}) \tag{1.3}$$

then the zeros of $F(z)$ have negative real parts.

Consequently, we can prove the following corollary.

Corollary 1.6. *Consider an entire function $F(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$. If the sequence $\{a_k\}_{k=0}^{\infty}$ satisfies condition (1.3), then the functions*

$$\sum_{k=0}^{\infty} (-1)^k a_{2k} z^{2k} \quad \textit{and} \quad \sum_{k=0}^{\infty} (-1)^k a_{2k+1} z^{2k+1}$$

have real, simple and interlacing zeros.

Proof. The proof follows at once by noting that for all $z \in \mathbb{C}$, $\Im z > 0$ if and only if $\mathrm{Re}(-iz) < 0$, and

$$F(iz) = \sum_{k=0}^{\infty}(-1)^k a_{2k} z^{2k} + i \sum_{k=0}^{\infty}(-1)^k a_{2k+1} z^{2k+1}.$$

□

Proofs of many results of this book depend on the Banach fixed point theorem, cf. [171]:

Theorem 1.7 (Banach fixed point Theory). *Let (X, d) be a Banach space and let $T : (X, d) \to (X, d)$ be a contraction mapping, i.e., there is a positive number w, $0 < w < 1$ such that*

$$d(Tx, Ty) \le w d(x, y) \quad \text{for all} \quad x,\ y \in X.$$

Then T has a unique fixed point u^*. Moreover, for any $u_0 \in X$,

$$\lim_{k\to\infty} T^k u_0 = u^*,$$

where

$$T^1 := T, \quad \text{and } T^{k+1} := T T^k \quad (k \in \mathbb{N}).$$

1.2 q-Notations and Results

In the following, unless otherwise stated, q is a positive number less than 1 and by the word “basic” we mean a q-analogue. In this section, we introduce some of the needed q-notations and results.

The q-shifted factorial, see [113], is defined for $a \in \mathbb{C}$ by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1}(1 - aq^i), & n \in \mathbb{N}. \end{cases} \tag{1.4}$$

The limit of $(a;q)_n$ as n tends to infinity exists and will be denoted by $(a;q)_\infty$. The multiple q-shifted factorial for complex numbers $a_1, \ldots, a_k$ is defined by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j;q)_n.$$

Let α be a complex number. We use the following notation for the q-binomial coefficients

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \begin{cases} 1, & k = 0, \\ \dfrac{(1-q^{\alpha})(1-q^{\alpha-1})\dots(1-q^{\alpha-k+1})}{(q;q)_k}, & k \in \mathbb{N}. \end{cases} \tag{1.5}$$

If $aq^{\alpha} \neq q^{-n}$ for all $n \in \mathbb{N}_0$, we define $(a;q)_{\alpha}$ to be

$$(a;q)_{\alpha} := \frac{(a;q)_{\infty}}{(aq^{\alpha};q)_{\infty}}, \tag{1.6}$$

where in (1.5) and (1.6) the principal branch of q^{α} is taken. We have the following series representations for $(a;q)_n$ $(n \in \mathbb{N}_0)$ and $(a;q)_{\infty}$, cf., e.g. [113, P. 11],

$$(a;q)_n := \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} a^k, \tag{1.7}$$

$$(a;q)_{\infty} = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k-1)}{2}} \frac{a^k}{(q;q)_k}. \tag{1.8}$$

For complex numbers $a_1, \dots, a_r$, $b_1, \dots, b_s$, let ${}_r\phi_s$ denote the q-hypergeometric series

$${}_r\phi_s\left(a_1, \dots, a_r; b_1, \dots, b_s; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r;q)_n}{(q, b_1, \dots, b_s;q)_n} z^n \left(-q^{(n-1)/2}\right)^{n(s-r+1)}. \tag{1.9}$$

The series representation of the function ${}_r\phi_s$ converges absolutely for all $z \in \mathbb{C}$ if $r \leq s$ and converges only for $|z| < 1$ if $r = s + 1$. The q-Vandermonde (q-Chu–Vandermonde) sums [113, Eq. (II.6)],

$${}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a, q)_n}{(c, q)_n} a^n, \tag{1.10}$$

and reversing the order of summation, [113, Eq. (II.7)]

$${}_2\phi_1\left(q^{-n}, a; c; q, \frac{cq^n}{a}\right) = \frac{(c/a, q)_n}{(c, q)_n}. \tag{1.11}$$

The following theorem, taken from [25, P. 491], will be needed in sequel.

Theorem 1.8. *Suppose λ and μ are real. Then*

$$\lim_{q \to 1^-} \frac{(q^{\lambda}x;q)_{\infty}}{(q^{\mu}x;q)_{\infty}} = (1-x)^{\mu-\lambda}, \tag{1.12}$$

uniformly on

$$\{x \in \mathbb{C} : |x| \leqslant 1\}, \text{ if } \mu \geqslant \lambda, \ \mu + \lambda \geqslant 1,$$

and uniformly on compact subsets of $\{x \in \mathbb{C} : \ |x| \leqslant 1, \ x \neq 1\}$ *for other choices of* λ *and* μ.

The q-translation operator ε^y is introduced by Ismail in [143] and is defined on monomials by

$$\varepsilon^y x^n := x^n \big(-y/x; q \big)_n, \tag{1.13}$$

and it is extended to polynomials as a linear operator. Thus

$$\varepsilon^y \left(\sum_{n=0}^{m} f_n x^n \right) := \sum_{n=0}^{m} f_n x^n \big(-y/x; q \big)_n. \tag{1.14}$$

The q-translation operator is defined for x^a $\big(a \in \mathbb{R}^+\big)$ to be

$$\varepsilon^y x^a := x^a \big(-y/x; q \big)_a, \tag{1.15}$$

provided that $y \neq -q^{-k-a} x$ for all $k \in \mathbb{N}_0$.

1.3 The *q*-Difference Operator

If $\mu \in \mathbb{R}$ is fixed, a subset A of $\mathbb{C}$ is called μ-geometric if $\mu z \in A$ whenever $z \in A$. If a subset A of $\mathbb{C}$ is a μ-geometric then it contains all geometric sequences $\{z\mu^n\}_{n=0}^{\infty}$, $z \in A$. Let f be a function, real or complex valued, defined on a q-geometric set A, $|q| \neq 1$. The q-difference operator, which was reintroduced by Jackson [158], and may go back to Heine [132] or Euler, is defined by

$$D_q f(z) := \frac{f(z) - f(qz)}{z - qz} \quad \text{for } z \in A \setminus \{0\}. \tag{1.16}$$

Jackson introduced a systematic study of this operator in [153–161]. The q-difference operator (1.16) sometimes called Jackson q-difference operator, Euler–Jackson q-difference operator or Euler–Heine–Jackson q-difference operator. If $0 \in A$, the q-derivative at zero is defined for $|q| < 1$ by

$$D_q f(0) := \lim_{n \to \infty} \frac{f(zq^n) - f(0)}{zq^n} \quad \text{for } z \in A \setminus \{0\}, \tag{1.17}$$

provided the limit exists and does not depend on z. In addition, the q-derivative at zero is defined for $|q| > 1$ by

$$D_q f(0) := D_{q^{-1}} f(0).$$

In some literature, the q-derivative at zero is defined to be $f'(0)$ if it exists, cf. e.g. [174, 280]. The definition in (1.17) is more suitable for our approaches.

Jackson [160] introduced an integral denoted by

$$\int_a^b f(t)\,d_qt$$

as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t)\,d_qt := \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt \quad (a, b \in A), \tag{1.18}$$

where

$$\int_0^x f(t)\,d_qt := (1-q)\sum_{n=0}^{\infty} xq^n f(xq^n) \quad (x \in A), \tag{1.19}$$

provided that the series at the right-hand side of (1.19) converges at $x = a$ and b.

Kac and Cheung [163, p. 68] proved that if $x^\alpha f(x)$ is bounded on $[0, a]$ for some $0 \leqslant \alpha < 1$, then $\int_0^x f(t)\,d_qt$ exists for all $x \in [0, a]$. Moreover, Bromwich [65, PP. 418–419] proved that if $\int_0^b f(x)\,dx$ converges, then

$$\int_0^b f(x)\,dx = \lim_{q\to 1^-} \int_0^b f(x)\,d_qx.$$

If $x > 0$ and f is a function defined on a q-geometric set A, Hahn [123] defined the q-integral of the function f on $[x, \infty)$ to be

$$\int_x^{\infty} f(t)\,d_qt = \sum_{k=1}^{\infty} xq^{-k}(1-q)f(xq^{-k}). \tag{1.20}$$

There is no unique canonical choice for the q-integration over $[0, \infty)$. Hahn in[123] defined the q-integration for a function f over $[0, \infty)$ by

$$\int_0^{\infty} f(t)\,d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

while in [210], Matsuo defined a q-integration on the interval $[0, \infty)$ by

$$\int_0^{\infty/b} f(t)\,d_qt := \frac{1-q}{b}\sum_{n=-\infty}^{\infty} q^n f(q^n/b) \quad (b > 0). \tag{1.21}$$

Consequently, the q-integration of a function f defined on $\mathbb{R}$ can be defined as

$$\int_{-\infty/b}^{\infty/b} f(t)\,d_qt = \frac{1-q}{b}\sum_{-\infty}^{\infty} q^n \left(f(q^n/b) + f(-q^n/b)\right),\ b > 0,$$

provided that the series converges absolutely.

Definition 1.3.1. Let f be a function defined on a q-geometric set A. We say that f is q-integrable on A if and only if $\int_0^z f(t)\, d_q t$ exists for all $z \in A$.

Definition 1.3.2. A function f which is defined on a q-geometric set A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(zq^n) = f(0) \;\text{ for all }\; z \in A.$$

Moreover, if A is also q^{-1}-geometric, then we say that f is q-regular at infinity if there exists a constant C such that

$$\lim_{n\to\infty} f(zq^{-n}) = C \quad \text{for all }\; z \in A.$$

From now on, if $A \subseteq \mathbb{R}$ is q-geometric and f is a q-regular at zero function defined on A, we define $f(0^+)$ and $f(0^-)$ by

$$f(0^+) := \lim_{\substack{k\to\infty\\ x>0}} f(xq^k), \qquad f(0^-) := \lim_{\substack{k\to\infty\\ x<0}} f(xq^k).$$

Clearly, if f is q-regular at zero, then

$$f(0) = f(0^+) = f(0^-).$$

The q-regularity at zero plays the role of continuity in the classical sense in some settings. However, continuity at zero implies q-regularity at zero, but the converse is not necessarily true. For example, the function $f\colon [0,1] \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 1, \; x = a_n = \dfrac{1}{\sqrt{n}}, \; n \;\text{ is prime,} \\ x, \;\text{otherwise.} \end{cases} \tag{1.22}$$

is q-regular at zero for rational q, but it is not continuous at zero.

Remarks.

1. If $x \in A \setminus \{0\}$ and A contains a neighborhood of the point x such that f is differentiable at x, then $\lim_{q\to 1} D_q f(x) = f'(x)$. If $x = 0$ and $f'(0)$ exists, then $D_q f(0) = f'(0)$.
2. If f is a function defined on a q-geometric set A which contains zero such that $D_q f(0)$ exists, then f is q-regular at zero. This is similar to the fact that differentiability implies continuity.
3. It may happen that $D_q f(0)$ exists for a function f without being differentiable or even continuous at zero. For example, the function f defined by (1.22) satisfies $D_q f(0) = 1$, while f is not continuous at zero.

4. The q-derivative of a function f defined on a q-geometric set A is zero if and only if $f(x) = f(qx)$ for all $x \in A$. To our best knowledge, these functions had been introduced by Boole who called them "periodic constants," see [13, 161]. Recently, they are called "q-periodic functions," see e.g. [281]. It is obvious that the constant functions are q-periodic functions. Another example of a q-periodic function, cf. [281, P. 856], is $f(x) = \sin(\alpha \log x)$ with $\alpha \log q = 2\pi$, which has singularity at $x = 0$.

Proposition 1.9. *Let f be a q-periodic function defined on a q-geometric set A. If f is q-regular at zero, then f is a constant function. In addition, if f is a q^{-1}-periodic function defined on a q^{-1}-geometric set A and f is q-regular at infinity, then f is constant on A.*

Proof. Since f is q-periodic on A, then $f(x) = f(qx) = \cdots = f(q^n x)$, for all $x \in A, n \in \mathbb{N}$. Thus,

$$f(x) \equiv \lim_{n\to\infty} f(xq^n) \equiv f(0).$$

The result in case of f is q-regular at ∞ follows similarly . ☐

We present here some of the basic rules concerning q-derivatives. By means of these rules, q-derivatives of commonly occurring functions can be found by straightforward manipulations.

In rules 1 through 4, two functions f and g are defined on a q-geometric set A such that the q-derivatives of f and g exist for all $x \in A$.

Rule 1. $D_q(f \pm g)(x) = D_q f(x) \pm D_q g(x)$.

This is an immediate consequence of (1.16)–(1.17).

Rule 2. [The q-product rule]

$$D_q(fg)(x) = D_q f(x)g(x) + f(qx)D_q g(x). \tag{1.23}$$

Symmetric rules are

$$D_q(fg)(x) = D_q f(x)g(x) + f(x)D_q g(x) - x(1-q)D_q f(x)D_q g(x)$$

and

$$D_q(fg)(x) = D_q f(x)A_q g(x) + A_q f(x)D_q g(x),$$

where A_q is the averaging operator which acting on a function h defined on a q-geometric set A by

$$A_q h(x) = \frac{h(x) + h(qx)}{2}.$$

Nevertheless, the non symmetric formula (1.23) is useful in computations. Another symmetric rule is

Rule 3. [q-type Leibniz rule]

$$D_q^n(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_{q,x}^{n-k} f\right)(q^k x)\ D_q^k g(x), \tag{1.24}$$

where $x \in A \setminus \{0\}$. If $x = 0$, we additionally assume that the kth q-derivatives of f and g exist at zero where $k = 1, 2, \ldots, n$.

In the following rule we assume that g does not vanish at x or qx.

Rule 4. If $g(x) \neq g(qx) \neq 0$ then

$$D_q\big(f/g\big)(x) = \frac{D_q f(x)g(x) - f(x)D_q g(x)}{g(x)g(qx)}.$$

Rule 5: The nth q-derivative, D_q^n, of a function f can be represented by its values at the points $\{q^j x, j = 0, 1, \cdots, n\}$ through the identity

$$D_q^n f(x) = (-1)^n (1-q)^{-n} x^{-n} q^{-n(n-1)/2} \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-1)/2} f(xq^{n-r}) \tag{1.25}$$

for every x in $A \setminus \{0\}$. After some straightforward manipulations, formula (1.25) can be written as

$$D_q^n f(x) = (1-q)^{-n} x^{-n} \sum_{r=0}^{n} q^r \frac{(q^{-n}; q)_r}{(q; q)_r} f(xq^r) \quad \text{for } x \in A \setminus \{0\}. \tag{1.26}$$

Moreover, the formula (1.25) can be inverted through the relation

$$f(xq^n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (1-q)^k x^k q^{\binom{k}{2}} D_q^k f(x). \tag{1.27}$$

Formulas (1.25) and (1.27) are well-known and follow easily by induction.

1.3.1 More Properties of *q*-Integrals

The rule of q-integration by parts is

$$\int_0^a g(t) D_q f(t)\, d_q t = (fg)(a) - \lim_{n\to\infty} (fg)(aq^n) - \int_0^a D_q g(t) f(qt)\, d_q t. \tag{1.28}$$

If f and g are q-regular at zero, then the limit on the right hand side of (1.28) can be replaced by $(fg)(0)$.

The following theorem is an analogue of the fundamental theorem of calculus. Its proof is simple and is omitted.

Theorem 1.10. *Let f be a q-regular at zero function defined on a q-geometric set A containing zero. Define*

$$F(z) := \int_c^z f(t)\, d_q t \quad (z \in A),$$

where c is a fixed point in A. Then F is q-regular at zero. Furthermore, $D_qF(z)$ exists for every $z \in A$ and

$$D_qF(z) = f(z) \quad \textit{for every} \quad z \in A.$$

Conversely, if a and b are two points in A, then

$$\int_a^b D_q f(t)\, d_qt = f(b) - f(a). \tag{1.29}$$

Theorem 1.11. *Let f be a function defined on $[a,b]$, $0 \le a \le b$. Assume that there exists γ, $0 \le \gamma < 1$ such that $x^\gamma f(x)$ is continuous on $[a,b]$. Let*

$$F(x) = \int_c^x f(t)\, d_qt \quad \textit{for } x \in [a,b],$$

where c is a fixed point in $[a,b]$. Then $F(x)$ is a continuous function on $[a,b]$.

Proof. Set

$$g(x) := x^\gamma f(x) \quad \text{for all } x \in [0,a].$$

Fix $x_0 \in [a,b]$ and assume that $x_0 \neq 0$. Then,

$$\begin{aligned} F(x) - F(x_0) &= (1-q)\sum_{k=0}^{\infty} xq^k f(xq^k) - (1-q)\sum_{k=0}^{\infty} x_0q^k f(x_0q^k) \\ &= (1-q)x^{1-\gamma}\sum_{k=0}^{\infty} xq^{k(1-\gamma)}\left[g(xq^k) - g(x_0q^k)\right] \\ &\quad + x_0^\gamma(x^{1-\gamma} - x_0^{1-\gamma})(1-q)\sum_{k=0}^{\infty} q^k g(x_0q^k). \end{aligned} \tag{1.30}$$

Since $g(x)$ is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$. Hence for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x,\ y \in [a,b]$

$$|x - y| < \delta \longrightarrow |g(x) - g(y)| < \epsilon.$$

Therefore, if $x \in [a,b]$, $|x - x_0| < \delta$ then $|xq^k - x_0q^k| < \delta$ for all $k \in \mathbb{N}_0$ and

$$|g(xq^k) - g(x_0q^k)| < \epsilon \;\; \text{for all} \;\; k \in \mathbb{N}_0.$$

Hence, $\lim\limits_{x \to x_0} g(xq^k) = g(x_0q^k)$ uniformly in k and we can calculate the limit as $x \to x_0$ on the series on (1.30) term by term to obtain $\lim\limits_{x \to x_0} F(x) = F(x_0)$. Now assume that $x_0 = 0$. Then,

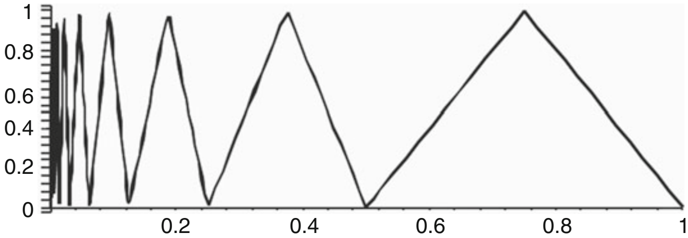

Fig. 1.1 The function $g(x)$ on $[0,1]$ with $q = 1/2$

$$F(x) - F(0) = \int_0^x f(t)\,d_qt = \int_0^x t^{1-\gamma}\,(g(t) - g(0)) + \frac{1-q}{1-q^{2-\gamma}}\,x^{2-\gamma} g(0).$$

Consequently,

$$|F(x) - F(0)| \le \left(\max_{k\in\mathbb{N}_0} |g(xq^k) - g(0)| + g(0) \right) \frac{1-q}{1-q^{2-\gamma}}\,x^{2-\gamma}.$$

Then, from the continuity of the function g at zero and since $0 < \gamma < 1$, we obtain $\lim_{x\to 0} F(x) = F(0)$. That is, $F(x)$ is continuous on $[a,b]$. □

Remarks.

(i) The next example indicates that the q-regularity at zero seems to be the best possible condition such that the first part of Theorem 1.10 holds.

Example 1.3.1. Define a function g on $[0,1]$ by

$$g(x) := \begin{cases} f_k(x), & x \in (q^{k+1}, q^k], \quad k \in \mathbb{N}_0, \\ 0, & x = 0, \end{cases} \tag{1.31}$$

where

$$f_k(x) := \begin{cases} \frac{2}{1-q}\left(-q + xq^{-k}\right), \; x \in \left(q^{k+1}, \dfrac{q^k(1+q)}{2}\right], \\ \frac{2}{1-q}\left(1 - xq^{-k}\right), \;\; x \in \left[\dfrac{q^k(1+q)}{2}, q^k\right]. \end{cases} \tag{1.32}$$

See Fig. 1.1. If $x \in (0,1]$, then x is represented uniquely in the form $x = tq^{k_0}$, for some $t \in (q,1]$ and $k_0 \in \mathbb{N}_0$. Consequently,

$$\lim_{k\to\infty} g(xq^k) = \lim_{k\to\infty} f_{k+k_0}(xq^k) = \begin{cases} 2\frac{t-q}{1-q}, \; q < t \le \frac{1+q}{2} \\ 2\frac{1-t}{1-q}, \; \frac{1+q}{2} \le t \le 1. \end{cases}$$

Since the value of the limit is dependent on the point x, then g is not q-regular at zero. But on the other hand, $g(x) = g(qx)$, for all $x \in (0, 1]$. Thus,

$$\int_0^1 D_q g(t)\, d_q t = \int_0^{(1+q)/2} D_q g(t)\, d_q t = 0,$$

implying

$$\int_{(1+q)/2}^1 D_q g(t)\, d_q t = 0 \neq g(1) - g((1+q)/2) = -1.$$

Therefore, if in Theorem 1.10 f is not q-regular at zero, then

$$\int_0^b D_q f(t)\, d_q t = f(b) - \lim_{n\to\infty} f(bq^n).$$

Consequently, (1.29) would take the form

$$\int_a^b D_q f(t)\, d_q t = \Big[f(b) - \lim_{n\to\infty} f(bq^n)\Big] - \Big[f(a) - \lim_{n\to\infty} f(aq^n)\Big].$$

(ii) The inequality

$$\left|\int_a^b f(t)\, d_q t\right| \leqslant \int_a^b |f(t)|\, d_q t \quad (0 \leqslant a \leqslant b < \infty) \tag{1.33}$$

is not always valid. For example, let us define a function $h\colon [0,1] \longrightarrow \mathbb{R}$ by, cf. Fig. 1.2,

$$h(x) = \begin{cases} \dfrac{1}{1-q}\left(4q^{-n}x - (1+3q)\right), & q^{n+1} \leqslant x \leqslant \dfrac{q^n(1+q)}{2},\ n \in \mathbb{N}_0, \\[2ex] \tfrac{4}{1-q}\left(-xq^{-n} + 1\right) - 1, & \dfrac{q^n(1+q)}{2} \leqslant x \leqslant q^n,\ n \in \mathbb{N}_0, \\[2ex] 0, & x = 0. \end{cases}$$

The function h is q-integrable on $[0, 1]$ such that

$$h(q^n) = -1 \quad \text{and} \quad h\left(\frac{q^n(1+q)}{2}\right) = 1 \text{ for all } n \in \mathbb{N}_0.$$

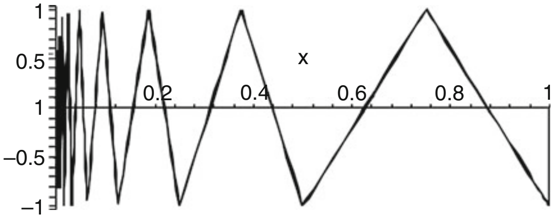

Fig. 1.2 The function $h(x)$ on $[0, 1]$ with $q = 1/2$

Therefore, a direct calculation yields

$$\int_{\frac{1+q}{2}}^{1} h(t)\, d_qt = -\frac{3+q}{2}, \quad \int_{\frac{1+q}{2}}^{1} |h(t)|\, d_qt = \frac{1-q}{2}.$$

Hence,

$$\left| \int_{\frac{1+q}{2}}^{1} h(t)\, d_qt \right| > \int_{\frac{1+q}{2}}^{1} |h(t)|\, d_qt.$$

However, inequality (1.33) holds when $a = 0$ or $b = \infty$ and when $a, b \in I$ $(a < b)$ are in the form $a = xq^n$ and $b = xq^m$; $n, m \in \mathbb{Z}$.

Lemma 1.12. *Let* $h(t, x)$ *be a function defined on* $[0, a] \times [0, a]$ *such that for each fixed* t *the functions*

$$D^j_{q,x} h(t, x) \quad (j = 0, 1, \ldots, k-1)$$

are q*-integrable on* $[0, a]$*. If for some* $x \in (0, a]$ *and* $k \in \mathbb{N}$

$$h(xq^r, xq^j) = 0 \quad (r = 0, 1, 2, \ldots, j-1;\ j = 1, 2, \ldots, k) \tag{1.34}$$

then

$$D^k_{q,x} \int_0^x h(t, x)\, d_qt = \int_0^x D^k_{q,x} h(t, x)\, d_qt.$$

Proof. From (1.25) we obtain

$$D^k_{q,x} \int_0^x h(t, x)\, d_qt = \sum_{j=0}^{j=k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2} - kj}}{x^k (1-q)^k} \int_0^{xq^j} h(t, xq^j)\, d_qt. \tag{1.35}$$

Letting f satisfy (1.34) implies

$$\int_0^{xq^j} h(t, xq^j)\, d_qt = \int_0^x h(t, xq^j)\, d_qt, \ j = 1, 2, \ldots, k.$$

Hence,

$$\begin{aligned}
D_{q,x}^{k}\int_{0}^{x}h(t,x)\,d_qt &= \sum_{j=0}^{j=k}(-1)^{j}\begin{bmatrix}k\\ j\end{bmatrix}_q\frac{q^{\frac{j(j+1)}{2}-kj}}{x^{k}(1-q)^{k}}\int_{0}^{x}h(t,xq^{j})\,d_qt\\
&= \int_{0}^{x}\left(\sum_{j=0}^{j=k}(-1)^{j}\begin{bmatrix}k\\ j\end{bmatrix}_q\frac{q^{\frac{j(j+1)}{2}-kj}}{x^{k}(1-q)^{k}}h(t,xq^{j})\right)d_qt\\
&= \int_{0}^{x}D_{q,x}^{k}h(t,x)\,d_qt.
\end{aligned}$$

□

Lemma 1.13. *Let $h(t,x)$ be a function defined on $[a,\infty)\times[a,\infty)$ such that for each fixed t the functions*

$$D_{q,x}^{j}h(t,x)\quad (j=0,1,\dots,k-1)$$

are q-integrable on $[a,\infty)$. If for some $x\in[a,\infty)$ and $k\in\mathbb{N}$

$$h(xq^{r},xq^{j})=0\quad (r=0,1,2,\dots,j-1;\ j=1,2,\dots,k)$$

then

$$D_{q,x}^{k}\int_{x}^{\infty}h(t,x)\,d_qt=\int_{x}^{\infty}D_{q,x}^{k}h(t,x)\,d_qt.$$

Proof. The proof is similar to the proof of Lemma 1.12 and is omitted. □

The proof of the following lemma is easy and is omitted.

Lemma 1.14. *Let I and J be intervals containing zero, such that $J\subseteq I$. Let f_n, f be functions defined in I, $n\in\mathbb{N}$, such that*

$$\lim_{n\to\infty}f_n(t)=f(t),\ \textit{for all } t\in I,\ \textit{and } f_n \textit{ tends uniformly to } f \textit{ on } J. \tag{1.36}$$

Then,

$$\lim_{n\to\infty}\int_{0}^{x}f_n(t)\,d_qt=\int_{0}^{x}f(t)\,d_qt\ \ \textit{for all } x\in I. \tag{1.37}$$

1.4 Function Spaces

Let $1\le p<\infty$, $a>0$ and η be a real number. Let $L_{q,\eta}^{p}(0,a)$ be the space of all equivalence classes of functions satisfying

$$\int_{0}^{a}t^{\eta}|f(t)|^{p}\,d_qt<\infty,$$

where we define two functions to be equivalent if they are equal on the sequence $\{aq^n : n \in \mathbb{N}_0\}$. With a slight abuse of notation, we denote by f both a function in $L^p_{q,\eta}(0,a)$ and its equivalence class. It is straightforward to prove that the space $L^p_{q,\eta}(0,a)$ associated with the norm function

$$\|f\|_{p,\eta,a} := \left(\int_0^a t^\eta |f(t)|^p \, d_q t\right)^{1/p}$$

is a Banach space. Moreover, if $p = 2$, then $L^2_{q,\eta}(0,a)$ associated with the inner product

$$\langle f, g\rangle := \int_0^a t^\eta f(t)\overline{g(t)}\, d_q t \quad \left(f, g \in L^2_{q,\eta}(0,a)\right) \tag{1.38}$$

is a separable Hilbert space. In fact,

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{x^{\eta+1}(1-q)}}, & x = aq^n;\, n \in \mathbb{N}_0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.39}$$

is an orthonormal basis of $L^2_{q,\eta}(0,a)$. The proof of this fact is just a straightforward generalization of the special case $\eta = 0$ proved in [26]. For any $q \in (0,1)$ and $0 < b < \infty$, we define the spaces

$$L^p(\mathbb{R}_{b,q}) := \left\{ f : \int_{-\infty/b}^{\infty/b} |f(x)|^p \, d_q x < \infty \right\} \quad (p \geq 1).$$

We shall use the particular notation $\mathbb{R}_q$ and $\widetilde{\mathbb{R}}_q$ to denote $\mathbb{R}_{1,q}$ and $\mathbb{R}_{\sqrt{1-q},q}$, respectively. One can verify that $L^2(\mathbb{R}_{b,q})$ associated with the inner product

$$< f, g >:= \int_{-\infty/b}^{\infty/b} f(t)\overline{g(t)}\, d_q t \quad \left(f, g \in L^2(\mathbb{R}_{b,q})\right)$$

is a Hilbert space.

Definition 1.4.1. For a real number η and a positive number p, we define the space $\mathscr{L}^p_{q,\eta}[0,a]$ to be the space of all functions f defined on $(0,a]$ satisfying

$$\|f\|_{p,\eta} := \sup_{x \in (0,a]} \left(\int_0^x t^\eta |f(t)|^p \, d_q t\right)^{1/p} < \infty. \tag{1.40}$$

For simplicity, we shall use the symbols $L^p_q[0,a]$, $\mathscr{L}^p_q[0,a]$ and $\|f\|_p$ to denote $L^p_{q,0}[0,a]$, $\mathscr{L}^p_{q,0}[0,a]$ and $\|f\|_{p,0}$, respectively.

Proposition 1.15. *The space* $\left(\mathscr{L}^p_{q,\eta}[0,a], \|\cdot\|_{p,\eta}\right)$ *is a Banach space.*

Proof. It is straightforward to prove that $\left(\mathscr{L}^p_{q,\eta}[0,a], \|\cdot\|_{p,\eta}\right)$ is a normed space. Therefore, we give only a proof for completeness. Let $(f_n)_n$ be a Cauchy sequence in $\left(\mathscr{L}^p_{q,\eta}[0,a], \|\cdot\|_{p,\eta}\right)$. Hence, for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$

$$n, m > n_0 \longrightarrow \sup_{x\in(0,a]} \sum_{k=0}^{\infty} (xq^k)^{\eta+1}(1-q)|f_n(xq^k) - f_m(xq^k)|^p < \epsilon. \tag{1.41}$$

Hence $x^{\frac{\eta+1}{p}} f_n(x)$ is a uniformly Cauchy sequence on $(0,a]$. Then there exists a function f defined on $(0,a]$ such that

$$\lim_{n\to\infty} x^{\frac{\eta+1}{p}} f_n(x) = x^{\frac{\eta+1}{p}} f(x), \text{ uniformly on } \ (0,a].$$

Fix $M > 0$ and $n > n_0$. Then from (1.41)

$$m > n_0 \longrightarrow \sum_{k=0}^{M} (xq^k)^{\eta+1}(1-q)|f_n(xq^k) - f_m(xq^k)|^p < \epsilon \tag{1.42}$$

for all $x \in (0,a]$. Then calculating the limit as $m \to \infty$ on (1.42) gives for all $M > 0$ and $n > n_0$

$$\sum_{k=0}^{M} (xq^k)^{\eta+1}(1-q)|f_n(xq^k) - f(xq^k)|^p \leqslant \epsilon \text{ for all } x \in [0,a].$$

Hence,

$$\|f_n - f\|_{p,\eta} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently,

$$f_{n_0+1} - f \in \left(\mathscr{L}^p_{q,\eta}[0,a], \|\cdot\|_{p,\eta}\right)$$

and since $f_{n_0+1} \in \left(\mathscr{L}^p_{q,\eta}[0,a], \|\cdot\|_{p,\eta}\right)$, then so is f. This completes the proof and yields the required result. □

Definition 1.4.2. For $\nu > 0$ and $A \subseteq \mathbb{R}$, $0 \in A$, let $H_\nu(A)$ be the space of all functions defined on A such that if $f \in H_\nu(A)$, then there exists $c > 0$ such that

$$|f(x) - f(0)| < c|x|^\nu \quad \text{for all } x \in A.$$

Definition 1.4.3. Let $L^2_q\big((0,a) \times (0,a)\big)$ be the space of all complex valued functions $f(x,t)$ defined on $[0,a] \times [0,a]$ such that

$$\|f(\cdot,\cdot)\|_2 := \left(\int_0^a \int_0^a |f(x,t)|^2 \, d_qx d_qt\right)^{1/2} < \infty.$$

The elements of $L_q^2\big((0,a)\times(0,a)\big)$ are equivalence classes where f and g are in the same equivalence class if $f(aq^m,aq^n)=g(aq^m,aq^n)$ for all $m,n\in\mathbb{N}_0$. The zero element is the equivalence class of all functions $f(x,t)$ satisfying $f(aq^m,aq^n)=0$ for all $m,n\in\mathbb{N}_0$.

Lemma 1.16. *The space $L_q^2\big((0,a)\times(0,a)\big)$ associated with the inner product*

$$\langle f,g\rangle_2 := \int_0^a\int_0^a f(x,t)\overline{g(x,t)}\,d_qx d_qt.$$

is a separable Hilbert space.

Proof. Similar to [26, pp. 217–218], the space $L_q^2\big((0,a)\times(0,a)\big)$ is a Banach space. To prove separability, it suffices to prove that

$$\phi_{ij}(x,t) := \phi_i(x)\phi_j(t) \ \ (i,j=1,2,\ldots)$$

is an orthonormal basis of $L_q^2\big((0,a)\times(0,a)\big)$ whenever $\{\phi_i(\cdot)\}_{i=1}^{\infty}$ is an orthonormal basis of $L_q^2(0,a)$. Indeed,

$$\begin{aligned}\big\langle\phi_{jk},\phi_{mn}\big\rangle_2 &= \int_0^a\int_0^a \phi_j(x)\phi_k(t)\overline{\phi_m(x)\phi_n(t)}\,d_qx\,d_qt\\ &= \int_0^a \phi_j(x)\overline{\phi_m(x)}\,d_qx \int_0^a \phi_k(t)\overline{\phi_n(t)}\,d_qt\\ &= \delta_{jm}\delta_{kn},\end{aligned}$$

proving orthogonality. To prove that $\{\phi_{ij}\}$ is a basis, we prove that if there exists

$$f\in L_q^2\big((0,a)\times(0,a)\big)$$

such that

$$\big\langle f,\phi_{ij}\big\rangle_2 = 0 \quad \text{for all } i,j\in\mathbb{N}_0,$$

then f is the zero element. Indeed,

$$\begin{aligned}0=\big\langle f,\phi_{ij}\big\rangle &= \int_0^a\int_0^a f(x,t)\overline{\phi_i(x)\phi_j(t)}\,d_qx\,d_qt\\ &= \int_0^a \overline{\phi_j(t)}\Big(\int_0^a f(x,t)\overline{\phi_i(x)}\,d_qx\Big)\,d_qt\\ &= \int_0^a h(t)\overline{\phi_j(t)}\,d_qt.\end{aligned}$$

Thus,

$$h(t) := \int_0^a f(x,t)\overline{\phi_i(x)}\, d_qx$$

is orthogonal to the ϕ_j's which implies that $h(aq^n) = 0$, for all $n \in \mathbb{N}_0$. So, $f(x, aq^n)$ is orthogonal to each ϕ_i. Consequently, $f(aq^m, aq^n) = 0$, for all $m, n \in \mathbb{N}_0$. □

Definition 1.4.4. Let $C_q^n[a,b]$ be the space of all continues functions with continuous q-derivatives up to order $n-1$ on the interval $[a,b]$.

The space $C_q^n[a,b]$ associated with the norm function

$$\|f\| := \sum_{k=0}^{n-1} \max_{a \le x \le b} |D_q^k f(t)| \quad \left(f \in C_q^n[a,b]\right)$$

is a Banach space as we shall see in the following lemma.

Lemma 1.17. *$\left(C_q^n[a,b], \|\cdot\|\right)$, $n \in \mathbb{N}$, is a Banach space.*

Proof. The proof that $\left(C_q^n[a,b], \|\cdot\|\right)$ is a normed space is straightforward and is omitted. Therefore, it remains to show that $C_q^n[a,b]$ is complete. Let $(f_m)_m$ be a Cauchy sequence in $C_q^n[a,b]$. Then for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $l, m \in \mathbb{N}$

$$l, m > n_0 \longrightarrow \sum_{k=0}^{n-1} \max_{x \in [a,b]} \left| D_q^k f_l(x) - D_q^k f_m(x) \right| < \epsilon.$$

Hence,

$$l, m > n_0 \longrightarrow \max_{x \in [a,b]} \left| D_q^k f_l(x) - D_q^k f_m(x) \right| < \epsilon.$$

That is $(D_q^k f_m)_m$ is a Cauchy sequence in $C[a,b]$ for $k = 0, 1, \ldots, n-1$. Thus, for each $k \in \{0, 1, \ldots, n-1\}$ there exists a function $g_k \in C[a,b]$ such that

$$\lim_{k \to \infty} \max_{x \in [a,b]} |D_q^k f(x) - g_k(x)| = 0, \ k = 0, 1, \ldots, n-1.$$

It is clear that

$$g_k(x) = D_q^k g_0(x) \quad \left(x \in [a,b] \setminus \{0\},\ k = 0, 1, 2, \ldots, n-1\right). \tag{1.43}$$

If $0 \in (a,b)$ then

$$\lim_{x \to 0} g_k(x) = \lim_{x \to 0} D_q^k g_0(x) = \lim_{r \to \infty} D_q^k g_0(tq^r),$$

for all $t \in (a, b)$, $t \neq 0$. Consequently,

$$\lim_{x\to 0} g_k(x) = \lim_{r\to\infty} \frac{D_q^{k-1} g_0(tq^r) - D_q^{k-1} g_0(tq^{r+1})}{tq^r(1-q)}$$
$$= \frac{1}{1-q} \lim_{r\to\infty} \left[\frac{D_q^{k-1} g_0(tq^r) - D_q^{k-1} g_0(0)}{tq^r} - q\frac{D_q^{k-1} g_0(tq^{r+1}) - D_q^{k-1} g_0(0)}{tq^{r+1}} \right]$$
$$= D_q^k g_0(0). \tag{1.44}$$

Hence, the identity in (1.43) holds for every $x \in [a, b]$ and hence $g_0 \in C_q^n[a, b]$. If $0 = a$ or b, we just replace the limit as $x \to 0$ in (1.44) by $x \to 0^+$ or $x \to 0^-$, respectively.

Definition 1.4.5. The space $C_\gamma[a, b]$ is the space defined for $\gamma \in \mathbb{R}$ by

$$C_\gamma[a, b] = \left\{ g(x) : \ x^\gamma g(x) \in C[a, b], \ \|g\|_{C_\gamma} := \max_{a\le x\le b} |x^\gamma g(x)| \right\}.$$

1.5 Some q-Functions

Euler proved that

$$(-z; q)_\infty := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n \quad (z \in \mathbb{C})$$

and

$$\frac{1}{(z;q)_\infty} := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_\infty} \quad (|z| < 1).$$

See [25, P. 490] and for a reference to Euler, see [24, P. 30]. The above two identities relate infinite products to infinite sums. Jackson [154] denoted $(-z; q)_\infty$ by $E_q(z)$ and $1/(z;q)_\infty$ by $e_q(z)$. See also [25, 113, 143]. Hence, $E_q(z)$ is an entire function with simple zeros at the points $\{-q^{-n}, n \in \mathbb{N}_0\}$, and

$$e_q(z) E_q(-z) \equiv 1 \quad \text{for } |z| < 1. \tag{1.45}$$

Therefore, the domain of the function $e_q(z)$ can be extended to $\mathbb{C}$ by defining $e_q(z)$, $z \in \mathbb{C}$, to be

$$e_q(z) := \frac{1}{(z;q)_\infty}.$$

Hence, the relation (1.45) holds in $\mathbb{C}$, and the function $e_q(z)$ has simple poles at the points $\{q^{-n},\ n \in \mathbb{N}_0\}$. See e.g. [113, 154]. Gasper and Rahman [113, P. 15] proved also that

$$e_q(z) = \frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q,z;q)_n} z^n \quad \text{for } z \in \mathbb{C} \setminus \left\{q^{-k}\ ,\ k \in \mathbb{N}_0\right\}.$$

Let the basic trigonometric functions $\sin_q z$, $\cos_q z$, $\operatorname{Sin}_q z$ and $\operatorname{Cos}_q z$ be defined by

$$\sin_q z := \frac{e_q(iz) - e_q(-iz)}{2i}, \qquad \cos_q z := \frac{e_q(iz) + e_q(-iz)}{2},\ |z| < 1, \tag{1.46}$$

$$\operatorname{Sin}_q z := \frac{E_q(iz) - E_q(-iz)}{2i}, \quad \operatorname{Cos}_q z := \frac{E_q(iz) + E_q(-iz)}{2},\ z \in \mathbb{C}, \tag{1.47}$$

respectively. See [154]. The functions $\sin_q z$ and $\cos_q z$ can be analytically continued through the identities

$$\sin_q z = \frac{\operatorname{Sin}_q z}{(-z^2;q^2)_\infty}, \quad \cos_q z = \frac{\operatorname{Cos}_q z}{(-z^2;q^2)_\infty} \quad \text{for } z \in \mathbb{C} \setminus \{\pm q^{-m} i\ , m \in \mathbb{N}_0\}.$$

Hence, the functions $\sin_q z$ and $\cos_q z$ are meromorphic functions with poles at the points $\{\pm q^{-m} i,\ m \in \mathbb{N}_0\}$. In addition, q-analogues of the hyperbolic functions $\sinh z$ and $\cosh z$ are defined by

$$\sinh_q z := -i \sin_q(iz), \qquad \cosh_q z := \cos_q(iz), \tag{1.48}$$

$$\operatorname{Sinh}_q z := -i \operatorname{Sin}_q(iz), \qquad \operatorname{Cosh}_q z := \operatorname{Cos}_q(iz). \tag{1.49}$$

The θ-function is defined for $z \in \mathbb{C} \setminus \{0\}$, $0 < |q| < 1$ to be

$$\theta(z;q) := \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \tag{1.50}$$

The following identity is introduced by Jacobi in 1829. It is called Jacobi's triple product identity, see [113]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty (-qz;q^2)_\infty (-qz^{-1};q^2)_\infty, \tag{1.51}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $0 < |q| < 1$. Hence, $\theta(z;q)$ has only real simple zeros at the points $\left\{-q^{2k+1},\ k \in \mathbb{Z}\right\}$. A generalization of (1.51) is Ramanujan's identity

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az, q/az, q, b/a; q)_\infty}{(z, b/az, b, q/a; q)_\infty}, \tag{1.52}$$

where $|q| < 1$ and $|ba^{-1}| < |z| < 1$. See [25, Theorem 10.5.1]. There are three known q-analogues of classical Bessel functions that are due to Jackson, see [143, 157]. These are designated by $J_\nu^{(k)}(z;q)$, $k = 1, 2, 3$ and defined by

$$J_\nu^{(1)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(z/2)^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n} \quad (|z| < 2)\,, \tag{1.53}$$

$$J_\nu^{(2)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(\nu+n)}(z/2)^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n} \quad (z \in \mathbb{C})\,, \tag{1.54}$$

$$J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)/2}z^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n} \quad (z \in \mathbb{C})\,. \tag{1.55}$$

The third Jackson q-Bessel function is also known as the Hahn–Exton q-Bessel function, see e.g. [156, 174]. These q-analogues satisfy that

$$\lim_{q\to 1^-} J_\nu^{(k)}((1-q)z;q) = J_\nu(z)\ (k = 1, 2), \qquad \lim_{q\to 1^-} J_\nu^{(3)}((1-q)z;q) = J_\nu(2z),$$

cf. [174]. The functions $J_\nu^{(k)}(z;q)$ $(k = 2, 3)$ are entire functions of order zero. Therefore, by Theorem 1.1, they have infinitely many zeros. Hahn [123] found that

$$J_\nu^{(1)}(z;q) = J_\nu^{(2)}(z;q)/\left(-z^2/4;q^2\right)_\infty \quad (|z| < 2). \tag{1.56}$$

Because of the finite radius of convergence of the series in (1.53), Ismail [141] remarked that $J_\nu^{(1)}(z;q)$ has only finitely many positive zeros. Furthermore, in [251], Rahman considered the identity in (1.56) as an analytic continuation of the function $J_\nu^{(1)}(z;q)$. Set

$$J_\nu^{(2)}(z|q) = J_\nu^{(2)}\left(2z(1-q^{1/2});q\right),$$

and

$$J_\nu^{(1)}(z|q) = J_\nu^{(2)}(z|q)/\left(-(1-q^{1/2})^2z^2;q^2\right),$$

so that

$$\lim_{q\to 1^-} J_\nu^{(1)}(z|q) = \lim_{q\to 1^-} J_\nu^{(2)}(z|q) = J_\nu(z).$$

Rahman, cf. [251], proved that $J_\nu^{(k)}(\lambda z|q)$ $(k = 1,\ 2)$ satisfies the q-difference equations

$$D_{q^{1/2}}\left(zD_{q^{1/2}}J_\nu^{(k)}(\lambda z|q)\right) - q^{-\nu/2}\left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 z^{-1}J_\nu^{(k)}\left(\lambda zq^{1/2}|q\right)$$
$$= -\lambda^2\begin{cases} zJ_\nu^{(1)}(\lambda z|q) \\ qzJ_\nu^{(2)}(\lambda zq|q) \end{cases}$$

and Exton, cf. [98], shows that $J_\nu^{(3)}(\lambda z;q)$ satisfies the q-difference equation

$$D_{q^{1/2}}\left(zD_{q^{1/2}}J_\nu^{(3)}(\lambda z;q)\right)+\left(\lambda^2 z-\frac{q^{-\nu/2}}{z}\left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2\right)J_\nu^{(3)}\left(\lambda z q^{1/2};q\right)=0.$$

1.6 The q-Gamma and q-Beta Functions

The q-gamma function was introduced by Thomae [282] and later by Jackson [155] and defined by

$$\Gamma_q(z):=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}\quad (0<|q|<1)\,, \tag{1.57}$$

where $z\in\mathbb{C}\backslash\{-n\ :\ n\in\mathbb{N}_0\}$. Here, we take the principal values of q^z and $(1-q)^{1-z}$. Then $\Gamma_q(z)$ is a meromorphic function with poles at $z=-n$, $n\in\mathbb{N}_0$. Since $\Gamma_q(z)$ has no zeros, then $1/\Gamma_q(z)$ is an entire function with zeros at $z=-n$, $n\in\mathbb{N}_0$. Obviously,

$$\Gamma_q(n)=\frac{(q;q)_{n-1}}{(1-q)^{n-1}}\quad (n\in\mathbb{N}).$$

It is known that for $x>0$, $\Gamma_q(x)$ is the unique logarithmically convex function that satisfies the functional equation

$$\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x),\quad \Gamma_q(1)=1.$$

See [25] and [113, P. 21] for more historical remarks about the q-gamma function. The q-beta function is defined by

$$B_q(x,y):=\int_0^1 t^{x-1}(qt;q)_{y-1}\,d_qt\quad (\mathrm{Re}(x)>0;\ \mathrm{Re}(y)>0)\,. \tag{1.58}$$

Using the q-binomial theorem, cf. [25, P. 488],

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}\quad (|z|<1), \tag{1.59}$$

Askey [40] proved

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\quad (\mathrm{Re}(x)>0;\ \mathrm{Re}(y)>0)\,,$$

see also [25, p. 494]. Many characterization theorems for the q-gamma and the q-beta functions are in [75, 89, 145, 164, 224].

Lemma 1.18. *Let α and β be two complex numbers. Then for every $n \in \mathbb{N}_0$*

$$(q^{\alpha+\beta};q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k\beta} (q^{\alpha};q)_k (q^{\beta};q)_{n-k}. \tag{1.60}$$

Proof. Using the q-analogue of Gauss'summation formula(1.10) and applying [113, Eq. (I.10)] we obtain

$$\begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k\beta} (q^{\alpha};q)_k (q^{\beta};q)_{n-k} &= (q^{\beta};q)_n \, {}_2\phi_1(q^{-n}, q^{\alpha}; q^{1-\beta-n}; q, q) \\ &= (q^{\beta};q)_n q^{n\alpha} \frac{(q^{1-\beta-\alpha-n};q)_n}{(q^{1-\beta-n};q)_n} = (q^{\alpha+\beta};q)_n. \end{aligned}$$

□

Corollary 1.19. *Let α and β be complex numbers, and let $n \in \mathbb{N}_0$. Then*

$$B_q(\alpha,\beta)(q^{n+1};q)_{\alpha+\beta-1} = \sum_{k=0}^{n} q^{k\beta}(1-q)(q^{k+1};q)_{\alpha-1}(q^{n-k+1};q)_{\beta-1}, \tag{1.61}$$

and

$$q^{-n\beta}(q^{n+1};q)_{\alpha+\beta-1} B_q(\alpha,\beta) = \sum_{k=0}^{n} q^{-k\beta}(1-q)(q^{n-k+1};q)_{\alpha-1}(q^{k+1};q)_{\beta-1}. \tag{1.62}$$

Proof. Using (1.6) and (1.57) one can easily see that for any $\gamma \in \mathbb{C}$, $j \in \mathbb{N}_0$

$$(q^{j+1};q)_{\gamma-1} = \frac{(q;q)_\infty (q^\gamma;q)_j}{(q^\gamma;q)_\infty (q;q)_j} = \Gamma_q(\gamma)(1-q)^{\gamma-1}\frac{(q^\gamma;q)_j}{(q;q)_j}. \tag{1.63}$$

Hence, (1.60) yields

$$\begin{aligned} &\sum_{k=0}^{n} q^{k\beta}(1-q)(q^{k+1};q)_{\alpha-1}(q^{n-k+1};q)_{\beta-1} \\ &\quad = \frac{(1-q)^{\alpha+\beta-1}\Gamma_q(\alpha)\Gamma_q(\beta)}{(q;q)_n} \sum_{k=0}^{n} q^{k\beta} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^{\alpha};q)_k \, (q^{\beta};q)_{n-k} \qquad (1.64) \\ &\quad = (1-q)^{\alpha+\beta-1}\Gamma_q(\alpha)\Gamma_q(\beta)\frac{(q^{\alpha+\beta};q)_n}{(q;q)_n}. \end{aligned}$$

But from (1.63)

$$\frac{(q^{\alpha+\beta};q)_n}{(q;q)_n} = \frac{(1-q)^{1-\alpha-\beta}}{\Gamma_q(\alpha+\beta)}(q^{n+1};q)_{\alpha+\beta-1}. \tag{1.65}$$

Combining (1.64) and (1.65) leads to (1.61). The proof of (1.62) follows from the proof of (1.61) by replacing each k in the right hand side of (1.61) by $n-k$. □

Lemma 1.20. *Let α and β be two complex numbers with positive real parts, and let $f \in L_q^1(0,a)$ for some $a > 0$. Then*

$$\begin{aligned} &a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\beta-1}\int_0^t (qu/t;q)_{\beta-1}f(u)\,d_qu\,d_qt \\ &\qquad = B_q(\alpha,\beta)a^{\alpha+\beta-1}\int_0^a \big(qt/a;q\big)_{\alpha+\beta-1}f(t)\,d_qt. \end{aligned} \tag{1.66}$$

Proof. Let α, $\beta > 0$ and $f \in L_q^1(0,a)$, $a > 0$. Then by the definition of the q-integration, cf. (1.19) we obtain

$$\begin{aligned} &a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\beta-1}\int_0^t (qu/t;q)_{\beta-1}f(u)\,d_qu\,d_qt \\ &= a^{\alpha+\beta}(1-q)^2\sum_{n=0}^{\infty} q^{n\beta}(q^{n+1};q)_{\alpha-1}\sum_{m=0}^{\infty} q^{n+m}(q^{m+1};q)_{\beta-1}f(aq^{n+m}) \\ &= a^{\alpha+\beta}(1-q)^2\sum_{n=0}^{\infty} q^{n\beta}(q^{n+1};q)_{\alpha-1}\sum_{k=n}^{\infty} q^{k}(q^{k-n+1};q)_{\beta-1}f(aq^{k}) \\ &= a^{\alpha+\beta-1}\sum_{k=0}^{\infty} aq^k(1-q)f(aq^k)\sum_{n=0}^{k} q^{n\beta}(1-q)(q^{n+1};q)_{\alpha-1}(q^{k-n+1};q)_{\beta-1}. \end{aligned}$$

Thus, from (1.61)

$$\begin{aligned} &a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\beta-1}\int_0^t (qu/t;q)_{\beta-1}f(u)\,d_qu\,d_qt \\ &= a^{\alpha+\beta-1}B_q\big(\alpha,\beta\big)\sum_{k=0}^{\infty} aq^k(1-q)(q^{k+1};q)_{\alpha+\beta-1}f(aq^k) \\ &= a^{\alpha+\beta-1}B_q\big(\alpha,\beta\big)\int_0^a (qt/a;q)_{\alpha+\beta-1}\,f(t)\,d_qt. \end{aligned}$$

□

1.7 q-Integral Representations of the q-Gamma, q-Beta and Some Related q-Functions

In [80], De Sole and Kac introduced the following theorem:

Theorem 1.21. *Let*

$$K_q(A,x) := A^x \frac{(-1/A,-qA;q)_\infty}{(-q^x/A,-q^{1-x}A;q)_\infty} \quad \text{and} \quad K_q(x) := K_q(1,x), \tag{1.67}$$

where $A, \operatorname{Re}(x) > 0$. *Then for* $x, y \in \mathbb{C}$ *such that* $\operatorname{Re} x > 0,\ \operatorname{Re} y > 0$

$$\Gamma_q(x) = K_q(A,x) \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t(1-q))\, d_qt, \tag{1.68}$$

and

$$B_q(x,y) = K_q(A,x) \int_0^{\infty/A} t^{x-1} (-tq^{x+y};q)_{-x-y}\, d_qt. \tag{1.69}$$

Moreover, if $\ln(1-q)/\ln(q) \in \mathbb{N}$ *then*

$$\Gamma_q(x) = \frac{K_q(x)}{(1-q)^x} \int_0^{\infty} t^{x-1} e_q(-t)\, d_qt.$$

Proof. Since

$$\int_0^{\infty/A(1-q)} t^{x-1} e_q(-t(1-q))\, d_qt = \frac{(1-q)^{1-x}}{A^x(-1/A;q)_\infty} \sum_{n=-\infty}^{\infty} q^{nx}(-1/A;q)_k,$$

then applying Ramanujan's identity (1.52) gives

$$\int_0^{\infty/A(1-q)} t^{x-1} e_q(-t(1-q))\, d_qt = \frac{(1-q)^{1-x}}{A^x(-1/A;q)_\infty} \frac{(q,-q^x/A,-q^{1-x}/A;q)_\infty}{(q^x,-1/A,-q/A;q)_\infty}.$$

Then (1.68) follows . Similarly, we can prove (1.69).

Proposition 1.22. *For* $\operatorname{Re}(x) > 0$

$$\Gamma_q(x) = \frac{1}{(1-q)^x} \int_0^1 t^{x-1} E_q(-qt)\, d_qt \tag{1.70}$$

$$= \int_0^{1/1-q} t^{x-1} E_q(-q(1-q)t)\, d_qt. \tag{1.71}$$

Proof. The q-integral representation of $\Gamma_q(x)$ in (1.70) follows at once by letting $\mathrm{Re}(y) \to \infty$ on (1.58). The second identity of the q-gamma functions follows from (1.70) by making the substitution $t = (1-q)\xi$ and then replacing ξ by t. ☐

The functions erf(z) and erfc(z) are respectively the error function and the complimentary error function encountered in integrating the normal distribution, see [127]. They are entire functions and defined by

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \quad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z) \qquad (z \in \mathbb{C})\,. \tag{1.72}$$

q-analogues of the error function and the complementary error function are introduced in [92] and defined by

$$\mathrm{Erf}(z;q) := \frac{1+q}{\Gamma_{q^2}(1/2)} \int_0^z E_{q^2}(-q^2t^2(1-q^2))\, d_qt \qquad (z \in \mathbb{C}) \tag{1.73}$$

and

$$\mathrm{Erfc}(z;q) := \frac{1+q}{\Gamma_{q^2}(1/2)} \int_z^{\frac{1}{\sqrt{1-q^2}}} E_{q^2}(-q^2t^2(1-q^2))\, d_qt \quad (z \in \mathbb{C})\,. \tag{1.74}$$

The function Erf($z;q$) has the series representation

$$\begin{aligned}\mathrm{Erf}(z;q) &= \frac{1-q^2}{\Gamma_{q^2}(1/2)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{z^{2n+1}}{(1-q^{2n+1})\Gamma_{q^2}(n+1)} \\ &= \frac{(1+q)z}{\Gamma_{q^2}(1/2)} {}_1\phi_1\left(q; q^3; q^2, q^2z^2(1-q^2)\right).\end{aligned}$$

The error function Erf($z;q$) has the following properties:

1. $\mathrm{Erf}(0;q) = 0$,
2. $\mathrm{Erf}(\frac{\pm q^{-k}}{\sqrt{1-q^2}};q) = \mathrm{Erf}(\frac{\pm 1}{\sqrt{1-q^2}};q) = \pm 1 \quad (k \in \mathbb{N}_0)$,
3. $\mathrm{Erf}(z;q) + \mathrm{Erfc}(z;q) = 1 \quad (z \in \mathbb{C})$.

Another q-analogue is introduced in [245] and defined by

$$\mathrm{Erf}_q(z) := \frac{2z}{\sqrt{\pi}} {}_2\phi_1(0, q^{1/2}; q^{3/2}; q, -z^2) \quad (|z| < 1)\,. \tag{1.75}$$

One can verify that

$$\mathrm{Erf}_q(z) := \frac{2}{\sqrt{\pi}} \int_0^z e_q(-t^2(1-q))\, d_{q^{1/2}}t. \tag{1.76}$$

Since the constant multiplier in (1.72) is a normalization factor so that

$$\lim_{z\to\infty} \operatorname{erf}(z) = 1,$$

the constant multiplier in (1.75) or in (1.76) does not make sense. In the following lemma we calculate the value of

$$\int_0^{\infty/\sqrt{1-q^2}} e_{q^2}(-t^2(1-q^2))\,d_qt,$$

to identify an appropriate multiplier constant in (1.76).

Lemma 1.23.

$$\int_0^{\infty/\sqrt{1-q^2}} e_{q^2}(-t^2(1-q^2))\,d_qt = \frac{1}{1+q}\frac{\Gamma_{q^2}(1/2)}{K_{q^2}(1/2)},$$

where

$$K_{q^2}(1/2) := K_{q^2}(1,1/2) = \frac{(-1,-q^2;q^2)_\infty}{(-q,-q;q^2)_\infty}.$$

Proof. Applying (1.21), we obtain

$$\int_0^{\infty/\sqrt{1-q^2}} e_{q^2}(-t^2(1-q^2))\,d_qt = \frac{1-q}{\sqrt{1-q^2}(-1;q^2)_\infty}\sum_{-\infty}^{\infty} q^k(-1;q^2)_k.$$

Therefore, applying Ramanujan identity (1.52) yields

$$\begin{aligned}\int_0^{\infty/\sqrt{1-q^2}} e_{q^2}(-t^2(1-q^2))\,d_qt &= \frac{1-q}{\sqrt{1-q^2}}\frac{\left(q^2,-q,-q;q^2\right)_\infty}{(-1,-q^2,q;q^2)_\infty}\\ &= \frac{\Gamma_{q^2}(1/2)}{(1+q)K_{q^2}(1/2)}.\end{aligned}$$

□

This motivates us to introduce a q-analogue of the error function by

$$\operatorname{erf}(z;q) := \frac{(1+q)K_{q^2}(1/2)}{\Gamma_q^2(1/2)}\int_0^z e_{q^2}(-t^2(1-q^2))\,d_qt, \tag{1.77}$$

so that

$$\lim_{k\to\infty} \operatorname{erf}(\frac{q^{-k}}{\sqrt{1-q^2}};q) = 1.$$

The function $\operatorname{erf}(z;q)$ is defined on $\mathbb{R}$ and as a complex function, it is meromorphic with simple poles at the points $z = \pm i \frac{q^{-k/2}}{\sqrt{1-q^2}}$, $k \in \mathbb{N}_0$. Moreover,

$$\operatorname{erf}(z;q) := \frac{1+q}{\Gamma_q^2(1/2)} K_{q^2}(1/2)\, z\, {}_2\phi_1\left(0, q; q^3; q^2, -z^2(1-q^2)\right) \text{ for } |z| < \frac{1}{1-q}.$$

We also define two q-analogues of the incomplete gamma function through the relations

$$\gamma_q(s,x) = \int_0^x t^{s-1}\, e_q(-t(1-q))\, d_qt = \sum_{n=0}^{\infty} (-1)^n \frac{(1-q)x^{n+s}}{(1-q^{n+s})\Gamma_q(n+1)}, \quad (1.78)$$

$$\Gamma_q(s,x) = \int_0^x t^{s-1}\, E_q(-t(1-q))\, d_qt = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\binom{n}{2}}(1-q)x^{n+s}}{(1-q^{n+s})\Gamma_q(n+1)}. \quad (1.79)$$

A straightforward manipulation gives

$$\gamma_q(s,x) = x^s \Gamma_q(s) \sum_{n=0}^{\infty} (-1)^n \frac{(q^s;q)_n}{(q;q)_n \Gamma_q(n+s+1)} x^n, \quad (1.80)$$

$$\Gamma_q(s,x) = x^s \Gamma_q(s) \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-1)^n \frac{(q^s;q)_n}{(q;q)_n \Gamma_q(n+s+1)} x^n. \quad (1.81)$$

Hence, for each fixed x, as functions of the variable s, the functions $\gamma_q(s,x)$ and $\Gamma_q(s,x)$ have poles at all non-positive integer values. Therefore, we extend the functions $\gamma_q(s,x)$ and $\Gamma_q(s,x)$ to the holomorphic functions

$$\gamma_q^*(s,x) = \frac{x^{-s}}{\Gamma_q(s)} \gamma_q(s,x), \quad \Gamma_q^*(s,x) = \frac{x^{-s}}{\Gamma_q(s)} \Gamma_q(s,x).$$

1.8 *q*-Analogue of the Wright Function

Wright [293] introduced the function

$$\phi(\alpha,\beta,z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)} \quad (\alpha > -1;\ \beta \in \mathbb{C}).$$

This function has many applications in fractional differential and partial differential equations. See, for example, [111, 115, 167, 193, 201, 269] for extensive study of the properties of the Wright function and its applications. The generalized Wright function ${}_r\psi_s(z)$ is defined by

$$ {}_r\psi_s(z) = {}_r\psi_s\left[\begin{matrix}(a_1,\alpha_1), \dots, & (a_r,\alpha_r)\\ (b_1,\beta_1) & ,\dots, (b_s,\beta_s)\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i+\alpha_i k)}{\prod_{l=1}^{s} \Gamma(b_l+\beta_l k)} \frac{z^k}{k!}, \tag{1.82} $$

where $a_i, b_l \in \mathbb{C}$, $\alpha_i, \beta_l \in \mathbb{R}$, $l = 1,\dots,r$ and $j = 1,\dots,s$. This generalization of the Wright function was investigated by Fox [109] and Wright [294–296], see also [169]. In some literature, the function ${}_r\psi_s$ is called the Fox–Wright function, see [79, 167, 170].

In [92], El-Shahed and Salem defined a q-analogue of the Wright function through the Barnes–Contour integral

$$ \phi_{\alpha,\beta}(z;q^k) = \frac{\ln q}{2\pi\,(q-1)} \int_C \frac{\Gamma_q(-s)}{\Gamma_q^k(\beta-\alpha\,s)} (-z)^s\,ds, $$

where $k \in \mathbb{N}$, $0 < q < 1$, z is not equal to zero and C is a suitable path in the complex s-plane that runs from $s = -i\infty$ to $s = i\infty$, so the points $s = n$, $n \in \mathbb{N}_0$ lie to the right of the contour C. It seems that there is an error in choosing the contour C passing through the pole $s = 0$. The authors of [92] also proved that

$$ \phi_{\alpha,\beta}(z;q^k) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+1)\Gamma_{q^k}(\alpha\,n+\beta)} q^{n(n+1)/2}. $$

In the following we introduce two q-analogues of the Wright function through the identities

$$ w_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha\,k+\beta)\Gamma_q(k+1)} \quad \text{for } |z| < (1-q)^{-\alpha}, \tag{1.83} $$

and

$$ W_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} q^{\alpha k(k-1)/2} \frac{z^k}{\Gamma_q(\alpha k+\beta)\Gamma_q(k+1)} \quad \text{for } z \in \mathbb{C}. \tag{1.84} $$

In the following definition, we introduce a q-analogue of the Fox–Wright function defined in (1.82).

Definition 1.8.1. A q-analogue of the Fox–Wright function is defined by

$$ \begin{aligned} {}_r\psi_s(z;q) &= {}_r\psi_s\left[\begin{matrix}(a_1,\alpha_1), \dots, (a_r,\alpha_r)\\ (b_1,\beta_1), \dots, (b_s,\beta_s)\end{matrix}\middle| q, z\right] \\ &= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma_q(a_i+\alpha_i k)}{\prod_{l=1}^{s} \Gamma_q(b_l+\beta_l k)} \left(q^{k(k-1)/2}\right)^{\sum_{i=1}^{s}\beta_i - \sum_{l=1}^{r}\alpha_l + 1} \frac{z^k}{\Gamma_q(k+1)}, \end{aligned} \tag{1.85} $$

where

$$\mathrm{Re}(a_l) > 0,\ \mathrm{Re}(b_i) > 0,\ l = 1, 2, \ldots, r,\ i = 1, 2, \ldots, s,$$

and

$$\sum_{i=1}^{s} b_i - \sum_{l=1}^{r} a_l + 1 \geq 0.$$

It is straightforward to prove that if

$$\sum_{i=1}^{s} \beta_i > \sum_{l=1}^{r} \alpha_l - 1,$$

then the domain of convergence is $\mathbb{C}$. If $\sum_{i=1}^{s} \beta_i = \sum_{l=1}^{r} \alpha_l - 1$, the series in (1.85) converges only for $|z| < 1$. If we set $a_l = b_i = 1$, we obtain

$$_r\psi_s(z;q) = {}_r\phi_s\left(\alpha_1, \ldots \alpha_r; \beta_1, \ldots, \beta_s; q, (-1)^{s-r+1}z\right).$$

1.9 *q*-Analogues of the Laplace Transform

In [123], Hahn defined the following two q-analogues of the Laplace transform

$$_qL_s f(x) = \phi(s) = \frac{1}{1-q} \int_0^{s^{-1}} E_q(-qsx)\ f(x)\, d_qx, \tag{1.86}$$

and

$$_q\mathscr{L}_s f(x) = \Phi(s) = \frac{1}{1-q} \int_0^{\infty} e_q(-sx) f(x)\, d_qx, \tag{1.87}$$

where $\mathrm{Re}(s) > 0$. Abdi [3], studied certain properties of these q-transforms. In [5], he used these analogues to solve linear q-difference equations with constant coefficients and certain allied equations. In this section, we present some properties of the transforms $_q\,L_s$ and $_q\,\mathscr{L}_s$. For extensive study of the properties of the Laplace transforms (1.86) and (1.87), cf. e.g. [3, 5, 6, 123–125, 214–218].

1.9.1 The $_q\,L_s$ Transform

We summarize here the most important results of the $_q\,L_s$ transform which we shall need in the sequel. Using the definition of the q-integration (1.19), the q-Laplace transform (1.86) is

$$_q\,L_s f(x) = \frac{(q;q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q;q)_j} f(s^{-1}q^j).$$

As an example, if

$$\phi_r(x) = \frac{x^r}{\Gamma_q(r+1)}\ (r > -1) \text{ then } {}_qL_s\phi_r(x) = \frac{(1-q)^r}{s^{r+1}},\ \mathrm{Re}(s) > 0. \tag{1.88}$$

In [123, Eq. (9.5)], Hahn defined the convolution of two functions $F,\ G$ to be

$$(F * G)(x) = \frac{x}{1-q}\int_0^1 F(tx)G[t - tqx]\, d_qt, \tag{1.89}$$

where $G[x-y]$, for

$$G(x) := \sum_{n=0}^{\infty} a_n x^n,$$

is defined to be

$$G[x-y] := \sum_{n=0}^{\infty} a_n [x-y]_n, \quad \text{with} \quad [x-y]_n := x^n \left(\frac{y}{x}; q\right)_n.$$

Using the definition of q-integration, $(F * G)$ is nothing but

$$(F * G)(x) = \frac{1}{1-q}\int_0^x F(t)G[x - qt]\, d_qt. \tag{1.90}$$

The translation operator (1.14) implies

$$G[x - qt] = \varepsilon^{-qt}G(x).$$

Ismail in [143] defined the convolution of two functions $F,\ G$ to be

$$\big(F * G\big) = \frac{1}{1-q}\int_0^x F(t)\varepsilon^{-qt}G(x)\, d_qt. \tag{1.91}$$

It is remarked by Hahn, cf. [123, P. 373] that the convolution theorem

$${}_qL_s(F * G) = {}_qL_sF\ {}_qL_sG, \tag{1.92}$$

holds only for ${}_qL_s$ transform and does not hold for the ${}_q\mathscr{L}_s$ transform. Abdi [3] also pointed out that whereas the ${}_qL_s$ transform has the above advantage over the ${}_q\mathscr{L}_s$ transform, it suffered from the disadvantage that a q-analogue of the well known Goldstein's theorem does not hold, see [3] and the references cited therein. Now we mention without proofs some of the basic properties of the ${}_qL_s$ transform. The interested reader could refer to [3] for proofs. In the following $\phi(s) = {}_qL_s(f(x))$.

Rule 1.

$$_q\,L_s\,f(ax) = (1/a)\phi(s/a) \quad (a \neq 0).$$

Rule 2.

$$_q\,L_s\left(D_q^n f(x)\right) = \left(\frac{s}{1-q}\right)^n \phi(s) - \sum_{m=1}^{n} D_q^{n-m} f(0)\frac{s^{m-1}}{(1-q)^m} \;\; (n \in \mathbb{N})\,. \tag{1.93}$$

The case $n = 1$ was proved by Hahn in [122].

Rule 3.

$$_q\,L_s\left(\frac{f(x)}{x}\right) = \frac{1}{1-q}\int_{qs}^{\infty} \phi(t)\,d_q t,$$

provided that $_q\,L_s\left(\frac{f(x)}{x}\right)$ exists.

Rule 4.

$$_q\,L_s\left(\int_x^{\infty} \frac{f(t)}{t}\,d_q t\right) = \frac{1}{s}\int_0^{qs} \phi(t)\,d_q t,$$

provided

(i) The two integrals exist,
(ii) $\phi(s)$ exists for all s,
(iii) $q^j \sum_{r=1}^{\infty} |f(q^j/s)| = O(|h|^j)$, for every fixed j and $|h| < 1$.

Rule 5. If

$$I_q^n f(x) = \int_0^x \int_0^{x_{n-1}} \int_0^{x_{n-1}} \ldots \int_0^{x_1} f(t)\,d_q t\,d_q x_1 d_q x_2 \ldots d_q x_{n-1} \tag{1.94}$$

then

$$_q\,L_s(I_q^n f(x)) = \left(\frac{1-q}{s}\right)^n \phi(s).$$

Rule 6. The addition theorem.

If $_q\,L_s(f_r(x)) = \phi_r(s)$ then

$$_q\,L_s\left(\sum_{r=0}^{m} f_r(x)\right) = \sum_{r=0}^{m} \phi_r(s) \tag{1.95}$$

provided any of the following conditions holds:

(a) m is finite
(b) m is infinite and

  (i) $\sum_{r=0}^{\infty} |f_r(xq^j)|$ is convergent for all $x_0 q^j$, x_0 being fixed,
  (ii) $q^j \sum_{r=0}^{\infty} |f_r(xq^j)| = O(h^j)$, where j is greater than some fixed J and h is a fixed quantity, $|h|$ being less than unity.

Rule 7. Set

$$F(x) = \int_0^x \frac{f(t)}{t}\, d_q t,$$

and assume that $\lim_{x\to 0} \dfrac{f(x)}{x} = 0$. Then

$$_q L_s\,(F(x)) = \frac{1}{s}\int_{qs}^{\infty} \phi(s)\, d_q s.$$

The following theorem is the inversion formula of the $_q L_s$ transform introduced by Hahn in [122, P. 373].

Theorem 1.24 (Hahn's inversion formula). *If $_q L_s(f(x)) = \phi(s)$ then*

$$f(x) = \int_C \phi(s) e_q(sx)\, ds, \tag{1.96}$$

where the path of integration C encircles the origin and can also be deformed into a loop, parallel to the imaginary axis. In particular, when $\phi(s)$ is analytic, the inversion is

$$f(x) = \frac{1}{x}\sum_{i=0}^{\infty}(-1)^i q^{i(i-1)/2}\frac{\phi(x^{-1}q^{-i})}{(q;q)_i}.$$

1.9.2 The $_q\mathscr{L}_s$ Transform

We mentioned in the previous section that although there is no q-analogue of the convolution theorem which holds for the $_q\mathscr{L}_s$ transform, the $_q\mathscr{L}_s$ transform has an advantage over the $_q L_s$ transform. This advantage is the verification of q-analogue of the Goldstein theorem. This q-analogue was proved by Abdi in [3, P. 400]. Using the definition of the q-integration (1.19), the q-Laplace transform (1.87) is

$$_q\mathscr{L}_s f(x) = \frac{1}{(-s;q)_\infty}\sum_{j=-\infty}^{\infty} q^j(-s;q)_j\, f(q^j).$$

The following is a brief account of the properties of the $_q\mathscr{L}_s$ transform introduced by Abdi in [3].

Rule 1. If $_q\mathscr{L}_s(f(x)) = \phi(s)$ and $a \in \mathbb{R}_{q,+}$, then

$$_q\mathscr{L}_s f(ax) = (1/a)\phi(s/a).$$

Rule 2.

$$
{}_q\mathscr{L}_s\left(D_q^n f(x)\right) = \left(\frac{s}{1-q}\right)^n \phi(s) - \sum_{m=1}^{n} D_q^{n-m} f(0) \frac{s^{m-1}}{q^{m-1}(1-q)^m}, \; n \in \mathbb{N}. \tag{1.97}
$$

Rule 3.

$$
{}_q\mathscr{L}_s\left(f(x)/x\right) = \frac{1}{1-q} \int_{qs}^{\infty} \phi(t)\, d_q t,
$$

provided that ${}_q\mathscr{L}_s\left(f(x)/x\right)$ exists.

Rule 4.

$$
{}_q\mathscr{L}_s\left(\int_x^{\infty} \frac{f(t)}{t}\, d_q t\right) = \frac{1}{s} \int_0^{qs} \phi(t)\, d_q t,
$$

provided

(i) The two integrals exist,
(ii) $\phi(s)$ exists for all s,
(iii) $q^j \sum_{r=1}^{\infty} |f(q^j/s)| = O(|h|^j)$, for every fixed j and $|h| < 1$.

Rule 5.

$$
{}_q\mathscr{L}_s(I_q^n f(x)) = \left(\frac{1-q}{s}\right)^n \phi(s),
$$

where I_q^n is defined in (1.94).

Rule 6. The addition theorem.

$$
\text{If } {}_q\mathscr{L}_s(f_r(x)) = \phi_r(s) \text{ then } {}_q\mathscr{L}_s\left(\sum_{r=0}^{m} f_r(x)\right) = \sum_{r=0}^{m} \phi_r(s),
$$

provided at least one of the following conditions holds:

(a) m is finite
(b) m is infinite and
  (i) $\sum_{r=0}^{\infty} |f_r(xq^j)|$ is convergent for all $x_0 q^j$, x_0 being fixed,
  (ii) $q^j \sum_{r=0}^{\infty} |f_r(xq^j)| = O(h^j)$, where j is greater than some fixed J and h is a fixed quantity, $|h|$ being less than unity.

Rule 7. If $\lim_{x \to 0} \dfrac{f(x)}{x} = 0$ then

$$
{}_q\mathscr{L}_s\left(\int_0^{x} \frac{f(x)}{x}\, d_q x\right) = \frac{1}{s} \int_{qs}^{\infty} \phi(t)\, d_q t.
$$

Rule 8. If ${}_q\mathscr{L}_s(f(x)) = \phi(s)$ then

$$
{}_q\mathscr{L}_s\left(x^n f(x)\right) = (q-1)^n D_{q,s}^n \phi(s). \tag{1.98}
$$

Abdi [3] introduced the identities

$$ {}_q\mathscr{L}_s\left(x^n D_q^m f(x)\right) = (-q^{-1})^m (q-1)^{n-m} D_{q,s}^n \left(s^m \phi(s/q^m)\right) \text{ for } m \le n \tag{1.99}$$

and

$$\begin{aligned} &{}_q\mathscr{L}_s\left(x^n D_q^m f(x)\right) \\ &= (q-1)^n D_q^m \left[\frac{s^m}{(q(1-q))^m}\phi(s/q^m) - \sum_{r=1}^{m} \frac{s^{r-1} D_q^{m-r} f(0)}{q^{r-1}(1-q)^r}\right], \end{aligned} \tag{1.100}$$

for all $m > n$. Then he obtained a generalization of Goldstein theorem which we state without proof.

Theorem 1.25 (q-Goldstein Theorem). *If* ${}_q\mathscr{L}_s f(x) = \phi(s)$ *and* ${}_q\mathscr{L}_s g(x) = \psi(s)$ *then*

$$\int_0^\infty f(x)\psi(x)\,d_qx = \int_0^\infty g(x)\phi(x)\,d_qx,$$

provided that

(i) *The two integral exists,*
(ii) $q^{j(j-1)/2} f(q^{-i}) = O(R^j)$ *when* $j \to \infty$,
(iii) $q^{j(j-1)/2} g(q^{-j}) = O(R^j)$ *when* $j \to \infty$,

where s *is greater than a fixed quantity* R, $R < 1$.

Remark 1.9.1. 1. It should be noted that Abdi derived Rules (1.93), (1.97) and (1.100) in terms of $f^{(r)}(0)$, the rth classical derivative at zero, since he defined the q-derivative at zero to be equal to $f'(0)$. But one can verify that the mentioned rules are still correct when $D_q f(0)$ is defined as in (1.17).

2. Abdi in [5] employed the ${}_q\mathscr{L}_s$ transform to solve q-difference equation with variable coefficients. As an illustration to his technique, he considered a q-difference equation of the form

$$(x^2 + a_0^2) D_q^2 f(x) + a_1 x D_q f(x) + a_2 f(x) = F(x), \tag{1.101}$$

where a_0 and a_2, are arbitrary constants and $a_1 := -\frac{1+q}{1-q}$. Then he applied the ${}_q\mathscr{L}_s$ transform to (1.101) to obtain

$$A(s)\phi(s) + B(s)\phi(\frac{s}{q}) + C(s)\phi(\frac{s}{q^2}) = D(s) + q^2(1-q)^2\psi(s), \tag{1.102}$$

where

$${}_q\mathscr{L}_s\left(f(x)\right)(s) = \phi(s), \quad {}_q\mathscr{L}_s\left(F(x)\right)(s) = \psi(s),$$

and

$$\begin{aligned}
A(s) &= q^2\left(q + a_1(1-q) + a_2(1-q)^2\right), \\
B(s) &= -q\left((1+q) + a_1(1-q)\right), \\
C(s) &= a_0^2 s^2 + 1, \\
D(s) &= q a_0^2 s f(0) + q^2(1-q)a_0^2 f^{(1)}(0).
\end{aligned}$$

He chose the constants a_0 and a_1 so that the coefficients of $\phi(s)$ and $\phi(\frac{s}{q})$ vanish. Thus, (1.102) becomes

$$\frac{1}{q^2}\phi(\frac{s}{q^2}) = \frac{a_0^2 q^{-1} s f(0) - (1-q)a_0^2 f^{(1)}(0)}{a_0 s^2 + 1} + \frac{(1-q)^2}{a_0 s^2 + 1}\psi(s).$$

Abdi proved in [3] that

$$\begin{aligned}
{}_q\mathscr{L}_s\left(\mathrm{Sin}_q(kx)\right)(s) &= \frac{qk}{k^2 + q^2 s^2}, \\
{}_q\mathscr{L}_s\left(\mathrm{Cos}_q(kx)\right)(s) &= \frac{q^2 s^2}{k^2 + q^2 s^2},
\end{aligned}$$

where $\mathrm{Sin}_q(\cdot)$ and $\mathrm{Cos}(\cdot)$ are q-analogues of the sine and cosine functions introduced by Jackson in [154], see also (1.47). Then by inversion (1.102), the solution function is defined through the relation

$$\begin{aligned}
f(q^2 x) = q^{-1} f(0)\,\mathrm{Cos}_q\left(\frac{x}{q a_0}\right) + (1-q)a_0 f^{(1)}(0)\,\mathrm{Sin}_q\left(\frac{x}{q a_0}\right) \\
+(1-q)^2 \frac{1}{2\pi i}\int_C \frac{\psi(s)}{a_0 s^2 + 1} E_q(qsx)\,ds.
\end{aligned}$$

1.10 *q*-Analogue of the Mellin Transform

The Mellin transform of a suitable function f defined over $(0,\infty)$ is defined by [243, Chap. 12]

$$\mathscr{M}(f)(s) = \int_0^\infty x^{s-1} f(x)\,dx. \tag{1.103}$$

In general, the integral (1.103) exists only for complex values $s = a + ib$ such that $a_1 < a < a_2$ where a_1 and a_2 depend on the function f. This introduces what is called the strip of definition of the Mellin transform. By $\langle\alpha,\beta\rangle$, we denote the largest open strip, called the fundamental strip, in which the integral (1.103) converges. It is well known that the inversion formula for the Mellin transform is given by the following line integral,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(f)(s) x^{-s}\, ds,$$

where $c \in \langle \alpha, \beta \rangle$. The definition of the Mellin convolution of suitable functions f and g is

$$f *_{\mathcal{M}} g(x) = \int_0^\infty f(y)\, g\left(\frac{x}{y}\right) \frac{dy}{y}.$$

The q-Mellin convolution rule is

$$\mathcal{M}\left(f *_{\mathcal{M}} g\right) = \mathcal{M}(f)\mathcal{M}(g).$$

Fitouhi et al. [102] defined q-analogue of the Mellin transform through the identity

$$\mathcal{M}_q(f)(s) = \int_0^\infty t^{s-1} f(t)\, d_q t, \;\; f \in L_q^1(\mathbb{R}_{q,+}).$$

Recall that $L_q^1(\mathbb{R}_{q,+})$ is the space of all functions defined over $(0, \infty)$ such that

$$\int_0^\infty |f(t)|\, d_q t < \infty.$$

It is straightforward to verify that the q-Mellin transform is a linear operator on $L_q^1(\mathbb{R}_{q,+})$. In other words, if f and g are two functions defined in $(0, \infty)$ such that their q-Mellin transforms exist then

$$\mathcal{M}_q(f + g) = \mathcal{M}_q(f) + \mathcal{M}_q(g).$$

By $\langle \alpha_{q,f}, \beta_{q,f} \rangle$, we mean the fundamental strip such that for $\alpha_{q,f} \leq \mathrm{Re}(s) \leq \beta_{q,f}$, the q-integral (1.103) is convergent. The authors of [102] introduced the following sufficient condition that guarantees the existence of the q-Mellin transform.

Proposition 1.26. *Let f be a function defined over $(0, \infty)$ and let u and v be real numbers such that $u > v$. Suppose*

$$f(x) = O(x^u) \text{ as } x \to 0, \quad \text{and} \quad f(x) = O(x^v) \text{ as } x \to \infty.$$

Then $\mathcal{M}_q(f)(s)$ exists in the strip $\langle -u, -v \rangle$.

Proof. See [102, P. 310]. □

The following rules of the q-Mellin transform is proved in [102].

Rule 1.

$$\mathcal{M}_q(t^a f(t))(s) = (\mathcal{M}_q f(t))(a + s). \tag{1.104}$$

Rule 2 . For $s \in \langle -\beta_{q,f}, -\alpha_{q,f} \rangle$, we have

$$\mathcal{M}_q\left[f\left(\frac{1}{t}\right)\right](s)=\mathcal{M}_q(f)(-s).$$

Rule 3. For $s\in\langle 1-\beta_{q,f},1-\alpha_{q,f}\rangle$, we have

$$\mathcal{M}_q\left[\frac{1}{t}f\left(\frac{1}{t}\right)\right]=\mathcal{M}_q(f)(1-s).$$

Rule 4. For $s\in\langle\alpha_{q,f}+1,\beta_{q,f}+1\rangle$, we have

$$\mathcal{M}_q\left[D_q^n f(t)\right]=\frac{\Gamma_q(n-s+1)}{\Gamma_q(1-s)}\mathcal{M}_q(f)(s-n)$$
$$+\sum_{k=0}^{n-1}(-1)^k\frac{\Gamma_q(s)}{\Gamma_q(s-k)}\left[\lim_{j\to-\infty}q^{j(s-k-1)}D_q^{n-k-1}(q^j)-\lim_{j\to\infty}q^{j(s-k-1)}D_q^{n-k-1}(q^j)\right].\tag{1.105}$$

If $f(t)$ and $\mathrm{Re}(s)$ are such that all the substitutions of the limit as $j\to\pm\infty$ in (1.105) give zero, then the formula (1.105) takes the simple form

$$\mathcal{M}_q\left[D_q^n f(t)\right]=\frac{\Gamma_q(n-s+1)}{\Gamma_q(1-s)}\mathcal{M}_q(f)(s-n).\tag{1.106}$$

Rule 5. For $s\in\langle\alpha_{q,f}-1,\beta_{q,f}-1\rangle$ we have

$$\mathcal{M}_q\left[\int_0^t f(x)\,d_qx\right](s)=\frac{1-q}{1-q^{-s}}\mathcal{M}_q(f)(s+1).$$

Rule 6. Given $\rho>0$ and $s\in\langle\rho\alpha_{q^\rho,f},\rho\beta_{q^\rho,f}\rangle$ we have

$$\mathcal{M}_q\left[f(t^\rho)(s)\right]=\frac{1-q}{1-q^\rho}\mathcal{M}_{q^\rho}(f)\left(\frac{s}{\rho}\right).$$

Rule 7. Let $(\mu_k)_k$ be a sequence of real numbers. Let $(\lambda_k)_k$ be a sequence of $\mathbb{C}$, and let f be a suitable function, then we have

$$\mathcal{M}_q\left[\sum_{k=0}^{\infty}\lambda_k f(\mu_k t)\right](s)=\left(\sum_{k=0}^{\infty}\frac{\lambda_k}{\mu_k^s}\right)\mathcal{M}_q(f)(s),$$

provided the sum converges.

Rule 8. If $a\in\mathbb{R}_{q,+}$ then

$$\mathcal{M}_q(f(at))(s)=\frac{1}{a^s}\mathcal{M}_q(f(t))(s).$$

Fitouhi et al. [102] proved the following inversion formula of the q-Mellin transform:

Theorem 1.27. *Let* f *be a function defined over* $(0,\infty)$ *and let* $c \in \langle \alpha_{q,f}, \beta_{q,f} \rangle$. *Then*

$$f(x) = \frac{\log(q)}{2\pi i(1-q)} \int_{c-i\frac{\pi}{\log q}}^{c+i\frac{\pi}{\log q}} \mathscr{M}_q(f)(s)x^{-s}\,ds \quad \textit{for all} \quad x \in \mathbb{R}_{q,+}.$$

Proof. See [102]. □

Fitouhi et al. defined the q-Mellin convolution of two functions f and g to be the function

$$f *_{\mathscr{M}_q} g(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{d_qy}{y}, \quad x \in \mathbb{R}_{q,+},$$

provided the q-integral exists. Then they proved that if the q-Mellin convolution product of f and g exists, then

$$f *_{\mathscr{M}_q} g = g *_{\mathscr{M}_q} f,$$

and

$$\mathscr{M}_q\left[f *_{\mathscr{M}_q} g\right] = \mathscr{M}_q(f)\mathscr{M}_q(g).$$

Chapter 2
q-Difference Equations

Abstract This chapter includes proofs of the existence and uniqueness of the solutions of first order systems of q-difference equations in a neighborhood of a point a, $a \geq 0$. Then, as applications of the main results, we study linear q-difference equations as well as the q-type Wronskian. These results are mainly based on (Mansour, q-Difference Equations, Master's thesis, Faculty of Science, Cairo University, Giza, Egypt, 2001). This chapter also includes a section on the asymptotics of zeros of some q-functions.

2.1 Introduction

The study of q-difference equations have been initiated by Jackson in [158]. The paper of Carmichael [73], to the best of our knowledge, is the first study of the problem of existence of solutions of linear q-difference equations using the technique established by Birkhoff in [56]. Mason [209] studied the existence of entire function solutions of homogeneous ($f = 0$) and nonhomogeneous linear q-difference equations of order n of the form

$$\sum_{j=0}^{n} a_j(x)y(q^{n-j}x) = f(x), \tag{2.1}$$

where the coefficients a_j are taken to be entire functions. Then Adams in [11–13] studied extensively the existence of solutions of (2.1) when the coefficients are analytic or have pole of finite order at the origin. More recently, Trjitzinsky [285] has developed an analytic theory of existence of solutions of homogenous linear q-difference equations and for their properties. The existence and uniqueness of solutions of first order linear q-difference equations in the space $C[0,\infty)$ and $L^p(\mathbb{R}^+)$ are discussed in [189]. Apart from this old history of q-difference equations, the subject received a considerable interest of many mathematicians and from many aspects, theoretical and practical. It is hard to encompass all such axes in a short notice, but to give the reader a reasonable idea we will mention some

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of these aspects. In the papers [136, 188, 190, 192, 198, 257, 258, 298] one finds general study of the theory of q-difference equations. The spectral analysis has also attracted the attention of many authors, like e.g. [14, 30, 35, 50, 66, 67, 98, 138, 140–142, 187, 194, 280, 281]. Since investigating q-difference equations using function theory tools explores more properties, this direction is also considered in many works, like [48, 184, 287, 299]. The solutions of difference equations and q-analogues of existing classical ones, especially orthogonal polynomials could be found in [38, 39, 43, 58, 74, 86, 107, 108, 144, 146]. Applicable problems involving q-difference equations and q-analogues of mathematical physical problems are studied extensively, see e.g. [1, 23, 63, 86, 87, 97, 100, 165, 194, 274, 286] for dynamical system, q-oscillator, q-classical and quantum models; [51, 101, 195, 225] for q-analogues of mathematical physical problems including heat and wave equations; [6, 8, 20, 25, 29, 36] for sampling theory of signal analysis.

In this chapter, we aim to develop a theory for q-difference equations similar to the one established for ordinary differential equations in [78, 88]. Therefore, we study the Cauchy problems of q-difference equations in a neighborhood of a point a, $0 \leqslant a < \infty$. We derive existence and uniqueness theorems for the cases $a = 0$ and $0 < a < \infty$. This is established by using a q-analogue of the Picard–Lindelöf method of differential equations and equations with deviating arguments, respectively. The range of validity of the solutions are investigated in each case and the existence and uniqueness theorem of solutions of q-difference equations of order n in a neighborhood of zero is also proved. The situation when the initial conditions are given at a point $a > 0$ is rather complicated. In [98, Sect. 5.1], it is claimed that the Cauchy problem

$$D_q\left\{K(x)D_q y\right\} - G(x)y(qx) = 0,\ a \leqslant x \leqslant b,\ y(c) = \gamma_0,\ D_q y(c) = \gamma, \quad (2.2)$$

where K and G are continuous function on $[a, b]$, c is an interior point of $[a, b]$ and γ_0, γ are complex numbers, has only one continuous solution with a continuous q-derivative. This is not necessarily true as the following counter example indicates.

Example 2.1.1. Let $g(t) := -(t - q^2c)(t - qc)$, $t \in [q^2c, qc]$ and c is a non zero real number. Since each $x \in [q^2c, \infty)$ is written uniquely in the form $x = tq^{-n}$; $t \in [q^2c, qc)$; $n \in \mathbb{N}_0$, the relation

$$\phi(x) = \phi(tq^{-n}) := g(t) \qquad \big(t \in [q^2, qc);\ n \in \mathbb{N}_0\big)$$

defines a function ϕ on $[q^2c, \infty)$. Clearly, ϕ is a continuous q-periodic function i.e., $D_q\phi(x) \equiv 0$. So, $D_q^2\phi(x) \equiv 0$, $D_q\phi(c) = 0$ and

$$\phi(c) = \phi\big((q^2c)q^{-2}\big) = g(q^2c) = 0.$$

Hence, the q-initial value problem $D_q^2 y(x) = 0$, $y(c) = D_q y(c) = 0$, has the functions

$$y(x) = \phi(x) \quad \text{and} \quad y(x) = 0$$

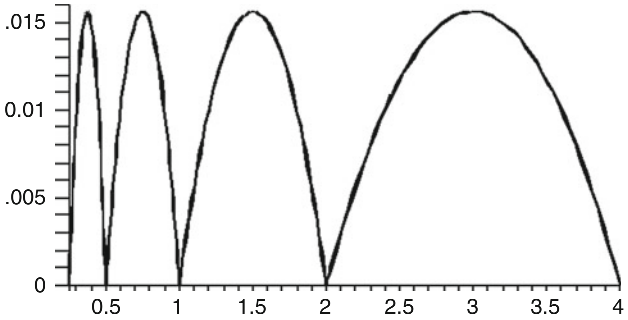

Fig. 2.1 The function $\phi(x)$ with $q = 1/2$, $c = 1$, and $x \in [0, 4]$

as two continuous solutions with continuous q-derivative over $[q^2c, \infty)$. See Fig. 2.1 for a graph of the function $\phi(x)$ when $q = \frac{1}{2}$.

In fact this setting is more complicated than the case when initial conditions are given at zero. The problem in this case behaves like differential equations with deviations, see e.g. [121]. We will have two types of problems, the forward and backward problems. This will be treated in Sect. 2.4.

2.2 q-Successive Approximations

In this section we define the q-successive approximations associated with q-Cauchy problems.

Definition 2.2.1. Let r, s and n_i, $i = 0, 1, \cdots, r$, be positive integers and let

$$N = (n_0 + 1) + \cdots + (n_r + 1) - 1.$$

Let $F_j(x, y_0, y_1, \cdots, y_N)$, $j = 0, 1, \cdots, s$, be real or complex-valued functions, where x is a real variable lying in some interval I and each y_i is a complex variable lying in some region D_i of the complex plane. If there is a sub-interval J of I and functions ϕ_i, $0 \le i \le r$, defined in J such that

1. ϕ_i has n_i q-derivatives in J for $i = 0, 1, \ldots, r$.
2. $D_q^m \phi_i$ exists and lies in the region D_i for all x in J, $0 \le m \le n_i$, and $0 \le i \le r$, for which the left-hand side in (2.3) below is defined.
3. For all x in J and $0 \le j \le s$, the following equations hold

$$F_j\left(x, \phi_0(x), D_q\phi_0(x), \cdots, D_q^{n_0}\phi_0(x), \cdots, \phi_r(x), \cdots, D_q^{n_r}\phi_r(x)\right) = 0, \quad (2.3)$$

then we say that $\{\phi_i\}_{i=0}^r$ is a solution of the system of the q-difference equations

$$F_j\left(x, y_0(x), D_q y_0(x), \cdots, D_q^{n_0} y_0(x), \cdots, y_r(x), \cdots, D_q^{n_r} y_r(x)\right) = 0, \tag{2.4}$$

$0 \leq j \leq s$, valid in J, or that the set $\{\phi_i\}_{i=0}^r$ satisfies (2.4) in J. If there are no such J and functions ϕ_i, we say that system (2.4) has no solutions. System (2.4) is said to be of order n, where $n := \max\limits_{0\leqslant i\leqslant p} n_i$.

We shall only consider first order systems (2.4) when $r = s = p$. If the functions F_j are such that (2.4) can be solved for the $D_q^{n_i} y_i$ in the form

$$D_q^{n_i} y_i(x) = f_i\left(x, y_0(x), D_q y_0(x), \cdots, y_1(x)\cdots\right) \quad (i = 0, 1, \ldots, p) \tag{2.5}$$

or in the form

$$D_q^{n_i} y_i(x) = f_i\left(qx, y_0(qx), D_{q,qx} y_0(qx), \cdots, y_1(qx)\cdots\right) \quad (i = 0, 1, \ldots, p), \tag{2.6}$$

the systems (2.5) and (2.6) will be called normal systems. The following are examples of normal first order systems:

$$D_q y_i(x) = f_i\left(qx, y_0(qx), y_1(qx), \ldots, y_p(qx)\right) \ \ (i = 0, 1, \ldots, p), \tag{2.7}$$
$$D_q y_i(x) = f_i\left(x, y_0(x), y_1(x), \ldots, y_p(x)\right) \ \ (i = 0, 1, \ldots, p). \tag{2.8}$$

The existence of a solution of system (2.7) or (2.8) in a neighborhood of a point a, $0 \leqslant a < \infty$, will be established by using a q-analogue of the Picard–Lindelöf method of successive approximations, see e.g. [78, 88]. The following is a description of this q-analogue for $a = 0$, and $0 < a < \infty$, respectively.

The Case $a = 0$

To establish the existence of solutions of the first order system (2.8), we define sequences of functions $\{\phi_{i,m}\}_{m=1}^\infty$, $i = 0, 1, \cdots, p$, by the equations

$$\phi_{i,m}(x) = \begin{cases} b_i, & m = 1, \\ b_i + \displaystyle\int_0^x f_i\left(t, \phi_{0,m}(t), \cdots, \phi_{p,m}(t)\right) d_q t, & m \geqslant 2, \end{cases} \tag{2.9}$$

where b_i are constants which lie in D_i. It is worth mentioning here that the constants $\{b_i\}_{i=0}^p$ should be replaced by q-periodic functions if we are looking for solutions which are not necessarily continuous at zero. We shall show in Theorem 2.1 that under suitable conditions on the f_i's, there are functions ϕ_i and an interval $J \subseteq I$ containing zero such that $\phi_{i,m}(x)$ tends uniformly to $\phi_i(x)$ on J as $m \to \infty$. Then, it will be possible to apply Lemma 1.14 to obtain

$$\phi_i(x) = b_i + \int_0^x f_i\left(t, \phi_0(t), \cdots, \phi_p(t)\right) d_q t. \tag{2.10}$$

Hence, $\phi_i(0) = b_i$, and $\{\phi_i\}_{i=0}^p$ is a solution of (2.8). Similarly, for (2.7).

The Case $0 < a < \infty$

This case differs from the case $a = 0$. For instance, instead of the initial values of (2.9), there should be initial functions defined on an initial interval $[qa, a)$ for (2.7) and $[a, aq^{-1})$ for (2.8). This is similar to the treatment of differential equations with deviations, see e.g. [121]. So, the successive approximations in case of (2.8) will be

$$\phi_{i,m}(tq^{-n}) = \begin{cases} c_i(t), & m = 1 \\ c_i(t) + \int_t^{tq^{-n}} f_i\left(qu, \phi_{0,m}(qu), \cdots, \phi_{p,m}(qu)\right) d_q u, & m \geq 2, \end{cases} \tag{2.11}$$

where $t \in [qa, a)$, $n \in \mathbb{N}_0$ and c_i are q-periodic functions. Thus, under suitable conditions on the functions f_i's which will be stated in Theorem 2.5, there exist functions $\{\phi_i\}_{i=0}^p$ such that $\lim\limits_{m\to\infty} \phi_{i,m}(x) = \phi_i(x)$ and

$$\phi_i(tq^{-k}) = c_i(t) + \int_t^{tq^{-k}} f_i(qu, \phi_0(qu), \ldots, \phi_m(qu))\, d_q t, \tag{2.12}$$

where $k \in \mathbb{N}_0$. Therefore, $\phi_i(t) \equiv c_i(t)$, $t \in [qa, a)$, and $\{\phi_i\}_{i=0}^p$ is a solution of (2.7) valid in $[qa, \infty)$. As for (2.7), the successive approximations will be

$$\phi_{i,m}(tq^n) = \begin{cases} c_i(t), & m = 1 \\ c_i(t) - \int_{tq^n}^{t} f_i\left(u, \phi_{0,m}(u), \ldots, \phi_{p,m}(u)\right) d_q u, & m \geqslant 2, \end{cases}$$

where c_i are q-periodic functions, $i = 0, 1, \ldots, p$ and $n \in \mathbb{N}$. It may seem that q-difference equations are equivalent to difference equations by the change of variable $x = q^y$. In fact, this transformation maps $x = 0$ to $\operatorname{Re} y = -\infty$ if $q > 1$ and $\operatorname{Re} y = \infty$ if $0 < q < 1$. Therefore, our analysis of q-regularity at $x = 0$ does not follow from existing theory of difference equations.

2.3 q-Initial Value Problems in a Neighborhood of Zero

Definition 2.3.1. Let I be an interval containing zero and E_r be disks of the form

$$E_r := \{y \in \mathbb{C} \; : \; |y - b_r| < \beta\}, \; \beta > 0, \; b_r \in \mathbb{C},$$

and $r = 0, 1, \cdots, p$. Let $f_i(x, y_0, y_1, \cdots, y_p)$, $i = 0, 1, \cdots, p$, be functions defined on $I \times E_0 \times \cdots \times E_p$. By a q-initial value problem in a neighborhood of zero we mean the problem of finding continuous at zero functions $\{y_i\}_{i=0}^{p}$ satisfying system (2.7) or (2.8) and the initial conditions

$$y_i(0) = b_i \quad (i = 0, 1, \ldots, p). \tag{2.13}$$

Theorem 2.1. *Let I be an interval containing zero and E_r be disks of the form*

$$E_r := \{y \in \mathbb{C} \; : \; |y - b_r| < \beta\}, \; \beta > 0, \; b_r \in \mathbb{C},$$

and $r = 0, 1, \cdots, p$. Let $f_i(x, y_0, y_1, \cdots, y_p)$, $i = 0, 1, \cdots, p$, be functions defined on $I \times E_0 \times \cdots \times E_p$ such that the following conditions are fulfilled.

(i) For $y_r \in E_r$, $0 \leqslant r \leqslant p$, $f_i(x, y_0, y_1, \cdots, y_p)$ is continuous at $x = 0$, $0 \leqslant i \leqslant p$.

(ii) There is a positive constant A such that, for $x \in I$ and $y_r, \tilde{y}_r \in E_r$, $0 \leqslant r, i \leqslant p$, the following Lipschitz' condition is fulfilled

$$\left|f_i(x, \tilde{y}_0, \cdots, \tilde{y}_p) - f_i(x, y_0, \cdots, y_p)\right| \le A\left(|\tilde{y}_0 - y_0| + \ldots + |\tilde{y}_p - y_p|\right). \tag{2.14}$$

Then, if 0 is not an end point of I, there exists $h > 0$ such that the Cauchy problem (2.8), (2.13) has a unique solution valid for $|x| \le h$. Moreover, if 0 is the left or right end point of I, the result holds, except that the interval $[-h, h]$ is replaced by $[0, h]$ or $[-h, 0]$, respectively.

Proof. We give a proof when 0 is an interior point of I. The proof when 0 is an end point of I is similar. Applying Lipschitz' condition (2.14),

$$|f_i(x, y_0, \ldots, y_p)| \leqslant |f_i(0, b_0, \ldots, b_p)|$$

$$+\left|f_i(x, y_0, \ldots, y_p) - f_i(x, b_0, \ldots, b_p)\right| + \left|f_i(x, b_0, \ldots, b_p) - f_i(0, b_0, \ldots, b_p)\right|$$

$$\leqslant A\sum_{j=0}^{p}|y_j - b_j| + \left|f_i(x, b_0, \ldots, b_p) - f_i(0, b_0, \ldots, b_p)\right| + |f_i(0, b_0, \ldots, b_p)|.$$

Since $f_i(x, b_0, \ldots, b_p)$ is continuous at zero, then there exists $\gamma > 0$ such that

$$|f_i(x, b_0, \ldots, b_p) - f_i(0, b_0, \ldots, b_p)| < 1 \;\text{ for } |x| \leqslant \gamma.$$

Hence,

$$|f_i(x, y_0, \ldots, y_p)| \leqslant A(p+1)\beta + 1 + \max_{0 \leqslant i \leqslant p} |f_i(0, b_0, \ldots, b_p)|,$$

for all $y_r \in E_r$, $r = 0, 1, \ldots, p$, and $|x| \leqslant \gamma$. Define the nonzero constants K, h to be

$$K := \max_{0 \le i \le p} \sup_{|x| \le \gamma, |y_i - b_i| \le \beta} |f_i(x, y_0, \cdots, y_p)|,$$

$$h := \min \left\{ \gamma, \frac{\beta}{K}, \frac{1}{A(p+1)(1-q)} \right\}.$$

We establish the existence of the solution $\{\phi_i\}_{i=0}^p$ of (2.8) and (2.13) on $J := [-h, h]$ using the method of successive approximations. We consider the sequence defined by (2.9).

Existence:- The proof of the existence is given in four steps. Fix $i \in \{0, 1, \cdots, p\}$.

(a) We show that $\phi_{i,m}$, $m \in \mathbb{N}$, are well defined. First

$$\phi_{i,m}(x) \in E_i \quad (x \in J;\ m \in \mathbb{N}). \tag{2.15}$$

Then, from (2.3), for $x \in J$ we have

$$|\phi_{i,m+1}(x) - b_i| \le \int_0^x \left| f_i\big(t, \phi_{0,m}(t), \cdots, \phi_{p,m}(t)\big) \right| d_q t \le K|x|. \tag{2.16}$$

Thus, each $\phi_{i,m}(x)$ is continuous at zero and (2.9) is well defined.

(b) For all $m \in \mathbb{N}$, $x \in J$, we can prove by induction on m that

$$|\phi_{i,m+1}(x) - \phi_{i,m}(x)| \le K B^{m-1} (1-q)^m \frac{|x|^m}{(q;q)_m}, \tag{2.17}$$

where $B := A(p+1)$.

(c) We show that $\phi_{i,m}$ tends to a function ϕ_i uniformly on J. From (2.17), the general term of the series

$$\phi_{i,1}(x) + \sum_{l=1}^{\infty} \phi_{i,l+1}(x) - \phi_{i,l}(x) \tag{2.18}$$

satisfies

$$|\phi_{i,l+1}(x) - \phi_{i,l}(x)| \leqslant \frac{K B^{l-1}}{(q;q)_l} h^l \quad (x \in J).$$

This implies the uniform convergence of the series on J by Weierstrass M-test. Since the mth partial sum of the series is $\phi_{i,m+1}(x)$, then $\phi_{i,m}(x)$ tends uniformly to $\phi_i(x)$ on J, where $\phi_i(x)$ is the sum of the series . Obviously $\phi_i(x) \in E_i$, $x \in J$ and it is continuous at zero.

(d) Now we show that $\{\phi_i\}_{i=0}^{p}$ satisfies (2.8) and (2.13). Indeed, from (2.14),

$$\begin{aligned}&|f_i(t, \phi_{0,m}(t), \cdots, \phi_{p,m}(t)) - f_i(t, \phi_0(t), \ldots, \phi_p(t))|\\ &\leqslant A\Big(|\phi_{0,m}(t) - \phi_0(t)| + \ldots + |\phi_{p,m}(t) - \phi_p(t)|\Big),\end{aligned}$$

for all $t \in J$ and for all $m \in \mathbb{N}$. Since by **(c)** the right-hand side tends uniformly to zero on J, as $m \to \infty$, it follows that

$$\lim_{m\to\infty} f_i\big(t, \phi_{0,m}(t), \ldots, \phi_{p,m}(t)\big) = f_i\big(t, \phi_0(t), \ldots, \phi_p(t)\big) \text{ uniformly on } J.$$

Letting $m \to \infty$ in (2.9) and using Lemma 1.14, we obtain

$$\phi_i(x) = b_i + \int_0^x f_i\left(t, \phi_0(t), \cdots, \phi_p(t)\right) d_qt \quad (0 \le i \le p;\ x \in J). \tag{2.19}$$

Using conditions (i), (ii), and the continuity of the functions $\{\phi_i\}_{i=0}^{p}$ at zero, one can verify that the functions $f_i(x, \phi_0(x), \cdots, \phi_p(x))$ are continuous at the point $(0, b_0, \cdots, b_p)$. Thus,

$$D_q\phi_i(x) = f_i\left(x, \phi_0(x), \cdots, \phi_p(x)\right) \quad (0 \le i \le p;\ \ x \in J)\,.$$

Hence, the set $\{\phi_i\}_{i=0}^{p}$ is a solution of problem (2.8), (2.13) valid in J since (2.13) is satisfied.

Uniqueness:- To prove the uniqueness, assume that $\{\psi_i\}_{i=0}^{p}$ is another solution valid in $|x| \le h_1 \le h$ and satisfies (2.13). Then, for $i = 0, 1, \cdots, p$, $|x| \leqslant h_1$

$$\psi_i(x) = \psi_i(qx) + x(1-q)f_i\left(x, \psi_0(x), \cdots, \psi_p(x)\right); \tag{2.20}$$

$$\phi_i(x) = \phi_i(qx) + x(1-q)f_i\left(x, \phi_0(x), \cdots, \phi_p(x)\right). \tag{2.21}$$

By subtracting (2.20), (2.21), and using the Lipschitz' condition (ii), we obtain

$$|\phi_i(x) - \psi_i(x)| \leqslant |\phi_i(qx) - \psi_i(qx)| + A|x|(1-q)\sum_{i=0}^{p} |\phi_i(x) - \psi_i(x)|. \tag{2.22}$$

Let $\sigma(x) := \sum_{i=0}^{p} |\phi_i(x) - \psi_i(x)|$, $|x| \leqslant h_1$. Hence, from (2.22) σ satisfies

$$(1 - B(1-q)|x|)\,\sigma(x) \le \sigma(qx) \quad (|x| \le h_1)\,.$$

Hence,

$$\sigma(x) \le \frac{\sigma(q^n x)}{\prod_{k=0}^{n-1}\left(1 - B(1-q)|x|q^k\right)} \qquad (|x| \le h_1),$$

and by calculating the limit as $n \to \infty$, we obtain

$$\sigma(x) \leqslant \frac{\sigma(0)}{\prod_{k=0}^{\infty}\left(1 - B(1-q)q^k|x|\right)} \quad (|x| \leqslant h_1).$$

Since $\sigma(0) = 0$, then $\sigma(x) \equiv 0$ in $|x| \leqslant h_1$ and the proof is complete. □

Theorem 2.2 (Range of validity). *Assume that all conditions of Theorem 2.1 are satisfied with $E_r = \mathbb{C}$ for all r; $r = 0, 1, \cdots, p$. Then problem (2.8), (2.13) has a unique solution valid at least in* $I \cap \left(-\frac{1}{A(p+1)(1-q)}, \frac{1}{A(p+1)(1-q)}\right)$.

Proof. We prove the theorem by proving the existence and uniqueness on any subinterval

$$[-h, h] \subseteq I^* := I \cap \left(-\frac{1}{A(p+1)(1-q)}, \frac{1}{A(p+1)(1-q)}\right),$$

where $h > 0$. Similar to the proof of Theorem 2.1, we can find a constant $\gamma \leqslant h$ such that $\phi_{i,m}$ tends uniformly to ϕ_i on $[-\gamma, \gamma]$, where $\phi_{i,m}$ are defined in (2.9). In addition to this, it is not hard to see that $\phi_{i,m}$ converges to ϕ_i pointwise on $[-h, h]$. Using Lemma 1.14 it can be shown that the solution $\{\phi_i\}_{i=0}^{p}$ could be extended throughout $[-h, h]$. □

Remark 2.3.1. Theorem 2.1 holds for the Cauchy problem (2.7), (2.13), but the solution will be valid throughout the whole interval I whenever the functions f_i's satisfy the conditions (i), (ii) of Theorem 2.1 with $E_r = \mathbb{C}$, $0 \leqslant r, i \leqslant p$.

The following corollary indicates that Theorem 2.1 can be applied to discuss the existence and uniqueness of the nth order q-initial value problem

$$\begin{aligned} D_q^n y(x) &= f\left(x, y(x), D_q y(x), \cdots, D_q^{n-1} y(x)\right) \\ D_q^{i-1} y(0) &= b_i \quad (b_i \in \mathbb{C};\ \ 1 \leqslant i \leqslant n). \end{aligned} \tag{2.23}$$

Corollary 2.3. *Let p, I, E_r, b_r be as in the Theorem 2.2. Let $f(x, y_0, y_1, \cdots, y_p)$ be a function defined on $I \times E_0 \times \cdots \times E_p$ such that the following conditions are satisfied:*

(i) For any fixed values of the y_r in E_r, $f(x, y_0, y_1, \cdots, y_p)$ is continuous at zero.
(ii) f satisfies Lipschitz' condition

$$\left|f(x,\tilde{y}_0,\ldots,\tilde{y}_p)-f(x,y_0,\ldots,y_p)\right| \le A\left(|\tilde{y}_0-y_0|+\ldots+|\tilde{y}_p-y_p|\right),$$

where $A>0$, y_i, $\widetilde{y_i}\in E_i$ *for* $i=0,1,\ldots,p$ *and* $x\in I$.

Then, if 0 *is an interior point of* I*, there exists* $h>0$ *such that Cauchy problem* (2.23) *has a unique solution* ϕ *valid for* $|x|\le h$*. If* 0 *is a left or right endpoint of* I *the result holds, except that the interval* $|x|\le h$ *is replaced by* $[0,h]$ *or* $[-h,0]$*, respectively.*

Proof. Assume that 0 is an interior point of I. The Cauchy problem (2.23) is equivalent to the first order q-initial value problem

$$D_q y_i(x)=f_i(x,y_0,\cdots,y_{n-1}),\quad y_i(0)=b_i,;0\leqslant i\leqslant n-1, \tag{2.24}$$

in the sense that $\{\phi_i\}_{i=0}^{n-1}$ is a solution of (2.24) if and only if ϕ_0 is a solution of (2.23). Here f_i are the functions

$$f_i(x,y_0,\cdots,y_{n-1})=\begin{cases} y_{i+1}, & 0\leqslant i\leqslant n-2,\\ f(x,y_0,\cdots,y_{n-1}), & i=n-1.\end{cases}$$

Hence, by Theorem 2.1, there exists $h>0$ such that system (2.24) has a unique solution valid in $|x|\leqslant h$. □

Corollary 2.4. *Consider the Cauchy problem*

$$\begin{gathered} a_0(x)D_q^n y(x)+a_1(x)D_q^{n-1}y(x)+\cdots+a_n(x)y(x)=b(x)\\ D_q^i y(0)=b_i\quad(0\leqslant i\leqslant n-1). \end{gathered} \tag{2.25}$$

Let the $a_j(x)$, $0\leqslant j\leqslant n$*, and* $b(x)$ *be defined on an interval* I *containing zero such that* $a_0(x)\neq 0$ *for all* x *in* I *and the functions* $a_j(x)$, $b(x)$ *are continuous at zero and bounded on* I*. Then, for any complex numbers* b_r*, there exists a subinterval* J *of* I *containing zero such that* (2.25) *has a unique solution.*

Proof. Dividing by $a_0(x)$, we get the equation

$$D_q^n y(x)=A_1(x)D_q^{n-1}y(x)+\cdots+A_n(x)y(x)+B(x), \tag{2.26}$$

where $A_j(x)=-a_j(x)/a_0(x)$, $1\leqslant j\leqslant n$, and $B(x)=b(x)/a_0(x)$. Equation (2.26) is of the form (2.24) with

$$f(x,y,\cdots,D_q^{n-1}y)=A_1(x)D_q^{n-1}y(x)+A_2(x)D_q^{n-2}y(x)+\cdots+A_n(x)y(x)+B(x).$$

Since $A_j(x)$ and $B(x)$ are continuous at zero and bounded on I, the function $f(x,y,\cdots,D_q^{n-1}y)$ satisfies the conditions of Theorem 2.3. Hence, there exists a subinterval J of I containing zero such that (2.26) has a unique solution valid in J. ☐

Remark 2.3.2. In a more restrictive manner, Carmichael [71] had defined the normal first order system to be a system of the form

$$f_i(qx) - q^{\mu_i}\lambda_i(x) f_i(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \ |x| \leqslant r \quad (i = 1, \dots, n) \tag{2.27}$$

where the μ_i's are arbitrary constants, the functions ψ_{ij} and λ_i are analytic functions for $|x| \leqslant r$ and the expansion of ψ_{ij} begins with a term in x^{s+1}. Moreover, the equation

$$f_i(qx) - q^{\mu_i}\lambda_i(x) f_i(x) = 0 \quad (i = 1, \dots, n)$$

has a solution in the form $f_i(x) = x^{\mu_i}\left(1 + c_{1,i}x + \cdots + c_{s,i}x^s\right)$. Similarly, normal systems in a neighborhood of ∞ were defined. These normal systems are more restrictive than those defined above. He considered linear systems of the form

$$G_i(qx) = q^{\alpha} x^{\alpha} \sum_{j=1}^{n} A_{ij}(x) G_j(x) \ \ (i = 1, \cdots, n), \tag{2.28}$$

where α is arbitrary and $|q| \neq 1$, $A_{ij}(x)$ are single valued functions for which

$$A_{ij}(x) = \begin{cases} A_{ij} + A_{ij}^{(1)}x^1 + A_{ij}^{(2)}x^2 + \cdots, & |x| \leqslant r, \\ A_{ij} + A_{ij}^{(1)}x^{-1} + A_{ij}^{(2)}x^{-2} + \cdots, & |x| \geqslant R. \end{cases}$$

The constants A_{ij} are such that the characteristic roots $\{A_i\}_{i=1}^{n}$ of the characteristic equation $|A_{ij} - \lambda I| = 0$ are different from zero and are of simple character, i.e., $q^{\gamma}A_i - A_j \neq 0$ for $i \neq j$ and $\gamma \in \mathbb{Z}$. Then, he proved under several transformations that the system (2.28) can be transformed into a normal system. To prove existence he defined the successive approximations $\left\{f_i^{(m)}\right\}_{i=1}^{n}$ of (2.28) throughout

$$\begin{aligned} &f_i^{(1)}(qx) - q^{\mu_i}\lambda_i(x) f_i^{(1)}(x) = 0 \\ &f_i^{(m)}(qx) - q^{\mu_i}\lambda_i(x) f_i^{(m)}(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j^{(m-1)}(x) \ \ (m \in \mathbb{N}). \end{aligned} \tag{2.29}$$

The most general solution of first equation on the preceding system is

$$f_i^{(1)}(x) = U_i(x) = C_i(x)u_i(x) = C_i(x)x^{\mu_i}\left(1 + c_i^{(1)}x + \cdots + c_i^{(s)}x^s\right), \tag{2.30}$$

where C_i is a q-periodic function. Then, it is shown that by solving system (2.29), the functions $\{f_i^m\}_{m=1}^\infty$ are

$$\begin{aligned} f_i^{(1)}(x) &= U_i(x), \\ f_i^{(m+1)}(x) &= U_i(x) + S_{xi}\left(\sum_{j=1}^{n} \psi_{ij}(x) f_j^{(m-1)}(x)\right) \quad (m \geqslant 2) \end{aligned} \tag{2.31}$$

where

$$S_{xi}(\xi(x)) = -U_i(x)\sum_{\nu=0}^{\infty} \frac{\xi(xq^\nu)}{U(xq^{\nu+1})}.$$

In fact these sequences of successive approximations coincide with those defined in Sect. 2.2. To see this, in (2.27), set $f_i = U_i g_i$, where $U_i := C_i u_i$, C_i is a q-periodic function and u_i is a solution of the equation

$$f_i(qx) - q^{\mu_i}\lambda_i(x) f_i(x) = 0.$$

Then (2.27) is equivalent to the normal system

$$D_q g_i(x) = -\frac{1}{x(1-q)}\sum_{j=1}^{n} \frac{U_j(x)}{U_i(qx)}\psi_{ij}(x) g_j(x) \quad (1 \leqslant i \leqslant n). \tag{2.32}$$

Applying the successive approximation method described in Sect. 2.2, we find that the successive approximation of (2.32) are nothing but

$$g_i^{(m)}(x) = \begin{cases} C_i, & m = 1, \\ C_i - \displaystyle\int_0^x \frac{1}{t(1-q)}\sum_{j=1}^{n} \frac{U_j(t)}{U_i(qt)}\psi_{ij}(t) g_j^{(m-1)}(t)\, d_q t, & m > 1, \end{cases} \tag{2.33}$$

$i = 1, \cdots, n$. One can easily verify that the successive functions $f_i^{(m)} := U_i g_i^{(m)}$ are exactly (2.31). We would like to mention that in [71] the problem of uniqueness has not been considered and moreover no concrete conditions are given to guarantee the existence.

Remark 2.3.3. We can use the power series method to find solutions of some linear q-difference equation. For example, if we assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the solution of the q-initial value problem

$$D_q y(x) = y(x), \qquad y(0) = 1, \tag{2.34}$$

then a_n is the solution of the difference equation

$$a_n = \frac{1-q^{n+1}}{1-q} a_{n+1}, \quad a_0 = 1.$$

That is

$$a_n = \frac{(1-q)^n}{(q;q)_n}, \quad f(x) = e_q(x(1-q)),$$

and by Theorem 2.2, the solution $f(x)$ is valid in $|x| < (1-q)^{-1}$. Another example is the q-initial value problem

$$D_q y(x) = y(qx), \quad y(0) = 1, \tag{2.35}$$

the coefficients b_n of the solution $g(x) := \sum_{n=0}^{\infty} b_n x^n$ satisfies the difference equation

$$b_{n+1} = q^n \frac{(1-q)}{(1-q^{n+1})} b_n, \quad a_0 = 1.$$

Therefore,

$$b_n := \frac{q^{\frac{n(n-1)}{2}}(1-q)^n}{(q;q)_n}, \quad g(x) = E_q(x(1-q)),$$

and by Remark 2.3.1, the solution of (2.35) holds throughout $\mathbb{C}$. The q-initial value problems (2.34) and (2.35) are q-analogues of the initial value problem

$$y'(x) = y(x), \quad y(0) = 1.$$

2.4 The Problem in a Neighborhood of Infinity

As we indicated in the introduction of this chapter, if the initial conditions are given at a point $a > 0$, the uniqueness of solutions of the q-initial value problem is not guaranteed. In this section we will study the problem of existence and uniqueness of initial-value problems with initial conditions at points differ from zero. In this case instead of an initial point c, we will have an initial interval and instead of the constants b_i's we will have initial q-periodic functions. Let us first define the Cauchy problem in this case. There will be two types of problems, the forward problem and the backward one depending on the q-difference equations.

Definition 2.4.1. By a forward value problem at $a > 0$, we mean the problem of finding a solution of (2.7) in an interval of the form $[a, \infty)$, such that the initial conditions $y_i(x) = g_i(x)$, $x \in [qa, a)$ are satisfied. The arbitrary functions g_i are called initial functions and the interval $[qa, a)$ is called the initial interval.

By a backward value problem at $b > 0$, we mean the problem of finding a solution of (2.8) in an interval of the form $(0, b]$, such that the backward conditions $y_i(x) = g_i(x),\ x \in]b, bq^{-1}]$ are satisfied. The arbitrary functions g_i are called backward functions and the interval $]b, bq^{-1}]$ is called the backward interval.

Theorem 2.5. *Let the functions $f_i(x, y_0, y_1, \cdots, y_p)$ be defined for $x \in [qa, \infty)$, $y_j \in \mathbb{C}$, where $0 \leqslant j, i \leqslant p$, and $a > 0$. Let $\{g_i(t)\}_{i=0}^{p}$ be a set of q-periodic functions such that:*

(i) $|f_i(x, g_0(x), \cdots, g_p(x))|$ is bounded function on $[qa, \infty)$,
(ii) There is a positive constant A such that

$$|f_i(x, \tilde{y_0}, \cdots, \tilde{y_p}) - f_i(x, y_0, \cdots, y_p)| \leqslant A\left(|\tilde{y_0} - y_0| + \cdots + |\tilde{y_p} - y_p|\right), \tag{2.36}$$

where $i = 0, 1, \ldots, p$, $x \in [qa, \infty)$ and $y_r, \tilde{y_r} \in \mathbb{C}$.

Then, the forward value problem

$$D_q y_i(x) = f_i(qx, y_0(qx), \ldots, y_p(qx)) \quad \text{for } x \in [a, \infty), \tag{2.37}$$

$$\phi_i(x) = g_i(x) \quad (x \in [qa, a),\ 0 \leqslant i \leqslant p)$$

has a unique solution $\{\phi_i\}_{i=0}^{p}$ valid in $[a, \infty)$.

Proof. From (i), there exists a positive constant C such that

$$C := \max_{0 \leqslant i \leqslant p} \sup_{x \geqslant qa} \left| f_i(x, g_0(x), \cdots, g_p(x)) \right|. \tag{2.38}$$

In this setting the successive approximations associated with problem (2.37), $\phi_{i,m}$, $0 \leqslant i \leqslant p$, $m \in \mathbb{N}$, will be the sequence (2.11). In (2.11) the integral over $[t, tq^{-k}]$ is understood to be

$$\int_t^{tq^{-k}} h(u) d_q u = (1-q) \sum_{j=1}^{k} tq^{-j} h(tq^{-j}).$$

Thus the q-integral in (2.11) is well defined. Now, let $t \in [qa, a)$ and $k \in \mathbb{N}_0$ be fixed.

Existence:- The existence of solutions of (2.11) is established in three steps.

(a) By induction on $m \in \mathbb{N}$, where $B = A(p+1)$, one finds

$$|\phi_{i,m}(tq^{-k}) - \phi_{i,m-1}(tq^{-k})| \leqslant CB^{m-1} q^{\frac{m(m-1)}{2}} \frac{t^m (1-q)^m}{q^{mk}\, (q;q)_m}, \tag{2.39}$$

for all $t \in [qa, a)$ and $k \in \mathbb{N}$.

(b) We show that $\phi_{i,m}(x)$ tends to functions $\phi_i(x)$, $x \in [qa, \infty)$. From (2.39), the general term of the series $\phi_{i,1}(tq^{-k}) + \sum_{l=1}^{\infty} \phi_{i,l+1}(tq^{-k}) - \phi_{i,l}(tq^{-k})$, satisfies

$$|\phi_{i,l+1}(tq^{-k}) - \phi_{i,l}(tq^{-k})| \leqslant C B^l \frac{t^{l+1}(1-q)^{l+1}}{q^{(l+1)k}(q;q)_{l+1}} q^{\frac{l(l+1)}{2}} \quad \text{for all} \quad l \in \mathbb{N}. \tag{2.40}$$

Since, $t \in [qa, a)$, $k \in \mathbb{N}_0$ are arbitrary, then (2.40) and the Weierstrass M-test imply the uniform convergence of the series on

$$\cup_{k\in\mathbb{N}_0}[aq^{-k+1}, aq^{-k}) = [a, \infty).$$

The mth partial sum of the series is $\phi_{i,m+1}(tq^{-k})$. Then denoting the sum of the series by $\phi_i(tq^{-k})$, we have

$$\lim_{m\to\infty} \phi_{i,m}(x) = \phi_i(x) \quad \text{uniformly on } [a, \infty). \tag{2.41}$$

(c) We now show that the set $\{\phi_i\}_{i=0}^{p}$ is a solution of the initial value problem (2.37) in $[a, \infty)$. By (2.36),

$$\begin{aligned} &\left| f_i\big(x, \phi_{0,m}(x), \ldots, \phi_{p,m}(x)\big) - f_i\big(x, \phi_0(x), \ldots, \phi_p(x)\big)\right| \\ &\leqslant A\left(|\phi_{0,m}(x) - \phi_0(x)| + \ldots + |\phi_{p,m}(x) - \phi_p(x)|\right). \end{aligned}$$

From (2.41) the right hand side of the last inequality tends to zero as $m \to \infty$ for all $x \in [qa, \infty)$. Then,

$$\lim_{m\to\infty} f_i\left(x, \phi_{0,m}(x), \cdots, \phi_{p,m}(x)\right) = f_i\left(x, \phi_0(x), \ldots, \phi_p(x)\right) \ \text{uniformly on} \ [a, \infty[.$$

Hence, letting $m \to \infty$ in (2.11), we obtain

$$\phi_i(tq^{-k}) = g_i(t) + \int_t^{tq^{-k}} f_i(qu, \phi_0(qu), \ldots, \phi_p(qu))\, d_q u \tag{2.42}$$

for all t in $[qa, a)$ and $k \in \mathbb{N}_0$. Putting $k = 0$ in (2.42), we see that the initial conditions on (2.37) are satisfied. From (1.16), one can see that the functions $\{\phi_i(x)\}_{i=0}^{p}$ is a solution of the q-initial value problem (2.37) valid in $[a, \infty)$.

Uniqueness:- Suppose that there is a second solution $\{\psi_i(x)\}_{i=0}^{p}$ valid in a subinterval $[a, b]$ of $[a, \infty)$ such that $\psi_i(x) = g_i(x)$ for all x in $[qa, a)$, $0 \leqslant i \leqslant p$. Thus,

$$D_q\psi_i(x) = f_i(qx, \psi_0(qx), \ldots, \psi_p(qx)) \quad (i = 0, 1, \cdots, p)$$

for all $x \in [a, b]$. Consequently,

$$\psi_i(x) = \psi_i(qx) + x(1-q) f_i\left(qx, \psi_0(qx), \ldots, \psi_p(qx)\right). \tag{2.43}$$

In addition,

$$\phi_i(x) = \phi_i(qx) + x(1-q) f_i\left(qx, \phi_0(qx), \ldots, \phi_p(qx)\right). \tag{2.44}$$

So, by subtracting (2.43), (2.44), and using the Lipschitz condition (ii), we obtain

$$|\phi_i(x) - \psi_i(x)| \leqslant |\phi_i(qx) - \psi_i(qx)| + A|x|(1-q)\sum_{i=0}^{p} |\phi_i(qx) - \psi_i(qx)|. \tag{2.45}$$

Let $\sigma(x) := \sum_{i=0}^{p} |\phi_i(x) - \psi_i(x)|$, $x \in [a,b]$. Hence, from (2.45), $\sigma(x)$ satisfies

$$\sigma(x) \leqslant (1 + A(1+p)(1-q)x)\sigma(qx) \text{ for all } x \in [a,b]. \tag{2.46}$$

But, for any x in $[a,b]$, there exists $t \in [qa,a)$ and $k \in \mathbb{N}_0$ such that $x = tq^{-k}$. Hence, substituting in (2.46) yields

$$\sigma(tq^{-k}) \leqslant (1 + A(1+p)(1-q)tq^{-k})\sigma(tq^{1-k}),$$

which leads to

$$\sigma(tq^{-k}) \leqslant \prod_{j=1}^{k} \left(1 + A(1+p)(1-q)q^k \frac{t}{q^j}\right)\sigma(t) \quad (t \in [qa,a)). \tag{2.47}$$

But $\sigma(t) = 0$ for all $t \in [qa,a)$ implies that $\sigma(tq^{-k}) = 0$ for all $t \in [qa,a)$ and $k \in \mathbb{N}_0$ such that $x = tq^{-k} \in [a,b]$. Thus, $\sigma(x) \equiv 0$ on $[a,b]$ and $\phi_i(x) = \psi_i(x)$ for all $x \in [a,b]$, $0 \leqslant i \leqslant p$, as required. □

Similarly, we can prove the following theorem.

Theorem 2.6. *Let the functions* $f_i(x, y_0, y_1, \cdots, y_p)$ *be defined for* $x \in (0,b]$ *and* $y_j \in \mathbb{C}$, *where* $0 \leqslant j,i \leqslant p$, $b > 0$. *Let* $\{g_i\}_{i=0}^{p}$ *be a set of q-periodic functions such that*

(i) $f_i(x, g_0(x), \cdots, g_p(x)))$ *are bounded function on* $(0,b]$.
(ii) There is a positive constant A such that, for $0 \leqslant i \leqslant p$,

$$|f_i(x, \tilde{y}_0, \cdots, \tilde{y}_p) - f_i(x, y_0, \cdots, y_p)| \leqslant A\left(|\tilde{y}_0 - y_0| + \cdots + |\tilde{y}_p - y_p|\right), \tag{2.48}$$

where $x \in (0,b]$ *and* $y_r, \tilde{y}_r \in \mathbb{C}$.

Then, there exists $0 < c < b$ *such that the system*

$$D_q y_i(x) = f_i(x, y_0(x), \cdots, y_p(x)),\ \phi_i(x) = g_i(x) \quad \left(x \in (c, cq^{-1}]\right). \tag{2.49}$$

has a unique solution $\{\phi_i\}_{i=0}^{p}$ *valid in* $(0,c]$.

2.5 Linear q-Difference Equations

Consider the non-homogeneous q-difference equation of order n

$$a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = b(x) \quad (x \in I) \tag{2.50}$$

for which a_i, $0 \leqslant i \leqslant n$, and b are continuous at zero functions defined on I and $a_0(x) \neq 0$ for all $x \in I$. Equation (2.50) together with the initial conditions

$$D_q^{i-1} y(0) = b_i, \qquad (b_i \in \mathbb{C};\ i = 1, \ldots, n) \tag{2.51}$$

form a q-type Cauchy problem. According to Corollary 2.4, there exists a unique solution of (2.50)–(2.51) in a subinterval J of I, $J = [-h, h]$, $h > 0$. In this section, we shall study the nth order homogeneous linear equation

$$a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = 0 \quad (x \in I). \tag{2.52}$$

Let M denote the set of solutions of (2.52) valid in a subset $J \subseteq I$ which contains zero. It it straightforward to prove that M is a linear space over $\mathbb{C}$. In addition to this, from the existence and uniqueness theorem of solutions of (2.52) subject to the initial conditions (2.51), if $\phi \in M$ and

$$D_q^i \phi(0) = 0 \quad (i = 0, 1, \ldots, n-1)$$

then $\phi(x) \equiv 0$ on J. Moreover, the functions $D_q^i \phi$, $0 \leqslant i \leqslant n-1$, are continuous at zero for any $\phi \in M$.

Definition 2.5.1. A set of n solutions of (2.52)–(2.51) is said to be a fundamental set for (2.52) valid in J or a fundamental set of M if it is linearly independent in J.

Lemma 2.7. *If b_{ij}, $1 \leq i, j \leq n$, are numbers, and, for each j, ϕ_j is the unique solution of (2.52) which satisfies the initial conditions*

$$D_q^{i-1} \phi_j(0) = b_{ij}, \quad (i, j = 1, \ldots, n),$$

then $\{\phi_j\}_{j=1}^n$ is a fundamental set of (2.52) if and only if $\det(b_{ij}) \neq 0$.

Proof. The proof is similar to the one of differential equation and is omitted. See [88]. □

Corollary 2.8. *The linear space M is of dimension n.*

Let M denote the set of solutions of (2.52) valid in a subset $J \subseteq I$, $0 \in J$. A set of n solutions of (2.52) is said to be a fundamental set for (2.52) valid in J or a fundamental set of M if it is linearly independent in J. Moreover, as in differential equations, if b_{ij}, $1 \leq i, j \leq n$, are numbers, and, for each j, ϕ_j is the unique solution of (2.52) which satisfies the initial conditions

$$D_q^{i-1}\phi_j(0) = b_{ij} \quad (i = 1, 2, \dots, n)$$

then $\{\phi_j\}_{j=1}^n$ is a fundamental set of (2.52) if and only if $\det(b_{ij}) \neq 0$. Hence M is a linear space of dimension n.

2.6 A Fundamental Set of Solution

In this section, we are concerned with constructing a fundamental set for (2.52) when it has constant coefficients, a_r, $0 \leqslant r \leqslant n$. Set

$$L := a_0 D_q^n + a_1 D_q^{n-1} + \dots + a_n.$$

Then, (2.52) can be written as

$$Ly(x) = a_0 D_q^n y(x) + a_1 D_q^{n-1} y(x) + \cdots + a_n y(x) = 0. \tag{2.53}$$

The characteristic polynomial $P(\lambda)$ of (2.53) is defined by

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n \quad (\lambda \in \mathbb{C}). \tag{2.54}$$

Let λ_i, $1 \leqslant i \leqslant k$, denote the distinct roots of $P(\lambda)$ and m_i denotes the multiplicity of λ_i, so that $\sum_{i=1}^k m_i = n$. Corresponding to each λ_i we define an m_i-dimensional subspace M_i by

$$M_i = \left\{v \in M : \left(D_q - \lambda_i\right)^{m_i} v = 0\right\}. \tag{2.55}$$

The construction of a fundamental set of (2.53) depends on the fact that, cf. [275],

$$M = M_1 \oplus \cdots \oplus M_k. \tag{2.56}$$

Lemma 2.9. *Let $(X, \mathbb{K})$ be a vector space, and let T be a linear operator on X. For any $\lambda \in \mathbb{K}$, if there exist $y_0, y_1, \dots, y_{m-1}$ in X such that*

$$\begin{aligned} Ty_0 &= \lambda y_0, \quad y_0 \neq 0 \\ Ty_i &= \lambda y_i + y_{i-1} \quad (1 \leqslant i \leqslant m-1), \end{aligned} \tag{2.57}$$

then $y_1, \dots, y_{m-1}$ are linearly independent.

Proof. The proof follows by finite induction on i, $0 \leqslant i \leqslant m-1$. □

Lemma 2.10. *If $\lambda_i \neq 0$, then the initial value problem*

$$\begin{aligned} D_q\phi_{0,i} &= \lambda_i\phi_{0,i}, \quad \phi_{0,i}(0) = 1, \\ D_q\phi_{r,i} &= \lambda_i\phi_{r,i} + \phi_{r-1,i}, \quad \phi_{r,i}(0) = 0 \ \ (r = 1, \dots, m_i - 1) \end{aligned} \tag{2.58}$$

has the solution

$$\phi_{r,i}(x) = \begin{cases} e_q(\lambda_i x(1-q)) := \sum_{k=0}^{\infty} \frac{(\lambda_i x(1-q))^k}{(q;q)_k}, & r = 0, \\ \frac{1}{\lambda_i^r} \sum_{k=r}^{\infty} \frac{k(k-1)\dots(k-r+1)}{r!} \frac{(\lambda_i x(1-q))^k}{(q;q)_k}, & r = 1, 2, \dots, m_i - 1, \end{cases} \tag{2.59}$$

which is valid for $|x| < \frac{1}{\lambda_i(1-q)}$. If $\lambda_i = 0$ then

$$\phi_{r,i}(x) = \frac{x^r(1-q)^r}{(q;q)_r} \quad (r = 0, 1, \dots, m_i - 1). \tag{2.60}$$

Proof. The proof follows by direct computations. □

It is worth noting that

$$(D_q - \lambda_i)^{m_i}\phi_{r,i} = 0 \quad (r = 0, 1, \dots, m_i - 1).$$

Thus,

$$\phi_{r,i} \in M_i \quad (r = 0, 1, \dots, m_i - 1).$$

Therefore, these functions form a basis for M_i since they are linearly independent by Lemma 2.9. This fact and (2.56) imply the following theorem.

Theorem 2.11. *The set $\{\phi_{i,r}\}_{r=0}^{m_i-1}$ of (2.59) when $\lambda_i \neq 0$ or of (2.60) when $\lambda_i = 0$ is a linearly independent set of solutions of (2.53). Moreover,*

$$\bigcup_{i=1}^{k} \{\phi_{i,r}\}_{r=0}^{m_i-1}$$

is a fundamental set of solutions of (2.53).

Example 2.6.1. The q-difference equation

$$D_q^3 y(x) - 4D_q^2 y(x) + 5D_q y(x) - 2y(x) = 0, \tag{2.61}$$

has the functions $e_q(2x(1-q))$, $e_q(x(1-q))$ and $\sum_{k=1}^{\infty} k\frac{(x(1-q))^k}{(q;q)_k}$ as a fundamental set of solutions.

2.7 A q-Type Wronskian

In the following we give a q-analogue of the Wronskian for nth order q-difference equations. A study for a second order Wronskian is established by Swarttouw and Meijer in [281]. The results of the present section are also included in [10, 203].

Definition 2.7.1. Let y_i, $1 \leq i \leq n$, be functions defined on a q-geometric set A. The q-Wronskian of the functions y_i which will be denoted by $W_q(y_1, \ldots, y_n)(x)$ is defined by

$$W_q(y_1, \ldots, y_n)(x) := \begin{vmatrix} y_1(x) & \ldots & y_n(x) \\ D_q y_1(x) & \ldots & D_q y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-1} y_1(x) & \ldots & D_q^{n-1} y_n(x) \end{vmatrix}, \tag{2.62}$$

provided that the q-derivatives exist in I. For convenience of the reader and when there is no ambiguity about the fundamental set, we write $W_q(x)$ instead of $W_q(y_1, \ldots, y_n)(x)$.

Lemma 2.12. *Let $y_1, \ldots, y_n$ be functions defined on a q-geometric set A. Then for any $x \in A$, $x \neq 0$,*

$$D_q W_q(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(qx) & y_2(qx) & \ldots & y_n(qx) \\ (D_q y_1)(qx) & (D_q y_2)(qx) & \ldots & (D_q y_n)(qx) \\ \vdots & \vdots & \ddots & \vdots \\ (D_q^{n-2} y_1)(qx) & (D_q^{n-2} y_2)(qx) & \ldots & (D_q^{n-2} y_n)(qx) \\ \\ D_q^n y_1(x) & D_q^n y_2(x) & \ldots & D_q^n y_n(x) \end{vmatrix}. \tag{2.63}$$

Proof. We prove the lemma by induction on n. The lemma is trivial when $n = 1$. Assume that (2.63) holds at $k \in \mathbb{N}$, then expanding $W_q(y_1, y_2, \ldots, y_{k+1})$ in terms of the first row we obtain

$$W_q(y_1, y_2, \ldots, y_{k+1})(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(x) W_q^{(j)}(x),$$

where

$$W_q^{(j)} := \begin{cases} W_q\big(D_q y_2, \ldots, D_q y_{k+1}\big), & j = 1; \\ W_q\big(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_{k+1}\big), & 1 \leqslant j \leqslant k+1; \\ W_q\big(D_q y_1, \ldots, D_q y_k\big), & j = k+1. \end{cases}$$

Consequently,

$$\begin{aligned} D_q W_q\big(y_1, y_2, \dots, y_{k+1}\big)(x) &= \sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) \\ &\quad + \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x). \end{aligned}$$

Now

$$\sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) = \begin{vmatrix} D_q y_1(x) & \dots & D_q y_{k+1}(x) \\ D_q y_1(x) & \dots & D_q y_{k+1}(x) \\ D_q^2 y_1(x) & \dots & D_q^2 y_{k+1}(x) \\ \vdots & \ddots & \vdots \\ D_q^{k-1} y_1(x) & \dots & D_q^{k-1} y_{k+1}(x) \\ D_q^k y_1(x) & \dots & D_q^k y_{k+1}(x) \end{vmatrix} = 0,$$

and from the induction hypothesis,

$$\sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) \times$$

$$\begin{vmatrix} (D_q y_1)(qx) & \dots & (D_q y_{j-1})(qx) & (D_q y_{j+1})(qx) & \dots & (D_q y_{k+1})(qx) \\ (D_q^2 y_1)(qx) & \dots & (D_q^2 y_{j-1})(qx) & (D_q^2 y_{j+1})(qx) & \dots & (D_q^2 y_{k+1})(qx) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (D_q^{k-1} y_1)(qx) & \dots & (D_q^{k-1} y_{j-1})(qx) & D_q^{k-1} y_{j+1}(qx) & \dots & D_q^{k-1} y_{k+1}(x)(qx) \\ \mathrm{d}_q^{k+1} y_1(x) & \dots & D_q^{k+1} y_{j-1}(x) & D_q^{k+1} y_{j+1}(x) & \dots & D_q^{k+1} y_{k+1}(x) \end{vmatrix}, \tag{2.64}$$

where when $i = 1$ the determinant of (2.64) starts with $D_q y_2(qx)$ and when $j = k + 1$, the determinant ends with $D_q^{k+1} y_k(x)$. Thus,

$$\sum_{j=1}^{k+1}(-1)^{j+1}y_j(qx)D_qW_q^{(j)}(x) = \begin{vmatrix} y_1(qx) & \dots & y_{k+1}(qx) \\ (D_qy_1)(qx) & \dots & (D_qy_{k+1})(qx) \\ (D_q^2y_1)(qx) & \dots & (D_q^2y_{k+1})(qx) \\ \vdots & \ddots & \vdots \\ (D_q^{k-1}y_1)(qx) & \dots & (D_q^{k-1}y_{k+1})(qx) \\ D_q^{k+1}y_1(x) & \dots & D_q^{k+1}y_{k+1}(x) \end{vmatrix},$$

proving (2.63) for $n = k+1$ and hence all $k \in \mathbb{N}$. □

Theorem 2.13. *If $y_1, y_2, \dots, y_n$ are solutions of (2.52) in $J \subseteq I$, then their q-Wronskian satisfies the first order q-difference equation*

$$D_qW_q(x) = -R(x)W_q(x),\ x \in J\setminus\{0\};\ R(x) = \sum_{k=0}^{n-1}(x-qx)^k\frac{a_{k+1}(x)}{a_0(x)}. \tag{2.65}$$

Proof. From the definition of the operator D_q, we have

$$(D_q^m y)(qx) = D_q^m y(x) - x(1-q)D_q^{m+1}y(x) \quad (m \in \mathbb{N}_0)\,.$$

Substituting in (2.63) yields

$$D_qW_q\big(y_1,\dots,y_n\big)(x) =$$

$$\begin{vmatrix} y_1(x)-x(1-q)D_qy_1(x) & \dots & y_n(x)-x(1-q)D_qy_n(x) \\ D_qy_1(x)-x(1-q)D_q^2y_1(x) & \dots & D_qy_n(x)-x(1-q)D_q^2y_n(x) \\ \dots & \ddots & \dots \\ D_q^{n-2}y_1(x)-x(1-q)D_q^{n-1}y_1(x) & \dots & D_q^{n-2}y_n(x)-x(1-q)D_q^{n-1}y_n(x) \\ D_q^ny_1(x) & \dots & D_q^ny_n(x) \end{vmatrix}.$$

We shall prove by induction on n that

$$D_qW_q\big(y_1,\dots,y_n\big)(x) =$$

$$\sum_{k=1}^{n}(-1)^{k-1}(x-qx)^{k-1}\begin{vmatrix} y_1(x) & \dots & y_n(x) \\ D_qy_1(x) & \dots & D_qy_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-k-1}y_1(x) & \dots & D_q^{n-k-1}y_n(x) \\ D_q^{n-k+1}y_1(x) & \dots & D_q^{n-k+1}y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^ny_1(x) & \dots & D_q^ny_n(x) \end{vmatrix}. \tag{2.66}$$

If (2.66) holds at $n = m$, then

$$D_q W_q(y_1, y_2, \ldots, y_{m+1})(x) = \sum_{j=1}^{m+1} (-1)^{j+1} \big(y_j(x) - x(1-q) D_q y_j(x)\big) A_{1j},$$

where

$$A_{1j} = D_q W_q\big(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_{m+1}\big) \;\; (j = 1, 2, \ldots, m).$$

Hence, from the previous hypothesis we obtain

$$\begin{aligned} & D_q W_q(y_1, y_2, \ldots, y_{m+1})(x) = \\ & \sum_{j=1}^{m+1} (-1)^{j+1} \big(y_j(x) - x(1-q) D_q y_j(x)\big) \sum_{k=1}^{m} (-1)^{k-1} \big(x(1-q)\big)^{k-1} B_{jk} \\ & \quad = \sum_{k=1}^{m} (-1)^{k-1} \big(x(1-q)\big)^{k-1} \sum_{j=1}^{m+1} (-1)^{j+1} y_j(x) B_{jk} \\ & \qquad + \sum_{k=1}^{m} (-1)^{k} \big(x(1-q)\big)^{k} \sum_{j=1}^{m+1} (-1)^{j+1} D_q y_j(x) B_{jk}, \end{aligned} \tag{2.67}$$

$$B_{jk} := \begin{vmatrix} D_q y_1(x) & \ldots & D_q y_{j-1}(x) & D_q y_j(x) & \ldots & D_q y_{m+1}(x) \\ D_q^2 y_1(x) & \ldots & D_q^2 y_{j-1}(x)(x) & D_q^2 y_j(x) & \ldots & D_q^2 y_{m+1}(x) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_q^{m-k} y_1(x) & \ldots & D_q^{m-k} y_{j-1}(x) & D_q^{m-k} y_j(x) & \ldots & D_q^{m-k} y_{m+1}(x) \\ D_q^{m-k+2} y_1(x) & \ldots & D_q^{m-k+2} y_{j-1}(x) & D_q^{m+k+1} y_{j+1}(x) & \ldots & D_q^{m+k+1} y_{m+1}(x) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_q^{m+1} y_1(x) & \ldots & D_q^{m+1} y_{j-1}(x) & D_q^{m+1} y_{j+1}(x) & \ldots & D_q^{m+1} y_{m+1}(x) \end{vmatrix},$$

$k = 1, 2, \ldots, m$, where when $j = 1$ the determinant B_{1k} start with $D_q y_2(x)$ and when $j = m + 1$, the determinant $B_{(m+1)k}$ ends with $D_q^{m+1} y_m(x)$. From the properties of the determinants we conclude that

$$\sum_{j=1}^{m+1}(-1)^{j+1}y_j(x)B_{jk} = \begin{vmatrix} y_1(x) & \dots & y_{m+1}(x) \\ D_q y_1(x) & \dots & D_q y_{m+1}(x) \\ \vdots & \ddots & \vdots \\ D_q^{m-k} y_1(x) & \dots & D_q^{m-k} y_{m+1}(x) \\ D_q^{m-k+2} y_1(x) & \dots & D_q^{m-k+2} y_{m+1}(x) \\ & & \\ \vdots & \ddots & \vdots \\ D_q^{m+1} y_1(x) & \dots & D_q^{m+1} y_{m+1}(x) \end{vmatrix}, \tag{2.68}$$

$$\sum_{j=1}^{m+1}(-1)^{j+1}D_q y_j(x)B_{jk} = 0 \quad (k = 1, 2, \dots, m-1), \tag{2.69}$$

and

$$\sum_{j=1}^{m+1}(-1)^{j+1}D_q y_j(x)B_{jm} = \begin{vmatrix} D_q y_1(x) & \dots & D_q y_{m+1}(x) \\ D_q^2 y_1(x) & \dots & D_q^2 y_{m+1}(x) \\ \vdots & \ddots & \vdots \\ D_q^{m+1} y_1(x) & \dots & D_q^{m+1} y_{m+1}(x) \end{vmatrix}. \tag{2.70}$$

Combining (2.68)–(2.70) with (2.7), we obtain (2.66) when $n = m + 1$. One can easily see that (2.66) holds at $n = 1$. Consequently it holds for all $n \in \mathbb{N}$. From (2.52), we have

$$D_q^n y_j(x) = -\sum_{i=1}^{i=n} \frac{a_i(x)}{a_0(x)} D_q^{n-i} y_j(x) \quad (j = 1, 2, \dots, n).$$

Then (2.66) is nothing but

$$D_q W_q(x) = -\left[\sum_{k=0}^{k=n-1} (x - qx)^k \frac{a_{k+1}(x)}{a_0(x)}\right] W_q(x) = -R(x) W_q(x).$$

This completes the proof of the theorem. □

The following theorems gives a q-type Liouville's formula, cf. [88], for the q-Wronskian.

Theorem 2.14. *Suppose that $x(1-q)R(x) \neq -1$ for all $x \in J$. Then the q-Wronskian of any set of solutions $\{\phi_i\}_{i=1}^n$ of (2.52) is given by*

$$W_q(x) = W_q(\phi_1, \dots, \phi_n)(x) = \frac{W_q(0)}{\prod_{k=0}^{\infty}(1 + x(1-q)q^k R(xq^k))} \quad (x \in J). \tag{2.71}$$

Proof. Since $W_q(x)$ satisfies the q-difference equation (2.65), we obtain

$$W_q(qx) = (1 + x(1-q)R(x))\, W_q(x) \quad (x \in J \setminus \{0\})\,.$$

Hence, under the assumption $1 + x(1-q)R(x) \neq 0$, we obtain

$$W_q(x) = \frac{W_q(qx)}{1 + x(1-q)R(x)}.$$

Consequently,

$$W_q(x) = \frac{W_q(xq^n)}{\prod_{m=0}^{n-1} (1 + xq^m(1-q)R(xq^m))}. \tag{2.72}$$

Since all functions a_j/a_0 are continuous at zero, then $\sum_{k=0}^{\infty} q^k |R(xq^k)|$ is convergent. Consequently, $\prod_{k=0}^{\infty} \left(1 + x(1-q)q^k R(xq^k)\right)$ converges for every $x \in I$. Thus, using the continuity of $W_q(x)$ at zero, (2.71) follows. □

Corollary 2.15. *Let $\{\phi_i\}_{i=1}^n$ be a set of solutions of (2.52) in some subinterval J of I which contains zero. Then $W_q(x)$ is either never zero or identically zero in I. The first case occurs when $\{\phi_i\}_{i=1}^n$ is a fundamental set of (2.52) and the second when it is not.*

Proof. From Lemma 2.7, the functions $\{\phi_i\}_{i=1}^n$ form a fundamental set of (2.52) if and only if $W_q(0) \neq 0$. Hence, the result is a direct consequence of Theorem 2.14. □

Example 2.7.1. We calculate the q-Wronskian of the solutions of the q-difference equation

$$-D_q^2 y(x) + y(x) = 0 \quad (x \in \mathbb{R}). \tag{2.73}$$

The functions $\sin_q x(1-q)$, $\cos_q x(1-q)$, $|x|(1-q) < 1$, are solutions of (2.73) subject to the initial conditions

$$y(0) = 0,\ D_q y(0) = 1 \quad \text{and} \quad y(0) = 1,\ D_q y(0) = 0,$$

respectively. Here $R(x) = x(1-q)$. So, $x(1-q)R(x) \neq -1$ for all x in $\mathbb{R}$. Hence,

$$W_q(x) = \frac{W_q(0)}{\prod_{n=0}^{\infty}(1 + q^{2n}\{x(1-q)\}^2)} \quad \text{for } |x|(1-q) < 1.$$

But

$$W_q(0) = \left(\mathrm{Cos}_q{}^2 x(1-q) + \mathrm{Sin}_q{}^2 x(1-q)\right)\Big|_{x=0} = 1.$$

Therefore, $W_q(x) \equiv 1/\prod_{n=0}^{\infty}(1 + q^{2n}\{x(1-q)\}^2)$, $|x|(1-q) < 1$.

We define a pair of basic trigonometric functions by studying the solutions of the second order q-difference equation

$$\frac{-1}{q} D_{q^{-1}} D_q y(x) + y(x) = 0 \quad (x \in \mathbb{R}) \tag{2.74}$$

subject to the initial conditions

$$y(0) = 1, \quad D_q y(0) = 0 \quad \text{and} \quad y(0) = 0,\ D_q y(0) = 1.$$

The solutions are the functions $\cos(x;q)$, $\sin(x;q)$, $x \in \mathbb{R}$, respectively. These functions are defined for $x \in \mathbb{C}$ by

$$\begin{aligned} \cos(x;q) &:= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}(x(1-q))^{2n}}{(q;q)_{2n}}, \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} (xq^{-1/2}(1-q))^{1/2} J_{-1/2}\left(x(1-q)/\sqrt{q}\,; q^2\right), \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} \sin(x;q) &:= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}(x(1-q))^{2n+1}}{(q;q)_{2n+1}} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} (x(1-q))^{1/2} J_{1/2}\left(x(1-q);\, q^2\right). \end{aligned} \tag{2.76}$$

q-analogues of the hyperbolic functions $\sinh x$ and $\cosh x$ are defined by

$$\sinh(x;q) := -i \sin(ix;q), \quad \cosh(x;q) := \cos(ix;q) \quad (x \in \mathbb{C}).$$

We shall calculate $W_q(\cos(x;q), \sin(x;q))$ in the next example.

Example 2.7.2. Equation (2.74) can be written as

$$D_q^2 y(x) + qx(1-q) D_q y(x) - q y(x) = 0. \tag{2.77}$$

Comparing with (2.50), $a_0(x) \equiv 1$, $a_1(x) = qx(1-q)$ and $a_2(x) = -q$. Thus $R(x) \equiv 0$ on $\mathbb{R}$ and $W_q(x) \equiv W_q(0)$. But

$$\begin{aligned} W_q(0) &= W_q\big(\cos(\cdot;q), \sin(\cdot;q)\big)(0) \\ &= \big(\cos(x;q)\cos(\sqrt{q}x;q) + \sqrt{q}\sin(x;q)\sin(\sqrt{q}x;q)\big)\big|_{x=0} = 1. \end{aligned}$$

Then, $W_q(x) \equiv 1$ for all $x \in \mathbb{R}$.

2.8 Zeros of q-Functions

This section includes some recent results about the zeros of some q-functions and q-transforms and their asymptotics. In the following, we state some results about the zeros of q-Bessel functions and their associated q-Hankel transforms. These results are taken from [9,31–34,66,129,141,174], see also [151]. As mentioned in Sect. 1.5, the functions $J_\nu^{(k)}(z:q)$ $(k=2,3)$ have infinitely many zeros. In [141], Ismail proved that the zeros of $J_\nu^{(2)}(z;q)$ are real and simple and the zeros of $z^{-\nu}J_\nu^{(2)}(z;q)$ and $z^{-\nu}J_{\nu+1}^{(2)}(z;q)$ interlace. Hayman proved in [129] that, for arbitrary k the positive zeros $\{z_{m,\nu}\}_{m=1}^\infty$ of $J_\nu^{(2)}(z;q)$ have the following asymptotic expansion

$$z_{m,\nu}=2q^{-m}q^{\frac{-\nu+1}{2}}\left\{1+\sum_{n=1}^{k}b_nq^{mn}+O(q^{(k+1)m})\right\}, \tag{2.78}$$

for sufficiently large m, where the constants b_n, $n=1,2,\dots,k$ depend on q and ν. In particular, we have

$$z_{m,\nu}=2q^{-m}q^{\frac{-\nu+1}{2}}\big(1+O(q^m)\big)\quad\text{as}\quad m\to\infty. \tag{2.79}$$

Therefore, the zeros $\{\xi_{m,\nu}\}_{m=1}^\infty$ of $J_\nu^{(2)}(z;q^2)$ have the asymptotics

$$\xi_{m,\nu}=2q^{-2m}q^{-\nu+1}\big(1+O(q^{2m})\big)\quad\text{as}\quad m\to\infty.$$

The functions $\operatorname{Cos}_q z$ and $\operatorname{Sin}_q z$ are related to the second Jackson q-Bessel functions via the relations

$$\operatorname{Cos}_q z=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}z^{1/2}J_{-1/2}^{(2)}(2z;q^2), \tag{2.80}$$

$$\operatorname{Sin}_q z=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}z^{1/2}J_{1/2}^{(2)}(2z;q^2). \tag{2.81}$$

Therefore, if we denote the positive zeros of $\operatorname{Cos}_q z$ and $\operatorname{Sin}_q z$ by $\left\{x_m^{(2)}\right\}_{m=1}^\infty$ and $\left\{y_m^{(2)}\right\}_{m=1}^\infty$, respectively, then

$$x_m^{(2)}=\frac{\xi_{m,-1/2}}{2},\qquad y_m^{(2)}=\frac{\xi_{m,1/2}}{2}\qquad(m\in\mathbb{N}).$$

Consequently, from (2.79), we have

$$x_m^{(2)}=q^{-2m+\frac{3}{2}}\big(1+O(q^{2m})\big),\qquad y_m^{(2)}=q^{-2m+\frac{1}{2}}\big(1+O(q^{2m})\big),$$

as $m \to \infty$. As for $J_\nu^{(3)}(z;q^2)$, Koelink and Swarttouw proved that its zeros are all real and simple, cf. [174], and that the zeros of $z^{-\nu}J_\nu^{(3)}(z;q)$ and $z^{-\nu}J_{\nu+1}^{(3)}(z;q)$ also interlace. Let $w_{m,\nu}$ $(\nu > -1)$ denote the positive zeros of $J_\nu^{(3)}(\cdot;q^2)$ in an increasing order of $m \in \mathbb{N}$. In [9], Abreu et al. proved that if $q^{2\nu+2} < (1-q^2)^2$, then

$$w_{m,\nu}(q) = q^{-m+2\epsilon_m(\nu)}, \qquad 0 < \epsilon_m(\nu) < \frac{\log\left(1 - \frac{q^{2m+2\nu}}{1-q^{2m}}\right)}{2\log q},$$

$$\sum_{m=0}^{\infty} \epsilon_m(\nu) = \frac{\log(q^{2\nu+2};q^2)_\infty}{4\log q}.$$

Moreover, $w_{m,\nu} \sim q^{-m}$ when $m \to \infty$ without the restriction

$$q^{2\nu+2} < (1-q^2)^2.$$

Applying[53, Theorem 2], we deduce that the positive zeros $\{w_{m,\nu}\}$ of $J_\nu^{(3)}(z;q^2)$ are given by

$$w_{m,\nu} = Aq^{-m}(1 + O(q^m)) \quad \text{as } m \to \infty. \tag{2.82}$$

Here, the non zero constant A depends on ν and q. In [34], the authors proved that $A = 1$. Let x_m and y_m, $m \in \mathbb{N}$, denote, respectively, the positive zeros of $\cos(z;q)$ and $\sin(z;q)$ in an increasing order of m. Since the functions $\cos(z;q)$ and $\sin(z;q)$ are related to the third Jackson q-Bessel function via the identities

$$\cos(z;q) := \frac{(zq^{-1/2}(1-q))^{1/2}(q^2;q^2)_\infty}{(q;q^2)_\infty} J_{-1/2}(z(1-q)/\sqrt{q}\,;q^2), \tag{2.83}$$

$$\sin(z;q) := \frac{(z(1-q))^{1/2}(q^2;q^2)_\infty}{(q;q^2)_\infty} J_{1/2}(z(1-q);\, q^2), \tag{2.84}$$

then

$$x_m = q^{1/2}(1-q)^{-1}w_{m,-1/2}, \quad y_m = (1-q)^{-1}w_{m,1/2}. \tag{2.85}$$

Substituting with $A = 1$ in (2.82), we obtain

$$x_m = \frac{q^{-m+1/2}}{1-q}\left(1 + O(q^m)\right), \quad y_m = \frac{q^{-m}}{1-q}\left(1 + O(q^m)\right), \tag{2.86}$$

as $m \to \infty$. In [66], the authors investigated the asymptotic behavior of the zeros of $\sin(z;q)$.

Now we state some results concerning the zeros of q-Hankel transforms. For $f \in L_q^1(0,1)$, we denote by $A_k(f)$ the q-moments of f, i.e.

$$A_k(f) := \int_0^1 t^k f(t)\, d_q t \quad (k \in \mathbb{N}_0).$$

We denote by $c_k(f)$, c_f, C_f the constants

$$\begin{aligned} c_k(f) &:= \tfrac{A_{2k+2}(f)}{A_{2k}(f)(1-q^{2k+1})(1-q^{2k+2})} \quad (k \in \mathbb{N}_0), \\ c_f &:= \inf_{k\in\mathbb{N}_0} c_k(f), \quad C_f := \sup_{k\in\mathbb{N}_0} c_k(f), \end{aligned} \tag{2.87}$$

and by $b_k(f)$, b_f, B_f to

$$\begin{aligned} b_k(f) &:= \tfrac{A_{2k+3}(f)}{A_{2k+1}(f)(1-q^{2k+3})(1-q^{2k+2})},\ k \in \mathbb{N}_0, \\ b_f &:= \inf_{k\in\mathbb{N}_0} b_k(f),\ B_f := \sup_{k\in\mathbb{N}_0} b_k(f). \end{aligned} \tag{2.88}$$

The constants c_f, C_f, b_f, B_f are finite positive numbers, cf. [31].

Theorem 2.16. *Let $f \in L_q^1(0,1)$ be positive on $\{0, q^n, n \in \mathbb{N}_0\}$. If*

$$q^{-1}(1-q)\frac{c_f}{C_f} > 1 \tag{2.89}$$

then the zeros of the entire function of order zero

$$U_f(z) := \int_0^1 f(t)\cos(tz;q)\, d_q t \quad (z \in \mathbb{C}),$$

are real, simple and infinite. Moreover, $U_f(z)$ is an even function with no zeros in the interval $\big[0, q^{-1/2}/\big(\sqrt{C_f}(1-q)\big)\big)$, and its positive zeros lie in the intervals

$$\left(\frac{q^{-r}}{(1-q)\sqrt{C_f}}, \frac{q^{-r-1}}{(1-q)\sqrt{C_f}}\right) \quad (r \in \mathbb{N}_0),$$

one zero in each interval.

Theorem 2.17. *Let $f \in L_q^1(0,1)$ be positive on $\{0, q^n, n \in \mathbb{N}_0\}$. If*

$$q^{-1}(1-q)\frac{b_f}{B_f} > 1 \tag{2.90}$$

then the zeros of the entire function of order zero

$$V_f(z) := \int_0^1 f(t)\sin(tz;q)\, d_q t \quad (z \in \mathbb{C})$$

are real, simple and infinite. Moreover the odd function $V_f(z)$ *has only one zero* $z = 0$ *in* $[0, \frac{q^{-1}}{\sqrt{B_f}(1-q)})$*, and its positive zeros are located in the intervals*

$$\left(\frac{q^{-r+1/2}}{(1-q)\sqrt{B_f}}, \frac{q^{-r-1/2}}{(1-q)\sqrt{B_f}} \right) \quad (r \in \mathbb{N}_0),$$

one zero in each interval.

Theorems 2.16, 2.17 give a basic analogue of Theorem 1.3 with the restrictions (2.89), (2.90) on q, respectively. The following theorems, which are introduced in [34], give another q-counterpart which is valid for any $q \in (0, 1)$, but in this case the entire functions $U_f(z)$ and $V_f(z)$ may have a finite number of complex zeros.

Theorem 2.18. *Let* $f \in L^1_q(0, a)$*,* $0 < q < 1$*. Then the function* $U_f(z)$ *has at most a finite number of non real zeros and it has an infinite number of real zeros* $\{\pm\zeta_m\}_{m=1}^{\infty}$*,* $\zeta_m > 0$*, such that* $\zeta_m \sim x_m$ *as* $m \to \infty$*. More precisely,*

$$\zeta_m = x_m(1 + O(q^m)) \quad \text{as} \quad m \to \infty.$$

Theorem 2.19. *Let* $f \in L^1_q(0, a)$*,* $0 < q < 1$*. Then the function* $V_f(z)$ *has at most a finite number of non real zeros and it has an infinite number of real zeros* $\{\pm\eta_m\}_{m=1}^{\infty}$*,* $\eta_m > 0$*, for all* $m \in \mathbb{N}$ *such that* $\eta_m \sim y_m$ *as* $m \to \infty$*. More precisely*

$$\eta_m = y_m(1 + O(q^m)) \quad \text{as} \quad m \to \infty.$$

There are two similar theorems for the q-cosine and q-sine transforms based on the basic trigonometric functions $\mathrm{Cos}_q\, z$ and $\mathrm{Sin}_q\, z$, cf. [34]. In the following two theorems, the number x_0 is the unique positive root of the polynomial $x^3 - x^2 - 2x - 1$ described in Theorem 1.5.

Theorem 2.20. *Let* $f \in L^1_q(0, a)$ *and* $U_f^{(2)}(z)$ *be defined for* $z \in \mathbb{C}$ *by*

$$U_f^{(2)}(z) := \int_0^1 f(t)\ \mathrm{Cos}_q\, tz\, d_qt \quad (0 < q < 1),$$

then $U_f^{(2)}(z)$ *has at most a finite number of non real zeros and has an infinite number of real zeros* $\{\pm\zeta_m\}_{m=1}^{\infty}$*,* $\zeta_m > 0$*, such that* $\zeta_m \sim x_m^{(2)}$ *as* $m \to \infty$*. More precisely,*

$$\zeta_m = x_m^{(2)}(1 + O(q^{2m})) = q^{-2m+\frac{3}{2}}\big(1 + O(q^{2m})\big), \quad \text{as} \quad m \to \infty.$$

In particular, if q *satisfies the condition*

$$x_0 \leqslant q^{-1}\frac{A_k A_{k+1}}{A_{k-1}A_{k+2}}\begin{cases}\dfrac{1-q^{m+1}}{1-q^m}, & k=2m,\\ \dfrac{1-q^{m+\frac{5}{2}}}{1-q^{m+\frac{3}{2}}}, & k=2m+1,\end{cases}$$

then all the zeros are real and simple.

Theorem 2.21. *Let $f \in L_q^1(0,a)$ and $V_f^{(2)}(z)$ be defined for $z \in \mathbb{C}$ by*

$$V_f^{(2)}(z) := \int_0^1 f(t)\ \mathrm{Sin}_q\, tz\, d_qt \quad (0 < q < 1).$$

Then $V_f^{(2)}(z)$ has at most a finite number of non real zeros and it has an infinite number of real zeros $\{\pm\tau_m\}_{m=1}^{\infty}$, $\tau_m > 0$, such that $\tau_m \sim y_m^{(2)}$ as $m \to \infty$. More precisely

$$\zeta_m = y_m^{(2)}(1 + O(q^{2m})) = q^{-2m+\frac{1}{2}}\big(1 + O(q^{2m})\big) \quad \text{as } m \to \infty.$$

In particular, if q satisfies the condition

$$x_0 \leqslant q^{-1}\frac{A_k A_{k+1}}{A_{k-1}A_{k+2}}\begin{cases}\dfrac{1-q^{m+1}}{1-q^m}, & k=2m,\\[2ex] \dfrac{1-q^{m+\frac{3}{2}}}{1-q^{m+\frac{1}{2}}}, & k=2m+1,\end{cases}$$

then all the zeros are real and simple.

Chapter 3
q-Sturm–Liouville Problems

Abstract In this chapter we introduce the study held by Annaby and Mansour in (J. Phys. A Math. Gen. 38(17), 3775–3797, 2005) of a self adjoint basic Sturm–Liouville eigenvalue problem in a Hilbert space. The last two sections of this chapter are about the q^2-Fourier transform introduced by Rubin in (J. Math. Anal. Appl. 212(2), 571–582, 1997; Proc. Am. Math. Soc. 135(3), 777–785, 2007), when q lies in a proper subset of $(0,1)$ and the generalization of Rubin's q^2-Fourier transform, introduced in (Mansour, Generalizations of Rubin's q^2-fourier transform and q-difference operator, submitted, 2012) for any $q \in (0,1)$.

3.1 Introduction

Let $[a,b] \subseteq \mathbb{R}$ be a finite closed interval and $\nu(\cdot)$ be a continuous real-valued function defined on $[a,b]$. By a Sturm–Liouville problem we mean the problem of finding a function $y(\cdot)$ and a number $\lambda \in \mathbb{C}$ satisfying the differential equation

$$Ly := -y'' + \nu(x)y(x) = \lambda y(x) \quad (a \leqslant x \leqslant b) \tag{3.1}$$

together with the boundary conditions

$$U_1(y) := a_1 y(a) + a_2 y'(a) = 0, \tag{3.2}$$

$$U_2(y) := b_1 y(b) + b_2 y'(b) = 0, \tag{3.3}$$

where a_i and b_i, $i = 1,2$ are real numbers for which

$$|a_1| + |a_2| \neq 0 \neq |b_1| + |b_2|. \tag{3.4}$$

This problem has been extensively studied. It is known that the differential equation (3.1) and the boundary conditions (3.2)–(3.3) determine a self adjoint operator in $L^2(a,b)$. There is a sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ with ∞ as the unique

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limit point such that corresponding to each λ_n there is one and only one linearly independent solution of the problem (3.1)–(3.3). The sequence $\{\lambda_n\}_{n=0}^{\infty}$ is called the sequence of eigenvalues and the sequence of corresponding solutions $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ is said to be a sequence of eigenfunctions. One of the most important properties of these eigenfunctions is that, $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ is an orthogonal basis of $L^2(a,b)$. For example, let $\nu(x) \equiv 0$ on $[a,b]$. If we take $a = 0$, $b = \pi$, $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, and $b_2 = 0$, we get

$$\lambda_n = n^2, \quad \phi_n(x) = \sin nx \quad (n \in \mathbb{N})$$

leading to the well known fact that $\{\sin nx\}_{n=1}^{\infty}$ is a complete orthogonal set of $L^2(0,\pi)$, while taking $a = 0$, $b = \pi$, $a_1 = 0$, $a_2 = 1$, $b_1 = 0$, and $b_2 = 1$, we get

$$\lambda_n = n^2, \quad \phi_n(x) = \cos nx \quad (n \in \mathbb{N}_0)$$

which leads to the completeness of $\{\cos nx\}_{n=0}^{\infty}$ in $L^2(0,\pi)$. We mention here that the Fourier orthogonal basis $\left\{e^{inx}\right\}_{n=0}^{\infty}$ of $L^2(-\pi,\pi)$ will not be extracted from this setting but from a simpler situation, namely the first order problem

$$-iy' = \lambda y, \quad y(-\pi) = y(\pi).$$

Among several references for the above mentioned facts we mention the monographs of Coddington and Levinson [78], Eastham [88], Levitan and Sargsjan [182, 183], Marchenko [208] and finally Titchmarsh [284].

The discrete analogue of the theory outlined above, i.e. when the differential operator d/dx is replaced by the forward difference operator $\Delta y(n) = y(n+1) - y(n)$ and the backward operator $\nabla y(n) = y(n) - y(n-1)$ where n is a positive integer belonging to a finite set of integers of the form $\{m, m+1, m+2, \cdots, m+N, m \in \mathbb{N}\}$, is treated in Atkinson's [45], see also [162].

The aim of this chapter is to study a basic analogue of Sturm–Liouville systems when the differential operator is replaced by the q-difference operator D_q. In [98, Chap. 5] and [99], a basic Sturm–Liouville system is defined. It is the system

$$D_q(r(x)D_q y) + \big(l(x) + \lambda w(x)\big)y(qx) = 0 \quad (a \leqslant x \leqslant b)\,, \tag{3.5}$$

$$h_1 y(a) + h_2 D_q y(a) = 0, \tag{3.6}$$

$$k_1 y(b) + k_2 D_q y(b) = 0, \tag{3.7}$$

where $r(\cdot)$, $l(\cdot)$ and $w(\cdot)$ are real-valued functions which posses appropriate q-derivatives, h_1, h_2, k_1, k_2 are constants. It is proved [98, pp. 164–170] that all eigenvalues of this system are real and the eigenfunctions satisfy an orthogonality relation [98, Eq. (5.1.5)]. The author in [98] considered only computational aspects to prove certain orthogonality relation of some q-special functions. There is no attention paid to several points, which may lead to several mistakes. First the existence of eigenvalues is not proved and it is not indicated how to determine the eigenvalues and the eigenfunctions. A basic point here is that if $a \neq 0 \neq b$, then it is not guaranteed that initial conditions at either a or b determine a unique solution of (3.5), see [203]. The geometric and algebraic simplicity of the eigenvalues, which plays a major rule in proving the reality of the eigenvalues and the orthogonality of the eigenfunctions are not proved or even assumed. Moreover, the space where the problem is defined is not specified. If an inner product is defined in the view of [98, Eq. (5.1.5)], there will be no orthogonality if h_1, h_2, k_1 and k_2 are not real. For more information concerning the monograph [98], see the review by Mourad Ismail in [301]. See also the review of [99] by Wolfgang Hahn in [302]. There are several physical models involving q-(basic) derivatives, q-integrals, q-functions and their related problems, see e.g. [76, 105, 106, 119, 277, 278]. In addition, the problem of expendability of functions in terms of q-orthogonal functions, which seems to be first discussed by Carmichael in [72, 73] has attracted the wok of several authors, see e.g. [66, 67, 142, 278, 279]. However, as far we know, there is no study of the general problem as we do in the present setting. At this point, it is worthy to mention that our work based on the q-difference operator which is attributed to Jackson, see [153], and a similar study of the Stum–Liouville systems generated by the Askey–Wilson derivative, cf. [41] is very much needed. The case of half-line singular Sturm–Liouville problem is investigated in [37], when the spectrum is discrete.

3.2 q-Sturm–Liouville Problem

In this section we investigate the fundamental solutions of the basic Sturm–Liouville equation

$$-\frac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x) = \lambda y(x) \quad (0 \leqslant x \leqslant a < \infty;\ \lambda \in \mathbb{C}) \tag{3.8}$$

where $v(\cdot)$ is defined on $[0, a]$ and continuous at zero. Let $C_q^2(0)$ be the space of all functions $y(\cdot)$ such that $y(\cdot)$, $D_q y(\cdot)$ are continuous at zero. Clearly, $C_q^2(0)$ is a subspace of the Hilbert space $L_q^2(0, a)$. By a solution of equation (3.8), we mean a continuous at zero function that satisfies (3.8) such that the function and its q-derivative have prescribed values at $x = 0$. It is proved in Sects. 2.5 and 2.8 that (3.8) has a fundamental set of solutions which consists of two linearly independent solutions $\{y_1(\cdot), y_2(\cdot)\}$ and $\{y_1, y_2\}$ forms a fundamental set of solutions if and only if their q-Wronskian does not vanish at any point of $[0, a]$. The following theorem is proved in [30].

Theorem 3.1. *For $c_1, c_2 \in \mathbb{C}$, (3.8) has a unique solution in $C_q^2(0)$ which satisfies*

$$\phi(0, \lambda) = c_1, \qquad D_{q^{-1}}\phi(0, \lambda) = c_2 \quad (\lambda \in \mathbb{C}). \tag{3.9}$$

Moreover, $\phi(x, \lambda)$ is entire in λ for all $x \in [0, a]$.

Proof. See [30] □

If we assume that the function $v(\cdot)$ in (3.8) is continuous on $[0, a]$, then the solution will be continuous on $[0, a]$ as we shall see in the following theorem.

Theorem 3.2. *Let* $0 \leq \gamma < 1$. *If the function* $v(\cdot)$ *defined in* (3.8) *is continuous on* $[0, a]$, *then* (3.8) *has a unique solution in* $C_q^2[0, a]$ *which satisfies*

$$\phi(0, \lambda) = c_1, \qquad D_{q^{-1}}\phi(0, \lambda) = c_2 \quad (\lambda \in \mathbb{C}) \tag{3.10}$$

where c_1 *and* c_2 *are arbitrary constants. Moreover,* $\phi(x, \lambda)$ *is entire in* λ *for all* $x \in [0, a]$.

Proof. The functions

$$\varphi_1(x, \lambda) = \cos(sx; q) \text{ and } \varphi_2(x, \lambda) = \begin{cases} \frac{\sin(sx;q)}{s}, & \lambda \neq 0 \\ x, & \lambda = 0 \end{cases},$$

where $s := \sqrt{\lambda}$ is defined with respect to the principal branch, form a fundamental set of

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + \lambda y(x) = 0,$$

with the q-Wronskian $W_q\big(\varphi_1(\cdot, \lambda), \varphi_2(\cdot, \lambda)\big) \equiv 1$, see Sect. 2.8. For all $x \in [0, a]$, $\lambda \in \mathbb{C}$, we define the sequence $\{y_m(\cdot, \lambda)\}_{m=1}^{\infty}$ of successive approximations by

$$y_1(x, \lambda) = c_1\varphi_1(x, \lambda) + c_2\varphi_2(x, \lambda), \tag{3.11}$$

$$y_{m+1}(x, \lambda) = c_1\varphi_1(x, \lambda) + c_2\varphi_2(x, \lambda) \tag{3.12}$$

$$-q \int_0^x \{\varphi_2(x, \lambda)\varphi_1(qt, \lambda) - \varphi_1(x, \lambda)\varphi_2(qt, \lambda)\}\, v(qt)\, y_m(qt, \lambda)\, d_q t.$$

We prove that for each fixed $\lambda \in \mathbb{C}$ the uniform limit of $y_m(\cdot, \lambda)$ as $m \to \infty$ exists and defines a solution of (3.8) and (3.9). Let $\lambda \in \mathbb{C}$ be fixed. There exist positive numbers $K(\lambda)$, $\widetilde{K}(\lambda)$, and A such that

$$|y_1(x, \lambda)| \leqslant \widetilde{K}(\lambda), \;\; |v(x)| \leqslant A, \;\; |\varphi_i(x, \lambda)| \leqslant \sqrt{\frac{K(\lambda)}{2}} \quad (i = 1, 2;\; x \in [0, a]).$$

Using mathematical induction, we have

$$|y_{m+1}(x, \lambda) - y_m(x, \lambda)| \leqslant \widetilde{K}(\lambda) q^{\frac{m(m+1)}{2}} \frac{(AK(\lambda)x(1-q))^m}{(q; q)_m} \quad (m \in \mathbb{N}). \tag{3.13}$$

Consequently by Weierstrass M-test the series

$$y_1(x, \lambda) + \sum_{m=1}^{\infty} y_{m+1}(x, \lambda) - y_m(x, \lambda) \tag{3.14}$$

convergence uniformly on [0,a]. Since the mth partial sums of the series is nothing but $y_{m+1}(\cdot,\lambda)$, then $y_{m+1}(\cdot,\lambda)$ approaches a function $\phi(\cdot,\lambda)$ uniformly on $[0,a]$ as $m\to\infty$, where $\phi(x,\lambda)$ is the sum of the series. Using Theorem 1.11 we can prove by induction on m that $y_m(x,\lambda)$ and $D_q y_m(x,\lambda)$ are continuous on $[0,a]$, where

$$\begin{aligned}D_q y_{m+1}(x,\lambda) &= c_1 D_q\varphi_1(x,\lambda)+c_2 D_q\varphi_2(x,\lambda)\\ &\quad -q\int_0^x \left\{D_q\phi_2(x,\lambda)\varphi_1(qt,\lambda)-D_q\varphi_1(x,\lambda)\varphi_2(qt,\lambda)\right\} y_m(qt,\lambda)\,d_qt,\end{aligned}$$

$m\in\mathbb{N}$. Hence, both $\phi(\cdot,\lambda)$ and $D_q\phi(\cdot,\lambda)$ are continuous on $[0,a]$. Consequently, $\phi(\cdot,\lambda)\in C_q^2(0)$. Because of the uniform convergence, letting $m\to\infty$ in (3.12) we obtain

$$\begin{aligned}\phi(x,\lambda) &= c_1\varphi_1(x,\lambda)+c_2\varphi_2(x,\lambda)\\ &\quad -q\int_0^x \{\varphi_2(x,\lambda)\varphi_1(qt,\lambda)-\varphi_1(x,\lambda)\varphi_2(qt,\lambda)\}\, v(qt)\,\phi(qt,\lambda)\,d_qt.\end{aligned}$$

Clearly, $\phi(\cdot,\lambda)$ satisfies (3.8) and (3.9). To prove that problem (3.8), (3.9) has a unique solution, suppose on the contrary that $\psi_i(\cdot,\lambda)$, $i=1,2$, are two solutions of (3.8), (3.9). Let

$$\chi(x,\lambda)=\psi_1(x,\lambda)-\psi_2(x,\lambda)\quad \text{for } x\in[0,a].$$

Then $\chi(\cdot,\lambda)$ is a solution of (3.8) subject to the initial conditions

$$\chi(0,\lambda)=D_{q^{-1}}\chi(0,\lambda)=0.$$

Applying the q-integration process on (3.8) twice yields

$$\chi(x,\lambda)=-q\int_0^x (x-qt)\big(\lambda-v(qt)\big)\chi(qt,\lambda)\,d_qt. \tag{3.15}$$

Since $\chi(x,\lambda)$, $v(x)$ are continuous on $[0,a]$, then there exist positive numbers N_λ, M_λ such that

$$N_\lambda=\max_{0\le x\le a}|\chi(x,\lambda)|,\quad M_\lambda=\max_{0\le x\le a}\big|\lambda-v(x)\big|. \tag{3.16}$$

Again we can prove by mathematical induction on k that

$$|\chi(x,\lambda)|\leqslant N_\lambda M_\lambda^k q^{k^2}(1-q)^{2k}\frac{x^{2k}}{(q;q)_{2k}}\quad (k\in\mathbb{N}_0;\ x\in[0,a])\,. \tag{3.17}$$

Since

$$\lim_{k\to\infty} N_\lambda M_\lambda^k q^{k^2}(1-q)^{2k}\frac{x^{2k}}{(q;q)_{2k}} = 0,$$

then $\chi(x,\lambda) = 0$, for all $x \in [0,a]$. This proves the uniqueness. Now, Let $M > 0$ be arbitrary but fixed. To prove that $\phi(x,\lambda)$, $x \in [0,a]$, is entire in λ, it is sufficient to prove that $\phi(x,\lambda)$ is analytic in each disk Ω_M; $\Omega_M := \{\lambda \in \mathbb{C} : |\lambda| \leqslant M\}$. We prove by induction on m that

$$\text{for all } \ x \in [0,a] \quad y_m(x,\lambda) \quad \text{is analytic on } \Omega_M, \tag{3.18}$$

$$\text{for all } \ \lambda \in \Omega_M \quad \frac{\partial}{\partial\lambda}y_m(x,\lambda) \quad \text{is continuous at } (0,\lambda). \tag{3.19}$$

Clearly, for each fixed $x \in [0,a]$, the functions $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ are entire functions of λ. Moreover, $\dfrac{\partial}{\partial\lambda}\phi_i(x,\lambda)$ is continuous at $(0,\lambda)$ for each $\lambda \in \mathbb{C}$. Then, the statements in (3.18) and (3.19) hold at $m = 1$. Now assume that (3.18) and (3.19) hold at $m \in \mathbb{N}$. Then, for $x_0 \in [0,a]$, $\lambda_0 \in \Omega_M$, we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}y_{m+1}(x_0,\lambda)\Big|_{\lambda=\lambda_0} &= \frac{\partial}{\partial\lambda}y_1(x_0,\lambda)\Big|_{\lambda=\lambda_0}\\ &\quad - q\frac{\partial}{\partial\lambda}\varphi_2(x_0,\lambda)\Big|_{\lambda=\lambda_0}\int_0^{x_0}\varphi_1(qt,\lambda)y_m(qt,\lambda)\,d_qt\\ &\quad + q\frac{\partial}{\partial\lambda}\varphi_1(x_0,\lambda)\Big|_{\lambda=\lambda_0}\int_0^{x_0}\varphi_2(qt,\lambda)y_m(qt,\lambda)\,d_qt\\ &\quad - q\varphi_2(x_0,\lambda)\frac{\partial}{\partial\lambda}\Big(\int_0^{x_0}\varphi_1(qt,\lambda)y_m(qt,\lambda)\,d_qt\Big)\Big|_{\lambda=\lambda_0}\\ &\quad + q\varphi_1(x_0,\lambda)\frac{\partial}{\partial\lambda}\Big(\int_0^{x_0}\varphi_2(qt,\lambda)y_m(qt,\lambda)\,d_qt\Big)\Big|_{\lambda=\lambda_0}\end{aligned} \tag{3.20}$$

From (3.19) we conclude that

$$\frac{\partial}{\partial\lambda}\big(\varphi_i(qt,\lambda)y_m(qt,\lambda)\big) \quad (i = 1,2)$$

are continuous at $(0,\lambda_0)$. Therefore, there exist constants $C,\delta > 0$ such that

$$\left|\frac{\partial}{\partial\lambda}\Big(\varphi_i(x_0q^n,\lambda)y_m(x_0q^n,\lambda)\Big)\right| \leqslant C \quad (n \in \mathbb{N};\ |\lambda-\lambda_0| \leqslant \delta).$$

Hence

$$x_0(1-q)q^n\left|\frac{\partial}{\partial\lambda}\Big(\varphi_i(x_0q^{n+1},\lambda)y_m(x_0q^{n+1}\lambda)\Big)\right| \leqslant x_0A(1-q)q^n \quad (n\in\mathbb{N}_0)$$

for all λ in the disk $|\lambda-\lambda_0|\leqslant\delta$. i.e. the series corresponding to the q-integrals

$$\int_0^{x_0}\frac{\partial}{\partial\lambda}\big(\varphi_i(qt,\lambda)y_m(qt,\lambda)\big)\,d_qt \quad (i=1,2) \tag{3.21}$$

are uniformly convergent in a neighborhood of $\lambda=\lambda_0$. Thus, we can interchange the differentiation and the q-integration processes in (3.20). Since x_0,λ_0 are arbitrary, we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}y_{m+1}(x,\lambda) &= \frac{\partial}{\partial\lambda}y_1(x,\lambda)-q\int_0^x\frac{\partial}{\partial\lambda}\Big(\varphi_2(x,\lambda)\varphi_1(qt,\lambda)y_m(qt,\lambda)\Big)v(qt)\,d_qt\\ &\quad +q\int_0^x\frac{\partial}{\partial\lambda}\Big(\varphi_1(x,\lambda)\varphi_2(qt,\lambda)y_m(qt,\lambda)\Big)v(qt)\,d_qt,\end{aligned} \tag{3.22}$$

for all $x\in[0,a]$, $\lambda\in\Omega_M$. From (3.19) the integrals in (3.22) are continuous at $(0,\lambda)$. Consequently $\frac{\partial}{\partial\lambda}y_{m+1}(x,\lambda)$ is continuous at $(0,\lambda)$. Let $x_0\in[0,a]$ be arbitrary. Then there exists $B(x_0),\widetilde{B}(x_0)>0$ such that

$$|\varphi_i(x_0,\lambda)|\leqslant\sqrt{\frac{B(x_0)}{2}},\ i=1,2,\ y_1(x,\lambda)|\leqslant\widetilde{B}(x_0)\quad(\lambda\in\Omega_M).$$

Finally the use of the mathematical induction yields

$$|y_{m+1}(x_0,\lambda)-y_m(x_0,\lambda)|\leqslant\widetilde{B}(x_0)q^{\frac{m(m+1)}{2}}\frac{(AB(x_0)\lambda(1-q))^m}{(q;q)_m}. \tag{3.23}$$

Consequently the series (3.14), with $x=x_0$, converges uniformly in Ω_M to $\phi(x_0,\lambda)$. Hence $\phi(x_0,\lambda)$ is analytic in Ω_M, i.e. it is entire. □

3.2.1 The Self Adjoint Problem

In this subsection we define a basic Sturm–Liouville problem and prove that it is self adjoint in $L_q^2(0,a)$. The following lemma which is needed in the sequel indicates that unlike the classical differential operator d/dx, D_q is neither self adjoint nor sqew self adjoint. Equation (3.25) below indicates that the adjoint of D_q is $-\frac{1}{q}D_{q^{-1}}$.

Lemma 3.3. *Let $f(\cdot)$, $g(\cdot)$ in $L_q^2(0,a)$ be defined on $[0,q^{-1}a]$. Then, for $x\in(0,a]$, we have*

$$(D_q g)(xq^{-1}) = D_{q,xq^{-1}} g(xq^{-1}) = D_{q^{-1}} g(x), \tag{3.24}$$

$$\langle D_q f, g\rangle = f(a)\overline{g(aq^{-1})} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^{n-1})} + \left\langle f, \frac{-1}{q} D_{q^{-1}} g\right\rangle, \tag{3.25}$$

$$\left\langle -\frac{1}{q} D_{q^{-1}} f, g\right\rangle = \lim_{n\to\infty} f(aq^{n-1})\overline{g(aq^n)} - f(aq^{-1})\overline{g(a)} + \langle f, D_q g\rangle. \tag{3.26}$$

Proof. Relation (3.24) follows from

$$\begin{aligned} D_{q^{-1}} g(x) &= \frac{g(x) - g(q^{-1}x)}{x(1-q^{-1})} = \frac{g(xq^{-1}) - g(x)}{xq^{-1}(1-q)} = (D_q g)(xq^{-1}) \\ &= D_{q,xq^{-1}} g(xq^{-1}). \end{aligned}$$

Using the formula (1.28) of q-integration by parts we obtain

$$\begin{aligned} \langle D_q f, g\rangle &= \int_0^a D_q f(x)\overline{g(x)}\, d_q x \\ &= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} - \int_0^a f(qt)\overline{D_q g(t)}\, d_q t \\ &= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} - \int_0^{qa} f(t)\overline{\frac{1}{q} D_{q^{-1}} g(t)}\, d_q t \\ &= f(a)\overline{g(a)} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^n)} + aq^{-1}(1-q) f(a)\overline{D_{q^{-1}} g(a)} \\ &\quad + \int_0^a f(t)\overline{\frac{-1}{q} D_{q^{-1}} g(t)}\, d_q t \\ &= f(a)\overline{g(aq^{-1})} - \lim_{n\to\infty} f(aq^n)\overline{g(aq^{n-1})} + \left\langle f, \frac{-1}{q} D_q g\right\rangle, \end{aligned}$$

proving (3.25). Equation (3.26) can be proved by using (3.25). □

Now consider the basic Sturm–Liouville problem

$$\ell(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x)y(x) = \lambda y(x) \ \ (0 \leqslant x \leqslant a < \infty,\ \lambda \in \mathbb{C}) \tag{3.27}$$

$$U_1(y) := a_{11} y(0) + a_{12} D_{q^{-1}} y(0) = 0, \tag{3.28}$$

$$U_2(y) := a_{21} y(a) + a_{22} D_{q^{-1}} y(a) = 0, \tag{3.29}$$

where $v(\cdot)$ is a continuous at zero real valued function and $\{a_{ij}\}, i, j \in \{1, 2\}$ are arbitrary real numbers such that the rank of the matrix $(a_{ij})_{1\leqslant i,j\leqslant 2}$ is 2.

Theorem 3.4. *The basic Sturm- Liouville eigenvalue problem (3.27)–(3.29) is self adjoint on* $C_q^2(0) \cap L_q^2(0, a)$.

Proof. We first prove that for $y(\cdot)$, $z(\cdot)$ in $L_q^2(0,a)$, we have the following q-Lagrange's identity

$$\int_0^a \Big(\ell y(x)\overline{z(x)} - y(x)\overline{\ell z(x)}\Big)\, d_q x = [y,z](a) - \lim_{n\to\infty}[y,z](aq^n), \tag{3.30}$$

where

$$[y,z](x) := y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}. \tag{3.31}$$

Applying (3.26) with $f(x) = D_q y(x)$ and $g(x) = z(x)$, we obtain

$$\begin{aligned}
&\big\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\big\rangle \\
&= -(D_q y)(aq^{-1})\overline{z(a)} + \lim_{n\to\infty}(D_q y)(aq^{n-1})\overline{z(aq^n)} + \big\langle D_q y, D_q z\big\rangle \\
&= -D_{q^{-1}}y(a)\overline{z(a)} + \lim_{n\to\infty} D_{q^{-1}}y(aq^n)\overline{z(aq^n)} + \big\langle D_q y, D_q z\big\rangle.
\end{aligned} \tag{3.32}$$

Applying (3.25) with $f(x) = y(x)$, $g(x) = D_q z(x)$,

$$\begin{aligned}
\big\langle D_q y, D_q z\big\rangle &= y(a)\overline{D_q z(aq^{-1})} - \lim_{n\to\infty} y(aq^n)\overline{D_q z(aq^{n-1})} + \big\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\big\rangle \\
&= y(a)\overline{D_{q^{-1}}z(a)} - \lim_{n\to\infty} y(aq^n)\overline{D_{q^{-1}}z(aq^n)} + \big\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\big\rangle.
\end{aligned}$$

Therefore,

$$\big\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\big\rangle = [y,z](a) - \lim_{n\to\infty}[y,z](aq^n) + \big\langle y, -\frac{1}{q}D_{q^{-1}}D_q z\big\rangle. \tag{3.33}$$

Lagrange's identity (3.30) results from (3.33) and the reality of $v(x)$. Letting $y(\cdot)$, $z(\cdot)$ in $C_q^2(0)$ and assuming that they satisfy (3.28)–(3.29), we obtain

$$a_{11}y(0) + a_{12}D_{q^{-1}}y(0) = 0, \quad a_{11}z(0) + a_{12}D_{q^{-1}}z(0) = 0. \tag{3.34}$$

The continuity of $y(\cdot)$, $z(\cdot)$ at zero implies that $\lim_{n\to\infty}[y,z](aq^n) = [y,z](0)$. Then (3.33) will be

$$\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\rangle = [y,z](a) - [y,z](0) + \langle y, -\frac{1}{q}D_{q^{-1}}D_q z\rangle.$$

Since a_{11} and a_{12} are not both zero, it follows from (3.34) that

$$[y,z](0) = y(0)\overline{D_{q^{-1}}z(0)} - D_{q^{-1}}y(0)\overline{z(0)} = 0.$$

Similarly,

$$[y, z](a) = y(a)\overline{D_{q^{-1}}z(a)} - D_{q^{-1}}y(a)\overline{z(a)} = 0.$$

Since $v(x)$ is real valued, then

$$\begin{aligned}\langle \ell(y), z\rangle &= \Big\langle -\frac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x), z(x)\Big\rangle \\ &= \Big\langle -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x)\Big\rangle + \langle v(x)y, z(x)\rangle \\ &= \Big\langle y, -\frac{1}{q}D_{q^{-1}}D_q z(x)\Big\rangle + \big\langle y, v(x)z(x)\big\rangle = \langle y, \ell(z)\rangle\,,\end{aligned}$$

i.e. ℓ is a self adjoint operator. □

Definition 3.2.1. A complex number λ^* is said to be an eigenvalue of problem (3.27)–(3.29) if there is a non trivial solution $\phi^*(\cdot)$ which satisfies the problem at this λ^*. In this case we say that $\phi^*(\cdot)$ is an eigenfunction of the basic Sturm–Liouville problem corresponding to the eigenvalue λ^*. The multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to it. In particular an eigenvalue is simple if and only if it has only one linearly independent solution.

Lemma 3.5. *The eigenvalues and the eigenfunctions of the boundary value problem (3.27)–(3.29) have the following properties:*

i. *The eigenvalues are real.*
ii. *Eigenfunctions that belong to different eigenvalues are orthogonal.*
iii. *All eigenvalues are simple.*

Proof. **i.** Let λ_0 be an eigenvalue with an eigenfunction $y_0(\cdot)$. Then,

$$\langle \ell(y_0), y_0\rangle = \langle y_0, \ell(y_0)\rangle\,. \tag{3.35}$$

Since $\ell(y_0) = \lambda_0 y_0$, then

$$\big(\lambda_0 - \bar{\lambda}_0\big)\int_0^a |y_0(x)|^2\, d_q x = 0. \tag{3.36}$$

Since $y_0(\cdot)$ is non-trivial then $\lambda_0 = \bar{\lambda}_0$, which proves **i**.

ii. Let λ, μ be two (real) distinct eigenvalues with corresponding eigenfunctions $y(\cdot)$, $z(\cdot)$, respectively. Then,

$$(\lambda - \mu)\int_0^a y(x)\overline{z(x)}\, d_q x = 0.$$

Since $\lambda \neq \mu$, then $y(\cdot)$ and $z(\cdot)$ are orthogonal, proving **ii**.

iii. Let λ_0 be an eigenvalue with two eigenfunctions $y_1(\cdot)$ and $y_2(\cdot)$. From Corollary 2.15, we can prove that the functions $\{y_1(\cdot),\ y_2(\cdot)\}$ are linearly dependent by proving that their q-Wronskian vanishes at $x = 0$. Indeed,

$$\begin{aligned} W_q(y_1, y_2)(0) &= y_1(0)D_q y_2(0) - y_2(0)D_q y_1(0) \\ &= y_1(0)D_q^{-1} y_2(0) - y_2(0)D_q^{-1} y_1(0) = [y_1, y_2] = 0, \end{aligned}$$

since both y_1 and y_2 satisfy (3.28). □

In the following we indicate how to obtain the eigenvalues and the corresponding eigenfunctions. Let $\phi_1(\cdot, \lambda)$, $\phi_2(\cdot, \lambda)$ be the linearly independent solutions of (3.27) determined by the initial conditions

$$D_q^{j-1}\phi_i(0, \lambda) = \delta_{ij} \quad (i, j = 1, 2;\ \lambda \in \mathbb{C}).$$

Thus $\phi_1(\cdot, \lambda)$ is determined by (3.11) by taking $c_1 = 1$, $c_2 = 0$ and $\phi_2(\cdot, \lambda)$ is determined by taking $c_1 = 0$, $c_2 = 1$. Then, every solution of (3.27) is of the form

$$y(x, \lambda) = A_1\phi_1(x, \lambda) + A_2\phi_2(x, \lambda),$$

where A_1 and A_2 does not depend on x. A solution $y(\cdot, \lambda)$ of (3.27) will be an eigenfunction if it satisfies the boundary conditions (3.28)–(3.29), i.e. if we can find a non trivial solution of the linear system

$$\begin{aligned} A_1U_1(\phi_1) + A_2U_1(\phi_2) = 0, \\ A_1U_2(\phi_1) + A_2U_2(\phi_2) = 0. \end{aligned} \tag{3.37}$$

Hence, $\lambda \in \mathbb{R}$ is an eigenvalue if and only if

$$\Delta(\lambda) = \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} = 0. \tag{3.38}$$

The function $\Delta(\lambda)$ is called the characteristic determinant associated with the basic Sturm- Liouville problem (3.27)–(3.29). The zeros of $\Delta(\lambda)$ are exactly the eigenvalues of the problem. Since $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ are entire in λ for each fixed $x \in [0, a]$, then $\Delta(\lambda)$ is also entire. Thus, the eigenvalues of the basic Sturm–Liouville system (3.27)–(3.29) are at most countable with no finite limit points. We have proved in Lemma 3.5 that all eigenvalues are simple from the geometric point of view. In the following theorem, we prove that the eigenvalues are also simple algebraically, i.e. they are simple zeros of $\Delta(\lambda)$.

Theorem 3.6. *The eigenvalues of the q-Sturm–Liouville problem (3.27)–(3.29) are simple zeros of $\Delta(\lambda)$.*

Proof. Let $\theta_1(\cdot,\lambda)$ and $\theta_2(\cdot,\lambda)$ be defined by the relations

$$\begin{aligned}\theta_1(x,\lambda) &:= U_1(\phi_2)\phi_1(x,\lambda) - U_1(\phi_1)\phi_2(x,\lambda),\\ \theta_2(x,\lambda) &:= U_2(\phi_2)\phi_1(x,\lambda) - U_2(\phi_1)\phi_2(x,\lambda).\end{aligned} \tag{3.39}$$

Hence, $\theta_1(\cdot,\lambda)$, $\theta_2(\cdot,\lambda)$ are solutions of (3.27) such that

$$\theta_1(0,\lambda) = a_{12},\ D_{q^{-1}}\theta_1(0,\lambda) = -a_{11};\ \ \theta_2(a,\lambda) = a_{22},\ D_{q^{-1}}\theta_2(a,\lambda) = -a_{21}. \tag{3.40}$$

One can verify that

$$W_q\big(\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda)\big)(x) = \varDelta(\lambda)W_q\big(\phi_1(\cdot,\lambda),\phi_2(\cdot,\lambda)\big)(x) = \varDelta(\lambda). \tag{3.41}$$

Let λ_0 be an eigenvalue of (3.27)–(3.29). Then λ_0 is a real number and therefore $\theta_i(x,\lambda_0)$ can be taken to be real valued, $i = 1,2$. From (3.41), we conclude that $\theta_1(x,\lambda_0)$, $\theta_2(x,\lambda_0)$ are linearly dependent eigenfunctions. So, there exists a non zero constant k_0 such that

$$\theta_1(x,\lambda_0) = k_0\theta_2(x,\lambda_0). \tag{3.42}$$

From (3.40) and (3.39)

$$\theta_1(a,\lambda_0) = k_0a_{22} = k_0\theta_1(a,\lambda),\ D_{q^{-1}}\theta_1(a,\lambda_0) = -k_0a_{21} = k_0D_{q^{-1}}\theta_1(a,\lambda). \tag{3.43}$$

In the q-Lagrange identity (3.30), taking $y(x) = \theta_1(x,\lambda)$, and $z(x) = \theta_1(x,\lambda_0)$ imply

$$\begin{aligned}&\big(\lambda-\lambda_0\big)\int_0^a \theta_1(x,\lambda)\theta_1(x,\lambda_0)\,d_qx\\ &\quad= \theta_1(a,\lambda)D_{q^{-1}}\theta_1(a,\lambda_n) - D_{q^{-1}}\theta_1(a,\lambda)\theta_1(a,\lambda_n)\\ &\quad= k_0\Big(\theta_1(a,\lambda)D_{q^{-1}}\theta_2(a,\lambda) - \theta_2(a,\lambda)D_{q^{-1}}\theta_1(a,\lambda)\Big)\\ &\quad= k_0W_q(\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda))(q^{-1}a) = k_0\varDelta(\lambda).\end{aligned}$$

Since $\varDelta(\lambda)$ is entire in λ,

$$\varDelta'(\lambda_0) := \lim_{\lambda\to\lambda_0}\frac{\varDelta(\lambda)}{\lambda-\lambda_0} = \frac{1}{k_0}\int_0^a \theta_1^2(x,\lambda_0)\,d_qx \neq 0. \tag{3.44}$$

Therefore λ_0 is a simple zero of $\varDelta(\lambda)$. □

Remark 3.2.1. For suitable functions $r(\cdot)$ and $w(\cdot)$, the q-difference equation (3.27) may be replaced by

$$-\frac{1}{q} D_{q^{-1}}(r(x) D_q y(x)) + v(x) y(x) = \lambda w(x) y(x).$$

We get similar results but the Hilbert space $L_q^2(0, a)$ will be replaced by a weighted one $L_q^2\big((0, a); w(\cdot)\big)$ with the inner product

$$\langle f, g\rangle_w = \int_0^a f(x)\overline{g(x)} w(x)\, d_q x \quad \Big(f, g \in L_q^2\left((0, a); w(\cdot)\right)\Big).$$

Accordingly the boundary conditions will be changed.

3.3 Basic Green's Function

The q-type Green's function arises when we seek a solution of the nonhomogeneous equation

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + \{-\lambda + v(x)\}\, y(x) = f(x) \quad (x \in [0, a];\ \lambda \in \mathbb{C}) \tag{3.45}$$

which satisfies the boundary conditions (3.28)–(3.29), where $f(\cdot) \in L_q^2(0, a)$ is given.

Lemma 3.7. *If λ is not an eigenvalue of the basic Sturm–Liouville problem (3.27)–(3.29), then the solution of (3.45), if it exists, would be unique.*

Proof. Assume that $\chi_1(x, \lambda)$, $\chi_2(x, \lambda)$ are two solutions of (3.45). Then $\chi_1(x, \lambda) - \chi_2(x, \lambda)$ is a solution of the problem (3.27)–(3.29). So, it is identically zero if λ is not an eigenvalue. ☐

Another proof of Lemma 3.7 is included in the proof of the next theorem.

Theorem 3.8. *Suppose that λ is not an eigenvalue of (3.27)–(3.29). Let $\phi(\cdot, \lambda)$ satisfy the q-difference equation (3.45) and the boundary conditions (3.28)–(3.29), where $f(\cdot) \in L_q^2(0, a)$. Then*

$$\phi(x, \lambda) = \int_0^a G(x, t, \lambda) f(t)\, d_q t \quad \textit{for } x \in \{aq^m;\ m \in \mathbb{N}_0\}, \tag{3.46}$$

where $G(x, t, \lambda)$ is the Green's function of problem (3.27)–(3.29) defined by

$$G(x, t, \lambda) := \frac{-1}{\Delta(\lambda)} \begin{cases} \theta_2(x, \lambda)\theta_1(t, \lambda)\, , \ 0 \leqslant t \leqslant x, \\[2ex] \theta_1(x, \lambda)\theta_2(t, \lambda)\, , \ x < t \leqslant a. \end{cases} \tag{3.47}$$

Conversely the function $\phi(\cdot,\lambda)$ defined by (3.46) satisfies (3.45) and (3.28)–(3.29). Green's function $G(x,t,\lambda)$ is unique in the sense that if there exists another function $\widetilde{G}(x,t,\lambda)$ such that (3.46) is satisfied, then

$$G(x,t,\lambda)=\widetilde{G}(x,t,\lambda) \quad \text{for all } x,t\in\{aq^m \mid m\in\mathbb{N}_0\}.$$

If $f(\cdot)$ is q-regular at zero, then (3.46) holds for all $x\in[0,a]$.

Proof. Using a q-analogue of the methods of variation of constants, a particular solution of the non-homogenous equation (3.45) may be given by

$$\phi(x,\lambda)=c_1(x)\theta_1(x,\lambda)+c_2(x)\theta_2(x,\lambda),$$

where $c_1(x)$, $c_2(x)$ are solutions of the first order q-difference equations

$$D_{q,x}c_1(x)=-\frac{q}{\Delta(\lambda)}\theta_2(qx,\lambda)f(qx),\;\; D_{q,x}c_2(x)=\frac{q}{\Delta(\lambda)}\theta_1(qx,\lambda)f(qx). \tag{3.48}$$

If the functions $D_{q,x}c_i(x)$, $i=1,2$, are q-integrable on $[0,t]$ then

$$\lim_{n\to\infty} tq^n\theta_i(tq^{n+1},\lambda)f(tq^{n+1})=0 \quad (i=1,2).$$

Define the q-geometric set A_f by

$$A_f:=\left\{x\in[0,a]:\quad \lim_{n\to\infty} xq^n|f(xq^n)|^2=0\right\}. \tag{3.49}$$

The set A_f is a q-geometric set containing $\{aq^m;\ m\in\mathbb{N}_0\}$ because $f\in L_q^2(0,a)$. Hence, $D_qc_i(\cdot)$, $i=1,2$, are q-integrable on $[0,x]$ for all $x\in A_f$ and appropriate solutions of (3.48) are given by

$$c_1(x)=c_1(0)+\frac{q}{\Delta(\lambda)}\int_0^x\theta_2(qt,\lambda)f(qt)\,d_qt \quad (x\in A_f), \tag{3.50}$$

$$c_2(x)=c_2(a)+\frac{q}{\Delta(\lambda)}\int_x^a\theta_1(qt,\lambda)f(qt)\,d_qt \quad (x\in A_f). \tag{3.51}$$

That is the general solution of (3.45) is given by

$$\begin{aligned}\phi(x,\lambda)=c_1\theta_1(x,\lambda)+c_2\theta_2(x,\lambda)+\frac{q}{\Delta(\lambda)}\theta_1(x,\lambda)\int_0^x\theta_2(qt,\lambda)f(qt)\,d_qt\\ +\frac{q}{\Delta(\lambda)}\theta_2(x,\lambda)\int_x^a\theta_1(qt,\lambda)f(qt)\,d_qt,\end{aligned} \tag{3.52}$$

where $x \in A_f$, and c_1, c_2 are arbitrary constants. Now, we determine c_1, c_2 for which $\phi(x, \lambda)$ satisfies (3.28)–(3.29). It is easy to see that

$$\phi(0, \lambda) = c_1\theta_1(0, \lambda) + \Big(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt, \lambda) f(qt)\, d_q t\Big)\theta_2(0, \lambda),$$

$$\begin{aligned} D_{q^{-1}}\phi(0, \lambda) &= \lim_{\substack{n\to\infty \\ x\in A_f}} \frac{\phi(xq^n, \lambda) - \phi(0, \lambda)}{xq^n} \\ &= c_1 D_{q^{-1}}\theta_1(0, \lambda) + \Big(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt, \lambda) f(qt)\, d_q t\Big) D_{q^{-1}}\theta_2(0, \lambda). \end{aligned}$$

The boundary condition $a_{11}\phi(0, \lambda) + a_{12}D_{q^{-1}}\phi(0, \lambda) = 0$ implies that

$$\Big(c_2 + \frac{q}{\Delta(\lambda)}\int_0^a \theta_1(qt, \lambda) f(qt)\, d_q t\Big) W_q(\theta_1, \theta_2)(0) = 0.$$

Therefore,

$$c_2 = \frac{-q}{\Delta(\lambda)}\int_0^a \theta_1(qt, \lambda) f(qt)\, d_q t.$$

Hence,

$$\phi(x, \lambda) = c_1\theta_1(x, \lambda) + \frac{q}{\Delta(\lambda)}\int_0^x \Big(\theta_1(x, \lambda)\theta_2(qt, \lambda) - \theta_2(x, \lambda)\theta_1(qt, \lambda)\Big) f(qt)\, d_q t. \tag{3.53}$$

Now we compute $\phi(a, \lambda)$ and $D_{q^{-1}}\phi(a, \lambda)$. Indeed, from the definition of the q-integration (1.19) and relation (3.52)

$$\begin{aligned} \phi(a, \lambda) &= c_1\theta_1(a, \lambda) + \frac{q}{\Delta(\lambda)}\int_0^a \Big(\theta_1(a, \lambda)\theta_2(qt, \lambda) - \theta_2(a, \lambda)\theta_1(qt, \lambda)\Big) f(qt)\, d_q t \\ &= c_1\theta_1(a, \lambda) + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a} \Big(\theta_1(a, \lambda)\theta_2(qt, \lambda) - \theta_2(a, \lambda)\theta_1(qt, \lambda)\Big) f(qt)\, d_q t \end{aligned}$$

and

$$\begin{aligned} D_{q^{-1}}\phi(a, \lambda) = D_{q^{-1}}\theta_1(a, \lambda)\Big(c_1 + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a} \theta_2(qt, \lambda) f(qt)\, d_q t\Big) \\ - \frac{q}{\Delta(\lambda)} D_{q^{-1}}\theta_2(a, \lambda)\int_0^{q^{-1}a} \theta_1(qt, \lambda) f(qt)\, d_q t. \end{aligned}$$

The boundary condition $a_{21}\phi_2(a, \lambda) + a_{22}D_{q^{-1}}\phi_2(a, \lambda) = 0$ implies

$$\Big(c_1 + \frac{q}{\Delta(\lambda)}\int_0^{q^{-1}a} \theta_2(qt, \lambda) f(qt)\, d_q t\Big) W_q(\theta_1, \theta_2)(a) = 0.$$

Hence

$$c_1 = \frac{-q}{\Delta(\lambda)} \int_0^{q^{-1}a} \theta_2(qt,\lambda) f(qt)\, d_q t.$$

So for $x \in A_f$

$$\begin{aligned}
&\phi(x,\lambda)\\
&= \frac{-q}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^x \theta_1(qt,\lambda) f(qt)\, d_q t - \frac{q}{\Delta(\lambda)}\theta_1(x,\lambda)\int_x^{q^{-1}a} \theta_2(qt,\lambda) f(qt)\, d_q t\\
&= \frac{-1}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^{qx} \theta_1(t,\lambda) f(t)\, d_q t - \frac{1}{\Delta(\lambda)}\theta_1(x,\lambda)\int_{qx}^{a} \theta_2(t,\lambda) f(t)\, d_q t\\
&= \frac{-1}{\Delta(\lambda)}\theta_2(x,\lambda)\int_0^{x} \theta_1(t,\lambda) f(t)\, d_q t - \frac{1}{\Delta(\lambda)}\theta_1(x,\lambda)\int_{x}^{a} \theta_2(t,\lambda) f(t)\, d_q t,
\end{aligned}$$

proving (3.46)–(3.47). Conversely, by direct computations, if $\phi(x,\lambda)$ is given by (3.46), then it is a solution of (3.45) and satisfies the boundary conditions (3.28)–(3.29). To prove the uniqueness, suppose that there exists another function, $\widetilde{G}(x,t,\lambda)$, such that

$$\psi(x,\lambda) = \int_0^a \widetilde{G}(x,t,\lambda)\, f(t)\, d_q t \tag{3.54}$$

is a solution of (3.45) which satisfies (3.28)–(3.29). For convenience, let

$$G(x,t,\lambda) = \begin{cases} G_1(x,t,\lambda),\ 0 \leqslant t \leqslant x,\\ G_2(x,t,\lambda),\ x \leqslant t \leqslant a. \end{cases}, \quad \widetilde{G}(x,t,\lambda) = \begin{cases} \widetilde{G}_1(x,t,\lambda),\ 0 \leqslant t \leqslant x,\\ \\ \widetilde{G}_2(x,t,\lambda),\ x \leqslant t \leqslant a. \end{cases}$$

By subtraction, we obtain

$$\int_0^a \left\{G(x,t,\lambda) - \widetilde{G}(x,t,\lambda)\right\} f(t)\, d_q t = 0 \quad \text{for all}\ \ x \in \{aq^m;\ m \in \mathbb{N}_0\} \tag{3.55}$$

and for all functions $f(t) \in L_q^2(0,a)$. Let us take

$$f(t) := \overline{G(x,t,\lambda) - \widetilde{G}(x,t,\lambda)} \quad (x = aq^m;\ m \in \mathbb{N}_0).$$

Then

$$\begin{aligned}
&\int_0^a \left|G(aq^m,t,\lambda)-\widetilde{G}(aq^m,t,\lambda)\right|^2\, d_qt\\
&\quad=\int_0^{aq^m} \left|G_1(aq^m,t,\lambda)-\widetilde{G_1}(aq^m,t,\lambda)\right|^2\, d_qt\\
&\qquad+\int_{aq^m}^a \left|G_2(aq^m,t,\lambda)-\widetilde{G_2}(aq^m,t,\lambda)\right|^2\, d_qt\\
&\quad=a(1-q)\sum\nolimits_{n=0}^{\infty} q^n \left|G(aq^m,aq^n,\lambda)-\widetilde{G}(aq^m,aq^n,\lambda)\right|^2=0
\end{aligned}\tag{3.56}$$

Therefore, from (3.56) we conclude that

$$G(aq^m,aq^n,\lambda)=\widetilde{G}(aq^m,aq^n,\lambda)\quad (m,\,n\in\mathbb{N}_0),$$

and proving the uniqueness. If $f(\cdot)$ is q-regular at zero, then $A_f\equiv[0,a]$ and (3.46) will be defined for all $x\in[0,a]$. ☐

Theorem 3.9. *Green's function has the following properties*

- **(i)** $G(x,t,\lambda)$ *is continuous at the point* $(0,0)$.
- **(ii)** $G(x,t,\lambda)=G(t,x,\lambda)$.
- **(iii)** *For each fixed* $t\in(0,qa]$, *as a function of* x, $G(x,t,\lambda)$ *satisfies the* q-*difference equation (3.27) in the intervals* $[0,t)$, $(t,a]$ *and it also satisfies the boundary conditions (3.28)–(3.29).*
- **(iv)** *Let* λ_0 *be a zero of* $\Delta(\lambda)$. *Then* λ_0 *can be a simple pole of the function* $G(x,t,\lambda)$, *and in this case*

$$G(x,t,\lambda)=\frac{-\psi_0(x)\psi_0(t)}{\lambda-\lambda_0}+\widetilde{G}(x,t,\lambda),$$

where $\widetilde{G}(x,t,\lambda)$ *is analytic function of* λ *in a neighborhood of* λ_0 *and* $\psi_0(\cdot)$ *is a normalized eigenfunction corresponding to* λ_0.

Proof. **(i)** Follows from the continuity of $\theta_1(\cdot,\lambda)$, $\theta_2(\cdot,\lambda)$ at zero for each fixed $\lambda\in\mathbb{C}$ and **(ii)** is easy to be checked. Now, we prove **(iii)**. Let $t\in(0,qa]$ be fixed. If $x\in[0,t]$, then

$$G(x,t,\lambda)=\frac{1}{\Delta(\lambda)}\theta_1(x,\lambda)\theta_2(t,\lambda).$$

So,

$$\ell G(x,t,\lambda)=\frac{1}{\Delta(\lambda)}\theta_2(t,\lambda)\ell\theta_1(x,\lambda)=\frac{\lambda}{\Delta(\lambda)}\theta_2(t,\lambda)\theta_1(x,\lambda)=\lambda G(x,t,\lambda).$$

Similarly if $x \in [t, a]$. From (3.40) and (3.47), we have

$$\begin{aligned}
&a_{11}G(0,t,\lambda) + a_{12}D_{q^{-1}}G(0,t,\lambda)\\
&\quad = \frac{\theta_2(t,\lambda)}{\Delta(\lambda)}\left\{a_{11}\theta_1(0,\lambda) + a_{12}D_{q^{-1}}\theta_1(0,\lambda)\right\} = 0,\\
&a_{21}G(a,t,\lambda) + a_{22}D_{q^{-1}}G(a,t,\lambda)\\
&\quad = \frac{\theta_1(t,\lambda)}{\Delta(\lambda)}\left\{a_{21}\theta_1(a,\lambda) + a_{22}D_{q^{-1}}\theta_1(a,\lambda)\right\} = 0.
\end{aligned}$$

(iv) Let λ_0 be a pole of $G(x,t,\lambda)$, and $R(x,t)$ be the residue of $G(x,t,\lambda)$ at $\lambda = \lambda_0$. From (3.42) and (3.44), we obtain

$$\begin{aligned}
R(x,t) &= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)G(x,t,\lambda) = k_0^{-1}\theta_1(x,\lambda_0)\theta_1(t,\lambda_0)\lim_{\lambda\to\lambda_0}\frac{\lambda-\lambda_0}{\Delta(\lambda)}\\
&= -\frac{\theta_1(x,\lambda_0)\theta_1(t,\lambda_0)}{\int_0^a|\theta_1(u,\lambda)|^2\,d_qu} = -\psi_0(x,\lambda_0)\psi_1(t,\lambda_0).
\end{aligned}$$

□

3.4 Eigenfunctions Expansion Formula

In this section, the existence of a countable sequence of eigenvalues of ℓ with no finite limit points will be proved by using the spectral theorem of compact self adjoint operators in Hilbert spaces, see e.g.[52]. Moreover, it will be proved that the corresponding eigenfunctions form an orthonormal basis of $L_q^2(0,a)$. We define the operator

$$\mathscr{L} : \mathscr{D}_{\mathscr{L}} \to L_q^2(0,a)$$

by

$$\mathscr{L}y = \ell y \quad \text{for all } y \in \mathscr{D}_{\mathscr{L}},$$

where $\mathscr{D}_{\mathscr{L}}$ is the subspace of $L_q^2(0,a)$ consisting of those complex valued functions y that satisfy (3.28)–(3.29) such that $D_q y(\cdot)$ is q-regular at zero and $D_q^2 y(\cdot)$ lies in $L_q^2(0,a)$. Thus $\mathscr{L}$ is the q-difference operator generated by the q-difference expression ℓ and the boundary conditions (3.28)–(3.29). By $\mathscr{L}(y) = \lambda y$, we mean that $\ell(y) = \lambda y$ and y satisfies (3.28)–(3.29). The operator $\mathscr{L}$ has the same eigenvalues of the basic Sturm–Liouville problem (3.27)–(3.29). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue. Thus $\ker\mathscr{L} = \{0\}$. From the previous section the solution of the problem

$$(\mathscr{L}y)(x) = f(x) \quad (f \in L_q^2(0,a)) \tag{3.57}$$

is given uniquely in $L_q^2(0,a)$ by

$$y(x) = \int_0^a G(x,t)f(t)\,d_qt, \tag{3.58}$$

where

$$G(x,t) = G(x,t,0) = \begin{cases} c\theta_1(t)\theta_2(x),\ 0 \leqslant t \leqslant x, \\ \\ c\theta_1(x)\theta_2(t)\ \ x \leqslant t \leqslant a, \end{cases},\ c := -\frac{1}{W_q(\theta_1,\theta_2)}.$$

Replacing $f(\cdot)$ by $\lambda y(\cdot)$ in (3.57). Then the eigenvalue problem

$$(\mathscr{L}y)(x) = \lambda y(x), \tag{3.59}$$

is equivalent to the following basic Fredholm integral equation of the second kind

$$y(x) = \lambda \int_0^a G(x,t)y(t)\,d_qt \quad \text{for } x \in \{aq^m;\ m \in \mathbb{N}_0\}. \tag{3.60}$$

Theorem 3.10. *Let $\mathscr{G}$ be the integral operator*

$$\mathscr{G} : L_q^2(0,a) \to L_q^2(0,a), \quad (\mathscr{G}f)(x) = \int_0^a G(x,t)f(t)\,d_qt. \tag{3.61}$$

Then

$$(\mathscr{L}\mathscr{G})f = f \quad (f \in L_q^2(0,a)), \tag{3.62}$$

and

$$(\mathscr{G}\mathscr{L})(y) = y \quad (y \in \mathscr{D}_{\mathscr{L}}). \tag{3.63}$$

Proof. We show first that $y = \mathscr{G}f \in \mathscr{D}_{\mathscr{L}}$. From (3.60) and (3.61)

$$y(x) = (\mathscr{G}f)(x) = \theta_2(x)y_1(x) + \theta_1(x)y_2(x),$$

where

$$y_1(x) = c\int_0^x \theta_1(t)f(t)\,d_qt \ \text{ and } \ y_2(x) = c\int_x^a \theta_2(t)f(t)\,d_qt.$$

Thus, for all $x \in A_f$, cf. (3.49),

$$\begin{aligned} D_qy(x) &= D_q\theta_2(x)y_1(qx) + D_q\theta_1(x)y_2(qx), \\ D_q^2y(x) &= -qv(qx)(y)(qx) - qf(qx) \in L_q^2(0,a). \end{aligned} \tag{3.64}$$

Since $D_q\theta_i(x,\lambda)$, $y_i(x)$, $i=1,2$, are q-regular at zero, then so is $D_q y(\cdot)$ and

$$D_{q^{-1}}y(0)=D_q y(0)=\lim_{\substack{n\to\infty\\ x\in A_f}}\frac{y(xq^n)-y(0)}{xq^n}=D_{q^{-1}}\theta_2(0)y_2(0),$$

$y_1(0)=0$, and $y_2(a)=0$, then

$$\bigl(a_{11}y(0)+a_{12}D_{q^{-1}}y(0)\bigr)=\bigl(a_{11}\theta_1(0)+a_{12}D_{q^{-1}}\theta_1(0)\bigr)y_2(0)=0,$$

and

$$a_{21}y(a)+a_{22}D_{q^{-1}}y(a)=\bigl(a_{21}\theta_2(a)+a_{22}D_{q^{-1}}\theta_2(a)\bigr)y_1(a)=0.$$

Thus $y\in\mathcal{D}_{\mathcal{L}}$. It follows from (3.64) that $\mathcal{L}y=(\mathcal{L}\mathcal{G})(f)=f$. Hence we have established (3.62). Now, we prove (3.63). Indeed, replacing f in (3.62) by $\mathcal{L}y$, we get $\mathcal{L}y=\mathcal{L}\mathcal{G}\mathcal{L}y$. Thus, $y=\mathcal{G}\mathcal{L}y$ since $\mathcal{L}$ is assumed to be injective. □

It follows from (3.62) and (3.63) that $\ker\mathcal{G}=\{0\}$ and ϕ is an eigenfunction of $\mathcal{G}$ with eigenvalue μ if and only if ϕ is an eigenfunction of $\mathcal{L}$ with eigenvalue $1/\mu$.

Recall that a bounded linear operator $T:X\to X$, X is a Banach Space, is called compact if for every bounded set S of vectors on X the set $T(S)$ is relatively compact. That is $\overline{\{Ts:\ s\in S\}}$ is compact. An equivalent definition of compact operators on Hilbert spaces may be given as follows:

Definition 3.4.1. An operator T on a Hilbert space H

$$T:H\to H$$

is said to be compact if it can be written as

$$T=\sum_{k=1}^{N}\lambda_k<f_k,\cdot>g_k,$$

where $1\le N\le\infty$, $f_1,\dots,f_N$ and $g_1,\dots,g_N$ are (not necessarily complete) orthonormal sets. Here $\lambda_1,\dots,\lambda_n$ are positive numbers.

Theorem 3.11. *The operator $\mathcal{G}$ is compact and self adjoint.*

Proof. Let $f,h,\in L_q^2(0,a)$. Since $G(x,t)$ is a real valued function defined on $[0,a]\times[0,a]$ and $G(x,t)=G(t,x)$, then for $f,h\in L_q^2(0,a)$,

$$\begin{aligned}\langle\mathcal{G}(f),h\rangle&=\int_0^a(\mathcal{G}f)(x)\overline{h(x)}\,d_qx=\int_0^a\int_0^a G(x,t)f(t)\overline{h(x)}\,d_qt\,d_qx\\&=\int_0^a f(t)\overline{\Bigl(\int_0^a G(t,x)h(x)\,d_qx\Bigr)}\,d_qt=\langle f,\mathcal{G}(h)\rangle,\end{aligned}$$

i.e. $\mathscr{G}$ is self adjoint. Let

$$\phi_{ij}(x,t) := \phi_i(x)\phi_j(t) \quad (i,j \in \mathbb{N})$$

be an orthonormal basis of $L_q^2\big((0,a)\times(0,a)\big)$. Consequently,

$$G = \sum_{i,j=1}^{\infty} \langle G, \phi_{ij}\rangle \phi_{ij}\,.$$

Let

$$G_n = \sum_{i,j=1}^{n} \langle G, \phi_{ij}\rangle \phi_{ij} \quad (n \in \mathbb{N}),$$

and let $\mathscr{G}_n$ be the finite rank integral operator defined on $L_q^2(0,a)$ by

$$\mathscr{G}_n(f)(x) := \int_0^a G_n(x,t) f(t)\, d_qt \quad \text{for } x \in \{0,\ aq^m;\ m \in \mathbb{N}_0\}\,.$$

Obviously $\mathscr{G}_n$ is compact for all $n \in \mathbb{N}$. From Cauchy–Schwarz' inequality

$$\begin{aligned}
\|(\mathscr{G}-\mathscr{G}_n)(f)\| &= \left(\int_0^a |(\mathscr{G}-\mathscr{G}_n)(f)(x)|^2\, d_qx\right)^{1/2}\\
&= \left(\int_0^a \left|\int_0^a (G-G_n)(x,t)(f)(t)\, d_qt\right|^2 d_qx\right)^{1/2}\\
&\leqslant \left(\int_0^a \int_0^a |(G-G_n)(x,t)|^2\, d_qt\, d_qx\right)^{1/2} \left(\int_0^a |f(x)|^2\, d_qx\right)^{1/2}\\
&= \|G-G_n\|_2\, \|f\|\,,
\end{aligned}$$

then

$$\|\mathscr{G}-\mathscr{G}_n\| \leqslant \|G-G_n\|_2 \to 0 \ \text{ as } \ n \to \infty.$$

This completes the proof. □

Corollary 3.12. *(i) The eigenvalues of the operator $\mathscr{L}$ form an infinite sequence $\{\lambda_k\}_{k=1}^{\infty}$ of real numbers which can be ordered so that*

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n| < \cdots \ \to \infty \ \text{ as } \ n \to \infty.$$

(ii) The set of all eigenfunctions of $\mathscr{L}$ form an orthonormal basis for $L_q^2(0,a)$.

Proof. First we prove (i). Since $\mathscr{G}$ is a compact self adjoint operator on $L_q^2(0,a)$, the operator $\mathscr{G}$ has an infinite sequence of non zero real eigenvalues $\{\mu_n\}_{n=1}^{\infty}$, $\mu_n \to 0$

as $n \to \infty$. Let $\{\phi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\mu_n\}_{n=1}^{\infty}$. From the spectral theorem of compact self adjoint operators, we have,

$$\mathscr{G}(f) = \sum_{n=1}^{\infty} \lambda_n \big\langle f, \phi_n \big\rangle \phi_n.$$

Since the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the operator $\mathscr{L}$ are the reciprocal of those of $\mathscr{G}$, we obtain

$$|\lambda_n| = \frac{1}{|\mu_n|} \to \infty \quad \text{as} \quad n \to \infty.$$

Now we prove (ii). Let $y \in \mathscr{D}_{\mathscr{L}}$. Then, $y = G(f)$, for some $f \in L_q^2(0,a)$. Consequently,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \lambda_n \, \langle f, \phi_n \rangle \, \phi_n = \sum_{n=0}^{\infty} \mu_n \, \langle ly, \phi_n \rangle \, \phi_n \\ &= \sum_{n=0}^{\infty} \mu_n \, \langle y, \ell\phi_n \rangle \, \phi_n = \sum_{n=0}^{\infty} \langle y, \phi_n \rangle \, \phi_n. \end{aligned}$$

If zero is an eigenvalue of $\mathscr{L}$. Then, we can choose $r \in \mathbb{R}$ such that r is not an eigenvalue of $\mathscr{L}$. Now, applying the above result on $\mathscr{L} - rI$ in place of $\mathscr{L}$ yields the corollary. ☐

Example 3.4.1. Consider the q-Sturm–Liouville boundary value problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) = \lambda y(x), \tag{3.65}$$

with the q-Dirichlet conditions

$$U_1(y) = y(0) = 0, \;\; U_2(y) = y(1) = 0. \tag{3.66}$$

A fundamental set of solutions of (3.65) is

$$\phi_1(x,\lambda) = \cos(\sqrt{\lambda}x; q), \qquad \phi_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x; q)}{\sqrt{\lambda}}. \tag{3.67}$$

Now, the eigenvalues of problem (3.65) are the zeros of the determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(\phi_1) & U_2(\phi_1) \\ U_1(\phi_2) & U_2(\phi_2) \end{vmatrix} = \phi_2(1,\lambda) = \frac{\sin(\sqrt{\lambda}; q)}{\sqrt{\lambda}}.$$

Hence, the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are the zeros of $\sin(\sqrt{\lambda};q)$. From (2.86),

$$\lambda_n = y_n^2 = \frac{q^{-2n}}{(1-q)^2}\,(1+O(q^n)) \quad (n \in \mathbb{N})$$

for sufficiently large n and the corresponding set of eigenfunctions $\left\{\frac{\sin(\sqrt{\lambda_n};q)}{\sqrt{\lambda_n}}\right\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. In the previous notations

$$\theta_1(x,\lambda) = \frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}},$$

and

$$\theta_2(x,\lambda) = \frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}\cos(\sqrt{\lambda}x;q) + \cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}.$$

So, if λ is not an eigenvalue, Green's function is given by

$$G(x,t,\lambda) = \frac{\sin(\sqrt{\lambda}t;q)}{\sin(\sqrt{\lambda};q)}\Big(\cos(\sqrt{\lambda}x;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}} - \cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\Big),$$

for $0 \leqslant t \leqslant x$, and

$$G(x,t,\lambda) = \frac{\sin(\sqrt{\lambda}x;q)}{\sin(\sqrt{\lambda};q)}\Big(\cos(\sqrt{\lambda}t;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}} - \cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}t;q)}{\sqrt{\lambda}}\Big),$$

for $x \leqslant t \leqslant 1$. Since $\lambda = 0$ is not an eigenvalue, then Green's function $G(x,t)$ is nothing but

$$G(x,t) = G(x,t,0) = \begin{cases} t(1-x), \ 0 \leqslant t \leqslant x, \\ x(1-t), \ x \leqslant t \leqslant 1. \end{cases}$$

Hence the boundary value problem (3.65)–(3.66) is equivalent to the basic Fredholm integral equation

$$y(x) = \lambda \int_0^1 G(x,t)y(t)\,d_qt.$$

Example 3.4.2. Consider (3.65) with the q-Neumann boundary conditions

$$U_1(y) = D_{q^{-1}}y(0) = 0, \quad U_2(y) = D_{q^{-1}}y(1) = 0. \tag{3.68}$$

In this case $\theta_1(x,\lambda) = \cos(\sqrt{\lambda}x;q)$ and

$$\theta_2(x,\lambda) = \cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}x;q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}x;q).$$

Since $\Delta(\lambda) = \sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)$. Then $\lambda_0 = 0$ and from (2.86) eigenvalue are given by

$$\lambda_n = q^{-1}y_n^2 = \frac{q^{-2n+1}}{(1-q)^2}\,(1 + O(q^n)) \quad (n \in \mathbb{N}).$$

Therefore, $\{1, \cos(\sqrt{\lambda_n}x;q)\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. If λ is not an eigenvalue, then the Green's function $G(x,t,\lambda)$ is defined for $x, t \in [0,1]\times[0,1]$ by

$$\begin{aligned} G(x,t,\lambda) = &-\frac{\cos(\sqrt{\lambda}t;q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)}\Big(\cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}x;q) \\ &+\sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}x;q)\Big) \ \text{ for } 0 \le t \le x \end{aligned}$$

and

$$\begin{aligned} G(x,t,\lambda) = &-\frac{\cos(\sqrt{\lambda}x;q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)}\Big(\cos(\sqrt{\lambda}q^{-1/2};q)\cos(\sqrt{\lambda}t;q) \\ &+\sqrt{q}\sin(\sqrt{\lambda}q^{-1/2};q)\sin(\sqrt{\lambda}t;q)\Big) \ \text{ for } x \leqslant t \leqslant 1. \end{aligned}$$

The operator $\mathscr{L}$ associated with problem (3.65), (3.68) is not invertible since zero is an eigenvalue.

Example 3.4.3. Consider (3.65) with the following boundary conditions

$$U_1(y) = y(0) = 0, \qquad U_2(y) = y(1) + D_{q^{-1}}y(1) = 0. \tag{3.69}$$

Then,

$$\Delta(\lambda) = \phi_2(1,\lambda) + D_{q^{-1}}\phi_2(1,\lambda) = \frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}} + \cos(\sqrt{\lambda}q^{-1/2};q).$$

The eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of this boundary value problem are the solutions of the equation

$$\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}} = -\cos(\sqrt{\lambda}q^{-1/2};q)$$

and the corresponding eigenfunctions are $\left\{\frac{\sin(\sqrt{\lambda_n};q)}{\sqrt{\lambda_n}}\right\}_{n=1}^{\infty}$. The functions $\theta_1(x,\lambda)$ and $\theta_2(x,\lambda)$ are

$$\begin{aligned} \theta_1(x,\lambda) &= \frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}, \\ \theta_2(x,\lambda) &= \Big(\cos(\sqrt{\lambda}q^{-1/2};q) + \frac{\sin(\sqrt{\lambda;q})}{\sqrt{\lambda}}\Big)\cos(\sqrt{\lambda x;q}) \\ &\quad -\big(-\sqrt{\lambda q}\sin(\sqrt{\lambda}q^{-1/2};q) + \cos(\sqrt{\lambda};q)\big)\frac{\sin(\sqrt{\lambda x;q})}{\sqrt{\lambda}}. \end{aligned}$$

If λ is not an eigenvalue, then the Green's function $G(x,t,\lambda)$ is defined to be

$$G(x,t,\lambda) = \frac{-1}{\sin(\sqrt{\lambda};q) + \sqrt{\lambda}\cos(\sqrt{\lambda}q^{-1/2};q)} \begin{cases} \sin(\sqrt{\lambda}t;q)\theta_2(x,\lambda), \ 0 \leqslant t \leqslant x, \\ \\ \sin(\sqrt{\lambda}x;q)\theta_2(t,\lambda), \ x \leqslant t \leqslant 1. \end{cases}$$

and

$$G(x,t) = \frac{-1}{2} \begin{cases} t(2-x), \ 0 \leqslant t \leqslant x, \\ x(2-t), \ x \leqslant t \leqslant 1. \end{cases}$$

Therefore, the boundary value problem (3.65), (3.69) is equivalent to the basic Fredholm integral equation

$$y(x) = \lambda \int_0^1 G(x,t)y(t)\,d_qt.$$

3.5 q^2-Analogue of the Fourier Transform

Koornwinder and Swarttouw introduced a q-analogue of the cosine and sine Fourier transform in [176] with the functions $\mathrm{Cos}(z;q^2)$ and $\mathrm{Sin}(z;q^2)$ defined by

$$\begin{aligned} \mathrm{Cos}(z;q^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}\,(z(1-q))^{2k}}{(q;q)_{2k}} \\ &= {}_1\phi_1\left(0;q;q^2,q^2z^2(1-q)^2\right), \\ \mathrm{Sin}(z;q^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}\,(z(1-q))^{2k+1}}{(q;q)_{2k+1}} \\ &= z\,{}_1\phi_1\left(0;q^3;q^2,q^2z^2(1-q)^2\right). \end{aligned} \tag{3.70}$$

They first introduced the pair of q-transforms

$$\begin{aligned} g(q^n) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} q^k \begin{cases} \mathrm{Cos}(\frac{q^{k+n}}{1-q};q^2) \\ \text{or} \\ \mathrm{Sin}(\frac{q^{k+n}}{1-q};q^2) \end{cases} f(q^k), \\ f(q^k) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^n \begin{cases} \mathrm{Cos}(\frac{q^{k+n}}{1-q};q^2) \\ \text{or} \\ \mathrm{Sin}(\frac{q^{k+n}}{1-q};q^2) \end{cases} g(q^n), \end{aligned} \tag{3.71}$$

where $0 < q < 1$ and f, g are in the space $L^2(\mathbb{R}_q)$. Now assume that $\frac{\log(1-q)}{1-q} \in 2\mathbb{Z}$ or equivalently

$$q \in \left\{q \in (0,1) : 1-q = q^{2m} \text{ for some integer } m\right\}. \tag{3.72}$$

Then by replacing q^k and q^n in (3.71) by $q^k\sqrt{1-q}$ and $q^n\sqrt{1-q}$, respectively, and then $f(q^k\sqrt{1-q})$ and $g(q^n\sqrt{1-q})$ by $f(q^k)$ and $g(q^k)$, respectively, Koornwinder and Swarttouw obtained the following q-analogue of the cosine and sine Fourier transforms

$$\begin{aligned} g(\lambda) &= \frac{\sqrt{1+q}}{\Gamma_q^2(1/2)} \int_0^\infty f(t) \left\{ \begin{array}{c} \mathrm{Cos}(t\lambda; q^2) \\ \text{or} \\ \mathrm{Sin}(t\lambda; q^2) \end{array} \right. d_q t, \\ f(x) &= \frac{\sqrt{1+q}}{\Gamma_q^2(1/2)} \int_0^\infty f(t) \left\{ \begin{array}{c} \mathrm{Cos}(x\lambda; q^2) \\ \text{or} \\ \mathrm{Sin}(x\lambda; q^2) \end{array} \right. d_q \lambda. \end{aligned} \tag{3.73}$$

Therefore if we let $q \to 1^-$ for such q's that satisfy (3.72) we obtain the cosine and sine Fourier transforms

$$g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(\lambda t)\,dt, \;\; f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda)\cos(t\lambda)\,d\lambda, \tag{3.74}$$

$$g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(\lambda t)\,dt, \;\; f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda)\sin(t\lambda)\,d\lambda. \tag{3.75}$$

A key point in Koornwinder and Swarttouw's investigations is the following orthogonality relation they proved in [176]:

Theorem 3.13. *Let $|z| < 1$ and n, m are integers. Then*

$$\delta_{mn} = \sum_{k=-\infty}^{\infty} z^{k+n} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1\big(0; z^2; q, q^{n+k+1}\big) z^{k+m} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1\big(0; z^2; q, q^{m+k+1}\big), \tag{3.76}$$

where the sum converges absolutely and uniformly on compact subsets of the open unit disc.

There is a relation connects the pairs of the q-cosine and q-sine functions

$$\left\{\mathrm{Cos}(x;q^2),\, \mathrm{Sin}(x;q^2)\right\}, \quad \{\cos(x;q),\, \sin(x;q)\},$$

where we introduced $\{\cos(x;q),\, \sin(x;q)\}$ in Sect. 2.8 as a fundamental set of solutions of the basic Sturm–Liouville problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) = y(x).$$

In fact

$$\mathrm{Cos}(x;q^2) = \cos(q^{1/2}x;q), \quad \mathrm{Sin}(x;q^2) = \sin(x;q)$$

Hence,

$$-\frac{1}{q}D_q^{-1}D_q y(x) = \begin{cases} \lambda^2 y(x), & \text{if } y(x) = \mathrm{Sin}(\lambda x;q^2) \\ \\ q\lambda^2 y(x), & \text{if } y(x) = \mathrm{Cos}(\lambda x;q^2). \end{cases}$$

Therefore, the eigenfunctions $\{\mathrm{Cos}(\lambda z;q^2),\ \mathrm{Sin}(\lambda z;q^2)\}$ have two different eigenvalues. Consequently, as remarked by Koornwinder and Swarttouw in [176] and illustrated in Examples 3.4.1 and 3.4.2, no q-exponential functions built from the functions $\{\mathrm{Cos}(x;q^2),\ \mathrm{Sin}(x;q^2)\}$ or $\{\cos(x;q),\ \sin(x;q)\}$ will satisfy an eigenfunction problem. This motivated Rubin in [265, 266] to introduce the q-difference operator

$$\partial_q f(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} \quad (z \neq 0). \tag{3.77}$$

It is straightforward to prove that if a function f is differentiable at a point z, then

$$\lim_{q\to 1^-} \partial_q f(z) = f'(z).$$

In addition,

$$\delta_q f(z) = \begin{cases} D_q f(z), & \text{if } f \text{ is odd,} \\ \frac{1}{q}D_{q^{-1}} f(z), & \text{if } f \text{ is even.} \end{cases}$$

The following properties of the ∂_q operator are from [54, 103, 266]:

Lemma 3.14. *Let f be a function defined on a set A where A satisfies*

$$z \in A \to \pm q^{\pm 1} z \in A,$$

and let f_e and f_o be the even and the odd parts of f, respectively. Then for all $z \in A \setminus \{0\}$ we have

i. $\partial_q f(z) = \frac{1}{q}D_{q^{-1}} f_e(z) + D_q f_o(z).$

ii. For two functions f and g,

- *If f is even and g is odd then*

$$\partial_q (fg)(z) = qg(z)(\partial_q f)(qz) + f(qz)\partial_q g(z),$$

- *If f and g are even then*

$$\partial_q (fg)(z) = \partial_q f(z)g(z) + f(z/q)\partial_q g(z),$$

- *If f and g are odd then*

$$\partial_q\left(fg\right)(x)=\frac{1}{q}\left(f(x)\left(\partial_q g\right)(x/q)+(\partial_q f)(x/q)g(x/q)\right).$$

- *For $a>0$, if $\int_{-a}^{a}\partial_q f(x)\,g(x)\,d_qx$ exists then*

$$\int_{-a}^{a}\partial_q f(x)\,g(x)\,d_qx=2\left[f_e(q^{-1}a)g_o(a)+f_o(a)g_e(q^{-1}a)\right]$$
$$-\int_{-a}^{a}f(x)\partial_q g(x)\,d_qx,$$

where g_o and g_e are the odd and the even parts of the function g, respectively.

- *For $a>0$ if $\int_{-\infty/a}^{\infty/a}\partial_q f(x)\,g(x)\,d_qx$ exists then*

$$\int_{-\infty/a}^{\infty/a}\partial_q f(x)\,g(x)\,d_qx=-\int_{-\infty/a}^{\infty/a}f(x)\partial_q g(x)\,d_qx.$$

Rubin [265] defined a q^2-analogue of the Fourier transform in the form

$$\begin{aligned}&\hat{f}(x;q^2):=\mathscr{F}_q(f)(x)=K\textstyle\int_{-\infty}^{\infty}f(t)e(-itx;q^2)\,d_qt,\\ &K=\frac{(1+q)^{1/2}}{2\Gamma_{q^2}(\frac{1}{2})}\quad\text{and}\quad f\in L^1(\mathbb{R}_q),\end{aligned}\tag{3.78}$$

and q satisfies (3.72). Rubin [265, 266] proved that the q^2-analogue of the Fourier transform satisfies the properties in the following theorem which we state without proof.

Theorem 3.15. *Let q satisfy the condition* (3.72). *Then the following properties of the q^2-Fourier transform are satisfied.*

(1) If $f(u)$ and $uf(u)\in L_q^1(\mathbb{R}_q)$, then

$$\partial_q\left(\mathscr{F}_q f\right)(x)=\mathscr{F}_q(-iuf(u))(x).$$

(2) If f and $\partial_q f\in L_q^1(\mathbb{R}_q)$, then

$$\mathscr{F}_q(\partial_q f)(x)=ix\mathscr{F}_q\,(f)(x).\tag{3.79}$$

(3) If $f\in L_q^1(\mathbb{R}_q)\cap L_q^2(\mathbb{R}_q)$, then

$$f(t)=K\int_{-\infty}^{\infty}\mathscr{F}_q(f)(x)e(itx;q^2)\,d_qx,\ \forall t\in\mathbb{R}_q.\tag{3.80}$$

3.6 A Reformulation of the q^2-Fourier Transform

In this section, we reformulate the definition of the q^2-Fourier transform for any $q \in (0,1)$. The results of this section are based on [207]. Rewrite the transform pairs in (3.71) as in the following:

$$\begin{aligned}
g\Big(\frac{q^n}{\sqrt{1-q}}\Big) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} q^k \begin{cases} \operatorname{Cos}(\frac{q^{k+n}}{1-q};q^2) \\ \text{or} \\ \operatorname{Sin}(\frac{q^{k+n}}{1-q};q^2) \end{cases} f\Big(\frac{q^k}{\sqrt{1-q}}\Big), \\
f\Big(\frac{q^k}{\sqrt{1-q}}\Big) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^n \begin{cases} \operatorname{Cos}(\frac{q^{k+n}}{1-q};q^2) \\ \text{or} \\ \operatorname{Sin}(\frac{q^{k+n}}{1-q};q^2) \end{cases} g\Big(\frac{q^n}{\sqrt{1-q}}\Big),
\end{aligned} \tag{3.81}$$

where here we assume that the functions f and g are in the space $L^1(\widetilde{\mathbb{R}}_q) \cap L^2(\widetilde{\mathbb{R}}_q)$. Using Matsuo definition of the q-integration on $(0,\infty)$, (1.21) with $a = \sqrt{1-q}$, the transformations in (3.81) can be written as

$$\begin{aligned}
g(x) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} f(t) \operatorname{Cos}(xt;q^2)\, d_qt, \\
f(t) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} g(x) \operatorname{Cos}(xt;q^2)\, d_qx,
\end{aligned} \tag{3.82}$$

and

$$\begin{aligned}
g(x) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} f(t) \operatorname{Sin}(xt;q^2)\, d_qt, \\
f(t) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} g(x) \operatorname{Sin}(xt;q^2)\, d_qx,
\end{aligned} \tag{3.83}$$

where $x,t \in \widetilde{\mathbb{R}}_q$ and f,g are in $L^1(\widetilde{\mathbb{R}}_q) \cap L^2(\widetilde{\mathbb{R}}_q)$. Consequently, we set the following generalization of Rubin's definition of the q^2-Fourier transform (3.78).

Definition 3.6.1. Let $0 < q < 1$. We define the q^2-Fourier transform for any function $f \in L^1(\widetilde{\mathbb{R}}_q)$ to be

$$\hat{f}(x;q^2) := \mathscr{F}_q(f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t) e(-itx;q^2)\, d_qt. \tag{3.84}$$

It is clear that Rubin's definition of the q^2-Fourier transform when q-satisfies the condition (3.72) is a special case of (3.6.1) since if $1-q = q^{2m}$ for some $m \in \mathbb{Z}$ then

$$\int_0^{\infty/\sqrt{1-q}} f(t)\,d_qt = \int_0^{\infty/q^m} f(t)\,d_qt = \int_0^{\infty} f(t)\,d_qt.$$

However, we get the classical Fourier transform only when $q \to 1^-$ and q satisfies the condition (3.72).

Now, replace q by q^2 in (3.76) and set $z = q^\alpha$, $\alpha > 0$, we obtain

$$\begin{aligned} &q^{-\alpha(n+m)} \frac{(q^2;q^2)_\infty^2}{(q^{2\alpha};q^2)_\infty^2} \delta_{mn} \\ &\quad = \sum_{k=-\infty}^{\infty} q^{2k\alpha}\, {}_1\phi_1\left(0;q^{2\alpha};q^2,q^{2n+2k+2}\right)\, {}_1\phi_1\left(0;q^{2\alpha};q^2,q^{2m+2k+2}\right). \end{aligned} \tag{3.85}$$

Theorem 3.16. *If $f \in L^1(\widetilde{\mathbb{R}}_q) \cap L^2(\widetilde{\mathbb{R}}_q)$, then*

$$f(t) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \mathscr{F}(f)(x)e(itx(1-q);q^2)\,d_qx \quad (t \in \widetilde{\mathbb{R}}_q). \tag{3.86}$$

Proof. Since

$$\begin{aligned} \mathscr{F}(f)(x) &= \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)e(itx(1-q);q^2)\,d_qt \\ &= \frac{\sqrt{1+q}}{4\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} (f_e(t) + f_o(t))\, e(itx(1-q);q^2)\,d_qt, \end{aligned}$$

where f_e and f_o are the even and the odd parts of the function f, respectively, we obtain

$$\begin{aligned} \mathscr{F}(f)(x) &= \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} f_e(t)\operatorname{Cos}(tx(1-q);q^2)\,d_qt \\ &\quad + \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} f_o(t)\operatorname{Sin}(tx(1-q);q^2)\,d_qt. \end{aligned}$$

Hence applying (3.82)–(3.83) we obtain

$$\begin{aligned} f(t) &= \frac{f_e(t)+f_o(t)}{2} \\ &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)} \int_0^{\infty/\sqrt{1-q}} \Big(f_e(x)\operatorname{Cos}(xt(1-q);q^2) + f_o(x)\operatorname{Sin}(xt(1-q);q^2)\Big)\,d_qx \\ &= \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(x)e(itx(1-q);q^2)\,d_qx. \end{aligned}$$

□

Similar to Theorem 3.15 we have the following theorem which we state without proof.

Theorem 3.17. *If $0 < q < 1$, then the following properties of the q^2-Fourier transform are satisfied.*

(1) If $f(u)$ and $uf(u) \in L_q^1(\widetilde{\mathbb{R}}_q)$, then

$$\partial_q \left(\mathscr{F}_q f\right)(x) = \mathscr{F}_q(-iuf(u))(x).$$

(2) If f and $\partial_q f \in L_q^1(\widetilde{\mathbb{R}}_q)$, then

$$\mathscr{F}_q(\partial_q f)(x) = ix\mathscr{F}_q\,(f)(x). \tag{3.87}$$

Now we prove the following orthogonality relations for any $q \in (0, 1)$.

Theorem 3.18. *For $0 < q < 1$*

$$\int_0^{\infty/\sqrt{1-q}} \operatorname{Sin}(\frac{q^n}{\sqrt{1-q}}z; q^2) \operatorname{Sin}(\frac{q^m}{\sqrt{1-q}}z; q^2)\, d_qz = \frac{\sqrt{1-q}(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} q^{-n}\delta_{n,m}, \tag{3.88}$$

$$\int_0^{\infty/\sqrt{1-q}} \operatorname{Cos}(\frac{q^n}{\sqrt{1-q}}z; q^2) \operatorname{Cos}(\frac{q^m}{\sqrt{1-q}}z; q^2)\, d_qz = \frac{\sqrt{1-q}(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} q^{-n}\delta_{n,m}, \tag{3.89}$$

and

$$\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e(i\frac{q^n}{\sqrt{1-q}}z; q^2) e(-i\frac{q^m}{\sqrt{1-q}}z; q^2)\, d_qz = 4\frac{\sqrt{1-q}(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} q^{-n}\delta_{n,m}. \tag{3.90}$$

Proof. We start with proving (3.88). Since

$$\begin{aligned}
&\int_0^{\infty/\sqrt{1-q}} \operatorname{Sin}(\frac{q^n}{\sqrt{1-q}}z; q^2) \operatorname{Sin}(\frac{q^m}{\sqrt{1-q}}z; q^2)\, d_qz \\
&= \sum_{k=-\infty}^{\infty} q^k \sqrt{1-q} \operatorname{Sin}(\frac{q^{n+k}}{1-q}; q^2) \operatorname{Sin}(\frac{q^{m+k}}{1-q}; q^2) \\
&= \frac{q^{n+m}}{(1-q)^{3/2}} \sum_{-\infty}^{\infty} q^{3k}\, {}_1\phi_1(0; q^3, q^2; q^{2+2n+2k})\, {}_1\phi_1(0; q^3, q^2; q^{2+2m+2k}).
\end{aligned} \tag{3.91}$$

In (3.85), set $\alpha = 3/2$ to obtain

$$\sum_{-\infty}^{\infty} q^{3k} {}_1\phi_1(0; q^3; q^2, q^{2+2n+2k}) {}_1\phi_1(0; q^3, q^2; q^{2+2m+2k})$$
$$= q^{-3(n+m)/2} \frac{(q^2; q^2)_\infty^2}{(q^3; q^2)_\infty^2} \delta_{n,m}. \tag{3.92}$$

Combining (3.91) and (3.92) yield (3.88). The proof of (3.89) follows similarly and the proof of (3.90) follows by combining (3.88) and (3.89). □

Theorem 3.19. *For any $q \in (0,1)$, the set $\left\{e(i \frac{q^n}{\sqrt{1-q}} x; q^2),\ n \in \mathbb{Z}\right\}$ is a complete orthogonal sets in $L_q^2(\widetilde{\mathbb{R}}_q)$.*

Proof. To prove that $\left\{e(i \frac{q^n}{\sqrt{1-q}} x; q^2),\ n \in \mathbb{Z}\right\}$ is complete in $L_q^2(\widetilde{\mathbb{R}}_q)$, it is equivalent to prove that if there exists a function $f \in L^2(\widetilde{\mathbb{R}}_q)$ such that

$$\langle f, e(i \frac{q^n}{\sqrt{1-q}} x; q^2)\rangle = 0 \quad \text{for all } n \in \mathbb{Z}, \tag{3.93}$$

then

$$f\left(\frac{iq^n}{1-q}\right) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

From (3.93) we deduce

$$\int_0^{\infty/\sqrt{1-q}} f_e(t) \operatorname{Cos}\left(\frac{q^n}{\sqrt{1-q}} t; q^2\right) d_q t = 0 \quad \text{for all } n \in \mathbb{Z},$$

$$\int_0^{\infty/\sqrt{1-q}} f_o(t) \operatorname{Sin}\left(\frac{q^n}{\sqrt{1-q}} t; q^2\right) d_q t = 0 \quad \text{for all } n \in \mathbb{Z},$$

where f_e and f_o are the even and the odd parts of the function f. Then from (3.82)–(3.83), we obtain

$$f_e(t) = f_o(t) = f(t) = 0 \quad \text{for all} \quad t \in \widetilde{\mathbb{R}}_q.$$

Hence, $\left\{e(i \frac{q^n}{\sqrt{1-q}} x; q^2),\ n \in \mathbb{Z}\right\}$ is a completer orthogonal set in $L^2(\widetilde{\mathbb{R}}_q)$. □

In the following we generalize the definitions of q^2-Fourier Multiplier and the q^2-Fourier convolution introduced by Rubin in [266] with the restriction (3.72) to any $q \in (0,1)$. Rubin defined the q^2-Fourier multiplier operator corresponding to translation by y to be

$$(T_y f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty}^{\infty} e(-ity; q^2)(\mathscr{F}_q f)(t) e(itx; q^2)\, d_q t,$$

whenever the q-integral makes sense and whenever q satisfies the condition (3.72).

If $f \in L^2(\mathbb{R}_q)$ and $g \in L^1(\mathbb{R}_q)$, Rubin defined the multiplier corresponding to q^2-Fourier convolution of f with g to be

$$(f * g)(z) = \int_{-\infty}^{\infty} [T_\lambda f](z) g(\lambda)\, d_q\lambda,$$

and he finally proved the following convolution formula

$$f, g \in \left(L^1 \cap L^2\right)(\mathbb{R}_q) \longrightarrow \mathscr{F}_q(f * g)(x) = \mathscr{F}_q(f)(x)\mathscr{F}_q(g)(x). \qquad (3.94)$$

Now we have the following generalizations:

Definition 3.6.2. Let $q \in (0,1)$. We define the q^2-Fourier multiplier operator corresponding to translation by y to be

$$(T_y f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e(-ity;q^2)(\mathscr{F}_q f)(t) e(itx;q^2)\, d_q t,$$

whenever the q-integral makes sense. If $f \in L^2(\widetilde{\mathbb{R}}_q)$ and $g \in L^1(\widetilde{\mathbb{R}}_q)$, we define the multiplier corresponding to Fourier convolution of f with g to be

$$(f * g)(z) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} [T_\lambda f](z) g(\lambda)\, d_q\lambda.$$

Theorem 3.20. *Let f and g be two functions in $\left(L^1 \cap L^2\right)(\widetilde{\mathbb{R}}_q)$. Then*

$$\mathscr{F}_q(f * g)(x) = \mathscr{F}_q(f)(x)\mathscr{F}_q(g)(x) \quad \left(x \in \widetilde{\mathbb{R}}_q\right). \qquad (3.95)$$

Proof. The proof of (3.95) is completely similar to the proof of (3.94) in [266, Theorem 8] and is omitted. □

Chapter 4
Riemann–Liouville *q*-Fractional Calculi

Abstract In this chapter we investigate q-analogues of the classical fractional calculi. We study the q-Riemann–Liouville fractional integral operator introduced by Al-Salam (Proc. Am. Math. Soc. 17, 616–621, 1966; Proc. Edinb. Math. Soc. 2(15), 135–140, 1966/1967) and by Agarwal (Proc. Camb. Phil. Soc. 66, 365–370, 1969). We give rigorous proofs of existence of the fractional q-integral and q-derivative. Therefore we establish a q-analogue of Abel's integral equation and its solutions.

4.1 Classical Fractional Calculi

In this section, we introduce some classical fractional calculi which we will consider their q-analogues. We also state a set of relations and properties of these fractional calculi for which we investigate their q-counterparts. First we denote by $\mathscr{A}C^{(n)}[a,b]$, $n \in \mathbb{N}$, to the set of functions f which have continuous derivatives up to order $n-1$ on $[a,b]$ with $f^{(n-1)} \in \mathscr{A}C[a,b]$, i.e. absolutely continuous on $[a,b]$. Functions of $\mathscr{A}C[a,b]$ can be characterized via the following Taylor's formula taken from [175].

Lemma A. The set $\mathscr{A}C^{(n)}[a,b]$ consists of those, and only those, functions f, which are represented in the form

$$f(x) = \sum_{k=0}^{n-1} c_k (x-a)^k + \frac{1}{n-1!}\int_a^x (x-t)^{n-1}\phi(t)\,dt,$$

where $\phi \in L_1(a,b)$ and the c_k's are arbitrary constants. Moreover, $\phi(x) = f^{(n)}(x)$ a.e., and $c_k = \dfrac{f^{(k)}(a)}{k!}$, $k = 0, 1, \ldots, n-1$.

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The first fractional integral operator we define in this section is that of Riemann and Liouville. It is connected to Abel's integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\phi(t)\,dt = f(x),\ x>a,\ \alpha>0,\quad f\in L_1(a,b), \tag{4.1}$$

cf. [269, P. 32], see also [213,227,262]. The following theorem, cf. [118,269], solves Abel's equation concretely.

Theorem 4.1. *The Abel's integral equation* (4.1) *with* $0<\alpha<1$ *has a unique solution in* $L_1(a,b)$ *if and only if the function* $f_{1-\alpha}$ *defined by*

$$f_{1-\alpha}(x) := \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} f(t)\,dt$$

is absolutely continuous on $[a,b]$ *and* $f_{1-\alpha}(a)=0$. *If these later conditions are fulfilled, then the unique solution* ϕ *is given explicitly by*

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)\,dt = \frac{d}{dx} f_{1-\alpha}(x),\ \textit{a.e.} \tag{4.2}$$

If $f\in\mathscr{A}C[a,b]$, then $f_{1-\alpha}\in\mathscr{A}C[a,b]$ and (4.2) becomes

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(s)}{(x-s)^{\alpha}}\,ds\Big].$$

The following Cauchy formulae can be considered as the nth primitive of a function $f\in L_1(a,b)$.

$$\int_a^x\int_a^{x_{n-1}}\ldots\int_a^{x_1} f(t)\,dt\,dx_1\ldots dx_{n-1} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt, \tag{4.3}$$

$$\int_x^b\int_{x_{n-1}}^b\ldots\int_{x_1}^b f(t)\,dt\,dx_1\ldots dx_{n-1} = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)\,dt, \tag{4.4}$$

$n\in\mathbb{N}$. Since the right hand sides of (4.3) and (4.4) exist also for non integer values of n, the Riemann–Liouville fractional integral can be considered as an extension of (4.3) and (4.4) when we replace n by $\alpha\in\mathbb{R}^+$ and $n-1!$ by $\Gamma(\alpha)$. Indeed, for $\alpha\in(0,\infty)$ and $f\in L_1(a,b)$, the fractional Riemann–Liouville integral is defined to be

$$\begin{aligned} I_{a+}^{\alpha} f(x) &:= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt,\\ I_{b-}^{\alpha} f(x) &:= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \end{aligned} \tag{4.5}$$

$x \in (a,b)$ with respect to $x = a$ and $x = b$, respectively. In some literature, they are called left-sided and right-sided Riemann–Liouville fractional integrals, respectively. It is known, see e.g [211, 227, 235, 269], that if $f \in L_1(a,b)$, then both of $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ exist a.e. and they are $L_1(a,b)$-functions. Moreover, for $f \in L_1(a,b)$, we have

$$\lim_{\alpha\to 0+} I^{\alpha}_{a+}f(x) = \lim_{\alpha\to 0+} I^{\alpha}_{b-}f(x) = f(x) \quad \text{a.e.} \tag{4.6}$$

We now pass to the definition of the fractional derivative of arbitrary order. The existence of the fractional derivative is connected to the solvability of the Abel's integral equation (4.1). For $f \in L_1(a,b)$, the left and right sided Riemann–Liouville fractional derivatives of order α, $\alpha \in \mathbb{R}^+$, are defined formally by

$$D^{\alpha}_{a+}f(x) := D^k I^{k-\alpha}_{a+}f(x) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dx^k}\int_a^x (x-t)^{k-\alpha-1}f(t)\,dt, \tag{4.7}$$

and

$$\begin{aligned} D^{\alpha}_{b-}f(x) &:= (-1)^k D^k I^{k-\alpha}_{b-}f(x) \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dx^k}\int_x^b (t-x)^{k-\alpha-1}f(t)\,dt, \end{aligned} \tag{4.8}$$

$x \in (a,b)$, $k = \lceil\alpha\rceil$, respectively, $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α. The fractional derivatives $D^{\alpha}_{a+}f$, $D^{\alpha}_{b-}f$ exist if $f \in L_1(a,b)$ and $I^{k-\alpha}_{a+}f$, $I^{k-\alpha}_{b-}f$ are of the class $\mathscr{A}C^{(k)}[a,b]$, see [269]. For example, the Riemann–Liouville fractional derivatives of x^c, $c \in \mathbb{R}$, is given by

$$D^{\alpha}_{a+}x^c = \frac{\Gamma(c+1)}{\Gamma(c-\alpha+1)}x^{c-\alpha} \quad (\alpha \ge 0)\,. \tag{4.9}$$

Of course if $\alpha < 0$, (4.9) gives the Riemann–Liouville fractional integral value of x^c.

Now we state some properties of the Riemann–Liouville fractional calculus for which we derive their q-analogues. We confine ourselves to the case of the left-sided Riemann–Liouville fractional calculus since we shall study its basic analogue. Let α, $\beta \in \mathbb{R}^+$. If $f \in L_1(a,b)$, then the semigroup property

$$I^{\beta}_{a+}I^{\alpha}_{a+}f(x) = I^{\alpha}_{a+}I^{\beta}_{a+}f(x) = I^{\alpha+\beta}_{a+}f(x), \tag{4.10}$$

holds for almost all $x \in [a,b]$. If $f(x)$ satisfies the conditions

$$f \in L_1(a,b) \ \text{ and } \ I^{k-\alpha}_{a+}f \in \mathscr{A}C^{(k)}[a,b], \ \ k = \lceil\alpha\rceil\,,$$

then

$$\begin{cases} D_{a+}^{\alpha-j} f \in L_1(a,b), & j = 0,1,\ldots,k, \\ \\ D_{a+}^{\alpha-j} f \in AC^{(j)}[a,b], & j = 1,2,\ldots,k-1. \end{cases} \tag{4.11}$$

Moreover, for functions satisfying the appropriate mentioned conditions

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) = f(x), \quad \text{a.e.} \tag{4.12}$$

$$D_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\beta-\alpha} f(x), \quad \text{a.e. if } \beta \geqslant \alpha \geqslant 0, \tag{4.13}$$

$$D_{a+}^{\alpha} I_{a+}^{\beta} f(x) = D_{a+}^{\alpha-\beta} f(x), \quad \text{a.e. if } \alpha > \beta \geqslant 0. \tag{4.14}$$

In addition,

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) - \sum_{j=1}^{k} D_{a+}^{\alpha-j} f(a^+) \frac{(x-a)^{\alpha-j}}{\Gamma(1+\alpha-j)}. \tag{4.15}$$

If $f \in L_1(a,b)$ and $D_{a+}^{-(n-\beta)} f \in \mathscr{AC}^{(n)}[a,b]$, $n = \lceil \beta \rceil$, then the equality

$$I_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\beta-\alpha} f(x) - \sum_{j=1}^{n} D_{a+}^{\beta-j} f(a^+) \frac{(x-a)^{\alpha-j}}{\Gamma(1+\alpha-j)} \tag{4.16}$$

holds almost everywhere in (a,b) for any $\alpha > 0$. Moreover, if

$$0 \leqslant k-1 \leqslant \alpha < k, \ \alpha + \beta < k, \ \text{and } D_{a+}^{-(n-\alpha)} f \in \mathscr{AC}^{(n)}[a,b],$$

then

$$D_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_{a+}^{\beta-j} f(a^+) \frac{(x-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \tag{4.17}$$

holds almost everywhere in (a,b). Finally,

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) \ \text{ and } \ I_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\beta-\alpha} f(x),$$

whenever $f \in I_{a+}^{\alpha}(L_1)$ and $f \in I_{a+}^{\beta}(L_1)$, respectively, cf. [269]. Here, by $I_{a+}^{\alpha}(L_1)$ we denote the set of functions f represented by the left sided fractional integral of order α ($\alpha > 0$) of a summable function, i.e. $f = I_{a+}^{\alpha}\phi$, $\phi \in L_1(a,b)$. For proofs of these and other properties, see e.g. [85, 235, 269].

The other types of fractional derivatives we discuss here are those of Grünwald–Letnikov and Caputo. It is known that the nth derivative of a function f defined on (a,b) is given by

$$f^{(n)}(x) = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n}, \tag{4.18}$$

where

$$\Delta_h^n f(x) = \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x - rh).$$

Starting from this formula, Grünwald [120] and Letnikov [180] developed an approach to fractional differentiation for which the formal definition of the fractional derivative $\mathbf{D}_{a+}^{\alpha} f(x)$ is the limit

$$\mathbf{D}_{a+}^{\alpha} f(x) := \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh),\ \alpha \in \mathbb{R}^+. \tag{4.19}$$

Grünwald in [120] gave a formal proof while Letnikov in [180] gave a rigorous proof of the fact that $\mathbf{D}^{-\alpha}$ coincides with the Riemann–Liouville fractional integral operator when $\alpha > 0$ and f is continuous on $[a, b]$, cf. [178]. The following theorem gives the conditions on which the Grünwald–Letnikov fractional derivative is nothing but the Riemann–Liouville fractional derivative.

Theorem 4.2 ([235]). *Assume that* $f \in \mathscr{AC}^{(n)}[a, b]$. *Then for every* α, $0 < \alpha < n$, *the Riemann–Liouville fractional derivative* $D_{a+}^{\alpha} f(t)$ *exists and coincides with the Grünwald–Letnikov derivative* $\mathbf{D}_{a+}^{\alpha} f(t)$, *i.e. if*

$$0 \leqslant m - 1 \leqslant \alpha \leqslant m \leqslant n$$

then for $a < t < b$ *the following holds:*

$$\begin{aligned}\mathbf{D}_{a+}^{\alpha} f(t) = D_{a+}^{\alpha} f(t) &= \sum_{j=0}^{m-1} \frac{f^{(j)}(a^+)(t-a)^{j-\alpha}}{j!} \\ &+ \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-1-\alpha} f^{(m)}(\tau)\, d\tau.\end{aligned} \tag{4.20}$$

It is known that in applied problems, we require definitions of fractional problems which allow the involvement of initial conditions with physical meanings, i.e. conditions like

$$f^{(j)}(a^+) = b_j, \quad j = 1, 2, \ldots, k, \tag{4.21}$$

where the b_j's and $k \in \mathbb{N}$ are given constants. The Riemann–Liouville approach leads to initial conditions containing the limit values of the Riemann–Liouville fractional derivatives at the lower end point $t = a$, for example

$$\lim_{t\to a+} D_{a+}^{\alpha-j} f(t) = c_j, \quad j = 1, 2, \ldots, k, \tag{4.22}$$

where the c_j's and $k \in \mathbb{N}$ are given constants and α is not an integer. It is frequently stated that the physical meaning of initial conditions of the form (4.22) is unclear or even non existent, see e.g. [269, P. 78]. The requirement for physical interpretation of such initial conditions was most clearly formulated recently by Diethelm et al. [83]. In [134], Heymans and Podlubny show that initial conditions of the form (4.22) for Riemann–Liouville fractional differential equations may have physical meanings, and that the corresponding quantities can be obtained from measurements. However, the problem of interpretation of initial conditions of the type (4.22) remains open. In [69, 70], Caputo gave a solution for this problem when he defined the fractional derivative of order α, ${}_a^C D_t^\alpha f(t)$, by the formula

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{4.23}$$

$n-1 < \alpha \leqslant n, n \in \mathbb{N}$. The Caputo approach leads to initial conditions of the form (4.21) which are physically accepted. Recently, several initial value problems based on the Caputo fractional operator have been studied, see e.g. [90, 91, 233, 236]. The relationship between the Caputo fractional derivative and the Riemann–Liouville fractional operator is given by

$$ {}_a^C D_t^\alpha f(t) = I_{a+}^{n-\alpha} f^{(n)}(t).$$

If the function $f(t)$ has $n+1$ continuous derivatives in $[a,b]$, then cf. [235],

$$\lim_{\alpha \to n} {}_a^C D_t^\alpha f(t) = f^{(n)}(t) \quad \text{for } t \in [a,b].$$

Another advantage of the Caputo fractional derivative is that ${}_a^C D_t^\alpha c = 0$, where c is a constant, while the Riemann–Liouville fractional derivative of a constant c is

$$ D_{a+}^\alpha c = D^n I_{a+}^{n-\alpha} c = \frac{c}{\Gamma(n-\alpha+1)} D^n (x-a)^{n-\alpha} = \frac{c}{\Gamma(1-\alpha)} (x-a)^{-\alpha},$$

where $n = \lceil \alpha \rceil$. Since we will establish a method for using the q-Laplace transform defined by Hahn in [123] to solve fractional q-difference equations, we state some results concerning the Laplace transform before closing this section.

Let $f \in L_1(0,b)$. If $I_{a+}^{n-\alpha} f \in \mathscr{A}C^{(n)}[0,b]$ for any $b > 0$ such that $I_{a+}^{n-\alpha} f(x)$ is of exponential order as $x \to \infty$. That is, there exist nonnegative constants K, T and $p_0 \geqslant 0$, such that

$$ |I_{a+}^{n-\alpha} f(x)| \leqslant K e^{p_0 x} \quad (x > T)\,.$$

Then the Laplace transform of the Riemann–Liouville fractional derivative $D_{a+}^\alpha f(x)$ exists, cf. [291], and satisfies the following identity

$$\begin{aligned}\mathscr{L}(D^{\alpha}_{a+}f)(p) &= \int_0^{\infty} e^{-pt} D^{\alpha}_{a+} f(t)\,dt \\ &= p^{\alpha}F(p) - \sum_{k=0}^{n-1} p^k D^{\alpha}_{a+} f(0^+) \quad (n = \lceil \alpha \rceil)\,,\end{aligned}$$

for all p that satisfies $\operatorname{Re}(p) > p_0$, cf. [269, 291].

Similarly, the Laplace transform of the Caputo derivative ${}^{C}_{a}D^{\alpha}_{t} f(x)$ exists if $f(x) \in \mathscr{A}C^{(n)}[0,b]$ for every $b > 0$ and $f(x)$ is of exponential order as $x \to \infty$. In this case we can find nonnegative real number p_0 such that the Laplace transform of the Caputo derivative satisfies the identity

$$\begin{aligned}\mathscr{L}({}^{C}_{0}D^{\alpha}_{t} f)(p) &= \int_0^{\infty} e^{-pt}\,{}^{C}_{0}D^{\alpha}_{t} f(t)\,dt \\ &= p^{\alpha}F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0),\end{aligned}$$

for all p satisfying $\operatorname{Re} p > \operatorname{Re}(p_0)$. The Laplace transform method is usually used for solving applied problems. So, it is clear that the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivative with known physical interpretations, while the Laplace transform of the Riemann–Liouville fractional derivative allows utilization of initial conditions of the form

$$D^{\alpha-j} f(t)|_{t=0} = b_j \qquad (j = 1, 2, \ldots, \lceil \alpha \rceil)\,,$$

which may form problems without known physical interpretations.

4.2 Recent History of Fractional q-Calculus

Over the last decades, fractional calculus became an area of intense research and developments. There are many surveys for the history of the fractional calculus, see the survey of Butzer and Westphal [68], the survey of Machado et al. [197] reported some of the major documents and events in the area of fractional calculus that took place since 1974 up to 2010. They also introduced a poster [196] illustrates the major contribution during the period 1966–2010. To the best of our knowledge, the recent developments in the theory of fractional q-calculus is not well reported. Therefore, we aim to cover in brief in this section the recent developments in the theory of fractional q-calculus. A q-analogue of the Riemann–Liouville fractional integral operator is introduced in [19] by Al-Salam through

$$I^{\alpha}_{q} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x;q\right)_{\alpha-1} f(t)\,d_qt, \tag{4.24}$$

$\alpha \notin \{-1,-2,\ldots\}$. Al-Salam defined this q-analogue as an extension of the following q-Cauchy formula who introduced in[18]

$$\begin{aligned} I_{q,a}^{n} f(x) &:= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(t)\, d_q t\, d_q x_1 \ldots d_q x_{n-1} \\ &= \frac{x^{n-1}}{\Gamma_q(n)} \int_a^x \big(qt/x;q\big)_{n-1} f(t)\, d_q t. \end{aligned} \tag{4.25}$$

For the convenience of the reader, we shall use the notation $I_q^n f(x)$ instead of $I_{q,0}^n f(x)$. This basic analogue of Riemann–Liouville fractional integral is also given independently by Agarwal, [17], who defined the q-fractional derivative to be

$$D_q^\alpha f(x) := I_q^{-\alpha} f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_q t.$$

Using the series representation of the q-integration (1.19), identity (4.24) reduces to

$$I_q^\alpha f(x) = x^\alpha (1-q)^\alpha \sum_{n=0}^{\infty} q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n). \tag{4.26}$$

Al-Salam, [19], defined a fractional q-integral operator $K_q^{-\alpha}$ by

$$\begin{aligned} &K_q^{-\alpha}\phi(x) := \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_x^\infty t^{\alpha-1} \big(x/t;q\big)_{\alpha-1} \phi(tq^{1-\alpha})\, d_q t, \\ &K_q^0 \phi(x) := \phi(x), \end{aligned} \tag{4.27}$$

where $\alpha \neq -1, -2, \ldots$ as a generalization of the q-Cauchy formula

$$\begin{aligned} K_q^{-n}\phi(x) &= \int_x^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_1}^\infty \phi(t)\, d_q t\, d_q x_1 \ldots d_q x_{n-1} \\ &= \frac{q^{-\frac{1}{2}n(n-1)}}{\Gamma_q(n)} \int_x^\infty t^{n-1}\, (x/t;q)_{n-1}\, \phi(tq^{1-n})\, d_q t, \end{aligned}$$

which he introduced in [18] for a positive integer n. The fractional q-integral operator $K_q^{-\alpha}$ is a q-analogue of Liouville fractional integral, K_-^α, defined by

$$K_-^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\phi(t)}{(t-x)^{1-\alpha}}\, dt \quad (x > 0;\ \mathrm{Re}(\alpha) > 0). \tag{4.28}$$

Al-Salam formally proved the following semigroup identity, cf. [19],

$$K_q^\alpha K_q^\beta \phi(x) = K_q^{\alpha+\beta}\phi(x),$$

for all α and β. It is remarked by Ismail, cf. [143, P. 553] that

$$\int_0^\infty f(x)K_q^{-\alpha}g(x)\,d_qx = \int_0^\infty g(xq^{-\alpha})I_q^\alpha f(x)\,d_qx. \tag{4.29}$$

There is a slight error on (4.29) as we shall see in Theorem 5.14. Al-Salam and Agarwal introduced their q-analogue with only zero as a lower point of the q-integration. Recently, Rajović et al. [254, 255] allowed the lower point of the q-integration in (4.24) to be nonzero and introduced the fractional definition:

Definition 4.2.1. The fractional q-integral is

$$I_{q,c}^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_c^x (qt/x;q)_{\alpha-1} f(t)\,d_qt,$$

and the fractional q-derivative is

$$D_{q,c}^\alpha f(x) = D_q^{\lceil\alpha\rceil} I_{q,c}^{\alpha-\lceil\alpha\rceil} f(x).$$

Rajvoicć et al. [255] proved the following properties

Proposition 4.3. *Let $\alpha,\beta \in \mathbb{R}^+$. The q-fractional integration has the following semigroup property*

$$I_{q,c}^\alpha I_{q,c}^\beta f(x) = I_{q,c}^{\alpha+\beta} f(x).$$

Proposition 4.4. *For $\alpha \in \mathbb{R}^+$*

$$D_{q,c}^\alpha I_{q,c}^\alpha f(x) = f(x).$$

Al-Salam [19] and Agarwal [17] defined a two parameter q-fractional operator by

$$K_q^{\eta,\alpha}\phi(x) := \frac{q^{-\eta}x^\eta}{\Gamma_q(\alpha)}\int_x^\infty \big(x/t;q\big)_{\alpha-1} t^{-\eta-1}\phi(tq^{1-\alpha})\,d_qt, \tag{4.30}$$

$\alpha \neq -1,-2,\ldots$. This is a q-analogue of the Erdélyi and Sneddon fractional operator, cf. [93, 96],

$$K^{\eta,\alpha}\phi(x) = \frac{x^\eta}{\Gamma(\alpha)}\int_x^\infty \big(t-x\big)^{\alpha-1} t^{-\eta-1}\phi(t)\,dt. \tag{4.31}$$

By means of (1.20), (4.30) can be written as

$$K_q^{\eta,\alpha}\phi(x) = (1-q)^\alpha\sum_{k=0}^\infty q^{k\eta}\frac{(q^\alpha;q)_k}{(q;q)_k}\phi(xq^{-\alpha-k}).$$

Al-Salam derived formally the following semigroup identity for η, α and β in $\mathbb{R}^+$.

$$K_q^{\eta,\alpha} K_q^{\eta+\alpha,\beta} f(x) = K_q^{\eta,\alpha+\beta} f(x).$$

Al-Salam did not consider the problem of the existence of $K_q^{-\alpha}$, $K_q^{\eta,\alpha}\phi$. In [112], Galue generalizes the Erdélyi–Kober fractional q-integral operator of arbitrary order α introduced by Agarwal in [17] as follows

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta)}}{\Gamma_q(\eta)} \int_0^x \left(qt^\beta/x^\beta; q\right)_{\mu-1} t^{\beta(\eta+1)-1} f(t)\, d_q t,$$

$$\operatorname{Re}(\beta) > 0,\ \operatorname{Re}(\mu) > 0,\ \eta \in \mathbb{C}.$$

Then, he investigated various rules of composition for the above-defined operator.

Recently, Purohit and Yadav in [250] introduced two new fractional q-integral operators by

$$\begin{aligned} I_q^{\alpha,\beta,\eta} f(x) = {} & \frac{x^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \\ & \times \int_0^x (qt/x;q)_{\alpha-1} \varepsilon^{-\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q\right)\right] \\ & \times f(t)\, d_q t, \end{aligned} \tag{4.32}$$

$$\begin{aligned} K_q^{\alpha,\beta,\eta} f(x) = {} & \frac{q^{-\eta(\alpha+\beta)-\frac{\alpha(\alpha+1)}{2}-2\beta}}{\Gamma_q(\alpha)} \\ & \times \int_x^\infty (x/t;q)_{\alpha-1} t^{-\beta-1} \varepsilon^{-\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q\right)\right] \\ & \times f(tq^{1-\alpha})\, d_q t \end{aligned} \tag{4.33}$$

where $\alpha > 0$, β is a real number, η is a nonnegative real number, and ε is the q-translation operator defined in (1.15). The series representations of the operators in (4.32) and (4.33) are

$$I_q^{\alpha,\beta,\eta} f(x) = x^{-\beta} q^{-\eta(\alpha+\beta)} (1-q)^\alpha \times$$

$$\sum_{n=0}^{\eta} q^n \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+n};q)_k}{(q;q)_k} f(xq^k),$$

and

$$K_q^{\alpha,\beta,\eta} = x^{-\beta} q^{-\eta(\alpha+\beta)-\frac{\alpha(\alpha+1)}{2}-\beta}(1-q)^{\alpha}\times$$
$$\sum_{n=0}^{\eta} q^n \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} \sum_{k=0}^{\infty} q^{k\beta} \frac{(q^{\alpha+n};q)_k}{(q;q)_k} f(xq^{-\alpha-k}).$$

It is straightforward to see that Purohit and Yadav integral operators can be regarded as extensions of Riemann–Liouville, Weyl and Kober fractional q-integral operators with the following fractional relations

$$\begin{aligned} I_q^{\alpha,0,\eta} &= I_q^{\alpha}, \\ K_q^{\alpha,0,\eta} &= q^{-\alpha(\alpha+1)/2} K_q^{\eta,\alpha}, \\ K_q^{\alpha,-\alpha,\eta} &= K_q^{\alpha}. \end{aligned}$$

Purohit and Yadav derived the fractional q-integration by parts formula:

$$\int_0^{\infty} f(x) K_q^{\alpha,\beta,\gamma} g(x)\, d_q x = q^{-\alpha(\alpha+1)/2-\beta} \int_0^{\infty} g(xq^{-\alpha}) I_q^{\alpha,\beta,\gamma} f(x)\, d_q x, \quad (4.34)$$

for $\alpha > 0$, β is real, and η is a nonnegative integer, provided that both of the q-integrals in (4.34) exist. The operators (4.32) and (4.33) are q-extensions of the operators

$$I_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha,1-\frac{t}{x}\right) f(t)\, dt, \quad (4.35)$$

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta,-\eta;\alpha,1-\frac{t}{x}\right) f(t)\, dt,$$

where ${}_2F_1$ is the Gaussian hypergeometric functions defined by

$${}_2F_1(a,b;c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

4.2.1 The Time Scale Fractional q-Calculus

The time scale fractional calculus was introduced by Hilger and Aulbad in [46, 135] then developed by Bohner and Petrson in [59–61]. It merges the theory of difference equations and the theory of differential equations into one theory. A time scale means a closed subset of the real line. A merger of the Riemann–Liouville fractional q-calculus and Riemann–Liouville fractional q-calculus on time scale was initiated in the paper of Atici and Eloe in [44, 2007]. They use the time scale calculus notations to develop Hahn q-Laplace transform operator, ${}_qL_s$, on the time scale $\mathbb{T} := \{q^n : n \in \mathbb{N}_0\} \cup \{0\}$ and define a fractional q-difference equation on $\mathbb{T}$ and finally apply the q-transform method to find solutions. A development for the Caputo fractional q-derivative in the time scale fractional calculus can be found in [2].

4.3 q-Abel Integral Equation

In this section, we define a q-analogue of Abel's integral equation. So we need the q-analogues of the classical classes $L_1(a,b)$ and $\mathscr{A}C[a,b]$ introduced in Sect. 1.2. In Sect. 2.2, we introduced a q-analogue of the class of integrable functions. Now, we introduce a q-analogue of $\mathscr{A}C[0,a]$ and we denote it by $\mathscr{A}C_q[0,a]$.

Definition 4.3.1. A function f defined on $[0,a]$ is called q-absolutely continuous if f is q-regular at zero, and there exists $K > 0$ such that

$$\sum_{j=0}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| \leq K \quad \text{for all } \ t \in (qa, a]. \tag{4.36}$$

If (4.36) holds then it can be extended throughout $(0,a]$. To see this, it suffices to investigate the case when $x \in (0,a]$. Indeed, if $x \in (0,a]$ then there exists $t \in (qa,a]$ and $k \in \mathbb{N}$ such that $x = tq^k$. Then

$$\begin{aligned} \sum_{j=0}^{\infty} \left| f(xq^j) - f(xq^{j+1}) \right| &= \sum_{j=k}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| \\ &\leqslant \sum_{j=0}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| < \infty. \end{aligned}$$

We shall use $\mathscr{A}C_q[0,a]$ to denote the class of q-absolutely continuous functions on $[0,a]$.

Theorem 4.5. *Let f be a function defined on $[0,a]$. Then, the function $f \in \mathscr{A}C_q[0,a]$ if and only if there exists a constant c and a function ϕ in $\mathscr{L}_q^1[0,a]$ such that*

$$f \in \mathscr{A}C_q[0,a] \Longleftrightarrow f(x) = c + \int_0^x \phi(u)\, d_qu \quad \textit{for all } x \in [0,a]. \tag{4.37}$$

Moreover, the constant c and the function ϕ are uniquely determined via $c = f(0)$ and

$$\phi(x) = D_q f(x) \quad \textit{for all } \ x \in (0,a].$$

Proof. Assume that $f \in \mathscr{A}C_q[0,a]$. Consequently, from the fundamental theorem of q-calculus, Theorem 1.10, we obtain

$$f(x) = f(0) + \int_0^x D_q f(t)\, d_q t, \quad x \in [0,a],$$

proving necessity. Assume that f is given by (4.37) for $\phi \in \mathscr{L}_q^1[0,a]$. Then for $x \in [0,a]$, we have

$$\lim_{n\to\infty} f(xq^n) = \lim_{n\to\infty} \left(c + x(1-q) \sum_{k=n}^{\infty} q^k \phi(xq^k) \right) = c.$$

Therefore, the function f is q-regular at zero. Applying Theorem 1.10 again, we obtain $D_q f(x) = \phi(x)$, $x \in (0,a]$. Hence, $D_q f(x) \in \mathscr{L}_q^1[0,a]$. This directly leads to (4.36), proving sufficiency. The uniqueness of c and ϕ holds by construction. ☐

Definition 4.3.2. Let $\mathscr{A}C_q^{(n)}[0,a]$, $n \in \mathbb{N}$, be the space of all functions f defined on $[0,a]$ such that f, $D_q f$, …, $D_q^{n-1} f$ are q-regular at zero and $D_q^{n-1} f(x) \in \mathscr{A}C_q[0,a]$.

When $n = 1$, we simply write $\mathscr{A}C_q[0,a]$ for $\mathscr{A}C_q^{(1)}[0,a]$.

Theorem 4.6. *A function $f : [0,a] \to \mathbb{C}$ lies in $\mathscr{A}C_q^{(n)}[0,a]$ if and only if there exists a function $\phi \in \mathscr{L}_q^1[0,a]$ such that*

$$f(x) = \sum_{k=0}^{n-1} c_k x^k + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x \big(qt/x;q\big)_{n-1} \phi(t)\, d_q t, \tag{4.38}$$

where

$$x \in (0,a], \quad n \in \mathbb{N}, \quad c_k = \frac{D_q^k f(0)}{\Gamma_q(k+1)},$$

and $\phi(x) = D_q^n f(x)$ for all $x \neq 0$.

Proof. First, we prove sufficiency. Assume that f has the form (4.38). Then, the q-derivative of order j for $x \neq 0$ is

$$\begin{aligned} D_q^j f(x) &= \sum_{k=j}^{n-1} c_k (1-q^k)(1-q^{k-1}) \ldots (1-q^{k-j+1})(1-q)^{-j} x^{k-j} \\ &\quad + \frac{x^{n-j-1}}{\Gamma_q(n-j)} \int_0^x (qt/x;q)_{n-j-1} \phi(t)\, d_q t, \end{aligned}$$

where $j = 0, 1, \dots, n-1$. We can prove by induction that

$$D_q^j f(0) = c_j \frac{(q;q)_j}{(1-q)^j} \quad (j = 0, 1, \dots, n-1).$$

Since

$$\left|\int_0^{xq^m} (qt/xq^m; q)_{n-j-1}\, \phi(t)\, d_qt\right| = \left|\sum_{r=0}^{\infty} xq^{m+r}(1-q)(q^{r+1};q)_{n-j-1}\phi(xq^{m+r})\right|$$
$$\leqslant \sum_{r=m}^{\infty} xq^r(1-q)|\phi(xq^r)|$$

and $\phi \in \mathscr{L}_q^1[0,a]$, then

$$\lim_{m\to\infty} D_q^j f(xq^m) = c_j \frac{(q;q)_j}{(1-q)^j} = D_q^j f(0) \quad (j = 0, 1, \dots, n-1)\,.$$

Hence, the functions $D_q^j f$, $j = 0, 1, \dots, n-1$, are q-regular at zero. Clearly,

$$D_q^n f(x) = \phi(x) \in \mathscr{L}_q^1[0,a],$$

proving sufficiency. As for necessity, we use mathematical induction. The case $n = 1$ is proved in Theorem 4.5. Now, assume that (4.38) holds at $n = m$. Let $f(x) \in \mathscr{AC}_q^{(m+1)}[0,a]$. That is, $f, \dots, D_q^m f$ are q-regular at zero and $D_q^m f \in \mathscr{AC}_q[0,a]$. Hence, $D_q f \in \mathscr{AC}_q^{(m)}[0,a]$. Then, from the induction hypothesis we have

$$D_q f(x) = \sum_{j=0}^{m-1} c_j x^j + \frac{x^{m-1}}{\Gamma_q(m)} \int_0^x \left(qt/x;q\right)_{m-1} \phi(t)\, d_qt, \tag{4.39}$$

where $x \in (0,a]$,

$$c_j = \frac{D_q^j D_q f(0)}{\Gamma_q(j+1)} = \frac{D_q^{j+1} f(0)}{\Gamma_q(j+1)} \quad \text{and} \quad \phi(x) = D_q^{m+1} f(x)$$

for all $x \neq 0$. Now, replacing x by t and t with u in (4.39); integrating from 0 to x and applying Lemma 1.20 yield

$$f(x) = \sum_{k=0}^{m} \frac{D_q^k f(0)}{\Gamma_q(k+1)} x^k + \frac{x^m}{\Gamma_q(m+1)} \int_0^x \left(qt/x;q\right)_m D_q^{m+1} f(t)\, d_qt,$$

which completes the proof. □

As in the case of the Riemann–Liouville fractional derivative, the existence of the fractional q-Riemann–Liouville derivative holds only for a more restrictive class of functions. For this reason, we study a q-analogue of Abel's integral equation.

Theorem 4.7. *The q-Abel integral equation*

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}\,\phi(t)\,d_qt = f(x) \quad (0<\alpha<1,\ x\in(0,a]) \tag{4.40}$$

has a unique solution $\phi(x)\in\mathscr{L}_q^1[0,a]$ if and only if

$$I_q^{1-\alpha}f(x)\in\mathscr{A}C_q[0,a] \quad \text{and} \quad I_q^{1-\alpha}f(0)=0. \tag{4.41}$$

Moreover, the unique solution $\phi(x)$ is given by

$$\phi(x)=D_{q,x}I_q^{1-\alpha}f(x). \tag{4.42}$$

Proof. First of all, we prove that the q-Abel integral equation cannot have more than one $\mathscr{L}_q^1[0,a]$-solution. Assume that (4.40) has a solution $\phi\in\mathscr{L}_q^1[0,a]$. Hence, replacing x with t and t with u in (4.40), multiplying both sides of the equation by $x^{-\alpha}\big(qt/x;q\big)_{-\alpha}$, and then integrating from 0 to x, we obtain

$$\frac{x^{-\alpha}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{-\alpha}\int_0^t t^{\alpha-1}\big(qu/t;q\big)_{\alpha-1}\phi(u)\,d_qu\,d_qt$$
$$= x^{-\alpha}\int_0^x \big(qt/x;q\big)_{-\alpha}f(t)\,d_qt.$$

But from (1.66),

$$\frac{x^{-\alpha}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{-\alpha}\int_0^t t^{\alpha-1}\big(qu/t;q\big)_{\alpha-1}\phi(u)\,d_qu\,d_qt$$
$$=\frac{B_q(1-\alpha,\alpha)}{\Gamma_q(\alpha)}\int_0^x \phi(t)\,d_qt.$$

Consequently,

$$I_q\phi(x)=I_q^{1-\alpha}f(x). \tag{4.43}$$

Therefore, any $\mathscr{L}_q^1[0,a]$-solution of (4.40) must satisfy the relation (4.43). Suppose that ϕ and ψ are two $\mathscr{L}_q^1[0,a]$-solutions of (4.40). From relation (4.43) and the linearity of q-integration, we obtain

$$I_q(\phi-\psi)(x)=\int_0^x(\phi(t)-\psi(t))\,d_qt=0 \quad \text{for all } x\in(0,a].$$

Hence, from the fundamental theorem of q-calculus, Theorem 1.10, we obtain $\phi \equiv \psi$ on $(0,a]$. That is, we have at most one solution in $\mathscr{L}_q^1[0,a]$. Now we prove necessity. Assume that (4.40) has a solution $\phi \in \mathscr{L}_q^1[0,a]$. Then, ϕ satisfies (4.43). Applying $D_q = D_{q,x}$ to (4.43), we obtain (4.42). By Theorem 4.5 and (4.42), $I_q^{1-\alpha} f \in \mathscr{A}C_q[0,a]$ and $I_q^{1-\alpha} f(0) = 0$.

Conversely, assume that f satisfies (4.41) and $\phi(x) := D_{q,x} I_q^{1-\alpha} f(x)$. Then $\phi \in \mathscr{L}_q^1(0,a)$. We prove that ϕ is a solution of (4.40). Indeed, let g be the function defined by

$$g(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x;q\big)_{\alpha-1} \phi(t)\, d_q t \quad \text{for all} \quad x \in (0,a]. \tag{4.44}$$

It suffices to show that $g(x) = f(x)$, $x \in (0,a]$. From the necessity part, ϕ should have the form (4.42) with g instead of f. Thus,

$$\begin{aligned}\phi(x) &= D_{q,x} I_q^{1-\alpha} f(x) = D_{q,x} I_q^{1-\alpha} g(x)\\ &= D_{q,x} \frac{x^{-\alpha}}{\Gamma_q(1-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha} g(t)\, d_q t,\end{aligned}$$

for all $x \in (0,a]$. The linearity of the q-integration implies

$$D_{q,x}\big(I_q^{1-\alpha}\big(f-g\big)(x)\big) = 0 \quad \text{for all} \quad x \in (0,a]. \tag{4.45}$$

Therefore, the function

$$c(x) := I_q^{1-\alpha}\big(f-g\big)(x) \quad \text{for all} \;\; x \in (0,a], \tag{4.46}$$

is a q-periodic function, i.e. $c(x) = c(qx)$. The function $I_q^{1-\alpha} g(x) \in \mathscr{A}C_q[0,a]$ because it satisfies (4.43) with $g(x)$ on the right hand side. Moreover, $I_q^{1-\alpha} g(0) = 0$. Thus,

$$I_q^{1-\alpha}\big(f-g\big)(x) \in \mathscr{A}C_q[0,a].$$

Accordingly, $I_q^{1-\alpha}\big(f-g\big)$ is q-regular at zero with $I_q^{1-\alpha}\big(f-g\big)(0) = 0$. Consequently, $c(x) \equiv 0$. That is,

$$\frac{x^{-\alpha}}{\Gamma_q(1-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha} (f(t)-g(t))\, d_q t = 0 \;\; \text{for all} \;\; x \in (0,a].$$

The previous equation is in the form (4.40). By the uniqueness of its solutions, we obtain $f(x) = g(x)$, for all $x \in (0,a]$. □

Notice that in the previous proof, we have used the fact that the only q-regular q-periodic functions are the constants. Indeed, if c has these two properties, then

$$c(x) = c(qx) = \ldots c(q^n x) \longrightarrow c(0) \quad \text{as}\, n \longrightarrow \infty.$$

We would like to mention also that another proof of sufficiency could be derived by direct computations using a technique similar to that applied in proving necessity. The following two examples confirm the validity of the theorem.

Example 4.3.1. Consider the q-Abel integral equation

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \bigl(qt/x;q\bigr)_{\alpha-1}\,\phi(t)\,d_qt = x \quad (0<\alpha<1,\ x\in(0,a])\,. \tag{4.47}$$

An easy computation gives that

$$I_q^{1-\alpha}f(x) = \frac{x^{2-\alpha}}{\Gamma_q(3-\alpha)}.$$

Then, $I_q^{1-\alpha}f(0) = 0$ and $I_q^{1-\alpha}f(x) \in \mathscr{A}C_q[0,a]$. Consequently, (4.47) has a unique solution given by

$$\phi(x) = \frac{x^{1-\alpha}}{\Gamma_q(2-\alpha)}.$$

Substitute with $\phi(t)$ on the left hand side of (4.47) and make the substitution $t/x = u$, we obtain

$$\begin{aligned}\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \bigl(qt/x;q\bigr)_{\alpha-1}\,\phi(t)\,d_qt &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_0^x (qt/x;q)_{\alpha-1}t^{1-\alpha}\,d_qt\\ &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(2-\alpha)}\int_0^1 (qu;q)_{\alpha-1}u^{1-\alpha}\,d_qu\\ &= \frac{x}{\Gamma_q(\alpha)\Gamma_q(2-\alpha)}B_q(\alpha,2-\alpha) = x.\end{aligned}$$

Example 4.3.2. Let $N\in\mathbb{N}_0$. Consider the q-Abel integral equation

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \bigl(qt/x;q\bigr)_{\alpha-1}\,\phi(t)\,d_qt = (x;q)_N, \tag{4.48}$$

$$0<\alpha<1, \quad \text{and} \quad x\in(0,a].$$

Since

$$f(x) := (x;q)_N = \sum_{j=0}^{N}(-1)^j \begin{bmatrix} N \\ j \end{bmatrix}_q q^{j(j-1/2)}x^j,$$

then

$$I_q^{1-\alpha}f(x) = \Gamma_q(N+1)\sum_{j=0}^{N}(-1)^j\frac{q^{j(j-1)/2}}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+2)}x^{j+1-\alpha}.$$

Hence, $I_q^{1-\alpha} f(0) = 0$ and it is a q-absolutely continuous function. Therefore, (4.48) has a unique solution give by

$$\phi(x) = \Gamma_q(N+1) \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)/2}}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} x^{j-\alpha}.$$

Substituting with $\phi(t)$ into the left hand side of (4.48), we obtain

$$\begin{aligned}
&\frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x;q\big)_{\alpha-1} \phi(t)\, d_qt \\
&= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)}/2}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} \int_0^x (qt/x;q)_{\alpha-1} t^{j-\alpha}\, d_qt \\
&= \frac{\Gamma_q(N+1)}{\Gamma_q(\alpha)} \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)}/2}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} B_q(\alpha, j-\alpha+1) \\
&= \sum_{j=0}^{N} (-1)^j \begin{bmatrix} N \\ j \end{bmatrix}_q q^{j(j-1)/2} x^j = f(x).
\end{aligned}$$

As in Theorem 4.1, the solution ϕ of q-Abel's equation (4.40) should be the fractional q-derivative of order α of f. Starting from this observation, we define the basic fractional derivative as follows:

Definition 4.3.3. Let f be a function defined on $[0,a]$. For $\alpha > 0$ the fractional q-derivative of order α is defined to be

$$D_q^\alpha f(x) := \phi(x) = D_q^k I_q^{k-\alpha} f(x) \quad (k = \lceil \alpha \rceil), \tag{4.49}$$

provided that

$$f(x) \in \mathscr{L}_q^1[0,a] \quad \text{and} \quad I_q^{k-\alpha} f(x) \in \mathscr{A}C_q^{(k)}[0,a]. \tag{4.50}$$

For example,

$$D_q^\alpha x^{\beta-1} = \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)} x^{\beta-\alpha-1} \ \ (\beta > 0;\ \alpha \in \mathbb{R}). \tag{4.51}$$

Theorem 4.7 guarantees that $D_q^\alpha f(x) \in \mathscr{L}_q^1[0,a]$, provided (4.50) is fulfilled.

The following lemma gives the relationship between the fractional q-derivative of order α and $I_q^{-\alpha}$, $\alpha > 0$.

Lemma 4.8. *Let* $f \in \mathscr{L}_q^1[0,a]$ *and* $\alpha > 0$*,* $k = \lceil \alpha \rceil$*, be such that* $I_q^{k-\alpha} f \in \mathscr{A}C_q^{(k)}[0,a]$*. Then*

$$D_q^\alpha f(x) = I_q^{-\alpha} f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_q t \quad \textit{for} \;\; x \in (0,a].$$

Proof. From (4.49), we have

$$\begin{aligned} D_q^\alpha f(x) &= D_q^k I_q^{k-\alpha} f(x) \\ &= D_q^k \left(\frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \big(qt/x;q\big)_{k-\alpha-1} f(t)\, d_q t \right). \end{aligned} \tag{4.52}$$

Set

$$h(t,x) := \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \big(qt/x;q\big)_{k-\alpha-1} f(t).$$

Hence,

$$h(xq^r, xq^j) = 0 \quad (r = 0,1,\ldots,j-1; \;\; j = 1,2,\ldots,k)\,.$$

Applying Lemma 1.12 to (4.52), we obtain

$$\begin{aligned} D_q^\alpha f(x) &= \int_0^x D_{q,x}^k \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \big(qt/x;q\big)_{k-\alpha-1} f(t)\, d_q t \\ &= \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_q t = I_q^{-\alpha} f(x) \end{aligned}$$

□

4.4 Some Properties of q-Riemann–Liouville Fractional Integral Operator

In this section, we derive basic properties of the Riemann–Liouville fractional q-integral operator in certain function spaces.

Lemma 4.9. *Let* $\alpha \in \mathbb{R}^+$ *and* $f : (0,a] \to \mathbb{C}$ *be a function. If* $f \in \mathscr{L}_q^1[0,a]$ *then* $I_q^\alpha f \in \mathscr{L}_q^1[0,a]$ *and*

$$\left\| I_q^\alpha f \right\|_1 \le \frac{a^\alpha}{\Gamma_q(\alpha+1)} \, \| f \|_1 \,. \tag{4.53}$$

Proof. Let f be a function defined on $(0,a]$ such that $f \in \mathscr{L}_q^1[0,a]$. Then from (4.24) we get for $x \in (qa,a]$

$$\begin{aligned}\int_0^x |I_q^\alpha f(t)|\, d_qt &= \frac{1}{\Gamma_q(\alpha)} \int_0^x t^{\alpha-1} \left|\int_0^t \left(qu/t;q\right)_{\alpha-1} f(u)\, d_qu\right| d_qt \\ &\leqslant \frac{1}{\Gamma_q(\alpha)} \int_0^x t^{\alpha-1} \int_0^t \left(qu/t;q\right)_{\alpha-1} |f(u)|\, d_qu\, d_qt \\ &= \frac{x^{\alpha+1}(1-q)^2}{\Gamma_q(\alpha)} \sum_{n=0}^\infty q^{n\alpha} \sum_{m=0}^\infty q^{n+m}\left(q^{m+1};q\right)_{\alpha-1} |f(xq^{n+m})| \qquad (4.54) \\ &= \frac{x^{\alpha+1}(1-q)^2}{\Gamma_q(\alpha)} \sum_{n=0}^\infty q^{n\alpha} \sum_{k=n}^\infty q^k\left(q^{k-n+1};q\right)_{\alpha-1} |f(xq^k)| \\ &= \frac{x^{\alpha+1}(1-q)}{\Gamma_q(\alpha)} \sum_{k=0}^\infty q^k |f(xq^k)| \sum_{n=0}^k (1-q)q^{n\alpha}\left(q^{k-n+1};q\right)_{\alpha-1}.\end{aligned}$$

But from (1.61) we obtain

$$\sum_{n=0}^k (1-q)q^{n\alpha}\left(q^{k-n+1};q\right)_{\alpha-1} = B_q(1,\alpha)\left(q^{k+1};q\right)_\alpha.$$

Since $(q^{k+1};q)_\alpha \leq 1$ for all $k \in \mathbb{N}_0$, then

$$\begin{aligned}\int_0^x |I_q^\alpha f(t)|\, d_qt &\leqslant B_q(1,\alpha)x^\alpha \sum_{k=0}^\infty q^k(1-q)|f(xq^k)| \\ &= \frac{x^\alpha}{\Gamma_q(\alpha)} B_q(1,\alpha) \int_0^x |f(t)|\, d_qt \qquad (4.55) \\ &\leqslant \frac{a^\alpha}{\Gamma_q(\alpha+1)} \|f\|_1 .\end{aligned}$$

Hence $I_q^\alpha f \in \mathscr{L}_q^1[0,a]$ and (4.53) is proved. ◻

We shall also need the following lemma:

Lemma 4.10. *Let $\gamma \in \mathbb{R}$, $\gamma < 1$, and let $g \in C_\gamma[0,a]$. Then*

(1) $I_q^\alpha g \in C_\gamma[0,a]$ and

$$\left\| I_q^\alpha g \right\|_{C_\gamma} \leq a^\alpha \frac{\Gamma_q(1-\gamma)}{\Gamma_q(1+\alpha-\gamma)} \|g\|_{C_\gamma} . \qquad (4.56)$$

(2) If we additionally assume that $\gamma \leq \alpha$, then $I_q^\alpha g \in C[0,a]$,

Proof. To prove that $I_q^\alpha g \in C_\gamma[0,a]$, it is equivalent to prove $x^\gamma I_q^\alpha g \in C[0,a]$. Since $x^\gamma g(x) \in C[0,a]$, there exits $M > 0$ such that

$$M := \max_{0\le x\le a} |x^\gamma g(x)|\,.$$

Hence,

$$\begin{aligned}\left|x^\gamma I_q^\alpha g(x)\right| &\le \frac{M}{\Gamma_q(\alpha)} x^{\gamma+\alpha-1}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\, d_qt\\ &= \frac{Mx^\alpha}{\gamma_q(\alpha)}\int_0^1 (q\xi;q)_{\alpha-1}\xi^{-\gamma}\, d_q\xi = \frac{Mx^\alpha}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma).\end{aligned}$$

Hence $\lim_{x\to 0} x^\gamma I_q^\alpha g(x) = 0$ and we can assume the continuity of the function $x^\gamma I_q^\alpha g$ at zero by assuming that its value at zero is zero. Now we prove the continuity of the function $x^\gamma I_q^\alpha g(x)$ at any $x_0 \ne 0$. Since $x^\gamma g(x) \in C[0,a]$, it is uniformly continuous on $[0,a]$. Hence, $\forall \epsilon > 0$ there exists $\delta > 0$ such that for all $x,\ y \in [0,a]$

$$|x-y| < \delta \longrightarrow |x^\gamma g(x) - y^\gamma g(y)| < \epsilon.$$

Assume that $|x - x_0| < \delta$ and we may assume that $x_0 - \delta > 0$. Consequently,

$$0 \ne (x_0-\delta, x_0+\delta).$$

Now

$$\begin{aligned}|x-x_0| < \delta &\longrightarrow |xq^k - x_0q^k| < \delta\\ &\longrightarrow \left|x(q^k)^\gamma g(xq^k) - (x_0q^k)^\gamma g(x_0q^k)\right| < \epsilon,\end{aligned}$$

for all $k \in \mathbb{N}_0$. Consequently,

$$\sum_{k=0}^\infty q^{k(1-\gamma)}\frac{(q^\alpha;q)_k}{(q;q)_k}\left|x(q^k)^\gamma g(xq^k) - (x_0q^k)^\gamma g(x_0q^k)\right| < \frac{(q^{\alpha+1-\gamma};q)_\infty}{(q^{1-\gamma};q)_\infty}\epsilon.$$

That is,

$$\left|x^{\gamma-\alpha} I_q^\alpha g(x) - x_0^{\gamma-\alpha} I_q^\alpha g(x_0)\right| < \frac{(q^{\alpha+1-\gamma};q)_\infty}{(q^{1-\gamma};q)_\infty}\epsilon.$$

Hence, $x^{\gamma-\alpha} I_q^\alpha g(x) \in C[0,a]$. Consequently,

$$x^\gamma I_q^\alpha g(x) = x^\alpha x^{\gamma-\alpha} I_q^\alpha g(x) \in C[0,a].$$

That is, $I_q^\alpha g \in C_\gamma[0,a]$.

Now we prove Inequality (4.56). Indeed,

$$\begin{aligned}\left\|I_q^\alpha g\right\|_{C_\gamma} &= \max_{0\le x\le a}\left|x^\gamma I_q^\alpha g(x)\right| \\ &= \max_{0\le x\le a}\left|\frac{x^{\gamma+\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} g(t)\, d_qt\right| \\ &\le \|g\|_{C_\gamma}\max_{0\le x\le a}\left|\frac{x^{\gamma+\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\, d_qt\right|. \qquad (4.57)\end{aligned}$$

Since $0 < \gamma < 1$, the q-integration on the most right side of (4.57) converges. Moreover, it can be calculated by making the substitution $t = x\xi$. This gives

$$\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\, d_qt = x^{1-\gamma}\int_0^1 (q\xi;q)_{\alpha-1}\xi^{-\gamma}\, d_q\xi = x^{-\gamma+1} B_q\left(\alpha, 1-\gamma\right). \qquad (4.58)$$

Substituting (4.58) into (4.57), we get the required inequality. The proof of (2) follows by using (1) and noting that

$$I_q^\alpha g(x) = x^{\alpha-\gamma} I_q^\gamma g(x) \quad \text{for all } x \in [0,a],$$

and the product of continuous functions is continuous. □

Lemma 4.11. *Let $\alpha > 0$, $\gamma < 1$ and $g \in C_\gamma[0,a]$. Then*

(i) If $\gamma < \alpha$, then $lim_{x\to 0^+} I_q^\alpha g(x) = 0$.
(ii) If $\alpha = \gamma$, then

$$\lim_{x\to 0+} x^\gamma g(x) = c \quad \textit{if and only if} \quad \lim_{x\to 0+} I_q^\alpha g(x) = c\Gamma_q(1-\alpha). \qquad (4.59)$$

Proof. Since

$$I_q^\alpha g(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(t^\gamma g(t) - c\right)\, d_qt + c\frac{x^{\alpha-\gamma}}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma)$$

and

$$\left|\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(t^\gamma g(t) - c\right)\, d_qt\right| \le \frac{B_q(\alpha, 1-\gamma)}{\Gamma_q(\alpha)} x^{\alpha-\gamma}\max_{0\le t\le x}|t^\gamma g(t) - c|,$$

we obtain from the continuity of the function $t^\gamma g$ at zero that

$$\lim_{x\to 0+}\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(f(t) - f(0)\right)\, d_qt = 0.$$

Hence

$$\lim_{x\to 0^+} I_q^{\alpha} g(x) = c \lim_{x\to 0^+} \frac{x^{\alpha-\gamma}}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma) = \begin{cases} 0, & \gamma < \alpha, \\ c\Gamma_q(1-\alpha), & \gamma = \alpha. \end{cases}$$

Now we prove the sufficient condition in (4.59). Therefore, we assume that

$$\lim_{x\to 0^+} I_q^{\alpha} g(x) = d\Gamma_q(1-\alpha),$$

where d is a constant. Since $\Gamma_q(1-\alpha)$ can be represented as

$$(1-q)^{\alpha} \sum_{k=0}^{\infty} q^{(1-\alpha)k} \frac{(q^{\alpha};q)_k}{(q;q)_k},$$

then

$$I_q^{\alpha} g(x) - c\Gamma_q(1-\alpha) = (1-q)^{\alpha} \sum_{k=0}^{\infty} q^{k(1-\alpha)} \frac{(q^{\alpha};q)_k}{(q;q)_k} \left((xq^k)^{\alpha} g(xq^k) - d\right). \quad (4.60)$$

Since $\lim_{x\to 0^+} x^{\alpha} g(x) = c$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\left|(xq^k)^{\alpha} g(xq^k) - d\right| \leq M \quad \text{for all} \quad x \in (-\delta, \delta).$$

Consequently,

$$q^{k(1-\alpha)} \frac{(q^{\alpha};q)_k}{(q;q)_k} \left|(xq^k)^{\alpha} g(xq^k) - d\right| \leq C q^{k(1-\alpha)},\ C := M \frac{(1-q)^{\alpha}}{(q^{\alpha};q)_{\infty}}.$$

Hence, the series on the right hand side of (4.60) is uniformly convergent to zero on $(-\delta, \delta)$, and we can calculate the limit as $x \to 0^+$ on (4.60) term by term. This gives $c - d = 0$. That is, $d = c$, completing the proof of (ii) and the lemma. □

In the following two lemmas, we study the limits $\lim_{q\to 1^-} I_q f(x)$ and $\lim\limits_{\alpha\to 0^+} I_q^{\alpha} f(x)$.

Lemma 4.12. *Let* $\alpha > 0$ *and* f *be a function defined on* $[0, a]$, $a > 0$. *If* f *is Riemann integrable on* $[0, x]$, *then*

$$\lim_{q\to 1^-} I_q^{\alpha} f(x) = I^{\alpha} f(x) \quad (x \in (0, a];\ \alpha > 0).$$

Proof. Let $x \in (0, a]$ be fixed. From (1.59) and (1.12), we obtain for $0 < t < x$

$$\begin{aligned}\lim_{q\to 1^-} x^{\alpha-1}\big(qt/x;q\big)_{\alpha-1} &= \lim_{q\to 1^-} x^{\alpha-1}\frac{\big(qt/x;q\big)_\infty}{\big(q^\alpha t/x;q\big)_\infty}\\ &= x^{\alpha-1}\left(1-\frac{t}{x}\right)^{\alpha-1} = (x-t)^{\alpha-1}\end{aligned}$$

uniformly for $0 < t < x$. Hence, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in \{q^k x,\ k \in \mathbb{N}\}$, we have

$$\left|x^{\alpha-1}\big(qt/x;q\big)_{\alpha-1} - (x-t)^{\alpha-1}\right| \leqslant \epsilon/2$$

whenever $0 < 1 - q < \delta$. That is,

$$\left|I_q^\alpha f(x) - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,d_qt\right| \leqslant \frac{\epsilon}{2\Gamma_q(\alpha)}\int_0^x |f(t)|\,d_qt.$$

Since

$$\begin{aligned}\left|I_q^\alpha f(x) - I^\alpha f(x)\right| &\leqslant \left|I_q^\alpha f(x) - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,d_qt\right|\\ &\quad + \left|\frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,d_qt - I_\alpha f(x)\right|,\end{aligned}$$

then the lemma follows if f is Riemann integrable on $[0, x]$. □

Lemma 4.13. *Let $\nu > 0$ and let $f \in H_\nu[0, a]$. Then for $\alpha > 0$*

$$I_q^\alpha f(x) = \frac{x^\alpha}{\Gamma_q(\alpha+1)} f(0) + F(x),$$

where

$$F(x) = O(x^{\alpha+\nu}).$$

Proof.

$$\begin{aligned}I_q^\alpha f(x) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\,d_qt\\ &= \frac{x^\alpha}{\Gamma_q(\alpha+1)} f(0) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}(f(t)-f(0))\,d_qt.\end{aligned}$$

Set

$$F(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}(f(t) - f(0))\, d_q t.$$

Since $f \in H_\nu[0,a]$, there exists $c > 0$ such that

$$|f(t) - f(0)| < ct^\nu \quad \text{for all} \quad t \in [0,a].$$

Therefore,

$$|F(x)| \leq \frac{cx^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} t^\nu \, d_q t = cx^{\alpha+\nu} \frac{\Gamma_q(\nu+1)}{\Gamma_q(\nu+\alpha+1)},$$

proving the lemma. □

Lemma 4.14. *Let* $f \in \mathscr{L}_q^1[0,a]$. *Then*

$$\lim_{\alpha\to 0+} I_q^\alpha f(x) = f(x) \quad \textit{for all} \quad x \in (0,a].$$

Proof. Let $x \in (0,a]$ be fixed. From (4.26), we have

$$\lim_{\alpha\to 0+} I_q^\alpha f(x) = \lim_{\alpha\to 0+} x^\alpha (1-q)^\alpha \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n). \tag{4.61}$$

Since

$$q^n \frac{(q^\alpha;q)_n}{(q;q)_n} |f(xq^n)| \leqslant q^n |f(xq^n)| \quad (\, n \in \mathbb{N}_0,\ \alpha \in (0,1])\,,$$

from Weierstrass M-test, the series in (4.61) is uniformly convergent for $\alpha \in (0,1)$. Thus,

$$\lim_{\alpha\to 0+} \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n) = f(x).$$

Hence,

$$\begin{aligned}\lim_{\alpha\to 0+} I_q^\alpha f(x) &= \lim_{\alpha\to 0+} x^\alpha (1-q)^\alpha \lim_{\alpha\to 0+} \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n) \\ &= f(x).\end{aligned}$$

□

From now on, we use the notation $I_q^0 f(x) = f(x)$.

4.5 q-Riemann–Liouville Fractional Calculus

In this section, we investigate the main properties of the Riemann–Liouville fractional q-integral and q-derivative. The property

$$I_q^\beta I_q^\alpha f(x) = I_q^\alpha I_q^\beta f(x) = I_q^{\alpha+\beta} f(x) \quad (\alpha;\, \beta \geqslant 0) \tag{4.62}$$

is a q-analogue of the semigroup property (4.10). It is established by Agarwal in [17]. For the convenience of the reader, we prove (4.62) in the following lemma.

Lemma 4.15. *If* $f \in \mathscr{L}_q^1[0,a]$ *then the semi group property* (4.62) *holds.*

Proof. Using (4.26) we obtain

$$I_q^\alpha(I_q^\beta f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta}\sum_{k=0}^{\infty} q^{k(1+\beta)}\frac{(q^\alpha;q)_k}{(q;q)_k}\sum_{m=0}^{\infty} q^m\frac{(q^\beta;q)_m}{(q;q)_m} f(xq^{n+m}).$$

Making the substitution $n = k + m$ we obtain

$$I_q^\alpha(I_q^\beta f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta}\sum_{k=0}^{\infty} q^{k(1+\beta)}\frac{(q^\alpha;q)_k}{(q;q)_k}\sum_{n=k}^{\infty} q^{n-k}\frac{(q^\beta;q)_{n-k}}{(q;q)_{n-k}} f(xq^{n}). \tag{4.63}$$

Since $f \in \mathscr{L}_q^1[0,a]$, the double series (4.63) is absolutely convergent. Then, we can interchange the order of summations to obtain

$$I_q^\alpha(I_q^\beta f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta}\sum_{n=0}^{\infty} q^n f(xq^n)\sum_{k=0}^{n} q^{k\beta}\frac{(q^\alpha;q)_k}{(q;q)_k}\frac{(q^\beta;q)_{n-k}}{(q;q)_{n-k}}.$$

It is straight forward to see that

$$\frac{(q^\beta;q)_{n-k}}{(q;q)_{n-k}} = \frac{(q^{-n};q)_k}{(q^{1-n-\beta};q)_k} q^{(1-\beta)k}.$$

Consequently,

$$I_q^\alpha(I_q^\beta f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta}\sum_{n=0}^{\infty} q^n f(xq^n)\frac{(q^\beta;q)_n}{(q;q)_n}\,{}_2\phi_1(q^{-n}, q^\alpha; q^{1-n-\beta}; q, q). \tag{4.64}$$

Applying (1.11) we obtain

$${}_2\phi_1(q^{-n}, q^\alpha; q^{1-n-\beta}; q, q) = \frac{(q^{1-n-\beta-\alpha};q)_n}{(q^{1-n-\beta};q)_n} q^{n\alpha} = \frac{(q^{\alpha+\beta};q)_n}{(q^\beta;q)_n}. \tag{4.65}$$

Substituting (4.65) into (4.64), we obtain the series representation of $I_q^{\alpha+\beta} f(x)$ and (4.62) follows. □

Similar to property (4.12), we have the following:

Lemma 4.16. *If* $f \in \mathscr{L}_q^1[0,a]$ *then*

$$D_q^\alpha I_q^\alpha f(x) = f(x) \quad (\alpha > 0;\ x \in (0,a])\,. \tag{4.66}$$

Proof. It is obvious that if $\alpha = n$, $n \in \mathbb{N}$, then $D_q^n I_q^n f(x) = f(x)$. For a non integer and positive α, $n-1 < \alpha < n$, $n \in \mathbb{N}$, we apply the semi group identity (4.62), to obtain

$$\begin{aligned} D_q^\alpha I_q^\alpha f(x) &= D_q^n I_q^{n-\alpha} I_q^\alpha f(x) \\ &= D_q^n I_q^n f(x) = f(x), \end{aligned}$$

for all $x \in (0,a]$. □

The converse of (4.66), which is a q-analogue of (4.15), is the following:

Lemma 4.17. *Let* $\alpha \in \mathbb{R}^+$ *and* $n := \lceil \alpha \rceil$. *If*

$$f \in \mathscr{L}_q^1[0,a] \quad \textit{such that} \quad I_q^{n-\alpha} f \in \mathscr{A}C_q^{(n)}[0,a],$$

then

$$I_q^\alpha D_q^\alpha f(x) = f(x) - \sum_{j=1}^{n} D_q^{\alpha-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha-j+1)},\ x \in (0,a]. \tag{4.67}$$

Proof. Set

$$h(t,x) := x^\alpha\, (qt/x;q)_\alpha\, D_q^\alpha f(t), \quad x \in (0,a], \quad 0 < t \leqslant x.$$

Hence, $h(x,qx) = 0$ for all $x \in (0,a]$. Accordingly, applying Lemma 1.12 yields

$$\begin{aligned} I_q^\alpha D_q^\alpha f(x) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x;q\big)_{\alpha-1} D_q^\alpha f(t)\, d_q t \\ &= D_{q,x}\left(\frac{x^\alpha}{\Gamma_q(\alpha+1)} \int_0^x \big(qt/x;q\big)_\alpha D_q^\alpha f(t)\, d_q t\right) \qquad (4.68) \\ &= D_{q,x}\left(\frac{x^\alpha}{\Gamma_q(\alpha+1)} \int_0^x \big(qt/x;q\big)_\alpha D_q^n I_q^{n-\alpha} f(t)\, d_q t\right) \end{aligned}$$

Now, applying the q-integration by parts (1.28) n times on the last q-integral of (4.68), we obtain

$$\begin{aligned}&\tfrac{x^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^x \left(qt/x;q\right)_{\alpha} D_q^{\alpha} f(t)\,d_qt\\ &\quad= -\sum_{j=1}^{n} D_{q,x}^{\alpha-j} f(0^+)\tfrac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)} + \tfrac{x^{\alpha-n}}{\Gamma_q(\alpha-n+1)}\int_0^x \left(qt/x;q\right)_{\alpha-n} I_q^{n-\alpha} f(t)\,d_qt\\ &\quad= -\sum_{j=1}^{n} D_{q,x}^{\alpha-j} f(0^+)\tfrac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)} + I_q^{\alpha-n+1} I_q^{n-\alpha} f(x).\end{aligned} \tag{4.69}$$

Applying the semigroup identity (4.62) on (4.69) yields

$$\begin{aligned}\frac{x^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^x \left(qt/x;q\right)_{\alpha} D_q^{\alpha} f(t)\,d_qt =&\\ I_q f(x) - \sum_{j=1}^{n} D_q^{\alpha-j} f(0^+)\frac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)}.&\end{aligned} \tag{4.70}$$

Computing the q-derivative of both sides of (4.70) for $x \in (0,a]$ and combining the result with (4.68) we end with (4.67). ☐

From (4.67), the equality

$$I_q^{\alpha} D_q^{\alpha} f(x) = f(x) \quad \text{for all} \quad x \in (0,a]$$

holds if and only if

$$D_q^{\alpha-j} f(0^+) = 0 \quad (j = 1,2,\dots,n)\,.$$

In the following four lemmas, we discuss the results of combining q-fractional integral and derivative of not necessarily equal orders. These results are q-analogues of relations (4.13), (4.14), (4.16), (4.17), and (4.11).

Lemma 4.18. *If $f \in \mathscr{L}_q^1[0,a]$ then*

$$D_q^{\alpha} I_q^{\beta} f(x) = I_q^{\beta-\alpha} f(x) \quad (\beta \geqslant \alpha \geqslant 0;\ x \in (0,a])\,. \tag{4.71}$$

If in addition $D_q^{\alpha-\beta} f(x)$ exists in $(0,a]$ then

$$D_q^{\alpha} I_q^{\beta} f(x) = D_q^{\alpha-\beta} f(x) \quad (\alpha > \beta \geqslant 0)\,. \tag{4.72}$$

Proof. First, assume that $\beta \geqslant \alpha$. Then, $\beta = \alpha + (\beta-\alpha)$ and from (4.62) we obtain

$$D_q^{\alpha} I_q^{\beta} f(x) = D_q^{\alpha} I_q^{\alpha} I_q^{\beta-\alpha} f(x) = I_q^{\beta-\alpha} f(x).$$

Now let $\beta < \alpha$, $m := \lceil \alpha \rceil$ and $n := \lceil \alpha - \beta \rceil$. Then $n \leqslant m$. Using (4.49) and (4.62), we get

$$\begin{aligned} D_q^{\alpha} I_q^{\beta} f(x) &= D_q^m I_q^{m-\alpha} I_q^{\beta} f(x) = D_q^m I_q^{m-n} I_q^{n-\alpha+\beta} f(x) \\ &= D_q^n I_q^{n-\alpha+\beta} f(x) = D_q^{\alpha-\beta} f(x). \end{aligned}$$

□

Lemma 4.19. *Let $f \in \mathscr{L}_q^1[0,a]$ such that $I_q^{n-\beta} f \in \mathscr{A}C_q^{(n)}[0,a]$, where $\beta > 0$ and $n := \lceil \beta \rceil$. Then for any $\alpha \geqslant 0$*

$$I_q^{\alpha} D_q^{\beta} f(x) = D_q^{-\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha-j+1)} \quad \textit{for } x \in (0,a]. \tag{4.73}$$

Proof. Since $\frac{1}{\Gamma_q(z)}$ has zeros at the negative integers, identity (4.73) holds for any $\beta > 0$ when $\alpha = 0$. Therefore, we assume that $\alpha > 0$. We distinguish between two cases. First, if $\alpha \geqslant \beta$ then from (4.62) and (4.67) we obtain

$$\begin{aligned} I_q^{\alpha} D_q^{\beta} f(x) &= I_q^{\alpha-\beta} \left(I_q^{\beta} D_q^{\beta} \right) f(x) \\ &= I_q^{\alpha-\beta} \left(f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta-j+1)} \right) \\ &= I_q^{\alpha-\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha-j+1)}, \end{aligned}$$

for all $x \in (0,a]$. If $\beta > \alpha$ then from (4.71) and (4.67), we get

$$\begin{aligned} I_q^{\alpha} D_q^{\beta} f(x) &= D_q^{\beta-\alpha} \big(I_q^{\beta} D_q^{\beta} f(x) \big) \\ &= D_q^{\beta-\alpha} \left(f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta-j+1)} \right) \\ &= D_q^{\beta-\alpha} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha-j+1)}, \end{aligned}$$

for all $x \in (0,a]$. □

Lemma 4.20. *Let $\beta > 0$ and $n := \lceil \beta \rceil$. Assume that*

$$f \in \mathscr{L}_q^1[0,a] \quad \textit{and} \quad I_q^{n-\beta} f(x) \in \mathscr{A}C_q^{(n)}[0,a].$$

Then

$$D_q^{\alpha} D_q^{\beta} f(x) = D_q^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{-\alpha-j}}{\Gamma_q(-\alpha-j+1)}$$

for all $x \in (0,a]$, provided that $D_q^{\alpha+\beta} f(x)$ exists for any $\alpha > 0$.

Proof. Let $x \in (0,a]$. From (4.67) and (4.72), we conclude that

$$\begin{aligned} D_q^{\alpha} D_q^{\beta} f(x) &= D_q^{\alpha+\beta} I_q^{\beta} D_q^{\beta} f(x) \\ &= D_q^{\alpha+\beta} \left[f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta-j+1)} \right] \\ &= D_q^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{-\alpha-j}}{\Gamma_q(-\alpha-j+1)}. \end{aligned}$$

□

Remark 4.5.1. Similar to (4.20), if $\alpha, \beta > 0$, $I_q^{k-\alpha} \in \mathscr{A}C_q^{(k)}[0,a]$ and $D_q^{\alpha+\beta} f(x)$ exists, where $k := \lceil \alpha \rceil$ then

$$D_q^{\beta} D_q^{\alpha} f(x) = D_q^{\beta+\alpha} f(x) - \sum_{i=1}^{k} D_q^{\alpha-i} f(0^+) \frac{x^{-\beta-i}}{\Gamma_q(-\beta-i+1)},$$

for all $x \in (0,a]$. If

$$D_q^{\beta-j} f(0^+) = 0 \quad (j = 1,2,\ldots,n)$$

and

$$D_q^{\alpha-j} f(0^+) = 0 \quad (j = 1,2,\ldots,k),$$

then we have the simplified relation

$$D_q^{\alpha} D_q^{\beta} = D_q^{\beta} D_q^{\alpha} = D_q^{\alpha+\beta}.$$

Lemma 4.21. *Let f be such that*

$$f \in \mathscr{L}_q^1(0,a) \quad \textit{and} \quad I_q^{k-\alpha} f \in \mathscr{A}C_q^{(k)}[0,a], \tag{4.74}$$

where $\alpha > 0$ and $k := \lceil \alpha \rceil$. Then

$$D_q^{\alpha-j} f(x) \in \mathscr{L}_q^1[0,a] \quad (j = 0,1,2,\ldots,k-1). \tag{4.75}$$

Moreover,

$$D_q^{\alpha-j} f(x) \in \mathscr{A}C_q^{(j)}[0,a] \quad (j = 1,2,\ldots,k-1)\,. \tag{4.76}$$

Proof. The proof of (4.75) follows immediately from (4.74) and Lemma 4.9. As for (4.76), we apply Lemma 4.18 to obtain

$$\begin{aligned} D_q^{\alpha-j} f(x) &= D_q^{k-j} I_q^{k-j-(\alpha-j)} f(x) \\ &= D_q^{k-j} I_q^{k-\alpha} f(x) \in \mathscr{A}C_q^{(j)}[0,a] \quad \text{for} \quad j = 1,2,\ldots,k-1. \end{aligned}$$

□

Similar to property (4.20), we have the following representation of the q-fractional derivative.

Lemma 4.22. *Let $\alpha > 0$, $k := \lceil \alpha \rceil$. If $f \in \mathscr{A}C_q^{(k)}[0,a]$ then*

$$D_q^{\alpha} f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(-\alpha+j+1)} x^{-\alpha+j} + \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \left(qt/x;q\right)_{k-\alpha-1} D_q^k f(t)\, d_qt, \tag{4.77}$$

for all $x \in (0,a]$. Furthermore, $D_q^{\alpha} f(0^+) = 0$ if and only if

$$D_q^j f(0^+) = 0 \quad (j = 0,1,\ldots,k-1)\,.$$

Proof. Since $f \in \mathscr{A}C_q^{(k)}[0,a]$, then from Theorem 4.6 we obtain

$$f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j+1)} x^j + \frac{x^{k-1}}{\Gamma_q(k)} \int_0^x \left(qt/x;q\right)_{k-1} D_q^k f(t)\, d_qt$$

for all $x \in (0,a]$. Consequently,

$$f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j+1)} x^j + I_q^k D_q^k f(x) \quad \text{for all } x \in (0,a]. \tag{4.78}$$

Applying the operator D_q^{α} to both sides of (4.78) and using (4.71) yield

$$\begin{aligned} D_q^{\alpha} f(x) &= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + D_q^{\alpha} I_q^k D_q^k f(x) \\ &= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + I_q^{k-\alpha} D_q^k f(x) \end{aligned}$$

$$= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \left(qt/x;q\right)_{k-\alpha-1} D_q^k f(t)\, d_q t,$$

which is (4.77). The rest of the proof arises from (4.77). □

4.6 A Complex Variable Approach for Riemann–Liouville Fractional q-Derivative

In this section, we represent Jackson q-derivative of order n and Riemann–Liouville fractional q-derivative of order α in terms of complex integration. The complex approach for Jackson q-derivative is already known. See for example [148, 149].

Theorem 4.23. *Let Γ be a simple closed positively oriented contour such that the zero point lies inside Γ. If $f(\xi)$ is analytic in some simply connected domain D containing Γ and z is any nonzero point lies inside Γ then*

$$D_q^n f(z) = \frac{\Gamma_q(n+1)}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - z)(\xi - qz)\dots(\xi - q^n z)}\, d\xi. \tag{4.79}$$

Proof. Since $0, z$ lie inside Γ, the set of points $\left\{zq^k,\ k \in \mathbb{N}_0\right\}$ lies inside Γ. Hence, from the Cauchy's integral formula, see for example [268, P. 143], we obtain

$$f(q^k z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - q^k z}\, d\xi \quad (k \in \mathbb{N}_0). \tag{4.80}$$

Substituting with the value of $f(q^k z)$ from (4.80) into (1.25), we obtain

$$\begin{aligned} D_q^n f(z) &= (-1)^n z^{-n} (1-q)^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk + \frac{k(k+1)}{2}} \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - q^k z}\, d\xi \\ &= \frac{(-1)^n z^{-n} (1-q)^{-n}}{2\pi i} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk + \frac{k(k+1)}{2}} \int_\Gamma \frac{f(\xi)}{\xi} \sum_{r=0}^{\infty} (\frac{q^k z}{\xi})^r\, d\xi, \end{aligned}$$

where we expand $1/(1 - (q^k z/\xi))$ and use that $|z| < |\xi|$. Hence, by changing the order of summations and applying the formula

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^k = (a;q)_m,$$

with $m = n$ and $a = q^{-n+r+1}$, we obtain

$$\begin{aligned} D_q^n f(z) &= \frac{(-1)^n z^{-n}(1-q)^{-n}}{2\pi i} \sum_{r=0}^{\infty} z^r \int_{\Gamma} \frac{f(\xi)}{\xi^{r+1}} (q^{-n+r+1}; q)_n \, d\xi \\ &= \frac{(-1)^n z^{-n}(1-q)^{-n}}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi} \left[\sum_{r=0}^{\infty} (q^{-n+r+1}; q)_n \left(\frac{z}{\xi}\right)^r \right] d\xi. \end{aligned}$$

But

$$\sum_{r=0}^{\infty} (q^{-n+r+1}; q)_n \left(\frac{z}{\xi}\right)^r = \sum_{r=n}^{\infty} (q^{-n+r+1}; q)_n \left(\frac{z}{\xi}\right)^r . \tag{4.81}$$

Making the substitution $k = r - n$, we obtain

$$\sum_{r=0}^{\infty} (q^{-n+r+1}; q)_n \left(\frac{z}{\xi}\right)^r = \sum_{k=0}^{\infty} (q^{k+1}; q)_n \left(\frac{z}{\xi}\right)^{k+n} .$$

Since

$$(q^{k+1}; q)_n = \frac{(q; q)_{n+k}}{(q; q)_k} = (q; q)_n \frac{(q^{n+1}; q)_k}{(q; q)_k},$$

from the q-binomial theorem we obtain

$$\begin{aligned} \sum_{r=0}^{\infty} (q^{-n+r+1}; q)_n \left(\frac{z}{\xi}\right)^r &= (q; q)_n \left(\frac{z}{\xi}\right)^n \frac{(q^{n+1} z/\xi; q)_\infty}{(z/\xi; q)_\infty} \\ &= \left(\frac{z}{\xi}\right)^n \frac{(q; q)_n}{(z/\xi; q)_{n+1}}. \end{aligned} \tag{4.82}$$

Substituting (4.81) into (4.82), we obtain (4.79). □

Observe that the denominator of the integrand of (4.79) can be replaced by $\xi^{n+1}(z/\xi; q)_{n+1}$. If we replace n by a non integer α, the function $\xi^{\alpha+1}$ no longer has a pole at $\xi = 0$ but a branch point. So, for $n \in \mathbb{N}$ we define Γ_n to be a simple closed positively oriented contour like the one in Fig. 4.1. We assume that the inner contour is a circle of radius ϵ_n and we denote it by c_{ϵ_n}. We choose ϵ_n so that $\epsilon_n \neq |z| q^k$ for all $k \in \mathbb{N}$. This leads to the following theorem

Theorem 4.24. *Let $R > 0$ and $h(\xi)$ be an analytic function in $|z| < R$. Let G be a branch domain of the logarithmic function and $f(\xi) = \xi^a h(\xi)$, $a > -1$, $\xi \in \Omega := D_R \cap G$. Then for each $z \in \Omega$*

$$D_q^\alpha f(z) = \lim_{n \to \infty} \int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1} (z/\xi; q)_{\alpha+1}} \, d\xi,$$

where $\Gamma_n \subset \Omega$ is the contour in Fig. 4.1.

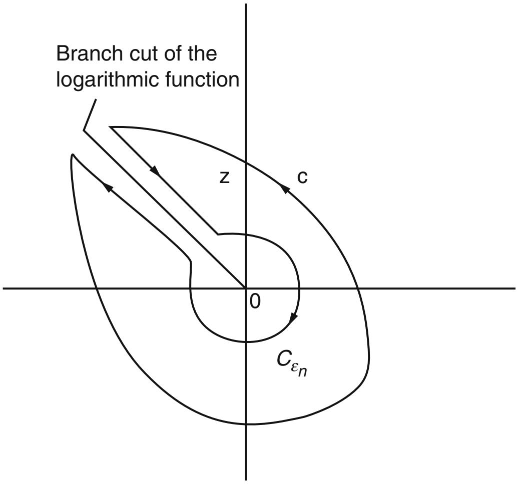

Fig. 4.1 Contour integration Γ_n

Proof. Since z is an interior point of Γ_n, and the set of points $\{zq^k,\ k = 0, 1, \dots, n\}$ lies inside Γ_n, from the Cauchy's residue theorem we obtain

$$\int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi = 2\pi i \sum_{k=0}^{n} Res(f;a_k);\ a_k := zq^k,$$

$$\begin{aligned}\operatorname{Res}(f;a_k) &= \lim_{\xi\to a_k} (\xi - a_k)\frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}} \\ &= q^k\, z^{-\alpha}\,\frac{(q^{-\alpha};q)_k(q^{\alpha+1};q)_\infty}{(q;q)_k(q;q)_\infty}.\end{aligned}$$

Hence,

$$\frac{\Gamma_q(\alpha+1)}{2\pi i}\int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi = (1-q)^{-\alpha}\sum_{k=0}^{n} q^k \frac{(q^{-\alpha};q)_k}{(q;q)_k} f(zq^k). \quad (4.83)$$

Since $f \in \mathscr{L}_q^1(G)$, the convergence of the finite sums in the previous equation as $n \to \infty$ is confirmed. Therefore, from (4.26) by letting $n \to \infty$ in (4.83) we get $D_q^\alpha f(z)$. ☐

In the following lemma, we need to integrate the function $f(\xi)/\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}$ on the branch cut of ξ^α or equivalently the logarithmic function. We apply the celebrated technique of integrating functions with branch cuts along their branch cuts. See for example [268]. Briefly, the value of the function on each side of the cut being determined by continuity from that side.

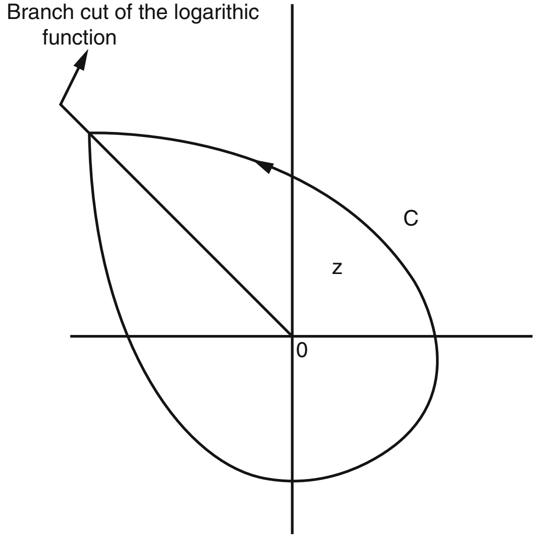

Fig. 4.2 Contour integration C

Lemma 4.25. *Let f, Ω, and α be as in Theorem 4.24. Then*

$$D_q^{\alpha} f(z) = \int_C \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi,$$

where C is a positively oriented contour in Fig. 4.2.

Proof. From Fig. 4.3, it is obvious that

$$\int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi = \left[\sum_{i=1}^{2}\int_{R_i} + \int_{C_{\epsilon_n}} + \int_C\right] \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi.$$

Since the function $\dfrac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}$ is analytic in the region bounded by R_i, $i = 1, 2$, its integration is zero on R_i, $i = 1, 2$. Hence, using

$$\left|\int_{C_{\epsilon_n}} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}\, d\xi}\right| \le M(\epsilon_n; f) \max_{|\xi|=\epsilon_n} \left|\frac{1}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\right| 2\pi\, \epsilon_n \to 0$$

as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}\, d\xi} = \int_C \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi,$$

and the lemma follows. □

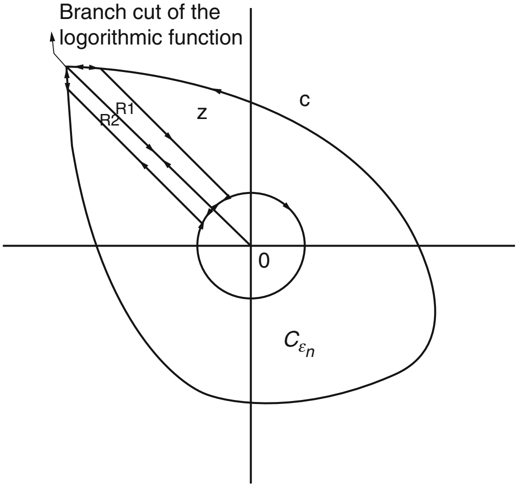

Fig. 4.3 The contour Γ_n can be represented as the sum of the contours C_{ϵ_n}, C and the contours R_1, R_2

Theorem 4.26. *Let $\alpha \in \mathbb{R}$ and G be an open simply connected domain of the complex plane containing the origin. Let f_n, $n \in \mathbb{N}$, be a family of complex functions defined on G, except may be at the origin. If for $z \in G \setminus \{0\}$*

(1) $D_q^\alpha f_n$ *exists for each* $z \in G$,

(2) $\sum_{n,m} q^m |f_n(zq^m)| < \infty$,

then

$$D_q^\alpha \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} D_q^\alpha f_n(z).$$

Proof. Set $F(z) := \sum_{n=0}^{\infty} f_n(z)$. For $z \in G$

$$\begin{aligned} D_q^\alpha F(z) &= z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} F(zq^m) \\ &= z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} \sum_{n=0}^{\infty} f_n(zq^m). \end{aligned}$$

But from the absolute convergence of the series $\sum_{n,m} q^m f_n(zq^m)$, we can interchange the order of summations to obtain

$$\begin{aligned} D_q^\alpha F(z) &= \sum_{n=0}^{\infty} z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} f_n(zq^m) \\ &= \sum_{n=0}^{\infty} D_q^\alpha f_n(z). \end{aligned}$$

□

4.7 The Law of Exponents

We have previously mentioned that Agarwal in [17] proved the semigroup property (4.62), which is

$$I_q^{\alpha}\left(I_q^{\beta} f(x)\right) = I_q^{\alpha+\beta} f(x) = I_q^{\beta}\left(I_q^{\alpha} f(x)\right) \quad (\alpha,\, \beta > 0).$$

Sometimes this rule is called the law of exponents. This rule may not be generalized when α and / or β are negative real numbers. As an example, take $\alpha = -1/2$ and $\beta = -3/2$ and $f(x) = x^{1/2}$. Then,

$$I_q^{-1/2} f(x) = D_q^{1/2} f(x) = \Gamma_q(3/2)$$
$$I_q^{-3/2} f(x) = D_q^{3/2} f(x) = 0$$

Therefore,

$$D_q^{3/2} D_q^{1/2} f(x) = \frac{\Gamma_q(3/2)}{\Gamma_q(-1/2)} x^{-3/2},$$

while

$$D_q^{1/2} D_q^{3/2} f(x) = 0.$$

That is,

$$D_q^{3/2} D_q^{1/2} x^{1/2} \neq D_q^{1/2} D_q^{3/2} x^{1/2}.$$

In the following theorem, we state precise conditions under which the law of exponents holds for arbitrary fractional operator.

Theorem 4.27. *Let α_1, α_2 and β be positive real numbers and let G be an open, simply connected domain of the complex plane containing the origin. If $f(z) = z^{\beta-1}\sum_{n=0}^{\infty} a_n z^n$ is defined on G, except may be at the origin then*

$$D_q^{\alpha_1} D_q^{\alpha_2} f(z) = D_q^{\alpha_2} D_q^{\alpha_1} f(z),$$

for all $z \in G \setminus \{0\}$ if

(a) $\alpha_1 < \beta$ and α_2 is arbitrary.

(b) $\alpha_1 \geq \beta$, α_2 is arbitrary, and $a_k = 0$ for $k = 0, 1, \ldots, m-1$, where m is the smallest integer greater than or equal to α_1.

Proof. The proof is similar to the proof of [213, Theorem 3] and is omitted. □

4.8 Miscellaneous Examples

It is known that

$$D_{0+}^{1/2}\exp(x) = \frac{1}{\sqrt{\pi\ x}} + \exp(x)\,\mathrm{erf}(\sqrt{x}), \tag{4.84}$$

$$I_{0+}^{1/2}\exp(x) = \exp(x)\,\mathrm{erf}(\sqrt{x}). \tag{4.85}$$

The following theorem gives q-analogues of (4.84) and (4.85).

Theorem 4.28. *For* $|x| < \frac{1}{1-q}$, *we have*

$$D_q^{1/2}e_q(x(1-q)) = \frac{1}{\sqrt{x}\Gamma_q(1/2)} + e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q}), \tag{4.86}$$

$$I_q^{1/2}e_q(x(1-q)) = e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q}). \tag{4.87}$$

For $x \in \mathbb{C}$, *we have*

$$D_q^{1/2}E_q(x(1-q)/\sqrt{q}) = \frac{1}{\sqrt{x}\Gamma_q(1/2)} + \frac{E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})}{\sqrt{q}K_q(1/2)}, \tag{4.88}$$

$$I_q^{1/2}E_q(x\sqrt{q}(1-q)) = \frac{1}{K_q(1/2)}E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q}). \tag{4.89}$$

Proof. First, we prove the identities in (4.86) and (4.87). From the series representation of the function e_q, we obtain

$$\begin{aligned} D_q^{1/2}e_q(x(1-q)) &= \sum_{n=0}^{\infty}\frac{x^{n-1/2}}{\Gamma_q(n+1/2)} \\ &= \frac{1}{\Gamma_q(1/2)\sqrt{x}} + \sum_{n=0}^{\infty}\frac{x^{n+1/2}}{\Gamma_q(n+3/2)}. \end{aligned} \tag{4.90}$$

On the other hand,

$$\begin{aligned} &\mathrm{Erf}(\sqrt{x};\sqrt{q})e_q(x(1-q)) \\ &= \frac{(1-q)}{\Gamma_q(1/2)}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\frac{x^{n+1/2}}{\Gamma_q(n+1)(1-q^{n+1/2})} \times \sum_{n=0}^{\infty}\frac{x^n}{\Gamma_q(n+1)} \\ &= \frac{(1-q)}{\Gamma_q(1/2)}\sum_{n=0}^{\infty}\frac{x^{n+1/2}}{\Gamma_q(n+1)}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\frac{q^k}{1-q^{k+1/2}}. \end{aligned}$$

Since

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \frac{q^k}{1-q^{k+1/2}} &= \frac{1}{1-q^{1/2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{1/2};q)_k}{(q^{3/2};q)_k (q;q)_k} q^{(n+1)k} \\ &= \frac{1}{1-q^{1/2}} {}_2\phi_1 \left(q^{1/2}, q^{-n}; q^{3/2}; q, q^{n+1}\right) \\ &= \frac{1}{1-q^{1/2}} \frac{(q;q)_n}{(q^{3/2};q)_n},\end{aligned}$$

where we used (1.11). Hence,

$$\begin{aligned}\mathrm{Erf}(\sqrt{x};\sqrt{q}) e_q(x(1-q)) &= \frac{1-q}{1-q^{1/2}\Gamma_q(1/2)} \sum_{n=0}^{\infty} \frac{(1-q)^n x^{n+1/2}}{(q^{3/2};q)_n} \\ &= \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+3/2)}.\end{aligned} \tag{4.91}$$

Combining (4.90) and (4.91) yield (4.86). The proof of (4.87) follows from (4.91) and the fact that

$$I_q^{1/2} e_q(x(1-q)) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+3/2)}.$$

□

As for the identities in (4.88) and (4.89), one can verify that

$$D_q^{1/2} E_q(x(1-q)/\sqrt{q}) = \frac{1}{\sqrt{x}\Gamma_q(1/2)} + q^{-1/2} \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}, \tag{4.92}$$

while

$$E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q}) = K_q(1/2) \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}. \tag{4.93}$$

Substituting (4.93) into (4.92), we obtain (4.88). The proof of (4.89) follows from (4.93) and

$$I_q^{1/2} E_q(x\sqrt{q}(1-q)) = \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}.$$

□

Corollary 4.29.

$$\begin{aligned}
D_q^{1/2}\left(e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})\right) &= e_q(x(1-q)),\\
I_q^{1/2}\left(e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})\right) &= \left(e_q(x(1-q))-1\right),\\
D_q^{1/2}\left(E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})\right) &= K_q(1/2)E_q(\sqrt{q}(1-q)x),\\
I_q^{1/2}\left(E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})\right) &= \sqrt{q}K_q(1/2)\left(E_q(x(1-q)/\sqrt{q})-1\right).
\end{aligned}$$

The following result is a q-analogue for the one proved in [178, P. 261].

Theorem 4.30. *For* $|x| < \frac{1}{1-q}$,

$$\gamma_q(a,x) = \Gamma_q(a)e_q(-x(1-q))I_q^aE_q(q^ax(1-q)), \tag{4.94}$$

$$\Gamma_q(a,x) = \Gamma_q(a)E_q(q^{-1}x(1-q))I_q^ae_q(q^{-1}x(1-q)). \tag{4.95}$$

Proof. First, we prove the first identity. A direct computation shows

$$I_q^aE_q(q^ax(1-q)) = \sum_{k=0}^{\infty} q^{ak}q^{\binom{k}{2}}\frac{x^{k+a}}{\Gamma_q(k+a+1)}.$$

Hence,

$$\begin{aligned}
&e_q(-x(1-q))I_q^aE_q(q^ax(1-q))\\
&= \frac{1}{\Gamma_q(a+1)}\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+a}}{\Gamma_q(n+1)}{}_2\phi_1(q^{-n},q;q^{a+1};q,q^{n+a}).
\end{aligned}$$

Using (1.11) we obtain

$$e_q(-x(1-q))I_q^aE_q(q^ax(1-q)) = \frac{\gamma_q(a,x)}{\Gamma_q(a)},$$

and (4.94) follows. The proof of (4.95) follows from

$$\begin{aligned}
&\Gamma_q(a+1)E_q(q^{-1}x(1-q))I_q^ae_q(q^{-1}x(1-q))\\
&= \sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{x^{n+a}}{\Gamma_q(n+1)}{}_2\phi_1(q^{-n},q;q^{a+1};q,q).
\end{aligned}$$

Then using (1.10), we obtain (4.95). □

Chapter 5

Other q-Fractional Calculi

Abstract In this chapter we investigate q-analogues of some known fractional operators. This chapter includes as well the fractional generalization of the Askey–Wilson operator introduced in (Ismail and Rahman, J. Approx. Theor. 114(2), 269–307, 2002). At the end of this chapter we introduce a fractional generalization of the q-difference operator introduced in (Rubin, J. Math. Anal. Appl. 212(2), 571–582, 1997).

5.1 Basic Grünwald–Letnikov Fractional Derivative

In this section we define a q-analogue of the Grünwald–Letnikov fractional derivative. If f is a function defined on $[0, a]$ $(a > 0)$ then from (1.25), $D_q^k f(x)$, $k \in \mathbb{N}_0$, $x \in (0, a]$ is given by

$$D_q^k f(x) = \frac{1}{x^k(1-q)^k} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{f(q^j x)}{q^{j(j-1)/2+j(k-j)}}. \tag{5.1}$$

Since $\begin{bmatrix} k \\ j \end{bmatrix}_q = 0$ for all $j > k$, then

$$D_q^k f(x) = \frac{1}{x^k(1-q)^k} \sum_{j=0}^{\infty} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{f(q^j x)}{q^{j(j-1)/2+j(k-j)}}. \tag{5.2}$$

The right hand side of (5.2) has a meaning when we replace k by any real number α. Therefore, we define the q-fractional operator

$$\mathbf{D}_q^\alpha f(x) = \frac{1}{x^\alpha(1-q)^\alpha} \sum_{j=0}^{\infty} (-1)^j \begin{bmatrix} \alpha \\ j \end{bmatrix}_q \frac{f(q^j x)}{q^{j(j-1)/2+j(\alpha-j)}}. \tag{5.3}$$

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The operator (5.3) is a q-analogue of the Grünwald–Letnikov fractional operator defined above. From (1.5)

$$(-1)^j \begin{bmatrix} \alpha \\ j \end{bmatrix}_q q^{j(j+1)/2-j\alpha} = q^j \frac{(q^{-\alpha};q)_j}{(q;q)_j} \quad (j \in \mathbb{N}_0)\,.$$

Then

$$\begin{aligned} \mathbf{D}_q^\alpha f(x) &= x^{-\alpha}(1-q)^{-\alpha} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2} q^{\alpha j} \begin{bmatrix} \alpha \\ j \end{bmatrix}_q f(q^j x) \\ &= x^{-\alpha}(1-q)^{-\alpha} \sum_{j=0}^{\infty} q^j \frac{(q^{-\alpha};q)_j}{(q;q)_j} f(q^j x). \end{aligned}$$

From (4.26), the q-fractional operator $\mathbf{D}_q^\alpha$, $\alpha \in \mathbb{R}$, coincides with the Riemann–Liouville fractional operator D_q^α.

5.2 Caputo Fractional q-Derivative

Definition 5.2.1. For $\alpha > 0$, the Caputo fractional q-derivative of order α is defined by

$$ {}^cD_q^\alpha f(x) := I_q^{n-\alpha} D_q^n f(x) \quad (n := \lceil \alpha \rceil)\,. \tag{5.4}$$

It is obvious form the definition that if α is a nonnegative integer, then

$$ {}^cD_q^n f(x) = I_q^0 D_q^n f(x) = D_q^n f(x).$$

Theorem 5.1. *Let* $\alpha > 0$ *and* $n = \lceil \alpha \rceil$. *If* $f \in \mathscr{A}C_q^{(n)}[0,a]$ *then* ${}^cD_q^\alpha f(x) \in \mathscr{L}_q^1[0,a]$.

Proof. The proof is straightforward since if $f \in \mathscr{A}C_q^{(n)}[0,a]$ then $D_q^n f \in \mathscr{L}_q^1[0,a]$. Hence, applying Lemma 4.9 yields

$$ {}^cD_q^\alpha f(x) = I_q^{n-\alpha} D_q^n f(x) \in \mathscr{L}_q^1[0,a].$$

□

Example 5.2.1. Let $f(x) = x^{\beta-1}$, $\beta > 0$. Then

$$ {}^cD_q^\alpha f(x) = \begin{cases} 0, & \text{if } \beta \in \{1,2,3,\dots,n\}\,, \\ \dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)} x^{\beta-\alpha-1}, & \beta > n\,. \end{cases} \tag{5.5}$$

A few additional examples of Caputo-type derivatives of certain important functions are collected for the readers convenience in Table A.5.

In the following theorem, we discuss the results of combining the Riemann–Liouville fractional q-derivative and Caputo fractional q-derivative of not necessarily equal orders.

Theorem 5.2. *Let α and β be positive numbers and let $n = \lceil \alpha \rceil$ and $m = \lceil \beta \rceil$. Then the following relations hold:*

1. *If $f \in \mathscr{A}C_q^{(m)}[0,a]$, then*

$$I_q^{\alpha}\, {}^cD_q^{\beta} f(x) = \begin{cases} I_q^{\beta-\alpha} f(x) - \sum_{j=0}^{m-1} \frac{D_q^j f(0^+)}{\Gamma_q(\beta-\alpha+j+1)} x^{\beta-\alpha+j}, & \beta \geq \alpha, \\ D_q^{\beta-\alpha} f(x) - \sum_{j=0}^{m-1} \frac{D_q^j f(0^+)}{\Gamma_q(\beta-\alpha+j+1)} x^{\beta-\alpha+j}, & \beta < \alpha, \end{cases} \tag{5.6}$$

for all $x \in (0,a]$.

2. *If $f \in \mathscr{L}_q^1[0,a]$ such that $I_q^{\alpha} f \in \mathscr{A}C_q^{(m)}[0,a]$, then*

$${}^cD_q^{\beta}\, I_q^{\alpha} f(x) = \begin{cases} I_q^{\alpha-\beta} f(x), & \alpha \geq m \geq \beta, \\ D_q^{\beta-\alpha} f(x) - \sum_{j=0}^{\lceil m-\alpha \rceil} \dfrac{D_q^{m-\alpha-j} f(0^+)}{\Gamma_q(m-\beta-j+1)} x^{m-\beta-j}, & \beta > \alpha, \end{cases} \tag{5.7}$$

for all $x \in (0,a]$.

Proof. First we prove (5.6).

$$I_q^{\alpha}\, {}^cD_q^{\beta} f(x) = I_q^{\alpha} I_q^{m-\beta} D_q^m f(x) = I_q^{m+\alpha-\beta} D_q^m f(x).$$

Then (5.6) follows for $\alpha \geq m \geq \beta$ by applying (4.71) and follows for $\beta > \alpha$ by applying (4.72), where α and β of (4.71) and (4.72) are replaced by $m + \alpha - \beta$ and m, respectively. Now, we prove (5.7). Since

$${}^cD_q^{\beta}\, I_q^{\alpha} f(x) = I_q^{m-\beta} D_q^m I_q^{\alpha} f(x),$$

then applying (4.71) and (4.72) give

$$D_q^m I_q^{\alpha} f(x) = \begin{cases} I_q^{\alpha-m}, & \text{if } \alpha \geq m \geq \beta, \\ D_q^{m-\alpha}, & \text{if } \alpha < m. \end{cases}$$

Hence, if $\alpha \geq m \geq \beta$ then from the semigroup property (4.62)

$${}^cD_q^{\beta} I_q^{\alpha} f(x) = I_q^{m-\beta} I_q^{\alpha-m} f(x) = I_q^{\alpha-\beta} f(x).$$

Now, if $\alpha < m$, then

$$ {}^{c}D_q^{\beta} I_q^{\alpha} f(x) = I_q^{m-\beta} D_q^{m-\alpha} f(x). $$

Hence, applying (4.73) (where α and β of (4.73) are replaced with $m-\beta$ and $m-\alpha$, respectively) yields the second part of (5.7). ☐

We need the following q-analogue of the integral mean value theorem.

Theorem 5.3. *(Mean Value Theorem for q-Integrals) Let f and g be two functions defined on $[a,b]$. Assume that*

(1) The function g is positive on $[a,b]$ and q-regular at zero,
(2) The function f is a continuous on $[0,a]$.

Then there exists $c \in (0,a)$ such that

$$ f(c) = \frac{\int_0^a f(t)g(t)\, d_qt}{\int_0^a g(t)\, d_qt}. $$

Proof. Let

$$ M := \max\{f(x) : x \in [0,a]\} \quad \text{and} \quad m := \min\{f(x) : x \in [0,a]\}. $$

Then

$$ m \le \frac{\int_0^a f(t)g(t)\, d_qt}{\int_0^a g(t)\, d_qt} \le M. $$

Hence, the theorem follows since $f([0,a]) = [m,M]$. ☐

Moreover, we shall also need the following q-Taylor formula. In [33], the authors proved a stronger q-Taylor formula.

Theorem 5.4. *Let $f \in \mathscr{A}C_q^{(n)}[0,a]$ be such that $D_q^n f \in C[0,a]$. Then*

$$ f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x (qt/x;q)_{n-1} D_q^n f(t)\, d_qt. $$

Proof. Since

$$ \sum_{k=0}^{n-1} I_q^{k+1} D_q^{k+1} f(x) - I_q^k D_q^k f(x) = I_q^n D_q^n f(x) - f(x), $$

from the fundamental theorem of q-calculus and Theorem 1.10 we obtain

$$\begin{aligned} I_q^{k+1} D_q^{k+1} f(x) - I_q^k D_q^k f(x) &= I_q^k \left(I_q D_q^{k+1} f(x) - D_q^k f(x) \right) \\ &= I_q^k \left(D_q^k f(0^+) \right) = D_q^k f(0^+) \frac{x^k}{\Gamma_q(k+1)}, \end{aligned}$$

$k = 0, 1, \ldots, n-1$. Consequently,

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k + I_q^n D_q^n f(x).$$

Hence, the result follows from the basic Cauchy integral formula (4.25). ☐

Remark 5.2.1. There are q-analogues of the mean value theorems in [253] for functions defined on an interval of the form $[a, b]$ and a is not necessarily equal to zero. These q-analogues are not valid for all q, $0 < q < 1$, but it is valid for $q \in [q_0, 1)$ for some $q_0 > 0$.

In the following theorem we introduce an identity relation between the Caputo fractional q-derivative and the Riemann–Liouville fractional q-derivative. This identity is a q-analogue of the result due to Diethelm in [82, Theorem 3.1].

Theorem 5.5. *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. Assume that $f \in \mathscr{A}C_q^{(n)}[0, a]$ and $D_q^n f \in C[0, a]$. Then*

$$^{c}D_q^{\alpha} f(x) = D_q^{\alpha} \left(f(x) - T_{n-1}[f; 0] \right) \quad (x \neq 0) \tag{5.8}$$

and

$$T_{n-1}[f; 0] := \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k.$$

Proof. Combining Theorems 5.4 and 5.3 yields

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k + D_q^n f(\xi) \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x (qt/x; q)_{n-1} \, d_q t,$$

for some $\xi \in [0, x]$. Using the substitution $t = x\xi$, one can easily prove that

$$\int_0^x (qt/x; q)_{n-1} \, d_q t = x B_q(n, 1).$$

Hence,

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k + \frac{x^n}{\Gamma_q(n+1)} D_q^n f(\xi). \tag{5.9}$$

Acting on the two sides of (5.9) by the operators ${}^cD_q^\alpha$ and D_q^α gives

$$ {}^cD_q^\alpha f(x) = \frac{x^{n-\alpha}}{\Gamma_q(n-\alpha+1)} D_q^n f(\xi) $$

and

$$ D_q^\alpha f(x) = D_q^\alpha \left(\sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k \right) + \frac{x^{n-\alpha}}{\Gamma_q(n-\alpha+1)} D_q^n f(\xi). $$

Hence,

$$ D_q^\alpha \left[f(x) - \left(\sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k \right) \right] = {}^cD_q^\alpha f(x). $$

□

One can observe that the existence of the left hand side of (5.8) requires weaker conditions than the conditions of Theorem 5.5. This is similar to the classical case which was introduced independently by many authors including Caputo. See the book of Diethelm, [82], and the references therein. Therefore, we introduce the following definition

Definition 5.2.2. Assume that $\alpha \geq 0$ and f is a function such that

$$ D_q^\alpha \left(f(x) - T_{n-1}[f;0] \right) \quad (n = \lceil \alpha \rceil) $$

exists. Then we define the function $D_{*q}^\alpha f$ by

$$ D_{*q}^\alpha f := D_q^\alpha \left(f(x) - T_{n-1}[f;0](x) \right). \tag{5.10} $$

The operator D_{*q} is called the Caputo fractional q-difference operator of order α.

If α is an integer, say m, then the integer n of (5.10) is equal to m and $D_{*q}^m f = D_q^m f(x)$ since $D_q^m \left[T_{m-1}[f;0] \right](x) = 0$.

Lemma 5.6. *Assume that* $\alpha \geq 0$ *and* $D_q^\alpha \left(f(x) - T_{n-1}[f;0] \right)$ *exists where* $n = \lceil \alpha \rceil$. *Then*

$$ D_{*q}^\alpha f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x (qt/x;q)_{-\alpha-1} \left(f(t) - T_{n-1}[f;0](t) \right) d_qt \;\; (\alpha \notin \mathbb{N}_0). $$

Proof. The proof follows directly from Lemma 4.8 and is omitted. □

Corollary 5.7. *Assume that* $\alpha > 0$ *and* $D_q^\alpha \left(f(x) - T_{n-1}[f;0] \right)$ *exists where* $n = \lceil \alpha \rceil$. *Then*

$$ D_{*q}^\alpha f(x) = x^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} \left(f - T_{n-1}[f;0] \right)(xq^m). \tag{5.11} $$

Proof. The proof follows directly from the definition of Jackson q-integration, see (1.19). ☐

Corollary 5.8. *Assume that* $\alpha > 0$ *and* $D_q^\alpha\left(f - T_{n-1}[f;0]\right)(x)$ *exists where* $n = \lceil\alpha\rceil$. *Then*

$$D_{*q}^\alpha f(x) = D_q^\alpha f(x) - \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k-\alpha+1)} x^{k+\alpha-1}. \tag{5.12}$$

Proof. Since from Example 4.51

$$\begin{aligned} D_q^\alpha T_{n-1}(x) &= D_q^\alpha \sum_{k=0}^{n-1} D_q^k f(0^+) \frac{x^k}{\Gamma_q(k+1)} \\ &= \sum_{k=0}^{n-1} D_q^k f(0^+) \frac{x^{k-\alpha}}{\Gamma_q(k-\alpha+1)}. \end{aligned} \tag{5.13}$$

and from (5.11),

$$D_{*q}^\alpha f(x) = D_q^\alpha f(x) - D_q^\alpha T_{n-1}[f;0](x), \tag{5.14}$$

then the corollary results by substituting with (5.13) in (5.14). ☐

Corollary 5.8 leads to the following immediate result.

Corollary 5.9. *Assume that* $\alpha \geq 0$ *and* $D_q^\alpha\left(f(x) - T_{n-1}[f;0]\right)$ *exists where* $n = \lceil\alpha\rceil$. *Then*

$$D_{*q}^\alpha f = D_q^\alpha f$$

if and only if

$$D_q^k f(0^+) = 0 \quad (k = 0, 1, \ldots, n-1).$$

5.3 Erdéli–Kober Fractional q-Integral Operator

Agarwal [17] defined a two parameter family of fractional q-operator corresponding to the basic integral (1.19). This operator is defined by

$$I_q^{\eta,\alpha} f(x) = \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x;q\right)_{\alpha-1} t^\eta f(t)\, d_q t \quad (\alpha \neq -1, -2, \ldots). \tag{5.15}$$

Using the definitions of the q-integration (1.19) and the q-gamma function (1.57), the definition of $I_q^{\eta,\alpha}$ can be extended to any $\alpha \in \mathbb{C}$ through the identity

$$I_q^{\eta,\alpha} f(x) = (1-q)^\alpha \sum_{k=0}^\infty q^{(\eta+1)k} \frac{\left(q^\alpha;q\right)_k}{(q;q)_k} f(xq^k). \tag{5.16}$$

The operator (5.15) is a q-analogue of the Erdéli–Kober fractional integral operator defined by

$$I^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt,$$

see [94, 173]. It is clear that

$$I_q^{\eta,\alpha} f(x) = x^{-\eta-\alpha} I_q^{\alpha} \left(x^{\eta} f(x) \right).$$

Agarwal [17] proved the following semigroup identity when η, λ, and μ are positive constants.

$$\begin{aligned} I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu} f(x) &= I_q^{\eta,\mu+\lambda} = I_q^{\eta+\lambda,\mu} I_q^{\eta,\lambda} f(x) \\ &= I_q^{\eta,\mu} I_q^{\mu+\eta,\lambda} f(x) = I_q^{\eta+\mu,\lambda} I_q^{\eta,\mu} f(x). \end{aligned} \tag{5.17}$$

It should also be mentioned here that in most of the proofs of [17, 19], the domain where the fractional integrals and the related properties hold is not determined precisely. A very restrictive condition might be found in [17, P. 366]. We give a sufficient for the existence of $I_q^{\eta,\alpha}$ in the following proposition.

Proposition 5.10. *Let η, r, α, and a be real numbers that satisfy*

$$\eta > -1, \; r \geq 1, \; \alpha > 0 \; \text{ and } \; a > 0.$$

If $f \in \mathscr{L}^r_{q,\eta}(0,a)$ $(a > 0)$ then $I_q^{\eta,\alpha} f(x)$ exists for all $x \in (0,a]$. Moreover,

$$I_q^{\eta,\alpha} f(x) \in \mathscr{L}^r_{q,\eta+\beta}(0,a) \;\text{ for all } \beta > 0.$$

Proof. First, assume that $r > 1$ and let $s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Assume that $f \in \mathscr{L}^r_{q,\eta}(0,a)$. Then using the Minkowski's inequality we obtain

$$\begin{aligned} \left| I_q^{\eta,\alpha} f(x) \right| &= \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \left| \int_0^x (qt/x;q)_{\alpha-1} t^{\eta} f(t)\, d_q t \right| \\ &\leq \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \left(\int_0^x t^{\eta} (qt/x;q)^s_{\alpha-1}\, d_q t \right)^{1/s} \left(\int_0^x t^{\eta} |f(t)|^r\, d_q t \right)^{1/r}. \end{aligned}$$

Using the substitution $t = x\xi$ we can prove that

$$\int_0^x t^{\eta} (qt/x;q)^s_{\alpha-1}\, d_q = x^{\eta+1} \int_0^1 \xi^{\eta} (q\xi;q)^s_{\alpha-1}\, d_q \xi.$$

Set

$$M^s := \frac{1}{\Gamma_q^s(\alpha)} \int_0^1 \xi^{\eta} (q\xi;q)^s_{\alpha-1}\, d_q \xi.$$

We obtain

$$\left|I_q^{\eta,\alpha} f(x)\right| \leq M x^{-\frac{\eta+1}{r}} \left(\int_0^x t^\eta |f(t)|^r \, d_q t\right)^{1/r} < \infty,$$

for all $x \in (0, a]$. Hence,

$$\begin{aligned}\left(\int_0^x t^{\eta+\beta} \left(I_q^{\eta,\alpha} f(t)\right)^r \, d_q t\right)^{1/r} &\leq M \left(\int_0^x t^\eta |f(t)|^r \, d_q t\right)^{1/r} \left(\int_0^x t^{\beta-1} \, d_q t\right)^{1/r} \\ &= M \left(\frac{1-q}{1-q^\beta} x^\beta\right)^{1/r} \left(\int_0^x t^\eta |f(t)|^r \, d_q t\right)^{1/r}.\end{aligned}$$

If $r = 1$, then one can prove that there exists $R > 0$ such that

$$\left|I_q^{\eta,\alpha} f(x)\right| \leq R x^{-\eta-1} \int_0^x t^\eta |f(t)| \, d_q t \leq R x^{-\eta-1} \|f\|_{\eta,1}.$$

Hence,

$$\int_0^x t^{\eta+\beta} |I_q^{\eta,\alpha} f(t)| \, d_q t \leq R a^\beta \frac{1-q}{1-q^\beta} \|f\|_{\eta,1}.$$

□

Now we prove the semigroup identity (5.17).

Proposition 5.11. *If η, λ, and μ, are positive numbers, and $f \in \mathscr{L}^r_{q,\eta}(0, a)$, $r \geq 1$, then the semigroup property (5.17) holds for all $x \in (0, a]$.*

Proof. It is enough to prove the most left identity in (5.17).

$$\begin{aligned}I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu} f(x) &= \frac{x^{-\eta-1}}{\Gamma_q(\lambda)} \int_0^x (qt/x; q)_{\lambda-1} t^\eta I_q^{\eta+\lambda,\mu} f(t) \, d_q t \\ = \frac{(1-q)^2 x}{\Gamma_q(\lambda)\Gamma_q(\mu)} &\sum_{k=0}^\infty q^{-k\lambda} (q^{k+1}; q)_{\lambda-1} \sum_{n=k}^\infty q^{n(\mu+\lambda+1)} (q^{n-k+1}; q)_{\mu-1} f(xq^n).\end{aligned} \tag{5.18}$$

Now if $f \in \mathscr{L}^r_{q,\eta}(0, a)$ then

$$\begin{aligned}\left|\sum_{n=k}^\infty q^{n(\mu+\lambda+1)} (q^{n-k+1}; q)_{\mu-1} f(xq^n)\right| &\leq M \sum_{n=k}^\infty q^{n(\mu+\lambda+1)} |f(xq^n)| \\ &\leq \left(\sum_{n=k}^\infty q^{n(\eta+1)} |f(xq^n)|^r\right)^{1/r} \left(\sum_{n=k}^\infty q^{n(\eta+1+\lambda s)}\right)^{1/s}.\end{aligned}$$

But

$$\left(\sum_{n=k}^{\infty} q^{n(\eta+1+\lambda s)}\right)^{1/s} = \frac{q^{k\lambda} q^{k(\eta+1)/s}}{(1-q^{\eta+1+\lambda s})^{1/s}}.$$

Hence, the double series in (5.18) is absolutely convergent. Then we can interchange the order of summation to obtain

$$I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu} f(x)$$
$$= \frac{(1-q)^2 x}{\Gamma_q(\lambda)\Gamma_q(\mu)} \sum_{n=0}^{\infty} q^{n(\eta+\lambda+1)} f(xq^n) \sum_{k=0}^{n} q^{-k\lambda} (q^{k+1};q)_{\lambda-1} (q^{n-k+1};q)_{\mu-1}.$$

Applying (1.62) with $\alpha = \mu$ and $\beta = \lambda$ gives

$$\sum_{k=0}^{n} q^{-n\lambda} (q^{k+1};q)_{\lambda-1} (q^{n-k+1};q)_{\mu-1} = B_q(\lambda,\mu)(q^{n+1};q)_{\lambda+\mu-1} q^{-\lambda n}.$$

This gives after straightforward manipulations that

$$I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu} f(x) = I_q^{\eta,\lambda+\mu} f(x).$$

□

5.4 Weyl Fractional q-Calculus

In previous chapters we dealt almost exclusively with the Riemann–Liouville fractional q-calculus introduced independently by Al-Salam and Agarwal. Here, we concentrate on the Weyl-fractional q-calculus. The q-analogue of the Weyl fractional operator (4.28) is defined in (4.27). Using (1.20) we can write (4.27) explicitly as

$$K_q^{-\alpha}\phi(x) = q^{-\frac{\alpha(\alpha+1)}{2}} x^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q \phi(xq^{-\alpha-k}), \tag{5.19}$$

or in a more simpler form

$$K_q^{-\alpha}\phi(x) = q^{-\frac{\alpha(\alpha+1)}{2}} x^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} q^{-k\alpha} \frac{(q^{\alpha};q)_k}{(q;q)_k} \phi(xq^{-\alpha-k}). \tag{5.20}$$

Using (1.25) we can prove

$$K_q^{n}\phi(x) = (-1)^n D_q^n \phi(x) \quad (n \in \mathbb{N}). \tag{5.21}$$

In the following we shall characterize a sufficient class of functions for which $K_q^{-\alpha}$ exists for some α.

Definition 5.4.1. Let $\alpha \in \mathbb{C}$ and let f be a function defined on a q^{-1}-geometric set A. We say that f is of class $S_{q,\alpha}$ if there exists $\mu \in \mathbb{C}$, $\operatorname{Re}\mu > \operatorname{Re}\alpha$ such that

$$f(xq^{-n}) = O(q^{n\mu}) \text{ as } n \to \infty,\ x \in A.$$

Proposition 5.12. *If $\alpha \in \mathbb{Z}^-$ then $K_q^{-\alpha} f$ exists for any function f defined on $(0,\infty)$. If $\alpha \notin \mathbb{Z}^-$ and $f \in S_{q,\alpha}$ then $K_q^{-\alpha} f$ exists.*

Proof. If $\alpha \in \mathbb{Z}^-$ then by (5.21), $K_q^{-\alpha} f$ exists for any functions f defined on a $(0,\infty)$. If $\alpha \notin \mathbb{Z}^-$ and $f \in S_{q,\alpha}$ then for each $x > 0$ there exists a constant $C > 0$, C depends on x and α such that

$$|f(xq^{-n-\alpha})| \le Cq^{n\mu}.$$

Applying the previous inequality in (5.20) gives

$$\begin{aligned}\left|K_q^{-\alpha} f(x)\right| &\le Cq^{-\alpha(\alpha+1)/2}|x|^\alpha(1-q)^\alpha \frac{(-q^{\operatorname{Re}\alpha};q)_\infty}{(q;q)_\infty}\sum_{k=0}^\infty q^{k(\operatorname{Re}\mu-\operatorname{Re}\alpha)}\\ &\le Cq^{-\alpha(\alpha+1)/2}|x|^\alpha(1-q)^\alpha \frac{(-q^{\operatorname{Re}\alpha};q)_\infty}{(1-q^{(\operatorname{Re}\mu-\operatorname{Re}\alpha)})(q;q)_\infty}.\end{aligned}$$

□

In the following we define a sufficient class of functions $\mathscr{S}_{q,\mu}$ for which $K_q^{-\alpha} f$ exists for all α.

Definition 5.4.2. Let f be a function defined on a q^{-1}-geometric set A. We say that f is in the class $\mathscr{S}_{q,\mu}$ if there exist $\mu > 0$ and $\nu \in \mathbb{R}$ such that for each $x \in A$

$$|f(xq^{-n})| = O(q^{\mu n(n+\nu)}) \text{ as } n \to \infty.$$

It is clear that if $f \in \mathscr{S}_{q,\mu}$ then $f \in S_{q,\alpha}$ for all α. The spaces $S_{q,\alpha}$ and $\mathscr{S}_{q,\mu}$ are q-analogues of the spaces of fairly good functions and good functions, respectively, introduced by Lighthill [185, P. 15], see also [213, Chap. VII].

Example 5.4.1. An example of a function in the class $S_{q,1/2,}$ is any function of the form

$$\frac{P_n(x)}{(ax;q)_\infty} \quad \text{for all } n \in \mathbb{N}_0,$$

where $P_n(x)$ is a polynomial of degree n and a is a constant such that $axq^k \neq 1$ for all $k \in \mathbb{N}_0$.

Theorem 5.13. *Let* α *and* β *be real numbers and* $\mu > 0$*. If* ϕ *is a function defined on* $(0,\infty)$ *and* $\phi \in \mathscr{S}_{q,\mu}$ *then*

$$K_q^{\alpha} K_q^{\beta} \phi(x) = K_q^{\alpha+\beta} \phi(x) \quad (x > 0).$$

Proof. From (5.20)

$$\begin{aligned} K_q^{\alpha} K_q^{\beta} \phi(x) &= q^{\frac{1}{2}(\alpha(1-\alpha)+\beta(1-\beta))-\alpha\beta} x^{-\alpha-\beta} (1-q)^{-\alpha-\beta} \\ &\times \sum_{k=0}^{\infty} q^{k(\alpha+\beta)} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \sum_{j=0}^{\infty} q^{j\beta} \frac{(q^{-\beta};q)_j}{(q;q)_j} \phi(xq^{\alpha+\beta-(k+j)}). \end{aligned} \tag{5.22}$$

On the inner sum of (5.22) make the substitution $m = k + j$. This gives

$$\begin{aligned} K_q^{\alpha} K_q^{\beta} \phi(x) &= q^{\frac{1}{2}(\alpha(1-\alpha)+\beta(1-\beta))-\alpha\beta} x^{-\alpha-\beta} (1-q)^{-\alpha-\beta} \\ &\times \sum_{k=0}^{\infty} q^{k\alpha} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \sum_{m=k}^{\infty} q^{m\beta} \frac{(q^{-\beta};q)_{m-k}}{(q;q)_{m-k}} \phi(xq^{\alpha+\beta-m}). \end{aligned} \tag{5.23}$$

Since $\phi \in \mathscr{S}_{q,\mu}$, we can interchange the order of summations in (5.23). This gives

$$\begin{aligned} K_q^{\alpha} K_q^{\beta} \phi(x) &= q^{\frac{1}{2}(\alpha(1-\alpha)+\beta(1-\beta))-\alpha\beta} x^{-\alpha-\beta} (1-q)^{-\alpha-\beta} \\ &\times \sum_{m=0}^{\infty} q^{m\beta} \phi(xq^{\alpha+\beta-m}) \sum_{k=0}^{m} q^{k\alpha} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \frac{(q^{-\beta};q)_{m-k}}{(q;q)_{m-k}}. \end{aligned} \tag{5.24}$$

Some tedious manipulation yields

$$\sum_{k=0}^{m} q^{k\alpha} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \frac{(q^{-\beta};q)_{m-k}}{(q;q)_{m-k}} = \frac{(q^{-\beta};q)_m}{(q;q)_m} {}_2\phi_1\left(q^{-m}, q^{-\alpha}; q^{1+\beta-m}; q, q^{1+\alpha+\beta}\right).$$

Hence, applying (1.11) yields

$$\begin{aligned} \sum_{k=0}^{m} q^{k\alpha} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \frac{(q^{-\beta};q)_{m-k}}{(q;q)_{m-k}} &= \frac{(q^{-\beta};q)_m (q^{\alpha+\beta+1-m})}{(q;q)_m (q^{1+\beta-m};q)_m} \\ &= \frac{(q^{-(\alpha+\beta)};q)_m}{(q;q)_m} q^{\alpha m}, \end{aligned}$$

where we used

$$\frac{(aq^{-m};q)_m}{(bq^{-m};q)_m} = \frac{(q/a;q)_m}{(q/b;q)_m} \left(\frac{a}{b}\right)^m.$$

Consequently,

$$K_q^{\alpha} K_q^{\beta}\phi(x)$$

$$= q^{\frac{(\alpha+\beta)(1-\alpha-\beta)}{2})} x^{-\alpha-\beta}(1-q)^{-\alpha-\beta}\sum_{m=0}^{\infty} q^{m(\alpha+\beta)}\frac{(q^{-(\alpha+\beta)};q)_m}{(q;q)_m}\phi(xq^{\alpha+\beta-m})$$

$$= K_q^{\alpha+\beta}\phi(x).$$

□

Theorem 5.14. *If* $f \in L(\mathbb{R}_q)$ *and* $g \in \mathscr{S}_{q,\mu}$ *for some* $\mu > 0$ *then*

$$\int_0^{\infty} f(x)K_q^{-\alpha}g(x)\,d_qx = q^{-\alpha(\alpha+1)/2}\int_0^{\infty} g(xq^{-\alpha})I_q^{\alpha}f(x)\,d_qx. \tag{5.25}$$

Proof. Since

$$\int_0^{\infty} f(x)K_q^{-\alpha}g(x)\,d_qx$$
$$= q^{-\frac{\alpha(\alpha+1)}{2}}(1-q)^{\alpha+1}\sum_{k=-\infty}^{\infty} q^{k(1+\alpha)}f(q^k)\sum_{r=0}^{\infty} q^{-r\alpha}\frac{(q^{\alpha};q)_r}{(q;q)_r}g(q^{-\alpha+k-r}). \tag{5.26}$$

Making the substitution $m = k - r$ on the most right series in (5.26) we obtain

$$\int_0^{\infty} f(x)K_q^{-\alpha}g(x)\,d_qx$$
$$= q^{-\frac{\alpha(\alpha+1)}{2}}(1-q)^{\alpha+1}\sum_{k=-\infty}^{\infty} q^k f(q^k)\sum_{m=-\infty}^{k} q^{m\alpha}\frac{(q^{\alpha};q)_{k-m}}{(q;q)_{k-m}}g(q^{-\alpha+m}). \tag{5.27}$$

The conditions on the functions f and g guarantee the absolute convergence of the double series in (5.27). Hence, we can interchange the order of summations to obtain

$$\int_0^{\infty} f(x)K_q^{-\alpha}g(x)\,d_qx$$
$$= q^{-\frac{\alpha(\alpha+1)}{2}}(1-q)^{\alpha+1}\sum_{m=-\infty}^{\infty} q^{m\alpha}g(q^{m-\alpha})\sum_{k=m}^{\infty} q^k f(q^k)\frac{(q^{\alpha};q)_{k-m}}{(q;q)_{k-m}}. \tag{5.28}$$

Making the substitution $r = k - m$ on the second series in (5.28) gives

$$\begin{aligned}
&\int_0^\infty f(x) K_q^{-\alpha} g(x)\, d_q x \\
&= q^{-\frac{\alpha(\alpha+1)}{2}} (1-q)^{\alpha+1} \sum_{m=-\infty}^{\infty} q^{m(\alpha+1)} g(q^{m-\alpha}) \sum_{r=0}^{\infty} q^r f(q^{r+m}) \frac{(q^\alpha;q)_r}{(q;q)_r} \\
&= q^{-\alpha(\alpha+1)/2} \int_0^\infty g(xq^{-\alpha}) I_q^\alpha f(x)\, d_q x.
\end{aligned}$$

□

5.5 Fractional q-Operator as a Generalization of a Right Inverse of Askey–Wilson Operator

Let f be an integrable function of period 2π. Weyl, see Zygmund's book [300], introduced a fractional operator which is more convenient for trigonometric series than the Riemann–Liouville fractional operator. This operator is defined by

$$(I_\alpha f)(x) \sim \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{(in)^\alpha} \quad \text{if} \quad f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, c_0 = 0, \tag{5.29}$$

where $i^\alpha = e^{i\alpha\pi/2}$. He then pointed out that

$$I_\alpha f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \Phi_\alpha(x-t)\, dt, \quad \Phi_\alpha(x) = \sum_{n \neq 0} \frac{e^{inx}}{(in)^\alpha}.$$

He also proved the semigroup identity

$$I_\alpha I_\beta = I_{\alpha+\beta} \quad (\alpha, \beta > 0).$$

In [147], Ismail and Rahman defined a q-analogue of the fractional operator I_α, so that $\alpha = 1$ represents a right inverse of the Askey-Wilson operator $\mathscr{D}_q$. The Askey-Wilson operator is defined by

$$(\mathscr{D}_q f)(x) := \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{(q^{1/2} - q^{-1/2}) \sin\theta}, \ x = \cos\theta,$$

where $f(x) = \breve{f}(z)$ with $x = (z + 1/z)/2$.

Ismail and Zhang [152] introduced the q-exponential function

$$\begin{aligned}&\varepsilon(\cos\theta, \cos\phi; \alpha)\\ &:= \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \left(-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q\right)_n \frac{(\alpha e^{-\phi})^n}{(q;q)_n} q^{n^2/4}.\end{aligned}$$

It is straightforward to see that

$$\mathscr{D}_q \varepsilon_q(\cos\theta, \cos\phi; \alpha) = \frac{2\alpha q^{1/4}}{1-q} \varepsilon_q(\cos\theta, \cos\phi; \alpha).$$

If we let $\alpha = 0$ and

$$\varepsilon_q(x; \alpha) := \varepsilon_q(x, 0; \alpha),$$

then

$$\varepsilon_q(0; 1) = 1 \quad \text{and} \quad \lim_{q \to 1^-} \varepsilon_q(x; (1-q)\alpha) = \exp(2\alpha\, x).$$

Bustoz and Suslov [67] used the notation

$$w_0 = 0, \quad w_n = \frac{1}{2} J_{1/2,n}(q) \quad (n \neq 0),$$

where $\{J_{\nu,k}(q)\}$ denotes the sequence of positive zeros of $J_\nu^{(2)}(z; q)$ and $J_{\nu,-k}(q) = -J_{\nu,k}(q)$. Hence,

$$w_{-n} = -w_n.$$

Ismail and Zhang [152] proved that the sequence $\left\{\varepsilon_q(x; iw_n)\right\}$ forms an orthogonal basis of $L^2(w(x, q^{1/2}|q), [-1, 1])$, where

$$w(x, \beta|q) = \frac{\left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{\sin\theta \left(\beta e^{2i\theta}, \beta e^{-2i\theta}; q\right)_\infty}, \ 0 < \theta < \pi, \ x = \cos\theta.$$

This motivates Ismail and Rahman in [147] to define in analogy to (5.29)

$$I_{\alpha,q} f(x) = \frac{(1-q)^\alpha}{2^\alpha q^{\alpha/4}} \sum_{n \neq 0} \frac{f_n}{(iw_n)^\alpha} \varepsilon_q(x; iw_n) \quad (\alpha > 0),$$

where $f \sim \sum_{-\infty}^{\infty} f_n \varepsilon(x; iw_n)$. Hence,

$$I_{\alpha,q} f(x) = \pi_0 \int_{-1}^{1} \Psi_\alpha(x, y|q) f(y) w(y; q^{1/2}|q)\, d_q y,$$

where

$$\Psi_\alpha(x,y|q)=\frac{(1-q)^\alpha}{2^\alpha q^{\alpha/4}\pi_0}\sum_{n\neq 0}\frac{\pi_n}{(iw_n)^\alpha}\varepsilon_q(x;w_n),$$

and π_n is the constant appeared in the orthogonality relations

$$\int_{-1}^{1}\varepsilon(x,iw_n)\overline{\varepsilon(x,iw_n)}w(x;q^{1/2}|q)\,dx=\frac{\delta_{m,n}}{\pi_n}.$$

Ismail and Rahman [147] proved the semigroup identity

$$I_{\alpha,q}I_{\beta,q}=I_{\alpha+\beta,q}\quad(\alpha>0;\ \beta>0).$$

They also proved that

$$\mathscr{D}_q I_{\alpha,q}=I_{\alpha-1,q}\quad(\alpha>1).$$

5.6 Fractional q-Operator as a Generalization of a q-Difference Operator

In this section, we introduce a q-analogue of the fractional operator (5.29) as a generalization of the q-difference operator defined by Rubin in [265]. From Theorem 3.19, if $f\in L^2(\widetilde{\mathbb{R}}_q)\cap L^1(\widetilde{\mathbb{R}}_q)$ then

$$f(x)=\sum_{-\infty}^{\infty}c_n e(ix\frac{q^n}{\sqrt{1-q}};q^2)+\sum_{-\infty}^{\infty}d_n e(-ix\frac{q^n}{\sqrt{1-q}})\ \left(x\in\widetilde{\mathbb{R}}_q\right),\tag{5.30}$$

where from the orthogonality relations (3.88)–(3.90)

$$c_n=\frac{q^n}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}f(t)e(-it\frac{q^n}{\sqrt{1-q}};q^2)\,d_qt,$$

$$d_n=\frac{q^n}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}f(t)e(it\frac{q^n}{\sqrt{1-q}};q^2)\,d_qt,$$

where $C:=\dfrac{4\sqrt{1-q}(q^2;q^2)^2_\infty}{(q;q^2)^2_\infty}$. Consequently,

$$f(x)=\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}f(t)\Psi_0(x,t)\,d_qt,$$

where

$$\Psi_0(x,t) = \sum_{-\infty}^{\infty} q^n e(ix\frac{q^n}{\sqrt{1-q}})e(-it\frac{q^n}{\sqrt{1-q}}) + q^n e(-ix\frac{q^n}{\sqrt{1-q}})e(it\frac{q^n}{\sqrt{1-q}})$$

$$= \sum_{-\infty}^{\infty} q^n \operatorname{Cos}(x\frac{q^n}{\sqrt{1-q}};q^2)\operatorname{Cos}(t\frac{q^n}{\sqrt{1-q}}) + q^n \operatorname{Sin}(x\frac{q^n}{\sqrt{1-q}};q^2)\operatorname{Sin}(t\frac{q^n}{\sqrt{1-q}};q^2).$$

Lemma 5.15. *The series*

$$\sum_{n=-\infty}^{\infty} q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)\ \big(x,t\in\widetilde{\mathbb{R}}_q\big) \tag{5.31}$$

is absolutely convergent only when Re $\alpha < 1$.

Proof. The series in (5.31) can be written as

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty}\right) q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2).$$

It is known that

$$\left|\frac{(z;q)_\infty}{(q;q)_\infty}{}_1\phi_1(0;z,q,q^{1+n})\right| \le \frac{(-|z|,-q;q)_\infty}{(q;q)_\infty}\begin{cases} 1, & n\ge 0,\\ |z|^{-n}q^{\frac{n(n+1)}{2}}, & n\le 0.\end{cases}$$

Hence, from the ${}_2\phi_1$ series representation of $\operatorname{Cos}(\cdot;q^2)$ and $\operatorname{Sin}(\cdot;q^2)$ we obtain

$$|\operatorname{Sin}(\frac{q^n}{1-q};q^2)| \le \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\begin{cases} 1, & n\ge 0,\\ q^{n^2}, & n<0,\end{cases} \tag{5.32}$$

and

$$|\operatorname{Cos}(\frac{q^n}{1-q};q^2)| \le \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\begin{cases} 1, & n\ge 0,\\ q^{n^2-2n}, & n<0,\end{cases} \tag{5.33}$$

Consequently,

$$|e(\frac{q^n}{1-q};q^2)| \le 2\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\begin{cases} 1, & n\ge 0,\\ q^{n^2}, & n<0.\end{cases} \tag{5.34}$$

Thus, the series $\sum_{n=0}^{\infty} q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)$ is absolutely convergent for Re $\alpha < 1$ and diverges for Re $\alpha \geq 1$ while the series

$$\sum_{n=-\infty}^{-1} q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)$$

is absolutely convergent for all $\alpha \in \mathbb{C}$. □

Set

$$\begin{aligned}\Psi_\alpha(x,t) &:= (1-q)^{\alpha/2} \sum_{n=-\infty}^{\infty} q^n \frac{e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)}{(iq^n)^\alpha} \\ &+ (1-q)^{\alpha/2} \sum_{n=-\infty}^{\infty} q^n \frac{e(-ix\frac{q^n}{\sqrt{1-q}};q^2)e(it\frac{q^n}{\sqrt{1-q}};q^2)}{(-iq^n)^\alpha},\end{aligned}$$

where $x, t \in \widetilde{\mathbb{R}}_q$ and i^α is defined with respect to the principal branch, i.e. $i^\alpha = e^{\frac{i\pi}{2}\alpha}$.

Lemma 5.16. *For* $k \in \mathbb{N}$ *and* Re $\alpha < 1$

$$\delta_{q,x}^k \Psi_\alpha(x,t) = \Psi_{\alpha-k}(x,t).$$

Proof. The proof follows directly by using that

$$\delta_{q,x}^k e(ix\frac{q^n}{\sqrt{1-q}};q^2) = \left(\frac{iq^n}{\sqrt{1-q}}\right)^k e(ix\frac{q^n}{\sqrt{1-q}};q^2).$$

□

A direct calculation yields that the following identities hold whenever Re $\alpha < 1$

$$(1-q)^{-\alpha/2}\Psi_\alpha(x,t) = 2\cos(\frac{\pi}{2}\alpha)A_\alpha(x,t) + 2\sin(\frac{\pi}{2}\alpha)B_\alpha(x,t), \tag{5.35}$$

where

$$\begin{aligned}&A_\alpha(x,t) \\ &:= \sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Cos}(\frac{q^{n+k}}{1-q};q^2)\operatorname{Cos}(\frac{q^{m+k}}{1-q};q^2) + q^{k(1-\alpha)} \operatorname{Sin}(\frac{q^{n+k}}{1-q};q^2)\operatorname{Sin}(\frac{q^{m+k}}{1-q};q^2)\end{aligned} \tag{5.36}$$

and

$$B_\alpha(x,t)$$

$$:= \sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Cos}(\frac{q^{m+k}}{1-q};q^2) \operatorname{Sin}(\frac{q^{n+k}}{1-q};q^2) - q^{k(1-\alpha)} \operatorname{Cos}(\frac{q^{n+k}}{1-q};q^2) \operatorname{Sin}(\frac{q^{m+k}}{1-q};q^2). \tag{5.37}$$

Theorem 5.17. *For* $\operatorname{Re}(\alpha) < 1$

$$(1-q)^{-\alpha/2} \Psi_\alpha(\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}})$$

$$= \begin{cases} 2\cos(\frac{\pi}{2}\alpha) q^{-m(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r} \\ -2\sin(\frac{\pi}{2}\alpha) q^{-m(1-\alpha)} \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} q^{\frac{\alpha r}{2}} \tau_r \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r}, \\ \qquad n > m + [\frac{1-\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha) q^{-n(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r} \\ -2\sin(\frac{\pi}{2}\alpha) q^{-n(1-\alpha)} \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} q^{\frac{\alpha r}{2}} \tau_r \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r}, \\ \qquad m > n + [\frac{1-\alpha}{2}], \end{cases} \tag{5.38}$$

where $\tau_r = \begin{cases} q^{-\frac{r}{2}} q^{-\frac{1-\alpha}{2}}, & r \text{ is odd}, \\ q^{\frac{r}{2}}, & r \text{ is even}. \end{cases}$

Moreover,

$$(1-q)^{-\alpha/2} \Psi_\alpha(\frac{q^n}{\sqrt{1-q}}, -\frac{q^m}{\sqrt{1-q}})$$

$$= \begin{cases} 2\cos(\frac{\pi}{2}\alpha) q^{-m(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} (-1)^r q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r} \\ +2\sin(\frac{\pi}{2}\alpha) q^{-m(1-\alpha)} \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} (-1)^r q^{\frac{\alpha r}{2}} \tau_r \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r}, \\ \qquad n > m + [\frac{1-\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha) q^{-n(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} (-1)^r q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r} \\ +2\sin(\frac{\pi}{2}\alpha) q^{-n(1-\alpha)} \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{r=0}^{\infty} (-1)^r q^{\frac{\alpha r}{2}} \tau_r \dfrac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r}, \\ \qquad m > n + [\frac{1-\alpha}{2}], \end{cases} \tag{5.39}$$

$$(1-q)^{-\alpha/2}\Psi_\alpha\Big(\frac{q^n}{\sqrt{1-q}},-\frac{q^m}{\sqrt{1-q}}\Big)$$
$$=\begin{cases} 2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\,\dfrac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty(-1)^r q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r} \\ -2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\,\dfrac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}, \\ \qquad n>m+[\frac{1-\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\,\dfrac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty(-1)^r q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r} \\ -2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\,\dfrac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}, \\ \qquad m>n+[\frac{1-\alpha}{2}]. \end{cases} \tag{5.40}$$

$$(1-q)^{-\alpha/2}\Psi_\alpha\Big(-\frac{q^n}{\sqrt{1-q}},-\frac{q^m}{\sqrt{1-q}}\Big)$$
$$=\begin{cases} 2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\,\dfrac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r} \\ +2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\,\dfrac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}, \\ \qquad n>m+[\frac{1-\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\,\dfrac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r} \\ +2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\,\dfrac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}, \\ \qquad m>n+[\frac{1-\alpha}{2}]. \end{cases} \tag{5.41}$$

Proof. Using the following formula from [176, P. 455]

$$\sum_{k=-\infty}^{\infty} q^{k(-\gamma+1)}J_\alpha(q^{m+k};q^2)J_\beta(q^{n+k};q^2)$$
$$=\begin{cases} q^{n\beta}q^{m(\gamma-\beta-1)}\,\dfrac{(q^{\alpha-\beta+\gamma+1},q^{2\beta+2};q^2)_\infty}{(q^{\alpha+\beta-\gamma+1},q^2;q^2)_\infty} \\ \times{}_2\phi_1\left(q^{\beta-\alpha-\gamma+1},q^{\alpha+\beta-\gamma+1};q^{2\beta+2};q^2,q^{2n-2m+\alpha-\beta+\gamma+1}\right), \\ q^{m\alpha}q^{n(\gamma-\alpha-1)}\,\dfrac{(q^{\beta-\alpha+\gamma+1},q^{2\alpha+2};q^2)_\infty}{(q^{\alpha+\beta-\gamma+1},q^2;q^2)_\infty} \\ \times{}_2\phi_1\left(q^{\alpha-\beta-\gamma+1},q^{\alpha+\beta-\gamma+1};q^{2\alpha+2};q^2,q^{2m-2n+\beta-\alpha+\gamma+1}\right), \end{cases}$$

where $\mathrm{Re}(\alpha+\beta-\gamma+1)>0$ we can prove that

$$\sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Cos}(\frac{q^{m+k}}{1-q}; q^2) \operatorname{Cos}(\frac{q^{n+k}}{1-q}; q^2)$$

$$= \begin{cases} q^{-m(1-\alpha)} \dfrac{(q^{\alpha}, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{2-\alpha}, q^{1-\alpha}; q; q^2, q^{2n-2m+\alpha}\right), \\ q^{-n(1-\alpha)} \dfrac{(q^{\alpha}, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{2-\alpha}, q^{1-\alpha}; q; q^2, q^{2m-2n+\alpha}\right), \end{cases}$$

where $\mathrm{Re}(1-\alpha)>0$ and

$$\sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Sin}(\frac{q^{m+k}}{1-q}; q^2) \operatorname{Sin}(\frac{q^{n+k}}{1-q}; q^2)$$

$$= \begin{cases} q^{n-m} \frac{q^{-m(1-\alpha)}}{1-q} \dfrac{(q^{\alpha}, q^2; q^2)_\infty}{(q^{3-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{2-\alpha}, q^{3-\alpha}; q^3; q^2, q^{2n-2m+\alpha}\right), \\ q^{m-n} \frac{q^{-n(1-\alpha)}}{1-q} \dfrac{(q^{\alpha}, q^2; q^2)_\infty}{(q^{3-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{2-\alpha}, q^{3-\alpha}; q^3; q^2, q^{2m-2n+\alpha}\right). \end{cases}$$

Also

$$\sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Sin}(\frac{q^{m+k}}{1-q}; q^2) \operatorname{Cos}(\frac{q^{n+k}}{1-q}; q^2)$$

$$= \begin{cases} q^{-m(1-\alpha)} \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{1-\alpha}, q^{2-\alpha}; q; q^2, q^{2n-2m+\alpha+1}\right), \\ q^{m-n} \frac{q^{-n(1-\alpha)}}{1-q} \dfrac{(q^{\alpha-1}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} {}_2\phi_1\left(q^{2-\alpha}, q^{3-\alpha}; q^3; q^2, q^{2m-2n+\alpha-1}\right). \end{cases}$$

Hence, if $x := \frac{q^n}{\sqrt{1-q}}$ and $t := \frac{q^m}{\sqrt{1-q}}$ then

$$(1-q)^{(1-\alpha)/2} \frac{(q^{1-\alpha}, q; q^2)_\infty}{(q^{\alpha}, q^2; q^2)_\infty} A_\alpha(x,t)$$

$$= \begin{cases} t^{-(1-\alpha)} \sum_{r=0}^{\infty} q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha}; q)_r}{(q;q)_r} \left(\frac{x}{t}\right)^r, \ n > m + [-\alpha/2], \\ x^{-(1-\alpha)} \sum_{r=0}^{\infty} q^{\alpha[\frac{r}{2}]} \dfrac{(q^{1-\alpha}; q)_r}{(q;q)_r} \left(\frac{t}{x}\right)^r, \ m > n + [-\alpha/2]. \end{cases} \tag{5.42}$$

Also

$$(1-q)^{(1-\alpha)/2}\frac{(q^{1-\alpha},q;q^2)_\infty}{(q^\alpha,q^2;q^2)_\infty}A_\alpha(x,-t)$$
$$=\begin{cases} t^{-(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}\left(\frac{x}{t}\right)^r, & n>m+[-\alpha/2], \\ x^{-(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\alpha[\frac{r}{2}]}\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}\left(\frac{t}{x}\right)^r, & m>n+[-\alpha/2]. \end{cases} \tag{5.43}$$

$$(1-q)^{(1-\alpha)/2}\frac{(q^{2-\alpha},q;q^2)_\infty}{(q^{1+\alpha},q^2;q^2)_\infty}B_\alpha(x,t)$$
$$=-\begin{cases} q^{-m(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}, & n\ge m+[\frac{1-\alpha}{2}], \\ q^{-n(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}, & m>n+[\frac{1-\alpha}{2}], \end{cases} \tag{5.44}$$

$$(1-q)^{(1-\alpha)/2}\frac{(q^{2-\alpha},q;q^2)_\infty}{(q^{1+\alpha},q^2;q^2)_\infty}B_\alpha(x,-t)$$
$$=\begin{cases} q^{-m(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}, & n\ge m+[\frac{1-\alpha}{2}], \\ q^{-n(1-\alpha)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{\alpha r}{2}}\tau_r\dfrac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}, & m>n+[\frac{1-\alpha}{2}]. \end{cases} \tag{5.45}$$

Substituting from (5.42)–5.45 into (5.35) yield the values $\Psi_\alpha(\pm x,\pm t)$ and the theorem follows. ☐

Remark 5.6.1. In the previous theorem we calculated the value of $\Psi_\alpha(x,t)$ $x,t\in\widetilde{\mathbb{R}}_q$ and for specific values of x,t. We can calculate the values of $\Psi_\alpha(x,t)$ for all x,t by using the identity

$$\sum_{-\infty}^{\infty}s^{k+m}y^{k+n}\frac{(y^2;q^2)_\infty}{(q^2;q^2)_\infty}{}_1\phi_1\left(0;y^2;q^2,q^{2k+2n+2}\right)x^{k+m}\frac{(x^2;q^2)_\infty}{(q^2;q^2)_\infty}$$
$${}_1\phi_1\left(0;x^2;q^2,q^{2k+2m+2}\right)$$
$$=y^{n-m}\frac{(s^{-1}xy^{-1},q^{2n-2m+2};q^2)_\infty}{(q^{2n-2m}s^{-1}xy^{-1},q^2;q^2)_\infty}$$
$${}_2\phi_1\left(q^{2n-2m}s^{-1}xy^{-1},s^{-1}yx^{-1};q^{2n-2m+2};q^2,sxy\right)$$

for $|sxy|<1$. See Proposition 4.1 of [176].

In this case we have

$$A_\alpha\Big(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\Big) := \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)}\operatorname{Cos}\Big(\frac{q^{n+r}}{1-q};q^2\Big)\operatorname{Cos}\Big(\frac{q^{m+r}}{1-q};q^2\Big)$$
$$= q^{-m(1-\alpha)}\frac{\big(q^\alpha,q^{2n-2m+2},q^2;q^2\big)_\infty}{(q^{2n-2m+\alpha},q,q;q^2)_\infty}\,{}_2\phi_1\left(q^{2n-2m+\alpha},q^\alpha;q^{2n-2m+2};q^2,q^{1-\alpha}\right), \tag{5.46}$$

$$B_\alpha\Big(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\Big) = \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)}\operatorname{Sin}\Big(\frac{q^{n+r}}{1-q};q^2\Big)\operatorname{Sin}\Big(\frac{q^{m+r}}{1-q};q^2\Big)$$
$$= q^{n-m}q^{-m(1-\alpha)}\frac{\big(q^\alpha,q^{2n-2m+2},q^2;q^2\big)_\infty}{(q^{2n-2m+\alpha},q,q;q^2)_\infty}\,{}_2\phi_1\left(q^{2n-2m+\alpha},q^\alpha;q^{2n-2m+2};q^2,q^{3-\alpha}\right), \tag{5.47}$$

$$C_\alpha\Big(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\Big) = \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)}\operatorname{Sin}\Big(\frac{q^{n+r}}{1-q};q^2\Big)\operatorname{Cos}\Big(\frac{q^{m+r}}{1-q};q^2\Big)$$
$$= q^{n-m}q^{-m(1-\alpha)}\frac{\big(q^{\alpha-1},q^{2n-2m+2},q^2;q^2\big)_\infty}{(q^{2n-2m+\alpha-1},q,q;q^2)_\infty}\,{}_2\phi_1\left(q^{2n-2m+\alpha-1},q^{\alpha+1};q^{2n-2m+2};q^2,q^{2-\alpha}\right).$$

Corollary 5.18. *For each fixed $x,t\in\widetilde{\mathbb{R}}_q$, the function $\Psi_\alpha(x,t)$ as a function of α can be extended to an entire function on $\mathbb{C}$.*

Proof. If α is a positive integer, then the series on the right hand sides of (5.38)–(5.41) are finite sums and hence are convergent. Since the zeros of the function $\cos(\frac{\pi}{2}\alpha)$ are the poles of the function $(q^{1-\alpha};q^2)_\infty$ with the same orders. In fact

$$\lim_{\alpha\to 2j+1}\frac{\cos\frac{\pi}{2}\alpha}{(q^{1-\alpha};q^2)_\infty} = -\frac{\pi}{2\ln q}\,\frac{q^{j^2+j}}{(q^2;q^2)_j\,(q^2;q^2)_\infty},\quad j\in\mathbb{N}_0.$$

Similarly the zeros of the function $\sin(\frac{\pi}{2}\alpha)$ are the poles of the function $(q^{2-\alpha};q^2)_\infty$ with the same orders and

$$\lim_{\alpha\to 2j}\frac{\sin\frac{\pi}{2}\alpha}{(q^{2-\alpha};q^2)_\infty} = -\frac{\pi}{2\ln q}\,\frac{q^{j^2-j}}{(q^2;q^2)_{j-1}(q^2;q^2)_\infty},\quad j\in\mathbb{N}.$$

Then the left hand side of (5.38)–(5.41) are entire functions. Hence as a function of α, the functions $\Psi_{-\alpha}(x,t)$ $(x,t\in\widetilde{\mathbb{R}}_q)$ can be analytically extended by defining its values when $\operatorname{Re}\alpha\geq 1$ by the left hand sides of (5.38)–(5.41). □

It also should be noted that for $\mathrm{Re}(\alpha) \geq 1$, the left hand sides of (5.38) and (5.39) determine $\Psi_{-\alpha}(x, t)$ for all $x, t \in \widetilde{\mathbb{R}}_q$ which is different from the case of $\mathrm{Re}(\alpha) < 1$, see Remark 5.6.1.

Definition 5.6.1. For $\mathrm{Re}\,\alpha > 0$, we define a fractional q-integral operator J_q^α on $L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$ by

$$J_q^\alpha f(x) := \frac{1}{C} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t) \Psi_\alpha(x, t)\, d_q t.$$

The following properties follow directly from the definition of J_q^α.

- If $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L(\widetilde{\mathbb{R}}_q)$ and f is even then

  $$J_q^\alpha f(x) = 2 \int_0^{\infty/\sqrt{1-q}} f(t) \varphi_\alpha(x, t)\, d_q t,$$

  where

  $$(1-q)^{-\alpha/2} \varphi_\alpha\Big(\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}}\Big)$$

  $$= \begin{cases} 2q^{-m(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \cos \frac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q; q)_{2j}} q^{j(2n-2m+\alpha)} \\ -2q^{n-m-m(1-\alpha)} \sin \frac{\pi}{2}\alpha \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j+1}}{(q; q)_{2j+1}} q^{j(2n-2m+\alpha-1)}, \\ 2q^{-n(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \cos \frac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q; q)_{2j}} q^{j(2m-2n+\alpha)} \\ +2q^{-n(1-\alpha)} \sin \frac{\pi}{2}\alpha \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q; q)_{2j}} q^{j(2m-2n+\alpha+1)}, \end{cases}$$

  and

  $$(1-q)^{-\alpha/2} \varphi_\alpha\Big(-\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}}\Big)$$

  $$= \begin{cases} 2q^{-m(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \cos \frac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{\alpha+1}; q)_{2j}}{(q; q)_{2j}} q^{i(2n-2m+\alpha)} \\ +2q^{n-m-m(1-\alpha)} \sin \frac{\pi}{2}\alpha \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j+1}}{(q; q)_{2j+1}} q^{j(2n-2m+\alpha-1)}, \\ 2q^{-n(1-\alpha)} \dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty} \cos \frac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q; q)_{2j}} q^{j(2m-2n+\alpha)} \\ -2q^{-n(1-\alpha)} \sin \frac{\pi}{2}\alpha \dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty} \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q; q)_{2j}} q^{j(2m-2n+\alpha+1)}. \end{cases}$$

- If $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L(\widetilde{\mathbb{R}}_q)$ and f is odd then

$$J_q^\alpha f(x) = 2\int_0^{\infty/\sqrt{1-q}} f(t)\psi_\alpha(x,t)\,d_qt,$$

where

$$(1-q)^{-\alpha/2}\psi_\alpha\Big(\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}}\Big)$$

$$= \begin{cases} 2q^{-m(1-\alpha)}q^{n-m}\dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty}\cos\tfrac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{2-\alpha}; q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha)} \\ -2q^{-m(1-\alpha)}\sin\tfrac{\pi}{2}\alpha\dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty}\sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q;q)_{2j}}q^{j(2n-2m+\alpha+1)}, \\ 2q^{-n(1-\alpha)}\dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty}\cos\tfrac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha)}, \\ +2q^{-(n+1)(1-\alpha)}q^{m-n}\sin\tfrac{\pi}{2}\alpha\dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty}\sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha-1)}, \end{cases}$$

and

$$(1-q)^{-\alpha/2}\psi_\alpha\Big(-\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}}\Big)$$

$$= \begin{cases} -2q^{-m(1-\alpha)}q^{n-m}\dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty}\cos\tfrac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{2-\alpha}; q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha)} \\ -2q^{-m(1-\alpha)}\sin\tfrac{\pi}{2}\alpha\dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty}\sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q;q)_{2j}}q^{j(2n-2m+\alpha+1)}, \\ 2q^{-n(1-\alpha)}\dfrac{(q^\alpha, q^2; q^2)_\infty}{(q^{1-\alpha}, q; q^2)_\infty}\cos\tfrac{\pi}{2}\alpha \sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha)} \\ -2q^{-(n+1)(1+\alpha)}q^{m-n}\sin\tfrac{\pi}{2}\alpha\dfrac{(q^{1+\alpha}, q^2; q^2)_\infty}{(q^{2-\alpha}, q; q^2)_\infty}\sum_{j=0}^\infty \dfrac{(q^{1-\alpha}; q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha-1)}. \end{cases}$$

Example 5.6.1. If $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$ then

$$J_q f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_1(x,t)\,d_qt,$$

where if $x, t \in \widetilde{\mathbb{R}}_q$ we obtain

$$\Psi_1(x,t) = -\frac{\pi\sqrt{1-q}}{\ln q} - \frac{C}{2},\ \Psi_1(x,-t) = \frac{\pi\sqrt{1-q}}{\ln q} - \frac{C}{2},$$

and

$$\Psi_1(-x,t) = -\frac{\pi\sqrt{1-q}}{\ln q} + \frac{C}{2},\ \Psi_1(-x,-t) = \frac{\pi\sqrt{1-q}}{\ln q} + \frac{C}{2}.$$

Hence

$$J_q f(x) = \frac{-2\pi\sqrt{1-q}}{C\ln q}\int_0^{\infty/\sqrt{1-q}} f_o(t)\,d_qt - \int_0^{\infty/\sqrt{1-q}} f_e(t)\,d_qt,$$

and

$$J_q f(-x) = \int_0^{\infty/\sqrt{1-q}} f_e(t)\,d_qt - \frac{2\pi\sqrt{1-q}}{C\ln q}\int_0^{\infty/\sqrt{1-q}} f_o(t)\,d_qt.$$

Definition 5.6.2. For $\alpha > 0$ we define a fractional q-difference operator δ_q^α on $L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$ by

$$\delta_q^\alpha f(x) := J_q^{-\alpha} f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{-\alpha}(x,t)\,d_qt. \tag{5.48}$$

Lemma 5.19. *The operator* δ_q^α *coincides with Rubin's* q*-difference operator when* α *ia a positive integer.*

Proof. Let $\alpha = k$ for some $k \in \mathbb{N}$ and let $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$. Using (3.88), we conclude

$$f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_0(x,t)\,d_qt.$$

Then from Lemma 5.16 we obtain

$$\delta_q^k f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{-k}(x,t)\,d_qt,$$

and the lemma follows. □

Lemma 5.20. *If* $\alpha > 0$ *and* $f \in L^2(\widetilde{\mathbb{R}}_q)$ *then*

$$\delta_q^\alpha f(x) = \delta_q^k J_q^{k-\alpha} f(x) \quad (\alpha > 0;\ k = \lceil\alpha\rceil;\ x \in \mathbb{R}_q). \tag{5.49}$$

Now if α is a positive integer then $\lceil\alpha\rceil = \alpha$ and from (5.49)

$$\partial_q^\alpha = \partial_q^k.$$

Proof. Since

$$\delta_q^k J_q^{k-\alpha} f(x) = \frac{1}{C}\delta_q^k \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{k-\alpha}(x,t)\, d_q t$$

and $\delta_{q,x}^k \Psi_{k-\alpha}(x,t) = \Psi_{-\alpha}(x,t)$, the proof follows. □

Theorem 5.21. *If* $f \in L^1(\widetilde{\mathbb{R}}_q) \cap L^2(\widetilde{\mathbb{R}}_q)$ *and* α, β *are complex numbers such that* $\mathrm{Re}(\alpha) < 1$ *and* $\mathrm{Re}(\beta) < 1$ *such that* $\mathrm{Re}(\alpha+\beta) < 1$ *then*

$$J_q^\alpha J_q^\beta f = J_q^\beta J_q^\alpha f = J_q^{\alpha+\beta} f.$$

Proof. Let $x \in \widetilde{\mathbb{R}}_q$. Since

$$\begin{aligned} J_q^\alpha (J_q^\beta f)(x) &= \frac{1}{C^2}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u)\Psi_\alpha(x,t)\Psi_\beta(t,u) d_q u\, d_q t \\ &= \frac{1}{C^2}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u) \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \Psi_\alpha(x,t)\Psi_\beta(t,u)\, d_q t\, d_q u, \end{aligned}$$

using the orthogonality relation (3.90) we obtain

$$\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \Psi_\alpha(x,t)\Psi_\beta(t,u)\, d_q t = C\Psi_{\alpha+\beta}(x;u).$$

Consequently,

$$J_q^\alpha (J_q^\beta f)(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u)\Psi_{\alpha+\beta}(x;u) d_q u = J_q^{\alpha+\beta} f(x).$$

□

Chapter 6
Fractional q-Leibniz Rule and Applications

Abstract This chapter includes analytic investigations on q-type Leibniz rules of q-Riemann–Liouville fractional operator introduced by Al-Salam and Verma in (Pac. J. Math. 60(2), 1–9, 1975). In this chapter, we provide a generalization of the Riemann–Liouville fractional q-Leibniz formula introduced by Agarwal in (Ganita 27(1–2), 25–32, 1976). Purohit (Kyungpook Math. J. 50(4), 473–482, 2010) introduced a Leibniz formula for Weyl q-fractional operator only when α is an integer. In this respect, We extend Purohit's result for any $\alpha \in \mathbb{R}$. We end the chapter with deriving some q-series and formulae by applying the fractional Leibniz formula mentioned and derived earlier in the chapter.

6.1 Leibniz Rule for Fractional Derivatives

Studies of Leibniz rule for derivatives of arbitrary order started with the work of Liouville [186] who introduced the Leibniz rule

$$D^{\alpha}u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} D^{\alpha-n}u(z)D^{n}v(z). \tag{6.1}$$

While Liouville used Fourier expansions in obtaining (6.1), Grünwald [120] and Letnikov [180] obtained (6.1) in a different technique. Other extensions and proofs could be found in the work of Watanabe [288], Post [242], Bassam [49] and Gaer–Rubel [110]. In Watanabe's paper, the author derived the generalization

$$D^{\alpha}u(z)v(z) = \sum_{-\infty}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-n+1)\Gamma(\gamma+n+1)} D^{\alpha-\gamma-n}u(z)D^{\gamma+n}v(z), \tag{6.2}$$

for any fixed γ. In [288], no precise domain of convergence is given [228, P. 659]. Osler in [228], see also [81, 229–231], determined the domain of convergence of (6.2) which includes (6.1) as the special case $\gamma = 0$.

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The expansion (6.2) can be used to derive some series expansions of some functions for example [228, P. 659]. If we take $u = z^{d-a-1}$, $v = z^{c-b-1}$, $\alpha = c-a-1$ and $\gamma = c-1$, we end with

$$\frac{\pi^2\Gamma(c+d-a-b-1)\csc\pi a\csc\pi b}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)} = \sum_{-\infty}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)}.$$

The identity

$$F(a,b;c.z) = \frac{\Gamma(c)\Gamma(b-c+1)\sin\pi(b-c)}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{F(a,b;b-n;z)}{n!(b-c-n)\Gamma(b-n)}$$

which holds for $\operatorname{Re} b > 0$ and $\operatorname{Re} z < 1/2$ is obtained from (6.2) by calculating $D^{b-c}z^{b-1}(1-z)^{-a}$ by two different methods. One method is by taking

$$U(z) = z^{b-1},\ V(z) = (1-z)^{-a},$$

and the other methods is by taking

$$U(z) = (1-z)^{-a}, \quad V(z) = z^{b-1}.$$

6.2 Al-Salam–Verma Fractional q-Leibniz Formula

In [20], Al-Salam and Verma derived formally a q-fractional Leibniz rule. They used the q-Taylor series

$$f(z) = \sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2} D_q^n f(aq^{-n})\frac{(1-q)^n}{(q;q)_n}[a-z]_n, \tag{6.3}$$

which is slightly different from the q-Newton series

$$f(z) = \sum_{n=0}^{\infty} D_q^n f(a)\frac{(1-q)^n}{(q;q)_n}[z-a]_n, \tag{6.4}$$

introduced formally by Jackson in [159]. According to Jackson's notations, $[y-t]_n$ denotes the function $\phi_n(y,t)$ defined in (6.7) below for $y,t \in \mathbb{C}$ and $n \in \mathbb{N}_0$. No proofs of (6.3) or (6.4) are given either in [20] or [159]. However, Al-Salam and Verma gave a formal verification of (6.3) under the condition that f can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}c_n[z-a]_n. \tag{6.5}$$

Again, no indication is given for the class of functions where (6.5) holds. To the best we know, Ismail and Stanton [149] are the first to give q-Taylor series in a rigorous analytic way. Ismail and Stanton result is proved for q-Askey Wilson operators and for functions analytic on bounded domains of $\mathbb{C}$ or entire functions whose maximum modulus $M(r;f)$ satisfies

$$\lim_{r\to\infty}\frac{|\ln M(r;f)|}{\ln^2 r}=C,\quad C<\frac{1}{\ln q^{-1}}.$$

See also [62, 148, 191, 289]. In [33], a rigorous proof of q-Taylor series based on Jackson q-difference operator is introduced for functions analytic in finite domains and also for entire functions that satisfy

$$\lim_{r\to\infty}\frac{|\ln M(r;f)|}{\ln^2 r}=C,\quad C<\frac{1}{2\ln q^{-1}}.$$

Returning to the fractional q-Leibniz rule of Al-Salam and Verma, the authors of [20] derived also in a formal way the relation

$$I_q^{\alpha}U(z)V(z)=\sum_{m=0}^{\infty}\begin{bmatrix}-\alpha\\ m\end{bmatrix}_q D_q^m U(zq^{-\alpha-m})I_q^{\alpha+m}V(z). \tag{6.6}$$

Then Al-Salam and Verma gave some applications which we will mention in the last section of this chapter. In Sect. 6.4 we provide an analytic proof of (6.6) indicating the conditions on U, V, α and the domain of convergence of the series in (6.6).

6.3 q-Taylor and Interpolation Series

In this section we exhibit the q-Taylor and interpolation series introduced in [33]. The q-Taylor expansion is given in terms of the polynomials $\phi_n(a,z)$ where for each $t\in\mathbb{C}$, the polynomials $\{\varphi_n(y,t)\}_{n=0}^{\infty}$ are defined for $y\in\mathbb{C}$ by

$$\begin{aligned}\varphi_0(y,t)&:=1,\\ \varphi_n(y,t)&:=\begin{cases}y^n\,(t/y;q)_n\,, & y\neq 0,\\ (-1)^n q^{\frac{n(n-1)}{2}}t^n, & y=0.\end{cases}\end{aligned} \tag{6.7}$$

We consider $\varphi_n(y,t)$ as a function of the two variables y and t.

Lemma 6.1. *For $n \in \mathbb{N}$ and $y, t \in \mathbb{C}$ we have*

$$\begin{aligned} D_{q,y}\varphi_n(y,t) &= \frac{1-q^n}{1-q}\varphi_{n-1}(y,t), \\ D_{q,t}\varphi_n(y,t) &= -\frac{1-q^n}{1-q}\varphi_{n-1}(y,qt), \\ I^n_{q,t}(1) &= \frac{\varphi_n(y,t)}{\Gamma_q(n+1)}. \end{aligned}$$

Let k be a positive real number and f be an entire function with the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{6.8}$$

Bezivin [55] called $f(z)$ q-Gevrey of order $1/k$, if there exist positive real numbers A and C such that

$$|a_n| < Aq^{\frac{n(n+1)}{2k}} C^n \qquad (n \in \mathbb{N}_0).$$

In [256], Ramis defined the q-Gevery series when $q > 1$. He also defined an entire function $f(z)$ to have a q-exponential growth of order k and a finite type δ, $\delta \in \mathbb{R}$, if there exists a positive number K such that

$$|f(z)| < K|z|^{\alpha} \exp\left(\frac{k \ln^2 |z|}{2 \ln^2 q}\right),$$

see also [149, P. 177]. The following lemma coincides with that introduced by Ramis [256] when $q > 1$.

Lemma 6.2. *Let f be an entire function with the power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (z \in \mathbb{C}).$$

The following are equivalent.

(i) There exist positive constants k, K and a real constant α such that for $r > 0$

$$M(r; f) \leqslant K \exp\left(\frac{k \ln^2 r}{2 \ln^2 q} + \alpha \ln r\right).$$

(ii) There exists a positive constant A such that

$$|a_n| \leqslant A \exp\left(\frac{-(n-\alpha)^2 \ln^2 q}{2k}\right) \qquad (n \in \mathbb{N}_0).$$

The following is one of the q-Taylor series introduced in [33] and it is essential in our investigations on the fractional q-Leibniz formula.

Theorem 6.3. *Let $f(z)$ be a function with q-exponential growth of order k, $k < \ln q^{-1}$, and a finite type δ, $\delta \in \mathbb{R}$. Then for $a \in \mathbb{C} \setminus \{0\}$, the function $f(z)$ has the expansion*

$$f(z) = \sum_{m=0}^{\infty} (-1)^m q^{-m(m-1)/2} \frac{D_q^m f(aq^{-m})}{\Gamma_q(m+1)} \varphi_m(a, z), \tag{6.9}$$

absolutely and uniformly on compact subsets $\mathbb{C}$.

Corollary 6.4. *Let f be an entire function with q-exponential growth of order k $\big(k < \ln q^{-1}\big)$ and a finite type δ, $\delta \in \mathbb{R}$. Then*

$$f(zq^n) = \sum_{m=0}^{\infty} (-1)^m \frac{q^{-m(m-1)/2}}{\Gamma_q(m+1)} D_q^m f(zq^{-m-\gamma})(q^{n+\gamma}, q)_m (zq^{-\gamma})^m, \tag{6.10}$$

where $\gamma \in \mathbb{R}$, $z \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}_0$. Moreover, for each $z \in \mathbb{C} \setminus \{0\}$, the convergence is absolute and uniform on $\{zq^m : m \in \mathbb{N}\}$.

Proof. Let $z \in \mathbb{C} \setminus \{0\}$ be fixed. Hence, $D := \{0, zq^m : m \in \mathbb{N}\}$ is a compact subset of $\mathbb{C}$. Applying Theorem 6.3 with $a = zq^{-\gamma}$ we obtain

$$f(y) = \sum_{m=0}^{\infty} (-1)^m q^{-m(m-1)/2} \frac{D_q^m f(zq^{-m-\gamma})}{\Gamma_q(m+1)} \varphi_m(zq^{-\gamma}, y) \quad (y \in D),$$

and the convergence ia absolute and uniform on D. Observing that

$$\varphi_m(zq^{-\gamma}, zq^n) = (zq^{-\gamma})^m (q^{n+\gamma}; q)_m,$$

then replacing y with zq^n, $n \in \mathbb{N}$ proves (6.10). □

6.3.1 A Fractional q-Taylor Formula

Purohit and Raina [249] derived a generalized q-Taylor's formula in fractional q-calculus using Riemann–Liouville fractional q-difference operator. We introduce Purohit and Raina's result in the theorem below, for the proof cf. [249].

Theorem 6.5. *Let η be an arbitrary complex number and* $\mathrm{Re}(p) > -1$. *If*

$$f(x) = \sum_{-\infty}^{\infty} A_n x^n,$$

then

$$x^p(t/x;q)_p F(x,t) = \sum_{-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2} t^{n+\eta}}{\Gamma_q(n+\eta+1)} D_{q,x}^{n+\eta}\{x^p f(x)\},$$

where

$$F(x,t) = \varepsilon^{-tq^p}(f(x)),$$

ε is the q-translation operator defined in (1.15)*,* $|t/x| < 1$*,* $|tq^p/x| < 1$*, and* $|q| < 1$.

The authors of [249] used the previous fractional q-Taylor formula to derive a q-generating function of the basic hypergeometric function ${}_r\phi_s$: They proved that

$$\begin{aligned} &x^p(t/x;q)_p\, \varepsilon^{-q^p t}\left[{}_r\phi_s\left(a_1,\ldots,a_r;b_1,\ldots,b_s;q,\rho x\right)\right] \qquad (6.11)\\ &= \sum_{-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}}{\Gamma_q(n+\eta+1)} t^{n+\eta} \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-n-\eta)}\\ &\quad \times x^{p-n-\eta}{}_{r+1}\phi_{s+1}\left(a_1,\ldots,a_r,q^{p+1};b_1,\ldots,b_s,q^{p+1-n-\eta};q,\rho x\right). \end{aligned}$$

Saxena et al. [270] introduced a basic analogue of the H-functions in terms of the Mellin–Barnes type basic contour in the following manner:

$$H_{A,B}^{m,n}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \frac{1}{2\pi i} \qquad (6.12)$$

$$\times \int_C \frac{\prod_{j=m+1}^{B}(q^{1-b_j+B_j s};q)_\infty \prod_{j=n+1}^{A}(q^{a_j-\alpha_j s};q)_\infty}{\prod_{j=1}^{m}(q^{b_j-B_j s};q)_\infty \prod_{j=1}^{n}(q^{1-a_j+\alpha_j s};q)_\infty} \frac{(q^{1-s};q)_\infty \pi z}{\sin \pi z}\, ds,$$

and $0 \le m \le B$; $0 \le n \le A$; a_j and α_j are all positive integers. The contour C is a line parallel to $\mathrm{Re}(ws) = 0$, with indentations, if necessary, in such a manner that all the zeros of $(q^{b_j-\beta_j s};q)_\infty$ $(1 \le j \le m)$ are to its right, and those of $(q^{1-a_j+\alpha_j s};q)_\infty$, $1 \le j \le n$ are to the left of C. The basic integral converges if $\mathrm{Re}\,[s\log(x) - \log\sin\pi s] < 0$, for large values of $|s|$ on the contour C, that is if $\left\{\arg(z) - w_2 w_1^{-1}\log|z|\right\} < \pi$, where $|q| < 1$, $\log q = -w = -(w_1 + i w_2)$, w_1 and w_2 are real numbers.

Purohit and Raina applied the fractional q-Taylor's formula to obtain

$$\begin{aligned} z^p(t/z;q)_p \varepsilon^{-tq^p}\left(H_{A,B}^{m,n}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right]\right) &= \sum_{-\infty}^{\infty} \frac{q^{(n+\eta)(n+\eta-1)/2}}{\Gamma_q(n+\eta+1)} \frac{z^p(t/z)^{n+\eta}}{(1-q)^{n+\eta}}\\ &\quad \times H_{A+1,B+1}^{m,n+1}\left[\rho z;q\,\middle|\,\begin{matrix}(-p,1), & (a,\alpha)\\(b,\beta), & (n+\eta-p,1)\end{matrix}\right], \end{aligned}$$

where η is arbitrary complex number, $0 \le m \le \beta$, $0 \le n \le A$, and the existence condition as in (6.12).

6.4 Analytic Proofs and Applications

In this section we give an analytic proof of the fractional q-Leibniz rule (6.6) of Al-Salam and Verma in [20].

Theorem 6.6. *Let $U(z)$ be an entire function with q-exponential growth of order k, $k < \ln q^{-1}$, and a finite type δ, $\delta \in \mathbb{R}$. Let V be a function that satisfies*

$$\sum_{j=0}^{\infty} q^j |V(zq^j)| < \infty \quad (z \in \mathbb{C}) .$$

Then

$$I_q^{\alpha}\,(UV)\,(z) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q D_q^m U(zq^{-\alpha-m}) I_q^{\alpha+m} V(z), \tag{6.13}$$

for $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

Proof. From (4.26) we have

$$\begin{aligned} I_q^{\alpha}(UV)(z) &= \frac{z^{\alpha}}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n (q^{n+1};q)_{\alpha-1} U(zq^n) V(zq^n) \\ &= z^{\alpha}(1-q)^{\alpha} \sum_{n=0}^{\infty} q^n \frac{(q^{\alpha};q)_n}{(q;q)_n} U(zq^n) V(zq^n). \end{aligned} \tag{6.14}$$

Since U has q-exponential growth of order k $\left(k < \ln q^{-1}\right)$ and a finite type δ, there exists a constant $K > 0$ such that

$$|M(r;U)| \leqslant K \, \exp\left(-\frac{l \ln^2 r}{2 \ln q} + \delta \ln\, r\right) \quad (r > 0;\; l < 1) .$$

So, from Lemma 6.2, the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies the inequality

$$|a_n| \leqslant C q^{(n-\delta)^2/2l} \quad (\, n \in \mathbb{N}_0) , \tag{6.15}$$

for some positive constant C, which is independent of n. Following Corollary 6.4, the function $U(\cdot)$ has the expansion

$$U(zq^n) = \sum_{m=0}^{\infty} (-1)^m \frac{q^{-\alpha m - m(m-1)/2}}{\Gamma_q(m+1)} (q^{n+\alpha};q)_m z^m D_q^m U(zq^{-\alpha-m}), \tag{6.16}$$

uniformly and absolutely on the sequence $\{zq^n : n \in \mathbb{N}_0\}, z \in \mathbb{C}\setminus\{0\}$. Substituting with (6.16) into (6.14), we obtain

$$\begin{aligned} I_q^\alpha(UV)(z) &= z^\alpha(1-q)^\alpha \sum_{n=0}^{\infty} q^n \frac{(q^\alpha;q)_n}{(q;q)_n} V(zq^n) \\ &\quad \times \sum_{m=0}^{\infty} (-1)^m q^{-\alpha m - m(m-1)/2} \frac{D_q^m U(zq^{-\alpha-m})}{\Gamma_q(m+1)} (q^{n+\alpha};q)_m z^m. \end{aligned} \tag{6.17}$$

If we prove that the double series in (6.17) is absolutely convergent, then we can interchange the order of summations in (6.17) and use

$$(q^\alpha;q)_n (q^{n+\alpha};q)_m = (q^\alpha;q)_{n+m} = (q^\alpha;q)_m (q^{\alpha+m};q)_n$$

and

$$I_q^{\alpha+m} V(z) = z^{m+\alpha}(1-q)^{m+\alpha} \sum_{n=0}^{\infty} q^n \frac{(q^{\alpha+m};q)_n}{(q;q)_n} V(zq^n)$$

to obtain

$$I_q^\alpha(UV)(z) = \sum_{m=0}^{\infty} (-1)^m \frac{q^{-\alpha m - m(m-1)/2}(q^\alpha;q)_m}{(q;q)_m} D_q^m U(zq^{-\alpha-m}) I_q^{\alpha+m} V(z). \tag{6.18}$$

Substituting with

$$\begin{bmatrix} -\alpha \\ m \end{bmatrix}_q = (-1)^m q^{-\alpha m - m(m-1)/2} \frac{(q^\alpha;q)_m}{(q;q)_m} \tag{6.19}$$

into (6.18), we obtain (6.13). Now we prove that the double series in (6.17) is absolutely convergent. We do this by proving that the series in (6.16) and (6.14) converge absolutely. Indeed, if $U(z)$ has the representation

$$U(z) = \sum_{r=0}^{\infty} a_r z^r \quad (z \in \mathbb{C})$$

then

$$\begin{aligned} D_q^m U(zq^{-\alpha-m}) &= (1-q)^{-m} \sum_{k=0}^{\infty} a_{k+m} (q^{k+1};q)_m z^k q^{-(\alpha+m)k} \\ &= \Gamma_q(m+1) \sum_{k=0}^{\infty} a_{k+m} \frac{(q^{m+1};q)_k}{(q;q)_k} q^{-(\alpha+m)k} z^k. \end{aligned}$$

Thus, from (6.15) we get

$$\left|D_q^m U(zq^{-\alpha-m})\right| \leqslant C\Gamma_q(m+1)q^{(m-\delta)^2/2l}\sum_{k=0}^{\infty}\frac{q^{k(k-2\delta)/2l}}{(q;q)_k}q^{-\alpha k}q^{(\frac{1}{l}-1)mk}|z|^k. \tag{6.20}$$

Since the series on the right hand side of (6.20) converges and its sum depends only on z, δ, and α, there exists a constant K_1 which depends on z, δ and α such that

$$\left|\frac{q^{-m(m-1)/2}}{\Gamma_q(m+1)}(zq^{-\alpha})^m D_q^m U(zq^{-\alpha-m})\right| \leqslant K_1 q^{(\frac{1}{l}-1)m^2/2}q^{m\left(\frac{1}{2}-\frac{\delta}{l}-\alpha\right)}.$$

Hence,

$$\sum_{m=0}^{\infty}\frac{q^{-\alpha m-\frac{m(m-1)}{2}}}{\Gamma_q(m+1)}\left(q^{n+\alpha};q\right)_m |z|^m \left|D_q^m U(zq^{-\alpha-m})\right| \leq K_2,$$

where

$$K_2 := K_1(-q^{\alpha};q)_{\infty}\sum_{m=0}^{\infty}q^{(l^{-1}-1)m^2/2}\,q^{m\left(\frac{1}{2}-\frac{\delta}{2l}-\alpha\right)}|z|^m.$$

Therefore,

$$\sum_{n=0}^{\infty}q^n\frac{(q^{\alpha};q)_n}{(q;q)_n}|U(zq^n)\,V(zq^n)| \leqslant \frac{K_2}{(q;q)_{\infty}}\sum_{n=0}^{\infty}q^n|V(zq^n)| < \infty,$$

i.e. the series in (6.17) is absolutely convergent. This completes the proof. □

It is worth noting that the condition

$$\sum_{j=0}^{\infty}q^j|V(zq^j)| < \infty \quad (z \in \mathbb{C})$$

is not restrictive. It is not hard to see that all q-regular functions as well as continuous at zero functions satisfy this integrability condition.

6.5 A Generalization for Agarwal's Fractional q-Leibniz Formula

In [15], Agarwal introduced the following fractional q-Leibniz formula:

Theorem 6.7 (Agarwal's fractional q-Leibniz formula). *Let U and V be two analytic functions which have power series representations at $z = 0$ with radii of* *convergence* R_1 *and* R_2*, respectively and* $R = \min\{R_1, R_2\}$*. Then,*

$$I_q^{\alpha}\left(UV\right)(z) = \sum_{n=0}^{\infty} \begin{bmatrix} -\alpha \\ n \end{bmatrix}_q D_q^n U(z) I_q^{\alpha+n} V(zq^n) \quad (|z| < R), \tag{6.21}$$

Proof. See [15]. □

Recently, Purohit [247] used (6.21) to derive a number of summation formulae for the generalized basic hypergeometric functions. In this section we present the generalization of Agarwal's fractional q-Leibniz formula (6.21) introduced in [206]. Let $0 < R < \infty$ and $D_R := \{z \in \mathbb{C} \ : \ |z| < R\}$. In the following we say that the function $f \in L_q^1(D_R)$ if

$$\sum_{j=0}^{\infty} q^j \, |f(zq^j)| < \infty \ \text{ for all } z \in D_R \setminus \{0\}\,.$$

Theorem 6.8. *Let* G *be a branch domain of the logarithmic function. Let* a, b *be complex numbers and* R *be a positive number. Let* u *and* v *be analytic functions in the disk* D_R*. Let* U *and* V *be defined in* $G \cap D_R$ *through the relations*

$$U(z) = z^a u(z), \quad V(z) = z^b v(z). \tag{6.22}$$

If $V(\cdot)$ *and* $UV(\cdot)$ *are in* $L_q^1(D_R)$*, then*

$$\begin{aligned} &I_q^{\alpha} UV(z) \\ &= z^{\gamma} \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)} \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha+\gamma \\ m \end{bmatrix}_q D_q^m \left((q^{\alpha-\gamma}\xi/z;q)_{\gamma} U(\xi)\right)|_{\xi=z} \left(I_q^{\alpha-\gamma+m}\right) V(zq^m), \end{aligned} \tag{6.23}$$

where $z \in G \cap D_R$*, and* $\alpha,\ \gamma \in \mathbb{R}$*.*

Remark 6.5.1. It is worthwhile to notice that if we set $\gamma = 0$ in (6.23), we obtain Agarwal's fractional Leibniz rule (6.21) with less restrictive conditions on the functions $U(z)$ and $V(z)$. Actually, the special case $\gamma = 0$ of Theorem 6.8 is an extension of the result given by Manocha and Sharma in [202].

Proof. Since V, UV are in $L_q^1(D_R)$, then

$$z^a V \in L_q^1(D_R), \ \operatorname{Re}(b) > -1 \text{ and } \ \operatorname{Re}(a+b) > -1.$$

From (4.26) we obtain

$$I_q^{\alpha}(U\,V)(z) = z^{\alpha}(1-q)^{\alpha} \sum_{n=0}^{\infty} q^n \frac{(q^{\alpha};q)_n}{(q;q)_n} U(zq^n)V(zq^n). \tag{6.24}$$

Substituting with

$$\frac{(q^{\alpha};q)_n}{(q;q)_n} = \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n}(q^{\alpha};q)_{-\gamma}(q^{\alpha-\gamma+n};q)_{\gamma}$$

into (6.24), we obtain

$$I_q^{\alpha}(UV)(z) = z^{\alpha}(1-q)^{\alpha}(q^{\alpha};q)_{-\gamma}\sum_{n=0}^{\infty} q^n \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n}(q^{\alpha-\gamma+n};q)_{\gamma}U(zq^n)V(zq^n). \tag{6.25}$$

The existence of UV in the space $L_q^1(D_R)$ guarantees that the series in (6.24) or in (6.25) converges absolutely for all $z \in D_R \setminus \{0\}$. Replace x in (1.27) by ξ and then let

$$f(\xi) = (q^{\alpha-\gamma}\xi/z;q)_{\gamma}\, U(\xi).$$

Consequently,

$$\begin{aligned} &(q^{\alpha-\gamma+n};q)_{\gamma}U(zq^n) \\ &= \sum_{k=0}^{n}(-1)^k(1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} z^k D_q^k \left((q^{\alpha-\gamma}\xi/z;q)_{\gamma}\, U(\xi)\right)\Big|_{\xi=z}. \end{aligned} \tag{6.26}$$

Then substituting (6.26) into (6.25), we get

$$\begin{aligned} I_q^{\alpha}(UV)(z) &= z^{\alpha}(1-q)^{\alpha}(q^{\alpha};q)_{-\gamma}\sum_{n=0}^{\infty} q^n \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n}V(zq^n) \\ &\times \sum_{k=0}^{n}(-1)^k(1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} z^k D_q^k \left((q^{\alpha-\gamma}\xi/z;q)_{\gamma}\, U(\xi)\right)\Big|_{\xi=z}. \end{aligned} \tag{6.27}$$

Using (1.25) we obtain

$$\begin{aligned} D_q^k\left((q^{\alpha-\gamma}\frac{\xi}{z};q)_{\gamma}U(\xi)\right)\Big|_{\xi=k} &= (-1)^k(1-q)^{-k}z^{a-k}q^{-\frac{k(k-1)}{2}}(q^{\alpha-\gamma};q)_{\gamma} \\ &\times \sum_{r=0}^{k} q^{\frac{r(r-1)}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \frac{(q^{\alpha};q)_{k-r}}{(q^{\alpha-\gamma};q)_{k-r}} q^{(k-r)a} u(zq^{k-r}) \end{aligned}$$

Therefore, since $u(z)$ is analytic in D_R, there exists $M > 0$ such that

$$\begin{aligned} &\left| D_q^k \left((q^{\alpha-\gamma}\tfrac{\xi}{z};q)_\gamma U(\xi) \right) \Big|_{\xi=k} \right| \\ &\leq (1-q)^{-k}|z|^{\mathrm{Re}(a)-k} q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}} M \textstyle\sum_{r=0}^{k} q^{\frac{r(r-1)}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q q^{-r\,\mathrm{Re}\,a} \\ &= M(1-q)^{-k}|z|^{\mathrm{Re}(a)-k} q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}} (-q^{-\mathrm{Re}(a)};q)_k \\ &\leq M(1-q)^{-k}|z|^{\mathrm{Re}(a)-k} q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}} (-q^{-\mathrm{Re}(a)};q)_\infty \end{aligned} \tag{6.28}$$

Consequently,

$$\begin{aligned} &\sum_{k=0}^{n} (1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \left| z^k D_q^k \left((q^{\alpha-\gamma}\frac{\xi}{z};q)_\gamma U(\xi) \right) \Big|_{\xi=z} \right| \\ &\qquad \leqslant M|z|^{\mathrm{Re}(a)} \frac{(-q^{\mathrm{Re}(a)};q)_\infty}{(q;q)_\infty^2} \frac{1-q^{(n+1)\,\mathrm{Re}(a)}}{1-q^{\mathrm{Re}(a)}}. \end{aligned} \tag{6.29}$$

Set $F(\xi) := (q^{\alpha-\gamma}\frac{\xi}{z};q)_\gamma U(\xi)$. Then substituting (6.29) into (6.5) we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{q^n|(q^{\alpha-\gamma};q)_n|}{(q;q)_n} |V(zq^n)| \sum_{k=0}^{n} (1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \left| z^k D_q^k F(z) \right| \\ &\leqslant M|z|^{\mathrm{Re}(a)} \frac{(-q^{\alpha-\gamma};q)_\infty (-q^{-\mathrm{Re}(a)};q)_\infty}{(q;q)_\infty^3} \sum_{n=0}^{\infty} \frac{1-q^{(n+1)\,\mathrm{Re}(a)}}{1-q^{\mathrm{Re}(a)}} q^n |V(zq^n)|. \end{aligned} \tag{6.30}$$

The last series converges for all $z \in D_R \setminus \{0\}$ since $V,\ z^a V \in L_q^1(D_R)$. Consequently, the series in (6.30) is absolutely convergent and we can interchange the order of summations in (6.5). This leads to

$$\begin{aligned} I_q^\alpha(UV)(z) &= z^\alpha (1-q)^\alpha (q^\alpha;q)_{1-\gamma} \\ &\quad \times \textstyle\sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} z^k \frac{(1-q)^k}{(q;q)_k} D_q^k F(z) \sum_{n=k}^{\infty} q^n \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_{n-k}} V(zq^n) \\ &= z^\alpha (1-q)^\alpha (q^\alpha;q)_{-\gamma} \textstyle\sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} z^k \frac{(1-q)^k}{(q;q)_k} D_q^k F(z) \\ &\quad \times \textstyle\sum_{j=0}^{\infty} q^j \frac{(q^{\alpha-\gamma};q)_{j+k}}{(q;q)_j} V(zq^{j+k}). \end{aligned} \tag{6.31}$$

Since

$$\left(I_q^{\alpha-\gamma+k} V \right)(zq^k) = (zq^k)^{\alpha-\gamma+k} (1-q)^{\alpha-\gamma+k} \sum_{j=0}^{\infty} q^j \frac{(q^{\alpha-\gamma+k};q)_j}{(q;q)_j} V(zq^{j+k}),$$

and

$$(q^{\alpha-\gamma};q)_{j+k} = (q^{\alpha-\gamma};q)_k (q^{\alpha-\gamma+k};q)_j,$$

the substitution with the last two identities in (6.31) gives

$$\begin{aligned} I_q^{\alpha}(UV)(z) &= z^{\gamma}(1-q)^{\gamma}(q^{\alpha};q)_{-\gamma} \\ &\quad \times \sum_{k=0}^{\infty}(-1)^k q^{-k(k-1)/2+k(-\alpha+\gamma)} \frac{(q^{\alpha-\gamma};q)_k}{(q;q)_k} \\ &\quad \times \left(I_q^{\alpha-\gamma+k}V\right)(zq^k) D_q^k \left((q^{\alpha-\gamma}\frac{\xi}{z};q)_{\gamma} U(\xi)\right)\Big|_{\xi=z} \\ &= z^{\gamma}(1-q)^{\gamma}(q^{\alpha};q)_{-\gamma} \\ &\quad \times \sum_{k=0}^{\infty} \begin{bmatrix} -\alpha+\gamma \\ k \end{bmatrix}_q \left(I_q^{\alpha-\gamma+k}V\right)(zq^k) D_q^k \left((q^{\alpha-\gamma}\frac{\xi}{z};q)_{\gamma} U(\xi)\right)\Big|_{\xi=z}, \end{aligned}$$

and the theorem follows. □

In the following theorem we derive Leibniz' formula for Caputo fractional q-difference operator.

Theorem 6.9. *Let f and g be analytic on $D_R := \{z \in \mathbb{C} : |z| < R\}$. Then*

$$D_{*q}^{\alpha}(fg)(z) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q D_q^m f(z) I_q^{\alpha+m} g(zq^m) - \sum_{k=0}^{n-1} D_q^k(fg)(0) \frac{z^{k-\alpha}}{\Gamma_q(k-\alpha+1)}.$$

Proof. The proof follows by combining (5.10) and Agarwal fractional q-Leibniz formula (6.21). □

6.6 A q-Extension of the Leibniz Rule via Weyl q-Fractional Derivative

In [248], Purohit derived a q-extension of the Leibniz rule via Weyl fractional q-difference operator defined in (4.27). He proved that for a nonnegative integer α

$$K_q^{\alpha}(UV)(z) = \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r} K_q^{\alpha-r} U(z) K_{q,z}^{r} V(zq^{r-\alpha}), \qquad (6.32)$$

where $U(z) = z^{-p_1}u(z)$, $V(z) = z^{-p_2}v(z)$, u and v are analytic functions having power series expansions at $z = 0$ with radius of convergence ρ, $\rho > 0$, and p_1, $p_2 \geq 0$. Purohit established some summation formulae as applications of the fractional Leibniz formula (6.32) which can be represented as

$$K_q^{\alpha}(UV)(z) = \sum_{r=0}^{\alpha} \frac{(q^{-\alpha};q)_r}{(q;q)_r} K_q^{\alpha-r} U(z) D_{q^{-1},z}^{r} \{V(zq^{\alpha})\},$$

where

$$D_{q^{-1},z}^{r} V(zq^{\alpha}) = (-1)^r q^{r(r+1)/2} K_{q,z}^{r} V(zq^{r-\alpha})$$

In [206], Mansour proved that the q-expansion in (6.32) can be derived for any $\alpha \in \mathbb{R}$. The proof introduced in [206] is completely different from the one introduced by Purohit for nonnegative integer values of α. The keynotes in proving the generalization of Purohit fractional q-Leibniz formula are two identities: the first one is

$$K_q^{\alpha} z^{-p} = q^{\alpha(1-\alpha)/2} q^{-\alpha p} z^{-\alpha-p} \frac{\Gamma_q(\alpha+p)}{\Gamma_q(p)}, \tag{6.33}$$

which holds for any $p \in \mathbb{R}$ when $\alpha \in \mathbb{N}$ or holds when $\alpha + p > 0$. The proof of (6.33) follows from (5.20) by replacing α with $-\alpha$, x with z, and setting $\phi(z) = z^{-p}$. The second identity follows from formula (1.27) with q replaced by q^{-1} and x by z. That is

$$\begin{aligned} f(zq^{-n}) &= \sum_{k=0}^{n} (q^{-1}-1)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} z^k D_{q^{-1}}^{k} f(z) \\ &= \sum_{k=0}^{n} q^{k(k-1)/2} (1-q)^k q^{-nk} \begin{bmatrix} n \\ k \end{bmatrix}_{q} z^k D_{q^{-1}}^{k} f(z), \end{aligned} \tag{6.34}$$

where we used, cf. [113, Eq. (I.47)]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_{q} q^{k^2-nk}.$$

The identity in (6.33) leads to the following result

Lemma 6.10. *Let p and α be such that $0 < \mathrm{Re}(p) < 1$ and $\alpha > \mathrm{Re}(p)$. Let G be the principal branch of the logarithmic function and let $D_R := \{z \in \mathbb{C} : |z| > R\} \cap G$. Assume that*

$$U(z) = z^p \sum_{j=0}^{\infty} a_j z^{-j}$$

is analytic on D_R. Let

$$\Omega_{\alpha} = \{z \in D_R : q^{\alpha} z \in D_R\}.$$

Then $K_q^\alpha U(z)$ exists for all $z \in \Omega_\alpha$. Moreover,

$$K_q^\alpha U(z) = q^{\alpha p} \frac{\Gamma_q(\alpha - p)}{\Gamma_q(-p)} z^{p-\alpha} \sum_{j=0}^{\infty} a_j \frac{(q^{-p};q)_j}{(q^{\alpha-p};q)_j} q^{-\alpha j} z^{-j} \quad (z \in \Omega_\alpha)\,. \tag{6.35}$$

Proof. From (5.20) we find that

$$K_q^\alpha U(z) = q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{k=0}^{\infty} q^{k(\alpha-p)} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \sum_{j=0}^{\infty} a_j q^{(-\alpha+k)j}. \tag{6.36}$$

From the assumptions of the present lemma we can easily deduce that the double series in (6.36) is absolutely convergent for all $z \in \Omega_\alpha$. Hence, we can interchange the order of summations in (6.36). This and the q-binomials theorem (1.59) give

$$\begin{aligned} K_q^\alpha U(z) &= q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{j=0}^{\infty} a_j q^{-\alpha j} z^{-j} \sum_{k=0}^{\infty} q^{k\alpha - kp + kj} \frac{(q^{-\alpha};q)_j}{(q;q)_j} \\ &= q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{j=0}^{\infty} a_j q^{-\alpha j} z^{-j} \frac{(q^{-p+j};q)_\infty}{(q^{\alpha-p+j};q)_\infty}. \end{aligned}$$

Simple manipulations lead to (6.35). □

Lemma 6.11. *Let p, α, G, U, D_R, and Ω_α as in Lemma 6.10. Then*

$$\sum_{j=0}^{\infty} q^{\alpha j} |U(zq^{\alpha-j})| < \infty \quad \textit{for all } z \in \Omega_\alpha.$$

Proof. The proof is easy and is omitted. □

Theorem 6.12. *Let U and V be functions defined on a q^{-1}-geometric set A and let $\alpha \in \mathbb{R}$. Assume that $UV \in S_{q,\alpha}$ and $U \in S_{q,\mu}$, $\mu > \frac{1}{2}$. Then*

$$K_q^\alpha UV(z) = \sum_{m=0}^{\infty} \frac{(q^{-\alpha};q)_m}{(q;q)_m} K_q^{\alpha-m} U(z) D_{q^{-1},z}^m \{V(zq^\alpha)\} \tag{6.37}$$

for all $z \in A$ and for all in $\alpha \in \mathbb{R}$. If $\mu = 1/2$ then (6.37) may not hold for all α on $\mathbb{R}$ but only for α in a subdomain of $\mathbb{R}$.

Proof. Let $z \in q^{-\alpha} A$ be arbitrary but fixed. Since $UV \in S_{q,\alpha}$, then

$$\sum_{k=0}^{\infty} q^{k\alpha} \left| U(zq^{\alpha-k}) V(zq^{\alpha-k}) \right| < \infty. \tag{6.38}$$

From (5.20)

$$\left(K_q^{\alpha} UV\right)(z) = q^{\alpha(1-\alpha)/2}(1-q)^{-\alpha} z^{-\alpha} \sum_{m=0}^{\infty} q^{\alpha m} \frac{(q^{-\alpha};q)_m}{(q;q)_m} U(zq^{\alpha-m}) V(zq^{\alpha-m}).$$

Applying (6.34) with $f(z) = V(zq^{\alpha})$ yields

$$\begin{aligned}\left(K_q^{\alpha} UV\right)(z) &= q^{\alpha(1-\alpha)/2}(1-q)^{-\alpha} z^{-\alpha} \sum_{m=0}^{\infty} q^{\alpha m} \frac{(q^{-\alpha};q)_m}{(q;q)_m} U(zq^{\alpha-m}) \\ &\times \sum_{j=0}^{m} q^{j(1-j)/2} (1-q)^j q^{-mj} \begin{bmatrix} m \\ j \end{bmatrix}_q z^j D_{q^{-1},z}^{j} \{V(zq^{\alpha})\}.\end{aligned} \tag{6.39}$$

From the assumptions on the function U, there exists a constant $C_1 > 0$ and $\nu \in \mathbb{R}$ such that

$$|U(zq^{\alpha-m})| \le C_1\, q^{\mu m(m+\nu)}.$$

Using (1.25) with q^{-1} instead of q, we obtain

$$z^j D_{q^{-1},z}^{j} \{V(zq^{\alpha})\} \le C_2.$$

Consequently, the double series on (6.39) is bounded from above by

$$\begin{aligned} & C_1 C_2 \frac{(-q^{-\operatorname{Re}\alpha};q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{m \operatorname{Re}\alpha} q^{\mu m(m+\nu)} \sum_{j=0}^{m} q^{j(1-j)/2} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{-mj} \\ &\le C_1 C_2 \frac{(-q^{-\operatorname{Re}\alpha};q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{m \operatorname{Re}\alpha} q^{\mu m(m+\nu)} (-q^{-m};q)_m \\ &\le C_1 C_2 \frac{(-q^{-\operatorname{Re}\alpha};q)_\infty (-q;q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{m \operatorname{Re}\alpha} q^{\mu m(m+\nu)} q^{-m(m+1)/2}, \end{aligned} \tag{6.40}$$

where we applied the identity in (1.7). Now it is clear that if $\mu > 1/2$, the series on the most right hand side of (6.40) is convergent for all $\alpha \in \mathbb{C}$. On the other hand, it is convergent only for $\operatorname{Re}\alpha > -\nu + \frac{1}{2}$ when $\mu = \frac{1}{2}$. Therefore, we can interchange the order of summation in the series on the right hand side of (6.39). This gives

$$\begin{aligned}\left(K_q^{\alpha} UV\right)(z) &= q^{\alpha(1-\alpha)/2}(1-q)^{-\alpha} z^{-\alpha} \\ \times \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j} z^j (1-q)^j D_{q^{-1},z}^{j} V(zq^{\alpha}) & \sum_{r=0}^{\infty} q^{(\alpha-j)r} \frac{(q^{-\alpha+j};q)_r}{(q;q)_r} U(zq^{\alpha-j-r}).\end{aligned} \tag{6.41}$$

But

$$\sum_{r=0}^{\infty} q^{(\alpha-j)r} \frac{(q^{-\alpha+j};q)_r}{(q;q)_r} U(zq^{\alpha-j-r}) = z^{\alpha-j}(1-q)^{\alpha-j} q^{(\alpha-j)(\alpha-j-1)/2} K_q^{\alpha-j} V(z).$$

Combining this latter identity with (6.41) yields the theorem. ☐

6.7 Applications

In this section we apply the fractional Leibniz formulae introduced through the chapter to derive transformations. Al-Salam and Verma [20] used the fractional q-Leibniz formula (6.6) to derive some identities as in the next two applications. In the remaining applications, we derive some identities using Agarwal's fractional Leibniz formula and its generalization.

Application 6.1 *Let* $N \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, *and* $\lambda \in \mathbb{C}$ *be such that* $\operatorname{Re}(\lambda) > 0$. *Then*

$$\begin{aligned} {}_2\Phi_1\left(q^{\alpha}, zq^N; z; q, q^{\lambda}\right) = {} & \frac{(q^{\lambda+\alpha};q)_\infty}{(q^{\lambda};q)_\infty} \frac{(zq^{-\alpha};q)_N}{(z;q)_N} \\ & \times {}_3\Phi_2\left(0, q^{\alpha}, q^{-N}; q^{\alpha-N+1}/z, q^{\lambda+\alpha}; q, q\right). \end{aligned} \tag{6.42}$$

Proof. Take

$$U(z) = (z;q)_N \quad \text{and} \quad V(z) = z^{\lambda-1}.$$

One can verify that for $m \in \mathbb{N}_0$ we have

$$D_q^m U(z) = \begin{cases} (1-q)^{-m} q^{Nm} (q^{-N};q)_m (zq^m;q)_{N-m}, & \text{if } m \le N, \\ 0, & \text{if } m > N. \end{cases}$$

Then

$$(D_q^m U)(zq^{-\alpha-m}) = (1-q)^{-m} q^{Nm} (q^{-N};q)_m (zq^{-\alpha};q)_{N-m}.$$

But from [113, Eq. (I.10)]

$$(zq^{-\alpha};q)_{N-m} = q^{\binom{m}{2}} \left(-\frac{q^{\alpha+1-N}}{z}\right)^m \frac{(zq^{-\alpha};q)_N}{(q^{\alpha-N+1}/z;q)_m}.$$

Consequently,

$$(D_q^m U)(zq^{-\alpha-m}) = q^{\binom{m}{2}} \left(-\frac{q^{\alpha+1}}{z(1-q)}\right)^m (q^{-N};q)_m \frac{(zq^{-\alpha};q)_N}{(q^{\alpha-N+1}/z;q)_m}. \tag{6.43}$$

Moreover,

$$I_q^{\alpha+m} z^{\lambda-1} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} \frac{(1-q)^m}{(q^{\lambda+\alpha};q)_m} z^{\lambda+\alpha+m-1}. \tag{6.44}$$

Combining (6.43), (6.44) and (6.19) yields

$$\begin{aligned} &I_q^{\alpha}(UV)(z) \\ &= z^{\alpha+\lambda-1} \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} (zq^{-\alpha};q)_N \sum_{m=0}^{N} \frac{(q^{\alpha};q)_m\,(q^{-N};q)_m}{(q;q)_m\,(q^{\alpha-N+1}/z;q)_m\,(q^{\lambda+\alpha};q)_m} q^m \\ &= z^{\alpha+\lambda-1} \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} (zq^{-\alpha};q)_N\ {}_3\Phi_2\left(0, q^{\alpha}, q^{-N}; q^{\alpha-N+1}/z, q^{\lambda+\alpha}; q, q\right). \end{aligned} \tag{6.45}$$

On the other hand, using (6.14) we obtain

$$\begin{aligned} I_q^{\alpha}(UV)(z) &= (1-q)^{\alpha} z^{\alpha+\lambda-1} (z;q)_N \sum_{m=0}^{\infty} q^{m\lambda} \frac{(q^{\alpha};q)_m (zq^N;q)_m}{(q;q)_m (z;q)_m} \\ &= (1-q)^{\alpha} z^{\alpha+\lambda-1} (z;q)_N\ {}_2\Phi_1\left(q^{\alpha}, zq^N; z; q, q^{\lambda}\right), \end{aligned} \tag{6.46}$$

whenever Re $(\lambda) > 0$. Combining (6.45) with (6.46) gives (6.42). □

Application 6.2 *Let* λ *and* α *be complex numbers satisfying* $\mathrm{Re}(\lambda-\alpha) > 0$. *Then*

$$\begin{aligned} {}_2\phi_1\left(q^{\alpha}, 0; q^{n+\lambda+\alpha}; q, -zq^{-\alpha}\right) &= (q^{-\alpha}z;q)_{\infty} q^{-n\alpha} \frac{(q^{\lambda+\alpha};q)_n}{(q^{\lambda};q)_n} \\ \times \sum_{k=0}^{n} \frac{q^{(\lambda+n)k}(q^{\alpha};q)_k\,(q^{-n};q)_k\,(zq^{n-k};q)_k}{(q;q)_k (q^{\alpha+\lambda};q)_k} &{}_2\phi_1\left(q^{\alpha+k}, 0; q^{\alpha+\lambda+k}; q, zq^{n-k}\right), \end{aligned} \tag{6.47}$$

holds for $|zq^{-\alpha}| < 1$ *and* $n \in \mathbb{N}_0$.

Proof. The identity in (6.47) follows by calculating

$$I_q^{\alpha}\left(z^{n+\lambda-1}(z;q)_{\infty}\right)$$

in two different ways. Firstly by letting

$$U(z) = z^n, \quad V(z) = z^{\lambda-1}(z;q)_{\infty}$$

and secondly by taking

$$U(z) = (z;q)_{\infty}, \quad V(z) = z^{n+\lambda-1}.$$

Equating the results of these two calculations we obtain (6.47). □

If we apply the Leibniz formula of Theorem 6.8 to the functions U and V defined on Application 6.1, we obtain the identity

$$ {}_2\phi_1\left(q^{\alpha}, zq^{n}; z; q, q^{\lambda}\right) = \frac{(q^{\lambda+\alpha};q)_\infty}{(q^{\lambda};q)_\infty}{}_2\phi_2\left(q^{-n}, q^{\alpha}; q^{\lambda+\alpha}, z; q, q^{\lambda+\alpha}\right). $$

If we set $a = q^{\alpha}$, $b = zq^{n}$, $c = z$, and $\xi = q^{\lambda}$, we obtain Jackson transformation of ${}_2\phi_1$ and ${}_2\phi_2$:

$$ {}_2\phi_1\left(a, b; c; q, \xi\right) = \frac{(a\xi;q)_\infty}{(\xi;q)_\infty}{}_2\phi_2\left(a, c/b; c, a\xi; q, b\xi\right). $$

See [113, Eq. (III.4)]

Application 6.3 *Let* $q \in (0,1)$ *and let* $\lambda, \mu \in \mathbb{C}$ *such that* $\operatorname{Re}(\lambda) > 0$. *If* $|zq^{\mu}| < 1$ *then*

$$ {}_2\Phi_1\left(q^{\alpha}, zq^{\mu}; z; q, q^{\lambda}\right) = \frac{(q^{\lambda+\alpha};q)_\infty}{(q^{\lambda};q)_\infty}{}_2\Phi_2\left(q^{\alpha}, q^{-\mu}; q^{\lambda+\alpha}, z; q, q^{\lambda+\mu}z\right). \tag{6.48} $$

Proof. Let

$$ U(z) = (z;q)_\mu, \quad V(z) = z^{\lambda-1}. $$

Then

$$ D_q^k U(z) = q^{\mu k}\left(q^{-\mu};q\right)_k \frac{(z;q)_\mu}{(z;q)_k}(1-q)^{-k}, $$

and

$$ I_q^{\alpha+k} V(zq^k) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}\frac{(1-q)^k}{(q^{\lambda+\alpha};q)_k}q^{k(\lambda+\alpha+k-1)}z^{\lambda+\alpha+k-1}. $$

Consequently, for $|zq^{\mu}| < 1$ we have

$$ \begin{aligned} &I_q^{\alpha} UV(z) \\ &= z^{\lambda+\alpha-1}(z;q)_\mu \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}\sum_{k=0}^{\infty}(-1)^k q^{k(k-1)/2}\frac{(q^{\alpha}, q^{-\mu};q)_k}{(q, q^{\lambda+\alpha}, z;q)_k}(zq^{\lambda+\mu})^k \qquad (6.49) \\ &= z^{\lambda+\alpha-1}(z;q)_\mu \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}{}_2\Phi_2\left(q^{\alpha}, q^{-\mu}; q^{\lambda+\alpha}, z; q, q^{\lambda+\mu}z\right). \end{aligned} $$

On the other hand, using the definition (6.14) we obtain for Re $(\lambda) > 0$

$$\begin{aligned} I_q^{\alpha} UV(z) &= z^{\lambda+\alpha-1}(1-q)^{\alpha}(z;q)_{\mu}\sum_{n=0}^{\infty} q^{n\lambda}\frac{(q^{\alpha};q)_n(z;q)_n}{(q;q)_n(z;q)_n} \\ &= z^{\lambda+\alpha-1}(1-q)^{\alpha}(z;q)_{\mu\,2}\Phi_1\left(q^{\alpha}, zq^{\mu}; z; q, q^{\lambda}\right). \end{aligned} \tag{6.50}$$

By equating (6.49) and (6.50) we obtain (6.48). □

Application 6.4 *Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$,* Re $(\lambda) > 0$. *Then*

$$\begin{aligned} &{}_2\phi_2\left(0, q^{\alpha}; z, q^{n+\lambda+\alpha}; q, zq^{\lambda+n}\right) \\ &= \frac{(q^{n+\lambda};q)_{\infty}}{(q^{n+\lambda+\alpha};q)_{\infty}}\sum_{k=0}^{n} q^{(\lambda+n)k}\frac{(q^{\alpha};q)_k\,(q^{-n};q)_k}{(q;q)_k(z;q)_k}\,{}_2\phi_1\left(0, q^{\alpha+k}; zq^k; q, q^{\lambda}\right), \end{aligned} \tag{6.51}$$

holds for all $z \in \mathbb{C} \setminus \{q^{-k} : k \in \mathbb{N}_0\}$.

Proof. Formula (6.51) follows by calculating $I_q^{\alpha} z^{n+\lambda-1}(z;q)_{\infty}$ by two different ways. Firstly, we take

$$U(z) = z^n \quad \text{and} \quad V(z) = z^{\lambda-1}(z;q)_{\infty}.$$

In this case

$$D_q^k U(z) = \begin{cases} (-1)^k q^{nk-k(k-1)/2}(1-q)^{-k}\,(q^{-n};q)_k\, z^{n-k}, & k = 0, 1, \ldots, n \\ 0, & k \in \{n, n+1, \ldots\}, \end{cases} \tag{6.52}$$

$$I_q^{\alpha+k} V(zq^k) = (zq^k)^{\alpha+k+\lambda-1}(zq^k;q)_{\infty}(1-q)^{\alpha+k}\sum_{r=0}^{\infty} q^{r\lambda}\frac{(q^{\alpha+k};q)_r}{(q;q)_r(zq^k;q)_r}. \tag{6.53}$$

The q-fractional integral $I_q^{\alpha+k} V(zq^k)$ exists for Re $(\lambda) > 0$ and for all $z \in \mathbb{C}$ because the poles of the series in (6.53) are zeros for $(zq^k;q)_{\infty}$. Moreover, the series in (6.53) is the function ${}_2\phi_1\left(0, q^{\alpha+k}; zq^k; q, q^{\lambda}\right)$. Hence, applying Agarwal's fractional Leibniz rule (6.21) gives

$$\begin{aligned} I_q^{\alpha} UV(z) &= (1-q)^{\alpha}\, z^{n+\lambda+\alpha-1}(z;q)_{\infty} \\ &\quad\times\sum_{k=0}^{n} q^{(\lambda+n)k}\frac{(q^{\alpha};q)_k\,(q^{-n};q)_k}{(q;q)_k(z;q)_k}\,{}_2\phi_1\left(0, q^{\alpha+k}; zq^k; q, q^{\lambda}\right). \end{aligned} \tag{6.54}$$

Secondly, if we take

$$U(z) = (z;q)_{\infty} \quad \text{and} \quad V(z) = z^{n+\lambda-1},$$

we obtain

$$I_q^{\alpha} U V(z) = z^{n+\lambda+\alpha-1}(z;q)_{\infty}(1-q)^{\alpha} \frac{(q^{n+\lambda+\alpha};q)_{\infty}}{(q^{n+\lambda};q)_{\infty}} {}_2\phi_2\left(q^{\alpha}; z, q^{n+\lambda+\alpha}; q, q^{\lambda+n}\right). \tag{6.55}$$

By equating (6.54) and (6.55) we obtain (6.51). □

Application 6.5 *Let a, b, and c be complex numbers such that* $\mathrm{Re}(b) > 0$*. Then for* $|zq^{-a}| < 1$*,*

$$\begin{aligned} {}_2\phi_1\left(q^a, q^b, q^c; q, q^{-a}z\right) &= \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \\ \times \sum_{m=0}^{\infty} q^{mb} &\frac{(q^{-b+1};q)_m}{(1-q^{c-b+m})(q;q)_m} {}_2\phi_1\left(q^a, q; q^{m+1+b-c}; q, zq^{-a+m}\right). \end{aligned}$$

Proof. The proof follows by calculating $I_q^{c-b}\left(z^{b-1}(z;q)_{-a}\right)$ by applying Theorem 6.8 in two different ways. The first one is by taking

$$U(z) = z^{b-1}, \ V(z) = (z;q)_{-a},$$

and the second one follows by taking

$$U(z) = (z;q)_{-a}, \ V(z) = z^{b-1}.$$

□

Application 6.6 *For* $n \in \mathbb{N}_0$*,* $\alpha \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > 0$

$$\begin{aligned} {}_2\phi_1\left(q^{-n}, q^{\alpha}; q^{\lambda+\alpha}; q, q^{\lambda+n}\right) &= q^{-n\alpha} {}_2\phi_1\left(q^{-n}, q^{\alpha}; q^{\lambda+\alpha}; q, q\right) \\ &= \frac{(q^{\lambda};q)_n}{(q^{\lambda+\alpha};q)_n}. \end{aligned} \tag{6.56}$$

Proof. Set $U(z) = z^n$ and $V(z) = z^{\lambda-1}$. Then applying Theorem 6.6 gives

$$I_q^{\alpha} UV(z) = z^{n+\lambda+\alpha-1} q^{-n\alpha} \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha+m)} {}_2\phi_1\left(q^{-n}, q^{\alpha}; q^{\lambda+\alpha}; q, q\right).$$

Now, applying Agarwal's fractional Leibniz formula (6.21) yields

$$I_q^{\alpha} UV(z) = z^{n+\lambda+\alpha-1} \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha+m)} {}_2\phi_1\left(q^{-n}, q^{\alpha}; q^{\lambda+\alpha}; q, q^{\lambda+n}\right).$$

But a direct computation gives

$$I_q^{\alpha} z^{n+\lambda-1} = \frac{\Gamma_q(n+\lambda)}{\Gamma_q(n+\lambda+\alpha)} z^{n+\lambda+\alpha-1}.$$

Hence, equating the last three identities gives (6.56). □

Application 6.7 *For complex numbers a, b, and c such that* $\mathrm{Re}(c-a) > 0$ *and* $\mathrm{Re}(c-a-b) > 0$

$$\frac{\Gamma_q(c-a-b)\Gamma_q(c)}{\Gamma_q(c-b)\Gamma_q(c-a)} = {}_2\phi_1\left(q^a, q^b; q^c; q, q^{c-a-b}\right) \tag{6.57}$$

Proof. The proof of (6.57) follows by taking

$$U(z) = z^{-b}, \quad V(z) = z^{c-a-1} \tag{6.58}$$

and applying the fractional Leibniz rule Theorem 6.8 with $\gamma = 0, \alpha = a$. This gives

$$I_q^{\alpha}(UV)(z) = z^{c-b-1} \frac{\Gamma_q(c-a)}{\Gamma_q(c)} \sum_{k=0}^{\infty} q^{k(c-a-b)} \frac{(q^a;q)_k (q^b;q)_k}{(q;q)_k (q^c;q)_k}. \tag{6.59}$$

On the other hand,

$$I_q^{\alpha} UV(z) = I_q^{a} z^{c-a-b-1} = \frac{\Gamma_q(c-a-b)}{\Gamma_q(c-b)} z^{c-b-1}. \tag{6.60}$$

Consequently, equating the identities in (6.60) and (6.59) yields (6.57). □

The expansion (6.57) is the q-Gauss summation formula introduced by Heine, cf. [130–133].

Application 6.8 *For complex numbers a, b, A, B, d, and D such that* $\mathrm{Re}(b) > -1$, $\mathrm{Re}(B) > 0$, *and* $\mathrm{Re}(b+B) > 1$

$$\begin{aligned} {}_2\phi_1\left(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)} z\right) &= (z;q)_{-a} \frac{\Gamma_q(d+D)\Gamma_q(B)}{\Gamma_q(b+B)\Gamma_q(d+B-b)} (1-q)^{B-D} \\ &\times \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} q^{mB} \frac{(q^{b-d};q)_m}{(q;q)_m (q^{d+D-b};q)_m} \\ \times {}_3\phi_2\left(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\right) &{}_2\phi_1\left(q^A, q^B; q^{d+B-b+m}; q, q^{-a-A+m}\right), \end{aligned} \tag{6.61}$$

for $|zq^{-a-A}| < 1$.

Proof. The previous identity follows by taking

$$U(z) = z^{b}(z;q)_{-a}, \quad V(z) = z^{B-1}(zq^{-a};q)_{-A},$$

and applying Theorem 6.8 with

$$\alpha = d + D - b - B, \quad \gamma = D - B.$$

Then using (1.26) we obtain

$$\begin{aligned} D_q^m &\left((q^{d-b}\frac{\xi}{z};q)_{D-B}\,\xi^{b}(\xi;q)_{-a} \right)\Big|_{\xi=z} \\ &= (1-q)^{-m} z^{-m+b}(z;q)_{-a}\frac{(q^{d-b};q)_\infty}{(q^{d-b+D-B};q)_\infty} \\ &\times {}_3\phi_2\left(q^{-m}, q^{d-b+D-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\right). \end{aligned} \tag{6.62}$$

In addition,

$$\begin{aligned} (I_q^{\alpha-\gamma+m}V)(zq^m) &= z^{d-b+\beta-1+m} q^{m^2+(d-b+\beta-1)m}\frac{\Gamma_q(\beta)}{\Gamma_q(B+m+d-b)} \\ &\times {}_2\phi_1\left(q^{A}, q^{\beta}; q^{m-b}; q, q^{m-a-A}\right), \end{aligned} \tag{6.63}$$

and

$$\begin{bmatrix} \gamma-\alpha \\ m \end{bmatrix}_q = (-1)^m q^{(\gamma-\alpha)m} q^{\binom{-m}{2}}\frac{(q^{\alpha-\gamma};q)_m}{(q;q)_m}. \tag{6.64}$$

Substituting with (6.62)–(6.64) into (6.23), we obtain

$$\begin{aligned} I_q^{\alpha}UV(z) &= z^{d+D-1}(1-q)^{D-B}(z;q)_{-a}\frac{\Gamma_q(\beta)}{\Gamma_q(B+d-b)} \\ &\times \sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}} q^{mB}\frac{(q^{b-d};q)_m}{(q;q)_m(q^{d+D-b};q)_m} \\ \times {}_3\phi_2&\left(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\right) {}_2\phi_1\left(q^{A}, q^{B}; q^{d+B-b+m}; q, q^{-a-A+m}\right) \end{aligned} \tag{6.65}$$

On the other hand,

$$I_q^{\alpha}UV(z) = \frac{\Gamma_q(b+B)}{\Gamma_q(d+D)} z^{d+D-1} {}_2\phi_1\left(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)}\right). \tag{6.66}$$

Combining (6.65) and (6.66) we obtain (6.61). □

Application 6.9 *Let γ, λ, μ, and α be complex numbers satisfying*

$$\mathrm{Re}(\lambda) > 0, \quad \mathrm{Re}(\lambda+\mu) > 0, \ \ \mu \notin \mathbb{N}_0 \ \ \textit{and} \ \ \mathrm{Re}(\alpha) > 0.$$

Then

$$\begin{aligned} &\frac{\Gamma_q(\lambda+\mu)\Gamma_q(\alpha)\Gamma_q(\lambda+\alpha-\gamma)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda)\Gamma_q(\lambda+\mu+\alpha)} \\ &= \sum_{m=0}^{\infty} q^{m(\lambda+\mu)} \frac{(q^{\alpha-\gamma}, q^{-\mu}; q)_m}{(q, q^{\lambda+\alpha-\gamma}; q)_m} {}_2\phi_1\left(q^{-\gamma}, q^{\mu+1}; q^{\mu-m+1}; q, q^{\alpha}\right). \end{aligned} \tag{6.67}$$

Proof. We prove the identity by using Theorem 6.8. Take $U(z) = z^{\mu}$ and $v(z) = z^{\lambda-1}$. Then,

$$\begin{aligned} D_q^m (q^{\alpha-\gamma}\frac{\xi}{z}; q)_\gamma \xi^{\mu} &= D_q^m \sum_{k=0}^{\infty} \frac{(q^{-\gamma}; q)_k}{(q;q)_k} \left(\frac{q^{\alpha}}{z}\right)^k \xi^{\mu+k} \\ &= \sum_{k=0}^{\infty} \frac{(q^{-\gamma}; q)_k}{(q;q)_k} \left(\frac{q^{\alpha}}{z}\right)^k \frac{\Gamma_q(\mu+k+1)}{\Gamma_q(\mu+k-m+1)} \xi^{\mu+k-m}. \end{aligned}$$

Hence,

$$\begin{aligned} D_q^m (q^{\alpha-\gamma}\tfrac{\xi}{z}; q)_\gamma \xi^{\mu}\Big|_{\xi=z} &= z^{\mu-m} \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-m+1)} {}_2\phi_1\left(q^{-\gamma}, q^{\mu+1}; q^{\mu-m+1}; q, q^{\alpha}\right) \\ &= (-1)^m \frac{(q^{-\mu}; q)_m}{(1-q)^m} q^{m\mu-\binom{m}{2}} z^{\mu-m} \\ &\quad \times {}_2\phi_1\left(q^{-\gamma}, q^{\mu+1}; q^{\mu-m+1}; q, q^{\alpha}\right). \end{aligned} \tag{6.68}$$

and

$$\begin{aligned} \left(I_q^{\alpha-\gamma+m} V\right)(zq^m) &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma+m)} (zq^m)^{\lambda+\alpha-\gamma+m-1} \\ &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma)} \frac{(1-q)^m}{(q^{\lambda+\alpha-\gamma}; q)_m} (zq^m)^{\lambda+\alpha-\gamma+m-1}. \end{aligned} \tag{6.69}$$

Then applying Theorem 6.8 gives

$$\begin{aligned} I_q^{\alpha}(UV)(z) &= z^{\mu+\lambda+\alpha-1} \frac{\Gamma_q(\alpha)\Gamma_q(\lambda)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda+\alpha-\gamma)} \\ &\quad \times \sum_{m=0}^{\infty} q^{m(\lambda+\mu)} \frac{(q^{\alpha-\gamma}, q^{-\mu}; q)_m}{(q, q^{\lambda+\alpha-\gamma}; q)_m} {}_2\phi_1\left(q^{-\gamma}, q^{\mu+1}; q^{\mu-m+1}; q, q^{\alpha}\right). \end{aligned} \tag{6.70}$$

On the other hand,

$$I_q^{\alpha} z^{\lambda+\mu-1} = \frac{\Gamma_q(\lambda+\mu)}{\Gamma_q(\lambda+\mu+\alpha)} z^{\lambda+\mu+\alpha-1}. \tag{6.71}$$

Equating (6.70) and (6.71) gives (6.67). □

Chapter 7
q-Mittag–Leffler Functions

Abstract The classical Mittag–Leffler function plays an important role in fractional differential equations. In this chapter we mention in brief the q-analogues of the Mittag–Leffler functions defined by mathematicians. We pay attention to a pair of q-analogues of the Mittag–Leffler function that may be considered as a generalization of the q-exponential functions $e_q(z)$ and $E_q(z)$. We study their main properties and give a Mellin–Barnes integral representations and Hankel contour integral representation for them. As in the classical case we prove that the q-Mittag–Leffler functions are solutions of q-type Volterra integral equations. Finally, asymptotics of zeros of one of the pair of the q-Mittag–Leffler function will be given at the end of the chapter.

7.1 Mittag–Leffler Functions

The special function of the form

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\rho + 1)} \quad (\rho > 0)\,, \tag{7.1}$$

and more general functions

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\rho + \mu)} \quad (\rho > 0;\ \mu \in \mathbb{C})\,, \tag{7.2}$$

where $z \in \mathbb{C}$, are known as Mittag–Leffler functions, see e.g. [95, 143, 168]. The one parameter function $E_{\rho,1}(z) = E_\rho(z)$ was first introduced by Mittag–Leffler in a sequence of five notes [219–223]. See also [200]. The two parameter function (7.2) was first introduced by Agarwal in [16], see also Humbert and Agarwal [137]. The function $E_{\rho,\mu}(z)$ is an entire function of z of order $1/\rho$ and type 1.

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The Mittag–Leffler function $E_{\rho,\mu}(z)$ is a special case of the Fox–Wright function (1.82). It is straightforward to verify

$$E_{\rho,\mu}(z) = {}_1\psi_1\left[\begin{matrix}(1,1)\\(\mu,\rho)\end{matrix}\middle| z\right].$$

These functions had been extensively studied by Kilbas et al. in [167] and in [168].

The two parameter Mittag–Leffler functions appeared as solutions of fractional integro differential equations, see e.g. [52, 85, 114, 117, 172, 200, 235]. See also [21, 22, 84, 85, 116, 235, 239–241, 246, 272, 273, 283] for properties and asymptotics of zeros, and more historical notes. In 1971, Prabhakar [244] introduced a generalization of the Mittag–Leffler function of two parameters as a kernel of certain fractional differential equations. The generalized Mittag–Leffler function is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n(\gamma)_n}{n!\Gamma(n\alpha+\beta)}.$$

Now, we give q-analogues of the Mittag–Leffler functions. Since we have two major q-exponential functions, namely, $e_q(z)$ and $E_q(z)$, we will define the following two q-analogues of the Mittag–Leffler functions

$$\begin{aligned} e_{\alpha,\beta}(z;q) &:= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n\alpha+\beta)} \qquad (|z(1-q)^{\alpha}|<1)\,,\\ E_{\alpha,\beta}(z;q) &:= \sum_{n=0}^{\infty} \frac{q^{\alpha n(n-1)/2}}{\Gamma_q(n\alpha+\beta)} z^n \qquad (z\in\mathbb{C})\,, \end{aligned} \tag{7.3}$$

where $\alpha > 0,\ \beta \in \mathbb{C}$. When $\beta = 1$, the functions $e_{\alpha,1}(z;q)$ and $E_{\alpha,1}(z;q)$ define families of q-exponential functions of one parameter. Another one-parameter family of q-exponential functions is defined by, cf. [42, 104, 143],

$$E_q^{(\alpha)}(z) := \sum_{k=0}^{\infty} \frac{q^{\alpha k^2/2}}{(q;q)_k} z^k \qquad (\alpha\in\mathbb{C})\,.$$

Rajković et al. in [255] introduced another pair of q-Mittag–Leffler functions. This pair is denoted by $e_{q;\alpha,\beta}(z;c)$, $E_{q;\alpha,\beta}(z;c)$, and defined by

$$e_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty} \frac{(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}} x^{\alpha n+\beta-1} \qquad (|x|>|c|)\,,$$

$$E_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty} \frac{q^{\binom{\alpha n+\beta-1}{2}}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}(-c;q)_{\alpha n+\beta-1}} \qquad (x\in\mathbb{C})\,,$$

where

$$\{q, x, c, \alpha\} \subset \mathbb{C},\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0 \text{ and } |q| < 1.$$

The authors of [255] called $e_{q;\alpha,\beta}(x;c)$ the small q-Mittag–Leffler function and $E_{q;\alpha,\beta}(x;c)$ the big q-Mittag–Leffler function. It is clear that

$$e_{q;\alpha,\beta}(x;0) = (1-q)^{-\beta} x^{\beta-1} e_{\alpha,\beta}(x^{\alpha}(1-q)^{-\alpha};q).$$

7.2 The q-Mittag–Leffler Function $e_{\alpha,\beta}(z;q)$

The following formulae follow directly from the definition of $e_{\alpha,\beta}(z;q)$:

$$\begin{aligned}
e_{2,1}(z;q) &= \cosh_q \sqrt{z}(1-q), \quad e_{2,2}(z;q) = \frac{\sinh_q \sqrt{z}(1-q)}{\sqrt{z}},\\
e_{1,1}(z;q) &= e_q(z(1-q)), \qquad e_{1,2}(z;q) = \frac{e_q\big(z(1-q)\big)-1}{z},\\
e_{\alpha,\beta}(z;q) &= \frac{1}{\Gamma_q(\beta)} + z e_{\alpha,\alpha+\beta}(z;q),\\
q^{\beta} z D_{q,z} e_{\alpha,\beta+1}(z^{\alpha};q) &= -\frac{1-q^{\beta}}{1-q} e_{\alpha,\beta+1}(z^{\alpha};q) + e_{\alpha,\beta}(z^{\alpha};q).
\end{aligned}$$

We also have the following set of properties ;

$$\begin{aligned}
D^m_{q,z} e_{m,1}(z^m;q) &= e_{m,1}(z^m;q),\\
D^m_{q,z} e_{m/n,1}(z^{m/n};q) &= e_{m/n,1}(z^{m/n};q) + \sum_{k=1}^{n-1} \frac{z^{\frac{m}{n}k-m}}{\Gamma_q(\frac{m}{n}k-m+1)},\\
e_{\alpha,\beta}(z;q) &= \frac{1-q^{\beta}}{1-q} e_{\alpha,\beta+1}(z;q) + q^{\beta}\frac{1-q^{\alpha}}{1-q} z D_{q^{\alpha},z} e_{\alpha,\beta+1}(z;q),\\
z^r e_{\alpha,\beta+r\alpha}(z;q) &= e_{\alpha,\beta}(z;q) + \sum_{j=0}^{r-1} \frac{z^j}{\Gamma_q(\alpha j+\beta)},\ r \in \mathbb{N}.
\end{aligned}$$

The last identity can be generalized to any $r > 0$ through the identity

$$z^{\lceil r \rceil} e_{\alpha,\beta+\alpha r}(z;q) = e_{\alpha,\beta+r-\lceil r \rceil}(z;q) + \sum_{j=0}^{\lceil r \rceil -1} \frac{z^j}{\Gamma_q(\alpha j + \beta + r - \lceil r \rceil)}.$$

7.2.1 Application of the ${}_qL_s$ Transform

Theorem 7.1. *Let α, β, and γ be complex numbers with positive real parts. Then*

$$\int_0^x t^{\beta-1}e_{\alpha,\beta}(at^\alpha;q)\varepsilon^{-qt}\left[x^{\gamma-1}e_{\alpha,\beta}(-ax^\alpha;q)\right]d_qt = x^{\alpha+\beta-1}e_{2\alpha,\gamma+\beta}(a^2x^{2\alpha};q),$$

where $a \in \mathbb{C}\setminus\{0\}$ and $|x| < \dfrac{1}{a^{1/\operatorname{Re}(\alpha)}(1-q)}$.

Proof. Let s and p be complex numbers related to the identity $s = p(1-q)$. If $|p| > |a|^{1/\operatorname{Re}(\alpha)}$, one directly obtains

$${}_qL_s\left(x^{\beta-1}e_{\alpha,\beta}(ax^\alpha;q)\right) = \frac{1}{1-q}\frac{p^{\alpha-\beta}}{p^\alpha-a},$$

and

$${}_qL_s\left(x^{\gamma-1}e_{\alpha,\gamma}(-ax^\alpha;q)\right) = \frac{1}{1-q}\frac{p^{\alpha-\gamma}}{p^\alpha+a}.$$

Hence,

$$\begin{aligned}{}_qL_s\left(x^{\beta-1}e_{\alpha,\beta}(ax^\alpha;q)\right){}_qL_s\left(x^{\gamma-1}e_{\alpha,\gamma}(-ax^\alpha;q)\right) &= \frac{1}{(1-q)^2}\frac{p^{2\alpha-(\gamma+\beta)}}{p^{2\alpha}-a^2}\\ &= \frac{1}{1-q}\,{}_qL_s\left(x^{\gamma+\beta-1}e_{2\alpha,\gamma+\beta}(a^2x^{2\alpha};q)\right).\end{aligned}$$

Therefore, the proof follows by applying the convolution theorem of the ${}_qL_s$ transform, cf. (1.91)–(1.92). □

7.3 The q-Mittag–Leffler Function $E_{\alpha,\beta}(z;q)$

The following set of formulae follow immediately from the definition of $E_{\alpha,\beta}(z;q)$.

$$E_{2,1}(z;q) = \cosh(q^{-1/2}\sqrt{z};q),\quad E_{2,2}(z;q) = \frac{\sinh(q^{-1}\sqrt{z};q)}{q^{-1}\sqrt{z}},$$

$$E_{1,1}(z;q) = E_q(z(1-q)),\ E_{1,2}(qz;q) = \frac{1}{z}(-1+E_q(z(1-q))),$$

$$E_{\alpha,\beta}(z;q) = \frac{1}{\Gamma_q(\beta)} + zE_{\alpha,\alpha+\beta}(qz,q)\quad(\operatorname{Re}(\alpha)>0),$$

$$z^\alpha E_{\alpha,\alpha+\beta}((qz)^\alpha;q) = \frac{1-q^\beta}{1-q}E_{\alpha,\beta+1}(z^\alpha;q) + q^\beta zD_qE_{\alpha,\beta+1}(z^\nu;q).$$

Lemma 7.2. *Let* α, μ, *and* β *be complex numbers with positive real parts. Then*

$$\frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z \big(qt/z;q\big)_{\alpha-1}E_{\beta,\mu}(\lambda t^{\beta};q)t^{\mu-1}\,d_qt = z^{\alpha+\mu-1}E_{\beta,\mu+\alpha}(\lambda z^{\beta};q). \tag{7.4}$$

Proof. Replace $E_{\beta,\mu}(\lambda t^{\beta};q)$ by its power series expansion on the left-hand side of (7.4). Consequently,

$$\begin{aligned}&\frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z \big(qt/z;q\big)_{\alpha-1}E_{\beta,\mu}(\lambda t^{\beta};q)t^{\mu-1}\,d_qt\\ &= \frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z \big(qt/z;q\big)_{\alpha-1}t^{\mu-1}\left(\sum_{n=0}^{\infty}q^{n(n-1)\beta/2}\frac{\big(\lambda t^{\beta}\big)^n}{\Gamma_q(n\beta+\mu)}\right)d_qt.\end{aligned} \tag{7.5}$$

Since the right hand side of (7.5) can be written as an absolutely convergent double series, we can interchange the order of summation and the q-integration in (7.5) to obtain

$$\begin{aligned}&\frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z \big(qt/z;q\big)_{\alpha-1}E_{\beta,\mu}(\lambda t^{\beta};q)t^{\mu-1}\,d_qt\\ &= \frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty}q^{n(n-1)\beta/2}\frac{\lambda^n}{\Gamma_q(n\beta+\mu)}\int_0^z \big(qt/z;q\big)_{\alpha-1}t^{n\beta+\mu-1}\,d_qt.\end{aligned} \tag{7.6}$$

Substituting with $u := t/z$ and $z \neq 0$ in (7.6) and using (1.58), we obtain

$$\begin{aligned}\int_0^z \big(qt/z;q\big)_{\alpha-1}t^{n\beta+\mu-1}\,d_qt &= z^{n\beta+\mu}\int_0^1 \big(qu;q\big)_{\alpha-1}u^{n\beta+\mu-1}\,d_qu\\ &= z^{n\beta+\mu}B_q\big(\alpha,n\beta+\mu\big).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty}q^{n(n-1)\beta/2}\frac{\lambda^n}{\Gamma_q(n\beta+\mu)}\int_0^z \big(qt/z;q\big)_{\alpha-1}t^{n\beta+\mu-1}\,d_qt\\ &= \frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty}q^{n(n-1)\beta/2}\frac{\lambda^n z^{n\beta+\mu}}{\Gamma_q(n\beta+\mu)}B_q\big(\alpha,n\beta+\mu\big)\\ &= \sum_{n=0}^{\infty}q^{n(n-1)\beta/2}\frac{\lambda^n z^{n\beta+\mu+\alpha-1}}{\Gamma_q(n\beta+\mu+\alpha)} = z^{\alpha+\mu-1}E_{\beta,\mu+\alpha}(\lambda z^{\beta};q),\end{aligned}$$

completing the proof. □

The previous lemma leads directly to the following corollary.

Corollary 7.3. *Let $\nu,\ \mu,\ \alpha$ be positive numbers. Then*

$$\int_0^z t^{\mu-1} E_{\nu,\mu}(t^\nu;q)\, d_qt = z^\mu E_{\nu,\mu+1}(z^\nu;q), \tag{7.7}$$

$$\frac{z^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^z \big(qt/z;q\big)_{\alpha-1} \cosh(\sqrt{\lambda}t;q)\, d_qt = z^\alpha E_{2,\alpha+1}(\lambda z^2 q;q), \tag{7.8}$$

$$\frac{z^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^z \big(qt/z;q\big)_{\alpha-1} \frac{\sinh(q^{-1}\sqrt{\lambda}t;q)}{\sqrt{\lambda}}\, d_qt = qz^{\alpha+1} E_{2,2+\alpha}(q^2\lambda z^2;q), \tag{7.9}$$

$$\frac{z^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^z \big(qt/z;q\big)_{\alpha-1} E_q(\lambda t(1-q))\, d_qt = z^\alpha E_{1,\alpha+1}(\lambda z;q). \tag{7.10}$$

Proof. Formula (7.7) follows from (7.4) by taking $\alpha = 1$. Formula (7.8) follows from (7.4) by taking $\beta = 2$, $\mu = 1$, and from (7.4) by replacing z in $E_{1,2}(z;q)$ by λt^2. As for formula (7.9) it follows from (7.4) by taking $\beta = \mu = 2$ and using (7.4) when z in $E_{2,2}(z;q)$ is replaced by λt^2. Finally, formula (7.9) follows from (7.4) by taking $\beta = \mu = 1$ and using (7.4) when z is replaced by λt. ☐

Lemma 7.4. *Let α and β be complex numbers such that* $\mathrm{Re}(\alpha) > 0$. *Then $E_{\alpha,\beta}(z;q)$ is an entire function of order zero.*

Proof. From (1.2) we have

$$\rho\big(E_{\alpha,\beta}(z;q)\big) = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)},$$

where c_n is the coefficient of z^n in (7.3), see [57], i.e.

$$c_n = \frac{q^{\frac{\alpha n(n-1)}{2}}}{\Gamma_q(n\alpha+\beta)} \quad (n \in \mathbb{N}_0)\,.$$

Hence,

$$\log\left(\frac{1}{|c_n|}\right) = -\frac{\mathrm{Re}(\alpha)}{2} n(n-1)\log q + \log\big|\Gamma_q(n\alpha+\beta)\big|\,.$$

From (1.8) and the definition of the q-gamma function, cf. (1.57) we obtain

$$\begin{aligned} \log\big|\Gamma_q(n\alpha+\beta)\big| &= \log\left|\tfrac{(q;q)_\infty}{(q^{n\alpha+\beta};q)_\infty}(1-q)^{1-n\alpha-\beta}\right| \\ &= \log\left|\tfrac{(q;q)_\infty}{(q^{n\alpha+\beta};q)_\infty}(1-q)^{1-n\alpha-\mathrm{Re}\,\beta}\right| \\ &= \log(q;q)_\infty + (1-n\,\mathrm{Re}(\alpha)-\mathrm{Re}(\beta))\log(1-q) \\ &\quad -\log\big|(q^{n\alpha+\beta};q)_\infty\big|, \end{aligned} \tag{7.11}$$

and

$$\log\left|(q^{n\alpha+\beta};q)_\infty\right| = \log\Big(\prod_{k=0}^{\infty}\left|1-q^{n\alpha+\beta+k}\right|\Big) = \log\Big(\lim_{m\to\infty}\prod_{k=0}^{m}\left|1-q^{n\alpha+\beta+k}\right|\Big)$$
$$= \lim_{m\to\infty}\sum_{k=0}^{m}\log\left|1-q^{n\alpha+\beta+k}\right| = \sum_{k=0}^{\infty}\log\left|1-q^{n\alpha+\beta+k}\right|.$$

Since

$$\log\left|1-q^{n\alpha+\beta+k}\right| \leqslant \log\big(1+|q^{n\alpha+\beta+k}|\big) \leqslant |q^{n\alpha+\beta+k}| = q^{n\,\mathrm{Re}(\alpha)+k+\mathrm{Re}(\beta)},$$

then

$$\sum_{k=0}^{\infty}\log\left|1-q^{n\alpha+\beta+k}\right| \leqslant \sum_{k=0}^{\infty}q^{n\,\mathrm{Re}(\alpha)+k+\mathrm{Re}(\beta)} = \frac{q^{n\,\mathrm{Re}(\alpha)+\mathrm{Re}(\beta)}}{1-q}.$$

Therefore,

$$\lim_{n\to\infty}\frac{\log\left|(q^{n\alpha+\beta};q)_\infty\right|}{n\log n} = 0.$$

Consequently, by (7.11)

$$\lim_{n\to\infty}\frac{\log\left|\Gamma_q(n\alpha+\beta)\right|}{n\log n} = 0.$$

Since

$$\lim_{n\to\infty}\frac{n-1}{\log n} = \infty,$$

then

$$\lim_{n\to\infty}\frac{\log(1/|c_n|)}{n\log n} = \infty,$$

i.e. $\rho\big(E_{\alpha,\beta}(z;q)\big) = 0$. □

7.4 q-Mellin–Barnes Contour Integrals for the q-Mittag–Leffler Functions

The Mellin–Barnes integral is an integral containing the gamma function in the integrand. A typical such integral is given by

$$f(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\prod_{j=1}^{n}\Gamma(a_j+A-js)\Gamma(b_j-B_js)}{\prod_{j=1}^{r}\Gamma(c_j+C_js)\prod_{j=1}^{s}\Gamma(d_j-D_js)}z^s\,ds,$$

where γ is real, A_j, B_j, C_j, and D_j are positive constants, and the contour is a straight line parallel to the imaginary axis with indentation if necessary to avoid poles of integrand. See [47, 212, 232]. To the best we know, Gasper and Rahman [113, Chap. 4] were the first to introduce q-analogues of the celebrated Mellin–Barnes integral. In this section we introduce q-analogue of the Mellin–Barnes integral of the classical Mittag–Leffler functions:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\,ds = -\sum_{n=0}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)},$$

where the contour of integration, L, starts at $c-i\infty$, ends at $c+i\infty$ $(0<c<1)$, and separates all the poles $s=-k$ $(k\in\mathbb{N}_0)$, to the left and all the poles $s=k+1$, $(k\in\mathbb{N}_0)$ to the right. See [169, PP. 41–44] and [127].

7.4.1 A q-Mellin–Barnes Contour Integral of $e_{\alpha,\beta}(z;q)$

Theorem 7.5. *Let* $0<d<1$, $\alpha>0$, *and* $\beta\in\mathbb{C}$. *Then*

$$e_{\alpha,\beta}(z;q) = \frac{-\Gamma_q(d)\Gamma_q(1-d)\ln q}{2\pi i\,(1-q)}\int_L \frac{(q^{-d}z)^{-s}\Gamma_q(s)\Gamma_q(1-s)}{\Gamma_q(\beta-\alpha s)\Gamma_q(d+s)\Gamma_q(1-d-s)}\,ds,$$

where $|z|<(1-q)^{-\alpha}$ *and the contour of integration, say L, starts at* $c-i\infty$, *ends at* $c+i\infty$ $(0<c<1)$, *and separates all the poles* $s=-k$ $(k\in\mathbb{N}_0)$ *to the left and all the poles* $s=k+1$ $(k\in\mathbb{N}_0)$ *to the right.*

Proof. We prove the theorem by using the Cauchy residue theorem. Set

$$f(s) := \frac{\Gamma_q(s)\Gamma_q(1-s)}{\Gamma_q(\beta-\alpha s)\Gamma_q(d+s)\Gamma_q(1-d-s)}(q^{-d}z)^{-s}. \tag{7.12}$$

We integrate f on the contour $\Gamma_R := \gamma_{1,R}\cup\gamma_{2,R}$ where $\gamma_{1,R}$ represents the line segment joining the points $c-iR$ and $c+iR$ and $\gamma_{2,R}$ is the curve

$$|s-c| = Re^{i\theta},\quad \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}.$$

Thus,

$$\int_{\Gamma_R} f(s)\,ds = 2\pi i\sum_{k=0}^{N_R}\operatorname{Res}\left(f(s);-k\right),$$

where N_R denotes the number of poles of $f(s)$ inside $\Gamma(R)$. Now we prove that

$$\lim_{R\to\infty}\int_{\gamma_{2,R}} f(s)\,ds = 0. \tag{7.13}$$

Hence, if

$$\sum_{k=0}^{\infty}\operatorname{Res}(f(s);-k) \tag{7.14}$$

is convergent then

$$\int_{c-i\infty}^{c+i\infty} f(s)\,ds = \lim_{R\to\infty}\int_{\Gamma_R} f(s)\,ds = 2\pi i\sum_{k=0}^{\infty}\operatorname{Res}\left(f(s);-k\right).$$

Since

$$\operatorname{Res}\left(f(s);-k\right) = \frac{1-q}{\Gamma_q(d)\Gamma_q(1-d)\ln\frac{1}{q}}\,\frac{z^k}{\Gamma_q(\alpha k+\beta)} \quad (k\in\mathbb{N}_0),$$

the series in (7.14) is convergent for $|z| < (1-q)^{-\alpha}$. Therefore, it remains to prove that the contour integration on $\gamma_{2,R}$ vanishes as $R\to\infty$. A parametrization of the contour $\gamma_{2,R}$ is given by

$$s(t) = c + Re^{it}, \quad \frac{\pi}{2}\le t\le\frac{3\pi}{2}.$$

Hence, on $\Gamma_{R,2}$, we deduce that $\operatorname{Re}(s)\to-\infty$ as $R\to\infty$. Since the function $f(s)$ can be represented as

$$f(s) = \frac{E_q(-q^{\beta-\alpha s})E_q(-q^{d+s})E_q(-q^{1-d-s})}{E_q(-q^s)E_q(-q^{1-s})}\,\frac{q^{ds}(1-q)^{\beta-1}}{(q;q)_\infty}\left(z(1-q)^{\alpha}\right)^{-s},$$

there exists a constant $c_1>0$ such that

$$|f(s)|\le c_1 q^{d\operatorname{Re}(s)}\frac{E_q(q^{d+\operatorname{Re}(s)})}{|E_q(-q^{\operatorname{Re}(s)})|}\left(|z|(1-q)^{\alpha}\right)^{-\operatorname{Re}s} \quad \text{for all } s\in\gamma_{2,R}.$$

Since the function $E_q(z)$ has the asymptotic relation

$$\lim_{r\to\infty}\frac{\ln M(r;f)}{\ln^2 r} = \frac{1}{2\ln q^{-1}},$$

see [148, 149, 256], there exists $c_2>0$ such that

$$q^{d\operatorname{Re}(s)}\frac{E_q(q^{d+\operatorname{Re}(s)})}{|E_q(q^{\operatorname{Re}(s)})|}\le c_2 \quad \text{for all } s\in\gamma_{2,R}.$$

Accordingly,

$$|f(s)| \leq c_1 c_2 \left(|z|(1-q)^{\alpha}\right)^{-\operatorname{Re} s} \quad \text{for all } s \in \gamma_{2,R}.$$

But

$$\lim_{\operatorname{Re}(s)\to-\infty} \left(|z|(1-q)^{\alpha}\right)^{-\operatorname{Re}(s)} = 0 \ \text{ for } \ |z|(1-q)^{\alpha} < 1.$$

Therefore, the contour integration on $\gamma_{2,R} \to 0$ as $R \to \infty$ and

$$\begin{aligned}\int_{c-i\infty}^{c+i\infty} f(s)\,ds &= \frac{1-q}{\Gamma_q(d)\Gamma_q(1-d)\ln(\frac{1}{q})} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k+\beta)} \\ &= \frac{1-q}{\Gamma_q(d)\Gamma_q(1-d)\ln(\frac{1}{q})} e_{\alpha,\beta}(z;q),\end{aligned}$$

completing the proof. □

Since

$$\operatorname{Res}\left(f(s);k\right) = \frac{1-q}{\Gamma_q(d)\Gamma_q(1-d)\ln\frac{1}{q}} \frac{z^{-k}}{\Gamma_q(\beta-\alpha k)} \quad (k \in \mathbb{N}),$$

then if we calculate the residues of the function $f(s)$ at the poles $s = k$, $k \in \mathbb{N}$ instead of the poles $s = -k$, $k \in \mathbb{N}_0$, we obtain the following analytic continuation formula of $e_{\alpha,\beta}(z;q)$.

$$e_{\alpha,\beta}(z;q) = \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma_q(\beta-\alpha k)} \quad \left(|z|(1-q)^{\alpha} > 1\right).$$

Consequently,

$$e_{\alpha,\beta}(z;q) = \sum_{k=1}^{N} \frac{z^{-k}}{\Gamma_q(\beta-\alpha k)} + O\left(\frac{1}{z^{n+1}}\right), \ \text{as } z \to \infty.$$

7.4.2 A q-Mellin–Barnes Contour Integral of $E_{\alpha,\beta}(z;q)$

Theorem 7.6. *Let* $0 < d < 1$, $\alpha > 0$, *and* $\beta \in \mathbb{C}$. *Then for* $z \in \mathbb{C}$

$$E_{\alpha,\beta}(z;q) =$$

$$\frac{-\Gamma_q(d)\Gamma_q(1-d)\ln q}{2\pi i(1-q)} \int_L \frac{q^{\frac{\alpha s^2}{2}}\Gamma_q(s)\Gamma_q(1-s)}{\Gamma_q(\beta-\alpha s)\Gamma_q(d+s)\Gamma_q(1-d-s)} (q^{\frac{-\alpha}{2}-d} z)^{-s}\,ds,$$

where the contour of integration, say L*, starts at* $c - i\infty$*, ends at* $c + i\infty$ $(0 < c < 1)$*, and separates all the poles* $s = -k$ $(k \in \mathbb{N}_0)$ *to the left and all the poles* $s = k + 1$ $(k \in \mathbb{N}_0)$ *to the right.*

Proof. Set $g(s) := f(s)q^{\frac{\alpha}{2}(s^2+s)}$ where $s \in \mathbb{C}$ and f is the function defined in (7.12). Then

$$\operatorname{Res}(g(s); -k) = \frac{1-q}{\Gamma_q(d)\Gamma_q(1-d)\ln\frac{1}{q}} q^{\frac{\alpha}{2}k(k-1)} \frac{z^k}{\Gamma_q(\alpha k + \beta)}.$$

The remaining of the proof is similar to the proof of Theorem 7.5 and is omitted. □

If we calculate the residues at the poles $s = 1 + n$, $n \in \mathbb{N}_0$ of $\Gamma_q(1-s)$, we obtain the following analytic continuation formula of $E_{\alpha,\beta}(z;q)$.

$$E_{\alpha,\beta}(z;q) = \sum_{k=1}^{\infty} \frac{q^{\frac{\alpha k(k-1)}{2}}}{\Gamma_q(\beta - \alpha k)} z^{-k}.$$

Consequently,

$$E_{\alpha,\beta}(z;q) = \sum_{k=1}^{N} \frac{q^{\frac{\alpha k(k-1)}{2}}}{\Gamma_q(\beta - \alpha k)} z^{-k} + O\left(\frac{q^{\frac{\alpha}{2}}(n+2)(n+3)}{z^{n+1}}\right), \text{ as } z \to \infty.$$

7.5 Hankel Contour Integral Representation of q-Mittag–Leffler Functions

In the survey of the Mittag–Leffler functions [127], the authors introduced an integral representation of $E_{\alpha,\beta}(z)$ of the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{Ha} \frac{t^{\alpha-\beta}\exp(t)}{t^\alpha - z}\, dt \quad (z \in \mathbb{C},\ \alpha > 0,\ \beta \in \mathbb{R}) \tag{7.15}$$

where the path of integration Ha is the Hankel path, i.e. a path starts and ends at $-\infty$ and encircles the circular disk $|t| \le |z|^{\frac{1}{\alpha}}$ in the positive sense:$-\pi < \arg(t) \le \pi$. The integrand has a branch cut along the negative real axis, and in the cut plane, the integrand is single valued. Hanneken et al. [126] used (7.15) to define $E_{\alpha,\beta}$ when $\alpha \le 0$ through the identity

$$E_{-\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} - E_{\alpha,\beta}\left(\frac{1}{z}\right).$$

In this section we shall introduce a Hankel contour integral representations of the q-Mittag–Leffler functions $e_{\alpha,\beta}(z;q)$ and $E_{\alpha,\beta}(z;q)$. To introduce these results, we first need to investigate the Hankel contour integral representation of $\dfrac{1}{\Gamma_q(x)}$.

Theorem 7.7.

$$\frac{1}{\Gamma_q(x)(1-q)^{x-1}} = \frac{1}{2\pi i}\int_C \frac{e_q(t)}{t^x}\,dt \qquad (\mathrm{Re}(x) > 0)\,,$$

where the path of integration C encircles the origin and can also be deformed into a loop, parallel to the imaginary axis.

Proof. Making the substitution $t = us$ in (1.70) gives

$$\Gamma_q(x)(1-q)^{x-1} = \frac{s^x}{1-q}\int_0^{1/s} u^{x-1}E_q(-qsu)\,d_qu = s^x\,{}_qL_s(u^{x-1}).$$

Using the Hahn's inversion formula, (1.96), of the ${}_qL_s$ transform we obtain

$$\frac{u^{x-1}}{\Gamma_q(x)(1-q)^x} = \frac{1}{2\pi\,i}\int_C \frac{e_q(su)}{s^x}\,ds,$$

and the theorem follows by making the substitution $t = su$ on the last integrand and using that the value of the integral is the same on all deformed contours. □

In the following theorem, we derive Hankel contour integral representation of $e_{\alpha,\beta}(z;q)$.

Theorem 7.8. *For $\alpha > 0$, $\beta \in \mathbb{R}$, we have*

$$e_{\alpha,\beta}(z;q) = \frac{(1-q)^{\beta-1}}{2\pi\,i}\int_{Ha} \frac{t^{\alpha-\beta}e_q(t)}{t^\alpha-(1-q)^\alpha z}\,dt \quad \textit{for } |z| < \frac{1}{(1-q)^\alpha}, \tag{7.16}$$

where the path of integration Ha is the Hankel path, a path starts and ends at $-\infty$ and encircles the circular disk $|t| \le |z|^{\frac{1}{\alpha}}$ in the positive sense:$-\pi < \arg(t) \le \pi$. The integrand has a branch cut along the negative real axis, and in the cut plane, the integrand is single valued.

Proof. From the series representation of $e_{\alpha,\beta}(z;q)$ and Theorem 7.7, we obtain

$$e_{\alpha,\beta}(z;q) = \frac{1}{2\pi\,i}\sum_{k=0}^{\infty}(1-q)^{\alpha k+\beta}z^k\int_C \frac{e_q(t)}{t^{\alpha k+\beta}}\,dt. \tag{7.17}$$

The contour C can be deformed to the Hankel contour Ha, see [290]. In addition, the series representation of $e_{\alpha,\beta}(z;q)$ is absolutely convergent for $|z|(1-q)^\alpha < 1$.

Accordingly, after replacing the contour C by Ha and interchanging the order of integration and summation in (7.17), we obtain

$$e_{\alpha,\beta}(z;q) = \frac{(1-q)^{\beta}}{2\pi\, i}\int_{Ha} t^{-\beta} e_q(t) \sum_{k=0}^{\infty}\left(\frac{z(1-q)^{\alpha}}{t^{\alpha}}\right)^{k} dt. \tag{7.18}$$

But

$$\frac{|z|(1-q)^{\alpha}}{|t|^{\alpha}} < 1 \quad \text{for all } t \in Ha.$$

Consequently,

$$\sum_{k=0}^{\infty}\left(\frac{z(1-q)^{\alpha}}{t^{\alpha}}\right)^{k} = \frac{t^{\alpha}}{t^{\alpha} - z(1-q)^{\alpha}}. \tag{7.19}$$

Substituting (7.19) into (7.18), we obtain (7.16) and the theorem follows. ☐

In the following theorem, we derive Hankel contour integral representation of $E_{\alpha,\beta}(z;q)$.

Theorem 7.9. *For $\alpha > 0$, and $\beta \in \mathbb{R}$*

$$E_{\alpha,\beta}(z;q) = \frac{(1-q)^{\beta} - 1}{2\pi\, i}\int_{Ha} \Psi\left(q^{\alpha}, \frac{(1-q)^{\alpha} q^{\alpha} z}{t^{\alpha}}\right)\frac{e_q(t)}{t^{\beta}}\, dt,$$

where H_a is the Hankel contour as in Theorem 7.7 and $\Psi(q,u)$ is the partial theta function defined by

$$\Psi(q,u) = \sum_{k=0}^{\infty} q^{\binom{k+1}{2}} u^{k}.$$

This function first appeared in [139, P. 330], see also [177].

Proof. The proof of this theorem is similar to the proof of Theorem 7.8 and is omitted. ☐

7.6 q-Volterra Integral Equations

The solution of the integral equation

$$u(x) = f(x) + \lambda \int_{c}^{x} K(x,t)u(t)dt \tag{7.20}$$

was originally attained by Volterra in 1896, cf. [264]. When $K(x,t)$ is a difference kernel of the form $K(x-t)$, the standard technique of solving the above equation when $c = 0$ is the use of Laplace transform provided that the Laplace transform (one sided) of the functions $f(x)$, $K(x)$, and $u(x)$ exist. Another technique is the use of Picard–Lindellöf method of successive approximations, see e.g. [77]. The following theorem gives explicit solutions of a basic Volterra integral equation of the second kind in terms of the translation operator defined by Ismail [143], cf. (1.14).

Theorem 7.10. *Let $\nu, a > 0$ and $\lambda \in \mathbb{C}$ be such that*

$$|\lambda| a^{\nu}(1-q)^{\nu} < 1. \tag{7.21}$$

If $f \in \mathscr{L}_q^1[0,a]$ then the q-integral equation

$$u(x) = f(x) + \frac{\lambda x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \big(qt/x; q\big)_{\nu-1} u(t)\, d_q t, \quad x \in (0,a], \tag{7.22}$$

has the unique solution

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} \lambda^k I_q^{k\nu} f(x) \\ &= f(x) + \lambda x^{\nu-1} \int_0^x \big(qt/x; q\big)_{\nu-1} \varepsilon^{-q^{\nu} t} \left(e_{\nu,\nu}(\lambda x^{\nu}; q)\right) f(t)\, d_q t, \end{aligned} \tag{7.23}$$

in the space $\mathscr{L}_q^1[0,a]$.

Proof. We prove the theorem in four steps.

i. We first prove uniqueness. Assume that U and V are solutions of (7.22) in $\mathscr{L}_q^1[0,a]$ valid in $(0,h]$, $h \leqslant a$. Set

$$Z(x) := U(x) - V(x) \quad \text{for all } \ x \in (0,h].$$

Then $Z(x)$ satisfies the functional equation

$$Z(x) = \frac{\lambda x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \big(qt/x; q\big)_{\nu-1} Z(t)\, d_q t. \tag{7.24}$$

Since $Z \in \mathscr{L}_q^1[0,a]$, then $Z \in \mathscr{L}_q^1[0,h]$ and there exists $K > 0$ such that

$$\int_0^x |Z(\xi)|\, d_q \xi \leq K \quad \text{for all } \ x \in (0,h].$$

Fix $w \in (qh,h]$ and $\tau \in \{wq^m,\ m \in \mathbb{N}_0\}$. From (7.24), we obtain

$$|Z(\tau)| \leqslant \frac{|\lambda|}{\Gamma_q(\nu)(q^{\nu}; q)_{\infty}} K \tau^{\nu-1}. \tag{7.25}$$

Applying (7.25) to (7.24) to estimate the value of $z(\tau)$, we get

$$\begin{aligned}|Z(\tau)| &\leqslant K\frac{|\lambda|^2}{\Gamma_q^2(\nu)(q^\nu;q)_\infty}\tau^{2\nu-1}B_q(\nu,\nu)\\ &= K\frac{|\lambda|^2}{\Gamma_q(2\nu)(q^\nu;q)_\infty}\tau^{2\nu-1}.\end{aligned}$$

Repeating the procedure n times we obtain

$$|Z(\tau)| \leqslant K\frac{|\lambda|^n}{\Gamma_q(n\nu)(q^\nu;q)_\infty}\tau^{n\nu-1},$$

Since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\lambda|^n}{\Gamma_q(n\nu)(q^\nu;q)_\infty}\tau^{n\nu-1} &= \frac{\lambda\tau^{\nu-1}}{(q^\nu;q)_\infty}\sum_{n=0}^{\infty}\frac{|\lambda|^n}{\Gamma_q(n\nu+\nu)}\tau^{n\nu}\\ &= \frac{\lambda\tau^{\nu-1}}{(q^\nu;q)_\infty}e_{\nu,\nu}(|\lambda|\tau^\nu;q),\end{aligned}$$

then

$$\lim_{n\to\infty}\frac{\lambda^n}{\Gamma_q(n\nu+\nu)}\tau^{n\nu} = 0.$$

Hence, $Z(\tau) = 0$ for all $\tau \in \{wq^m,\ m \in \mathbb{N}_0\}$ and for all $w \in (qh,h]$. That is $Z(x) = 0$ for all $x \in (0,h]$ and the uniqueness is proved.

ii. Now we prove that a solution exists by using the q-analogue of the method of successive approximations introduced in Sect. 2.2. Define the sequence $\{u_m\}_{m=0}^{\infty}$ recursively by

$$\begin{aligned}u_0(x) &= f(x),\\ u_m(x) &= f(x) + \frac{\lambda x^{\nu-1}}{\Gamma_q(\nu)}\int_0^x \left(qt/x;q\right)_{\nu-1}u_{m-1}(t)\,d_qt,\end{aligned} \tag{7.26}$$

where $x \in (0,a]$, $m \in \mathbb{N}$. We prove by mathematical induction that

$$u_m(x) = \sum_{k=0}^{m}\lambda^k I_q^{k\nu}f(x) \tag{7.27}$$

for all $m \in \mathbb{N}$ and for all $x \in (0,a]$. Indeed, if u_m satisfies the identity (7.27), then from (7.26) we obtain

$$\begin{aligned} u_{m+1}(x) &= f(x) + \frac{\lambda x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \big(qt/x;q\big)_{\nu-1} \sum_{k=0}^{m} \lambda^k I_q^{k\nu} f(t)\, d_qt \\ &= f(x) + \sum_{k=0}^{m} \lambda^{k+1} \frac{x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \big(qt/x;q\big)_{\nu-1} I_q^{k\nu} f(t)\, d_qt \qquad (7.28) \\ &= f(x) + \sum_{k=0}^{m} \lambda^{k+1} I_q^{\nu} I_q^{k\nu} f(x), \end{aligned}$$

for all $x \in (0,a]$. Hence, from the semigroup identity (4.62) we have

$$u_{m+1}(x) = f(x) + \sum_{k=0}^{m} \lambda^{k+1} I_q^{k\nu+\nu} f(x) = \sum_{k=0}^{m+1} \lambda^k I_q^{k\nu} f(x),$$

for all $x \in (0,a]$. Thus, the identity in (7.27) is true for all $m \in \mathbb{N}_0$ because it is true at $m = 0$. Now we prove that the series $\sum_{k=0}^{\infty} \lambda^k I_q^{k\nu} f(x)$ is absolutely and uniformly convergent on $(0,a]$ and defines a function $u(x)$ on $(0,a]$. Since $f \in \mathscr{L}_q^1[0,a]$, there exists $K > 0$ such that

$$\int_0^x |f(u)|\, d_qu \le K \quad \text{for all } x \in (0,a].$$

Let $y \in (qa,a]$ and $t \in \{yq^m,\ m \in \mathbb{N}_0\}$ be fixed numbers. Since

$$I_q^{\nu k} f(t) = \frac{t^{\nu k}}{\Gamma_q(\nu k)} \sum_{r=0}^{\infty} q^r (1-q)(q^{r+1};q)_{\nu k-1} f(tq^r)$$

and $f \in \mathscr{L}_q^1[0,a]$, we obtain

$$\begin{aligned} \Big|\lambda^k I_q^{k\nu} f(t)\Big| &\leqslant \frac{|\lambda|^k t^{k\nu-1}}{\Gamma_q(k\nu)} \sum_{r=0}^{\infty} tq^r(1-q)\big(q^{r+1};q\big)_{k\nu-1} |f(tq^r)| \\ &\leqslant K \frac{|\lambda|^k y^{k\nu-1}}{(q^\nu;q)_\infty \Gamma_q(k\nu)}, \end{aligned} \qquad (7.29)$$

for all $k \in \mathbb{N}_0$. Since the series

$$\sum_{k=1}^{\infty} \frac{\lambda^k y^{k\nu-1}}{\Gamma_q(k\nu)} = \lambda y^{\nu-1} \sum_{k=0}^{\infty} \frac{\lambda^k y^{k\nu}}{\Gamma_q(k\nu+\nu)} = \lambda y^{\nu-1} e_{\nu,\nu}(\lambda y^\nu;q),$$

is convergent for all y such that $0 < |y| \le a$ and a satisfies the condition (7.21), the series $\sum_{k=0}^{\infty} \Big|\lambda^k I_q^{k\nu} f(t)\Big|$ is absolutely and uniformly convergent on $\{yq^m,\ m \in \mathbb{N}\}$, $y \in (qa,a]$. Hence, the convergence holds throughout $(0,a]$.

The function $u(x)$ defined by

$$u(x) := \sum_{k=0}^{\infty} \lambda^k I_q^{k\nu} f(x), \quad x \in (0, a] \tag{7.30}$$

is a solution of (7.22).

iii. We prove that $u \in \mathscr{L}_q^1[0, a]$, i.e. $u \in L_q^1(0, y)$ for all $y \in (qa, a]$. Indeed from (7.30) and the semigroup identity (4.62) we obtain

$$\int_0^y |u(\xi)|\, d_q\xi = I_q(|u|)(y) \leqslant \sum_{k=0}^{\infty} |\lambda|^k \left| I_q^{\nu k+1} f(y) \right|.$$

But from (7.29) we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} |\lambda|^k \left| I_q^{\nu k+1} f(y) \right| &\leqslant \sum_{k=0}^{\infty} |\lambda|^k \int_0^y \left| I_q^{\nu k} f(t) \right| d_q t \\ &\leqslant \frac{\int_0^y |f(u)|\, d_q u}{(q^\nu; q)_\infty} \sum_{k=0}^{\infty} \frac{|\lambda|^k y^{k\nu}}{\Gamma_q(k\nu + 1)} \\ &= \frac{\int_0^y |f(u)|\, d_q u}{(q^\nu; q)_\infty} e_{\nu,1}(|\lambda| a^\nu; q),\end{aligned}$$

for all $y \in (qa, a]$. Since a satisfies the condition (7.21) and $f \in \mathscr{L}_q^1(0, y)$, for all $y \in (qa, a]$, then so is u.

iv. Finally, we prove that u has the representation (7.23). Since

$$\begin{aligned}u(x) &= \sum_{k=0}^{\infty} I_q^{k\nu} f(x) = f(x) + \sum_{k=1}^{\infty} \frac{\lambda^k x^{k\nu-1}}{\Gamma_q(k\nu)} \int_0^x \left(qt/x; q\right)_{k\nu-1} f(t)\, d_q t \\ &= f(x) + \lambda x^{\nu-1} \sum_{k=0}^{\infty} \frac{\lambda^k x^{k\nu}}{\Gamma_q(k\nu + \nu)} \int_0^x \left(qt/x; q\right)_{k\nu+\nu-1} f(t)\, d_q t\end{aligned}$$

and

$$(qt/x; q)_{k\nu+\nu-1} = (qt/x; q)_{\nu-1} (q^\nu t/x; q)_{k\nu},$$

then

$$\begin{aligned}u(x) &= f(x) + \lambda x^{\nu-1} \int_0^x \left(\sum_{k=0}^{\infty} \frac{\lambda^k x^{k\nu}}{\Gamma_q(k\nu)} \left(q^\nu t/x; q\right)_{k\nu} \right) \left(qt/x; q\right)_{\nu-1} f(t)\, d_q t \\ &= f(x) + \lambda x^{\nu-1} \int_0^x \left(qt/x; q\right)_{\nu-1} \varepsilon^{-q^\nu t} e_{\nu,\nu}(\lambda x^\nu) f(t)\, d_q t,\end{aligned}$$

where the q-translation operator ε^y is defined in (1.15). ☐

Similarly, we prove the following theorem.

Theorem 7.11. *Let* $\nu, a > 0$ *and* $\lambda \in \mathbb{C}$. *If* $f \in \mathscr{L}_q^1[0,a]$ *then the* q*-integral equation*

$$u(x) = f(x) + \frac{\lambda x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \left(qt/x;q\right)_{\nu-1} u(qt)\, d_qt$$

has the unique solution

$$\begin{aligned} u(x) &= f(x) \\ &+\lambda q^{\nu} x^{\nu-1} \int_0^x \left(qt/x;q\right)_{\nu-1} \sum_{j=0}^{\infty} \frac{\lambda^j q^{\frac{\nu j(j-1)}{2}}}{\Gamma_q(\nu j+\nu)} \left(q^{\nu} t/x;q\right)_{\nu j-1} f(q^{j+1}t)\, d_qt, \end{aligned} \tag{7.31}$$

in the space $\mathscr{L}_q^1[0,a]$.

Proof. First, we prove that a solution of (7.31) exists. Let $\{u_m(x)\}_{m=0}^{\infty}$ be the sequence defined recursively by

$$\begin{aligned} u_0(x) &= f(x), \\ u_m(x) &= f(x) + \tfrac{\lambda x^{\nu-1}}{\Gamma_q(\nu)} \int_0^x \left(qt/x;q\right)_{\nu-1} u_{m-1}(qt)\, d_qt, \end{aligned} \tag{7.32}$$

for all $x \in (0,a]$ and for all $m \in \mathbb{N}_0$. We can prove by mathematical induction that

$$u_m(x) = \sum_{j=0}^{m} \lambda^j q^{\frac{j(j-1)}{2}\nu} I_q^{j\nu} f(q^j x) \quad (m \in \mathbb{N}_0)\,. \tag{7.33}$$

Indeed, from (7.32) we obtain

$$u_{m+1}(x) = f(x) + \lambda I_q^{\nu} \left(\sum_{j=0}^{m} \lambda^j q^{\frac{j(j-1)}{2}\nu} g(qt) \right)(x),$$

where

$$g(t) = I_q^{j\nu}\left(f(q^j t)\right).$$

Hence,

$$u_{m+1}(x) = f(x) + \sum_{j=0}^{m} \lambda^{j+1} q^{\frac{j(j-1)}{2}\nu} I_q^{\nu}(g(qt))(x).$$

A straightforward manipulation gives

$$(I_q^{\nu} h)(qt) = q^{\nu} I_q^{\nu} h(qt) \quad \text{for all } \ h \in \mathscr{L}_q^1[0,a].$$

Consequently,

$$I_q^{\nu}\left(g(qt)\right) = q^{j\nu} I_q^{(j+1)\nu} f(q^{j+1}t),$$

and from the semigroup property (4.62) we obtain

$$\begin{aligned} U_{m+1}(x) &= f(x) + \sum_{j=0}^{m} \lambda^j q^{\frac{j(j+1)}{2}\nu} I_q^{(j+1)\nu} f(q^{j+1}x) \\ &= \sum_{j=0}^{m+1} \lambda^j q^{\frac{j(j-1)}{2}\nu} I_q^{\nu j} f(q^j x). \end{aligned} \tag{7.34}$$

This proves (7.33) because it holds at $m = 0$. Fix $x \in (0, a]$. Now we prove that the series

$$\sum_{j=0}^{\infty} \lambda^j q^{\frac{j(j-1)}{2}\nu} I_q^{\nu j} f(q^j t)$$

is absolutely and uniformly convergent on $\left\{xq^j,\ \ j \in \mathbb{N}_0\right\}$. Since

$$I_q^{\nu j} f(t) = \frac{t^{\nu j}}{\Gamma_q(\nu j)} \sum_{k=0}^{\infty} q^k (1-q)(q^{k+1}; q)_{\nu j-1} f(tq^k)$$

and $f \in \mathscr{L}_q^1[0, a]$, we obtain

$$|I_q^{\nu j} f(tq^j)| \leq \frac{1}{(q^{\nu}; q)_{\infty} \Gamma_q(\nu j)} t^{\nu j-1} \, \|f\| \quad \text{for } t \in \left\{xq^k,\ k \in \mathbb{N}_0\right\}.$$

Hence,

$$\begin{aligned} \left| \sum_{j=1}^{\infty} \lambda^j q^{\frac{j(j-1)}{2}\nu} I_q^{\nu j}\left(f(tq^j)\right) \right| &\leqslant \frac{\|f\|}{x(q^{\nu}; q)_{\infty}} \sum_{j=1}^{\infty} |\lambda x^{\nu}|^j \frac{q^{\frac{j(j-1)}{2}\nu}}{\Gamma_q(j\nu)} \\ &= \frac{|\lambda| x^{\nu-1} \, \|f\|}{(q^{\nu}; q)_{\infty}} \sum_{j=0}^{\infty} |\lambda x^{\nu}|^j \frac{q^{\frac{j(j+1)}{2}\nu}}{\Gamma_q(j\nu + \nu)} \\ &\leqslant \frac{|\lambda| x^{\nu-1} \, \|f\|}{(q^{\nu}; q)_{\infty}} E_{\nu,\nu}(|\lambda x^{\nu}|; q) \end{aligned}$$

for all $t \in \left\{xq^k,\ k \in \mathbb{N}_0\right\}$. Since the function $E_{\nu,\nu}(|\lambda x^{\nu}|; q)$ is convergent for all $\lambda \in \mathbb{C}$, the series $\sum_{j=0}^{\infty} \lambda^j q^{\frac{j(j-1)}{2}\nu} I_q^{\nu j} f(q^j t)$ is uniformly and absolutely convergent on $t \in \left\{xq^j,\ j \in \mathbb{N}_0\right\}$. Thus we can pass the limit as $m \to \infty$ through the q-integral in (7.32) to obtain (7.31). The rest of the proof is similar to that of Theorem 7.10 and is omitted. ☐

7.7 Zeros of q-Mittag–Leffler Functions

From Theorem 1.1 and Lemma 7.4 we conclude that $E_{\nu,\mu}(z;q)$ has infinitely many zeros. In this section we prove that for specific values of ν and μ, $E_{\nu,\mu}(z;q)$, $0 < q < 1$, may have only a finite number of non-real zeros. Moreover, if q satisfies an additional condition then the zeros of $E_{\nu,\mu}(z;q)$ are all real. The results of this section are from [204] and they are q-extensions of the results of Wiman [292].

Consider the function

$$\mathscr{E}_{\gamma,\sigma}(z;q) := E_{2,1+\gamma}(-\sigma^2 z;q) \quad (z \in \mathbb{C}),$$

where σ is a fixed positive number and $0 \le \gamma < 2$. Recall that $\{\pm x_n\}_{n=1}^{\infty}$ and $\{0, \pm y_n\}_{n=1}^{\infty}$, $x_n,\ y_n > 0$, denote the zeros of $\cos(z;q)$ and $\sin(z;q)$, respectively, cf. (2.85). Then we have the following theorem

Theorem 7.12. *Let $0 \leqslant \gamma < 2$. Then*

1. *For any $q \in (0,1)$ the function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ has at most a finite number of non real zeros and it has an infinite number of real, simple and symmetric zeros, $\left\{\pm\xi_n^{(\gamma)}\right\}_{n=1}^{\infty}$, $\xi_n^{(\gamma)} > 0$, with the asymptotic behavior*

$$\xi_n^{(\gamma)} = \begin{cases} \dfrac{q^{1/2}x_n}{\sigma}\big(1+O(q^n)\big), \ 0<\gamma<1, \\[2ex] \dfrac{q y_n}{\sigma}\big(1+O(q^n)\big), \quad 1<\gamma<2, \end{cases} \tag{7.35}$$

   as $n \to \infty$.

2. *If q satisfies the condition*

$$q^{-1}(1-q)(1-q^{\gamma+1})(1-q^{\gamma+2}) > 1, \quad \gamma \in (0,2), \quad \gamma \neq 1, \tag{7.36}$$

   then the zeros of $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ are all real, simple and symmetric with respect to the point $z = 0$ such that for $n \in \mathbb{N}$

$$\xi_n^{(\gamma)} \in \begin{cases} \left(\dfrac{q^{-n+3/2}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}, \dfrac{q^{-n+1/2}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}\right), \gamma \in (0,1), \\[3ex] \left(\dfrac{q^{-n+5/2}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}, \dfrac{q^{-n+3/2}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}\right), \gamma \in (1,2). \end{cases}$$

3. *If $\gamma \in \{0,1\}$ then $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ has only real, simple and symmetric zeros such that*

$$\xi_m^{(0)} := \frac{q^{1/2}x_m}{\sigma}, \quad \xi_m^{(1)} := \frac{q y_m}{\sigma} \quad (m \in \mathbb{N}),$$

   where x_m and y_m are defined in (2.85).

Proof. If $\gamma \in \{0, 1\}$ then from (7.4) we obtain

$$\mathscr{E}_{0,\sigma}(z^2;q) = \cos(q^{-1/2}\sigma z;q), \quad \mathscr{E}_{1,\sigma}(z^2;q) = \frac{\sin(q^{-1}\sigma z;q)}{q^{-1}\sigma z}.$$

Since the zeros of $\cos(\cdot;q)$ and $\sin(\cdot;q)$ are real and simple, then the desired statement follows for $\gamma \in \{0, 1\}$. From formula (7.4) with $\beta = 2$, $\lambda = -\sigma^2$ we have

$$E_{2,\mu+\alpha}(-\sigma^2 z^2;q) = \frac{z^{-\mu}}{\Gamma_q(\alpha)} \int_0^z \big(qt/z;q\big)_{\alpha-1} E_{2,\mu}\big(-\sigma^2 t^2;q\big) t^{\mu-1}\, d_q t. \tag{7.37}$$

Substitute with $u := t/z$, $z \neq 0$, on the q-integral of (7.37). Then

$$\begin{aligned}\int_0^z \big(qt/z;q\big)_{\alpha-1} E_{2,\mu}\big(-\sigma^2 t^2;q\big) t^{\mu-1}\, d_q t \\ = z^{\mu} \int_0^1 (qu;q)_{\alpha-1} E_{2,\mu}(-\sigma^2 u^2 z^2;q) t^{\mu-1}\, d_q t.\end{aligned}$$

Equation (7.37) is nothing but

$$E_{2,\mu+\alpha}(-\sigma^2 z^2;q) = \int_0^1 (qu;q)_{\alpha-1} E_{2,\mu}(-\sigma^2 u^2 z^2;q) t^{\mu-1}\, d_q t. \tag{7.38}$$

If $0 < \gamma < 1$ then substituting in (7.38) with $\mu = 1$ and $\alpha = \gamma$ yields

$$\mathscr{E}_{\gamma,\sigma}(z^2;q) = \frac{1}{\Gamma_q(\gamma)} \int_0^1 (qt;q)_{\gamma-1} \cos(q^{-1/2}\sigma t z;q)\, d_q t, \tag{7.39}$$

while if $1 < \gamma < 2$, the substitution into (7.38) by $\mu = 2$ and $\alpha = \gamma - 1$ leads to

$$q^{-1}\sigma z \mathscr{E}_{\gamma,\sigma}(z^2;q) = \frac{1}{\Gamma_q(\gamma-1)} \int_0^1 (qt;q)_{\gamma-2} \sin(q^{-1}\sigma t z;q)\, d_q t. \tag{7.40}$$

For $\delta \in (0, 1)$, let f_δ be the function defined on $[0, 1]$ by

$$f_\delta(t) := \frac{(qt;q)_{\delta-1}}{\Gamma_q(\delta)}.$$

Since

$$\frac{f_\delta(q^k)}{f_\delta(q^{k+1})} = \frac{1-q^{k+1}}{1-q^{k+\delta}} > 1 \quad (k \in \mathbb{N}_0)$$

then

$$f_\delta(1) > f_\delta(q) > f_\delta(q^2) > \ldots > f_\delta(q^n) > \ldots > f_\delta(0) = \frac{1}{\Gamma_q(\delta)}.$$

That is f_δ, $\delta \in (0,1)$ is increasing and is positive on the sequence $\{0, q^n, n \in \mathbb{N}\}$. Moreover, $f_\delta \in L_q^1(0,1)$ because

$$\int_0^1 f_\delta(t)\, d_qt = \frac{B_q(\delta,1)}{\Gamma_q(\delta)} = \frac{1}{\Gamma_q(\delta+1)}.$$

Hence, applying Theorem 2.18 to (7.39) and Theorem 2.19 to (7.40) we conclude that for any $q \in (0,1)$, $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ has at most a finite number of non-real zeros, and it has an infinite number of real, simple and symmetric zeros, say $\left\{\xi_m^{(\gamma)}\right\}$, such that $\left\{\xi_m^{(\gamma)}\right\}$ satisfies the asymptotic properties (7.35). From (2.87) and (2.88) we have

$$\begin{aligned} c_{k,f_{\gamma-1}} &= \frac{1}{(1-q^{2k+\gamma+1})(1-q^{2k+\gamma+2})} \quad (0<\gamma<1)\,,\\ b_{k,f_{\gamma-2}} &= \frac{1}{(1-q^{2k+\gamma+1})(1-q^{2k+\gamma+2})} \quad (1<\gamma<2)\,, \end{aligned}$$

for all $k \in \mathbb{N}_0$. Consequently,

$$\begin{aligned} c_{f_{\gamma-1}} &= 1, \quad C_{f_{\gamma-1}} = \frac{1}{(1-q^{\gamma+1})(1-q^{\gamma+2})} \quad (0<\gamma<1)\,,\\ b_{f_{\gamma-2}} &= 1, \quad C_{f_{\gamma-2}} = \frac{1}{(1-q^{\gamma+1})(1-q^{\gamma+2})} \quad (1<\gamma<2)\,. \end{aligned}$$

Thus, the condition (2.89) of Theorem 2.16 and condition (2.90) of Theorem 2.17 are nothing but the condition (7.36). Therefore, the proof of the point (2) of the theorem follows from Theorems 2.16 and 2.17 of Sect. 2.8. ◻

Chapter 8
Fractional q-Difference Equations

Abstract As in the classical theory of ordinary fractional differential equations, q-difference equations of fractional order are divided into linear, nonlinear, homogeneous, and inhomogeneous equations with constant and variable coefficients. This chapter is devoted to certain problems of fractional q-difference equations based on the basic Riemann–Liouville fractional derivative and the basic Caputo fractional derivative. In this chapter, we investigate questions concerning the solvability of these equations in a certain space of functions. A special class of Cauchy type q-fractional problems is also developed at the end of this chapter.

8.1 Equations with the Riemann–Liouville Fractional q-Derivatives

In this section we shall study the existence and uniqueness of solutions of the q-Cauchy type problem

$$D_q^{\alpha} y(x) = f(x, y(x)) \quad (\alpha > 0), \tag{8.1}$$

$$D_q^{\alpha-k} y(0^+) = b_k, \quad b_k \in \mathbb{R} \quad (k = 1, \ldots, \lceil\alpha\rceil)\,. \tag{8.2}$$

The result of this section is an extension of the results derived by Kilbas et al. in [169, Chap. 3].

8.1.1 Solutions in the Space $\mathscr{L}_q^1[0, a]$

Theorem 8.1. *Let* $\alpha > 0$*,* $n = \lceil\alpha\rceil$*. Let* G *be an open set in* $\mathbb{C}$ *and let* $f : (0, a] \times G \to \mathbb{R}$ *be a function such that* $f(x, y) \in \mathscr{L}_{q,1}[0, a]$ *for any* $y \in G$*. Assume that*

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$y(x) \in \mathscr{L}_q^1[0,a]$. *Then* $y(x)$ *satisfies* (8.1)–(8.2) *for all* $x \in (0,a]$ *if and only if* $y(x)$ *satisfies the* q*-integral equation*

$$y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y(t))\, d_qt, \quad (8.3)$$

for all $x \in (0,a]$.

Proof. First, we prove the necessity condition. Let $y(x)$ satisfy (8.1)–(8.2), then $D_q^\alpha y(x) \in \mathscr{L}_q^1[0,a]$. Hence, applying Lemma 4.17 gives

$$I_q^\alpha D_q^\alpha y(x) = y(x) - \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k} \quad (0 < x \leq a). \quad (8.4)$$

On the other hand,

$$I_q^\alpha D_q^\alpha y(x) = I_q^\alpha f(x, y(x)) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y(t))\, d_qt, \quad (8.5)$$

for all $x \in (0,a]$. Consequently, combining (8.4) and (8.5) proves the necessity condition. Now we prove the sufficiency. Let $y(x)$ satisfy (8.3) for all $x \in (0,a]$, then from Lemma 4.9, $y(x) \in \mathscr{L}_q^1[0,a]$. Moreover,

$$I_q^{n-\alpha} y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(n - k + 1)} x^{n-k} + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x (qt/x;q)_{n-1} f(t, y(t))\, d_qt,$$

for all $x \in (0,a]$. Hence, from Theorem 4.6, $I_q^{n-\alpha} y(x) \in \mathscr{A}C_q^{(n)}[0,a]$. That is $D_q^\alpha y(x)$ exists for all $x \in (0,a]$. Since

$$D_q^\alpha \frac{x^{\alpha-k}}{\Gamma_q(\alpha-k+1)} := D_q^n I_q^{n-\alpha} \frac{x^{\alpha-k}}{\Gamma_q(\alpha-k+1)} = D_q^n \frac{x^{n-k}}{\Gamma_q(n-k+1)} = 0,$$

for $k = 1, 2, \ldots, n$, then applying the operator D_q^α on the two sides of (8.3) gives

$$D_q^\alpha y(x) = D_q^\alpha I_q^\alpha f(x, y) = f(x, y(x)), \text{ for all } x > 0.$$

Now

$$\begin{aligned} D_q^{\alpha-k} y(0^+) &= \lim_{j\to\infty} D_q^{\alpha-k} y(xq^j) = b_k + \lim_{j\to\infty} I_q^k f(xq^j, y(xq^j)) \\ &= b_k + \lim_{j\to\infty} \frac{(xq^j)^{k-1}}{\Gamma_q(k)} \int_0^{xq^j} (qt/x;q)_{k-1} f(t, y(t))\, d_qt, \end{aligned} \quad (8.6)$$

where the limit on the most left hand side of (8.6) vanishes because of $f(x, y(x)) \in \mathscr{L}_q^1[0, a]$. □

The proof of the existence and uniqueness theorem of this section depends on the Banach fixed point Theorem 1.7.

Theorem 8.2. *Let* $\alpha > 0$, $n = \lceil \alpha \rceil$. *Let* G *be an open set in* $\mathbb{C}$ *and*

$$f : (0, a] \times G \longrightarrow R$$

be a function satisfying the following conditions:

1. *For any* $y \in G$, $f(x, y) \in \mathscr{L}_q^1[0, a]$.
2. *There exists a constant* $A > 0$ *such that for all* $x \in (0, a]$ *and for all* $y_1, y_2 \in G$, *the following Lipschitz condition is satisfied:*

$$|f(x, y_1(x)) - f(x, y_2(x))| \le A|y_1(x) - y_2(x)|.$$

If $0 < h \le a$ *satisfies the condition*

$$\frac{Ah^\alpha}{\Gamma_q(\alpha + 1)} < 1$$

then the fractional q*-Cauchy problem* (8.1)–(8.2) *has a unique solution in* $\mathscr{L}_q^1[0, h]$.

Proof. From Theorem 8.1, the Cauchy type problem (8.1)–(8.2) is equivalent to the Volterra q-integral equation (8.3). Consequently, a solution of (8.1)–(8.2) in $\mathscr{L}_{q,1}[0, h]$ is a fixed point of the operator $T : \mathscr{L}_{q,1}[0, h] \longrightarrow \mathscr{L}_{q,1}[0, h]$ defined by

$$Ty(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha - k} + \frac{x^{\alpha - 1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha - 1} f(t, y(t))\, d_q t.$$

Therefore, to prove the theorem, we prove

(1) If $y \in \mathscr{L}_{q,1}[0, h]$, then $Ty \in \mathscr{L}_{q,1}[0, h]$,
(2) T is a contraction mapping.

The proof of (1) follows from Lemma 4.9, moreover, from (4.53) we obtain

$$\|Ty_1 - Ty_2\|_1 \le A \left\| I_q^\alpha (y_1 - y_2) \right\|_1 \le w \|y_1 - y_2\|_1, \; w := A \frac{h^\alpha}{\Gamma_q(\alpha + 1)} < 1.$$

Hence from the Banach fixed point theorem, there exists a unique $y^* \in \mathscr{L}_{q,1}[0, h]$ such that $Ty^* = y^*$. Therefore, the result follows by applying the sufficient part of Theorem 8.1. □

By Theorem 1.7, the solution y^* is obtained as a limit of a convergent sequence $(T^m y_0)(x)$:

$$\lim_{m\to\infty} \|T^m y_0 - y^*\|_1 = 0,$$

in the space $\mathscr{L}_{q,1}[0,h]$, where y_0 is any function in $\mathscr{L}_{q,1}[0,h]$. If at least one $b_k \neq 0$ in the initial condition (8.2), we can take

$$y_0(x) := \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k}.$$

Consequently, the sequence $T^m y_0$ is defined by the recurrence relation

$$T^m y_0(x) = y_0(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, T^{m-1} y_0(t))\, d_q t \quad (m \in \mathbb{N}).$$

If we denote $y_m(x) = (T^m y_0)(x)$, then the last relation takes the form

$$y_m(x) = y_0(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y_{m-1}(t))\, d_q t \quad (m \in \mathbb{N}).$$

This means that the successive approximation method can be used to find a unique solution of (8.1)–(8.2) whenever $b_k \neq 0$ for some $k \in \{1, 2, \ldots, n\}$.

The following example is a q-extension of the example given in [168, Example 3.1].

Example 8.1.1. Consider the Cauchy type q-fractional equation:

$$\begin{aligned} D_q^{\alpha} y(x) &= \lambda x^{\beta} y^2(x) \quad (0<\alpha<1,\ x>0; \lambda, \beta \in \mathbb{R};\ \ \lambda \neq 0)\,, \\ I_q^{1-\alpha} y(0^+) &= 0. \end{aligned} \tag{8.7}$$

Set

$$f(x, y(x)) = \lambda x^{\beta} y^2(x) \quad (x>0),$$

where

$$(x,y) \in (0,a] \times G := \{0 < x \le a;\ 0 < |y| \le K\,x^{w};\ K > 0\}.$$

Let $x \in (0,a]$ and $y_1, y_2 \in G$. Then

$$|f(x,y_1) - f(x,y_2)| \le |\lambda| x^{\beta} |y_1^2 - y_2^2|.$$

Therefore, using

$$|y_1^2 - y_2^2| \le 2\max\{|y|_1, |y_2|\}\,|y_1 - y_2| \le 2Kx^{w}|y_1 - y_2|,$$

we obtain

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|, \quad A := 2|\lambda| K a^{\beta+w},$$

provided that $\beta + w \geq 0$. That is, f satisfies the Lipschitz condition. If we also assume that $\beta + 2w > -1$ then $f(x, y) \in \mathscr{L}_q^1[0, a]$. Hence, the conditions of Theorem 8.2 is satisfied and the fractional q-Cauchy problem (8.7) has a unique solution in $\mathscr{L}_{q,1}[0, h]$, where h is a positive number satisfying the inequality

$$2|\lambda| K \frac{h^{\beta+w+\alpha}}{\Gamma_q(\alpha + 1)} < 1.$$

Hence, w should satisfy the conditions

$$w + \beta \geq 0, \quad 2w + \beta > -1. \tag{8.8}$$

Assume that $w = r - (\beta + \alpha)$. Consequently, $w + \beta = r - \alpha$ and we should assume that $r \geq \alpha$. In addition, the condition $2w + \beta > -1$ implies

$$2r - \beta - 2\alpha > -1.$$

I.e

$$2r > -1 + (\beta + 2\alpha).$$

Therefore, if we assume that $\beta + 2\alpha < 1$ then the conditions in (8.8) are satisfied and (8.7) has a unique solution in $\mathscr{L}_{q,1}[0, h]$. One can verify that this solution is given by

$$y(x) = \frac{\Gamma_q(1 - \alpha - \beta)}{\lambda \Gamma_q(1 - 2\alpha - \beta)} x^{-(\alpha+\beta)}.$$

8.1.2 Solutions in the Space of Weighted Continuous Functions

In this subsection, we provide conditions by which the Cauchy type problem (8.1)–(8.2) has a unique solutions in the space $C_{n-\alpha}[0, a]$, $n - 1 < \alpha \leq n$ $(n \in \mathbb{N})$.

Theorem 8.3. *Let* $\alpha > 0$, $n = \lceil \alpha \rceil$, *and* $\gamma < 1$. *Let* G *be an open set in* $\mathbb{C}$ *and let* $f : (0, a] \times G \to \mathbb{R}$ *be a function such that* $f(x, y) \in C_\gamma[0, a]$ *for any* $y \in G$. *If* $y \in C_{n-\alpha}[0, a]$ *then* $y(x)$ *satisfies* (8.1)–(8.2) *for all* $x \in (0, a]$ *if and only if* $y(x)$ *satisfies the* q*-integral equation*

$$y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y(t))\, d_q t, \quad (8.9)$$

for all $x \in (0, a]$.

Proof. The theorem follows from Theorem 8.1, because if $g \in C_\gamma[0, a]$ for some $\gamma < 1$, then $g \in \mathscr{L}_q^1[0, a]$. □

Theorem 8.4. *Let* $\alpha > 0$ *and* $n = \lceil \alpha \rceil$. *Let* G *be an open set in* $\mathbb{C}$ *and let* $f : (0, a] \times G \longrightarrow \mathbb{R}$ *be a function satisfying the following conditions:*

1. $f(x, y) \in C_{n-\alpha}[0, a]$ *for any* $y \in G$.
2. *There exists a constant* $A > 0$ *such that for all* $x \in (0, a]$ *and for all* $y_1, y_2 \in G$, *the following Lipschitz condition is satisfied:*

$$|f(x, y_1(x)) - f(x, y_2(x))| \le A|y_1(x) - y_2(x)|.$$

If $0 < h \le a$ *satisfies the condition*

$$Ah^\alpha \frac{\Gamma_q(1 - n + \alpha)}{\Gamma_q(1 - n + 2\alpha)} < 1$$

then the fractional q*-Cauchy problem* (8.1)–(8.2) *has a unique solution in* $C_{n-\alpha}[0, h]$.

Proof. From Theorem 8.3, the fractional q-Cauchy problem (8.1)–(8.2) is equivalent to (8.9). Hence, a solution of (8.1)–(8.2) in $C_{n-\alpha}[0, h]$ is a fixed point of the operator $T : \mathscr{L}_{q,1}[0, h] \longrightarrow \mathscr{L}_{q,1}[0, h]$ defined by

$$Ty(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y(t))\, d_q t.$$

To prove the theorem, we prove

1. If $y \in C_{n-\alpha}[0, h]$, then $Ty \in C_{n-\alpha}[0, h]$,
2. T is a contraction mapping.

Since

$$x^{n-\alpha} Ty(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha - k + 1)} x^{n-k} + x^{n-\alpha} I_q^\alpha f(x, y(x)),$$

the proof of (1) follows by applying Lemma 4.10. In addition, from (4.56) we conclude that

$$\|Ty_1 - Ty_2\|_{C_{n-\alpha}} \le A \left\| I_q^\alpha (y_1 - y_2) \right\|_{C_{n-\alpha}} \le w \, \|y_1 - y_2\|_{C_{n-\alpha}},$$

where

$$w := Ah^{\alpha}\frac{\Gamma_q(1-n+\alpha)}{\Gamma_q(1-n+2\alpha)} < 1.$$

Therefore, from Banach fixed point theorem there exists a unique $y^* \in C_{n-\alpha}[0,h]$ such that $Ty^* = y^*$. Thus, the result follows by applying the sufficient part of Theorem 8.3. □

One can verify that if $g \in C_r[0,a]$ $(r \le \gamma)$ then $g \in C_\gamma[0,a]$. Hence, a generalized form of Theorem 8.4 is in the following theorem:

Theorem 8.5. *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. Let G be an open set in $\mathbb{R}$ and let $f : (0,a] \times G \longrightarrow \mathbb{C}$ be a function such that*

(1) There exists $\gamma \le n-\alpha$ such that $f(x,y) \in C_\gamma[0,a]$ for any $y \in G$,
(2) There exists a constant $A > 0$ such that for all $x \in (0,a]$ and for all $y_1, y_2 \in G$, the following Lipschitz condition is satisfied:

$$|f(x,y_1(x)) - f(x,y_2(x))| \le A|y_1(x) - y_2(x)|.$$

If $0 < h \le a$ satisfies the condition

$$Ah^{\alpha}\frac{\Gamma_q(1-n+\alpha)}{\Gamma_q(1-n+2\alpha)} < 1$$

then the fractional q-Cauchy problem (8.1)–(8.2) has a unique solution in $C_{n-\alpha}[0,h]$.

Theorem 8.6. *Let $0 < \alpha < 1$, G be an open set in $\mathbb{C}$ and $f : (0,a] \times G \longrightarrow \mathbb{R}$ be a function such that the following conditions are fulfilled:*

1. *There exists $\gamma \le n-\alpha$ such that $f(x,y) \in C_\gamma[0,a]$ for any $y \in G$.*
2. *There exists a constant $A > 0$ such that for all $x \in (0,a]$ and for all $y_1, y_2 \in G$, the following Lipschitz condition is satisfied:*

$$|f(x,y_1(x)) - f(x,y_2(x))| \le A|y_1(x) - y_2(x)|.$$

If $0 < h \le a$ satisfies the condition

$$Ah^{\alpha}\frac{\Gamma_q(1-n+\alpha)}{\Gamma_q(1-n+2\alpha)} < 1$$

then the fractional q-Cauchy problem

$$D_q^{\alpha} y(x) = f(x,y(x)) \quad (0 < x \le a)\,, \tag{8.10}$$

$$\lim_{x\to 0^+} x^{1-\alpha} y(x) = \frac{b}{\Gamma_q(1-\alpha)} \quad (b \in \mathbb{R})\,, \tag{8.11}$$

has a unique solution in $C_{1-\alpha}[0,h]$.

Proof. The proof follows by noting that from (ii) of Lemma 4.11, the q-fractional Cauchy problem (8.10)–(8.11) is equivalent to the fractional q-Cauchy problem (8.1)–(8.2) with $0 < \alpha < 1$. Therefore, the theorem follows by applying the sufficient part of Theorem 8.5. □

Example 8.1.2. Consider the Cauchy type q-fractional equation:

$$\begin{aligned} &D_q^{\alpha} y(x) \quad = \lambda x^{\beta} y^m(x) \quad (m > 1,\, 0 < \alpha < 1,\, x > 0; \lambda, \beta \in \mathbb{R};\ \ \lambda \neq 0)\,, \\ &I_q^{1-\alpha} y(0^+) = 0. \end{aligned} \tag{8.12}$$

Set

$$f(x, y(x)) = \lambda x^{\beta} y^m(x) \quad (x > 0),$$

where

$$(x, y) \in (0, a] \times G := \{0 < x \le a;\ 0 < |y| \le K\, x^{w};\ K > 0\}\,.$$

Let $x \in (0, a]$ and $y_1, y_2 \in G$. Then

$$|f(x, y_1) - f(x, y_2)| \le |\lambda| x^{\beta} |y_1^m - y_2^m|.$$

Hence, using

$$|y_1^m - y_2^m| \le m \max\left\{|y_1|^{m-1}, |y_2|^{m-1}\right\} |y_1 - y_2| \le 2K^{m-1} x^{(m-1)w} |y_1 - y_2|,$$

we obtain

$$|f(x, y_1) - f(x, y_2)| \le A|y_1 - y_2|, \quad A := m|\lambda| K^{m-1} a^{\beta+(m-1)w},$$

provided that $\beta + (m-1)w \ge 0$. That is, f satisfies the Lipschitz condition. If $f(x, y) \in C_{1-\alpha}[0, a]$ then $x^{1-\alpha} f(x, y)$ is continuous and hence bounded on $[0, a]$. Therefore, we should assume that $1 - \alpha + \beta + mw \ge 0$. Hence the conditions of Theorem 8.4 are satisfied and the Cauchy type q-fractional problem (8.12) has a unique solution in $C_{1-\alpha}[0, h]$, where h is a positive number satisfying the inequality

$$m|\lambda| K^{m-1} \frac{h^{\beta+(m-1)w+\alpha} \Gamma_q(\alpha)}{\Gamma_q(2\alpha)} < 1.$$

Therefore, w should satisfy the conditions

$$\beta + (m-1)w \ge 0, \quad 1 - \alpha + mw + \beta \ge 0. \tag{8.13}$$

Assume that $(m-1)w = r - (\beta + \alpha)$. Then $(m-1)w + \beta = r - \alpha$. From (8.13), $r \ge \alpha$. Also, the condition $1 - \alpha + mw + \beta \ge 0$ implies

$$\frac{m}{m-1}r + 1 - \frac{\beta + (2m-1)\alpha}{m-1} \geq 0.$$

I.e

$$\frac{m}{m-1}r \geq -1 + \frac{(2m-1)\alpha + \beta}{m-1}.$$

So, if we assume that $\beta + (2m-1)\alpha < m-1$ then the conditions in (8.13) are satisfied and problem (8.12) has a unique solution in $C_{1-\alpha}[0,h]$ and this solution is given by

$$y(x) = \left(\frac{\Gamma_q(1 - \frac{\alpha+\beta}{m-1})}{\lambda \Gamma_q(1 - \frac{m\alpha+\beta}{m-1})} \right)^{\frac{1}{m-1}} x^{-\frac{\alpha+\beta}{m-1}}.$$

8.2 q-Riemann–Liouville Fractional Order Systems

This section deals with two systems of q-Riemann–Liouville fractional derivatives, namely

$$D_q^\alpha y_i(x) = f_i(x, y_1(x), \ldots, y_n(x)) \quad (i = 1, 2, \ldots, n)\,, \tag{8.14}$$

and

$$D_q^\alpha y_i(x) = f_i(qx, y_1(qx), \ldots, y_n(qx)) \quad (i = 1, 2, \ldots, n)\,, \tag{8.15}$$

where $\alpha > 0$. System (8.14) together with the initial conditions

$$D_q^{\alpha-k} y_i(0^+) = b_{ik} \quad (b_{ik} \in \mathbb{R};\ i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, \lceil \alpha \rceil)\,, \tag{8.16}$$

is called an initial value problem.

In this section, by a solution of the initial value problem (8.14), (8.16) we mean a set of functions $\{y_i\}_{i=1}^n$ that satisfies (8.14) and the initial conditions (8.16) in an interval $I := (0, a]$ such that

$$y_i \in \mathscr{L}_q^1[0,a] \quad (i = 1, 2, \ldots, n)\,.$$

Similarly, we define solutions of the initial value problem (8.15)–(8.16). We begin this section with the existence and uniqueness theorems of solutions of the initial value problem (8.14), (8.16). The following lemma is needed in the derivations of the main results of this section.

Lemma 8.7. *Let n be a positive integer and $f(x, y_1, \ldots, y_n)$ be functions defined for $x \in (0, a]$, $a > 0$, and y_i in the domain $G_i \subseteq \mathbb{C}$, $1 \leq i \leq n$, satisfying the following conditions.*

(i) There is a positive constant A such that for $x \in [a, b]$ and $y_i, \tilde{y}_i \in G_i$, $1 \leqslant i \leqslant n$, the following Lipschitz' condition is fulfilled

$$|f(x, y_1, \ldots, y_n) - f(x, \tilde{y}_1, \ldots, \tilde{y}_n)| \leqslant A\Big(|y_1 - \tilde{y}_1| + \ldots + |y_n - \tilde{y}_n|\Big).$$

(ii) There is $\gamma > 0$ *such that* $f(x, y_1, y_2, \ldots, y_n) \in C_\gamma[a, b]$ *for any* $y_i \in G_i$.

Then, the function $x^\gamma f(x, y_1, y_2, \ldots, y_n)$ *is continuous on* $[a, b] \times \prod_{i=1}^n G_i$. *Moreover, if each* G_i, $i = 1, 2, \ldots, n$, *is compact subset of* $\mathbb{C}$, *then* $x^\gamma f(x, y_1, y_2, \ldots, y_n)$ *is bounded on* $[a, b] \times \prod_{i=1}^n G_i$.

Proof. We shall prove the lemma for the case $n = 1$. The proof for any $n > 1$ is similar. Now

$$\begin{aligned}\left|x^\gamma f(x, y) - x_0^\gamma f(x_0, y_0)\right| &\leq x^\gamma |f(x, y) - f(x, y_0)| + |x^\gamma f(x, y_0) - x_0^\gamma f(x_0, y_0)| \\ &\leq A x^\gamma |y - y_0| + |x^\gamma f(x, y_0) - x_0^\gamma f(x_0, y_0)|.\end{aligned} \tag{8.17}$$

Hence, the lemma follows since the limit of the most right hand side of (8.17) tends to zero as $(x, y) \to (x_0, y_0)$. □

Theorem 8.8. *Let* n *be a positive integer and let* $f_i(x, y_1, \ldots, y_n)$ *be functions defined for* $x \in (0, a]$, $a > 0$, *and* y_i *in the domain* $G_i \subseteq \mathbb{C}$, $1 \leq i \leq n$, *satisfying the following conditions.*

(i) There is a positive constant A *such that for* $x \in (0, a]$ *and* $y_i, \tilde{y}_i \in G_i$, $1 \leqslant i \leqslant n$, *the following Lipschitz' condition is fulfilled*

$$|f_i(x, y_1, \ldots, y_n) - f_i(x, \tilde{y}_1, \ldots, \tilde{y}_n)| \leqslant A\Big(|y_1 - \tilde{y}_1| + \ldots + |y_n - \tilde{y}_n|\Big).$$

(ii) There exists $\gamma < 1$ *and* $\gamma \leq \alpha$ *such that*

$$f_i(x, y_1, \ldots, y_n) \in C_\gamma[0, a], \quad \text{for any } y_i \in G_i, \quad i = 1, \ldots, n. \tag{8.18}$$

Let K *be a constant satisfying*

$$\frac{M a^{\alpha-\gamma} \Gamma_q(1-\gamma)}{\Gamma_q(\alpha - \gamma + 1)} \leq K, \tag{8.19}$$

where

$$M := \max\left\{|x^\gamma f_i(x, y_1, y_2, \ldots, y_n)|,\ x \in [0, a],\ y_i \in D_i(a, K),\ i = 1, 2, \ldots, n\right\},$$

and $D_i(a, K) \subset G_i$ *is the set of points* $y_i \in G_i$ *satisfying the relation*

$$\left| y_i - \sum_{k=1}^{\lceil \alpha \rceil} b_{ik} \frac{x^{\alpha-k}}{\Gamma_q(\alpha - k + 1)} \right| \leq K,$$

for all $x \in (0, a]$. *Then, there exists* $h \in (0, a]$ *such that the initial value problem* (8.14), (8.16) *has a unique solution* $\{y_i(\cdot)\}_{i=1}^{n}$ *valid in* $(0, h]$, $y_i \in C_{\lceil\alpha\rceil-\alpha}[0, a]$ *for* $i = 1, 2, \ldots, n$.

Proof. Existence. Define the sequences $\{\phi_{i,m}(\cdot)\}_{m=1}^{\infty}$, $i = 1, \ldots, n$, $x \in (0, a]$ by the following relationships

$$\begin{aligned}\phi_{i,1}(x) &= \sum_{k=1}^{\lceil\alpha\rceil} b_{ik} \frac{x^{\alpha-k}}{\Gamma_q(\alpha-k+1)},\\ \phi_{i,m}(x) &= \sum_{k=1}^{\lceil\alpha\rceil} b_{ik} \frac{x^{\alpha-k}}{\Gamma_q(\alpha-k+1)}\\ &\quad + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x;q\right)_{\alpha-1} f_i\left(t, \phi_{1,m-1}(t), \ldots, \phi_{n,m-1}(t)\right) d_qt,\end{aligned} \tag{8.20}$$

where $m \geqslant 2$. We will show that $\lim\limits_{m\to\infty} \phi_{i,m}(x)$ exists and gives the required solution $\{\phi_i(\cdot)\}_{i=1}^{n}$ of the initial value problem (8.14), (8.16). We prove the existence in four steps.

i. We prove by induction on m that

$$\phi_{i,m}(x) \in D_i(a, k) \quad (m \in \mathbb{N};\ i = 1, \ldots, n;\ x \in (0, a]). \tag{8.21}$$

Clearly, $\phi_{i,1}(x) \in D_i(a, K)$ for $i = 1, \ldots, n$ and $x \in (0, a]$. If we assume that

$$\phi_{i,m}(x) \in D_i(a, k) \quad \text{for } i = 1, \ldots, n \quad \text{and } x \in (0, a]$$

then from Lemma 8.7, there exists $M > 0$ such that

$$|f_i(x, \phi_{1,m}(x), \ldots, \phi_{n,m}(x))| \leqslant M x^{-\gamma} \quad \text{for all } \ x \in (0, a].$$

Thus, by (1.58)

$$\begin{aligned}&\left|\phi_{i,m+1}(x) - \textstyle\sum_{k=1}^{\lceil\alpha\rceil} \tfrac{b_{ik}}{\Gamma_q(\alpha-k+1)} x^{\alpha-k}\right|\\ &\leqslant \tfrac{x^{\alpha-1}}{\Gamma_q(\alpha)} \textstyle\int_0^x \left(qt/x;q\right)_{\alpha-1} |f_i(t, \phi_{1,m}(t), \ldots, \phi_{n,m}(t))|\, d_qt\\ &\leqslant \tfrac{Mx^{\alpha-1}}{\Gamma_q(\alpha)} \textstyle\int_0^x \left(qt/x;q\right)_{\alpha-1} t^{-\gamma}\, d_qt = \tfrac{M}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma) x^{\alpha-\gamma}\\ &= \tfrac{Mx^{\alpha-\gamma}\Gamma_q(1-\gamma)}{\Gamma_q(\alpha-\gamma+1)} \leqslant \tfrac{Ma^{\alpha-\gamma}\Gamma_q(1-\gamma)}{\Gamma_q(\alpha-\gamma+1)} \leqslant K.\end{aligned} \tag{8.22}$$

So,

$$\phi_{i,m+1}(x) \in D_i(a, K) \quad \text{for all} \quad x \in (0, a].$$

This completes the induction steps and proves (8.21).

ii. Applying Lemma 4.10 gives that $I_q^\alpha f_i(t, \phi_{1,m}(t), \ldots, \phi_{n,m}(t)) \in C[0, a]$ for all $m \in \mathbb{N}$. Since $\sum_{k=1}^{n} \frac{b_{ik} x^{\alpha-k}}{\Gamma_q(\alpha-k+1)} \in C_{\lceil\alpha\rceil-\alpha}[0, a]$, the function $\phi_{i,m+1} \in C_{\lceil\alpha\rceil-\alpha}[0, a]$ for all $m \in \mathbb{N}_0$ and $i \in \{1, 2, \ldots, n\}$.

iii. We prove by induction on m that

$$|\phi_{i,m+1}(x) - \phi_{i,m}(x)| \leqslant \frac{MB^{m-1} x^{m\alpha-\gamma} \Gamma_q(1-\gamma)}{\Gamma_q(m\alpha - \gamma + 1)}, \tag{8.23}$$

where

$$B := An, \quad m \in \mathbb{N}, \quad \text{and for any } x \in (0, a].$$

From (8.22), inequality (8.23) is true at $m = 1$. Assume that (8.23) is true at $m = k$, that is,

$$|\phi_{i,k+1}(x) - \phi_{i,k}(x)| \leqslant \frac{MB^{k-1} x^{k\alpha-\gamma} \Gamma_q(1-\gamma)}{\Gamma_q(k\alpha - \gamma + 1)} \quad (i = 1, 2, \ldots, n).$$

Hence, for $x \in (0, a]$ we have

$$\begin{aligned}
&|\phi_{i,k+2}(x) - \phi_{i,k+1}(x)| \\
&\quad \leqslant \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x; q\right)_{\alpha-1} \\
&\qquad \times |f_i(t, \phi_{1,k+1}(t), \ldots, \phi_{n,k+1}(t)) - f_i(t, \phi_{1,k}(t), \ldots, \phi_{n,k}(t))| \, d_q t \\
&\quad \leqslant \frac{Ax^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x; q\right)_{\alpha-1} \sum_{j=1}^{n} |\phi_{j,k+1}(t) - \phi_{j,k}(t)| \, d_q t \\
&\quad = \frac{MB^k x^{\alpha-1} \Gamma_q(1-\gamma)}{\Gamma_q(\alpha)\Gamma_q(k\alpha - \gamma + 1)} \int_0^x \left(qt/x; q\right)_{\alpha-1} t^{k\alpha-\gamma} \, d_q t \\
&\quad = \frac{MB^k \Gamma_q(1-\gamma)}{\Gamma_q(\alpha)\Gamma_q(k\alpha + 1)} B_q(\alpha, k\alpha - \gamma + 1) x^{(k+1)\alpha-\gamma} \\
&\quad = \frac{MB^k \Gamma_q(1-\gamma)}{\Gamma_q\big((k+1)\alpha - \gamma + 1\big)} x^{(k+1)\alpha-\gamma}.
\end{aligned}$$

That is (8.23) is true at $m = k + 1$ and hence it is true for all $m \geqslant 1$.

iv. We prove that $\lim_{m\to\infty}\phi_{i,m}(x)$ exists for $i = 1,2,\dots,n$, and for any $x \in (0,a]$ such that $\{\phi_i(\cdot)\}_{i=1}^n$,

$$\phi_i(x) := \lim_{m\to\infty}\phi_{i,m}(x) \quad (i = 1,2,\dots,n;\ \ x \in (0,a])\,,$$

defines a solution of (8.14), (8.16). Consider the infinite series

$$\phi_{i,1}(x) + \sum_{m=1}^{\infty}\phi_{i,m+1}(x) - \phi_{i,m}(x). \tag{8.24}$$

From (8.23) we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}|\phi_{i,m+1}(x) - \phi_{i,m}(x)| &\leqslant \frac{M\Gamma_q(1-\gamma)}{B}x^{-\gamma}\sum_{m=1}^{\infty}\frac{(Bx^{\alpha})^m}{\Gamma_q(m\alpha-\gamma+1)}\\ &\leqslant \frac{M\Gamma_q(1-\gamma)x^{-\gamma}}{B}\sum_{m=0}^{\infty}\frac{(Bx^{\alpha})^m}{\Gamma_q(m\alpha-\gamma+1)}\\ &= \frac{M\Gamma_q(1-\gamma)x^{-\gamma}}{B}e_{\alpha,1-\gamma}(Bx^{\alpha};q).\end{aligned}$$

Set $h := \min\left\{a, \frac{1}{B^{1/\alpha}(1-q)}\right\}$. Since $e_{\alpha,1-\gamma}(Bx^{\alpha};q)$ is defined only for $|x| \leqslant h$, the series in (8.24) is uniformly convergent on $(0,h]$ to a function ϕ_i, i.e. $\phi_i(x) = \lim_{m\to\infty}\phi_{i,m}(x)$. Now $\phi_{i,m}(x) \in D_i(a,k)$ implies that $\phi_i(x) \in D_i(a,k)$, $x \in (0,h]$. Since $x^{\lceil\alpha\rceil-\alpha}\phi_{i,m}(x) \in C[0,a]$ for each $m \in \mathbb{N}$, then so is $x^{\lceil\alpha\rceil-\alpha}\phi_i(x)$, $i = 1,\dots,n$. The uniform convergence of the sequences $\{\phi_{i,m}(x)\}$ on $(0,h]$ allows us to let $m \to \infty$ in the relationship (8.20), which gives for $x \in (0,h]$

$$\phi_i(x) = \sum_{k=1}^{\lceil\alpha\rceil}\frac{b_{ik}}{\Gamma_q(\alpha-k+1)}x^{\alpha-k} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \left(qt/x;q\right)_{\alpha-1}f_i\left(t,\phi_1(t),\dots,\phi_n(t)\right)d_qt.$$

Therefore, we obtain

$$D_q^{\alpha-r}\phi_i(x) = \sum_{k=1}^{r}\frac{b_{ik}}{\Gamma_q(\alpha-k+1)}x^{r-k} + I_q^r f_i(x,y_1(x),\dots,y_n(x)).$$

But from Lemma 4.9 we obtain

$$|I_q^r f_i(x)| \le \frac{Mx^r}{\Gamma_q(r+1)} \quad (i = 1,2,\dots,n).$$

Hence,

$$\lim_{x\to 0+} D_q^{\alpha-r}\phi_i(x) = b_{ir} \quad (i = 1, 2, \ldots, n;\; r = 1, 2, \ldots, \lceil\alpha\rceil).$$

Uniqueness. To prove the uniqueness we assume that $\{\psi_i\}_{i=1}^n$ is another solution valid in an interval $(0, b]$, $b \leqslant h$. Set

$$\chi_i(x) := \phi_i(x) - \psi_i(x),$$

and

$$g_i(x) := f_i\left(x, \phi_1(x), \ldots, \phi_n(x)\right) - f_i\left(x, \psi_1(x), \ldots, \psi_n(x)\right),$$

for $i = 1, 2, \ldots, n$ and for any $x \in (0, h]$. Hence,

$$D_q^\alpha \chi_i(x) = g_i(x) \quad (i = 1, \ldots, n).$$

From (4.67) we obtain

$$\chi_i(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x; q\right)_{\alpha-1} g_i(t)\, d_q t, \quad i = 1, \ldots, n. \tag{8.25}$$

Now $x^{1-\alpha}\chi_i$ is continuous at zero for $i = 1, 2, \ldots, n$. Then, for each $x \in (qb, b]$ there exists a constant $C_x > 0$ such that

$$t^{1-\alpha}|\chi_i(t)| \leqslant \frac{C_x}{\Gamma_q(\alpha)} \quad \text{for } t \in \{xq^m,\ m \in \mathbb{N}_0\}.$$

Fix $x \in (qh, h]$ and $t \in \{xq^m,\ m \in \mathbb{N}\}$. Hence, from (8.25) we obtain

$$\begin{aligned}
|\chi_i(t)| &\leqslant \frac{At^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{i=1}^n \int_0^t \left(qu/t; q\right)_{\alpha-1} \chi_i(u)\, d_q u \\
&\leqslant \frac{AnC_x t^{\alpha-1}}{\Gamma_q^2(\alpha)} \int_0^t u^{\alpha-1}\left(qu/t; q\right)_{\alpha-1} d_q u \\
&= \frac{AnC_x t^{2\alpha-1}}{\Gamma_q^2(\alpha)} B_q(\alpha, \alpha) = \frac{BC_x}{\Gamma_q(2\alpha)} t^{2\alpha-1}.
\end{aligned}$$

Repeating the previous process k times, $k \in \mathbb{N}$, we obtain

$$|\chi_i(t)| \leqslant C_x \frac{B^{k+1} t^{\alpha k+\alpha-1}}{\Gamma_q(\alpha k + \alpha)} \quad (i = 1, 2, \ldots, n). \tag{8.26}$$

Since $\dfrac{B^k t^{\alpha k}}{\Gamma_q(\alpha k+\alpha)}$ is the general term of the series of $e_{\alpha,\alpha}(\beta t^\alpha; q)$ and

$$\beta t^\alpha \leqslant \beta a^\alpha < (1-q)^{-\alpha},$$

we have $\lim\limits_{k\to\infty} \frac{B^k t^{\alpha k}}{\Gamma_q(\alpha k+\alpha)} = 0$. Therefore, $\chi_i(t) = 0$ for all $t \in \{xq^m,\ m \in \mathbb{N}_0\}$ and for all $x \in (qb, b]$. That is $\chi(x) = 0$ for all $x \in (0, b]$, proving the uniqueness.

□

Now we investigate the initial value problem (8.15)–(8.16). We state the existence and uniqueness theorem. We give a brief account on the proof by mentioning the main differences from the previous proof.

Theorem 8.9. *Let $f_i(x, y_1, \dots, y_n)$ be functions defined for $x \in (0, a]$ and y_i in the domain G_i, satisfying the following conditions.*

(i) There is a positive constant A such that for $x \in (0, a]$ and $y_r, \tilde{y}_i \in G_i$, $1 \leqslant r, i \leqslant n$, the following Lipschitz condition is fulfilled

$$\left| f_i(x, \tilde{y}_1, \dots, \tilde{y}_n) - f_i(x, y_1, \dots, y_n) \right| \leqslant A\left(|\tilde{y}_1 - y_1| + \dots + |\tilde{y}_n - y_n|\right).$$

(ii) There exists $\gamma < 1$ and $\gamma \le \alpha$ such that

$$f_i(x, y_1, \dots, y_n) \in C_\gamma[0, a] \quad \textit{for any} \quad y_i \in G_i;\ i = 1, \dots, n. \tag{8.27}$$

Let K be a constant that satisfies

$$K \geqslant \frac{M a^{\alpha+w} \Gamma_q(w+1)}{\Gamma_q(\alpha+w+1)},$$

where

$$M := \max\left\{|x^\gamma f_i(x, y_1, y_2, \dots, y_n)|,\ x \in [0, a],\ y_i \in D_i(a, K),\ i = 1, 2, \dots, n\right\},$$

and $D_i(a, K) \subset G_i$ is the set of points $y_i \in G_i$ that satisfies

$$\left| y_i - \sum_{k=1}^{\lceil \alpha \rceil} b_{ik} \frac{x^{\alpha-k}}{\Gamma_q(\alpha-k+1)} \right| \le K \quad \textit{for all} \quad x \in (0, a].$$

Then the initial value problem (8.15)–(8.16) has a unique solution $\{y_i(\cdot)\}_{i=1}^n$ valid in $(0, a]$, $y_i \in C_{\lceil \alpha \rceil - \alpha}[0, a]$ for $i = 1, 2, \dots, n$.

Proof. Consider the system of functions

$$\{\phi_{i,m}(x)\}_{m=1}^{\infty} \quad (i = 1, \dots, n;\ x \in (0, a])$$

defined by

$$\begin{aligned} \phi_{i,1}(x) &= \sum_{k=1}^{\lceil\alpha\rceil} \frac{b_{ik}\, x^{\alpha-k}}{\Gamma_q(\alpha-k+1)}, \\ \phi_{i,m}(x) &= \sum_{k=1}^{\lceil\alpha\rceil} \frac{b_{ik} x^{\alpha-k}}{\Gamma_q(\alpha-k+1)} \\ &\quad + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x; q\big)_{\alpha-1} f_i\big(qt, \phi_{1,m-1}(qt), \dots, \phi_{n,m-1}(qt)\big)\, d_q t, \end{aligned} \tag{8.28}$$

where $m \geqslant 2$. We prove that $\lim\limits_{m\to\infty} \phi_{i,m}(x)$ exists and give the required solution $\{\phi_i(\cdot)\}_{i=1}^n$ of the initial value problem (8.15)–(8.16) in $(0, a]$. As in the proof of Theorem 8.8, we can prove by induction on m that $\phi_{i,m}(x) \in D_i(a, k)$, $m \in \mathbb{N}$ for all $x \in (0, a]$ and

$$|\phi_{i,m+1}(x) - \phi_{i,m}(x)| \leqslant \frac{M B^{m-1} q^{\frac{m(m-1)\alpha}{2} - (m-1)\nu} x^{m\alpha-\gamma} \Gamma_q(1-\nu)}{\Gamma_q(m\alpha - \gamma + 1)}, \tag{8.29}$$

where $B := An$. According to the estimate (8.29), for $x \in (0, a]$, the absolute value of the terms of the series

$$\phi_{i,1}(x) + \sum_{m=1}^{\infty} \phi_{i,m+1}(x) - \phi_{i,m}(x) \tag{8.30}$$

is less than the corresponding terms of the convergent numeric series

$$\frac{M}{B} q^{\gamma} \Gamma_q(1-\gamma) \sum_{m=0}^{\infty} \frac{B^m q^{\frac{m(m-1)\alpha}{2} - m\gamma} a^{m\alpha-\gamma}}{\Gamma_q(m\alpha-\gamma+1)} = \frac{M}{B} \left(\frac{q}{a}\right)^{\gamma} \Gamma_q(1-\gamma) E_{\alpha,1-\gamma}(q^{-\gamma} B a^{\alpha}; q).$$

Therefore, the series (8.30) is uniformly convergent on $(0, a]$ to a function ϕ_i, where $\phi_i(x) = \lim\limits_{m\to\infty} \phi_{i,m}(x)$. Since $x^{\lceil\alpha\rceil-\alpha}\phi_{i,m} \in C[0, a]$ for all $m \in \mathbb{N}$ and $i = 1, \dots, n$, then so is $x^{\lceil\alpha\rceil-\alpha}\phi_i$. In addition, $\phi_{i,m}(x) \in D_i(a, k)$ implies that $\phi_i(x) \in D_i(a, k)$, $x \in (0, a]$. The uniform convergence of the sequences $\{\phi_{i,m}(x)\}$, $x \in (0, a]$, allows us to let $m \to \infty$ in the relationship (8.28), this gives that

$$\phi_i(x) = \sum_{k=1}^{\lceil\alpha\rceil} \frac{b_{ik}}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x; q\big)_{\alpha-1} f_i(t, \phi_1(qt), \dots, \phi_n(qt))\, d_q t.$$

Consequently,

$$\begin{aligned} D_q^{\alpha-k}\phi_i(x) &= b_{ik} + D_q^{\alpha-k} I_q^{\alpha} f_i(x, y_1(qx), \ldots, y_n(qx)) \\ &= b_{ik} + I_q^{k} f_i(x, y_1(qx), \ldots, y_n(qx)). \end{aligned}$$

Therefore,

$$D_q^{1-\alpha}\phi_i(0^+) = b_{ik} \quad (i = 1, 2, \ldots, n),$$

i.e. $\{\phi_i(\cdot)\}_{i=1}^n$ satisfies the initial conditions (8.16). The proof of the uniqueness of the solution is similar to that of Theorem 8.8 so it is omitted. □

Remark 8.2.1. When $\alpha = 1$, the initial value problems (8.14), (8.16) and equations (8.15)– (8.16) are reduced to the initial value problems (2.7), (2.13) and (2.8), (2.13), respectively. Therefore, the results of Sect. 8.2 generalize the ones of Sect. 2.3.

Example 8.2.1. Consider the q-fractional first order system

$$\begin{aligned} D_q^{\alpha} y_1(x) &= y_2(x), & D_q^{\alpha} y_2(x) &= -y_1(x), \\ I_q^{1-\alpha} y_1(0^+) &= 0, & I_q^{1-\alpha} y_2(0^+) &= 1, \end{aligned}$$

where $x \in \mathbb{R}$ and $0 < \alpha < 1$. Substituting into (8.20) with $b_1 = 0$ and $b_2 = 1$, we get

$$\begin{aligned} &\phi_{1,1}(x) = 0, \\ &\phi_{1,2m}(x) = \phi_{1,2m+1}(x) = x^{2\alpha-1}\sum_{j=0}^{m-1}(-1)^j \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + 2\alpha)},\ m \in \mathbb{N}, \end{aligned}$$

and

$$\phi_{2,2m-1}(x) = \phi_{2,2m}(x) = x^{\alpha-1}\sum_{j=0}^{m-1}(-1)^j \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + \alpha)},\ m \in \mathbb{N}.$$

One can verify that the functions $\phi_i(x) := \lim_{m\to\infty} \phi_{i,m}(x)$, $i = 1, 2$, are defined only for $|x| < (1-q)^{-1}$ by

$$\phi_1(x) = x^{2\alpha-1} e_{2\alpha,2\alpha}(-x^{2\alpha}; q),\ \phi_2(x) = x^{\alpha-1} e_{2\alpha,\alpha}(-x^{2\alpha}; q). \tag{8.31}$$

If we take $\alpha = 1$ in (8.31), we get

$$\phi_1(x) = \sin_q x(1-q),\ \text{and}\ \phi_2(x) = \cos_q x(1-q),$$

which is the solution of the q-initial value problem

$$\begin{aligned} D_q y_1(x) &= y_2(x), & D_q y_2(x) &= -y_1(x), \\ y_1(0^+) &= 0, & y_2(0^+) &= 1. \end{aligned}$$

Example 8.2.2. Using the successive approximation method, one can prove that the successive functions $\{\phi_{i,m}(x)\}_{m=1}^{\infty}$, $i = 1, 2$, corresponding to the initial value problem

$$\begin{aligned} D_q^{\alpha} y_1(x) &= y_2(x) & D_q^{\alpha} y_2(x) &= -q^{1-2\alpha} y_1(qx) \\ I_q^{1-\alpha} y_1(0^+) &= 0 & I_q^{1-\alpha} y_2(0^+) &= 1 \end{aligned}$$

are

$$\begin{aligned} \phi_{1,1}(x) &= 0, \\ \phi_{1,2m+1}(x) &= \phi_{1,2m}(x) = x^{2\alpha-1} \sum_{j=0}^{m-1} (-1)^j q^{j(j-1)\alpha} \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + 2\alpha)}, \\ \phi_{2,2m-1}(x) &= \phi_{2,2m}(x) = x^{\alpha-1} \sum_{j=0}^{m-1} (-1)^j q^{j(j-1)\alpha} \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + \alpha)}, \end{aligned}$$

$m \in \mathbb{N}$. Hence, the functions

$$\phi_i(x) := \lim_{m\to\infty} \phi_{i,m}(x) \quad (i = 1, 2)$$

are defined on $\mathbb{R}$ by

$$\phi_1(x) = x^{2\alpha-1} E_{2\alpha,2\alpha}(-x^{2\alpha}; q), \quad \phi_2(x) = x^{\alpha-1} E_{2\alpha,\alpha}(-x^{2\alpha}; q).$$

If we set $\alpha = 1$, we obtain

$$\phi_1(x) = q^{-1} \sin(q^{-1}x; q), \quad \text{and} \quad \phi_2(x) = \cos(q^{-1/2}x; q), \; x \in \mathbb{R},$$

which is the solution of the q-initial value problem

$$\begin{aligned} D_q y_1(x) &= y_2(x), & D_q y_2(x) &= -q^{-1} y_1(qx), \\ y_1(0) &= 0, & y_2(0) &= 1, \end{aligned}$$

valid on $\mathbb{R}$.

8.3 Equations with Caputo Fractional q-Derivatives

In this section we shall study the existence and uniqueness of solutions of the Cauchy type problem

$$ {}^cD_q^\alpha y(x) = f(x, y(x)) \quad (\alpha > 0), \tag{8.32}$$

$$ D_q^k y(0^+) = b_k \quad (b_k \in \mathbb{R};\ k = 0, 1, \ldots, \lceil \alpha \rceil - 1)\,. \tag{8.33}$$

The result of this section is an extension of the results derived by Kilbas et al. in [169, Sect. 3.5]. In the following, we prove the existence and uniqueness of solutions in the space $C_q^n[0,a]$,

Lemma 8.10. *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. If there exists $\gamma \le \alpha - n + 1$ such that $f \in C_\gamma[0,a]$ then $I_q^\alpha f \in C_q^n[0,a]$.*

Proof. From Lemma 4.10, $I_q^{\alpha-k} f \in C[0,a]$ for $k = 0, 1, \ldots, n-1$. Since

$$ D_q^k I_q^\alpha f = I_q^{\alpha-k} f(x) \quad (k = 0, 1, \ldots, \lceil \alpha \rceil - 1). $$

Hence $D_q^k I_q^\alpha f \in C[0,a]$ and $f \in C_q^n[0,a]$. ☐

Theorem 8.11. *Let $\alpha > 0$, $n = \lceil \alpha \rceil$. Let G be an open set in $\mathbb{C}$ and $f : (0,a] \times G \to \mathbb{R}$ be a function such that $f(x,y) \in C_\gamma[0,a]$ for any $y \in G$, $\gamma \le \alpha - n + 1$. If $y \in C_q^n[0,a]$ then $y(x)$ satisfies (8.32)–(8.33) for all $x \in (0,a]$ if and only if $y(x)$ satisfies the q-integral equation*

$$ y(x) = \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} f(t, y(t))\, d_qt, \tag{8.34}$$

for all $x \in [0,a]$.

Proof. First, we prove the necessity condition. Let $y(x)$ satisfy (8.32)–(8.33), then ${}^cD_q^\alpha y(x) \in C_\gamma[0,a]$. Hence, applying (5.6) (with $\beta = \alpha$) gives

$$ I_q^{\alpha\,c} D_q^\alpha y(x) = y(x) - \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k \quad (0 \le x \le a). \tag{8.35}$$

On the other hand,

$$ I_q^{\alpha\,c} D_q^\alpha y(x) = I_q^\alpha f(x, y(x)) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} f(t, y(t))\, d_qt \tag{8.36}$$

for all $x \in [0,a]$. Consequently, combining (8.35) and (8.36) proves the necessity condition. Now we move to prove the sufficiency. Let $y(x)$ satisfy (8.34) for all

$x \in [0,a]$, then from Lemma 8.10, $y(x) \in C_q^n[0,a]$. Consequently, acting on the two sides of (8.34) by the operator ${}^cD_q^\alpha$ and applying (5.8) give

$$ {}^cD_q^\alpha y(x) = D_q^\alpha \left[y(x) - \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k \right] = D_q^\alpha I_q^\alpha f(x,y(x)) = f(x,y(x)) $$

for all $x \in [0,a]$. For $x \in (0,a]$ and $k \in \{0,1,\ldots,n-1\}$ we obtain

$$ \begin{aligned} D_q^k y(0^+) &= \lim_{j\to\infty} D_q^k y(xq^j) = b_k + \lim_{j\to\infty} I_q^{\alpha-k} f(xq^j, y(xq^j)) \\ &= b_k + \lim_{j\to\infty} \frac{(xq^j)^{\alpha-k-1}}{\Gamma_q(\alpha-k)} \int_0^{xq^j} (qt/x;q)_{\alpha-k-1} f(t,y(t))\, d_qt \qquad (8.37) \\ &= b_k + \lim_{j=\infty} \frac{(xq^j)^{\alpha-k}}{\Gamma_q(\alpha-k)} \sum_{r=j}^{\infty} q^r \frac{(q^{\alpha-k};q)_r}{(q;q)_r} f(xq^r, y(xq^r)), \end{aligned} $$

where the limit on the most left hand side of (8.37) vanishes because of $f(x,y(x)) \in C_\gamma[0,a]$. □

Theorem 8.12. *Let $\alpha > 0$, $n = \lceil \alpha \rceil$ and G be an open set in $\mathbb{C}$. Let*

$$ f : (0,a] \times G \longrightarrow R $$

be a function satisfying the following conditions:

1. *There exists γ, $\gamma \le \alpha - n + 1$, such that $f(x,y) \in C_\gamma[0,a]$ for any $y \in G$.*
2. *There exists a constant $A > 0$ such that for all $x \in (0,a]$ and for all $y_1, y_2 \in G$, the following Lipschitz condition is satisfied:*

$$ |f(x,y_1(x)) - f(x,y_2(x))| \le A|y_1(x) - y_2(x)|. $$

If $0 < h \le a$ satisfies the condition

$$ w := A \sum_{k=0}^{n-1} \frac{h^{\alpha-k}\Gamma_q(1-\gamma)}{\Gamma_q(\alpha-k-\gamma+1)} < 1, $$

then the fractional Cauchy problem (8.32)–(8.33) has a unique solution in $C_q^n[0,a]$.

Proof. From Theorem 8.11, the Cauchy type problem (8.32)–(8.33) is equivalent to the Volterra q-integral equation (8.34). Hence, a solution of (8.32)–(8.33) in $C_q^n[0,a]$ is a fixed point of the operator $T : C_q^n[0,a] \longrightarrow C_q^n[0,a]$ defined by

$$ Ty(x) = \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t,y(t))\, d_qt. $$

To prove the theorem, we prove

1. If $y \in C_q^n[0,a]$, then $Ty \in C_q^n[0,a]$
2. T is a contraction mapping.

The proof of (1) follows from Lemma 8.10, and from (4.53) we obtain

$$\begin{aligned}\|Ty_1 - Ty_2\|_1 &= \sum_{k=0}^{n-1} \max_{0\le t\le h} \left| I_q^{\alpha-k}\left(f(t,y_1(t)) - f(t,y_2(t))\right)\right| \\ &\le A \sum_{k=0}^{n-1} \max_{t\in[0,h]} I_q^{\alpha-k}(y_1-y_2) \le w\,\|y_1-y_2\|_1 .\end{aligned}$$

Consequently, from the Banach fixed point theorem, there exists a unique $y^* \in C_q^n[0,a]$ such that $Ty^* = y^*$. So, the result follows by applying the sufficient part of Theorem 8.11. □

By Theorem 1.7, the solution y^* is obtained as a limit of a convergent sequence $(T^m y_0)(x)$:

$$\lim_{m\to\infty} \|T^m y_0 - y^*\| = 0,$$

in the space $C_q^n[0,a]$, where y_0 is any function in $C_q^n[0,a]$. If at least one $b_k \neq 0$ in the initial condition (8.33), we can take

$$y_0(x) := \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k .$$

Consequently, the sequence $T^m y_0$ is defined by the recurrence relation

$$T^m y_0(x) = y_0(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, T^{m-1}y_0(t))\, d_q t \quad (m \in \mathbb{N}).$$

If we denote $y_m(x) = (T^m y_0)(x)$ then the last relation takes the form

$$y_m(x) = y_0(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y_{m-1}(t))\, d_q t \quad (m \in \mathbb{N}).$$

This means that the successive approximation method can be used to find a unique solution of (8.32)–(8.33) whenever $b_k \neq 0$ for some $k \in \{0,1,\ldots,n-1\}$.

Example 8.3.1. Consider the Cauchy type q- fractional equation:

$$\begin{aligned} {}^cD_q^\alpha y(x) &= \lambda x^\beta y^m(x) \quad (m>1;\ 0<\alpha<1;\ x>0;\ \lambda,\ \beta\in\mathbb{R};\ \lambda\neq 0)\,, \\ y(0^+) &= 0. \end{aligned} \tag{8.38}$$

Set

$$f(x, y(x)) = \lambda x^{\beta} y^{m}(x) \quad (x > 0),$$

where

$$(x, y) \in (0, a] \times G := \{0 < x \leq a;\ 0 < |y| \leq K\, x^{w};\ K > 0\}.$$

Then, similar to Example 8.1.2, we obtain

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|, \quad A := m|\lambda| K^{m-1} a^{\beta+(m-1)w},$$

whenever $\beta+(m-1)w \geq 0$. That is, f satisfies the Lipschitz condition. If $f(x, y) \in C_{\gamma}[0, a]$ then $x^{\gamma} f(x, y)$ is continuous and hence bounded on $[0, a]$. Therefore we should assume that $\gamma + \beta + mw \geq 0$. Hence the conditions of Theorem 8.12 is satisfied and the Cauchy type q-fractional problem (8.38) has a unique solution in $C[0, h]$, where h is a positive number satisfying the inequality

$$m|\lambda| K^{m-1} \frac{h^{\beta+(m-1)w+\alpha}}{\Gamma_q(\alpha + 1)} < 1.$$

Hence w should satisfy the conditions

$$w + (m - 1)\beta \geq 0, \quad \gamma + mw + \beta \geq 0. \tag{8.39}$$

Assume that $(m-1)w = r - (\beta + \alpha)$. Hence $(m-1)w + \beta = r - \alpha$ and we should assume that $r \geq \alpha$. In addition to this the condition $\gamma + mw + \beta \geq 0$ implies

$$\frac{m}{m-1} r + \gamma - \frac{\beta + (m)\alpha}{m-1} \geq 0.$$

That is

$$\frac{m}{m-1} r \geq -\gamma + \frac{m\alpha + \beta}{m-1}.$$

So if we assume that $\gamma \leq \frac{\beta+m\alpha}{m-1}$, then the conditions in (8.39) are satisfied and the Cauchy fractional initial problem has a unique solution in $C_{\gamma}[0, h]$. One can verify that this solution is given by

$$y(x) = \left[\frac{\Gamma_q(1 - \frac{\alpha+\beta}{m-1})}{\lambda \Gamma_q(1 - \frac{m\alpha+\beta}{m-1})}\right]^{\frac{1}{m-1}} x^{-\frac{\alpha+\beta}{m-1}}.$$

8.4 q-Caputo Fractional Order Systems

This section deals with two systems of q-Caputo fractional derivatives, namely

$$ {}^{c}D_q^{\alpha} y_i(x) = f_i(x, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n), \tag{8.40}$$

and

$$ {}^{c}D_q^{\alpha} y_i(x) = f_i(qx, y_1(qx), \dots, y_n(qx)) \quad (i = 1, 2, \dots, n), \tag{8.41}$$

where $\alpha > 0$. We call system (8.40) together with the initial conditions of the form

$$D_q^j y_i(0) = b_{ij} \quad \big(b_{ij} \in \mathbb{R};\ i = 1, 2, \dots, n;\ j = 0, 1, \dots, \lceil\alpha\rceil - 1\big), \tag{8.42}$$

an initial value problem. The same applies to system (8.41) with the initial conditions (8.42).

By a solution of the initial value problem (8.40), (8.42) we mean a set of functions $\{y_i\}_{i=1}^n$ that satisfies (8.40) and the initial conditions (8.42) in an interval I containing zero such that the functions $y_i(x), i = 1, 2, \dots, n$ are continuous on I. Similarly, we can define the solution of the initial value problem (8.41)–(8.42).

In the following theorem we investigate the conditions under which the initial value problem (8.40), (8.42) has a unique solution.

Theorem 8.13. *Let $f_i(x, y_1, \dots, y_n)$ be functions defined for $x \in (0, a]$, and y_i in the domain G_i, satisfying the following conditions.*

(i) There is a positive constant A such that for $x \in (0, a]$ and $y_i, \tilde{y}_i \in G_i$, $1 \leqslant i \leqslant n$, the following Lipschitz' condition is fulfilled

$$|f_i(x, y_1, \dots, y_n) - f_i(x, \tilde{y}_1, \dots, \tilde{y}_n)| \leqslant A\Big(|y_1 - \tilde{y}_1| + \dots + |y_n - \tilde{y}_n|\Big).$$

(ii) There exists $\gamma \le \alpha - \lceil\alpha\rceil + 1$ such that

$$f_i(x, y_1, \dots, y_n) \in C_\gamma[0, a] \ \text{ for any } y_i \in G_i;\ i = 1, 2, \dots, n.$$

Let

$$M := \max\{|x^\gamma f_i(x, y_1, y_2, \dots, y_n)|,\ x \in [0, a],\ y_i \in D_i(a, K),\ i = 1, 2, \dots, n\},$$

and $D_i(a, K) \subset G_i$ be the set of points $y_i \in G_i$ satisfying

$$\left| y_i(x) - \sum_{j=0}^{\lceil\alpha\rceil - 1} b_{ij} \frac{x^j}{\Gamma_q(j+1)} \right| \le K \ \text{ for all } \ x \in [0, a].$$

Then there exists $h \in (0, a]$,

$$\frac{M h^{\alpha} \Gamma_q(1-\gamma)}{\Gamma_q(\alpha-\gamma+1)} \le K$$

such that the initial value problem (8.40), (8.42) *has a unique solution* $\{y_i(x)\}_{i=1}^{n}$ *in* $C_q^{\lceil\alpha\rceil}[0, h]$.

Proof. Existence. Define the iterations

$$\{\phi_{i,m}(x)\}_{m=1}^{\infty} \quad (1 \le i \le n, \quad x \in (0, a])$$

by the following relationships

$$\begin{aligned}
\phi_{i,1}(x) &= \sum_{j=0}^{\lceil\alpha\rceil-1} b_{ij} \frac{x^j}{\Gamma_q(j+1)}, \\
\phi_{i,m}(x) &= \sum_{j=0}^{\lceil\alpha\rceil-1} b_{ij} \frac{x^j}{\Gamma_q(j+1)} \\
&\quad + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} f_i\left(t, \phi_{i,m-1}(t), \ldots, \phi_{n,m-1}(t)\right) d_q t,
\end{aligned} \tag{8.43}$$

where $m \geqslant 2$. We show that $\lim_{m\to\infty} \phi_{i,m}(x)$ exists and gives the required solution $\{\phi_i(x)\}_{i=1}^{n}$ of the initial value problem (8.40), (8.42) in subinterval $(0, h]$ of $(0, a]$. Similar to (8.21), one can prove that

$$\phi_{i,m}(x) \in D_i(a, k) \quad (m \in \mathbb{N}; i = 1, 2, \ldots, n; \; x \in (0, a]) .$$

Applying Lemma 8.10, we conclude that $\phi_{i,m}(x) \in C_q^{\lceil\alpha\rceil}[0, a]$ for $i = 1, 2, \ldots, n$ and $m \in \mathbb{N}$. We can also prove by induction on m that for $r = 0, 1, \ldots, \lceil\alpha\rceil - 1$

$$\left| D_q^r \phi_{i,m+1}(x) - D_q^r \phi_{i,m}(x) \right| \leqslant \frac{M B^{m-1} x^{m(\alpha-r)-\gamma} \Gamma_q(1-\gamma)}{\Gamma_q(m(\alpha-r)-\gamma+1)}, \tag{8.44}$$

where $B := An$ and $m \in \mathbb{N}_0$. Now, consider the infinite series

$$D_q^r \phi_{i,1}(x) + \sum_{m=1}^{\infty} D_q^r \phi_{i,m+1}(x) - D_q^r \phi_{i,m}(x). \tag{8.45}$$

Let $h = \min\left\{a, \frac{1}{B^{1/\alpha}(1-q)}\right\}$. According to the estimate (8.44), for $0 \leqslant x \leqslant h$, the absolute value of the terms of (8.45) is less than the corresponding terms of the convergent numerical series

$$\frac{Mh^{-\gamma}}{B}\Gamma_q(1-\gamma)\sum_{m=0}^{\infty}\frac{B^m h^{m(\alpha-r)}}{\Gamma_q(m(\alpha-r)-\gamma+1)}=\frac{Mh^{-\gamma}}{B}\Gamma_q(1-\gamma)e_{\alpha-r,1-\gamma}(Bh^{\alpha-r};q).$$

This means that the series (8.45) is uniformly convergent on $[0,h]$ to a function $g_{r,i}(x)$, and $g_{r,i}(x)=\lim_{m\to\infty}D_q^r\phi_{i,m}(x)$. Since $D_q^r\phi_{i,m}$ is continuous for all $m\in\mathbb{N}$, $i=1,\dots,n$, then so is $g_{r,i}$. One can verify that

$$g_{r,i}(x)=D_q^r\phi_i(x),\quad \phi_i(x):=g_{0,i}(x).$$

Hence,

$$\phi_i\in C_q^{\lceil\alpha\rceil}[0,a]\quad (i=1,2,\dots,n)\,.$$

In addition, $\phi_{i,m}\in D_i(a,k)$ implies that $\phi_i(x)\in D_i(a,k)$. The uniform convergence of the sequences $\{\phi_{i,m}\}$ allows us to let $m\to\infty$ in identity (8.43), which gives

$$\phi_i(x)=\sum_{j=0}^{\lceil\alpha\rceil-1}b_{ij}\frac{x^j}{\Gamma_q(j+1)}+\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x\left(qt/x;q\right)_{\alpha-1}f_i\left(t,\phi_1(t),\dots,\phi_n(t)\right)\,d_qt.$$

Obviously, the initial conditions (8.42) holds. The proof of the uniqueness of the solution is similar to of that of Theorem 8.8 and is omitted. □

Similar to Theorem 8.13 we have the following theorem

Theorem 8.14. *Let $f_i(x,y_1,\dots,y_n)$ be functions defined for $x\in[0,a]$ and y_i in the domain G_i, satisfying the following conditions.*

(i) There is a positive constant A such that

$$|f_i(x,y_1,\dots,y_n)-f_i(x,\tilde{y}_1,\dots,\tilde{y}_n)|\leqslant A\left(|y_1-\tilde{y}_1|+\dots+|y_n-\tilde{y}_n|\right),$$

for all $x\in(0,a]$ and for any $y_r,\tilde{y}_i\in G_i$, $1\leqslant r,i\leqslant n$.

(ii) There exists $\gamma<\alpha-\lceil\alpha\rceil+1$ such that

$$f_i(x,y_1,\dots,y_n)\in C_\gamma[0,a]\ \textit{for any}\ y_i\in G_i,\ i=1,2,\dots,n.$$

Let

$$M:=\max\left\{|x^\gamma f_i(x,y_1,y_2,\dots,y_n)|\quad (x\in[0,a];\ y_i\in D_i(a,K);\ i=1,2,\dots,n)\right\},$$

where $D_i(a,K)\subset G_i$ is the set of points $y_i\in G_i$ satisfying

$$\left|y_i(x)-\sum_{j=0}^{\lceil\alpha\rceil-1}b_{ij}\frac{x^j}{\Gamma_q(j+1)}\right|\le K\ \textit{for all}\ x\in(0,a],$$

and

$$K \geq \frac{Ma^{\alpha}\Gamma_q(1-\gamma)}{\Gamma_q(\alpha-\gamma+1)}.$$

Then, the initial value problem (8.41)– (8.42) *has a unique solution* $\{y_i\}_{i=1}^{n}$ *in* $C_q^{\lceil\alpha\rceil}[0,a]$.

Proof. Consider the system of successive functions $\{\phi_{i,m}\}_{m=1}^{\infty}$, $i = 1,\dots,n$,

$$\phi_{i,1}(x) = \sum_{j=0}^{\lceil\alpha\rceil-1} b_{ij}\frac{x^j}{\Gamma_q(j+1)}$$

and for $m \in \{2,3,\dots\}$

$$\begin{aligned}\phi_{i,m}(x) &= \textstyle\sum_{j=0}^{\lceil\alpha\rceil-1} b_{ij}\frac{x^j}{\Gamma_q(j+1)}\\ &\quad+\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \left(qt/x;q\right)_{\alpha-1} f_i\left(qt,\phi_{1,m-1}(qt),\dots,\phi_{n,m-1}(qt)\right)\, d_qt.\end{aligned} \tag{8.46}$$

We show that $\lim_{m\to\infty}\phi_{i,m}(x)$ exists and gives the required solution $\{\phi_i\}_{i=1}^{n}$ of the initial value problem (8.41)–(8.42). As in the proof of Theorem 8.13, we can prove by induction on m that $\phi_{i,m}(x) \in D_i(a,k)$ for $i = 1,\dots,n$, $m \in \mathbb{N}$, $x \in [0,a]$, and

$$|\phi_{i,m+1}(x)-\phi_{i,m}(x)| \leqslant \frac{MB^{m-1}q^{m(m-1)\alpha/2}x^{m\alpha}}{\Gamma_q(m\alpha+1)}, \tag{8.47}$$

where $B := An$ and $m \in \mathbb{N}$. According to the estimate (8.47), for $0 \leqslant x \leqslant a$, the absolute values of the terms of the series

$$\phi_{i,1}(x) + \sum_{m=1}^{\infty}\phi_{i,m+1}(x)-\phi_{i,m}(x) \tag{8.48}$$

are less than the corresponding terms of the convergent series

$$\frac{M}{B}\sum_{m=0}^{\infty}\frac{q^{m(m-1)\alpha/2}(Ba^{\alpha})^m}{\Gamma_q(m\alpha+1)} = \frac{M}{B}E_{\alpha,1}(Ba^{\alpha};q).$$

This means that the series (8.48) is uniformly convergent on $(0,a]$ to a function ϕ_i, where $\phi_i(x) = \lim_{m\to\infty}\phi_{i,m}(x)$. Since $\phi_{i,m} \in \mathscr{L}_q^1[0,a]$, for all $m \in \mathbb{N}$, $i = 1,\dots,n$, then so is ϕ_i. In addition, $\phi_{i,m}(x) \in D_i(a,k)$ implies that $\phi_i(x) \in D_i(a,k)$, $x \in [0,a]$. The uniform convergence of the sequences $\{\phi_{i,m}(x)\}_{m=1}^{\infty}$, $x \in (0,a]$, allows us to let $m \to \infty$ in the relationship (8.46), this gives that

$$\phi_i(x) = \sum_{j=0}^{n-1} b_{ij} \frac{x^j}{\Gamma_q(j+1)} + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x;q\right)_{\alpha-1} f_i\left(t, \phi_1(t), \ldots, \phi_n(t)\right) d_q t,$$

which are the required solutions. Uniqueness can be proved as in Theorem 8.8. □

Example 8.4.1. Consider the first order system

$$\begin{aligned} &{}^cD_q^\alpha y_1(x) = y_2(x), && {}^cD_q^\alpha y_2(x) = -y_1(x) \quad (0 < \alpha < 1),\\ &y_1(0^+) = 0, && y_2(0^+) = 1. \end{aligned}$$

Using the method of successive approximations, the functions $\{\phi_{i,m}(\cdot)\}_{m=1}^\infty$ are given by

$$\begin{aligned} \phi_{1,1}(x) &= 0,\\ \phi_{1,2m}(x) = \phi_{1,2m+1}(x) &= x^\alpha \sum_{j=0}^{m-1} (-1)^j \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + \alpha + 1)},\\ \phi_{2,2m-1}(x) = \phi_{2,2m}(x) &= \sum_{j=0}^{m-1} (-1)^j \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + 1)}, \end{aligned}$$

where $m \in \mathbb{N}$. One can see that the limits

$$\phi_i(x) := \lim_{m\to\infty} \phi_{i,m}(x) \quad (i = 1, 2)$$

exist only for $|x(1-q)| < 1$ and are defined by

$$\phi_1(x) = x^\alpha e_{2\alpha,\alpha+1}(-x^{2\alpha};q), \quad \phi_2(x) = e_{2\alpha,1}(-x^{2\alpha};q). \tag{8.49}$$

Substituting $\alpha = 1$ into (8.49), we obtain

$$y_1(x) = \sin_q x(1-q),\ y_2(x) = \cos_q x(1-q),\ |x(1-q)| < 1,$$

forming the solution of the initial value problem

$$\begin{aligned} &D_q y_1(x) = y_2(x), && D_q y_2(x) = -y_1(x),\\ &y_1(0) = 0, && y_2(0) = 1. \end{aligned}$$

Example 8.4.2. Consider the first order system

$$\begin{aligned} &{}^cD_q^\alpha y_1(x) = y_2(x), && {}^cD_q^\alpha y_2(x) = -q^{-\alpha} y_1(qx),\\ &y_1(0^+) = 0, && y_2(0^+) = 1. \end{aligned} \tag{8.50}$$

The successive approximating functions $\{\phi_{i,m}(x)\}_{m=1}^{\infty}$, $i = 1, 2$, are defined for $m \in \mathbb{N}$ by

$$\begin{aligned}
\phi_{1,1}(x) &= 0,\\
\phi_{1,2m}(x) = \phi_{1,2m+1}(x) &= x^{\alpha} \sum_{j=0}^{m-1} (-1)^j q^{j(j-1)\alpha} \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + \alpha + 1)},\\
\phi_{2,2m-1}(x) = \phi_{2,2m}(x) &= \sum_{j=0}^{m-1} (-1)^j q^{j(j-1)\alpha} \frac{x^{2\alpha j}}{\Gamma_q(2\alpha j + 1)}.
\end{aligned}$$

Hence, the solutions

$$\phi_i(x) := \lim_{m\to\infty} \phi_{i,m}(x) \quad (i = 1, 2)$$

are defined on $\mathbb{R}$ by

$$\phi_1(x) = x^{\alpha} E_{2\alpha,\alpha+1}(-x^{2\alpha}; q), \quad \phi_2(x) = E_{2\alpha,1}(-x^{2\alpha}; q). \tag{8.51}$$

If we set $\alpha = 1$ in (8.51), we get

$$y_1(x) = q \sin(q^{-1}x; q), \quad \text{and} \quad y_2(x) = \cos(q^{-1/2}x; q),\ x \in \mathbb{R},$$

which form a solution of the first order system

$$\begin{aligned}
D_q y_1(x) &= y_2(x), & D_q y_2(x) &= -q^{-1} y_1(qx),\\
y_1(0^+) &= 0, & y_2(0^+) &= 1.
\end{aligned}$$

8.5 A q-Cauchy Fractional Problem

In this section we investigate the existence and uniqueness of solutions of a q-Cauchy fractional problem defined in terms of a set of real parameters $\{\gamma_j\}_{j=0}^{3}$ and a set $\{L_j\}_{j=0}^{3}$ of q-integro difference operators of fractional order. This problem is a q-analogue of a Cauchy problem given by Djrbashian [85]. As in the classical case, the solutions will be given explicitly in terms of q-Mittag–Leffler functions.

Let the set of parameters $\{\gamma_j\}_{j=0}^{3}$ satisfy the conditions

$$\frac{1}{2} < \gamma_0,\ \gamma_3 \leqslant 1, \qquad 0 \leqslant \gamma_1,\ \gamma_2 \leqslant 1, \quad \sum_{j=0}^{3} \gamma_j = 3. \tag{8.52}$$

Then obviously

$$1 < \gamma_0 + \gamma_3 \leqslant 2, \qquad 1 \leqslant \gamma_1 + \gamma_2 < 2.$$

We shall also consider the two special cases

$$\begin{aligned} \gamma_0 = \gamma_2 = \gamma_3 = 1 \text{ when } \gamma_1 = 0, \\ \gamma_0 = \gamma_1 = \gamma_3 = 1 \text{ when } \gamma_2 = 0. \end{aligned} \tag{8.53}$$

We associate a given set of parameters $\{\gamma_j\}_{j=0}^{3}$ satisfying (8.52) with in the set $\{L_j\}_{j=0}^{3}$ of q-integro-difference operators of fractional order setting

$$\begin{aligned} L_0 y(x) &:= I_q^{1-\gamma_0} y(x), \\ L_1 y(x) &:= I_q^{1-\gamma_1} D_q L_0 y(x) = I_q^{1-\gamma_1} D_q^{\gamma_0} y(x), \\ L_2 y(x) &:= I_q^{1-\gamma_2} D_q L_1 y(x) = I_q^{1-\gamma_2} D_q^{\gamma_1} D_q^{\gamma_0} y(x), \\ \mathbb{L}_{1/2} y(x) &:= L_3 y(x) = I_q^{1-\gamma_3} D_q L_2 y(x) = I_q^{1-\gamma_3} D_q^{\gamma_2} D_q^{\gamma_1} D_q^{\gamma_0} y(x). \end{aligned}$$

In the following we introduce the domain of the definitions of the operators $\{L_j\}_{j=0}^{3}$.

Definition 8.5.1. For $a > 0$ let $\mathscr{A}C_{\{\gamma\}}[0,a]$, be the set of functions $y(x)$ satisfying the conditions

1. $y(x) \in \mathscr{L}_q^1[0,a]$.
2. $L_j y(x) \in \mathscr{A}_q C[0,a]$, $j = 0, 1, 2$.

Obviously, if $L_j y(x) \in \mathscr{A}_q C[0,a]$ then $D_q L_j y(x) \in \mathscr{L}_q^1(0,a)$, for $j = 0, 1, 2$, and $\mathbb{L}_{1/2} y(x) \in \mathscr{L}_q^1[0,a]$. Now, let $y(x) \in \mathscr{A}_q C_\gamma[0,a]$ be a given function. We introduce the following constants:

$$m_j(y) = L_j y(0^+), \quad \mu_j = \sum_{k=0}^{j} \gamma_k, \quad \text{for } j = 0, 1, 2.$$

Recall that since $y \in \mathscr{A}C_{\{\gamma\}}[0,a]$, the operator $L_j y$ is q-regular at zero for $j = 0, 1, 2$, and

$$L_j y(0^+) = \lim_{k\to\infty} L_j y(xq^k) \quad (x > 0).$$

One can easily verify that

$$\gamma_{j1} + \gamma_{j2} \leqslant 2, \quad j_1 \neq j_2,$$

for any pair of numbers of the set $\{\gamma_j\}_{j=0}^{3}$. The q-Cauchy problem which we are interested in is

$$\mathbb{L}_{1/2}y(x) + \lambda y(qx) = 0,\ 0 < x \le a, \tag{8.54}$$

$$L_j y(0^+) = m_j(y),\ j = 0, 1, 2, \tag{8.55}$$

where $\{m_j(y)\}_{j=0}^{2}$ and λ are arbitrary complex numbers. We solve (8.54) by repeatedly applying the q-Riemann–Liouville fractional operator of appropriate orders until (8.54) is transformed into a solvable q-difference equation. This technique is used by Ross and Djrbashian to solve certain integral operators, cf. e.g. [85, 263, 264].

Lemma 8.15. *If* $y \in \mathscr{A}C_{\{\gamma\}}[0,a]$ *then*

$$I_q^2\mathbb{L}_{1/2}y(x) = y(x) - \sum_{k=0}^{2} m_k(y)\frac{x^{\mu_k-1}}{\Gamma_q(\mu_k)},\ x \in (0,a]. \tag{8.56}$$

Proof. Since $\mathbb{L}_{1/2}y(x) \in \mathscr{L}_q^1(0,a)$, then using (4.62) we obtain the following equality for all $x \in (0,a]$

$$\begin{aligned} I_q^2\mathbb{L}_{1/2}y(x) &= I_q^2 I_q^{1-\gamma_3} D_q L_2 y(x) \\ &= I_q^{2-\gamma_3}\left\{I_q D_q L_2 y(x)\right\} = I_q^{2-\gamma_3}\left\{L_2 y(x) - m_2(y)\right\} \\ &= I_q^{2-\gamma_3} L_2 y(x) - m_2(y)\frac{x^{2-\gamma_3}}{\Gamma_q(3-\gamma_3)}. \end{aligned} \tag{8.57}$$

Furthermore,

$$\begin{aligned} I_q^{2-\gamma_3} L_2 y(x) &= I_q^{2-\gamma_3}\left\{I_q^{1-\gamma_2} D_{q,x} L_1 y(x)\right\} \\ &= I_q^{2-\gamma_2-\gamma_3}\left\{I_q D_q L_1 y(x)\right\} = I_q^{2-\gamma_2-\gamma_3}\left\{L_1 y(x) - m_1(y)\right\} \\ &= I_q^{2-\gamma_2-\gamma_3} L_1 y(x) - m_1(y)\frac{x^{2-\gamma_2-\gamma_3}}{\Gamma_q(3-\gamma_2-\gamma_3)}. \end{aligned} \tag{8.58}$$

Similarly,

$$\begin{aligned} I_q^{2-\gamma_2-\gamma_3} L_1 y(x) &= I_q^{\gamma_0} D_{q,x} L_0 y(x) \\ &= I_q^{\gamma_0} D_{q,x}\left\{I_q^{1-\gamma_0} y(x)\right\} = I_q^{\gamma_0} D_q^{\gamma_0} y(x). \end{aligned}$$

But

$$I_q^{\gamma_0} D_q^{\gamma_0} y(x) = y(x) - I_q^{1-\gamma_0} y(0^+).$$

Therefore,

$$I_q^{2-\gamma_2-\gamma_3} L_1 y(x) = y(x) - m_0(y)\frac{x^{\gamma_0-1}}{\Gamma_q(\gamma_0)} \quad \text{for } x \in (0,a].$$

This formula together with (8.57) and (8.58) imply the desired representation (8.56). ☐

Lemma 8.16. *If $y(x) \in \mathscr{A}C_\gamma[0,a]$ then*

$$\mathbb{L}_{1/2} y(x) = \begin{cases} D_q^2 y(x) - \sum_{k=0}^{2} m_k(y)\frac{x^{\mu_k-3}}{\Gamma_q(\mu_k-2)}, & \text{if } \mu_k \notin \{1,2\}, \\ D_q^2 y(x), & \text{otherwise.} \end{cases} \tag{8.59}$$

Proof. Applying the operation D_q^2 to both sides of identity (8.56), we arrive at formula (8.59). ☐

Lemma 8.17. *The q-Cauchy problem* (8.54)–(8.55) *may have only a unique solution in the class $\mathscr{A}C_\gamma[0,a]$.*

Proof. If there exist two solutions $y_j(x) \in \mathscr{A}C_\gamma[0,a]$, $j = 1,2$, then $y(x) := y_1(x)-y_2(x)$ is of the same class, and it is a solution of the homogeneous q-Cauchy problem

$$\begin{aligned} \mathbb{L}_{1/2} y(x) + \lambda y(qx) &= 0 \ \ (x \in (0,a])\,, \\ L_k y(0^+) &= m_k(y) \ \ (k = 0,1,2)\,. \end{aligned} \tag{8.60}$$

By (8.56) and (8.60),

$$I_q^2 \mathbb{L}_{1/2} y(x) = y(x) = -\lambda I_q^2 y(qx) \text{ for } x \in (0,a).$$

In other words, the function $y(x) \in \mathscr{L}_q^1(0,a)$ is a solution of the homogeneous q-Volterra equation

$$y(x) = -\lambda \int_0^x (x-qt)y(qt)\, d_qt, \quad x \in (0,a].$$

Hence, from Theorem 7.11, $y(x) = 0$ for all $x \in (0,a]$. ☐

The next lemma deals with the function

$$\begin{aligned} y_\mu(x,\lambda) &= x^{\mu-1} E_{2,\mu}(-\lambda q^{\mu-1}x^2; q) \\ &= \sum_{k=0}^{\infty} (-1)^k q^{k^2+(\mu-2)k} (-\lambda)^k \frac{x^{2k+\mu-1}}{\Gamma_q(2k+\mu)}, \end{aligned} \tag{8.61}$$

which obviously satisfies

$$y_\mu(x,\lambda) \in \begin{cases} \mathscr{L}_q^1(0,a),\ \mu > 0, \\ \\ \mathscr{L}_q^2(0,a),\ \mu > 1/2. \end{cases}$$

Lemma 8.18. *If*

$$\mu_r = \sum_{j=0}^{r} \gamma_j \quad (r = 0,1,2) \tag{8.62}$$

the functions $\left\{y_{\mu_r}(\cdot,\lambda)\right\}_{r=0}^{2}$ *are solutions of the* q*-Cauchy problem*

$$\mathbb{L}_{1/2}y(x) + \lambda y(qx) = 0 \quad \textit{for } x \in (0,a], \tag{8.63}$$

$$L_r y(x)|_{x=0} = \delta_{k,r} \equiv \begin{cases} 1,\ k = r, \\ 0,\ k \neq r, \end{cases} \tag{8.64}$$

in the space $\mathscr{A}C_{\{\gamma\}}[0,a]$.

Proof. We consider three cases

1. If $\mu = \mu_0 = \gamma_0$ then

$$\begin{aligned} L_0 y_{\mu_0}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_0-2)k} \frac{x^{2k}}{\Gamma_q(1+2k)}, \\ L_1 y_{\mu_0}(x;\lambda) &= -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{(k+1)^2+(\mu_0-2)(k+1)} \frac{x^{2k+2-\gamma_1}}{\Gamma_q(3-\gamma_1+2k)}, \\ L_2 y_{\mu_0}(x;\lambda) &= -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{(k+1)^2+(\mu_0-2)(k+1)} \frac{x^{2k+2-\gamma_1-\gamma_2}}{\Gamma_q(3-\gamma_1-\gamma_2+2k)}. \end{aligned}$$

Since $\gamma_1 + \gamma_2 < 2$ and

$$3 - \gamma_1 - \gamma_2 - \gamma_3 = \gamma_0 = \mu_0,$$

we obtain

$$\begin{aligned} \mathbb{L}_{1/2}y(x) &= L_3 y(x) \\ &= -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{(k+1)^2+(\mu_0-2)(k+1)} \frac{x^{2k+2-\gamma_1-\gamma_2-\gamma_3}}{\Gamma_q(3-\gamma_1-\gamma_2-\gamma_3+2k)} \\ &= -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_0-2)k} \frac{(qx)^{2k+\mu_0-1}}{\Gamma_q(2k+\mu_0)} = -\lambda\, y(qx). \end{aligned}$$

That is, the function $y_{\mu_0}(x;\lambda)$ is a solution of the Problem (8.63)–(8.64).

2. If $\mu = \mu_1 = \gamma_0 + \gamma_1$ then

$$\begin{aligned}
L_0 y_{\mu_1}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_1-2)k} \frac{x^{2k+\gamma_1}}{\Gamma_q(2k+\gamma_1+1)},\\
L_1 y_{\mu_1}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_1-2)k} \frac{x^{2k}}{\Gamma_q(2k+1)},\\
L_2 y_{\mu_1}(x;\lambda) &= -\lambda \sum_{k=0}^{\infty} q^{(k+1)^2+(\mu_1-2)(k+1)} \frac{x^{2k-\gamma_2+2}}{\Gamma_q(3-\gamma_2+2k)},\\
L_3 y_{\mu_1}(x;\lambda) &= \mathbb{L}_{1/2} y_{\mu_1}(x;\lambda)\\
&= -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_1-2)k} \frac{(qx)^{2k+2-\gamma_2-\gamma_3}}{\Gamma_q(3-\gamma_2-\gamma_3+2k)}.
\end{aligned} \tag{8.65}$$

But in this case, $3-\gamma_2-\gamma_3 = \gamma_0+\gamma_1 = \mu_1$ according to (8.52). Therefore, the last identity may be written in the form

$$\mathbb{L}_{1/2} y_{\mu_1}(x;\lambda) + \lambda y_{\mu_1}(qx;\lambda) = 0 \quad \text{for } x \in (0,a].$$

By formulae (8.65), $y_{\mu_1}(x;\lambda) \in \mathscr{A}C_{\{\gamma\}}[0,a]$ and, in addition, this function satisfies (8.63) with the initial condition (8.64) with $r = 1$.

3. If $\mu = \mu_2 = \gamma_0 + \gamma_1 + \gamma_2$ then

$$\begin{aligned}
L_0 y_{\mu_2}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_2-2)k} \frac{x^{2k+\gamma_1+\gamma_2}}{\Gamma_q(2k+\gamma_1+\gamma_2+1)},\\
L_1 y_{\mu_2}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_2-2)k} \frac{x^{2k+\gamma_2}}{\Gamma_q(2k+\gamma_2+1)},\\
L_2 y_{\mu_2}(x;\lambda) &= \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_2-2)k} \frac{x^{2k}}{\Gamma_q(2k+1)},\\
L_3 y_{\mu_2}(x;\lambda) &= \mathbb{L}_{1/2} y_{\mu_2}(x;\lambda) = -\lambda \sum_{k=0}^{\infty} (-\lambda)^k q^{k^2+(\mu_2-2)k} \frac{(qx)^{2k+2-\gamma_3}}{\Gamma_q(3-\gamma_3+2k)}.
\end{aligned} \tag{8.66}$$

Since $\mu_2 = 3 - \gamma_3$ and according to (8.52), the last identity may be written in the form

$$\mathbb{L}_{1/2} y_{\mu_2}(x;\lambda) + \lambda y_{\mu_2}(qx;\lambda) = 0, \ x \in (0,a).$$

By formulae (8.66),

$$y_{\mu_2}(x;\lambda) \in \mathscr{A}C_\gamma[0,a]$$

and $y_{\mu_2}(x;\lambda)$ satisfies (8.63) with the initial condition (8.64) with $r = 1$. □

The next theorem indicates that the set of solutions $\left\{y_{\mu_r}(\cdot,\lambda)\right\}_{r=0}^{2}$ plays the role of fundamental set of solutions of q-difference equations.

Theorem 8.19. *Let $a > 0$ be an arbitrary number. Then the function*

$$Y(x;\ \lambda) = \sum_{j=0}^{2} m_j(y) y_{\mu_j}(x;\lambda) \in \mathscr{L}_q^1[0,a] \tag{8.67}$$

is the unique solution of the q-cauchy problem (8.54)–(8.55) *in the class* $\mathscr{A}C_\gamma[0,a]$.

Proof. By Lemma 8.18,

$$\mathbb{L}_{1/2} Y(x;\ \lambda) + \lambda Y(qx;\lambda)$$
$$= \sum_{j=0}^{2} m_j(y)\left\{\mathbb{L}_{1/2} y_{\mu_j}(x;\lambda) + \lambda Y(qx;\lambda)\right\} = 0, \ x \in (0,a]$$

and

$$\begin{aligned} L_k Y(0^+;\lambda) &= \sum_{j=0}^{2} a_j L_{\mu_j}(0^+;\lambda) \\ &= \sum_{j=0}^{2} m_j(y)\delta_{k,j} = a_k \quad (k = 0,1,2). \end{aligned}$$

The uniqueness of the solution follows from Lemma 8.17. □

Finally, we present the forms taken by the operators $\left\{L_j\right\}_{j=0}^{3}$, the corresponding Cauchy type problems and their solutions in the special cases (8.53).

(1) $\gamma_1 = 0$ and $\gamma_0 = \gamma_2 = \gamma_3 = 1$. In this case we have

$$\begin{aligned} &L_0 y(x) = y(x), \quad L_1 y(x) = I_q D_q y(x) = y(x) - y(0), \\ &L_2 y(x) = D_q y(x), \quad \mathbb{L}_{1/2} y(x) = L_3 y(x) = D_q^2 y(x), \end{aligned}$$

$\mu_0 = 1,\ \mu_1 = \mu_2 = 2$, and

$$m_0(y) = y(0^+),\ m_1(y) = 0,\ \text{and}\, m_2(y) = D_q y(0^+).$$

(2) $\gamma_2 = 0$ and $\gamma_0 = \gamma_1 = \gamma_3 = 1$. In this case we have

$$\begin{aligned} L_0 y(x) &= y(x), \quad L_1 y(x) = D_q y(x), \\ L_2 y(x) &= I_q D_q^2 y(x) = D_q y(x) - D_q y(0), \\ \mathbb{L}_{1/2} y(x) &= L_3 y(x) = D_q^2 y(x), \end{aligned}$$

$\mu_0 = 1,\ \mu_1 = \mu_2 = 2$ and

$$m_0(y) = y(0^+),\ m_1(y) = D_q y(0^+),\ \text{and}\, m_2(y) = 0.$$

In both cases we reach the same Cauchy problem

$$D_q^2 y(x) + \lambda y(qx) = 0 \quad \text{for } x \in (0,a), \tag{8.68}$$

$$y(0^+) = \alpha, \quad D_q y(0^+) = \beta. \tag{8.69}$$

Therefore, the solution of (8.68)–(8.69) is

$$Y(x;\lambda) = \alpha E_{2,1}(-\lambda x^2; q) + \beta x E_{2,2}(-q\lambda x^2; q).$$

Remark 8.5.1. 1. The Cauchy type problem

$$\mathbb{L}_{1/2} y(x) + \lambda y(x) = 0 \quad \text{for}\, x \in (0,a), \tag{8.70}$$

$$L_j y(0^+) = m_j(y),\ j = 0,1,2, \tag{8.71}$$

can be treated similarly. In this case, we shall get the following existence and uniqueness theorem.

Theorem 8.20. *Let* $\lambda \in \mathbb{R}$. *Then there exists* $a > 0$ *such that the function*

$$Z(x;\ \lambda) = \sum_{j=0}^{2} m_j(y) z_{\mu_j}(x;\lambda) \in \mathscr{L}_q^1[0,a], \tag{8.72}$$

$$z_\mu(x;\lambda) = x^{\mu-1} e_{2,\mu}\left(-\lambda x^2; q\right) \ \textit{for}\ |\lambda x^2 (1-q)^{2\mu}| < 1,$$

is the unique solution of the cauchy type problem (8.70)–(8.71) *in the class* $\mathscr{A}C_\gamma[0,a]$.

2. Theorems 8.19 and 8.20 of this section can be used to find particular solutions of the q-difference equation

$$D_q^2(y)(x) + \lambda y(qx) = f(x),$$

or

$$D_q^2 y(x) + \lambda y(x) = f(x),$$

where

$$f(x) = \sum_{j=0}^{2} a_j \frac{x^{\mu_j - 1}}{\Gamma_q(\mu_j)},$$

and the μ_j's are the constants defined in (8.62).
3. The studies of the special types of q-fractional integro-differential problems of [84, 85] lead to formulation of eigenvalue problems whose eigenvalues are zeros of Mittag–Leffler functions and whose eigenfunctions are Riesz bases of families of Mittag–Leffler functions in some Hilbert spaces. This is applied in [28] to obtain sampling theorems in Paley–Wiener-Type spaces. There are several works in sampling theory and in the q-setting, see e.g. [7, 26, 27, 150]. To derive sampling theories associated with q-fractional integro-difference equations, we have to establish q-fractional eigenvalue problems and investigate the properties of eigenvalues and eigenfunctions. This is not expected to be an easy straightforward task because of the properties of the q-integration over $[a, b]$, $a \neq 0$ which leads to difficulties in defining right-sided q-fractional integrals and derivative. This has an impact on defining the boundary conditions at the right end points.

8.6 Linear Sequential Riemann–Liouville Fractional q-Difference Equations

In this section we investigate the existence and uniqueness theorem for solutions of linear sequential q-difference equation. We also introduce and prove some essential properties of the q, α Wronskian, see [169, Chap. 7] for the definition of the α Wronskian. The results of this section are mostly based on [205]. Throughout this section we assume that $0 < \alpha < 1$.

Definition 8.6.1. Let $n \in \mathbb{N}$. We shall call linear sequential Riemann–Liouville fractional q-difference equation of order $n\alpha$ the equation of the form

$$\sum_{k=0}^{n} b_k(x) \mathscr{D}_q^{k\alpha} y(x) = g(x), \quad 0 < x \leq a, \tag{8.73}$$

where b_k, $k = 0, 1, \ldots, n$, and g are given real functions, $b_n(x) \neq 0$ for all $x \in (0, a)$, and $\mathscr{D}_q^{k\alpha} y$ is the sequential Riemann–Liouville fractional q-derivative defined by:

$$\mathscr{D}_q^{\alpha} y := D_q^{\alpha} y, \quad \mathscr{D}_q^{k\alpha} y := D_q^{\alpha} \mathscr{D}_q^{(k-1)\alpha} y \quad (k \in \mathbb{N}). \tag{8.74}$$

If $b_n(x) \neq 0$ for all $x \in [0, a]$, (8.73) may be expressed in its normal form as follows:

$$\left(L_{q,n\alpha} y\right)(x) := \mathscr{D}_q^{n\alpha} y(x) + \sum_{k=0}^{n-1} a_k(x) \mathscr{D}_q^{k\alpha} y(x) = f(x). \tag{8.75}$$

Theorem 8.21. *Let $a_j(x)$, $j = 0, 1, \ldots, n-1$, and $f(x)$ be in the space $C_\gamma[0, a]$, $\gamma \leq \alpha$. Then there exists $0 < h \leqslant a$ such that the equation*

$$L_{n\alpha,q} y(x) = f(x) \tag{8.76}$$

has a unique solution in $C_{1-\alpha}[0, h]$ satisfying

$$I_q^{1-\alpha} \mathscr{D}_q^{k\alpha} y(0) = c_k \tag{8.77}$$

or equivalently

$$\lim_{x \to 0+} x^{1-\alpha} \mathscr{D}_q^{k\alpha} y(x) = \frac{c_k}{\Gamma_q(\alpha)}, \tag{8.78}$$

where $c_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n-1$.

Proof. From Theorem 8.8, there exists $h > 0$ such that the first order system

$$\begin{aligned} D_q^{\alpha} y_i &= y_{i+1}, \quad i = 1, 2, \ldots, n-1, \\ D_q^{\alpha} y_n &= -a_0 y_1 - a_1 y_2 - \ldots - a_{n-1} y_n + f(x) \end{aligned} \tag{8.79}$$

together with the initial conditions

$$I_q^{1-\alpha} y_k(0^+) = c_k \quad (k = 0, 1, \ldots, n-1) \tag{8.80}$$

has a unique solution in $C_{1-\alpha}[0, h]$. But $\left\{y_j\right\}_{j=1}^{n}$ is a solution of (8.79) with the initial conditions (8.80) if and only if y_1 is a solution of (8.76) with the initial conditions (8.77) or (8.78). ☐

The following two propositions are straight forward results of Theorems 8.21 and their proof is omitted .

Proposition 8.22. *Let $a_j \in C_\gamma[0, a]$, $\gamma \leq \alpha$, $j = 1, 2, \ldots, n$. Then the homogeneous fractional q-difference equation*

$$L_{n\alpha,q} y(x) = 0 \tag{8.81}$$

together with the initial conditions

$$\lim_{x\to 0^+} x^{1-\alpha}\mathscr{D}_q^{j\alpha}y(x)=0 \quad or \quad I_q^{1-\alpha}\mathscr{D}_q^{j\alpha}y(0^+)=0 \quad (j=0,1,\dots,n-1)$$

has only the trivial continuous solution $y(x)=0$.

Proposition 8.23. *Any linear combination of solutions of the homogeneous equation*

$$L_{q,n\alpha}y(x)=0$$

is also a solution of this equation.

Definition 8.6.2. We call q,α Wronskian of n functions $u_j,\ j\in\mathbb{N}$, having fractional Riemann–Liouville sequential q-derivative up to order $(n-1)\alpha$ in $(0,a]$, the following determinant

$$\left|W_{q,\alpha}\left(u_1,\dots,u_n\right)(x)\right| = \begin{vmatrix} u_1(x) & u_2(x) & \dots & u_n(x) \\ \mathscr{D}_q^{\alpha}u_1(x) & \mathscr{D}_q^{\alpha}u_2(x) & \dots & \mathscr{D}_q^{\alpha}u_n(x) \\ \vdots & & & \vdots \\ \mathscr{D}_q^{(n-1)\alpha}u_1(x) & \mathscr{D}_q^{(n-1)\alpha}u_2(x) & \dots & \mathscr{D}_q^{(n-1)\alpha}u_n(x) \end{vmatrix}.$$

To simplify the notation, this will be represented by $|W_{q,\alpha}(x)|$. We shall use $W_{q,\alpha}$ for the corresponding q,α Wronskian matrix.

Proposition 8.24. *Let* $\{u_j(x)\}_{j=1}^n$ *be a family of functions which admits Riemann–Liouville fractional sequential q-derivatives up to order* $(n-1)\alpha$ *in* $(0,a]$, *satisfying for* $j=1,2,\dots,n$ *and* $k=0,\dots,n-1$, *the condition*

$$\lim_{x\to 0^+} x^{n-n\alpha}\mathscr{D}_q^{k\alpha}u_j(x) \quad exists. \tag{8.82}$$

If the functions $\{x^{1-\alpha}u_j(x)\}_{j=1}^n$ *are linearly dependent on* $[0,a]$ *then*

$$x^{n-n\alpha}|W_{q,\alpha}(x)|=0 \quad for\ all \quad x\in[0,a].$$

Proof. Since $\{x^{1-\alpha}u_j(x)\}_{j=1}^n$ are linearly dependent on $[0,a]$, then there exist n constants $\{c_j\}_{j=1}^n$, not all zeros, such that for all $x\in[0,a]$, $\sum_{j=1}^n c_j x^{1-\alpha}u_j(x)=0$. Therefore, for all $x\in(0,a]$ we obtain

$$\sum_{k=1}^{n} c_j u_j(x)=0. \tag{8.83}$$

Successive applications of the sequential q-derivative $\mathscr{D}_q^{k\alpha}$, $k \in \{1,\dots,n-1\}$, to (8.83) lead to the following relation:

$$W_{q,\alpha}(x)\widetilde{C} = \widetilde{0}, \quad \widetilde{C} = (c_1, c_2, \dots, c_n)^\top \neq \widetilde{0}, \tag{8.84}$$

and $\widetilde{0}$ is the zero $n \times 1$ matrix. Consequently, $\left|W_{q,\alpha}(x)\right| = 0$ for all $x \in (0,a]$. In addition, by (8.82), we can also conclude that

$$\lim_{x\to 0+} x^{1-\alpha} W_{q,\alpha}(x)\widetilde{C} = \widetilde{0}.$$

Hence, $\lim_{x\to 0+} x^{n-n\alpha}|W_{q,\alpha}(x)| = 0$, which completes the proof. □

Proposition 8.25. *Let $\{u_j(x)\}_{j=1}^n$ be a set of solutions of (8.81) satisfying the initial conditions (8.77) or equivalently (8.78). Then $\{u_j(x)\}_{j=1}^n$ is a linearly independent set if and only if*

$$\lim_{x\to 0+} x^{n-n\alpha} \left|W_{q,\alpha}(u_1,\dots,u_n)(x)\right| \neq 0. \tag{8.85}$$

Proof. The proof of the sufficient part follows from Proposition 8.24 by reductio ad absurdum. To prove the necessary part, we suppose on the contrary that

$$\lim_{x\to 0+} x^{n-n\alpha}|W_{q,\alpha}(x)| = 0.$$

Hence, the system

$$x^{1-\alpha} W_{q,\alpha}(x)\Big|_{x=0} \widetilde{C} = \widetilde{0}$$

has a non zero solution $\widetilde{C}$. For the function

$$y(x) := \sum_{j=1}^n c_j u_j(x) \quad (x \in (0,a])$$

which is a solution of (8.81) in $(0,a]$, it holds that

$$x^{1-\alpha}\mathscr{D}_q^{k\alpha} y(x)\Big|_{x=0} = 0, \quad k = 0,1,\dots,n-1.$$

Therefore, using the uniqueness of solutions, the function $x^{1-\alpha}y(x)$ vanishes identically on $[0,a]$ and consequently $\{x^{1-\alpha}u_j(x)\}_{j=1}^n$ are linearly dependent on $[0,a]$, which is a contradiction. Hence, (8.85) should hold. □

Let M be the vector space of all solutions of the homogeneous equation (8.81). In the following, any set of linearly independent solutions forming a basis of M is called a fundamental set of solutions of (8.81).

Theorem 8.26. *M is a vector space of dimension n.*

Proof. Let $\{\phi_i\}_{i=1}^n$ be n solutions of (8.81) satisfying the initial conditions

$$\lim_{x\to 0^+} x^{1-\alpha}\mathscr{D}_q^{(k-1)\alpha}\phi_j(x) = \frac{\delta_{kj}}{\Gamma_q(\alpha)} \quad (k,j = 1,\dots,n),$$

or equivalently

$$I_q^{1-\alpha}\mathscr{D}_q^{(k-1)\alpha}\phi_j(0) = \delta_{kj} \quad (k,j = 1,\dots,n).$$

From proposition 8.24 since

$$\lim_{x\to 0^+} x^{n-n\alpha} W_{q,\alpha}(\phi_1,\dots,\phi_n)(x) = \frac{1}{\Gamma_q^n(\alpha)},$$

the functions $\{\phi_i\}_{i=1}^n$ are linearly independent. Let $y \in M \setminus \{0\}$. Then there exist constants $\{b_k\}_{k=1}^n$, not all zeros, such that

$$\lim_{x\to 0^+} x^{1-\alpha}\mathscr{D}_q^{(k-1)\alpha} y(x) = b_k.$$

Set $\widetilde{b} = (b_1, b_2, \dots, b_n)^\top$. Then the system

$$\lim_{x\to 0^+} x^{1-\alpha} W_q(\phi_1,\dots,\phi_n)(x)\widetilde{C} = \widetilde{b}$$

has a non zero solution $\widetilde{C}$, $\widetilde{C} = (c_1, c_2, \dots, c_n)$. Let

$$z(x) = \sum_{j=1}^n c_j\phi_j(x) \quad \text{for } x \in (0,a].$$

Then

$$\lim_{x\to 0^+} x^{1-\alpha}\mathscr{D}_q^{(k-1)\alpha}(y-z)(x) = 0 \quad (k = 1,2,\dots,n).$$

Consequently,

$$z(x) \equiv y(x) \quad \text{for all } x \in (0,a].$$

Thus, the solution y is written uniquely as a linear combination of the functions $\{\phi_i\}_{i=1}^n$. Hence, $\{\phi_i\}_{i=1}^n$ is a basis of M. □

Corollary 8.27. *A set $\left\{u_j(x)\right\}_{j=1}^n$ of n linearly independent solutions of (8.81) is a fundamental set if and only if*

$$\lim_{x\to 0^+} x^{n-n\alpha}\left|W_{q,\alpha}(u_1,\dots,u_n)(x)\right| \neq 0.$$

Proof. The proof is a direct consequence of Propositions 8.24 and 8.25 and is omitted. □

Proposition 8.28. *If y_p is a particular solution to the* (8.76)*, then the general solution to this equation is given by $y_g = y_h + y_p$, where y_h is the general solution to the associated homogeneous equation* (8.81).

Proof. The proof is straightforward and is omitted. □

8.7 Linear Sequential Caputo Fractional q-Difference Equations

In this section, we assume that α is a positive number which is less than one.

Definition 8.7.1. Let $n \in \mathbb{N}$. We shall call linear sequential Caputo fractional q-difference equation of order $n\alpha$ the equation of the form

$$\sum_{k=0}^{n} b_k(x) {}^c\mathscr{D}_q^{k\alpha} y(x) = g(x), \quad 0 \le x \le a, \tag{8.86}$$

where b_k, $k = 0, 1, \dots, n$, and g are given real functions and ${}^c\mathscr{D}_q^{k\alpha} y$ is the sequential fractional Caputo q-derivative defined by:

$${}^c\mathscr{D}_q^{\alpha} y := {}^cD_q^{\alpha} y, \quad {}^c\mathscr{D}_q^{k\alpha} y := {}^cD_q^{\alpha}\, {}^c\mathscr{D}_q^{(k-1)\alpha} y \quad (k \in \mathbb{N}). \tag{8.87}$$

If $b_n(x) \neq 0$ for all $x \in [0, a]$, (8.86) may be expressed in its normal form as follows:

$$\left(L_{q,n\alpha}^{C} y\right)(x) := {}^c\mathscr{D}_q^{n\alpha} y(x) + \sum_{k=0}^{n-1} a_k(x) {}^c\mathscr{D}_q^{k\alpha} y(x) = f(x). \tag{8.88}$$

Theorem 8.29. *Let $a_j \in C_\gamma[0, a]$ for $j = 0, 1, \dots, n-1$ and for some $\gamma \le \alpha$. Then there exists $0 < h \leqslant a$ such that the equation*

$$L_{n\alpha,q}^{C} y(x) = f(x) \tag{8.89}$$

has a unique solution in $C[0, h]$ satisfying

$${}^c\mathscr{D}_q^{k\alpha} y(0^+) = c_k, \tag{8.90}$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \dots, n-1$.

Proof. From Theorem 8.13, there exists $h > 0$ such that the first order system

$$\begin{aligned} {}^cD_q^{\alpha} y_i &= y_{i+1}, \quad i = 1, 2, \ldots, n-1, \\ {}^cD_q^{\alpha} y_n &= -a_0 y_1 - a_1 y_2 - \ldots - a_{n-1} y_n + f(x) \end{aligned} \tag{8.91}$$

together with the initial conditions

$$y_k(0^+) = c_k \qquad (k = 1, \ldots, n) \tag{8.92}$$

has a unique solution in $C[0,h]$. But $\{y_j\}_{j=1}^n$ is a solution of (8.91) with the initial conditions (8.92) if and only if y_1 is a solution of (8.89) with the initial conditions (8.90). ☐

Similar to Propositions 8.23 and 8.30, we have the following two propositions which we state without proof.

Proposition 8.30. *Let $a_j \in C_\gamma[0,a]$ for $j = 1, 2, \ldots, n$, and for some $\gamma \leq \alpha$. Then the homogeneous fractional Caputo q-difference equation (8.86) together with the initial conditions*

$${}^c\mathscr{D}_q^{j\alpha} y(0^+) = 0 \quad (j = 0, 1, \ldots, n-1)$$

has only the trivial continuous solution $y(x) = 0$.

Proposition 8.31. *Any linear combination of solutions of the homogeneous equation*

$$L_{q,n\alpha}^C y(x) = 0 \tag{8.93}$$

is also a solution of this equation.

Definition 8.7.2. We call q, α Wronskian of n functions u_j, $j \in \mathbb{N}$, having fractional Caputo sequential q-derivative up to order $(n-1)\alpha$ in $(0,a]$, the following determinant

$$\left| W_{q,\alpha}^C (u_1, \ldots, u_n)(x) \right| = \begin{vmatrix} u_1(x) & u_2(x) & \ldots & u_n(x) \\ {}^c\mathscr{D}_q^{\alpha} u_1(x) & {}^c\mathscr{D}_q^{\alpha} u_2(x) & \ldots & {}^c\mathscr{D}_q^{\alpha} u_n(x) \\ \vdots & & & \vdots \\ {}^c\mathscr{D}_q^{(n-1)\alpha} u_1(x) & {}^c\mathscr{D}_q^{(n-1)\alpha} u_2(x) & \ldots & {}^c\mathscr{D}_q^{(n-1)\alpha} u_n(x) \end{vmatrix}.$$

To simplify the notation, this will be represented by $|W_{q,\alpha}^C(x)|$. We shall use $W_{q,\alpha}^C$ for the corresponding q, α Caputo Wronskian matrix.

Proposition 8.32. *Let $\{u_j(x)\}_{j=1}^n$ be a family of functions which admit Caputo fractional sequential q-derivatives up to order $(n-1)\alpha$ in $(0,a]$ such that*

${}^{c}\mathscr{D}_q^{k\alpha}u_j(0^+)$ exists for $j = 1, 2, \ldots, n$ and $k = 0, \ldots, n-1$, If the functions $\{u_j(x)\}_{j=1}^n$ are linearly dependent on $[0, a]$ then

$$|W_{q,\alpha}^C(x)| = 0, \quad \textit{for all} \quad x \in [0, a].$$

Proof. Since $\{u_j(x)\}_{j=1}^n$ are linearly dependent on $[0, a]$, then there exist n constants $\{c_j\}_{j=1}^n$, not all zeros, such that for all $x \in [0, a]$,

$$\sum_{k=1}^{n} c_j u_j(x) = 0. \tag{8.94}$$

Successive applications of the Caputo sequential q-derivative

$${}^{c}\mathscr{D}_q^{k\alpha} \quad (k = 1, \ldots, n-1)$$

to (8.94) lead to the following relation:

$$W_{q,\alpha}^C(x)\widetilde{C} = \widetilde{0}, \quad \widetilde{C} = (c_1, c_2, \ldots, c_n)^\top \neq \widetilde{0}, \tag{8.95}$$

and $\widetilde{0}$ is the zero $n \times 1$ matrix. Consequently, $\left|W_{q,\alpha}^C(x)\right| = 0$ for all $x \in [0, a]$, which completes the proof. ☐

Proposition 8.33. *Let $\{u_j(x)\}_{j=1}^n$ be solutions of the equation (8.89) satisfying the initial conditions (8.90). Then $\{u_j(x)\}_{j=1}^n$ are linearly independent if and only if*

$$\left|W_{q,\alpha}^C\left(u_1, \ldots, u_n\right)\left(0^+\right)\right| \neq 0. \tag{8.96}$$

Proof. The proof of the sufficient part follows from Proposition 8.32 by reductio ad absurdum. To prove the necessary part, we suppose on the contrary that $|W_{q,\alpha}^C|(0^+) = 0$. Hence, the system

$$W_{q,\alpha}^C(0^+)\widetilde{C} = \widetilde{0}$$

has a non zero solution $\widetilde{C}$. For the function

$$y(x) := \sum_{j=1}^{n} c_j u_j(x), \quad x \in [0, a],$$

which is a solution of (8.89) in $[0, a]$, it holds that

$${}^{c}\mathscr{D}_q^{k\alpha}y(0^+) = 0 \quad (k = 0, 1, \ldots, n-1)\,.$$

Therefore, using the uniqueness of solutions, the function $y(x)$ vanishes identically on $[0, a]$ and consequently $\left\{u_j(x)\right\}_{j=1}^{n}$ are linearly dependent on $[0, a]$, which is a contradiction. Hence, (8.96) should hold. □

Let H_C be the vector space of all solutions of the homogeneous equation (8.89). In the following, any set of linearly independent solutions forming a basis of H_C is called a fundamental set of solutions of (8.89).

Theorem 8.34. *H_C is a vector space of dimension n.*

Proof. The proof is similar to the proof of Theorem 8.26 and is omitted. □

The proofs of the following results are direct and are omitted.

Corollary 8.35. *A set $\left\{u_j(x)\right\}_{j=1}^{n}$ of n linearly independent solutions of (8.89) is a fundamental set if and only if*

$$\left|W_{q,\alpha}^{C}\left(u_1, \ldots, u_n\right)\left(0^{+}\right)\right| \neq 0.$$

Proof. The proof is a direct consequence of Proposition 8.33 and is omitted. □

Proposition 8.36. *If y_p is a particular solution to the (8.76), then the general solution to this equation is given by $y_g = y_h + y_p$, where y_h is the general solution to the associated homogeneous equation (8.81).*

Proof. The proof is straightforward and is omitted. □

8.8 Fundamental Set of Solutions of a Linear Sequential Riemann–Liouville Fractional q-Difference Equation

In this section we are concerned with constructing a fundamental set of solutions of (8.75) when it has constant coefficients. Let

$$L_{q,n\alpha}\, y(x) := \sum_{k=0}^{n} a_k \mathscr{D}_q^{k\alpha} y(x) = 0, \tag{8.97}$$

where the coefficients $\{a_k\}_{k=0}^{n}$ are real constants and $a_n \neq 0$.

The characteristic polynomial $P(\lambda)$ of (8.97) is defined by

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\,, \quad \lambda \in \mathbb{C}. \tag{8.98}$$

From now on λ_i $(i = 1, 2, \ldots, K)$ denote the distinct roots of $P(\lambda)$ and μ_i denotes the multiplicity of λ_i, so that $\sum_{i=1}^{K} \mu_i = n$. As in the classical case, we shall seek a solution in the form

$$y(x,\lambda) = x^{\alpha-1}e_{\alpha,\alpha}(\lambda x^{\alpha};q) = x^{\alpha-1}\sum_{j=0}^{\infty}\frac{(\lambda x^{\alpha})^{j}}{\Gamma_q(\alpha j+\alpha)}$$

where $(|\lambda(x(1-q))^{\alpha}| < 1)$. Since

$$\mathscr{D}_q^{k\alpha}y(x,\lambda) = \lambda^{k}y(x,\lambda) \quad (k \in \mathbb{N}_0),$$

we obtain

$$L_{q,n\alpha}y(x,\lambda) = P(\lambda)y(x,\lambda). \tag{8.99}$$

Therefore, $y(x,\lambda)$ is a solution of (8.97) if and only if $P(\lambda) = 0$. The following assertion is true for complex $\lambda \in \mathbb{C}$.

Lemma 8.37. *If λ is a complex root of the characteristic polynomial* (8.98)*, then for all $l \in \mathbb{N}_0$ we obtain*

$$\frac{\partial^{l}}{\partial\lambda^{l}}\left(L_{q,n\alpha}y(x,\lambda)\right) = L_{q,n\alpha}\left(\frac{\partial^{l}}{\partial\lambda^{l}}y(x,\lambda)\right) \tag{8.100}$$

and

$$\frac{\partial^{l}}{\partial\lambda^{l}}y(x,\lambda) = x^{l\alpha+\alpha-1}\sum_{m=0}^{\infty}(m+l)(m+l-1)\dots(m+1)\frac{(\lambda x^{\alpha})^{m}}{\Gamma_q(m\alpha+l\alpha+\alpha)}.$$

Proof. The lemma follows from the linearity of the operators $\frac{\partial^{l}}{\partial\lambda^{l}}$ and $L_{q,n\alpha}$. □

Lemma 8.38. *The functions*

$$\phi_{\alpha,l}(x,\lambda_i) := x^{l\alpha+\alpha-1}\sum_{m=0}^{\infty}(m+l)(m+l-1)\dots(m+1)\frac{(\lambda_i x^{\alpha})^{m}}{\Gamma_q(m\alpha+l\alpha+\alpha)},$$
$$i = 1,\dots,K,\ l = 1,\dots,\mu_i,\ \textit{and}\ |\lambda_i||x(1-q)|^{\alpha} < 1, \tag{8.101}$$

are linearly independent solutions of (8.97)*.*

Proof. From (8.99)–(8.100) and the classical Leibniz rule, we obtain

$$\begin{aligned}L_{q,n\alpha}\left(\phi_{\alpha,l}(x,\lambda_i)\right) &= L_{q,n\alpha}\left(\frac{\partial^{l}}{\partial\lambda^{l}}y(x,\lambda)\right)\Big|_{\lambda=\lambda_i}\\ &= \sum_{r=0}^{l}\binom{l}{r}P^{(r)}(\lambda)\frac{\partial^{l-r}}{\partial\lambda^{l-r}}y(x,\lambda)\Big|_{\lambda=\lambda_i} = 0,\end{aligned}$$

for $l = 0, 1, \ldots, \mu_i - 1$. So the functions defined in (8.101) are the solutions of (8.97). The linear independence follows from Proposition 8.24 since

$$\lim_{x\to 0+} x^{k-\alpha k} \left|W_{q,\alpha}\left(\phi_{\alpha,1}, \phi_{\alpha,2}, \ldots, \phi_{\alpha,\mu_i}\right)(x, \lambda_i)\right| = \frac{1}{\Gamma_q^k(\alpha)}.$$

□

This lemma and the above discussion lead to the following theorem which we state without proof.

Theorem 8.39. *For $i = 1, 2, \ldots, K$, the set $\{\phi_{\alpha,r}(x, \lambda_i)\}_{r=0}^{\mu_i-1}$ of (8.101) is a linearly independent set of solutions of (8.97). Moreover, the set*

$$\left\{\{\phi_{\alpha,r}(x, \lambda_i)\}_{r=0}^{m_i-1},\ i = 1, \ldots, K\right\}$$

is a fundamental set of solutions of (8.97).

Example 8.8.1. Consider the sequential Riemann–Liouville fractional q-difference equation

$$\mathscr{D}_q^{2\alpha} y(x) - y(x) = 0.$$

The characteristic polynomial $P(\lambda) = \lambda^2 - 1$ has two distinct roots $\lambda_1 = 1$, $\lambda_2 = -1$. Hence, the general solution is given by

$$y(x) = c_1 x^{\alpha-1} e_{\alpha,\alpha}(x; q) + c_2 x^{\alpha-1} e_{\alpha,\alpha}(-x; q),$$

where c_1 and c_2 are arbitrary constants.

Example 8.8.2. Consider the sequential Riemann–Liouville fractional q-difference equation

$$\mathscr{D}_q^{3\alpha} y(x) - 4\mathscr{D}_q^{2\alpha} y(x) + 5\mathscr{D}_q^{\alpha} y(x) - 2y(x) = 0.$$

The characteristic polynomial

$$P(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda - 1)^2(\lambda - 2)$$

has the root $\lambda = 1$ which is of multiplicity 2 and the simple root $\lambda = 2$. Hence, the general solution is given for $2|x(1-q)|^\alpha < 1$ by

$$y(x) = c_1 x^{\alpha-1} e_{\alpha,\alpha}(x; q) + c_2 \phi_{\alpha,1}(x, 1) + c_3 x^{\alpha-1} e_{\alpha,\alpha}(2x; q).$$

Recall that $\phi_{\alpha,1}(x, 1) = x^{2\alpha-1} \sum_{m=0}^{\infty} (m+1) \dfrac{x^{m\alpha}}{\Gamma_q(m\alpha + 2\alpha)}$.

8.9 Fundamental Set of Solutions of a Linear Sequential Caputo Fractional q-Difference Equation

In this section we are concerned with constructing a fundamental set of solutions of the fractional q-difference equation

$$L^C_{q,n\alpha}y(x) := \sum_{k=0}^{n} a_k\, {}^c\mathscr{D}_q^{k\alpha} y(x) = 0, \tag{8.102}$$

where the coefficients $\{a_k\}_{k=0}^n$ are real constants and $a_n \neq 0$. The characteristic polynomial $Q(\lambda)$ of (8.102) is defined by

$$Q(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\,, \quad \lambda \in \mathbb{C}. \tag{8.103}$$

From now on λ_i $(i = 1, 2, \ldots, L)$ denote the distinct roots of $Q(\lambda)$ and μ_i denotes the multiplicity of λ_i, so that $\sum_{i=1}^{L} \mu_i = n$. We shall seek solutions of the form

$$z(x,\lambda) = e_{\alpha,1}(\lambda x^\alpha; q) = \sum_{j=0}^{\infty} \frac{(\lambda x^\alpha)^j}{\Gamma_q(\alpha j + 1)} \quad \text{for } |\lambda x^\alpha| < (1-q)^{-\alpha}.$$

Since

$${}^c\mathscr{D}_q^{k\alpha} z(x,\lambda) = \lambda^k z(x,\lambda) \quad \text{for all } \ k \in \mathbb{N}_0,$$

we obtain

$$L^C_{q,n\alpha} z(x,\lambda) = Q(\lambda) z(x,\lambda). \tag{8.104}$$

Therefore, $z(x,\lambda)$ is a solution of (8.102) if and only if $Q(\lambda) = 0$.

The following assertion is true for complex $\lambda \in \mathbb{C}$.

Lemma 8.40. *If* λ *is a complex root of the characteristic polynomial* (8.103)*, then for all* $l \in \mathbb{N}$ *we obtain*

$$\frac{\partial^l}{\partial\lambda^l}\left(L^C_{q,n\alpha} z(x,\lambda)\right) = L^C_{q,n\alpha}\left(\frac{\partial^l}{\partial\lambda^l} z(x,\lambda)\right)$$

and

$$\frac{\partial^l}{\partial\lambda^l} z(x,\lambda) = x^{l\alpha} \sum_{m=0}^{\infty} (m+l)(m+l-1)\ldots(m+1)\frac{(\lambda x^\alpha)^m}{\Gamma_q(m\alpha + l\alpha + 1)}.$$

Proof. The lemma follows from the linearity of the operators $\frac{\partial^l}{\partial\lambda^l}$ and $L^C_{q,n\alpha}$. □

Lemma 8.41. *The functions*

$$\psi_{\alpha,l}(x,\lambda_i) := x^{l\alpha}\sum_{m=0}^{\infty}(m+l)(m+l-1)\ldots(m+1)\frac{(\lambda_i x^{\alpha})^m}{\Gamma_q(m\alpha+l\alpha+1)}, \quad (8.105)$$

where

$$i=1,\ldots,L,\ l=1,\ldots,\mu_i,\ \textit{and}\ |\lambda_i||x(1-q)|^{\alpha}<1,$$

are linearly independent solutions of (8.102).

Proof. The proof is similar to the proof of Lemma 8.38 and is omitted. □

Lemma 8.41 and the above discussion lead to the following theorem.

Theorem 8.42. *For* $i = 1,2,\ldots,L$, *the set* $\{\psi_{\alpha,r}(x,\lambda_i)\}_{r=0}^{\mu_i-1}$ *of* (8.105) *is a linearly independent set of solutions of* (8.102). *Moreover, the set*

$$\left\{\{\psi_{\alpha,r}(x,\lambda_i)\}_{r=0}^{m_i-1},\ i=1,\ldots,L\right\}$$

is a fundamental set of solutions of (8.102).

Example 8.9.1. Consider the linear sequential Caputo fractional q-difference equation

$${}^{c}D_q^{2\alpha}y(x)-y(x)=0.$$

The characteristic polynomial $Q(\lambda)=\lambda^2-1$ has two distinct roots $\lambda_1=1$, $\lambda_2=-1$. Hence, the general solution is given by

$$y(x)=c_1e_{\alpha,1}(x;q)+c_2e_{\alpha,1}(-x;q),$$

where c_1 and c_2 are arbitrary constants.

Example 8.9.2. Consider the linear sequential Caputo fractional q-difference equation

$${}^{c}\mathscr{D}_q^{3\alpha}y(x)-5{}^{c}\mathscr{D}_q^{2\alpha}y(x)+8{}^{c}\mathscr{D}_q^{\alpha}y(x)-4y(x)=0.$$

Then $Q(\lambda)=\lambda^3-5\lambda^2+8\lambda-4=(\lambda-2)^2(\lambda-1)$. Hence, the general solution is given for $2|x(1-q)|^{\alpha}<1$ by

$$y(x)=c_1e_{\alpha,1}(x;q)+c_2e_{\alpha,1}(2x;q)+\psi_{\alpha,1}(x,2),$$

where

$$\psi_{\alpha,1}(x,2)=x^{\alpha}\sum_{m=0}^{\infty}\frac{m+1}{\Gamma_q(m\alpha+\alpha+1)}(2x^{\alpha})^m,\ |2x^{\alpha}(1-q)^{\alpha}|<1.$$

Chapter 9

q-Integral Transforms for Solving Fractional q-Difference Equations

Abstract Integral transforms like Laplace, Mellin, and Fourier transforms are used in finding explicit solutions for linear differential equations, linear fractional differential equations, and diffusion equations. See for example (Kilbas et al., Theory and Applications of Fractional Differential Equations, Elsevier, London, first edition, 2006; Mainardi, Appl. Math. Lett. 9(6), 23–28, 1996; Nikolova and Boyadjiev, Fract. Calc. Appl. Anal. 13(1), 57–67, 2010; Wyss, J. Math. Phys. 27(11), 2782–2785, 1986). This chapter is devoted to the use of the q-Laplace, q-Mellin, and q^2-Fourier transforms to find explicit solutions of certain linear q-difference equations, linear fractional q-difference equations, and certain fractional q-diffusion equations.

9.1 q-Laplace Transform of Fractional q-Integrals and q-Derivatives

In the following theorem we compute the ${}_q L_s$ transform of the Riemann–Liouville fractional q-integral and q-derivatives.

Theorem 9.1. *If* $F \in \mathscr{L}_q^1[0,a]$ *and* $\Phi(s) := {}_q L_s F(x)$ *then*

$$ {}_q L_s I_q^{\alpha} F(x) = \frac{(1-q)^{\alpha}}{s^{\alpha}} \Phi(s) \quad \textit{for any } \alpha > 0. \tag{9.1}$$

If $n-1 < \alpha \leq n$ *and* $I_q^{n-\alpha} F(x) \in \mathscr{A}C_q^{(n)}[0,a]$ *then*

$$ {}_q L_s D_q^{\alpha} F(x) = \frac{s^{\alpha}}{(1-q)^{\alpha}} \Phi(s) - \sum_{m=1}^{n} D_q^{\alpha-m} F(0^+) \frac{s^{m-1}}{(1-q)^m}. \tag{9.2}$$

M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations*,
Lecture Notes in Mathematics 2056, DOI 10.1007/978-3-642-30898-7_9,

Proof. Since

$$I_q^\alpha F(x) = (1-q)\frac{x^{\alpha-1}}{\Gamma_q(\alpha)} *_q F(x) = (1-q)\left(\phi_{\alpha-1}(x) *_q F(x)\right),$$

we obtain (9.1) from (1.88), (1.91), and (1.92). Now we prove (9.2). From (4.49), (1.93), and (9.1) we get

$$\begin{aligned} {}_qL_s\big(D_q^\alpha F(x)\big) &= {}_qL_s\big(D_q^n I_q^{n-\alpha} F(x)\big) \\ &= \frac{s^n}{(1-q)^n}\,{}_qL_s I_q^{n-\alpha} F(x) - \sum_{m=1}^{n} D_q^{n-m} I_q^{n-\alpha} F(0^+)\frac{s^{m-1}}{(1-q)^m} \\ &= \frac{s^\alpha}{(1-q)^\alpha}\Phi(s) - \sum_{m=1}^{n} D_q^{\alpha-m} F(0^+)\frac{s^{m-1}}{(1-q)^m}, \end{aligned} \tag{9.3}$$

proving the theorem. ☐

Theorem 9.2. *If* ${}_qL_s f(x) = \Phi(s)$ *then the* q*-Laplace transform of the Caputo fractional* q*-derivative is given by*

$${}_qL_s\,{}^cD_q^\alpha f(x) = \frac{s^\alpha}{(1-q)^\alpha}\left(\Phi(s) - \sum_{r=0}^{n-1} D_q^r f(0^+)\frac{(1-q)^r}{s^{r+1}}\right).$$

Proof. Since ${}^cD_q^\alpha f(x) = (1-q)D_q^n f(x) * \phi_{n-\alpha-1}(x)$, then by (1.88) and (1.93) we obtain

$$\begin{aligned} {}_qL_s\,{}^cD_q^\alpha f(x) &= \frac{(1-q)^{n-\alpha}}{s^{n-\alpha}}\,{}_qL_s(D_q^n f(x)) \\ &= \frac{(1-q)^{n-\alpha}}{s^{n-\alpha}}\left(\left(\frac{s}{1-q}\right)^n \Phi(s) - \sum_{m=1}^{n} D_q^{n-m} f(0^+)\frac{s^{m-1}}{(1-q)^m}\right) \\ &= \left(\frac{s^\alpha}{(1-q)^\alpha}\Phi(s) - \sum_{r=0}^{n-1} D_q^r f(0^+)\frac{s^{-r+\alpha-1}}{(1-q)^{-r+\alpha}}\right). \end{aligned}$$

☐

Lemma 9.3. *If* ${}_qL_s f(x) = \Phi(s)$ *then the* q*-Laplace transform of the Riemann–Liouville sequential* q*-derivative of order* $m\alpha$*,* $0 < \alpha < 1$*, is given by*

$${}_qL_s\mathscr{D}_q^{m\alpha} f(x) = p^{m\alpha}\phi(s) - \frac{1}{1-q}\sum_{r=0}^{m-1} p^{\alpha(m-1-r)} I_q^{1-\alpha}\mathscr{D}_q^{\alpha r} f(0^+), \quad p = \frac{s}{1-q},$$

where the operator $\mathscr{D}_q^{k\alpha}$ is defined in (8.74).

Proof. The proof of this lemma follows by induction on m and by using (9.2). ☐

Similarly, we have the following lemma:

Lemma 9.4. *If* ${}_qL_s f(x) = \Phi(s)$ *then the* q*-Laplace transform of the Caputo sequential* q*-derivative of order* $m\alpha$*,* $0<\alpha<1$*, is given by*

$$ {}_qL_s\, {}^c\mathscr{D}_q^{m\alpha} f(x) = p^{m\alpha}\phi(s) - \frac{1}{1-q}\sum_{r=0}^{m-1} p^{\alpha(m-1-r)\,c}\mathscr{D}_q^{\alpha r} f(0^+). $$

The next lemma will be needed in the proof of the main result of the next section. In the remaining of this chapter by $f^{(k)}(u)$, $k\in\mathbb{N}$, we mean $\frac{d^k}{du^k}f(u)$.

Lemma 9.5. *Let* $\alpha,\ \beta,\ a\in\mathbb{R}^+$ *and* $k\in\mathbb{N}$*. Then the identity*

$$ {}_qL_s\left(t^{k\alpha+\beta-1}e^{(k)}_{\alpha,\beta}(\pm a t^{\alpha};q)\right) = \frac{p^{\alpha-\beta}}{1-q}\frac{k!}{\left(p^{\alpha}\mp a\right)^{k+1}}, \quad |p|^{\alpha}>a \tag{9.4} $$

is valid in the disk $\{t\in\mathbb{C}\,:\,a|t(1-q)|^{\alpha}<1\}$.

Proof. From (7.3) we obtain

$$ e^{(k)}_{\alpha,\beta}(u;q) = \sum_{n=0}^{\infty}\frac{n(n-1)\ldots(n-k+1)}{\Gamma_q(\alpha n+\beta)}u^{n-k}, \quad |u|<(1-q)^{-\alpha}. $$

Thus, for $|at^{\alpha}|<(1-q)^{-\alpha}$ we have

$$ t^{k\alpha+\beta-1}e^{(k)}_{\alpha,\beta}(\pm a t^{\alpha};q) = \sum_{n=0}^{\infty}\frac{n(n-1)\ldots(n-k+1)}{\Gamma_q(\alpha n+\beta)}(\pm a)^{n-k}t^{\alpha n+\beta-1}. $$

Consequently,

$$ \begin{aligned} {}_qL_s\left(t^{k\alpha+\beta-1}e^{(k)}_{\alpha,\beta}(\pm a t^{\alpha};q)\right) &= \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)(\pm a)^{n-k}\,\frac{(1-q)^{\alpha n+\beta-1}}{s^{\alpha n+\beta}} \\ &= \frac{p^{-k\alpha-\beta}}{1-q}\sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)(\pm a p^{-\alpha})^{n-k} \\ &= \frac{p^{-k\alpha-\beta}}{1-q}\,\frac{d^k}{d\xi^k}\sum_{n=0}^{\infty}\xi^n,\ \xi=\pm a p^{-\alpha}. \end{aligned} $$

By choosing $a^{-1}|p|^{\alpha}>1$ we obtain $|\xi|:=|ap^{-\alpha}|<1$. Since

$$\frac{p^{-k\alpha-\beta}}{1-q}\frac{d^k}{d\xi^k}\sum_{n=0}^{\infty}\xi^n = \frac{p^{-k\alpha-\beta}}{1-q}\frac{d^k}{d\xi^k}\frac{1}{1-\xi} = \frac{p^{-k\alpha-\beta}}{1-q}\frac{k!}{(1-\xi)^{k+1}}$$
$$= \frac{p^{-k\alpha-\beta}}{1-q}\frac{k!}{\left(1\mp ap^{-\alpha}\right)^{k+1}} = \frac{p^{\alpha-\beta}}{1-q}\frac{k!}{\left(p^{\alpha}\mp a\right)^{k+1}}.$$

Equation (9.4) follows and the lemma is proved. □

Lemma 9.6. *Let $\phi_{\alpha,l}(x,\lambda_i)$ be the function defined for $l = 1,2,\dots,\mu_i$, $i = 1,\dots,k$, and $\lambda_i \in \mathbb{C}$ by* (8.101)*. Then*

$$_qL_s\left(\phi_{\alpha,l}(x,\lambda_i)\right) = \frac{l!}{1-q}\left(p^{\alpha}-\lambda_i\right)^{-l-1}, \tag{9.5}$$

which is valid for $|p|^{\alpha} > |\lambda_i|$ where $p = \frac{s}{1-q}$.

Proof. From the properties of the q-Laplace transform, we obtain

$$_qL_s\phi_{\alpha,l}(x,\lambda_i) = {}_qL_s\sum_{m=0}^{\infty}(m+l)(m+l-1)\dots(m+1)\frac{\lambda_i^m x^{m\alpha+l\alpha+\alpha-1}}{\Gamma_q(m\alpha+l\alpha+\alpha)}$$
$$= \frac{p^{-l\alpha-\alpha}}{1-q}\sum_{m=0}^{\infty}(m+l)(m+l-1)\dots(m+1)(\lambda_i\, p^{-\alpha})^m.$$

So, for $|\lambda_i| < p^{\alpha}$ we get

$$_qL_s\phi_{\alpha,l}(x,\lambda_i) = \frac{p^{-l\alpha-\alpha}}{1-q}\frac{d^l}{dz^l}\sum_{m=0}^{\infty}z^m\Big|_{z=\lambda_i p^{-\alpha}}$$
$$= \frac{p^{-l\alpha-\alpha}}{(1-q)}\frac{d^l}{dz^l}\frac{1}{1-z}\Big|_{z=\lambda_i p^{-\alpha}}$$
$$= \frac{l!}{(1-q)(p^{\alpha}-\lambda_i)^{l+1}}.$$

□

9.2 A General Solution of Nonhomogeneous Linear Fractional q-Difference Equations

In this section n is a positive integer greater than one, and $\{a_j\}_{j=0}^n$ is a set of complex numbers such that $a_n \neq 0$. Let $\{\beta_j\}_{j=0}^n$ be a set of real numbers such that

$$\beta_n > \beta_{n-1} > \dots > \beta_1 > \beta_0 \geqslant 0. \tag{9.6}$$

Here we are concerned with finding a general solution of the nonhomogeneous linear fractional q-initial value problem

$$\sum_{k=0}^{n} a_k\, D_q^{\beta_k} y(x) = f(x) \quad (x > 0), \tag{9.7}$$

$$D_q^{\beta_k - r} y(0^+) = \begin{cases} 0, \text{ if } \lceil \beta_k \rceil \geq 2, \ r = 2, \dots, \lceil \beta_k \rceil \\ b_k, \qquad\qquad \text{if } r = 1 \end{cases} \qquad (k = 0, 1, \dots, n), \tag{9.8}$$

by using the q-Laplace transform method. Let $p := \frac{s}{1-q}$ and $R > 0$ be such that

$$\sum_{j=0}^{n} a_j\, p^{\beta_j} \neq 0 \quad \text{for } |p| \geqslant R_0 > 0.$$

Let $g_n(p)$ be the function defined for $|p| \geqslant R$ by

$$g_n(p) := \frac{1}{\sum_{j=0}^{n} a_j\, p^{\beta_j}}.$$

Theorem 9.7. *Let $\phi_n(x)$ be the sequence of functions defined for $n = 1, 2, \dots$ and $|x(1-q)| < \left|\frac{a_n}{a_{n-1}}\right|^{1/(\beta_n - \beta_{n-1})}$ by*

$$\begin{aligned} \phi_n(x) := {} & \frac{1-q}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0 + \dots + k_{n-2} = m \\ k_0 \geqslant 0, \dots, k_{n-2} \geqslant 0}} \binom{m}{k_0, k_1, \dots, k_{n-2}} \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \\ & \times x^{(\beta_n - \beta_{n-1})m + \beta_n - \sum_{i=0}^{n-2} (\beta_i - \beta_n) k_i - 1} \\ & \times e^{(m)}_{\beta_n - \beta_{n-1}, \beta_n - \sum_{i=0}^{n-2} (\beta_i - \beta_n) k_i} \left(-\frac{a_{n-1}}{a_n} x^{\beta_n - \beta_{n-1}}; q\right), \end{aligned} \tag{9.9}$$

and defined for $n = 1$ by

$$\phi_1(x) := \frac{1-q}{a_1} x^{\beta_1 - 1} e_{\beta_1 - \beta_0, \beta_1} \left(-\frac{a_0}{a_1} x^{\beta_1 - \beta_0}; q\right), \ |x(1-q)| < \left|\frac{a_1}{a_0}\right|^{1/(\beta_1 - \beta_0)}.$$

Then

$$ {}_q L_s(\phi_n(x)) = g_n(p), \quad n \geqslant 1, \quad g_n(p) := \frac{1}{\sum_{j=0}^{n} a_j p^{\beta_j}}, \quad |s| > R \geqslant R_0. $$

Proof. We choose $R \geqslant R_0$ such that for $|s| \geqslant R$,

$$ \left| \sum_{j=0}^{n-2} a_j p^{\beta_j} \right| < \left| a_{n-1} p^{\beta_{n-1}} + a_n p^{\beta_n} \right|. \tag{9.10} $$

For $|p| \geqslant R$, we have

$$ Y(s) = c g_n(p) + F(s) g_n(p), \ c := \frac{1}{1-q} \sum_{j=0}^{n} a_j b_j. \tag{9.11} $$

Using (9.10) we get

$$ \begin{aligned} g_n(p) &= \frac{1}{a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}} + \ldots + a_0 p^{\beta_0}} = \frac{\left(a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}}\right)^{-1}}{1 + \dfrac{\sum_{j=0}^{n-2} a_j p^{\beta_j}}{a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}}}} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{\left(\sum_{j=0}^{n-2} a_j p^{\beta_j}\right)^m}{\left(a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}}\right)^{m+1}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{a_n p^{\beta_{n-1}}} \frac{\left(\sum_{j=0}^{n-2} \dfrac{a_j}{a_n} p^{\beta_j - \beta_{n-1}}\right)^m}{\left(\dfrac{a_{n-1}}{a_n} + p^{\beta_n - \beta_{n-1}}\right)^{m+1}}. \end{aligned} $$

Applying the multinomial theorem on

$$ \left(\sum_{j=0}^{n-2} \frac{a_j}{a_n} p^{\beta_j - \beta_{n-1}} \right)^m $$

we obtain

$$ \begin{aligned} &\left(\sum_{j=0}^{n-2} \tfrac{a_j}{a_n} p^{\beta_j - \beta_{n-1}}\right)^m \\ &\quad = \sum_{\substack{k_0 + \ldots + k_{n-2} = m \\ k_0 \geqslant 0, \ldots, k_{n-2} \geqslant 0}} \binom{m}{k_0, k_1, \ldots, k_{n-2}} \prod_{j=0}^{n-2} \left(\tfrac{a_j}{a_n}\right)^{k_j} p^{(\beta_j - \beta_{n-1}) k_j} \\ &\quad = \sum_{\substack{k_0 + \ldots + k_{n-2} = m \\ k_0 \geqslant 0, \ldots, k_{n-2} \geqslant 0}} \binom{m}{k_0, k_1, \ldots, k_{n-2}} \left(\prod_{j=0}^{n-2} \left(\tfrac{a_j}{a_n}\right)^{k_j} \right) p^{\sum_{j=0}^{n-2} (\beta_j - \beta_{n-1}) k_j}. \end{aligned} $$

Thus,

$$g_n(p) = \sum_{m=0}^{\infty} \frac{(-1)^m}{a_n} \sum_{\substack{k_0+\ldots+k_{n-2}=m \\ k_0 \geqslant 0,\ldots,k_{n-2} \geqslant 0}} \binom{m}{k_0, k_1, \ldots, k_{n-2}} \prod_{j=0}^{n-2} (\frac{a_j}{a_n})^{k_j} \frac{p^{-\beta_{n-1}+\sum_{j=0}^{n-2}(\beta_j-\beta_n)k_j}}{\left(\frac{a_{n-1}}{a_n} + p^{\beta_n-\beta_{n-1}}\right)^{m+1}}. \tag{9.12}$$

Since the last series in (9.12) is absolutely convergent for $|s| \geqslant R$, then using (1.95), and (9.4) we obtain $g_n(p) = {}_q L_s \phi_n(x)$, $|s| \geqslant R$. □

Theorem 9.8. *The function $y(\cdot)$ defined by*

$$\begin{aligned} y(x) &= \frac{1}{1-q} \left(\sum_{j=0}^{n} a_j I_q^{1-\beta_j} y(0^+) \right) \phi_n(x) + \phi_n(x) * f(x) \\ &= \frac{1}{1-q} \left(\sum_{j=0}^{n} a_j I_q^{1-\beta_j} y(0^+) \right) \phi_n(x) + \frac{1}{1-q} \int_0^x f(t) \varepsilon^{-qt} \phi_n(x) \, d_q t \end{aligned} \tag{9.13}$$

is the general solution of (9.7)–(9.8) *valid in*

$$|x(1-q)| < \left| \frac{a_n}{a_{n-1}} \right|^{1/(\beta_n - \beta_{n-1})}.$$

Proof. Assume that ${}_q L_s\, y(x) = Y(s)$ and ${}_q L_s\, f(x) = F(s)$ for $|s| \geqslant R_0 > 0$. Let $n \in \mathbb{N}$. From (9.2) the q-Laplace transform of (9.7) is

$$\left(\sum_{j=0}^{n} a_j \frac{s^{\beta_j}}{(1-q)^{\beta_j}} \right) Y(s) - \frac{1}{1-q} \sum_{j=0}^{n} a_j I_q^{1-\beta_j} y(0^+) = F(s).$$

Thus,

$$Y(s) = \frac{g_n(p)}{1-q} \sum_{j=0}^{n} a_j I_q^{1-\beta_j} y(0^+) + g_n(p) F(s), \quad |s| > R_0.$$

Consequently, from Theorem 9.7

$$y(x) = \frac{1}{1-q} \left(\sum_{j=0}^{n} a_j I_q^{1-\beta_j} y(0^+) \right) \phi_n(x) + f(x) * \phi_n(x),$$

which is (9.13). □

Example 9.2.1. Consider the fractional q-difference equation

$$D_q^{1/2} y(x) - \lambda y(x) = f(x), \quad I_q y(0^+) = D_q^{-1/2} y(0^+) = 0,\ 0 \leqslant x \leqslant a, \tag{9.14}$$

where f is q-integrable on $[0,a]$, $a > 0$, and $\lambda \in \mathbb{C}$. We have

$$n = 1,\ \beta_1 = 1/2,\ \beta_0 = 0,\ a_1 = 1,\ \text{and } a_0 = -\lambda.$$

Hence, the solution of (9.14) is given for $x \in [0,a]$, $|a(1-q)| < |\lambda|^{-2}$ by

$$y(x) = \int_0^x f(t)\, \varepsilon^{-qt} \left(x^{-1/2} e_{1/2,1/2}(\lambda x^{1/2}; q) \right) d_q t.$$

Example 9.2.2. Consider the q-fractional equation

$$D_q y(x) - D_q^{1/2} y(x) + \lambda y(x) = f(x), \quad 0 \leqslant x \leqslant a,\ \lambda \in \mathbb{C},$$

and f is q-integrable on $[0,a]$. In this case

$$n = 2,\ \beta_2 = 1,\ \beta_1 = 1/2,\ \beta_0 = 0,\ a_2 = a_0 = 1,\ a_1 = -\lambda.$$

Then

$$\phi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, x^{\frac{3m}{2}} e^{(m)}_{1/2,m+1}\left(-\lambda\, x^{1/2}; q \right),$$

and

$$y(x) = c\phi(x) + f(x) * \phi(x), \quad x \in [0,a], \quad |a(1-q)| < |\lambda|^2,$$

where

$$c := \frac{1}{1-q} \sum_{j=0}^{2} a_j\, I_q^{(1-\beta_j)} y(0^+).$$

As for the q-Caputo fractional derivative, we have the following theorems.

Theorem 9.9. *Let $n \in \mathbb{N}$ and let $\beta_k, k = 0, 1, \dots, n$ be real numbers satisfying (9.6). Consider the nth term fractional Caputo q-difference equation*

$$\sum_{k=0}^{n} a_k\, {}^c D_q^{\beta_k} y(x) = f(x),\ x > 0, \tag{9.15}$$

associated with the initial condition

$$y(0^+) = b, \quad y^{(k)}(0^+) = 0 \quad \textit{for } k = 1, \dots, \lceil \beta_n \rceil.$$

Then

$$
\begin{aligned}
y(x) &= \frac{b}{1-q}\phi_n(x) + \phi_n(x) * f(x) \\
&= \frac{b}{1-q}\phi_n(x) + \frac{1}{1-q}\int_0^x f(t)\varepsilon^{-qt}\phi_n(x)\,d_qt
\end{aligned}
$$

is the general solution of (9.15) *valid in*

$$
|x(1-q)| < \left|\frac{a_n}{a_{n-1}}\right|^{1/(\beta_n-\beta_{n-1})}.
$$

Proof. The proof is similar to the proof of Theorem 9.8 and is omitted. □

9.3 General Solutions of Nonhomogeneous Linear Sequential Fractional *q*-Difference Equation

In this section we seek a general solution for the non-homogeneous equation

$$
L_{q,n\alpha}y(x) = \mathscr{D}_q^{n\alpha}y(x) + \sum_{k=0}^{n-1} a_k\mathscr{D}_q^{k\alpha}y(x) = f(x), \tag{9.16}
$$

where $0 < \alpha < 1$, by applying the q-Laplace transform method to derive a particular solution $y_p(x)$ of (9.16).

We need the following lemma in our investigations.

Theorem 9.10. *Let* $\{\lambda_j\}_{j=1}^K$ *be the* K *distinct roots of the multiplicity* $\{\mu_j\}_{j=1}^K$ *of the characteristic polynomial* $P_n(\lambda)$ *associated with the homogeneous equation* $L_{q,n\alpha}y(x) = 0$. *Let* $Q_{n-1}(\lambda)$ *be the polynomial defined by*

$$
Q_{n-1}(\lambda) = \sum_{l=0}^{n-1} d_l\lambda^l, \quad d_l := \sum_{k=0}^{n-l-1} a_k I_q^{1-\alpha}\mathscr{D}_q^{k\alpha}y(0^+).
$$

Let $\{\gamma_{j,r}\}$, $\{\delta_{j,r}\}$, $j = 1,\dots,k$, $r = 1,\dots,\mu_j$ *be the constants satisfying the identities*

$$
\frac{1}{P_n(\lambda)} = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{\delta_{j,r}}{(\lambda-\lambda_j)^r}, \quad \frac{Q_{n-1}(p)}{P_n(p)} = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{\gamma_{j,r}}{(\lambda-\lambda_j)^r}. \tag{9.17}
$$

Then the general solution of (9.16) *is given by*

$$y(x)=\sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{(1-q)}{r-1!}\Big(\gamma_{j,r}\phi_{\alpha,r}(x,\lambda_j)+\delta_{j,r}\,f(x)*\phi_{\alpha,r}(x,\lambda_j)\Big), \tag{9.18}$$

where $\phi_{\alpha,r}(x,\lambda_j)$ *are the functions defined in* (8.101)*.*

Proof. First. It should be noted that the constants $\{\gamma_{j,r}\}$, $\{\delta_{j,r}\}$, $j=1,\dots,k,\ r=1,\dots,\mu_j$ are uniquely determined by applying the method of partial fractions on the functions $\dfrac{1}{P_n(\lambda)}$ and $\dfrac{Q_{n-1}(\lambda)}{P_n(\lambda)}$, respectively. Applying the q-Laplace transform on the two sides of (9.16) gives

$$\left(\sum_{r=0}^{n}a_r\,p^{r\alpha}\right)\phi(s)-\sum_{m=1}^{n}\sum_{l=0}^{m-1}a_{m-1-l}\left(I_q^{1-\alpha}\mathscr{D}_q^{\alpha l}\,y(0^+)\right)\,p^{l\alpha}=F(s),$$

where $p=\frac{s}{1-q}$. Since

$$\begin{aligned}&\textstyle\sum_{m=1}^{n}\sum_{l=0}^{m-1}a_{m-1-l}\left(I_q^{1-\alpha}\mathscr{D}_q^{\alpha l}\,y(0^+)\right)\,p^{l\alpha}\\&\quad=\textstyle\sum_{l=0}^{n-1}\left(\sum_{m=l+1}^{n}a_{m-1-l}\,\left(I_q^{1-\alpha}\mathscr{D}_q^{\alpha(m-l-1)}y(0^+)\right)\right)p^{l\alpha}\\&\quad=\textstyle\sum_{l=0}^{n-1}d_l\,p^{l\alpha}=Q_{n-1}(p^{\alpha}),\end{aligned}$$

we obtain $\phi(s)=\dfrac{Q_{n-1}(p^{\alpha})}{P_n(p^{\alpha})}+\dfrac{F(s)}{P_n(p^{\alpha})}$. In addition to this from (9.17) and (1.92) we obtain

$$\phi(s)=\sum_{j=1}^{k}\sum_{r=1}^{\mu_j}\frac{\gamma_{j,r}}{\left(p^{\alpha}-\lambda_j\right)^r}+\sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{\delta_{j,r}}{\left(p^{\alpha}-\lambda_j\right)^r}F(s).$$

$$\text{Thus,}\quad y(x)=\sum_{j=1}^{K}\sum_{r=1}^{\mu_j}{}_qL_s^{-1}\left(\frac{\gamma_{j,r}}{\left(p^{\alpha}-\lambda_j\right)^r}+F(s)\frac{\delta_{j,r}}{\left(p^{\alpha}-\lambda_j\right)^r}\right). \tag{9.19}$$

From (9.4) and (1.92) we get, for $|p^{\alpha}|>\max\limits_{1\leqslant j\leqslant K}|\lambda_j|$,

$$\frac{1}{\left(p^{\alpha}-\lambda_j\right)^r}={}_qL_s\left(\frac{(1-q)}{r-1!}\phi_{\alpha,r}(x,\lambda_j)\right), \tag{9.20}$$

$${}_qL_s^{-1}\frac{F(s)}{\left(p^{\alpha}-\lambda_j\right)^r}=f(x)*\phi_{\alpha,r}(x,\lambda_j). \tag{9.21}$$

Then substituting from (9.20) and (9.21) in (9.19) gives (9.18) and completes the proof. □

Example 9.3.1. Consider the sequential fractional q-difference equation of order 2α

$$\mathscr{D}_q^{2\alpha} y(x) - \mathscr{D}_q^{\alpha} y(x) - 2y(x) = f(x). \tag{9.22}$$

The characteristic polynomial $P_2(\lambda)$ of (9.22) is given by

$$P_2(\lambda) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

It has two distinct roots $\lambda_1 = -1$ and $\lambda_2 = 2$. Therefore, the general solution of the homogeneous equation associated with (9.22) is given by

$$y(x) = c_1 x^{\alpha-1} e_{\alpha,\alpha}(-x^{\alpha}; q) + c_2 x^{\alpha-1} e_{\alpha,\alpha}(2x^{\alpha}; q),$$

where c_1 and c_2 are arbitrary constants. One can verify that

$$\gamma_{1,1} = \frac{1}{3}(d_1 - d_0),\ \gamma_{2,1} = \frac{1}{3}(2d_1 - d_0),\ \delta_{1,1} = -\frac{1}{3},\ \text{ and } \delta_{2,1} = \frac{1}{3}.$$

That is

$$y_p(x) = \gamma_{1,1}\phi_{\alpha,1}(x,-1) + \gamma_{2,1}\phi_{\alpha,1}(x,2) + \frac{1}{3} f(x) * \left(\phi_{\alpha,1}(x,2) - \phi_{\alpha,1}(x,-1)\right).$$

Now we compute y_p in case of $f(x) = x$. Set $g(x) := \phi_{\alpha,1}(x,2) - \phi_{\alpha,1}(x,-1)$. Then

$$f(x) * g(x) = \frac{1}{1-q}\int_0^x g(t)\varepsilon^{-qt} x\, d_q t = \frac{1}{1-q}\int_0^x g(t)(x - qt)\, d_q t.$$

Applying the q-integration by part rule (1.28) with

$$a = 0,\ b = x,\ u(t) = x - t, \quad \text{and} \quad D_q v(t) = g(t)$$

gives

$$\begin{aligned} f(x) * g(x) &= \frac{1}{1-q}\sum_{m=0}^{\infty}(m+1)\left(2^m + (-1)^{m+1}\right)\frac{x^{m\alpha+2\alpha}}{\Gamma_q(m\alpha + 2\alpha + 1)} \\ &= \frac{x^{2\alpha+1}}{1-q}\left(e^{(1)}_{\alpha,\alpha+2}(2x^{\alpha}; q) - e^{(1)}_{\alpha,\alpha+2}(-x^{\alpha}; q)\right), \end{aligned}$$

where by $f^{(k)}(z)$, $k \in \mathbb{N}$, we mean $\dfrac{d^k}{dz^k} f(z)$. Thus,

$$y_p(x) = \gamma_{1,1}\phi_{\alpha,1}(x,-1) + \gamma_{2,1}\phi_{\alpha,1}(x,2) + \frac{x^{2\alpha+1}}{3(1-q)}\left(e^{(1)}_{\alpha,\alpha+2}(2x^{\alpha}; q) - e^{(1)}_{\alpha,\alpha+2}(-x^{\alpha}; q)\right).$$

Remark 9.3.1. The results of this section hold if we consider linear sequential q-difference equations where the fractional q-derivative is the Caputo q-derivative.

9.4 q-Laplace Transform Method for Solving Certain q-Integral Equations

In Sect. 7.6, we solved q-analogue of Volterra integral equations by using q-analogue of Picard–Lindelöf method of successive approximations. In this section, we use the q-Laplace transform to solve certain q-integral equations. It is known that one of the most important applications of Laplace transform is solving integral equations. Abdi in [4] used the q-Laplace transform to solve q-integral equations of the form

$$\frac{1}{1-q}\int_0^x \varepsilon^{-qy}\,(K(x))\,F(y)\,d_qy = G(x), \tag{9.23}$$

where ε is the q-translation operator defined in (1.13)–(1.14) and

$$K(x) = \sum_{j=0}^{\infty} A_j x^j \quad (|x| < R;\ R > 0).$$

That is

$$\varepsilon^{-qy}\,(K(x)) = \sum_{j=0}^{\infty} A_j x^j\,(qy/x;q)_j\,.$$

Abdi observed that the left hand side of (9.23) is the q-convolution of the functions $K(x)$ and $F(x)$. Hence, (9.23) is

$$K(x) *_q\ F(x) = G(x). \tag{9.24}$$

Assume that the functions K, F, and G are analytic functions and

$$k := {}_qL_sK, \quad f := {}_qL_sF, \quad \text{and} \quad g := {}_qL_sG$$

exist. Using ${}_qL_s\big(K(x) *_q\ F(x)\big) = {}_qL_sK\,{}_qL_sF = fg$, and then applying the q-Laplace transform on (9.24) give

$$f(s) = \frac{g(s)}{k(s)}, \quad \text{or} \quad F(x) = {}_qL_s^{-1}\left(\frac{g(s)}{k(s)}\right).$$

Then using the inversion formula introduced by Hahn in [122], leads to

$$F(x) = \frac{1}{2\pi i}\int_C \frac{g(s)}{k(s)} e_q(sx)\,ds,$$

where the contour of integration C, is a contour encircling the origin and can be modified into a loop parallel to the imaginary axis. If we additionally assume that there exists $R > 0$ such that $g(s)/k(s)$ is analytic in $|s| > R$ then from the inversion formula of Hahn q-Laplace transform, cf. [122, P. 373],

$$F(x) = \frac{1}{x}\sum_{i=0}^{\infty}(-1)^i \frac{q^{\binom{i}{2}}}{(q;q)_i}\frac{g(x^{-1}q^{-i})}{k(x^{-1}q^{-i})} \quad \text{for} \quad |x| < \frac{1}{R}.$$

Abdi introduced the following example in [4].

Example 9.4.1. Consider the q-analogue of Abel's integral equation

$$\frac{x^{-a}}{1-q}\int_0^x (qy/x;q)_{-a} D_q F(y)\, d_q y = G(x) \quad (0 < a < 1). \tag{9.25}$$

Calculating the q-Laplace transform of the two sides of (9.25) gives

$${}_qL_s\, F(x) = \frac{F(0^+)}{s} + s^{-a} g(s)\frac{(1-q)^{a+1}}{\Gamma_q(-a+1)},$$

where we used (1.88) and (9.1) with $n = 1$. Consequently,

$$F(x) = (1-q)F(0^+) + \frac{(1-q)}{\Gamma_q(a)\Gamma_q(-a+1)}\left(x^{a-1} * G(x)\right).$$

Abdi [4] applied the same technique to solve q-integral equations of the second kind of the form

$$F(x) = G(x) + \frac{1}{1-q}\int_0^x \varepsilon^{-qy} K(x)F(y)\, d_q y. \tag{9.26}$$

Assume that the q-Laplace transforms of the functions F, G, and K exist and are analytic in $|S| > R$ for some $R > 0$ and denote them by lowercase letters f, g and k. Then applying the q-Laplace transform on (9.26) gives

$$f(s) = \frac{g(s)}{1-k(s)}.$$

By inversion

$$F(x) = \frac{1}{2\pi i}\int_C \frac{g(s)}{1-k(s)} e_q(sx) ds.$$

In particular, if $\frac{1}{1-k(s)}$ is the q-Laplace transform of some function $H(x)$ then

$$F(x) = G(x) *_q H(x).$$

Remark 9.4.1. Mishra in [214] defined a generalization of the q-Laplace transform (1.87) as follows

$$\begin{aligned} G(s) = {} & \frac{\Gamma_q(\eta+\beta+1)}{(1-q)\Gamma_q(\alpha+\eta+\beta+1)} \\ & \times \int_0^\infty (sx)^\beta {}_2\phi_1\left(q^{\eta+\beta+1}, 0; q^{\alpha+\eta+\beta+1}, q, -sx\right) f(x)\, d_q x, \end{aligned} \tag{9.27}$$

provided that

(i) $\operatorname{Re}\beta > -1$,
(ii) $\sum_{j=0}^\infty \left|q^{(\beta+1)j} f(q^j)\right|$ converges,
(iii) $f(q^{-j}) = O(R^j)$ $(|q| < 1, \quad j > j_0)$.

Then in [217] he made an attempt to solve certain q-integral equation analogues to Fredholm and Volterra types with the help of the generalized q-Laplace transform (9.27).

9.5 q-Mellin Transform of Riemann–Liouville and Weyl Fractional q-Derivatives

Proposition 9.11. *Let* $\alpha > 0$ *and let* f *be a function defined on* $(0,\infty)$. *Assume that* $I_q^\alpha f(t)$ *exists for all* $t \in \mathbb{R}_{q,+}$. *If* $I_q^\alpha f$ *satisfies the sufficient condition on Proposition 1.26 then*

$$\mathscr{M}_q(I_q^\alpha f)(s) = \frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)}(\mathscr{M}_q f)(s+\alpha) \quad (\operatorname{Re}(s+\alpha) < 1)\,. \tag{9.28}$$

Proof.

$$\begin{aligned} \mathscr{M}_q(I^\alpha f)(s) &= \int_0^\infty t^{s-1} I_q^\alpha f(t)\, d_q t \\ &= (1-q)\int_0^\infty \frac{t^{s+\alpha-1}}{\Gamma_q(\alpha)} \sum_{k=0}^\infty q^k (q^{k+1};q)_{\alpha-1} f(tq^k)\, d_q t. \end{aligned}$$

The conditions on the function f allow us to interchange the order of summation and the q-integration on the last identity. Therefore, applying property (1.104) of the q-Mellin transform gives

$$\begin{aligned}\mathscr{M}_q(I^\alpha f)(s) &= \frac{1}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} q^k(1-q)(q^{k+1};q)_{\alpha-1}\mathscr{M}_q(f(tq^k))(s+\alpha)\\ &= \frac{1}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} q^k(1-q)(q^{k+1};q)_{\alpha-1}q^{k(-s-\alpha)}\mathscr{M}_q(f)(s+\alpha)\\ &= \frac{B_q(\alpha,1-s-\alpha)}{\Gamma_q(\alpha)}\mathscr{M}_q(f)(s+\alpha) = \frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)}\mathscr{M}_q(f)(s+\alpha),\end{aligned}$$

and completes the proof. □

Proposition 9.12. *For* $\alpha > 0$ *and* $n = \lceil\alpha\rceil$

$$\mathscr{M}_q(D_q^\alpha f)(s) = \frac{\Gamma_q(1-s+\alpha)}{\Gamma_q(1-s)}(\mathscr{M}_q f)(s-\alpha) + \sum_{k=0}^{n-1}(-1)^k\frac{\Gamma_q(s)}{\Gamma_q(s-k)} \tag{9.29}$$

$$\times\left[\lim_{j\to\infty} q^{-j(s-k-1)}D_q^{n-k-1}I_q^{n-\alpha}f(q^{-j}) - \lim_{j\to\infty} q^{j(s-k-1)}D_q^{n-k-1}I_q^{n-\alpha}(q^j)\right],$$

If $f(t)$ *and* $\mathrm{Re}(s)$ *are such that substitutions of the limits values in* (9.29) *are zeros, then*

$$\mathscr{M}_q(D_q^\alpha f)(s) = \frac{\Gamma_q(1-s+\alpha)}{\Gamma_q(1-s)}(\mathscr{M}_q f)(s-\alpha). \tag{9.30}$$

Proof. Since $\mathscr{M}_q(D_q^\alpha f)(s) = \mathscr{M}_q(D_q^n I_q^{n-\alpha} f)(s)$, the proof of (9.29) follows from (9.28) and (1.105) and the proof of (9.30) follows from (9.28) and (1.106). □

Proposition 9.13.

$$\mathscr{M}_q\left(x^\alpha D_q^\alpha f\right)(s) = \frac{\Gamma_q(1-s)}{\Gamma_q(1-s-\alpha)}(\mathscr{M}_q f)(s). \tag{9.31}$$

Proof. The proof follows by applying (9.30) and (1.104). □

Proposition 9.14. *Let* $\alpha \in \mathbb{R}$ *and let* f *be a function defined on* $(0,\infty)$ *such that* $f \in \mathscr{S}_{q,\mu}$, $\mu > 0$. *If* $K_q^\alpha f$ *satisfies the sufficient condition on Proposition 1.26 then*

$$\mathscr{M}_q\left((K_q^\alpha f)(xq^{-\alpha})\right)(s) = q^{\frac{\alpha(\alpha+1)}{2}}\frac{\Gamma_q(s)}{\Gamma_q(s-\alpha)}(\mathscr{M}_q f)(s-\alpha) \tag{9.32}$$

and

$$\mathscr{M}_q\left(x^\alpha\left(K_q^\alpha f\right)(xq^{-\alpha})\right)(s) = q^{\frac{\alpha(\alpha+1)}{2}}\frac{\Gamma_q(s+\alpha)}{\Gamma_q(s)}(\mathscr{M}_q f)(s), \tag{9.33}$$

where either $\mathrm{Re}\, s > 0$ *or* $s \in \mathbb{C}$ *and* $\alpha \in \mathbb{N}_0$.

Proof. Applying (5.20) with α is replaced by $-\alpha$. This gives

$$\left(K_q^{\alpha} f\right)(xq^{-\alpha}) = q^{\frac{\alpha(\alpha+1)}{2}}(1-q)^{-\alpha}x^{-\alpha}\sum_{k=0}^{\infty} q^{k\alpha}\frac{(q^{-\alpha};q)_k}{(q;q)_k} f(xq^{-k}).$$

Hence,

$$\begin{aligned}&\mathscr{M}_q\left((K_q^{\alpha} f)(xq^{-\alpha})\right)(s)\\ &= q^{\frac{\alpha(\alpha+1)}{2}}(1-q)^{-\alpha}\sum_{k=0}^{\infty} q^{k\alpha}\frac{(q^{-\alpha};q)_k}{(q;q)_k}\int_0^{\infty} x^{s-\alpha-1} f(xq^{-k})\, d_qx,\end{aligned} \tag{9.34}$$

where the conditions on the function f allow us to interchange the order of the q-integration and the summation. Applying the substitution $y = xq^{-k}$ on the q-integral in the right hand side of (9.34), we obtain

$$\begin{aligned}\mathscr{M}_q\left((K_q^{\alpha} f)(xq^{-\alpha})\right)(s) &= q^{\frac{\alpha(\alpha+1)}{2}}(1-q)^{-\alpha}\left(\mathscr{M}_q f\right)(s-\alpha)\sum_{k=0}^{\infty} q^{ks}\frac{(q^{-\alpha};q)_k}{(q;q)_k}\\ &= q^{\frac{\alpha(\alpha+1)}{2}}(1-q)^{-\alpha}\frac{(q^{s-\alpha};q)_\infty}{(q^s;q)_\infty}\left(\mathscr{M}_q f\right)(s-\alpha)\\ &= q^{\frac{\alpha(\alpha+1)}{2}}\frac{\Gamma_q(s)}{\Gamma_q(s-\alpha)}\left(\mathscr{M}_q f\right)(s-\alpha),\end{aligned}$$

completing the proof of (9.32). The proof of (9.33) follows by applying (1.104) and (9.32). □

9.5.1 q-Mellin Transform Method for Solving Nonhomogeneous Fractional q-Difference Equation

In this section we use the q-Mellin transform to solve fractional q-difference equation of the forms

$$\sum_{k=0}^{m} A_k x^{\alpha+k} D_q^{\alpha+k} y(x) = f(x) \quad (x > 0,\ \alpha > 0), \tag{9.35}$$

$$\sum_{k=0}^{m} B_k x^{\alpha+k}\left(K_q^{\alpha+k} y\right)(xq^{-\alpha-k}) = f(x) \quad (x > 0,\ \alpha > 0). \tag{9.36}$$

Assume that the $\mathscr{M}_q(f)(s)$ exists for $s \in \mathbb{C}$ such that $\alpha_{q,f} < \mathrm{Re}(s) < \beta_{q,f}$, where $\langle \alpha_{q,f}, \beta_{q,f} \rangle$ is the fundamental strip of the function f. Applying the q-Mellin transform to (9.35) and (9.36) and using the linearity of the operator $\mathscr{M}_q$, (9.31) and (9.33) give

$$\left[\sum_{k=0}^{m} A_k \frac{\Gamma_q(1-s)}{\Gamma_q(1-s-\alpha-k)}\right](\mathscr{M}_q y)(s) = (\mathscr{M}_q f)(s) \tag{9.37}$$

and

$$\left[\sum_{k=0}^{m} B_k q^{\frac{(\alpha+k)(\alpha+k+1)}{2}} \frac{\Gamma_q(s+\alpha+k)}{\Gamma_q(s)}\right](\mathscr{M}_q y)(s) = (\mathscr{M}_q f)(s), \tag{9.38}$$

respectively. Set

$$P_\alpha(s) = \sum_{k=0}^{m} A_k \frac{\Gamma_q(1-s)}{\Gamma_q(1-s-\alpha-k)},$$

and

$$Q_\alpha(s) = \sum_{k=0}^{m} B_k q^{\frac{(\alpha+k)(\alpha+k+1)}{2}} \frac{\Gamma_q(s+\alpha+k)}{\Gamma_q(s)}.$$

Then using the inverse q-Mellin transform in (9.37) and (9.38), we derive the following solutions to the equations (9.35) and (9.36) in respective forms:

$$\begin{aligned} y(x) &= \mathscr{M}_q^{-1}\left[\frac{1}{P_\alpha(s)}\mathscr{M}_q(f)(s)\right] = \mathscr{M}_q^{-1}\left(\mathscr{M}_q(G_\alpha(x) *_{\mathscr{M}_q} f(x))\right) \\ &= \int_0^\infty G_\alpha(t) f\left(\frac{x}{t}\right)\frac{d_q t}{t}, \end{aligned}$$

and

$$\begin{aligned} y(x) &= \mathscr{M}_q^{-1}\left[\frac{1}{Q_\alpha(s)}\mathscr{M}_q(f)(s)\right] = \mathscr{M}_q^{-1}\left(\mathscr{M}_q(G_\alpha(x) *_{\mathscr{M}_q} f(x))\right) \\ &= \int_0^\infty H_\alpha(t) f\left(\frac{x}{t}\right)\frac{d_q t}{t}, \end{aligned}$$

where

$$G_\alpha(x) = \mathscr{M}_q^{-1}\left(\frac{1}{P_\alpha(s)}\right), \quad H_\alpha(x) = \mathscr{M}_q^{-1}\left(\frac{1}{Q_\alpha(s)}\right),$$

9.5.2 Equation with Riemann–Liouville Fractional q-Derivative

We consider in this section as an illustrative example the fractional Riemann–Liouville q-difference equation

$$x^{\alpha+1} D_q^{\alpha+1} y(x) + \lambda x^{\alpha} D_q^{\alpha} y(x) = f(x) \quad (x > 0,\ \alpha > 0,\ \lambda \in \mathbb{R})\,. \tag{9.39}$$

Applying the q-Mellin transform method and using (9.31), we obtain

$$\left[\frac{\Gamma_q(1-s)}{\Gamma_q(-s-\alpha)} + \lambda \frac{\Gamma_q(1-s)}{\Gamma_q(1-s-\alpha)}\right] \mathscr{M}_q(y)(s) = F(s).$$

We shall consider two cases: $\lambda(1-q) \neq -1$ and $\lambda(1-q) = -1$.

The Case $\boldsymbol{\lambda(1-q) \neq -1}$. In this case we have

$$(\mathscr{M}_q y)(s) = \frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)} \frac{1-q}{1+\lambda(1-q)-q^{-s-\alpha}} (\mathscr{M}_q f)(s).$$

We set

$$\frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)} = \mathscr{M}_q(G_1)(s), \qquad \frac{1-q}{1+\lambda(1-q)-q^{-s-\alpha}} = \mathscr{M}_q(G_2)(s).$$

One can prove by a direct application of the q-Mellin transform definition that for $\mathrm{Re}(s) \leq \min\{\mathrm{Re}(p)-\alpha, 1-\alpha\}$

$$G_1(t) = \begin{cases} \dfrac{t^{\alpha-1}(q/t;q)_{\alpha-1}}{\Gamma_q(\alpha)}, & t \geq 1, \\ 0, & t < 1, \end{cases}$$

and

$$G_2(t) = \begin{cases} \dfrac{t^{\alpha-p}}{1+\lambda(1-q)}, \quad q^p = \frac{1}{1+\lambda(1-q)}, & t \geq 1, \\ 0, & t < 1. \end{cases}$$

Hence, for $x \in \mathbb{R}_{q,+}$

$$\begin{aligned} G_\alpha(x) = G_1(x) *_{\mathscr{M}_q} G_2(x) &= \int_0^\infty G_1(t) G_2\left(\frac{x}{t}\right) \frac{d_q t}{t} \\ &= \int_1^\infty G_1(qt) G_2\left(\frac{x}{qt}\right) \frac{d_q t}{t}. \end{aligned}$$

One can prove that

$$G_\alpha(x) = \begin{cases} 0, & x \in \{q^n,\ n \in \mathbb{N}\}, \\ \dfrac{1-q}{1+\lambda(1-q)} \dfrac{x^{\alpha-p}}{\Gamma_q(\alpha)} \sum_{k=0}^{n} q^{k(1-p)}(q^{k+1};q)_{\alpha-1}, & x \in \{q^{-n},\ n \in \mathbb{N}_0\}. \end{cases}$$

If we assume that $p < 1$ then the function $G(x)$ can be defined on $(0,\infty)$ through the formula

$$G_\alpha(x) = 0 \quad \text{if } x < 1$$

and

$$G_\alpha(x) = \frac{1-q}{1+\lambda(1-q)} x^{\alpha-p} \frac{\Gamma_q(1-p)}{\Gamma_q(1-p+\alpha)}$$
$$-\frac{1-q}{1+\lambda(1-q)} x^{\alpha-1} \frac{(q^{\alpha+1-p};q)_\infty}{(q^{1-p};q)_\infty} {}_2\phi_1\left(q^{1-\alpha}, q^{1-p}; q^{\alpha+1-p}; q, q^\alpha/x\right)$$

if $x \geq 1$.

The Case $\boldsymbol{\lambda(1-q) = -1}$. In this case applying the q-Mellin transform on (9.39) gives

$$\mathscr{M}_q(y(s)) = -q^{s+\alpha}(1-q)\frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)}\mathscr{M}_q(f)(s).$$

It is straightforward to see that

$$\mathscr{M}_q(G_\alpha)(s) = -q^{s+\alpha}(1-q)\frac{\Gamma_q(1-s-\alpha)}{\Gamma_q(1-s)},\ \operatorname{Re}(s) < 1-\alpha,$$

where

$$G_\alpha(x) = \begin{cases} -\frac{q(1-q)}{\Gamma_q(\alpha)} x^{\alpha-1}(q^2/x;q)_{\alpha-1}, & x > 1, \\ 0, & x \leq 1. \end{cases}$$

Therefore,

$$y(x) = -\frac{q(1-q)}{\Gamma_q(\alpha)} \int_1^\infty t^{\alpha-1}(q^2/t;q)_{\alpha-1} f\,(t/x)\; d_qt \quad (x > 0).$$

9.5.3 Equation with Weyl Fractional q-Derivative

In this subsection we will discuss , as an illustrating example, the Weyl fractional q-difference equation

$$x^{\alpha+1}\left(K_q^{\alpha+1}y\right)(xq^{-\alpha-1}) + \lambda x^\alpha \left(K_q^\alpha y\right)(xq^{-\alpha}) = f(x), \tag{9.40}$$

where

$$x > 0, \quad \alpha > 0, \quad \text{and} \quad \lambda \leq \frac{q^{2\alpha+1}}{1-q}.$$

Applying the q-Mellin transform method gives

$$q^{\frac{\alpha(\alpha+1)}{2}} \frac{\Gamma_q(s+\alpha)}{\Gamma_q(s)} \frac{q^{\alpha+1} + \lambda(1-q) - q^{s+2\alpha+1}}{1-q} \mathscr{M}_q(y)(s) = (\mathscr{M}_q f)(s).$$

We will consider two cases: $\lambda \neq -\frac{q^{\alpha+1}}{1-q}$ and $\lambda = -\frac{q^{\alpha+1}}{1-q}$.

The Case $\boldsymbol{\lambda(1-q) \neq -q^{\alpha+1}}$. In this case we have

$$(\mathscr{M}_q y)(s) = q^{-\frac{\alpha(\alpha+1)}{2}} \frac{\Gamma_q(s)}{\Gamma_q(s+\alpha)} \frac{q^p(1-q)}{(1-q^{s+\alpha+p})} (\mathscr{M}_q f)(s),$$

where

$$q^p := \frac{q^{\alpha+1}}{q^{\alpha+1} + \lambda(1-q)}.$$

We set

$$\frac{\Gamma_q(s)}{\Gamma_q(s+\alpha)} = \mathscr{M}_q(H_1)(s), \quad q^{-\frac{\alpha(\alpha+1)}{2}} \frac{q^p(1-q)}{(1-q^{s+\alpha+p})} = \mathscr{M}_q(H_2)(s).$$

One can prove by a direct application of the q-Mellin transform definition that for $\mathrm{Re}(s) > \max(0, -\alpha - \mathrm{Re}(p))$

$$H_1(t) = \begin{cases} \frac{(qt;q)_{\alpha-1}}{\Gamma_q(\alpha)}, & 0 \leq t \leq 1, \\ 0, & t > 1, \end{cases}$$

$$H_2(t) = \begin{cases} q^{-\frac{\alpha(\alpha+1)}{2}} q^p t^{\alpha+p}, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

Hence, for $x \in \mathbb{R}_{q,+}$

$$\mathscr{M}_q(H_\alpha(x)) = \mathscr{M}_q(H_1(x))(s) \mathscr{M}_q(H_2(x))(s) = \mathscr{M}_q(H_1(x) *_{\mathscr{M}_q} H_2(x))(s).$$

Thus,

$$H_\alpha(x) = \int_0^\infty H_1(t) H_2(x/t) \frac{d_q t}{t} = \int_0^1 H_1(t) H_2(x/t) \frac{d_q t}{t},$$

where $x \in \mathbb{R}_{q,+}$. One can prove that $H_\alpha(x) = 0$ for all $x > 1$ and

$$H_\alpha(q^n) = q^p q^{-\frac{\alpha(\alpha+1)}{2}} \frac{(q^{n+1};q)_\infty}{(q^{n+\alpha};q)_\infty \Gamma_q(\alpha)} \sum_{k=0}^{n} (1-q) q^{(\alpha+p+1)k} \frac{(q^{-n};q)_k}{(q^{-n-\alpha+1};q)_k}.$$

If we assume that $p > -1$, then the function $H_\alpha(x)$ can be defined on $(0,\infty)$ through the formula

$$H_\alpha(x) = 0 \quad \text{if} \;\; x > 1$$

and

$$H_\alpha(x) = q^p(1-q)q^{-\frac{\alpha(\alpha+1)}{2}} \frac{(qx;q)_{\alpha-1}}{\Gamma_q(\alpha)} {}_2\phi_1\left(1/x, q; q^{-\alpha+1}/x; q, q^{p+1}\right), \text{ if } \; x \leq 1.$$

The Case $q^{\alpha+1} + \lambda(1-q) = 0$. In this case, applying the q-Mellin transform on (9.40) gives

$$\mathscr{M}_q(y(s)) = -q(1-q)q^{-\frac{\alpha(\alpha+5)}{2}} q^{-s} \frac{\Gamma_q(s)}{\Gamma_q(s+\alpha+1)} \mathscr{M}_q(f)(s) \; (\mathrm{Re}(s) > 0).$$

It is straightforward to see that in this case

$$H_\alpha(x) = \begin{cases} -q(1-q)q^{-\frac{\alpha(\alpha+5)}{2}} \dfrac{(q^2x;q)_\alpha}{\Gamma_q(\alpha+1)}, & x \leq q^{-1}, \\ \qquad 0, & x > q^{-1}. \end{cases}$$

Therefore,

$$y(x) = \frac{-q(1-q)q^{-\frac{\alpha(\alpha+5)}{2}}}{\Gamma_q(\alpha+1)} \int_0^{1/q} (q^2t;q)_\alpha f\,(t/x)\; d_qt \;\; (x > 0)\,.$$

9.6 q^2-Fourier Transform Method for Solving Fractional q-Difference Equations

In [136], Ho explored the possibility of using the classical Fourier and Mellin integral transforms to solve the class of q-difference differential equations.

$$D_{q,t}^n u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \;\; x \in \mathbb{R}, \;\; t > 0, \;\; n \in \mathbb{N}, \tag{9.41}$$

with the initial conditions

$$y(x,0) = f(x), \quad D_{q,t}^k y(x,t)|_{t=0^+} = g_k(x) \quad (k = 1, \ldots, n-1)\,,$$

where the functions $f(x)$ and $g_k(x)$ are assumed to vanish as $x \to \pm\infty$. In [64] Brahim and Quanes used the q^2-Fourier transform and the q-Mellin transform to solve (9.41) in case of $n = 1$ and 2 and only for

$$q \in \left\{q \in (0,1) : 1-q = q^{2m}, \; \text{for some} \;\; m \in \mathbb{Z}\right\}.$$

In this section, we use the q^2-Fourier transform with the ${}_q L_s$ transform to solve the q-fractional diffusion equation

$$D^{\alpha}_{q,t} u(x,t) = \lambda \partial^2_{q,x} u(x,t), \tag{9.42}$$

$$x \in \widetilde{\mathbb{R}}_q,\ t \in \mathbb{R},\ 0 < \alpha < 1, ; 0 < q < 1,$$

with the initial conditions

$$D^{\alpha-1}_{q,t} u(x,t)|_{t=0+} = \phi(x), \quad \partial^k_{q,x} u(x,t) \in L^1_q(\mathbb{R}_q) \quad (k = 0,1), \tag{9.43}$$

$$\phi \in \left(L^1_q \cap L^2_q\right)(\mathbb{R}_q). \tag{9.44}$$

Theorem 9.15. *The solution of the q-fractional diffusion equation* (9.42) *subject to the initial conditions* (9.43)–(9.44) *is given by*

$$u(x,t) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \left[T_y G(x,t)\right](x) \phi(y)\, d_q y,$$

where

$$\left[T_y G(x,t)\right](x) = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(\frac{1}{2})} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e(-iy\xi; q^2) g(\xi,t) e(ix\xi; q^2)\, d_q \xi,$$

and

$$g(\xi,t) := \begin{cases} t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2 t^{\alpha}; q), & |\lambda\xi^2 t^{\alpha}| < \frac{1}{(1-q)^{\alpha}}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. First we calculate the q^2-Fourier transform of (9.42) with respect to the variable x. Hence, applying (3.79) yields

$$D^{\alpha}_{q,t} U(\xi,t) = -\lambda\xi^2 U(\xi,t), \tag{9.45}$$

where

$$U(\xi,t) := \mathscr{F}_{q,x}(u(x,t))(\xi).$$

Now we calculate the ${}_q L_s$ transform of (9.45) with respect to the variable t. Using (9.2) we obtain

$$\left(P^{\alpha} + \lambda\xi^2\right) V(\xi,s) = \frac{\mathscr{F}_q(\phi)(\xi)}{1-q},$$

where

$$V(\xi,s) = {}_{q,t}\mathrm{L}_s\left(U(\xi,t)\right)(s) \text{ and } p = \frac{s}{1-q}.$$

One can verify that

$$ {}_{q,t}\mathrm{L}_s \left(t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2\, t^{\alpha}; q)\right) = \frac{1}{1-q}\,\frac{1}{p^{\alpha}+\lambda\,\xi^2}, $$

for $\left|\lambda\xi^2 p^{-\alpha}\right| < 1$. Consequently,

$$ U(\xi,t) = \mathscr{F}_q(\phi)(\xi)\, t^{\alpha-1} e_{\alpha,\alpha}\left(-\lambda\,\xi^2\, t^{\alpha}\right), \left|\lambda\xi^2\, t^{\alpha}\right| < \frac{1}{(1-q)^{\alpha}}. $$

It follows from the inversion formula of the q^2-Fourier transform that

$$ t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2\, t^{\alpha}; q) = \mathscr{F}_{q,x}\left(G(x,t)\right)(\xi), $$
$$ G(x,t) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2\, t^{\alpha}; q) e(i\,\xi\, x; q^2)\, d_q\xi, $$

where the variable of the q-integration ξ runs only over all $\xi \in \widetilde{\mathbb{R}}_q$ such that

$$ \left|\lambda\,\xi^2\, t^{\alpha}\right| < \frac{1}{(1-q)^{\alpha}}. $$

Consequently,

$$ u(x,t) = \phi(x) * G(x,t). $$

Applying the q^2- Fourier convolution formula gives

$$ u(x,t) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \left[T_y G(x,t)\right](x)\phi(y)\, d_q y, $$

where

$$ \left[T_y G(x,t)\right](x) $$
$$ = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e(-iy\xi; q^2) t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2\, t^{\alpha}; q) e(ix\xi; q^2)\, d_q\xi. $$

□

Appendix A
Tables of Fractional Derivatives and q-Derivatives

In this appendix, we collect the Riemann–Liouville fractional derivative and Caputo fractional of some q-analogues of the celebrated special functions and we also include a table of Riemann–Liouville fractional derivative for comparison.

A.1 Table of Riemann–Liouville Fractional Derivatives

Table A.1 Riemann–Liouville fractional derivatives

$\phi(x)$	$\big(D_{0+}^{\alpha}\phi\big)(x),\ x>0,\ \alpha>0$
$x^{\beta-1}$	$\dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1},\ \beta>0$
$e^{\lambda x}$	$(x)^{-\alpha}E_{1,1-\alpha}(\lambda x)$
$x^{\beta-1}e^{\lambda x}$	$\dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1}{}_1F_1(\beta;\beta-\alpha;\lambda x)$
$\cos(\lambda x)$	$x^{-\alpha}E_{1/2,1-\alpha}(-\lambda^2x^2)$
$\sin(\lambda(x-a))$	$x^{1-\alpha}\lambda E_{1/2,2-\alpha}(-\lambda^2x^2)$
$x^{\beta-1}E_{\mu,\beta}(\lambda x^{\mu})$	$x^{\beta-\alpha-1}E_{\mu,\beta-\alpha}(\lambda x^{\mu}),\ \beta,\ \mu>0$
$x^{\beta-1}\,{}_2F_1\left(\mu,\nu;\beta;\lambda x\right)$	$\dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}x^{\beta-\alpha+1}\,{}_2F_1\left(\mu,\nu;\beta-\alpha;\lambda x\right),\ \beta>0$

M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations*,
Lecture Notes in Mathematics 2056, DOI 10.1007/978-3-642-30898-7,

A.2 Table of Riemann–Liouville Fractional q-Derivatives

Table A.2 Riemann–Liouville fractional q-derivatives

ϕ	$D_q^\alpha \phi \quad x > 0, \quad \alpha > 0$
$x^{\beta-1},\ \beta > 0$	$x^{\beta-\alpha-1}\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}$
$e_q(\lambda x)$	$x^{-\alpha}e_{1,1-\alpha}(\lambda x(1-q)^{-1};q)$
$E_q(\lambda x)$	$x^{-\alpha}E_{1,1-\alpha}(\lambda x(1-q)^{-1};q)$
$x^{\beta-1}e_q(\lambda x)$	$\frac{x^{\beta-\alpha-1}\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}{}_2\phi_1(0,q^\beta;q^{\beta-\alpha},q,\lambda x),\ \beta > 0$
$x^{\beta-1}E_q(\lambda x)$	$\frac{x^{\beta-\alpha-1}\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}{}_1\phi_1(q^\beta;q^{\beta-\alpha};q,\lambda x),\ \beta > 0$
$\cos_q \lambda x$	$x^{-\alpha}e_{2,1-\alpha}(-\lambda^2x^2(1-q)^{-2};q)$
$\sin_q \lambda x$	$\lambda(1-q)^{-1}x^{1-\alpha}e_{2,2-\alpha}(-\lambda^2x^2(1-q)^{-2};q)$
$\mathrm{Cos}_q\lambda x$	$\frac{x^{-\alpha}}{\Gamma_q(1-\alpha)}{}_1\phi_2(q^2;q^{2-\alpha},q^{1-\alpha};q^2,-q\lambda^2x^2)$
$\mathrm{Sin}_q\lambda x$	$\frac{\lambda x^{1-\alpha}}{\Gamma_q(1-\alpha)}{}_1\phi_2(q^2;q^{2-\alpha},q^{1-\alpha};q^2,-q^3\lambda^2x^2)$
$\cos(\lambda x;q)$	$x^{-\alpha}E_{2,1-\alpha}(-q\lambda^2x^2;q)$
$\sin(\lambda x;q)$	$\lambda x^{1-\alpha}E_{2,2-\alpha}(-q^2\lambda^2x^2;q)$
$x^{\beta-1}E_{\mu,\beta}(\lambda x^\mu;q)$	$x^{\beta-\alpha-1}E_{\mu,\beta-\alpha}(\lambda x^\mu;q),\ \beta,\ \mu > 0$
$x^{\beta-1}e_{\mu,\beta}(\lambda x^\mu;q)$	$x^{\beta-\alpha-1}e_{\mu,\beta-\alpha}(\lambda x^\mu;q),\ \beta,\mu > 0$
$x^{\beta-1}\,{}_2\phi_1\left(a,b;q^\beta;q,\lambda x\right)$	$\dfrac{\Gamma_q(\beta)x^{\beta-\alpha-1}}{\Gamma_q(\beta-\alpha)}\,{}_2\phi_1\left(a,b;q^{\beta-\alpha};q,\lambda x\right),\ \beta > 0$
$x^{\beta-1}\,{}_2\phi_1\left(a,b;c;q,\lambda x\right)$	$\dfrac{\Gamma_q(\beta)x^{\beta-\alpha-1}}{\Gamma_q(\beta-\alpha)}\,{}_3\phi_2\left(a,b,q^\beta;c,q^{\beta-\alpha};q,\lambda x\right),\ \beta > 0$

A.3 Table of the Erdéli–Kober Fractional q-Integral Operator

The next table contains the Erdéli–Kober fractional integrals for some q-functions. An extended table can be found in [271].

Table A.3 The integral operator $I_q^{\eta,\alpha}$

ϕ	$I_q^{\eta,\alpha}\phi \quad (x > 0)$
$x^{\beta-1}$	$x^{\beta-1}\dfrac{\Gamma_q(\eta+\beta)}{\Gamma_q(\eta+\beta+\alpha)}$, $\mathrm{Re}\,(\beta+\eta) > 0$
$x^{\beta-1}e_q(\lambda x)$, $\mathrm{Re}\,(\beta+\eta) > 0$	$x^{\beta-1}\dfrac{\Gamma_q(\eta+\beta)}{\Gamma_q(\eta+\beta+\alpha)}{}_2\phi_1\left(0, q^{\eta+\beta}; q^{\eta+\beta+\alpha}; q, \lambda x\right)$
$x^{\beta-1}E_q(\lambda x)$, $\mathrm{Re}\,(\beta+\eta) > 0$	$x^{\beta-1}\dfrac{\Gamma_q(\eta+\beta)}{\Gamma_q(\eta+\beta+\alpha)}{}_1\phi_1\left(q^{\eta+\beta}; q^{\eta+\alpha+\beta}; q, -\lambda x\right)$
$x^{\beta-1}\cos_q \lambda x$, $\mathrm{Re}\,(\beta+\eta) > 0$	$x^{\beta-1}\dfrac{\Gamma_q(\eta+\beta)}{\Gamma_q(\eta+\beta+\alpha)}\times$ ${}_4\phi_3(0, 0, q^{\eta+\beta}, q^{\eta+\beta+1}; q^{\beta+\alpha+\eta}, q^{\beta+\alpha+\eta+1}; q^2, -\lambda^2x^2)$
$x^{\beta-1}\,\mathrm{Cos}_q(\lambda x)$, $\mathrm{Re}\,(\beta+\eta) > 0$	$\dfrac{x^{\beta-1}\Gamma_q(\eta+\beta)}{\Gamma_q(\eta+\beta+\alpha)}\times$ ${}_2\phi_3(q^{\eta+\beta}, q^{\eta+\beta+1}; q, q^{\eta+\alpha+\beta}, q^{\eta+\alpha+\beta+1}; q^2, -q\lambda^2x^2)$,
$x^{\beta-1}\cos(\lambda x; q)$, $\mathrm{Re}\,(\beta+\eta) > 0$	$x^{\beta-1}\dfrac{\Gamma_q(\beta+\eta)}{\Gamma_q(\beta+\eta+\alpha)}\times$ ${}_3\phi_3\left(0, q^{\beta+\eta}, q^{\beta+\eta+1}; q, q^{\beta+\eta+\alpha}, q^{\beta+\eta+\alpha+1}; q^2, q\lambda^2(1-q)^2x^2\right)$
$x^{\beta-1}\sin_q \lambda x$, $\mathrm{Re}(\beta+\eta) > -1$	$\dfrac{\lambda x^{\beta}\Gamma_q(\eta+\beta+1)}{(1-q)\Gamma_q(\eta+\beta+\alpha+1)}\times$ ${}_4\phi_3\left(0, 0, q^{\eta+\beta+1}, q^{\eta+\beta+2}; q^3, q^{\eta+\beta+\alpha+1}, q^{\eta+\beta+\alpha+2}; q^2, -\lambda^2x^2\right)$
$x^{\beta-1}\,\mathrm{Sin}_q \lambda x$, $\mathrm{Re}(\beta+\eta) > -1$	$\dfrac{x^{\beta}\lambda\Gamma_q(\eta+\beta+1)}{(1-q)\Gamma_q(\eta+\beta+\alpha+1)}\times$ ${}_2\phi_3\left(q^{\eta+\beta+1}; q^{\eta+\beta+2}; q^3, q^{\eta+\beta+1+\alpha}, q^{\eta+\beta+\alpha+2}; q^2, -q^3\lambda^2x^2\right)$
$x^{\beta-1}\sin(\lambda x; q)$ $\mathrm{Re}(\beta+\eta) > -1$	$\lambda x^{\beta}\dfrac{\Gamma_q(\beta+\eta+1)}{\Gamma_q(\beta+\eta+\alpha+1)}\times$ ${}_3\phi_3\left(0, q^{\beta+\eta+1}, q^{\beta+\eta+2}; q^3, q^{\beta+\eta+\alpha+1}, q^{\beta+\eta+\alpha+2}; q^2,\right.$ $\left.q^2\lambda^2(1-q)^2x^2\right)$
$x^{\beta-1}E_{\mu,\beta+\eta}(\lambda x^{\mu}; q)$, $\mathrm{Re}(\beta+\eta) > 0$	$x^{\beta-1}E_{\mu,\beta+\eta+\alpha}(\lambda x^{\mu}; q)$, $\mathrm{Re}(\mu) > 0$
$x^{\beta-1}e_{\mu,\beta+\eta}(\lambda x^{\mu}; q)$ $\mathrm{Re}(\beta+\eta) > 0$	$x^{\beta-1}e_{\mu,\beta+\eta+\alpha}(\lambda x^{\mu}; q)$, $\mathrm{Re}(\mu) > 0$
$x^{\beta-1}{}_2\phi_1(a, b; c; q, \lambda x)$, $\mathrm{Re}(\beta+\eta) > 0$	$x^{\beta-1}\dfrac{\Gamma_q(\beta+\eta)}{\Gamma_q(\beta+\eta+\alpha)}{}_3\phi_2\left(a, b, q^{\beta+\eta}; c, q^{\beta+\eta+\alpha}; q, \lambda x\right)$

A.4 Table of Erdéli–Sneddon q-Fractional Integral Operator

Most of the results in the following table are proved in [19].

Table A.4 The integral operator $K_q^{\eta,\alpha}$

$\phi(x)$	$K_q^{\eta,\alpha}\phi(x)$
$x^{\lambda},\ \operatorname{Re}\eta > \operatorname{Re}\lambda$	$q^{-\alpha\lambda}\dfrac{\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\alpha)}x^{\lambda}$
$x^{\eta+\alpha+\lambda}\,(b/x;q)_{\lambda}\,,\ \operatorname{Re}(\alpha+\lambda) < 0$	$\dfrac{\Gamma_q(-\alpha-\lambda)}{\Gamma_q(-\lambda)}q^{-\alpha(\eta+\lambda+\alpha)}x^{\eta+\alpha+\lambda}(b/x;q)_{\lambda+\alpha}$
$x^{-\lambda+\eta}e_q(c/x),\ \operatorname{Re}\eta > \operatorname{Re}\lambda$	$x^{\eta-\lambda}q^{\alpha(\lambda-\eta)}\dfrac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}{}_2\phi_1$ $\big(0,q^{\lambda};q^{\lambda+\alpha};q,cq^{\alpha}/x\big)\,,\ \lvert\frac{cq^{\alpha}}{x}\rvert < 1$
$x^{\eta+\alpha}e_q(x)$	$x^{\eta+\alpha}q^{-\alpha(\eta+\alpha)}(1-q)^{\alpha}e_q(xq^{-\alpha})(q/x;q)_{\alpha}$
$x^{\mu+\eta-1}(ax;q)_{\nu},\ \operatorname{Re}(\nu+\mu) < 1$	$x^{\mu+\eta-1}q^{-\alpha(\mu+\eta-1)}(1-q)^{\alpha}(axq^{-\alpha};q)_{\nu}\times$ ${}_2\phi_1\left(q^{\alpha},q^{\alpha+1}/ax;q^{\alpha+1-\nu}/ax;q;q^{-\mu-\nu+1}\right),$ $\lvert x\rvert < 1$
$x^{\lambda+\eta-1}{}_2\phi_1\,(a,b;c;q,1/x)\,,$ $\lvert x\rvert > \min(1,\operatorname{Re}(\alpha)),\ \operatorname{Re}\lambda < 1$	$q^{-\alpha(\lambda+\eta-1)}\dfrac{\Gamma_q(1-\lambda)}{\Gamma_q(1-\lambda+\alpha)}\times$ ${}_3\phi_2\left(a,b,q^{1-\lambda};c,q^{1-\lambda+\alpha};q,q^{\alpha}/x\right)$

A.5 Table of Caputo Fractional q-Derivatives

Table A.5 Caputo fractional q-derivative

ϕ	${}^{c}D_q^{\alpha}\phi\quad x > 0,\quad \alpha > 0,\ \lceil\alpha\rceil = n,\ \beta > n,\ \mu > 0$
$x^{\beta-1},\ \ \beta\notin\mathbb{N}$	$x^{\beta-\alpha-1}\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}$
$x^{\beta-1}$	$zero$ if $\beta\in\{1,2,\dots,n-1\}$
$e_q(\lambda x)$	$\lambda^{n}(1-q)^{-n}x^{n-\alpha}e_{1,n+1-\alpha}(\lambda x(1-q)^{-1};q)$
$E_q(\lambda x)$	$\lambda^{n}(1-q)^{-n}x^{n-\alpha}E_{1,n+1-\alpha}(q^{n}\lambda x(1-q)^{-1};q)$
$x^{\beta-1}e_q(\lambda x),\ \beta\notin\mathbb{N}$	$\dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}x^{\beta-\alpha-1}{}_2\phi_1(0,q^{\beta};q^{\beta-\alpha};q,\lambda x)$

(continued)

Table A.5 (continued)

$x^{\beta-1}E_q(\lambda x),\ \beta \notin \mathbb{N}$	$\dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)} x^{\beta-\alpha-1}\,{}_1\phi_1(q^\beta; q^{\beta-\alpha}; q, \lambda x)$
$\cos_q \lambda x$	$\left(\frac{-\lambda^2x^2}{(1-q)^2}\right)^{\lceil n/2\rceil} x^{-\alpha} e_{2,2\lceil n/2\rceil-\alpha+1}(-\lambda^2x^2(1-q)^{-2}; q)$
$\sin_q \lambda x$	$(-1)^{[n/2]}x^{-\alpha}\left(\frac{\lambda x}{1-q}\right)^{2[n/2]+1} e_{2,2+2[n/2]-\alpha}$ $(-\lambda^2x^2(1-q)^{-2}; q)$
$\mathrm{Cos}_q \lambda x$	$\left(\frac{-\lambda^2x^2}{(1-q)^2}\right)^{\lceil n/2\rceil} x^{-\alpha} \dfrac{q^{\lceil n/2\rceil(2\lceil n/2\rceil-1)}}{\Gamma_q(2\lceil n/2\rceil-\alpha)} \times$ ${}_1\phi_2(q^2; q^{2\lceil n/2\rceil-\alpha+1}, q^{\lceil n/2\rceil-\alpha+2}; q^2, -q^{4\lceil n/2\rceil+1}\lambda^2x^2)$
$\mathrm{Sin}_q \lambda x$	$\left(\frac{\lambda x}{1-q}\right)^{2[n/2]+1} x^{-\alpha} \dfrac{(-q^{2[n/2]+1})^{[n/2]}}{\Gamma_q\,(1-\alpha+2[n/2])} \times$ ${}_1\phi_2(q^2; q^{2[n/2]-\alpha+2}, q^{2[n/2]-\alpha+3}; q^2, -q^{3+4[n/2]}\lambda^2x^2)$
$\cos(\lambda x; q)$	$\left(-q^{\lceil n/2\rceil}\lambda^2x^2\right)^{\lceil n/2\rceil} x^{-\alpha} \times$ $E_{2,2\lceil n/2\rceil-\alpha+1}(-q^{\lceil n/2\rceil+2}\lambda^2x^2; q)$
$\sin(\lambda x; q)$	$(-1)^{[n/2]}q^{[n/2]([n/2]+1)}(\lambda x)^{1+2[n/2]}x^{-\alpha} \times$ $E_{2,2[n/2]-\alpha+2}\,(-q^{1+2[n/2]}\lambda^2x^2; q)$
$x^{\beta-1}E_{\mu,\beta}(\lambda x^\mu; q),\ \beta \notin \mathbb{N}$	$x^{\beta-\alpha-1}E_{\mu,\beta-\alpha}(\lambda x^\mu; q)$
$x^{\beta-1}e_{\mu,\beta}(\lambda x^\mu; q),\ \beta \notin \mathbb{N}$	$x^{\beta-\alpha-1}e_{\mu,\beta-\alpha}(\lambda x^\mu; q)$
${}_2\phi_1\,(a, b; c; q, \lambda x)$	$\dfrac{\lambda^n x^{n-\alpha}(a, b; q)_n}{(c; q)_n\Gamma_q(1-\alpha+n)} \times$ ${}_3\phi_2\left(aq^n, bq^n, q; cq^n, q^{1-\alpha+n}; q, \lambda x\right)$

A.6 The ±1/2 Riemann–Liouville Fractional q-Derivatives

It is known that in fractional calculus, the Riemann–Liouville fractional derivative of order $1/2$ is very suitable for describing some physicals phenomena. Therefore, we collect here the Riemann–Liouville fractional q-derivative and fractional q-integral of order $1/2$ of some q-functions.

Table A.6 The $\pm 1/2$ Riemann–Liouville fractional derivatives

$f(x)$	$D_q^{1/2} f(x)$	$I_q^{1/2} f(x)$
$e_q(x(1-q))$	$\dfrac{1}{\sqrt{x}\Gamma_q(1/2)}+$ $e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})$	$e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})$
$e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})$	$e_q(x(1-q))$	$\big(e_q(x(1-q))-1\big)$
$E_q(xq^{-1/2}(1-q))$	$\dfrac{1}{\sqrt{x}\Gamma_q(1/2)}+$ $\dfrac{E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})}{\sqrt{q}K_q(1/2)}$	$\frac{1}{K_q(1/2)}E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})$
$E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})$	$q^{-1/4}\mathrm{e}_q(q^{1/2}x(1-q))$	$\sqrt{q}K_q(1/2)\,\big(E_q(x(1-q)/\sqrt{q})-1\big)$
$J_0^{(1)}(2\sqrt{z};q)$	$\dfrac{1}{\Gamma_q(1/2)\sqrt{z}}\cos_{\sqrt{q}}(\sqrt{z})$	$\dfrac{(1-q)}{\Gamma_q(1/2)}\sin_{\sqrt{q}}(\sqrt{z})$
$J_0^{(2)}(2\sqrt{z};q)$	$\dfrac{1}{\Gamma_q(1/2)\sqrt{z}}\mathrm{Cos}_{\sqrt{q}}(q\sqrt{z})$	$\dfrac{q^{1/4}(1-q)}{\Gamma_q(1/2)}\mathrm{Sin}_{\sqrt{q}}(\sqrt{z}/\sqrt{q})$
$J_0^{(3)}(\sqrt{z};q)$	$\dfrac{1}{\Gamma_q(1/2)\sqrt{z}}\mathrm{Cos}(\sqrt{z/1-q};q)$ or $\dfrac{1}{\Gamma_q(1/2)\sqrt{z}}$ $\times\cos(\sqrt{z\sqrt{q}/1-q};\sqrt{q})$	$\dfrac{(1-q)}{\Gamma_q(1/2)}\mathrm{Sin}(\sqrt{z/1-q};q)$ or $\dfrac{(1-q)}{\Gamma_q(1/2)}\sin(\sqrt{z/1-q};\sqrt{q})$

A.7 Generalized Rodrigues q-Type Formulae

The results mentioned in the table below are q-analogues of the results introduced by Lavoie, Osler, and Tremblay in [178]. The derivations of these formulas follow by applying Theorem 4.26. In some times we use some transformation. For example in deriving the Rodrigues formula of ${}_2\phi_1(q^a, q^b; q^c; z)$ we use Heine's transformation of q-series, cf. [113, Eq. (III.1)]. All the functions in the table are defined in the book except the little Legendre function which is defined by

$$P_\nu(x|q) = {}_2\phi_1\left(q^{-\nu}, q^{\nu+1}; q; q, qx\right),$$

where ν is a nonnegative integer, it is called little Legendre polynomial. See [252, Eq. (1.26)].

Table A.7 Generalized Rodrigues type formulae

First Jackson q-Bessel	$J_\nu^{(1)}(2\sqrt{x};q^2) = \frac{(1-q^2)^{-\nu+1}}{\Gamma_{q^2}(1/2)} x^{-\nu/2} D_{q^2}^{-\nu+1/2} \sin_q(\sqrt{x})$
Second Jackson q-Bessel	$J_\nu^{(2)}(2\sqrt{x};q) = q^{\frac{1-2\nu}{4}} \frac{(1-q^2)^{-\nu+1}}{\Gamma_{q^2}(1/2)} x^{-\nu/2} D_{q^2}^{-\nu+1/2} \mathrm{Sin}_q(q^{\frac{2\nu-1}{4}}\sqrt{x})$
Third Jackson q-Bessel	$J_\nu^{(3)}(\sqrt{x};q) = \frac{(1-q^2)^{-\nu+1}}{\Gamma_{q^2}(1/2)} x^{-\nu/2} D_q^{-\nu+1/2} \sin(x/1-q;q)$
Little Legendre function	$P_\nu(x\vert q) = \frac{1}{\Gamma_q(\nu+1)} D_q^\nu \left(x^\nu (q^{-\nu+1}x;q)_\nu\right)$
Little incomplete q-gamma	$\gamma_q(a,x) = \Gamma_q(a) e_q(-x(1-q)) D_q^{-a} E_q(q^a x(1-q)$
Big incomplete q-gamma	$\Gamma_q(a,x) = \Gamma_q(a) E_q(q^{-1}x(1-q)) D_a^{-a} e_q(q^{-1}x(1-q))$
${}_2\phi_1(q^a,q^b;q^c;x),\ \vert x\vert < 1$	$\frac{\Gamma_q(c)}{\Gamma_q(b)} x^{1-c} D_q^{b-c} \left(x^{b-1}(x;q)_{-a}\right)$
${}_1\phi_1(q^a;q^c;q,x),\ x \in \mathbb{C}$	$\frac{\Gamma_q(c)}{\Gamma_q(a)} x^{1-c} D_q^{a-c} \left(x^{a-1}(x;q)_\infty\right)$

References

1. H.I. Abdel-Gawad, A.A. Aldailami, On q-dynamic equations modelling and complexity. Appl. Math. Model. **34**(3), 697–709 (2010)
2. T. Abdeljawad, D. Baleanu, Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. Comm. Nonlinear Sci. Numer. Simul. **16**(12), 4682–4688 (2011)
3. W.H. Abdi, On q-Laplace transforms. Proc. Natl. Acad. Sci. India (Sect. A) **29**, 389–408 (1960)
4. W.H. Abdi, Application of q-Laplace transform to the solution of certain q-integral equations. Rend. Circ. Mat. Palermo **11**(2), 245–257 (1962)
5. W.H. Abdi, On certain q-difference equations and q-Laplace transforms. Proc. Natl. Acad. Sc. India A **28**, 1–15 (1962)
6. W.H. Abdi, Certain inversion and representation formulae for q-Laplace transforms. Math. Z. **83**, 238–249 (1964)
7. L.D. Abreu, A q-sampling theorem related to the q-Hankel transform. Proc. Am. Math. Soc. **133**, 1197–1203 (2005)
8. L.D. Abreu, Sampling theory associated with q-difference equations of the Sturm-Liouville type. J. Phys. A **38**(48), 10311–10319 (2005)
9. L.D. Abreu, J. Bustoz, J.L. Caradoso, The roots of the third Jackson q-Bessel functions. Int. J. Math. Math. Sci. **2003**(67), 4241–4248 (2003)
10. M.H. Abu-Risha, M.H. Annaby, M.E.H. Ismail, Z.S. Mansour, Linear q-difference equations. Z. Anal. Anwend. **26**(4), 481–494 (2007)
11. C.R. Adams, Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial. Ann. Math. **27**(2), 73–83 (1925)
12. C.R. Adams, Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial. Ann. Math. (2) **30**(1–4), 626 (1928/1929)
13. C.R. Adams, On the linear ordinary q-difference equation. Ann. Math. (2) **30**(1–4), 195–205 (1928/1929)
14. M. Adivar, M. Bohner, Spectral analysis of q-difference equations with spectral singularities. Math. Comput. Modell. **43**(7–8), 695–703 (2006)
15. R.P. Agarwal, Fractional q-derivative and q-integrals and certain hypergeometric transformations. Ganita **27**(1–2), 25–32 (1976)
16. R.P. Agarwal, A propos d'une note de M. Pierre Humbert (French). C. R. Acad. Sci. Paris **236**, 2031–2032 (1953)
17. R.P. Agarwal, Certain fractional q-integrals and q-derivatives. Proc. Camb. Phil. Soc. **66**, 365–370 (1969)
18. W.A. Al-Salam, q-analogues of Cauchy's formulas. Proc. Am. Math. Soc. **17**, 616–621 (1966)

M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations*,
Lecture Notes in Mathematics 2056, DOI 10.1007/978-3-642-30898-7,

19. W.A. Al-Salam, Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **2**(15), 135–140 (1966/1967)
20. W.A. Al-Salam, A. Verma, A fractional Leibniz q-formula. Pac. J. Math. **60**(2), 1–9 (1975)
21. T.S. Aleroev, On the problem of the zeros of the Mittag-Leffler function and the spectrum of a fractional-order differential operator. Differ. Equat. **36**(9), 1414–1415 (2000). Translation from Differ. Uravn. **36**(9), 1278–1279 (2000)
22. T.S. Aleroev, H.T. Aleroeva, A problem on the zeros of the Mittag–Leffler function and the spectrum of a fractional-order differential operator. Electron. J. Qual. Theor. Differ. Equat. **2009**(25), 18 (2009)
23. A. Algin, A comparative study on q-deformed fermion oscillators. Int. J. Theor. Phys. **50**(5), 1554–1568 (2011)
24. G.E. Andrews, *The Theory of Partitions*, vol. 2 of Encyclopedia of Mathematics and Applications (Addison-Wesley, London, 1976)
25. G.E. Andrews, R. Askey, R. Roy, *Special Functions* (Cambridge University Press, Cambridge, 1999)
26. M.H. Annaby, q-type sampling theorems. Result. Math. **44**(3–4), 214–225 (2003)
27. M.H. Annaby, J. Bustoz, M.E.H. Ismail, On sampling theory and basic Sturm–Liouville systems. J. Comput. Appl. Math. **206**(1), 73–85 (2007)
28. M.H. Annaby, P.L. Butzer, Sampling in Paley-Wiener spaces associated with fractional integro-differential operators. J. Comput. Appl. Math. **171**(1–2), 39–57 (2004)
29. M.H. Annaby, H.A. Hassan, Z.S. Mansour, Sampling theorems associated with singular q-Sturm Liouville problems. doi:10.1007/s00025-011-0134-9
30. M.H. Annaby, Z.S. Mansour, Basic Sturm Liouville problems. J. Phys. A Math. Gen. **38**(17), 3775–3797 (2005). Corrigendum, J. Phys. A Math. Gen. **39**(27), 8747 (2006)
31. M.H. Annaby, Z.S. Mansour, On the zeros of basic finite Hankel transforms. J. Math. Anal. Appl. **323**(2), 1091–1103 (2006)
32. M.H. Annaby, Z.S. Mansour, A basic analog of a theorem of Pólya. Math. Z. **258**(2), 363–379 (2008)
33. M.H. Annaby, Z.S. Mansour, q-Taylor and interpolation series for Jackson q-difference operators. J. Math. Anal. Appl. **334**(1), 472–483 (2008)
34. M.H. Annaby, Z.S. Mansour, On zeros of second and third Jackson q-Bessel functions and their q-integral transforms. Math. Proc. Camb. Philos. Soc. **147**(1), 47–67 (2009)
35. M.H. Annaby, Z.S. Mansour, Asymptotic formulae for eigenvalues and eigenfunctions of q-Sturm-Liouville problems. Math. Nachr. **284**(4), 443–470 (2011)
36. M.H. Annaby, Z.S. Mansour, O.A. Ashour, Sampling theorems associated with biorthogonal q-Bessel functions. J. Phys. A Math. Theor. **43**(29), 15 (2010)
37. M.H. Annaby, Z.S. Mansour, I.A. Soliman, $q-$titchmarsh-weyl theory: Series expansion. Nagoya Math. J. **205**, 67–118 (2012)
38. I. Area, E. Godoy, F. Marcellán, J.J. Moreno-Balcázar, Inner products involving q-differences: The little q-Laguerre-Sobolev polynomials. Higher transcendental functions and their applications. J. Comput. Appl. Math. **118**(1–2), 1–22 (2000)
39. I. Area, E. Godoy, A. Ronveaux, A. Zarzo, Hypergeometric type q-difference equations: Rodrigues type representation for the second kind solution. J. Comput. Appl. Math. **173**(1), 81–92 (2005)
40. R. Askey, The q-gamma and q-beta functions. Appl. Anal. **8**(2), 125–141 (1978/1979)
41. R. Askey, J. Wilson, Some basic hypergeometric polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **54**(319), iv+55 (1985)
42. N.M. Atakishiyev, On a one-parameter family of q-exponential functions. J. Phys. A Math. Gen. **29**(10), L223–L227 (1996)
43. M.K. Atakishiyeva, N.M. Atakishiyev, On continuous q-Hermite polynomials and the classical Fourier transform. J. Phys. A Math. Theor. **41**(12), 9 (2008)
44. F. Atici, P. Eloe, Fractional q-calculus on time scale. J. Nonlinear Math. Phys. **14**(3), 333–344 (2007)

45. F.V. Atkinson, *Discrete and Continuous Boundary Problems*, vol. 8 of Mathematics in Science and Engineering (Academic, New York, 1964)
46. B. Aulbach, S. Hilger, *A Unified Approach to Continuous and Discrete Dynamics*. Qualitative Theory of Differential Equations (Szegad, 1988), vol. 53 of Colloq. Math. Soc. János Bolayi (North-Holland, Amsterdam, 1990), pp. 37–56
47. E.W. Barnes, A new development of the theory of the hypergeometric functions. Proc. Lond. Math. Soc. **s2–6**(1), 141–177 (1908)
48. D.C. Barnett, R.G. Halburd, W. Morgan, R.J. Korhonen, Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations. Proc. R. Soc. Edinb. Sect. A **137**(3), 457–474 (2007)
49. M.A. Bassam, Some properties of Holmgren-Riez transform. Ann. Scuola Norm. Sup. Pisa **15**(3), 1–24 (1961)
50. M.B. Bekker, M.J. Bohner, A.N. Herega, H. Voulov, Spectral analysis of a q-difference operator. J. Phys. A Math. Theor. **43**(14) (2010) pp. 15
51. M.S. Ben Hammouda, A. Nemri, Polynomial expansions for solutions of higher-order q-Bessel heat equation. Tamsui Oxf. J. Math. Sci. **24**(2), 153–174 (2008)
52. Y.M. Berezansky, Z.G. Shaftl, G.F. US, *Functional Analysis*, vol. I (Birkhäuser, Basel, 1996)
53. W. Bergweiler, W.K. Hayman, Zeros of solutions of a functional equation. Comput. Meth. Funct. Theor. **3**(1–2), 55–78 (2003)
54. N. Bettaibi, R.H. Bettaieb, q-analogue of the Dunkl transform on the real line. Tamsui Oxf. J. Math. Sci. **26**(2), 178–206 (2009)
55. J.P. Bezivin, Sur les équations fonctionelles p-adiques aux q-différences. Collect. Math. **43**(2), 125–140 (1992)
56. G.D. Birkhoff, Note on a canonical form for the linear q-difference system. Proc. Natl. Acad. Sci. **27**, 218–222 (1941)
57. R.P. Boas, *Entire Functions* (Academic, New York, 1954)
58. L. Boelen, C. Smet, W. Van Assche, q-discrete painlevé equations for recurrence coefficients of modified q-Freud orthogonal polynomials. J. Difference Equ. Appl. **16**(1), 37–53 (2010)
59. M. Bohner, G. Sh. Guseinov, The convolution on time scales. Abstr. Appl. Anal. (2007). Art. ID 58373 pp. 24
60. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications* (Birkhäuser, Boston, 2001)
61. M. Bohner, A. Peterson, Laplace transform and z-transform: Unification and extension methods. Appl. Anal. **9**(1), 151–157 (2002)
62. F. Bouzeffour, Interpolation of entire functions, product formula for basic sine function. J. Nonlinear Math. Phys. **13**(2), 293–301 (2006)
63. F. Bouzeffour, Inversion formulas for q-Riemann-Liouville and q-Weyl transforms. J. Math. Anal. Appl. **336**(2), 833–848 (2007)
64. K. Brahim, R. Quanes, Some applications of the q-Mellin transform. Tamsui Oxf. J. Math. Sci. **26**(3), 335–343 (2010)
65. T.J.I'A. Bromwich, *An Introduction to the Theory of Infinite Series*, 1st edn. (Macmillan, London, 1908)
66. J. Bustoz, J.L. Cardoso, Basic analog of Fourier series on a q-linear grid. J. Approx. Theor. **112**(1), 134–157 (2001)
67. J. Bustoz, S.K. Suslov, Basic analog of Fourier series on a q-quadratic grid. Meth. Appl. Anal. **5**(1), 1–38 (1998)
68. P.L. Butzer, U. Westphal, in *An Introduction to Fractional Calculus*, Chap. I. Applications of Fractional Calculus in Physics (World Scientific, Singapore, 2000), pp. 3–85
69. M. Caputo, *Elasticitàe Dissipazione* (Zanichelli, Bologna, 1969)
70. M. Caputo, Linear models of dissipation whose Q is almost frequency independent–II. Fract. Calc. Appl. Anal. **11**(1), 4–14 (2008). Reprinted from Geophys. J. R. Astron. Soc. **13**(5), 529–539 (1976)
71. R.D. Carmichael, The general theory of linear q-difference equations. Am. J. Math. **34**, 147–168 (1912)

72. R.D. Carmichael, Linear difference equations and their analytic solutions. Trans. Am. Math. Soc. **12**(2), 99–134 (1912)
73. R.D. Carmichael, On the theory of linear difference equations. Am. J. Math. **35**(2), 163–182 (1913)
74. B.Q. Chen, X.Z. Chen, S. Li, Properties on solutions of some q-difference equations. Acta Math. Sin. (Engl. Ser.) **26**(10), 1877–1886 (2010)
75. Y. Chen, M.E.H. Ismail, K.A. Muttalib, Asymptotics of basic Bessel functions and q-Laguerre polynomials. J. Comp. Appl. Math. **54**, 263–273 (1994)
76. K. Chung, W. Chung, S. Nam, H. Kang, New q-derivative and q-logarithm. Int. J. Theoret. Phys. **33**(10), 2019–2029 (1993)
77. J.A. Cochran, *The Analysis of Linear Integral Equations* (McGraw Hill, New York, 1972)
78. E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955)
79. T. Craven, G. Csordas, The Fox–Wright functions and Laguerre multiplier sequences. J. Math. Anal. Appl. **314**(1), 109–125 (2006)
80. A. De Sole, V.G. Kac, On integral representations of q-gamma and q-beta functions. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **16**(1), 11–29 (2005)
81. J.B. Diaz, T.J. Osler, Differences of fractional order. Math. Comp. **28**(125), 185–202 (1974)
82. K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*. Lecture Notes of Mathematics (Springer, Berlin, 2010)
83. K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, Algorithms for the fractional calculus: A selection of numerical methods. Comput. Meth. Appl. Mech. Eng. **194**(6–8), 743–773 (2005)
84. M.M. Djrbashian, *Integral Transforms and Representations of Functions in the Complex Domain* (Nauka, Moscow, 1962). In Russian
85. M.M. Djrbashian, *Harmonic Analysis and Boundary Value Problem in the Complex Domain* (Birkhäuser, Basel, 1993)
86. A. Dobrogowska, A. Odzijewicz, Second order q-difference equations solvable by factorization method. J. Comput. Appl. Math. **193**(1), 319–346 (2006)
87. A. Dobrogowska, A. Odzijewicz, Solutions of the q-deformed Schrödinger equation for special potentials. J. Phys. A Math. Theor. **40**(9), 2023–2036 (2007)
88. M.S.P. Eastham, *Theory of Ordinary Differential Equations* (Van Nostrand, Reinhold, 1970)
89. A.M.I. El-Guindy, Z.S.I. Mansour, Functional definitions for q-analogues of Eulerian functions and applications. Aquationes Math. DOI:10.1007/s00010-012-0141-2
90. A.M.A. El-Sayed, Fractional order evolution equations. J. Fract. Calc. **7**, 89–100 (1995)
91. A.M.A. El-Sayed, Multivalued fractional differential equations. Appl. Math. Comput. **86**(1), 15–25 (1995)
92. M. El-Shahed, A. Salem, q-analogue of Wright function. Abstr. Appl. Anal. (2008). Art. ID 962849 pp. 11
93. A. Erdélyi, On some functional transformations. Rend. Sem. Mat. Univ. Politec. Torino **10**, 217–234 (1951)
94. A. Erdélyi, H. Kober, Some remarks on Hankel transforms. Quart. J. Math. **11**, 212–221 (1940). Oxford Series
95. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vol. III. (McGraw-Hill, New York, 1955)
96. A. Erdélyi, I.N. Sneddon, Fractional integration and dual integral equations. Can. J. Math. **14**(4), 685–693 (1962)
97. V.V. Eremin, A.A. Meldianov, The q-deformed harmonic oscillator, coherent states, and the uncertainty relation. Theor. Math. Phys. **147**(2), 709–715 (2006). Translation from Teor. Mat. Fiz. **147**(2), 315–322 (2006)
98. H. Exton, *q-Hypergeometric Functions and Applications* (Ellis-Horwood, Chichester, 1983)
99. H. Exton, Basic Sturm-Liouville theory. Rev. Tecn. Fac. Ingr. Univ. Zulia **1**, 85–100 (1992)

100. R.A.C. Ferreira, Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theor. Differ. Equat. **10**(70), 10 (2010)
101. C.M. Field, N. Joshi, F.W. Nijhoff, q-difference equations of KdV type and Chazy-type second-degree difference equations. J. Phys. A Math. Theor. **41**(33), 13 (2008)
102. A. Fitouhi, N. Bettaibi, K. Brahim, The Mellin transform in quantum calculus. Constr. Approx. **23**(3), 305–323 (2006)
103. A. Fitouhi, A. Safraoui, Paley-Wiener theorem for the q^2-Fourier-Rubin transform. Tamsui Oxf. J. Math. Sci. **26**(3), 287–304 (2010)
104. R. Floreanini, J . LeTourneux, L. Vinet, More on the q-oscillator algebra and q-orthogonal polynomials. J. Phys. A Math. Gen. **28**, L287–L293 (1995)
105. R. Floreanini, L. Vinet, A model for the continuous q-ultraspherical polynomials. J. Math. Phys. **36**, 3800–3813 (1995)
106. R. Floreanini, L. Vinet, More on the q-oscillator algebra and q-orthogonal polynomials. J. Phys. A **28**, L287–L293 (1995)
107. M. Foupouagnigni, W. Koepf, A. Ronveaux, Fourth-order difference equation for the associated classical orthogonal polynomials. J. Comput. Appl. Math. **92**(2), 103–108 (1998)
108. M. Foupouagnigni, A. Ronveaux, W. Koepf, Fourth order q-difference equation for the first associated of the q-classical orthogonal polynomials. J. Comput. Appl. Math. **101**(1–2), 231–236 (1999)
109. C. Fox, The asymptotic expansion of generalized hypergeometric functions. Proc. Lond. Math. Soc. **S2–27**(1), 389–400 (1928)
110. M.C. Gaer, L.A. Rubel, in *The Fractional Derivative and Entire Functions*. Fractional Calculus and Its Applications. Proceedings of International Conference, University of New Haven, West Haven, CT, 1974, vol. 457 of Lecture Notes in Math. (Springer, Berlin, 1975), pp. 171–206
111. Lj. Gajić, B. Stanković, Some properties of Wright's function. Publ. Inst. Math. (Beograd) (N.S.) **20**(34), 91–98 (1976)
112. L. Galué, Generalized Erdélyi-Kober fractional q-integral operator. Kuwait J. Sci. Eng. **36**(2A), 21–34 (2009)
113. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edn. (Cambridge university Press, Cambridge, 2004)
114. R. Gorenflo, A.A. Kilbas, S. Rogozin, On the generalized Mittag-Leffler type functions. Integr. Transforms Spec. Funct. **7**(3–4), 215–224 (1998)
115. R. Gorenflo, Y. Luchko, F. Mainardi, Analytical properties and applications of the Wright function. Fract. Calc. Appl. Anal. **2**(4), 383–414 (1999)
116. R. Gorenflo, Yu. Luchko, J. Loutchko, Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative. Frac. Calc. Appl. Anal. **5**(4), 491–518 (2002)
117. R. Gorenflo, F. Mainardi, in *Fractional Calculus: Integral and Differential Equations of Fractional Order*, ed. by A. Carpinteri, F. Mainardi. Fractals and Fractional Calculus in Continuum Mechanics, vol. 378 of CISM Courses and Lecture Notes (Springer, New York, 1997), pp. 223–276
118. R. Gorenflo, S. Vessella, in *Abel Integral Equations: Analysis and Applications*. Lecture Notes in Mathematics, vol. 1461 (Springer, Berlin, 1991)
119. R.W. Gray, C.A. Nelson, A completeness relation for the q-analogue coherent states by q-integration. J. Phys. A. **23**, L945–L950 (1990)
120. A.K. Grünwald, Über begrenzte derivationen und deren anwedung. Z. Math. Phys. **12**, 441–480 (1867)
121. I. Győri, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications* (Clarendon press, Oxford, 1991)
122. W. Hahn, Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der Hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-transformation (German). Math. Nachr. **2**, 340–379 (1949)
123. W. Hahn, Beiträge zur Theorie der Heineschen Reihen (German). Math. Nachr. **2**, 340–379 (1949)

124. W. Hahn, Über die zerlegung einer klasse von polynomen in irreduzible faktoren (German). Math. Nachr. **3**, 257–294 (1950)
125. W. Hahn, Über uneigentliche Lösungen linearer geometrischer Differenzengleichungen (German). Math. Ann. **125**, 67–81 (1952)
126. J.W. Hanneken, B.N.N. Achar, R. Puzio, D.M. Vaught, Properties of the Mittag-Leffler function for negative alpha. Phys. Scr. **T136**, 5 (2009)
127. H.J. Haubold, A.M. Mathai, R.K. Saxena, Mittag-Leffler functions and their applications. J. Appl. Math. **2011**, 51 (2011)
128. W.K. Hayman, *Subharmonic Functions*, vol. II (Academic, London, 1989)
129. W.K. Hayman, in *On the Zeros of a q-Bessel Function*, ed. by M. Agranovsky, L. Karp, D. Shoikhet. Complex Analysis and Dynamical Systems II: A Conference in Honor of Professor Lawrence Zalcman's Sixtieth Birthday, June 9–12, 2003, vol. 382 of Contemp. Math. (Amer. Math. Soc., Providence, 2005), pp. 205–216
130. E. Heine, Über die reihe ... J. Reine Angew. Math. **32**, 210–212 (1846)
131. E. Heine, Untersuchngen über die reihe ... J. Reine Angew. Math. **34**, 285–328 (1847)
132. E. Heine, *Handbuch der Kugelfunctionen, Theorie und Anwendungen*, vol. 1 (G. Reimer, Berlin, 1878)
133. E. Heine, *Handbuch der Kugelfunctionen, Theorie und Anwendungen*, vol. 2 (G. Reimer, Berlin, 1881)
134. N. Heymans, I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. Rheol. Acta **45**(5), 765–771 (2006)
135. S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus. Results Math. **18**(1–2), 18–56 (1990)
136. C.L. Ho, On the use of Mellin transform to a class of q-difference-differential equations. Phys. Lett. A **268**(4–6), 217–223 (2000)
137. P. Humbert, R.P. Agarwal, Sur la fonction de Mittag–Leffler et quelques-unes de ses généralisations (French). Bull. Sci. Math. **77**(2), 180–185 (1953)
138. A. Huseynov, E. Bairamov, An eigenvalue problem for quadratic pencils of q-difference equations and its applications. Appl. Math. Lett. **22**(4), 521–527 (2009)
139. J.I. Hutchinson, On a remarkable class of entire functions. Trans. Am. Math. Soc. **25**(3), 325–332 (1923)
140. M.E.H. Ismail, The basic Bessel functions and polynomials. SIAM J. Math. Anal. **12**(3), 454–468 (1981)
141. M.E.H. Ismail, The zeros of basic Bessel functions, the functions $J_{\nu+\alpha}(x)$ and associated orthogonal polynomials. J. Math. Anal. Appl. **86**(1), 11–19 (1982)
142. M.E.H. Ismail, On Jackson's third q-Bessel function. preprint (1996)
143. M.E.H Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable* (Cambridge University Press, Cambridge, 2005)
144. M.E.H. Ismail, S.J. Johnston, Z.S. Mansour, Structure relations for q-polynomials and some applications. Appl. Anal. **90**(3–4), 747–767 (2011)
145. M.E.H. Ismail, L. Lorch, M.E. Muldoon, Completely monotonic functions associated with the gamma functions and its q-analogues. J. Math. Anal. Appl. **116**, 1–9 (1986)
146. M.E.H. Ismail, Z.S.I. Mansour, q-analogues of Freud weights and nonlinear difference equations. Adv. Appl. Math. **45**(4), 518–547 (2010)
147. M.E.H. Ismail, M. Rahman, Inverse operators, q-fractional integrals and q-Bernoulli polynomials. J. Approx. Theor. **114**(2), 269–307 (2002)
148. M.E.H. Ismail, D. Stanton, Applications of q-Taylor theorems. J. Comp. Appl. Math. **153**(1–2), 259–272 (2003)
149. M.E.H. Ismail, D. Stanton, q-Taylor theorems, polynomial expansions, and interpolation of entire functions. J. Approx. Theor. **123**(1), 125–146 (2003)
150. M.E.H. Ismail, A.I. Zayed, A q-analogue of the Whittaker-Shannon-Koteĺnikov sampling theorem. Proc. Am. Math. Soc. **131**(2), 3711–3719 (2003)

151. M.E.H. Ismail, C. Zhang, Zeros of entire functions and a problem of Ramanujan. Adv. Math. **209**(1), 363–380 (2007)
152. M.E.H. Ismail, R. Zhang, Diagonalization of certain integral operators. Adv. Math. **109**(1), 1–33 (1994)
153. F.H. Jackson, On generalized functions of Legendre and Bessel. Trans. Roy. Soc. Edinb. **41**, 1–28 (1903)
154. F.H. Jackson, A basic-sine and cosine with symbolical solutions of certain differential equations. Proc. Edinb. Math. Soc. **22**, 28–38 (1903/1904)
155. F.H. Jackson, A generalization of the function $\Gamma(n)$ and x^n. Proc. Roy. Soc. Lond. **74**, 64–72 (1904)
156. F.H. Jackson, The applications of basic numbers to Bessel's and Legendre's equations (second paper). Proc. Lond. Math. Soc. **3**(2), 1–23 (1905)
157. F.H. Jackson, The basic gamma function and elliptic functions. Proc. Roy. Soc. A **76**, 127–144 (1905)
158. F.H. Jackson, On q-functions and a certain difference operator. Trans. Roy. Soc. Edinb. **46**, 64–72 (1908)
159. F.H. Jackson, A q-form of Taylor's theorem. Messenger Math. **38**, 62–64 (1909)
160. F.H. Jackson, On q-definite integrals. Quart. J. Pure Appl. Math. **41**, 193–203 (1910)
161. F.H. Jackson, The q-integral analogous to Borel's integral. Messenger Math. **47**, 57–64 (1917)
162. A. Jirari, Second-order Sturm Liouville difference equations and orthogonal polynomials. Mem. Am. Math. Soc. **115**(542)x, pp. 138 (1995)
163. V. Kac, P. Cheung, *Quantum Calculus* (Springer, New York, 2002)
164. H.H. Kairies, M.E. Muldoon, Some characterizations of q-factorial functions. Aequationes Mathematicae **25**, 67–76 (1982)
165. H. Karabulut, Distributed Gaussian polynomials as q-oscillator eigenfunctions. J. Math. Phys. **47**(1), 13 (2006)
166. O.M. Katkova, A.M. Vishnyakova, A sufficient condition for a polynomial to be stable. J. Math. Anal. Appl. **347**(1), 81–89 (2008)
167. A.A. Kilbas, Fractional calculus of the generalized Wright function. Fract. Calc. Appl. Anal. **8**(2), 113–126 (2005)
168. A.A. Kilbas, M. Saigo, R.K Saxena, Generalized Mittag-Leffler functions and generalized fractional calculus operator. Integr. Transforms Spec. Funct. **15**(1), 31–49 (2004)
169. A.A. Kilbas, H.M. Srivastava, J.I. Trujillo, *Theory and Applications of Fractional Differential Equations*, 1st edn. (Elsevier, London, 2006)
170. Y.O. Kim, A proof of the Pólya-Wiman conjecture. Proc. Am. Math. Soc. **109**(4), 1045–1052 (1990)
171. W.A. Kirk, M.A. Khamsi, *An Introduction to Metric Spaces and Fixed Point Theory* (Wiley, New York, 2001)
172. V.S. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. Higher transcendental functions and their applications. J. Comput. Appl. Math. **118**(1–2), 241–259 (2000)
173. H. Kober, On fractional integrals and derivatives. Quart. J. Math. Oxford Ser. **11**, 193–211 (1940)
174. H.T. Koelink, R.F. Swarttouw, On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials. J. Math. Anal. Appl. **186**(3), 690–710 (1994)
175. A.N. Kolmogorov, S.V. Fomin, *Introductory Real Analysis* (Dover Publications, New York, 1957)
176. T.H. Koornwinder, R.F. Swarttouw, On q-analogues of Fourier and Hankel transforms. Trans. Am. Math. Soc. **333**(1), 445–461 (1992)
177. V.P. Kostov, B. Shapiro, Hardy-Petrovitch-Hutchinson's problem and partial theta function. arXiv:1106.6262v1[math. CA] (2011)
178. J.L. Lavoie, T.J. Osler, R. Tremblay, Fractional derivative and special functions. SIAM Rev. **18**(2), 240–268 (1976)

179. C.W. Leibniz, *Mathematische Schriften* (Georg Olms Verlagsbuchhandlung, Hildesheim, 1962)
180. A.V. Letinkov, Theory of differentiation of fractional order. Mat. Sb. **3**, 1–68 (1868)
181. B.Ja. Levin, *Distribution of Zeros of Entire Functions*, vol. 5 of *Translations of Mathematical Monographs* (Amer. Math. Soc., Providence, 1980)
182. B.M. Levitan, I.S. Sargsjan, *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators* (Amer. Math. Soc., Providence, 1975)
183. B.M. Levitan, I.S. Sargsjan, *Sturm Liouville and Dirac Operators* (Kluwer Academic, Dordrecht, 1991)
184. X. Li, C. Zhang, Existence of analytic solutions to analytic nonlinear q-difference equations. J. Math. Anal. Appl. **375**(2), 412–417 (2011)
185. M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge University Press, New York, 1960)
186. J. Liouville, Mèmoire sur le calcul des différentielles à indices quelconques. J. de I'Ecole Polytechnique **13**, 71–162 (1832)
187. J.E. Littelwood, On the asymptotic approximation to integral functions of order zero. Proc. Lond. Math. Soc. **5**(2), 361–410 (1907)
188. H.K. Liu, Application of a differential transformation method to strongly nonlinear damped q-difference equations. Comput. Math. Appl. **61**(9), 2555–2561 (2011)
189. Y.K. Liu, The linear q-difference equation $y(x) = ay(qx) + f(x)$. Appl. Math. Lett. **8**(1), 15–18 (1995)
190. Z.G. Liu, Two q-difference equations and q-operator identities. J. Differ. Equat. Appl. **16**(11), 1293–1307 (2010)
191. B. López, J.M. Marco, J. Parcet, Taylor series for the Askey-Wilson operator and classical summation formulas. Proc. Am. Math. Soc. **134**(8), 2259–2270 (2006)
192. D.Q. Lu, q-difference equation and the Cauchy operator identities. J. Math. Anal. Appl. **359**(1), 265–274 (2009)
193. Yu. Luchko, Asymptotics of zeros of the Wright function. Z. Anal. Anwendungen **19**(2), 583–595 (2000)
194. I. Lutzenko, V. Spiridonov, A. Zhedanov, On the spectrum of a q-scillator with a linear interaction. Phys. Lett. A **204**(3–4), 236–242 (1995)
195. H. Mabrouk, q-heat operator and q-Poisson's operator. Fract. Calc. Appl. Anal. **9**(3), 265–286 (2006)
196. J.T. Machado, V. Kiryakova, F. Mainardi, A poster about the recent history of fractional calculus. Fract. Calc. Appl. Anal. **13**(3), 329–334 (2010)
197. J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus. Comm. Nonlinear Sci. Numer. Simul. **16**(3), 1140–1153 (2011)
198. N.I. Mahmudov, P. Sabancıgil, q-parametric Bleimann Butzer and Hahn operators. J. Inequal. Appl. **2008**, 15 (2008) (Art. ID 816367)
199. F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation. Appl. Math. Lett. **9**(6), 23–28 (1996)
200. F. Mainardi, R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes. J. Comput. Appl. Math. **118**(1–2), 283–299 (2000)
201. F. Mainardi, G. Pagnini, The Wright functions as solutions of the time-fractional diffusion equation. Appl. Math. Comput. **141**(1), 51–62 (2003)
202. H.L. Manocha, B.L. Sharma, Fractional derivatives and summation. J. Indian Math. Soc. **38**(1–4), 371–382 (1974)
203. Z.S.I. Mansour, q-Difference Equations. Master's thesis, Faculty of Science, Cairo University, Giza, Egypt, 2001
204. Z.S.I. Mansour, On q-Difference and Fractional Equations and Their Solutions. PhD thesis, Faculty of Science, Cairo University, Giza, Egypt, 2006
205. Z.S.I. Mansour, Linear sequential q-difference equations of fractional order. Fract. Calc. Appl. Anal. **12**(2), 160–178 (2009)
206. Z.S.I. Mansour, Generalizations of fractional q-Leibniz formulae. submitted (2012)

207. Z.S.I. Mansour, Generalizations of Rubin's q^2-fourier transform and q-difference operator. submitted (2012)
208. V.A. Marchenko, *Sturm–Liouville Operators and Applications* (Birkhäuser, Basel, 1986)
209. T.E. Mason, On properties of the solutions of linear q-difference equations with entire function coefficients. Am. J. Math. **37**(4), 439–444 (1915)
210. A. Matsuo, Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations. Comm. Math. Phys. **151**(2), 263–273 (1993)
211. A.C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions* (Pitman Research Notes in Mathematics, London, 1979)
212. H. Mellin, Om definita integraler. Acta Societatis Scientiarum Fennicae **20**(7), 1–39 (1895)
213. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993)
214. R. Mishra, On certain generalized q-Laplace and Stieltjes transforms. J. Math. Sci. **5**(2), 22–37 (1970)
215. R. Mishra, A few inversion formulae for generalized q-Laplace transforms. Indian J. Math. **14**(3), 157–172 (1972)
216. R. Mishra, Applications of fractional q-integration on the theory of a generalized q-Stieltjes transform. Bull. Calcutta. Math. Soc. **69**(5), 247–254 (1977)
217. R. Mishra, Application of generalized q-Laplace transforms to the solutions of certain q-integral equations. Indian J. Math. **23**(1–3), 115–120 (1981)
218. R. Mishra, On a relation between a q-Laplace transform and two generalized q-Laplace transforms. Proc. Natl. Acad. Sci. India Sect. A **53**(4), 379–385 (1983)
219. G. Mittag-Leffler, Sur l'intégrale de Laplace-Abel. C. R. Acad. Sci. Paris (Ser. II) **136**, 937–939 (1902)
220. G. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$. C. R. Acad. Sci. Paris **137**, 554–558 (1903)
221. G. Mittag-Leffler, Une généralisation de l'intégrale de Laplace-Abel. C. R. Acad. Sci. Paris (Ser. II) **137**, 537–539 (1903)
222. G. Mittag-Leffler, Sopra la funzione $E_\alpha(x)$. R. Accad. Lincei, Rend. (Ser. V) **13**, 3–5 (1904)
223. G. Mittag-Leffler, Sur la reprwesentation analytique dune branche uniforme dune fonction monogéne. Acta Math. **29**, 101–182 (1905)
224. D.S. Moak, The q-gamma function for $q > 1$. Aequationes Mathematicae **20**(2–3), 278–285 (1980)
225. A. Nemri, A. Fitouhi, Polynomial expansions for solution of wave equation in quantum calculus. Matematiche **65**(1), 73–82 (2010)
226. Y. Nikolova, L. Boyadjiev, Integral transform method to solve a time-space fractional diffusion equation. Fract. Calc. Appl. Anal. **13**(1), 57–67 (2010)
227. K.B. Oldham, J. Spanier, *The Fractional Calculus* (Academic, London, 1974)
228. T.J. Osler, Leibniz rule for fractional derivatives generalized and an application to infinite series. SIAM J. Appl. Math. **18**(3), 658–674 (1970)
229. T.J. Osler, Fractional derivatives and Leibniz rule. Am. Math. Mon. **78**(6), 645–649 (1971)
230. T.J. Osler, A further extension of the Leibniz rule to fractional derivatives and its relation to Parserval's formula. SIAM J. Math. Anal. **3**(1), 1–16 (1972)
231. T.J. Osler, The integral analogue of the Leibniz rule. Math. Comp. **26**(120), 903–915 (1972)
232. R.B. Paris, D. Kaminski, *Asymptotics and Mellin-Barnes Integrals* (Cambridge University Press, Cambridge, 2001)
233. I. Podlubny, in *Solution of Linear Fractional Differential Equations with Constant Coefficients*, ed. by P. Rusev, I. Dimovski, V. Kiryakova. Transform Methods and Special Functions (SCT Publications, Singapore, 1995), pp. 217–228
234. I. Podlubny, *Fractional Differential Equations* (Academic, New York, 1999)
235. I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. **5**(4), 367–386 (2002)
236. I. Podlubny, A.M.A. El-Sayed, On two definitions of fractional derivatives. UEF-03-96, Inst. Exp. Phys, Slovak Acad. Sci., 1996. ISBN 80-7099-252-2

237. G. Pólya, Über die Nullstellengewisser ganzer Functions(German). Math. Z. **2**(3–4), 352–383 (1918)
238. G. Pólya, G. Szegö, *Problems and Theorems in Analysis*, vol. I (Springer, New York, 1972). Translated from the German by D. Aeppli Die Grundlehren der mathematischen Wissenschaften, Band 193
239. A.Yu. Popov, On the spectral values of a boundary value problem and the zeros of Mittag-Leffler functions. Differ. Equat. **38**(5), 642–653 (2002). Translation from Differ. Uravn. **38**(5), 611–621 (2002)
240. A.Yu. Popov, On zeroes of Mittag–Leffler functions with parameter $\rho < 1/2$. Anal. Math. **32**(3), 207–246 (2006)
241. A.Yu. Popov, A.M. Sedletskiĭ, Distribution of zeros of the Mittag-Leffler function. Dokl. Math. **67**(3), 336–339 (2003). Translation from Dokl. Akad. Nauk **390**(2), 165–168 (2003)
242. E.L. Post, Generalized differentiation. Trans. Am. Math. Soc. **32**, 723–781 (1930)
243. A.D. Poularikas (ed.), *The Transforms and Applications Handbook*, 2nd edn. The Electrical Engineering Handbook Series (CRC Press, FL, 2000)
244. T.R. Prabhakar, A singular integral equation with a generalized Mittag–Leffler function in the kernel. Yokohama Math. J. **19**, 7–15 (1971)
245. S.D. Prasad, Certain q-transforms. Math. Scand. **30**, 324–330 (1972)
246. A.V. Pskhu, On the real zeros of functions of Mittag-Leffler type. Math. Notes **77**(3–4), 546–552 (2005). Translation from Mat. Zametki **77**(4), 592–599 (2005)
247. S.D. Purohit, Summation formulae for basic hypergeometric functions via fractional q-calculus. Mathematica (Catania) **64**(1), 67–75 (2009)
248. S.D. Purohit, On a q-extension of the Leibniz rule via Weyl type of q-derivative operator. Kyungpook Math. J. **50**(4), 473–482 (2010)
249. S.D. Purohit, R.K. Raina, Generalized q-Taylor's series and applications. Gen. Math. **18**(3), 19–28 (2010)
250. S.D. Purohit, R.K. Yadav, On generalized fractional q-integral operators involving the q-Gauss hypergeometric function. Bull. Math. Anal. Appl. **2**(4), 35–44 (2010)
251. M. Rahman, A note on the orthogonality of Jackson's q-Bessel functions. Can. Math. Bull. **32**(3), 369–376 (1989)
252. M. Rahman, Q.M. Tariq, Addition formulas for q-Legendre-type functions. Meth. Appl. Anal. **6**(1), 3–20 (1999)
253. P.M. Rajković, M. Stanković, S.D. Marinković, Mean value theorems in q-calculus. Mat. Vesnik **54**, 171–178 (2002)
254. P.M. Rajković, S.D. Marinković, M.S. Stanković, Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**(1), 311–323 (2007)
255. P.M. Rajković, S.D. Marinković, M.S. Stanković, On q-analogues of Caputo derivative and Mittag–Leffler function. Fract. Calc. Appl. Anal. **10**(4), 359–373 (2007)
256. J.P. Ramis, About the growth of entire functions solutions of linear algebraic q-difference equations. Ann. Fac. Sci. Toulouse Math. **1**(6), 53–94 (1992)
257. C. Richard, On q-functional equations and excursion moments. Discrete Math. **309**(1), 207–230 (2009)
258. E.Yu. Romanenko, Differential-difference equations reducible to difference and q-difference equations. Comput. Math. Appl. **42**(3–5), 615–626 (2001)
259. P.G. Rooney, On the ranges of certain fractional integrals. Can. J. Math. **24**, 1198–1216 (1972)
260. P.G. Rooney, A survey of mellin multipliers. fractional calculus. Res. Notes Math. **138**, 176–187 (1985)
261. B. Ross (ed.), *Fractional Calculus and Its Applications*, vol. 457 of Lecture Notes of Mathematics. Proceeding of the International Conference held at the University of New Haven, June 15-16, 1974 (Springer, Berlin, 1975)
262. B. Ross, Fractional Calculus. An historical apologia for the development of a calculus using differentiation and antidifferentiation of non-integral orders. Math. Mag. **50**(3), 115–122 (1977)

263. B. Ross, Origins of fractional calculus and some applications. Int. J. Math. Statist. Sci. **1**(1), 21–34 (1992)
264. B. Ross, B.K. Sachdeva, The solution of certain integral equations by means of operators of arbitrary order. Am. Math. Mon. **97**(6), 498–503 (1990)
265. R.L. Rubin, A q^2-analogue operator for q^2-analogue Fourier analysis. J. Math. Anal. Appl. **212**(2), 571–582 (1997)
266. R.L. Rubin, Duhamel solutions of non-homogeneous q^2-analogue wave equations. Proc. Am. Math. Soc. **135**(3), 777–785 (2007)
267. J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado (eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering* (Springer, London, 2007)
268. E.B. Saff, A.D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 3rd edn. (Prentice-Hall, Englewood Cliffs, 2003)
269. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications* (Gordan and Breach Science Publisher, New York, 1993)
270. R.K. Saxena, G.C. Modi, S.L Kalla, A basic analogue of Fox's H-function. Rev. Técn. Fac. Ingr. Univ. Zulia **6**, 139–143 (1983). Special Issue
271. R.K. Saxena, R.K. Yadav, S.D. Purohit, Kober fractional q-integral operator of the basic analogue of the H-function. Rev. Téc. Ing. Univ. Zulia, ago **28**(2), 154–158 (2005)
272. A.M. Sedletskiĭ, Asymptotic formulae for zeros of a function of Mittag-Leffler's type. Anal. Math. **20**(2), 117–132 (1994)
273. A.M. Sedletskiĭ, On the zeros of functions of Mittag–Leffler type. Math. Notes **68**(5), 602–613 (2000). Translation from Mat. Zametki **68**(5), 710–724 (2000)
274. S. Sergeev, Quantum curve in q-oscillator model. Int. J. Math. Math. Sci. **2006**, 31 (2006) (Art. ID 92064)
275. G.E. Shilov, *Linear Algebra* (Dover, New York, 1971)
276. I.N. Sneddon, *Mixed Boundary ValueProblems* (North Holland, Amsterdam, 1966)
277. S.K. Suslov, Another addition theorem for the q-exponential function. J. Phys. A Gen. **33**, L375–L380 (2000)
278. S.K. Suslov, Some expansions in basic Fourier series and related topics. J. Approx. Theor. **115**, 289–353 (2002)
279. S.K. Suslov, *An Introduction to Basic Fourier Series*, vol. 9 of Kluwer Series Developments in Mathematics (Kluwer Academic Publishers, Boston, 2003)
280. R.F. Swarttouw, The Hahn-Exton q-Bessel Function. PhD thesis, The Technical University of Delft, 1992
281. R.F. Swarttouw, H.G. Meijer, A q-analogue of the Wronskian and a second solution of the Hahn-Exton q-Bessel difference equation. Proc. Am. Math. Soc. **120**(3), 855–864 (1994)
282. J. Thomae, Beitrage zur theorie der durch die heinesche reihe . . .. J. Reine Angew. Math. **70**, 258–281 (1869)
283. I.V. Tikhonov, Yu.S. Éĭdeĺman, An inverse problem for a differential equation in a Banach space and the distribution of zeros of an entire Mittag-Leffler function. Differ. Equat. **38**(5), 669–677 (2002). Translation from Differ. Uravn. **38**(5), 637–644 (2002)
284. E. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations* (Clarendon Press, Oxford, 1962)
285. W.J. Trjitzinsky, Analytic theory of linear q-difference equations. Acta Math. **61**(1), 1–38 (1933)
286. T. Tsuda, On an integrable system of q-difference equations satisfied by the universal characters: Its lax formalism and an application to q-painlevé equations. Comm. Math. Phys. **293**(2), 347–359 (2010)
287. L. Di Vizio, Local analytic classification of q-difference equations with $|q| = 1$. J. Noncommut. Geom. **3**(1), 125–149 (2009)
288. Y. Watanabe, Notes on the generalized derivative of Riemann–Liouville and its application to Leibniz's formula. I and II. Tôhoku Math. J. **34**, 8–41 (1931)

289. M. Welter, Interpolation of entire functions on regular sparse sets and q-Taylor series. J. Théor. Nombres Bordeaux **17**(1), 397–404 (2005)
290. E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, 4th edn. (Cambridge University Press, London, 1927). Reprinted (1973)
291. D.V. Widder, *Laplace Transform* (Princeton University Press, Princeton, 1941)
292. A. Wiman, Über den fundamentalsatz in der teorie der funktionen$E_\alpha(x)$. Acta Math. **29**, 191–201 (1905)
293. E.M. Wright, On the coefficients of power series having exponential singularities. J. Lond. Math. Soc. **8**, 71–79 (1933)
294. E.M. Wright, The asymptotic expansion of the generalized hypergeometric function. J. Lond. Math. Soc. **s1–10**(4), 286–293 (1935)
295. E.M. Wright, The asymptotic expansion of integral functions defined by Taylor series. Philos. Trans. Roy. Soc. Lond. Ser. A **238**, 423–451 (1940)
296. E.M. Wright, The asymptotic expansion of the generalized hypergeometric function II. Proc. Lond. Math. Soc. **46**(2), 389–408 (1940)
297. W. Wyss, The fractional diffusion equation. J. Math. Phys. **27**(11), 2782–2785 (1986)
298. B. Xu, W. Zhang, Small divisor problem for an analytic q-difference equation. J. Math. Anal. Appl. **342**(1), 694–703 (2008)
299. X. Zheng, Z. Chen, Some properties of meromorphic solutions of q-difference equations. J. Math. Anal. Appl. **361**(2), 472–480 (2010)
300. A. Zygmund, *Trigonometric Series*, vol. II (Cambridge University Press, Cambridge, 1959)
301. M. Ismail, SIAM Rev. **16**, 279–281 (1985)
302. W. Hahn, Math. Rev. **84i**, 39002 (1982)

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