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Marten Hillebrand

Pension Systems, Demographic Change, and the Stock Market

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To my father

Preface

The goal of this thesis is to study pension systems and their interactions with real and financial markets in the presence of demographic change and randomness due to a stochastic asset market. Most existing contributions in the literature are confined to a deterministic or simplified stochastic setting. This type of approach precludes the incorporation of randomness and uncertainty typically observed in stock markets. It also does not facilitate a discerning study of the feedback structure between asset markets and the pension system within a stochastic dynamic model. The aim of this dissertation is therefore twofold. The first goal is the conception and development of a suitable theoretical framework that complements the existing approaches. The second goal is to present a simulation study which employs the previously developed framework to analyze the role of a pension system and the impact of demographic change on the dynamic behavior of real and financial markets as well as on the welfare of consumers.

The Department of Business Administration and Economics at Bielefeld University, Germany, has accepted this work as dissertation in partial fulfillment of the requirements for the degree of Doctor in Economic Sciences (Doktor der Wirtschaftswissenschaften, Dr. rer. pol.). The final oral examination was held on December 18, 2006 and passed successfully.

I am deeply indebted to my advisor Volker Böhm who has not only inspired this research but has also made numerous suggestions and valuable comments. I greatly benefitted from the fruitful research atmosphere stimulated by him and also by the entire Ph.D. program at the Bielefeld Graduate School of Economics and Management (BiGSEM). Likewise, I am equally grateful to my second advisor Jan Wenzelburger for his constant support and our most seminal collaboration over the past years. Many of the results on multiperiod investment problems developed under his supervision in my diploma thesis defined the starting point for this work.

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Bielefeld, December 2007 *Marten Hillebrand*

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1

Introduction

One of the greatest challenges faced by virtually all industrialized nations is the demographic change within their populations. There are two major developments which are responsible for this phenomenon. Firstly, a constant improvement in health care and medical advancements resulting in a continually increasing life expectancy over the past decades. Secondly, low fertilities and a substantial decline in birth rates that arose from the late sixties onwards. Both effects put together induce a significant change in the age-structure of the population and put increasing pressure on existing pay-as-you-go pension systems in Europe, in particular on the German system. While the first effect increases the number of pensioners and, thus, the beneficiaries from the system, the second one reduces the aggregated labor force and the number of contributors to the system. Although the consequences of this development are observable already today, there is general agreement that they will become much more dramatic in the near future.

Against this background, the past years have witnessed a vivid debate about the efficiency and sustainability of pension systems. In this regard, the design of the existing pay-as-you-go systems has been questioned and numerous proposals to reform these systems have been investigated. However, the pay-as-you-go structure of the German as well as of numerous other pension systems implies a fundamental trade-off between maintaining a sufficiently high level of pension incomes and keeping contributions at a reasonably low level (cf. [20] and Section 2.6 of this work). Hence, a mere adjustment of contribution rates or pension payments could merely shift the demographic burden between contributors and beneficiaries but could not solve the demographic problem.

To ameliorate this dilemma, many economists have suggested supplementing the existing public pension systems by a so-called pre-funding component consisting of an increased share of private savings for retirement. This additional pillar has been implemented in the German pension system as part of the so-called Riester reform in 2001 (see, e.g., [58] for details). The economic reasoning behind this measure to potentially attenuate the consequences of demographic change is that an increase in private savings for retirement potentially fosters the accumulation of capital (cf. [35]). This in turn would enhance the production possibilities of the economy providing a potential way to overcome the loss in aggregated workforce induced by the demographic change. Opponents of such a reform, however, typically argue that private savings for retirement are exposed to capital market risk. Hence, a shift in retirement provision towards more pre-funding would necessarily increase the risk to which pension incomes are exposed due to the volatility and unpredictability of capital markets in general and stock markets in particular. This view has to some extent been supported by the drastic decline in the stock market observed at the beginning of this century.

The latter argument stresses the importance to pay adequate respect to the role of uncertainty and financial risk when studying pension reforms. In this regard, it seems natural to assume that adjustments in the pension system affect consumers' savings behavior which in turn affect prices on financial markets. A comprehensive theoretical analysis therefore requires a framework which incorporates not only the issue of demographic change but also the mutual *interactions* between the pension system and asset/stock markets. Conceptually, this calls for a macroeconomic model which incorporates the following three building blocks: Firstly, a population model to study the impact of demographic changes in the population structure. Secondly, a description of the production side of the economy to analyze the consequences of pension reforms on real variables such as capital stock, real wages, etc. Thirdly, a stochastic asset market in order to study the role of financial risk and the impact of pension parameters on financial variables such as stock prices, interest rates, etc.

The development and study of such a framework forms the core of this work. Some of the issues to be analyzed within the model are:

- Does the randomness in stock markets necessarily transmit to the real sector and the pension income?
- What are the general interactions between pension systems, the production sector and asset markets, in particular with stock markets?

- How does a reduction in public pensions and a shift towards more pre-funding affect real and financial markets and the welfare of consumers?
- Which impact does demographic change exert on the evolution of real and financial markets?
- How does demographic change affect the welfare of consumers and how should the pension system be adjusted in response to this development?

The literature on pension systems mostly confines itself to a deterministic framework. In this regard, the multiperiod overlapping generations model developed in [5] has been employed by numerous authors to study pension reforms within so-called computable general equilibrium (CGE) models. Examples may be found in [3], [18] or in the survey by [25]. In these models, the behavior of households and firms is typically derived from a deterministic intertemporal decision problem based on expectations for future economic variables. Usually these models are then solved numerically such that market clearing prices are determined from the behavior of agents and consistency between expectations and actual realizations is obtained. This approach permits a numerical study of alternative pension systems and their impact on certain economic variables in a deterministic world. As a consequence of the deterministic setting, it is not possible to incorporate randomness and to study the role of risk within these models. In addition, the proposed CGE approach essentially restricts attention to a particular path of the system along which agents' expectations are fully rational. As a consequence, it is not possible to study the dynamic behavior of the system and the influence of parameter changes using mathematical methods from deterministic dynamical systems and bifurcation theory. This limits the theoretical possibilities to analyze the impact of pension reforms on the evolution and long run dynamic behavior of the economy. Further deterministic studies dealing with the efficiency of pay-as-you-go pension systems can be found, e.g., in [21], [22] and [45].

While deterministic pension models seem to be pervasive in the literature, similar models which incorporate randomness and a stochastic asset market appear to have received much less attention. More importantly, many of the existing ones treat prices and returns on asset markets or consumers' income processes as given stochastic processes which follow an exogenously determined probability law. Examples may be found in [29], [30] or [33]. While this approach permits a study of risk to which pension incomes and savings for retirement are exposed, it neglects the mutual interactions between pension systems and asset markets as exhibited above. As a consequence, a theoretical study of the feedback structure between pension systems and certain real and financial variables is not possible in this setup.

Stochastic models which partly overcome the afore-mentioned problem of a mere exogenous asset price process can be found, for instance, in [27], [38], and [48]. These studies employ a particular stochastic setting where the underlying probability space is finite and the randomness can be represented by a so-called date-event tree. A distinctive advantage of this approach is that the theoretical framework of incomplete markets (see, e.g., [50]) becomes applicable. This permits the formulation of various normative concepts such as interim Pareto optimality and ex-ante efficiency which may be applied to study and compare pension systems and their impact on consumer welfare. On the other hand, the proposed structure makes it difficult to characterize the evolution of the model on a time series level using tools and methods from (random) dynamical systems theory and time series analysis. A comparison of the long-run dynamic behavior of real and financial variables such as aggregate output or asset prices and their statistical properties depending on the population structure and/or the parameters of the pension system is therefore not possible. As a further obstacle, it is almost impossible to obtain explicit solutions within the proposed setting such that the analysis is typically confined to numerical simulations. In this regard, the solution of these models becomes computationally very involved with a large number of generations and/or stochastic states. As a consequence, many of the models within this class restrict attention to an overlapping generations setting with only few generations and/or assume a constant population. This limits the theoretical possibilities to account for demographic effects and changes in the population structure. In addition, it becomes difficult to calibrate these models and compare them to (annual) empirical data due to the relatively coarse time structure. Further studies which focus on the interactions between real capital markets (interpreted as stock markets) and the evolution of the population can be found in [1], [2] and [37]. Again these models employ a deterministic or simplified stochastic setting. None of them incorporates the uncertainty and randomness typically observed in stock markets.

With these findings the aim of this dissertation is twofold. The first goal is the conception and development of a suitable theoretical model that complements the existing approaches in the literature. The second goal is to employ this framework and present a theoretical study which investigates the efficiency of pension systems and their interactions with real and financial variables in the presence of demographic change of the population. Given these two objectives the dissertation is divided into two parts. The first part comprises the conception and development of a suitable theoretical framework. In this regard, the explicit modeling strategy successfully applied in the asset market models by [12], [13], [44] and [60] is adopted. These models provide an explicit description of the formation and dynamic evolution of prices and allocations on financial markets and extend the class of rational expectations models by allowing for arbitrary and hence possibly non-rational expectations of investors. For our purposes the conceptual challenge is to join these financial models with a real sector describing endogenously the production and investment activities of firms and the income streams of consumers generated through the production process. A similar objective has been pursued in [10] and has further been developed in [14], [15]. These authors combine the dynamic version of the Capital Asset Pricing Model (CAPM) studied in [12] with the theoretical framework of a neoclassical growth model providing a description of the production side of the economy and the income streams of consumers. Like most of the afore-mentioned models, they maintain the assumption of a stationary OLG population with two-period lived consumers who consume only in their terminal period of life. A first challenging task of this work is to extend the underlying population model by allowing for consumers with multiperiod lives and for changes in the population structure to incorporate the impact of demographic change. In addition, it seems desirable to extend the individual consumption-savings decision by allowing for consumption in each period of the life cycle. In this regard, the multiperiod asset market models developed in [43] and [44] combined with the class of multiperiod consumption-investment problems studied in [40], [41] provides a satisfactory starting point.

As with the deterministic models described above the approach put forward in this dissertation employs an overlapping generations structure of consumers who live for a finite number of time periods. Assuming that this number remains constant over time, the demographic change of the population is exclusively governed by the number of young consumers born in any one period which is assumed to be exogenously determined. The latter assumption allows to hypothesize various demographic scenarios corresponding to different birth processes and to analyze their impact on real and financial variables as well as on the efficiency and sustainability of the pension system.

Within this framework the subsequent model building may be subdivided into three stages. The first stage comprises a sound microeconomic foundation of the demand behavior of consumers and firms in the economy which are derived from suitably defined decision problems. Apart from the usual microeconomic characteristics like preferences, technologies, etc., the resulting demand behavior is mainly determined by agents' subjective expectations for uncertain economic variables (future prices, dividends, real wages, etc.). At this stage, the impact of a pension system on individual behavior can be studied offering a possibility to assess the consequences of alternative pension reforms on the micro-level. For example, one could analyze under which circumstances a reduction in public pension payments leads to an increase in private savings as is claimed with the German Riester reform.

Utilizing the demand behavior obtained at the first stage the second stage describes the interactions of consumers and firms on real and financial markets in the economy. In this regard, the concept of a temporary equilibrium is used to model the formation of prices and allocations on the respective markets in each period. This provides a possibility to study the impact of pension systems on real and financial markets in a certain time period for given expectations and a given population structure. At this stage, for instance, the comparative-static effect of pension reforms on stock prices in a given time period could be studied.

The third stage models the evolution of the model's variables over time. This is achieved by introducing the concept of a forecasting rule providing a description of how consumers and firms form their expectations based on the available information. In addition, a population law is specified which describes the birth process and changes in the population structure. The evolution of the economy is then essentially governed by the interaction of market forces determining prices and allocations in any one period and the prediction behavior of agents as well as the evolution of the population. In addition, the system may be subjected to exogenous random shocks of an arbitrary stochastic nature in each time period.

Following this modeling strategy the first part of this dissertation develops a dynamic macroeconomic model which complements the existing approaches in the literature by incorporating the three building blocks described above. To make the employed framework sufficiently flexible and amendable to future extensions, the model is derived first under general assumptions on the microeconomic characteristics of consumers and firms. This is followed by a parameterized version in which specific assumptions on these characteristics are made to obtain explicit functional forms of the model's equations. The latter provides the basis to study the model's dynamic behavior using numerical simulations.

Building upon this framework the second part of this work comprises a comprehensive study of pension systems and their interactions with real and financial variables with and without demographic change of the population. A major advantage of the previously described modeling strategy is that the tools and methods from dynamical systems theory, in particular those of random dynamical systems [4], become available. These concepts offer a powerful mathematical framework to study the long run evolution of the economy depending on the prevailing pension system and to assess and compare the efficiency and sustainability of alternative pension policies.

Due to the complexity of the model the analysis in the second part is mainly carried out with the help of numerical simulations. To obtain a benchmark scenario, the study first considers the case with a constant population. In a second step this is extended to the case with demographic transitions and a shrinking population. Proceeding in this fashion allows one to carefully separate the effects induced by the pension system and those which are due to demographic changes in the population. The analysis in both scenarios is carried out by first taking a purely descriptive view that analyzes the qualitative and quantitative impact of pension systems on prices and allocations on real and financial markets. This is followed by a normative part that seeks to judge and compare alternative pension policies in terms of efficiency by analyzing their impact on consumer welfare. The policies under scrutiny also comprise various types of adjustment that have been suggested or applied as part of the reform of the German pension system (cf. [56]).

The dissertation is organized as follows. Part I consisting of Chapters 2 and 3 develops the underlying theoretical framework. In this regard, Chapter 2 introduces the basic setup under general assumptions on the microeconomic characteristics of consumers and firms. These assumptions are specialized in Chapter 3 in order to obtain a particular parametrization of the model. This provides the basis for the simulation study which is presented in Part II consisting of Chapters 4 and 5. In this regard, Chapter 4 studies the special case where the population is constant over time. Chapter 5 extends the study to the case with demographic transitions and a changing population structure. The final Chapter 6 draws some conclusions and outlines possible extensions.

Part I

The Model

2

The General Model

Following the modeling strategy proposed in the introduction the following two chapters develop a theoretical framework that forms the basis for the subsequent study of pension systems and the issues motivated above. In this regard, the present chapter introduces the basic setup of the model under a general class of assumptions on the microeconomic characteristics of consumers and producers. This ensures that the model is sufficiently flexible and amendable to future extensions. The primary goal is to derive the demand behavior of consumers and firms from suitably defined decision problems. In a second step, the market structure and the formation of prices and allocations on the respective market is formulated by employing the concept of a temporary equilibrium.

The chapter is organized as follows: Section 2.1 introduces the OLG population structure. A typical consumer's decision problem and the existence of demand functions are considered in Sections 2.2 and 2.3. Section 2.4 studies the production and investment behavior of firms while the market structure and the formation of prices is considered in Section 2.5. The final Section 2.6 derives some general results on pension systems and the consequences of demographic change. Mathematical proofs are contained in the Appendices 2.A and 2.B.

2.1 Overlapping Generations of Consumers

Consider an economy with a discrete time structure and a population consisting of overlapping generations (OLG) of homogeneous consumers who live for $J+1$ consecutive periods. In each time period $t \in \mathbb{N}_0$, each generation is identified by the index $j \in \{0, 1, \dots, J\}$ describing the remaining lifetime of the consumers in this generation. In particular, $j = J$ refers to the young generation of consumers born at the beginning of period t and $j = 0$ identifies the old generation whose members die at the end of the period under consideration.

Let $N_t^{(j)} > 0$ denote the number of consumers in generation j at time t and define for each t the population vector $N_t := (N_t^{(j)})_{j=0}^J$. Each consumer in generation $j \in \{j_L, \dots, J\}$ supplies $\bar{L}^{(j)} > 0$ units of labor inelastically to the labor market where the threshold $j_L > 0$ defines the retirement age. The total amount of labor supplied at time t is thus exclusively determined by the structure of the population and given by

$$L_t^S := \sum_{j=j_L}^{J} \bar{L}^{(j)} N_t^{(j)}. \tag{2.1}$$

There is a single consumption good in the economy which serves as numeraire such that all prices and payments are denominated in terms of the consumption good. Let $\omega_t > 0$ denote the gross real wage per unit of labor at time t out of which a fraction $\tau_t \in [0, 1[$ has to be contributed to the public pension system. Then each working consumer in generation $j \in \{j_L, \dots, J\}$ earns net labor income

$$e_t^{(j)} = (1 - \tau_t)\omega_t \, \bar{L}^{(j)} > 0 \tag{2.2}$$

at time t. The pension system is a pure pay-as-you-go system where contributions are divided up equally between current retirees. It follows that the non-capital income of each consumer in generation $j \in \{0, \dots, j_L - 1\}$ at time t is given by

$$e_t^{(j)} = e_t^R := \tau_t \frac{\omega_t \, L_t^S}{N_t^R} \geq 0 \tag{2.3}$$

with $N_t^R := \sum_{j=0}^{j_L - 1} N_t^{(j)}$ denoting the number of pensioners at time t.

To transfer income between different periods there exist $M + 1$ assets in the economy, indexed $m = 0, 1, \dots, M$. The first asset $m = 0$ is a one-period lived bond which is traded at a price of unity at time t and pays a non-random return $R_t > 0$ in the following period $t + 1$. Since R_t is determined at time t, the bond provides a riskless investment possibility between any two consecutive periods. The remaining M assets correspond to retradeable shares of firms $m = 1, \dots, M$ which are traded at strictly positive asset prices $p_t = (p_t^{(1)}, \dots, p_t^{(M)})^\top \in \mathbb{R}_{++}^M$ and pay a non-negative random dividend $d_t = (d_t^{(1)}, \dots, d_t^{(M)})^\top \in \mathbb{R}_+^M$

(prior to trading) in each period t. The dividends are generated endogenously from the production activities of the respective firm. The total number of shares issued by firms is constant and given by the vector $\bar{x} = (\bar{x}^{(1)}, \dots, \bar{x}^{(M)})^\top \in \mathbb{R}^M_{++}$ with $\bar{x}^{(m)} > 0$ denoting the number of shares issued by firm m.

Let $\mathbb{Y} \subset \mathbb{R}$ denote the set of feasible bond investments and $\mathbb{X} \subset \mathbb{R}^M$ the set of feasible risky portfolios for each consumer. In the sequel we assume that bonds may be sold short without bound but exclude short selling of shares $m = 1, \dots, M$ such that $\mathbb{Y} = \mathbb{R}$ and $\mathbb{X} = \mathbb{R}^M_+$. The bond thus provides the sole possibility for consumers to obtain credit. The space $\mathbb{Z} := \mathbb{Y} \times \mathbb{X}$ defines the set of feasible portfolios for each consumer. In addition, we assume that firms can issue bonds to obtain credit for their capital investment but are not allowed to purchase bonds.

Denote by $z_t^{(j)} := (y_t^{(j)}, x_t^{(j)}) \in \mathbb{Z}$ the portfolio purchased by a consumer in generation $j \in \{1, \dots, J\}$ at time t consisting of a bond investment $y_t^{(j)} \in \mathbb{Y}$ and a non-negative vector $x_t^{(j)} \in \mathbb{X}$ defining the number of shares in the portfolio.[1] The wealth of a consumer in generation j at time t consists of his current *non-capital income* defined by (2.2) and (2.3), respectively and his *capital income* corresponding to the return on his previous investment $z_{t-1}^{(j+1)} = (y_{t-1}^{(j+1)}, x_{t-1}^{(j+1)})$. The latter comprises the return on the bond investment $y_{t-1}^{(j+1)}$ and the return on the stock portfolio $x_{t-1}^{(j+1)}$ consisting of dividend earnings and the selling revenue at time t. Since the capital income of young consumers is zero, we define the wealth of a consumer in generation j at time t as

$$w_t^{(j)} := \begin{cases} e_t^{(j)}, & j = J \\ e_t^{(j)} + R_{t-1}\, y_{t-1}^{(j+1)} + x_{t-1}^{(j+1)\top}(p_t + d_t) & j = 0, \dots, J-1. \end{cases} \tag{2.4}$$

2.2 Decision Problem of Consumers

To derive the consumption and investment behavior of consumers consider a typical consumer belonging to generation $j > 0$ in an arbitrary period t who dies at the end of period $t+j$. To alleviate the time script notation we set $t = 0$ for the current period and use the time index

[1] We adopt the convention that the generational index j always refers to the time index of the respective variable. For example, $z_t^{(j)}$ is the portfolio held by a consumer at time t who dies at the end of time $t+j$. If $j < J$, $z_{t-1}^{(j+1)}$ identifies his portfolio from the previous period $t-1$ and if $j > 1$, $z_{t+1}^{(j-1)}$ is the portfolio purchased by him during the following period $t+1$

$n \in \{0, 1, \ldots, j\}$ to refer to periods within the consumer's remaining lifetime. The index j identifying the consumer's generation will be suppressed for convenience.

In each period $n \in \{0, 1, \ldots, j\}$ the consumer can consume part of his wealth and use the investment possibilities described in the previous section to transfer wealth into future periods. Let $\mathbb{C} \subset \mathbb{R}$ denote the consumption set describing feasible consumption plans in each period. In the sequel we set $\mathbb{C} = \mathbb{R}_+$ requiring that consumption be nonnegative but may be arbitrarily large. It is assumed that the decision in $t = 0$ is made *after* the dividend payment $d_0 \in \mathbb{R}^M_+$ and the consumer's current non-capital income $e_0 \geq 0$ are observed but *prior* to trading, i.e., before the bond return R_0 and asset prices p_0 are determined. Hence, the consumer treats these variables as parameters $R > 0$ and $p \in \mathbb{R}^M_{++}$. Likewise, his current wealth position defined by (2.4) is treated as parameter $w \in \mathbb{R}$ in the decision problem. Although the latter value will generically (whenever $j < J$) depend on current asset prices, it will be convenient to suppress this dependence for the following derivations and treat current wealth as a separate parameter.

In $t = 0$ the consumer holds expectations $\hat{e} := (\hat{e}_1, \ldots, \hat{e}_j) \in \mathbb{R}^j_+$ for his future non-capital income with $\hat{e}_n \geq 0$ denoting the non-capital income expected to receive in period $n \in \{1, \ldots, j\}$. Likewise he holds expectations $\hat{R} := (\hat{R}_1, \ldots, \hat{R}_{j-1}) \in \mathbb{R}^{j-1}_{++}$ for future bond returns where $\hat{R}_n$ is his point forecast for the bond return R_n between future periods n and $n+1$, $n \in \{1, \ldots, j-1\}$. For the following derivations the consumer's planning horizon j as well as his expectations will be assumed to be fixed quantities and will therefore be suppressed as arguments of functions, etc. to alleviate the notation.

For each $n \in \mathbb{N}_0$ define the pair $s_n := (p_n, d_n) \in \mathbb{S} := \mathbb{R}^M_{++} \times \mathbb{R}^M_+$ of prices and dividends in period n. At time $t = 0$ there is uncertainty about all future s_n, $n > 0$ which are treated as random variables in the decision problem. More specifically, the consumer considers future prices and dividends as an $\mathbb{S}$-valued stochastic process $\{s_n\}_{n \geq 1}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which are adapted to a suitable filtration $\{\mathcal{F}_n\}_{n \geq 1}$ of sub-σ-algebras of $\mathcal{F}$.[2]

Equipped with these prerequisites the following definition introduces the concept of a strategy which will be the crucial object in the con-

[2] This kind of behavior suggests that the consumer perceives future asset prices and dividends to be the primary source of randomness and uncertainty while future non-capital incomes and bond returns can be relatively precisely predicted. It will be shown later that this assumption is actually consistent with the dynamic behavior of the model.

sumer's decision problem. In the sequel the notation $\mathcal{B}(\mathbb{A})$ is used to denote the Borel σ–algebra on a given topological space $\mathbb{A}$.

Definition 2.2.1 *Let the planning horizon $j \geq 1$ be given.*

(i) A consumption strategy is a list $C = (c_0, c_1(\cdot), \ldots, c_j(\cdot))$ consisting of a consumption decision $c_0 \in \mathbb{C}$ for $t = 0$ and a list of $\mathcal{B}(\mathbb{S}^n)$–$\mathcal{B}(\mathbb{C})$ measurable functions (consumption plans) $c_n : \mathbb{S}^n \longrightarrow \mathbb{C}$ for each $n = 1, \ldots, j$.

(ii) An investment strategy is a list $Z = (z_0, z_1(\cdot), \ldots, z_{j-1}(\cdot))$ consisting of an investment decision $z_0 = (y_0, x_0) \in \mathbb{Z}$ for $t = 0$ and a list of $\mathcal{B}(\mathbb{S}^n) - \mathcal{B}(\mathbb{Z})$ measurable functions (investment plans) $z_n = (y_n, x_n) : \mathbb{S}^n \longrightarrow \mathbb{Z}$ for each $n = 1, \ldots, j-1$.

(iii) The pair (C, Z) is called a consumption investment strategy *or simply a* strategy.

A consumption strategy specifies the current consumption decision c_0 and consumption plans for all future periods $n = 1, \ldots, j$ within the consumer's planning horizon which are made conditional on the random variables $s_1, \ldots, s_n$ observed up to time n.[3] Likewise, the investment strategy Z specifies the current investment $z_0 = (y_0, x_0)$ and planned investments in bonds and shares for all future periods. Since the consumer's planning horizon ends in period j, no portfolio is carried over to period $j+1$ such that we may set $z_j \equiv 0$ for the investment plan in period j.

In the sequel we adopt the notation $s_1^n := (s_1, \ldots, s_n) \in \mathbb{S}^n$, $n \geq 1$ and set $\hat{R}_0 := R$. Furthermore, the arguments of a plan for period n are frequently suppressed by writing c_n, y_n, and x_n instead of $c_n(s_1^n)$, $y_n(s_1^n)$, and $x_n(s_1^n)$. Given these conventions, the following definition characterizes the strategies which are feasible from the initial situation at time $t = 0$.

Definition 2.2.2 *Given the bond return $R > 0$, prices $p \gg 0$ and initial wealth w at time $t = 0$, a strategy (C, Z) is called feasible if*

(i) $c_0 + y_0 + x_0^\top p = w$

[3] The literature often defines a strategy as an adapted stochastic process defined on a probability space representing uncertain future states of the world. While the definition given here is equivalent from a mathematical point of view, the uncertainty here rests on future asset prices and dividends rather than states of the world. This formulation appears more suitable from an economic point of view since prices and dividends are the relevant quantities which are directly observable.

(ii) for each $n = 1, \ldots, j$

$$c_n + y_n + x_n^\top p_n = \hat{e}_n + \hat{R}_{n-1}\, y_{n-1} + x_{n-1}^\top (p_n + d_n) \qquad \forall s_1^n \in \mathbb{S}^n.$$

The set of all strategies which are feasible from (R, p, w) *is denoted by* $\mathscr{B}(R, p, w)$.

Throughout we shall assume that the strategy set $\mathscr{B}(R, p, w)$ is non-empty. Conditions under which this is the case are stated in Lemma 2.2.1 below. Associated with the choice of an investment strategy Z is the induced wealth process $\{W_n(Z, s_1^n)\}_{n=1}^j$ where

$$W_n(Z, s_1^n) := \hat{e}_n + \hat{R}_{n-1}\, y_{n-1} + x_{n-1}^\top\, (p_n + d_n)\,. \tag{2.5}$$

The value $W_n(Z, s_1^n)$ describes the consumer's future wealth at time $n \geq 1$ depending on the investment strategy $Z = (z_n)_{n=0}^{j-1} = (y_n, x_n)_{n=0}^{j-1}$ and the random variables $s_1, \ldots, s_n$ observed up to period n. Note that wealth may well become negative. However, the following Lemma 2.2.1 shows that there exist lower bounds on planned bond investments and on the wealth process which are essentially determined by the discounted non-capital income stream. To obtain a compact notation define the values

$$\begin{aligned} \hat{\epsilon}_0 &:= \hat{e}_1 + \frac{\hat{e}_2}{\hat{R}_1} + \ldots + \frac{\hat{e}_j}{\hat{R}_1 \cdots \hat{R}_{j-1}} \geq 0 \\ \hat{E}_n &:= \frac{\hat{e}_{n+1}}{\hat{R}_n} + \ldots + \frac{\hat{e}_j}{\hat{R}_n \cdots \hat{R}_{j-1}} \geq 0,\ n = 1, \ldots, j \end{aligned} \tag{2.6}$$

derived from the expectations $\hat{e}$ and $\hat{R}$ with the understanding that $\hat{E}_j := 0$. For each future period $n = 1, \ldots, j$, the value $\hat{E}_n \geq 0$ defines the discounted non-capital income stream expected to receive after period n. Likewise, given the current bond return $R > 0$, the value $\hat{\epsilon}_0/R$ defines the expected future non-capital income stream discounted to the decision period $t = 0$. Utilizing these values, the following lemma establishes the desired properties of the consumer's investment behavior and the wealth process and provides conditions under which the strategy set is non-empty.

Lemma 2.2.1 *Let* $\hat{\epsilon}_0 \geq 0$ *and* $\hat{E}_n \geq 0$, $n = 1, \ldots, j$ *be defined as in (2.6). Then for each strategy* $(C, Z) \in \mathscr{B}(R, p, w)$ *the following holds true:*

(i) The bond investments $(y_0, y_1(\cdot), \ldots, y_{j-1}(\cdot))$ *associated with* Z *satisfy*

$$y_0 \geq -\hat{\epsilon}_0/R \quad \textit{and} \quad y_n(s_1^n) \geq -\hat{E}_n \quad \textit{for all } s_1^n \in \mathbb{S}^n.$$

(ii) The associated wealth process $\{W_n(Z, s_1^n)\}_{n=1}^j$ *defined as in (2.5) satisfies:*

$$W_n(Z, s_1^n) \geq -\hat{E}_n \quad \textit{for all } s_1^n \in \mathbb{S}^n.$$

(iii) The strategy set $\mathscr{B}(R, p, w)$ *is non-empty if and only if* $w \geq -\hat{\epsilon}_0/R$.

Proof. We show the claim in (i) for $n = j - 1$ and then apply an induction argument. Let $(C, Z) \in \mathscr{B}(R, p, w)$ be an arbitrary strategy. Since $\mathbb{C} = \mathbb{R}_+$ and there is no investment in the terminal period, the consumption plan for period j must satisfy

$$c_j = \hat{e}_j + \hat{R}_{j-1}\, y_{j-1} + x_{j-1}^\top (p_j + d_j) \geq 0 \tag{2.7}$$

for all $s_1^j \in \mathbb{S}^j$. If $j > 1$, let $s_1^{j-1} \in \mathbb{S}^{j-1}$ be arbitrary but fixed. Then (2.7) must hold for any $s_j \in \mathbb{S}$. Recalling that $\mathbb{S} = \mathbb{R}_{++}^M \times \mathbb{R}_+^M$, the last term on the r.h.s. of (2.7) is non-negative but may become arbitrarily small. It follows that (2.7) can only be satisfied for *all* $s_j \in \mathbb{S}$ if $\hat{R}_{j-1} y_{j-1} \geq -\hat{e}_j$. If $j > 1$ this requires $y_{j-1}(s_1^{j-1}) \geq -\hat{E}_{j-1}$ for all $s_1^{j-1} \in \mathbb{S}^{j-1}$ while for $j = 1$ one must have $y_0 \geq -\hat{\epsilon}_0/R$.

Now let $n \in \{0, 1, \ldots, j-2\}$ be arbitrary and assume that the claim is true for $n + 1$, i.e., $y_{n+1}(s_1^{n+1}) \geq -\hat{E}_{n+1}$ for all $s_1^{n+1} \in \mathbb{S}^{n+1}$. By Definition 2.2.2

$$\begin{aligned} c_{n+1} &= \hat{e}_{n+1} + \hat{R}_n\, y_n + x_n^\top (p_{n+1} + d_{n+1}) - y_{n+1} - x_{n+1}^\top p_{n+1} \\ &\leq \hat{e}_{n+1} + \hat{R}_n\, y_n + x_n^\top (p_{n+1} + d_{n+1}) + \hat{E}_{n+1}. \end{aligned} \tag{2.8}$$

Using a similar argument as in the first step equation (2.8) requires that for all $s_{n+1} \in \mathbb{S}$

$$\hat{e}_{n+1} + \hat{R}_n y_n + (p_{n+1} + d_{n+1})^\top x_n + \hat{E}_{n+1} \geq 0$$

and, therefore, $\hat{e}_{n+1} + \hat{R}_n y_n + \hat{E}_{n+1} \geq 0$. For $n > 0$ this is equivalent to $y_n(s_1^n) \geq -\hat{E}_n$ for all $s_1^n \in \mathbb{S}^n$. If $n = 0$ the above inequality requires $y_0 \geq -\hat{\epsilon}_0/R$. This proves claim (i). The assertion (ii) is an immediate consequence of (i) and (2.5). The 'only if' part in (iii) can be proved by using Definition 2.2.2 (i) and the result from (i) to see that $w < -\hat{\epsilon}_0/R$ implies $c_0 = w - y_0 - x^\top p \leq w - y_0 \leq w + \hat{\epsilon}_0/R < 0$ such that the condition $w \geq -\hat{\epsilon}_0/R$ is necessary. The 'if' part in (iii) follows from the fact that as soon as $w \geq -\hat{\epsilon}_0/R$ the set $\mathscr{B}(R, p, w)$ will contain the pure-bond investment strategy which never invests in any risky assets $m > 0$ and consumes only in the terminal period of life. ■

Remark 2.2.1 *One observes from Definition 2.2.2 and Lemma 2.2.1 that if* $w = -\hat{e}_0/R$ *the strategy set* $\mathscr{B}(R, p, w)$ *contains a single strategy* (C, Z) *defined by the decision* $c_0 = 0$, $y_0 = -\hat{e}_0/R$, $x_0 = 0$ *and plans* $c_n \equiv 0$, $y_n \equiv -\hat{E}_n$ *and* $x_n \equiv 0$ *for each* $n = 1, \ldots, j$.

Based on the set $\mathscr{B}(R, p, w)$ of possible strategies available to the consumer at time $t = 0$, the next goal is to set up a corresponding decision problem. This requires assumptions on the consumer's preferences about alternative strategies as well as on the perceived stochastic nature of future asset prices and dividends. In this regard, recall that at time $t = 0$ the consumer treats future prices and dividends $s_1, s_2, \ldots$ as random variables. It is now assumed that he forms expectations for those future prices and dividends which are within his planning horizon. These expectations are characterized in the following assumption.

Assumption 2.2.1 *Given the planning horizon* $j > 0$ *the consumer's expectations at time* $t = 0$ *for future asset prices and dividends are given by a probability measure* ν *on the measurable product space* $\big(\mathbb{S}^j, \mathcal{B}(\mathbb{S}^j)\big)$ *defining a subjective joint probability distribution of the random variables* $s_1, \ldots, s_j$.

In the sequel we denote the class of all probability measures on $\big(\mathbb{S}^j, \mathcal{B}(\mathbb{S}^j)\big)$, $j \geq 1$, by $\mathrm{Prob}(\mathbb{S}^j)$. The following assumption characterizes the consumer's preferences over alternative consumption streams.

Assumption 2.2.2 *Given the planning horizon* $j \geq 1$, *the consumer's preferences over consumption within his remaining lifetime possess an expected utility representation which is defined by the utility function*

$$(c_0, c_1, \ldots, c_j) \longmapsto u(c_0) + \sum_{n=1}^{j} \beta^n u(c_n), \quad \beta \in]0, 1[. \tag{2.9}$$

The instantaneous utility function $u : \mathbb{C} \longrightarrow \mathbb{R}$ *is continuous, strictly increasing, and strictly concave.*

For each strategy $(C, Z) \in \mathscr{B}(R, p, w)$ and $s_1^j \in \mathbb{S}^j$, define the utility attained over the remaining lifetime

$$U_0(C, s_1^j) := u(c_0) + \sum_{n=1}^{j} \beta^n u(c_n(s_1^n)).$$

The expected utility induced by strategy $(C, Z) \in \mathscr{B}(R, p, w)$ is thus given by

$$\mathbb{E}_\nu \left[U_0(C, \cdot)\right] = \int_{\mathbb{S}^j} U_0(C, s_1^j)\ \nu(ds_1^j). \tag{2.10}$$

Letting for each (R, p, w) for which $\mathscr{B}(R, p, w) \neq \emptyset$

$$V_0(R, p, w) := \sup\left\{ \int_{\mathbb{S}^j} U_0(C, s_1^j)\ \nu(ds_1^j) \middle| (C, Z) \in \mathscr{B}(R, p, w) \right\}, \tag{2.11}$$

the definition of an optimal strategy is then straightforward.

Definition 2.2.3 *Given a triple (R, p, w) with $\mathscr{B}(R, p, w) \neq \emptyset$, a strategy $(C^\star, Z^\star) \in \mathscr{B}(R, p, w)$ is termed an* optimal strategy *if*

$$\int_{\mathbb{S}^j} U_0(C^\star, s_1^j)\ \nu(ds_1^j) = V_0(R, p, w), \tag{2.12}$$

where $V_0(R, p, w)$ is defined in (2.11).

The consumer pursues the objective to choose an optimal strategy $(C^\star, Z^\star) \in \mathscr{B}(R, p, w)$ in the sense of Definition 2.2.3. Formally his decision problem at time $t = 0$ may be stated as

$$\max_{(C,Z)} \left\{ \int_{\mathbb{S}^j} U_0(C, s_1^j)\ \nu(ds_1^j) \middle| (C, Z) \in \mathscr{B}(R, p, w) \right\}. \tag{2.13}$$

Note, however, that the problem (2.13) is only well-defined if the supremum in (2.11) is finite, i.e., if $V_0(R, p, w) < \infty$. While this is trivially satisfied for the case where the utility function u is bounded (as is often assumed in the literature), this requirement turns out to be too strong in many scenarios. This is for example the case with a logarithmic utility function which is studied in Chapter 3. The following assumption offers an alternative providing two sufficient conditions for the decision problem (2.13) to be well-defined.

Assumption 2.2.3 *Either of the following two conditions is satisfied:*

(i) The utility function $u : \mathbb{C} \longrightarrow \mathbb{R}$ in Assumption 2.2.2 is bounded.
(ii) The measure ν defined in Assumption 2.2.1 is supported on the compact set[4]

$$\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \ldots \times \bar{\mathbb{S}}_j \in \mathcal{B}(\mathbb{S}^j) \tag{2.14}$$

with each $\bar{\mathbb{S}}_n$, $n = 1, \ldots, j$, being compact.

[4] The assumption that $\bar{\mathbb{S}}$ is of the product form stated in equation (2.14) is made just for convenience. In principle, it suffices to have a compact set $\bar{\mathbb{S}}$.

The following lemma shows that (ii) in Assumption 2.2.3 is indeed sufficient for the decision problem to be well-defined even if u is not bounded. The proof is given in Section 2.A.1 in Appendix 2.A of this chapter.

Lemma 2.2.2 *Let the utility function u satisfy the conditions stated in Assumption 2.2.2 and assume that the joint distribution ν satisfies Assumption 2.2.3 (ii). Then the supremum in (2.11) satisfies $V_0(R,p,w) < \infty$ for all (R,p,w) for which $\mathscr{B}(R,p,w) \neq \emptyset$.*

In the sequel we assume that Assumption 2.2.3 is satisfied such that the consumer's decision problem (2.13) is well-defined. Associated with a solution to (2.13) defining an optimal strategy $(C^\star, Z^\star)$ is an optimal decision $(c_0^\star, z_0^\star) \in \mathbb{C} \times \mathbb{Z}$ for $t = 0$. The main goal of the following section is to state conditions under which this optimal decision is well-defined and can be represented by a continuous function describing the consumer's demand behavior in the decision period.

2.3 Consumer Demand

The goal of this section is to investigate the existence of an optimal strategy defining a solution to the consumer's decision problem. Since this task turns out to be trivial in the case where $w = -\hat{\epsilon}_0/R$ (see Remark 2.2.1), assume for the following derivations that $w > -\hat{\epsilon}_0/R$. Employing a recursive solution technique from stochastic dynamic programming we show that under some mild additional restrictions a continuous demand function describing the optimal consumption and investment decision for $t = 0$ can be defined.

To alleviate the subsequent notation define the following function $W : \mathbb{Z} \times \mathbb{S} \times \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$,

$$W\big((y,x),(p,d),\hat{e},\hat{R}\big) := \hat{e} + \hat{R}\,y + x^\top(p+d). \tag{2.15}$$

Given expectations $\hat{e}_{n+1} \geq 0$, $\hat{R}_n > 0$ and a portfolio $z_n \in \mathbb{Z}$ purchased at time n the value $W(z_n, s_{n+1}, \hat{e}_{n+1}, \hat{R}_n)$ describes the consumer's wealth in the following period depending on prices p_{n+1} and dividends d_{n+1}. The following definition introduces the concept of non-redundant assets that will become important in the sequel.

Definition 2.3.1 *A probability measure $\hat{\nu}$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is said to induce non-redundant assets if for any two portfolios $z', z'' \in \mathbb{Z}$, $z' \neq z''$ the set $A(z',z'') := \big\{s \in \mathbb{S} | W(z',s,0,1) \neq W(z'',s,0,1)\big\} \in \mathcal{B}(\mathbb{S})$ satisfies $\hat{\nu}(A(z',z'')) > 0$.*

Note that the set $A(z', z'')$ in Definition 2.3.1 is indeed measurable due to the continuity of the function $W(\cdot)$. The property of non-redundancy ensures that two distinct portfolios can not induce the same return with probability one. This condition will turn out to be necessary in order to obtain demand functions. Intuitively, it is clear that otherwise the consumer's portfolio decision may not be uniquely determined. Note that the property of non-redundancy depends neither on the expected non-capital income nor on the bond return which is the reason why they have been set to zero and unity in the definition.[5]

The subsequent recursive solution of the consumer's decision problem requires to characterize for each $n \geq 1$ the conditional distribution of the random variable s_n depending on the previous observations $s_1, \dots, s_{n-1}$. Clearly, if $j = 1$, this task is trivial. If $j > 1$, the following lemma describes how the joint probability distribution ν introduced in Assumption 2.2.1 can be factorized into conditional probabilities and a marginal probability. Here $\mathbf{1}_B$ denotes the characteristic function on the set B which satisfies $\mathbf{1}_B(x) = 1$ if $x \in B$ and $\mathbf{1}_B(x) = 0$ otherwise. The proof is given in Section 2.A.2 in Appendix 2.A of this chapter.

Lemma 2.3.1 *Let $j > 1$ be arbitrary and let $\mathbb{S} = \mathbb{R}^M_{++} \times \mathbb{R}^M_{+}$ as before. Then there exists a factorization of the measure ν into conditional probabilities $Q_n : \mathbb{S}^{n-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$, $n = 2, \dots, j$, and a marginal probability $\nu_1 : \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ such that for each $B \in \mathcal{B}(\mathbb{S}^j)$:*

$$\nu(B) = \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \mathbf{1}_B(s_1^j)\, Q_j(s_1^{j-1}, ds_j) \cdots Q_2(s_1, ds_2)\nu_1(ds_1). \quad (2.16)$$

The factorization is ν–a.s. unique.

For each $n = 2, \dots, j$ the map $Q_n(s_1^{n-1}, \cdot) : \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ defines the conditional distribution of the random variable s_n depending on the previous observations $s_1, \dots, s_{n-1}$. Similarly, the measure ν_1 defines the marginal distribution of the random variable s_1. The following assumption imposes some additional restrictions on these distributions.

Assumption 2.3.1 *Each of the conditional probability distributions $Q_n(s_1^{n-1}, \cdot)$, $n = 2, \dots, j$ as well as the marginal probability distribution ν_1 from Lemma 2.3.1 each defined on the measurable space $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ satisfy the following conditions:*

[5] It can be shown that a sufficient condition for $\hat{\nu}$ to induce non-redundant assets is that for each $a \in \mathbb{R}^{2M}$ and $b \in \mathbb{R}$ the hyperplane $\mathbb{H}(a,b) := \{s \in \mathbb{S} | s^\top a = b\}$ satisfies $\hat{\nu}(\mathbb{H}(a,b)) < 1$. This condition is always satisfied if the variance-covariance matrix $\hat{\Sigma} := \mathbb{E}_{\hat{\nu}}\left[(s - \mathbb{E}_{\hat{\nu}}[s])(s - \mathbb{E}_{\hat{\nu}}[s])^\top\right]$ is positive definite and shows that the requirement of non-redundancy is not too strong.

(i) Each Q_n has the following Feller-property (see [59] for details): For each bounded and continuous function $h : \mathbb{H} \times \mathbb{S}^n \longrightarrow \mathbb{R}$, $\mathbb{H} \subset \mathbb{R}^m$ the integral function $H(x, s_1^{n-1}) := \int_{\mathbb{S}} h(x, s_1^{n-1}, s) Q_n(s_1^{n-1}, ds)$ is again bounded and continuous on $\mathbb{H} \times \mathbb{S}^{n-1}$.

(ii) The assets are non-redundant in the sense of Definition 2.3.1 with respect to each conditional distribution $Q_n(s_1^{n-1}, \cdot)$ as well as with respect to the marginal distribution ν_1.

Equipped with these technical preparations consider now the existence of a solution to the consumer's decision problem (2.13). The idea of the following recursive solution technique is to split the original j-period decision problem into a sequence of j one-stage problems. Since this task is again trivial if $j = 1$, assume for the following derivations that $j > 1$. Define the following list of value functions $V_n : [-\hat{E}_n, \infty[\times \mathbb{S}^n \longrightarrow \mathbb{R}$, $n = 1, \ldots, j$ recursively by setting $V_j(w_j, s_1^j) := u(w_j)$ and for each $n = 1, \ldots, j-1$

$$V_n(w_n, s_1^n) := \max_{(c,z) \in \mathbb{B}_n(w_n, p_n)} \left\{ u(c) + \beta \int_{\mathbb{S}} V_{n+1}\big(W(z, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s\big) Q_{n+1}(s_1^n, ds) \right\} \tag{2.17}$$

where the budget set at time n is defined as

$$\mathbb{B}_n(w_n, p_n) := \left\{ (c, y, x) \in \mathbb{C} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, c + y + x^\top p_n = w_n,\ y \geq -\hat{E}_n \right\}. \tag{2.18}$$

In the literature the recursion (2.17) is called Bellmann's equation. The following proposition ensures that the functions in (2.17) are indeed well-defined objects.

Proposition 2.3.1 *Let Assumptions 2.2.1– 2.3.1 be satisfied. Then the following holds true:*

(i) The value functions $V_n : [-\hat{E}_n, \infty[\times \mathbb{S}^n \longrightarrow \mathbb{R}$, $n = 1, \ldots, j$ defined recursively by (2.17) are all well-defined and continuous. If u is bounded, so is each V_n.

(ii) Each $V_n(\cdot, s_1^n)$ is strictly increasing and strictly concave for $s_1^n \in \mathbb{S}^n$.

(iii) At each stage $n = 1, \ldots, j-1$ the solutions to the maximization problem in (2.17) can be represented by a pair of continuous functions $(c_n^\star, z_n^\star) : [-\hat{E}_n, \infty[\times \mathbb{S}^n \longrightarrow \mathbb{C} \times \mathbb{Z}$.

Proof. Since $V_j(w_j, s_1^j) = u(w_j)$, properties (i) and (ii) are obviously true for $n = j$. Hence assume by way of induction that there exists $n \in \{1, \ldots, j-1\}$ such that V_{n+1} satisfies properties (i) and (ii). We show that this implies the properties (i) and (ii) for the function V_n and the solution to problem (2.17) satisfies (iii). In this regard, the first part of the proof covers the case where Assumption 2.2.3 (i) is satisfied and the utility function u is bounded. The second part extends the argument to an unbounded utility function by assuming that the measure ν satisfies Assumption 2.2.3 (ii).

The first induction hypothesis is that $V_{n+1}(\cdot)$ is well-defined, continuous, and bounded and $V_{n+1}(\cdot, s_1^{n+1}) : [-\hat{E}_{n+1}, \infty[\longrightarrow \mathbb{R}$ is strictly increasing and strictly concave for each fixed $s_1^{n+1} \in \mathbb{S}^{n+1}$. We show by induction that this implies that (i) and (ii) hold true for V_n and the solution at stage n satisfies (iii). To enhance readability the remainder of this proof is organized in five steps.

Step 1. Let $s_1^n \in \mathbb{S}^n$ and $w_n \geq -\hat{E}_n$ be arbitrary but fixed. Define the set $\mathbb{B}_n(w_n, p_n)$ as in (2.18) which is non-empty since $w_n \geq -\hat{E}_n$. For each $(c, z) \in \mathbb{B}_n(w_n, p_n)$ let

$$U_n(c, z; s_1^n) := u(c) + \beta \int_{\mathbb{S}} V_{n+1}\left(W(z, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s\right) Q_{n+1}(s_1^n, ds). \tag{2.19}$$

The integral in (2.19) is well-defined since for all $(c, y, x) \in \mathbb{B}_n(w_n, p_n)$ one has by (2.15) $W((y, x), (p, d), \hat{e}_{n+1}, \hat{R}_n) = \hat{e}_{n+1} + \hat{R}_n y + x^\top(p+d) \geq \hat{e}_{n+1} - \hat{R}_n \hat{E}_n = -\hat{E}_{n+1}$ for each $s = (p, d) \in \mathbb{S}$. Moreover, Assumptions 2.2.2 and 2.3.1 (i) together with the induction hypothesis imply that $U_n(\cdot)$ is continuous and bounded. Since $p_n \gg 0$ the set $\mathbb{B}_n(w_n, p_n)$ is compact implying that the maximization problem

$$\max_{(c,z)\in\mathbb{C}\times\mathbb{Z}} \left\{ U_n(c, z; s_1^n) \middle| (c, z) \in \mathbb{B}_n(w_n, p_n) \right\} \tag{2.20}$$

possesses a solution $(c^\star, z^\star)$.

Step 2. We show that the solution to (2.20) is unique. Since $\mathbb{B}_n(w_n, p_n)$ is convex, it suffices to show that the map $U_n(\cdot, s_1^n)$ is strictly concave. Let $(c', z'), (c'', z'') \in \mathbb{B}_n(w_n, p_n)$, $(c', z') \neq (c'', z'')$ and $\lambda \in]0, 1[$ be arbitrary and define $(c_\lambda, z_\lambda) := \lambda(c', z') + (1 - \lambda)(c'', z'')$. We show that $U_n(c_\lambda, z_\lambda, s_1^n) > \lambda U_n(c', z', s_1^n) + (1 - \lambda) U_n(c'', z'', s_1^n)$. If $z' = z''$, this is trivially satisfied, for in this case $c' \neq c''$ and the assertion follows immediately from the strict concavity of $u(\cdot)$. So assume $z' \neq z''$. The non-redundancy condition (ii) from Assumption 2.3.1 implies that the set $A_n(z', z'') := \left\{ s \in \mathbb{S} | W(z', s, \hat{e}_{n+1}, \hat{R}_n) \neq W(z'', s, \hat{e}_{n+1}, \hat{R}_n) \right\}$ has positive measure, i.e., $Q_{n+1}(s_1^n, A_n(z', z'')) > 0$. Furthermore, the linearity of $W(\cdot, \cdot, \hat{e}_{n+1}, \hat{R}_n)$ gives

$$W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n) = \lambda W(z', s, \hat{e}_{n+1}, \hat{R}_n) + (1 - \lambda) W(z'', s, \hat{e}_{n+1}, \hat{R}_n)$$

for all $s \in \mathbb{S}$. This together with the strict concavity of the function $V_{n+1}(\cdot, s_1^{n+1})$ yields the inequality

$$
\begin{aligned}
V_{n+1}(W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) \geq{}& \lambda V_{n+1}(W(z', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) \\
&+ (1-\lambda) V_{n+1}(W(z'', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s)
\end{aligned}
\tag{2.21}
$$

for all $s \in \mathbb{S}$ whereas the inequality is strict for $s \in A_n(z', z'')$. Furthermore, the strict concavity of u implies that $u(c_\lambda) \geq \lambda u(c') + (1-\lambda) u(c'')$. Integrating both sides of (2.21) and applying Lemma 2.B.1 from Appendix 2.B yields the desired inequality

$$
\begin{aligned}
& U_n(c_\lambda, z_\lambda; s_1^n) \\
&= u(c_\lambda) + \beta \int_{\mathbb{S}} V_{n+1}(W(z_\lambda, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \\
&> \lambda \Big(u(c') + \beta \int_{\mathbb{S}} V_{n+1}(W(z', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \Big) \\
&\quad + (1-\lambda) \Big(u(c'') + \beta \int_{\mathbb{S}} V_{n+1}(W(z'', s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds) \Big) \\
&= \lambda U_n(c', z'; s_1^n) + (1-\lambda) U_n(c'', z''; s_1^n).
\end{aligned}
$$

This result permits us to define the solution to the maximization problem (2.20) as a pair of functions $(c_n^\star, z_n^\star) : [-\hat{E}_n, \infty[\times \mathbb{S}^n \longrightarrow \mathbb{C} \times \mathbb{Z}$,

$$
(c_n^\star, z_n^\star)(w_n, s_1^n) := \arg\max_{c,z} \Big\{ U_n(c, z; s_1^n) \Big| (c, z) \in \mathbb{B}_n(w_n, p_n) \Big\}. \tag{2.22}
$$

Step 3. We claim that the mappings $(c_n^\star, z_n^\star)(\cdot)$ and the function $V_n(\cdot)$ are both continuous. To see this, define the budget set for alternative (w_n, p_n) as a correspondence $\mathbb{B}_n : [-\hat{E}_n, \infty[\times \mathbb{R}_{++}^M \rightrightarrows \mathbb{C} \times \mathbb{Z}$, which is non-empty-, compact- and convex-valued on its domain of definition. Furthermore, we show in Lemma 2.B.4 in Appendix 2.B that $\mathbb{B}_n$ is continuous. This together with the continuity of the function U_n implies the continuity of the solution functions $(c_n^\star, z_n^\star)(\cdot)$ by virtue of the Theorem of the Maximum (cf. [59, p. 57, 62]). Substituting the solution (2.22) into (2.20) yields the value function defined for all $w_n \geq -\hat{E}_n$ and $s_1^n \in \mathbb{S}^n$ as

$$
V_n(w_n, s_1^n) = U_n \left(c_n^\star(w_n, s_1^n), z_n^\star(w_n, s_1^n); s_1^n \right), \tag{2.23}
$$

the continuity of which follows immediately from the continuity of the functions $U_n(\cdot)$ and $(c_n^\star, z_n^\star)(\cdot)$.

Step 4. We show that the map $V_n(\cdot, s_1^n)$ is strictly concave. Let w', w'' and $\lambda \in]0,1[$ be arbitrary such that $w' \geq -\hat{E}_n$, $w'' \geq -\hat{E}_n$ and $w' \neq w''$. Set $w_\lambda := \lambda w' + (1-\lambda) w''$. Let $(c'^\star, z'^\star) := (c^\star, z^\star)(w', s_1^n) \in \mathbb{B}_n(w', p_n)$

and $(c''^{\star}, z''^{\star}) := (c^{\star}, z^{\star})(w'', s_1^n) \in \mathbb{B}_n(w'', p_n)$ be the optimal solutions pertaining to w' and w'' and let $(c_{\lambda}^{\star}, z_{\lambda}^{\star}) := \lambda(c'^{\star}, z'^{\star}) + (1-\lambda)(c''^{\star}, z''^{\star})$ be their convex combination. Note that $(c_{\lambda}^{\star}, z_{\lambda}^{\star}) \in \mathbb{B}_n(w_{\lambda}, p_n)$ but possibly $(c_{\lambda}^{\star}, z_{\lambda}^{\star}) \neq (c^{\star}, z^{\star})(w_{\lambda}, s_1^n)$, i.e., $(c_{\lambda}^{\star}, z_{\lambda}^{\star})$ does not have to be the optimal solution at w_{λ}. Also note from the budget set that $w' \neq w''$ implies that $(c'^{\star}, z'^{\star}) \neq (c''^{\star}, z''^{\star})$. This together with (2.23) and the strict concavity of the function $U_n(\cdot, s_1^n)$ therefore implies the desired inequality

$$\begin{aligned} V_n(w_{\lambda}, s_1^n) &\geq U_n\left(c_{\lambda}^{\star}, z_{\lambda}^{\star}; s_1^n\right). \\ &= U_n\left(\lambda(c'^{\star}, z'^{\star}) + (1-\lambda)(c''^{\star}, z''^{\star}); s_1^n\right) \\ &> \lambda U_n\left(c'^{\star}, z'^{\star}; s_1^n\right) + (1-\lambda) U_n\left(c''^{\star}, z''^{\star}; s_1^n\right) \\ &= \lambda V_n(w', s_1^n) + (1-\lambda) V_n(w'', s_1^n). \end{aligned}$$

Step 5. We are left to show that $V_n(\cdot, s_1^n)$ is strictly increasing. To this end, let $w' > w'' \geq -\hat{E}_n$ be arbitrary. Let $(c'^{\star}, z'^{\star})$ and $(c''^{\star}, z''^{\star})$ be defined as in the previous step and set $\delta := w' - w'' > 0$ and $(c_{\delta}^{\star}, z_{\delta}^{\star}) := (c''^{\star} + \delta, z''^{\star})$. Note that $(c_{\delta}^{\star}, z_{\delta}^{\star}) \in \mathbb{B}_n(w', p_n)$ but possibly $(c'^{\star}, z'^{\star}) \neq (c_{\delta}^{\star}, z_{\delta}^{\star})$. Hence, exploiting the strict monotonicity of u:

$$\begin{aligned} V_n(w', s_1^n) &\geq U_n\left(c_{\delta}^{\star}, z_{\delta}^{\star}; s_1^n\right) = U_n\left(c''^{\star} + \delta, z''^{\star}; s_1^n\right) \\ &> U_n\left(c''^{\star}, z''^{\star}; s_1^n\right) = V_n(w'', s_1^n). \end{aligned}$$

Consider now the second case where u is not necessarily bounded but ν satisfies Assumption 2.2.3 (ii). The previous induction proof may then be repeated under the induction hypothesis that $V_{n+1}(\cdot)$ is well-defined and continuous and $V_{n+1}(\cdot, s_1^{n+1}) : [-\hat{E}_{n+1}, \infty[\longrightarrow \mathbb{R}$ is strictly increasing and strictly concave for each $s_1^{n+1} \in \mathbb{S}^{n+1}$. From Lemma 2.B.3 it follows that for each $s_1^n \in \mathbb{S}^n$ the measure $Q_{n+1}(s_1^n, \cdot)$ is supported on a subset of the compact set $\bar{\mathbb{S}}_{n+1}$. From Lemma 2.B.2 and the Feller-property of Q_{n+1} it then follows that for each $(c, z) \in \mathbb{B}_n(w_n, p_n)$ and $s_1^n \in \mathbb{S}^n$ the function

$$U_n(c, z; s_1^n) := u(c) + \beta \int_{\bar{\mathbb{S}}_{n+1}} V_{n+1}\left(W(z, s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s\right) Q_{n+1}(s_1^n, ds) \tag{2.24}$$

is well-defined (i.e., $U_n(c, z; s_1^n) < \infty$) and continuous. Repeating the steps 1–5 of the previous argument then shows the claim. ∎

Utilizing the value function V_1 obtained in the final recursion step, consider the following one-stage decision problem defined for all $(R,p) \gg 0$ and $w > -\hat{e}_0/R$:

$$\max_{(c,z)\in\mathbb{B}_0(R,p,w)} \left\{ u(c) + \beta \int_{\mathbb{S}} V_1\big(W(z,s,\hat{e}_1,R),s\big)\nu_1(ds) \right\} \tag{2.25}$$

where the budget set at time $t=0$ is defined as

$$\mathbb{B}_0(R,p,w) := \left\{ (c,y,x) \in \mathbb{C}\times\mathbb{Y}\times\mathbb{X} \,\middle|\, c+y+x^\top p = w, y \geq -\hat{e}_0/R \right\}. \tag{2.26}$$

The following theorem shows that the solution to (2.25) is unique and can be used together with the functions defined by (2.22) to construct an optimal strategy.

Theorem 2.1. *Let the hypotheses of Proposition 2.3.1 be satisfied and let the functions* $(c_n^\star, z_n^\star)$*,* $n = 1,\dots,j$ *be defined as in (2.22) where* $c_j^\star(w_j,s_1^j) := w_j$ *and* $z_j^\star(s_1^j) := 0$ $\forall s_1^j \in \mathbb{S}^j$*. Then for each* $(R,p)\gg 0$ *and* $w > -\hat{e}_0/R$ *the following holds true:*

(i) The problem (2.25) has a unique solution $(c_0^\star, z_0^\star) \in \mathbb{C}\times\mathbb{Z}$*.*

(ii) The strategy $(C^\star, Z^\star)$ *defined by the optimal decision* $(c_0^\star, z_0^\star)$ *from (i) and plans* $(c_n^{\star\prime}, z_n^{\star\prime}) : \mathbb{S}^n \longrightarrow \mathbb{C}\times\mathbb{Z}$*,* $n=1,\dots,j$ *defined recursively as*[6]

$$\begin{aligned} c_n^{\star\prime}(s_1^n) &:= c_n^\star(W(z_{n-1}^\star{}'(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}), s_1^n) \\ z_n^{\star\prime}(s_1^n) &:= z_n^\star(W(z_{n-1}^\star{}'(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}), s_1^n) \end{aligned}$$

is an optimal strategy in the sense of Definition 2.2.3.

(iii) For any strategy $(C,Z) \in \mathscr{B}(R,p,w)$*:*

$$(c_0,z_0) \neq (c_0^\star, z_0^\star) \quad \Rightarrow \quad \mathbb{E}_\nu\left[U_0(C,\cdot)\right] < \mathbb{E}_\nu\left[U_0(C^\star,\cdot)\right].$$

The proof of Theorem 2.1 can be found in Section 2.A.3 in Appendix 2.A of this chapter. It asserts that an optimal strategy exists and can be constructed from the solutions obtained from the recursive definition (2.17). More importantly, however, it ensures that the optimal decision for $t=0$ is uniquely defined by the solution to the one-stage problem (2.25). For alternative prices (R,p) and wealth w determined by (2.4) this optimal decision defines the demand behavior of the consumer in the decision period.

The main result of this section is stated in the following theorem.

[6] By abuse of notation we set $z_{n-1}^\star{}'(s_1^{n-1}) := z_0^\star$ if $n=0$.

Theorem 2.2. *Let the consumer's planning horizon* $j \in \{1, \dots, J\}$ *and expectations* $\hat{e} = (\hat{e}_1, \dots, \hat{e}_j) \in \mathbb{R}^j_+$, $\hat{R} = (\hat{R}_1, \dots, \hat{R}_{j-1}) \in \mathbb{R}^{j-1}_{++}$ *and* $\nu \in \mathrm{Prob}(\mathbb{S}^j)$ *be given and let Assumptions 2.2.1–2.3.1 be satisfied. Then there exists a continuous demand function* $\varphi^{(j)}(\cdot; \nu, \hat{e}, \hat{R})$ *defined for all* $(R,p) \gg 0$ *and* $w \geq -\hat{\epsilon}_0/R$ *as*

$$\varphi^{(j)}(R,p,w;\nu,\hat{e},\hat{R}) = \begin{pmatrix} \varphi^{(j)}_c(R,p,w;\nu,\hat{e},\hat{R}) \\ \varphi^{(j)}_y(R,p,w;\nu,\hat{e},\hat{R}) \\ \varphi^{(j)}_x(R,p,w;\nu,\hat{e},\hat{R}) \end{pmatrix} \tag{2.27}$$

$$:= \operatorname*{arg\,max}_{(c,y,x) \in \mathbb{B}_0(R,p,w)} \left\{ u(c) + \beta \int_{\mathbb{S}} V_1\big(W(y,x,s,\hat{e}_1,R),s\big)\nu_1(ds) \right\}.$$

Proof. Existence of the demand function follows immediately from Theorem 2.1 (i). Continuity can be proved as in step 3 in the proof of Proposition 2.3.1. In this regard, continuity of the budget correspondence $(R,p,w) \longmapsto \mathbb{B}_0(R,p,w)$ with $\mathbb{B}_0(R,p,w)$ being defined as in (2.26) follows by applying Lemma 2.B.4, noting that for each $R > 0$ $\mathbb{B}_0(R,p,w) = \{(c,y,x) \in \mathbb{C} \times \mathbb{Y} \times \mathbb{X} \,|\, c + y + x^\top p = Rw, y \geq -\hat{\epsilon}_0\}$. ■

Given expectations, the function $\varphi^{(j)}(\cdot; \nu, \hat{e}, \hat{R})$ describes the consumer's optimal consumption and investment in bonds and stocks in the decision period $t = 0$ for alternative prices $(R,p) \gg 0$ and wealth $w \geq -\hat{\epsilon}_0/R$ determined by (2.4). At this point, some remarks about the consumer's wealth w and the condition $w \geq -\hat{\epsilon}_0/R$ are in order. The previous derivations treated initial wealth w as a separate parameter. However, by virtue of equation (2.4), whenever $j < J$ the consumer's wealth will generically depend on prices p of the current period. Hence, for a consumer with planning horizon $j < J$ one could alternatively define his demand as a function $\tilde{\varphi}^{(j)}(\cdot)$ of current prices $(R,p) \gg 0$ taking as given his current non-capital income $e_0 \geq 0$, the dividend payment $d_0 \in \mathbb{R}^M_+$ and his previous portfolio $z_{-1} = (y_{-1}, x_{-1}) \in \mathbb{Z}$ together with the previous bond return $R_{-1} > 0$. Using (2.4), (2.15) and (2.27) yields

$$\begin{aligned} &\tilde{\varphi}^{(j)}(R,p;d_0,e_0,z_{-1},R_{-1},\nu,\hat{e},\hat{R}) \\ &:= \varphi^{(j)}(R,p,W(z_{-1},p,d_0,e_0,R_{-1});\nu,\hat{e},\hat{R}). \end{aligned}$$

In the sequel, however, the form (2.27) will be more convenient.

Another issue concerns the condition $w \geq -\hat{\epsilon}_0/R$ which was required to obtain a non-empty strategy set $\mathscr{B}(R,p,w)$. From equation (2.4) we see that this constraint is automatically satisfied for any young consumer with wealth $w = e_0$. For any non-young consumer with $j < J$

who owns an initial portfolio $z_{-1} = (y_{-1}, x_{-1}) \in \mathbb{Z}$ from the previous period equation (2.4) implies that the condition $w \geq -\hat{\epsilon}_0/R$ can be written as

$$e_0 + R_{-1} y_{-1} + (p + d_0)^\top x_{-1} \geq -\hat{\epsilon}_0/R. \tag{2.28}$$

Since by (2.6) $\hat{\epsilon}_0 \geq 0$, it is clear that (2.28) can only be violated if $y_{-1} < 0$, i.e., if the consumer has taken credit by selling bonds in the previous period. Observe that a sufficient condition for (2.28) to hold is that $y_{-1} \geq -(e_0/R_{-1} + \hat{\epsilon}_0/(R\,R_{-1}))$. However, Lemma 2.2.1 has shown that there exist lower bounds on the consumers credit taking behavior which are determined by the expectations for his discounted future non-capital income stream. Given these subjective expectations the consumer chooses a strategy which ensures his solvency at any point in time for any possible realization of future prices and dividends. Hence, we see that if during the previous $t = -1$ the consumer has correctly anticipated his non-capital income e_0 and the bond return $R > 0$ at time $t = 0$ (and continues to hold the same expectations for his discounted future non-capital income stream), the inequality $y_{-1} \geq -(e_0/R_{-1} + \hat{\epsilon}_0/(R\,R_{-1}))$ is automatically satisfied as a consequence of the consumer's credit taking behavior during the previous period. From a sequential point of view, it is therefore clear that as long as consumers' predictions for future non-capital income and future bond returns are correct or at least sufficiently precise, bankruptcy is excluded by the behavior of consumers themselves. Clearly, if expectations fail to be correct and actual realizations deviate too much from their predicted values, a potential problem of bankruptcy comes into play (which may still be avoided if dividend payments and/or asset prices are sufficiently large). In the sequel we will set aside this issue by restricting attention to those cases where consumers remain solvent at each point in time.

Returning to the general time structure of the model as introduced in Section 2.1, assume that in each period t each consumer solves a decision problem of the form (2.13). The expectations held by a consumer belonging to generation $j \in \{1, \ldots, J\}$ at time t will be denoted as $\hat{e}_t^{(j)} := (e_{t,t+n}^{(j)})_{n=1}^{j} \in \mathbb{R}_+^j$, $\hat{R}_t^{(j)} := (\hat{R}_{t,t+n})_{n=1}^{j-1} \in \mathbb{R}_{++}^{j-1}$ and $\nu_t^{(j)} \in \text{Prob}(\mathbb{S}^{\mathrm{j}})$. Here $e_{t,t+n}^{(j)} \geq 0$ and $\hat{R}_{t,t+n} > 0$ denote the consumer's expectations held at time t for his non-capital income and the bond return at time $t+n$. Observe that the latter forecasts are homogeneous among consumers such that the lists $\hat{R}_t^{(j)}$ differ only in their length due to the different remaining lifetimes. Also note that $R_t^{(1)}$ is empty.

Each consumer's preferences are assumed to be of the form stated in Assumption 2.2.2 for some common utility function $u : \mathbb{C} \longrightarrow \mathbb{R}$ and discount factor $\beta \in]0,1[$. For all t and j the measure $\nu_t^{(j)}$ is assumed to satisfy the conditions listed in Assumptions 2.2.1 and 2.3.1. In addition, preferences and expectations are assumed to satisfy Assumption 2.2.3. It then follows from Theorem 2.2 that for each $j = 1, \ldots, J$ each consumer's demand behavior can be described by some continuous demand function $(R, p, w) \longmapsto \varphi^{(j)}(R, p, w; \nu_t^{(j)}, \hat{e}_t^{(j)}, \hat{R}_t^{(j)})$ with wealth w being determined by (2.4). Since the members of the old generation $j = 0$ do not invest and consume their terminal wealth defined by (2.4) the demand function of consumers in generation $j = 0$ are defined as

$$\varphi^{(0)}(R,p,w) = \begin{pmatrix} \varphi_c^{(0)}(R,p,w) \\ \varphi_y^{(0)}(R,p,w) \\ \varphi_x^{(0)}(R,p,w) \end{pmatrix} := \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix}. \tag{2.29}$$

2.4 Decision Problem of Firms

Next consider the production and investment behavior of firms in the economy.[7] Recall that there are $M \geq 1$ firms, indexed $m = 1, \ldots, M$. Assume that each firm produces a homogeneous consumption good using labor and capital as input factors. In addition, the production process of each firm is subjected to random shocks.

More specifically, consider a typical firm m at time t. Let $K_t^{(m)} > 0$ denote the firm's capital stock and $L_t^{(m)} \geq 0$ labor input employed at time t. The production shock $\eta_t^{(m)}$ is modeled as a random variable taking values in the interval $[0, \eta_{max}^{(m)}]$. Based on these quantities the production output produced at time t is determined by the production function $F^{(m)} : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, \eta_{max}^{(m)}] \longrightarrow \mathbb{R}_+$

$$F^{(m)}(L_t^{(m)}, K_t^{(m)}, \eta_t^{(m)}) = f^{(m)}(L_t^{(m)}, K_t^{(m)}) + \eta_t^{(m)}, \quad m = 1, \ldots, M. \tag{2.30}$$

The particular functional form of the production process (2.30) with additive noise is assumed mainly for technical reasons. While this structure will be convenient for the subsequent derivations, it should be possible to relax this specification and to allow for more general functional forms $F^{(m)}$ in future extensions of the model.

The following assumption specifies additional properties of each firm's production technology as well as the perceived stochastic nature of the noise process.

[7] The following section draws on ideas from [2], [10] and [51].

Assumption 2.4.1 *For each firm* $m = 1, \dots, M$ *the following hypotheses are satisfied:*

(i) The map $(L, K) \longmapsto f^{(m)}(L, K)$ *in (2.30) is* C^2*, linear homogeneous, strictly increasing, and strictly quasi-concave. The derivatives satisfy* $\partial_L f^{(m)}(\cdot) := \frac{\partial f^{(m)}}{\partial L}(\cdot) > 0$*,* $\partial_K f^{(m)}(\cdot) := \frac{\partial f^{(m)}}{\partial K}(\cdot) > 0$*,* $\partial_{LL} f^{(m)}(\cdot) := \frac{\partial^2 f^{(m)}}{(\partial L)^2}(\cdot) < 0$ *and* $\partial_{KK} f^{(m)}(\cdot) := \frac{\partial^2 f^{(m)}}{(\partial K)^2}(\cdot) < 0$.

(ii) Each firm treats the noise terms in its production technology (2.30) as an i.i.d. process $\{\eta_t^{(m)}\}_t$ *of random variables taking values in some compact interval* $[0, \eta_{max}^{(m)}]$*. The expected value* $\bar{\eta}^{(m)} := \mathbb{E}\big[\eta_t^{(m)}\big]$ *is known to firm* m.

As before, denote by $\omega_t > 0$ the real wage per unit of labor at time t. A typical firm m in period t takes its current capital stock $K_t^{(m)} > 0$ as given and decides about labor input $L_t^{(m)} \geq 0$ and investment $I_t^{(m)} \geq 0$, the latter being measured in units of capital. Assuming that capital depreciates at a constant rate $\delta \in]0, 1[$ which is common to all firms, any investment decision $I_t^{(m)}$ made at time t determines the firm's capital stock in the following period according to

$$K_{t+1}^{(m)} = I_t^{(m)} + (1 - \delta) K_t^{(m)}, \quad m = 1, \dots, M. \tag{2.31}$$

In order to extend its capital stock each firm can transform consumption goods into capital goods. As in [2], suppose that given the current capital stock $K_t^{(m)} > 0$, the amount of consumption goods firm m needs to produce $I_t^{(m)} \geq 0$ units of new capital is determined by the adjustment cost function $G^{(m)} : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_+$,

$$G^{(m)}(I, K) := K\, g^{(m)}(I/K), \quad m = 1, \dots, M. \tag{2.32}$$

The function $G^{(m)}$ may therefore be viewed as an input requirement function for an underlying capital adjustment technology. Its properties are mainly determined by the function $g^{(m)}$ which depends on the investment ratio $i := I/K$. The properties of this function are stated in the following assumption.[8]

Assumption 2.4.2 *For each firm* $m = 1, \dots, M$ *the adjustment cost function* $g^{(m)} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is strictly increasing and satisfies* $g^m(0) = 0$*. Moreover,* $g^{(m)}$ *is* C^2 *and strictly convex on* $\mathbb{R}_{++}$ *and the first derivative satisfies* $\lim_{i \to \infty} g^{(m)\prime}(i) = \infty$.

[8] To alleviate the terminology, the function $g^{(m)}$ will also be referred to as the firm's adjustment cost function. Likewise we shall call $f^{(m)}$ the firm's production function. No confusion shall arise.

The property of strict convexity excludes the traditional one-to-one relation between consumption and capital goods in which the identity $g^{(m)}(i) = i$ holds. Also observe that the function $g^{(m)}$ may have a discontinuity at zero. Economically, this may be justified by some fixed costs to investment which have to be incurred whenever the investment is positive. A more detailed account on this and the theory of adjustment costs in general may be found in [54].

Assume that the firm's investment decision $I_t^{(m)}$ at time t is exclusively financed by issuing bonds $B_t^{(m)} \geq 0$ inducing the obligation to repay $R_t B_t^{(m)}$ units of output/consumption good at time $t+1$. Recalling that the bond price is normalized to unity, one immediately finds that investment decision and bond supply at time t are related as

$$B_t^{(m)} = G^{(m)}(I_t^{(m)}, K_t^{(m)}) = K_t^{(m)}\, g^{(m)}(I_t^{(m)}/K_t^{(m)}), \quad m = 1, \ldots, M. \tag{2.33}$$

After paying for labor and the bond debt incurred in the previous period, each firm distributes its excess output as a dividend payment to its shareholders. Letting as before $\bar{x}^{(m)} > 0$ denote the total number of shares in the market, the dividend payment (per share) of firm m at time t is given by

$$d_t^{(m)} = \frac{f(L_t^{(m)}, K_t^{(m)}) + \eta_t^{(m)} - \omega_t L_t^{(m)} - R_{t-1} B_{t-1}^{(m)}}{\bar{x}^{(m)}}, \quad m = 1, \ldots, M. \tag{2.34}$$

In order to derive the firm's labor demand $L_t^{(m)}$ and investment decision $I_t^{(m)}$ from a suitably defined decision problem consider a typical firm m in an arbitrary period t. To alleviate the notation we adopt a similar convention as in Section 2.2 setting $t = 0$ and suppressing the index m and the time index referring to the decision period. Let the firm's current capital stock $K_0 > 0$ and the bond debt $R_{-1}B_{-1} \geq 0$ resulting from the previous investment decision be given. The current real wage and the current bond return enter the decision problem as parameters $\omega > 0$ and $R > 0$, respectively. Furthermore, it is assumed that the decision problem is solved after the current shock η_0 has been observed.

Given these quantities, assume that the firm seeks to act in favor of its shareholders by maximizing expected dividend payments. For simplicity, suppose that the firm has a one-period planning horizon such that only today's and tomorrow's dividend payment are considered in the decision problem (a related setting has been adopted in [51]). Due to the assumption that investment is exclusively financed by issuing bonds the labor demand and investment decision can be determined separately. Suppose first that the firm chooses labor input at time $t = 0$ to maximize the current dividend payment. Using (2.34) this implies that current labor demand is a solution to the optimization problem

$$\max_{L \geq 0} \left\{ \frac{f(L, K_0) + \eta_0 - \omega L - R_{-1} B_{-1}}{\bar{x}} \right\}. \tag{2.35}$$

In particular, labor demand is independent of the bond return R. This property will become crucial for the derivation of equilibrium prices in Section 2.5. Also note that neither the additive terms η_0 and $R_{-1}B_{-1}$ nor the scaling factor $\bar{x}^{-1}$ affect the solution to (2.35).

As a second step, suppose that the firm chooses its investment at time $t = 0$ to maximize the expected dividend payment of the following period. For this purpose, the firm holds expectations $\hat{\omega}_1 > 0$ for the real wage prevailing in the following period $t = 1$. The decision involves an investment decision I made at time $t = 0$ and planned labor demand L_1 for the following period. These are chosen subject to the constraint that next period's dividend be non-negative for any possible realization of the production shock. Using equations (2.31)–(2.34) the corresponding maximization problem reads

$$\max_{(I, L_1) \geq 0} \left\{ \begin{array}{l} \dfrac{f(L_1, I + (1-\delta)K_0) + \bar{\eta} - \hat{\omega}_1 L_1 - R K_0 g(I/K_0)}{\bar{x}} \\ \text{s.t.} \qquad f(L_1, I + (1-\delta)K_0) - \hat{\omega}_1 L_1 \geq R K_0 g(I/K_0) \end{array} \right\}. \tag{2.36}$$

Similar to (2.35) neither the additive term $\bar{\eta}$ nor the scaling factor $\bar{x}^{-1}$ affect the solution to (2.36).

Before we state conditions under which solutions to (2.35) and (2.36) exist, some technical preparations are required. First note that f being linear homogeneous implies that the map $(L, K) \longmapsto \partial_L f(L, K) = \partial_L f(L/K, 1)$ is homogeneous of degree zero (cf. [52, p. 928, Theorem M.B.1]). This together with our assumptions made on f ensure that the limits $\underline{\omega} := \lim_{L \to \infty} \partial_L f(L, K)$ and $\overline{\omega} := \lim_{L \to 0} \partial_L f(L, K)$ do not depend on K and satisfy $0 \leq \underline{\omega} < \overline{\omega} \leq \infty$. It follows that the map $\partial_L f(\cdot, 1) : \mathbb{R}_{++} \longrightarrow]\underline{\omega}, \overline{\omega}[$ is bijective and strictly monotonically decreasing with an inverse denoted by $l :]\underline{\omega}, \overline{\omega}[\longrightarrow \mathbb{R}_{++}$ which is also strictly decreasing. Let $g_0 := \lim_{i \searrow 0} g(i) \geq 0$ and $g_0' := \lim_{i \searrow 0} g'(i) \geq 0$. Note that $g_0 = 0$ if and only if the function g is continuous on $\mathbb{R}_+$. The properties of the function g imply that the map $g' : \mathbb{R}_{++} \longrightarrow]g_0', \infty[$ is bijective and therefore has an inverse $(g')^{-1} :]g_0', \infty[\longrightarrow \mathbb{R}_{++}$. Moreover, both maps are strictly monotonically increasing. Finally, denote by $E_g(i) := \frac{g'(i)\, i}{g(i)}$ the elasticity of the cost function g defined for all $i > 0$. Equipped with these prerequisites the following theorem establishes the existence and the functional form of a solution to the firm's decision problem.

Proposition 2.4.1 *Let Assumptions 2.4.1 and 2.4.2 be satisfied and let* $R_{-1}B_{-1} \geq 0$, $K > 0$, *and* $\eta_0 \in [0, \eta_{max}]$ *be given. Assume that the values* $\omega > 0$, $\hat{\omega}_1 > 0$ *and* $R > 0$ *satisfy the following two hypotheses:*

(a) Both ω *and* $\hat{\omega}_1$ *lie within the interval* $]\underline{\omega}, \overline{\omega}[$.

(b) If $g_0' > 0$, *the pair* $(R, \hat{\omega}_1)$ *satisfies* $R < \dfrac{\partial_K f(l(\hat{\omega}_1), 1)}{g_0'}$.

Then the solutions to the firm's optimization problems are as follows:

(i) Current labor demand derived from (2.35) is determined as

$$L^\star = l(\omega) K_0 \tag{2.37}$$

where the function $l(\cdot)$ *is defined as above. Moreover, the current dividend payment is non-negative if and only if*

$$\partial_K f(l(\omega), 1)\, K_0 + \eta_0 \geq R_{-1}B_{-1}. \tag{2.38}$$

(ii) The optimal investment decision and planned labor demand derived from (2.35) are given by

$$\begin{aligned} I^\star &= \begin{cases} i(R; \hat{\omega}_1) K_0, & \text{if } E_g(i(R; \hat{\omega}_1)) > 1 \\ 0 & \text{otherwise} \end{cases} \\ L_1^\star &= l(\hat{\omega}_1)(I^\star + (1 - \delta) K_0) \end{aligned} \tag{2.39}$$

with the function $l(\cdot)$ *being defined as above and*

$$i(R; \hat{\omega}_1) := (g')^{-1}\big(R^{-1} \partial_K f(l(\hat{\omega}_1), 1)\big).$$

Moreover, the solution is unique if and only if $E_g(i(R; \hat{\omega}_1)) \neq 1$.

Proof. We prove both claims (i) and (ii) separately.
(i). Consider first the solution to (2.35). Exploiting (a) it follows from the properties of the function f that any solution has to satisfy the (necessary and sufficient) first order condition

$$\partial_L f(L^\star, K_0) - \omega \overset{!}{=} 0 \tag{2.40}$$

which is equivalent to (2.37). An application of Euler's formula (cf. [52, p. 928, Theorem M.B.2]) gives

$$f(L^\star, K_0) + \eta_0 - \omega L^\star - R_{-1}B_{-1} = \partial_K f(l(\omega), 1)\, K_0 + \eta_0 - R_{-1}B_{-1}$$

proving the second statement in (i).
(ii). To alleviate the notation define the map $H : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$, $H(L_1, I) := f(L_1, I + (1-\delta)K_0) - \hat{\omega}_1 L_1 - RK_0 g(I/K_0)$. Omitting the constant $\bar{\eta}$ and the scaling factor $\bar{x}^{-1}$ the solution to (2.36) may be obtained from the following equivalent problem

$$\max_{L_1 \geq 0, I \geq 0} \Big\{ H(L_1, I) \quad \text{s.t.} \quad H(L_1, I) \geq 0 \Big\}. \tag{2.41}$$

Noting that $H(0,0) = f(0, (1-\delta)K_0) \geq 0$ the constraint $H(L_1, I) \geq 0$ in (2.41) can be dispensed with since it will automatically be satisfied by any solution $(I^\star, L_1^\star)$. Since H need not be continuous we partition its domain $\mathbb{R}^2_+$ into the disjoint subsets $\Lambda_+ := \{(L, I) \in \mathbb{R}^2_+ | I > 0\}$ and $\Lambda_0 := \{(L, I) \in \mathbb{R}^2_+ | I = 0\}$ and consider the maximum of H on each of the two sets separately. To enhance readability the remainder of the proof is organized in three steps.
Step 1. We show that H has a unique global maximum on Λ_+ where the maximizer is denoted $(L_+^\star, I_+^\star)$. To this end, note by Assumptions 2.4.1 (i) and 2.4.2 that H is C^2 and strictly quasi-concave on the convex set Λ_+. The claim will follow if we show the existence of a pair $(I_+^\star, L_+^\star) \gg 0$ which solves the necessary and sufficient first order conditions

$$\begin{aligned} \partial_{L_1} H(L_+^\star, I_+^\star) &= \partial_L f(L_+^\star, I_+^\star + (1-\delta)K_0) - \hat{\omega}_1 &&\overset{!}{=} 0 \\ \partial_I H(L_+^\star, I_+^\star) &= \partial_K f(L_+^\star, I_+^\star + (1-\delta)K_0) - Rg'(I_+^\star/K_0) &&\overset{!}{=} 0. \end{aligned} \tag{2.42}$$

Exploiting (i), (iii) and Assumption 2.4.2 together with Euler's formula implying that $f(l(\hat{\omega}_1), 1) - \hat{\omega}_1 l(\hat{\omega}_1) = \partial_K f(l(\hat{\omega}_1), 1)$ the solution to (2.42) can be calculated explicitly as

$$\begin{aligned} L_+^\star &= l(\hat{\omega}_1)(i(R; \hat{\omega}_1) + (1-\delta))K_0 > 0 \\ I_+^\star &= i(R; \hat{\omega}_1)K_0 > 0. \end{aligned} \tag{2.43}$$

Exploiting the linear homogeneity of the function f and Euler's formula the (possibly negative) value of the objective function H at the point $(I_+^\star, L_+^\star) \in \Lambda_+$ computes

$$H(I_+^\star, L_+^\star) = \big(\partial_K f(l(\hat{\omega}_1), 1)\big(i(R; \hat{\omega}_1) + 1 - \delta\big) - R\, g \circ i(R; \hat{\omega}_1)\big) K_0. \tag{2.44}$$

Step 2. We show that H has a unique global maximum on Λ_0 where the maximizer is denoted $(L_0^\star, 0)$. To this end, note that Λ_0 is a convex set and the map $L_1 \longmapsto H(L_1, 0) = f(L_1, (1-\delta)K_0) - \hat{\omega}_1 L_1$ is C^2 and strictly concave (a routine calculation shows that the second derivative satisfies $\partial_{LL} H(L,0) = \partial_{LL} f(L,(1-\delta)\,K_0) < 0)$ by virtue of Assumption 2.4.1. The necessary and sufficient condition for H to have a global maximum at the point $(L_0^\star, 0) \in \Lambda_0$ is thus

$$\partial_{L_1} H(L_0^\star, 0) = \partial_L f\,(L_0^\star, (1-\delta)K_0) - \hat{\omega}_1 \overset{!}{=} 0 \;\Leftrightarrow\; L_0^\star = l(\hat{\omega}_1)(1-\delta)K_0. \tag{2.45}$$

It follows that the point $(L_0^\star, 0) \in \Lambda_0$ is the unique maximizer of H on the set Λ_0. The corresponding maximum value is given by

$$H(L_0^\star, 0) = \partial_K f(l(\hat{\omega}_1), 1)(1-\delta)K_0 > 0. \tag{2.46}$$

Step 3. We compare the maximum values (2.44) and (2.46) derived in the previous steps. To this end, note from (2.42) and (2.43) that $\partial_K f(l(\hat{\omega}_1), 1) = R\, g' \circ i(R;\hat{\omega}_1)$. Using this one obtains

$$H(L_+^\star, I_+^\star) \gtreqqless H(L_0^\star, 0) \Leftrightarrow E_g(i(R;\hat{\omega}_1)) = \frac{g'(i(R;\hat{\omega}_1))i(R;\hat{\omega}_1)}{g(i(R;\hat{\omega}_1))} \gtreqqless 1. \tag{2.47}$$

Utilizing equations (2.43) and (2.45) a global maximizer $(L_1^\star, I^\star)$ of H on $\mathbb{R}_+^2$ is obtained by setting $(L_1^\star, I^\star) := (L_+^\star, I_+^\star)$ if $E_g(i(R;\hat{\omega}_1)) > 1$ and $(L_1^\star, I^\star) := (L_0^\star, 0)$ otherwise. Note that since $H(L_0^\star, 0) > 0$ the maximum value $H(L_1^\star, I^\star)$ will always be strictly positive. The solution $(L_1^\star, I^\star)$ satisfies the claim (ii) stated in the proposition. The uniqueness assertion follows immediately from (2.47). ∎

The requirements imposed by Proposition 2.4.1 appear not to be too strong. Condition (a) is automatically satisfied if f satisfies the Inada type conditions $\lim_{L \to 0} \partial_L f(L,K) = \infty$ and $\lim_{L \to \infty} \partial_L f(L,K) = 0$, which, for example, hold with a Cobb-Douglas production technology. Likewise, the second condition (b) can be dispensed with if one assumes $g_0' = 0$, which is often done in the literature. Here we relax this assumption by supposing instead that the value g_0' is sufficiently small such that the upper bound $\partial_K f(l(\hat{\omega}_1), 1) g_0'^{-1}$ is sufficiently large. Also note that if $g_0 = 0$ and the function g is strictly convex on $\mathbb{R}_+$, the elasticity condition $E_g(i(R,\hat{\omega}_1)) > 1$ ensuring an interior investment solution is automatically satisfied.[9] The condition (2.38) in (i) of Proposition 2.4.1 ensures that the current dividend payment is non-negative. If this condition were violated, the firm would be unable to repay its previous bond debt $R_{-1}B_{-1}$ from the excess production output in $t = 0$. In

[9] To see this, note that by strict convexity of g one has $g(i_0) > g(i) + g'(i)(i_0 - i)$ for any $i, i_0 \in \mathbb{R}_+ \backslash \{i\}$, $i \neq i_0$. Setting $i_0 = 0$ shows that $E_g(i) > 1$ for all $i > 0$.

this regard, recall that the decision behavior derived from (2.36) implies that in each period the firm anticipates next period's real wage and chooses its current investment such that next period's dividend payment is non-negative for any possible realization of the production shock. Hence, it is clear that (2.38) will be satisfied if during period $t = -1$ the firm has chosen its investment (and thus the payment obligation $R_{-1}B_{-1}$) based on a sufficiently precise forecast for the real wage at time $t = 0$. Alternatively, if the firm has severely underestimated this variable, the production shock η_0 could still be sufficiently large to compensate for the erroneous forecast. From a sequential perspective, it is therefore clear that condition (2.38) will be satisfied in each period as long as the firm's wage expectations are correct or at least sufficiently precise. In the sequel we will assume that this is the case and will not further deal with this issue.

The results from Proposition 2.4.1 may now be used to characterize the demand behavior of all firms in each period $t \geq 0$. For this purpose, define from (2.30) for each $m = 1, \ldots, M$ the values

$$\underline{\omega}^{(m)} := \lim_{L \to \infty} \partial_L f^{(m)}(L, K) \quad \text{and} \quad \overline{\omega}^{(m)} := \lim_{L \to 0} \partial_L f^{(m)}(L, K) \tag{2.48}$$

noting that neither value depends on $K > 0$. Furthermore, define from equations (2.30) and (2.32) for each $\omega, \hat{\omega}_1 \in]\underline{\omega}^{(m)}, \overline{\omega}^{(m)}[$ and for each[10] $R \in \big]0, (g_0^{(m)\prime})^{-1} \partial_K f^{(m)}(l^{(m)}(\hat{\omega}_1), 1)\big[$ the mappings

$$\begin{aligned} l^{(m)}(\omega) \quad &:= (\partial_L f^{(m)})^{-1}(\omega, 1) \\ i^{(m)}(R; \hat{\omega}_1) &:= \left(g^{(m)\prime}\right)^{-1} \left(R^{-1} \partial_K f^{(m)}(l^{(m)}(\hat{\omega}_1), 1)\right). \end{aligned} \tag{2.49}$$

Finally, let $\hat{\omega}_{t,t+1} > 0$ denote the point forecast commonly made by firms at time t for the real wage of the following period. The main result of this section is now stated in the following theorem.

Theorem 2.3. *For each firm $m = 1, \ldots, M$ let Assumptions 2.4.1 and 2.4.2 be satisfied. Define the values $0 \leq \underline{\omega}^{(m)} < \overline{\omega}^{(m)} \leq \infty$ as in (2.48) and the functions $l^{(m)}(\cdot)$ and $i^{(m)}(\cdot)$ as in (2.49). Then given the values $R_{t-1}B_{t-1}^{(m)} \geq 0$, $\eta_t^{(m)} \in [0, \eta_{max}^{(m)}]$ and $K_t^{(m)} > 0$ the firm's demand behavior at time t is as follows:*

(i) Labor demand is determined by the function

$$L^{(m)}(\omega; K_t^{(m)}) = l^{(m)}(\omega) K_t^{(m)}$$

defined for all $\omega \in]\underline{\omega}^{(m)}, \overline{\omega}^{(m)}[$.

[10] If $g_0^{(m)\prime} = 0$ define $(g_0^{(m)\prime})^{-1} \partial_K f^{(m)}(l^{(m)}(\hat{\omega}_1), 1) := \infty$.

(ii) Given expectations $\hat{\omega}_{t,t+1} \in]\underline{\omega}^{(m)}, \overline{\omega}^{(m)}[$ *investment is determined for each* $0 < R < (g_0^{(m)'})^{-1}\partial_K f^{(m)}(l^{(m)}(\hat{\omega}_{t,t+1}), 1)$ *by the function*

$$I^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) := \begin{cases} i^{(m)}(R; \hat{\omega}_{t,t+1})K_t^{(m)} & \text{if } E_{g^{(m)}}(i^{(m)}(R; \hat{\omega}_{t,t+1})) > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The firm's bond supply derived from (ii) is defined by the function

$$B^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) := G^{(m)}\big(I^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}), K_t^{(m)}\big). \tag{2.50}$$

Proof. Given the results stated in Proposition 2.4.1 the proof of (i) and (ii) is straightforward using the definitions in (2.48) and (2.49). The assertion in (iii) follows immediately from (ii) and (2.33). ■

2.5 Temporary Equilibrium

Based on the demand behavior of consumers and firms derived in the previous sections consider now their interactions and the formation of prices on real and financial markets. For the following derivations consider an arbitrary period $t \in \mathbb{N}_0$. Let the previous population vector $N_{t-1} = (N_{t-1}^{(j)})_{j=1}^J$ as well as the capital stocks $K_t = (K_t^{(m)})_{m=1}^M$ of firms resulting from the investment decisions made during the previous period be given at the beginning of period t. Furthermore, let the initial distribution of shares and bonds be defined by the asset allocations $z_{t-1} := (x_{t-1}^{(j)}, y_{t-1}^{(j)})_{j=1}^J$ and $B_{t-1} = (B_{t-1}^{(m)})_{m=1}^M$ determined during the previous period together with the previous bond return $R_{t-1} > 0$. We assume that initially all shares are distributed among (non-young) consumers such that $\sum_{j=1}^J N_{t-1}^{(j)} x_{t-1}^{(j)} = \bar{x}$ and that the previous bond allocation satisfies $\sum_{j=1}^J N_{t-1}^{(j)} y_{t-1}^{(j)} - \sum_{m=1}^M B_{t-1}^{(m)} = 0$. Given these quantities, the following five steps describe the sequential structure of the model in period t.

Step 1. Recalling the overlapping generations structure introduced in Section 2.1, suppose that the number of young consumers born at the beginning of period t is determined from the previous population vector by some map $\mathcal{N} : \mathbb{R}_{++}^{J+1} \longrightarrow \mathbb{R}_{++}$ such that $N_t^{(J)} = \mathcal{N}(N_{t-1})$. In addition, since all generations live identically for $J + 1$ periods one has $N_t^{(j)} = N_{t-1}^{(j+1)}$ for $j = 0, 1, \dots, J-1$. It follows that the population vector $N_t = (N_t^{(j)})_{j=1}^{J}$ at time t is determined from the previous population N_{t-1} according to the population law

$$\begin{cases} N_t^{(j)} = N_{t-1}^{(j+1)}, & j = 0, 1, \dots, J-1 \\ N_t^{(J)} = \mathcal{N}(N_{t-1}). \end{cases} \tag{2.51}$$

Given the amount of labor $\bar{L}^{(j)} > 0$ supplied by each consumer in generation $j \in \{j_L, \dots, J\}$, the population vector N_t determines aggregate labor supply $L_t^S > 0$ at time t according to equation (2.1).

Step 2. Utilizing the labor demand functions of firms as characterized in (i) of Theorem 2.3 and labor supply determined in the previous step, assume that the real wage ω_t at time t is determined such that market clearing on the labor market obtains. This leads to the following definition of a temporary equilibrium on the labor market.

Definition 2.5.1 *Let the firms' capital stocks $K_t = (K_t^{(m)})_{m=1}^{M}$ and aggregate labor supply L_t^S as defined in (2.1) be given. A temporary equilibrium on the labor market is a real wage $\omega_t > 0$ and an allocation $(L_t^{(m)})_{m=1}^{M} \in \mathbb{R}_+^M$ of labor force such that:*

(i) $\sum_{m=1}^{M} L_t^{(m)} = L_t^S$

(ii) For all $m = 1, \dots, M$:

$$L_t^{(m)} = L^{(m)}\left(\omega_t; K_t^{(m)}\right).$$

Combining conditions (i) and (ii) yields the following implicit condition for the equilibrium real wage at time t:

$$\sum_{m=1}^{M} L^{(m)}\left(\omega_t; K_t^{(m)}\right) = L_t^S. \tag{2.52}$$

Setting apart the existence issue and supposing that given K_t and L_t^S equation (2.52) can be inverted with respect to ω_t, the equilibrium real wage at time t is determined by an implicitly defined temporary equilibrium map $\mathcal{W}$ such that

$$\omega_t = \mathcal{W}\left((K_t^{(m)})_{m=1}^{M}, L_t^S\right). \tag{2.53}$$

Let the contribution rate $\tau_t \in [0, 1[$ be determined by the pension system. Then the equilibrium real wage ω_t defined in (2.53) determines the non-capital income distribution $e_t = (e_t^{(j)})_{j=0}^J$ among consumers according to (2.2) and (2.3). Here $e_t^{(j)}$ denotes the non-capital income of each consumer in generation $j \in \{0, 1, \ldots, J\}$. Furthermore, given the production shocks $\eta_t = (\eta_t^{(m)})_{m=1}^M \in \prod_{m=1}^M [0, \eta_{max}^{(m)}]$ determined exogenously the dividend payments $d_t = (d_t^{(m)})_{m=1}^M$ of firms follow from equation (2.34).

Step 3. Based on the equilibrium values determined in the previous steps, each consumer in generation $j \in \{1, \ldots, J\}$ forms her point predictions $\hat{e}_t^{(j)} = (\hat{e}_{t,t+m}^{(j)})_{m=1}^j$, $\hat{R}_t^{(j)} = (\hat{R}_{t,t+m})_{m=1}^{j-1}$ for future non-capital income and bond returns and determines her subjective probability distribution $\nu_t^{(j)} \in Prob(\mathbb{S}^j)$ for future asset prices and dividends. To obtain a compact notation the expectations of all consumers are collected in the vectors $\hat{e}_t := (\hat{e}_t^{(j)})_{j=1}^J$, $\hat{R}_t := (\hat{R}_t^{(j)})_{j=1}^J$ and $\nu_t := (\nu_t^{(j)})_{j=1}^J$. In addition, firms make a homogeneous prediction $\hat{\omega}_{t,t+1}$ about next period's real wage. Using these expectations each consumer determines her demand functions by solving a decision problem as described in Sections 2.2 and 2.3. Furthermore, each firm determines its investment and bond supply function as described in Section 2.4.

Step 4. Given the asset demand functions of consumers from Theorem 2.2 and the bond supply functions of firms from Theorem 2.3 (iii), assume that the bond return R_t and asset prices p_t at time t are determined such that market clearing obtains on all asset markets $m = 0, 1, \ldots, M$. This leads to the notion of a temporary financial equilibrium which is introduced in the following definition.

Definition 2.5.2 *Let the fundamentals (N_t, K_t, e_t, d_t), expectations $(\nu_t, \hat{R}_t, \hat{e}_t, \hat{\omega}_{t,t+1})$, and the previous values (z_{t-1}, R_{t-1}) be given. A temporary financial equilibrium at time t is a pair of prices $(R_t, p_t) \gg 0$ and an allocation $\big((x_t^{(j)}, y_t^{(j)})_{j=1}^J, (B_t^{(m)})_{m=1}^M\big)$ such that:*

(i) $\displaystyle\sum_{j=1}^J N_t^{(j)} x_t^{(j)} - \sum_{j=0}^{J-1} N_t^{(j)} x_{t-1}^{(j+1)} = 0$

(ii) $\displaystyle\sum_{j=1}^J N_t^{(j)} y_t^{(j)} - \sum_{m=1}^M B_t^{(m)} = 0$

(iii) For all $j = 1, \ldots, J$*:*

$$w_t^{(j)} = \begin{cases} e_t^{(J)}, & j = J \\ e_t^{(j)} + R_{t-1} y_{t-1}^{(j+1)} + x_{t-1}^{(j+1)\top}(p_t + d_t), & j = 1, \ldots, J-1. \end{cases}$$
$$x_t^{(j)} = \varphi_x^{(j)}\big(R_t, p_t, w_t^{(j)}; \nu_t^{(j)}, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}\big)$$
$$y_t^{(j)} = \varphi_y^{(j)}\big(R_t, p_t, w_t^{(j)}; \nu_t^{(j)}, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}\big)$$

(iv) For all $m = 1, \ldots, M$*:*

$$B_t^{(m)} = B^{(m)}\big(R_t; \hat{\omega}_{t,t+1}, K_t^{(m)}\big).$$

Combining (i)–(iv) and exploiting that $\sum_{j=0}^{J-1} N_t^{(j)} x_{t-1}^{(j+1)} = \bar{x}$ the market clearing asset price p_t and the equilibrium bond return R_t have to satisfy the implicit conditions

$$\sum_{j=1}^{J} N_t^{(j)} \varphi_x^{(j)}\big(R_t, p_t, w_t^{(j)}; \nu_t^{(j)}, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}\big) = \bar{x} \tag{2.54}$$

$$\sum_{j=1}^{J} N_t^{(j)} \varphi_y^{(j)}\big(R_t, p_t, w_t^{(j)}; \nu_t^{(j)}, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}\big) = \sum_{m=1}^{M} B^{(m)}\big(R_t; \hat{\omega}_{t,t+1}, K_t^{(m)}\big) \tag{2.55}$$

where each $w_t^{(j)}$ is determined by (2.4) and thus may also depend on p_t. Setting apart the existence problem, suppose that there exists a pair of mappings $\mathcal{R}$ and $\mathcal{P}$ which determine the unique solution to (2.54) and (2.55) as

$$R_t = \mathcal{R}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \nu_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \tag{2.56}$$

$$p_t = \mathcal{P}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \nu_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}). \tag{2.57}$$

The existence and form of these mappings will be established in the following chapter for a special parametrization of the model. The equilibrium prices (R_t, p_t) together with conditions (iii) and (iv) from Definition 2.5.2 define the new asset allocation

$$z_t := (y_t^{(j)}, x_t^{(j)})_{j=1}^{J} \quad \text{and} \quad B_t := (B_t^{(m)})_{m=1}^{M}.$$

Step 5. In the final step, consumers realize their consumption decisions and firms use the consumption goods collected from their bond supply to adjust their capital stock for the following period. Given the consumption functions of consumers from Theorem 2.2 and the investment functions of firms from Theorem 2.3 (ii) together with equation (2.31), the final step is described by the equations

$$\begin{aligned} c_t^{(j)} &= \varphi_c^{(j)}\left(R_t, p_t, w_t^{(j)}; \mu_t, \Sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}\right), && j = 0, 1, \ldots, J \\ I_t^{(m)} &= I^{(m)}(R_t; \hat{\omega}_{t,t+1}, K_t^{(m)}), && m = 1, \ldots, M \\ K_{t+1}^{(m)} &= I_t^{(m)} + (1-\delta)K_t^{(m)}, && m = 1, \ldots, M. \end{aligned}$$

From Definition 2.2.2 (i) one observes that the consumption and investment decision of each consumer satisfies the ex-post budget constraint

$$w_t^{(j)} = c_t^{(j)} + y_t^{(j)} + x_t^{(j)\top} p_t, \quad j = 0, 1 \ldots, J \tag{2.58}$$

with wealth $w_t^{(j)}$ being determined by (2.4). Utilizing equation (2.58), in conjunction with (2.2), (2.3) and (2.34) and the market clearing conditions in Definition 2.5.2 we see that aggregate consumption at time t satisfies

$$\begin{aligned} \sum_{j=0}^{J} N_t^{(j)} c_t^{(j)} &= \sum_{j=0}^{J} N_t^{(j)} \left(w_t^{(j)} - y_t^{(j)} - x_t^{(j)\top} p_t\right) \\ &= \sum_{j=0}^{J} N_t^{(j)} e_t^{(j)} + \sum_{j=0}^{J-1} N_t^{(j)} \left(R_{t-1} y_t^{(j+1)} + x_{t-1}^{(j+1)\top}(p_t + d_t)\right) \\ &\quad - \sum_{m=1}^{M} B_t^{(m)} - \bar{x}^\top p_t \\ &= \sum_{m=1}^{M} \left(\omega_t L_t^{(m)} + R_{t-1} B_{t-1}^{(m)}\right) + \bar{x}^\top d_t - \sum_{m=1}^{M} B_t^{(m)} \\ &= \sum_{m=1}^{M} F^{(m)}(L_t^{(m)}, K_t^{(m)}, \eta_t^{(m)}) - \sum_{m=1}^{M} B_t^{(m)}. \end{aligned}$$

Letting $Y_t := \sum_{m=1}^{M} F^{(m)}(L_t^{(m)}, K_t^{(m)}, \eta_t^{(m)})$ denote the aggregate production output at time t, $C_t := \sum_{j=0}^{J} N_t^{(j)} c_t^{(j)}$ aggregate consumption, and $S_t := \sum_{m=1}^{M} B_t^{(m)}$ aggregate savings (understood as production output used for the formation of capital) one obtains the familiar macroeconomic identity

$$Y_t = C_t + S_t. \tag{2.59}$$

In particular, this shows that the model is closed in the sense that the consumption good market also clears. This is of course a consequence of Walras' law.

To complete the description of the model we are left to specify how consumers determine their predictions for future non-capital income and bond returns as well as their subjective probability distribution of future asset prices and dividends. Similarly, one needs to model how firms make their prediction for next period's real wage. We will postpone this task to Section 3.4 in Chapter 3 when a more specific parametrization of the model is available.

2.6 Pension Systems and Demographic Change

We close this chapter by reviewing some basic properties of pension systems and the problem of demographic change within our overlapping generations framework. For the following derivations we identify the pension system at time t with the contribution rate τ_t.[11]

The way it has been modeled the pension system described by equations (2.2) and (2.3) is a pure pay-as-you-go system where all contributions collected at time t are used to finance the payment of current pension incomes. Hence, by definition the budget of the pension system is always balanced. In particular, the system has no own resources to cover any imbalance between revenues and payments. This corresponds to the design of the German as well as many other pension systems in Europe as an unfunded system (see [20] for a survey).

From equation (2.3) it follows that the pension income e_t^R and the contribution rate τ_t at time t are uniquely related as

$$e_t^R = \frac{\omega_t \, L_t^S}{N_t^W} \left[\frac{N_t^R}{N_t^W} \right]^{-1} \tau_t \tag{2.60}$$

where as before $N_t^R := \sum_{j=0}^{j_L - 1} N_t^{(j)}$ denotes the number of pensioners at time t and $N_t^W := \sum_{j=j_L}^{J} N_t^{(j)}$ is the corresponding number of workers. The first term in (2.60) is the average gross labor income at time t. The second term in brackets is the so-called *dependency ratio* which is denoted and defined as

$$\Delta_t := \frac{N_t^R}{N_t^W}. \tag{2.61}$$

---

[11] In our setup contributions are exclusively paid by workers out of their gross labor income. In other models (see, e.g., [18]) as well as in the German pension system half of the contribution is paid by the employee (out of gross wage income) while the other half is paid by the employer (in addition to the gross wage). Since there is a one-to-one correspondence between the two formulations we will stick to our simpler version.

The variable Δ_t captures the ratio between the number of pensioners relative to the number of contributors (workers) and thus plays a pivotal role to describe and assess the problem of demographic change.[12] The following Figure 2.6 portrays the predicted evolution of the dependency ratio for Germany until 2050. The values are taken from [18].

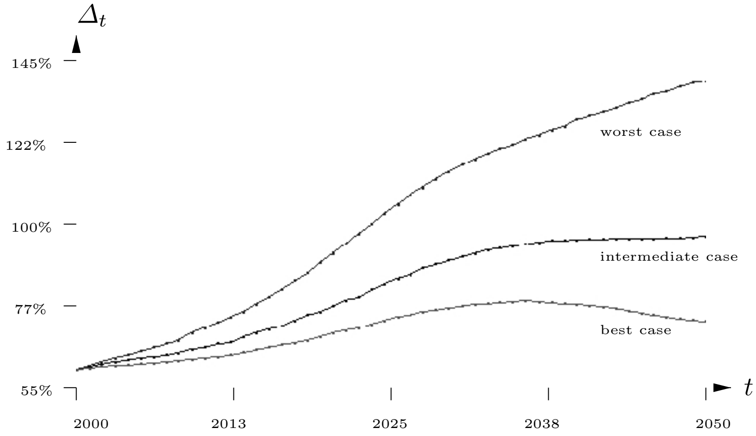


Fig. 2.1. Predicted evolution of the German dependency ratio.

Depending on the assumptions made, e.g., with respect to fertility, mortality, labor force participation and migration, three scenarios emerge which are depicted in the figure. One observes that even in the most optimistic case the dependency ratio will significantly increase from a little more than 60% in 2006 to more than 75% in 2050 while it will more than double in the worst case.

To illuminate the consequences of this development on the pension system, define the so-called *pension-level* (cf. [20], p. 9) as

$$\varsigma_t := \frac{e_t^R}{\frac{\omega_t \cdot L_t^S}{N_t^W}} = \frac{\tau_t}{\Delta_t}. \tag{2.62}$$

The pension level describes current pension payments as a percentage share of current average gross labor income. One observes that given

[12] The authors in [20] distinguish between the *demographic dependency ratio* being defined as the number of retired persons relative to the number of persons in working age and the *economic dependency ratio* being defined as the number of pensioners relative to the number of employed people. Since there is no unemployment in our model, both definitions coincide in the present case.

the demographic change of the population, equation (2.62) defines a *fundamental trade-off* between the contribution rate τ_t and the pension level ς_t. With the predicted increase of the dependency ratio Δ_t, a sufficiently high level of pensions can only be maintained through a very large contribution rate. Conversely, any attempt to keep contributions at a reasonable level will result in a very low pension level. Any pension policy controlling contributions and pension payments of the system is therefore confined to shift the burden of demographic change between contributors and pensioners. In the two extreme cases, only the contribution rate τ_t or the pension level ς_t is adjusted while the respective other variable is kept fixed at a prescribed target value.

As already pointed out in the introduction, one possible way out of this dilemma is to supplement public pension payments by a funded component consisting of private savings for retirement. This increase in private savings may potentially increase the aggregate capital stock of the economy thus offering a possibility to counteract the demographic problem. The details of such a change on consumer behavior and the impact on real and financial variables of the model will be studied in Chapters 4 and 5 of this work.

Summary of Chapter 2

The model developed in this chapter provides a consistent macroeconomic framework featuring an endogenous description of the formation of prices and allocations on real and financial markets. With these structural features the model incorporates the full feedback structure between real and financial sectors and their interactions with the pension system through the demand behavior of consumers and firms. The demand behavior was derived from a suitably defined decision problem providing a sound microeconomic foundation under general assumptions on individual characteristics. The latter cover most cases found in the literature offering a broad range of possible parameterizations of the model. In addition, all results were obtained for arbitrary expectations thereby relaxing the assumption of fully rational expectations predominant in the literature. This type of approach permits to model alternative forms of expectations formation and to study their impact on the model's behavior. In addition, the employed setup allows for arbitrary random shocks to the system as well as for demographic change of the population. For the latter purpose, the employed population model with an arbitrary number of generations opens up the possibility to study a broad range of demographic scenarios.

2.A Mathematical Appendix

2.A.1 Proof of Lemma 2.2.2

It suffices to show that there exists an upper bound $\bar{U}$ such that $\mathbb{E}_\nu[U_0(C,\cdot)] \leq \bar{U}$ for all $(C,Z) \in \mathscr{B}(R,p,w)$. Let $(C,Z) \in \mathscr{B}(R,p,w)$ be arbitrary. It suffices to show that $U_0(C,s_1^j) < \bar{U}$ for all $s_1^j \in \bar{\mathbb{S}}$. Since $\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \ldots, \times \bar{\mathbb{S}}_j$ with each $\bar{\mathbb{S}}_n$ being compact there exist upper and lower bounds $\underline{s}_n \in \mathbb{S}$ and $\bar{s}_n \in \mathbb{S}$ such that each random variable $s_n = (p_n, d_n)$ satisfies $\underline{s}_n \leq s_n \leq \bar{s}_n$ ν–a.s. for each $n = 1, \ldots, j$. Setting e.g. $\underline{s} = (\underline{s}^{(k)})_{k=1}^{2M}$ and $\bar{s} = (\bar{s}^{(k)})_{k=1}^{2M}$ where $\underline{s}^k := \min\{\underline{s}_n^k | n = 1, \ldots, j\}$ and $\bar{s}^{(k)} := \max\{\bar{s}_n^{(k)} | n = 1, \ldots, j\}$ these bound can be chosen independently of n. That is, each random variable s_n satisfies $\underline{s} \leq s_n \leq \bar{s}$ ν–a.s., $n = 1, \ldots, j$. Moreover, defining the values $\underline{q} := \underline{p} + \underline{d}$, $\bar{q} := \bar{p} + \bar{d}$ derived from $\underline{s} = (\underline{p}, \underline{d})$ and $\bar{s} = (\bar{p}, \bar{d})$ and letting $q_n := p_n + d_n$ it follows from the definition of $\mathbb{S}$ that $0 \ll \underline{p} \leq p_n \leq \bar{p}$ ν–a.s. and $0 \ll \underline{q} \leq \bar{q}_n \leq \bar{q}$ ν–a.s. for each $n = 1, \ldots, j$.

Define the wealth process $\{W_n(Z, s_1^n)\}_{n=1}^{j}$ associated with strategy $(C,Z) \in \mathscr{B}(R,p,w)$ as in (2.5). By virtue of Definition 2.2.2 and Lemma 2.2.1 we have $c_0 \leq w + \hat{\epsilon}_0/R$ and $c_n(s_1^n) \leq W_n(Z, s_1^n) + \hat{E}_n$ for each $s_1^n \in \mathbb{S}^n$, $n = 1, \ldots, j$. It is therefore sufficient to prove that for each $n = 1, \ldots, j$ there exists $\bar{W}_n \in \mathbb{R}$ such that

$$W_n(Z, s_1^n) \leq \bar{W}_n \quad \nu - a.s. \tag{2.63}$$

We prove (2.63) for $n = 1$ and then apply an induction argument. To this end, note that any investment decision (y_0, x_0) made at stage $n = 0$ satisfies the budget constraint $c_0 + y_0 + x_0^\top p = w$. Utilizing Lemma 2.2.1 one obtains bounds on this investment as $-\hat{\epsilon}_0/R \leq y_0 \leq w =: \bar{y}_0$ and $x_0^{(m)} \leq (w + \hat{\epsilon}_0/R)/p^{(m)} =: \bar{x}_0^{(m)}$ for each $m = 1, \ldots, M$. Setting $\bar{x}_0 := (\bar{x}_0^{(1)}, \ldots, \bar{x}_0^{(M)})^\top$ gives

$$\begin{aligned} W_1(Z, s_1) &= \hat{e}_1 + R\,y_0 + x_0^\top q_1 \\ &\leq \hat{e}_1 + R\,\bar{y}_0 + \bar{x}_0^\top \bar{q}_1 =: \bar{W}_1 \quad \nu - a.s. \end{aligned}$$

By way of induction, suppose that for some $n \in \{1, \ldots, j-1\}$ there exists an upper bound $\bar{W}_n$ such that $W_n(Z, s_1^n) \leq \bar{W}_n$ ν–a.s. Lemma 2.2.1 and the induction hypothesis imply that the investment plans $y_n(\cdot)$ and $x_n(\cdot)$ satisfy $-\hat{E}_n \leq y_n(s_1^n) \leq W_n(Z, s_1^n) \leq \bar{W}_n =: \bar{y}_n$ ν–a.s. and $x_n^{(m)}(s_1^n) \leq (W_n(Z, s_1^n) + \hat{E}_n)/p_n^{(m)} \leq (\bar{W}_n + \hat{E}_n)/\underline{p}^{(m)} =: \bar{x}_n^{(m)}$ ν–a.s. for each $m = 1, \ldots, M$. Setting $\bar{x}_n := (\bar{x}_n^{(1)}, \ldots, \bar{x}_n^{(M)})^\top$ gives

$$\begin{aligned} W_{n+1}(Z, s_1^{n+1}) &= \hat{e}_{n+1} + \hat{R}_n\, y_n(s_1^n) + x_n(s_1^n)^\top q_{n+1} \\ &\le \hat{e}_{n+1} + \hat{R}_n\, \bar{y}_n + \bar{x}_n^\top \bar{q} =: \bar{W}_{n+1} \quad \nu - a.s. \end{aligned}$$

which proves the claim (2.63). Equation (2.63) and Lemma 2.2.1 imply that

$$c_n(s_1^n) \le \bar{c}_n := \bar{W}_n + \hat{E}_n \quad \nu - a.s.$$

for each $n = 1, \ldots, j$ and, exploiting the monotonicity of u

$$U_0(C, s_1^j) = u(c_0) + \sum_{n=1}^{j} \beta^n u(c_n(s_1^n)) \le \bar{U} := \sum_{n=0}^{j} \beta^n u(\bar{c}_n) \quad \nu - a.s.$$

The last equation implies that $\mathbb{E}_\nu\left[U_0(C, \cdot)\right] \le \bar{U}$ for each strategy $(C, Z) \in \mathscr{B}(R, p, w)$ and, therefore, $V_0(R, p, w) \le \bar{U} < \infty$. ■

2.A.2 Proof of Lemma 2.3.1

For each $n = 1, \ldots, j$ let $\pi_n : \mathbb{S}^j \longrightarrow \mathbb{S}$ denote the nth projection mapping defined as $\pi_n(s_1, \ldots, s_n, \ldots, s_j) = s_n$ and by $\pi_{1,n} : \mathbb{S}^j \longrightarrow \mathbb{S}^n$, $\pi_{1,n}(s_1, \ldots, s_n, \ldots, s_j) = (s_1, \ldots, s_n)$ the projection of $\mathbb{S}^j$ onto its first n components. In the sequel we will utilize the following factorization lemma the proof of which can be found in [36, p. 198, Satz 5.3.21] and [4, p. 23, Satz 1.4.3].

Lemma 2.A.1 *Let $(\Omega_1, \mathcal{A}_1)$ be a measurable space and $(\Omega_2, \mathcal{B}(\Omega_2))$ be a Polish space equipped with its Borelian σ-Algebra generated by the open subsets of Ω_2. Then for each probability measure $\nu_{1,2}$ on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{B}(\Omega_2))$ there exists a transition probability $Q_2 : \Omega_1 \times \mathcal{B}(\Omega_2) \longrightarrow [0, 1]$ and a marginal probability $\nu_1 : \mathcal{A}_1 \longrightarrow [0, 1]$ such that one has the factorization*

$$\nu_{1,2}(A) = \int_{\Omega_1} \int_{\Omega_2} 1_A(\omega_1, \omega_2) Q_2(\omega_1, d\omega_2) \nu_1(d\omega_1). \tag{2.64}$$

for each $A \in \mathcal{A}_1 \otimes \mathcal{B}(\Omega_2)$. The measure ν_1 on $(\Omega_1, \mathcal{A}_1)$ is defined by the projection mapping $\pi_1 : \Omega_1 \times \Omega_2 \longrightarrow \Omega_1$, $\pi_1(\omega_1, \omega_2) = \omega_1$ such that $\nu_1 = \pi_1 \nu := \nu \circ \pi_1^{-1}$. The factorization in (2.64) is $\nu_{1,2}$–a.s. unique.

To apply Lemma 2.A.1 we first show that the space $\mathbb{S} = \mathbb{R}^M_{++} \times \mathbb{R}^M_{+}$ is Polish. Following [6, p. 179, Beispiele 1–4], the Euclidean space $\mathbb{R}^M$ is Polish, hence $\mathbb{R}^M_{++}$ and $\mathbb{R}^M_{+}$ being open and closed subspaces of a Polish space are Polish and hence also their product. This shows that the space $\mathbb{S}$ satisfies the requirements of Lemma 2.A.1.

The desired factorization (2.16) of the measure ν in Assumption 2.2.1 is now achieved by a repeated application of Lemma 2.A.1. In a first step, using the notation of Lemma 2.A.1, set $\Omega_1 = \mathbb{S}^{j-1}$, $\Omega_2 = \mathbb{S}$ and $\nu_{1,2} = \nu$ to obtain for each $B \in \mathcal{B}(\mathbb{S}^j)$ the factorization

$$\nu(B) = \int_{\mathbb{S}^{j-1}} \int_{\mathbb{S}} 1_B(s_1^{j-1}, s_j) Q_j(s_1^{j-1}, ds_j) \nu_{j-1}(ds_1^{j-1})$$

with a transition probability $Q_j : \mathbb{S}^{j-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ and a marginal probability $\nu_{j-1} = \pi_{1,j-1}\nu := \nu \circ \pi_{1,j-1}^{-1}$. In a second step, set $\Omega_1 = \mathbb{S}^{j-2}$, $\Omega_2 = \mathbb{S}$ and $\nu_{1,2} = \nu_{j-1}$ to obtain a factorization of ν_{j-1} into transition probability $Q_{j-1} : \mathbb{S}^{j-2} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ and marginal probability $\nu_{j-2} = \pi_{1,j-2}\nu_{j-1} := \nu_{j-1} \circ \pi_{1,j-2}^{-1}$.

Continuing in this fashion one obtains a sequence of transition probabilities Q_n, $n = j, \dots, 2$ and marginal probabilities ν_n, $n = j-1, \dots, 1$. At each stage $n > 1$ the measure ν_n is factorized into transition probability $Q_n : \mathbb{S}^{n-1} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ and marginal probability $\nu_{n-1} = \pi_{1,n-1}\nu_n := \nu_n \circ \pi_{1,n-1}^{-1}$. In the final step one obtains a factorization of the measure ν_2 into transition probability $Q_2 : \mathbb{S} \times \mathcal{B}(\mathbb{S}) \longrightarrow [0,1]$ and marginal probability ν_1 which satisfies $\nu_1 = \pi_1\nu := \nu \circ \pi_1^{-1}$ completing the proof of Lemma 2.3.1. ■

2.A.3 Proof of Theorem 2.1

(i). Utilizing the properties of the value function $V_1(\cdot)$ stated in Proposition 2.3.1 and the non-redundancy of the measure ν_1 stated in Assumption 2.3.1 (ii) the proof is straightforward by following steps 1–3 in the proof of Proposition 2.3.1.

(ii). To alleviate the notation we adopt the conventions that if $n = 0$ we set $c_n(s_1^n) := c_0$, $z_n(s_1^n) := z_0$ and $W(z_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) := w$ for any strategy $(C, Z) \in \mathscr{B}(R, p, w)$ as well as $\hat{R}_n := R$. Also recall from Definition 2.2.2 that the terminal plan of *any* feasible strategy satisfies $c_j(s_1^j) = W(z_{j-1}(s_1^{j-1}), s_j, \hat{e}_j, \hat{R}_{j-1})$ and $z_j(s_1^j) = 0 \ \forall s_1^j \in \mathbb{S}^j$. Given the optimal decision $(c_0^\star, z_0^\star)$ from (i) and the functions $(c_n^\star, z_n^\star)(\cdot)$ defined in Proposition 2.3.1(iii) let the strategy $(C^\star, Z^\star)$ be defined as in (ii) of Theorem 2.1 and set $c_n^\star(w_n, s_1^n) := c_0^\star$ and $z_n^\star(w_n, s_1^n) := z_0^\star$ if $n = 0$. Finally, recall from the definition (2.17) of the value functions V_n, $n = 1, \dots, j-1$, that for each $\hat{w} \geq -\hat{E}_n$ and $s_1^n \in \mathbb{S}^n$

$$V_n(\hat{w}, s_1^n) = u(c_n^\star(\hat{w}, s_1^n)) + \beta \int_{\mathbb{S}} V_{n+1}(W(z_n^\star(\hat{w}, s_1^n), s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s) Q_{n+1}(s_1^n, ds). \quad (2.65)$$

Let $(\hat{C},\hat{Z}) = (\hat{c}_0, \hat{c}_1(\cdot), \ldots, \hat{c}_j(\cdot), \hat{z}_0, \hat{z}_1(\cdot), \ldots, \hat{z}_{j-1}(\cdot)) \in \mathscr{B}(R,p,w)$ be an arbitrary strategy. The claim follows if we show that $\mathbb{E}_\nu\left[U_0(C^\star,\cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C},\cdot)\right]$. If $j=1$, this task is trivial since any strategy reduces to a decision for $n=0$ and the inequality is therefore implied by (i). Hence the remainder of the proof assumes that $j \geq 2$. The idea is to construct a list of induced strategies $(\hat{C}^{(n)},\hat{Z}^{(n)})$, $n=1,\ldots,j-1$ obtained by successively replacing the plans $(\hat{c}_n,\hat{z}_n)(\cdot)$ in $(\hat{C},\hat{Z})$ with the potentially optimal plans for stage n defined by the functions $(c^\star_n, z^\star_n)(\cdot)$ from Proposition 2.3.1(iii) and to show that

$$\mathbb{E}_\nu\left[U_0(C^\star,\cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(1)},\cdot)\right] \geq \ldots \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(j-1)},\cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C},\cdot)\right]. \quad (2.66)$$

Following the above conventions, define for each $n=0,1,\ldots,j$ the strategy $(\hat{C}^{(n)},\hat{Z}^{(n)}) = (\hat{c}^{(n)}_0, \hat{c}^{(n)}_1(\cdot), \ldots, \hat{c}^{(n)}_j(\cdot), \hat{z}^{(n)}_0, \hat{z}^{(n)}_1(\cdot), \ldots, \hat{z}^{(n)}_j(\cdot))$ as follows:

$$\begin{aligned}
\hat{c}^{(n)}_m(s_1^m) &:= \hat{c}_m(s_1^m), && m=0,1,\ldots,n-1\\
\hat{z}^{(n)}_m(s_1^m) &:= \hat{z}_m(s_1^m), && m=0,1,\ldots,n-1\\
\hat{c}^{(n)}_m(s_1^m) &:= c^\star_m(W(\hat{z}^{(n)}_{m-1}(s_1^{m-1}), s_m, \hat{e}_m, \hat{R}_{m-1}), s_1^m) && m=n,\ldots,j\\
\hat{z}^{(n)}_m(s_1^m) &:= z^\star_m(W(\hat{z}^{(n)}_{m-1}(s_1^{m-1}), s_m, \hat{e}_m, \hat{R}_{m-1}), s_1^m) && m=n,\ldots,j.
\end{aligned} \quad (2.67)$$

Observe that each strategy $(\hat{C}^{(n)},\hat{Z}^{(n)})$ is feasible and coincides with the original strategy $(\hat{C},\hat{Z})$ up to period stage $n-1$ (the same is obviously true for any strategy $(\hat{C}^{(m)},\hat{Z}^{(m)})$ with $m>n$). In particular, the strategies induce the same wealth process until stage n and hence yield the same random variable $W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$ defining wealth at stage n. From stage n onwards the plans of the strategy $(\hat{C}^{(n)},\hat{Z}^{(n)})$ are defined by the functions $((c^\star_m, z^\star_m)(\cdot))_{m=n}^j$. Observe, however, that this does *not* imply that $(\hat{C}^{(n)},\hat{Z}^{(n)})$ coincides with $(\hat{C}^\star,\hat{Z}^\star)$ from stage n onwards because in general $W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) \neq W(z^\star_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$.

Since the above conventions imply that $(\hat{C}^{(j)},\hat{Z}^{(j)}) = (\hat{C},\hat{Z})$ and $(\hat{C}^{(0)},\hat{Z}^{(0)}) = (\hat{C}^\star,\hat{Z}^\star)$, the claim (2.66) will follow if we show that $\mathbb{E}_\nu\left[U_0(\hat{C}^{(n)},\cdot)\right] \geq \mathbb{E}_\nu\left[U_0(\hat{C}^{(n+1)},\cdot)\right]$ for each $n=0,\ldots,j-1$. Since by definition (2.67) the strategies $(\hat{C}^{(n)},\hat{Z}^{(n)})$ and $(\hat{C}^{(n+1)},\hat{Z}^{(n+1)})$ coincide until stage $n-1$, it suffices to show that for each $n=0,\ldots,j-1$

$$\mathbb{E}_\nu\left[\sum_{m=n}^{j} \beta^{m-n} u(\hat{c}^{(n)}_m(\cdot))\right] \geq \mathbb{E}_\nu\left[\sum_{m=n}^{j} \beta^{m-n} u(\hat{c}^{(n+1)}_m(\cdot))\right]. \quad (2.68)$$

Case 1. Assume that $n > 0$. Let $s_1^n \in \mathbb{S}^n$ be arbitrary but fixed and set $\hat{w}_n := W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1}) \geq -\hat{E}_n$. Recall from (2.67) that the strategy $(\hat{C}^{(n)}, \hat{Z}^{(n)})$ is defined by the functions $((c_m^\star, z_m^\star)(\cdot))_{m=n}^j$ from stage n onwards. Using the conditional distributions Q_m, $m = n+1, \ldots, j$ from Lemma 2.3.1 and equation (2.65) this implies that

$$\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n}^{j} \beta^{m-n} u(\hat{c}_m^{(n)}(s_1^m)) Q_j(s_1^{j-1}, ds_j) \cdots Q_{n+1}(s_1^n, ds_{n+1}) = V_n(\hat{w}_n, s_1^n). \quad (2.69)$$

For fixed $s_1^{n+1} \in \mathbb{S}^{n+1}$ and $\hat{w}_{n+1} := W(\hat{z}_n(s_1^n), s_{n+1}, \hat{e}_{n+1}, \hat{R}_n) \geq -\hat{E}_{n+1}$ the plans of strategy $(\hat{C}^{(n+1)}, \hat{Z}^{(n+1)})$ satisfy

$$\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n+1}^{j} \beta^{m-n} u(\hat{c}_m^{(n+1)}(s_1^m)) Q_j(s_1^{j-1}, ds_j) \ldots Q_{n+2}(s_1^{n+1}, ds_{n+2}) = \beta\, V_{n+1}(\hat{w}_{n+1}, s_1^{n+1}). \quad (2.70)$$

Combining (2.69) and (2.70) and recalling that $\hat{c}_n^{(n+1)}(s_1^n) = \hat{c}_n(s_1^n)$ and $\hat{z}_n^{(n+1)}(s_1^n) = \hat{z}_n(s_1^n)$ from (2.67) one has for each fixed $s_1^n \in \mathbb{S}^n$

$$\begin{aligned}
&\int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n}^{j} \beta^{m-n} u(\hat{c}_m^{(n)}(s_1^m)) Q_j(s_1^{j-1}, ds_j) \ldots Q_{n+1}(s_1^n, ds_{n+1}) \\
&- \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \sum_{m=n}^{j} \beta^{m-n} u(\hat{c}_m^{(n+1)}(s_1^m)) Q_j(s_1^{j-1}, ds_j) \ldots Q_{n+1}(s_1^n, ds_{n+1}) \\
&= V_n(\hat{w}_n, s_1^n) - u(\hat{c}_n(s_1^n)) \\
&- \beta \int_{\mathbb{S}} V_{n+1}(W(\hat{z}_n(s_1^n), s, \hat{e}_{n+1}, \hat{R}_n), s_1^n, s)) Q_{n+1}(s_1^n, ds) \geq 0 \qquad (2.71)
\end{aligned}$$

where as before $\hat{w}_n = W(\hat{z}_{n-1}(s_1^{n-1}), s_n, \hat{e}_n, \hat{R}_{n-1})$. Clearly, (2.71) holds with equality if and only if $\hat{c}_n(s_1^n) = c_n^\star(\hat{w}_n, s_1^n)$ and $\hat{z}_n(s_1^n) = z_n^\star(\hat{w}_n, s_1^n)$. Using the notation introduced in the proof of Lemma 2.3.1 let $\nu_n := \pi_{1,n}\nu$ denote the joint marginal distribution of the random variables $s_1, \ldots, s_n$. The inequality in (2.71) being true for all $s_1^n \in \mathbb{S}^n$ implies that it is preserved under integration with respect to ν_n.[13] Hence, integrating both sides of (2.71) with respect to ν_n and exploiting the factorization Lemma 2.3.1 gives

[13] Alternatively, one could integrate (2.71) successively over the conditional distributions $Q_n, \ldots, Q_2$ and the marginal distribution ν_1 defined in Lemma 2.3.1.

$$
\begin{aligned}
&\mathbb{E}_\nu\left[\sum_{m=n}^{j}\beta^{m-n}(u(\hat{c}_m^{(n)}(\cdot))\right] - \mathbb{E}_\nu\left[\sum_{m=n}^{j}\beta^{m-n}u(\hat{c}_m^{(n+1)}(\cdot))\right]\\
&= \int_{\mathbb{S}^n}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=n}^{j}\beta^{m-n}u(\hat{c}_m^{(n)}(s_1^m))Q_j(s_1^{j-1},ds_j)\cdots\\
&\qquad\qquad\qquad\qquad\cdots Q_{n+1}(s_1^n,ds_{n+1})\nu_n(ds_1^n)\\
&- \int_{\mathbb{S}^n}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=n}^{j}\beta^{m-n}u(\hat{c}_m^{(n+1)}(s_1^m))Q_j(s_1^{j-1},ds_j)\cdots\\
&\qquad\qquad\qquad\qquad\cdots Q_{n+1}(s_1^n,ds_{n+1})\nu_n(ds_1^n)\\
&= \int_{\mathbb{S}^n}V_n(W(\hat{z}_{n-1}(s_1^{n-1}),s_n,\hat{e}_n,\hat{R}_{n-1}),s_1^n)\nu_n(ds_1^n) - \int_{\mathbb{S}^n}u(\hat{c}_n(s_1^n))\nu_n(ds_1^n)\\
&- \beta\int_{\mathbb{S}}V_{n+1}(W(\hat{z}_n(s_1^n),s,\hat{e}_{n+1},\hat{R}_n),s_1^n,s))Q_{n+1}(s_1^n,ds)\nu_n(ds_1^n) \geq 0.
\end{aligned}
$$

This proves (2.68) for the case $n > 0$.

Case 2. Assume that $n = 0$. In this case, one has $(\hat{C}^{(0)}, \hat{Z}^{(0)}) = (\hat{C}^\star, \hat{Z}^\star)$ and the strategies may only differ with respect to the decisions $(c_0^\star, z_0^\star)$ and $(\hat{c}_0^{(1)}, \hat{z}_0^{(1)}) = (\hat{c}_0, \hat{z}_0)$. Using an analogous reasoning as in the previous case one has from (2.65) and (2.67) for each fixed $s_1 \in \mathbb{S}$

$$
\begin{aligned}
&\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=1}^{j}\beta^m u(\hat{c}_m^{(0)}(s_1^m))Q_j(s_1^{j-1},ds_j)\ldots Q_2(s_1,ds_2)\\
&\qquad\qquad\qquad = \beta\, V_1(W(z_0^\star,s_1,\hat{e}_1,R),s_1)\\
&\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=1}^{j}\beta^m u(\hat{c}_m^{(1)}(s_1^m))Q_j(s_1^{j-1},ds_j)\ldots Q_2(s_1,ds_2)\\
&\qquad\qquad\qquad = \beta\, V_1(W(\hat{z}_0,s_1,\hat{e}_1,R),s_1).
\end{aligned}
$$

Hence one obtains, utilizing the factorization Lemma 2.3.1

$$
\begin{aligned}
&\mathbb{E}_\nu\left[\sum_{m=0}^{j}\beta^m u(\hat{c}_m^{(0)}(\cdot))\right] - \mathbb{E}_\nu\left[\sum_{m=0}^{j}\beta^m u(\hat{c}_m^{(1)}(\cdot))\right]\\
&= \int_{\mathbb{S}}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=0}^{j}\beta^m(u(\hat{c}_m^{(0)}(s_1^m))Q_j(s_1^{j-1},ds_j)\ldots Q_2(s_1,ds_2)\nu_1(ds_1)\\
&- \int_{\mathbb{S}}\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\sum_{m=0}^{j}\beta^m u(\hat{c}_m^{(1)}(s_1^m))Q_j(s_1^{j-1},ds_j)\ldots Q_2(s_1,ds_2)\nu_1(ds_1)
\end{aligned}
$$

$$= u(c_0^\star) + \beta \int_{\mathbb{S}} V_1(W(z_0^\star, s, \hat{e}_1, R), s_1)\nu_1(ds)$$
$$-u(\hat{c}_0) - \beta \int_{\mathbb{S}} V_1(W(\hat{z}_0, s, \hat{e}_1, R), s_1)\nu_1(ds) \geq 0,$$

where equality holds if and only if $(\hat{c}_0, \hat{z}_0) = (c_0^\star, z_0^\star)$.
(iii). This is an immediate consequence of the last statement. ■

2.B Technical Lemmas

Lemma 2.B.1 *Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and $f, g : \Omega \longrightarrow \mathbb{R}$ be measurable functions which are ν–integrable and satisfy the conditions $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$ and $\nu(\{\omega \in \Omega | f(\omega) > g(\omega)\}) > 0$. Then*

$$\int_\Omega f(\omega)\nu(d\omega) > \int_\Omega g(\omega)\nu(d\omega).$$

Proof. Define the map $h := f - g$ which is measurable and ν–integrable. By definition $h(\omega) \geq 0$ for all $\omega \in \Omega$ and $\nu(\{\omega \in \Omega | h(\omega) > 0\}) > 0$. By the monotonicity of integrals one has $\int_\Omega h(\omega)\nu(d\omega) \geq 0$. Assume by way of contradiction that $\int_\Omega h(\omega)\nu(d\omega) = 0$. Following [6, Satz 13.2, p. 81], this is equivalent to $\nu(\{\omega \in \Omega | h(\omega) > 0\}) = 0$, which is a contradiction. Hence $\int_\Omega h(\omega)\nu(d\omega) > 0$ and, by the linearity of integrals, $\int_\Omega f(\omega)\nu(d\omega) > \int_\Omega g(\omega)\nu(d\omega)$. ■

Lemma 2.B.2 *Let Ω be a topological space and $(\Omega, \mathcal{B}(\Omega), \nu)$ be a probability space with ν being supported on the compact subset $\bar{\Omega} \in \mathcal{B}(\Omega)$, i.e., $\nu(\bar{\Omega}) = 1$. Let $\Theta \subset \mathbb{R}^m_{++}$ be a compact set and $h : \Theta \times \Omega \longrightarrow \mathbb{R}$ be a continuous function. Then h is ν–integrable and the map $H : \Theta \longrightarrow \mathbb{R}$ defined as*

$$H(\theta) := \int_\Omega h(\theta, \omega)\nu(d\omega)$$

is continuous.

Proof. The proof makes use of Lemma 16.1 in [6, p. 101]. We show that the requirements (a)–(c) of this lemma are satisfied.
(a). Since ν is supported on the compact set $\bar{\Omega}$, the integral function H can be written as

$$H(\theta) = \int_{\bar{\Omega}} h(\theta, \omega)\nu(d\omega).$$

For each fixed $\theta \in \Theta$ the continuous mapping $h(\theta, \cdot) : \bar{\Omega} \longrightarrow \mathbb{R}$ is Borel-measurable and bounded from above by $\bar{h}(\theta) := \max\{h(\theta, \omega) | \omega \in \bar{\Omega}\}$ and from below by $\underline{h}(\theta) := \min\{h(\theta, \omega) | \omega \in \bar{\Omega}\}$. Note that both bounds are well-defined due to the compactness of $\bar{\Omega}$. This implies that the map $h(\theta, \cdot)$ is ν–integrable for all $\theta \in \Theta$.
(b). By assumption, for each fixed $\omega \in \Omega$ the map $h(\cdot, \omega) : \Theta \longrightarrow \mathbb{R}$ is continuous and hence continuous at each point $\theta_0 \in \Theta$.
(c). Tychonoff's Theorem (cf. [49, p. 171, Theorem 12.9]) implies compactness of the product set $\Theta \times \bar{\Omega}$. Hence, by continuity of the function h the value $\bar{h} := \max\{|h(\theta, \omega)| : (\theta, \omega) \in \Theta \times \bar{\Omega}\}$ is well-defined and satisfies $|h(\theta, \omega)| \leq \bar{h}$ for all $(\theta, \omega) \in \Theta \times \bar{\Omega}$.

Since the map h fulfills the requirements (a)–(c) the map $H(\cdot)$ is continuous. ■

Lemma 2.B.3 *Suppose that the measure ν defined in Assumption 2.2.1 is supported on the measurable subset $\bar{\mathbb{S}} = \bar{\mathbb{S}}_1 \times \ldots \times \bar{\mathbb{S}}_j \in \mathcal{B}(\mathbb{S}^j)$. Then each conditional distribution $Q_n(s_1^{n-1}, \cdot)$, $n = 2, \ldots, j$ is supported on a subset of $\bar{\mathbb{S}}_n$ while the marginal distribution ν_1 is supported on $\bar{\mathbb{S}}_1$.*

Proof. For each set $B \in \mathcal{B}(\mathbb{S}^j)$ we have $B = (B \cap \bar{\mathbb{S}}) \uplus (B \cap \bar{\mathbb{S}}^c)$ where $\uplus$ denotes the union of disjoint sets and $\bar{\mathbb{S}}^c$ is the complement of $\bar{\mathbb{S}}$. Since ν is supported on $\bar{S}$ and $B \cap \bar{\mathbb{S}}^c \subset \bar{\mathbb{S}}^c$ we have $\nu(B \cap \bar{\mathbb{S}}^c) = 0$ and therefore

$$\nu(B) = \nu(B \cap \bar{\mathbb{S}}) + \nu(B \cap \bar{\mathbb{S}}^c) = \nu(B \cap \bar{\mathbb{S}}) \quad \forall B \in \mathcal{B}(\mathbb{S}^j). \tag{2.72}$$

Noting that $\mathbf{1}_{B \cap \bar{\mathbb{S}}}(s_1^j) = \mathbf{1}_B(s_1^j) \cdot \mathbf{1}_{\bar{\mathbb{S}}}(s_1^j)$ and $\mathbf{1}_{\bar{\mathbb{S}}}(s_1^j) = \prod_{n=1}^j \mathbf{1}_{\bar{\mathbb{S}}_n}(s_n)$ one obtains the factorization (2.16) as

$$\begin{aligned}
\nu(B) &= \nu(B \cap \bar{\mathbb{S}}) \qquad (2.73)\\
&= \int_{\mathbb{S}} \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \mathbf{1}_B(s_1^j)\, \mathbf{1}_{\bar{\mathbb{S}}_j}(s_j) Q_j(s_1^{j-1}, ds_j) \cdots \\
&\qquad\qquad \cdots \mathbf{1}_{\bar{\mathbb{S}}_2}(s_2) Q_2(s_1, ds_2) \mathbf{1}_{\bar{\mathbb{S}}_1}(s_1) \nu_1(ds_1) \\
&= \int_{\bar{\mathbb{S}}_1} \int_{\bar{\mathbb{S}}_2} \cdots \int_{\bar{\mathbb{S}}_j} \mathbf{1}_B(s_1^j)\, Q_j(s_1^{j-1}, ds_j) \cdots Q_2(s_1, ds_2) \nu_1(ds_1).
\end{aligned}$$

This equality has to be satisfied for all $B \in \mathcal{B}(\mathbb{S}^j)$. Comparing (2.73) with (2.16) it follows that for each $n = 2, \ldots, j$ the set $\bar{\mathbb{S}}_n$ must have full measure, i.e., $Q_n(s_1^{n-1}, \bar{\mathbb{S}}_n) = 1$. Hence the support of $Q_n(s_1^{n-1}, \cdot)$ must be a subset of $\bar{\mathbb{S}}_n$. The fact that the support of ν_1 is the set $\bar{\mathbb{S}}_1$ follows from the fact that $\nu_1(\cdot) = \nu \circ \pi_1^{-1}(\cdot)$ is induced by the projection π_1 (cf. proof of Lemma 2.3.1). ■

Lemma 2.B.4 *Let the sets $\mathbb{C}$, $\mathbb{Y}$ and $\mathbb{X}$ be defined as in Section 2.1 and let the real number $\hat{E} \geq 0$ be given. Then the budget correspondence $\mathbb{B} : \left[-\hat{E}, \infty\right[\times \mathbb{R}^M_{++} \rightrightarrows \mathbb{C} \times \mathbb{Y} \times \mathbb{X}$,*

$$\mathbb{B}(w,p) := \left\{ (c,y,x) \in \mathbb{C} \times \mathbb{Y} \times \mathbb{X} \middle| c + y + x^\top p = w,\ y \geq -\hat{E} \right\}$$

is (upper- and lower-hemi-) continuous.

Proof. Noting that $\mathbb{B}$ is non-empty- and compact-valued, we may apply the definitions given in [59, p. 56] to show that $\mathbb{B}$ is lower- and upper-hemi-continuous at each point $(w_0, p_0) \in [-\hat{E}, \infty[\times \mathbb{R}^M_{++}$.

L.h.c. Let the point $(w_0, p_0) \in [-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$ be arbitrary and let $(w_n, p_n)_{n \geq 1}$ be a sequence taking values in $[-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$ which satisfies $\lim_{n\to\infty}(w_n, p_n) = (w_0, p_0)$. Let (c_0, y_0, x_0) be an arbitrary point in $\mathbb{B}(w_0, p_0)$. We prove existence of a sequence $(c_n, y_n, x_n)_{n \geq 1}$ satisfying $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(c_n, y_n, x_n) = (c_0, y_0, x_0)$. Assume first that $w_0 > -\hat{E}$ and define for each $n \geq 1$

$$c_n := \frac{w_n + \hat{E}}{w_0 + \hat{E}} c_0, \quad x_n^{(m)} := \frac{w_n + \hat{E}}{w_0 + \hat{E}} \frac{p_0^{(m)}}{p_n^{(m)}} x_0^{(m)},\ m = 1, \ldots, M$$

$$y_n := w_n - c_n - x_n^\top p_n.$$

Then we have $\lim_{n\to\infty} c_n = c_0 \geq 0$, $\lim_{n\to\infty} x_n^{(m)} = x_0^{(m)} \geq 0$ for each $m = 1, \ldots, M$ and $\lim_{n\to\infty} y_n = y_0 \geq -\hat{E}$. By construction, $c_n + y_n + x_n^\top p_n = w_n$ for all n and

$$y_n = w_n - \frac{w_n + \hat{E}}{w_0 + \hat{E}}(c_0 + p_0^\top x_0) \geq w_n - \frac{w_n + \hat{E}}{w_0 + \hat{E}}(w_0 + \hat{E}) = -\hat{E}$$

which shows that indeed $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n. In the second case where $w_0 = -\hat{E}$ one observes from the definition of $\mathbb{B}$ that the set $B(w_0, p_0)$ is a singleton and the point (c_0, y_0, x_0) must satisfy $c_0 = 0$, $y_0 = -\hat{E}$ and $x_0 = 0$. In this case, define $c_n = 0$, $y_n = w_n$ and $x_n = 0$ for each $n \geq 1$ to see that $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(c_n, y_n, x_n) = (0, -\hat{E}, 0) = (c_0, y_0, x_0)$. This proves that $\mathbb{B}$

is indeed l.h.c. on its domain of definition.

U.h.c. Let $(w_n, p_n)_{n\geq 1}$ and $(c_n, y_n, x_n)_{n\geq 1}$ be arbitrary sequences taking values in $[-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$ and $\mathbb{C}\times\mathbb{Y}\times\mathbb{X}$, respectively which satisfy $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n and $\lim_{n\to\infty}(w_n, p_n) = (w_0, p_0) \in [-\hat{E}_n, \infty[\times \mathbb{R}^M_{++}$. The claim will follow if we show the existence of a convergent subsequence $(c_{n_k}, y_{n_k}, x_{n_k})_{k\geq 1}$ of $(c_n, y_n, x_n)_{n\geq 1}$ which satisfies $\lim_{k\to\infty}(c_{n_k}, y_{n_k}, x_{n_k}) = (c_0, y_0, x_0)$ where $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. Note first that $(w_n, p_n)_{n\geq 1}$ being a convergent sequence implies that it must be bounded. In particular there exist values $\bar{w} \geq -\hat{E}$ and $\underline{\mathrm{p}}^{(m)} > 0$, $m = 1, \ldots, M$ such that $w_n \leq \bar{w}$ and $\underline{\mathrm{p}}^{(m)} \leq p_n^{(m)}$ for all $n \geq 1$. Since $(c_n, y_n, x_n) \in \mathbb{B}(w_n, p_n)$ for all n this implies $0 \leq c_n \leq \bar{w} + \hat{E}$, $-\hat{E} \leq y_n \leq \bar{w}$ and $0 \leq x_n^{(m)} \leq (\bar{w} + \hat{E})/\underline{\mathrm{p}}^{(m)}$, $m = 1, \ldots, M$. Hence the sequence $(c_n, y_n, x_n)_{n\geq 1}$ is bounded implying the existence of a convergent subsequence $(c_{n_k}, y_{n_k}, x_{n_k})_{k\geq 1}$. Let $(c_0, y_0, x_0) := \lim_{k\to\infty}(c_{n_k}, y_{n_k}, x_{n_k})$. It therefore remains to show that $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. Since $c_{n_k} \geq 0$, $y_{n_k} \geq -\hat{E}$ and $x_{n_k}^{(m)} \geq 0$, $m = 1, \ldots, M$ for all $k \geq 1$ one has $c_0 \geq 0$, $y_0 \geq -\hat{E}$ and $x_0^{(m)} \geq 0$, $m = 1, \ldots, M$. Furthermore, since $(w_n, p_n)_{n\geq 1}$ converges to (w_0, p_0), so does the subsequence $(w_{n_k}, p_{n_k})_{k\geq 1}$ cf. [49, p. 65]. Therefore, since $\lim_{k\to\infty}(w_{n_k}, p_{n_k}) = (w_0, p_0)$ and $c_{n_k} + y_{n_k} + x_{n_k}^\top p_{n_k} - w_{n_k} = 0$ for each $k \geq 1$ this yields $\lim_{k\to\infty}(c_{n_k} + y_{n_k} + x_{n_k}^\top p_{n_k} - w_{n_k}) = c_0 + y_0 + x_0^\top p_0 - w_0 = 0$ and therefore $(c_0, y_0, x_0) \in \mathbb{B}(w_0, p_0)$. ∎

3

The Parameterized Model

This chapter refines the general framework developed in the previous sections by making specific assumptions on the microeconomic characteristics of consumers and firms. The goal is to obtain a particular parametrization which admits closed form solutions to the model's equations. This is desirable for several reasons. Firstly, it provides a possibility to gain additional insights into the structural properties of the demand functions derived in the previous chapter. Secondly, it allows one to prove the existence and uniqueness of temporary equilibria and to obtain additional results on the formation of prices and allocations on real and financial markets. Thirdly, in combination with a population model and a description of the expectations formation of consumers and firms, the proposed parametrization offers a dynamic macroeconomic framework which is tailor-made to study the role of pension systems and the consequences of demographic change within a random environment. In this regard, due to the modeling strategy proposed in this work the model possesses an explicitly defined sequential structure such that its dynamic properties may be analyzed theoretically and with the help of numerical simulations. A comprehensive simulation study of this type will be presented in Chapters 4 and 5 of this work.

The present chapter is structured as follows: Sections 3.1 and 3.2 study the consumption and investment behavior of consumers for the special case with logarithmic utilities and expectations taken from the class of elliptical distributions. Section 3.3 specializes the production and investment behavior of firms by making certain assumptions on the underlying technologies. Section 3.4 establishes the existence, uniqueness and properties of temporary equilibria and describes the expectation formation process of consumers and firms. The chapter closes with

Section 3.5 providing a survey of the model's sequential structure. All technical proofs together with an introduction to the theory of elliptical distributions can be found in the Appendices 3.A and 3.B.

3.1 Consumer Demand with Logarithmic Utility

In this section we study the demand behavior of consumers for the special case with logarithmic preferences. It has long been recognized in the literature that this class implies many desirable properties of the demand functions (2.27) many of which can even be derived explicitly. The present section modifies the results obtained by [40], [41] and extends them to the case with an overlapping generations setting.

As in Sections 2.2 and 2.3, consider the decision problem of a consumer at time $t = 0$ with planning horizon $j > 0$. Let the consumer's expectations $\hat{e} = (\hat{e}_1, \ldots, \hat{e}_j) \in \mathbb{R}^j_+$, $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_{j-1}) \in \mathbb{R}^{j-1}_{++}$, and $\nu \in \text{Prob}(\mathbb{S}^j)$ for future non-capital income, future bond returns and future prices and dividends be given and define $\hat{\epsilon}_0 \geq 0$ and $\hat{E}_n \geq 0$, $n = 1, \ldots, j$ as in (2.6). Furthermore, let the current bond return $R > 0$, asset prices $p \gg 0$, and wealth w as defined by (2.4) be given parametrically and assume as in Section 2.3 that $w > -\hat{\epsilon}_0/R$. It then follows from Theorem 2.2 that the consumer's consumption and investment behavior can be described by continuous demand functions as defined in (2.27). In the sequel we derive additional properties of these demand function by making the following assumption.

Assumption 3.1.1 *The consumer's preferences and expectations as introduced in Assumption 2.2.1 and 2.2.2*[1] *satisfy the following additional hypotheses:*

(i) The instantaneous utility function u is of the form $u(c) := \ln(c)$.
(ii) The measure ν satisfies Assumption 2.2.3 (ii) of a compact support.

Given (i) of Assumption 3.1.1, (ii) is a sufficient condition to ensure that the consumer's decision problem (2.13) remains well-defined, i.e., the supremum in (2.11) satisfies $V_0(R, p, w) < \infty$. Since the logarithmic function satisfies $\lim_{c \searrow 0} \ln(c) = -\infty$, it follows from Lemma 2.2.1 that the wealth process $(W_n(Z, s_1^n))_{n=1}^j$ induced by any potentially optimal strategy $(C, Z) \in \mathscr{B}(R, p, w)$ has $W_n(Z, s_1^n) > -\hat{E}_n$ ν–a.s. To see this,

[1] As a consequence of the specification in (i) we implicitly relax Assumption 2.2.2 by requiring the properties stated there to hold only on the interior of the consumption set $\mathbb{C}$.

note from Lemma 2.2.1 and Definition 2.2.1 that $W_n(Z, s_1^n) = -\hat{E}_n$ implies $c_n(s_1^n) = 0$. As a consequence, any strategy $(C, Z) \in \mathscr{B}(R, p, w)$ where $W_n(Z, s_1^n) = -\hat{E}_n$ for some n with positive probability yields expected utility $\mathbb{E}_\nu\left[U_0(C, \cdot)\right] = -\infty < V_0(R, p, w)$.

Consider now the recursive solution to the consumer's decision problem as studied in Section 2.3. The following proposition shows that in the present case the value functions $V_n(\cdot)$ defined by (2.17) possess a particularly convenient form. To alleviate the subsequent notation we define $\beta_n := 1 + \beta + \ldots + \beta^{j-n}$ for each $n = 0, 1, \ldots, j$.

Proposition 3.1.1 *In addition to the hypotheses of Proposition 2.3.1 let Assumption 3.1.1 be satisfied. Then for $n = 1, \ldots, j$ and $w_n > -\hat{E}_n$ the value functions V_n defined by (2.17) are of the form*

$$V_n(w_n, s_1^n) = \beta_n \ln\left(w_n + \hat{E}_n\right) + h_n(s_1^n)$$

for some continuous function $h_n : \mathbb{S}^n \longrightarrow \mathbb{R}$ with $h_j \equiv 0$.

The proof of Proposition 3.1.1 employs the following lemma that will also be used in the sequel. Both proofs are found in Sections 3.A.1 and 3.A.2 in the appendix of this chapter.

Lemma 3.1.1 *Let $\hat{\nu}$ be a probability measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ supported on the compact set $\bar{\mathbb{S}} \subset \mathbb{S}$ which induces non-redundant assets in the sense of Definition 2.3.1. Let the values $\hat{\epsilon} \geq 0$, $\hat{R} > 0$, $w > -\hat{\epsilon}/\hat{R}$, $p \gg 0$ and $\hat{\beta} > 0$ be given. Then the solution to the problem*

$$\max_{(c,y,x)\in\hat{\mathbb{B}}} \left\{ \ln(c) + \hat{\beta} \int_{\mathbb{S}} \ln\big(W(y, x, s, \hat{\epsilon}, \hat{R})\big)\hat{\nu}(ds) \right\} \tag{3.1}$$

where $\hat{\mathbb{B}} := \left\{(c, y, x) \in \mathbb{C} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, c + y + x^\top p = w, y \geq -\hat{\epsilon}/\hat{R}\right\}$ is uniquely defined and takes the form

$$\begin{aligned}
c^\star &= \tfrac{1}{1+\hat{\beta}}\big(w + \hat{\epsilon}/\hat{R}\big), \\
x^\star &= \tfrac{\hat{\beta}}{1+\hat{\beta}}\big(w + \hat{\epsilon}/\hat{R}\big)\theta^\star \\
y^\star &= \tfrac{\hat{\beta}}{1+\hat{\beta}}\big(w + \hat{\epsilon}/\hat{R}\big)(1 - p^\top\theta^\star) - \hat{\epsilon}/\hat{R} \\
\theta^\star &:= \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\mathbb{S}} \ln\big(W(1 - \theta^\top p, \theta, s, 0, \hat{R})\big)\hat{\nu}(ds) \,\middle|\, \theta^\top p \leq 1 \right\}.
\end{aligned} \tag{3.2}$$

Utilizing Proposition 3.1.1 and the fact that equation (2.6) implies $\hat{\epsilon}_0 = \hat{e}_1 + \hat{E}_1$, one obtains from (2.25) the following one-stage decision problem for $t = 0$ defined for all $(R, p) \gg 0$ and $w > -\hat{\epsilon}_0/R$:

$$\max_{(c,y,x,)\in\mathbb{B}_0(R,p,w)} \left\{ \ln(c) + (\beta_0 - 1) \int_{\mathbb{S}} \ln\bigl(W(y, x, s, \hat{\epsilon}_0, R)\bigr) \nu_1(ds) \right\}. \quad (3.3)$$

Here $\mathbb{B}_0(R, p, w)$ is defined as in (2.26), $\beta_0 := (1 + \beta + \ldots + \beta^j)$ and the additive constant $h_0 := \int_{\mathbb{S}} h_1(s)\nu_1(ds)$ has been omitted since it does not affect the solution to the problem. Utilizing Lemma 3.1.1 the optimal consumption and investment decision for $t = 0$ can be determined from (3.3) as

$$\begin{aligned} c^\star &= \bar{c}(w + \hat{\epsilon}_0/R) \\ x^\star &= (1 - \bar{c})(w + \hat{\epsilon}_0/R)\, \theta^\star \\ y^\star &= (1 - \bar{c})(w + \hat{\epsilon}_0/R)(1 - p^\top \theta^\star) - \hat{\epsilon}_0/R \end{aligned} \quad (3.4)$$

where $\bar{c} := \beta_0^{-1} = [1 + \beta + \ldots + \beta^j]^{-1}$ and

$$\theta^\star := \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\mathbb{S}} \ln\bigl(W(1 - \theta^\top p, \theta, s, 0, R)\bigr)\nu_1(ds) \,\middle|\, \theta^\top p \le 1 \right\}. \quad (3.5)$$

We now exploit the structure of (3.5) to transform this problem into a more convenient form. To this end, note from the integral in (3.5) and the definition (2.15) of the function W that only the sum of next period's prices and dividends $s_1 = (p_1, d_1)$ enter the problem defining the cum-dividend price $q := p_1 + d_1$ of the following period. This sum being a measurable function of the random variable s_1 permits us to define an induced measure ν_q for the random variable q corresponding to the image measure induced by ν_1. In the sequel it will be more convenient to work with ν_q rather than with ν_1. Since by (ii) of Assumption 3.1.1 and Lemma 2.B.3, ν_1 is supported on the compact set $\bar{\mathbb{S}}_1 \subset \mathbb{S}$, the support of ν_q will be a compact subset $\bar{\mathbb{Q}} \subset \mathbb{R}^M_{++}$. Using the definition (2.15) of the function W and exploiting the change-of-variable formula the solution $\theta^\star$ to (3.5) can equivalently be defined as

$$\theta^\star := \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\bar{\mathbb{Q}}} \ln\bigl(R(1 - \theta^\top p) + \theta^\top q\bigr)\nu_q(dq) \,\middle|\, \theta^\top p \le 1 \right\}. \quad (3.6)$$

Given these results, we are now in a position to state the main result of this section in the following theorem which characterizes the demand functions derived from (3.3). Given the previous derivations the proof is straightforward.

Theorem 3.1. *In addition to the hypotheses of Proposition 2.3.1 let Assumption 3.1.1 be satisfied and let the planning horizon* $j \in \{1, \ldots, J\}$ *and expectations* $\hat{e} = (\hat{e}_1, \ldots, \hat{e}_j) \in \mathbb{R}^j_+$, $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_{j-1}) \in \mathbb{R}^{j-1}_{++}$ *and* $\nu \in \mathrm{Prob}(\mathbb{S}^j)$ *be given. Furthermore, define* $\hat{\epsilon}_0 \geq 0$ *as in (2.6) and* ν_q *as above. Then for all* $(R, p) \gg 0$ *and* $w > -\hat{\epsilon}_0/R$ *determined by (2.4) the consumer's demand behavior can be described by the functions*

$$\begin{aligned} \varphi_c^{(j)}(R, p, w; \nu_q, \hat{e}, \hat{R}) &= \bar{c}^{(j)}(w + \hat{\epsilon}_0/R) \\ \varphi_x^{(j)}(R, p, w; \nu_q, \hat{e}, \hat{R}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_0/R)\, \theta(R, p; \nu_q) \\ \varphi_y^{(j)}(R, p, w; \nu_q, \hat{e}, \hat{R}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_0/R)(1 - p^\top \theta(R, p; \nu_q)) - \hat{\epsilon}_0/R \end{aligned} \tag{3.7}$$

where $\bar{c}^{(j)} := [1 + \beta + \ldots + \beta^j]^{-1}$ *and*

$$\theta(R, p; \nu_q) := \arg\max_{\theta \in \mathbb{X}} \left\{ \int_{\bar{\mathbb{Q}}} \ln(R + \theta^\top (q - Rp))\, \nu_q(d\,q) \;\Big|\; \theta^\top p \leq 1 \right\}. \tag{3.8}$$

The structure of the demand functions (3.7) is quite remarkable. A first observation is that the optimal consumption decision is independent of expectations for future asset prices and dividends. It is given by a constant fraction $\bar{c}^{(j)}$ of the sum of current wealth w and the discounted expected non-capital income stream $\hat{\epsilon}_0/R$. This sum will be called *lifetime income*. Such a consumption behavior strongly supports the so-called permanent income hypothesis, see, e.g., [53]. Clearly, only the quantity w is directly available to the consumer while the quantity $\hat{\epsilon}_0/R$ has to be borrowed by issuing bonds. This is the reason for the appearance of the term $-\hat{\epsilon}_0/R$ in the bond demand function. The number $\bar{c}^{(j)}$ defines the marginal propensity to consumption (out of lifetime income) and depends exclusively on the subjective discount factor β and the consumer's remaining lifetime j. Also note that the map $j \longmapsto \bar{c}^{(j)}$ is decreasing such that the share of lifetime income consumed increases with age. The remaining lifetime income $(1-\bar{c}^{(j)})(w+\hat{\epsilon}_0/R)$ is invested into the safe asset $m = 0$ and shares $m = 1, \ldots, M$. In this regard, the amount invested in shares is determined by the solution θ to (3.8). The structure of this problem corresponds exactly to the portfolio decision problem solved by a consumer with a one-period planning horizon who is endowed with one unit of wealth and who is neither allowed to take short sales nor credit. In particular, no expectations for asset prices and dividends which lie further than one period ahead enter the problem. This property is called complete myopia or myopic investment behavior and is well-known to hold with logarithmic utility, see, e.g.,

[41] or [47]. Note though that expectations for future bond returns and future non-capital income strongly influence the decision through the term $\hat{\epsilon}_0$. Also observe that the problem in (3.8) is independent of the planning horizon j. As a consequence the solution to (3.8) may be seen as a reference portfolio defining an optimal mix of shares which is independent of j. All consumers who hold the same expectations ν_q will - irrespective of their age - hold a portfolio which is collinear to this reference portfolio. The corresponding scale is determined by the fraction $(1 - \bar{c}^{(j)})$ times the consumer's lifetime income, implying in particular that asset demand is linear homogeneous in lifetime income.

These results show that the logarithmic utility function implies many desirable properties of the induced demand functions from a technical as well as from an economic point of view.[2] Similar results have been derived in the literature, e.g., by [41] in the presence of an infinite planning horizon and by [40], [47] for the case with a finite planning horizon. The major innovation here is that we extend the class of expectations by allowing the measure ν to induce arbitrary correlations between future asset prices and dividends. This avoids the restrictive assumption of i.i.d. or Markovian asset returns which is made in most examples found in the literature. In addition, the decision problem is formulated within an overlapping generations setting and allows for an uncertain non-capital income stream and time-varying bond returns for which arbitrary subjective expectations are formed.

3.2 Asset Demand with Elliptical Distributions

Given the functional form of the demand functions (3.7), the next goal is to derive some additional properties of the function $\theta(\cdot)$ defined in (3.8). For this purpose, we assume that the probability distribution ν_q belongs to the class of elliptical distributions which play a major role in portfolio theory, e.g., see [47, pp. 104]. A detailed introduction to the underlying theory and the properties of spherical and elliptical distributions used in the sequel is given in Appendix 3.A.3 of this chapter.

For the following derivations, we identify the distribution ν_q with the random variable q which is distributed according to ν_q on the compact set $\bar{\mathbb{Q}}$. Proceeding in this fashion permits to state properties of the measure ν_q in terms of the random variable q and vice versa.[3]

[2] For a general account on the logarithmic utility function and a justification of its use see [55].

[3] For example, the statement that ν_q is a uniform distribution on $\bar{\mathbb{Q}}$ would be equivalent to the statement that q is uniformly distributed on $\bar{\mathbb{Q}}$.

The first step will be to obtain a more favorable form of the objective function

$$U(\theta; R, p, \nu_q) := \int_{\bar{\mathbb{Q}}} \ln(R + \theta^\top (q - Rp))\, \nu_q(d\,q) \tag{3.9}$$

derived from (3.8). Denoting by $\mathbb{M}_M$ the set of all symmetric and positive definite $M \times M$ matrices the following assumption will be made.

Assumption 3.2.1 *The distribution ν_q of the random variable q is taken from a fixed class of elliptical distributions parameterized in $\mu \in \mathbb{R}^M$ and $\Sigma \in \mathbb{M}_M$. The spherical random variable $\hat{\varepsilon} = (\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_M)$ with distribution $\nu_{\hat{\varepsilon}}$ generating this class is supported on the closed sphere $\bar{\mathcal{E}} := \{e \in \mathbb{R}^M \,|\, \|e\| \leq \bar{\varepsilon}\}$ where $\bar{\varepsilon} > 0$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^M$.*

A consequence of Assumption 3.2.1 is that the random variable q has the so-called stochastic representation

$$q \stackrel{d}{=} \mu + A\,\hat{\varepsilon}. \tag{3.10}$$

where the relation $\stackrel{d}{=}$ indicates that both sides have the same distribution and A is some non-singular $M \times M$ matrix satisfying $AA^\top = \Sigma$ (see Appendix 3.A.3 for more details). The invariance of spherical random variables with respect to orthogonal transformations implies that the representation (3.10) of q is unique up to orthogonal transformations of the matrix A. For this reason, we may without loss of generality assume that A is symmetric by setting $A := \Sigma^{\frac{1}{2}}$ where $\Sigma^{\frac{1}{2}}$ denotes the square root of the matrix Σ (a definition may be found in [60]).[4]

Equation (3.10) implies that the measure ν_q may be represented as the image measure induced by the distribution $\nu_{\hat{\varepsilon}}$ under the affine-linear map $e \longmapsto \mu + \Sigma^{\frac{1}{2}} e$, $e \in \mathbb{R}^M$. For this reason we associate the distribution ν_q with the corresponding parameters μ and Σ writing $\nu_q \equiv \nu_{\mu,\Sigma}$. The support of the random variable q with distribution $\nu_{\mu,\Sigma}$ is then denoted by the set $\bar{\mathbb{Q}}(\mu, \Sigma) := \{\mu + \Sigma^{\frac{1}{2}} e | e \in \bar{\mathcal{E}}\}$. Figure 3.2 illustrates the relationship between the random variables $\hat{\varepsilon}$ and q for the case where $M = 2$.

By Corollary 3.A.1 which may be found in Appendix 3.A.3, the parameters $(\mu, \Sigma) \in \mathbb{R}^M_{++} \times \mathbb{M}_M$ define the mean and variance-covariance matrix of the distribution $\nu_{\mu,\Sigma}$. More specifically, $\mu = \mathbb{E}_{\nu_{\mu,\Sigma}}[q]$ and $\Sigma = \lambda\, \mathbb{V}_{\nu_{\mu,\Sigma}}[q]$ for some scalar $\lambda > 0$. The pair (μ, Σ) will be called the

[4] I owe this observation to Jan Wenzelburger.

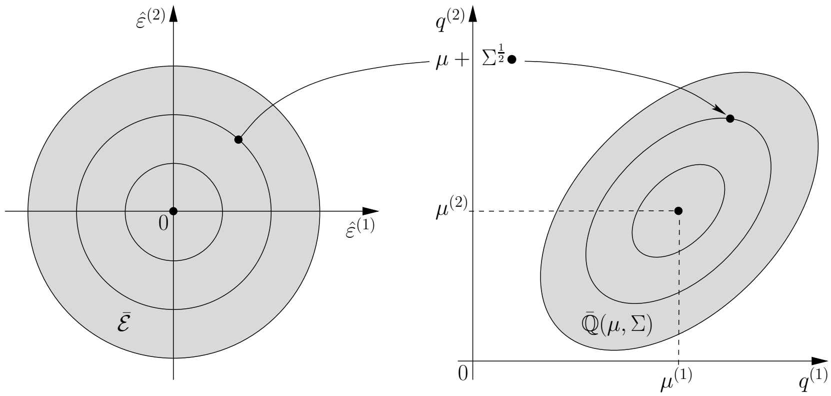


Fig. 3.1. A spherically distributed random variable $\hat{\varepsilon}$ generating an elliptical random variable q via an affine-linear transformation ($M = 2$).

consumer's *beliefs* about q. Note that the dispersion matrix Σ equals the variance covariance matrix of q only up to a strictly positive multiplicative constant which is independent of beliefs. Clearly, the requirement that the support of q be a strictly positive subset of $\mathbb{R}^M$, i.e., $\bar{\mathbb{Q}}(\mu, \Sigma) \subset \mathbb{R}^M_{++}$ restricts the set of beliefs, leading to the following definition of feasibility.

Definition 3.2.1 *Given Assumption 3.2.1, beliefs* $(\mu, \Sigma) \in \mathbb{R}^M_{++} \times \mathbb{M}_M$ *defining the distribution* $\nu_{\mu,\Sigma}$ *of the random variable* q *will be called feasible if the induced support of* q *is strictly positive, i.e.,* $\bar{\mathbb{Q}}(\mu, \Sigma) \subset \mathbb{R}^M_{++}$. *The set of feasible beliefs will be denoted by* $\mathfrak{B} \subset \mathbb{R}^M_{++} \times \mathbb{M}_M$.

Loosely speaking, feasibility of beliefs requires that the components of μ are sufficiently large relative to the entries of the matrix Σ. In the special case where $\Sigma^{\frac{1}{2}} = \text{diag}\left(\sigma^{(1)}, \ldots, \sigma^{(\mathrm{M})}\right)$ (or if $M = 1$) it is necessary and sufficient to require $\mu^{(m)} > \sigma^{(m)}\bar{\varepsilon}$ for each $m = 1, \ldots, M$.

We are now in a position to state the desired representation of the function (3.9). The proof of the following proposition relies on the properties of elliptical distributions and is given in Section 3.A.4 in Appendix 3.A of this chapter.

Proposition 3.2.1 *Let Assumption 3.2.1 be satisfied. Then there exists a real-valued random variable* ε *with symmetric distribution* ν_ε *supported on the interval* $[-\bar{\varepsilon}, \bar{\varepsilon}]$ *such that for each* $(R, p) \gg 0$ *and each pair* $(\mu, \Sigma) \in \mathfrak{B}$ *the expected utility function (3.9) can be written as*

$$U(\theta; R, p, \nu_{\mu,\Sigma}) = \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \ln(R + \theta^\top(\mu - Rp) + (\theta^\top \Sigma\, \theta)^{\frac{1}{2}}\, \varepsilon)\; \nu_\varepsilon(d\,\varepsilon). \tag{3.11}$$

Remark 3.2.1 *The random variable ε corresponds to the first component of the random variable $\hat{\varepsilon}$ defined in Assumption 3.2.1, i.e. $\varepsilon = \hat{\varepsilon}_1$ and the distribution ν_ε is the corresponding marginal distribution. See Appendix 3.A.3 and in particular Lemma 3.A.3 for more details.*

The result from Proposition 3.2.1 has first been proved by [26] for a general class of elliptical distributions and utility functions which are defined on the entire real line. Due to the logarithmic form of the utility function the present study restricts attention to the subclass of elliptical distributions with compact and strictly positive support.

In the sequel, a representation of the form (3.11) will be called a mean-variance representation since utility depends essentially on the mean $R + \theta^\top(\mu - Rp)$ and the term $\theta^\top \Sigma\, \theta$ measuring dispersion of the wealth induced by the portfolio θ.[5] Utilizing the representation (3.11), the maximization problem (3.8) defining the function $\theta(\cdot)$ may now be restated as

$$\max_{\theta \in \mathbb{X}} \left\{ \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \ln(R + \theta^\top(\mu - Rp) + (\theta^\top \Sigma\, \theta)^{\frac{1}{2}}\, \varepsilon)\; \nu_\varepsilon(d\,\varepsilon) \;\middle|\; \theta^\top p \leq 1 \right\}. \tag{3.12}$$

In the sequel we are particularly interested in interior solutions to problem (3.12), i.e., solutions $\theta^\star$ which satisfy $\theta^\star \gg 0$ and $p^\top \theta^\star < 1$. The following theorem establishes conditions under which such a solution obtains and characterizes its form.

Theorem 3.2. *Let Assumptions 2.3.1[6] and 3.2.1 be satisfied and let beliefs $(\mu, \Sigma) \in \mathfrak{B}$ be feasible. Then for all $(R, p) \gg 0$ the following holds true:*

(i) The solution to (3.12) is interior if and only if

$$\Sigma^{-1}(\mu - Rp) \gg 0 \quad \textit{and} \quad \mathbb{E}_{\nu_{\mu,\Sigma}} \left[\frac{Rp^\top \Sigma^{-1}\,(\mu - Rp)}{q^\top \Sigma^{-1}\,(\mu - Rp)} \right] > 1.$$

---

[5] Since the matrix Σ is only proportional to the variance covariance matrix of q, it would be more accurate to call (3.11) a mean-dispersion representation, a terminology adopted, e.g., by [47].

[6] One can show that requiring the distribution ν_1 to induce non-redundant assets in the sense of Definition 2.3.1 is equivalent to the positive definiteness of the matrix Σ and, therefore, of the variance-covariance matrix of the (derived) distribution $\nu_q = \nu_{\mu,\Sigma}$.

(ii) Any interior solution to (3.12) takes the form

$$\theta^\star = \frac{R\,\lambda^\star}{((\mu - Rp)^\top \Sigma^{-1}(\mu - Rp))^{\frac{1}{2}}} \Sigma^{-1}\,(\mu - Rp)$$

where the scalar $\lambda^\star$ is the unique (interior) solution to the problem

$$\max_{0 \le \lambda \le \bar{\lambda}} \left\{ \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \ln(1 + \left((\mu - Rp)^\top \Sigma^{-1}(\mu - Rp)\right)^{\frac{1}{2}}\ \lambda + \lambda\,\varepsilon)\ \nu_\varepsilon(d\,\varepsilon) \right\}$$

with $\bar{\lambda} := \dfrac{\left((\mu - Rp)^\top \Sigma^{-1}(\mu - Rp)\right)^{\frac{1}{2}}}{Rp^\top \Sigma^{-1}\,(\mu - Rp)} > 0.$

Proof. Let beliefs (μ, Σ) and $(R, p) \gg 0$ be arbitrary but fixed. By Theorem 3.1 and (3.8), problem (3.12) has a unique solution $\theta^\star$. Now assume that this solution is interior, i.e., $\theta^\star \gg 0$ and $p^\top \theta^\star < 1$. By Lemma 3.B.2 the objective function $U(\cdot; R, p, \nu_{\mu,\Sigma})$ defined on the compact set $B(p) := \{\theta \in \mathbb{R}^M_+ | p^\top \theta \le 1\}$ is differentiable on the interior of $B(p)$ and the corresponding gradient reads

$$D_\theta U(\theta; R, p, \nu_{\mu,\Sigma}) = \tag{3.13}$$
$$\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \left(\mu - Rp + \frac{\varepsilon}{(\theta^\top \Sigma\,\theta)^{\frac{1}{2}}} \Sigma\,\theta\right) \frac{\nu_\varepsilon(d\,\varepsilon)}{R + \theta^\top(\mu - R\,p) + (\theta^\top \Sigma\,\theta)^{\frac{1}{2}}\,\varepsilon}.$$

Exploiting the strict concavity of $U(\cdot; R, p, \nu_{\mu,\Sigma})$ (cf. proof of Lemma 3.1.1), $\theta^\star$ is an interior solution if and only if is satisfies the (necessary and sufficient) first order condition $D_\theta U(\theta^\star; R, p, \nu_{\mu,\Sigma}) \overset{!}{=} 0$. Using (3.13) this condition can be written as

$$\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{(\theta^{\star\top} \Sigma\,\theta^\star)^{\frac{1}{2}}}{R + \theta^{\star\top}(\mu - R\,p) + (\theta^{\star\top} \Sigma\,\theta^\star)^{\frac{1}{2}}\,\varepsilon} \Sigma^{-1}\left(\mu - Rp\right)\,\nu_\varepsilon(d\,\varepsilon)$$
$$- \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{\varepsilon}{R + \theta^{\star\top}(\mu - R\,p) + (\theta^{\star\top} \Sigma\,\theta^\star)^{\frac{1}{2}}\,\varepsilon}\,\theta^\star\,\nu_\varepsilon(d\,\varepsilon) \overset{!}{=} 0. \tag{3.14}$$

Since the fractions appearing on both sides are scalars, equation (3.14) implies that the solution $\theta^\star$ must be of the form $\theta^\star = \tilde{\lambda}\Sigma^{-1}(\mu - Rp)$ for some $\tilde{\lambda} \in \mathbb{R}$. For notational brevity, set $m := (\mu - Rp)^\top \Sigma^{-1}(\mu - Rp)$. It then follows from (3.14) that $\tilde{\lambda}$ has to satisfy the condition

$$\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{m^{\frac{1}{2}} + \varepsilon}{R + m\tilde{\lambda} + m^{\frac{1}{2}}\tilde{\lambda}\,\varepsilon}\,\nu_\varepsilon(d\,\varepsilon) = 0. \tag{3.15}$$

One easily shows that $\tilde{\lambda} \leq 0$ can never be a solution to (3.15). Hence, $\tilde{\lambda} > 0$ and the first condition $\Sigma^{-1}(\mu - Rp) \gg 0$ in (i) is necessary for $\theta^\star$ to be an interior solution. We show that if in addition the second condition in (i) holds, (3.15) will have a unique solution $\tilde{\lambda}$ which satisfies $0 < \tilde{\lambda} < [p^\top \Sigma^{-1}(\mu - Rp)]^{-1}$. Setting $\lambda^\star := R^{-1} m^{\frac{1}{2}} \tilde{\lambda}$ this condition can equivalently be restated as

$$\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{m^{\frac{1}{2}} + \varepsilon}{1 + m^{\frac{1}{2}} \lambda^\star + \lambda^\star \varepsilon} \, \nu_\varepsilon(d\,\varepsilon) = 0 \tag{3.16}$$

for some $\lambda^\star$ satisfying $0 < \lambda^\star < \bar{\lambda}$ with $\bar{\lambda}$ being defined as above. Define the map $H(\cdot; Rp, \mu, \Sigma) : [0; \bar{\lambda}] \longrightarrow \mathbb{R}$ as

$$H(\lambda; Rp, \mu, \Sigma) := \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \ln(1 + m^{\frac{1}{2}} \lambda + \lambda\, \varepsilon) \, \nu_\varepsilon(d\,\varepsilon).$$

A repeated application of Lemma 3.B.1 shows that $H(\cdot; Rp, \mu, \Sigma)$ is C^2, and the first and second derivatives take the form

$$\begin{aligned}
\frac{\partial H}{\partial \lambda}(\lambda; Rp, \mu, \Sigma) &= \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{m^{\frac{1}{2}} + \varepsilon}{1 + m^{\frac{1}{2}} \lambda + \lambda\, \varepsilon} \, \nu_\varepsilon(d\,\varepsilon) \\
\frac{\partial^2 H}{(\partial \lambda)^2}(\lambda; Rp, \mu, \Sigma) &= - \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \left(\frac{m^{\frac{1}{2}} + \varepsilon}{1 + m^{\frac{1}{2}} \lambda + \lambda\, \varepsilon} \right)^2 \nu_\varepsilon(d\,\varepsilon) < 0
\end{aligned}$$

showing that $H(\cdot; Rp, \mu, \Sigma)$ is strictly concave. Condition (3.16) can now be restated as

$$\frac{\partial H}{\partial \lambda}(\lambda^\star; Rp, \mu, \Sigma) \stackrel{!}{=} 0 \quad \text{for some } \ \lambda^\star \in]0, \bar{\lambda}[. \tag{3.17}$$

Since $\frac{\partial H}{\partial \lambda}(\cdot; Rp, \mu, \Sigma)$ is continuous and strictly monotonically decreasing, it is necessary and sufficient to require $\frac{\partial H}{\partial \lambda}(0; Rp, \mu, \Sigma) > 0$ and $\frac{\partial H}{\partial \lambda}(\bar{\lambda}; Rp, \mu, \Sigma) < 0$ for (3.17) to have a unique solution. As for the first condition, observe from Lemma 3.A.4 that $\mathbb{E}_{\nu_\varepsilon}[\varepsilon] = 0$, and, hence, $\frac{\partial H}{\partial \lambda}(0; Rp, \mu, \Sigma) = m^{\frac{1}{2}} > 0$ due to $\Sigma^{-1}(\mu - Rp) \gg 0$ which implies $\mu \neq Rp$ and, therefore, $m = (\mu - Rp)^\top \Sigma^{-1}(\mu - Rp) > 0$ since Σ is positive definite. The second condition gives

$$\begin{aligned}
\frac{\partial H}{\partial \lambda}(\bar{\lambda}; Rp, \mu, \Sigma) &= \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{m^{\frac{1}{2}} + \varepsilon}{1 + m^{\frac{1}{2}} \bar{\lambda} + \bar{\lambda}\, \varepsilon} \, \nu_\varepsilon(d\,\varepsilon) < 0 \\
&\Leftrightarrow \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{\pm 1 + m^{\frac{1}{2}} \bar{\lambda} + \bar{\lambda}\, \varepsilon}{1 + m^{\frac{1}{2}} \bar{\lambda} + \bar{\lambda}\, \varepsilon} \, \nu_\varepsilon(d\,\varepsilon) < 0
\end{aligned}$$

$$\Leftrightarrow \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{1}{1+m^{\frac{1}{2}}\bar{\lambda}+\bar{\lambda}\,\varepsilon}\;\nu_\varepsilon(d\,\varepsilon) > 1$$

$$\Leftrightarrow \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{Rp^\top \Sigma^{-1}(\mu - Rp)}{\mu^\top \Sigma^{-1}(\mu - Rp) + m^{\frac{1}{2}}\,\varepsilon}\;\nu_\varepsilon(d\,\varepsilon) > 1.$$

In the last step, the definition of $\bar{\lambda}$ has been used and the l.h.s. has been multiplied by $\frac{Rp^\top \Sigma^{-1}(\mu-Rp)}{Rp^\top \Sigma^{-1}(\mu-Rp)}$. The claim will follow if we show that

$$\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{\nu_\varepsilon(d\,\varepsilon)}{\mu^\top \Sigma^{-1}(\mu - Rp) + m^{\frac{1}{2}}\,\varepsilon} = \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{1}{q^\top \Sigma^{-1}\,(\mu - Rp)}\right]. \quad (3.18)$$

From the stochastic representation (3.10) and Lemma 3.B.3 one obtains

$$q^\top \Sigma^{-1}(\mu - Rp) \overset{d}{=} \mu^\top \Sigma^{-1}(\mu - Rp) + m^{\frac{1}{2}}\varepsilon. \quad (3.19)$$

Utilizing Lemma 3.A.1 and the change-of-variable-formula proves (3.18) and gives the desired result

$$\frac{\partial H}{\partial \lambda}(\bar{\lambda}; Rp, \mu, \Sigma) < 0 \Leftrightarrow \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{Rp^\top \Sigma^{-1}(\mu - Rp)}{q^\top \Sigma^{-1}\,(\mu - Rp)}\right] > 1. \quad (3.20)$$

Note that if condition (3.17) were violated and $\frac{\partial H}{\partial \lambda}(\bar{\lambda}; Rp, \mu, \Sigma) \geq 0$ one would have $\frac{\partial H}{\partial \lambda}(\lambda; Rp, \mu, \Sigma) > 0$ for all $0 \leq \lambda < \bar{\lambda}$ showing that any solution $\lambda^\star$ to (3.17) (if it exists at all) will satisfy $\lambda^\star \geq \bar{\lambda}$. This shows that the second condition in (i) is also necessary for an interior solution. Either of the two conditions in (i) is therefore necessary while both of them together are sufficient for (3.17) to have an interior solution. The claimed form of the interior solution $\theta^\star$ in (ii) is now readily inferred from the relation $\theta^\star = \tilde{\lambda}\Sigma^{-1}\,(\mu - Rp)$ and the above definition of $\lambda^\star$. The fact that $\lambda^\star$ can be written as the solution to the maximization problem stated in (ii), follows immediately from the strict concavity of the function $H(\cdot; Rp, \mu, \Sigma)$. It implies that any solution $\lambda^\star$ to (3.17) is the unique maximizer of $H(\cdot; Rp, \mu, \Sigma)$ on the convex set $[0, \bar{\lambda}]$. ■

The result in (ii) of Theorem 3.2 parallels the separation theorem from classical CAPM theory (cf. Lemma 2.2 in [9]). From (i) of Theorem 3.2 one observes that for any pair $(R, p) \gg 0$ the question whether or not the solution to (3.12) is interior depends only on the term $\pi := Rp$. If we interpret the fraction $\frac{1}{R}$ as the price for bonds in the decision period, π may be seen as the relative price between stock and bond investment.[7] For each pair $(\mu, \Sigma) \in \mathfrak{B}$, define the set of relative prices which satisfy the two conditions listed in (i) of Theorem 3.2 as

$$\mathbb{I}(\mu, \Sigma) := \left\{ \pi \in \mathbb{R}^M \,\middle|\, \Sigma^{-1} (\mu - \pi) \gg 0,\ \mathbb{E}_{\nu_{\mu,\Sigma}} \left[\frac{\pi^\top \Sigma^{-1} (\mu - \pi)}{q^\top \Sigma^{-1} (\mu - \pi)} \right] > 1 \right\}. \tag{3.21}$$

By Theorem 3.2 (ii), $(R, p) \gg 0$ induce an interior solution to (3.12) if and only if $Rp \in \mathbb{I}(\mu, \Sigma)$. The next goal is a characterization of the set $\mathbb{I}(\mu, \Sigma)$. For the special case where $M = 1$ it is straightforward to show that $\mathbb{I}(\mu, \Sigma) = \left] \left[\mathbb{E}_{\nu_{\mu,\Sigma}} \left[q^{-1} \right] \right]^{-1}, \mu \right[$. Noting that the map $q \longmapsto \frac{1}{q}$ defined on the compact set $\bar{\mathbb{Q}}(\mu, \Sigma)$ is strictly convex and therefore, exploiting Jensen's inequality, $\mathbb{E}_{\nu_{\mu,\Sigma}} \left[q^{-1} \right] > \frac{1}{\mathbb{E}_{\nu_{\mu,\Sigma}}[q]} = \frac{1}{\mu} > 0$, we see that if $M = 1$ the set $\mathbb{I}(\mu, \Sigma)$ is a non-empty, strictly positive, open interval. For the general case where $M \geq 1$, the following lemma provides a characterization of the set. The proof involves some tedious calculations and is given in Section 3.A.5 in Appendix 3.A of this chapter.

Lemma 3.2.1 *For each pair $(\mu, \Sigma) \in \mathfrak{B}$ of feasible beliefs the set $\mathbb{I}(\mu, \Sigma) \subset \mathbb{R}^M$ defined in (3.21) is a non-empty, bounded, open subset of $\mathbb{R}^M_{++}$ for which μ is a limit point.*

The assertion from Lemma 3.2.1 permits the main result of this section to be stated in the following theorem.

Theorem 3.3. *Let the hypotheses of Theorem 3.2 be satisfied and let beliefs $(\mu, \Sigma) \in \mathfrak{B}$ be feasible. Then the following holds true:*

(i) There exists a differentiable function $\lambda^\star(\cdot; \mu, \Sigma) : \mathbb{I}(\mu, \Sigma) \longrightarrow \mathbb{R}_{++}$ which is strictly decreasing and which is defined implicitly for each $\pi \in \mathbb{I}(\mu, \Sigma)$ as the solution to

$$\frac{\partial H}{\partial \lambda}(\lambda; \pi, \mu, \Sigma) := \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \frac{((\mu - \pi)^\top \Sigma^{-1} (\mu - \pi))^{\frac{1}{2}} + \varepsilon}{1 + ((\mu - \pi)^\top \Sigma^{-1} (\mu - \pi))^{\frac{1}{2}} \lambda + \lambda \varepsilon} \, \nu_\varepsilon(d\varepsilon) = 0.$$

[7] This can be justified as follows. In our setup we have normalized the current bond price to unity while varying its return $R > 0$ promised for the following period. Note however that it would be equivalent to normalize the bond's pay-off to unity and to consider alternative bond prices, say $p^{(0)}$, at time $t = 0$. The return per unit of the consumption good invested at time $t = 0$ would then be $1/p^{(0)}$ implying the relation $R = 1/p^{(0)}$ or, equivalently, $p^{(0)} = R^{-1}$. Hence we may legitimately interpret R^{-1} as a bond price and the term $\pi = Rp$ as a relative asset price.

(ii) For each pair $(R,p) \gg 0$ *satisfying* $Rp \in \mathbb{I}(\mu,\Sigma)$ *the function* $\theta(\cdot,\nu_{\mu,\Sigma})$ *defined in (3.8) takes the form*

$$\theta\left(R,p,;\nu_{\mu,\Sigma}\right) = \frac{R\,\lambda^{\star}\left(Rp;\mu,\Sigma\right)}{\left((\mu - Rp)^{\top}\Sigma^{-1}(\mu - Rp)\right)^{\frac{1}{2}}}\Sigma^{-1}\left(\mu - Rp\right).$$

Proof. The existence of the map $\lambda^{\star}(\cdot;\mu,\Sigma)$ is an immediate consequence of Theorem 3.2 and Lemma 3.2.1. As shown in the proof of Theorem 3.2, the derivative of $\frac{\partial H}{\partial\lambda}(\cdot;\pi,\mu,\Sigma)$ with respect to λ satisfies $\frac{\partial^2 H}{(\partial\lambda)^2}(\lambda;\pi,\mu,\Sigma) < 0$. Utilizing Lemma 3.B.2 one observes that for each fixed λ the function $H'(\lambda;\cdot,\mu,\Sigma) := \frac{\partial H}{\partial\lambda}(\lambda;\cdot,\mu,\Sigma) : \mathbb{I}(\mu,\Sigma) \longrightarrow \mathbb{R}$ is differentiable with respect to each component of π and the gradient reads

$$\begin{aligned} D_{\pi}H'(\lambda;\pi,\mu,\Sigma) = {} & D_{\pi}\left((\mu-\pi)^{\top}\Sigma^{-1}(\mu-\pi)\right)^{\frac{1}{2}} \\ & \times\int_{[-\bar{\varepsilon},\bar{\varepsilon}]}\left(1 + \left((\mu-\pi)^{\top}\Sigma^{-1}(\mu-\pi)\right)^{\frac{1}{2}}\lambda + \lambda\,\varepsilon\right)^{-2}\nu_{\varepsilon}(d\,\varepsilon). \end{aligned} \tag{3.22}$$

Using equation (3.22) and noting that for all $\pi \in \mathbb{I}(\mu,\Sigma)$

$$\begin{aligned} D_{\pi}\left((\mu-\pi)^{\top}\Sigma^{-1}(\mu-\pi)\right)^{\frac{1}{2}} &= -\frac{1}{\left((\mu-\pi)^{\top}\Sigma^{-1}(\mu-\pi)\right)^{\frac{1}{2}}}\Sigma^{-1}\left(\mu-\pi\right) \\ &\ll 0. \end{aligned}$$

This proves that $\frac{\partial H'}{\partial\pi^{(m)}}(\lambda;\pi,\mu,\Sigma) < 0$ for all $\pi \in \mathbb{I}(\mu,\Sigma)$ and for all $\lambda \in \left]0;\frac{\left((\mu-\pi)^{\top}\Sigma^{-1}(\mu-\pi)\right)^{\frac{1}{2}}}{\pi^{\top}\Sigma^{-1}(\mu-\pi)}\right[$. Furthermore, an application of the implicit function theorem shows that the mapping $\lambda^{\star}(\cdot;\mu,\Sigma) : \mathbb{I}(\mu,\Sigma) \longrightarrow \mathbb{R}_{++}$ is differentiable in each component with derivatives

$$\frac{\partial\lambda^{\star}(\pi;\mu,\Sigma)}{\partial\pi^{(m)}} = -\left.\frac{\frac{\partial H'}{\partial\pi^{(m)}}(\lambda;\pi,\mu,\Sigma)}{\frac{\partial H'}{\partial\lambda}(\lambda;\pi,\mu,\Sigma)}\right|_{\lambda=\lambda^{\star}(\pi;\mu,\Sigma)} < 0.$$

This proves the claim (i) of Theorem 3.3. The second claim in (ii) is an immediate consequence of Theorem 3.2. ■

In the sequel we assume that Assumptions 3.1.1 and 3.2.1 are satisfied for each generation $j \in \{1,\dots,J\}$ and all times t. It then follows from Theorem 3.1 that the demand behavior of all consumers can be characterized by demand functions of the form (3.7) which are independent of any expectations for future asset prices and dividends which lie further than one period ahead of the decision period. Thus, we can restrict attention to consumers' expectations $\nu_q = \nu_{\mu_t,\Sigma_t}$ for next period's cum-dividend prices which are determined by the pair $(\mu_t, \Sigma_t) \in \mathfrak{B}$ of feasible beliefs. Suppose that this pair is given and common to all consumers. In addition, let the expectations $\hat{e}_t^{(j)} = (\hat{e}_{t,t+n})_{n=1}^{j}$ and $\hat{R}_t^{(j)} = (\hat{R}_{t,t+n})_{n=1}^{j-1}$ of each consumer in generation $j \in \{1, \ldots, J\}$ at time t be given and define

$$\hat{\epsilon}_t^{(j)} := \hat{e}_{t,t+1}^{(j)} + \frac{\hat{e}_{t,t+2}^{(j)}}{\hat{R}_{t,t+1}} + \ldots + \frac{\hat{e}_{t,t+j}^{(j)}}{\hat{R}_{t,t+1} \cdots \hat{R}_{t,t+j-1}}, \quad j = 0, 1, \ldots, J \quad (3.23)$$

with the understanding that $\hat{\epsilon}_t^{(0)} = 0$. Then for each $(R, p) \gg 0$ and wealth $w > -\hat{\epsilon}_t^{(j)}/R$ being determined by (2.4), the consumer's demand behavior in period t is defined by the functions

$$\begin{aligned}
\varphi_c^{(j)}(R, p, w; \mu_t, \Sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= \bar{c}^{(j)}(w + \hat{\epsilon}_t^{(j)}/R) \\
\varphi_y^{(j)}(R, p, w; \mu_t, \Sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_t^{(j)}/R) \\
&\quad \times (1 - p^\top \theta(R, p; \mu_t, \Sigma_t)) - \hat{\epsilon}_t^{(j)}/R \\
\varphi_x^{(j)}(R, p, w; \mu_t, \Sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_t^{(j)}/R)\, \theta(R, p; \mu_t, \Sigma_t)
\end{aligned} \quad (3.24)$$

where we write $\theta(R, p; \mu_t, \Sigma_t)$ instead of $\theta(R, p; \nu_{\mu_t,\Sigma_t})$ by a slight abuse of notation. Note from the definition of $\bar{c}^{(j)}$ given in Theorem 3.1 that $\bar{c}^{(0)} = 1$. Using this and $\hat{\epsilon}_t^{(0)} = 0$ in (3.24) shows that the demand functions in (3.24) also include the old generation $j = 0$ and are consistent with (2.29).

3.3 Demand Behavior of Firms

Next consider the demand behavior of firms $m = 1, \ldots, M$ as studied in Section 2.4 under specific assumptions on the production technologies $f^{(m)}$ and the adjustment cost functions $g^{(m)}$. To this end, assume that for each $m = 1, \ldots, M$ the production function $f^{(m)}$ in (2.30) is of the Cobb-Douglas form

$$f^{(m)}(L, K) = \kappa^{(m)}\, L^{\alpha^{(m)}} K^{1-\alpha^{(m)}}, \quad \kappa^{(m)} > 0, \quad \alpha^{(m)} \in]0, 1[. \quad (3.25)$$

Note from (3.25) that each function $f^{(m)}$ satisfies the Inada-type conditions $\lim_{L \to 0} \partial_L f^{(m)}(L, K) = \infty$ and $\lim_{L \to \infty} \partial_L f^{(m)}(L, K) = 0$. As a consequence, the condition $\omega, \hat{\omega}_{t,t+1} \in]\underline{\omega}^{(m)}, \overline{\omega}^{(m)}[$ required by Theorem 2.3 (i) and (ii) will automatically be satisfied by each firm m.

As for the adjustment cost function in (2.32), suppose that for each $m = 1, \dots, M$ the function $g^{(m)}$ is of the exponential form

$$g^{(m)}(i) = \begin{cases} 0 & i = 0 \\ \gamma_0^{(m)} \exp\left\{\gamma_1^{(m)} i\right\} & i > 0 \end{cases}, \quad \gamma_0^{(m)} > 0, \quad \gamma_1^{(m)} > \frac{1}{\delta} \tag{3.26}$$

where, as in Section 2.4, $\delta \in]0,1[$ denotes the depreciation rate of capital. Since $\lim_{i \searrow 0} g^{(m)}(i) = \gamma_0^{(m)} > 0 = g^{(m)}(0)$, the function $g^{(m)}$ has a discontinuity at zero. As already argued in Section 2.4, this jump may be due to some fixed costs to investment. The restriction $\gamma_1^{(m)} > \frac{1}{\delta}$ will be explained below.

Given these specifications the following theorem provides a characterization of the firms' labor demand and investment behavior.

Theorem 3.4. *For each firm $m = 1, \dots, M$ let the functions $f^{(m)}$ and $g^{(m)}$ be specified as in (3.25) and (3.26). Then the following holds true:*

(i) The labor demand function from Theorem 2.3 (i) is of the form

$$L^{(m)}(\omega; K_t^{(m)}) = \left(\frac{\alpha^{(m)}\kappa^{(m)}}{\omega}\right)^{\frac{1}{1-\alpha^{(m)}}} K_t^{(m)}. \tag{3.27}$$

(ii) Given $\hat{\omega}_{t,t+1} > 0$ the investment function in Theorem 2.3 (ii) will be positive if and only if $0 < R < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ where for each $\hat{\omega} > 0$

$$\bar{R}^{(m)}(\hat{\omega}) := \frac{(1-\alpha^{(m)})\kappa^{(m)}}{\gamma_0^{(m)}\gamma_1^{(m)} \exp\{1\}} \left(\frac{\alpha^{(m)}\kappa^{(m)}}{\hat{\omega}}\right)^{\frac{\alpha^{(m)}}{1-\alpha^{(m)}}}. \tag{3.28}$$

In this case, $I^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) = i^{(m)}(R; \hat{\omega}_{t,t+1})\, K_t^{(m)}$ where

$$i^{(m)}(R; \hat{\omega}_{t,t+1}) = \frac{1}{\gamma_1^{(m)}} \ln\left(\frac{(1-\alpha^{(m)})\kappa^{(m)}}{\gamma_0^{(m)}\gamma_1^{(m)} R} \left(\frac{\alpha^{(m)}\kappa^{(m)}}{\hat{\omega}_{t,t+1}}\right)^{\frac{\alpha^{(m)}}{1-\alpha^{(m)}}}\right). \tag{3.29}$$

Proof. (i). Note that $f^{(m)}$ and $g^{(m)}$ satisfy Assumptions 2.4.1 and 2.4.2. The assertion is therefore a special case of Theorem 2.3. For each $\omega > 0$ define $l^{(m)}(\omega)$ as in equation (2.49) using the specification (3.25). Then it is straightforward to show that Theorem 2.3 (i) implies the form of the labor demand function (3.27).
(ii). Let $\hat{\omega}_{t,t+1} > 0$ be arbitrary and suppose that $0 < R < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$. In order to apply Theorem 2.3 (ii) we show that this implies

$$R < \left(g_0^{(m)'}\right)^{-1} \partial_K f^{(m)}(l^{(m)}(\hat{\omega}_{t,t+1}), 1).$$

To this end, note that from (3.25) and (3.26) that in the present case

$$g_0^{(m)'} := \lim_{i \searrow 0} g^{(m)'}(i) = \gamma_0^{(m)} \gamma_1^{(m)}$$

and

$$\partial_K f^{(m)}(l^{(m)}(\hat{\omega}_{t,t+1}), 1) = \kappa^{(m)}(1 - \alpha^{(m)})\big(l^{(m)}(\hat{\omega}_{t,t+1})\big)^{\alpha^{(m)}}.$$

This together with $\exp\{1\} > 1$ yields

$$\begin{aligned}\bar{R}^{(m)}(\hat{\omega}_{t,t+1}) &< \frac{\kappa^{(m)}(1-\alpha^{(m)})}{\gamma_0^{(m)} \gamma_1^{(m)}}\big(l^{(m)}(\hat{\omega}_{t,t+1})\big)^{\alpha^{(m)}} \\ &= \frac{\partial_K f^{(m)}(l^{(m)}(\hat{\omega}_{t,t+1}), 1)}{g_0^{(m)'}}.\end{aligned}$$

Define the value $i^{(m)}(R; \hat{\omega}_{t,t+1})$ as in (2.49). Using the specifications from (3.25) and (3.26) it is straightforward to show that $i^{(m)}(R; \hat{\omega}_{t,t+1})$ is of the form stated in (3.29). Furthermore, note that the adjustment cost function (3.26) has a linear elasticity, i.e., $E_{g^{(m)}}(i) = \gamma_1^{(m)} i$ for all $i > 0$. Using (3.29) one easily verifies that $E_{g^{(m)}}(i^{(m)}(R; \hat{\omega}_{t,t+1})) > 1$ if and only if $R < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$. This proves (ii). ■

Theorem 3.4 (ii) restricts attention to interior investment solutions by imposing an upper bound $\bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ on the current bond return. Note that this does not imply that the firm necessarily increases its capital stock. In fact, we have

$$\lim_{R \nearrow \bar{R}^{(m)}(\hat{\omega}_{t,t+1})} i(R; \hat{\omega}_{t,t+1}) = \gamma_1^{(m)-1} < \delta \tag{3.30}$$

due to our restriction $\gamma_1^{(m)} > \frac{1}{\delta}$. Hence, for R sufficiently close to the boundary $\bar{R}^{(m)}(\hat{\omega}_{t,t+1})$, the firm will disinvest in the sense that the investment at time t will not be sufficient to compensate the effect of depreciation and the capital stock will be smaller during the following period.

It would be possible to extend the domain of the investment function (3.29) in Theorem 3.4 by setting $I^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) := 0$ if $R \geq \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$. However, equation (3.30) shows that this extended investment function fails to be continuous. This would cause technical problems in the subsequent derivation of a temporary equilibrium.

To avoid these difficulties the remainder of this chapter restricts attention to the case of interior investment solutions by assuming that $0 < R < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ for all $m = 1, \dots, M$. Formally, this can be justified by supposing that the parameter $\gamma_0^{(m)} > 0$ is sufficiently small such that the upper bound $\bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ is sufficiently large for all m.

Utilizing (3.29) the firm's bond supply function $B^{(m)}(\cdot; \hat{\omega}_{t,t+1}, K_t^{(m)})$ defined in (2.50) can be written for each $R < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ as

$$B^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) = \frac{(1 - \alpha^{(m)})\kappa^{(m)}}{\gamma_1^{(m)} R} \Big(\frac{\alpha^{(m)}\kappa^{(m)}}{\hat{\omega}_{t,t+1}} \Big)^{\frac{\alpha^{(m)}}{1-\alpha^{(m)}}} K_t^{(m)}. \tag{3.31}$$

From (3.31) we see that the bond supply function is homogeneous of degree -1 in the bond return R, i.e.,

$$B^{(m)}(R; \hat{\omega}_{t,t+1}, K_t^{(m)}) = R^{-1} B^{(m)}(1; \hat{\omega}_{t,t+1}, K_t^{(m)}) \tag{3.32}$$

which will be exploited in the sequel. In fact, it is this homogeneity property that will allow us to prove the existence and uniqueness of a temporary financial equilibrium in the following section and to calculate the corresponding temporary equilibrium map. Its validity is exclusively due to the specification (3.26) of the adjustment cost function and would continue to hold with any alternative specification of the production technology. As a consequence, it would easily be possible to modify the present parametrization by replacing equation (3.25) with a more general production technology, e.g., a CES production function.

3.4 Temporary Equilibrium and Expectations Formation

Building upon the previous results consider now the temporary equilibrium problem as introduced for the general case in Section 2.5. Suppose that Assumptions 3.1.1 and 3.2.1 on the preferences and expectations of consumers as well as the hypotheses of Theorem 3.4 describing the firms' technologies are satisfied. It will be shown that in this case the temporary equilibrium maps $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{P}$ introduced in (2.53), (2.56) and (2.57) are well-defined and many properties can even be derived explicitly.

For the following derivations, consider an arbitrary period $t \in \mathbb{N}_0$. Let the population vector $N_t = (N_t^{(j)})_{j=0}^J$ defining aggregated labor supply $L_t^S > 0$ according to (2.1) as well as the firms' capital stocks $K_t = (K_t^{(m)})_{m=1}^M$ be given. Following the sequential structure of the model introduced in Section 2.5 consider first the existence of a temporary equilibrium on the labor market as introduced in Definition 2.5.1. Utilizing (3.27) the aggregated labor demand function of firms can be defined for each $\omega > 0$ as

$$L(\omega; K_t) := \sum_{m=1}^{M} L^{(m)}(\omega; K_t^{(m)}) = \sum_{m=1}^{M} \Big(\frac{\alpha^{(m)} \kappa^{(m)}}{\omega} \Big)^{\frac{1}{1-\alpha^{(m)}}} K_t^{(m)}. \quad (3.33)$$

One observes that the function $L(\cdot; K_t)$ being strictly monotonically decreasing and surjective has an inverse $L^{-1}(\cdot; K_t) : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$. Inverting the market clearing condition $L(\omega_t; K_t) \overset{!}{=} L_t^S$ the equilibrium real wage $\omega_t > 0$ can therefore be obtained as

$$\omega_t = \mathcal{W}(K_t, L_t^S) := L^{-1}(L_t^S; K_t). \quad (3.34)$$

This shows that in the present case the equilibrium map $\mathcal{W}$ from (2.53) is indeed well-defined. Given the contribution rate $\tau_t \in [0, 1[$ to the pension system the non-capital income distribution $e_t := (e_t^{(j)})_{j=0}^{J}$ among consumers follows from (3.34) using (2.2) and (2.3). Likewise, given the realization of the production shock η_t at time t the firms' dividend payments $d_t = (d_t^{(m)})_{m=1}^{M}$ can be determined from (2.34).

Next consider a temporary equilibrium on the asset markets as introduced in Definition 2.5.2. Let the fundamentals (N_t, K_t, e_t, d_t), expectations $(\mu_t, \Sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1})$ and the previous equilibrium asset allocation $z_{t-1} = (y_{t-1}^{(j)}, x_{t-1}^{(j)})_{j=1}^{J}$ and $B_{t-1} = (B_{t-1}^{(m)})_{m=1}^{M}$ among consumers and firms together with the previous bond return $R_{t-1} > 0$ be given. The pair (z_{t-1}, B_{t-1}) being an equilibrium allocation implies $\sum_{j=0}^{J-1} N_t^{(j)} x_{t-1}^{(j+1)} = \bar{x}$ and $\sum_{j=0}^{J-1} N_t^{(j)} y_{t-1}^{(j+1)} = \sum_{m=1}^{M} B_{t-1}^{(m)}$ by virtue of Definition 2.5.2. Consider the market clearing conditions (2.54) and (2.55) supposing that the equilibrium bond return $R_t > 0$ induces an interior investment decision for each firm, i.e. $R_t < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ for all $m = 1, \dots, M$ with the upper bound being defined as in Theorem 3.4. Furthermore, for notational brevity, define from (3.23) and (3.32)

$$\begin{aligned} \hat{m}_t &= \hat{m}(N_t, K_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \quad (3.35) \\ &:= \sum_{j=1}^{J} N_t^{(j)} \hat{e}_t^{(j)} + \sum_{m=1}^{M} B^{(m)}\Big(1; \hat{\omega}_{t,t+1}, K_t^{(m)}\Big). \end{aligned}$$

Then, utilizing the form (3.24) and (3.31) of the asset demand functions of consumers and firms together with (3.23) and (3.32) the market clearing conditions (2.54) and (2.55) can be written as[8]

$$\theta\big(R_t, p_t; \mu_t, \Sigma_t\big) \sum_{j=1}^{J} N_t^{(j)} (1 - \bar{c}^{(j)}) \Big(w_t^{(j)} + \frac{\hat{\epsilon}_t^{(j)}}{R_t} \Big) \stackrel{!}{=} \bar{x} \qquad (3.36)$$

$$\big(1 - p_t^\top \theta\,(R_t, p_t; \mu_t, \Sigma_t)\big) \sum_{j=1}^{J} N_t^{(j)} (1 - \bar{c}^{(j)}) \Big(w_t^{(j)} + \frac{\hat{\epsilon}_t^{(j)}}{R_t} \Big) \stackrel{!}{=} \frac{\hat{m}_t}{R_t}. \qquad (3.37)$$

Here the equilibrium wealth levels $w_t^{(j)}$, $j = 1, \ldots, J$ are determined by (2.4) (and thus also depend on p_t). Since the r.h.s. of (3.36) and (3.37) are strictly positive, an immediate observation is that the equilibrium reference portfolio $\theta_t := \theta\,(R_t, p_t; \mu_t, \Sigma_t)$ must be interior, i.e., $\theta_t \gg 0$ and $\theta_t^\top p_t < 1$. By virtue of Theorem 3.2 (i) and equation (3.21) this is equivalent to $\pi_t := R_t\, p_t \in \mathbb{I}(\mu_t, \Sigma_t)$. In this case, one obtains from Theorem 3.3 (ii)

$$\theta\,(R_t, p_t; \mu_t, \Sigma_t) = \frac{R_t\ \lambda^\star\,(R_t\, p_t; \mu_t, \Sigma_t)}{\big((\mu_t - R_t\, p_t)^\top \Sigma_t^{-1} (\mu_t - R_t\, p_t)\big)^{\frac{1}{2}}} \Sigma_t^{-1}\,(\mu_t - R_t\, p_t) \qquad (3.38)$$

with $\lambda^\star\,(\cdot; \mu_t, \Sigma_t)$ given in Theorem 3.3. Using (3.38) in (3.36) yields

$$\frac{R_t\ \lambda^\star\,(R_t\, p_t; \mu_t, \Sigma_t)}{\big((\mu_t - R_t\, p_t)^\top \Sigma_t^{-1} (\mu_t - R_t\, p_t)\big)^{\frac{1}{2}}} \Sigma_t^{-1}\,(\mu_t - R_t\, p_t) \times \sum_{j=1}^{J} N_t^{(j)} (1 - \bar{c}^{(j)}) \Big(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t \Big) \stackrel{!}{=} \bar{x}. \qquad (3.39)$$

From this equation one immediately observes that the equilibrium π_t must satisfy the collinearity condition

$$\pi_t \in \mathbb{L}\,(\mu_t, \Sigma_t, \bar{x}) := \Big\{ \mu_t - v\, \Sigma_t\, \bar{x} \in \mathbb{R}^M | v \geq 0 \Big\}.$$

Utilizing the definition (3.21) of the set $\mathbb{I}\,(\mu_t, \Sigma_t)$ and combining these two conditions gives

$$\pi_t \in \underbrace{\mathbb{I}\,(\mu_t, \Sigma_t)}_{\text{interiority}} \cap \underbrace{\mathbb{L}\,(\mu_t, \Sigma_t, \bar{x})}_{\text{collinearity}} \Leftrightarrow \pi_t = \mu_t - v_t\, \Sigma_t\, \bar{x} \qquad (3.40)$$

8 At this point the homogeneity property (3.32) of firms' bond supply functions is crucial to obtain the r.h.s. of equation (3.37). Otherwise it would be impossible to derive the subsequent temporary equilibrium maps explicitly while, in addition, all of the following computations would become much more involved.

for some $v_t \in \left]0, \bar{v}(\mu_t, \Sigma_t)\right[$ where the upper bound depending on beliefs is determined by the map $\bar{v} : \mathfrak{B} \longrightarrow \mathbb{R}$

$$\bar{v}(\mu, \Sigma) := \frac{1}{\bar{x}^\top \Sigma \bar{x}} \left(\mu^\top \bar{x} - \left[\mathbb{E}_{\nu_{\mu,\Sigma}} \left[\frac{1}{q^\top \bar{x}} \right] \right]^{-1} \right). \tag{3.41}$$

Note that the condition $v_t > 0$ is owed to the first inequality in the definition (3.21) while $v_t < \bar{v}(\mu_t, \Sigma_t)$ is required by the second one. Also note that $\bar{v}(\mu_t, \Sigma_t) > 0$ by Jensen's inequality. The construction of the equilibrium v_t is illustrated in Figure 3.4 which depicts the sets $\mathbb{L}(\mu_t, \Sigma_t, \bar{x})$ and $\mathbb{I}(\mu_t, \Sigma_t)$ for the special case where $M = 2$.

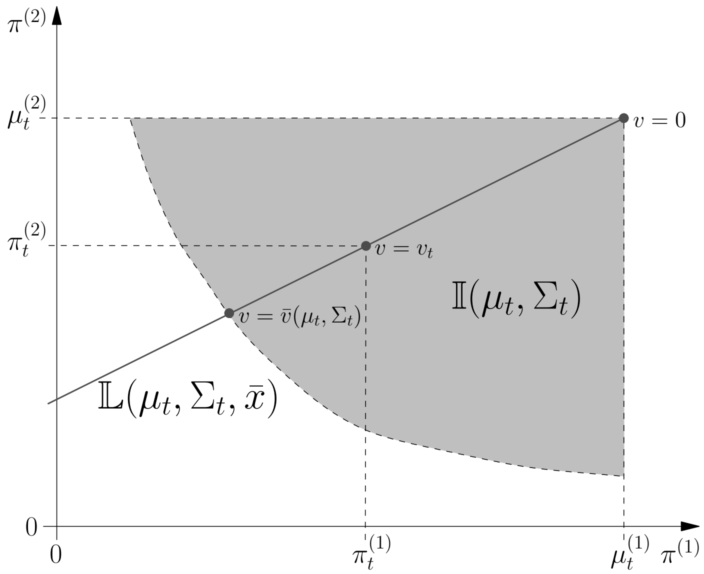


Fig. 3.2. Construction of the equilibrium relative price $\pi_t := R_t\, p_t$ for $M = 2$

Substituting (3.40) into (3.38) and using the result in (3.36) and (3.37) one finds by eliminating from both equations the common term $\sum_{j=1}^{J} N_t^{(j)}(1-\bar{c}^{(j)})(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t)$ that the equilibrium v_t has to satisfy the implicit condition:

$$1 - \frac{\lambda^\star(\mu_t - v_t \Sigma_t\, \bar{x}; \mu_t, \Sigma_t)}{(\bar{x}^\top \Sigma_t\, \bar{x})^{\frac{1}{2}}} \left(\mu_t^\top \bar{x} - v_t \bar{x}^\top \Sigma_t\, \bar{x} + \hat{m}_t \right) = 0. \tag{3.42}$$

The following lemma claims that in the present case (3.42) possesses a unique solution. The tedious proof is relegated to Section 3.A.6 in Appendix 3.A of this chapter.

Lemma 3.4.1 *Let the function $\lambda^\star(\cdot;\mu_t,\Sigma_t)$ be defined as in Theorem 3.3 (i). Then there exists a differentiable function*

$$v:\mathfrak{B}\times\mathbb{R}_{++}\longrightarrow\mathbb{R}_{++},\quad(\mu,\Sigma,\hat{m})\longmapsto v(\mu,\Sigma,\hat{m})$$

which determines the unique solution $v_t=v(\mu_t,\Sigma_t,\hat{m}_t)\in]0,\bar{v}(\mu_t,\Sigma_t)[$ to equation (3.42). Moreover, $v(\cdot)$ is strictly decreasing in each component of μ and $\hat{m}$.

Utilizing the result of Lemma 3.4.1 yields the equilibrium relative price

$$\pi_t=R_t\,p_t=\mu_t-v(\mu_t,\Sigma_t,\hat{m}_t)\Sigma_t\bar{x}\gg 0 \tag{3.43}$$

with $\hat{m}_t>0$ being defined by (3.35).

To derive the form of equilibrium prices (R_t,p_t) recall that $\bar{c}^{(0)}=1$ and, by assumption $\sum_{j=0}^{J-1}N_t^{(j)}x_{t-1}^{(j+1)}=\bar{x}$. This permits us to write $\bar{x}-\sum_{j=1}^{J-1}N_t^{(j)}(1-\bar{c}^{(j)})x_{t-1}^{(j+1)}=\sum_{j=0}^{J-1}N_t^{(j)}\bar{c}^{(j)}x_{t-1}^{(j+1)}$. Using this and equations (3.38), (3.42) and (3.43) together with (2.4) in (3.36) and (3.37) and solving for R_t yields the following map $\mathcal{R}$ which determines the equilibrium bond return at time t as

$$R_t=\mathcal{R}(N_t,K_t,e_t,d_t,z_{t-1},R_{t-1},\mu_t,\Sigma_t,\hat{e}_t,\hat{R}_t,\hat{\omega}_{t,t+1}) \tag{3.44}$$

$$:=\frac{\displaystyle\sum_{m=1}^{M}B^{(m)}(1;\hat{\omega}_{t,t+1},K_t^{(m)})+\sum_{j=1}^{J}N_t^{(j)}\bar{c}^{(j)}\hat{\epsilon}_t^{(j)}+\pi_t^\top\sum_{j=0}^{J-1}N_t^{(j)}\bar{c}^{(j)}x_{t-1}^{(j+1)}}{\displaystyle\sum_{j=1}^{J}N_t^{(j)}(1-\bar{c}^{(j)})e_t^{(j)}+R_{t-1}\sum_{j=1}^{J-1}N_t^{(j)}(1-\bar{c}^{(j)})\big(y_{t-1}^{(j+1)}+d_t^\top x_{t-1}^{(j+1)}\big)}$$

where

$$\pi_t=\mu_t-v\big(\mu_t,\Sigma_t,\hat{m}(N_t,K_t,\hat{e}_t,\hat{R}_t,\hat{\omega}_{t,t+1})\big)\;\Sigma_t\,\bar{x}.$$

Here the maps $\hat{m}(\cdot)$ and $v(\cdot)$ are given by equation (3.35) and Lemma 3.4.1, respectively. Likewise, utilizing (3.43) and (3.44), the temporary equilibrium map $\mathcal{P}$ determining equilibrium asset prices at time t can be calculated as

$$p_t=\mathcal{P}(N_t,K_t,e_t,d_t,z_{t-1},R_{t-1},\mu_t,\Sigma_t,\hat{e}_t,\hat{R}_t,\hat{\omega}_{t,t+1}) \tag{3.45}$$

$$:=\frac{1}{\mathcal{R}\Big(N_t,K_t,e_t,d_t,z_{t-1},R_{t-1},\mu_t,\Sigma_t,\hat{e}_t,\hat{R}_t,\hat{\omega}_{t,t+1}\Big)}\,\pi_t.$$

These results suggest that the temporary equilibrium maps $\mathcal{R}$ and $\mathcal{P}$ from (2.56) and (2.57) are indeed well-defined in the present case.

However, recall that the derivations of (3.44) and (3.45) were based on the presumption of interior investment solutions for each firm. Hence, consistency requires that the solution R_t determined by (3.44) satisfies $R_t < \bar{R}^{(m)}(\hat{\omega}_{t,t+1})$ for each $m = 1, \dots, M$ or, equivalently, $R_t < \bar{R}(\hat{\omega}_{t,t+1})$ where $\bar{R}(\hat{\omega}) := \min_m \left\{ \bar{R}^{(m)}(\hat{\omega}) | m = 1, \dots, M \right\}$ for $\hat{\omega} > 0$.

Due to the complexity of the map $\mathcal{R}$, it seems difficult to derive explicit conditions under which this requirement is satisfied. However, one observes from Theorem 3.4 (ii) that the upper bound $\bar{R}(\hat{\omega}_{t,t+1})$ is crucially determined by the parameters $(\gamma_0^{(m)}, \gamma_1^{(m)})$ of the adjustment cost function of each firm m introduced in (3.26). Assuming a sufficiently small value $\gamma_0^{(m)}$ for all firms will make the upper bound sufficiently large while one observes from the bond supply function (3.31) that this parameter does not enter the interest law (3.44). This observation suggests that there exists a robust set of parameters such that the condition $R_t < \bar{R}(\hat{\omega}_{t,t+1})$ is indeed satisfied for all times t.[9] In the sequel, we will therefore assume that the solution to (3.44) satisfies this condition and the values determined by (3.44) and (3.45) are indeed the equilibrium prices at time t.

Given this assumption, the mappings (3.34), (3.44) and (3.45) define economic laws in the sense of [16]. These determine equilibrium prices on real and financial markets at time t from the given fundamentals of the economy, the previous asset allocation and the subjective expectations of consumers and firms. In this regard, note that the expectations which enter the laws (3.44) and (3.45) refer to future periods such that these mappings possess an expectational lead of length J, cf. [17].

From (3.36) and the form of the asset demand function $\varphi_x^{(j)}(\cdot)$ in (3.24) one observes that the equilibrium portfolio $x_t^{(j)} \in \mathbb{X}$ held by a consumer in generation $j \in \{1, \dots, J\}$ after trading can be written as

$$x_t^{(j)} = \vartheta_t^{(j)} \, \bar{x} \tag{3.46}$$

where

$$\vartheta_t^{(j)} := \frac{(1 - \bar{c}^{(j)})(w_t^{(j)} + \hat{\epsilon}_t^{(j)}/R_t)}{\sum_{i=1}^{J} N_t^{(i)} (1 - \bar{c}^{(i)})(w_t^{(i)} + \hat{\epsilon}_t^{(i)}/R_t)}.$$

Equation (3.46) recovers the classical result from CAPM theory claiming that each investor holds a share $\vartheta_t^{(j)} \in]0, 1[$ of the market portfolio $\bar{x}$. In our model this share is determined by the consumer's expected

[9] The numerical simulations of the model presented in Chapters 4 and 5 confirm this conjecture.

lifetime income relative to aggregate expected lifetime income, each weighted by the respective savings propensities.

As already noted in Section 2.5, the description of the model is only complete once the way in which consumers and firms form their expectations has been defined. In the present case, this requires a description of how consumers determine their beliefs (μ_t, Σ_t) about next period's cum-dividend price and their point forecasts $\hat{e}_t^{(j)} = (\hat{e}_{t,t+n}^{(j)})_{n=1}^{j}$ and $\hat{R}_t^{(j)} = (\hat{R}_{t,t+n})_{n=1}^{j-1}$, $j = 1, \ldots, J$ for future non-capital income and future bond returns. Similarly, it has to be specified how firms determine their real wage prediction $\hat{\omega}_{t,t+1}$ from information available at time t. In this regard, recall from Section 2.5 that expectations are formed *after* the real wage ω_t and the random production shocks η_t have been determined but *prior* to trading on the asset markets. Hence, the information set upon which expectations at time t are based contains the current non-capital income distribution e_t and the dividend payment d_t but not the current price p_t and the bond return R_t. Clearly, all observations made during previous periods $t-1, t-2, \ldots$ are also available.

Due to the complexity of the model this work refrains from the assumption of fully rational expectations but instead assumes that both consumers and firms form their expectations according to simple prediction rules. To this end, suppose that each firm assumes that the current real wage will continue to prevail in the following period such that the wage prediction at time t satisfies

$$\hat{\omega}_{t,t+1} = \omega_t \tag{3.47}$$

for all times $t \in \mathbb{N}_0$. Consumers in generation $j \in \{1, \ldots, J\}$ derive their non-capital income expectation $\hat{e}_{t,t+n}^{(j)}$ for period $t+n$ from the current income of generation $j-n$ (corresponding to their age at $t+n$) such that for each $t \in \mathbb{N}_0$

$$\hat{e}_{t,t+n}^{(j)} = e_t^{(j-n)}, \quad n = 1, \ldots, j, \; j = 1, \ldots, J. \tag{3.48}$$

The prediction for future bond returns is assumed to be uniformly equal to the last observed bond return such that for each $t \in \mathbb{N}_0$

$$\hat{R}_{t,t+n} = R_{t-1}, \quad n = 1, \ldots, J-1. \tag{3.49}$$

Finally, suppose that second moment beliefs for cum-dividend prices are constant such that $\Sigma_t \equiv \Sigma$ for all times t and that first moment beliefs μ_t are updated using an error-correction principle of the form

$$\mu_t = \mu_{t-2} + \varrho(q_{t-1} - \mu_{t-2}), \quad 0 \leq \varrho \leq 1. \tag{3.50}$$

This kind of prediction behavior is known as adaptive expectations formation and is widely used in the literature, see, e.g. [12] or [23], [24]. The idea is that if the previous expectations error $(q_{t-1} - \mu_{t-2})$ is positive such that the price q_{t-1} has been underestimated there is a tendency to increase the current forecast μ_t. Likewise, if the previous price q_{t-1} has been overestimated, there is a tendency to decrease the current forecast μ_t. Note that (3.50) includes the cases of naive expectations $(\varrho = 1 \Rightarrow \mu_t = q_{t-1})$ and of static expectations $(\varrho = 0 \Rightarrow \mu_t = \mu_{t-2})$.

Despite their simplicity the following chapter will reveal that the proposed prediction rules (3.47) to (3.50) are capable of generating unbiased, i.e., on average correct predictions with relatively small forecast errors. Hence, their use may be seen as a form of near-rational behavior. Due to the explicit modeling strategy proposed in this work it is straightforward to integrate more sophisticated forecasting rules into the present framework. In this regard, the existence and form of prediction rules generating fully rational and correct expectations as studied in [61] is a challenging exercise for future research.

At this point, a general remark on the forecasting behavior of consumers and firms must be made. The hypothesized expectations formation distinguished between point predictions and forecasts made for moments of subjective probability distributions. In this regard, it was assumed that consumers treat future asset prices and dividends as random variables in their decision problems and attempt to predict the first two moments of their hypothesized distribution. In contrast to that, they form point predictions for future non-capital incomes and future bond returns. Similarly, firms form point predictions for future real wages. This kind of behavior suggests that the latter variables can relatively safely be predicted and that, e.g., fluctuations in real wages are negligible compared to fluctuations in asset prices. Although this assumption may be reasonable from an empirical point of view, a theoretical justification requires to show that the actual behavior of the model is consistent with this assumption. While a general theoretical argument is difficult due to the complexity of the model, the numerical simulations presented in the following chapters will show that fluctuations in asset prices and dividends are indeed much larger as compared to those variables for which point predictions are made. This will be taken as a justification for the hypothesized forecasting behavior.

To complete the description of the model we are left to specify how a government authority that controls the pension system determines the pension income e_t^R and the contribution rate τ_t at time t. In accordance with the sequential structure introduced in Section 2.5 we assume that these parameters are determined *after* the labor market has cleared and the equilibrium real wage ω_t has been determined but *prior* to trading on asset markets. By construction it is sufficient to determine one of these parameters while the other one will follow from the budget equation (2.60). In the simplest case contributions are constant over time such that $\tau_t \equiv \tau$ for all t and pension incomes adjust accordingly. This case will be referred to as a *static* pension system and will mainly be studied in the following chapter for different values of τ. The case where contributions and pension payments are adjusted dynamically over time according to a well-specified pension policy will be studied in Section 4.6 of Chapter 4 and in Sections 5.3 and 5.4 of Chapter 5.

3.5 The Model in Period *t*

We close this chapter with a summary of the sequential structure of the model in period t. The following five steps correspond to those in Section 2.5. The superscripts above the equality symbol identify the equation in which the respective variable is determined.

Step 1: Population and labor force.

$$\begin{aligned} N_t^{(j)} &\overset{(2.51)}{=} N_{t-1}^{(j+1)}, \qquad\qquad j = 1, \ldots, J-1 \\ N_t^{(J)} &\overset{(2.51)}{=} \mathcal{N}(N_{t-1}) \\ L_t^S &\overset{(2.1)}{=} \sum_{j=j_L}^{J} \bar{L}^{(j)} N_t^{(j)}. \end{aligned}$$

Step 2: Labor market clearing, non-capital income, dividends.

$$\begin{aligned} \eta_t &= \left(\eta_t^{(m)}\right)_{m=1}^{M} && \text{exogenous shock} \\ \omega_t &\overset{(3.34)}{=} \mathcal{W}(K_t, L_t^S), && \\ e_t^{(j)} &\overset{(2.2)}{=} (1-\tau_t)\,\omega_t\,\bar{L}^{(j)}, && j = j_L, \ldots, J \\ e_t^{(j)} &\overset{(2.3)}{=} \frac{\tau_t\,\omega_t\,L_t^S}{N_t^R}, && j = 0, \ldots, j_L - 1 \\ d_t^{(m)} &\overset{(2.34)}{=} \frac{F^{(m)}(L_t^{(m)}, K_t^{(m)}, \eta_t^{(m)}) - \omega_t L_t^{(m)}}{\bar{x}^{(m)}} && \\ &\quad - \frac{R_{t-1} B_{t-1}^{(m)}}{\bar{x}^{(m)}}, && m = 1, \ldots, M. \end{aligned}$$

Step 3: Expectations formation of firms and consumers.

$$\begin{aligned}
\hat{\omega}_{t,t+1} &\overset{(3.47)}{=} \omega_t && \\
\hat{e}^{(j)}_{t,t+n} &\overset{(3.48)}{=} e^{(j-n)}_t, && n = 1,\dots,j,\ j = 1,\dots,J \\
\hat{R}_{t,t+n} &\overset{(3.49)}{=} R_{t-1}, && n = 1,\dots,J-1 \\
\mu_t &\overset{(3.50)}{=} \mu_{t-2} + \varrho(q_{t-1} - \mu_{t-2}) &&
\end{aligned}$$

Step 4: Asset market clearing, wealth, asset holdings.

$$\begin{aligned}
R_t &\overset{(3.44)}{=} \mathcal{R}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \Sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \\
p_t &\overset{(3.45)}{=} \mathcal{P}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \Sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}) \\
w^{(j)}_t &\overset{(2.4)}{=} \begin{cases} e^{(J)}_t & j = J \\ e^{(j)}_t + R_{t-1}\, y^{(j+1)}_{t-1} + x^{(j+1)\top}_{t-1}(p_t + d_t) & j = 0,\dots,J-1 \end{cases} \\
x^{(j)}_t &\overset{(3.46)}{=} \varphi^{(j)}_x\big(R_t, p_t, w^{(j)}_t; \mu_t, \Sigma_t, \hat{e}^{(j)}_t, \hat{R}_t\big), && j = 1,\dots,J \\
y^{(j)}_t &\overset{(3.24)}{=} \varphi^{(j)}_y\big(R_t, p_t, w^{(j)}_t; \mu_t, \Sigma_t, \hat{e}^{(j)}_t, \hat{R}_t\big), && j = 1,\dots,J \\
B^{(m)}_t &\overset{(3.31)}{=} B^{(m)}\left(R_t; \hat{\omega}_{t,t+1}, K^{(m)}_t\right), && m = 1,\dots,M.
\end{aligned}$$

Step 5: Consumption, investment and capital formation.

$$\begin{aligned}
c^{(j)}_t &\overset{(3.24)}{=} \varphi^{(j)}_c\left(R_t, p_t, w^{(j)}_t; \mu_t, \Sigma_t, \hat{e}^{(j)}_t, \hat{R}_t\right), && j = 1,\dots,J \\
I^{(m)}_t &\overset{(3.29)}{=} I^{(m)}(R_t; \hat{\omega}_{t,t+1}, K^{(m)}_t), && m = 1,\dots,M \\
K^{(m)}_{t+1} &\overset{(2.31)}{=} I^{(m)}_t + (1-\delta)K^{(m)}_t, && m = 1,\dots,M.
\end{aligned}$$

Summary of Chapter 3

The parameterized version of the model developed in this chapter provides an explicit description of the demand behavior of consumers and firms as well as of the formation of prices and allocations on real and financial markets. Combined with a description of the forecasting behavior of consumers and firms the model offers a flexible theoretical framework within which the role of pension systems as well as the issue of demographic change can be studied explicitly. In this regard, the developed parametrization is tailor-made to hypothesize alternative pension policies describing the adjustment of contribution rates over time.

Likewise the employed population model allows one to consider various demographic scenarios corresponding to different births processes. Moreover, since the proposed setup permits an arbitrary number of assets an extensive analysis of risk and diversification becomes possible. In this regard, due to the properties of elliptical distributions, the present setup preserves many of the insights from classical CAPM theory which offer a wide array of results on this issue and which become accessible in our extended macroeconomic framework.

Several modifications of the present parametrization are straightforward. As an example, it would easily be possible to modify the assumptions made with respect to the production technologies of firms. For simplicity, these have been assumed to be of the Cobb-Douglas type, however, all of the structural results derived in this chapter remain intact if one allows for more general, e.g., CES, production technologies.

A more challenging modification of the proposed setup concerns the hypothesized expectations formation of consumers and firms which has been kept quite simple. Due to the explicit modeling strategy, it is straightforward to integrate alternative forecasting rules into the model. This opens the possibility to analyze and compare the role of the prediction behavior of consumers and firms on the properties and the dynamic behavior of the model. In particular, the existence and form of forecasting rules generating correct, or in some sense rational, expectations is left as a challenging exercise for future research. In this regard, the framework developed in [61] sets the stage to study the existence of rational forecasting rules for agents with multiperiod planning horizons in a random environment.

Summarizing one finds that the present parametrization incorporates a broad range of possible scenarios and modifications. The simulation study carried out in the subsequent second part of this dissertation should therefore be viewed as being only a first step towards a comprehensive exploration of the model and its dynamic features.

3.A Mathematical Appendix

3.A.1 Proof of Lemma 3.1.1

Let the parameter values $\hat{\epsilon} > 0$, $\hat{R} > 0$, $w > -\hat{\epsilon}/\hat{R}$, $p \gg 0$, and $\hat{\beta} > 0$ be arbitrary but fixed. Define the values $c^\star$ and $\theta^\star$ as follows:

$$c^\star := \arg\max_{c\in\mathbb{C}} \left\{ \ln(c) + \hat{\beta}\ln(w + \hat{\epsilon}/\hat{R} - c) \middle| c \leq w + \hat{\epsilon}/\hat{R} \right\} \tag{3.51}$$

$$\theta^\star := \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\mathbb{S}} \ln\left(W(1-\theta^\top p, \theta, s, 0, \hat{R}) \right) \hat{\nu}(ds) \middle| p^\top\theta \leq 1 \right\}. \tag{3.52}$$

We first show that these values are indeed well-defined, i.e., both problems in (3.51) and (3.52) possess unique solutions. By straightforward calculations one can easily verify that the value $c^\star = \frac{1}{1+\hat{\beta}}\big(w + \hat{E}/\hat{R}\big)$ is the unique maximizer of problem (3.51). Consider the second problem (3.52). For each $\theta \in B(p) := \{\theta \in \mathbb{R}^M_+ | p^\top\theta \leq 1\}$, define the function

$$U(\theta; \hat{R}, p, \hat{\nu}) := \int_{\mathbb{S}} \ln\left(W(1-\theta^\top p, \theta, s, 0, \hat{R}) \right) \hat{\nu}(ds). \tag{3.53}$$

By Lemma 2.B.2, the function $U(\cdot; \hat{R}, p, \hat{\nu})$ being defined on the compact set $B(p)$ is continuous implying the existence of a solution to (3.52). Uniqueness will follow if we show that $U(\cdot; \hat{R}, p, \nu)$ is strictly concave. For this purpose, let $\theta', \theta'' \in B(p)$ be arbitrary with $\theta' \neq \theta''$. Set $z' := (1 - p^\top\theta', \theta') \in \mathbb{Z}$ and $z'' := (1 - p^\top\theta'', \theta'') \in \mathbb{Z}$ and observe that the non-redundancy property of $\hat{\nu}$ implies that the set $A(z', z'', \hat{R}) := \{s \in \mathbb{S} | W(z', s, 0, \hat{R}) \neq W(z'', s, 0, \hat{R})\}$ has positive measure, i.e., $\hat{\nu}(A(z', z'', \hat{R})) > 0$. Employing a similar argument as in step 2 in the proof of Proposition 2.3.1 and exploiting the linearity of the function W and the strict concavity of the logarithmic function it is now straightforward to show that $U(\lambda\theta' + (1-\lambda)\theta''; \hat{R}, p, \hat{\nu}) > \lambda U(\theta'; \hat{R}, p, \hat{\nu}) + (1-\lambda)U(\theta''; \hat{R}, p, \hat{\nu})$ for all $\lambda \in]0, 1[$. This proves the strict concavity of the function $U(\cdot; \hat{R}, p, \hat{\nu})$ which together with the convexity of the set $B(p)$ ensures the uniqueness of the solution $\theta^\star$. This shows that both values in (3.51) and (3.52) are indeed well-defined.

Let $x^\star := \frac{\hat{\beta}}{1+\hat{\beta}}\big(w + \hat{\epsilon}/\hat{R}\big)\theta^\star$, $y^\star := \frac{\hat{\beta}}{1+\hat{\beta}}\big(w + \hat{\epsilon}/\hat{R}\big)(1 - p^\top\theta^\star) - \hat{\epsilon}/\hat{R}$ and let $c^\star$ be defined as above. We show that the triple $(c^\star, y^\star, x^\star)$ is the unique maximizer to (3.1). Note first that $0 < c^\star < w + \hat{\epsilon}/\hat{R}$, $c^\star + p^\top x^\star + y^\star = w$ and $y^\star = w - c^\star - p^\top x^\star \geq -\hat{\epsilon}/\hat{R}$. Hence the constraints in (3.1) are satisfied. Substituting $y^\star = w - c^\star - p^\top x^\star$ and exploiting that the definition (2.15) of the function W implies that $W(w - c^\star - p^\top x^\star, x^\star, s, \hat{\epsilon}, \hat{R}) = (w + \hat{\epsilon}/\hat{R} - c^\star)W(1 - \theta^{\star\top} p, \theta^\star, s, 0, \hat{R})$ the value of the objective function in (3.1) at the point $(c^\star, y^\star, x^\star)$ can be written as

$$\begin{aligned} V^\star &:= \ln(c^\star) + \hat{\beta} \int_{\mathbb{S}} \ln\bigl(W(y^\star, x^\star, s, \hat{\epsilon}, \hat{R})\bigr)\hat{\nu}(ds) \\ &= \ln(c^\star) + \hat{\beta} \int_{\mathbb{S}} \ln\bigl(W(w - c^\star - p^\top x^\star, x^\star, s, \hat{\epsilon}, \hat{R})\bigr)\hat{\nu}(ds) \\ &= \ln(c^\star) + \hat{\beta} \ln(w + \hat{\epsilon}/\hat{R} - c^\star) \\ &\quad + \hat{\beta} \int_{\mathbb{S}} \ln\bigl(W(1 - \theta^{\star\top} p, \theta^\star, s, 0, \hat{R})\bigr)\hat{\nu}(ds). \end{aligned}$$

Let $(\tilde{c}, \tilde{y}, \tilde{x}) \neq (c^\star, y^\star, x^\star)$ be another triple satisfying $\tilde{c} + \tilde{x}^\top p + \tilde{y} = w$ and $\tilde{y} \geq -\hat{\epsilon}/\hat{R}$. Defining

$$\tilde{V} := \ln(\tilde{c}) + \hat{\beta} \int_{\mathbb{S}} \ln\left(W(\tilde{y}, \tilde{x}, s, \hat{\epsilon}, \hat{R})\right) \hat{\nu}(ds)$$

we show that $V^\star > \tilde{V}$.

If $\tilde{c} = 0$ or $\tilde{c} = w + \hat{\epsilon}/\hat{R}$ we have $\tilde{V} = -\infty < V^\star$ and the claim is trivially satisfied in either case. Hence assume $0 < \tilde{c} < w + \hat{\epsilon}/\hat{R}$. Define $\tilde{\theta} := [w + \hat{\epsilon}/\hat{R} - \tilde{c}]^{-1}\tilde{x}$ and observe that $\tilde{\theta} \in B(p)$. Since $\tilde{y} = w - \tilde{c} - \tilde{x}^\top p$ and $W(w - \tilde{c} - p^\top \tilde{x}, \tilde{x}, s, \hat{\epsilon}, \hat{R}) = (w + \hat{\epsilon}/\hat{R} - \tilde{c})W(1 - \tilde{\theta}^\top p, \tilde{\theta}, s, 0, \hat{R})$ the value $\tilde{V}$ can be written as

$$\tilde{V} = \ln(\tilde{c}) + \hat{\beta} \ln(w + \hat{\epsilon}/\hat{R} - \tilde{c}) + \hat{\beta} \int_{\mathbb{S}} \ln\left(W(1 - \tilde{\theta}^\top p, \tilde{\theta}, s, 0, \hat{R})\right) \hat{\nu}(ds).$$

Note that our assumption $(\tilde{c}, \tilde{y}, \tilde{x}) \neq (c^\star, y^\star, x^\star)$ implies $(\tilde{c}, \tilde{\theta}) \neq (c^\star, \theta^\star)$. But since $c^\star$ and $\theta^\star$ are the unique maximizers to problems (3.51) and (3.52) we have

$$\tilde{c} \neq c^\star \Rightarrow \ln(\tilde{c}) + \hat{\beta} \ln(w + \hat{\epsilon}/\hat{R} - \tilde{c}) < \ln(c^\star) + \hat{\beta} \ln(w + \hat{\epsilon}/\hat{R} - c^\star)$$

$$\begin{aligned} \tilde{\theta} \neq \theta^\star \Rightarrow & \int_{\mathbb{S}} \ln\left(W(1 - \tilde{\theta}^\top p, \tilde{\theta}, s, 0, \hat{R})\right) \hat{\nu}(ds) \\ & < \int_{\mathbb{S}} \ln\left(W(1 - \theta^{\star\top} p, \theta, s, 0, \hat{R})\right) \hat{\nu}(ds). \end{aligned}$$

Since at least one of the two inequalities must be satisfied, this shows that $V^\star > \tilde{V}$ and, since $(\tilde{c}, \tilde{y}, \tilde{x})$ was arbitrary, that $(c^\star, y^\star, x^\star)$ are indeed the maximizers of (3.1). ■

3.A.2 Proof of Proposition 3.1.1

Since $h_j \equiv 0$ and $\hat{E}_j = 0$, the claim is obviously true for $n = j$ and hence in the special case where $j = 1$. The remainder of this proof therefore employs an induction argument assuming that $j > 1$.

To this end, suppose there exists $n \in \{1, \dots, j-1\}$ such that the claim is true for $V_{n+1}(\cdot)$. Let $s_1^n \in \mathbb{S}^n$ and $w_n > -\hat{E}_n$ be arbitrary but fixed. Using the recursive definition (2.17) yields the value function V_n under the induction hypothesis as

$$V_n(w_n, s_1^n) = \tilde{V}_n(w_n, s_1^n) + \beta \int_{\mathbb{S}} h_{n+1}(s_1^n, s) Q_{n+1}(s_1^n, ds) \tag{3.54}$$

where, noting that $\beta\,\beta_{n+1} = (\beta_n - 1)$ and that $\hat{e}_{n+1} + \hat{E}_{n+1} = \hat{R}_n \hat{E}_n$ by virtue of (2.6) and defining $\mathbb{B}_n(w_n, p_n)$ as in (2.18)

$$\begin{aligned}&\tilde{V}_n(w_n, s_1^n) := \\ &\max_{(c,z)\in\mathbb{B}_n(w_n,p_n)} \left\{ \ln(c) + (\beta_n - 1) \int_{\mathbb{S}} \ln\big(W(z, s, \hat{R}_n\hat{E}_n, \hat{R}_n)\big) Q_{n+1}(s_1^n, ds) \right\}.\end{aligned} \tag{3.55}$$

Utilizing Lemma 3.1.1 (setting $\hat{E} = \hat{R}_n\hat{E}_n$, $\hat{R} = \hat{R}_n$ and $\hat{\beta} = (\beta_n - 1)$ the solution $c_n^\star$ and $z_n^\star = (y_n^\star, x_n^\star)$ to the maximization problem in (3.55) takes the form

$$\begin{aligned}
c_n^\star &= \tfrac{1}{\beta_n}\left(w_n + \hat{E}_n\right) \\
y_n^\star &= (1 - \tfrac{1}{\beta_n})\left(w_n + \hat{E}_n\right)(1 - p_n^\top \theta_n^\star) - \hat{E}_n \\
x_n^\star &= (1 - \tfrac{1}{\beta_n})\left(w_n + \hat{E}_n\right)\theta_n^\star \\
\theta_n^\star &:= \arg\max_{\theta\in\mathbb{X}} \left\{ \int_{\mathbb{S}} \ln\big(W(1 - \theta^\top p_n, \theta, s, 0, \hat{R}_n)\big) Q_{n+1}(s_1^n, ds) \,\Big|\, \theta^\top p_n \le 1 \right\}.
\end{aligned} \tag{3.56}$$

Substituting (3.56) into (3.55) yields

$$\tilde{V}_n(w_n, s_1^n) = \ln(c_n^\star) + (\beta_n - 1) \int_{\mathbb{S}} \ln\big(W(z_n^\star, s, \hat{R}_n\hat{E}_n, \hat{R}_n)\big) Q_{n+1}(s_1^n, ds). \tag{3.57}$$

Equation (3.56) and the definition of the function W in (2.15) implies $W(y_n^\star, x_n^\star, s, \hat{R}_n\hat{E}_n, \hat{R}_n) = \big(w_n + \hat{E}_n - c_n^\star\big)\big(W(1 - \theta_n^{\star\top} p_n, \theta_n^\star, s, 0, \hat{R}_n)\big)$ and therefore

$$\begin{aligned}&\int_{\mathbb{S}} \ln\big(W(y_n^\star, x_n^\star, s, \hat{R}_n\hat{E}_n, \hat{R}_n)\big) Q_{n+1}(s_1^n, ds) \\ &= \ln\big(w_n + \hat{E}_n - c_n^\star\big) + \int_{\mathbb{S}} \ln\big(W(1 - \theta_n^{\star\top} p_n, \theta_n^\star, s, 0, \hat{R}_n)\big) Q_{n+1}(s_1^n, ds).\end{aligned} \tag{3.58}$$

Substituting (3.58) into (3.57) using (3.56) and using the obtained result in (3.55) yields finally

$$V_n(w_n, s_1^n) = \beta_n \ln\left(w_n + \hat{E}_n\right) + h_n(s_1^n)$$

where the function h_n is defined as

$$\begin{aligned} & h_n(s_1^n) := \\ & \max_{\theta\in\mathbb{X}}\left\{(\beta_n - 1)\int_{\mathbb{S}} \ln\bigl(W(1-\theta^\top p_n, \theta, s, 0, \hat{R}_n)\bigr) Q_{n+1}(s_1^n, ds)\middle|\theta^\top p_n \le 1\right\} \\ & \quad +\beta\int_{\mathbb{S}} h_{n+1}(s_1^n, s) Q_{n+1}(s_1^n, ds) + (\beta_n - 1)\ln(\beta_n - 1) - \beta_n \ln(\beta_n). \end{aligned}$$

The fact that h_n is continuous is a consequence of Proposition 2.3.1. ■

3.A.3 Properties of Elliptical Distributions

This section reviews some of the basic properties of elliptical distributions which were used in Section 3.2. All proofs of the subsequent results may be found, e.g., in the textbooks [31] or [32].

For the following derivations we fix an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and equip the Euclidean space $\mathbb{R}^N$ with its Borelian-σ-algebra $\mathcal{B}(\mathbb{R}^N)$. For any random variable (i.e., $\mathcal{F}-\mathcal{B}(\mathbb{R}^N)$ measurable mapping) $\xi : \Omega \to \mathbb{R}^N$ we denote by ν_ξ the probability distribution (image measure) of ξ defined by $\nu_\xi(B) := \mathbb{P}(\xi^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R}^N)$. Often properties of the random variable ξ will be stated in terms of its image measure ν_ξ and vice versa.[10] The characteristic function[11] of ξ is defined as

$$\psi_\xi : \mathbb{R}^N \longrightarrow \mathbb{C} \qquad \psi_\xi(t) := \int_{\mathbb{R}^N} \exp\left\{i\, t^\top x\right\} \nu_\xi(dx) \tag{3.59}$$

where $i := \sqrt{-1}$. It is well known that there is a one-to-one correspondence between the distribution ν_ξ of a random variable ξ and its characteristic function ψ_ξ, cf. [7, pp. 186]. For each $N \geq 1$ the space $\mathscr{X}_N$ of all random variables defined on Ω with values in $\mathbb{R}^N$ will be equipped with the equivalence relation $\overset{d}{=} \subset \mathscr{X}_N \times \mathscr{X}_N$ indicating that two random

[10] For example, the statement that ξ is normally distributed is equivalent to the statement that ν_ξ is a multivariate normal distribution.

[11] Within this section the symbol $\mathbb{C}$ is used to denote the complex numbers. Although this notation was used in other sections to identify the consumers' consumption set, no confusion should arise.

variables ξ and ξ' have the same distribution, i.e. $\xi \overset{d}{=} \xi' :\Leftrightarrow \nu_\xi = \nu_{\xi'}$. The following Lemma (cf. [31, p. 13]) will be used frequently.

Lemma 3.A.1 *Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^L$ be a $\mathcal{B}(\mathbb{R}^N) - \mathcal{B}(\mathbb{R}^L)$ measurable transformation. Then for all $\xi, \xi' \in \mathscr{X}_N$:*

$$\xi \overset{d}{=} \xi' \Rightarrow f(\xi) \overset{d}{=} f(\xi').$$

Equipped with these prerequisites we now turn to the basic properties of elliptical distributions. Since these are generated by spherical distributions, we begin with the definition of a spherical distribution.

Definition 3.A.1 *A random variable $\hat{\varepsilon} \in \mathscr{X}_N$ with distribution $\nu_{\hat{\varepsilon}}$ is said to be* spherically distributed *and $\nu_{\hat{\varepsilon}}$ is called a* spherical distribution *if for any orthogonal matrix $\Gamma \in \mathbb{R}^{N \times N}$, i.e. $\Gamma^\top \Gamma = I_N$, it is true that $\hat{\varepsilon} \overset{d}{=} \Gamma \hat{\varepsilon}$.*

A spherical distribution is thus invariant under any orthogonal transformation. The most prominent example of a spherical distribution is the standard normal distribution on $\mathbb{R}^N$. Another example is the uniform distribution supported on the closed unit sphere $\{u \in \mathbb{R}^N | \, \|u\| \leq 1\}$, cf. [31, Example 2.2, p. 28]. For $N = 1$ the class of spherical distributions coincides with the symmetric distributions (about the origin). The following lemma states an important property of the characteristic function of spherical random variables. In this regard, recall that the characteristic function of a random variable $\xi \in \mathscr{X}_N$ is real-valued, if and only if $\xi \overset{d}{=} -\xi$, see, e.g., [7, p. 201, Beispiel 1]. Since $\Gamma := -I_N$ is an orthogonal matrix, the characteristic function of a spherically distributed random variable is obviously real-valued.

Lemma 3.A.2 *A random variable $\hat{\varepsilon} \in \mathscr{X}_N$ is spherically distributed if and only if there exists a real-valued function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that the characteristic function $\psi_{\hat{\varepsilon}}$ can be written in the form*

$$\psi_{\hat{\varepsilon}}(t) = \psi \circ \|t\|^2 = \psi(t^\top t). \tag{3.60}$$

where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^N$.

Lemma 3.A.2 asserts that a spherical distribution on $\mathbb{R}^N$ can be constructed by specifying a suitable function ψ which defines the characteristic function (3.59) of this distribution according to (3.60). Consequently, the function ψ is called the *characteristic generator* of the spherical distribution. For each $N \geq 1$, let Ψ_N denote the class of all such generators, i.e., $\psi \in \Psi_N$ if and only if $\psi \circ \|\cdot\|^2$ defines a characteristic function on $\mathbb{R}^N$ (meaning that it satisfies the properties of characteristic functions which can be found e.g. in [7, p. 189, Satz 22.3]. In the sequel we shall write $\hat{\varepsilon} \sim \mathscr{S}_N(\psi)$ to mean that the random variable $\hat{\varepsilon} \in \mathscr{X}_N$ has a spherical distribution on $\mathbb{R}^N$ generated by the function $\psi \in \Psi_N$ according to (3.60). For example, choosing $\psi(u) = \exp\{-\frac{u}{2}\}$ generates the standard normal distribution on $\mathbb{R}^N$ for every $N \geq 1$. It is well-known that the sequence $(\Psi_n)_{n\geq 1}$ is decreasing, i.e. $\Psi_1 \supset \Psi_2 \supset \ldots$. This implies that any characteristic generator $\psi \in \Psi_N$, $N \geq 1$ defining a spherical distribution on $\mathbb{R}^N$ also defines a spherical distribution on $\mathbb{R}^L$ for $L \leq N$ and, in particular, on $\mathbb{R}$. The following lemma establishes an important relation between spherical distributions on $\mathbb{R}^N$ and $\mathbb{R}$. It may be found as Theorem 2.4 on p. 31 in [31].

Lemma 3.A.3 *A random variable* $\hat{\varepsilon} = (\hat{\varepsilon}^{(1)}, \ldots, \hat{\varepsilon}^{(N)})^\top \in \mathscr{X}_N$ *taking values in* $\mathbb{R}^N$ *is spherically distributed if and only if* $a^\top \hat{\varepsilon} \stackrel{d}{=} \|a\|\, \hat{\varepsilon}^{(1)}$ *for every vector* $a \in \mathbb{R}^N$*. In particular:*

$$\hat{\varepsilon} \sim \mathscr{S}_N(\psi) \Rightarrow \hat{\varepsilon}^{(1)} \sim \mathscr{S}_1(\psi).$$

The second statement says that the marginal distribution of the first component $\hat{\varepsilon}^{(1)}$ of $\hat{\varepsilon}$ has a spherical distribution on $\mathbb{R}$ defined by the same generator ψ.

We close our remarks on spherical distributions with the following Lemma describing the moments of spherical distributions (cf. the Corollary on p. 34 in [31]).

Lemma 3.A.4 *The first two moments of any spherically distributed random variable* $\hat{\varepsilon} \sim \mathscr{S}_N(\psi)$ *with distribution* $\nu_{\hat{\varepsilon}}$ *satisfy:*

$$\mathbb{E}_{\nu_{\hat{\varepsilon}}}\left[\hat{\varepsilon}\right] = 0 \quad \text{and} \quad \mathbb{V}_{\nu_{\hat{\varepsilon}}}\left[\hat{\varepsilon}\right] = \mathbb{E}_{\nu_{\hat{\varepsilon}}}\left[\hat{\varepsilon}^2\right] = \lambda_\psi \, I_N$$

where the scalar $\lambda_\psi > 0$ *is exclusively determined by the generator* ψ.[12]

Given the framework of spherical distributions, we now turn to the class of elliptical random variables which are introduced in the following definition.

Definition 3.A.2 *A random variable* $\xi \in \mathscr{X}_N$ *taking values in* $\mathbb{R}^N$ *is said to be* elliptically distributed *with parameters* $(\mu, \Sigma) \in \mathbb{R}^N \times \mathbb{M}_N$ *and its distribution* ν_ξ *is called an* elliptical distribution *if there exists*

[12] In fact, if ψ is differentiable at the origin, one has $\lambda_\psi = -2\psi'(0)$, cf. Theorem 2.17, p. 43 in [31]. This is easily shown to be true for the standard normal distribution.

a spherically distributed random variable $\hat{\varepsilon} \sim \mathscr{S}_N(\psi)$ *such that* ξ *has the stochastic representation*

$$\xi \stackrel{d}{=} \mu + A\,\hat{\varepsilon} \tag{3.61}$$

where the $N \times N$ *matrix* A *satisfies* $A\,A^\top = \Sigma$.

The spherical random variable $\hat{\varepsilon} \sim \mathscr{S}_N(\psi)$ thus generates a class $\mathscr{E}_N(\cdot,\cdot,\psi)$ of elliptically distributed $\mathbb{R}^N$-valued random variables in the sense that each member ξ can be represented as an affine linear transformation of $\hat{\varepsilon}$. Consequently, $\hat{\varepsilon}$ is called the generating variate of this class. We write $\xi \sim \mathscr{E}_N(\mu,\Sigma,\psi)$ to mean that ξ is a member of this class with parameters (μ,Σ). The representation (3.61) is well-defined for any symmetric and positive definite matrix Σ, although it is generically non-unique. However, one can show that if $\xi \stackrel{d}{=} \tilde{\mu} + \tilde{A}\,\hat{\varepsilon}$ is another representation, then $\mu = \tilde{\mu}$ and there exists an orthogonal matrix $O \in \mathbb{R}^{N\times N}$, i.e., $O^\top O = I_N$, such that $\tilde{A} = A\,O$. As a consequence of this result, one can without loss of generality choose $A := \Sigma^{\frac{1}{2}}$ in (3.61).

One of the most useful properties of elliptical random variables is that they are robust with respect to affine-linear transformations. This property is stated in the following proposition the proof of which may be found in [31].

Proposition 3.A.1 *Let* $\xi \sim \mathscr{E}_N(\mu,\Sigma,\psi)$ *be an elliptically distributed random variable with values in* $\mathbb{R}^N$ *and parameters* $(\mu,\Sigma) \in \mathbb{R}^N \times \mathbb{M}_N$. *Then for any matrix* $\Lambda \in \mathbb{R}^{L\times N}$ *with rank* $L \le N$ *and any vector* $\lambda \in \mathbb{R}^L$ *the random variable* $\xi' := \lambda + \Lambda\,\xi \in \mathscr{L}_L$ *satisfies:*

$$\xi' \sim \mathscr{E}_L(\lambda + \Lambda\mu, \Lambda\Sigma\Lambda^\top, \psi).$$

The assertion of this proposition is that any affine-linear transformation of an $\mathbb{R}^N$-valued elliptical random variable ξ defines an $\mathbb{R}^L$-valued random variable ξ' within the same class (i.e., which is generated by the same ψ) with corresponding transformed parameters. In particular, if ξ is elliptical on $\mathbb{R}^N$, any affine-linear transformation from $\mathbb{R}^N$ into $\mathbb{R}$ induces an elliptical distribution on $\mathbb{R}$ of the same class. It is in fact this property that makes elliptical distributions so useful in the theory of portfolio choice.

We close this section with the following corollary describing the moments of elliptical random variables. The proof is immediate from Lemma 3.A.4 and (3.10).

Corollary 3.A.1 *For any elliptical random variable $\xi \sim \mathscr{E}_N(\mu, \Sigma, \psi)$ with distribution ν_ξ the first two moments satisfy*

$$\mathbb{E}_{\nu_\xi}[\xi] = \mu \quad \textit{and} \quad \mathbb{V}_{\nu_\xi}[\xi] = \lambda_\psi \, \Sigma,$$

where the scalar $\lambda_\psi > 0$ is exclusively determined by the generator ψ (see Lemma 3.A.4 and the footnote thereafter).

3.A.4 Proof of Proposition 3.2.1

The following proof draws on the properties of elliptical distributions presented in the previous section. Let Assumption 3.2.1 be satisfied such that the random variable q with distribution ν_q is taken from a fixed class of elliptical random variables generated by the spherical random variable $\hat{\varepsilon} \sim \mathscr{S}_M(\psi)$ with characteristic generator $\psi \in \Psi_M$. By definition, for each pair $(\mu, \Sigma) \in \mathbb{R}^M_{++} \times \mathbb{M}_M$ of feasible beliefs, q has the stochastic representation

$$q \stackrel{d}{=} \mu + \Sigma^{\frac{1}{2}} \hat{\varepsilon} \tag{3.62}$$

where the non-singular matrix $\Sigma^{\frac{1}{2}} \in \mathbb{R}^{M \times M}$ denotes the square root of Σ. By Lemma 3.A.1, for each $\theta \in \mathbb{R}^M_+$ satisfying $p^\top \theta \leq 1$, one has

$$R + \theta^\top (q - Rp) \stackrel{d}{=} R + \theta^\top (\mu - Rp) + \theta^\top \Sigma^{\frac{1}{2}} \hat{\varepsilon}. \tag{3.63}$$

Since $\hat{\varepsilon} = (\hat{\varepsilon}^{(1)}, \ldots, \hat{\varepsilon}^{(M)})$ is a spherical random variable, an application of Lemma 3.A.3 yields, noting that $\|\Sigma^{\frac{1}{2}\top} \theta\| = \big(\theta^\top \Sigma \, \theta\big)^{\frac{1}{2}}$

$$\theta^\top \Sigma^{\frac{1}{2}} \hat{\varepsilon} \stackrel{d}{=} \big(\theta^\top \Sigma \, \theta\big)^{\frac{1}{2}} \varepsilon \tag{3.64}$$

where $\varepsilon := \hat{\varepsilon}^{(1)}$ is the first component of the random variable $\hat{\varepsilon}$ with distribution ν_ε which, by Lemma 3.A.3 is spherically distributed on $\mathbb{R}$, i.e. $\varepsilon \sim \mathscr{S}_1(\psi)$. Furthermore, since the support of $\hat{\varepsilon}$ is the compact set $\bar{\mathcal{E}} := \{e \in \mathbb{R}^M | \|e\| \leq \bar{\varepsilon}\}$, the support of ε will be the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Since $\stackrel{d}{=}$ is an equivalence relation and hence transitive one obtains from (3.63) and (3.64)

$$R + \theta^\top (q - Rp) \stackrel{d}{=} R + \theta^\top (\mu - Rp) + \big(\theta^\top \Sigma \, \theta\big)^{\frac{1}{2}} \varepsilon.$$

The map $\ln : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ being a continuous and hence Borel-measurable transformation implies by Lemma 3.A.1 that

$$\ln\big(R + \theta^\top (q - Rp)\big) \stackrel{d}{=} \ln\big(R + \theta^\top (\mu - Rp) + \big(\theta^\top \Sigma \, \theta\big)^{\frac{1}{2}} \varepsilon\big).$$

Integrating both sides of the last equation making use of the change-of-variable formula finally gives the desired result

$$\int_{\bar{\mathbb{Q}}(\mu,\Sigma)} \ln\big(R + \theta^\top (q - Rp)\big)\nu_{\mu,\Sigma}(dq)$$
$$= \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \ln\big(R + \theta^\top (\mu - Rp) + \big(\theta^\top \Sigma\, \theta\big)^{\frac{1}{2}}\varepsilon\big)\nu_\varepsilon(d\varepsilon).$$

■

3.A.5 Proof of Lemma 3.2.1

Given beliefs $(\mu, \Sigma) \in \mathfrak{B}$ define the set $\mathbb{I}(\mu, \Sigma)$ as in (3.21). Define the M-dimensional simplex $S_1 := \{\Delta \in \mathbb{R}^M_{++} \,|\, \|\Delta\| = 1\}$ and let $\Delta \in S_1$ be arbitrary. Consider the line segment $\mathbb{L}(\mu, \Sigma, \Delta) := \{\mu - v\Sigma\Delta \,|\, v \geq 0\}$ passing through μ. Let $\pi \in \mathbb{L}(\mu, \Sigma, \Delta)$. Then by definition $\pi = \mu - v\Sigma\Delta$ for some $v \geq 0$ or, equivalently, $\Sigma^{-1}(\mu - \pi) = v\Delta$. This implies that $\Sigma^{-1}(\mu - \pi) \gg 0$ if and only if $v > 0$. Furthermore,

$$\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\pi^\top \Sigma^{-1}(\mu - \pi)}{q^\top \Sigma^{-1}(\mu - \pi)}\right] = \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\mu^\top \Delta - v\Delta^\top \Sigma\Delta}{q^\top \Delta}\right]$$
$$= \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\mu^\top \Delta}{q^\top \Delta}\right] - v\,\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\Delta^\top \Sigma\Delta}{q^\top \Delta}\right].$$

Therefore, noting that by Jensen's inequality $\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\mu^\top \Delta}{q^\top \Delta}\right] > 1$,

$$\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\pi^\top \Sigma^{-1}(\mu - \pi)}{q^\top \Sigma^{-1}(\mu - \pi)}\right] > 1$$

if and only if

$$v < \bar{v}(\Delta) := \frac{\mu^\top \Delta}{\Delta^\top \Sigma\Delta} - \frac{1}{\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\Delta^\top \Sigma\Delta}{q^\top \Delta}\right]}. \tag{3.65}$$

It follows that $\pi \in \mathbb{I}(\mu, \Sigma)$ provided that $v \in]0, \bar{v}(\Delta)[$. Defining the set $\bar{\mathbb{L}}(\mu, \Sigma, \Delta) := \{\mu - v\Sigma\Delta \,|0 < v < \bar{v}(\Delta)\}$ it is clear from the previous observation that $\bar{\mathbb{L}}(\mu, \Sigma, \Delta) \subset \mathbb{I}(\mu, \Sigma)$. Since Δ was arbitrary, this inclusion has to be true for all $\Delta \in S_1$. Consequently

$$\bar{\mathbb{L}}(\mu, \Sigma) := \bigcup_{\Delta \in S_1} \bar{\mathbb{L}}(\mu, \Sigma, \Delta) \subset \mathbb{I}(\mu, \Sigma) \tag{3.66}$$

proving in particular that $\mathbb{I}(\mu, \Sigma)$ is non-empty. We claim that the reverse inclusion $\supset$ also holds in (3.66). Let $\pi \in \mathbb{I}(\mu, \Sigma)$ be arbitrary. Then $\Sigma^{-1}(\mu - \pi) =: \tilde{\Delta} \gg 0$. Now set $\Delta := \frac{1}{\|\tilde{\Delta}\|}\tilde{\Delta}$ and $v := \|\tilde{\Delta}\| > 0$ to see that π can be written in the from $\pi = \mu - v\Sigma\Delta$ with $\Delta \in S_1$. Clearly, the second condition $\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\pi^\top \Sigma^{-1}(\mu-\pi)}{q^\top \Sigma^{-1}(\mu-\pi)}\right] > 1$ then requires $v < \bar{v}(\Delta)$ with $\bar{v}(\cdot)$ being defined as in (3.65). This implies that $\pi \in \bar{\mathbb{L}}(\mu, \Sigma, \Delta)$ and, therefore, $\pi \in \bar{\mathbb{L}}(\mu, \Sigma)$. This proves that $\bar{\mathbb{L}}(\mu, \Sigma) = \mathbb{I}(\mu, \Sigma)$.

Next we show that $\mathbb{I}(\mu, \Sigma) \subset \mathbb{R}^M_{++}$. It suffices to show that for all $\Delta \in S_1$ one has $\bar{\mathbb{L}}(\mu, \Sigma, \Delta) \subset \mathbb{R}^M_{++}$. To this end, let $\Delta \in S_1$ be arbitrary. Since the set $\bar{\mathbb{L}}(\mu, \Sigma, \Delta)$ is just the line segment between the two points $\mu \gg 0$ and $\bar{\pi} := \mu - \bar{v}(\Delta)\Sigma\Delta$, it suffices to show that $\bar{\pi} \gg 0$. Utilizing the definition of $\bar{v}(\Delta)$ yields

$$\bar{\pi} = \mu + \Sigma^{\frac{1}{2}}\zeta \quad \text{where} \quad \zeta := -\left(\frac{\mu^\top \Delta}{\Delta^\top \Sigma \Delta} - \frac{1}{\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\Delta^\top \Sigma \Delta}{q^\top \Delta}\right]}\right)\Sigma^{\frac{1}{2}\top}\Delta \in \mathbb{R}^M. \tag{3.67}$$

Note from Definition 3.2.1 that the feasibility of beliefs (μ, Σ) implies that $\mu + \Sigma^{\frac{1}{2}}\tilde{\zeta} \gg 0$ for all $\tilde{\zeta} \in \mathbb{R}^M$ satisfying $\|\tilde{\zeta}\| \leq \bar{\varepsilon}$. It therefore suffices to show that the vector ζ defined in (3.67) satisfies $\|\zeta\| \leq \bar{\varepsilon}$. Noting that $\|\Sigma^{\frac{1}{2}\top}\Delta\| = (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}$ and that $\mu^\top \Delta > \left[\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{1}{q^\top \Delta}\right]\right]^{-1}$ by Jensen's inequality one obtains:

$$\|\zeta\| = \frac{1}{(\Delta^\top \Sigma \Delta)^{\frac{1}{2}}}\left(\mu^\top \Delta - \left[\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{1}{q^\top \Delta}\right]\right]^{-1}\right) \leq \bar{\varepsilon}$$

if and only if

$$\mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{1}{q^\top \Delta}\right] \leq \frac{1}{\mu^\top \Delta - (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}\bar{\varepsilon}} \tag{3.68}$$

where we have exploited that $\mu^\top \Delta - (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}\bar{\varepsilon} > 0$ by virtue of Lemma 3.B.3 (ii). On the other hand, Lemma 3.B.3 (i) shows that $q^\top \Delta \stackrel{d}{=} \mu^\top \Delta + (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}\varepsilon$ where the random variable ε is symmetrically distributed on $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Combining these two properties yields

$$q^\top \Delta \geq \mu^\top \Delta - (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}\bar{\varepsilon} > 0 \quad \nu_\varepsilon - a.s.$$
$$\Rightarrow \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{1}{q^\top \Delta}\right] \leq \frac{1}{\mu^\top \Delta - (\Delta^\top \Sigma \Delta)^{\frac{1}{2}}\bar{\varepsilon}}$$

which proves that $\|\zeta\| \leq \bar{\varepsilon}$ by virtue of equation (3.68). We have shown that $\bar{\pi} \gg 0$ and therefore $\mathbb{I}(\mu, \Sigma) \subset \mathbb{R}^M_{++}$. The fact that $\mathbb{I}(\mu, \Sigma)$ is an open set follows immediately from the strictness of the defining inequalities and the continuity of the mappings $\pi \longmapsto \Sigma^{-1}(\mu - \pi)$ and $\pi \longmapsto \mathbb{E}_{\nu_{\mu,\Sigma}}\left[\frac{\pi^\top \Sigma^{-1}(\mu-\pi)}{q^\top \Sigma^{-1}(\mu-\pi)}\right]$ for $\pi \neq \mu$.

Next we show that the set $\mathbb{I}(\mu, \Sigma)$ is bounded. Let $P_m : \mathbb{R}^M \longrightarrow \mathbb{R}$ denote the projection mapping of $\mathbb{R}^M$ onto its mth coordinate space. For $\Delta \in S_1$ let $\overline{\pi}^{(m)}(\Delta) := \max\{\mu^{(m)}, \mu^{(m)} - \bar{v}(\Delta)P_m(\Sigma\Delta)\}$, and $\underline{\pi}^{(m)}(\Delta) := \min\{\mu^{(m)}, \mu^{(m)} - \bar{v}(\Delta)P_m(\Sigma\Delta\}$. Let $\pi \in \mathbb{I}(\mu, \Sigma)$ be arbitrary. Then, as shown above, there exists some $\Delta \in S_1$ such that $\pi^{(m)} = \mu^{(m)} - vP_m(\Sigma\Delta)$ where $0 < v < \bar{v}(\Delta)$ and, by definition, $\underline{\pi}^{(m)}(\Delta) < \pi^{(m)} < \overline{\pi}^{(m)}(\Delta)$. Let $\underline{\pi}^{(m)} := \min_\Delta\{\underline{\pi}^{(m)}(\Delta) | \Delta \in \mathbb{R}^M_+\}$ and $\overline{\pi}^{(m)} := \max_\Delta\{\overline{\pi}^{(m)}(\Delta) | \Delta \in \mathbb{R}^M_+\}$. Observe that both values are well-defined due to the compactness of the set $\{\Delta \in \mathbb{R}^M_+ | \|\Delta\| = 1\}$ and the continuity of the functions $\bar{v}(\cdot)$, $P_m(\cdot)$ and of the map $\Delta \longmapsto \Sigma\Delta$. It follows that for all $\pi = (\pi^{(1)}, \ldots, \pi^{(M)})^\top \in \mathbb{I}(\mu, \Sigma)$ we have $\underline{\pi}^{(m)} < \pi^{(m)} < \overline{\pi}^{(m)}$. Hence the vectors $\overline{\pi} := (\overline{\pi}^{(1)}, \ldots, \overline{\pi}^{(M)})^\top$ and $\underline{\pi} := (\underline{\pi}^{(1)}, \ldots, \underline{\pi}^{(M)})^\top$ define an upper and a lower bound of the set $\mathbb{I}(\mu, \Sigma)$.

Finally, we show that μ is a limit point. Let $\mathcal{O} \subset \mathbb{R}^M$ be an open set containing μ. Let $\Delta \in S_1$ be arbitrary and set $\pi = \mu - v\Sigma\Delta$. It is clear that choosing $0 < v < \bar{v}(\Delta)$ implies that $\pi \in \mathbb{I}(\mu, \Sigma)$. On the other hand, choosing v sufficiently small, we will also have $\pi \in \mathcal{O}$. Since $\mathcal{O}$ was arbitrary, this proves that any open set containing μ also contains a point of $\mathbb{I}(\mu, \Sigma)$. We have shown that μ is a limit point of $\mathbb{I}(\mu, \Sigma)$. ■

3.A.6 Proof of Lemma 3.4.1

Let beliefs $(\mu_t, \Sigma_t) \in \mathfrak{B}$ be arbitrary but fixed. Define from equation (3.41) $\bar{v}_t := \bar{v}(\mu_t, \Sigma_t)$ and let $\bar{\lambda}_t := \frac{\sqrt{\bar{x}^\top \Sigma_t \bar{x}}}{\mu_t^\top \bar{x} - \bar{v}_t \bar{x}^\top \Sigma_t \bar{x}}$. Note from (3.41) that $\mu_t^\top \bar{x} - \bar{v}_t \bar{x}^\top \Sigma_t \bar{x} > 0$ such that $\bar{\lambda}_t > 0$ is indeed well-defined. The first goal will be to show that the map $\lambda^\star(\cdot; \mu_t, \Sigma_t)$ defined in Theorem 3.3 satisfies the boundary conditions

$$\lim_{v \searrow 0} \lambda^\star(\mu_t - v\Sigma_t\bar{x}; \mu_t, \Sigma_t) = 0 \tag{3.69}$$

$$\lim_{v \nearrow \bar{v}_t} \lambda^\star(\mu_t - v\Sigma_t\bar{x}; \mu_t, \Sigma_t) = \bar{\lambda}_t. \tag{3.70}$$

Define the function $\hat{H} : [0, \bar{\lambda}_t] \times [0, \bar{v}_t] \longrightarrow \mathbb{R}$,

$$\hat{H}(\lambda, v) := \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \frac{v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} + \varepsilon}{1 + v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} \lambda + \lambda\,\varepsilon}\, \nu_\varepsilon(d\,\varepsilon). \tag{3.71}$$

Observe by Lemma 3.B.1 that $\hat{H}(\cdot, v) : [0, \bar{\lambda}_t] \longrightarrow \mathbb{R}$, $v \in [0, \bar{v}_t]$ and $\hat{H}(\lambda, \cdot) : [0, \bar{v}_t] \longrightarrow \mathbb{R}$, $\lambda \in [0, \bar{\lambda}_t]$ are both continuously differentiable. Hence the map $\hat{H}(\cdot)$ is continuously differentiable and the derivatives satisfy for all $(\lambda, v) \in [0, \bar{\lambda}_t] \times [0, \bar{v}_t]$

$$\frac{\partial \hat{H}(\lambda, v)}{\partial \lambda} = -\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \left(\frac{v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} + \varepsilon}{1 + v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} \lambda + \lambda\,\varepsilon} \right)^2 \nu_\varepsilon(d\,\varepsilon) < 0 \tag{3.72}$$

$$\frac{\partial \hat{H}(\lambda, v)}{\partial v} = \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}}}{\left(1 + v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} \lambda + \lambda\,\varepsilon\right)^2} \nu_\varepsilon(d\,\varepsilon) > 0. \tag{3.73}$$

For each $v \in [0, \bar{v}_t]$ define the value $\hat{\lambda} \in [0, \bar{\lambda}_t]$ implicitly by the condition $\hat{H}(\hat{\lambda}, v) = 0$. We show that this value is indeed well-defined and can be described by a continuous function $\hat{\lambda} : [0, \bar{v}_t] \longrightarrow [0, \bar{\lambda}_t]$. Due to the strict monotonicity of the function $\hat{H}(\cdot, v)$ implied by (3.72), it suffices to show for each $v \in [0, \bar{v}_t]$ the existence of a value $\hat{\lambda}$ that satisfies $\hat{H}(\hat{\lambda}, v) = 0$.

Note from Theorem 3.3 (i) and equation (3.71) that for all interior $(\lambda, v) \in]0, \bar{\lambda}_t[\times]0, \bar{v}_t[$ we have $\hat{H}(\lambda, v) = \frac{\partial H}{\partial \lambda}(\lambda; \mu - v\Sigma\bar{x}, \mu, \Sigma)$. For all $v \in]0, \bar{v}_t[$ define $\hat{\lambda}(v) := \lambda^\star(\mu - v\Sigma\bar{x}; \mu, \Sigma)$. Then we have $\hat{H}(\hat{\lambda}(v), v) = \frac{\partial H}{\partial \lambda}(\lambda^\star(\mu - v\Sigma\bar{x}; \mu, \Sigma); \mu - v\Sigma\bar{x}, \mu, \Sigma) = 0$ showing that for all interior $v \in]0, \bar{v}_t[$ the function $\hat{\lambda}(\cdot)$ is well-defined. We claim that for the boundary cases $\hat{\lambda}(0) = 0$ and $\hat{\lambda}(\bar{v}_t) = \bar{\lambda}_t$. Again it suffices to show that $\hat{H}(0, 0) = 0$ and $\hat{H}(\bar{\lambda}_t, \bar{v}_t) = 0$, which can be done by direct substitution into (3.71). This proves that $\hat{\lambda}$ is indeed a well-defined function the continuity of which follows from the continuity of the function $\hat{H}(\cdot)$. Furthermore, making use of the implicit function theorem, the function $\hat{\lambda}$ is even differentiable and the derivative satisfies

$$\frac{d\hat{\lambda}(v)}{dv} = -\left.\frac{\frac{\partial \hat{H}(\lambda,v)}{\partial v}}{\frac{\partial \hat{H}(\lambda,v)}{\partial \lambda}}\right|_{\lambda=\hat{\lambda}(v)} \tag{3.74}$$

$$= \frac{\left.\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}}}{\left(1 + v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} \lambda + \lambda\,\varepsilon\right)^2} \nu_\varepsilon(d\,\varepsilon)\right|_{\lambda=\hat{\lambda}(v)}}{\left.\int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \left(\frac{v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} + \varepsilon}{1 + v\,(\bar{x}^\top \Sigma_t \bar{x})^{\frac{1}{2}} \lambda + \lambda\,\varepsilon} \right)^2 \nu_\varepsilon(d\,\varepsilon)\right|_{\lambda=\hat{\lambda}(v)}} > 0.$$

Let $(v_n)_n$ be a sequence in $]0, \bar{v}_t[$ with $\lim_{n\to\infty} v_n = 0$. Then, by continuity of $\hat{\lambda}(\cdot)$, one has $\lim_{n\to\infty} \hat{\lambda}(v_n) = \hat{\lambda}(0) = 0$. On the other hand, as shown above, for each $n \geq 1$ one has $\hat{\lambda}(v_n) = \lambda^\star(\mu - v_n \Sigma \bar{x}; \mu, \Sigma)$ which proves that $\lim_{n\to\infty} \lambda(\mu - v_n \Sigma \bar{x}, \mu, \Sigma) = 0$. Since $(v_n)_n$ was arbitrary, this proves the claim in (3.69). The other limit in (3.70) can be verified analogously.

Now consider the existence of a solution v_t to (3.42). Defining for each $v \in]0, \bar{v}_t[$ the function

$$\varrho(v; \mu_t, \Sigma_t, \hat{m}_t) := 1 - \frac{\lambda^\star(\mu_t - v\Sigma_t\,\bar{x}; \mu_t, \Sigma_t)}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}}\left(\mu_t^\top \bar{x} - v\bar{x}^\top \Sigma_t\,\bar{x} + \hat{m}_t\right) \tag{3.75}$$

we see from (3.69) and (3.70) and the definition of $\bar{\lambda}_t$ that

$$\lim_{v \searrow 0} \varrho(v; \mu_t, \Sigma_t, \hat{m}_t) = 1 > 0$$

and

$$\begin{aligned}\lim_{v \nearrow \bar{v}_t} \varrho(v; \mu_t, \Sigma_t, \hat{m}_t) &= 1 - \frac{\bar{\lambda}_t}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}}\left(\mu_t^\top \bar{x} - \bar{v}_t\bar{x}^\top \Sigma_t\,\bar{x} + \hat{m}_t\right)\\ &= 1 - \frac{\mu_t^\top \bar{x} - \bar{v}_t\bar{x}^\top \Sigma_t\,\bar{x} + \hat{m}_t}{\mu_t^\top \bar{x} - \bar{v}_t\bar{x}^\top \Sigma_t\,\bar{x}} < 0.\end{aligned}$$

By continuity of the function $\varrho(\cdot)$ this proves the existence of an interior solution $v_t \in]0, \bar{v}_t[$ to (3.42). To show that the solution is unique we prove that $\varrho(\cdot; \mu_t, \Sigma_t, \hat{m}_t)$ is strictly monotonically decreasing. Utilizing that, as shown previously,

$$\varrho(v; \mu_t, \Sigma_t, \hat{m}_t) = 1 - \frac{\hat{\lambda}(v)}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}}\left(\mu_t^\top \bar{x} - v\bar{x}^\top \Sigma_t\,\bar{x} + \hat{m}_t\right)$$

the first derivative reads:

$$\begin{aligned}\frac{\partial \varrho(v; \mu_t, \Sigma_t, \hat{m}_t)}{\partial v} &= -\frac{d\hat{\lambda}(v)}{dv}\frac{\hat{m}_t}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}} \\ &\quad -\left(\frac{d\hat{\lambda}(v)}{dv}\frac{\left(\mu_t^\top \bar{x} - v\bar{x}^\top \Sigma_t\,\bar{x}\right)}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}} - \hat{\lambda}(v)\left(\bar{x}^\top \Sigma_t\,\bar{x}\right)^{\frac{1}{2}}\right).\end{aligned} \tag{3.76}$$

Recalling from (3.74) that $\frac{d\hat{\lambda}(v)}{dv} > 0$ we see that the first term in (3.76) will always be negative. We therefore show that

$$\frac{d\hat{\lambda}(v)}{dv}\frac{\left(\mu_t^\top \bar{x} - v\bar{x}^\top \Sigma_t\,\bar{x}\right)}{(\bar{x}^\top \Sigma_t\,\bar{x})^{\frac{1}{2}}} - \hat{\lambda}(v)\left(\bar{x}^\top \Sigma_t\,\bar{x}\right)^{\frac{1}{2}} > 0.$$

Utilizing the definition of the derivative (3.74) and carrying out some tedious although trivial transformations yields finally the desired result

$$\frac{d\hat{\lambda}(v)}{dv}\frac{\left(\mu_t^\top\bar{x}-v\bar{x}^\top\Sigma_t\,\bar{x}\right)}{(\bar{x}^\top\Sigma_t\,\bar{x})^{\frac{1}{2}}}-\hat{\lambda}(v)\left(\bar{x}^\top\Sigma_t\,\bar{x}\right)^{\frac{1}{2}}$$

$$=\frac{\left.\int_{[-\bar{\varepsilon},\bar{\varepsilon}]}\frac{\mu^\top\bar{x}+(\bar{x}^\top\Sigma\bar{x})^{\frac{1}{2}}\,\varepsilon}{\left(1+v\,(\bar{x}^\top\Sigma\bar{x})^{\frac{1}{2}}\lambda+\lambda\,\varepsilon\right)^2}\,\nu_\varepsilon(d\,\varepsilon)\right|_{\lambda=\hat{\lambda}(v)}}{\left.\int_{[-\bar{\varepsilon},\bar{\varepsilon}]}\left(\frac{v\,(\bar{x}^\top\Sigma_t\bar{x})^{\frac{1}{2}}+\varepsilon}{(1+v\,(\bar{x}^\top\Sigma_t\bar{x})^{\frac{1}{2}}\lambda+\lambda\,\varepsilon}\right)^2\,\nu_\varepsilon(d\,\varepsilon)\right|_{\lambda=\hat{\lambda}(v)}}>0.$$

Note in this regard that the numerator will be positive since, by feasibility of beliefs and Lemma 3.B.3 (ii) $\mu_t^\top\bar{x}+(\bar{x}^\top\Sigma_t\bar{x})^{\frac{1}{2}}\,e>0$ for all $e\in[-\bar{\varepsilon},\bar{\varepsilon}]$. This proves that $\frac{\partial\varrho(v;\mu_t,\Sigma_t,\hat{m}_t)}{\partial v}<0$ for all $v\in[0,\bar{v}_t]$ and, therefore, the uniqueness of v_t. This implies that one may legitimately define the function $v(\cdot)$ as claimed in the Lemma. The partial derivatives of $\varrho(\cdot)$ with respect to $\mu_t^{(m)}$ and $\hat{m}_t$ are given by

$$\frac{\partial\varrho(v;\mu_t,\Sigma_t,\hat{m}_t)}{\partial\mu_t^{(m)}}=-\frac{\hat{\lambda}(v)}{(\bar{x}^\top\Sigma_t\bar{x})^{\frac{1}{2}}}\bar{x}^{(m)}<0$$

$$\frac{\partial\varrho(v;\mu_t,\Sigma_t,\hat{m}_t)}{\partial\hat{m}_t}=-\frac{\hat{\lambda}(v)}{(\bar{x}^\top\Sigma_t\bar{x})^{\frac{1}{2}}}<0.$$

Applying the implicit function theorem gives

$$\frac{\partial v(\mu_t,\Sigma_t,\hat{m}_t)}{\partial\mu_t^{(m)}}=-\left.\frac{\frac{\partial\varrho(v_t;\mu_t,\Sigma_t,\hat{m}_t)}{\partial\mu_t^{(m)}}}{\frac{\partial\varrho(v_t;\mu_t,\Sigma_t,\hat{m}_t)}{\partial v_t}}\right|_{v_t=v(\mu_t,\Sigma_t,\hat{m}_t)}<0$$

$$\frac{\partial v(\mu_t,\Sigma_t,\hat{m}_t)}{\partial\hat{m}_t}=-\left.\frac{\frac{\partial\varrho(v_t;\mu_t,\Sigma_t,\hat{m}_t)}{\partial\hat{m}_t}}{\frac{\partial\varrho(v_t;\mu_t,\Sigma_t,\hat{m}_t)}{\partial v_t}}\right|_{v_t=v(\mu_t,\Sigma_t,\hat{m}_t)}<0$$

showing that the function $v(\cdot)$ is strictly decreasing in the respective arguments as claimed. ■

3.B Technical Lemmas

Lemma 3.B.1 *Let Ω be a topological space and $(\Omega, \mathcal{B}(\Omega), \nu)$ be a probability space with ν being supported on the compact subset $\bar{\Omega} \in \mathcal{B}(\Omega)$, i.e., $\nu(\bar{\Omega}) = 1$. Let $\mathbb{I} \subset \mathbb{R}$ be an uncountable compact interval and let the map $h : \mathbb{I} \times \Omega \longrightarrow \mathbb{R}$ be continuous and differentiable with respect to i for all $\omega \in \Omega$ with continuous derivative $h' := \frac{dh}{di} : \mathbb{I} \times \Omega \longrightarrow \mathbb{R}$. Then the map $H : \mathbb{I} \longrightarrow \mathbb{R}$*

$$H(i) := \int_{\Omega} h(i, \omega)\, \nu(d\omega)$$

is continuously differentiable and the derivative $H' : \mathbb{I} \longrightarrow \mathbb{R}$ satisfies

$$H'(i) = \int_{\Omega} h'(i, \omega)\, \nu(d\omega). \tag{3.77}$$

Proof. The proof is an application of Lemma 16.2, p. 102 in [6]. We check that the requirements (a)–(c) of this lemma are satisfied. Since ν is supported on $\bar{\Omega}$ we can restrict attention to this subset.
(a). For each $i \in I$ the map $h(i, \cdot) : \bar{\Omega} \longrightarrow \mathbb{R}$ is continuous and hence measurable. Furthermore, $\bar{\Omega}$ being compact implies that $h(i, \cdot)$ is bounded and therefore ν–integrable.
(b). By assumption $h(\cdot, \omega) : \mathbb{I} \longrightarrow \mathbb{R}$ is continuously differentiable for each $\omega \in \Omega$.
(c). The set $\mathbb{I}$ being compact and the derivative $h' : \mathbb{I} \times \bar{\Omega} \longrightarrow \mathbb{R}$ being continuous implies that h' is bounded from above by the value $\bar{h}' := \max_{(i,\omega)}\{h'(i, \omega) | (i, \omega) \in \mathbb{I} \times \bar{\Omega}\}$ and from below by the value $\bar{h}' := \min_{(i,\omega)}\{h'(i, \omega) | (i, \omega) \in \mathbb{I} \times \bar{\Omega}\}$.

Applying Lemma 16.2 in [6] this shows that H is differentiable and the derivative H' satisfies (3.77). The continuity of H' then follows from Lemma 2.B.2. ■

Lemma 3.B.2 *Let Ω be a topological space and $(\Omega, \mathcal{B}(\Omega), \nu)$ be a probability space where ν is supported on the compact subset $\bar{\Omega} \in \mathcal{B}(\Omega)$, i.e., $\nu(\bar{\Omega}) = 1$. Let $\Theta \subset \mathbb{R}^N_{++}$ be a bounded open set and $h : \Theta \times \Omega \longrightarrow \mathbb{R}$ be a continuous function which is differentiable with respect to each component of θ with continuous partial derivatives $h_n := \frac{\partial h}{\partial \theta_n} : \Theta \times \Omega \longrightarrow \mathbb{R}$, $n = 1, \ldots, N$. Suppose that h and each h_n can be continuously extended to the closure $\bar{\Theta} \supset \Theta$. Then the map $H : \Theta \longrightarrow \mathbb{R}$*

$$H(\theta) := \int_{\Omega} h(\theta, \omega)\, \nu(d\omega)$$

is continuously differentiable and the gradient reads

$$D_\theta H(\theta) := \begin{pmatrix} \frac{\partial H(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial H(\theta)}{\partial \theta_N} \end{pmatrix} = \int_\Omega D_\theta h(\theta, \omega)\, \nu(d\omega)$$

Proof. Noting that the closure $\bar{\Theta}$ is a compact set it can easily be verified that the requirements (a)–(c) of Korollar 16.3, p. 103 in [6] are satisfied using the same arguments as in the proof of Lemma 3.B.1. ■

Lemma 3.B.3 *Let Assumption 3.2.1 be satisfied and let the pair of beliefs $(\mu, \Sigma) \in \mathfrak{B}$ be feasible. Then for any vector $\theta \in \mathbb{R}^M_{++}$ we have*

(i) $q^\top \theta \overset{d}{=} \mu^\top \theta + (\theta^\top \Sigma \theta)^{\frac{1}{2}} \varepsilon$
(ii) $\mu^\top \theta + (\theta^\top \Sigma \theta)^{\frac{1}{2}} e > 0$ *for all* $e \in [-\bar{\varepsilon}, \bar{\varepsilon}]$.

Proof. The first claim is obvious from Lemma 3.A.3 and Proposition 3.A.1. Feasibility now implies that $q \gg 0$ ν_ε–a.s. and, therefore, $q^\top \theta \overset{d}{=} \mu^\top \theta + (\theta^\top \Sigma \theta)^{\frac{1}{2}} \varepsilon > 0$ ν_ε–a.s. Since ε is supported on the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$ and the map $e \longmapsto \mu^\top \theta + (\theta^\top \Sigma \theta)^{\frac{1}{2}} e$ is continuous this proves the second claim. ■

Part II

The Simulation Study

4

Pension Systems in the Presence of a Stationary Population

The parameterized framework developed in the previous chapter offers a rich set of possibilities to study the role of a pension system as well as the issue of demographic change. In this regard, the explicit modeling approach adopted in this work entails two distinctive advantages. Firstly, it permits the study to be placed within the theory of random dynamical systems. This framework provides powerful mathematical concepts to describe the long run and transient behavior of dynamical systems which are subjected to random perturbations. Secondly, it sets the stage for a systematic exploration of the model's dynamic features with the help of numerical simulations. The latter possibility appears particularly desirable due to the complexity of the model.

A comprehensive numerical simulation study is carried out in the following two chapters which constitute the second part of this dissertation. The present chapter restricts attention to the special case with a stationary population where the sizes of all generations are identical and constant over time. Proceeding in this fashion allows one to isolate those effects which are entirely due to the presence of a pension system while abstracting from any demographic effects. Taking demographic change as a temporary phenomenon, the stationary situation may be viewed as a scenario to which the population adjusts in the long run. From this perspective the present chapter is concerned with the *long-run efficiency* of pension systems and their impact on the *long-run behavior* of real and financial markets.

The structure of the chapter is as follows: Section 4.1 embeds the model into the framework of random dynamical systems theory. Section 4.2 motivates the parametrization of the simulation model and establishes a benchmark for its dynamic behavior. The impact of alternative pension systems characterized by different contribution rates on the long-run evolution of the economy and on the welfare of consumers are studied in Sections 4.3 and 4.4, respectively. Section 4.5 presents a sensitivity analysis showing that all results are robust against parameter changes. In a final step, the consequences of a gradual reduction of the public pension system are studied in Section 4.6. The required technical concepts from the theory of random dynamical systems may be found in the Appendix 4.A.

4.1 Dynamics of the Model

The present chapter seeks to analyze how the presence of a public pension system affects the dynamic behavior of real and financial variables and their statistical properties as well as the welfare of consumers. As before, we identify the pension system at time t with the contribution rate τ_t. The following study considers alternative contribution rates τ_t assumed to be taken from some subinterval $[0, \bar{\tau}] \subset [0, 1]$ where the bound $\bar{\tau} < 1$ is determined below. In what follows it will mostly be assumed that the pension system is static in the sense that $\tau_t \equiv \tau$ for all t. The case where contributions are adjusted dynamically over time will be studied in Section 4.6 and in the following chapter.

As the following study is based on the theory of random dynamical systems, the first step is to embed the model developed in the previous chapter into this framework. The underlying theory is comprehensively presented in [4]. A formal description of the technical concepts employed here is contained in Appendix 4.A. Formally, we seek to describe the evolution of the model as a so-called random difference equation. For this purpose, define the state vector

$$\xi_t := (N_t, K_t, \mu_{t-1}, \mu_t, R_t, q_t, y_t, x_t) \tag{4.1}$$

where for each $t \geq 0$

$$\begin{aligned} N_t &:= (N_t^{(j)})_{j=0}^J \quad & K_t &:= (K_t^{(m)})_{m=1}^M \quad & \mu_t &:= (\mu_t^{(m)})_{m=1}^M \\ q_t &:= (q_t^{(m)})_{m=1}^M \quad & y_t &:= (y_t^{(j)})_{j=1}^J \quad & x_t &:= (x_t^{(j)})_{j=1}^J. \end{aligned} \tag{4.2}$$

As before, the lists in (4.2) denote the population vector, the firms' capital stocks, consumers' cum-dividend expectations, cum-dividend prices and the allocations of bonds and shares among consumers. In addition, the noise processes which enter the firms' production technologies (2.30) are summarized as

$$\eta_t := (\eta_t^{(m)})_{m=1}^M. \tag{4.3}$$

Based on (4.1) and (4.3) the following lemma provides a formal characterization of the forward-recursive structure of the model.

Lemma 4.1. *Let* $\tau \in [0, \bar{\tau}]$ *be constant and define* ξ_t *and* η_t *as in (4.1) and (4.3). Then there exists a continuous map* ϕ_τ *defined on* $\Xi \times \prod_{m=1}^{M} [0, \eta_{max}^{(m)}]$ *where* $\Xi \subset \mathbb{R}_{++}^{J+1} \times \mathbb{R}_{++}^{M} \times \mathbb{R}_{++}^{M} \times \mathbb{R}_{++}^{M} \times \mathbb{R}_{++} \times \mathbb{R}_{++}^{M} \times \mathbb{R}^{J} \times \mathbb{R}_{+}^{J \cdot M}$ *which determines the state of the model at time* t *as*

$$\xi_t = \phi_\tau\big(\xi_{t-1}; \eta_t\big). \tag{4.4}$$

Proof. It is straightforward to show by following the steps 1 - 5 from Section 3.5 that the vectors ξ_{t-1} and $\eta_t \in \prod_{m=1}^{M} [0, \eta_{max}^{(m)}]$ uniquely determine the vector ξ_t. The continuity of all involved functions imply the continuity of the map $\phi_\tau(\cdot)$. ■

The vector $\xi_t \in \Xi$ represents the state of the model at time t and will be called the *state vector* while the set Ξ will be called the *state space*. Note that all other variables of the model at time t not included in ξ_t can be determined from the pair (ξ_{t-1}, η_t). It follows from (4.4) that given some initial state ξ_0 the evolution of the model is obtained by successively iterating the map ϕ_τ defined in (4.4).[1] Clearly, this requires a description of the probabilistic nature of the exogenous noise process $\{\eta_t\}_t$ which enters (4.4). In accordance with Assumption 2.4.1 (ii) we specify the properties of this process as follows.

Assumption 4.1.1 *The noise process* $\{\eta_t\}_t$ *with* η_t *being defined as in (4.3) consists of i.i.d. random variables defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *which is adapted to some filtration* $\{\mathcal{F}_t\}_t$*. Each* η_t *is distributed according to the measure* ν_η *supported on* $\prod_{m=1}^{M} [0, \eta_{max}^{(m)}]$.

Note that the independence assumption only holds between distinct periods such that the production shocks of distinct firms may well be correlated in any one period. Also note from Section 3.5 and the sequential structure of the model that the price process $\{p_t\}_{t \geq 0}$ and the dividend processes $\{d_t\}_{t \geq 0}$ will be adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Hence Assumption 4.1.1 is consistent with our setup in Section 2.2.

[1] For (4.4) to define a dynamical system it is required that the set Ξ can be chosen forward-invariant, i.e., $\xi \in \Xi \Rightarrow \phi_\tau\big(\xi; \eta\big) \in \Xi$ for all $\eta \in \prod_{m=1}^{M} [0, \eta_{max}^{(m)}]$. In this regard, several constraints including no bankruptcy on the part of consumers (see the discussion at the end of Section 2.3) and non-negative dividend payments of firms (Section 2.4) must be respected in the definition of Ξ. For the following derivations we simply assume the forward invariance of Ξ. The justification will be derived from the numerical simulations of the model.

Based on the random difference equation (4.4) it follows that given some initial state $\xi_0 \in \Xi$ the evolution of the model is obtained by successively iterating the map ϕ_τ. Hence, for each $\tilde{\omega} \in \Omega$ defining the path of the noise process the state of the model at time t is determined by some map Φ_τ (defined in equation (4.14) in the appendix) such that

$$\xi_t = \Phi_\tau(t, \tilde{\omega}, \xi_0), \quad t \geq 0. \tag{4.5}$$

Equation (4.5) implies that each $\tau \in [0, \bar{\tau}]$ defines a different stochastic process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t\geq 0}$ describing the stochastic evolution of the state vector (4.1) for a given initial state ξ_0. To describe the properties of this process we assume that for each $\tau \in [0, \bar{\tau}]$ there exists a unique stable random fixed point which governs the long run behavior of the system. The concept of a random fixed point is the analogue of a deterministic fixed point for dynamical systems which are perturbed by random noise. A definition is given in Appendix 4.A of this chapter. Formally, a random fixed point is composed of a random variable $\xi^\star_\tau : \Omega \longrightarrow \Sigma$ and a map $\vartheta : \Omega \longrightarrow \Omega$ called the left shift on Ω. The composition of both ingredients defines a stationary and ergodic stochastic process of the form $\{\xi^\star_\tau \circ \vartheta^t\}_{t\geq 0}$.

In the stable case, the behavior of the process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t\geq 0}$ is eventually indistinguishable from the behavior of the random fixed point $\{\xi^\star_\tau \circ \vartheta^t\}_{t\geq 0}$ and independent of the initial state ξ_0. An illustration of this property is provided in Section 4.3 (cf. Figure 4.7). It follows that the long run behavior of the model depending on τ can be analyzed by comparing the associated random fixed points. In this regard, the property of ergodicity implies that the statistical properties of the random fixed point (moments, frequencies, etc.) may be obtained from time averages along the observed sample paths of the system. Formally, if $\mathbb{E}\left[\|\xi^\star_\tau\|\right] < \infty$

$$\mathbb{E}\left[\xi^\star_\tau\right] = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \Phi_\tau(t, \tilde{\omega}, \xi_0) \quad \mathbb{P} - a.s. \tag{4.6}$$

for any ξ_0 taken from some suitable set $U(\tilde{\omega})$. More general, for any measurable map $\Psi : \Xi \longrightarrow \mathbb{R}^L$

$$\mathbb{E}\left[\Psi \circ \xi^\star_\tau\right] = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \Psi \circ \Phi_\tau(t, \tilde{\omega}, \xi_0) \quad \mathbb{P} - a.s. \tag{4.7}$$

whenever the expectation in (4.7) exists.

Since in the following numerical simulations only a finite number T of observations $\{\Phi_\tau(t,\tilde{\omega},\xi_0)\}_{t=0}^T$ is available, the subsequent study uses

$$\hat{\mathbb{E}}\left[\Psi \circ \xi_\tau^\star\right] := \frac{1}{T}\sum_{t=1}^{T} \Psi \circ \Phi_\tau(t,\tilde{\omega},\xi_0) \tag{4.8}$$

as an approximation for (4.7).

Due to the complexity of the model, a theoretical proof of the existence and stability of a unique random fixed point for each contribution rate τ turns out to be very difficult. As a consequence, we will establish these properties with the help of numerical simulations. In this regard, the subsequent simulations indicate that despite its complexity the model is very well-behaved in terms of its dynamic features. A deeper theoretical exploration of the numerical results is therefore left as a challenging exercise for future research.

Since we are interested in the long run properties of the model the remainder of this chapter restricts attention to the special case of a stationary population where the number of consumers in each generation remains constant over time. In particular, this holds true along the random fixed point (cf. Section 4.A). Mathematically, the scenario with a constant population amounts to assuming the existence of a steady state of the population law defined in (2.51). Observe that any such steady state will be of the form $N^\star = (\bar{N})_{j=0}^J$ where $\bar{N}$ satisfies the condition $\bar{N} = \mathcal{N}((\bar{N})_{j=0}^J)$. While the following sections simply assume the existence of such a steady state that can be varied exogenously, the following chapter will introduce a specific functional form of the map $\mathcal{N}$ which actually justifies this presumption.

4.2 The Simulation Model

The following simulation study[2] employs a standard set of parameter values which are justified on the grounds of empirical studies and which will be motivated first. Variations of these parameters are studied as part of a sensitivity analysis in Section 4.5.

Consider first the OLG structure which is specified as follows. By assumption the population is constant with each generation consisting of $N_t^{(j)} \equiv \bar{N} = 1000$ consumers. The parameters defining consumers'

[2] All simulations are carried out using the software package MACRODYN which is tailor-made for the numerical investigation of deterministic and stochastic dynamical systems. See [11] for a detailed introduction to the software.

life expectancy and retirement age are set to $J = 14$ and $j_L = 6$. This implies that in each period there are fifteen generations nine of which work while six are retired. Assuming as in [18] that economic life starts at the age of 20 and ends at the age of 80 years, each consumer thus lives for 60 years and one time unit in our simulations corresponds to four years.

Each consumer in working age supplies one unit of labor such that $\bar{L}^{(j)} = 1$ for all $j \in \{j_L, \ldots, J\}$. The choice of the subjective discount factor β describing consumers' preferences (2.9) is based on an empirical study by [46]. He reports a subjective discount rate of 0.011 corresponding to an annual discount factor of 1/1.011. Given a time unit of four years we therefore set $\beta = (1/1.011)^4 \approx 0.96$ in our simulations.

The production sector of the economy is assumed to consist of a single firm such that $M = 1$ and a single share is traded in the stock market. The firm index m will therefore be suppressed in the sequel. The consumers' subjective expectations for future asset prices are specified as follows. The random variable ε with probability distribution ν_ε introduced in Assumption 3.2.1 which defines the class of elliptical distributions under consideration is assumed to have a truncated standard normal distribution supported on the symmetric interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$. The corresponding density function reads

$$g(z) := \frac{1}{c_{\bar{\varepsilon}} \cdot \sqrt{2\pi}} \cdot \mathbf{1}_{[-\bar{\varepsilon},\bar{\varepsilon}]}(z) \cdot \exp\Big\{-\frac{1}{2} \cdot z^2\Big\}, \quad z \in \mathbb{R}$$

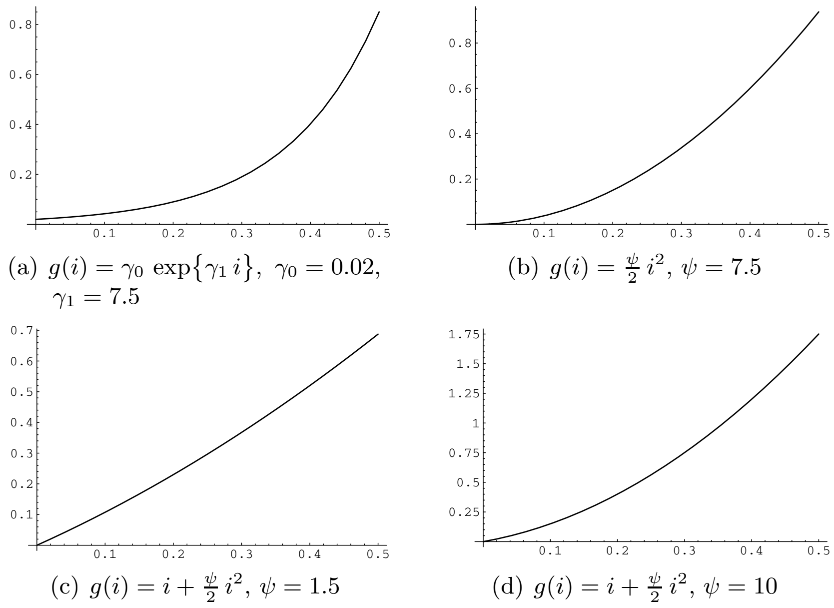
where

$$c_{\bar{\varepsilon}} := \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{1}{\sqrt{2\pi}} \cdot \exp\Big\{-\frac{1}{2} \cdot z^2\Big\}\, d\,z > 0.$$

Observe that ε is symmetrically distributed around zero ($\varepsilon \overset{d}{=} -\varepsilon$) and hence spherical (see Chapter 3, Appendix 3.A.3). In the sequel we use a value $\bar{\varepsilon} = 0.92$ implying that the perceived variance of cum-dividend prices is constant over time and equal to $\mathbb{V}_{\nu_q}[q_t] = \Sigma^2/4$. The dispersion parameter Σ (see Section 3.2) is set to $\Sigma = 0.97$ which matches the unconditional variance of the (stationary) cum-dividend price process for the benchmark case where $\tau = 0.1$ studied below. The parameter ϱ describing the adaptive updating of the subjective mean μ_t over time is set to $\varrho = 0.5$.

The firm's production and capital adjustment technologies defined in equations (3.25) and (3.26) are specified as follows. The elasticity of labor α is set to $\alpha = 0.66$. This is justified by most empirical studies which typically suggest a value of $\alpha \in]0.6, 0.7[$. Total factor productivity κ serves mainly as a scaling parameter and is set to $\kappa = 2.5$. The average rate of depreciation per year for the German economy between 1960 and 1990 is reported by [18] to equal ≈ 0.053. With our four years time unit we therefore set $\delta = 1 - (1 - 0.053)^4 \approx 0.2$.

While the theory of adjustment costs is widely used in the literature, there are many authors who suggest that the adjustment cost function g, as defined in (2.32) respectively (3.26), is of the quadratic form $g(i) = \frac{\psi}{2} \cdot i^2$ (cf. [2]) or $g(i) = i + \frac{\psi}{2} \cdot i^2$ (cf. [3, 19]) for some parameter $\psi > 0$. For our purposes the functional form (3.26) turned out to be superior in order to obtain closed form solutions to the model. To justify its use Figure 4.1 compares the graph of the function introduced in (3.26) with the quadratic functional forms described above. The parameters of our function have been set to $\gamma_0 = 0.02$ and $\gamma_1 = 7.5$ corresponding to the values used in the subsequent simulations. The parameter ψ of the quadratic specifications has been set to the empirical estimate $\psi = 10$ reported by [3] and, alternatively, to $\psi = 1.5$ corresponding to the value used by [19]. Since the argument i represents the fraction of the capital stock that is replaced, it will typically take values within the unit interval and fluctuate around the depreciation rate $\delta = 0.2$. For this reason the functions are compared on the interval $i \in]0, 0.5[$.



(a) $g(i) = \gamma_0 \exp\{\gamma_1 i\}$, $\gamma_0 = 0.02$, $\gamma_1 = 7.5$

(b) $g(i) = \frac{\psi}{2} i^2$, $\psi = 7.5$

(c) $g(i) = i + \frac{\psi}{2} i^2$, $\psi = 1.5$

(d) $g(i) = i + \frac{\psi}{2} i^2$, $\psi = 10$

Fig. 4.1. Different adjustment cost functions

Figure 4.2 shows that the functional properties of the different specifications (in particular on the subinterval $[0.1, 0.3]$) do not differ too much which justifies our non-standard form. In fact, as [39, p. 49] states, the primary cause for using the quadratic adjustment cost function is its technical simplicity and not its justification on empirical grounds.

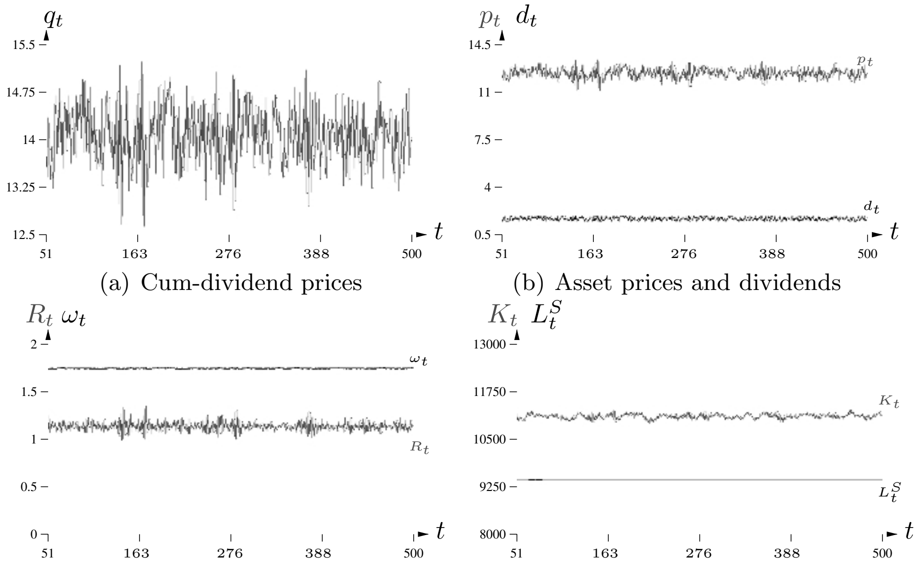
To complete the description of the parameter set, suppose that each random variable η_t of the stochastic process $\{\eta_t\}_t$ is uniformly distributed on the compact interval $[0, \eta_{max}]$ where we chose $\eta_{max} = 2000$. The initial values for capital stock, cum-dividend price and bond return are set to $K_0 = 11,500$, $q_0 = 14.5$ and $R_0 = 1.14$, however, as will be shown below, all results are independent of any initial choices of state variables. All parameter values used in the simulations are summarized in the following Table 4.1.

Table 4.1. Standard parameter set for the numerical simulations

Parameter	Value	Description
J	14	Life expectancy
j_L	6	Number of retired generations
$\bar{N}$	1000	Consumers per generation
$\bar{L}^{(j)}$	1	Individual labor supply
β	0.96	Subjective discount factor
$\bar{\varepsilon}$	0.92	Expectations parameter
Σ	0.97	Expectations parameter
ϱ	0.5	Expectations parameter
$\bar{x}$	5000	Total number of shares
α	0.66	Production parameter
κ	2.5	Production parameter
γ_0	0.02	Adjustment cost parameter
γ_1	7.5	Adjustment cost parameter
δ	0.2	Depreciation rate
η_{max}	2000	Upper bound for real noise
K_0	11,500	Initial capital stock
q_0	14.5	Initial cum-dividend price
R_0	1.14	Initial bond return

To establish a benchmark for the dynamic behavior of the model under the above parametrization the remainder of this section assumes a moderate contribution rate setting $\tau = 0.1$. The goal will be to study the evolution of certain variables and to reveal some of their correlation structure.

As a first step, Figure 4.2 displays time windows of the first 500 realizations of selected real and financial variables of the model. The financial variables are the ex- and cum-dividend prices p_t and q_t, the dividend payment d_t, and the bond-return R_t. The production part of the economy is represented by the capital stock K_t, the real wage ω_t and labor supply L_t^S. Clearly, the latter is constant due to the assumption of a stationary population. The table below the figures displays the corresponding sample moments of these variables. To avoid possible dependence on initial conditions, only the realizations from $t = 51$ until $t = 500$ are used in the calculations of these quantities. As will be shown in the following section, the behavior of the model is independent of initial conditions and convergence to a unique sample path is obtained within the first fifty periods as the initial state is varied.



(a) Cum-dividend prices (b) Asset prices and dividends (c) Real wage and bond return (d) Capital stock and labor force

	q_t	p_t	d_t	ω_t	R_t	K_t
Mean	14.06	12.39	1.668	1.745	1.135	11102.0
Variance	0.236	0.152	0.013	0.000	0.003	3756.7
Maximum	15.22	13.35	1.869	1.754	1.343	11265.9
Minimum	12.63	11.12	1.461	1.736	0.991	10931.4

Fig. 4.2. Stationary population and constant contribution rates; $\tau = 0.1$

All time series in Figure 4.2 fluctuate within a constant range around a time-invariant level. A first comparison of Figures 4.2(a) and 4.2(b) reveals that most fluctuations in cum-dividend prices are due to fluctuations in ex-dividend prices which are much larger than the volatility of the dividend process. In fact, as is seen from the sample moments, the variance in ex-dividend prices is almost ten times as large as the variance of the dividend process. This phenomenon is called *excess-volatility* and is a stylized fact typically observed in asset markets. In addition, the variance of cum-dividend prices exceeds the sum of variances of ex-dividend prices and dividends indicating – as one would expect – a positive correlation between asset prices and dividends.

Similar to the stock market variables, the time series of bond returns and real wages displayed in Figure 4.2(c) fluctuate about a constant mean where the level of bond returns is ≈ 1.135 and thus exceeds unity. If the bond is interpreted as a safe asset with interest rate $r_t := R_t - 1$, the latter will be positive on average corresponding to an annual level of $\approx 3\%$ which seems quite reasonable from an empirical point of view. One also observes that throughout bond returns fluctuate significantly more than real wages. In fact, fluctuations in real wages are almost negligible. However, as the corresponding sample moments reveal, the variance in bond returns is still much smaller as compared to fluctuations in asset prices and dividends. This phenomenon is consistent with our assumptions and provides a justification of the prediction behavior of consumers (and firms) who form point-predictions with respect to bond returns and non-capital incomes while treating asset prices and dividends as random variables (see Section 2.2).

In the present case with constant labor supply, fluctuations in the capital stock displayed in Figure 4.2(d) translate directly into fluctuations of aggregate output defined in (2.30) which, in addition, is perturbed by the exogenous noise process. Similar to the financial variables the time series of capital is stationary again. These observations indicate that the long-run outcome in real and financial markets is governed by stationary stochastic processes which is a necessary condition for the hypothesized existence of a stable random fixed point.

In order to reveal some of the correlation structure of the model's variables, define for any two lists $(x_t)_{t=1}^T$ and $(y_t)_{t=1}^T$ of observations the empirical correlation coefficient

$$\hat{\rho}(x_t, y_t) := \frac{1}{T} \sum_{t=1}^{T} \frac{(x_t - \bar{x}) \cdot (y_t - \bar{y})}{s_x \cdot s_y}.$$

Here $\bar{x}$, $\bar{y}$ and s_x, s_y denote the mean and the standard deviation of the samples, respectively. The value $\hat{\rho}(x_t, y_t) \in [-1, 1]$ captures the strength of co-movement between the observed data. A value $\hat{\rho}(x_t, y_t) = 1$ (-1) indicates a perfectly positive (negative) correlation while $\hat{\rho}(x_t, y_t) = 0$ suggests that the two variables are independent. In the sequel we will take ± 0.5 as a critical value to indicate a significant correlation.

Our next aim is to gain some insights into the direction of causality between real and financial markets. In this regard, the real sector at time t is represented by the capital stock K_t while the financial sector is identified with the cum-dividend price q_t and the bond return R_t. Figures 4.3 and 4.4 display the correlation structure of these variables at different lags and leads. The diagrams show the frequencies of hits in the plane of the respective variables where dark areas represent low hit frequencies while light areas imply a large frequency of visits in this part of the plane. All figures are based on $T = 25000$ iterations of the model. The corresponding correlation coefficient is displayed below each figure.

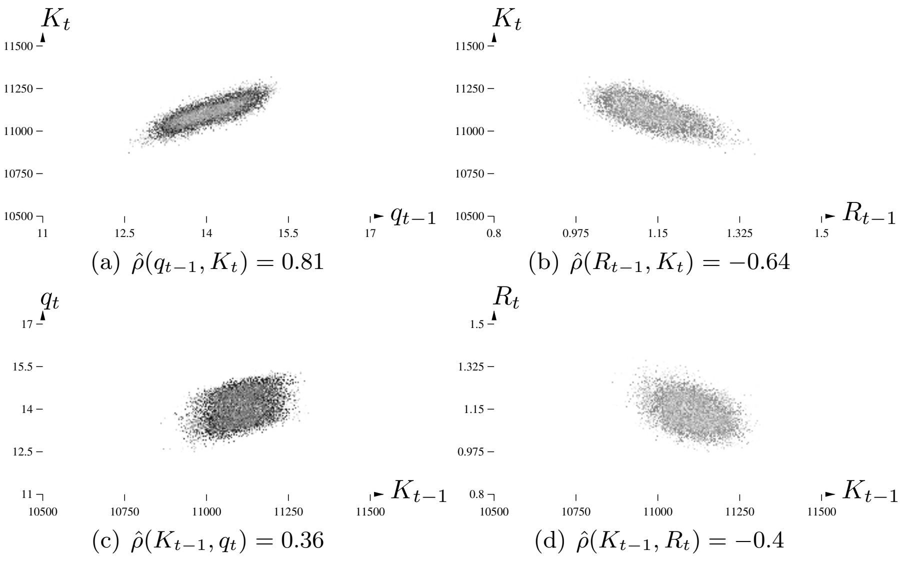
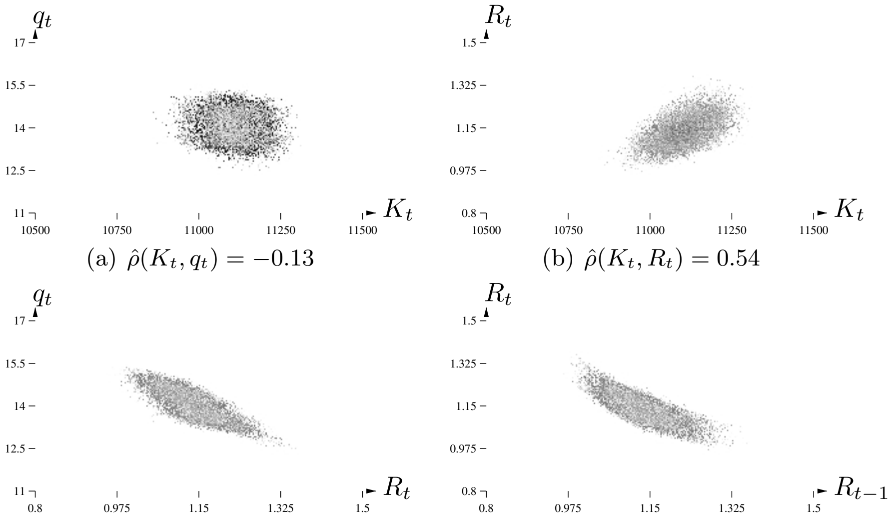


Fig. 4.3. Correlation patterns of state variables ; $\tau = 0.1$, $T = 25000$

The upper row in Figure 4.3 represents the impact of the financial sector at time $t-1$ on the real sector at time t. In this regard, Figure 4.3(a) suggests a highly positive correlation between the price q_{t-1} at time $t-1$ and the capital stock K_t at time t ($\hat{\rho}(q_{t-1}, K_t) > 0.8$). This indicates that a boom/crash in asset prices is transmitted to the real sector causing a corresponding increase/decrease in the capital stock. Furthermore, Figure 4.3(b) indicates a negative relation between the previous bond return R_{t-1} and the capital stock K_t reflecting the investment behavior of the firm. The corresponding correlation coefficient is slightly lower ($\hat{\rho}(R_{t-1}, K_t) = -0.64$) in absolute value but still significant. In summary, both figures convey the impression that the state of the financial sector at time $t-1$, notably of the stock market, exhibits a significant impact on the production sector of the economy at time t.

The second row in Figure 4.3 represents the other direction of causality showing the impact of the previous capital stock K_{t-1} on the financial variables q_t and R_t. One observes that the corresponding correlation coefficients maintain the same sign, however, the significance of the correlation (measured by the absolute value of the correlation coefficient) is much smaller. In particular, the positive correlation between the previous capital stock and current asset prices is less significant and more than twice as large, in the opposite direction.

Next consider the correlations between real and financial variables within period t depicted in Figure 4.4. In addition, the first order autocorrelations of the bond return series are displayed.



(a) $\hat{\rho}(K_t, q_t) = -0.13$ (b) $\hat{\rho}(K_t, R_t) = 0.54$

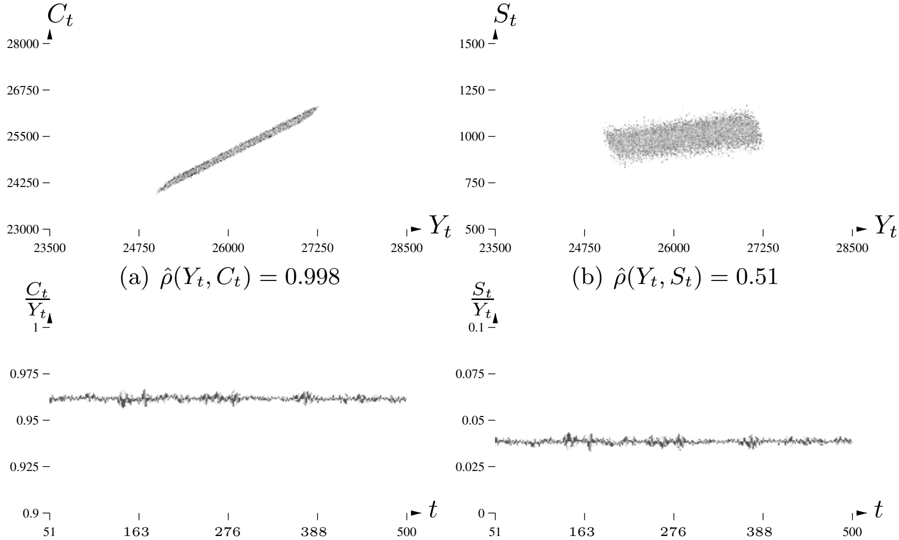
(c) $\hat{\rho}(R_t, q_t) = -0.81$ (d) $\hat{\rho}(R_{t-1}, R_t) = -0.79$

Fig. 4.4. Correlation patterns of state variables; $\tau = 0.1$, $T = 25000$

The correlation diagram in Figure 4.4(a) provides very little evidence for a significant impact of the current capital stock on the current asset price: the corresponding correlation coefficient is negative but very close to zero indicating that there is a negligible influence of the capital stock on the current stock price. In contrast to that, Figure 4.4(b) reveals that the correlation between current capital stock and bond return is larger and positive. In the lower row Figure 4.4(c) suggests a strongly negative correlation between asset prices and bond returns in any one period. This follows immediately from the asset price law (3.45) and is in line with empirical observations. In addition, Figure 4.4(d) indicates a large negative autocorrelation in bond returns. In fact, this may to some extent explain the strong correlation between K_t and R_t in Figure 4.4(b) which was not observed with q_t and K_t.

In summary, the properties derived from Figures 4.3 and 4.4 tend to support the view that the direction of causality is more from the financial to the real sector and that fluctuations on asset markets translate into fluctuations of capital and production while such a linkage is less significant in the opposite direction. Loosely speaking, it is more the financial sector which drives the real sector than vice versa. As a consequence, the subsequent simulation experiments will first exhibit the impact of parameter variations on financial markets. Clearly, none of the subsequent results depends on this interpretation.

We close our correlation analysis by investigating the correlation structure between some important macroeconomic variables. These are aggregated output Y_t, aggregated consumption C_t and aggregated savings S_t which have been defined in Section 2.5 already. The upper row in Figure 4.5 shows the corresponding correlation diagram, the lower row depicts time series' of the aggregated consumption and investment ratios, respectively. Figure 4.5(a) suggests a highly positive correlation between aggregated output and aggregated consumption with a correlation coefficient of 0.997. Moreover, the shape of the attractor supports the hypothesis of a linear aggregated consumption function. This is quite remarkable in so far as such a linear relation is often assumed in the macroeconomic literature without providing a corresponding sound microeconomic foundation (see, e.g., [34]). Likewise, Figure 4.5(b) suggests a positive correlation between aggregated savings and aggregated output which is less significant but still exceeds the critical 0.5 value. The corresponding consumption and investment ratios in the second row indicate that on average approximately 96% of output is consumed while the remaining 4% is used for the formation of new capital.

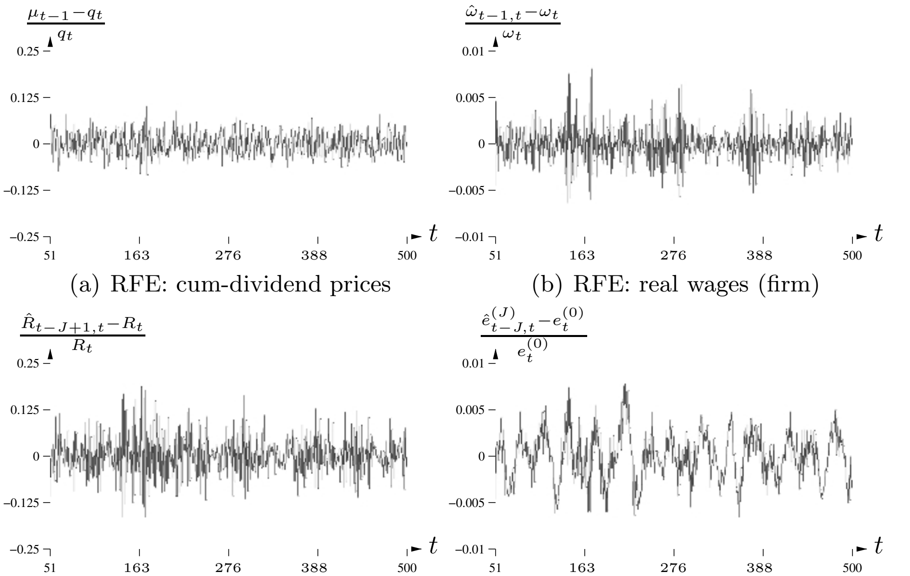


(a) $\hat{\rho}(Y_t, C_t) = 0.998$ (b) $\hat{\rho}(Y_t, S_t) = 0.51$

(c) Consumption ratio (d) Investment ratio

Fig. 4.5. Production, consumption and investment; $\tau = 0.1$, $T = 25000$

We close this section by analyzing the forecast errors made by consumers and firms. In this regard, recall from Section 3.4 that the rules according to which consumers and the firm predict the future evolution of the economy are relatively simple and, in particular, need not generate fully correct forecasts over time. Hence, it is necessary to justify their use by showing that the associated forecast errors remain sufficiently small such that agents can reasonably be expected to abide by these simple rules. This has been done in Figure 4.6 showing the evolution of relative forecast errors (RFE) made by consumers and firms over time. Figure 4.6(a) shows consumers' relative forecast errors for the next period's cum dividend price while Figure 4.6(b) displays the relative forecast error associated with the firm's real wage prediction. Furthermore, Figures 4.6(c) and 4.6(d) show the relative forecast errors pertaining to the consumers' prediction for future bond returns and non-capital income which are made $J-1$ and J periods in advance, respectively. These two forecasts correspond to the predictions made furthest in advance in terms of the time between the period when the forecast is made and when the actual realization is observed. It is reasonable to expect that forecast errors arising for predictions concerning shorter time periods would be smaller such that the two errors under scrutiny represent the extreme cases. All figures



(a) RFE: cum-dividend prices

(b) RFE: real wages (firm)

(c) RFE: bond returns

(d) RFE: non-capital income

	$\frac{\mu_{t-1}-q_t}{q_t}$	$\frac{\hat{\omega}_{t-1,t}-\omega_t}{\omega_t}$	$\frac{\hat{R}_{t-J+1,t}-R_t}{R_t}$	$\frac{\hat{e}^{(J)}_{t-J,t}-e^{(0)}_t}{e^{(0)}_t}$
Mean	0.000910	-0.000005	0.001880	-0.000046
Maximum	0.100565	0.008101	0.186926	0.007762
Minimum	-0.083243	-0.006348	-0.163741	-0.006564

Fig. 4.6. Relative forecast errors of consumers and the firm; $\tau = 0.1$

are supplemented by the corresponding sample moments displayed in the table below. One observes that, throughout, forecast errors fluctuate symmetrically about zero indicating that all forecasting rules are unbiased in the sense that they generate on average correct forecasts. The forecast errors made for real wages and non-capital income are in fact negligible since they are smaller than 1% for the entire sample. The largest relative errors are generated by consumers' bond return prediction made $J-1$ periods in advance. This is not too surprising due to the large time interval between the forecast and the actual realization and since our previous results have shown that bond returns fluctuate significantly more than real wages. Nevertheless, we find that these results and the relatively small forecast errors provide a convincing justification for the hypothesized prediction behavior.

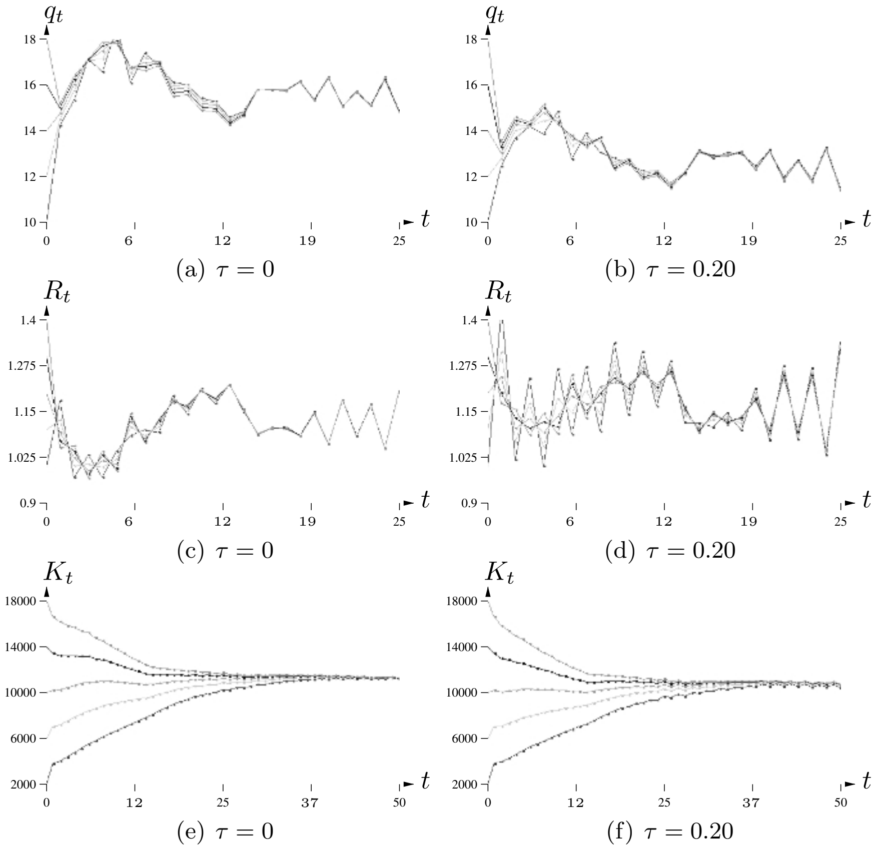
4.3 Impact of Pension Systems on Real and Financial Markets

Based on the previous parametrization the present section studies the impact of a public pension system on real and financial variables by hypothesizing different contribution rates. All simulation parameters are the same as in the previous section corresponding to the values listed in Table 4.1 while the values assumed for τ are restricted to lie in the interval $[0, 0.208]$. The reason for this will be explained below. For each experiment, the dynamic behavior of real and financial variables and their statistical properties will be analyzed.

To verify the applicability of the concepts introduced in Section 4.1, the first aim is to provide numerical evidence for the existence of a stable random fixed point. In the present case, this is achieved by demonstrating that the model's dynamic behavior is stationary and independent of the initial state $\xi_0 \in \Xi$ for all values of τ under consideration. Figure 4.7 depicts time series' of the cum-dividend price, bond return and the capital stock, respectively for five different initial values and the two cases $\tau = 0$ and $\tau = 0.2$. The realization of the noise process is the same in all five cases. Moreover, the qualitative result derived from the figure is exactly the same as the experiment is repeated with other values of τ studied in this section.

In all six cases depicted, the respective state variable converges to a unique stationary sample path, independently of its initial value. Moreover, in either case convergence occurs within the first fifty periods and is in fact even much faster for the financial variables q_t and R_t. It can be shown that the same result holds true for all other state variables of the model. These observations provide a convincing justification for our assumption of a stable random fixed point which governs the long-run behavior of the model in the sense that all sample paths pertaining to different initial conditions eventually behave like the corresponding path of the random fixed point.

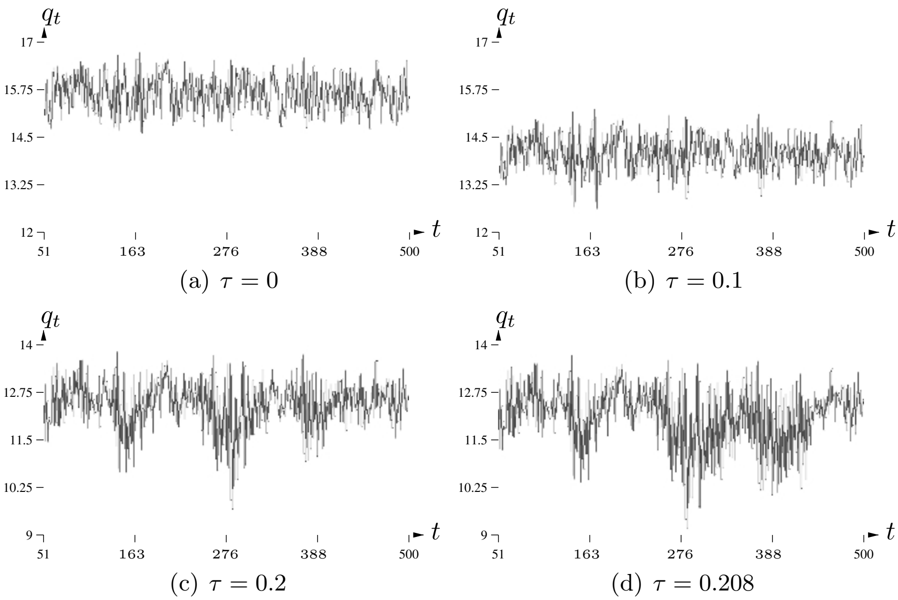
Building upon this result, consider next the impact of alternative contribution rates on the stock market. Figure 4.8 displays the evolution of cum-dividend asset prices depending on τ together with the corresponding empirical moments calculated from the sample $\{q_t\}_{t=51}^{500}$. As in the previous section, all time series depicted in Figures 4.8(a) – 4.8(d) fluctuate about a stationary level which decreases as the contribution rate τ is increased (note the different scaling of the vertical axes). This observation is confirmed by the corresponding sample means suggesting a strictly negative relationship between the level of asset prices and contributions to the pension system. Apart from that, the quali-



	Value 1	Value 2	Value 3	Value 4	Value 5
q_0	10	12	14	16	18
R_0	1.0	1.1	1.2	1.3	1.4
K_0	2000	6000	10000	14000	18000

Fig. 4.7. Convergence of selected state variables for different initial values

tative impression of the upper two figures where $\tau = 0$ and $\tau = 0.1$ is similar. In particular, fluctuations of the processes appear to be of the same magnitude, which is also confirmed by the corresponding sample variances. However, as the contribution rate is further increased to $\tau = 0.2$ and $\tau = 0.208$, the process becomes considerably more volatile. More importantly, several crashes are observable, e.g., during the time window $t \in \{270, 290\}$ showing a drastic decline in asset prices during these periods. In this regard, note that all time series are based on the

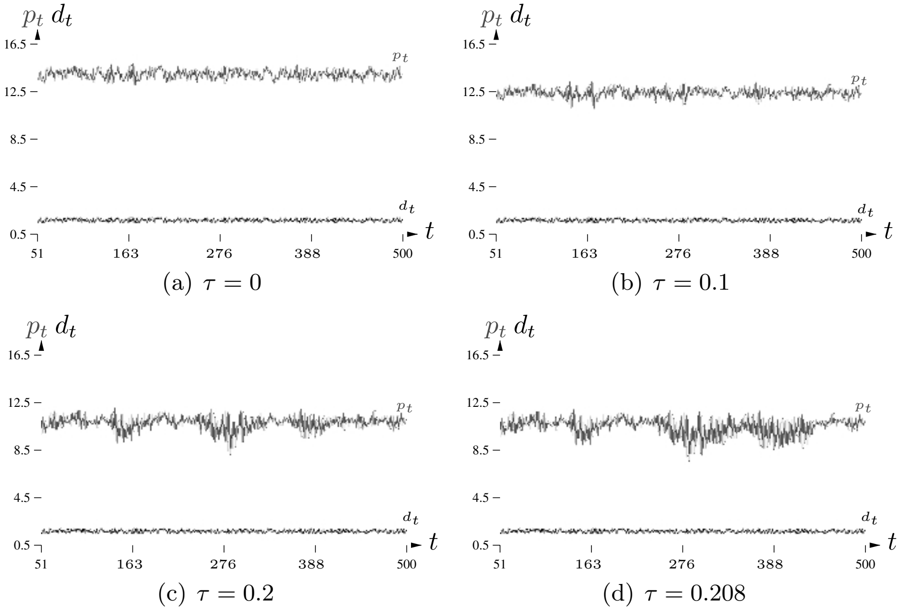


	Cum-dividend price q_t			
	$\tau = 0$	0.1	0.2	0.208
Mean	15.64	14.06	12.44	12.13
Variance	0.232	0.236	0.466	0.752
Maximum	16.73	15.22	13.81	13.71
Minimum	14.61	12.63	9.68	9.15

Fig. 4.8. Impact of contribution rates on the stock market

same realization of the noise process. Hence, the additional volatility observed in asset prices can solely be attributed to the public pension system and the larger contribution rate. Increasing τ further to a value larger than $\bar{\tau} = 0.208$ leads to bankruptcy problems on the part of consumers and is therefore not possible. The value $\bar{\tau}$ therefore defines an upper bound for contributions.

More insight into the structural cause for the additional volatility generated by the pension system is provided by Figure 4.9 showing the ex-dividend asset price process together with the dividend payments during the same time window. The figure reveals that the additional fluctuations in cum-dividend prices can fully be attributed to an increased volatility in ex-dividend prices while no change in fluctuations of the dividend process can be observed. Moreover, as in the previous

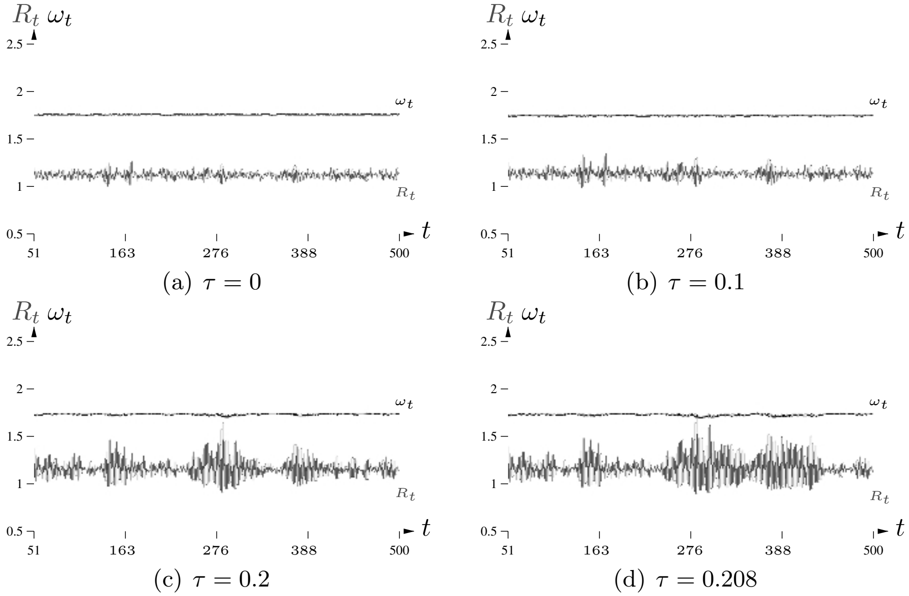


	Asset prices p_t				Dividends d_t			
	$\tau = 0$	0.1	0.2	0.208	$\tau = 0$	0.1	0.2	0.208
Mean	13.97	12.39	10.78	10.48	1.678	1.668	1.655	1.652
Variance	0.141	0.152	0.391	0.677	0.013	0.013	0.013	0.013
Maximum	14.85	13.35	11.98	1.88	1.878	1.869	1.857	1.855
Minimum	13.10	11.12	8.09	7.57	1.474	1.460	1.451	1.449

Fig. 4.9. Impact of contribution rates on asset prices and dividends

section, the model generates strong excess volatility in the sense that the volatility of asset prices exceeds the volatility of dividends by a factor of ten in the case $\tau = 0$ and by a factor of more than fifty (!) in the case where $\tau = 0.208$. In addition, the levels of both ex-dividend prices and dividends decreases as the contribution rate τ increases whereas the former effect is much larger than the second one.

Consider next the impact of the pension system on the bond market and the labor market. Figure 4.10 portrays the evolution of bond returns and of real wages for the four scenarios $\tau \in \{0, 0.1, 0.2, 0.208\}$. The table below shows the sample moments of the two series which are calculated from the samples $\{R_t\}_{t=51}^{450}$ and $\{\omega_t\}_{t=51}^{450}$, respectively. In all four cases depicted, both series fluctuate about a stationary level.



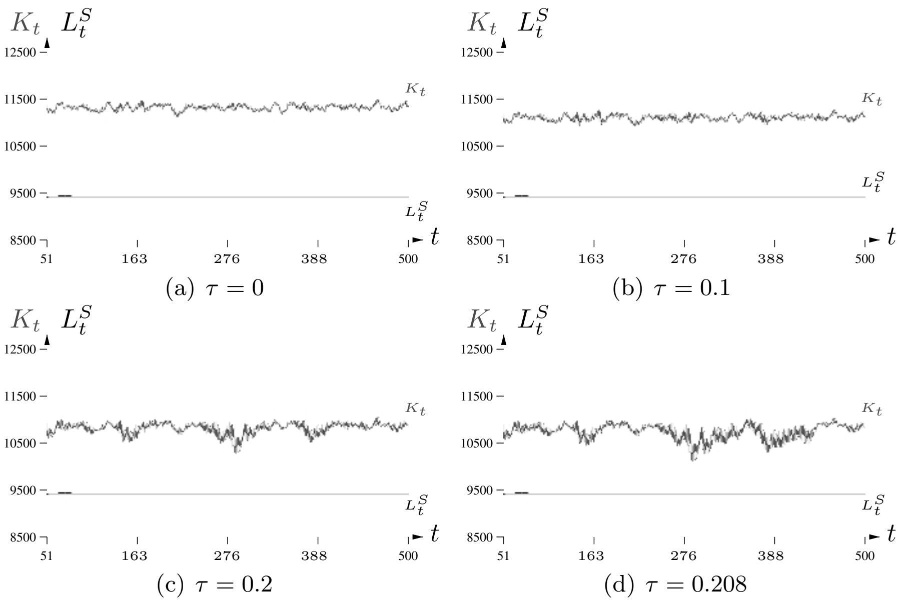
	Bond return R_t				Real wage ω_t			
	$\tau = 0$	0.1	0.2	0.208	$\tau = 0$	0.1	0.2	0.208
Mean	1.12	1.35	1.156	1.167	1.756	1.754	1.729	1.725
Variance	0.001	0.003	0.012	0.022	0.000	0.000	0.000	0.000
Maximum	1.255	1.343	1.634	1.71	1.765	1.754	1.741	1.741
Minimum	1.005	0.991	0.910	0.896	1.747	1.736	1.700	1.691

Fig. 4.10. Impact of contribution rates on bond returns and real wages

As in the previous section, the level of bond returns slightly exceeds unity and increases with the contribution rate τ. As a consequence, the interest rate $r_t := R_t - 1$ will be positive in each case corresponding to an annual level of $\approx 3\%$ in case $\tau = 0$ and to $\approx 4\%$ in case $\tau = 0.2$. Apart from that, the same volatility effect as before is present: As the contribution rate τ is increased, this results in a significantly higher variance of bond returns over time. Furthermore, the bond return series exhibits strong volatility clustering for sufficiently large contribution rates, a phenomenon typically observed in empirical financial time series. In contrast to that, fluctuations in the real wage process remain small throughout all time windows. The level of the series slightly decreases with a larger τ while the variance remains negligible in all cases (the standard deviation slightly increases with the contribution rate).

Nevertheless, the response of the process to changes in τ appears to be much smaller than with the financial variables before.

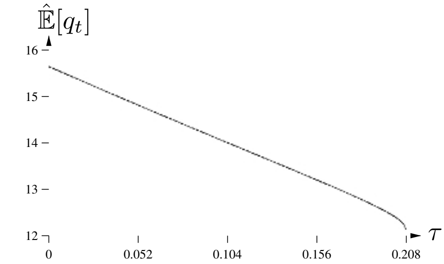
At first sight, the last observation suggests that fluctuations in the real sector are smaller than those in the financial sector. However, as one would expect from the previous correlation analysis, the increased volatility in financial variables with large τ is also observable in the capital stock. This suspicion is confirmed by Figure 4.11 displaying the evolution of the capital stock depending on τ together with labor supply.



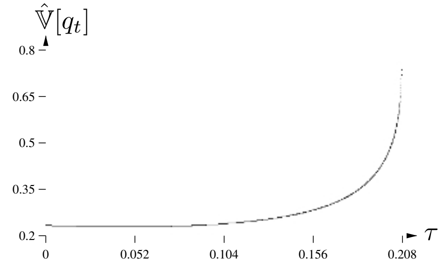
	Capital stock K_t			
	$\tau = 0$	0.1	0.2	0.208
Mean	11320.0	11101.9	10815.3	10742.7
Variance	3572.6	3756.7	14451.6	27446.5
Maximum	11482.5	11265.9	11035.1	11022.8
Minimum	11136.2	10931.4	10277.9	10119.7

Fig. 4.11. Impact of contribution rates on the capital stock

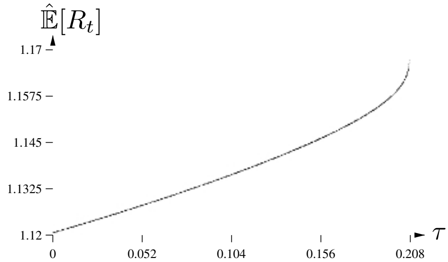
One observes from the figures and the corresponding sample moments that any reduction in the contribution rate increases the level of the capital stock. This observation confirms the insights obtained, e.g., by [35] that a reduction in public pension payments increases private savings and, thus, the aggregated capital stock. The previous result therefore suggests that the basic economic mechanism which makes a capital-based system with private retirement provisions potentially superior to a pay-as-you-go system is present in our model. Apart from that, one observes a similar volatility effect as for the financial sector: A higher contribution rate increases the volatility of the capital stock and hence fluctuations in aggregate production output. Furthermore, the crashes in asset prices for large τ are also transmitted to the real sector resulting in a substantial temporary decline in the capital stock during these periods.



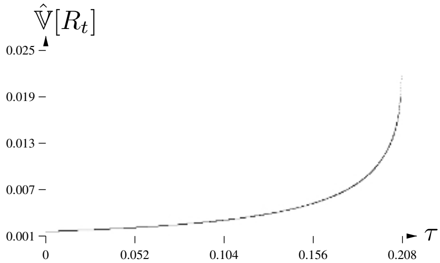
(a) Mean of cum-dividend price



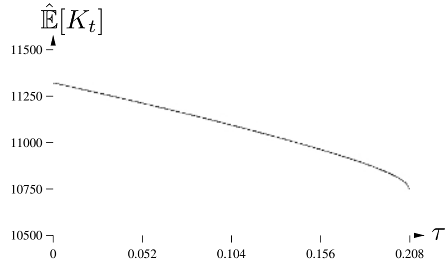
(b) Variance of cum-dividend price



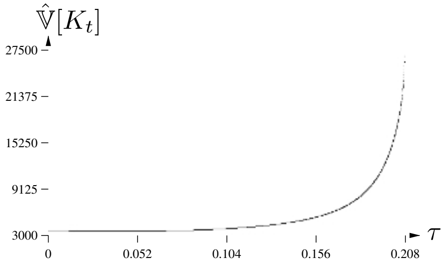
(c) Mean of bond return



(d) Variance of bond return



(e) Mean of capital

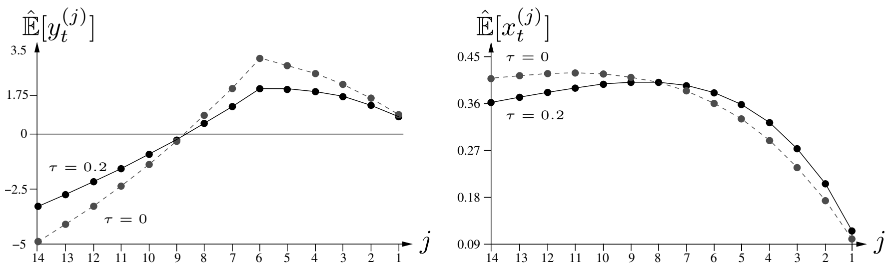


(f) Variance of capital

Fig. 4.12. Impact of contribution rates on real and financial variables

The previous simulation experiments compared up to four distinct values of the contribution rate. To further substantiate the previous results, Figure 4.12 depicts a collection of bifurcation diagrams showing the sample mean and the sample variance of the cum-dividend price q_t, bond return R_t and the capital stock K_t depending on $\tau \in [0, 0.208]$. The result supports the previous findings that a reduction in τ, throughout, leads on average to larger asset prices, larger capital stock and a lower interest rate. Moreover, a sufficiently large contribution rate generates additional volatility and increases fluctuations in all three series.

We close this section by analyzing how changes in the contribution rate affect the savings and investment behavior of consumers over their life cycle. Figure 4.13 displays consumers' investment in bonds and shares for the two cases where $\tau \in \{0, 0.2\}$. Figure 4.13(a) depicts the average bond investments of a consumer in generation $j \in \{1, \dots, J\}$ whereas Figure 4.13(b) shows the corresponding average share holdings. By ergodicity, these values approximate the expected value of the portfolio process along the random fixed point and thus provide a description of the investment behavior of consumers over their life cycle.



(a) Average bond investments (b) Average income invested in stocks

Fig. 4.13. Investment profile depending on the contribution rate; $\tau \in \{0, 0.2\}$

A first observation form Figure 4.13(a) is that consumers take credit by issuing bonds during their first periods of life where $j \geq 9$. The amount of credit taken strictly decreases with age and the bond investment becomes positive as soon as $j < 9$. Qualitatively, this behavior is independent of the contribution rate, however, the absolute bond investments of all generations are monotonically decreasing in τ. As a consequence, an increase in the contribution rate leads to a decrease in the number of bonds traded between generations causing a reduction in the trading volume on the bond market. This result indicates that the presence of a pension system leads to a crowding out of private investment resulting in a lower trading activity on financial markets. In addition, Figure 4.13(b) suggests that the pension system entails a significant impact on consumers' share holdings over their life cycle. In the absence of a pension system, share holdings are largest during the earlier periods of life and decrease sharply during the retirement age. The presence of a pension system changes this distribution by shifting more weight to the middle part of life and increasing share holdings during the retirement age. The latter seems reasonable as the pension system implies a wealth transfer from the working age to the retirement age which is reflected in the investment behavior.

In summary, the results from Figure 4.13 indicate that a change in contribution rates exerts a strong impact on the savings behavior of consumers and the portfolios held over the life cycle. The following statement summarizes the main results of this section.

Numerical Result 4.3.1 *Let the contribution rate $\tau \in [0, 0.208]$ be constant and the model be parameterized as described in Table 4.1. Then the following properties hold true:*

(i) The long run behavior of the economy is independent of initial conditions and determined by a stationary stochastic process (random fixed point) to which all sample paths converge.
(ii) A reduction in τ increases the levels of asset prices and the capital stock while it decreases the level of bond returns.
(iii) Increasing the contribution rate beyond a critical value leads to stock market crashes and a significant increase in the volatility of stock prices. Volatilities in bond returns and capital stock also increase.

4.4 Impact of Pension Systems on Consumer Welfare

The previous study focused on the qualitative and quantitative impact of a pension system on real and financial markets as well as on the savings behavior of consumers. In this section we take a normative point of view by analyzing the impact of alternative contribution rates on the long run welfare of consumers. Based on the concepts introduced in Section 4.1 we first derive the consumption and utility processes of consumers and show that each choice of the contribution rate τ gives rise to a stationary stochastic process describing consumption and lifetime utility along the random fixed point. Utilizing this result, a notion of long-run efficiency is developed which amounts to comparing the expected lifetime utility of consumers along the random fixed point depending on the contribution rate.

As before let the population be constant such that $N_t^{(j)} \equiv \bar{N}$. Then given the initial state $\xi_0 \in \Xi$ and a fixed contribution rate $\tau \in [0, \bar{\tau}]$ the stochastic evolution of the system is described by the stochastic process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t \geq 0}$ which defines the state ξ_t of the system at time t according to (4.14). To alleviate the notation we denote the components of the state process as $\{K_t(\tau, \cdot, \xi_0)\}_{t \geq 0}$, $\{q_t(\tau, \cdot, \xi_0)\}_{t \geq 0}$, etc. The process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t \geq 0}$ defines a consumption process $\{c_t(\tau, \cdot, \xi_0)\}_{t > 0}$ where $c_t(\tau, \cdot, \xi_0) = \left(c_t^{(j)}(\tau, \cdot, \xi_0)\right)_{j=0}^{J} : \Omega \longrightarrow \mathbb{C}^{J+1}$ describes individual consumption in generations $j \in \{0, \dots, J\}$ at time t. Utilizing equations (2.4) and (3.23) together with the forecasting rules (3.48) and (3.49) in the consumption function (3.24) the components of the random variable $c_t(\tau, \cdot, \xi_0)$ may explicitly be written as[3]

$$c_t^{(j)}(\tau, \tilde{\omega}, \xi_0) := \bar{c}^{(j)} \Big(e_t^{(j)}(\tau, \tilde{\omega}, \xi_0) + R_{t-1}(\tau, \tilde{\omega}, \xi_0)\, y_{t-1}^{(j+1)}(\tau, \tilde{\omega}, \xi_0) \tag{4.9}$$

$$+ q_t(\tau, \tilde{\omega}, \xi_0)^\top x_{t-1}^{(j+1)}(\tau, \tilde{\omega}, \xi_0) + \frac{1}{R_t(\tau, \tilde{\omega}, \xi_0)} \sum_{m=1}^{j} \frac{e_t^{(j-m)}(\tau, \tilde{\omega}, \xi_0)}{R_{t-1}(\tau, \tilde{\omega}, \xi_0)^{m-1}} \Big)$$

for each $t > 0$ where the non-capital income process is defined as

$$e_t^{(j)}(\tau, \tilde{\omega}, \xi_0) := \begin{cases} \dfrac{\tau\, \mathcal{W}(K_t(\tau, \tilde{\omega}, \xi_0), L^S)\, L^S}{N^R} & j = 0, \dots, j_L - 1 \\[2ex] (1-\tau)\, \mathcal{W}(K_t(\tau, \tilde{\omega}, \xi_0), L^S)\, L^{(j)} & j = j_L, \dots, J \end{cases} \tag{4.10}$$

for each $\tilde{\omega} \in \Omega$. Here L^S denotes constant labor supply as defined in (2.1), $N^R := j_L \bar{N}$ is the number of pensioners and $\mathcal{W}$ is defined as in (3.34). Associated with the consumption process $\{c_t(\tau, \cdot, \xi_0)\}_{t > 0}$ is the induced lifetime utility process $\{U_t(\tau, \cdot, \xi_0)\}_{t > J}$ where for each $t > J$ the random variable $U_t(\tau, \cdot, \xi_0) : \Omega \longrightarrow \mathbb{R}$ is defined as

$$U_t(\tau, \tilde{\omega}, \xi_0) := \sum_{j=0}^{J} \beta^{J-j} \ln c_{t-j}^{(j)}(\tau, \tilde{\omega}, \xi_0), \quad \tilde{\omega} \in \Omega. \tag{4.11}$$

For each $t > J$ the quantity in (4.11) describes the lifetime utility attained by the consumers who were born in period $t - J$ and die at the end of period t.

[3] We define $y_{t-1}^{(j+1)}(\tau, \cdot, \xi_0) \equiv 0$ and $x_{t-1}^{(j+1)}(\tau, \cdot, \xi_0) \equiv 0$ for $j = J$ and $\sum_{m=1}^{j} \frac{e_t^{(j-m)}(\tau, \cdot, \xi_0)}{R_{t-1}(\tau, \cdot, \xi_0)^m} \equiv 0$ for $j = 0$.

As before, suppose that for each $\tau \in [0, \bar{\tau}]$ the system (4.14) possesses a stable random fixed point corresponding to the stationary and ergodic process $\{\xi^\star_\tau \circ \vartheta^t\}_{t\geq 0}$. Similar to the previous derivations denote the components of the map $\xi^\star_\tau : \Omega \longrightarrow \Xi$ as $K^\star_\tau(\cdot)$, $R^\star_\tau(\cdot)$, $q^\star_\tau(\cdot)$, etc. In particular, $y^\star_\tau(\cdot) = (y^{(j)\star}_\tau(\cdot))^J_{j=1}$ and $x^\star_\tau(\cdot) = (x^{(j)\star}_\tau(\cdot))^J_{j=1}$ define the portfolio processes along the random fixed point. Then each choice of $\tau \in [0, \bar{\tau}]$ gives rise to an induced process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ describing consumption of generations along the random fixed point $\{\xi^\star_\tau \circ \vartheta^t\}_{t\geq 0}$. Utilizing the same procedure as in the derivation of (4.9) the components of the map $c^\star_\tau = (c^{(j)\star}_\tau)^J_{j=0} : \Omega \longrightarrow \mathbb{C}^{J+1}$ take the explicit form[4]

$$\begin{aligned} c^{(j)\star}_\tau(\tilde{\omega}) = \bar{c}^{(j)} \Big(& e^{(j)\star}_\tau(\tilde{\omega}) + R^\star_\tau(\vartheta^{-1}\tilde{\omega})\, y^{(j+1)\star}_\tau(\vartheta^{-1}\tilde{\omega}) \\ & + q^\star_\tau(\tilde{\omega})^\top x^{(j+1)\star}_\tau(\vartheta^{-1}\tilde{\omega}) + \frac{1}{R^\star_\tau(\tilde{\omega})} \sum_{m=1}^{j} \frac{e^{(j-m)\star}_\tau(\tilde{\omega})}{R^\star_\tau(\vartheta^{-1}\tilde{\omega})^{m-1}} \Big) \end{aligned} \tag{4.12}$$

where for each $\tilde{\omega} \in \Omega$

$$e^{(j)\star}_\tau(\tilde{\omega}) := \begin{cases} \dfrac{\tau\, \mathcal{W}(K^\star_\tau(\tilde{\omega}), L^S) L^S}{N^R} & j = 0, \ldots, j_L - 1 \\ (1-\tau)\, \mathcal{W}(K^\star_\tau(\tilde{\omega}), L^S) \bar{L}^{(j)} & j = j_L, \ldots, J. \end{cases}$$

The consumption process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ is again stationary and ergodic and defines an induced lifetime utility process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ describing lifetime utility attained by consumers along the path of the random fixed point. The random variable $U^\star_\tau : \Omega \longrightarrow \mathbb{R}$ is defined as

$$U^\star_\tau(\tilde{\omega}) := \sum_{j=0}^{J} \beta^{J-j} \ln c^{(j)\star}_\tau(\vartheta^{-j}\tilde{\omega}). \tag{4.13}$$

The process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ inherits the properties of stationarity and ergodicity from the consumption process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$. As a consequence, the expected value

$$\mathbb{E}\left[U^\star_\tau \circ \vartheta^t\right] := \int_\Omega U^\star_\tau \circ \vartheta^t(\tilde{\omega}) \mathbb{P}(d\tilde{\omega})$$

is independent of t implying that $\mathbb{E}\left[U^\star_\tau\right] = \mathbb{E}\left[U^\star_\tau \circ \vartheta^t\right]$ for all t. The real number $\mathbb{E}\left[U^\star_\tau\right]$ describes the expected lifetime utility attained by

[4] Similar to the previous convention we define $y^{(j+1)\star}_\tau(\cdot) \equiv 0$ and $x^{(j+1)\star}_\tau(\cdot) \equiv 0$ for $j = J$ and $\sum_{m=1}^{j} \frac{e^{(j-m)\star}_\tau(\cdot)}{R^\star_\tau \circ \vartheta^{-1}(\cdot)^m} \equiv 0$ for $j = 0$.

consumers along the random fixed point depending on the contribution rate. The efficiency criterion adopted in the sequel is to choose $\tau \in [0, \bar{\tau}]$ such that $\mathbb{E}[U_\tau^\star]$ becomes maximal. For this purpose, the following lemma shows that stability of the random fixed point implies that the paths of the consumption and utility processes converge to the corresponding paths of consumption and utility along the random fixed point. The proof is given in Section 4.A.2 in the Appendix.

Lemma 4.4.1 *Let $\tau \in [0, \bar{\tau}]$ be arbitrary and suppose that the random dynamical system (4.14) possesses a stable random fixed point. Then both the consumption process $\{c_\tau^\star \circ \vartheta^t\}_{t>0}$ defined by (4.12) and the lifetime utility process $\{U_\tau^\star \circ \vartheta^t\}_{t>J}$ defined by (4.13) are stable in the sense that for all initial states $\xi_0 \in U(\tilde{\omega})$ $\mathbb{P}$–a.s.*

(i) $\lim_{t\to\infty} \| c_\tau^{(j)\star}(\vartheta^t\tilde{\omega}) - c_t^{(j)}(\tau, \tilde{\omega}, \xi_0)\| = 0,\ j = 0, \ldots, J$

(ii) $\lim_{t\to\infty} \| U_\tau^\star(\vartheta^t\tilde{\omega}) - U_t(\tau, \tilde{\omega}, \xi_0)\| = 0.$

Exploiting ergodicity of the process $\{U_\tau^\star \circ \vartheta^t\}_{t>J}$ and the stability result from Lemma 4.4.1 the expected utility along the random fixed point may again be obtained as the limit of time averages such that

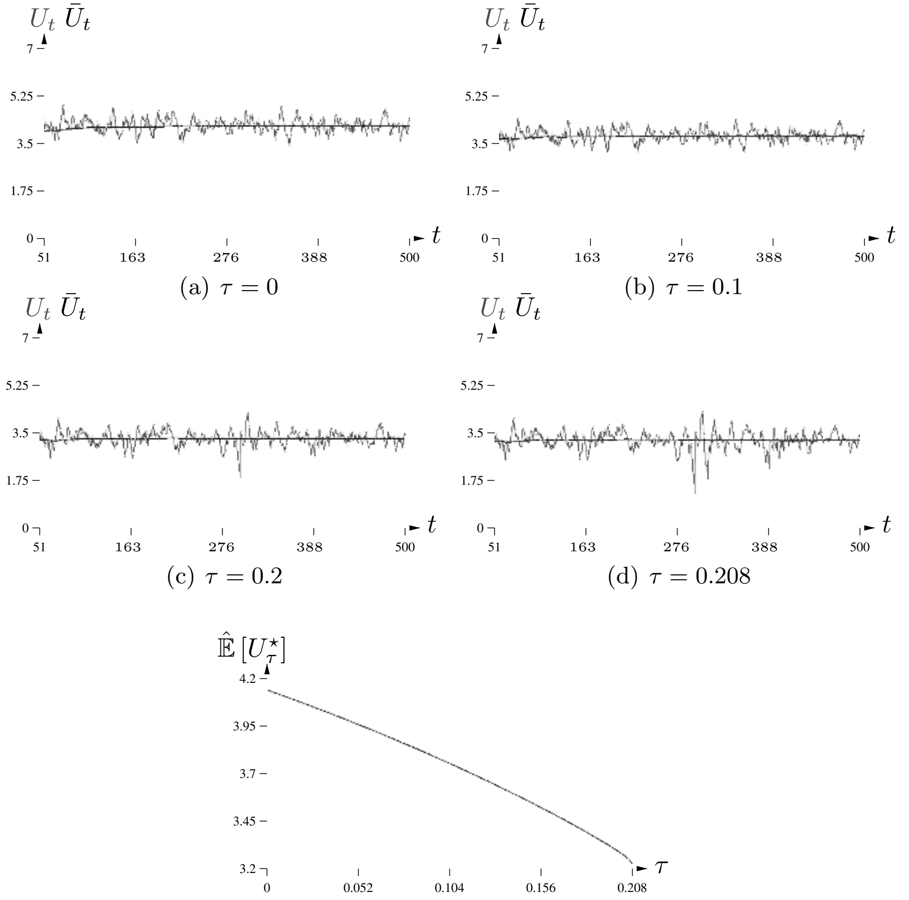
$$\mathbb{E}[U_\tau^\star] = \lim_{T\to\infty} \frac{1}{T-J} \sum_{t=J+1}^{T} U_t(\tau, \tilde{\omega}, \xi_0) \quad \mathbb{P} - a.s.$$

In the sequel the expected value $\mathbb{E}[U_\tau^\star]$ will again be approximated by the sample mean

$$\hat{\mathbb{E}}[U_\tau^\star] = \frac{1}{T-J} \sum_{t=J+1}^{T} U_t(\tau, \tilde{\omega}, \xi_0).$$

Based on these concepts we are now in a position to continue our numerical case study by analyzing the consumption and lifetime utility processes and the welfare of consumers depending on the contribution rate τ. All parameters as well as the realization of the noise process are as in the previous sections. For given $\xi_0 \in \Xi$ and fixed $\tau \in [0, \bar{\tau}]$ the following notation frequently suppresses the dependence of variables on these arguments writing, e.g., U_t as a shorthand for the realization $U_t(\tau, \tilde{\omega}, \xi_0)$.

Given this convention, Figure 4.14 depicts a time window of the lifetime utility process defined in (4.11) for $\tau \in \{0, 0.1, 0.2, 0.208\}$ together with the recursive mean $\bar{U}_t := \frac{1}{t-J} \sum_{n=J+1}^{t} U_n$, $t > J$ of the sample $\{U_n\}_{n=J+1}^{T}$. In the stationary situation, the recursive mean converges to the sample mean $\bar{U}_T = \hat{\mathbb{E}}\left[U^\star_\tau\right]$ which approximates the expected lifetime utility of consumers along the random fixed point.



(a) $\tau = 0$ (b) $\tau = 0.1$

(c) $\tau = 0.2$ (d) $\tau = 0.208$

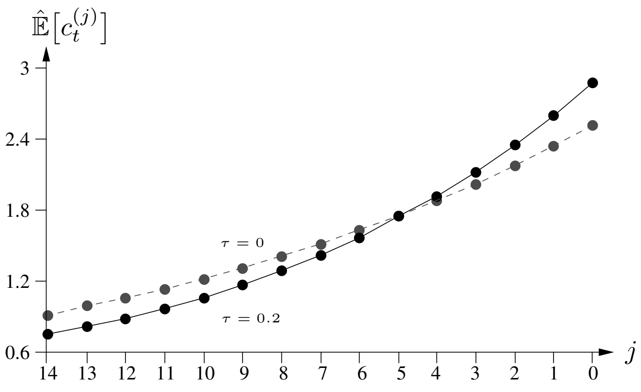
(e) Expected utility depending on τ

Fig. 4.14. Impact of contribution rates on lifetime utilities

The time series plots in Figure 4.14 confirm the previous result of a stationary utility process which fluctuates about a time-invariant level. Moreover, a comparison of Figures 4.14(a) – 4.14(d) reveals that the level of the process represented by the sample mean $\hat{\mathbb{E}}\left[U^\star_\tau\right]$ is strictly decreasing in τ. By ergodicity of the utility process and Lemma 4.4.1, this observation suggests that consumer welfare will be higher in the long run as the contribution rate is reduced. These properties are further confirmed by Figure 4.14(e) showing a bifurcation diagram of the sample mean $\hat{\mathbb{E}}\left[U_\tau^\star\right]$ for alternative contribution rates $\tau \in [0, 0.208]$. The figure supports the conjecture of a strictly negative relationship between the contribution rate τ and the long-run expected utility $\mathbb{E}\left[U_\tau^\star\right]$.

In summary, the results derived from Figure 4.14 unequivocally suggest that in the present scenario of a stationary population any persistent increase in the contribution rate reduces the long-run welfare of all generations. Conversely, any reduction of the pension system leads to a long-run welfare improvement of consumers. Clearly, this is a very strong statement since it essentially asserts the long-run inefficiency of a pay-as-you-go system. However, the sensitivity analysis carried out in the following section shows that this result is robust and continues to hold even with a different parametrization of the model.

In a final step we investigate how the pension system affects the distribution of consumption over the life cycle. For this purpose, Figure 4.15 depicts average consumption of consumers during their life-cycle for the two cases $\tau \in \{0, 0.2\}$. For each $j \in \{0, 1, \dots, J\}$ the corresponding values are calculated as averages of the sample $\{c_t^{(j)}\}_{t=51}^{500}$. By ergodicity, these values approximate the expected value of the consumption process along the random fixed point as defined in (4.12).



	Expected consumption $\hat{\mathbb{E}}[c_t^{(j)}]$			
	$\tau = 0$	0.1	0.2	0.208
working age	11.21	10.62	9.97	9.91
retirement age	12.73	13.18	13.64	13.65

Fig. 4.15. Average consumption over the life-cycle depending on τ

In both cases depicted in Figure 4.15 expected consumption strictly increases with age. While the qualitative result does not depend on the contribution rate, a reduction in τ changes the distribution over the life cycle by shifting more weight to the earlier periods of life. Hence, a reduction in τ fosters consumption during the working years and reduces consumption when retired. This effect is quantified in the table below which contrasts average consumption during the working age and the retirement age depending on $\tau \in \{0, 0.1, 0.2, 0.208\}$. It reveals that even for $\tau = 0$ retired consumers have on average higher consumption than consumers in working age. An increase in contributions broadens this gap by increasing consumption when retired and reducing it during the working years. In this regard, recall that the retirement age corresponds to six periods of the life-cycle while the working age is nine periods long. A reduction in τ thus leads to a more uniform distribution of consumption over the life cycle which, as shown before, has a positive impact on expected lifetime utility and increases the welfare of consumers.

The main results obtained in this section are summarized in the following statement.

Numerical Result 4.4.1 *Let the contribution rate $\tau \in [0, 0.208]$ be constant and the model be parameterized as described in Table 4.1. Then the following properties hold true:*

(i) A reduction in τ increases the level of the lifetime utility process and thus the expected lifetime utility of consumers along the random fixed point.

(ii) A reduction in τ reduces expected consumption in the retirement age while increasing it during the working years. Even if $\tau = 0$, average consumption in the retirement age is larger than in the working age.

4.5 Robustness of Results

The previous sections delivered some strong numerical results concerning the inefficiency of pension systems and the impact of changes in the contribution rate on the long-run behavior of real and financial markets. Since a theoretical foundation is not yet available, the goal of this section is to demonstrate that all of the previous results are robust against parameter changes and continue to hold for a large set of economically meaningful specifications of the model. The following sensitivity analysis is organized in the same spirit as in [18].

Recall from the Numerical Results 4.3.1 and 4.4.1 that for the parameter set in Table 4.1 and a constant contribution rate $\tau \in [0, 0.208]$ the following effects could be observed:

(i) Any reduction in the contribution rate increases the levels of asset prices and capital while it decreases the level of bond returns
(ii) A sufficiently large contribution rate leads to a significant increase in the volatilities of asset prices and bond returns
(iii) Any reduction in the contribution rate has a positive welfare effect by increasing the long-run expected utility attained by consumers.

In the sequel we shall refer to (i) as the *level effect* and to (ii) as the *volatility effect* while (iii) will be called the *welfare effect*. The objective is to demonstrate that all three effects are robust against (small) parameter changes. For this purpose we employ the standard parameter set listed in Table 4.1 and consider variations of the model's most important parameters to a higher and a lower value, respectively. For each of these modified parameter sets the same numerical experiments as before are repeated such that the model is iterated for $T = 500$ periods at different contribution rates τ. For each experiment the corresponding sample moments of the variables of interest are calculated as before.

Some preliminary remarks must be made before we begin our analysis. In Section 4.3 it was shown that increasing the contribution rate up to a critical value $\bar{\tau} = 0.208$ results in a dramatic increase in the fluctuations of asset prices and bond returns leading to phases of crashes and extremely high volatility. It was argued that, as τ is further increased this volatility effect becomes so large that it causes bankruptcy problems on the part of the consumers (see Section 2.3 for a discussion of this issue). Since the model's forward dynamics are no longer defined in the case of bankruptcy and the simulation is terminated, the previous parametrization did not allow for values of τ larger than the critical value $\bar{\tau} = 20.8\%$. It can be shown that such a critical contribution rate exists for all parameter sets studied below which lies sometimes above and sometimes below 20% but throughout exceeds 10%. To keep the subsequent results comparable to the previous ones, each of the following parameter variations compares the three cases where $\tau = 0$, $\tau = 0.1$ and $\tau = \bar{\tau}$ where $\bar{\tau}$ depends on the respective parameter set.[5]

[5] Since only variations of a single parameter are considered, the feasible range is sometimes restricted by previous assumptions on parameters. For example, the constraint $\delta > \frac{1}{\gamma_1}$ precludes to study, e.g., the case $\delta = 0.1$ which would require a simultaneous increase in the parameter γ_1, e.g., to $\gamma_1 = 11$.

Given this convention, consider first the impact of a change in contribution rates on the levels of cum-dividend prices and bond returns. The following table displays the sample means $\hat{\mathbb{E}}[q_t]$ and $\hat{\mathbb{E}}[R_t]$ which are calculated from the samples $\{q_t\}_{t=51}^{500}$ and $\{R_t\}_{t=51}^{500}$, respectively.

Table 4.2. Robustness of the level effect on financial variables

		$\hat{\mathbb{E}}[q_t]$			$\hat{\mathbb{E}}[R_t]$			$\bar{\tau}$ (%)
			τ			τ		
Parameter	Value	0	0.1	$\bar{\tau}$	0	0.1	$\bar{\tau}$	
α	0.63	16.59	15.01	11.32	1.129	1.144	1.209	31.9
	0.69	14.84	13.23	12.30	1.112	1.127	1.143	14.65
β	0.90	12.23	11.00	9.73	1.157	1.177	1.220	16.6
	0.99	17.46	15.71	12.97	1.107	1.120	1.155	24.85
γ_0	0.01	22.05	19.80	17.30	1.117	1.132	1.157	20.65
	0.03	12.83	11.54	9.87	1.123	1.138	1.174	21.10
γ_1	6.5	18.34	16.46	14.52	1.117	1.132	1.154	20.20
	9.5	11.62	10.47	8.93	1.126	1.142	1.181	21.10
δ	0.17	17.48	15.71	14.12	1.119	1.134	1.153	19.00
	0.25	13.02	11.71	9.98	1.1227	1.138	1.175	21.10
κ	2.0	11.36	10.22	8.83	1.124	1.140	1.172	20.95
	3.0	20.39	18.32	15.98	1.118	1.133	1.159	20.65
ϱ	0.1	15.65	14.09	11.46	1.121	1.135	1.178	26.5
	0.9	15.64	14.05	12.45	1.121	1.136	1.162	18.70
$\bar{N}$	750	11.38	10.23	8.85	1.124	1.140	1.172	20.95
	2000	29.66	26.62	23.27	1.116	1.130	1.154	20.80
j_L	5	15.92	14.29	12.98	1.131	1.148	1.170	16.60
	7	15.21	13.70	11.23	1.111	1.125	1.163	25.30

One observes that throughout the previous level effect is confirmed: a decrease in τ leads to an increase in the level of q_t and a decrease in the level of R_t. While the qualitative result holds true in each case under scrutiny, variations in the parameter set exert a crucial effect on the respective level as well as on the critical contribution rate $\bar{\tau}$. As an example, one observes that a decrease in the production elasticity α to 0.63 increases the critical contribution rate $\bar{\tau}$ to almost 32% while if α is increased to 0.69 the critical rate decreases to approximately 14.6% (it will be shown in the next table that at these critical values the previously observed volatility effect occurs). In particular, this observation suggests that one should be careful to interpret the 20% level of contributions in general as a critical value.

Next consider the robustness of the volatility effect on financial markets which is displayed in Table 4.3. For each of the parameter variations the sample variances $\hat{\mathbb{V}}[q_t]$ and $\hat{\mathbb{V}}[R_t]$ of cum-dividend prices and bond returns are calculated from the samples $\{q_t\}_{t=51}^{500}$ and $\{R_t\}_{t=51}^{500}$.

Table 4.3. Robustness of the volatility effect on financial variables

		$\hat{\mathbb{V}}[q_t]$			$\hat{\mathbb{V}}[R_t]$			$\bar{\tau}$ (%)
			τ			τ		
Parameter	Value	0	0.1	$\bar{\tau}$	0	0.1	$\bar{\tau}$	
α	0.63	0.228	0.216	0.946	0.0013	0.0021	0.0349	31.9
	0.69	0.237	0.277	0.638	0.0018	0.0045	0.0171	14.65
β	0.90	0.160	0.193	0.857	0.0021	0.0052	0.04496	16.6
	0.99	0.282	0.265	0.786	0.0014	0.0022	0.0195	24.85
γ_0	0.01	0.232	0.237	0.778	0.0008	0.0015	0.0122	20.65
	0.03	0.232	0.235	0.670	0.0023	0.0043	0.0286	21.10
γ_1	6.5	0.229	0.231	0.492	0.0011	0.0021	0.0098	20.20
	9.5	0.236	0.242	0.686	0.0029	0.0054	0.0352	21.10
δ	0.17	0.232	0.236	0.394	0.0012	0.0023	0.0078	19.00
	0.25	0.232	0.235	0.733	0.0022	0.0041	0.0306	21.10
κ	2.0	0.232	0.235	0.510	0.0029	0.0054	0.0254	20.95
	3.0	0.232	0.237	0.755	0.0009	0.0017	0.0136	20.65
ϱ	0.1	0.230	0.221	0.606	0.0011	0.0017	0.0189	26.5
	0.9	0.243	0.260	0.841	0.0019	0.0037	0.02100	18.70
$\bar{N}$	750	0.232	0.235	0.511	0.0029	0.0054	0.0254	20.95
	2000	0.232	0.239	1.001	0.0004	0.0008	0.0092	20.80
j_L	5	0.219	0.257	0.711	0.0016	0.0038	0.0189	16.60
	7	0.252	0.236	0.701	0.0016	0.0025	0.0234	25.30

For each of the different parameter sets the calculations support the existence of a critical level of contribution rates needed for the emergence of the volatility effect. Increases in τ not exceeding the critical value have only a minor impact and may – in some cases – even reduce the volatility of asset prices. In contrast to that, the variance in bond returns increases monotonically with τ even for small contribution rates, however, the effect is still relatively small. These results change dramatically as soon as the contribution rate exceeds a certain critical level. In this case, a dramatic increase in the volatility of asset prices accompanied by temporary stock market crashes is observable. The same effect is present with bond returns. These observations confirm the previous result that the presence of a pension system may generate additional fluctuations and increase the volatility observed in financial markets. Conversely, a reduction in contributions to a value below a certain threshold has a stabilizing effect on asset markets.

As a final experiment, consider the impact of changes in τ on the levels of the capital process and the lifetime utility process. For each of the parameter variations under study Table 4.4 reports the mean values $\hat{\mathbb{E}}[K_t]$ and $\hat{\mathbb{E}}[U_t]$ of the samples $\{K_t\}_{t=51}^{500}$ and $\{U_t\}_{t=51}^{500}$, respectively.

Table 4.4. Robustness of the level effect on the capital stock and of the welfare effect

		$\hat{\mathbb{E}}[K_t]$			$\hat{\mathbb{E}}[U_t]$			$\bar{\tau}$ (%)
			τ			τ		
Parameter	Value	0	0.1	$\bar{\tau}$	0	0.1	$\bar{\tau}$	
α	0.63	12896.0	12638.3	11749.2	4.520	4.113	2.751	31.9
	0.69	9935.7	9748.5	9607.4	3.875	3.528	3.310	14.65
β	0.90	10790.1	10525.0	10143.1	2.885	2.487	2.074	16.6
	0.99	11523.3	11327.3	10900.9	5.291	4.923	4.148	24.85
γ_0	0.01	32473.7	31858.0	30968.2	8.184	7.813	7.298	20.65
	0.03	6107.5	5988.2	5776.8	1.819	1.439	0.835	21.10
γ_1	6.5	19111.7	18746.5	18264.4	6.114	5.745	5.258	20.20
	9.5	4286.1	4201.6	4045.9	0.536	0.148	-0.495	21.10
δ	0.17	15939.5	15634.3	15294.9	5.465	5.091	4.667	19.00
	0.25	6397.5	6272.6	6044.8	1.995	1.615	1.005	21.10
κ	2.0	8035.0	7876.3	7617.4	0.375	-0.009	-0.599	20.95
	3.0	14965.3	14680.7	14262.3	7.271	6.900	6.381	20.65
ϱ	0.1	11316.7	11093.5	10592.4	4.165	3.778	2.912	26.5
	0.9	11322.7	11102.7	10809.0	4.164	3.787	3.334	18.70
$\bar{N}$	750	8048.3	7889.4	7630.0	4.245	3.861	3.271	20.95
	2000	22094.5	21679.1	21089.5	4.055	3.689	3.175	20.80
j_L	5	12407.2	12144.5	11872.8	5.055	4.621	4.241	16.60
	7	10188.4	10010.1	9603.1	3.098	2.775	2.073	25.30

The values in Table 4.4 reveal that the previously observed level effect on the capital stock is again robust: any decrease in the contribution rate leads to an increase in the sample mean of the capital stock. Hence, the basic economic mechanism stressed, e.g., by [35] that a reduction in public pension payments fosters the formation of capital is present in our model. Conversely, the presence of a pension system has a crowding-out effect on private savings which diminishes the capital stock. The result from Table 4.4 thus implies that for all parameter sets considered here a transition towards a capital-based pension system through a decrease in contribution rates fosters the accumulation of capital and increases the capital stock in the long run.

As a second result, Table 4.4 confirms the welfare effect observed with the standard parameter set: For all parameter changes under scrutiny, a reduction in the contribution rate increases the expected lifetime utility of consumers in the long run. Hence, the long-run inefficiency of a pay-as-you-go pension system obtained as a main result in the previous section is confirmed and does not depend on the previous specification of the model.

The previous results provide overwhelming numerical evidence that the installation of a pension system is not desirable and should be avoided. Conversely, if the pension system is already installed, any reduction of contributions leads to a welfare improvement of generations in the long run. Hence, from a long-run perspective, any social authority that controls the contribution rate should seek to abandon the pension system by ultimately reducing contributions to zero. Note, however, that this reduction will necessarily lead to a loss of welfare on the part of retirees during the transition. Hence, from a political point of view, the important issue is how a transition towards a lower contribution rate should be organized such that these losses remain sufficiently small. This question will be discussed in the next section.

4.6 Reducing the Public Pension System

As the previous results unanimously suggest the inefficiency of a public pension system in the presence of a stationary population, this section discusses various possibilities as to how one can organize a transition towards a lower (zero) contribution rate. To this end, suppose that initially a public pension system exists which is characterized by a constant contribution rate $\tau_t \equiv 0.2$. Assume that the long-run goal of the social authority which controls the pension system is to reach a contribution rate $\tau_t \equiv 0$. To achieve this goal, the present section discusses two different policies. In the first one, the pension system is immediately abandoned while it is gradually reduced in the second one.[6] More specifically, assume that the adjustment starts in period $t = 51$ such that $\tau_t = 0.2$ for $t \leq 50$. In the first scenario, where the system is instantaneously abandoned we have $\tau_t = 0$ for $t \geq 51$. This type of adjustment is denoted symbolically as $\tau_t = 0.2 \downarrow 0$. In the second scenario, we assume that from $t = 51$ onwards the contribution rate is gradually lowered by 0.5% points in every period. This implies that the target value of $\tau_t = 0$ is reached in period $t = 90$. The gradual adjustment policy is denoted as $\tau_t = 0.2 \searrow 0$. Figure 4.16 visualizes the adjustment policies for either scenario.

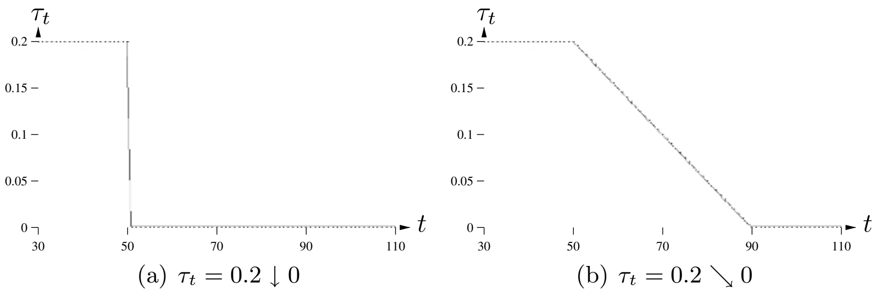
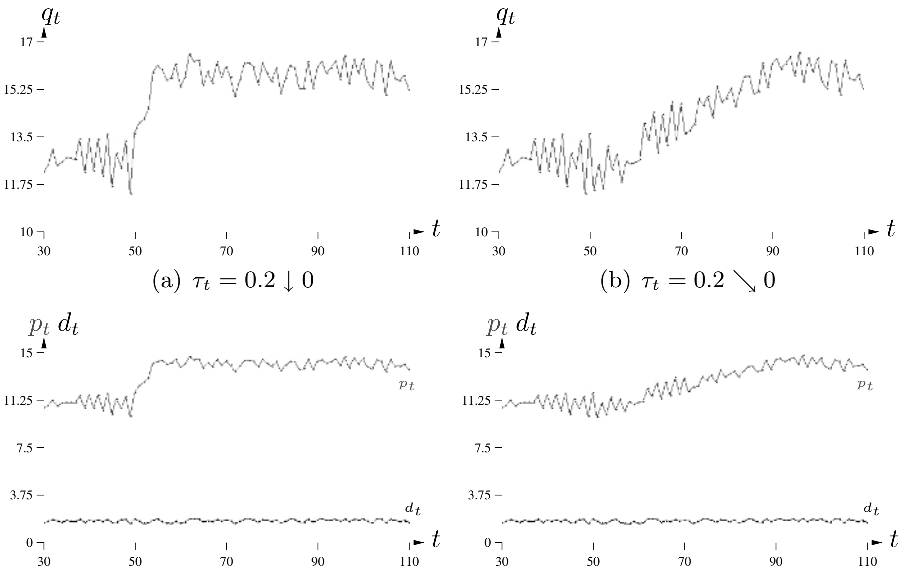


Fig. 4.16. Instantaneous and gradual reduction of contribution rates

Consider first the impact of either adjustment on the stock market. Figure 4.17 depicts time windows of asset prices and dividends for both scenarios. For $t \geq 120$ the evolution of the series' is the same in both scenarios and similar to the case where $\tau_t \equiv 0$ as studied in Section 4.3. The table below displays the levels of asset prices and dividends pertaining to the initial case where $\tau_t = 0.2$ and the long-run case where $\tau_t = 0$ together with the associated percentage change. Clearly, these values do not depend on the adjustment policy.

As one would expect from the results of Section 4.3, the reduction in contributions induces an upward shift in the level of asset prices which triggers a temporary boom on the stock market. This boom is much steeper with an instantaneous reduction where the new long run level is reached within ≈ 10 periods. With a gradual adjustment the transition is much smoother resulting in a constant increase in asset prices over a time window of ≈ 40 periods. The lower row in Figure 4.17 and also

[6] Clearly, from a political point of view an instantaneous abandonment of the pension system seems unrealistic since it implies that some generations do not receive any pension payments although they have paid contributions. Nevertheless, this scenario provides an interesting benchmark case.



(a) $\tau_t = 0.2 \downarrow 0$ (b) $\tau_t = 0.2 \searrow 0$

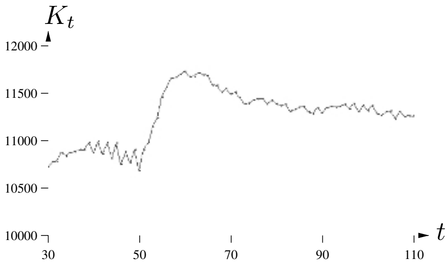
(c) $\tau_t = 0.2 \downarrow 0$ (d) $\tau_t = 0.2 \searrow 0$

	Initial level	Long run level	Change
q_t	15.64	12.44	+ 25.7%
p_t	13.97	10.78	+ 29.6%
d_t	1.678	1.655	+ 1.4%

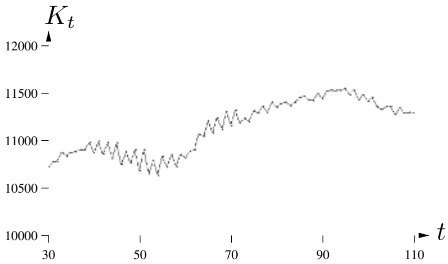
Fig. 4.17. Impact of a reduction in contribution rates on the stock market

the values in the table below show that the boom in cum-dividend prices is almost exclusively due to an increase in ex-dividend prices which increase by almost 30% while dividends increase only slightly by 1.4%. Another observation from the figure is that during the adjustment phase asset prices fluctuate more with a gradual adjustment than with an instantaneous reduction. This can be explained as follows: In the previous sections we have seen that a reduction in contribution rates may reduce the volatility in asset prices. This effect is delayed with a gradual adjustment policy such that asset prices fluctuate more during the adjustment phase. With an instantaneous adjustment some of the volatility on asset markets is therefore 'avoided'.

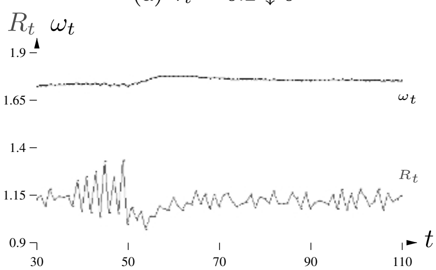
The impact of the two adjustment policies on the capital stock, real wages and bond returns are depicted in Figure 4.18. Again it can be shown that for both scenarios the respective time series' coincide at the end of the time window. The table below displays the long-run effect of the adjustment on the level of the respective variable.



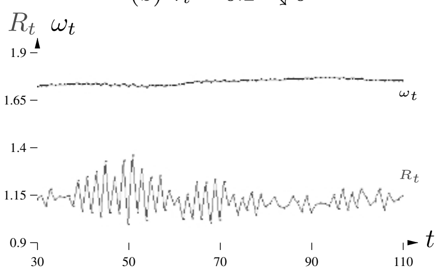
(a) $\tau_t = 0.2 \downarrow 0$



(b) $\tau_t = 0.2 \searrow 0$



(c) $\tau_t = 0.2 \downarrow 0$



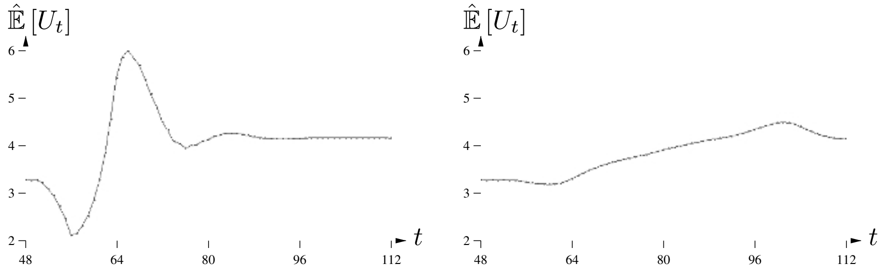
(d) $\tau_t = 0.2 \searrow 0$

	Initial level	Long run level	Change
K_t	11320	10815.3	+ 4.7%
R_t	1.12	1.156	- 3.1%
ω_t	1.756	1.729	+ 1.6%

Fig. 4.18. Impact of a reduction in contribution rates on real markets

Qualitatively, the effects observed on the stock market translate – with a slight delay – into a similar behavior of the capital stock. The sample moments reveal that the reduction in contribution rates increases the long-run capital level by a little less than 5%. Again the adjustment to the new level is much faster with an instantaneous reduction which is accompanied by a significant overshooting. The gradual adjustment causes a delay in the level-effect as well as in the volatility-effect resulting in larger fluctuations of the capital stock during the adjustment phase. Apart from that, the impact on the level of real wages and bond returns are again as one would expect from our previous results: The reduction in the contribution rate decreases the bond return by $\approx -3\%$ while it increases the real wage by $\approx 1.6\%$. One also observes that the overshooting in the capital stock causes a slight overshoot in real wages which temporarily increase above their long-run level. As in the previous cases the adjustment to the new level is much slower with a gradual adjustment which again results in a temporarily higher volatility of bond returns as compared to the case with an instantaneous adjustment. Hence, we find again that the smooth transition comes at a cost of increased volatility during the adjustment phase which is somewhat avoided in the case with an instantaneous adjustment.

In a final step, we display the consequences of either adjustment policy on consumer welfare by comparing the expected lifetime utilities attained by the different generations during the transition phase. Clearly, it is not possible in this case to make use of the ergodic theorem and to calculate expected utilities from time averages during the transition phase. We therefore simulate the model drawing $N = 800$ independent realizations of the noise process. For each t the expected lifetime utility is calculated by taking averages of lifetime utilities observed for each realization of the noise process. The result is shown in Figure 4.19 which depicts the expected lifetime utilities attained by consumers during the transition phase.



(a) $\tau_t = 0.2 \downarrow 0$ (b) $\tau_t = 0.2 \searrow 0$

Fig. 4.19. Impact of a reduction in contributions on expected lifetime utilities

For the instantaneous adjustment policy one observes a dramatic loss in utilities during the time window $t \in \{51, \ldots, 56\}$. These are the generations which lose their pension income but do not benefit from the reduction in contribution rates during their working years. From $t = 57$ utility starts to increase again. These are now the generations which have increasingly benefitted from the reduced contribution rate during their working years. In this regard, recall that the reduction in contributions causes a shift of consumption/utility from the retirement age to the working age. Again we observe a large overshooting in utility which is most likely due to the stock market boom and the temporary overshooting in real wages. After the peak in $t = 66$ utility decreases to reach its long-run level from $t = 85$ onwards.

In contrast to this, the initial downturn as well as the following overshoot is almost entirely avoided with a gradual reduction in contribution rates. One also observes that with the gradual adjustment policy all generations are better-off from $t = 64$ while with an instantaneous adjustment this is the case from $t = 61$ onwards. Nevertheless, from a political point of view, the gradual reduction of contributions appears to be more favorable since it generates a much smoother transition and improves the welfare of all generations in the long run.

Summary of Chapter 4

This chapter analyzed the long-run effects of a static pension system by comparing different levels of contribution rates under the assumption of a stationary population. It was demonstrated that any reduction in contributions fosters the accumulation of capital and leads to a higher capital stock in the long run. This shows that the results obtained, e.g., by [35], carry over to the present case with randomness and a stochastic asset market. In addition, any reduction in contributions was shown to increase the level of asset prices while decreasing interest rates. Qualitatively, the latter result is in line with the findings by [18] who claim that a smaller contribution rate reduces the return on savings respectively capital. For the stochastic setting employed here, an additional volatility effect was established by showing that a reduction in the pension system may exert a stabilizing effect on financial markets by reducing the volatility in asset prices and avoiding stock market crashes.

From a normative point of view, the study revealed that the presence of a pension system exhibits a distorting effect on the long-run welfare of consumers. As a consequence, any decrease in contribution rates increases expected lifetime utility and thus leads to a welfare improvement of consumers in the long-run. For the case with an initially existing pension system, a gradual reduction of contributions was shown to induce a smooth transition with only small welfare losses on the part of current retirees and higher expected utility of all generations in the long run.

One possible explanation for the inefficiency of a public pension system in our model may be obtained by comparing the returns on asset markets with the (internal) rate of return earned by the pension system. In a deterministic framework, the latter corresponds to the sum of the population growth rate and the rate of technical progress which would be zero in the scenario studied in this chapter. Since the interest rate on the bond market was shown to be positive in our study, the previous results may not be too surprising. On the other hand, one could conjecture from these insights that the results may change as one incorporates changes in the population and technical progress. While the first modification will be studied in the following chapter, the other extension of the model is straightforward and left for future research.

4.A Mathematical Appendix

4.A.1 Concepts from Random Dynamical Systems Theory

In this section we supplement the discussion from Section 4.1 by the formal concepts from random dynamical systems theory. In particular, a formal definition of a stable random fixed point is provided.

In order to write the dynamic evolution defined by (4.4) as a random dynamical system we reformulate the noise process introduced in Assumption 4.1.1 as a so-called metric dynamical system with two sided time.[7] This essentially amounts to using the (equivalent) canonical representation of the process $\{\eta_t\}_t$.[8] Define the cube $[0, \eta_{max}] := \prod_{m=1}^{M}[0, \eta_{max}^{(m)}]$ where $\eta_{max} := (\eta_{max}^{(m)})_{m=1}^{M}$ and endow the product space $\Omega := \prod_{t\in\mathbb{Z}}[0, \eta_{max}]$ with its Borel-σ algebra $\mathcal{F} := \mathcal{B}(\Omega)$ and the product measure $\mathbb{P} := \otimes_{t\in\mathbb{Z}}\nu_\eta$. The left shift on Ω is defined as the map $\vartheta : \Omega \longrightarrow \Omega$, $(\tilde{\omega}_t)_{t\in\mathbb{Z}} \longmapsto \vartheta(\tilde{\omega}_t)_{t\in\mathbb{Z}} := (\tilde{\omega}_{t+1})_{t\in\mathbb{Z}}$ the inverse of which is denoted as ϑ^{-1}. For each $t \in \mathbb{Z}$ let ϑ^t denote the tth iterate of the map ϑ if $t > 0$ and of ϑ^{-1} if $t < 0$, respectively. With the help of the evaluation map $\eta : \Omega \longrightarrow [0, \eta_{max}]$, $\eta\big((\tilde{\omega}_t)_{t\in\mathbb{Z}}\big) := \tilde{\omega}_0$ the original process $\{\eta_t\}_{t\in\mathbb{Z}}$ can be written in the form $\big\{\eta(\vartheta^t)\big\}_{t\in\mathbb{Z}}$, i.e., for each $t \in \mathbb{Z}$ and $\tilde{\omega} \in \Omega$, $\eta_t(\tilde{\omega}) = \eta(\vartheta^t\tilde{\omega})$. Note that by construction the measure $\mathbb{P}$ is invariant with respect to ϑ, i.e. for all $A \in \mathcal{F}$ one has $\mathbb{P}(\vartheta A) = \mathbb{P}(A)$. In addition we shall assume that $\mathbb{P}$ is ergodic with respect to ϑ, i.e., for each $A \in \mathcal{F}$ which is invariant under the map ϑ (i.e. $\vartheta A = A$) one has $\mathbb{P}(A) \in \{0, 1\}$. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t)_{t\in\mathbb{Z}})$ defines an ergodic

[7] Note that the noise process is modeled with two-sided time although the induced random dynamical system introduced below is modeled with one-sided time. The same approach is adopted, e.g., in [57].

[8] In this section the symbol $\mathbb{Z}$ denotes the set of integers rather than the portfolio space. Since the use is restricted to the present section, no confusion should arise.

metric dynamical system in the sense of [4] which constitutes the first building block of a random dynamical system.

Using the previous representation (4.4) may now be written as

$$\xi_t = \phi_\tau(\xi_{t-1}, \eta(\vartheta^t \tilde{\omega})).$$

To alleviate the notation write $\phi_\tau(\eta(\tilde{\omega})) := \phi_\tau(\cdot; \eta(\tilde{\omega})) : \Xi \longrightarrow \Xi$, $\xi \longmapsto \phi_\tau(\eta(\tilde{\omega}))\,\xi := \phi_\tau(\xi; \eta(\tilde{\omega}))$ for each $\tilde{\omega} \in \Omega$ and let $\mathbb{T} := \mathbb{N}_0$. For each fixed $\tau \in [0, \bar{\tau}]$ the iteration of the map $\phi_\tau(\cdot)$ defines a measurable flow $\Phi_\tau : \mathbb{T} \times \Omega \times \Xi \longrightarrow \Xi$

$$\Phi_\tau(t, \tilde{\omega}, \xi) := \begin{cases} \xi & t = 0 \\ \phi_\tau\left(\eta(\vartheta^t \tilde{\omega})\right) \circ \ldots \circ \phi_\tau\left(\eta(\vartheta \tilde{\omega})\right) \xi & t \geq 1 \end{cases} \tag{4.14}$$

Equation (4.14) in conjunction with the noise process represented by the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t)_{t \in \mathbb{Z}})$ defines a random dynamical system in the sense of [4] with one-sided time $\mathbb{T} = \mathbb{N}_0$. The value $\Phi_\tau(t, \tilde{\omega}, \xi_0) \in \Xi$ determines the state of the system at time $t \in \mathbb{T}$ from the initial state $\xi_0 \in \Xi$ and the path $\tilde{\omega} \in \Omega$ of the perturbation. The following definition introduces the concept of a stable random fixed point which plays a crucial role in this chapter.

Definition 4.A.1 *Let $\tau \in [0, \bar{\tau}]$ be fixed. A random fixed point of the random dynamical system defined by (4.14) is defined by a random variable $\xi^\star_\tau : \Omega \longrightarrow \Xi$ with the property that for each $\tilde{\omega} \in \Omega$*

$$\xi^\star_\tau(\vartheta \tilde{\omega}) = \phi_\tau(\eta(\vartheta \tilde{\omega}), \xi^\star_\tau(\tilde{\omega})) = \phi_\tau(\eta(\vartheta \tilde{\omega}))\xi^\star_\tau(\tilde{\omega}). \tag{4.15}$$

A random fixed point $\xi^\star_\tau$ is said to be asymptotically stable if for each fixed $\tilde{\omega} \in \Omega$ there exists a random set $U(\tilde{\omega}) \subset \Xi$ such that

$$\lim_{t \to \infty} \|\xi^\star_\tau(\vartheta^t \tilde{\omega}) - \Phi_\tau(t, \tilde{\omega}, \xi_0)\| = 0 \tag{4.16}$$

for all $\xi_0 \in U(\tilde{\omega})$ $\mathbb{P}$–a.s.

In the stable case, the path $t \longmapsto \Phi_\tau(t, \tilde{\omega}, \xi)$ will asymptotically move as the corresponding path $t \longmapsto \xi^\star_\tau(\vartheta^t \tilde{\omega})$ of the random fixed point ($\mathbb{P}$–a.s.). Clearly, for those components of the map ϕ_τ which are deterministic and thus independent of $\tilde{\omega} \in \Omega$, the property (4.15) coincides with that of a deterministic fixed point. Therefore, since the population law (2.51) is deterministic, the population will necessarily be constant along any path of the random fixed point. As an immediate consequence, we obtain as a necessary condition for the existence of a random fixed point the existence of a steady state of the population law (2.51). Asymptotic stability of this random fixed point then requires, in addition, the asymptotic stability of the fixed point of the population dynamics.

4.A.2 Proof of Lemma 4.4.1

Adopting a similar argument as in the proof of Lemma 4.1, the sequential structure of the model (cf. Section 3.5) implies the existence of continuous mappings $C^{(j)} : \Xi \times \prod_{m=1}^{M}[0, \eta_{max}^{(m)}] \longrightarrow \mathbb{R}_{++}$, $j \in \{0, 1, \ldots, J\}$ which determine consumption at time t as

$$c_t^{(j)} = C^{(j)}(\xi_{t-1}, \eta_t). \tag{4.17}$$

Let $\tilde{\omega} \in \Omega$ be fixed and let $\xi_0 \in U(\tilde{\omega})$ be such that, by stability of the random fixed point

$$\lim_{t\to\infty} \|\xi_\tau^\star(\vartheta^t\tilde{\omega}) - \Phi_\tau(t, \tilde{\omega}, \xi_0)\| = 0. \tag{4.18}$$

Fix $j \in \{0, 1, \ldots, J\}$. Using (4.17) and the canonical representation of the noise process (cf. Section 4.A.1) consumption at time $t > 0$ as defined in (4.9) may be written as

$$c_t^{(j)}(\tau, \tilde{\omega}, \xi_0) = C^{(j)}\big(\Phi_\tau(t-1, \tilde{\omega}, \xi_0), \eta(\vartheta^t\tilde{\omega})\big). \tag{4.19}$$

Similarly, consumption at time $t > 0$ along the random fixed point as defined in (4.12) takes the form

$$c_\tau^{(j)\star}(\vartheta^t\tilde{\omega}) = C^{(j)}(\xi^\star(\vartheta^{t-1}\tilde{\omega}), \eta(\vartheta^t\tilde{\omega})). \tag{4.20}$$

Let $\varepsilon > 0$ be arbitrary. We show that there exists $t_0 > 0$ such that

$$t > t_0 \ \Rightarrow \ \|c_t^{(j)}(\tau, \tilde{\omega}, \xi_0) - c_\tau^{(j)\star}(\vartheta^t\tilde{\omega})\| < \varepsilon. \tag{4.21}$$

The map $C^{(j)}(\cdot, \eta(\vartheta^t\tilde{\omega}))$ being continuous at each point $\xi \in \Xi$ implies for each $\varepsilon > 0$ the existence of a $\delta > 0$ such that

$$\|\xi - \xi'\| < \delta \Rightarrow \|C^{(j)}(\xi, \eta(\vartheta^t\tilde{\omega})) - C^{(j)}(\xi', \eta(\vartheta^t\tilde{\omega}))\| < \varepsilon. \tag{4.22}$$

On the other hand, by stability of the random fixed point it follows from (4.18) that for each $\delta > 0$ there exists $t_0 > 0$ such that

$$t > t_0 \Rightarrow \|\xi_\tau^\star(\vartheta^t\tilde{\omega}) - \Phi_\tau(t, \tilde{\omega}, \xi_0)\| < \delta. \tag{4.23}$$

Combining the definitions (4.19) and (4.20) with (4.22) and (4.23) proves (4.21). Since j was arbitrary, this gives the claim in (i) for each $j \in \{0, 1, \ldots, J\}$. The assertion in (ii) is then implied by continuity of the logarithm. ■

5

Pension Systems in the Presence of Demographic Change

The introduction as well as the discussion in Section 2.6 has revealed that the predicted change in the German population structure over the next fifty years entails serious consequences for the public pension system. Against this background, the question how the pension system should be adjusted to meet the demographic challenge has been the subject of numerous political debates in Germany as well as in most other industrialized countries. Although these discussions have resulted in several reforms of the German pension system, many economists and demographic experts claim that the measures enacted yet are not sufficient to meet the demographic problem, cf. [56].

The goal of this chapter is to contribute to the debate and to study the performance of various pension policies describing possible adjustments of the pension system during demographic transitions. The policies under scrutiny also comprise reform proposals which have been suggested for or implemented in the German pension system. To obtain a suitable demographic scenario the present chapter uses a particular form of the general population model introduced before. The employed specification is suitable to describe demographic transitions during which the population shrinks and the dependency ratio increases. Since ultimately the population becomes stationary again, all results obtained in the previous chapter remain valid in the long run.

The chapter is organized as follows: Section 5.1 refines the underlying population model and develops the demographic scenario used in the sequel. Section 5.2 considers the case with a static pension system and constant contributions while Section 5.3 studies the effect of a gradual reduction in contributions. Section 5.4 analyzes the consequences of a temporary increase in contributions according to the Riester-Rürup formula which is currently implemented in the German pension system.

Finally, Section 5.5 analyzes a temporary increase in the retirement age as another option to adjust the pension system during demographic transitions.

5.1 Population Dynamics and Demographic Change

To incorporate the issue of demographic change the present chapter refines the general population law (2.51). In this regard, recall from (2.51) that the evolution of the population is essentially governed by the map $\mathcal{N}$ which determines the number of births $N_t^{(J)}$ at time t from the previous population vector N_{t-1}. In the sequel we assume that the map $\mathcal{N}$ is of the form

$$\mathcal{N}(N_{t-1}) = \sum_{j=0}^{J} N_{t-1}^{(j)} \Big(n_0^{(j)} + n_1^{(j)} \exp\Big\{ -n_2 \sum_{i=0}^{J} N_{t-1}^{(i)} \Big\} \Big) \tag{5.1}$$

where $n_0^{(j)}, n_1^{(j)} \geq 0$, $j = 0, \ldots, J$ and $n_2 \geq 0$. If $n_2 = 0$, the population law (5.1) is linear with constant fertilities of generations $j = 0, \ldots, J$. If $n_2 > 0$, fertility is a decreasing function of the previous population size $\sum_{i=0}^{J} N_{t-1}^{(i)}$. If $J = 0$ (corresponding to the degenerate case where consumers live for one period only), equation (5.1) reduces to the following Ricker-type map where the population evolves as

$$N_t = \mathcal{N}(N_{t-1}) = N_{t-1}\big(n_0 + n_1 \exp\{-n_2 N_{t-1}\}\big). \tag{5.2}$$

The dynamic equation (5.2) is widely used in biological sciences to model the evolution of (human and non-human) populations, see, e.g., [28] or [42]. Apart from this justification the functional form (5.1) turns out to be convenient in order to model demographic transitions of the population due to its dynamic properties.

Let $\hat{\mathcal{N}} : \mathbb{R}_+^{J+1} \longrightarrow \mathbb{R}_+^{J+1}$, $\hat{\mathcal{N}}((N^{(j)})_{j=0}^{J}) := ((N^{(j)})_{j=1}^{J}, \mathcal{N}((N^{(j)})_{j=0}^{J}))$ denote the time-one map of the population law defined by (2.51) and (5.1). Given an initial population vector $N_0 = (N_0^{(j)})_{j=0}^{J}$ the map $\hat{\mathcal{N}}$ defines the evolution of the population as a deterministic dynamical system of the form $N_t = \hat{\mathcal{N}}(N_{t-1})$ for each $t \geq 1$. Some properties of the population dynamics are stated in the following lemma.

Lemma 5.1.1 *In addition to the previous restrictions suppose that the parameters of the map (5.1) satisfy* $1 - n_1 < n_0 < 1$ *and* $n_2 > 0$ *where* $n_0 := \sum_{j=0}^{J} n_0^{(j)}$ *and* $n_1 := \sum_{j=0}^{J} n_1^{(j)}$. *Then the following holds true:*

(i) The dynamical system defined by the map $\hat{\mathcal{N}}$ has a unique positive steady state $N^\star = (\bar{N})_{j=0}^J$ where

$$\bar{N} := \frac{1}{(J+1)n_2} \ln \frac{n_1}{1-n_0} > 0.$$

(ii) The steady state in (i) is asymptotically stable if all eigenvalues of the Jacobian matrix

$$D\hat{\mathcal{N}}(N^\star) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ \delta^{(0)} & \delta^{(1)} & \delta^{(2)} & \dots & \delta^{(J)} \end{bmatrix}$$

where $\delta^{(j)} := n_0^{(j)} + n_1^{(j)} \frac{1-n_0}{n_1} + \frac{(1-n_0)}{J+1} \ln\left(\frac{1-n_0}{n_1}\right)$, $j = 0, 1, \dots, J$ lie inside the complex unit disc.

Proof. Solving the condition $\bar{N} \stackrel{!}{=} \mathcal{N}((\bar{N})_{j=0}^J)$ the stated form of the steady state in (i) can be verified by direct calculations. Noting that $\hat{\mathcal{N}}$ is differentiable the Jacobian matrix evaluated at the steady state can be shown to be of the form stated in (ii) from which the second assertion follows. ■

In the sequel we model demographic change as a transitory phenomenon which is due to shifts in the steady state of the population dynamics. In this regard, an immediate observation from Lemma 5.1.1 is that the parameter n_2 is crucial to determine the steady state value $\bar{N}$ in (i) while the stability condition (ii) is independent of it. This property allows us to vary the value $\bar{N}$ by varying n_2 without affecting its asymptotic stability.

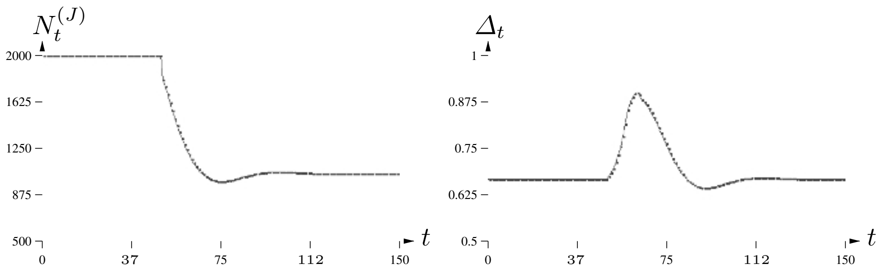
For the following numerical investigation suppose for simplicity that $n_0^{(j)} \equiv n_1^{(j)} =: n^{(j)}$ for all $j = 0, \dots, J$ and assume the map $j \longmapsto n^{(j)}$ to be non-decreasing. Recalling that a consumer's life starts at the age of 20 ($j = J$) and ends at the age of 80 ($j = 0$) with each period corresponding to four years this property reflects the plausible assumption that fertility is a decreasing function of age. The parameters used in the subsequent simulations are summarized in the following table. The values in parenthesis describe the age of agents in the respective generation.[1]

[1] It should be noted that neither the employed population model nor the parameter choices are justified on empirical grounds. For our purpose the prescribed speci-

Table 5.1. Parameter values for the population dynamics

Parameter	Value	Description	Parameter	Value	Description
J	14	Number of gen.	$n_0^{(11)}$	0.1	Fertility (32-35)
$n^{(14)}$	0.275	Fertility (20-23)	$n^{(10)}$	0.05	Fertility (36-39)
$n^{(13)}$	0.25	Fertility (24-27)	$n^{(9)}$	0.01	Fertility (40-43)
$n^{(12)}$	0.2	Fertility (28-31)	$n^{(j)}, j \leq 8$	0	Fertility (≥ 44)

The parameter values in Table 5.1 satisfy the stability condition (ii) in Lemma 5.1.1 such that for any choice $n_2 > 0$ the induced steady state will be asymptotically stable. The demographic transition is now modeled as follows. Assume that initially ($t \leq 50$) the population is in a steady state such that $N_t \equiv (\bar{N})_{j=0}^J$ where $\bar{N} \approx 2000$ corresponding to a parameter choice $n_2 = 0.000067$. Note that the model's behavior at the initial steady has already been studied in the sensitivity analysis in Section 4.5. In period $t = 51$ the parameter n_2 changes to $n_2 = 0.00013$ shifting the steady state to a lower value $\bar{N}' \approx 1000$. The associated adjustment process of the population towards the new steady state value then defines a demographic transition period during which the number of births decreases. Note that the shifted steady state value $\bar{N}'$ corresponds to the generation size used in the simulations of the previous chapter. Hence, the previous results remain valid in the long run as soon as the population has reached the new steady state. Figure 5.1 depicts the evolution of the population represented by the number of births as well as the associated dependency ratio as defined in (2.61).



(a) Number of young consumers (b) Dependency ratio

Fig. 5.1. Demographic transition of the population

fications offer a simple way to model demographic transitions of the population and to study the consequences for the public pension system.

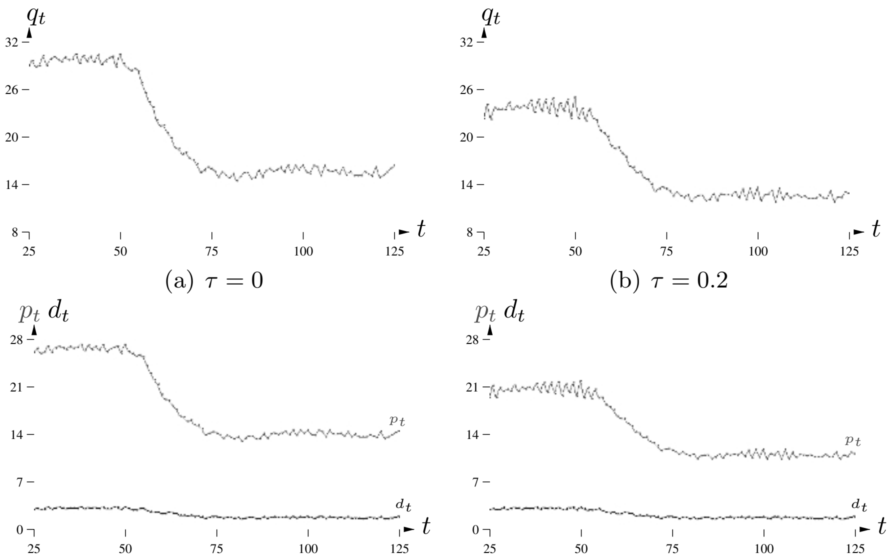
One observes from Figure 5.1(a) that the level of births converges to its new steady state value within slightly less than 50 periods. As a consequence it can be shown that for $t \geq 110$ the population will be constant again such that $N_t \equiv (\bar{N}')_{j=0}^{J}$ and the dynamic behavior is as described in the previous chapter. In the sequel we will therefore restrict attention to the demographic transition period where $t \in \{51, \ldots, 110\}$. As is seen from Figure 5.1(b) the demographic transition is accompanied by a significant temporary increase in the dependency ratio. The latter reaches a maximum of $\approx 90\%$ in $t = 63$, i.e., 12 periods (≈ 50 years) after the shift before it eventually returns to its initial value of 66%. This range corresponds roughly to the predicted evolution for the German population over the next 50 years which was presented in Section 2.6 (cf. Figure 2.6). Thus, the simple demographic scenario developed in this section mimics roughly the predicted demographic change of the German population over the next 50 years.

5.2 Constant Contributions

Based on the demographic scenario developed in the previous section the remainder of this chapter studies how demographic change affects real and financial markets and consumer welfare. Moreover, different adjustments of the pension system during the transition and their consequences are investigated.

The present section studies the case of a constant contribution rate $\tau_t \equiv \tau$ where the three cases $\tau \in \{0, 0.1, 0.2\}$ are compared. All other parameters of the model are kept at the same values as in the previous chapter which are listed Table 4.1.[2] As a first step, consider the impact of demographic change on real and financial markets. Figure 5.2 depicts the evolution of the stock market during the demographic transition period for the two boundary cases where $\tau = 0$ and $\tau = 0.2$. The table below displays the level of cum-dividend prices associated with the initial ($\bar{N} = 2000$) as well as with the shifted ($\bar{N}' = 1000$) steady state value of the population together with the associated percentage change. Note that these values correspond to the ones which were calculated in the sensitivity analysis in Section 4.5 (cf. Table 4.2).

[2] Due to the larger initial population size the initial values for the capital stock and cum-dividend price are adjusted to $K_0 = 22,000$ and $q_0 = 25$. This improves the model's behavior during the first simulation periods. Since, as shown in the previous chapter, the model's behavior (for $t \geq 50$) does not depend on initial conditions, this does not impact the following results.

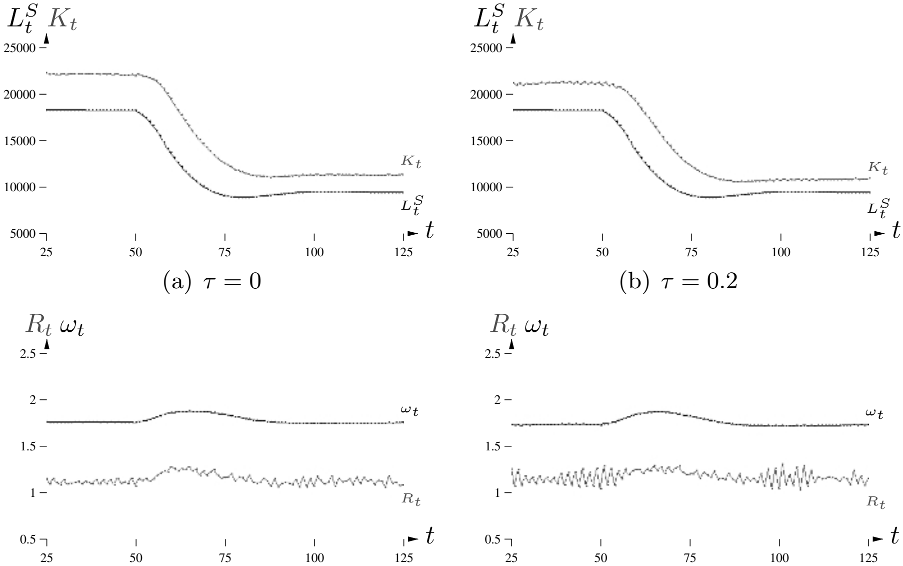


Level of cum-dividend prices q_t			
	$\bar{N} = 2000$	$\bar{N}' = 1000$	Change
$\tau = 0$	29.66	15.64	- 47.27%
$\tau = 0.1$	26.62	14.06	- 47.18%
$\tau = 0.2$	23.65	12.44	- 47.40%

Fig. 5.2. Impact of demographic change on the stock market

The most striking phenomenon observed from Figure 5.2 is the dramatic decline in the level of asset prices. The table below reveals that the shrinking population size triggers a decline in asset prices resulting in a loss of approximately 50%! Although the level of the respective price process depends on the contribution rate, the percentage loss is roughly the same for all three scenarios. Furthermore, as the lower row in Figure 5.2 shows, the decline in cum-dividend prices is almost entirely due to a decrease in ex-dividend prices. This phenomenon of a so-called asset market meltdown induced by demographic change of the population has been predicted by various models in the literature, see, e.g., [1], [2]. In fact, this result could have been anticipated from our sensitivity analysis in Section 4.5 (cf. Table 4.2) which showed that a smaller population size leads to a lower level of asset prices. In the present scenario the percentage change in the level of asset prices turns out to of roughly the same magnitude as the percentage change in the size of the population.

Next consider Figure 5.3 showing the evolution of labor supply and the capital stock as well as real wages and bond returns. The table below compares the initial and long run level of the capital stock together with its percentage change.



(a) $\tau = 0$ (b) $\tau = 0.2$

(c) $\tau = 0$ (d) $\tau = 0.2$

Level of the capital stock K_t			
	$\bar{N} = 2000$	$\bar{N}' = 1000$	Change
$\tau = 0$	22094.5	11320.0	- 48.8%
$\tau = 0.1$	21679.1	11101.9	- 48.8%
$\tau = 0.2$	21173.1	10815.3	- 48.9%

Fig. 5.3. Impact of demographic change on real and financial variables

Due to the absence of technical progress in our model the decline in birth rates translates directly into a decline in aggregated labor supply (cf. equation (2.1)). The capital stock mirrors this development with a slight delay resulting again in a loss of almost 50%. While the percentage loss is the same in all three scenarios, the initial as well as the long-run level is – as one would expect from the results of the previous chapter – higher with a lower contribution rate. In contrast to that, the long run level of real wages and bond returns is not affected by the decline in the population and both series return to their initial value after the population becomes constant again. However, a temporary increase in both series is observable during the transition periods. For real wages this is due to the increased scarcity of labor which causes a temporary increase in the capital-labor ratio. Ultimately the capital stock adjusts resulting in the same capital-labor ratio as initially. Similar to the results obtained in the previous chapter the effect of a larger volatility in bond returns due to a larger contribution rate is present.

In a second step consider the impact of demographic change on the welfare of consumers. Figure 5.4 shows the expected lifetime utilities of generations during the demographic transition period for all three scenarios. As in Section 4.4 the expected utility during each transition period is calculated by taking averages over $L = 800$ independent realizations of the noise process.

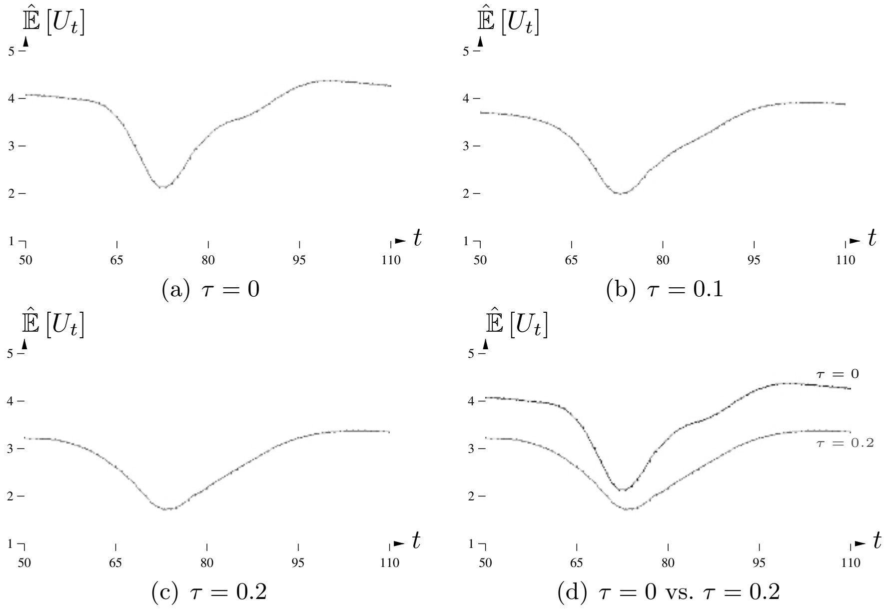


Fig. 5.4. Impact of demographic change on expected lifetime utilities

One observes that demographic change causes a significant decline in consumer welfare which inversely mirrors – with a slight delay – the evolution of the dependency ratio (cf. Figure 5.1(b)). This decline occurs independently of the prevailing contribution rate and is only slightly attenuated if the contribution rate is lower (although expected utility remains higher with a lower contribution rate throughout the entire time window). After the demographic transition expected utility returns again to a stationary level which is in fact higher than the initial level confirming the result of the sensitivity analysis carried out in Section 4.5 (cf. Table 4.4).

These observations suggest that without further adjustments of the pension system, demographic change has a serious impact on consumer welfare resulting in large welfare losses during the demographic transition periods. The following statement summarizes the main results of this section.

Numerical Result 5.2.1 *Let the model be parameterized as in Table 4.1 and the demographic scenario as introduced in Section 5.1. Furthermore, let the contribution rate be constant. Then the following holds true:*

(i) Demographic change causes a permanent decline in the levels of asset prices and the capital stock reducing their initial levels by approximately 50%*. This phenomenon occurs independently of the level of contribution rates.*
(ii) Consumers suffer large welfare losses during the demographic transition period which are only slightly attenuated with a lower contribution rate τ*.*

5.3 Reducing Contributions

With a constant contribution rate the demographic transition entails a dramatic loss in consumer welfare and a substantial decline in the levels of asset prices and the capital stock. This raises the natural question how the pension system should be adjusted to avoid or at least attenuate these effects. To analyze these questions, the following part takes a slightly different view than before by assuming that initially (i.e., before demographic change starts) a public pension system with a constant contribution rate $\tau_t = 0.2$ exists.

In this section we consider a policy that aims at reducing contribution rates during the demographic transition. To this end, consider the following three experiments:

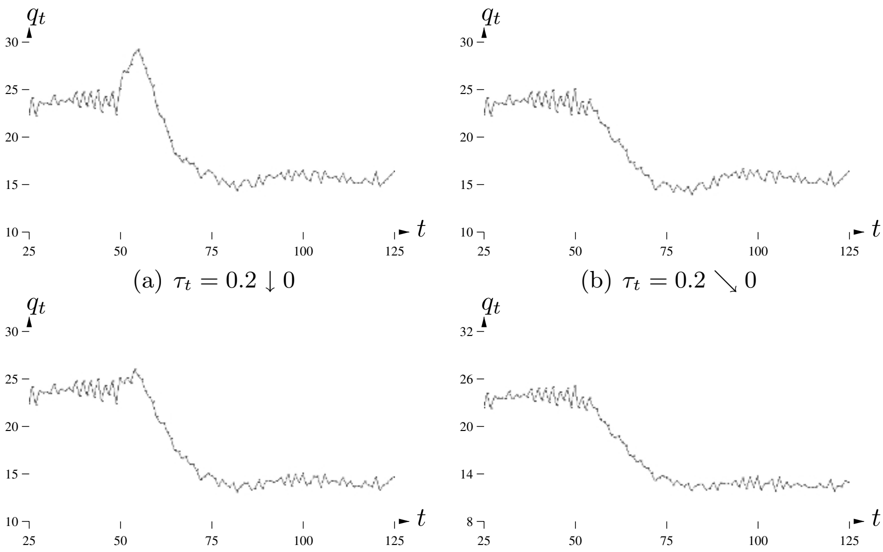
1. From $t = 51$ onwards the initial contribution rate $\tau_t = 0.2$ is instantaneously set to $\tau_t = 0$. As in Section 4.6 we denote this symbolically as $\tau_t = 0.2 \downarrow 0$.

2. From $t = 51$ onwards the initial contribution rate $\tau_t = 0.2$ is gradually lowered by 0.5% points in every period such that $\tau_t = 0$ for $t \geq 90$, we write $\tau_t = 0.2 \searrow 0$.
3. From $t = 51$ onwards the initial contribution rate $\tau_t = 0.2$ is instantaneously set to $\tau_t = 0.1$, written $\tau_t = 0.2 \downarrow 0.1$.

The first two adjustment scenarios correspond to the ones which were studied in Section 4.6 for the case with a stationary population. In addition, a third scenario is considered which does not fully abolish the pension system but only reduces the contribution rate to a lower value. For all three cases the scenario where the contribution rate remains at its initial level of 20% (as studied in the previous section) serves as a reference case.

Consider first the impact of either policy on asset markets which is depicted in Figure 5.5. As before the corresponding initial and long-run levels of cum-dividend prices together with the percentage change are calculated in the table below. The left hand side in Figure 5.5 shows that an instantaneous reduction of contributions to zero triggers a temporary boom in asset prices confirming the results obtained in Section 4.6. This boom is still slightly present with a moderate reduction to 10% but vanishes with a gradual reduction in contributions. From $t = 60$ onwards the initial increase is upset by the previously observed decline in asset prices due to the shrinking population size. However, the table below indicates that the percentage loss is now slightly reduced (from $\approx -47\%$ to $\approx -34\%$ and $\approx -40.5\%$ with a long-run contribution rate of 0% and 10%, respectively). This is due to the fact that the reduction in contributions has an increasing effect on the level of asset prices which partly upsets the demographic impact. However, these measures can only attenuate but not avoid the asset market meltdown.

The impact of either policy on the capital stock is depicted in Figure 5.6. Again the case where $\tau_t \equiv 0.2$ serves as a reference case. By and large, the capital stock mirrors the behavior of the stock market. For the case with an instantaneous reduction an initial boom is observable which is much more present if contributions are reduced to zero and which vanishes entirely with a gradual reduction policy. From $t = 60$ onwards the capital stock decreases to a lower long-run level. From the table below one finds that a (gradual or instantaneous) adjustment of contributions to zero reduces the loss in capital by roughly two percentage points and by a little more than one percentage point if the contribution is reduced to 10%. These findings are in line with our results from the previous chapter showing that a decrease in con-



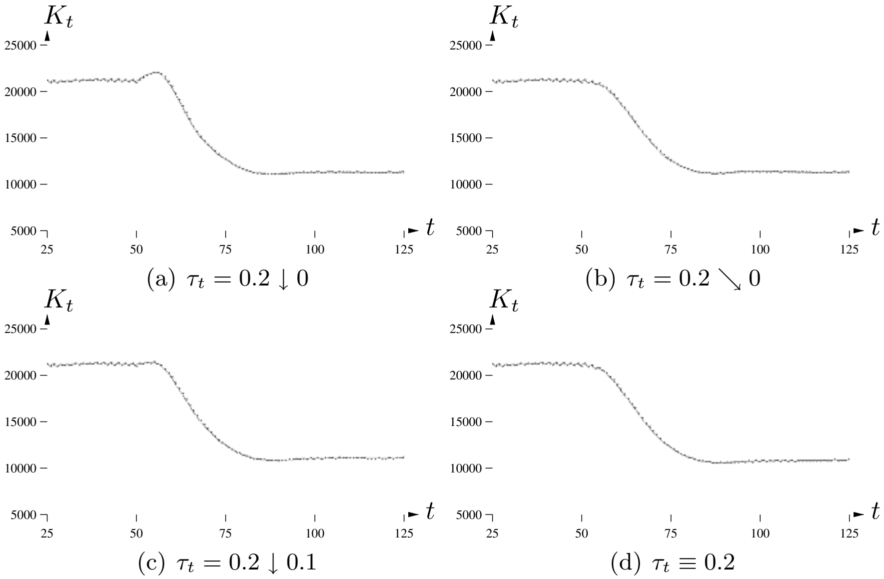
(a) $\tau_t = 0.2 \downarrow 0$ (b) $\tau_t = 0.2 \searrow 0$ (c) $\tau_t = 0.2 \downarrow 0.1$ (d) $\tau_t \equiv 0.2$

Level of cum-dividend prices q_t			
Reform:	$\bar{N} = 2000$	$\bar{N} = 1000$	Change
$\tau_t = 0.2 \downarrow 0$	23.65	15.64	- 33.87%
$\tau_t = 0.2 \searrow 0$	23.65	15.64	- 33.87%
$\tau_t = 0.2 \downarrow 0.1$	23.65	14.06	- 40.55%
$\tau_t \equiv 0.2$	23.65	12.44	- 47.4%

Fig. 5.5. Impact of demographic change on the stock market

tributions increases the level of the capital stock and thus may slightly reduce the loss due to the demographic development.

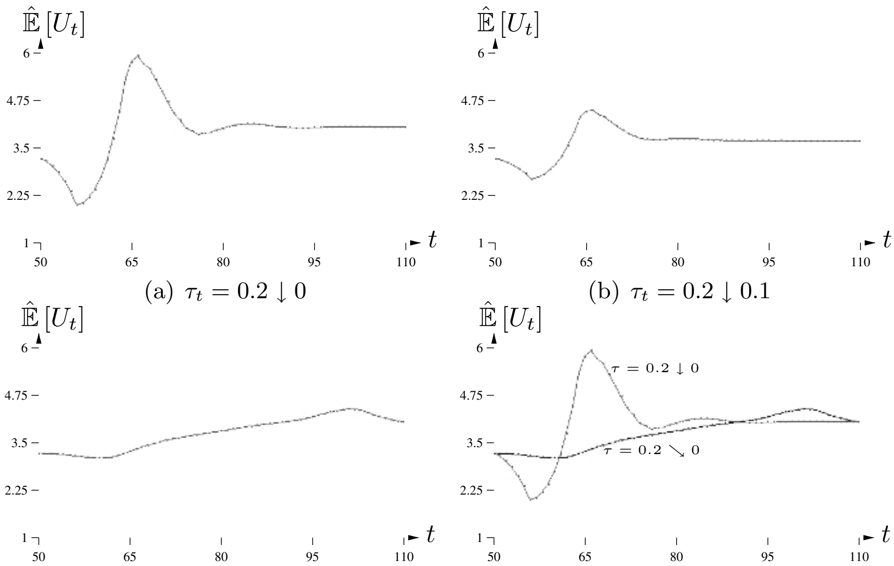
Finally, consider the impact of either adjustment policy on consumer welfare. To carefully separate the demographic effect and the contribution effect of the adjustment policy the analysis proceeds in two steps. The first step investigates the impact of either policy on consumer welfare under the assumption that no demographic change occurs and the population remains at its initial steady state where $\bar{N} = 2000$. This reveals the pure contribution effect and repeats the experiment carried out in Section 4.6 with a different population size and an additional scenario where τ is decreased to a moderate value of 10%. In a second step the overall welfare effect on consumers' lifetime utility resulting from demographic change *and* the reduction in contribution rates is investigated.



Level of capital stock K_t			
Reform:	$\bar{N} = 2000$	$\bar{N}' = 1000$	Change
$\tau_t = 0.2 \downarrow 0$	21173.1	11320.0	- 46.5%
$\tau_t = 0.2 \searrow 0$	21173.1	11320.0	- 46.5%
$\tau_t = 0.2 \downarrow 0.1$	21173.1	11101.9	- 47.6%
$\tau_t \equiv 0.2$	21173.1	10815.3	- 48.9%

Fig. 5.6. Impact of demographic change on the stock market

The first experiment is visualized in Figure 5.7 showing the pure contribution-effect on expected lifetime utilities for either scenario. For the case with an instantaneous reduction in contributions to 0% and 10%, respectively, the outcome is qualitatively the same as in Section 4.6 (cf. Figure 4.19(a)): The reduction in $t = 51$ leads to a significant decline in expected utilities until $t = 56$. These are the generations which suffer large losses in their pension income but do not benefit from reduced contributions during their working age. Clearly, this decline is less dramatic if the contribution rate is only reduced to 10% since then pension incomes do not completely drop to zero. From $t = 57$ onwards utility increases again since now the respective generations have increasingly benefitted from the reduction in contributions during their working years. Again the same overshooting phenomenon as observed in Section 4.6 occurs which is again less severe with a moderate re-



(a) $\tau_t = 0.2 \downarrow 0$

(b) $\tau_t = 0.2 \downarrow 0.1$

(c) $\tau_t = 0.2 \searrow 0$

(d) $\tau_t = 0.2 \downarrow 0$ vs. $\tau_t = 0.2 \searrow 0$

Fig. 5.7. Contribution effect on expected lifetime utilities with a stationary population

duction to 10%. In contrast to that, the gradual adjustment is able to smooth out these two effects almost completely: Utilities decrease only slightly during periods $t \in \{52, \ldots, 59\}$ while from $t = 60$ onwards one observes a steady increase. In all three cases expected utility becomes constant again for $t \geq 110$ at a level which is highest if $\tau_t = 0$ and which throughout exceeds the initial one where $\tau_t = 0.2$.

In a second step, consider the overall effect on consumer welfare depicted in Figure 5.8 for either scenario. Comparing the upper row in Figure 5.8 with Figure 5.4 one observes that the instantaneous reduction of contributions during the demographic transition phase adds a second crash to the utility of generations. The first downturn which occurs from period $t = 51$ until $t = 56$ is due to the contribution-effect as exhibited in Figure 5.7. However, the reduction in contributions still does not avoid the second decline which is induced by the demographic change of the population as studied in Figure 5.4. Note that both crashes are more serious when τ is reduced to zero rather than to 10%. This is to some extent surprising as the previous analysis has shown that the pure demographic effect is slightly attenuated with a lower τ.

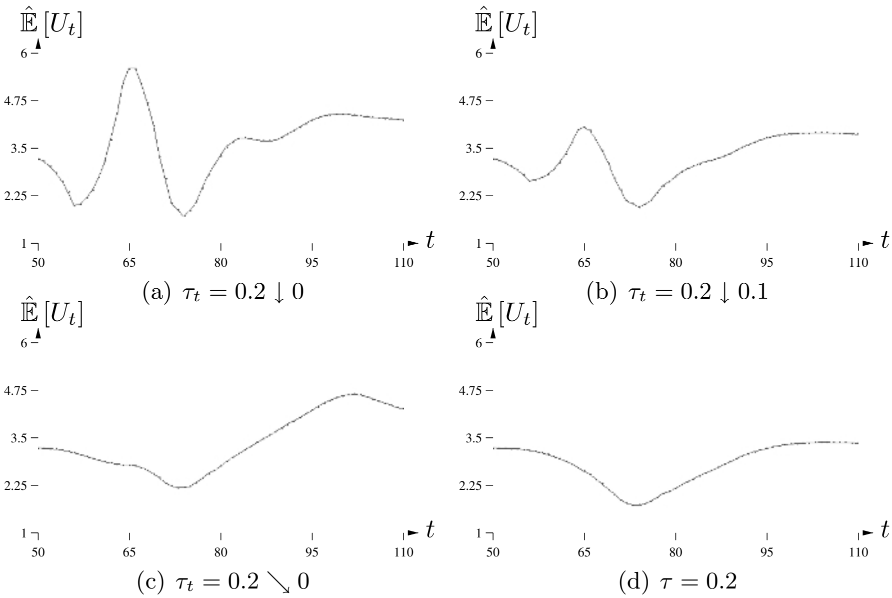


Fig. 5.8. Overall effect on expected lifetime utilities

From the second row in Figure 5.8 it is seen that the first crash can almost entirely be smoothed out by adopting a gradual change in contributions. In addition, a comparison with Figure 5.8(d) shows that this policy attenuates the decline in welfare during the demographic transition. Nevertheless, even a gradual reduction of contributions can not avoid a significant welfare loss during the transition phase.

These observations show quite clearly that an instantaneous abolishment or reduction of the pension system during demographic transitions does more harm than good: It does not avoid (but even amplifies!) the welfare losses, thereby adding a second crash to consumers' utilities due to the sudden reduction in pension payments. Although the former is avoided with a gradual reduction in contributions, the second crash is still present. Summarizing we find that among the four policies discussed a gradual abolishment of the pension system is the most promising one although none of the measures under consideration can fully avoid the welfare loss during the demographic transition.

5.4 Increasing Contributions

The previous section has studied a permanent reduction in contribution rates as one possibility to counteract the consequences of a shrinking population. It was demonstrated that such an adjustment policy can only slightly attenuate the decline in asset prices and the welfare losses induced by the demographic development. This result raises the natural question whether a temporary increase in contributions can do better and provides an appropriate political measure to avoid the welfare losses during the demographic transition.

In order to analyze this issue, the present section restricts attention to a particular pension formula which plays an important role in the adjustment of the German pension system. To this end, suppose that pension payments and contributions at time t are determined as

$$\begin{cases} e_t^R = e_{t-1}^R \, \dfrac{\omega_{t-1}}{\omega_{t-2}} \, \dfrac{b^{(1)} - \tau_{t-1}}{b^{(1)} - \tau_{t-2}} \left(1 + b^{(2)} \left(1 - \dfrac{\Delta_{t-1}}{\Delta_{t-2}}\right)\right) \\ \tau_t \;\;= \dfrac{e_t^R \, N_t^R}{\omega_t \, L_t^S} \end{cases} \tag{5.3}$$

where $0 \le b^{(i)} \le 1$, $i = 1, 2$. The adjustment policy in (5.3) will be referred to as the *Riester-Rürup formula*. It determines the pension income e_t^R at time t essentially from three determinants while contributions τ_t are adjusted accordingly. The first factor $\frac{\omega_{t-1}}{\omega_{t-2}}$ accounts for the previous change in gross real wages. The second factor $\frac{b^{(1)} - \tau_{t-1}}{b^{(1)} - \tau_{t-2}}$ captures changes in the contribution rate over the last two periods where the parameter $b^{(1)}$ is assumed to be sufficiently close to unity such that $b^{(1)} > \tau_t$ for all times t. Note that a decrease in $b^{(1)}$ increases the sensitivity to changes in previous contribution rates. Finally, the third factor $1 + b^{(2)} \left(1 - \frac{\Delta_{t-1}}{\Delta_{t-2}}\right)$ accounts for demographic changes of the population measured by the previous change in the dependency ratio. With this specification an increase in real wages has a positive impact on current pensions while a previous increase in contribution rates has a diminishing impact. Likewise an increase in the factor $\frac{\Delta_{t-1}}{\Delta_{t-2}}$ corresponding to an accelerated aging of the population decreases the pension income at time t.

Based on the general adjustment formula (5.3) the present section studies four different pension policies corresponding to different parameter choices $b^{(1)}$ and $b^{(2)}$ in (5.3). A more detailed description of these policies and their application to the German pension system can be found in [56] and also in [18] who conduct a study in the same spirit.

Case 1: Net Wage Adjustment ($b^{(1)} = 1$, $b^{(2)} = 0$).
With these parameter choices one observes from (5.3) that the growth rates of pension incomes correspond to the previous increase in net wages. Consequently, this type of adjustment is referred to as net wage adjustment. This formula has been in effect in Germany from 1992 until 2001. Note that the demographic structure of the population does not enter the adjustment formula.

Case 2: Riester Adjustment ($b^{(1)} = 0.86$, $b^{(2)} = 0$).
With this specification the demographic development of the population continues to be irrelevant for the evolution of pension incomes and contributions. However, compared to the previous case the reduction in $b^{(1)}$ leads to an increased sensitivity to changes in contribution rates such that this formula dampens the growth of pension payments. The formula corresponds to the one which was introduced for the German pension system as part of the so-called Riester reform in 2001.[3]

Case 3: Rürup Adjustment I ($b^{(1)} = 0.86$, $b^{(2)} = 0.25$).
This adjustment policy not only takes into account the previous increase in contribution rates but also the demographic change of the population. The parameter $b^{(2)}$ may be interpreted as a weight that shifts the demographic burden between retirees and workers. A larger value of $b^{(2)}$ reduces the growth rate of pensions thus shifting more of the burden to retired generations. In addition the increased sensitivity to changes in previous contribution rates is maintained by choosing the same value $b^{(1)}$ as in the previous case. This parametrization corresponds to the adjustment formula suggested by the so-called Rürup Kommission for the German pension system (see [56, pp. 98]).

Case 4: Rürup Adjustment II ($b^{(1)} = 0.86$, $b^{(2)} = 0.5$).
The final adjustment policy corresponds to a reinforced version of the previous one which increases the sensitivity to changes in the population structure. The increase in $b^{(2)}$ thus has a diminishing impact on pension incomes and dampens the growth in contribution rates.

Given these four specifications, consider the induced evolution of contribution rates depicted in Figure 5.9. The initial contribution rate at time $t = 0$ has been set to $\tau_0 = 0.2$ and is represented by the black line in each of the figures. All parameters of the model as well as the underlying demographic scenario are the same as in the previous sections.

[3] In fact, the actual Riester formula is slightly more complicated involving a gradual adjustment of the parameter $b^{(1)}$ over a time period of 10 years. Our specification corresponds to the long-run Riester formula after the year 2011, see [56, p. 98]

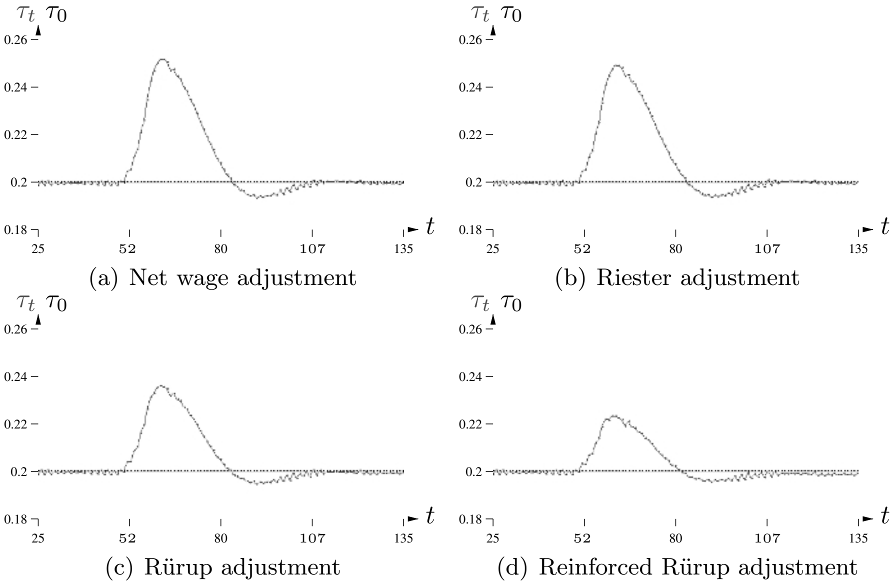
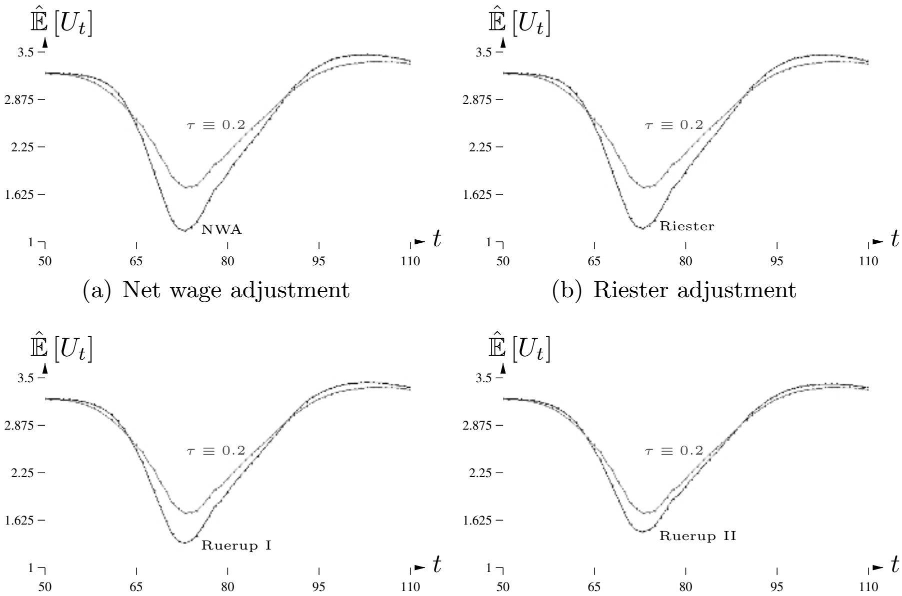


Fig. 5.9. Contribution rates under different adjustment policies

In all four cases depicted in Figure 5.9 contributions are more or less constant for $t \leq 50$, i.e., before the demographic transition. From $t = 51$ onwards the accelerating demographic change causes a significant increase in contribution rates which become largest in the NWA case (up to $\approx 25.5\%$) and least in the Rürup II scenario (up to $\approx 22.5\%$). After the demographic transition period contributions decrease again and eventually return to their initial value of 20% in all four cases. From these observations it is clear that the policies under scrutiny may have an effect only during the demographic transition whereas the long run behavior will be exactly the same as with a constant contribution rate $\tau \equiv 0.2$. Likewise, it is clear that none of these adjustments will avoid the asset meltdown and the decrease in the capital stock observed during the previous sections. In fact, it can be shown that all four policies exhibit only a minor impact on the qualitative behavior of real and financial markets during the demographic transition phase. Consequently, we confine the following analysis to the induced welfare effects of the four reforms and the question, whether the dramatic loss in expected utilities during the demographic transition observed in the previous sections is avoided by either of the four policies.

Figure 5.10 depicts the impact of each policy on consumers' expected lifetime utilities during the transition phase. In addition, the case where contributions remain at their initial level, i.e., $\tau \equiv 0.2$ is depicted as a reference case for all four scenarios.



(a) Net wage adjustment

(b) Riester adjustment

(c) Rürup adjustment

(d) Reinforced Rürup adjustment

Fig. 5.10. Consumer welfare under different adjustment policies

One observes that both the net wage adjustment as well as the Riester policy depicted in Figures 5.10(a) and 5.10(b) yield almost identical results. Both cases suggest that during the first phase of the demographic transition period ($t \leq 64$), it is still possible to stabilize pension incomes through an increase in contribution rates. Hence, both adjustment policies yield slightly higher utility during this time window as compared to the reference case where contributions remain at the initial level of 20%. However, this initial gain comes at a cost of a dramatic loss in the welfare of later generations such that from $t = 65$ onwards, the utility levels in both scenarios are lower than in the reference case. More importantly, the increase in contribution rates leads to a much more dramatic downturn of utility during the time window $t \in \{65, \ldots, 90\}$ when the demographic structure of the population is most unfavorable.

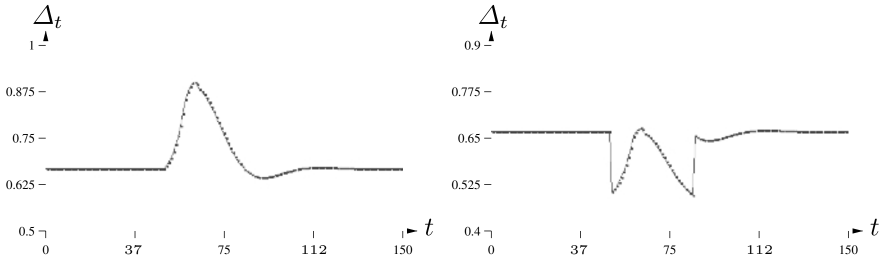
The evolution of expected utility for the remaining two scenarios Rürup I and II are displayed in Figures 5.10(c) and 5.10(d). At first sight, they convey a similar impression as the previous ones: The increase in contribution rates can attenuate the decrease in expected utilities only at the very beginning ($t \leq 64$). After this period, expected utility is lower than in the reference case and suffers a dramatic downturn. However, the dampened increase in contribution rates implied by the Rürup adjustment policy slightly attenuates this loss as compared to the previous cases. Nevertheless, both of the two reforms fail to ameliorate and in fact – as in the previous case – even amplify the welfare loss during the demographic transition.

The results from Figure 5.10 show quite clearly that none of the four adjustment policies studied in this section is capable of attenuating the welfare loss during the demographic transition. In fact, the loss becomes more dramatic as compared to the reference scenario due to the increase in contribution rates during the demographic transition phase which is only somewhat avoided by the reinforced Rürup II reform. These findings unanimously suggest that an adjustment policy leading to a temporarily larger contribution rate is inappropriate to encounter the demographic problem and in fact does more harm than good.

5.5 Increasing the Retirement Age

So far the analysis has focused on changes in contribution rates as a possibility to adjust the pension system during demographic transitions. In this section, we study a temporary increase in the retirement age as another political measure to counteract the demographic problem. In this regard, recall that the retirement age is essentially determined by the parameter j_L which has been set to $j_L = 6$ during all the previous simulations. However, if we regard this parameter as being chosen by government authorities (rather than being determined, e.g., by the physical capabilities of consumers) we can study the impact of a temporary decrease in j_L during the demographic transition phase. A huge advantage of this measure is that it is the only reform option which directly affects and decreases the dependency ratio in (2.61) by simultaneously increasing the number of workers and lowering the number of pensioners.

For the following experiments we assume the same demographic scenario as before and consider a temporary decrease to $j_L = 5$ during the time periods $t \in \{51, \dots, 85\}$. For $t < 51$ and $t > 85$ we have $j_L = 6$ as before. The impact of this policy on the structure of the population is depicted in Figure 5.11 showing the dependency ratio with and without a temporary adjustment of the retirement age.



(a) Constant retirement age (b) Adjusted retirement age

Fig. 5.11. Evolution of the dependency ratio

One observes that the reduction in j_L leads to a sudden decline in the dependency ratio in $t = 51$. In the subsequent periods, the series starts to increase again due to the accelerating demographic change. However, even in period $t = 63$ when the demographic effect reaches its maximum the dependency ratio only slightly exceeds its initial level of 66%. After this, it starts to decrease again since the population now gradually adjusts to its new long-run level. The increase of j_L in $t = 86$ back to its initial value then shifts the ratio upwards to its initial (and new) long-run level.

To study the impact of the proposed adjustment on consumer welfare during the demographic transition period assume first that contributions are constant such that $\tau_t \equiv \tau$. Figures 5.12(a) to 5.12(c) depict the evolution of expected utilities with and without a temporary increase in the retirement age for the three cases where $\tau \in \{0, 0.1, 0.2\}$. In each case the increase in the retirement age leads to a jump in utility from $t = 56$ onwards. These are the generations which now have one additional period of labor income in exchange for one period of pension income. It is therefore clear that the jump increases with a lower τ and is thus largest with $\tau = 0$. In addition, if $\tau > 0$ pension incomes increases during $t \in \{51, \ldots, 55\}$ because now there are fewer retirees who receive benefits from the pension system. This explains the gradual increase in utilities during this time window in the scenarios where $\tau = 0.1$ and, even more striking, where $\tau = 0.2$. From $t = 63$ onwards the demographic effect leads to a decrease in utility which nevertheless is much less serious than without an adjustment of the retirement

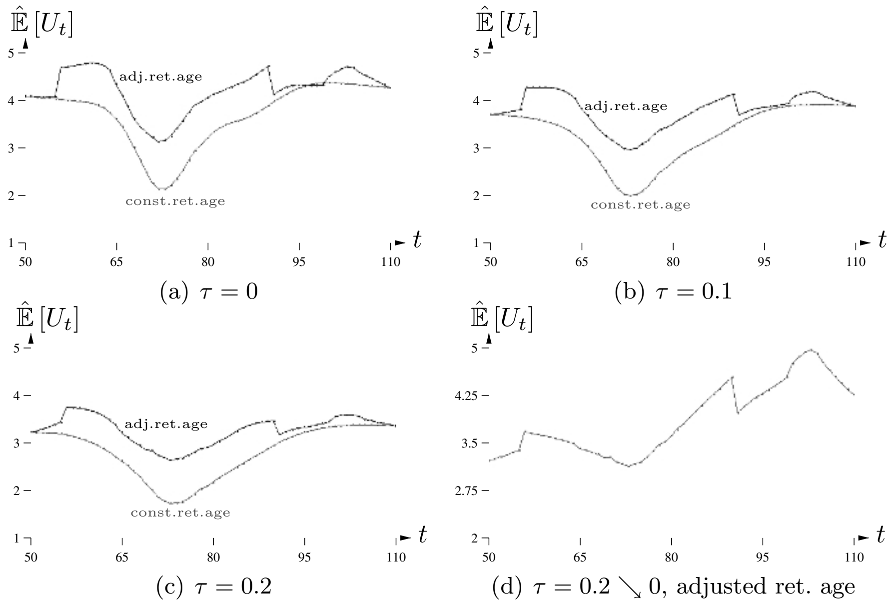


Fig. 5.12. Impact of adjustments in the retirement age on expected lifetime utilities

age. In fact, the proposed adjustment leads to higher expected utility throughout the entire time window in each case.

A striking phenomenon in Figures 5.12(a) to 5.12(c) is that the increase in the retirement age causes an initial overshooting in utility. This raises the natural question whether this excess can somehow be transferred to counteract the loss in subsequent periods. The previous observations motivate one final experiment where the adjustment in the retirement age is supplemented by a gradual reduction in contribution rates. In this regard, we assume that initially, for $t \leq 50$, the contribution rate is equal to $\tau_t = 20\%$. As in Section 5.3, suppose that in addition to the increase in the retirement age from $t = 51$ until $t = 85$, the initial contribution rate $\tau_t = 0.2$ is lowered by 0.5% points in every period from $t = 51$ onwards. The impact of this combined policy on consumers' expected utility is depicted in Figure 5.12(d). The result seems very convincing: the combination of the two measures is capable of almost entirely eliminating the consequences of demographic change. Throughout the entire time window expected utility never falls below its initial level but is in fact much larger in most periods.

With these findings, it seems that a combination of a gradual decrease in contributions accompanied by a temporary increase in the retirement age is the most promising political measure to encounter the problems induced by the demographic change of the population. However, several remarks must be made at this point. Firstly, in our model, consumers do not derive disutility from labor. As a consequence, the additional period they have to work as the retirement age is increased does not have a diminishing impact on utility. Clearly, this may be debatable. If one takes into account the utility-diminishing effect of an increase in the retirement age, the result may change. A second point is that the utility levels in Figure 5.12 are still far from being equally distributed across the time window. This is to some extent due to our coarse time scale with one time unit corresponding to four years. An adjustment of the retirement age on an annual basis could lead to an even smoother distribution of utilities. This could be accompanied by an improved adjustment policy where contributions are not necessarily decreased in constant steps but the decrease changes in each period. These two refinements should lead to a smoother distribution of utility over the transition periods.

Summary of Chapter 5

The numerical results obtained in this chapter revealed that demographic change entails significant consequences for real and financial markets as well as for the welfare of consumers. If no adjustment of the pension system takes place, the change in the population structure leads to large welfare losses on the part of consumers accompanied by a substantial decline in asset prices and the capital stock. The latter confirms existing results found in the literature (e.g., see [1], [2]) claiming the emergence of a so-called asset market meltdown as a consequence of a shrinking population.

Against this background, the previous sections studied several adjustment possibilities of the pension system which may potentially counteract these developments. It was shown that a mere adjustment of contributions are, at best, capable of attenuating the consequences of demographic change. In particular, an adjustment policy as suggested by the German Riester-Rürup formula results in a temporary increase of contributions which even amplifies the negative effects. These insights are an immediate consequence of the fundamental trade-off faced by any pay-as-you-go pension system as discussed in Section 2.6.

From a political point of view, the most promising measure to overcome the loss in welfare was shown to be a temporary increase in the retirement age. In fact, this measure has been part of the political debate in Germany ("Rente mit 67") and has also been suggested by the Rürup commission (cf. [56]) to supplement the adjustment in contributions. Opponents of such a reform, however, have argued that any increase in the retirement age will only add to the high unemployment rate but will not solve the demographic problem. Since the framework employed in this dissertation does not allow for unemployment but instead assumes market clearing on the labor market, it is not possible to study the validity of the latter argument in our model. All that can be said is that the previous results hold in an idealized situation in which wages are sufficiently flexible. The extension of the model to incorporate the issue of unemployment is one of the challenging tasks of future research. In this regard, the disequilibrium approach developed in [8] may provide a suitable starting point to study situations with unemployment and a non-market clearing real wage.

6

Conclusions and Outlook

The macroeconomic model developed in the first part of this dissertation provides a comprehensive theoretical framework to analyze the interactions between a public pension system and real and financial markets as well as the consequences of demographic change. The model complements existing approaches found in the literature by allowing for consumers with multiperiod lives of arbitrary finite length and by providing an explicit description of the evolution of prices and allocations on real and financial markets in the presence of exogenous perturbations and changes in the population structure. To ensure that the framework is sufficiently flexible and amendable to future extensions the model was first formulated under a general class of assumptions on the microeconomic characteristics of consumers and firms. This was followed by a particular parametrization that permitted closed form solutions to the model's equations. Based on this parametrization the model was shown to possess an explicitly defined forward-recursive structure which may be subjected to random perturbations in each period. This permitted the model to be formulated as a random dynamical system as introduced in [4]. Within this framework, the concept of a stable random fixed point is available to describe the long-run evolution of the model in the presence of random shocks to the system. In particular, the impact of parameter changes on the long-run dynamic behavior of the model's variables and its statistical properties may be analyzed. In the present case, the concept was used to develop a notion of long-run efficiency of pension systems by comparing the expected lifetime utilities of consumers along the random fixed point induced by different contribution rates to the pension system.

Utilizing this theoretical framework, the second part of this work presented a numerical case study which analyzed the role of a pension system and the impact of demographic change. To ensure that the employed parametrization was compatible with empirical observations, the model was calibrated by using parameter values that were justified from various empirical studies. In addition, all results were proven to be robust in the sense that they continue to hold as parameters are varied. The analysis itself was carried out in two steps. The first scenario hypothesized a constant population which may be viewed as a long-run scenario of the population structure. In a second step, the consequences of demographic changes modeled as a transitory phenomenon due to a shrinking population size were investigated.

In the absence of demographic change, a lower contribution rate was shown to foster the accumulation of capital and to increase the levels of the capital stock and asset prices while decreasing the interest rate. In addition, a reduction in contributions may exhibit a stabilizing impact on financial markets by reducing the volatility and avoiding crashes in stock prices. The latter result stresses the importance to incorporate the mutual interactions between pension systems and asset markets into the analysis. From a normative point of view the long-run inefficiency of a public pension system was proved by showing that a lower contribution rate leads to higher expected utility of consumers. For the scenario with an initially existing pension system, a gradual reduction of contributions was shown to induce a smooth transition with only small welfare losses on the part of retirees during the transition and higher expected utility of all generations in the long run.

For the second scenario with a shrinking population it was shown that demographic change causes large welfare losses on the part of consumers accompanied by a substantial decline in asset prices and the capital stock. The latter confirms the so-called asset market meltdown hypothesis claimed in the literature (cf. [1], [2]). Several reform proposals to adjust the pension system during the demographic transition were studied. It was shown that a reduction of contribution rates may ameliorate, but not avoid the consequences of demographic change, while a temporary increase as suggested by the German Riester–Rürup adjustment formula will even reinforce them. The most promising political measure was shown to be a temporary increase in the retirement age combined with a gradual reduction in contribution rates. This measure proved capable of almost entirely avoiding the welfare losses during the demographic transition phase while at least attenuating the meltdown in asset prices.

Although the numerical results were shown to be robust against parameter changes, a thorough theoretical foundation is not available yet. At first sight, this task seems very difficult due to the large scale and complexity of the model. However, many observations and results collected from the numerical simulations indicate that the model is very well-behaved in terms of its dynamic properties and there is hope to gain - at least partly - a better theoretical understanding of the results. This is in particular true because many of the previous findings are in line with existing results found in the literature for the class of deterministic models. A more elaborate theoretical investigation would therefore be a challenging task for future research to complement the existing numerical results.

In addition, several possible extensions of the model have already been suggested above. These comprise straightforward modifications such as replacing the firms' production technologies with more general functional forms and the extension of the numerical simulations to the case with more than one firm. By construction, the latter increases the number of shares traded in the stock market and would therefore allow for a more intensive study of risk and diversification.

A more challenging modification would be to replace the simple expectations formation behavior of consumers and firms by more elaborate forecasting rules. In this regard, the existence and form of forecasting rules which generate in some sense rational expectations would be of particular interest. This would not only extend the scope of the model from a theoretical point of view, but would also allow one to contrast the results with those found in the literature where the paradigm of rational expectations is predominant

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