FIRST YEAR ALGEBRA

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BY

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FIRST YEAR ALGEBRA

E. P. I

PREFACE

THIS book has been written to meet the growing demand for a High School Algebra that contains only the first year's work. While the order of topics resembles in general that found in the author's other algebras, yet a number of changes have been made, for the purpose of *simplifying* the work and deferring difficulties until the pupil is able to cope with them.

One of the hardest ideas for the young student to grasp is that of negative numbers; and the common practice of presenting them at the very beginning of the book results not only in the bewilderment but also in the discouragement of the student. In this book, therefore, the pupil is first taught the symbols and the fundamental operations as applied to positive numbers, and not until he has become thoroughly familiar with these is he introduced to negative numbers. He can thus concentrate his entire attention on the one new idea, and it becomes a pleasure to him to extend his knowledge by applying the principles he has already learned to the new concept. Again, the troublesome operation of removing and inserting signs of aggregation is deferred until the pupil's gain in power of manipulating algebraic numbers renders the work comparatively easy.

On the other hand, in order to arouse from the first the interest of the pupil, simple problems to be solved both arithmetically and algebraically, as well as easy solutions of simultaneous equations and of quadratic equations by factoring, are presented very early in the course, while the more difficult phases of these subjects are discussed later. Throughout the work, indeed, the greatest emphasis is placed on equations and problems, which furnish the most apt illustrations of the *practical* uses of algebra.

The treatment of every principle is based on the pupil's knowledge of *arithmetic*. This close correlation of the two subjects not only illuminates both of them, but adds further to the simplicity of the book.

The *problems* are based on interesting facts gathered from a variety of sources, including physics, geometry, and commercial life. A few problems of the older style are included for the purpose of familiarizing the pupil with them and for their disciplinary value.

Graphs are presented in a simple and comprehensive manner, but the chapters are introduced in such a way as to render practicable their omission, without disturbing the continuity of the course.

Factoring is thoroughly taught, and the study is greatly simplified by the careful classifying and summarizing of the various cases.

New terms are illustrated or defined wherever they are needed, the object of this plan being to prevent the confusion that results in the pupil's mind from the massing of large collections of definitions at the beginning of each chapter. Formal definitions of all terms are placed at the end of the book in a glossary arranged in alphabetical order.

Abstract and concrete work is well balanced, so that the drills in algebraic processes and representation are as plentiful as the exercises for the development of the reasoning faculties.

Accuracy is secured by the numerous checks, tests, and verifications that are required of the student, and thoroughness is acquired through the frequent and exhaustive reviews.

In the preparation of the work, careful consideration has been given to the courses of study outlined by the Regents of the State of New York and by educational authorities elsewhere. The book will be found to meet the requirements of these courses in every particular.

WILLIAM J. MILNE.

CONTENTS

FIRST YEAR ALGEBRA

INTRODUCTION

1. In passing from arithmetic to algebra, the student will not find the change a very marked one. He will meet signs, definitions, principles, and processes with which he is already familiar. The fundamental principles of arithmetic and algebra are identical, but in algebra their application is broader.

Algebra uses the same number symbols as arithmetic, namely, 1, 2, 3, 4, 5, etc., but from time to time, as need for them arises, various new symbols will be introduced. While arithmetic, to a limited extent, uses letters to represent numbers, their use is a distinctive feature of algebra.

The terms, addition, subtraction, multiplication, division, fraction, etc.; the associated terms such as addend, subtrahend, multiplier, quotient, numerator; and the signs, $+$, $-$, $\times$, $\div$, $=$, have the same meanings that they have in arithmetic; but it will be seen that algebra gives to some of them additional meanings.

In algebra, multiplication is also indicated by the dot $(\cdot)$ or by the absence of sign; thus, $a \times b$, $a \cdot b$, and ab all mean the same.

Division is often indicated by a fraction; thus, $a \div b$ and $\frac{a}{b}$ have the same meaning.

EXERCISES

2. Read, and tell the meaning of each of the following **algebraic expressions**:

1. $2 + 3$.
2. $a + b$.
3. $8 - 5$.
4. $x - y$.
5. 2×5.
6. $m \times n$.
7. $8 \div 4$.
8. $w \div v$.
9. $4 \cdot 5$.
10. $x \cdot y$.
11. pq.
12. $ab - rs$.
13. $3v + 5z$.
14. $a + m - n$.
15. $\frac{m}{n}$.
16. $\frac{ab}{3x}$.
17. $\frac{a}{b} + \frac{r}{s}$.
18. $\frac{a - r}{b + s}$.

Indicate the

19. Sum of 5 and 2; of x and y.

20. * Difference of 9 and 6; of m and n.

21. Product of 3 and 4 in two ways; product of r and s in three ways.

22. Quotient of 8 divided by 5 in two ways; quotient of p divided by q.

23. Sum of 5 times d and 2 times c.

24. Difference of a times b and 2 times 4.

25. Product of $3m$ and n.

26. Quotient of $v - w$ divided by c times d.

27. Product of $2x + 7$ and $3y - 2$.

The product of $2x + 7$ and $3y - 2$ is indicated thus:

$$(2x + 7)(3y - 2).$$

NOTE. — Parentheses, ( ), are used to group numbers, when the numbers in each group are to be considered as a single number.

28. Product of $a - b$ and $5m + 2$.

29. Product of a and $a + b$ divided by the product of b and $a - b$.

30. A boy had a apples and his brother gave him b more. How many apples had he then?

* In this book, the 'difference' of two numbers means the first mentioned less the second.

31. Edith is 14 years old. How old was she 4 years ago? a years ago? How old will she be in 3 years? in b years?

32. At x cents each, how much will 5 oranges cost?

33. If z caps cost 10 dollars, how much will 1 cap cost?

34. At y cents each, how many pencils can be bought for x cents?

35. George won a race by running the distance in t seconds. Represent Elmer's time, if he took 2 seconds longer.

36. James weighs p pounds. Represent Edward's weight, if he weighs 10 pounds less than James.

37. A boy who had p marbles lost q marbles and afterward bought r marbles. How many marbles did he then have?

38. If m represents the number of miles a boy can walk in a certain time, indicate the distance his father, who walks twice as fast, can walk in the same time.

39. Mary paid c cents for a pin and half as much for a belt. Represent the number of cents she paid for the belt.

40. What two whole numbers are nearest to 9? to x, if x is a whole number? to a, if a is a whole number?

41. If y is an even number, what are the two nearest even numbers? the two nearest odd numbers?

3. Unite **terms** as indicated by their signs:

20	2 tens	2×10	$2t$	$2x$	$2z$
$+40$	$+4$ tens	$+4 \times 10$	$+4t$	$+4x$	$+4z$
$+30$	$+3$ tens	$+3 \times 10$	$+3t$	$+3x$	$+3z$
90	9 tens	9×10	$9t$		

$2t + 4t + 3t = 9t.$ $\quad 2x + 4x + 3x = ?$ $\quad 2z + 4z + 3z = ?$

Such terms as $2x$, $+4x$, and $+3x$ are called **like, or similar, terms** because they have the same unit, x.

The multipliers, 2, 4, and 3 are called **coefficients** of x.

Such terms as $2t$, $+4x$, and $3z$ are **unlike, or dissimilar, terms** because they have different units, t, x, and z.

ALGEBRAIC SOLUTIONS

4. The numbers in this chapter do not differ in character from the numbers with which the student is already familiar in arithmetic.

The following solutions and problems, however, serve to illustrate how the solution of an arithmetical problem may often be made easier and clearer by the *algebraic* method, in which the numbers sought are represented by *letters*, than by the ordinary arithmetical method.

Letters that are used for numbers are usually called **literal numbers.**

5. Illustrative Problem. — A man had 400 acres of corn and oats. If there were 3 times as many acres of corn as of oats, how many acres were there of each?

Arithmetical Solution

A certain number = the number of acres of oats.
Then, 3 times that number = the number of acres of corn,
and 4 times that number = the number of acres of both;
therefore, 4 times that number = 400.
Hence, the number = 100, the number of acres of oats,
and 3 times the number = 300, the number of acres of corn.

Algebraic Solution

Let $x =$ the number of acres of oats.
Then, $3x =$ the number of acres of corn,
and $4x =$ the number of acres of both;
therefore, $4x = 400$.
Hence, $x = 100$, the number of acres of oats,
and $3x = 300$, the number of acres of corn.

Observe that in the algebraic solution x is used to stand for 'a certain number' or 'that number,' and thus the work is abbreviated.

6. A statement of the equality of two numbers or expressions is called an **equation.**

$5x = 30$ is an equation.

Problems

7. Solve the following problems:

1. A bicycle and suit cost $54. How much did each cost, if the bicycle cost twice as much as the suit?

2. Two boys dug 160 clams. If one dug 3 times as many as the other, how many did each dig?

3. Find a number whose double equals 52.

4. If 3 times a number equals 75, find the number.

5. A certain number added to 3 times itself equals 96. What is the number?

6. The average length of a fox's life is twice that of a rabbit's. If the sum of these averages is 21 years, what is the average length of a rabbit's life?

7. The battleship fleet that sailed for the Pacific consisted of 20 ships. The number of warships was 4 times the number of the auxiliary ships. How many warships were there?

8. The water and steam in a boiler occupied 120 cubic feet of space, and the water occupied twice as much space as the steam. How many cubic feet of space did each occupy?

9. One year the United States exported 24 million pounds of butter and cheese. If this included twice as much butter as cheese, how many pounds of each were exported?

10. Porto Rico and the Philippines together produce 400,000 tons of sugar each year. If the latter produces 3 times as much as the former, how much does Porto Rico produce?

11. Canada and Alaska together annually export furs worth 3 million dollars. If Canada exports 5 times as much as Alaska, find the value of Alaska's export.

12. The poultry and dairy products of this country amount to 520 million dollars a year, or 4 times the value of the potato crop. What is the value of the potato crop?

13. At Portland, Oregon, recently vessels were loaded with 25 million feet of lumber for home and foreign ports. Find the foreign shipment, if it was 4 times that to home ports.

14. In constructing the Galveston sea wall 10,000 loads of sand and crushed granite were used. If there were 3 times as many loads of sand as of granite, how many loads of each were used?

15. The Weather Bureau of the United States yearly saves the country 30 million dollars, or 20 times its cost. What is the annual cost of the Weather Bureau?

16. One year in continental Europe 6 million watches were made, and this number was $\frac{1}{2}$ of a million more than twice the number made in the United States. How many were made in this country?

SUGGESTION. $2x = 6 - \frac{1}{2}.$

17. Probably Ceylon has the oldest tree in the world, and its age is about 2200 years. If this is 70 years more than 6 times the age of the *Powhatan Oak* in Virginia, find the age of the latter.

18. The value of the *King's Cup*, the challenge trophy for yachting, is twice as great as that of the *Bennett Cup*, the prize for long-distance balloon racing. If the difference in value is $2500, find the value of each.

19. The owner of a piano found that the annual cost of keeping it in tune and insuring it against fire was $12.50, and that the cost of keeping it in tune was 9 times the cost of insuring it. Find the cost of each item.

20. The Quebec bridge that collapsed was 1800 feet long, and twice the length of the Forth bridge was $\frac{1}{10}$ of the length of that at Quebec. Find the length of the Forth bridge.

21. One year 1500 violins were made in the United States. Twice as many were made in New York as in Massachusetts, and these two states made half of all that were made in this country. How many violins were made in New York?

22. The sides of any square (Fig. 1) are equal in length. How long is one side of a square, if the perimeter (distance around it) is 36 inches?

FIG. 1.

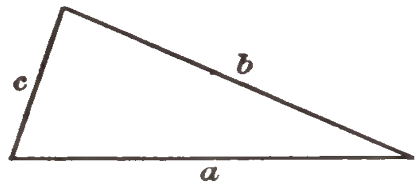


FIG. 2.

FIG. 3.

23. The length of each of the sides, a and b, of the triangle (Fig. 2) is twice the length of the side c. If the perimeter is 40 inches, what is the length of each side?

24. The opposite sides of any rectangle (Fig. 3) are equal. If a rectangle is twice as long as it is wide and its perimeter is 48 inches, how wide is it? how long?

25. Divide 21 into three parts, such that the first is twice the second, and the second is twice the third.

SUGGESTION. — Let $x =$ the third part; then, $2x =$ the second part, and $2 \cdot 2x =$ the first part; that is, $x + 2x + 2 \cdot 2x = 21$.

26. Three newsboys sold 60 papers. If the first sold twice as many as the second, and the third sold 3 times as many as the second, how many did each sell?

27. The battleship *Connecticut* has twice as many 8-inch as 12-inch guns, and the sum of the two equals the number of its 7-inch guns. If it has in all 24 guns of these sizes, find the number of each.

28. One winter the Borough of Richmond had four falls of snow amounting in all to $16\frac{1}{2}$ inches. The second and third falls were each 4 times the first. Find the depth of the fourth fall, if it was twice the first.

29. In a recent year Massachusetts produced twice as many barrels of cranberries as New Jersey, and New Jersey 5 times as many as Wisconsin. Find the production of each of these states, if their total crop was 400,000 barrels.

30 A plumber and two helpers together earned \$ 7.50 per day. How much did each earn per day, if the plumber earned 4 times as much as each helper?

31. James bought a pony and a saddle for \$ 60. If the saddle cost $\frac{1}{3}$ as much as the pony, find the cost of each.

SUGGESTION. — Let $x =$ the number of dollars the saddle cost.

32. Separate 72 into two parts, one of which shall be $\frac{1}{3}$ of the other.

33. Separate 78 into two parts, one of which shall be $\frac{1}{5}$ of the other.

34. A skating rink accommodated 4000 persons. If there were $\frac{1}{3}$ as many skaters as spectators, find the number of each.

35. The total production of sulphur averages 625,000 tons per year. How much is produced by the rest of the world, if it is $\frac{1}{4}$ the amount produced by Sicily?

36. The average height of the land above sea level is $\frac{1}{12}$ as great as the average depth of the ocean, and the sum of the two is 13,000 feet. Find the average height of the land and the average depth of the ocean.

37. The first issue of Christmas stamps by the Delaware Red Cross Society was $\frac{1}{2}$ as much as the second, which was $\frac{1}{2}$ as much as the third. If the three issues amounted to 350,000 stamps, how many were there in each issue?

38. Sand and clay road costs $\frac{1}{6}$ as much per mile as macadam. If the former costs \$ 400 per mile, find the cost of the latter.

SOLUTION

Let $x =$ the number of dollars macadam costs per mile.
Then, $\frac{1}{6}x = 400.$
Therefore, $x = 6$ times $400 = 2400.$
Hence, macadam road costs \$ 2400 per mile.

39. The gold output of the United States for a recent year was 110 million dollars, or $\frac{1}{4}$ that of the entire world. What was the world's output for that year?

40. A man in New York rented his house and lived in an apartment costing him $2000 a year. This was $\frac{1}{3}$ as much as the rent of his house. For how much did his house rent?

41. The Pennsylvania Railroad station in New York is 780 feet long, and this is $404\frac{1}{3}$ feet more than $\frac{1}{2}$ the length of the Capitol at Washington. Find the length of the Capitol.

42. A basketball team won 16 games, or $\frac{2}{3}$ of the games it played. Find the number of games it played.

SOLUTION

Let $x =$ the number of games it played.
Then, $\frac{2}{3}x = 16,$
and $\frac{1}{3}x = 8.$
Therefore, $x = 24$, the number of games it played.

43. The largest thermometer in the world has a glass tube 16 feet long. Find the length of the thermometer, if the tube is $\frac{4}{5}$ of the entire length.

44. What is the annual rainfall of Hawaii, if at least 56 inches, or $\frac{4}{5}$ of it, passes off without rendering any service?

45. Of the inhabitants of Guam, $\frac{9}{10}$, or 8100, can read and write. What is the population of the island?

46. The average annual fire loss in Berlin is $\frac{3}{10}$ of that in Chicago. If the fire loss in Berlin is $150,000, what is the fire loss in Chicago?

47. The largest stone ever quarried in the South was dressed down to weigh 60,000 pounds. If this was $\frac{3}{4}$ of its weight as originally blocked out, find its original weight.

48. Find the amount of lumber on hand in San Francisco at the time of the earthquake, if $\frac{2}{5}$ of it, or 36 million feet, were consumed by the fire that followed the earthquake.

49. The manufacturing industries of Great Britain use 150 million tons of coal per year. If this is $\frac{5}{6}$ of the total amount used, what is that country's annual consumption of coal?

50. The number of German-speaking people in the world is 75 million, or $\frac{3}{5}$ the number that speak English. What is the number of English-speaking people?

51. The United States sent to Germany one year 135,000 pairs of shoes. This was $\frac{3}{5}$ of the number sent the next year. How many pairs of shoes were sent the second year?

52. If $\frac{1}{5}$ of a number is added to the number, the sum is 12. What is the number?

SUGGESTION. $x + \frac{1}{5}x = 12$; that is, $\frac{6}{5}x = 12$.

53. If $\frac{1}{3}$ of a number is subtracted from twice the number, the difference is 35. What is the number?

SUGGESTION. $2x - \frac{1}{3}x = 35$; that is, $\frac{5}{3}x = 35$.

54. The total cost of the Pennsylvania Capitol was 13 million dollars. If the furnishings cost $2\frac{1}{4}$ times as much as the construction, what was the cost of each?

55. The retail dressmaking trade each day uses $\frac{1}{3}$ of the total daily output of spool silk. If the manufacturing trade uses the remainder, or 16,000 miles, how much does the dressmaking trade use per day?

56. Out of the average daily output of stamped envelopes $\frac{3}{10}$ are plain stamped. The remainder, 2,800,000, bear the return address. What is the daily output?

57. In one year, 5600 tons of dynamite were required for the Panama Canal. If the amount for the Culebra division was $1\frac{4}{5}$ as much as that for the rest of the canal, find the amount required for the Culebra division.

58. In the first twenty-one hours after the institution of regular wireless service, $6\frac{1}{2}$ times as many words were sent to Europe as were received, and the number sent was 11,000 more than the number received. Find the number sent; the number received.

59. The Pacific battleship fleet carried twice as much ham as it did salt pork, and $2\frac{1}{2}$ times as much beef as it did ham. The weight of the beef was 800,000 pounds more than that of the salt pork. Find the weight of each.

FACTORS, POWERS, AND POLYNOMIALS

8. Since the product of 2 and 6 or of 3 and 4 is 12, each of the numbers 2, 6, 3, and 4 is a **factor** of 12. So also, each of the numbers 3, a, b, $3a$, $3b$, and ab is a factor of $3ab$.

9. In algebra, as in arithmetic, such a product as $2 \times 2 \times 2 \times 2$, called a **power** of 2, may be more briefly written 2^4.

The small figure 4, placed at the right of, and a little above, the 2 to indicate the number of times 2 is used as a factor, is called an **exponent**.

Since a^1 means the same as a, the exponent 1 is usually omitted.

a^2 is read 'a second power' or 'a square'; a^3 is read 'a third power' or 'a cube'; a^4 is read 'a fourth power,' 'a fourth,' or 'a exponent 4.'

The terms 'coefficient' and 'exponent' should be distinguished.

Thus, $5a$ means $a + a + a + a + a$, but a^5 means $a \cdot a \cdot a \cdot a \cdot a$.

EXERCISES

10. Read, and tell the meaning of:

1. x^6.	4. x^2y^2.	7. $3zw^2$.	10. $9ab^3c^2d^4$.
2. y^4.	5. a^3b^3.	8. $4p^3q^5$.	11. $5p^2q^3s^4t^5$.
3. z^8.	6. r^2s^2.	9. $2m^3n^2$.	12. $7x^3ym^7n^4$.

Express in abbreviated form by using exponents:

13. $2 \cdot 2$.	16. $3aaa$.	19. $2 \cdot 2 \cdot 2 \cdot x \cdot x$.
14. $3 \cdot 3 \cdot 3$.	17. $8llll$.	20. $7 \cdot 7 \cdot z \cdot z \cdot z \cdot z$.
15. $5 \cdot 5 \cdot 5 \cdot 5$.	18. $9ssrrr$.	21. $3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot b \cdot b \cdot b$.

22. What is the coefficient of x in $3x$? in ax? in $3ax$?

NOTE. — A coefficient is *numerical*, *literal*, or *mixed* according as it is composed of figures, letters, or both.

When not otherwise specified 'coefficient' means numerical coefficient.

Since $1a$ means the same as a, the coefficient 1 is usually omitted.

23. What is the literal coefficient of t^2 in at^2? in gt^2? in n^3t^2? of y^3 in ny^3? in x^2y^3? in lmy^3?

Name the various factors of:

24. ax.	26. x^3.	28. $6n$.	30. $pqrs$.
25. $3mn$.	27. $5r^2s^2$.	29. $15z^2$.	31. $24vt$.

32. In each of the exercises 24–31, name the factors in sets such that the product of the factors in each set shall equal the given number.

11. An algebraic expression is called a **monomial, binomial,** or **trinomial** according as it has *one, two,* or *three* terms.

Thus, $3a$ is a monomial; $2x + y^3$, a binomial; and $x^2 + 2xy + y^2$, a trinomial.

The name **polynomial** is often applied to any algebraic expression of more than one term.

EXERCISES

12. From the algebraic expressions given below select the:

1. Binomials.	3. Monomials.	5. Similar terms.
2. Trinomials.	4. Polynomials.	6. Dissimilar terms.

$2ax$; $3x^2y$; $2a + 3b$; $3x + 2b$; $3ax + 2y^2$; $6a - c + d$; $3a^2x^2 - 4ax + 2d - y^2$; $2x^2y - xy + a^2x^2$.

7. Find the value of $3 + 4 - 2 + 3$; of $3 \times 4 \div 2 \times 3$.

SOLUTIONS. $3 + 4 - 2 + 3 = 7 - 2 + 3 = 5 + 3 = 8$;
$3 \times 4 \div 2 \times 3 = 12 \div 2 \times 3 = 6 \times 3 = 18$.

When only + and − occur in any expression, or only × and ÷, the operations are performed in order from left to right.

Find the value of:

8. $3 - 2 - 1 + 8 - 3 + 4$.	10. $10 \div 2 \times 8 \div 4 \div 2 \times 6$.
9. $5 + 1 - 4 + 3 - 2 + 6$.	11. $35 \div 7 \div 5 \times 3 \times 4 \div 2$.

12. Find the value of $7 + 10 - 6 \div 3 \times 4$.

SOLUTION. $7 + 10 - 6 \div 3 \times 4 = 7 + 10 - 2 \times 4 = 7 + 10 - 8 = 9$.

Unless otherwise indicated, as by the use of parentheses, *when ×, ÷, or both, occur in connection with +, −, or both, the indicated multiplications and divisions are performed first.*

Find the value of:

13. $5 \times 10 - 7$.

14. $5 \times (10 - 7)$.

15. $2 \times 5 + 3 \times 4$.

16. $(25 - 13) \div 4 \times 2$.

17. $16 - 2 \times 2 \times 12 \div 4$.

18. $6 + 2 \times 8 - 4 \div 2$.

19. $(6 + 2) \times 8 - 4 \div 2$.

20. $(6 + 2 \times 8 - 4) \div 2$.

21. $6 + 2 \times (8 - 4) \div 2$.

22. $6 + 2 \times (8 \div 4 \div 2)$.

Read, and tell the meaning of each of these polynomials:

23. $2x^2 + y^2$.

24. $3x - 4y$.

25. $4ab - c^3$.

26. $a + d(ax - y)$.

27. $3 + 4(y - 3z)$.

28. $c(l^2 + t^2) - 4d$.

29. $3x^2 + 2y - 3z$.

30. $a^2x^2 - 3xy + 2z^2$.

31. $5b^2y + x^2y^2 + 5cz^2$.

Represent algebraically:

32. The sum of five times a and three times the square of x.

33. Three times b less twice the fifth power of a.

34. The product of a, b, and $a - c$.

35. Three times x, divided by five times the sum of a, b, and c.

36. Seven times the product of x and y, increased by three times the cube of z.

37. Six times the square of m, increased by the product of m and n.

38. The product of a used five times as a factor, multiplied by the sum of b and c.

39. Twelve times the square of a, diminished by five times the cube of b.

40. Eight times the product of a and b, divided by four times the seventh power of c.

41. Six times the product of a second power and n, increased by five times the product of a and the second power of n.

42. The fourth power of the sum of a and b, increased by three times the product of the square of a and the square of b, diminished by the cube of d.

NUMERICAL SUBSTITUTION

13. When a particular number takes the place of a letter, or **general number**, the process is called **substitution.**

EXERCISES

14. 1. When $a = 2$ and $b = 3$, find the numerical value of $3\,ab$; of a^4.

SOLUTIONS. $3\,ab = 3 \cdot 2 \cdot 3 = 18$; also, $a^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

When $a = 5$, $b = 3$, $c = 10$, $m = 4$, find the value of:

2. $10\,a$.	6. $5\,m^2$.	10. am^4.	14. $\frac{1}{3}ab^2$.
3. $2\,ab$.	7. $2\,a^2b$.	11. $(ab)^2$.	15. $\frac{1}{2}bm$.
4. $3\,cm$.	8. $3\,bm^3$.	12. a^2b^2.	16. $\frac{1}{5}abc$.
5. $6\,bc$.	9. $4\,a^3b$.	13. a^bc.	17. $3\,b^2cm^2$.

18. When $m = 0$ and $n = 4$, find the value of $3\,m^2n$.

SOLUTION. $3\,m^2n = 3 \cdot 0^2 \cdot 4 = 3 \cdot 0 \cdot 4 = 0$.

NOTE. — When any factor of a product is 0, the product is 0; therefore, any power of 0 is 0.

When $a = 4$, $b = 2$, $r = 0$, and $s = 5$, find the value of:

19. $7\,b^2r$.	21. $3\,s^bb^a$.	23. $\frac{3}{8}a^3bs$.	25. $2^ab^3s^2r^4$.
20. $\dfrac{3\,a^2b}{s\,b}$.	22. $\dfrac{a^2bs}{abs^2}$.	24. $\dfrac{6\,a^sb^a}{b^{2s}a^4}$.	26. $\dfrac{24\,a^bb^ss^a}{6\,a^3b^3s^3}$.

27. When $x = 3$ and $y = 2$, find the value of $(x + y)^2$; of $x^2 + 2\,xy + y^2$.

SOLUTIONS

$$(x + y)^2 = (3 + 2)^2 = 5 \cdot 5 = 25.$$
$$x^2 + 2\,xy + y^2 = 3 \cdot 3 + 2 \cdot 3 \cdot 2 + 2 \cdot 2 = 9 + 12 + 4 = 25.$$

28. Show that $2\,x + 3\,x = 5\,x$ when $x = 2$; when $x = 3$. Giving x any value you choose, find whether $2\,x + 3\,x = 5\,x$.

29. Show that $m(a + b) = ma + mb$ when $m = 5$, $a = 4$, and $b = 3$. Find whether the same relation holds true for other values of m, a, and b.

30. Show that $(a-b)^2 = a^2 - 2ab + b^2$ when $a = 4$ and $b = 2$. Find whether this is true for other values of a and b.

When $a = 5$, $b = 3$, $m = 4$, $n = 1$, find the value of:

31. $a^2 + b^2$. **33.** $n^5 - 1$. **35.** m^{a-b}.

32. $(a+b)^2$. **34.** $(n-1)^5$. **36.** $(bm)^{b-1}$.

37. $ab - bn + mb^2 \div 3mn^2$.

38. $(ab - bn + mb^2) \div 3mn^2$.

39. $2^a m^2 n^2 - abmn \div 4bn - m^3 n^7$.

40. $ambn^2 - \frac{3}{4}b^2 m + \frac{5}{8}m^2 n^3 - \frac{1}{5}m^3$.

REVIEW

15. Read the following; classify each expression according to the number of terms it contains; find the number represented by each expression when $t = 10$.

1. $6t$. **4.** t^2. **7.** t^3.

2. $7t$. **5.** $t^2 + 2t + 1$. **8.** $t^3 + 2t^2 + 5t + 4$.

3. $9t + 9$. **6.** $3t^2 + 6t + 5$. **9.** $5t^3 + 3t^2 + 8t + 6$.

10. Write 25 as a polynomial in t, t representing 10; letting t represent 10, and, using exponents to represent powers of t, express in polynomial form:

732 523 893 4867 6248 72,565

11. What does $2a$ denote? a^2?

Illustrate the difference in meaning between $2a$ and a^2 when $a = 1$; when $a = 2$; when $a = 3$; when $a = \frac{1}{2}$; when $a = \frac{1}{3}$. For what value of a are $2a$ and a^2 equal?

12. Which is the greater, 2^3 or 3^2? 4^2 or 2^4? 2^5 or 5^2?

13. Compare also 2^3 and 2^2; $(\frac{1}{2})^3$ and $(\frac{1}{2})^2$; 1^3 and 1^2.

14. Find, for $x = 1$, the value of:

$3x$ $4x^2$ $6x^3$ $8x^5 - 4x^4 + 2x^3 - x + 5$

Name the exponent of x in each term that contains x.

15. Name the coefficient of n in each of these monomials:

$2n$ n $\frac{1}{2}n$ bn $3b^2n$ a^2b^3n

16. Write three similar monomials; four dissimilar monomials.

17. If n is a whole number greater than 1 and a is any number, what is the meaning of a^n?

Find the value of each of the following expressions when $a = 5$, $b = 4$, $c = 3$, $d = 2$, $e = 1$, and $n = 3$.

18. $6\,ab$; $2\,cd$; $4\,nbd$; $\frac{1}{2}\,ea$; nd^{n+1}.

19. $3\,a^2b$; $3\,ab^2$; $3(ab)^2$; d^2n^3; $(dn)^3$; d^{n-2}.

20. $a + b \div d - n \div e$.

21. $a(b - d) + a - n \div c$.

22. $10 \div d + 3 \div n - e$.

23. $10 \div (d + 3) + ac \div n$.

24. $c^5 + c^4 + 2\,c^3 - 2\,c^2 - 3\,c + 3$.

25. $d^7 + d^6 + 3\,d^5 - 5\,d^4 + 2\,d^3 - 4\,d^2 + 8\,d - 1$.

26. For what value of x is $12\,x$ equal to 72?

Write '$12\,x$ is equal to 72' as an equation. Solve the equation.

Express in algebraic form; solve equations when you can:

27. Three times a certain number, x, is 21.

28. The sum of a certain number and three times the number is 40.

29. Six times a number, less 4 times that number, is 13.

30. The distance around a square lot, each side a feet long, is 1280 feet.

31. Half of a certain number is 17.

32. Twice a certain number, less $\frac{1}{3}$ of the number, equals 15.

33. Mary had m books and James had twice as many, the two together having 18 books.

34. John had 50 cents, spent c cents, and earned d cents. How much money had he then?

35. I bought 2 bottles of olives at b cents per bottle, 3 packages of crackers at p cents per package, and a small cheese for c cents. How much did I expend for all? How much money had I left out of a dollar?

FUNDAMENTAL OPERATIONS

16. In this chapter the student will use numbers he has used in arithmetic and letters to represent such numbers. He will notice that the processes of addition, subtraction, multiplication, and division here are performed as in arithmetic.

ADDITION

17. To add monomials.

1. How many are 2 plus 5? How many times a number are 2 times the number plus 5 times the number?

2. If n stands for a number, how many times n are 2 times n plus 5 times n? $2n + 5n = ?$

3. $2x + 5x = ?$ 4. $2r + 5r = ?$ 5. $2t + 5t = ?$

6. How many are $3 + 4 + 6$?

7. How many days are 3 days + 4 days + 6 days?

8. $3d + 4d + 6d = ?$ 9. $3y + 4y + 6y = ?$

EXERCISES

18. 1. Add $4a$ and $3a$.

PROCESS

$$\begin{array}{r} 4a \\ \underline{3a} \\ 7a \end{array}$$

EXPLANATION. — Just as 3 a's and 4 a's are 7 a's, so $3a + 4a = 7a$; that is, when the monomials are similar the sum may be obtained by adding the numerical coefficients and annexing to their sum the common literal part.

Add:

2. $\begin{array}{r} 3 \\ \underline{6} \end{array}$ 3. $\begin{array}{r} 3x \\ \underline{6x} \end{array}$ 4. $\begin{array}{r} 7 \\ \underline{1} \end{array}$ 5. $\begin{array}{r} 7m \\ \underline{m} \end{array}$ 6. $\begin{array}{r} 3y \\ \underline{4y} \end{array}$

Add:

7. $2n$ $5n$	8. $3x$ $8x$	9. $4xy$ $7xy$	10. $3mn^2$ $9mn^2$
11. $5r$ $2r$ $4r$	12. $9n$ $4n$ $6n$	13. $2ab$ ab $4ab$	14. $6c^2d^3$ $8c^2d^3$ c^2d^3

Perform the additions indicated:

15. $8a + 2a + a + 3a + a + 7a$.

16. $5y + 3y + 8y + 10y + 6y + y + 2y$.

17. $8m + 3m + 5m + 2m + 6m + 4m$.

18. $7bc + bc + 4bc + 5bc + 8bc + 3bc$.

19. $4x^2y^2 + 5x^2y^2 + 3x^2y^2 + x^2y^2 + 10x^2y^2 + 6x^2y^2$.

20. $3(ab)^2 + 9(ab)^2 + (ab)^2 + 7(ab)^2 + 9(ab)^2 + 2(ab)^2 + (ab)^2$.

21. $5(x+y) + 2(x+y) + 3(x+y) + 8(x+y) + 2(x+y) + (x+y)$.

22. $4(a+b)^2 + 11(a+b)^2 + 7(a+b)^2 + 2(a+b)^2 + 5(a+b)^2$.

Only similar terms can be united into a single term. Dissimilar terms are considered to have been added when the addition is indicated.

23. Add $6a$, $5b$, $2a$, $3b$, $2c$, and a.

Solution. — Sum $= 6a + 2a + a + 5b + 3b + 2c = 9a + 8b + 2c$.

Add:

24. $2x$, $4a$, $3x$, and a.

25. m, $3c$, $6m$, and $4c$.

26. $4u$, v, $3u$, and $10v$.

27. $5r$, $\frac{3}{4}t$, $2r$, and $\frac{1}{4}t$.

28. $\frac{1}{2}p$, $\frac{2}{3}q$, $\frac{1}{4}p$, and $\frac{1}{6}q$.

29. d, $.4b$, $.5d$, and $.6b$.

30. $2m$, mn, n, $2mn$, $3m$, $4n$, and $5mn$.

31. $3b$, $2a$, $2b$, $2c$, $2d$, a, c, b, $4d$, and $3c$.

32. rs, $3r^2s$, $4rs^2$, $2rs$, rs^2, $4r^2s$, $2rs^2$, and $3rs$.

33. $3xy$, $2pq$, $7cd$, pq, $2cd$, $8pq$, $4cd$, and $2xy$.

34. x^2, $4xy$, $7y^2$, $2xy$, $3y^2$, $6x^2$, y^2, xy, $5x^2$, and $4y^2$.

19. To add polynomials.

EXERCISES

1. Add $x + 2y + 3z$, $x + y$, and $x + 4y + z$.

PROCESS

$$\begin{array}{r} x + 2y + 3z \\ x + y \\ x + 4y + z \\ \hline 3x + 7y + 4z \end{array}$$

EXPLANATION. — For convenience, similar terms may be written in the same column.

The sum of the first column is $3x$, of the second $7y$, of the third $4z$; the sum of these dissimilar terms is then indicated.

Add:

2. $$\begin{array}{r} 2a + 4b \\ 6a + 2b \\ a + 3b \\ \hline \end{array}$$

3. $$\begin{array}{r} 4r + 3s \\ r + s \\ 3r + 2s \\ \hline \end{array}$$

4. $$\begin{array}{r} x^2 + 2xy + y^2 \\ x^2 + y^2 \\ 3xy + y^2 \\ \hline \end{array}$$

5. Add $2c + 5d$, $7c + d$, $d + 4c$, and $2d + c$.

6. Add $6m + 4n$, $2m + 3n$, $5n + 7m$, and $2n + 3m$.

7. Add $ab + a^2c + 5$, $3ab + 3a^2c + 7$, and $2a^2c + 2ab + 3$.

Express in simplest form:

8. $2a + 2b + 3c + 4b + 4a + 6a + 2c$.

9. $3w + 4x + 7y + 2v + 2w + x + 3y + 4v + 3x + 4w + v$.

10. $x^2z + 5xz^2 + 7xy + 6xz^2 + 2x^2z + 4xy + 4x^2z + xz^2 + xy$.

Add:

11. $6m + 8n + x + y$, $2m + 2n + 3x + 4y$, and $m + x + y$.

12. $3x + 7y + 4z + 6w$, $7z + 4x + 2y + w$, and $x + y + z + w$.

13. $x^2 + 2xy + y^2$, $2x^2 + xy + y^2$, $x^2 + xy + y^2$, $3xy + y^2 + x^2$, $2x^2 + 3xy + y^2$, $x^2 + xy + 2y^2$, and $2xy + 3x^2 + 4y^2$.

14. $2c + 7d + 6n$, $11m + 3c + 5n$, $7n + 2d + 8c$, $d + 3m + 10c$, $4d + 3n + 8m$, $m + 6n$, and $2m + 3d$.

15. $3x^m + 2y^n$, $4x^m + 5y^n$, $2x^m + 7y^n$, and $2x^m + y^n$.

16. $4y^a + z^c + w^b$, $y^a + 2w^b + 3z^c$, $5z^c + 3w^b$, $2y^a + w^b$, and $y^a + 4z^c + 2w^b$.

SUBTRACTION

20. 1. How many are 8 less 3? How many times a number are 8 times the number less 3 times the number?

2. Letting n stand for a number, how many times n are 8 times n less 3 times n? $8n - 3n = ?$

3. $8z - 3z = ?$ 4. $8s - 3s = ?$ 5. $8a - 3a = ?$

EXERCISES

21. 1. From $10a$ subtract $4a$.

PROCESS

$$\begin{array}{r} 10a \\ 4a \\ \hline 6a \end{array}$$

EXPLANATION. — Just as 10 a's less 4 a's are 6 a's, so $10a - 4a = 6a$; that is, when terms are similar their difference may be obtained by subtracting the numerical coefficients and annexing the common literal part.

	2.	3.	4.	5.	6.	7.
From	9	$9x$	7	$7ab$	$18m^2$	$20xy$
Take	4	$4x$	3	$3ab$	$13m^2$	$16xy$

	8.	9.	10.	11.
From	$16ax^2$	$14r^2s^3$	$8x^2y^2z$	$21(a+b)$
Take	$9ax^2$	$7r^2s^3$	$6x^2y^2z$	$11(a+b)$

	12.	13.	14.	15.
From	$3p+8q$	$4l+2t$	$9x+7y$	$5r+8s$
Take	$2p+4q$	$4l+t$	$2x+3y$	$2r+5s$

	16.	17.	18.
From	$n+5n^2+2n^3$	$3r+2s+t$	$8a^2+2ab+3b^2$
Take	$n+n^2+n^3$	$r+s+t$	$5a^2+ab+2b^2$

19. From $12x+7y$ subtract $8x+3y$.

20. From $10ab+3c$ subtract $5ab+2c$.

21. From $7r+5s+6t$ subtract $3r+2s+5t$.

22. From $9x^2+8y^2+6xy$ subtract $5x^2+3y^2+2xy$.

23. From $5m + 7n + 8l + 6$ subtract $5m + 4n + 4l + 5$.

24. From $7x^2 + 3y + 6z + 4v$ subtract $3x^2 + 2y + 5z + 3v$.

Subtract:

25. $2x^3 + y^3 + 3r^3$ from $4x^3 + 7y^3 + 5r^3$.

26. $4ab + 2b^2 + 2cd$ from $6ab + 3b^2 + 6cd$.

27. $3x^2y + xy + 5$ from $9x^2y + 6xy^2 + xy + 8$.

28. $2v^3w^3 + 2vw + 4v^5$ from $12v^5 + 9v^3w^3 + 6vw$.

29. $5m^2nx^2 + abd$ from $18m^2nx^2 + 12a^2b^2c^2 + 4abd$.

30. $4x^m + 2x^my^n + 5y^n$ from $7x^m + 2x^my^n + 9y^n$.

31. $6m^s + 11m^sn^t + 5n^t$ from $10m^s + 11m^sn^t + 8n^t$.

32. $a^{m+n} + 3b^{m-n} + 7c^{2n}$ from $3a^{m+n} + 5b^{m-n} + 9c^{2n}$.

33. $10(m + n^2) + 5(m^2 + n)$ from $12(m + n^2) + 8(m^2 + n)$.

Simplify, adding or subtracting in order as signs indicate:

34. $9x - 4x + 6x$.

35. $5n + 3n - 7n$.

36. $8a - 5a - 3a$.

37. $2s + 8s - 5s$.

38. $3b - 2b + 7b$.

39. $8r - 6r + 5r - 2r$.

40. $7y + 8y - 6y + 7y$.

41. $5z + 7z - 2z - 4z$.

42. $9v - 3v + 2v - 5v$.

43. $7n - 2n - 3n + 4n$.

44. $8x + 7x - 3x + 4x - 2x - 3x + 6x$.

45. $2y + 3y - y + 7y - 3y + 9y + 2y - 6y$.

46. $9z - 5z + 6z - 3z + 4z + 2z - 7z + 3z$.

47. $5v + 6v + 2v - 5v + 4v - 6v + 9v - 5v - 3v$.

48. $7m + 6n - 3m + 5n + 7m - 4n + 3m + 4n + 5m$.

49. $9r + 8s + 7r - 2s + 9s - 3r + 2r - 7s + 6s - 5r + 4r$.

50. $2l + 9t + 3l - l + 3t + 2t + 8l - 5t + 9l - 6l + 2t - 4t + 7t$.

51. $10(a - x) + 15(a - x) + 7(a - x) - 13(a - x) - 12(a - x)$.

MULTIPLICATION

22. Product of two monomials.

In algebra, as in arithmetic, the product of two numbers contains all the factors of both numbers, arranged or grouped in any way we please.

Then, since $a^2 = aa$ and $a^3 = aaa$,

$$a^2 \cdot a^3 = (aa)(aaa) = aaaaa = a^5.$$

That is, $a^2 \cdot a^3 = a^{2+3} = a^5.$ (Add exponents)

Similarly, $3\,a^2 \cdot 5\,a^3 = (3 \cdot 5)(a^2 \cdot a^3) = 15\,a^5.$

(Multiply coefficients)

Again, $3\,a^2b \cdot 5\,a^3b^2 = (3 \cdot 5)(a^2 \cdot a^3)(b^1 \cdot b^2) = 15\,a^5b^3.$

Hence, for multiplication:

23. Law of exponents. — *The exponent of a number in the product is equal to the sum of its exponents in multiplicand and multiplier.*

24. Law of coefficients. — *The coefficient of the product is equal to the product of the coefficients of multiplicand and multiplier.*

EXERCISES

25. Tell products quickly:

1.	2.	3.	4.
$7\,a$	$3\,x$	$5\,m$	$8\,ab^2x^3$
$3\,a$	$4\,y$	$2\,m^3n$	b^4cx^4
$21\,a^2$	$12\,xy$	$10\,m^4n$	$8\,ab^6cx^7$

5.	6.	7.	8.
$3\,y$	$4\,x^2$	$8\,av$	$12\,a^3bc$
$4\,y$	$7\,x^2$	$3\,aw$	$3\,a^2b^2d^2$

9.	10.	11.	12.
ab^2	$3\,xy^3$	$2\,ax$	$16\,c^2d^4m$
a^3b	$9\,xz$	$2\,by$	$2\,c^5d^3n$

13.	14.	15.	16.
x^4y	$7\,pq^2$	$8\,c^2d$	$2\,axy^5t$
xy	p^5q	$4\,d^3z$	$6\,a^6yz^2s$

26. To multiply a polynomial by a monomial.

Multiplying as in arithmetic, we have:

1.
$$\begin{array}{r}43\\2\\\hline 86\end{array}\qquad \begin{array}{r}40+3\\2\\\hline 80+6\end{array}\qquad \begin{array}{r}4\text{ tens}+3\text{ units}\\2\\\hline 8\text{ tens}+6\text{ units}\end{array}\qquad \begin{array}{l}4\,t+3\,u\\2\\\hline 8\,t+6\,u\end{array}$$

2.
$$\begin{array}{r}321\\3\\\hline 963\end{array}\qquad \begin{array}{r}300+20+1\\3\\\hline 900+60+3\end{array}\qquad \begin{array}{l}3\,h+2\,t+u\\3\\\hline 9\,h+6\,t+3\,u\end{array}\qquad \begin{array}{l}x+y+z\\a\\\hline ax+ay+az\end{array}$$

27. *The product of a polynomial by a monomial is equal to the sum of the partial products obtained by multiplying each term of the polynomial by the monomial.*

EXERCISES

28. Multiply:

1. $\begin{array}{l}x+2\\4\\\hline\end{array}$ 2. $\begin{array}{l}ax^2+y\\ax\\\hline\end{array}$ 3. $\begin{array}{l}5\,m^2s+2\,t\\3\,st^2\\\hline\end{array}$

4. $\begin{array}{l}m^2+n^3\\mn\\\hline\end{array}$ 5. $\begin{array}{l}x^2+2\,xy+y^2\\xy\\\hline\end{array}$ 6. $\begin{array}{l}xy+yz+xz\\xyz\\\hline\end{array}$

7. $\begin{array}{l}1+2\,x+6\,x^2+4\,x^3\\x^3\\\hline\end{array}$ 8. $\begin{array}{l}4\,x^3+6\,x^2+2\,x+1\\3\,x^2\\\hline\end{array}$

In exercise 7, the multiplicand is **arranged** according to the **ascending powers** of x; in exercise 8, according to the **descending powers** of x.

Arrange according to the ascending or descending powers of some letter and perform the multiplications indicated:

9. $ab(6\,a^2+a^4+1+4\,a^3+4\,a)$.

10. $2\,xy(8\,x^3y+2\,x^4+2\,y^4+12\,x^2y^2+8\,xy^3)$.

11. $a^2bc(3\,a^4+16\,b^4+2\,ab^3+4\,a^3b+5\,a^2b^2)$.

12. $8\,t^3s^3(t^6+6\,ts^5+20\,t^3s^3+15\,t^4s^2+s^6+15\,t^2s^4)$.

13. $5\,x^2y^3(x^5+y^5+5\,x^4y+5\,xy^4+10\,x^3y^2+10\,x^2y^3)$.

29. To multiply a polynomial by a polynomial.

EXERCISES

1. Multiply $x+5$ by $x+2$; test the result.

	PROCESS	TEST
	$x+5$	$=6$ when $x=1$
	$x+2$	$=3$
x times $(x+5)=$	x^2+5x	
2 times $(x+5)=$	$2x+10$	
$(x+2)$ times $(x+5)=$	$x^2+7x+10$	$=18$

TEST. — The product must equal the multiplicand multiplied by the multiplier, regardless of what value x may represent. To test the result, therefore, we may assign to x any value we choose and observe whether, for that value, *product obtained = multiplicand × multiplier*. When $x=1$, multiplicand $=6$, multiplier $=3$, and $x^2+7x+10=18$; since $6\times 3=18$, it may be assumed that $x^2+7x+10$ is the correct product.

RULE. — *Multiply the multiplicand by each term of the multiplier and find the sum of the partial products.*

2. Multiply $x+4$ by $x+6$; test the result when $x=1$.
3. Multiply $x+1$ by $x+2$; test the result when $x=5$.
4. Multiply $2x+3$ by $4x+1$; test the result when $x=1$.
5. Multiply x^2+x+1 by $x+1$; test the result when $x=2$.
6. Multiply $2a+b+c$ by $3a+b$; test the result when $a=1$, $b=1$, and $c=1$.

In like manner the multiplication of any two literal expressions may be tested arithmetically by assigning any values we please to the letters.

While it is usually most convenient to substitute 1 for each letter, since this may be done readily by adding the numerical coefficients, the student should bear in mind that this really tests the coefficients and not necessarily the exponents, for any power of 1 is 1.

Multiply, and test each result:

7. $\dfrac{x+y}{x+y}$ 8. $\dfrac{x+4}{x+9}$ 9. $\dfrac{2x+1}{3x+5}$ 10. $\dfrac{5y+3z}{4y+z}$

Multiply, and test each result:

11. $2x+3$ by $x+2$.
12. $4x+1$ by $3x+4$.
13. $5n+1$ by $4n+5$.
14. $h+2k$ by $3h+k$.
15. $3l+5t$ by $2l+6t$.
16. $4y+6b$ by $2y+b$.
17. $2b+5c$ by $5b+2c$.
18. $ax+by$ by $ax+by$.

An indicated product is said to be **expanded** when the multiplication is performed.

Expand, and test each result:

19. $(c^2+d^2)(c^2+d^2)$.
20. $(3a+b)(3a+b)$.
21. $(2n^2+l)(n^2+2l)$.
22. $(2b+5c)(3b+8c)$.
23. $(\frac{1}{2}a+\frac{1}{3}b)(\frac{1}{2}a+\frac{1}{3}b)$.
24. $(\frac{2}{3}x+\frac{1}{4}y)(\frac{3}{4}x+\frac{1}{2}y)$.
25. $(2xy+3y)(4xy+7y)$.
26. $(4ax+3by)(4ax+3by)$.

Multiply, and test each result:

27. $\begin{array}{l} x^2+2xy+y^2 \\ \underline{x+y} \end{array}$

28. $\begin{array}{l} a+b+c \\ \underline{a+b+c} \end{array}$

29. $\begin{array}{r} 2t+3s+6 \\ \underline{t+2s+1} \end{array}$

30. a^2+ay+y^2 by $a+y$.
31. x^3+3x^2+x by $x+1$.
32. $(3l+n+1)(1+l+n)$.
33. $(r^2+2rs+s^2)(r+s+1)$.
34. $2a^2+3b^2+ab$ by $3a^2+4ab+5b^2$.
35. $3n^2+3m^2+mn$ by $m^3+2mn^2+m^2n$.
36. $a^5+a^4+4a^3+a^2+a$ by $a+1$.
37. $31x^3+27x^2+9x+3$ by $3x+1$.
38. $4x^3+3x^2y+5xy^2+6y^3$ by $5x+6y$.
39. $a^2+b^2+c^2+ab+ac+bc$ by $a+b+c$.
40. $m^8+m^6n^3+m^4n^6+m^2n^9+n^{12}$ by m^2+n^3.
41. Multiply $a^{6n}+a^{4n}b^{2c}+a^{2n}b^{4c}+b^{6c}$ by $a^{2n}+b^{2c}$.
42. Multiply $x^{2n}+2x^ny^n+y^{2n}$ by $x^{2n}+2x^ny^n+y^{2n}$.

DIVISION

30. In multiplication two numbers are given and their product is to be found. In division the product of two numbers and one of the numbers are given, and the other number is to be found.

Division is thus the **inverse** of multiplication.

Thus, $3 \times 4 = 12$ illustrates multiplication;
but $12 \div 4 = 3$ illustrates division, the *inverse* process.

31. To divide a monomial by a monomial.

Because $a^2 \cdot a^3 = a^{2+3} = a^5$,

$a^5 \div a^3 = a^{5-3} = a^2$. (Subtract exponents)

Similarly, because

$$3\,a^2 \cdot 5a^3 = (3 \cdot 5)\,a^{2+3} = 15\,a^5,$$

$$15\,a^5 \div 5\,a^3 = (15 \div 5)\,a^{5-3} = 3\,a^2. \quad \text{(Divide coefficients)}$$

The quotient may be obtained, just as in arithmetic, by removing equal factors from dividend and divisor by cancellation, thus:

$$\frac{15\,a^5}{5\,a^3} = \frac{3 \cdot \not{5} \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot a \cdot a}{\not{5} \cdot \not{a} \cdot \not{a} \cdot \not{a}} = 3\,a^2.$$

Again, $$\frac{21\,a^5b^3}{3\,a^3b^2} = \frac{\not{3} \cdot 7 \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot a \cdot a \cdot \not{b} \cdot \not{b} \cdot b}{\not{3} \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot \not{b} \cdot \not{b}} = 7\,a^2b,$$

or $$\frac{21\,a^5b^3}{3\,a^3b^2} = \frac{21}{3}\,a^{5-3}b^{3-2} = 7\,a^2b^1 = 7\,a^2b.$$

Hence, for division:

32. Law of exponents. — *The exponent of a number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

Since a number divided by itself equals 1, $a^5 \div a^5 = a^{5-5} = a^0 = 1$; that is, a number whose exponent is 0 is equal to 1.

33. Law of coefficients. — *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

EXERCISES

34. Tell quotients quickly:

1. $5\overline{)5^3}$, quotient 5^2
2. $7\,c^2d^4\overline{)35\,c^4d^5}$, quotient $5\,c^2d$.
3. $2\,a^3\overline{)a^5x}$, quotient $\frac{1}{2}\,a^2x$
4. $2^2\overline{)2^3}$
5. $3^4 \div 3^4$.
6. $4^0\,m^3\overline{)4^5\,m^4n^2}$
7. $\dfrac{12\,a^3b^3}{4\,ab^2}$.
8. $\dfrac{18\,x^4y^4}{3\,x^2y}$.
9. $\dfrac{21\,ab^3c^2}{7\,b^3}$.
10. $\dfrac{28\,a^4b^2c}{4\,abc}$.
11. $\dfrac{16\,x^3y^3z^3}{4\,xy^3z}$.
12. $\dfrac{24\,x^2y^2z^3}{8\,x^2z^2}$.
13. $\dfrac{20\,a^4b^5y^2}{4\,a^2b^2y^2}$.
14. $\dfrac{36\,a^4y^2z^3}{9\,a^4z^2}$.
15. $\dfrac{3\,ab(a+b)^2}{2(a+b)}$.
16. $\dfrac{4\,a^4b^3c^5}{20\,a^2bc^3}$.
17. $\dfrac{4\,x^6y^3z^4}{32\,x^4y^2z^3}$.
18. $\dfrac{2\,a^2(x-y)^3}{a(x-y)^2}$.

35. To divide a polynomial by a monomial.

Dividing as in arithmetic, we have

1. $2\overline{)86}$, quotient 43 $\qquad 2\overline{)80+6}$, quotient $40+3$ $\qquad 2\overline{)8\text{ tens}+6\text{ units}}$, quotient $4\text{ tens}+3\text{ units}$ $\qquad 2\overline{)8\,t+6\,u}$, quotient $4\,t+3\,u$

2. Since, § 27, $\qquad (a+b)\,x = ax + bx,$
if $ax + bx$ is regarded as the dividend and x as the divisor,

$$(ax+bx) \div x = a + b\text{; that is,}$$

36. *The quotient of a polynomial divided by a monomial is equal to the sum of the partial quotients obtained by dividing each term of the polynomial by the monomial.*

EXERCISES

37. 1. Divide $9\,x^2y^2 + 15\,xy^2z^2$ by xy^2; by $3\,xy$.

PROCESS

$$xy^2\overline{)9\,x^2y^2 + 15\,xy^2z^2}$$
$$9\,x \quad + 15\,z^2$$

PROCESS

$$3\,xy\overline{)9\,x^2y^2 + 15\,xy^2z^2}$$
$$3\,xy \quad + \quad 5\,yz^2$$

Find quotients:

2. $4\,cd\overline{)4\,c^2d + 20\,cd^2}$

3. $\dfrac{xz^2 + 3\,xz + x^2z^2}{xz}$.

4. $\dfrac{5\,x^2y + 10\,x^2y^2 + 15\,xy^2}{5\,xy}$.

5. $\dfrac{4\,a^2b^3 + 12\,a^3b^2 + 16\,a^4b}{4\,a^2b}$.

6. $\dfrac{24\,a^6b^2 + 32\,a^5b^3 + 40\,a^4b^4}{8\,a^4b^2}$.

7. $7\,ab\overline{)14\,a^4b^3 + 49\,a^2b}$

8. $\dfrac{35\,x^2y^3z^4 + 45\,x^4y^3z^2}{5\,x^2y^2z}$.

9. $\dfrac{36\,a^3b^4c^6 + 60\,a^2b^5c^7}{12\,a^2b^4c^6}$.

10. $\dfrac{24\,r^3s^2 + 30\,r^2s^2 + 42\,r^2s^3}{6\,r^2s^2}$.

11. $\dfrac{9\,x^2yz + 36\,xy^2z^3 + 45\,xyz^5}{9\,xyz}$.

12. $(8\,a^7b^3 + 28\,a^6b^4 + 16\,a^5b^5 + 4\,a^4b^6) \div 4\,a^4b^3$.

13. $(3\,x^3yz^2 + 15\,x^5y^2z^3 + 6\,x^4yz^3 + 18\,x^6y^3z) \div 3\,x^3yz$.

38. To divide a polynomial by a polynomial.

EXERCISES

1. Divide $3\,x^2 + 35 + 22\,x$ by $x + 5$; test the result.

	PROCESS		TEST
	$3\,x^2 + 22\,x + 35$	$x + 5$	$60 \div 6$
$3\,x$ times $(x + 5)$	$3\,x^2 + 15\,x$	$3\,x + 7$	$= 10$
	$7\,x + 35$		
7 times $(x + 5)$	$7\,x + 35$		

EXPLANATION. — For convenience, the divisor is written at the right of the dividend and the quotient below the divisor. Both dividend and divisor are arranged according to the descending powers of x.

Since the dividend is the product of the quotient and divisor, it is the sum of all the partial products formed by multiplying each term of the quotient by each term of the divisor. Hence, if $3\,x^2$, the first term of the dividend as arranged, is divided by x, the first term of the divisor, the result, $3\,x$, is the first term of the quotient.

Subtracting $3\,x$ times $(x + 5)$ from the dividend, leaves $7\,x + 35$, the part of the dividend still to be divided.

Proceeding, then, as before we find, $7x \div x = 7$, the next term of the quotient. 7 times $(x+5)$ equals $7x+35$. Subtracting, we have no remainder. Hence, the quotient is $3x+7$.

TEST. — When $x = 1$, the dividend equals 60 and the divisor 6. The quotient then should equal $60 \div 6$, or 10. On substituting 1 for x, we find that the quotient is equal to 10. Presumably, then, the result is correct.

2. Divide $x^3 + 6x^2 + 12x + 10$ by $x + 2$.

PROCESS

$$\begin{array}{r|l} x^3 + 6x^2 + 12x + 10 & x + 2 \\ \cline{2-2} x^3 + 2x^2 \qquad\qquad & x^2 + 4x + 4 + \dfrac{2}{x+2} \\ \cline{1-1} 4x^2 + 12x \qquad & \\ 4x^2 + 8x \qquad & \\ \cline{1-1} 4x + 10 & \\ 4x + 8 & \\ \cline{1-1} 2 & \end{array}$$

TEST

$$29 \div 3 = 9\tfrac{2}{3}$$

As in arithmetic, the whole of the undivided part of the dividend is not brought down for each division, but only so much of it as may be needed each time.

The remainder 2 is written over the divisor in the form of a fraction which is then added to the quotient as in arithmetic.

RULE. — *Arrange both dividend and divisor according to the ascending or the descending powers of a common letter.*

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend. The remainder will be a new dividend.

Divide the new dividend as before, and continue to divide in this way until the first term of the divisor is not contained in the first term of the new dividend.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and add the fraction to the part of the quotient previously obtained.

Divide, and test each result:

3. $x^2 + 2x + 1$ by $x + 1$.

4. $a^2 + 5a + 6$ by $a + 2$.

5. $5r + r^2 + 4$ by $r + 4$.

6. $3 + 7y^2 + 2y^4$ by $y^2 + 3$.

7. $6x^3 + x^6 + 7$ by $x^3 + 1$.

8. $6t^2 + 20t + 23$ by $3t + 7$.

9. $y^3 + 3y^2 + 3y + 1$ by $y + 1$.

10. $6z^4 + 4 + 10z^2 + 4z^6$ by $4z^2 + 2$.

11. $b^9 + 6b^6 + b^4 + 9b^3 + 4b + 8$ by $b^3 + 4$.

12. Divide $a^4 + 6a^3 + 27a^2 + 54a + 81$ by $a^2 + 3a + 9$.

PROCESS

$$
\begin{array}{l|l}
a^4 + 6a^3 + 27a^2 + 54a + 81 & a^2 + 3a + 9 \\
\underline{a^4 + 3a^3 + 9a^2} & \overline{\underline{a^2 + 3a + 9}} \\
\qquad 3a^3 + 18a^2 + 54a & \\
\qquad \underline{3a^3 + 9a^2 + 27a} & \\
\qquad\qquad 9a^2 + 27a + 81 & \\
\qquad\qquad \underline{9a^2 + 27a + 81} &
\end{array}
$$

TEST

$169 \div 13$
$= 13$

Divide, and test each result:

13. $x^4 + 4x^3 + 12x^2 + 16x + 16$ by $x^2 + 2x + 4$.

14. $4l^8 + 4l^6 + 13l^4 + 6l^2 + 9$ by $2l^4 + l^2 + 3$.

15. $4y^4 + 5y^3 + y^6 + 11y + 3y^2 + 6$ by $y^3 + 3y + 2$.

16. $6r^4 + 26r^2 + 18 + 15r + 7r^3$ by $2r + 3r^2 + 3$.

17. $x^5 + x^4y + 2x^3y^2 + 2x^2y^3 + xy^4 + y^5$ by $x + y$.

18. $2a^5 + 6a^3 + 3a^2 + 2a^4 + 5a + 2$ by $a^2 + a + 2$.

19. $x^7 + 2x^6y + 4x^5y^2 + 3x^4y^3 + 2x^3y^4$ by $x^2 + xy + y^2$.

20. $z^7 + 8z^6 + 3z^9 + z^4 + 6z + 20z^3 + 30$ by $z + 3z^3 + 5$.

21. $5s^3t + 4s^2t^2 + t^4 + 3s^4 + 3st^3$ by $3s^3 + 2s^2t + 2st^2 + t^3$.

22. $4a^4 + 28a^3b + 61a^2b^2 + 45ab^3 + 12b^4$ by $2a^2 + 7ab + 3b^2$.

23. $p^6 + 4p^4q^2 + 9p^2q^4 + 6q^6 + 2p^5q + 6p^3q^3 + 12pq^5$ by $p^4 + 3p^2q^2 + 6q^4$.

EQUATIONS AND PROBLEMS

39. How many pounds added to 25 pounds will give 30 pounds?

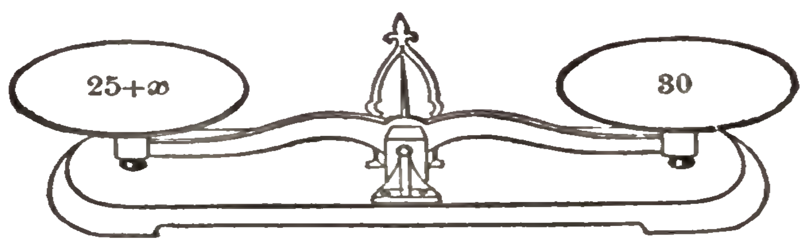
The statement of the problem may be condensed to

$$\begin{array}{r} 25 \text{ pounds} \\ +\ ?\ \text{pounds} \\ \hline 30 \text{ pounds} \end{array} \quad \text{or} \quad \begin{array}{r} 25 \text{ pounds} \\ +\ x\ \text{pounds} \\ \hline 30 \text{ pounds} \end{array} \quad \text{or} \quad 25 + x = 30$$

The letter x is only a convenient symbol for the **unknown number** (of pounds), or the number (of pounds) to be found. 25 and 30, on the other hand, are **known numbers.**

The equation, $25 + x = 30$, is the briefest possible statement of the relation between the known and unknown numbers in the problem. Finding the value of x is called **solving** the equation, $25 + x$ is the **first member** of the equation, and 30 is the **second member.**

Equations



40. 1. If 25 pounds are taken from the weight in each scale pan, the balance will be preserved.

In the same way, if 25 is subtracted from each member of the equation $25 + x = 30$, the equality will be preserved.

$$\begin{array}{rcr} 25 + x & = & 30 \\ 25 & & 25 \\ \hline x & = & 5 \end{array}$$

2. What number subtracted from $x + 10$ will give x?

If the first member of $x + 10 = 12$ is decreased to x by subtracting 10, what must be done to the second member to preserve the equality?

Tell how the equation $x + 10 = 12$ may be solved.

3. Suppose that $x - 4 = 3$ and we wish to find the value of x. How much greater is x than $x - 4$?

If the first member of $x - 4 = 3$ is increased to x by adding 4, what must be done to the second member to preserve the equality? Tell how the equation may be solved.

The same number may be added to both members of an equation, or subtracted from both, without destroying the equality.

EXERCISES

41. State what must be done to both members to change one member to x without destroying the equality; solve:

1. $x + 6 = 8$.	5. $x + 2 = 10$.	9. $12 = 10 + x$.
2. $x - 3 = 2$.	6. $x - 5 = 11$.	10. $15 = 11 + x$.
3. $x - 4 = 5$.	7. $x + 1 = 12$.	11. $30 = 20 + x$.
4. $x + 7 = 9$.	8. $x - 7 = 10$.	12. $14 = x + 10$.

42. 1. If $x = 8$, what is the value of $2x$? of $3x$? of $\frac{5}{2}x$?

2. If $6x = 12$, what is the value of $1x$, or of x?

3. If $\frac{1}{3}x = 10$, what is the value of 3 times $\frac{1}{3}x$, or of x?

4. What must be done to both members of each of the following equations to give an equation whose first member is x?

$$\tfrac{1}{2}x = 3 \qquad \tfrac{1}{3}x = 5 \qquad 4x = 12 \qquad 5x = 35$$

Both members of an equation may be multiplied or divided by the same number without destroying the equality.

EXERCISES

43. State what must be done to both members to change one member to x without destroying the equality; solve:

1. $2x = 6$.	5. $\frac{1}{2}x = 5$.	9. $\frac{1}{6}x = 5$.
2. $5x = 5$.	6. $\frac{1}{3}x = 2$.	10. $\frac{1}{8}x = 4$.
3. $4x = 8$.	7. $\frac{1}{4}x = 3$.	11. $8x = 24$.
4. $3x = 9$.	8. $\frac{1}{5}x = 7$.	12. $9x = 18$.

44. The equations solved so far in this chapter have been solved each by a single one of the following steps:

1. *By adding the same number to both members.*
2. *By subtracting the same number from both members.*
3. *By multiplying both members by the same number.*
4. *By dividing both members by the same number.*

The equations that follow may be solved by two or more of these steps taken separately.

EXERCISES

45. 1. Solve the equation $2x + 20 = 80 - 4x$.

SOLUTION

The first step in solving an equation is to get the unknown terms into one member, usually the first, and the known terms into the other.

$$2x + 20 = 80 - 4x.$$

Adding $4x$ to both members,

$$2x + 4x + 20 = 80 + 4x - 4x,$$

or, uniting terms, $\quad 6x + 20 = 80.$

Subtracting 20 from both members,

$$6x + 20 - 20 = 80 - 20,$$

or, uniting terms, $\quad 6x = 60.$

Dividing both members by 6, $\quad x = 10.$

VERIFICATION

We should always test the answer by finding whether the value obtained is such as to make the members of the original equation equal.

Thus, substituting $x = 10$ in the given equation, we have

$$20 + 20 = 80 - 40,$$

or $\quad 40 = 40.$

Hence, 10 is the true value of x.

Solve and verify:

2. $7x + 12 = 5x + 16.$
3. $5x - 20 = 2x + 13.$
4. $4x - 11 = 19 - 2x.$
5. $13x + 4 = 5x + 12.$
6. $4x - 11 + 2x = 2x - 5.$
7. $3x + 14 + 7x = 78 + 2x.$
8. $9x + 23 + 2x = 4x + 37.$
9. $5x - 7 + 15x = 13x + 14.$

10. Solve the equation $\frac{3}{2}x = 15$.

FIRST SOLUTION

$\frac{3}{2}x = 15.$

Dividing both members by 3, $\frac{1}{2}x = 5.$

Multiplying both members by 2, $x = 10.$

SECOND SOLUTION

By multiplying by 2 *before* dividing by 3, fractions may be avoided.

$\frac{3}{2}x = 15.$

Multiplying both members by 2, $3x = 30.$

Dividing both members by 3, $x = 10.$

VERIFICATION. $\frac{3}{2}$ of $10 = 15.$

Solve and verify:

11. $\frac{3}{2}x = 9.$

12. $\frac{4}{3}x = 8.$

13. $\frac{3}{5}x = 21.$

14. $\frac{2}{3}x = 30.$

15. $\frac{5}{8}x = 15.$

16. $\frac{7}{3}x = 21.$

Solve by the method best adapted; verify results:

17. $9x - 17 = 23 + x.$

18. $2x + 3x - 2x = 21.$

19. $9x - 4x + 2x = 14.$

20. $8x + 5x - 5x = 48.$

21. $22 - 6x = 40 - 8x.$

22. $7x + 6x - 7x = 42.$

23. $\frac{5}{2}x = 10.$

24. $\frac{2}{3}x = 14.$

25. $\frac{4}{5}x = 28.$

26. $\frac{5}{6}x = 20.$

27. $\frac{7}{8}x = 63.$

28. $\frac{6}{7}x = 48.$

29. $2x - 4 + 6x = 22 - 15 + 21.$

30. $5x - 10 - 4x = 46 + 3x - 60.$

31. $6x + 5x - 70 = 5x + 54 - 70.$

32. $5x + 16 - 6x = 16 + 24 - 6x.$

33. $9x + 15 - 2x = 32 + 4x - 11.$

34. $10x - 39 + 12x - 9x + 42 - 4x = 42 - 4x.$

35. $16x + 12 - 75 + 2x - 12 - 110 = 8x - 50 - 25.$

36. $3x - 18 + 27 + 10x - 11 = 25 + 4x - 7x + 12 + 3x.$

37. $18x + 16 = 8 + 12x + 8 - 13 + 25x - 9 + 100 - 25x.$

Algebraic Representation

46. 1. Express the sum of 2, $\frac{1}{3}$, and $\frac{1}{5}$; of x, $\frac{1}{2}y$, and $\frac{1}{4}z$.

2. What number is 4 less than 12? n less than 25?

3. Express the number that exceeds 5 by 3; a by b.

4. Represent in the shortest way the sum of five x's; the product of five x's.

5. Mary read 10 pages of a book. On what page did she begin to read, if she stopped at the top of page 21? of page a?

6. Express 10 dollars in terms of cents; 10 cents in terms of dollars; m dollars in terms of cents; m cents in terms of dollars.

7. A has 12 dollars and B, 8 dollars. How much will each have if A gives B 4 dollars? m dollars?

8. At 3 dollars per day, how much will a man earn in 4 days? in x days? At a dollars per day, how much will he earn in b days? in c days? in a days?

9. By what number must 25 be multiplied to produce 300? 10 to produce x? r to produce s?

10. What are the two odd numbers nearest to 5? If $n+3$ is an odd number, what are the two odd numbers nearest to $n+3$? the two even numbers?

11. How many square rods are there in a square field one of whose sides is 2 rods long? $(x+y)$ rods long?

12. How many square rods are there in a field 6 rods long and 4 rods wide? $(m+n)$ rods long and m rods wide?

13. If it takes 4 men 5 days to do a piece of work, how long will it take 1 man to do it? 2 men? x men? If it takes b men c days to do a piece of work, how long will it take 1 man? z men?

14. The number 25 may be written $20+5$. Write the number whose first digit is x and second digit y.

15. Represent $(a+b)$ times the number whose tens' digit is m and units' digit n.

Problems

47. **1.** What number increased by 6 is equal to 44?

SOLUTION

Let $x =$ the number.
Then, $x + 6 = 44.$
Solving the equation, $x = 38$, the number.

2. What number increased by 15 is equal to 51?

3. What number decreased by 32 is equal to 60?

4. What number multiplied by 3 is equal to 78?

5. What number divided by 8 is equal to 62?

6. If 20 is added to a certain number and 14 is subtracted from the sum, the result is 19. Find the number.

7. One half of a number, and 11 more, is equal to 37. Find $\frac{1}{2}$ of the number, then find the number.

8. If $\frac{3}{4}$ of a certain number is 18, what is the number?

9. The sum of two numbers is 55 and the larger is 4 times the smaller. What are the numbers?

SOLUTION

Let $x =$ the smaller number.
Then, $4x =$ the larger number,
and $x + 4x =$ the sum of the two numbers.
But $55 =$ the sum of the two numbers.
$\therefore x + 4x = 55.$
Solving the equation, $x = 11$, the smaller number,
and $4x = 44$, the larger number.

NOTE. — The sign $\therefore$ means 'therefore.'

10. Separate 116 into two parts, one of which shall be 3 times the other.

11. Separate 72 into two parts, one of which shall be $\frac{1}{3}$ of the other.

12. What number increased by $\frac{1}{2}$ of itself equals 54?

13. What number decreased by $\frac{1}{5}$ of itself equals 84?

14. Five times a number exceeds 3 times the number by 14. What is the number?

15. The double of a number is 64 less than 10 times the number. What is the number?

16. Four times a certain number exceeds 12 as much as 3 times the number is less than 72. What is the number?

17. Of the steam vessels built on the Great Lakes one year, 21, or 5 less than $\frac{1}{3}$ of all, were of steel. How many steam vessels were built on the Lakes that year?

SOLUTION

Let $x =$ the number of steam vessels built.

Then, $\frac{1}{3}x - 5 =$ the number of steel vessels.

But $21 =$ the number of steel vessels.

$\therefore \frac{1}{3}x - 5 = 21.$

Adding 5 to both members of the equation,

$$\frac{1}{3}x + 5 - 5 = 21 + 5,$$

or

$$\frac{1}{3}x = 26.$$

Multiplying both members by 3,

$x = 78$, the number of steam vessels built.

NOTE. — The equation $\frac{1}{3}x - 5 = 21$ is called the **equation of the problem.**

General Directions for Solving Problems. — 1. *Represent one of the unknown numbers by some letter, as x.*

2. *From the conditions of the problem find an expression for each of the other unknown numbers.*

3. *Find from the conditions two expressions that are equal and write the equation of the problem.*

4. *Solve the equation.*

18. Two cars together contained 400 bales of cotton. If one car had 6 bales more than the other, how many had each?

19. The playgrounds of two cities occupy 183 acres. One city has 27 acres less than the other. How many acres has each?

20. The height of the big tree *Wawona* in California is 8 feet more than 9 times its diameter. If the height is 260 feet, what is the diameter of the tree?

21. Yellowstone Park contains 3400 antelope and deer. If the antelope number 200 less than twice the number of deer, how many deer are there in the Park?

22. A department store restaurant serves luncheon daily to 5000 people. If the number served lacks 1000 of being 3 times the number seated at once, find the seating capacity.

23. One year the Bureau of Engraving and Printing employed 2400 people. The number of women was 400 greater than the number of men. Find the number of each employed.

24. In a recent year the Lake Superior region furnished 38,400,000 tons of iron ore, or $\frac{4}{5}$ of all that was mined in the United States. How much was mined in the United States?

25. Denmark produces 44,000 tons of beet sugar annually. If this is 4000 more than $\frac{1}{2}$ the number of tons consumed, what is the annual consumption of beet sugar in Denmark?

26. One year the government spent $60,000 in operating a flag factory. The material cost $8000 less than 3 times the amount expended for labor. What was the cost of each?

27. Two power companies together use 27,200 cubic feet of water per second from Niagara Falls. Find the average discharge of the falls per second, if these companies use $\frac{2}{15}$ of it.

28. The whalebone in one whale was worth $\frac{5}{16}$ as much as that in another, and the value of the whalebone in the two was $525. Find the value of the whalebone in each.

29. In a fossil bed in Switzerland 470 species of insects were found and this was 30 less than $\frac{5}{7}$ of the number found in a bed in Colorado. Find the number in the latter bed.

30. The largest cask in the world contains a number of hogsheads that is 1 less than 25 times the number of feet in its diameter. If it contains 649 hogsheads, find its diameter.

31. The railroads consume $\frac{3}{10}$ of the total annual production of coal in the United States. Their annual expenses for coal are 240 million dollars with the average price $2 per ton. How many tons are produced in the United States each year?

32. The United States uses 101 million files each year. The number of files made in this country lacks 15 million of being 3 times the number of those imported. How many files are imported?

33. The Jamestown Exposition pier inclosed a rectangular lagoon, the length of which was 1000 feet more than its width. If its perimeter was 6800 feet, how long was it?

34. In one year the output of scrap mica was 5 tons more than twice the output of sheet mica and there were $430\frac{1}{2}$ tons more of the former than of the latter. Find the number of tons of each.

35. The largest concrete chimney in the world contains 1460 tons of steel and sand. The weight of the steel used was $\frac{3}{70}$ of the weight of the sand. Find the number of tons of each that were used.

36. Mt. Whitney, the highest point in the United States, is 14,500 feet above sea level. This is 700 feet more than 50 times the depth below sea level of Death Valley, the lowest point of dry land in the country. How far below sea level is Death Valley?

37. The lilies sent to the United States annually from Bermuda are worth $\frac{1}{20}$ as much as all our imported floral products. If the other floral products are worth \$1,900,000, find the value of the lilies imported from Bermuda.

38. The distance from Cuba to Haiti is 31 miles less than the distance to Jamaica, and from Cuba to Yucatan, which is 130 miles, is 9 miles less than the sum of the distances to Haiti and Jamaica. Find the distance from Cuba to Jamaica.

39. In field and track events at the Olympic games in London, America won $35\frac{1}{2}$ points more than Great Britain and Sweden together, and Sweden won 54 points less than Great Britain. Find the score of each, if the total score of the three countries was $193\frac{1}{2}$ points.

REVIEW

48. 1. Tell how similar terms are added; subtracted. Tell what to do with dissimilar terms in addition; in subtraction.

2. Write a polynomial arranged according to the ascending powers of some letter; the descending powers.

3. State the law of exponents for multiplication; for division; the law of coefficients for each. $3^0 = ?$ $8^0 = ?$ $a^0 = ?$

4. What is an equation? Write one and point out the unknown numbers in it; the known numbers; its first member; its second member.

5. What is meant by 'solving an equation'? Give four methods by one or more of which equations may be solved. How may the value of an unknown number, obtained by solving an equation, be verified?

Solve and verify:

6. $3x = 21$.

7. $\frac{5}{6}x = 15$.

8. $7x - 3 + 4x = 21 - 2x + 2$.

9. $10 - 2x = 3x + 5 + 9 - 6x$.

10. Add $x + y + z$, $7x + 2z + 3y$, $4z + 5y$, and $9x + 3y + 2z$.

11. $11a + 5b + 2c - 4b + 2a - c + 4c - 9a + 5b - 3c + a = ?$

12. Subtract $5a^m + 7b^n + 18c$ from $7a^m + 25c + 8b^n + 8d$.

Expand:

13. $7p^3q^5r^7(pqr^2 + 4p^2qr + 2qr^3 + p^5 + 5p^3q^2r^4 + 3pq)$.

14. $(x^4 + 7x^3y + 4x^2y^2 + 3xy^3 + 2y^4)(x^3 + 4x^2y + 2y^3)$.

15. $(3z^4 + 4z^3w + 6z^2w^2 + 4zw^3 + 13w^4)(2z^2 + 4zw + 3w^2)$.

16. $(a^5b + 3a^4b^2 + 6a^3b^3 + 5a^2b^4 + 11ab^5)(a^3 + 3a^2b + 4ab^2)$.

Divide, and test each result:

17. $12l^6m^4n^2 + 18l^5m^2n^4 + 15l^4m^3n^3 + 3l^3mn^2$ by $3l^3mn^2$.

18. $35r^4 + 30r^3s + 69r^2s^2 + 12rs^3 + 22s^4$ by $5r^2 + 2s^2$.

19. $22x^2y^2 + 24xy^3 + 27x^3y^3 + 36x^2y^4 + 3x^3y$ by $3xy + 4y^2$.

20. $4a^5c + 8a^4bc + 11a^3b^2c + 24a^2b^3c + 24ab^4c + 7b^5c$ by $2a^2 + 3ab + b^2$.

POSITIVE AND NEGATIVE NUMBERS

49. The student of arithmetic knows the meaning of such an expression as $10 - 4$, but as yet an expression like $4 - 10$ has no meaning to him. It is the purpose of this chapter to extend the idea of number so that subtracting a larger number from a smaller one will have as much meaning as subtracting a smaller number from a larger one, to show that there is a practical demand for a new kind of number, and finally to show how operations involving this new kind of number are performed.

50. Suppose that at noon the temperature is 10° above 0 and that at 6 P.M. it has fallen 4°. The temperature is then $10° - 4°$, or 6° above 0, but if it has fallen 15° instead of 4°, it is then $10° - 15°$, and because the numbers on a thermometer extend below as well as above 0, we see that $10° - 15°$ means that the temperature is 5° below 0, 10° of the 15° of fall taking it to 0 and the other 5° of fall taking it to 5° below 0.

For convenience and brevity degrees 'above 0' are marked with the sign + and degrees 'below 0' with the sign −.

Such statements may be abbreviated algebraically, thus:

$$+10° - 4° = +6°,$$

and

$$+10° - 15° = -5°.$$

Similarly, if a ship now at 20° north latitude (latitude, +20°) sails south 30°, it will cross the equator (latitude, 0°) and be at 10° south latitude (latitude, −10°).

Again, a tourist in going from Lake Lucerne 1435 feet above sea level (altitude +1435 feet) to the Dead Sea 1295 feet below sea level (altitude −1295 feet) goes not only *to* 0 altitude (sea level), but *through* 0 altitude.

51. Such illustrations as those on the preceding page show a practical need of extending the number scale of arithmetic,

$$1, 2, 3, 4, 5, \cdots,^*$$

below 0, as on the thermometer.

The scale of **algebraic numbers**, then, including 0, is

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots,$$

the numbers in the scale increasing by 1 from left to right.

52. Numbers greater than 0, called **positive numbers**, are written either with or without the sign + prefixed.

Numbers less than 0, called **negative numbers**, *always* have the sign − prefixed.

53. By repeating the **positive unit**, +1, any positive integer may be obtained, and by repeating the **negative unit**, −1, any negative integer may be obtained.

Fractions are measured by positive or negative *fractional* units.

54. It is seen, then, that while in arithmetic the signs + and − are used to indicate operations to be performed, they have an extended meaning and use in algebra, namely, to denote **opposition**. In this sense they are called **quality**, or **direction, signs**.

Thus, if gains are considered *positive*, indicated by +, *losses* are *negative*, indicated by − ; if *credits* are +, *debts* are − ; if distances *north* or *west* or *upstream* are +, distances *south* or *east* or *downstream* are −.

55. When it is necessary to distinguish between signs of operation and signs of quality the number with its sign of quality may be inclosed in parentheses.

Thus, the sum of 2 and −3 may be written $2 + (-3)$.

56. Positive and negative numbers, whether integers or fractions, are called **algebraic numbers**.

Arithmetical numbers are positive numbers.

* The sign of continuation, ⋯, is read 'and so on' or 'and so on to.'

57. The value of a number without regard to its sign is called its **absolute value**.

Thus, the absolute value of both + 4 and − 4 is 4.

ADDITION AND SUBTRACTION

58. Addition and subtraction of positive and negative numbers may be performed by counting along the scale of algebraic numbers.

To illustrate, + 3 is *added* to − 2 by beginning at − 2 in the scale and counting 3 units in the *ascending*, or *positive*, direction, arriving at + 1; consequently, + 3 is subtracted from + 1 by beginning at + 1 and counting 3 units in the *descending*, or *negative*, direction, arriving at − 2.

59. The result of adding two or more algebraic numbers is called their **algebraic sum**.

This differs from their *arithmetical sum*, which is the sum of their absolute values.

Unless otherwise specified 'sum' in this book means 'algebraic sum'.

60. In addition, two numbers are given, and their algebraic sum is to be found. In subtraction, the algebraic sum and one of the numbers are given, and the other number is to be found.

Subtraction is thus the *inverse* of addition.

The **difference** is the algebraic number that *added to the subtrahend gives the minuend.*

Sum of Two or More Numbers

EXERCISES

61. Give algebraic sums:

1.	+ 5	− 5	+ 5	+ 5	+ 5	− 5	− 5
	+ 5	− 5	− 5	− 4	− 9	+ 8	+ 2

SUGGESTIONS.—The sum of 5 positive units and 5 positive units is 10 positive units; of 5 negative units and 5 negative units, 10 negative units; of 5 negative units and 5 positive units, 0; of 4 negative units and 5 positive units, 1 positive unit; of − 9 and + 5, − 4; etc.

To add two algebraic numbers:

RULE. — *If they have* **like signs,** *add the absolute values and prefix the common sign; if they have* **unlike signs,** *find the difference of the absolute values and prefix the sign of the numerically greater.*

By successive applications of the above rule any number of numbers may be added.

Add:

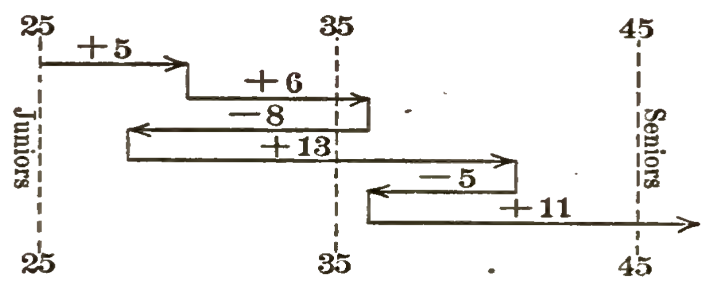
2.	3.	4.	5.	6.
8	$+8$	-8	$+4$	-5
2	$+2$	-2	-7	$+8$

7.	8.	9.	10.	11.
6	$+7$	-5	$+8$	-9
-3	-3	-3	-9	3
7	$+2$	-8	$+1$	2

12. $10+(-4)+(-6)+(-7)$.
13. $10-4-6-7$.
14. $-40+6+8+7+6$.
15. $-12+8+2+(-6)+2$.
16. $-12+8+2-6+2$.
17. $8-2+3+6-8+7-9$.

18. Julius Cæsar was born in the year -100 (100 B.C.), and was 56 years old when assassinated. In what year was he assassinated?

19. In a football game the ball was advanced 5 yards from the Juniors' 25-yard line toward the Seniors' goal, then 6 yards, then -8 yards (*i.e.* it went back 8 yards), and so on, as shown in the diagram. What was the position of the ball after 3 plays? after 4 plays? after 5 plays? after 6 plays?



20. Plot the following and find the last position of the ball: On 15-yard line; gained 4 yards; gained 5 yards; lost 2 yards; gained 30 yards; lost 6 yards; lost 2 yards; gained 12 yards.

21. How far from port is a vessel, if it sails 50 miles, -10 miles (driven back 10 miles), 40 miles, -30 miles, and 80 miles?

62. By doing the work in §§ 17–19, the student has become familiar with adding literal expressions in which the terms are all positive. When some of the terms are negative, the method is essentially the same, the only difference being in the matter of signs, which has just been explained.

EXERCISES

63. Add:

1. $\begin{array}{r} 2x \\ \underline{3x} \end{array}$ 2. $\begin{array}{r} a \\ \underline{5a} \end{array}$ 3. $\begin{array}{r} -a \\ \underline{4a} \end{array}$ 4. $\begin{array}{r} -4c \\ \underline{-3c} \end{array}$

5. $\begin{array}{r} 4v \\ -2v \\ \underline{-7v} \end{array}$ 6. $\begin{array}{r} -y \\ 4y \\ \underline{-9y} \end{array}$ 7. $\begin{array}{r} 12mb \\ -2mb \\ \underline{-6mb} \end{array}$ 8. $\begin{array}{r} 40x^2 \\ -10x^2 \\ \underline{-60x^2} \end{array}$

9. $\begin{array}{r} 6a-2b \\ 2a+3b \\ \underline{-5a-4b} \end{array}$ 10. $\begin{array}{r} 3xy+2y^2 \\ -2xy+6y^2 \\ \underline{7xy-4y^2} \end{array}$ 11. $\begin{array}{r} 10x+3y+z \\ -x-\ \ y\ \ \ \ \ \\ \underline{2x+2y+z} \end{array}$

12. Simplify $11a^2b-7ab^2+2ac^2+10ab+ac^2-2a^2b+b^3+5ab^2-2b^3+2ab^2-8ab-6a^2b$.

PROCESS

$$\begin{array}{rrrrr} 11a^2b & -7ab^2 & +2ac^2 & +10ab & +b^3 \\ -2a^2b & +5ab^2 & +ac^2 & & -2b^3 \\ -6a^2b & +2ab^2 & & -8ab & \\ \hline 3a^2b & & +3ac^2 & +2ab & -b^3 \end{array}$$

RULE. — *Arrange the terms so that similar terms shall stand in the same column.*

Find the algebraic sum of each column, and write the results in succession with their proper signs.

13. Simplify $2a+2b+3c+4b-4a+6a-2c$.
14. Simplify $7w+4x-4y-x-2w+3y-3x+4w$.
15. Simplify $7l-6m+3n-8l+4m+11n+m-2l$.
16. Simplify $15r+6s-11t+r-9s+t-2s+5r-2t-10r$.

Simplify the following polynomials:

17. $7x - 11y + 4z - 7z + 11x - 4y + 7y - 11z - 4x + y - x - z.$

18. $a + 3b + 5c - 6a + d + 4b - 2c - 2b + 5a - d + a - b.$

19. $4x^2 - 3xy + 5y^2 + 10xy - 17y^2 - 11x^2 - 5xy + 12x^2 - 2xy.$

20. $2xy - 5y^2 + x^2y^2 - 7xy + 3y^2 - 4x^2y^2 + 5xy + 4y^2 + x^2y^2.$

21. Add $2a - 3b$, $2b - 3c$, $5c - 4a$, $10a - 5b$, and $7b - 3c$.

22. Add $x + y + z$, $x - y + z$, $y - z - x$, $z - x - y$, and $x - z$.

23. Add $4x^3 - 2x^2 - 7x + 1$, $x^3 + 3x^2 + 5x - 6$, $4x^2 - 8x^3 + 2 - 6x$, $2x^3 - 2x^2 + 8x + 4$, and $2x^3 - 3x^2 - 2x + 1$.

24. Add $5x - 3y - 2z$, $4y - 2x + 6z$, $3a - 2x - 4y$, $4b - 2z - 5y$, $a - 5b$, $5y - 6x$, $8x + 2y - 5a - 2b$, and $6x - y - 2z + 4b$.

25. Add $.12x^3 - 4x^2 + x + 2$, $.4x^2 - 4x + .4 - x^3$, $3\frac{1}{2}x - .6 + 3x^2 + 2x^3$, and $1 - \frac{1}{2}x + 1.2x^2 + \frac{22}{25}x^3$.

26. Add $20x^{2m} - 4x^my^n + 36y^{2n}$, $4x^my^n - 15y^{2n} - 12x^{2m}$, $3y^{2n} + 3x^{2m}$, $4x^my^n - 11x^{2m} - 16y^{2n}$, and $x^{2m} - y^{2n}$.

Difference of Two Numbers

EXERCISES

64. On account of the extension of the scale of numbers below zero (§ 51), subtraction is always possible in algebra.

When the subtrahend is positive, algebraic subtraction is like arithmetical subtraction, and consists in counting *backward* along the scale of numbers, as illustrated in § 58.

Subtract the lower number from the upper one:

1.	6	6	6	6	6	6	6
	3	4	5	6	7	8	9
2.	−3	−3	−3	−4	−5	−6	−7
	0	1	2	3	4	5	6

Observe that *subtracting a positive number is equivalent to adding a numerically equal negative number.*

When the subtrahend is negative, it is no longer possible to subtract as in arithmetic by counting backward.

3. Subtract -2 from 8.

PROCESS

$$\begin{array}{r} 8 \\ -2 \\ \hline 8+2=10 \end{array}$$

EXPLANATION. — If 0 were subtracted from 8, the result would be 8, the minuend itself.

The subtrahend, however, is not 0, but is a number 2 units below 0 in the scale of numbers. Hence, the difference is not 8, but is $8 + 2$, or the minuend plus the subtrahend with its sign changed.

Or, -2 is subtracted from 8 by beginning at 8 in the scale of numbers and counting 2 units in the direction *opposite* to that indicated by the sign of the subtrahend, arriving at 10.

REMARK. — Notice that any number is *added* by counting along the scale of numbers *in the direction indicated by its sign;* and any number is subtracted by counting *in the direction opposite to that indicated by its sign.*

Subtract the lower number from the upper one:

4.	4	4	4	4	5	7	9
	0	-1	-2	-3	-6	-7	-5
5.	-5	-5	-5	-5	-1	-4	-6
	0	-1	-2	-6	-3	-7	-5

Observe that:

PRINCIPLE. — *Subtracting any number (positive or negative) is equivalent to adding it with its sign changed.*

Subtract the lower number from the upper one:

6.	10	7.	12	8.	20	9.	16	10.	40
	-2		5		-6		-4		-8
11.	0	12.	-3	13.	-7	14.	10	15.	-5
	-2		-6		4		-5		10
16.	4	17.	4	18.	-4	19.	-9	20.	-7
	4		-4		4		3		8

21. Subtract 12 from -1.

22. Subtract -4 from 14.

23. From 0 subtract -3.

24. From -3 subtract 0.

25. From 0 subtract -7; from the result subtract -4; then add -2; add -3; add 7; subtract 11; and add -6.

26. Which is greater and how much, 3 or -5? -2 or -5? 6 or $8-3$? $-2+(-8)$ or $-2-(-8)$?

A weather map for January 16 gave the following minimum and maximum temperatures (Fahrenheit):

	Chicago	Duluth	Helena	Montreal	New Orleans	New York
Minimum	24°	$-6°$	$-12°$	$-12°$	64°	20°
Maximum	30°	2°	$-4°$	18°	76°	42°

27. The range of temperature in Chicago was 6°. Find the range of temperature in each of the other cities.

28. The freezing point is 32° F. How far below the freezing point did the temperature fall in Montreal?

29. How much colder was it in Duluth than in Chicago? in Montreal than in New York? in Helena than in New Orleans?

30. An elevator runs from a basement, -22 feet above the first floor, to the tenth story, 105 feet above the first floor. Express its total *rise* from the basement to the tenth floor; from the tenth floor to the basement.

65. From the work of this chapter, the student will have discovered that negative numbers give the definitions of addition, subtraction, sum, and difference a wider range of meaning than they had in arithmetic. In algebra addition does not always imply an increase, nor subtraction a decrease.

In §§ 20, 21, the student learned how to subtract one literal expression from another, all the terms being positive and the subtrahend being less than the minuend. This is arithmetical subtraction. He will now apply the *broader algebraic idea* of subtraction to literal expressions.

EXERCISES

66. 1. From $10x$ subtract $15x$.

PROCESS

$$\begin{array}{r} 10\,x \\ 15\,x \\ \underline{-\qquad} \\ -\ 5\,x \end{array}$$

EXPLANATION. — Since (§ 64, Prin.) subtracting any number is equivalent to adding it with its sign changed, $15x$ may be subtracted from $10x$ by changing the sign of $15x$ and adding $-15x$ to $10x$.

	2.	3.	4.	5.	6.
From	$5a$	$5x$	$9am$	$-8mn$	$3x^2y^2$
Take	$2a$	$-2x$	$21am$	$-4mn$	$-10x^2y^2$

7. From $8x-3y$ subtract $5x-7y$.

PROCESS

$$\begin{array}{r} 8\,x - 3\,y \\ 5\,x - 7\,y \\ \underline{-\qquad +\ } \\ 3\,x + 4\,y \end{array}$$

EXPLANATION. — Since (§ 64, Prin.) subtracting any number is equivalent to adding it with its sign changed, subtracting $5x$ from $8x$ is equivalent to adding $-5x$ to $8x$, and subtracting $-7y$ from $-3y$ is equivalent to adding $+7y$ to $-3y$.

RULE. — *Change the sign of each term of the subtrahend, and add the result to the minuend.*

After a little practice the student should make the change of sign mentally.

	8.	9.	10.	11.
From	$9a+7b$	$5r-10s$	$10x+2y$	$3m-3n$
Take	$2a+3b$	$7r+4s$	$6x-4y$	$2m-5n$

12. From $5x-3y+z$ take $2x-y+8z$.

13. From $3a^2b+b^3-a^3$ take $4a^2b-8a^3+2b^3$.

14. From $13a^2+5b^2-4c^2$ take $8a^2+9b^2+10c^2$.

15. From $15x-3y+2z$ subtract $3x+8y-9z$.

16. From a^2-ab-b^2 subtract $ab-2a^2-2b^2$.

17. From m^2-mn+n^2 subtract $2m^2-3mn+2n^2$.

18. From $4x^2+3xy+y^2$ subtract $2x^2-5xy+2y^2$.

19. From $3ab + a^2 + b^2$ subtract $a^2 + 4ab + b^2$.

20. From $6x^2 + 4xy - 3y^2$ subtract $4y^2 - 3xy + 6x^2$.

21. From the sum of $3a^2 - 2ab - b^2$ and $3ab - 2a^2$ subtract $a^2 - ab - b^2$.

22. From $3x - y + z$ subtract the sum of $x - 4y + z$ and $2x + 3y - 2z$.

23. From $a + b + c$ subtract the sum of $a - b - c$, $b - c - a$, and $c - a - b$.

24. Subtract the sum of $m^2n - 2mn^2$ and $2m^2n - m^3 - n^3 + 2mn^2$ from $m^3 - n^3$.

25. Subtract the sum of $2c - 9a - 3b$ and $3b - 5a - 5c$ from $b - 3c + a$.

26. From the sum of $3x^m + 4y^n + z^{m+n}$ and $2z^{m+n} + 2x^m - 3y^n$ subtract $4x^m - 2y^n + z^{m+n}$.

If $x = a^2 + b^2$, $y = 2ab$, $z = a^2 - b^2$, and $v = a^2 - 2ab + b^2$,

27. $x + y + z + v = ?$

28. $x - y - z + v = ?$

29. $x - y + z - v = ?$

30. $y - x - v + z = ?$

TRANSPOSITION IN EQUATIONS

67. In the solution of equations the student has used certain principles, stated in § 40 and § 42 and summed up in § 44.

They are usually stated in somewhat broader terms as in the following section and are so simple as to be self-evident. Such self-evident principles are called **axioms**.

68. Axioms. — 1. *If equals are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.*

In the application of axiom 4, it is not allowable to divide by zero or any number equal to zero, because the result cannot be determined.

EXERCISES

69. 1. Solve $x - 6 = 4$ by adding 6 to both members (Ax. 1).

2. Solve the equation $x + 3 = 11$ by employing Ax. 2.

3. Solve $\frac{1}{3}x = 10$ by employing Ax. 3.

4. Solve $7x = 21$. Explain how Ax. 4 applies.

5. Solve $\frac{2}{3}x = 16$ in two steps, first finding the value of $\frac{1}{3}x$ by Ax. 4, then the value of x by Ax. 3.

Solve, and give the axiom applying to each step:

6. $2x = 6$.	13. $x + 2 = 10$.	20. $\frac{3}{2}m = 9$.
7. $5x = 5$.	14. $w - 5 = 11$.	21. $\frac{4}{3}n = 8$.
8. $4y = 8$.	15. $w + 1 = 12$.	22. $\frac{5}{2}x = 10$.
9. $3y = 9$.	16. $s - 7 = 10$.	23. $\frac{3}{5}x = 21$.
10. $\frac{1}{2}z = 5$.	17. $9 + s = 12$.	24. $\frac{5}{6}z = 20$.
11. $\frac{1}{3}z = 2$.	18. $5 + y = 15$.	25. $\frac{5}{8}z = 15$.
12. $\frac{1}{4}v = 3$.	19. $10 + x = 12$.	26. $\frac{7}{8}w = 49$.

70. 1. Adding 7 to both members of the equation

$$x - 7 = 3,$$

we obtain, by Ax. 1, $x = 3 + 7$.

-7 has been removed from the first member, but reappears in the second member with the opposite sign.

2. Subtracting 5 from both members of the equation

$$x + 5 = 9,$$

we obtain, by Ax. 2, $x = 9 - 5$.

When plus 5 is removed, or **transposed**, from the first member to the second, its sign is changed.

3. Explain the transposition of terms in each of the following:

$2x - 1 = 5$;	$3x + 2 = 11$;	$4x = 14 - 3x$;
$2x = 5 + 1$.	$3x = 11 - 2$.	$4x + 3x = 14$.

71. PRINCIPLE. — *Any term may be transposed from one member of an equation to the other, provided its sign is changed.*

EXERCISES

72. 1. Solve the equation $6 - 5x + 18 = 6 + 3x - 30$.

SOLUTION

By Ax. 2, $\quad \not{6} - 5x + 18 = \not{6} + 3x - 30.$
Transposing, § 71, $\quad -5x - 3x = -30 - 18.$
Uniting terms, $\quad -8x = -48.$
Changing signs, $\quad 8x = 48.$
Dividing by 8, Ax. 4, $\quad x = 6.$

VERIFICATION. — Substituting 6 for x in the given equation,

$$6 - 5 \cdot 6 + 18 = 6 + 3 \cdot 6 - 30, \text{ or } -6 = -6.$$

Hence, 6 is the true value of x; that is, the value 6 substituted for x *satisfies* the equation.

SUGGESTIONS. — 1. By Ax. 2 the same number may be subtracted, or *canceled*, from both members.

2. By Ax. 2 the signs of all the terms of an equation may be changed, for each member may be subtracted from the corresponding member of the equation $0 = 0$.

Solve and verify:

2. $3 = 5 - x$.
3. $9 - 5x = -1$.
4. $10 + v = 18 - v$.
5. $2z + 2 = 32 - z$.
6. $7x + 2 = x + 14$.
7. $3p + 2 = p + 30$.
8. $6y - 2 = 3y + 7$.
9. $5m - 5 = 2m + 4$.
10. $8 + 7a = 5a + 20$.
11. $2 + 13h = 50 - 9$.
12. $50 - n = 20 + n$.
13. $3x - 23 = x - 17$.
14. $4x + 12 = 2x + 36$.
15. $2x + \frac{1}{2}x = 30 - \frac{1}{2}x$.
16. $3x - \frac{1}{4}x = 30 + \frac{1}{4}x$.
17. $5w - 10 = \frac{2}{3}w + 16$.

Simplify as much as possible before transposing terms, solve, and verify:

18. $10x + 30 - 4x - 9x + 33 + 12x = 90 + 12 - 4x$.
19. $16x + 12 - 75 + 2x - 12 - 70 = 8x - 50 - 25$.
20. $11s - 60 + 5s + 17 - 2s + 41 - 3s = 2s + 97$.
21. $10z - 35 - 12z + 16 + 32 = 4z - 35 + 10z + 32$.
22. $14n - 25 = 19 - 11n + 4 + 16 - 10n + n + 136 - 16n$.

Algebraic Representation

73. 1. How much does 8 exceed $3+2$? z exceed $10+y$?

2. What number must be added to 5 so that the sum shall be 9? to m so that the sum shall be 4?

3. George rode a miles on his bicycle; then b miles on the cars; and walked 3 miles. How far did he travel?

4. A man bought a house for m dollars; spent n dollars for improvements; and then sold it for s dollars less than the entire cost. How much did he receive for it?

5. If 40 is separated into two parts, one of which is x, represent the other part.

6. A man made three purchases of a, b, and 2 dollars, respectively, and tendered a 20-dollar bill. Express the number of dollars in change due him.

7. Represent three times a number plus five times the double of the number.

8. What two integers are nearest to 8? to x, if x is an integer? to $a+b$, if $a+b$ is an integer?

9. What are the two even numbers nearest to 8? What are the two even numbers nearest to the even number $2n$?

10. Express the two odd numbers nearest to the odd number $2n+1$; the two even numbers nearest to $2n+1$.

11. There is a family of three children, each of whom is 2 years older than the next younger. When the youngest is x years old, what are the ages of the others? When the oldest is y years old, what are the ages of the others?

12. A boy who had 2 dollars spent 25 cents of his money. How much money had he left? If he had x dollars and spent y cents of his money, how much money had he left?

13. The number 376 may be written $300+70+6$. Write the number whose first digit is x, second digit y, and third digit z.

Problems

74. If $3x =$ a certain number and $x + 10 =$ the same number, then,

$$3x = x + 10.$$

This illustrates another axiom to be added to the list that is given in § 68.

AXIOM 5. — *Numbers that are equal to the same number, or to equal numbers, are equal to each other.*

This axiom is useful in the solution of problems, for its application is always involved in writing the *equation of the problem.*

75. The student is advised to review the *general directions for solving problems* given on page 45.

1. The Borough of Manhattan contains 22,000 elevators. If 2000 more are for freight than for passengers, how many freight elevators are there?

2. The total height of a certain brick chimney in St. Louis is 172 feet. Its height above ground is 2 feet more than 16 times its depth below. How high is the part above ground?

3. There are 3141 of the Philippine Islands, of which the number that has been named is 195 more than the number that is nameless. Find the number of each.

4. The value of the toys made in Germany one year was $22,500,000, or $100 more than 4 times the amount purchased by the United States. Find the value of the latter's purchase.

5. The Canadian Falls in the Niagara River are 158 feet high. This is 8 feet more than $\frac{10}{11}$ of the height of the American Falls. Find the height of the American Falls.

6. The summer bird population of Illinois is estimated at 30,750,000 and the number of English sparrows is 19,750,000 less than the number of other birds. Find the number of sparrows.

7. The porch of a temple in India is 876 feet in perimeter, and $\frac{1}{3}$ of its length is 6 feet more than its width. Find its length and width.

8. With the machines of the present time a pin maker can turn out 1,500,000 pins a day, or 60,000 more than 300 times the daily output of a pin maker of early times. How many pins did the early pin maker turn out per day?

9. The cost of dressing the fur of a beaver is 2 cents more than 8 times that for a muskrat. For a muskrat the cost is 9 cents less than for a mink. If the cost of dressing all three furs is 71 cents, find the cost of dressing a beaver's fur.

10. The daily consumption of water per person in New York City is 22 gallons less than that in Boston. The daily consumption in Pittsburg is 250 gallons, or 30 gallons less than that in New York and Boston together. Find the daily consumption per person in Boston.

11. A carpenter, a plumber, and a mason together earn $12.70 a day. If the carpenter earns $1.70 less than the mason, and he and the plumber together earn $7.50, how much does each earn?

12. A letter sent from Indianapolis to Point Barrow, Alaska, travels 6800 miles. It goes 900 miles more by train than by steamer and 200 miles more by dog sleds than by train. How far does it travel by each?

13. In the first three years of excavation, 313,356 cubic yards more were taken from the Panama Canal than from the New York Barge Canal. The amount taken from the former was 6,364,484 cubic yards less than twice that from the latter. How much was excavated from each?

14. Of the wood used for pulp in New York State one year, 500,000 cords were supplied by the state. The amount imported was $\frac{6}{5}$ of that used by Maine. If New York used twice as much as Maine, how many cords were used by each?

15. At one time the coffee stored at the docks of Havre, France, was $\frac{1}{5}$ of the total yearly production of the world and $\frac{1}{3}$ that of Brazil. If Brazil produces $3\frac{1}{2}$ million bags more than all the rest of the world, find the amount stored at Havre.

MULTIPLICATION

76. Primarily *multiplication* is the process of taking one number as many times as there are units in another.

Thus, $3 \times 5 = 5 + 5 + 5 = 15.$

In this section and the next, the sign $\times$ is to be read 'times.'

Even in arithmetic multiplication is extended to cases that cannot by any stretch of language be brought under the original definition.

Thus, strictly, in $3\frac{2}{3} \times 4$, 4 cannot be taken $3\frac{2}{3}$ times any more than a revolver can be fired $3\frac{2}{3}$ times.

So in algebra there are still other cases, like -3×4, that do not come under the original definition.

What we are concerned with, however, is the method of finding the product (consistent with the laws of operation used in arithmetic) and the interpretation of the results obtained.

77. Sign of the product.

1. Just as in arithmetic 3 times 4 are 12, so 3 times 4 *positive* units are 12 *positive* units;

that is, $+3 \times +4 = +12.$ (1)

2. Also, 3 times 4 *negative* units are 12 *negative* units;

that is, $+3 \times -4 = -12.$ (2)

3. Just as in arithmetic $4 \times 3 = 3 \times 4,$

so $-3 \times 4 = 4 \times -3;$

and since $-12 = 4 \times -3,$

$-3 \times 4 = -12.$ (3)

4. Again, since $6 - 4 = 2,$

by Ax. 3, $-3(6-4) = -3 \times 2 = -6.$

Also, § 27, $-3(6-4) = -3 \times 6 - 3 \times -4 = -18 - 3 \times -4.$

Now, since both $-18 - 3 \times -4$ and -6 are equal to $-3(6-4)$,

by Ax. 5, $-18 - 3 \times -4 = -6.$

Transposing -18, § 71,

$-3 \times -4 = -6 + 18;$

that is, $-3 \times -4 = +12.$ (4)

5. The preceding conclusions may be written as follows:

From (1), $+a \times +b = +ab,$

from (2), $+a \times -b = -ab,$

from (3), $-a \times +b = -ab,$

and from (4), $-a \times -b = +ab.$

Hence, for multiplication:

78. Law of signs.—*The sign of the product of two factors is* $+$ *when the factors have* **like signs,** *and* $-$ *when they have* **unlike signs.**

EXERCISES

79. 1. Multiply each of the following by $+2$; then by -2:

$$3,\ 5,\ -6,\ 10,\ -8,\ -9,\ -12,\ a,\ x,\ -b.$$

2.

Multiply	-8	9	6	4	-2
By	6	3	-5	-7	10

3.

Multiply	a	$-b$	$-x$	$-y$	n
By	4	6	-8	-1	-12

80. When there are several factors, by the law of signs,

$$-a \times -b = +ab;$$
$$-a \times -b \times -c = +ab \times -c = -abc;$$
$$-a \times -b \times -c \times -d = -abc \times -d = +abcd; \text{ etc.}$$

Hence,

The product of an even number of negative factors is positive; of an odd number of negative factors, negative.

Positive factors do not affect the sign of the product.

EXERCISES

81. Find the products indicated:

1. $(-1)(-1)(-1).$
2. $(-2)(-a)(-b)c.$
3. $(-1)(x)(y)(-3b).$
4. $(-3)(2)(-2a)b.$
5. $(-2a)(-3b)(-c)x.$
6. $(-1)(-2)(-3)(-4).$
7. $(-a)(-b)(+c)(-d).$
8. $(-x)(-y)(-1)z(-v).$
9. $(-x)(y)(z)(-v)(-w).$
10. $(-r)(-s)(-t)(x)(-3y).$

82. To multiply when the numbers are either positive or negative.

Having learned to multiply when the numbers are positive, §§ 22–29, and having just learned about the sign of the product when there are negative factors, the student is now prepared to multiply whether the terms of the factors are positive or negative.

EXERCISES

83. 1. Multiply $-4ax^2$ by $2a^3x^4$.

PROCESS

$$\begin{array}{r} -4ax^2 \\ 2a^3x^4 \\ \hline -8a^4x^6 \end{array}$$

EXPLANATION. — Since the signs of the monomials are unlike, the sign of the product is $-$ (Law of Signs, § 78).

$4 \cdot 2 = 8$ (Law of Coefficients, § 24).

$a \cdot a^3 = a^1 \cdot a^3 = a^{1+3} = a^4$ (Law of Exponents, § 23).

$x^2 \cdot x^4 = x^{2+4} = x^6$ (Law of Exponents).

Hence, the product is $-8a^4x^6$.

Multiply:

2. $\begin{array}{r} 10a^5 \\ 5a^3 \\ \hline \end{array}$	3. $\begin{array}{r} -5m^3n^2 \\ 3mn \\ \hline \end{array}$	4. $\begin{array}{r} -4abc \\ 2a^2b \\ \hline \end{array}$	5. $\begin{array}{r} 3a^2bc^3 \\ -7ab^2c \\ \hline \end{array}$
6. $\begin{array}{r} x^2y^2 \\ xy^3 \\ \hline \end{array}$	7. $\begin{array}{r} 5pq^2x^2 \\ -2rq^4x \\ \hline \end{array}$	8. $\begin{array}{r} -8ab \\ -1 \\ \hline \end{array}$	9. $\begin{array}{r} -5a^2x^2 \\ -2ax^3 \\ \hline \end{array}$
10. $\begin{array}{r} -2x \\ 2x^2 \\ \hline \end{array}$	11. $\begin{array}{r} -6a^2c^2x \\ -4a^3bn \\ \hline \end{array}$	12. $\begin{array}{r} -3ab \\ 2ba \\ \hline \end{array}$	13. $\begin{array}{r} -2a^6x^2 \\ -4ax^4 \\ \hline \end{array}$
14. $\begin{array}{r} -3n^3 \\ 6b^3 \\ \hline \end{array}$	15. $\begin{array}{r} 4a^2b^3y^4 \\ 3a^3b^2y \\ \hline \end{array}$	16. $\begin{array}{r} -1 \\ -1 \\ \hline \end{array}$	17. $\begin{array}{r} -5m^3d^2 \\ -2m^{10}cd^3 \\ \hline \end{array}$
18. $\begin{array}{r} 2a^{m+1} \\ 3a^2 \\ \hline \end{array}$	19. $\begin{array}{r} -2a^rb^{2z} \\ 7a^{3r}b^{4z} \\ \hline \end{array}$	20. $\begin{array}{r} -x^ny^n \\ xy \\ \hline \end{array}$	21. $\begin{array}{r} 4x^{n-1} \\ -2x^{n+1} \\ \hline \end{array}$
22. $\begin{array}{r} d^{s-n} \\ d^{2s+n} \\ \hline \end{array}$	23. $\begin{array}{r} 8r^{x-y} \\ 3r^ys^{a-2b} \\ \hline \end{array}$	24. $\begin{array}{r} -x^{1-n} \\ -x^n \\ \hline \end{array}$	25. $\begin{array}{r} y^{n-m} \\ y^{m-n+1} \\ \hline \end{array}$

26. $3x^2 - 2xy$ by $5xy^2$.

27. $3a^3 - 6a^2b$ by $-2b$.

28. $m^2n^3 - 3mn^4$ by $2mn$.

29. $p^2q^2 - 2pq^3$ by $-pq$.

30. $4a^2 - 5b^2c - c^3$ by abc^2.

31. $-2ac + 4ax$ by $-5acx$.

Expand, and test each result:

32. $(a+b+c)(a+b+c)$.

33. $(a^2-ab+b^2)(a^2+ab+b^2)$.

34. $(a^3+3a^2b+3ab^2+b^3)(a+b)$.

35. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$.

36. $(x^4-x^3y+x^2y^2-xy^3+y^4)(x+y)$.

37. $(a^2+b^2+c^2+d^2)(a^2-b^2+c^2-d^2)$.

38. $(x^2-xy+y^2+x+y+1)(x+y+1)$.

39. $(a^3+3a^2b+3ab^2+b^3)(a^2+2ab+b^2)$.

40. $(a^2-ab-ac+b^2-bc+c^2)(a+b+c)$.

Arrange both multiplicand and multiplier according to the ascending or the descending powers of the letter involved; multiply, and test each result.

41. $x+x^3+1+x^2$ by $x-1$.

42. $x^3+10-7x-4x^2$ by $x-2$.

43. $14-9x-6x^2+x^3$ by $x+1$.

44. $a^3-30-11a+4a^2$ by $a-1$.

45. $4a^2-2a^3-8a+a^4-3$ by $2+a$.

46. $2m-3+2m^3-4m^2$ by $2m-3$.

47. $x+x^2-5$ by x^2-3-2x.

48. b^2+5b-4 by $-4+2b^2-3b$.

49. $4n^3+6-2n^4+16n-8n^2+n^5$ by $n+2$.

50. $1+x+4x^2+10x^3+46x^5+22x^4$ by $2x^2+1-3x$.

51. $5x^2+7-4x^3-6x+x^6+3x^4-2x^5$ by $2x+x^2+1$.

Multiply:

52. $ax^{2n}+ay^{2n}$ by $ax^{2n}-ay^{2n}$.

53. $ax^{n-1}+y^{n-1}$ by $3ax^{n-1}+2y^{n-1}$.

54. $x^{2n}+2x^ny^n+y^{2n}$ by $x^{2n}-2x^ny^n+y^{2n}$.

55. $a^{6n}+a^{4n}b^{2c}+a^{2n}b^{4c}+b^{6c}$ by $a^{2n}-b^{2c}$.

56. $m^{x+1}n^{x-1}+m^{x-1}n^{x+1}+1$ by $m^{x-1}n^{x+1}-m^{x+1}n^{x-1}+1$.

Special Cases in Multiplication

84. The square of the sum of two numbers.

1. Multiply $a+b$ by $a+b$; find the square of $x+y$.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;\; +ab+b^2 \\ \hline a^2+2\,ab+b^2 \end{array} \qquad\qquad \begin{array}{l} x+y \\ x+y \\ \hline x^2+xy \\ \quad\;\; +xy+y^2 \\ \hline x^2+2\,xy+y^2 \end{array}$$

2. How is the first term of the product, that is, of the square of the sum of two numbers, obtained from the numbers? How is the second term obtained? the third term?

3. What signs have the terms of the result?

85. PRINCIPLE. — *The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the first and second, plus the square of the second.*

Since $5\,a^3 \times 5\,a^3 = 25\,a^6$, $3\,a^4b^5 \times 3\,a^4b^5 = 9\,a^8b^{10}$, etc., it is evident that in squaring a number the exponents of literal factors are doubled.

EXERCISES

86. Expand by inspection, and test each result:

1. $(p+q)(p+q)$.
2. $(r+s)(r+s)$.
3. $(a+x)(a+x)$.
4. $(x+4)(x+4)$.
5. $(a+6)(a+6)$.
6. $(y+7)(y+7)$.
7. $(z+1)(z+1)$.
8. $(c+9)(c+9)$.
9. $(v+3)(v+3)$.
10. $(w+5)(w+5)$.
11. $(5\,x+z)^2$.
12. $(2\,a+x)^2$.
13. $(ab+cd)^2$.
14. $(5\,x+2\,y)^2$.
15. $(7\,z+3\,c)^2$.
16. $(3\,b+4\,x)^2$.
17. $(2\,m+3\,n)^2$.
18. $(3\,c+7\,d)^2$.
19. $(8\,s+5\,t)^2$.
20. $(5\,w+3\,u)^2$.
21. $(a^3+b^3)^2$.
22. $(a^5+b^5)^2$.
23. $(a^n+b^n)^2$.
24. $(x^m+y^n)^2$.
25. $(3\,a^2+5\,b^3)^2$.
26. $(1+2\,a^3b)^2$.
27. $(3\,xy^2+4\,x^2y)^2$.
28. $(9\,a^{2m}+2\,b^{2n})^2$.
29. $(4\,x^{2r}+y^{2t+1})^2$.
30. $(x^{n-1}+y^{n-1})^2$.

87. The square of the difference of two numbers.

1. Multiply $a - b$ by $a - b$; find the square of $x - y$.

$$\begin{array}{r} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad - ab + b^2} \\ a^2 - 2\,ab + b^2 \end{array} \qquad\qquad \begin{array}{r} x - y \\ \underline{x - y} \\ x^2 - xy \\ \underline{\quad - xy + y^2} \\ x^2 - 2\,xy + y^2 \end{array}$$

2. How is the first term of the square of the difference of two numbers obtained from the numbers? How is the second term obtained? the third term?

3. What signs have the terms of the result?

88. PRINCIPLE. — *The square of the difference of two numbers is equal to the square of the first number, minus twice the product of the first and second, plus the square of the second.*

EXERCISES

89. Expand by inspection, and test each result:

1. $(x - m)(x - m)$.
2. $(m - n)(m - n)$.
3. $(x - 6)(x - 6)$.
4. $(p - 8)(p - 8)$.
5. $(q - 7)(q - 7)$.
6. $(a - c)(a - c)$.
7. $(a - x)(a - x)$.
8. $(x - 1)(x - 1)$.
9. $(b - 5)(b - 5)$.
10. $(st - 2)(st - 2)$.
11. $(x - 4)(x - 4)$.
12. $(z - 3)(z - 3)$.
13. $(w - 9)(w - 9)$.
14. $(2\,a - x)^2$.
15. $(3\,m - n)^2$.
16. $(4\,x - y)^2$.
17. $(m - 4\,n)^2$.
18. $(p - 3\,q)^2$.
19. $(a - 7\,b)^2$.
20. $(4\,a - 3)^2$.
21. $(5\,x - 4)^2$.
22. $(ab - 3)^2$.
23. $(2\,a - 3\,b)^2$.
24. $(2\,z - 7\,y)^2$.
25. $(8\,x - 5\,y)^2$.
26. $(9\,w - 2\,v)^2$.
27. $(3\,x - 2)^2$.
28. $(2\,x - 5\,y)^2$.
29. $(5\,m - 3\,n)^2$.
30. $(3\,p - 5\,q)^2$.
31. $(a^n - b^n)^2$.
32. $(x^m - y^n)^2$.
33. $(a^2 - 2\,b^2)^2$.
34. $(y^3 - 6\,x)^2$.
35. $(ab - 2\,c^2)^2$.
36. $(4\,x^2 - 5\,y^2)^2$.
37. $(x^{t+r} - y^{2r})^2$.
38. $(mx^m - ny^n)^2$.
39. $(x^{m-1} - y^{n-1})^2$.

90. The square of any polynomial.

1. Find the square of $a+b+c$.

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad +ab \qquad +b^2+bc \\ \qquad\quad +ac \qquad +bc+c^2 \\ \hline a^2+2ab+2ac+b^2+2bc+c^2 \end{array}$$

That is, $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.

2. Show by actual multiplication that

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

3. Similarly, by squaring any polynomial by multiplication, it will be observed that:

91. PRINCIPLE. — *The square of a polynomial is equal to the sum of the squares of the terms and twice the product of each term by each term, taken separately, that follows it.*

When some of the terms are negative, some of the double products will be negative, but the squares will always be positive. For example, since $(-b)^2=+b^2$, $(a-b+c)^2=a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c$ $=a^2+b^2+c^2-2ab+2ac-2bc$.

EXERCISES

92. Expand by inspection, and test each result:

1. $(x+y+z)^2$.
2. $(x+y-z)^2$.
3. $(x-y-z)^2$.
4. $(x-y+z)^2$.
5. $(x+y-3z)^2$.
6. $(x-y+3z)^2$.
7. $(a-2b+c)^2$.
8. $(2a-b-c)^2$.
9. $(rs+st-rt)^2$.
10. $(qb-pc-rd)^2$.
11. $(ax-by+cz)^2$.
12. $(xy-3c-ab)^2$.
13. $(3x-2y+4z)^2$.
14. $(2a-5b+3c)^2$.
15. $(2m-4n-r)^2$.
16. $(12-2x+3y)^2$.
17. $(a+m+b+n)^2$.
18. $(a-m+b-n)^2$.

93. The product of the sum and difference of two numbers.

1. Find the product of $a+b$ and $a-b$; of $x-y$ and $x+y$.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array} \qquad\qquad \begin{array}{l} x-y \\ x+y \\ \hline x^2-xy \\ \quad +xy-y^2 \\ \hline x^2 \qquad -y^2 \end{array}$$

2. How are the terms of the product of the sum and difference of two numbers obtained from the numbers?

3. What sign connects the terms?

94. PRINCIPLE. — *The product of the sum and difference of two numbers is equal to the difference of their squares.*

EXERCISES

95. Expand by inspection, and test each result:

1. $(x+y)(x-y)$.
2. $(a+c)(a-c)$.
3. $(p+q)(p-q)$.
4. $(p+5)(p-5)$.
5. $(x+1)(x-1)$.
6. $(x^2+1)(x^2-1)$.
7. $(x^3+1)(x^3-1)$.
8. $(x^4-1)(x^4+1)$.
9. $(x^5-1)(x^5+1)$.
10. $(x^n+y^3)(x^n-y^3)$.
11. $(ab+5)(ab-5)$.
12. $(cd+d^2)(cd-d^2)$.
13. $(ab-c^2)(ab+c^2)$.
14. $(4y+z^2)(4y-z^2)$.
15. $(lx-5m)(lx+5m)$.
16. $(2x+3y)(2x-3y)$.
17. $(3m+4n)(3m-4n)$.
18. $(12+xy)(12-xy)$.
19. $(ab+cd)(ab-cd)$.
20. $(3m^2n-b)(3m^2n+b)$.
21. $(2x^3+5y^2)(2x^3-5y^2)$.
22. $(3x^6+2y^6)(3x^6-2y^6)$.
23. $(2a^2+2b^2)(2a^2-2b^2)$.
24. $(-5n-b)(-5n+b)$.
25. $(-x-2y)(-x+2y)$.
26. $(3x^m+7y^n)(3x^m-7y^n)$.
27. $(mx^a+2y^b)(mx^a-2y^b)$.
28. $(a^nb^m+a^mb^n)(a^nb^m-a^mb^n)$.
29. $(x^{m-1}+y^{n+1})(x^{m-1}-y^{n+1})$.
30. $(5a^3b^2+2x^z)(5a^3b^2-2x^z)$.

96. The product of two binomials that have a common term.

Let $x+a$ and $x+b$ represent any two binomials having a common term, x. Multiplying $x+a$ by $x+b$,

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\qquad bx+ab} \\ x^2+(a+b)x+ab \end{array}$$

97. PRINCIPLE. — *The product of two binomials having a common term is equal to the sum of: the square of the common term, the product of the sum of the unlike terms and the common term, and the product of the unlike terms.*

EXERCISES

98. 1. Expand $(x+2)(x+5)$ and test the result.

SOLUTION

The square of the common term is x^2;
the sum of 2 and 5 is 7;
the product of 2 and 5 is 10;
$\therefore (x+2)(x+5)=x^2+7x+10.$

TEST. — If $x=1$, we have $3\cdot 6=1+7+10$, or $18=18$.

2. Expand $(a+1)(a-4)$ and test the result.

SOLUTION

The square of the common term is a^2;
the sum of 1 and -4 is -3;
the product of 1 and -4 is -4;
$\therefore (a+1)(a-4)=a^2-3a-4.$

TEST. — If $a=4$, we have $5\cdot 0=16-12-4$, or $0=0$.

3. Expand $(n-2)(n-3)$ and test the result.

SOLUTION

The square of the common term is n^2;
the sum of -2 and -3 is -5;
the product of -2 and -3 is 6;
$\therefore (n-2)(n-3)=n^2-5n+6.$

TEST. — If $n=3$, we have $1\cdot 0=9-15+6$, or $0=0$.

Expand by inspection, and test each result:

4. $(x+5)(x+6)$.
5. $(x+7)(x+8)$.
6. $(x-7)(x+8)$.
7. $(x+7)(x-8)$.
8. $(x-5)(x-4)$.
9. $(x-3)(x-2)$.
10. $(x-5)(x-1)$.
11. $(x+5)(x+8)$.
12. $(p-4)(p+1)$.
13. $(r-3)(r-1)$.
14. $(n-8)(n-12)$.
15. $(n-6)(n+15)$.
16. $(x^2+5)(x^2-3)$.
17. $(x^3-7)(x^3+6)$.
18. $(x^n-5)(x^n+4)$.
19. $(x^n-a)(x^n-2a)$.
20. $(y-2b)(y+3b)$.
21. $(z-4a)(z+3a)$.
22. $(2x+5)(2x+3)$.
23. $(2x-7)(2x+5)$.
24. $(3y-1)(3y+2)$.
25. $(4x^2+1)(4x^2-7)$.
26. $(ab-6)(ab+4)$.
27. $(x^2y^2-a)(x^2y^2+2a)$.
28. $(3xy+y^2)(y^2-xy)$.
29. $(b^2c^2+ef)(b^2c^2-ef)$.
30. $(5ab+2c^2)(5ab-2c^2)$.
31. $(3x^2+2y^2)(3x^2-2y^2)$.

By an extension of the method given above, the product of any two binomials having similar terms may be written.

32. Expand $(2x-5)(3x+4)$.

PROCESS

$$\begin{array}{l} 2x-5 \\ \quad\times \\ 3x+4 \\ \hline 6x^2-7x-20 \end{array}$$

EXPLANATION. — The product must have a term in x^2, a term in x, and a numerical, or **absolute**, term.

The x^2-term is the product of $2x$ and $3x$; the x-term is the sum of the partial products $-5\cdot 3x$ and $2x\cdot 4$, called the **cross-products**; and the absolute term is the product of -5 and 4.

The process should not be used except as an aid in explanation.

Expand by inspection, and test each result:

33. $(2x+5)(3x+4)$.
34. $(3x-2)(2x-3)$.
35. $(3a-4)(4a+3)$.
36. $(3x-y)(x-3y)$.
37. $(7z-a)(3z+2a)$.
38. $(2a+5b)(5a+2b)$.
39. $(7n^2-2p)(2n^2-7p)$.
40. $(ab^2-m^3)(ab^2-4m^3)$.
41. $(4r^m-3s^n)(2r^m-5s^n)$.
42. $(x^{t-1}+5y)(2x^{t-1}-3y)$.

SIMULTANEOUS EQUATIONS

99. If 4 bananas and 9 oranges cost 35¢, and 4 bananas and 6 oranges cost 26¢, and it is required to find the cost of 1 of each, we may simplify the problem thus:

	4 bananas and 9 oranges cost 35¢	(1)
	4 bananas and 6 oranges cost 26¢	(2)
Subtracting,	3 oranges cost 9¢	(3)

By thus *eliminating* the cost of the bananas, we have obtained a relation, (3), more simple than either of the two given relations, (1) and (2), for it involves only one unknown cost.

Or, let x represent the number of cents 1 banana costs, and y the number of cents 1 orange costs.

Then, 4 bananas will cost $4x$ cents, 9 oranges $9y$ cents, etc.

$$4x + 9y = 35 \quad (1)$$

$$4x + 6y = 26 \quad (2)$$

Eliminating the x's, $3y = 9$ (3)

$y = 3$, or 1 orange costs 3¢.

Since $y = 3$, $9y$ in the first equation is equal to 27.

Substituting 3 for y in the first equation,

$$4x + 27 = 35, \quad (4)$$

$x = 2$, or 1 banana costs 2¢.

100. In eliminating the x's in the preceding section, equal numbers, $4x + 6y$ and 26, were subtracted from the members of (1). Hence, Ax. 2, the results are equal, giving a true equation, $3y = 9$.

This method of elimination is called **elimination by subtraction.**

101. How must the equations $2x + 3y = 16$ and $5x - 3y = 19$ be combined to eliminate the y's?

$$2x + 3y = 16$$

$$5x - 3y = 19$$

Adding, Ax. 1, $7x = 35$

This method of elimination is called **elimination by addition.**

102. Equations like those discussed in § 99 or in § 101, in which the same unknown numbers have the same values, are called **simultaneous equations.**

EXERCISES

103. 1. If $2x+3y=18$ and $2x+y=10$, what is the value of each unknown number?

SOLUTION

$$2x+3y=18 \qquad (1)$$

$$2x+y=10 \qquad (2)$$

Subtracting, Ax. 2, $2y=8$; $\therefore y=4$.

Substituting 4 for y in (2), $2x+4=10$; $\therefore x=3$.

TEST. — Substituting 3 for x and 4 for y in (1) and (2),

(1) becomes $6+12=18$, or $18=18$;

(2) becomes $6+4=10$, or $10=10$.

NOTE. — The value of y may be substituted in *either* of the given equations.

Solve, and test results:

2. $\begin{cases} 5x+2y=22, \\ 5x+y=21. \end{cases}$

3. $\begin{cases} 3x+4y=23, \\ 3x+2y=19. \end{cases}$

4. $\begin{cases} 4x+5y=32, \\ 2x+5y=26. \end{cases}$

5. $\begin{cases} 7x-2y=22, \\ 3x+2y=18. \end{cases}$

6. $\begin{cases} 6x+7y=13, \\ 6x+y=7. \end{cases}$

7. $\begin{cases} 9x-2y=41, \\ 7x-2y=31. \end{cases}$

8. $\begin{cases} 5x+3y=16, \\ 2x+3y=10. \end{cases}$

9. $\begin{cases} 6x+2y=22, \\ 6x-7y=4. \end{cases}$

10. $\begin{cases} 3x-4y=16, \\ 5x+4y=48. \end{cases}$

11. $\begin{cases} 6x+5y=70, \\ x-5y=0. \end{cases}$

12. $\begin{cases} 5x-4y=8, \\ 3x-4y=0. \end{cases}$

13. $\begin{cases} 8x+5y=18, \\ 8x+3y=14. \end{cases}$

14. $\begin{cases} 7x-3y=39, \\ 5x-3y=27. \end{cases}$

15. $\begin{cases} 5y-4x=9, \\ 6y+4x=46. \end{cases}$

16. $\begin{cases} 8x-3y=39, \\ 8x-4y=36. \end{cases}$

17. $\begin{cases} 7x+5y=83, \\ 7x-4y=47. \end{cases}$

18. If $2x+3y=16$ and $5x+4y=33$, find x and y.

SOLUTION

$$2x+3y=16, \quad (1)$$
$$5x+4y=33. \quad (2)$$

We may eliminate either x or y. If we choose to eliminate x, we must first prepare the equations, so that x may have the same coefficient in each. Multiplying both members of (1) by 5, and both members of (2) by 2,

$$10x+15y=80 \quad (3)$$

and $$10x+8y=66 \quad (4)$$

Subtracting (4) from (3), $7y=14$; $\therefore y=2$.

Substituting 2 for y in (1), $2x+6=16$; $\therefore x=5$.

TEST. — These values, in (1) and (2), give $10+6=16$ and $25+8=33$.

NOTE. — To eliminate y instead of x, proceed as follows:

Multiplying (1) by 4, $8x+12y=64$.

Multiplying (2) by 3, $15x+12y=99$.

Subtracting the *upper equation from the lower*, thus avoiding negative coefficients,

$$7x=35; \quad \therefore x=5.$$

Substituting 5 for x in (1), $10+3y=16$; $\therefore y=2$.

Solve, and test results:

19. $\begin{cases} 9x+2y=20, \\ 3x+y=7. \end{cases}$

20. $\begin{cases} 6x+5y=28, \\ 2x+3y=12. \end{cases}$

21. $\begin{cases} x+y=10, \\ x-y=2. \end{cases}$

22. $\begin{cases} 5x+2y=49, \\ 3x-2y=23. \end{cases}$

23. $\begin{cases} 4x-y=27, \\ x-y=3. \end{cases}$

24. $\begin{cases} 2x+y=13, \\ x+4y=17. \end{cases}$

25. $\begin{cases} 4y-3x=30, \\ 5y-6x=33. \end{cases}$

26. $\begin{cases} 10x+3y=62, \\ 6x+4y=46. \end{cases}$

27. $\begin{cases} 11x+8y=37, \\ 5x+6y=18. \end{cases}$

28. $\begin{cases} y+2x=18, \\ y-2x=2. \end{cases}$

29. $\begin{cases} 2y-3x=5, \\ 5y+4x=93. \end{cases}$

30. $\begin{cases} 4x-7y=12, \\ 3x+5y=50. \end{cases}$

31. $\begin{cases} 8x+7y=37, \\ 4x-3y=-1. \end{cases}$

32. $\begin{cases} 11x-5y=29, \\ 3x+2y=18. \end{cases}$

Problems

104. 1. The sum of two numbers is 8 and their difference is 2. Find the numbers.

SOLUTION

Let $x =$ the larger number,
and $y =$ the smaller number.

Then, $x + y = 8$, (1)
and $x - y = 2$. (2)

Adding (1) and (2), $2x = 10$; $\therefore x = 5$.

Subtracting (2) from (1), $2y = 6$; $\therefore y = 3$.

Hence, the numbers are 5 and 3.

Find two numbers related to each other as follows:

2. Sum $= 14$; difference $= 8$.

3. Sum of 2 times the first and 3 times the second $= 34$; sum of 2 times the first and 5 times the second $= 50$.

4. Sum $= 18$; sum of the first and 2 times the second $= 20$.

5. A cotton tent is worth \$10 less than a linen one of the same size, and 3 cotton ones cost \$2 more than 2 linen ones. Find the cost of each.

SOLUTION

Let $x =$ the number of dollars a linen tent costs,
and $y =$ the number of dollars a cotton tent costs.

Then, $x - y = 10$, (1)
and $3y - 2x = 2$. (2)

Multiplying (1) by 2,

$2x - 2y = 20$. (3)

Adding (2) and (3), $y = 22$. (4)

Substituting (4) in (1),

$x - 22 = 10$; $\therefore x = 32$.

Hence, a linen tent costs \$32 and a cotton one \$22.

6. A steam train took 10 minutes longer to pass through the Simplon tunnel than an electric train. What was the time of each, if the steam train lacked 8 minutes of taking twice as long as the electric train?

7. During one month the number of arrivals and departures of vessels at the port of Seattle was 183. There were 5 more arrivals than departures. Find the number of each.

8. At one time the United States Navy had 17 coaling stations on the Atlantic and Pacific coasts. If the Atlantic had had 1 less, it would have had 3 times as many as the Pacific coast. How many coaling stations were there on each coast?

9. The length of the Grand Canal in China is 13 times the approximate length of the Panama Canal, and the difference in their lengths is 600 miles. Find the length of each.

10. A steam train is 25 tons heavier than an electric train that carries as many passengers. If 9 such steam trains weigh as much as 14 of the electric ones, find the weight of each train.

11. If at Christmas time 3 dozen carnations and 2 dozen orchids cost \$30, and 2 dozen carnations and 3 dozen orchids cost \$40, find the cost of each per dozen.

12. Small goldfish are worth \$4 per hundred less than large ones. If 3 hundred of the former and 2 hundred of the latter together cost \$18, find the cost of each per hundred.

13. A stock car will hold 45 more sheep than hogs. If the number of animals in 16 cars of sheep is the same as in 25 cars of hogs, find the number in a car load of each.

14. One sugar factory employs 2200 men in the factory and fields. If the number of field hands is 100 more than twice the number of factory workers, find the number of each.

15. The Roosevelt dam in Arizona is 17 feet lower than the Croton dam, and twice the height of the latter plus 3 times the height of the former is 1434 feet. Find the height of each.

16. The duty paid on 7 pianos entering Italy was \$173.70. If the duty paid on each upright piano was \$17.37, or $\frac{1}{2}$ that on each grand piano, how many pianos of each kind were imported?

17. Prospect Park is $326\frac{3}{4}$ acres smaller than Central Park, and twice the area of the former is $189\frac{1}{2}$ acres more than the area of the latter. Find the area of each.

18. A woman picked 5 crates of Brussels sprouts, containing in all 192 quarts. If the crates hold 32 and 48 quarts respectively, how many crates of each size did she pick?

19. One year a jeweler had 693 broken watch springs brought to him to renew. The number broken in summer lacked 39 of being twice the number broken in winter. How many were broken in summer? in winter?

20. An engine at Sharon, Pa., weighed 40 tons less than 5 times as much as two of its castings. The weight of the whole engine, minus twice the weight of the castings, was 314 tons. Find the weight of the whole engine and of the castings.

21. In a typewriting contest in Paris a woman in a given time wrote 500 words less than a man, and twice the number that the man wrote is 15,500 less than 3 times the number that the woman wrote. Find the number of words written by each.

22. In United States money, 2 marks, German money, and 3 francs, French money, are valued at $1.055, and 1 mark and 5 francs at $1.203. What is the value in United States money of a mark? of a franc?

23. The expense of running a small automobile is estimated at 51 ¢ a week more than the expense of keeping a horse and carriage. The former can be run for 3 weeks for $2.22 less than the latter can be kept for 4 weeks. What is the weekly expense of each?

SUGGESTION. — Both equations should be expressed in terms of cents, or both in terms of dollars.

24. It cost 42 cents to stop a certain train and get it back to its former speed. Another train of less speed cost 35 cents to stop and start. If in all both trains made 5 stops, at a cost of $1.89, find the number of stops made by each.

DIVISION

105. Sign of the quotient.

Since division is the inverse of multiplication, the following are direct consequences of the law of signs for multiplication given in (§ 78):

$$+a \times +b = +ab; \quad \therefore \quad +ab \div +a = +b.$$
$$+a \times -b = -ab; \quad \therefore \quad -ab \div +a = -b.$$
$$-a \times +b = -ab; \quad \therefore \quad -ab \div -a = +b.$$
$$-a \times -b = +ab; \quad \therefore \quad +ab \div -a = -b.$$

Hence, for division:

106. Law of signs. — *The sign of the quotient is + when the dividend and divisor have* **like signs,** *and — when they have* **unlike signs.**

EXERCISES

107. 1. Divide each of the following numbers by 2:

$6, \quad -6, \quad 10, \quad -10, \quad 14, \quad -12, \quad -18, \quad 22, \quad -8.$

2. Divide each of the foregoing numbers by -2.

Perform the indicated divisions:

3. $7\overline{)-14}$	4. $-3\overline{)15}$	5. $-3\overline{)-12}$	6. $-1\overline{)9}$
7. $4 \div (-4)$.	8. $22 \div (-2)$.	9. $-1 \div (-1)$.	10. $-6 \div 3$.
11. $9 \div (-3)$.	12. $-21 \div 3$.	13. $45 \div (-5)$.	14. $-8 \div 2$.
15. $\frac{36}{4}$.	16. $\frac{28}{-7}$.	17. $\frac{-42}{6}$.	18. $\frac{-20}{-5}$.
19. $\frac{-99}{11}$.	20. $\frac{48}{-6}$.	21. $\frac{-63}{-7}$.	22. $\frac{72}{8}$.

108. To divide when the numbers are either positive or negative.

Division, when the numbers involved are positive, was treated in §§ 30–38. The student is now prepared to divide whether the numbers are positive or negative, since the only new point involved is the matter of signs, just discussed.

EXERCISES

109. 1. Divide $-8a^4x^6$ by $2a^3x^4$.

PROCESS

$$2a^3x^4)\underline{-8a^4x^6}$$
$$-4ax^2$$

EXPLANATION. — Since the signs of dividend and divisor are unlike, the sign of the quotient is — (Law of Signs, § 106).

$8 \div 2 = 4$ (Law of Coefficients, § 33).

$a^4 \div a^3 = a^{4-3} = a^1 = a$ (Law of Exponents, § 32).

$x^6 \div x^4 = x^{6-4} = x^2$ (Law of Exponents).

Hence, the quotient is $-4ax^2$.

Divide:

2. $30m^2n^2$ by $5mn$.
3. $-24x^2y^2z^3$ by $8x^2y$.
4. $21ax^2y$ by $-7ay$.
5. $-9abc^3$ by $-3abc$.
6. $42n^2x^2$ by $-6n^2$.
7. $-12p^3q^3$ by $12pq$.
8. $-20r^2s^4$ by $-10rs^2$.
9. $40mnv^2$ by $-8mnv$.
10. Divide $4a^3b - 6a^2b^2 + 4ab^3$ by $2ab$; by $-2ab$.

PROCESS

$$2ab)\underline{4a^3b - 6a^2b^2 + 4ab^3}$$
$$2a^2 - 3ab + 2b^2$$

PROCESS

$$-2ab)\underline{4a^3b - 6a^2b^2 + 4ab^3}$$
$$-2a^2 + 3ab - 2b^2$$

TEST OF SIGNS. — When the divisor is positive, the signs of the quotient should be *like* those of the dividend. When the divisor is negative, the signs of the quotient should be *unlike* those of the dividend.

Divide:

11. $a^2y - 2ay^2$ by ay; by $-ay$.
12. $9x^2y^2 + 15xy^2$ by $3xy^2$; by $-3xy^2$.
13. $-xz^3 - 3xz + x^2z^2$ by $-xz$; by xz.
14. $3x^3 - 6x^5 + 9x^7 - 12x^9$ by $3x^2$; by $-3x^2$.
15. $30r^3s^3 + 15r^2s^2 - 45rs^4 + 75r$ by $15r$; by $-15r$.
16. $-t^5u - t^4uv + tu^4v - t^3u^3v^2 + tu^2v^3$ by tu; by $-tu$.
17. $x^a + 2x^{a+1} - 5x^{a+2} - x^{a+3} + 3x^{a+4}$ by x^a; by $-x^a$.
18. $x^n - x^{n-1} + x^{n-2} - x^{n-3} + x^{n-4} - x^{n-5}$ by x^2; by $-x^2$.
19. $y^{n+1} - 2y^{n+2} + y^{n+3} - 3y^{n+4} + y^{n+5}$ by y^{n+1}; by $-y^{n+1}$.
20. $a(x+y)^2 - ab(x+y)^3 + a^2b^2(x+y)^4$ by $-a$; by $a(x+y)^2$.

21. Divide $81 + 9a^2 + a^4$ by $a^2 - 3a + 9$; test the result.

PROCESS | TEST

$$
\begin{array}{l|l}
a^4 + 9a^2 + 81 & a^2 - 3a + 9 \\
\underline{a^4 - 3a^3 + 9a^2} & \overline{a^2 + 3a + 9} \\
\quad 3a^3 + 81 & \\
\quad \underline{3a^3 - 9a^2 + 27a} & \\
\qquad 9a^2 - 27a + 81 & \\
\qquad \underline{9a^2 - 27a + 81} &
\end{array}
\qquad
\begin{array}{l}
91 \div 7 \\
\quad = 13
\end{array}
$$

NOTE. — The test is made by substituting 1 for a; similarly, the result may be tested by substituting any other value for a, except such as gives for the result $0 \div 0$ or any number divided by 0, because we are unable to determine the numerical value of such results.

Divide, and test:

22. $a^4 + 16 + 4a^2$ by $2a + a^2 + 4$.

23. $x^5 - 61x - 60$ by $x^2 - 2x - 3$.

24. $a^5 - 41a - 120$ by $a^2 + 4a + 5$.

25. $25x^5 - x^3 - 8x - 2x^2$ by $5x^2 - 4x$.

26. $a^8 + a^6 + a^4 + a^2 + 3a - 1$ by $a + 1$.

27. $4y^4 - 9y^2 - 1 + 6y$ by $3y + 2y^2 - 1$.

28. $2a^4 - 5a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - ab + b^2$.

PROCESS | TEST

$$
\begin{array}{l|l}
2a^4 - 5a^3b + 6a^2b^2 - 4ab^3 + b^4 & a^2 - ab + b^2 \\
\underline{2a^4 - 2a^3b + 2a^2b^2} & \overline{2a^2 - 3ab + b^2} \\
\quad - 3a^3b + 4a^2b^2 - 4ab^3 & \\
\quad \underline{- 3a^3b + 3a^2b^2 - 3ab^3} & \\
\qquad a^2b^2 - ab^3 + b^4 & \\
\qquad \underline{a^2b^2 - ab^3 + b^4} &
\end{array}
\qquad
\begin{array}{l}
0 \div 1 \\
\quad = 0
\end{array}
$$

NOTE. — It will be observed from the test that $0 \div 1 = 0$. In general, $0 \div a = 0$; that is, zero divided by any number equals zero.

29. Divide $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$.

30. Divide $20x^2y - 25x^3 - 18y^3 + 27xy^2$ by $6y - 5x$.

31. Divide $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ by $a^2 - 2ax + x^2$.

32. Divide $c^3 - 8$ by $c + 2$.

PROCESS

$$
\begin{array}{l|l}
c^3 - 8 & c + 2 \\
\underline{c^3 + 2c^2} & \overline{c^2 - 2c + 4 + \dfrac{-16}{c+2}} \\
\quad -2c^2 - 8 & \\
\quad \underline{-2c^2 - 4c} & \\
\qquad\quad 4c - 8 & \\
\qquad\quad \underline{4c + 8} & \\
\qquad\qquad -16 &
\end{array}
$$

Divide, and test results :

33. $x^5 + 32$ by $x + 2$.

34. $x^6 - y^6$ by $x^2 + y^2$.

35. $m^5 - n^5$ by $m + n$.

36. $m^5 + n^5$ by $m + n$.

37. $x^7 + 2x^6 - 2x^4 + 2x^2 - 1$ by $x + 1$.

38. $y^5 + 3y^4 + 5y^3 + 3y^2 + 3y + 5$ by $y + 1$.

39. $2n^5 - 4n^4 - 3n^3 + 7n^2 - 3n + 2$ by $n - 2$.

40. $y^4 + 7y - 10y^2 - y^3 + 15$ by $y^2 - 2y - 3$.

41. $7x^3 + 2x^4 - 27x^2 + 16 - 8x$ by $x^2 + 5x - 4$.

42. $28x^4 + 6x^3 + 6x^2 - 6x - 2$ by $2 + 2x + 4x^2$.

43. $25v^2 - 20v^3 + 3v^4 + 16v - 6$ by $3v^2 - 8v + 2$.

44. $4 - 18x + 30x^2 - 23x^3 + 6x^4$ by $2x^2 - 5x + 2$.

45. $32x^3 + 24x^4 - 25x - 4 - 16x^2$ by $6x^2 - x - 4$.

46. 1 by $1 + x$ to five terms of the quotient.

47. 1 by $1 - x$ to five terms of the quotient.

48. $a^3 - 6a^2 + 12a - 8 - b^3$ by $a - 2 - b$.

49. $y^5 + 32x^5$ by $16x^4 + y^4 - 2xy^3 - 8x^3y + 4x^2y^2$.

50. $2 - 3n^x + 13n^{2x} + 23n^{3x} - 11n^{4x} + 6n^{5x}$ by $2 + 3n^x$.

51. $6a^{2m} + 5a^{2m-1} - 10a^{2m-2} + 20a^{2m-3} - 16a^{2m-4}$ by $2a^m + 3a^{m-1} - 4a^{m-2}$.

Special Cases in Division

110. 1. By actual division,

$(x^2 - y^2) \div (x - y) = x + y.$

$(x^3 - y^3) \div (x - y) = x^2 + xy + y^2.$

$(x^4 - y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3.$

Observe that the *difference* of the same powers of two numbers is *divisible* by the *difference* of the numbers.

'Divisible' means 'exactly divisible.'

2. By actual division,

$(x^2 - y^2) \div (x + y) = x - y.$

$(x^3 - y^3) \div (x + y) = x^2 - xy + y^2$, rem., $-2y^3$.

$(x^4 - y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3.$

$(x^5 - y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$, rem., $-2y^5$.

Observe that the *difference* of the same powers of two numbers is *divisible* by the *sum* of the numbers *only when the powers are even*.

3. By actual division,

$(x^2 + y^2) \div (x - y) = x + y$, rem., $2y^2$.

$(x^3 + y^3) \div (x - y) = x^2 + xy + y^2$, rem., $2y^3$.

$(x^4 + y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3$, rem., $2y^4$.

Observe that the *sum* of the same powers of two numbers is *not divisible* by the *difference* of the numbers.

4. By actual division,

$(x^2 + y^2) \div (x + y) = x - y$, rem., $2y^2$.

$(x^3 + y^3) \div (x + y) = x^2 - xy + y^2.$

$(x^4 + y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$, rem., $2y^4$.

$(x^5 + y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$

Observe that the *sum* of the same powers of two numbers is *divisible* by the *sum* of the numbers *only when the powers are odd*.

111. Hence, the preceding conclusions may be summarized as follows, n being a positive integer:

PRINCIPLES. — 1. *$x^n - y^n$ is always divisible by $x - y$.*

2. *$x^n - y^n$ is divisible by $x + y$ only when n is even.*

3. *$x^n + y^n$ is never divisible by $x - y$.*

4. *$x^n + y^n$ is divisible by $x + y$ only when n is odd.*

112. The following **law of signs** may be inferred readily:

When $x - y$ is the divisor, the signs in the quotient are plus.

When $x + y$ is the divisor, the signs in the quotient are alternately plus and minus.

113. The following **law of exponents** also may be inferred:

In the quotient the exponent of x decreases and that of y increases by 1 in each successive term.

EXERCISES

114. Find quotients by inspection:

1. $\frac{a^3 - b^3}{a - b}$. 2. $\frac{m^3 - n^3}{m - n}$. 3. $\frac{a^3 - 8}{a - 2}$.

4. Devise a rule for dividing the difference of the cubes of any two numbers by the difference of the numbers.

Find quotients by inspection:

5. $\frac{a^3 + b^3}{a + b}$. 6. $\frac{m^3 + n^3}{m + n}$. 7. $\frac{c^3 + 27}{c + 3}$.

8. Devise a rule for dividing the sum of the cubes of any two numbers by the sum of the numbers.

Find quotients by inspection:

9. $\frac{a^3 - 125}{a - 5}$. 12. $\frac{r^4 - s^4}{r - s}$. 15. $\frac{1 + a^5}{1 + a}$.

10. $\frac{n^3 + 64}{n + 4}$. 13. $\frac{n^6 - 1}{n - 1}$. 16. $\frac{x^5 - 32}{x - 2}$.

11. $\frac{m^5 + n^5}{m + n}$. 14. $\frac{x^6 - 1}{x + 1}$. 17. $\frac{a^4 - 81}{a + 3}$.

PARENTHESES

115. The student has seen how *parentheses,* ( ), are used to group numbers that are to be regarded as a single number. Other signs used in the same way are *brackets,* [ ]; *braces,* { }; the *vinculum,* $\overline{\quad\quad}$; and the *vertical bar,* |.

Thus, all of the forms, $(a+b)c$, $[a+b]c$, $\{a+b\}c$, $\overline{a+b}\cdot c$, and $\left.\begin{matrix}a\\+b\end{matrix}\right| c$, have the same meaning.

These signs have the general name, **signs of aggregation.**

When numbers are included by any of the signs of aggregation, they are commonly said to be *in parenthesis*, *in a parenthesis*, or *in parentheses.*

116. Removal of parentheses preceded by + or —.

EXERCISES

1. Remove parentheses and simplify $3a+(b+c-2a)$.

SOLUTION

The given expression indicates that $(b+c-2a)$ is to be *added* to $3a$. This may be done by writing the terms of $(b+c-2a)$ after $3a$ in succession, each with its proper sign, and uniting terms.

$$\therefore 3a+(b+c-2a)=3a+b+c-2a=a+b+c.$$

2. Remove parentheses and simplify $4a-(2a-2b)$.

SOLUTION

The given expression indicates that $(+2a-2b)$ is to be *subtracted* from $4a$. Proceeding as in subtraction, that is, changing the sign of each term of the subtrahend and adding, we have

$$4a-(2a-2b)=4a-2a+2b=2a+2b.$$

PRINCIPLES. — 1. *A parenthesis preceded by a plus sign may be removed from an expression without changing the signs of the terms in parenthesis.*

2. *A parenthesis preceded by a minus sign may be removed from an expression, if the signs of all the terms in parenthesis are changed.*

Simplify by removing parentheses:

3. $a + (b - c)$.
4. $x - (y - z)$.
5. $x - (-y + z)$.
6. $m - n - (-a)$.
7. $m - (n - 2a)$.
8. $5x - (2x + y)$.
9. $a - m + (n - m)$.
10. $a - b - (c - d)$.
11. $5a - 2b - (a - 2b)$.
12. $a - (b - c + a) - (c - b)$.
13. $2xy + 3y^2 - (x^2 + xy - y^2)$.
14. $m + (3m - n) - (2n - m) + n$.

When an expression contains parentheses within parentheses, they may be removed *in succession*, beginning with either the outermost or the innermost, preferably the latter.

15. Simplify $6x - \{3a + (9b - 2a) + 4x - 10b\}$.

SOLUTION

	$6x - \{3a + (9b - 2a) + 4x - 10b\}$
Prin. 1,	$= 6x - \{3a + 9b - 2a + 4x - 10b\}$
Uniting terms,	$= 6x - \{a - b + 4x\}$
Prin. 2.	$= 6x - a + b - 4x$
Uniting terms,	$= 2x - a + b$.

Simplify:

16. $4a + b - \{x + 4a + b - 2y - (x - y)\}$.
17. $ab - \{ab + ac - a - (2a - ac) + \overline{2a - 2ac}\}$.
18. $a + [y - \{5 + 4a - 6y - 3\} - (7y - 4a - 1)]$.
19. $4m - [p + 3n - (m + n) + 3 - (6p - 3n - 5m)]$.
20. $ab - \{5 + x - (b + c - ab + x)\} + [x - (b - c - 7)]$.
21. $1 - x - \{1 - x - [1 - x - (1 - x) - (x - 1)] - x + 1\}$.
22. Simplify $a^2 + a(b - a) - b(2b - 3a)$.

SOLUTION

The expression indicates the sum of a^2, $a(b - a)$, and $-b(2b - 3a)$.

Expanding, $a(b - a) = ab - a^2$ and $-b(2b - 3a) = -2b^2 + 3ab$.

Therefore, writing the terms in order with their proper signs,

$$a^2 + a(b - a) - b(2b - 3a) = a^2 + ab - a^2 - 2b^2 + 3ab = 4ab - 2b^2.$$

Simplify:

23. $x^2 + x(y - x)$.

24. $c^2 - c(c - d)$.

25. $5 - 2(x - 3)$.

26. $x^2 - y^2 - (x - y)^2$.

27. $c(a - b) - c(a + b)$.

28. $a^3 - b^3 - 3ab(a - b)$.

29. $-2(x^2y - xy^2) - 5(xy^2 - x^2y)$.

30. $(3a - 2)(2a - 3) - 6(a - 2)(a - 1)$.

31. $(3m - 1)(m + 2) - 3m(m + 3) + 2(m + 1)$.

32. $(a - b)^2 - 2(a^2 - b^2) - 2a(-a - b) - 4b^2$.

33. $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2) - (x^2 + y^2)(x^2 + y^2)$.

34. $y^3 - [2x^3 - xy(x - y) - y^3] + 2(x - y)(x^2 + xy + y^2)$.

35. When $a = -2$, $b = 3$, $c = 4$, find the value of

$$a^2 - (a - c)(b + c) + 2b.$$

SOLUTION

$$\begin{aligned} a^2 - (a - c)(b + c) + 2b &= (-2)^2 - (-2 - 4)(3 + 4) + 2 \cdot 3 \\ &= 4 - (-6)(7) + 6 \\ &= 4 - (-42) + 6 \\ &= 4 + 42 + 6 = 52. \end{aligned}$$

When $x = 3$, $y = -4$, $z = 0$, $m = 6$, $n = 2$, find the value of:

36. $m(x - y) + z^2$.

37. $z + m^2 - (y^3 - 1)$.

38. $x^2 - y^2 - m^2 + n^2$.

39. $(x + y)(m - n) + 3z$.

40. $(m + x)^2 - (n - y)^2 - y^4$.

41. $xyz - n(x - m)^3 - (nx)^3$.

42. $3m(m - n) + 4n(y - x) - 7(y + z)$.

43. $x^2y^2(m - n)^2(m + n) + (m + n)^2(m - n)$.

44. $3m(x - y - n)^2 - (y - n - x)(n - x - y)$.

45. $(x + y + z)^2 - xy(y + z - x)(x + z - y) - z(x + y - z)$.

46. $(2x + y)^n - (x^2 - 2y)^x - (m + n)^2(x + y + z).^3$

47. $(m + n + x)^n - (m + n - x)^n - (m - n + x)^n(-m + n + x)^n$.

48. Show that $(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$, when $a = 1$, $b = 2$, and $c = 3$; when $a = 4$, $b = 2$, and $c = -1$.

117. Grouping terms by means of parentheses.

It follows from § 116 that:

PRINCIPLES. — 1. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a plus sign without changing the signs of the terms to be inclosed.*

2. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a minus sign, if the signs of the terms to be inclosed are changed.*

EXERCISES

118. 1. In $a^2 + 2ab + b^2$, group the terms involving b.

SOLUTION

$$a^2 + 2ab + b^2 = a^2 + (2ab + b^2).$$

2. In $a^2 - x^2 - 2xy - y^2$, group as a subtrahend the terms involving x and y.

SOLUTION

$$a^2 - x^2 - 2xy - y^2 = a^2 - (x^2 + 2xy + y^2).$$

3. In $ax^2 + ab + 2x^2 + 2b$, group the terms involving x^2, and also the terms involving b, as addends.

4. In $a^3 + 3a^2b + 3ab^2 + b^3$, group the first and fourth terms, and also the second and third terms, as addends.

In each of the following expressions group the last three terms as a subtrahend:

5. $a^2 - b^2 - 2bc - c^2$.

6. $a^2 - b^2 + 2bc - c^2$.

7. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

8. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.

9. In $a^2 + 2ab + b^2 - 4a - 4b + 4$, group the first three terms as an addend and the fourth and fifth as a subtrahend.

Errors like the following are common. Point them out.

10. $x^3 - x^2 + x - 1 = (x^3 - 1) - (x^2 + x)$.

11. $x^2 - y^2 + 2yz - z^2 = x^2 - (y^2 + 2yz - z^2)$.

119. The use of parentheses in grouping numbers enables us to extend the application of certain cases in multiplication.

Thus, in § 94 and in § 97, one or both numbers may consist of more than one term.

EXERCISES

120. 1. Expand $(a+m-n)(a-m+n)$.

SOLUTION

$a+m-n=a+(m-n)$ and $a-m+n=a-(m-n)$.

$\therefore [a+m-n][a-m+n]=[a+(m-n)][a-(m-n)]$

§ 94, $\quad = a^2-(m-n)^2$

§ 88, $\quad = a^2-(m^2-2\,mn+n^2)$

$\quad = a^2-m^2+2\,mn-n^2$.

Expand:

2. $(r+p-q)(r-p+q)$.
3. $(r+p+q)(r-p-q)$.
4. $(x+b+n)(x-b-n)$.
5. $(x^2+2\,x+1)(x^2+2\,x-1)$.
6. $(x^2+2\,x-1)(x^2-2\,x+1)$.
7. $(x^2+3\,x-2)(x^2-3\,x+2)$.
8. $[(a+b)+(c+d)][(a+b)-(c+d)]$.
9. $(a+b+x+y)(a+b-x-y)$.
10. $(a+b+m-n)(a+b-m+n)$.
11. $(x-m+y-n)(x-m-y+n)$.
12. $(a-m-b-n)(a+m-b+n)$.

13. Expand $(x+y+1)(x+y-3)$.

SOLUTION

$(x+y+1)(x+y-3)=(\overline{x+y}+1)(\overline{x+y}-3)$

§ 97, $\quad =(x+y)^2-2\,(x+y)-3$

§ 85, $\quad =x^2+2\,xy+y^2-2\,x-2\,y-3$.

Expand:

14. $(x-y-2)(x-y-8)$.
15. $(x^2+x-1)(x^2+x+3)$.
16. $(m-n+2)(m-n-4)$.
17. $(t^4-2\,t^2-5)(t^4-2\,t^2+2)$.
18. $(2\,s+3\,r+4)(2\,s+3\,r-3)$.
19. $(2\,a+5\,b+6)(2\,a+5\,b-8)$.

121. Collecting literal coefficients.

EXERCISES

Add:

1. $\begin{array}{r} ax \\ \underline{bx} \\ (a+b)x \end{array}$
2. $\begin{array}{r} bm \\ \underline{-cm} \\ (b-c)m \end{array}$
3. $\begin{array}{r} -cx \\ \underline{-dx} \\ -(c+d)x \end{array}$
4. $\begin{array}{r} (t+r)x \\ \underline{(t+2r)x} \\ (2t+3r)x \end{array}$
5. $\begin{array}{r} ax \\ \underline{nx} \end{array}$
6. $\begin{array}{r} cy \\ \underline{-dy} \end{array}$
7. $\begin{array}{r} -mp \\ \underline{-np} \end{array}$
8. $\begin{array}{r} (a+b)x \\ \underline{(2a+c)x} \end{array}$

Subtract the lower expression from the upper one:

9. $\begin{array}{r} mx \\ \underline{nx} \end{array}$
10. $\begin{array}{r} dy+az \\ \underline{ey-bz} \end{array}$
11. $\begin{array}{r} ax-by \\ \underline{2x-cy} \end{array}$
12. $\begin{array}{r} a^2x+aby \\ \underline{b^2x+aby} \end{array}$
13. $\begin{array}{r} mx-ny \\ \underline{nx-my} \end{array}$
14. $\begin{array}{r} (2r-s)y \\ \underline{(r+2s)y} \end{array}$

15. Collect the coefficients of x and of y in $ax-ay-bx-by$.

SOLUTION. — The total coefficient of x is $(a-b)$; the total coefficient of y is $(-a-b)$, or $-(a+b)$.

$$\therefore ax-ay-bx-by=(a-b)x-(a+b)y.$$

Collect in alphabetical order the coefficients of x and of y in each of the following, giving each parenthesis the sign of the first coefficient to be inclosed therein:

16. $ax-by-bx-cy+dx-ey$.
17. $5ax+3ay-2dx+ny-5x-y$.
18. $cx-2bx+7ay+3ax-lx-ty$.
19. $bx+cy-2ax+by-cx-dy$.
20. x^2+ax-y^2+ay.
21. $x^2-ay-ax-y^2$.
22. $bx-cy-2ay+by$.
23. $rx-ay-sx+2cy$.

Group the same powers of x in each of the following:

24. $ax^3+bx^2-cx+ex^3-dx^2-fx$.
25. $x^3+3x^2+3x-ax^2-3ax^3+bx$.
26. $x^2-abx-x^3-bx^2-cx-mnx^3+dx$.
27. $ax^4-x^4-ax^2+x^2+ax-x-abx^3+x^3$.

EQUATIONS AND PROBLEMS

122. 1. Given $5(2x-3)-7(3x+5)=-72$, to find the value of x.

SOLUTION

$$5(2x-3)-7(3x+5)=-72.$$

Expanding, $10x-15-21x-35=-72.$

Transposing, $10x-21x=15+35-72.$

Uniting terms, $-11x=-22.$

Multiplying by -1, $11x=22.$

$\therefore x=2.$

VERIFICATION. — Substituting 2 for x in the given equation,

$$5(4-3)-7(6+5)=-72.$$
$$5-77=-72.$$

Hence, 2 is a true value of x.

Find the value of x, and verify the result, in:

2. $2=2x-5-(x-3).$
3. $10x-2(x-3)=22.$
4. $1=5(2x-4)+5x+6.$
5. $7(5-3x)=3(3-4x)-1.$
6. $4x-x^2=x(2-x)+2.$
7. $7(2x-3)=2-3(2x+1).$
8. $3(2-4x)-(x-1)=-6.$
9. $4x^3-4(x^3-x^2+x-2)=4x^2.$
10. $5+7(x-5)=15(x+2-36).$
11. $2(x-5)+7=x+30-9(x-3).$
12. $(x-2)(x-2)=(x-3)(x-3)+7.$
13. $(x-4)(x+4)=(x-6)(x+5)+25.$
14. $x^2-(2x+3)(2x-3)+(2x-3)^2=(x+9)(x-2)-2.$
15. $3(4-x)^2-2(x+3)=(2x-3)^2-(x+2)(x-2)+1.$
16. $20(2-x)+3(x-7)-2[x+9-3\{9-8+28\}]=23.$
17. $(2x-4)^2-2\{x-6-3x(4+5)\}=4(x+2)^2+72.$
18. $3(x-7)-(x-9)+136=x-2[3x+4-(2x+6+9x)].$

Literal Equations

123. 1. Find the value of x in the equation $bx - b^2 = cx - c^2$.

SOLUTION

$$bx - b^2 = cx - c^2.$$

Transposing, $bx - cx = b^2 - c^2$.

Collecting coefficients of x, $(b - c)x = b^2 - c^2$.

Dividing by $b - c$, $x = \dfrac{b^2 - c^2}{b - c} = b + c.$

2. Find the value of x in the equation $x - a^3 = 2 - ax$.

SOLUTION

$$x - a^3 = 2 - ax.$$

Transposing, $ax + x = a^3 + 2$.

Collecting coefficients of x, $(a + 1)x = a^3 + 2$.

Dividing by $a + 1$, $x = \dfrac{a^3 + 2}{a + 1} = a^2 - a + 1 + \dfrac{1}{a + 1}.$

Find the value of x in:

3. $3(x - a - 2b) = 3b$.

4. $5b = 3(2x - b) - 4b$.

5. $cx - c^3 - d^3 + dx = 0$.

6. $x - 1 - c = cx - c^3 - c^4$.

7. $a^2 - ax + 5x = 7a - 10$.

8. $2m^3 - mx + nx - 2n^3 = 0$.

9. $a^2 - ax - 2ab + bx + b^2 = 0$.

10. $2n^2 + 5n + x = n^3 - nx - 2$.

11. $3ab - a^2 - 2bx = 2b^2 - ax$.

12. $a^2x - a^3 + 2a^2 + 5x - 5a + 10 = 0$.

13. $ax - 2bx + 3cx = ab - 2b^2 + 3bc$.

14. $cx - c^4 - 2c^3 - 2c^2 = 2c - x + 1$.

15. $9a^2 + 4mx = -(3ax - 16m^2)$.

16. $x + 6n^4 - 4n^3 = 1 - 3nx + 2n - n^2$.

17. $n^2x - 3m^2n^3 + nx + 3m^2 + x = 0$.

18. $x - 3b^2 - 192b^2c^3 - 4cx + 16c^2x = 0$.

19. $r(x - sx - 1) + r^2(x - r + s^2) = -1 - x - r(sx + 1) + r^2s^2$.

20. $a^4 - c - ax - bx + cx - b^4c = 2a^2b^2 + c(x - 1) - b^4(1 + c)$.

Algebraic Representation

124. 1. Find the value of x that will make $6x$ equal to 48.

2. Indicate the product when the sum of x, y, and $-d$ is multiplied by xy.

3. If a man earns a dollars per month, and his expenses are b dollars per month, how much will he save in a year?

4. Indicate the sum of x and z multiplied by m times the sum of x and y.

5. From x subtract m times the sum of the squares of $(a+b)$ and $(a-b)$.

6. A number x is equal to $(y-c)$ times $(d+c)$. Write the equation.

7. What is the number of square rods in a rectangular field whose length is $(a+b)$ rods and width $(a-b)$ rods?

8. At a factory where N persons were employed, the weekly pay roll was P dollars. Find the average earnings of each person per week.

9. How many seconds are x days $+c$ hours $+d$ minutes?

10. Express in cents the interest on y dollars for x years, if the interest for one month is z cents on one dollar.

11. If it takes b men c days to dig part of a well, and d men e days to finish it, how long will it take one man to dig the well alone?

12. Find an expression for 5 per cent of x; y per cent of z.

13. A train ran M miles in H hours and m miles in the succeeding h hours. Find its average rate per hour during each period and during the whole time.

14. A farmer has hay enough to last m cows for n days. How long will it last $(a-b)$ cows?

15. A dealer bought n 50-gallon barrels of paint at c cents per gallon. He sold the paint and gained g dollars. Find the selling price per gallon.

Problems

125. Solve the following problems and verify the solutions:

1. I bought 40 stamps for 95 cents. If part of them were 2-cent stamps and part 3-cent stamps, how many of each did I buy?

SOLUTION

Let $x =$ the number of 2-cent stamps.
Then, $40 - x =$ the number of 3-cent stamps.

$$\therefore 2x + 3(40 - x) = 95.$$

Solving, $x = 25$, the number of 2-cent stamps,
and $40 - x = 15$, the number of 3-cent stamps.

VERIFICATION. — The results obtained may satisfy the *equation of the problem* and still be incorrect, because the equation may be incorrect. If, however, the results satisfy the *conditions of the problem*, the solution is presumably correct.

1st condition: The whole number of stamps bought is 40.

$$25 + 15 = 40.$$

2d condition: The total cost of the stamps = 95 ¢.
The cost of 25 stamps @ 2 ¢ + 15 stamps @ 3 ¢ = 95 ¢.

2. A certain paper mill produces 350 tons of paper from sawdust each week. Of this 50 tons more is used for newspapers than for wrapping paper. How many tons are used for each?

3. The roadway of the Connecticut Avenue concrete bridge in Washington, D.C., together with two sidewalks, is 52 feet wide. How wide is the roadway, if it is 8 feet less than twice the combined width of the sidewalks?

4. One year the box factories of New England used 6,000,000 feet of boards. The amount of white pine used less 1,200,000 feet was 3 times that of the other timber. How much white pine was used?

5. It costs $2\frac{1}{2}$ ¢ more a day to feed an immigrant than it does to feed a United States private soldier. If it costs as much to feed 44 immigrants as it does to feed 49 privates, find the cost of the daily rations of each.

6. Of the 160,000 inhabitants of Hawaii, twice as many were Japanese as Chinese. The rest of the inhabitants, or $\frac{1}{4}$ of the total, were Americans and Europeans. Find the number of Chinese.

7. The combined capacity of two ice factories is 264 tons a day. If the capacity of the smaller one is increased 57 tons, its capacity will be half that of the larger one. Find the capacity of each.

8. It cost a man 60¢ to send a telegram at '30–2', that is, 30¢ for the first 10 words and 2¢ for each additional word. How many words did the message contain?

SUGGESTION. — Let x be the number of words in the message.
Then, $x - 10$ will represent the number of words in excess of 10 words.

$$\therefore 30 + 2(x - 10) = 60$$

9. How many words can be sent by telegraph from New Haven to New York for 75¢ at the day rate, '25–2'?

10. A long-distance telephone message cost me $1.25. The rate was 50¢ for the first 3 minutes and 15¢ for each additional minute. How long did the conversation last?

11. The day rate for a telegram between New Orleans and New York is '60–4' and the night rate is '40–3.' A message of a certain number of words cost 25¢ less to send, at night than in the daytime. Find the number of words.

12. A boy was twice as old as his sister 4 years ago. Now his sister is $\frac{2}{3}$ as old as he is. Find the age of each.

13. During one month the Dead Letter Office received 1,000,000 pieces of mail matter. If the number remaining in the office was $\frac{2}{3}$ as many as the number returned to the senders, how many pieces were returned?

14. An eighteen-hour train between New York and Chicago was late 91 times during its first year's run. It was late at Chicago 10 times more than 50 % as many times as it was late at Jersey City. How many times was it late at Jersey City?

15. In China, one woman earned 3¢ and another 8¢ a day by embroidering. The former worked 28 days on a piece of work, and then the two finished it. If the labor cost $5.02, how long did each work?

16. The shed that sheltered an airship was 544 feet in perimeter. If twice its length was 52 feet more than 4 times its width, what was its width? its length?

17. The average life of 5-dollar bills is $\frac{3}{4}$ of a year longer than that of 1-dollar bills, and $\frac{2}{3}$ as long as that of 10-dollar bills. If a 10-dollar bill lasts $1\frac{3}{4}$ years longer than a 1-dollar bill, find the average life of a bill of each denomination.

18. A farmer's net receipts from hens in a year were $90.15. The eggs sold for $92.55 more than the chickens, and the expenses were $72.65 less than the selling price of the eggs. What did the eggs sell for? the chickens?

19. Upon the floor of a room 4 feet longer than it is wide is laid a rug whose area is 112 square feet less than the area of the floor. There are 2 feet of bare floor on each side of the rug. What is the area of the floor? of the rug?

20. A party of 8 traveled second class from London to Paris for $5.70 less than twice the amount paid by a party of 3 traveling first class. If a first-class ticket cost $4.15 more than a second-class ticket, find the price of each.

21. A military cable and telegraph system between Seattle and Alaska covers 4044 miles. The length of the submarine cable is 272 miles less than twice that of the land telegraph. The land telegraph is 12 miles longer than 13 times the wireless. How long is the wireless?

22. The United States has 280 life-saving stations, 1 being situated at the falls of the Ohio River. Of the remainder, the Atlantic coast has $11\frac{1}{6}$ times as many as the Pacific. Find the number on the Pacific coast, if it lacks 2 of being $\frac{1}{3}$ the number on the Great Lakes.

REVIEW

126. 1. What are positive numbers? negative numbers? In the following expression point out the positive numbers; the negative numbers. Perform the indicated operations:
$3\,ax + 7\,by - 9\,bx + 10\,by - 4\,ax - 3\,bx + 4\,ax - 2\,ax - 12\,by.$

2. What two meanings has the minus sign in algebra? If distance north is positive, what is the meaning of -150 miles? $+75$ miles?

3. Distinguish between arithmetical numbers and algebraic numbers.

4. Instead of subtracting a number (positive or negative), what may be done to secure the same result? Illustrate by subtracting -7 from $+12$. What is the absolute value of each of these numbers?

5. What is transposition? Give the principle relating to transposition.

6. State the law of signs for multiplication; for division.

7. What is the sign of the product of an even number of negative factors? of an odd number of negative factors?

8. In what respect do $(a - b)$ and $(b - a)$ differ? Expand $(a - b)^2$ and $(b - a)^2$ and compare the results.

9. For what values of n is $x^n + y^n$ divisible by $x + y$? by $x - y$? When is $x^n - y^n$ divisible by $x + y$? by $x - y$?

10. State the law of signs for the quotient when $x^n + y^n$ or $x^n - y^n$ is divided by $x + y$ or $x - y$; the law of exponents.

11. What must be added to $x^2 - 10\,x$ to make it the square of $x - 5$? to $a^2 + b^2$ to make it the square of $a + b$? to $x^4 + x^2y^2 + y^4$ to make it the square of $x^2 + y^2$?

12. How may a parenthesis preceded by a minus sign be removed from an algebraic expression without changing the value of the expression?

13. Add $3a + 5b - 11c$, $b - 2a + c$, $2c + 3a - b$, $7c - b + 6a$, $5b - 4a - 2c$, $b - a$, $c + b - a$, and $c - 4a$.

14. Subtract the sum of $x - 2y + 3z - 5w$ and $7x + w - 2z$ from $10x - y + z - 8w$.

15. If $x = r^2 + rs - s^2$, $y = 2r^2 + 4rs + 2s^2$, and $z = r^2 - 3rs - s^2$, find the value of $x + y - z$.

Expand, and test each result:

16. $(r^3 + 7r^2s - 3rs^2 + 2s^3)(r^2 + 2rs + s^2)$.

17. $(3l^{3r} + 6l^{2r}m - 12l^rm^2 + 3m^3)(4l^{2r} + 3l^rm + 2m^2)$.

18. $(x^4 + y^4 - 4xy^3 + 5x^2y^2 + 3x^3y)(x^3 + 3x^2y + 3xy^2 + y^3)$.

Expand by inspection, and test each result:

19. $(3a + 7b)^2$.

20. $(9w - 2v)^2$.

21. $(x + 2y)(x - 2y)$.

22. $(a - 3)(a + 10)$.

23. $(7r + 4s)(7r - 4s)$.

24. $(3x - 5y + z)^2$.

25. $(2c + d)(3c + 2d)$.

26. $(5a - 3b)(2a + 2b)$.

Divide, and test each result:

27. $2l^6 + 5l^4r^2 - 3l^5r - 6l^3r^3 + 3lr^5 - r^6$ by $2l^2 - 3lr + r^2$.

28. $3x^4 + 8y^2 - 10yz - 8x^2z - 3z^2 + 10x^2y$ by $x^2 + 2y - 3z$.

29. $4x^{4n} - 25x^{2n}y^{2n} - 10x^ny^{3n} - y^{4n}$ by $2x^{2n} - 5x^ny^n - y^{2n}$.

Find quotients by inspection:

30. $\dfrac{y^4 - 1}{y + 1}$.

31. $\dfrac{x^3 - 64}{x - 4}$.

32. $\dfrac{a^5 + 32}{a + 2}$.

33. $\dfrac{9x^2 - 16y^2}{3x + 4y}$.

34. $\dfrac{c^3 + 125d^3}{c + 5d}$.

35. $\dfrac{243 - x^5}{3 - x}$.

36. Simplify $17x - \{3y + 4z - [z + 5a + x - 3a - 2y]\}$.

37. Simplify $a + 2b - [4c + 2(a + 2b) - \overline{b + 4c} - a] + b$.

When $x = 2$, $y = -3$, and $z = 5$, find the value of:

38. $xz - (x + y + z)$.

39. $3(x - y) + 2(y - x) - zy$.

40. $x^2 - 3x(y + z) + y^2 - z$.

41. $(x - y)(y + z) - z^2(y - z)$.

42. When $a = 2$ and $b = 3$, prove that

$$b(ab + b - 2a) = ab^2 + b^2 - 2ab.$$

43. Collect similar terms within parentheses:

$$ax^2 - cy + ax - 2ax^2 + 2cy^2 - ax - cy^2 + ax^2 + cy.$$

44. Collect the coefficients of x and of y in

$$7ax - 3by - 22a^2x + 5ay - 17bx + cy - 4x + 13y.$$

Solve for x, and test results:

45. $3x + 7(x - 2) - 13 = 12 - 3x$.

46. $20 = 7 - 5(3 - x) + 9(x + 2)$.

47. $x^3 - 1 = x(x^2 - x) + x + 3 + x^2$.

48. $(x - 4)(x + 3) = (x + 6)^2 - 3 + 2x$.

49. $4a - 3(b + x) - 5a = 7b + 4a - 5(x + a)$.

50. $(a - x)(b - x) - b\{x - (a - x) - x\} = x^2 - 2bx + ab$.

Solve, and test:

51. $\begin{cases} 3x + 2y = 13, \\ 2x + y = 8. \end{cases}$

52. $\begin{cases} 7x + 4y = 39, \\ x - y = 4. \end{cases}$

53. $\begin{cases} x - y = 3, \\ 3x - 2y = 13. \end{cases}$

54. $\begin{cases} 5x + 7y = 24, \\ 3x - 2y = 2. \end{cases}$

55. $\begin{cases} 3x + 5y = 13, \\ 2x - 3y = -4. \end{cases}$

56. $\begin{cases} 9x - 2y = 15, \\ 5x + 7y = 57. \end{cases}$

Supply the missing coefficients in the following equations:

57. $3a - *b + 6a + 5b - *xy = *a + b - 2xy$.

58. $x^2 + 2xy + 3y^2 - [2x^2 + *y^2] = *x^2 + *xy$.

59. $6m^2 + 9mn - 3n^2 - [3m^2 + *mn] + n^2 = *m^2 - mn - *n^2$.

FACTORING

127. In *multiplication*, we find the product of two or more given numbers; in **factoring**, we have given the product to find the numbers that were multiplied to produce it.

These numbers are called the **factors** of the product.

128. A number that has no factors except itself and 1 is called a **prime number**.

MONOMIALS

129. To factor a monomial.

While in factoring it is usually the *prime* factors that are sought, this is not generally true in the case of monomials, because the factors of a monomial, except those of the coefficient, are evident.

Thus, $2a^3b^2$ shows its prime factors as well as though written $2 \cdot a \cdot a \cdot a \cdot b \cdot b$, but $81a^3b^2$ should be written $3^4a^3b^2$ to be considered in factored form.

However, it is often desirable to separate a monomial into two factors, one being given or both being specified in some way.

EXERCISES

130. 1. In each of the following, if xy is one factor, find the other: $6x^5y$, $15x^4y^2$, $2x^3y^3$, $a^2x^2b^2y^2$, $-mnxy$, $-xy$.

2. In each of the following, if abc is one factor, find the other: a^2bc, ab^2c, abc^2, $-a^2b^2c^2$, $-a^2bc$, $-\frac{1}{3}abc$.

3. Find two equal positive factors of x^2; of $9a^2x^2$; of $64m^4$.

4. Find two equal negative factors of $25x^2$; of $16a^2$; of $9a^6$.

131. A factor of two or more numbers is called a **common factor** of them.

132. One of the two equal factors of a number is called its **square root.**

Every number has two square roots, one *positive* and the other *negative.*

The square root of 25 is 5 or -5, for $5 \cdot 5 = 25$ and $(-5)(-5) = 25$.

In factoring, usually only the *positive* square root is taken.

133. To factor an expression whose terms have a common monomial factor.

EXERCISES

1. Find the factors of $3xy - 6x^2y + 9xy^2$.

PROCESS

$$3xy - 6x^2y + 9xy^2$$
$$= 3xy(1 - 2x + 3y)$$

EXPLANATION. — By examining the terms of the expression, it is seen that the monomial $3xy$ is a factor of every term. Dividing by this common factor gives the other factor.

Hence, the factors of $3xy - 6x^2y + 9xy^2$ are $3xy$ and $1 - 2x + 3y$.

TEST. — The product of the factors should equal the given expression; thus, $3xy(1 - 2x + 3y) = 3xy - 6x^2y + 9xy^2$.

Factor, and test each result:

2. $5x^3 - 5x^2$.
3. $8x^2 + 2x^4$.
4. $3x^3 - 6x^2y$.
5. $4a^2 - 6ab$.
6. $5m^2 - 3mn$.
7. $3x^3y^2 - 3x^2y^3$.
8. $4a^3b - 6a^2b^2$.
9. $5m^4n - 10m^3n^2$.
10. $3x^4 - 9x^3 - 6x^2$.
11. $3a^4 - 2a^3b + a^2$.
12. $x^{12} + x^{11} + x^{10} - x^9$.
13. $ac - bc - cy - abc$.
14. $3x^3y^3 - 3x^2y^2 + 12xy$.
15. $3m^5 - 12m^3n^2 + 6mn^4$.
16. $9a^2by - 18ab^2y^2 + 24a^3b^2y^2$.
17. $12x^2y^2z^3 - 16x^2y^2z^2 - 20x^3y^3z^3$.
18. $25c^2dx^3 + 35c^3d^2x^4 - 55c^2d^2x^5$.
19. $16a^2b^3c^4 - 24a^3b^2c^3 + 32a^3b^4c^3$.
20. $14a^2mn^2 - 21a^3m^2n^3 - 49a^4mn^2$.
21. $60m^2n^3r^2 - 45m^3n^2r^3 + 90m^4n^3r^2$.

BINOMIALS

134. To factor the difference of two squares.

By multiplication, § 94, $(a+b)(a-b)=a^2-b^2$.

Therefore, $\qquad \boldsymbol{a^2-b^2=(a+b)(a-b)}$.

Hence, to factor the difference of two squares,

RULE. — *Find the square roots of the two terms, and make their sum one factor and their difference the other.*

EXERCISES

135. Factor, and test each result:

1. x^2-m^2.
2. a^2-y^2.
3. x^2-1.
4. a^2-3^2.
5. n^2-4^2.
6. $1-\frac{1}{c^2}$.
7. Factor $a^2x^2-4c^2$.

SOLUTION

$$a^2x^2-4c^2=(ax)^2-(2c)^2$$
$$=(ax+2c)(ax-2c).$$

Factor, and test each result:

8. x^2-81.
9. b^4-49.
10. $25y^2-1$.
11. m^4-16n^2.
12. a^8-b^2.
13. m^2-n^{12}.
14. $81-x^6y^6$.
15. $9a^2-49b^2$.
16. $1-144m^2$.
17. $64x^2-a^2c^2$.
18. $81a^4-100$.
19. $121n^2-36r^2$.

136. To factor the sum or the difference of two cubes.

By applying the principles of §§ 111–113,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2 \text{ and } \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Then, § 30, $\qquad \boldsymbol{a^3+b^3=(a+b)(a^2-ab+b^2)}$,

and $\qquad \boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2)}$.

By use of these forms any expression that can be written as the sum or the difference of two cubes may be factored.

EXERCISES

137. Factor, and test each result:

1. $x^3 + y^3$.
2. $x^3 - y^3$.
3. $m^3 - 1$.
4. $1 + m^3$.
5. $r^3 + 2^3$.
6. $\frac{a^3}{8} - 1$.
7. Factor $x^6 + y^6$.

SOLUTION

$$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

8. Factor $a^9 - 8\,b^3$.

SOLUTION

$$a^9 - 8\,b^3 = (a^3)^3 - (2\,b)^3 = (a^3 - 2\,b)(a^6 + 2\,a^3b + 4\,b^2).$$

Factor, and test each result:

9. $x^3 - y^6$.
10. $8\,r^9 + s^9$.
11. $a^3b^3 - 27$.
12. $v^6 + 64\,t^3$.
13. $x^3y^6z^9 + 1$.
14. $n^3 - 1000$.

By applying §§ 111–113, as in § 136, any expression that can be written as the sum or the difference of the same odd powers of two numbers may be resolved into two factors.

Thus, $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$,
and $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Factor:

15. $m^5 + n^5$.
16. $m^5 - n^5$.
17. $a^5 + 32$.
18. $32 - a^5$.
19. $1 + s^5t^5$.
20. $x^5 - y^{10}$.

TRINOMIALS

138. To factor a trinomial that is a perfect square.

By multiplication, § 85,

$$(a + b)(a + b) = a^2 + 2\,ab + b^2.$$

Then, $\boldsymbol{a^2 + 2\,ab + b^2 = (a + b)(a + b)}$.

Also, § 88, $(a - b)(a - b) = a^2 - 2\,ab + b^2$.

Then, $\boldsymbol{a^2 - 2\,ab + b^2 = (a - b)(a - b)}$.

These trinomials, $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, are **perfect squares,** for each may be separated into two *equal* factors.

They are types, showing the form of all **trinomial squares,** for a and b may represent any two numbers.

Hence, to factor a trinomial square,

RULE. — *Connect the square roots of the terms that are squares with the sign of the other term, and indicate that the result is to be taken twice as a factor.*

EXERCISES

139. Factor, and test each result:

1. $x^2 + 2xy + y^2$.
2. $p^2 - 2pq + q^2$.
3. $c^2 + 2cd + d^2$.
4. $t^2 - 2tu + u^2$.
5. $x^2 - 2x + 1$.
6. $x^2 + 2x + 1$.

140. From the forms in the preceding discussion and exercises it is seen that a trinomial is a perfect square, if these two conditions are fulfilled:

1. Two terms, as $+a^2$ and $+b^2$, must be perfect squares.

2. The other term must be numerically equal to twice the product of the square roots of the terms that are squares.

Thus, $25x^2 - 20xy + 4y^2$ is a perfect square, for $25x^2 = (5x)^2$, $4y^2 = (2y)^2$, and $-20xy = -2(5x)(2y)$.

EXERCISES

141. Discover which of the following are perfect squares, factor such as are, and test each result:

1. $x^2 + 6x + 9$.
2. $4 - 4a + a^2$.
3. $r^2 - 8r + 16$.
4. $m^2 - mn + n^2$.
5. $1 + 4b + 4b^2$.
6. $1 - 6a^3 + 9a^6$.
7. $m^2 + m + \frac{1}{4}$.
8. $3x^2 + 8xy + 2y^2$.
9. $16p^2 - 24p + 9$.
10. $x^4 + 2x^2y^2 + y^4$.
11. $9 + 42b^3 + 49b^6$.
12. $1 - 6m^4 + 9m^8$.
13. $4x^2y^2 - 20xy + 25$.
14. $4x^2 + 12xyz + 9y^2z^2$.

142. To factor a trinomial of the form $x^2 + px + q$.

By multiplication, § 97,

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Then, $$\boldsymbol{x^2 + (a + b)x + ab = (x + a)(x + b).}$$

This trinomial consists of an x^2-term, an x-term, and a term without x, that is, an **absolute term**. Therefore, it has the type form $x^2 + px + q$.

Hence, if a trinomial of this form is factorable, it may be factored by *finding two factors of q (the absolute term) such that their sum is p (the coefficient of x), and adding each factor of q to x.*

Thus,
$$x^2 + 8x + 15 = (x + 3)(x + 5),$$
$$x^2 - 8x + 15 = (x - 3)(x - 5),$$
$$x^2 + 2x - 15 = (x - 3)(x + 5),$$
$$x^2 - 2x - 15 = (x + 3)(x - 5).$$

EXERCISES

143. 1. Resolve $x^2 - 13x - 48$ into two binomial factors.

SOLUTION. — The first term of each factor is evidently x.

Since the product of the second terms of the two binomial factors is -48, the second terms must have opposite signs; and since their algebraic sum, -13, is negative, the negative term must be numerically larger than the positive term.

The two factors of -48 whose sum is negative may be 1 and -48, 2 and -24, 3 and -16, 4 and -12, or 6 and -8. Since the algebraic sum of 3 and -16 is -13, 3 and -16 are the factors of -48 sought.

$$\therefore x^2 - 13x - 48 = (x + 3)(x - 16).$$

2. Factor $72 - m^2 - m$.

SOLUTION. — Arranging the trinomial according to the descending powers of m,

$$72 - m^2 - m = -m^2 - m + 72$$

Making m^2 positive,
$$= -(m^2 + m - 72)$$
$$= -(m - 8)(m + 9)$$
$$= (-m + 8)(m + 9)$$
$$= (8 - m)(9 + m).$$

Separate into factors, and test each result by assigning a numerical value to each letter:

3. $x^2 + 7x + 12$.
4. $y^2 - 7y + 12$.
5. $p^2 - 8p + 12$.
6. $r^2 + 8r + 12$.
7. $15 + 2a - a^2$.
8. $b^2 + b - 12$.
9. $30 - r^2 + r$.
10. $c^2 - c - 72$.
11. $c^2 - 5c - 14$.
12. $x^2 - x - 110$.
13. $x^2 + 5ax + 6a^2$.
14. $x^2 - 6ax + 5a^2$.
15. $y^2 - 4by - 12b^2$.
16. $y^2 - 3ny - 28n^2$.
17. $z^2 - anz - 2a^2n^2$.
18. $-x^2 + 25x - 100$.
19. $x^4 + 19cx^2 + 90c^2$.
20. $x^6 + 12ax^3 + 20a^2$.
21. $x^{10} - 11b^2x^5 + 24b^4$.
22. $n^2x^2 - 11nxy + 30y^2$.

144. To factor a trinomial of the form $ax^2 + bx + c$.

EXERCISES

1. Factor $3x^2 + 11x - 4$.

SOLUTION. — If this trinomial is the product of two binomial factors, they may be found by reversing the process of multiplication illustrated in exercise 32, page 73.

Since $3x^2$ is the product of the *first terms* of the binomial factors, the first terms, each containing x, are $3x$ and x.

Since -4 is the product of the last terms, § 78, they must have unlike signs, and the only possible *last terms* are 4 and -1, -4 and 1, or 2 and -2.

Hence, associating these pairs of factors of -4 with $3x$ and x in all possible ways, the possible binomial factors of $3x^2 + 11x - 4$ are:

$$\left.\begin{matrix}3x+4\\x-1\end{matrix}\right\},\quad \left.\begin{matrix}3x-1\\x+4\end{matrix}\right\},\quad \left.\begin{matrix}3x-4\\x+1\end{matrix}\right\},\quad \left.\begin{matrix}3x+1\\x-4\end{matrix}\right\},\quad \left.\begin{matrix}3x+2\\x-2\end{matrix}\right\},\quad \left.\begin{matrix}3x-2\\x+2\end{matrix}\right\}.$$

Of these we select *by trial* the pair that will give $+11x$ (the middle term of the given trinomial) for the algebraic sum of the 'cross-products,' that is, the second pair.

$$\therefore 3x^2 + 11x - 4 = (3x - 1)(x + 4).$$

By a reversal of the law of signs for multiplication and from the preceding solution it may be observed that:

1. *When the sign of the last term of the trinomial is +, the last terms of the factors must be both + or both —, and like the sign of the middle term of the trinomial.*

2. *When the sign of the last term of the trinomial is —, the sign of the last term of one factor must be +, and of the other —.*

Factor, and test each result:

2. $2y^2 + 3y + 1$.
3. $5x^2 + 9x - 2$.
4. $3x^2 - 7x - 6$.
5. $4r^2 + 8r + 3$.
6. $6x^2 - 7x + 2$.
7. $2x^2 - 5x - 12$.
8. $10t^2 + t - 3$.
9. $6n^2 - 13n + 6$.
10. $2x^2 + x - 15$.
11. $5x^2 + 13x + 6$.
12. $3x^2 - 17x + 10$.
13. $6x^2 - 11x - 35$.
14. $15x^2 + 17x - 4$.
15. $15x^2 - 14x - 8$.
16. $2x^2 + 3xy - 2y^2$.
17. $3x^2 - 10xy + 3y^2$.

When the coefficient of x^2 is a square, and when the square root of the coefficient of x^2 is exactly contained in the coefficient of x, the trinomial may be factored as follows:

18. Factor $9x^2 + 30x + 16$.

Solution

$$\begin{aligned}&9x^2 + 30x + 16\\ &= (3x)^2 + 10(3x) + 16\\ &= (3x + 2)(3x + 8).\end{aligned}$$

Separate into factors, and test each result:

19. $9x^2 - 9x + 2$.
20. $4x^2 - 4x - 15$.
21. $9x^2 - 42x + 40$.
22. $25x^2 + 15x + 2$.
23. $16x^2 + 16x + 3$.
24. $36x^2 - 36x + 5$.
25. $49x^2 - 42x - 55$.
26. $25x^2 + 25x - 24$.
27. $16x^2 - 32x + 15$.
28. $64x^2 - 32x - 77$.
29. $100x^2 + 40x + 3$.
30. $81x^2 - 108x + 35$.

POLYNOMIALS

145. To factor a polynomial whose terms may be grouped to show a common polynomial factor.

EXERCISES

1. Factor $ax + ay + bx + by$.

SOLUTION

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (a + b)(x + y). \end{aligned}$$

2. Factor $ax + by - ay - bx$.

SOLUTION

$$ax + by - ay - bx = ax - ay - bx + by$$

§ 117, Prin. 2,

$$\begin{aligned} &= (ax - ay) - (bx - by) \\ &= a(x - y) - b(x - y) \\ &= (a - b)(x - y). \end{aligned}$$

REMARK. — The given polynomial must be arranged and grouped in such a way that after the monomial factor is removed from each group the polynomial factors in all the groups will be *alike in every respect.*

Factor, and test each result, especially for signs:

3. $am - an + mx - nx$.
4. $bc - bd + cx - dx$.
5. $pq - px - rq + rx$.
6. $ay - by - ab + b^2$.
7. $x^2 - xy - 5x + 5y$.
8. $b^2 - bc + ab - ac$.
9. $x^2 + xy - ax - ay$.
10. $c^2 - 4c + ac - 4a$.
11. $1 - m + n - mn$.
12. $2x - y + 4x^2 - 2xy$.
13. $2p + q + 6p^2 + 3pq$.
14. $ar - rs - ab + bs$.
15. $x^3 + x^2 + x + 1$.
16. $y^3 + y^2 - 3y - 3$.
17. $x^5 + x^3 + x^2y + y$.
18. $2 - n^2 - 2n + n^3$.
19. $ax - x - a + x^2$.
20. $12a^2 - 8b - 3a^3 + 2ab$.
21. $8ax + 6ay - 4bx - 3by$.
22. $3x^3 - 15x + 10y - 2x^2y$.
23. $3r^2t - 9rt^2 + ar - 3at$.
24. $ax - a - bx + b - cx + c$.

146. To factor by the factor theorem.

Zero multiplied by any number is equal to 0.

Conversely, if a product is equal to zero, at least one of the factors must be 0 or a number equal to 0.

If $5x = 0$, since 5 is not equal to 0, x must equal 0.

If $5(x-3) = 0$, since 5 is not equal to 0, x must have such a value as to make $x-3$ equal to 0; that is, $x = 3$.

If $5(x-3)$, or $5x - 15$, or any other polynomial in x reduces to 0 when $x = 3$, $x - 3$ is a factor of the polynomial.

Sometimes a polynomial in x reduces to 0 for more than one value of x. For example, $x^2 - 5x + 6$ equals 0 when $x = 3$ and also when $x = 2$; or when $x - 3 = 0$ and $x - 2 = 0$. In this case both $x - 3$ and $x - 2$ are factors of the polynomial.

$$\therefore\ x^2 - 5x + 6 = (x-3)(x-2).$$

147. Factor Theorem. — *If a polynomial in x, having positive integral exponents, reduces to zero when r is substituted for x, the polynomial is exactly divisible by $x - r$.*

The letter r represents any number that may be substituted for x.

EXERCISES

148. 1. Factor $x^3 - x^2 - 4x + 4$.

Solution. — When $x = 1$, $x^3 - x^2 - 4x + 4 = 1 - 1 - 4 + 4 = 0$.
Therefore, $x - 1$ is a factor of the given polynomial.
Dividing $x^3 - x^2 - 4x + 4$ by $x - 1$, the quotient is found to be $x^2 - 4$.
By § 134, $x^2 - 4 = (x+2)(x-2)$.

$$\therefore\ x^3 - x^2 - 4x + 4 = (x-1)(x+2)(x-2).$$

Suggestions. — 1. Only factors of the absolute term of the polynomial need be substituted for x in seeking factors of the polynomial of the form $x - r$, for if $x - r$ is one factor, the absolute term of the polynomial is the product of r and the absolute term of the other factor.

2. In substituting the factors of the absolute term, try them in order beginning with the numerically smallest.

2. Factor $x^3 + x^2 - 9x - 9$.

Suggestion. — When $x = 1$, $x^3 + x^2 - 9x - 9 = -16$.
Therefore, $x - 1$ is not a factor of the given polynomial.
When $x = -1$, $x^3 + x^2 - 9x - 9 = 0$.
Therefore, $x - (-1)$, or $x + 1$, is a factor of the given polynomial.

Factor by the factor theorem:

3. $13x^2 - 5x - 8$.
4. $x^3 - 7x + 6$.
5. $x^3 - 9x^2 + 23x - 15$.
6. $x^3 - 4x^2 - 7x + 10$.
7. $x^3 - 6x^2 - 9x + 14$.
8. $x^3 - 11x^2 + 31x - 21$.
9. $x^3 - 10x^2 + 29x - 20$.
10. $x^3 - 19x + 30$.
11. $x^3 - 67x - 126$.
12. $m^3 + 7m^2 + 2m - 40$.
13. $x^4 - 25x^2 + 60x - 36$.
14. $x^4 + 13x^2 - 54x + 40$.
15. $x^4 + 22x^2 + 27x - 50$.
16. $x^4 - 9x^3 + 21x^2 + x - 30$.

17. Factor $2x^3 + x^2y - 5xy^2 + 2y^3$.

SUGGESTION. — When $x = y$,

$$2x^3 + x^2y - 5xy^2 + 2y^3 = 2y^3 + y^3 - 5y^3 + 2y^3 = 0.$$

Therefore, $x - y$ is a factor of $2x^3 + x^2y - 5xy^2 + 2y^3$.

Factor by the factor theorem:

18. $x^3 - 13xy^2 + 12y^3$.
19. $x^3 - 31xy^2 - 30y^3$.
20. $x^4 - 9x^2y^2 + 12xy^3 - 4y^4$.
21. $x^4 - 9x^2y^2 - 4xy^3 + 12y^4$.

MISCELLANEOUS EXERCISES

149. In the exercises under the preceding cases, except those under the factor theorem, the expressions given have been *completely* factored by *one* application of a *single* case, but frequently it is necessary to apply *two* or *more* cases in succession or *one* case *more than once* to factor the given expression *completely*.

Monomial factors should usually first be removed, as they often disguise a familiar type form.

1. Factor $x^3 + 3x^2 - 10x$.

SOLUTION

By § 133, $x^3 + 3x^2 - 10x = x(x^2 + 3x - 10)$

By § 142, $= x(x + 5)(x - 2)$.

2. Factor $x^6 - y^6$.

Solution

Writing the expression as the difference of two squares, we have,

$$x^6 - y^6 = (x^3)^2 - (y^3)^2$$

§ 134, $$= (x^3 + y^3)(x^3 - y^3)$$

§ 136, $$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

3. Factor $x^8 - y^8$.

Solution

By successive applications of § 134,

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

Factor completely, and test each result:

4. $x^3 - xy^2$.
5. $m - m^4$.
6. $a^4 - b^4$.
7. $x^8 - 1$.
8. $w^4 - 81$.
9. $3t^5 - 3t$.
10. $z^6 - 1$.
11. $5c^8 - 5$.
12. $4a - 4a^2 + a^3$.
13. $2x^2 + 5xy + 2y^2$.
14. $10x^2 - 20x + 10$.
15. $11a^2x - 55ax + 66x$.
16. $n^3 + 2n - 3n^2 - 6$.
17. $r^5 + 2r^2 + 5r^3 + 10$.
18. $18a^2b + 60ab^2 + 50b^3$.
19. $w^3 + uw^2 + vw^2 + uvw$.
20. $x^6 - y^3$.
21. $l^5 - 16l$.
22. $3r^4 - 3s^6$.
23. $4ax + 2ax^2 - 48a$.
24. $18m^2 - 3m - 36$.
25. $y + 10b^2y + 25b^4y$.
26. $27n + n^7$.
27. $64 - 2a^5$.
28. $m^7 + m^2n^5$.
29. $21a^2 - a - 10$.
30. $-a^2 - 4a + 45$.
31. $16x^2 + 20x - 66$.
32. $36x^2 - 48x - 20$.
33. $x^3 - 21xy^2 + 20y^3$.
34. $3a^2bx^2 - 3a^2bx - 6a^2b$.
35. $x^3 - 12x^2 + 41x - 30$.
36. $30 - m - 6m^2 + m^3$.
37. $3r^2s - 9rs^2 + br - 3bs$.
38. $15tl^2 - 9l^2n - 35tl + 21ln$.
39. $w^4 - w^3 - 7w^2 + w + 6$.
40. $m^4 - 15m^2 + 10m + 24$.

SPECIAL APPLICATIONS AND DEVICES

150. The method of factoring by grouping the terms of an expression in certain ways is very important. Polynomials may often be arranged in some one of the type forms already studied, and even many of these types themselves may be factored by grouping to show a common polynomial factor.

151. To factor a polynomial that may be grouped to form the difference of two squares.

Just as, § 134, $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$,

so
$$a^2 - (b+c)^2 = [a + (b+c)][a - (b+c)]$$
$$= (a+b+c)(a-b-c).$$

EXERCISES

152. Factor:

1. $(a+b)^2 - c^2$.
2. $r^2 - (s+t)^2$.
3. $x^2 - (y-z)^2$.
4. $(l+m)^2 - n^2$.
5. $(a-b)^2 - c^2$.
6. $1 - (v+w)^2$.
7. Factor $x^4 - (3x^2 - 2y)^2$.

SOLUTION

$$\begin{aligned} x^4 - (3x^2 - 2y)^2 &= [x^2 + (3x^2 - 2y)][x^2 - (3x^2 - 2y)] \\ &= (x^2 + 3x^2 - 2y)(x^2 - 3x^2 + 2y) \\ &= (4x^2 - 2y)(2y - 2x^2) \\ &= 2(2x^2 - y)2(y - x^2) \\ &= 4(2x^2 - y)(y - x^2). \end{aligned}$$

TEST. — When $x = 2$ and $y = 3$,
$x^4 - (3x^2 - 2y)^2 = 2^4 - (3 \cdot 2^2 - 2 \cdot 3)^2 = 16 - (12 - 6)^2 = 16 - 36 = -20$,
and $4(2x^2 - y)(y - x^2) = 4(2 \cdot 2^2 - 3)(3 - 2^2) = 4(8-3)(3-4) = -20$.

Factor, and test each result:

8. $4c^2 - (b+c)^2$.
9. $(2a+b)^2 - b^2$.
10. $9t^2 - (2t-5)^2$.
11. $25a^2 - (b+c)^2$.
12. $49r^2 - (5r - 4s)^2$.
13. $36z^2 - (3z - 7y)^2$.
14. $(6w - 3k)^2 - 64k^2$.
15. $(3m + 8n)^2 - 16m^2$.

16. Factor $a^2 + 4 - c^2 - 4a$.

SUGGESTION. — The given expression contains three square terms and a term that is not a square. The latter may be the middle term of a trinomial square. If so, it contains as factors the square roots of two of the square terms and these are the other terms of the trinomial square. Then, arranging and grouping terms, we have

$$a^2 + 4 - c^2 - 4a = (a^2 - 4a + 4) - c^2 = (a - 2)^2 - c^2.$$

NOTE. — It will be observed that the term that is *not a perfect monomial square* furnishes a *key* to the *grouping*.

Factor, and test each result:

17. $a^2 - 2ax + x^2 - n^2$.

18. $b^2 + 2by + y^2 - n^2$.

19. $1 - 4q + 4q^2 - a^2$.

20. $r^2 - 2rx + x^2 - 16t^2$.

21. $c^2 - a^2 - b^2 - 2ab$.

22. $b^2 - x^2 - y^2 + 2xy$.

23. $4c^2 - x^2 - y^2 - 2xy$.

24. $9a^2 - 6ab + b^2 - 4c^2$.

153. The principle by which the difference of two squares is factored may be extended to expressions that may be *written* as the difference of two squares by *adding* and *subtracting* the *same monomial perfect square.*

EXERCISES

154. 1. Factor $a^4 + a^2b^2 + b^4$.

SUGGESTION. — Since $a^4 + a^2b^2 + b^4$ lacks $+a^2b^2$ of being a perfect square, and since the value of the polynomial will not be changed by adding a^2b^2 and also subtracting a^2b^2, the polynomial may be written

$$a^4 + 2a^2b^2 + b^4 - a^2b^2, \text{ or } (a^2 + b^2)^2 - a^2b^2.$$

2. Factor $x^4 - 21x^2 + 36$.

SUGGESTION. — This expression has two square terms, but in order that it shall be a perfect trinomial square it must fulfill another condition, namely (§ 140), the other term must be numerically equal to twice the product of the square roots of the terms that are squares; that is, the middle term must be either $+12x^2$ or $-12x^2$.

Hence, the number to be added and subtracted is either $33x^2$ or $9x^2$, but the former will not give the difference of two squares, for $33x^2$ is not a perfect square; then, $9x^2$ is the number to be added and subtracted, giving

$$x^4 - 12x^2 + 36 - 9x^2, \text{ or } (x^2 - 6)^2 - 9x^2.$$

Separate into prime factors, and test each result:

3. $x^4 + x^2y^2 + y^4$.
4. $a^8 + a^4b^4 + b^8$.
5. $9x^4 + 20x^2y^2 + 16y^4$.
6. $4a^4 + 11a^2b^2 + 9b^4$.
7. $16a^4 - 17a^2x^2 + x^4$.
8. $x^4 + x^2 + 1$.
9. $n^8 + n^4 + 1$.
10. $16x^4 + 4x^2y^2 + y^4$.
11. $25a^4 - 14a^2b^4 + b^8$.
12. $9a^4 + 26a^2b^2 + 25b^4$.

13. Factor $a^4 + 4$.

SUGGESTION. $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2 = (a^2 + 2)^2 - 4a^2$.

Factor completely, and test each result:

14. $a^4 + 4b^4$.
15. $m^4 + 64$.
16. $x^8 - 16$.
17. $4a^4 + 81$.
18. $m^5 + 4mn^4$.
19. $x^5y^2 + 4xy^2$.

155. The method of factoring by grouping to show a common polynomial factor applied to cases already solved by other methods.

The student has learned how to factor several type forms by special methods. He will now see how many of these forms may be factored by the method of § 145, which is of importance because its application is so general.

EXERCISES

156. *The forms* $a^2 + 2ab + b^2$ *and* $a^2 - 2ab + b^2$.

1. Factor $x^2 + 2xy + y^2$; also $9a^2m^2 - 6am + 1$.

SOLUTION

$$\begin{aligned} &x^2 + 2xy + y^2 \\ &= x^2 + xy + xy + y^2 \\ &= x(x + y) + y(x + y) \\ &= (x + y)(x + y). \end{aligned}$$

SOLUTION

$$\begin{aligned} &9a^2m^2 - 6am + 1 \\ &= 9a^2m^2 - 3am - 3am + 1. \\ &= 3am(3am - 1) - (3am - 1) \\ &= (3am - 1)(3am - 1). \end{aligned}$$

Factor by separating and grouping; test each result:

2. $l^2 + 14l + 49$.
3. $r^2 - 18r + 81$.
4. $4x^2 + 12xy + 9y^2$.
5. $16a^2 - 24ab + 9b^2$.

The form $a^2 - b^2$.

6. Factor $x^2 - y^2$; also $9r^2 - 4s^2$.

SOLUTION

$$\begin{aligned} & x^2 - y^2 \\ &= x^2 - xy + xy - y^2 \\ &= x(x-y) + y(x-y) \\ &= (x+y)(x-y). \end{aligned}$$

SOLUTION

$$\begin{aligned} & 9r^2 - 4s^2 \\ &= 9r^2 - 6rs + 6rs - 4s^2 \\ &= 3r(3r-2s) + 2s(3r-2s) \\ &= (3r+2s)(3r-2s). \end{aligned}$$

Factor by grouping; test each result:

7. $a^2 - z^2$.

8. $x^2 - 4$.

9. $n^2 - 1$.

10. $9x^2 - 25$.

11. $36z^2 - 25v^2$.

12. $49n^2 - 100l^2$.

The form $x^2 + px + q$.

13. Factor $x^2 + 8x + 15$; also $x^2 - 2x - 15$.

SOLUTION

$$\begin{aligned} & x^2 + 8x + 15 \\ &= x^2 + 5x + 3x + 15 \\ &= x(x+5) + 3(x+5) \\ &= (x+3)(x+5). \end{aligned}$$

SOLUTION

$$\begin{aligned} & x^2 - 2x - 15 \\ &= x^2 - 5x + 3x - 15 \\ &= x(x-5) + 3(x-5) \\ &= (x+3)(x-5). \end{aligned}$$

Factor by separating and grouping; test each result:

14. $x^2 + 12x + 20$.

15. $m^2 + 9m + 18$.

16. $y^2 + 8y - 20$.

17. $n^2 - 5n - 14$.

The form $ax^2 + bx + c$.

18. Factor $2x^2 + 11x + 12$; also $2x^2 + x - 15$.

SOLUTION

$$\begin{aligned} & 2x^2 + 11x + 12 \\ &= 2x^2 + 8x + 3x + 12 \\ &= 2x(x+4) + 3(x+4) \\ &= (2x+3)(x+4). \end{aligned}$$

SOLUTION

$$\begin{aligned} & 2x^2 + x - 15 \\ &= 2x^2 + 6x - 5x - 15 \\ &= 2x(x+3) - 5(x+3) \\ &= (2x-5)(x+3). \end{aligned}$$

Factor by separating and grouping; test each result:

19. $2x^2 + 3x + 1$.

20. $9y^2 + 21y + 10$.

21. $3m^2 + 5m - 22$.

22. $10r^2 - 3rs - 18s^2$.

Factor by separating and grouping:

23. $12n^2 + 31n + 9$.

24. $5z^2 + 48z - 77$.

25. $20x^2 + 13xy - 15y^2$.

26. $14a^2 - 23ab - 30b^2$.

REVIEW OF FACTORING

157. Summary of Cases. — In the previous pages the student has learned to factor expressions of the following types:

Monomial Factors

I. **Of monomials; as a^2b^2c.** (§ 129)

II. **Of expressions whose terms have a common factor; as**
$nx + ny + nz$. (§ 133)

Binomials

III. **Difference of two squares; as**
$a^2 - b^2$. (§§ 134, 151, 155)

IV. **Sum or difference of two cubes; as**
$a^3 + b^3$ or $a^3 - b^3$. (§ 136)

V. **Sum or difference of same odd powers; as**
$a^n + b^n$ or $a^n - b^n$ (when n is odd). (§ 137)

Trinomials

VI. **That are perfect squares; as**
$a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$. (§§ 138, 155)

VII. **Of the form** $x^2 + px + q$. (§§ 142, 155)

VIII. **Of the form** $ax^2 + bx + c$. (§§ 144, 155)

Polynomials

IX. **Whose terms may be grouped to show a common polynomial factor; as**
$ax + ay + bx + by$. (§§ 145, 155)

X. **Having binomial factors** (Factor Theorem). (§ 146)

158. General Directions for Factoring Polynomials.—1. *Remove monomial factors if there are any.*

2. *Then endeavor to bring the polynomial under some one of the cases II–IX.*

3. *When other methods fail, try the factor theorem.*

4. *Resolve into prime factors.*

Each factor should be divided out of the given expression as soon as found in order to simplify the discovery of the remaining factors.

EXERCISES

159. Factor, and test each result:

1. $y^4 - 1$.
2. $1 - x^8$.
3. $x^{10} - 1$.
4. $x^6 - 1$.
5. $a - a^7$.
6. $b^7 + b$.
7. $p^4 + 4$.
8. $1 + x^{12}$.
9. $y - a^4 y$.
10. $x^2 y - y^3$.
11. $a^{13} - ab^{12}$.
12. $64 - 2y^5$.
13. $7n^7 - 7n$.
14. $4x^4 - 4x$.
15. $8 - 27a^3x^3$.
16. $32x - 2x^3$.
17. $6b^4 + 24$.
18. $a^5 + 27a^2$.
19. $b^2 - 196$.
20. $450 - 2a^2$.
21. $7y^4 - 175$.
22. $x^2 - xy - 132y^2$.
23. $ax^2 - 3ax - 4a$.
24. $x^3 + 5x^2 - 6x$.
25. $3x^2 + 30x + 27$.
26. $128a^2 - 250a^5$.
27. $6x^2 - 19x + 15$.
28. $x^{2n} + 2x^n y^p + y^{2p}$.
29. $7x^2 - 77xy - 84y^2$.
30. $y^2 - 25yx + 136x^2$.
31. $9x^2 - 24xy + 16y^2$.
32. $x^2 - ax - 72a^2$.
33. $n^2 - an - 90a^2$.
34. $a^2b^2 + ab - 56$.
35. $a^{2n} - 2a^nb^n + b^{2n}$.
36. $25x^2 + 60xy + 36y^2$.
37. $6ax^2 + 5axy - 6ay^2$.
38. $169x^4 - 26ax^3 + a^2x^2$.
39. $a^4c^4 + a^2b^2c^2 + b^4$.
40. $16x^4 + 4x^2y^2 + y^4$.
41. $b^4c - 13b^2c + 42c$.

Factor, and test each result:

42. $9x^2 + 21x + 10$.

43. $5x^2 - 26xy + 5y^2$.

44. $y^2 + 16ay - 36a^2$.

45. $8a^2 - 21ab - 9b^2$.

46. $9x^2 - 15x - 50$.

47. $30x^2 - 37x - 77$.

48. $2x^3 + 28x^2 + 66x$.

49. $a^2 + b^2 - c^2 - 2ab$.

50. $ax^2 + 10ax - 39a$.

51. $n^4 + n^2a^2b^4 + a^4b^8$.

52. $a^2z^4 + a^2z^2 + a^2$.

53. $a^{2m} - 16a^m - 17$.

54. $a^2x^2 - 4ax + 3$.

55. $b^8 + b^4y^2 + y^4$.

56. $xy - 3y + x - 3y^2$.

57. $ax^2 - axy - ax + ay$.

58. $9c^2 - x^2 - y^2 + 2xy$.

59. $x^3 - a^2x - 4b^2x - 4abx$.

60. $bc^2 - 9a^2b - b^3 - 6ab^2$.

61. $ab^2 - 4a^3 - 12a^2c - 9ac^2$.

62. $x^2 - cx + 2dx - 2cd$.

63. $x^3y + 4x^2y - 31xy - 70y$.

64. $x^2 - 3ax + 4bx - 12ab$.

65. $ax^3 - 9ax^2 + 26ax - 24a$.

66. $12ax - 8bx - 9ay + 6by$.

67. $25x^2 - 9y^2 - 24yz - 16z^2$.

68. $2b^2t - 3ab^2 + 2btx - 3abx$.

69. $x^3y + 14x^2y + 43xy + 30y$.

70. $x^3y - 15x^2y + 38xy - 24y$.

71. $abx^3 + 3abx^2 - abx - 3ab$.

72. $3bmx + 2bm - 3anx - 2an$.

73. $20ax^3 - 28ax^2 + 5a^2x - 7a^2$.

74. $(a + b)^3 - 1$.

75. $a^3 - 2a^2 + 1$.

76. $b^3 - 4b^2 + 8$.

77. $3x^6 + 96x$.

78. $8x^4 - 6x^2 - 35$.

79. $x^5 - x^2 - x^4 + x^3$.

80. $x^3 - xy - x^2y + y^2$.

81. $12x^3 + 3x^2 - 8x - 2$.

82. $2x^2 + 10x + ax + 5a$.

83. $m^3 + m^2 - mn - mn^2$.

84. Factor $16 + 5x - 11x^2$ by the factor theorem.

85. Factor $x^3 - 6bx^2 + 12b^2x - 8b^3$ by the factor theorem.

EQUATIONS SOLVED BY FACTORING

160. Equations thus far solved have been such as involved, when in simplest form, only the *first power* of the unknown number.

Such equations are called **simple equations.**

161. A valuable application of factoring is found in the solution of equations that involve the unknown number in powers higher than the first.

An equation that in simplest form involves the second, but no higher, power of the unknown number is called a **quadratic equation.**

Thus, $x^2 = 4$ and $x^2 + 2x + 1 = 0$ are quadratic equations; but $x^2 + 3x = x^2 + 4$ is a simple equation, for in its simplest form, $3x = 4$, it has only the first power of x.

162. An equation that contains a higher power of the unknown number than the second is called a **higher equation.**

163. To solve quadratic equations by factoring.

EXERCISES

1. Find the values of x that satisfy $x^2 + 1 = 10$.

SOLUTION

$$x^2 + 1 = 10. \quad (1)$$

Transposing so that all terms are in the first member and uniting terms,

$$x^2 - 9 = 0. \quad (2)$$

Factoring the first member, § 134,

$$(x - 3)(x + 3) = 0. \quad (3)$$

Since the product of the two factors is 0, one of them must equal 0; that is, the equation is satisfied for any value of x that will make either factor equal to 0.

If $x - 3 = 0$, $x = 3$; if $x + 3 = 0$, $x = -3$.

Hence, the values of x that satisfy (3) and therefore (1) are 3 or -3.

VERIFICATION. — When $x = 3$, (1) becomes $9 + 1 = 10$, or $10 = 10$.

When $x = -3$, (1) becomes $9 + 1 = 10$, or $10 = 10$.

Solve, and verify results:

2. $x^2 + 3 = 28.$

3. $x^2 + 1 = 50.$

4. $x^2 - 5 = 59.$

5. $x^2 - 7 = 29.$

6. $x^2 + 3 = 84.$

7. $x^2 - 24 = 120.$

8. $x^2 + 11 = 180.$

9. $x^2 - 11 = 110.$

10. Solve $x^2 + 4x = 45.$

SOLUTION

$$x^2 + 4x = 45.$$

Transposing, $x^2 + 4x - 45 = 0.$

Factoring, § 142, $(x - 5)(x + 9) = 0.$

Hence, $x - 5 = 0$ or $x + 9 = 0$;

whence, $x = 5$ or $-9.$

Solve, and verify results:

11. $x^2 - 6x = 40.$

12. $x^2 - 8x = 48.$

13. $x^2 - 5x = -4.$

14. $x^2 + 4x + 3 = 0.$

15. $r^2 + 6r + 8 = 0.$

16. $y^2 + 42 = 13y.$

17. $t^2 + 63 = 16t.$

18. $v^2 - 60 = 11v.$

19. $x^2 - 7x = 18.$

20. $x^2 + 10x = 56.$

21. Solve $6x^2 + 5x - 21 = 0.$

SOLUTION

$$6x^2 + 5x - 21 = 0.$$

Factoring, § 144, $(2x - 3)(3x + 7) = 0.$

Hence, $2x - 3 = 0$ or $3x + 7 = 0$;

whence, $x = \frac{3}{2}$ or $-\frac{7}{3}.$

Solve, and verify results:

22. $3x^2 + 2x - 1 = 0.$

23. $5x^2 + 4x - 1 = 0.$

24. $3y^2 + y - 10 = 0.$

25. $7x^2 + 6x - 1 = 0.$

26. $2x^2 + 9x - 18 = 0.$

27. $2v^2 - 9v - 35 = 0.$

28. $6y^2 - 22y + 20 = 0.$

29. $6z^2 - 11z - 21 = 0.$

30. $4x^2 - 15x + 14 = 0.$

31. $5x^2 - 48x - 20 = 0.$

Solve, and verify results:

32. $x^2 - 21 = 4.$

33. $x^2 - 56 = 8.$

34. $x^2 - 9x = 36.$

35. $x^2 + 11x = 26.$

36. $x^2 - 12x = 45.$

37. $y^2 - 15y = 54.$

38. $y^2 - 21y = 46.$

39. $3y^2 - 4y - 4 = 0.$

40. $4y^2 + 9y - 9 = 0.$

41. $32 - x^2 = 28.$

42. $65 - x^2 = 16.$

43. $x^2 + x - 132 = 0.$

44. $32 = 4w + w^2.$

45. $3s = 88 - s^2.$

46. $160 = x^2 - 6x.$

47. $4y = y^2 - 192.$

48. $3x^2 + 13x - 30 = 0.$

49. $4x^2 + 13x - 12 = 0.$

50. $(2x + 3)(2x - 5) - (3x - 1)(x - 2) = 1.$

51. $(2x - 6)(3x - 2) - (5x - 9)(x - 2) = 4.$

Other methods of solving quadratics will be given in §§ 336–348.

164. To solve higher equations by factoring.

Any higher equation may be solved by the method just given for quadratic equations, whenever the expression obtained by transposing all of its terms to one member is factorable.

EXERCISES

165. 1. Solve the equation $x^3 - 2x^2 - 5x + 6 = 0.$

SOLUTION

$$x^3 - 2x^2 - 5x + 6 = 0.$$

Factoring, § 146, $(x - 1)(x - 3)(x + 2) = 0.$

Hence, $x - 1 = 0$ or $x - 3 = 0$ or $x + 2 = 0$;

whence, $x = 1$ or 3 or $-2.$

2. $x^3 - 15x^2 + 71x - 105 = 0.$

3. $x^3 + 10x^2 + 11x - 70 = 0.$

4. $x^3 - 12x + 16 = 0.$

5. $x^3 - 19x - 30 = 0.$

6. $x^4 + x^3 - 21x^2 - x + 20 = 0.$

7. $x^4 - 7x^3 + x^2 + 63x - 90 = 0.$

8. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

Problems

166. 1. A sealing fleet carries 4000 men, and the number of men on each ship is 40 less than 8 times the number of ships. Find the number of ships and the number of men on each ship.

SOLUTION

Let $x =$ the number of ships in the fleet.

Then, $8x - 40 =$ the number of men on each ship,

and $x(8x - 40) =$ the total number of men with the fleet.

$\therefore x(8x - 40) = 4000.$

Expanding, dividing by 8, and transposing,

$$x^2 - 5x - 500 = 0.$$

Factoring, $(x - 25)(x + 20) = 0.$

Hence, $x - 25 = 0$ or $x + 20 = 0$;

whence, $x = 25$ or -20,

and $8x - 40 = 160$ or -200.

The second value of x and of $8x - 40$ is evidently inadmissible, since neither the number of ships nor the number of men on a ship can be negative.

Hence, there are 25 ships in the fleet, and 160 men on a ship.

Solve the following problems and verify (§ 125) each solution:

2. The gold mined in a recent year would fill a square room, the height of which is 1 foot less than its length. If the area of one wall is 90 square feet, find the dimensions of the room.

3. Certain wooden paving blocks are twice as long as they are wide and the thickness of each is 4 inches. Find the length and width, if the volume of each block is 128 cubic inches.

4. A rectangular swimming tank on board a ship is 3 times as long as it is wide. If it were divided into 3 square tanks, the area of each would be 225 square feet. Find the dimensions of the tank.

5. A man bought as many tons of crude borax as it is worth dollars a ton, and crude borax is worth $\frac{1}{7}$ as much as refined borax. If the same amount of refined borax would be worth \$ 2800, find the value of crude borax a ton.

6. A farmer keeps his chickens in a rectangular lot that is 4 times as long as it is wide. If its area is 2500 square feet, find its length and width.

7. At a luncheon with Menelik of Abyssinia there was a pile of bread containing 448 cubic feet. Its height was twice its width and its length was 14 feet. Find its height and width.

8. A large rectangular freight station in Atlanta, Georgia, covers an area of 41,750 square feet. If the length is 16.7 times the width, what is the width of the station?

9. An automobilist paid $3.60 for gasoline. If the number of cents he paid per gallon was 2 less than the number of gallons he bought, find how many gallons he bought and the price per gallon.

10. St. Louis has the largest steam whistle in the world. The number of times it is blown each day is 3 more than the number of dollars it costs to blow it once. How many times is it blown a day, if the cost for 12 days is $48?

11. In the Panama Canal Zone a washerwoman washed as many dozen pieces as she received dollars a dozen for her labor. If she had washed 2 dozen more, she would have received $15. How much did she receive a dozen?

12. The number of pounds of duck feathers that a man bought was the same as the number of cents that he paid a pound for them. If he had bought 10 pounds more, they would have cost $20. How many pounds did he buy?

13. The area of one of the largest photographic prints ever made is 180 square feet. Its dimensions are 18 times those of the picture from which it was enlarged. Find the dimensions of the picture, if its length is 2 inches greater than its width.

14. The cages holding canaries imported into this country are arranged in rows in crates. The number of rows in a crate is 2 less than 5 times the number of cages in a row. If there are 231 cages in a crate, how many rows are there?

FRACTIONS

167. In algebra, an indicated division is called a **fraction**.

The fraction $\frac{a}{b}$ means $a \div b$ and is read 'a divided by b.'

168. The *dividend*, written above a line, is the **numerator**; the *divisor*, written below the line, is the **denominator**; the numerator and denominator are called the **terms** of the fraction.

An arithmetical fraction also indicates division, but the arithmetical notion is that a fraction is one or more of the *equal* parts of a unit; that is, in arithmetic, the terms of a fraction are *positive integers*, while in algebra they may be *any numbers whatever*.

169. The student will find no difficulty with algebraic fractions, if he will bear in mind that they are essentially the same as the fractions he has met in arithmetic. He will have occasion to change fractions to higher or lower terms; to write integral and mixed expressions in fractional form; to change fractions to integers or mixed numbers; to add, subtract, multiply, and divide with algebraic fractions just as he has learned to do with arithmetical fractions, except that *signs* must be considered in dealing with positive and negative numbers.

Signs in Fractions

170. There are *three* signs to be considered in connection with a fraction; namely, the sign of the numerator, the sign of the denominator, and the sign written before the dividing line, called the **sign of the fraction**.

In $-\frac{x}{3z}$ the sign of the fraction is $-$, while the signs of its terms are $+$.

171. An expression like $\frac{-a}{-b}$ indicates a process in division, in which the quotient is to be found by dividing a by b and prefixing the sign according to the law of signs in division; that is,

$$\frac{-a}{-b}=+\frac{a}{b}, \qquad \frac{+a}{+b}=+\frac{a}{b},$$

$$\frac{-a}{+b}=-\frac{a}{b}, \qquad \frac{+a}{-b}=-\frac{a}{b}.$$

By observing the above fractions and their values the following principles may be deduced:

172. PRINCIPLES. — 1. *The signs of both the numerator and the denominator of a fraction may be changed without changing the sign of the fraction.*

2. *The sign of either the numerator or the denominator of a fraction may be changed, provided the sign of the fraction is changed.*

When either the numerator or the denominator is a polynomial, its sign is changed by changing the sign of each of its terms. Thus, the sign of $a-b$ is changed by writing it $-a+b$, or $b-a$.

EXERCISES

173. Reduce to fractions having positive numbers in both terms:

1. $\frac{-3}{-4}$.
2. $\frac{2}{-5}$.
3. $\frac{+a-x}{2x}$.
4. $\frac{-4c}{-b-y}$.
5. $-\frac{-a-b}{c+d}$.
6. $-\frac{-2}{-a-y}$.
7. $-\frac{-2-m}{2+n}$.
8. $\frac{-4(a+b)}{5(-x-y)}$.

174. In accordance with § 80,

PRINCIPLES. — 3. *The sign of either term of a fraction is changed by changing the signs of an odd number of its factors.*

4. *The sign of either term of a fraction is not changed by changing the signs of an even number of its factors.*

EXERCISES

175. 1. Show that $\dfrac{(a-b)(d-c)}{(c-a)(b-c)}=\dfrac{(a-b)(c-d)}{(a-c)(b-c)}$.

Solution or Proof

Changing $(d-c)$ to $(c-d)$ changes the sign of *one* factor of the numerator and therefore changes the sign of the numerator (Prin. 3).

Similarly, changing $(c-a)$ to $(a-c)$ changes the sign of the denominator (Prin. 3).

We have changed the signs of both terms of the fraction. Therefore, the sign of the fraction is not affected (Prin. 1).

2. Show that $\dfrac{(b-a)(d-c)}{(c-b)(a-c)}=-\dfrac{(a-b)(c-d)}{(b-c)(a-c)}$.

Solution or Proof

Changing the signs of *two* factors of the numerator does not change the sign of the numerator (Prin. 4).

Changing the sign of *one* factor of the denominator changes the sign of the denominator (Prin. 3).

Since we have changed the sign of only *one* term of the fraction, we must change the sign of the fraction (Prin. 2).

3. Show that $\dfrac{-b}{b-a}$ may be properly changed to $\dfrac{b}{a-b}$.

4. From $\dfrac{-a}{b-a+c}$ derive $\dfrac{a}{a-b-c}$ by proper steps.

5. Prove that $\dfrac{3}{1-x}=-\dfrac{3}{x-1}$; that $-\dfrac{2}{4-x^2}=\dfrac{2}{x^2-4}$.

6. Prove that $\dfrac{1}{(b-a)(c-b)}=\dfrac{1}{(a-b)(b-c)}$.

7. Prove that $\dfrac{(m-n)(m+n)}{(a-c)(b-a)}=\dfrac{-m^2+n^2}{(a-c)(a-b)}$.

8. Prove that $\dfrac{(a-b)(b-a+c)}{(y-x)(z-y)(z-x)}=\dfrac{(a-b)(a-b-c)}{(x-y)(y-z)(x-z)}$.

REDUCTION OF FRACTIONS

176. The process of changing the form of an expression without changing its value is called **reduction**.

177. An expression, some of whose terms are integral and some fractional, is called a **mixed number,** or a **mixed expression.**

Thus, $a - \frac{a-b}{c}$, $\frac{x^2}{a^2} - 2 + \frac{a^2}{x^2}$, and $a - b + \frac{1}{ab}$ are mixed expressions.

178. To reduce a fraction to an integer or a mixed expression.

This change in form is made in algebra in the same manner as in arithmetic.

EXERCISES

179. 1. Reduce to a mixed number: $\frac{13}{4}$; $\frac{ax+b}{x}$.

PROCESS

$$\frac{13}{4} = 13 \div 4 = 3 + \frac{1}{4} = 3\tfrac{1}{4}.$$

PROCESS

$$\frac{ax+b}{x} = (ax+b) \div x = a + \frac{b}{x}.$$

EXPLANATION. — Since a fraction is an indicated division, by performing the division indicated the fraction is changed into the form of a mixed number.

Reduce to an integral or a mixed expression:

2. $\frac{27\,a}{9}$.

3. $\frac{26\,b}{13\,b}$.

4. $\frac{45\,x^2y^3z}{15\,xy}$.

5. $\frac{36\,ac + 9\,c}{9\,c}$.

6. $\frac{a^2 + c^2}{a}$.

7. $\frac{12\,x^2 - 6\,x}{6\,x}$.

8. Reduce $\frac{a^3 - 3\,a^2 - a + 1}{a^2}$ to a mixed number.

SOLUTION. $\frac{a^3 - 3\,a^2 - a + 1}{a^2} = a - 3 + \frac{-a+1}{a^2} = a - 3 - \frac{a-1}{a^2}$.

NOTE. — It is not necessary to write the step $(a^3 - 3\,a^2 - a + 1) \div a^2$. The division should be continued until the undivided part of the numerator no longer contains the denominator.

Reduce to an integral or a mixed expression:

9. $\frac{4x^3 - 8x^2 + 2x - 1}{2x}$.

10. $\frac{ab - bc - cd + d^2}{b}$.

11. $\frac{a^2x^3 - ax^2 - x - 1}{ax}$.

12. $\frac{x^2 - x - 12}{x - 4}$.

13. $\frac{x^3 - 2xy - y^3}{x - y}$.

14. $\frac{x^2 - 6xy + 4y^2}{2xy}$.

15. $\frac{x^3 - 6x^2 + 14x - 9}{x - 2}$.

16. $\frac{x^4 - 3x^2 + 5x - 1}{x - 3}$.

17. $\frac{x^3 + 5x^2 + 3x - 6}{x + 2}$.

18. $\frac{x^4 + 2x^3 - x^2 + 5}{x^3 + x^2}$.

19. $\frac{a^3 + 9a^2 + 24a + 18}{a + 3}$.

20. $\frac{4a^5 + 12a^3 - a^2 + 34}{2a^3 + 5}$.

21. $\frac{a^2 + 3ab - 5b^2 + bc}{a - 2b}$.

22. $\frac{x^3 - 7x^2 - 4x + 40}{x^2 - 3}$.

23. $\frac{a^3 + 3a^2b - ab^2 + ab}{a^2 + b}$.

24. $\frac{x^2 - xy - 3y^2 - z}{x + y}$.

25. $\frac{x^{3n} - 3x^{2n} + 5x^n - 3}{x^n - 1}$.

26. $\frac{l^3 - l^2t - 7lt^2 + 3t^3}{l - 3t}$.

27. $\frac{x^2 + x^2y - y^2 - xy^2 - 3}{x - y}$.

28. $\frac{3x^2y^2 - 12x^2 + 2y^2 - 3}{3x^2 + 2}$.

180. To reduce a fraction to its lowest terms.

Just as $\frac{3}{4} = \frac{9}{12}$ or $\frac{9}{12} = \frac{3}{4}$,

so $\frac{a}{b} = \frac{am}{bm}$ or $\frac{am}{bm} = \frac{a}{b}$. That is,

181. PRINCIPLE. — *Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction.*

182. A fraction is in its **lowest terms** when its terms have no *common* factor.

183. The product of all the common prime factors of two or more numbers is called their **highest common factor (h. c. f.)**.

Thus, to find the h. c. f. of two expressions, as $9ax^2y + 3bx^2y$ and $6x^4y^2 - 6x^2y^4$, we first factor them. Since $9ax^2y + 3bx^2y = 3 \cdot x^2 \cdot y(3a+b)$ and $6x^4y^2 - 6x^2y^4 = 2 \cdot 3 \cdot x^2 \cdot y^2(x+y)(x-y)$, their *common* factors are 3, x, x, and y; hence, their h. c. f. $= 3x^2y$. (See also §§ 407–411.)

NOTE. — The number of *literal* factors in a term determines its **degree**, and the term of an expression that has the greatest number of literal factors determines the *degree of the expression*. Thus, $2a^2b$ is of the *third degree* and is *higher* than $5ab$, which is of the *second degree*; the expression $2a^2b + 5ab$, then, is of the *third degree*.

EXERCISES

184. 1. Reduce to lowest terms: $\frac{20}{24}$; $\frac{21a^2x^2y}{30a^3xz}$.

PROCESS

$$\frac{20}{24} = \frac{2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 6} = \frac{5}{6}$$

PROCESS

$$\frac{21a^2x^2y}{30a^3xz} = \frac{3 \cdot 7a^2x^2y}{3 \cdot 10a^3xz} = \frac{7xy}{10az}$$

EXPLANATION. — Since a fraction is in its lowest terms when its terms have no common factor, a fraction may be reduced to its lowest terms by removing in succession all common factors of its numerator and denominator; or by dividing the terms by their *highest common factor*.

Reduce to lowest terms:

2. $\frac{a^2xy^2}{a^3xy}$.

3. $\frac{m^3n^3}{am^2n^4}$.

4. $\frac{a^2b^2x^2}{b^3xy^2}$.

5. $\frac{16m^2nx^2z^2}{40am^3yz^3}$.

6. $\frac{210bc^2d}{750ab^2c}$.

7. $\frac{-25x^2y^5z^2}{-100x^4y^3}$.

8. $\frac{-7a^2bcd^3}{42ab^2cd^4}$.

9. $\frac{tr^2 + ts^2}{3t}$.

10. $\frac{2x}{4x^2 - 6ax}$.

11. Reduce $\frac{bx - ax}{a^2 - b^2}$ to its lowest terms.

SOLUTION. $$\frac{bx - ax}{a^2 - b^2} = \frac{x(b-a)}{(a+b)(a-b)}$$

§ 174, Prin. 4, $$= \frac{-x(a-b)}{(a+b)(a-b)} = \frac{-x}{a+b} = -\frac{x}{a+b}.$$

Reduce to lowest terms:

12. $\dfrac{a^2-b^2}{a^2+2\,ab+b^2}$.

13. $\dfrac{a^2-2\,ab+b^2}{b^2-a^2}$.

14. $\dfrac{4\,a^2-9\,x^2}{8\,a^3+27\,x^3}$.

15. $\dfrac{6\,xy-3\,x^2y}{x^4y-8\,xy}$.

16. $\dfrac{3\,a^2b-3\,b^3}{2\,b^4-2\,a^3b}$.

17. $\dfrac{x^4y+x^2y^3+y^5}{x^6-y^6}$.

18. $\dfrac{x^4y-x^2y^3+y^5}{x^6+y^6}$.

19. $\dfrac{x^3-6\,x^2+5\,x}{x^3+2\,x^2-35\,x}$.

20. $\dfrac{7\,x-2\,x^2-3}{4-7\,x-2\,x^2}$.

21. $\dfrac{x^2-2\,x^4+x^6}{x^6-x^2}$.

22. $\dfrac{a^3+2\,a^2b+ab^2}{a^5-2\,a^3b^2+ab^4}$.

23. $\dfrac{x^3+5\,x^2-6\,x}{2\,x^2-2}$.

24. $\dfrac{x^3-7\,x+6}{x^4-10\,x^2+9}$.

25. $\dfrac{20-21\,x+x^3}{x^4-26\,x^2+25}$.

26. $\dfrac{3\,a^2+4\,ax-4\,x^2}{9\,a^2-12\,ax+4\,x^2}$.

27. $\dfrac{x^3+1+3\,x^2+3\,x}{4+4\,x-x^2-x^3}$.

28. $\dfrac{m-m^2-n+mn}{m-mn+n^2-n}$.

29. $\dfrac{am-an-m+n}{am-an+m-n}$.

30. $\dfrac{a^3-b^3-3\,a^2b+3\,ab^2}{3\,ab^2-3\,a^2b}$.

31. $\dfrac{x^3+5\,x^2-9\,x-45}{x^3+3\,x^2-25\,x-75}$.

32. $\dfrac{2\,ax-ay-4\,bx+2\,by}{4\,ax-2\,ay-2\,bx+by}$.

33. $\dfrac{x^3+2\,x^2-23\,x-60}{x^3-11\,x^2-10\,x+200}$.

185. To reduce a fraction to an equal fraction having a given denominator or a given numerator.

In order to change $\frac{3}{4}$ to a fraction whose denominator is 12, both terms must be multiplied by $12 \div 4$, or 3; similarly, to change $\dfrac{x}{z}$ to a fraction whose denominator is nz^2, both terms must be multiplied by $nz^2 \div z$, or nz.

EXERCISES

186. 1. Reduce $\frac{a}{a+b}$ to a fraction whose denominator is $a^2 - b^2$.

PROCESS

$$(a^2 - b^2) \div (a + b) = a - b.$$

Then, $$\frac{a}{a+b} = \frac{a(a-b)}{(a+b)(a-b)} = \frac{a^2 - ab}{a^2 - b^2}.$$

EXPLANATION. — Since the required denominator is $(a - b)$ times the given denominator, in order that the value of the fraction shall not be changed (§ 181) both terms of the fraction must be multiplied by $(a - b)$.

2. Reduce $\frac{5a}{6}$ to a fraction whose denominator is 42.

3. Reduce $\frac{3x}{11b}$ to a fraction whose denominator is $55b$.

4. Reduce $\frac{3a}{14x}$ to a fraction whose denominator is $84xy$.

5. Reduce $\frac{4a^2}{5y}$ to a fraction whose denominator is $20y^3$.

6. Reduce $\frac{x-3}{x-1}$ to a fraction whose denominator is $(x-1)^2$.

7. Reduce $\frac{2x-5}{2x+5}$ to a fraction whose denominator is $(2x+5)^2$.

8. Reduce $\frac{a}{3-a}$ to a fraction whose numerator is $3a + a^2$.

9. Reduce $\frac{x-y}{2x+y}$ to a fraction whose numerator is $x^2 - y^2$.

10. Reduce $\frac{-1}{x-2}$ to a fraction whose denominator is $4 - x^2$.

11. Reduce $\frac{ab}{3-b}$ to a fraction whose denominator is $b^2 - 9$.

12. Reduce $x - 5$ to a fraction whose denominator is $x + 5$.

13. Reduce $3t + 2l$ to a fraction whose denominator is $2l - 3t$.

187. Reduction to lowest common denominator.

In algebra, as in arithmetic, it is frequently desirable to reduce fractions that have different denominators to respectively equal fractions that have a *common* denominator.

188. In algebra, **lowest common denominator** corresponds to *least common denominator* in arithmetic.

The word 'lowest' has reference to the *degree* of the denominator.

189. It is not always easy to discover by inspection the *lowest common denominator* (l. c. d.), that is, the *lowest common multiple* (l. c. m.) of the given denominators. However, it may be found, as in arithmetic, by factoring the denominators, for it is the product of all their *different* prime factors, each factor used the greatest number of times that it occurs in any denominator. (See also §§ 412–414.)

Thus, if the given denominators are $ax - bx$, $a^2 - b^2$, and $a^2 - 2ab + b^2$, on factoring we find: $ax - bx = x(a - b)$; $a^2 - b^2 = (a + b)(a - b)$; and $a^2 - 2ab + b^2 = (a - b)(a - b)$.

Then, the factors of the l. c. d. are x, $a + b$, $a - b$, and $a - b$.

Hence, the l. c. d. $= x(a + b)(a - b)^2$.

EXERCISES

190. 1. Reduce to fractions having their lowest common denominator: $\frac{5}{6}$ and $\frac{3}{8}$; $\frac{a}{3bc}$ and $\frac{c}{6ab}$.

PROCESS

$$\text{l. c. d.} = 24$$

$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24}$$

$$\frac{3}{8} = \frac{3 \times 3}{8 \times 3} = \frac{9}{24}$$

PROCESS

$$\text{l. c. d.} = 6abc$$

$$\frac{a}{3bc} = \frac{a \times 2a}{3bc \times 2a} = \frac{2a^2}{6abc}$$

$$\frac{c}{6ab} = \frac{c \times c}{6ab \times c} = \frac{c^2}{6abc}$$

EXPLANATION. — The l. c. m. of the given denominators is found for the l. c. d. in accordance with § 189. Then, each fraction is reduced to an equal fraction having this denominator, as in § 185.

NOTE. — All fractions should first be reduced to lowest terms.

2. Reduce $2m$ and $\frac{m+n}{m-n}$ to fractions having their lowest common denominator.

Suggestion. — First write $2m$ as a fraction with the denominator 1.

Reduce to fractions having their lowest common denominator:

3. $\frac{x}{2}$ and $\frac{3y}{5}$.

4. $\frac{2a}{5b}$ and $3x$.

5. $\frac{a^2b}{c^2d}$ and $\frac{ab^2}{cd^2}$.

6. $\frac{x^2}{2xy}$ and $\frac{3}{4ay}$.

7. $\frac{3xy}{cx}$ and $\frac{2ay}{3by}$.

8. $\frac{3}{x^2y^4}, \frac{-6}{x^3y^3}, \frac{3}{x^4y^2}$.

9. $\frac{3ab}{8a^2c}, \frac{7a^2}{4b^2c}, \frac{5a^3}{a^3bc^2}$.

10. $\frac{m-n}{a}, 2, \frac{a}{m+n}$.

11. $\frac{x+y}{2}, \frac{x-y}{4}, \frac{x^2-y^2}{6}$.

12. $\frac{x^2}{x^2-1}, \frac{x}{x+1}, \frac{x}{x-1}$.

13. $\frac{a^3}{a^4-16}, \frac{a}{a^2+4}, \frac{2a}{4-a^2}$.

Suggestion. — By § 172, Prin. 1, $\frac{2a}{4-a^2} = \frac{-2a}{a^2-4}$.

14. $\frac{4a}{b-a}, \frac{3b}{a+b}, \frac{1}{a^2-b^2}$.

15. $\frac{a}{1-ax}, \frac{x}{1+ax}, \frac{-ax}{ax-1}$.

16. $\frac{1}{x^2+7x+10}, \frac{1}{x^2+x-2}, \frac{1}{x^2+4x-5}$.

17. $\frac{3x}{x^2-3x+2}, \frac{x-1}{x^2-5x+6}, \frac{x+3}{x^2-4x+3}$.

18. $\frac{a+5}{a^2-4a+3}, \frac{a-2}{a^2-8a+15}, \frac{a+1}{a^2-6a+5}$.

ADDITION AND SUBTRACTION OF FRACTIONS

191. The method of adding and subtracting fractions is the same in algebra as in arithmetic. In algebra, however, subtraction of fractions practically reduces to addition of fractions, for every fraction to be subtracted is really added with its sign changed (§ 64, Prin.).

Just as $$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{4+3}{12},$$

so $$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

Also as $$\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{4-3}{12},$$

so $$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

EXERCISES

192. 1. Add $\frac{3x}{4}$, $\frac{7x}{10}$, and $\frac{5z}{12}$.

Solution. — Since the fractions have unlike denominators, they must be reduced to fractions having a common denominator. By § 189, the l. c. d. = 60.

$$\frac{3x}{4}+\frac{7x}{10}+\frac{5z}{12}=\frac{45x}{60}+\frac{42x}{60}+\frac{25z}{60}$$
$$=\frac{45x+42x+25z}{60}=\frac{87x+25z}{60}.$$

2. Subtract $\frac{x-2}{7}$ from $\frac{5x-1}{8}+\frac{x}{4}$.

Solution

$$\frac{5x-1}{8}+\frac{x}{4}-\frac{x-2}{7}=\frac{35x-7}{56}+\frac{14x}{56}-\frac{8x-16}{56}$$
$$=\frac{35x-7+14x-(8x-16)}{56}$$
$$=\frac{35x-7+14x-8x+16}{56}=\frac{41x+9}{56}.$$

Suggestion. — When a fraction is preceded by the sign −, it is well for the beginner to inclose the numerator in a parenthesis, if it is a polynomial, as shown above.

RULE. — *Reduce the fractions to similar fractions having their lowest common denominator.*

Change the signs of all the terms of the numerators of fractions preceded by the sign —, then find the sum of all the numerators, and write it over the common denominator.

Reduce the resulting fraction to its lowest terms, if necessary.

Add:

3. $\frac{2x}{5}$ and $\frac{3x}{2}$.

4. $\frac{4a}{3}$ and $\frac{6b}{5}$.

5. $\frac{2a}{3b}$ and $\frac{3a}{2b}$.

6. $\frac{-5}{7x}$ and $\frac{-2}{3x}$.

Subtract:

7. $\frac{5m}{6}$ from $\frac{4m}{3}$.

8. $\frac{4x}{9}$ from $-\frac{x}{2}$.

9. $-\frac{y}{3}$ from $\frac{3}{y}$.

10. $\frac{a+b}{3}$ from $\frac{a-b}{2}$.

Simplify:

11. $\frac{2x+1}{3}+\frac{x-2}{4}-\frac{x-3}{6}+\frac{5-x}{2}$.

12. $\frac{x-2}{6}-\frac{x-4}{9}+\frac{2-3x}{4}-\frac{2x+1}{12}$.

13. $\frac{x-1}{3}-\frac{x-2}{18}-\frac{4x-3}{27}+\frac{1-x}{6}$.

14. $\frac{2-6x}{5}+\frac{4x-1}{2}-\frac{5x-3}{6}-\frac{1-x}{3}$.

15. $\frac{x+3}{4}-\frac{x-2}{5}+\frac{x-4}{10}-\frac{x+3}{6}$.

16. $\frac{1-2a}{5}+\frac{2a-1}{4}-\frac{2a-a^2+1}{8}$.

17. $\frac{3+x-x^2}{4}-\frac{1-x+x^2}{6}-\frac{1-2x-2x^2}{3}$.

18. Reduce $\frac{5a^2+b^2}{a^2-b^2}-2$ to a fraction.

SOLUTION

$$\frac{5a^2+b^2}{a^2-b^2}-\frac{2}{1}=\frac{5a^2+b^2-2(a^2-b^2)}{a^2-b^2}$$

$$=\frac{5a^2+b^2-2a^2+2b^2}{a^2-b^2}$$

$$=\frac{3(a^2+b^2)}{a^2-b^2}.$$

Reduce the following **mixed expressions** to fractions:

19. $a+\frac{b}{2}$.

20. $x-\frac{y}{2}$.

21. $\frac{a^2-c^2}{c}+5c$.

22. $\frac{1-x}{3}-4x$.

23. $a-\frac{a^2-ab}{b}$.

24. $a-\frac{a-b-c}{2}$.

25. $a+x-\frac{x^2}{a-x}$.

26. $a^2-ab+b^2-\frac{b^3}{a+b}$.

Perform the additions and subtractions indicated:

27. $\frac{a-b}{ab}+\frac{b-c}{bc}$.

28. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

29. $x+y-\frac{x^2+y^2}{x-y}$.

30. $\frac{x}{x-2}-\frac{x-2}{x+2}$.

31. $x+1+\frac{x^3-3}{x-1}$.

32. $m-\frac{m^2+n^2}{m-n}+n$.

33. $\frac{1}{x}+1+\frac{2x}{1+x}-2$.

34. $3a-2x-\frac{8a^2-4x^2}{3a+2x}$.

35. $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2}$.

36. $\frac{1}{a+b}-\frac{1}{a-b}+\frac{2a}{a^2-b^2}$.

37. $\frac{a+x}{a-x}+\frac{a-x}{a+x}+\frac{4ax}{a^2-x^2}$.

38. $3x+\frac{5}{ax}-\left(2x+\frac{3}{ax}\right)$.

39. $\frac{a}{a-2}-\frac{a-2}{a+2}+\frac{3}{4-a^2}$.

SUGGESTION. — By § 172, Prin. 1, $\frac{3}{4-a^2}=\frac{-3}{a^2-4}$.

40. $\frac{a+1}{a-1}+\frac{2}{a+1}+\frac{4a}{1-a^2}$.

41. $\frac{5x+2}{x^2-4}+\frac{2}{x-2}-\frac{3}{2-x}$.

42. $\frac{x(a+x)}{a-x}-\frac{3ax-x^2}{x-a}+4a$.

43. $\frac{a}{a-b}-\frac{a}{a+b}-\frac{2ab}{a^2+b^2}-\frac{4ab^3}{a^4+b^4}$.

SUGGESTION. — Combine the first two fractions, then the result and the third fraction, then this result and the fourth fraction.

44. $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2+b^2}+\frac{8ab^3}{a^4+b^4}$.

45. $\frac{1}{a-b}-\frac{1}{a+b}-\frac{2b}{a^2+b^2}+\frac{2b^3}{a^4+b^4}$.

46. $\frac{x+y}{(y-z)(z-x)}-\frac{y+z}{(x-z)(x-y)}+\frac{z+x}{(y-x)(z-y)}$.

SOLUTION

$$\begin{aligned}\text{Sum} &= \frac{x+y}{(y-z)(z-x)}+\frac{y+z}{(z-x)(x-y)}+\frac{z+x}{(x-y)(y-z)}\\ &= \frac{(x^2-y^2)+(y^2-z^2)+(z^2-x^2)}{(x-y)(y-z)(z-x)}\\ &= \frac{0}{(x-y)(y-z)(z-x)}=0.\end{aligned}$$

47. $\frac{1}{(b-c)(a-c)}+\frac{1}{(c-a)(a-b)}+\frac{1}{(b-a)(b-c)}$.

48. $\frac{a+1}{(a-b)(a-c)}+\frac{b+1}{(b-c)(b-a)}+\frac{c+1}{(a-c)(b-c)}$.

MULTIPLICATION OF FRACTIONS

193. Fractions are multiplied in algebra just as they are in arithmetic.

Thus, $$\frac{3}{4}\times\frac{5}{2}=\frac{3\times 5}{4\times 2}.$$

In general, $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$ That is,

PRINCIPLE: — *The product of two or more fractions is equal to the product of their numerators divided by the product of their denominators.*

EXERCISES

194. 1. Multiply $\dfrac{x-5}{x+5}$ by x^2-25.

SOLUTION

$$\frac{x-5}{\cancel{x+5}}\cdot\frac{\overset{x-5}{\cancel{x^2-25}}}{1}=(x-5)^2=x^2-10x+25.$$

2. Multiply $\dfrac{x+3}{x+2}$ by $1+\dfrac{1}{x+1}$.

SOLUTION

$$\left(\frac{x+3}{x+2}\right)\left(1+\frac{1}{x+1}\right)=\frac{x+3}{x+2}\left(\frac{x+1}{x+1}+\frac{1}{x+1}\right)$$
$$=\frac{x+3}{\cancel{x+2}}\cdot\frac{\cancel{x+2}}{x+1}$$
$$=\frac{x+3}{x+1}.$$

GENERAL SUGGESTIONS. — 1. Any integer may be written with the denominator 1.

2. After finding the product of the numerators and the product of the denominators the resulting fraction may be reduced to lowest terms, in many cases, by canceling common factors from numerator and denominator. It is, however, more convenient to remove the common factors before performing the multiplications.

3. Generally, mixed numbers should be reduced to fractions.

Multiply:

3. $\frac{2}{3}$ by $\frac{5}{6}$.

4. $\frac{3}{4}$ by $\frac{4}{15}$.

5. $\frac{3ab}{4xy}$ by $\frac{2y}{3a^2}$.

6. $\frac{5xy}{2ac}$ by $\frac{3ax}{10y^2}$.

7. $\frac{4ab}{10c^2}$ by $\frac{3bc}{a^3}$.

8. $\frac{4mn}{3xy}$ by $-\frac{15bx}{16m^2}$.

9. $\frac{2ax}{12by}$ by $-\frac{10b^2}{x^2}$.

10. $\frac{a}{a+b}$ by $\frac{b}{a-b}$.

11. $\frac{xy^2}{20-8x}$ by $\frac{25-10x}{x^2y}$.

12. $\frac{1-6x+5x^2}{x^2-3x+2}$ by $\frac{2-x}{1-x}$.

Simplify each of the following:

13. $\frac{(a-b)^2}{a+b} \times \frac{b}{a^2-ab} \times \frac{(a+b)^2}{a^2-b^2}$.

14. $\frac{a^4-x^4}{a^3+x^3} \times \frac{a+x}{a^2-x^2} \times \frac{a^2-ax+x^2}{(a+x)^2}$.

15. $\frac{4a-b}{2x+y} \times \frac{2a}{4a^2-ab} \times \frac{4x^2-y^2}{4}$.

16. $\frac{p+2}{x-3} \times \frac{3x^2-27}{2p^2-8} \times \frac{4}{px+3p}$.

17. $\frac{p^4-q^4}{(p-q)^2} \times \frac{p-q}{p^2+pq} \times \frac{p^2}{p^2+q^2}$.

18. $\frac{a^3+8}{a^3-8} \times \frac{a^2+2a+4}{a^2-2a+4}$.

19. $\frac{a^4+a^2x^2+x^4}{a^4-ax^3} \times \frac{x}{a^2-ax+x^2}$.

20. $\frac{a^4+4}{a^4+a^2+1} \times \frac{a^2+a+1}{a^2+2a+2}$.

21. $\frac{4r^2-4s^2}{5r^2+10rs+5s^2} \times \frac{10r+10s}{8r^3-8s^2}$.

22. $\left(1-\frac{x-1}{x^2+6x+5}\right)\left(1-\frac{2}{x^2+7x+12}\right).$

23. $\left(1+\frac{7x+11}{x^2-4x-21}\right)\left(1-\frac{17x-11}{x^2+7x+10}\right).$

24. $\frac{4ax^2-4ay^2}{3ax^2+3axy+bxy+by^2}\cdot\frac{3ax^2-3axy+bxy-by^2}{5ax^2-10axy+5ay^2}.$

25. $\frac{x^3-5x^2+8x-4}{x^3-8x^2+19x-12}\cdot\frac{x^3-10x^2+33x-36}{x^3-6x^2+11x-6}.$

26. $\frac{x^4-3x^3-23x^2+75x-50}{x^4-5x^3-21x^2+125x-100}\cdot\frac{x^3-10x^2+29x-20}{x^3-12x^2+45x-50}.$

27. $\frac{a^2+ab+ac+bc}{ax-ay-x^2+xy}\cdot\frac{a^2-ax+ay-xy}{a^2+ac+ax+cx}\cdot\frac{x^2-x(y-a)-ay}{a^2-a(y-b)-by}.$

DIVISION OF FRACTIONS

195. The **reciprocal** of a fraction is the fraction *inverted*, or 1 divided by the fraction.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$; of $\frac{x}{z}$ is $\frac{z}{x}$; of m, or $\frac{m}{1}$, is $\frac{1}{m}$.

196. The **reciprocal** of a number is 1 divided by the number.

197. Just as $$\frac{3}{4}\div\frac{2}{3}=\frac{3}{4}\times\frac{3}{2}=\frac{3\times 3}{4\times 2},$$

so $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

PRINCIPLE. — *Dividing by a fraction is equivalent to multiplying by its reciprocal.*

EXERCISES

198. Write the reciprocal of:

1. $\frac{a}{b}$.
2. $\frac{3m}{p}$.
3. rs.
4. $\frac{2a^2}{bc}$.
5. $3tu$.
6. $\frac{1}{3m}$.
7. $\frac{4}{ab}$.
8. $\frac{a-x}{b-y}$.

9. Divide $\frac{x^2-4}{x^2-1}$ by $\frac{x+2}{x-1}$.

SOLUTION

$$\frac{x^2-4}{x^2-1} \div \frac{x+2}{x-1} = \frac{\cancel{(x+2)}(x-2)}{(x+1)\cancel{(x-1)}} \times \frac{\cancel{x-1}}{\cancel{x+2}} = \frac{x-2}{x+1}.$$

Simplify:

10. $\frac{5\,mn}{6\,bx} \div \frac{10\,m^2n}{3\,ax^2}$.

11. $\frac{3\,abm}{7} \div abx$.

12. $\frac{my-y^2}{(m+y)^2} \div \frac{y^2}{m^2-y^2}$.

13. $\frac{(a-b)^2}{a+b} \div \frac{a^2-ab}{b}$.

14. $(4\,a+2) \div \frac{2\,a+1}{5\,a}$.

15. $\frac{a^4-b^4}{a^2-2\,ab+b^2} \div \frac{a^2+b^2}{a^2-ab}$.

16. $\frac{x^3+y^3}{x^2-y^2} \div \frac{x^2+xy+y^2}{x-y}$.

17. $\left(x \div \frac{1}{y}\right) \div \left(y^2 \div \frac{1}{x^2}\right)$.

18. $\left(\frac{a^3}{b} \div b^2\right) \div \left(\frac{a^2}{b^2} \times ab\right)$.

19. $(a+c) \div \left(\frac{a^2-c^2}{1+x} \div \frac{a-c}{1-x^2}\right)$.

20. $\left(y-x+\frac{x^2}{y}\right) \div \left(\frac{x}{y^2}+\frac{y}{x^2}\right)$.

SUGGESTION. — Reduce the dividend to a fraction.

21. $\left(x^2-\frac{1}{x^2}\right) \div \left(x-\frac{1}{x}\right)$.

22. $\left(1-\frac{y^2}{x^2}\right) \div \left(1-\frac{2\,x}{y}+\frac{x^2}{y^2}\right)$.

23. $\left(1+\frac{1}{y^2}+\frac{1}{y^4}\right) \div \left(1+\frac{1}{y}+\frac{1}{y^2}\right)$.

24. $\frac{a^2+b^2-c^2+2\,ab}{a^2-b^2-c^2+2\,bc} \div \frac{a^2-b^2+c^2-2\,ac}{a^2-b^2+c^2+2\,ac}$.

25. $\frac{x^3-6\,x^2+11\,x-6}{x^3+2\,x^2-19\,x-20} \div \frac{x^3-13\,x+12}{x^3+10\,x^2+29\,x+20}$.

26. $\left(x-4+\frac{9}{x+2}\right)\div\left(1-\frac{4\ x-7}{x^2-4}\right)$.

27. $\left(x+\frac{3x+6}{x^2-1}+2\right)\div\left(x+3+\frac{1}{x+1}\right)$.

Complex Fractions

199. A fraction one or both of whose terms contains a fraction is called a **complex fraction.**

EXERCISES

200. 1. Simplify the expression $\dfrac{\frac{a}{b}}{\frac{x}{y}}$.

SOLUTION. $$\frac{\frac{a}{b}}{\frac{x}{y}}=\frac{a}{b}\div\frac{x}{y}=\frac{a}{b}\times\frac{y}{x}=\frac{ay}{bx}.$$

Simplify:

2. $\dfrac{\frac{x+y}{ab}}{\frac{x^2-y^2}{ab^2}}$.

3. $\dfrac{a+\frac{b}{c}}{b+\frac{c}{a}}$.

4. $\dfrac{2+\frac{3a}{4b}}{3a+\frac{8b}{3}}$.

5. $\dfrac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}$.

6. $\dfrac{b-\frac{c}{2}}{\frac{b}{2}-c}$.

7. $\dfrac{ax-\frac{x^2}{2}}{\frac{a^2}{2}-ax}$.

8. Simplify the expression $\dfrac{\frac{x^2}{y^2}-\frac{x}{y}+1}{\frac{x^2}{y^2}+\frac{x}{y}+1}$.

SOLUTION. — On multiplying the numerator and denominator of the fraction by y^2, which is the l. c. d. of the fractional parts of the numerator and denominator, the expression becomes $\frac{x^2-xy+y^2}{x^2+xy+y^2}$.

Simplify:

9. $\dfrac{\dfrac{x^2-1}{x}}{\dfrac{x+1}{x^2}}$.

11. $\dfrac{\dfrac{x^3+y^3}{xy}}{\dfrac{x^2-xy+y^2}{xy}}$.

13. $\dfrac{\dfrac{x^2+y^2}{2y}-x}{\dfrac{x}{y}-\dfrac{y}{x}}$.

10. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}$.

12. $\dfrac{\dfrac{1}{a+1}}{1-\dfrac{1}{a+1}}$.

14. $-\dfrac{\dfrac{1}{1-a}}{\dfrac{a}{a-1}}$.

15. $\dfrac{\dfrac{1}{x+1}}{1-\dfrac{1}{1+x}}+\dfrac{\dfrac{1}{x+1}}{\dfrac{x}{1-x}}+\dfrac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}$.

16. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\div\dfrac{1}{1+\dfrac{b^2+c^2-a^2}{2bc}}$.

17. Simplify the expression $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$.

Solution. — By successive reductions and divisions,

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{1}{\dfrac{x+1}{x}}}=\frac{1}{1+\dfrac{x}{x+1}}=\frac{x+1}{x+1+x}=\frac{x+1}{2x+1}.$$

Simplify:

18. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

20. $\dfrac{2}{2-\dfrac{2}{2-\dfrac{2}{2-x}}}$.

19. $\dfrac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}$.

21. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$.

EQUATIONS AND PROBLEMS

201. Since the student has learned how to perform operations when fractions are involved, he is now prepared to solve certain equations that heretofore he could solve only by a roundabout method, and others that he could not solve at all.

Clearing Equations of Fractions

202. The process of changing an equation containing fractions to an equation without fractions is called **clearing the equation of fractions.**

EXERCISES

203. 1. Solve the equation $\frac{x}{2} = 10 - \frac{x}{3}$.

SOLUTION

$$\frac{x}{2} = 10 - \frac{x}{3}.$$

Since the first fraction will become an integer if the members of the equation are multiplied by 2 or some number of times 2, and since the second fraction will become an integer if the members are multiplied by 3 or some multiple of 3, the equation may be *cleared of fractions* in a single operation by multiplying both members by some *common multiple* of 2 and 3, as 6, or 12, or 18, etc.

It is best to multiply by the l. c. m. of the denominators, that is, by the l. c. d. of the fractions, which in this case is 6.

Multiplying by 6, Ax. 3, $3x = 60 - 2x$.
Transposing, etc., § 71, $5x = 60$.
Hence, Ax. 4, $x = 12$.

VERIFICATION. — When 12 is substituted for x, the given equation becomes $6 = 6$; that is, the equation is satisfied for $x = 12$.

2. Solve the equation $\frac{x-1}{2} - \frac{x-2}{3} = \frac{2}{3} - \frac{x-3}{4}$.

SUGGESTION. — Multiplying both members of the equation by the l. c. d., which in this case is 12, we obtain

$$6(x-1) - 4(x-2) = 8 - 3(x-3).$$

To clear an equation of fractions:

RULE. — *Multiply both members of the equation by the lowest common denominator of the fractions.*

CAUTIONS. — 1. To insure correct results in *solving* equations:

Before clearing, reduce all fractions to lowest terms, and unite fractions that have like denominators.

Test results and reject such as do not satisfy the equation.

2. If a fraction is negative, the sign of each term of the numerator must be changed when the denominator is removed.

Solve, and verify each result:

3. $2x + \frac{x}{3} = \frac{35}{3}.$

4. $\frac{x}{4} + 10 = 13.$

5. $\frac{x}{2} + \frac{x}{6} = \frac{10}{3}.$

6. $7\frac{1}{2} - \frac{3x}{14} = \frac{x}{7}.$

7. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{3x}{10} - \frac{5x}{12} = 7.$

8. $\frac{25x}{18} - \frac{5x}{9} + \frac{2x}{3} - \frac{5x}{6} = 2.$

9. $\frac{7z+2}{6} - \frac{12-z}{4} + \frac{z+2}{2} = 6.$

10. $\frac{u-3}{7} + \frac{u+5}{3} - \frac{u+2}{6} = 4.$

11. $\frac{y-1}{2} + \frac{y-2}{3} + \frac{y-3}{4} = \frac{5y-1}{6}.$

12. $\frac{x-5}{3} + \frac{2x+2}{8} - \frac{x-1}{4} = \frac{x+4}{6}.$

13. $1.07x + .32 = .15x + 10.12 + .675x.$

SUGGESTION. — Clear of decimal fractions by multiplying by 1000.

14. $.604x - 3.16 - .7854x + 7.695 = 0.$

15. $\frac{.2x}{7} - \frac{.1x}{4} - \frac{.1x}{2} + \frac{.4x}{7} = \frac{.3}{14}.$

16. Solve $\dfrac{9x+5}{14}+\dfrac{8x-7}{6x+2}=\dfrac{36x+15}{56}+\dfrac{10\frac{1}{4}}{14}$.

SUGGESTION. — The equation may be written,

§ 36, $\dfrac{9x}{14}+\dfrac{5}{14}+\dfrac{8x-7}{6x+2}=\dfrac{36x}{56}+\dfrac{15}{56}+\dfrac{41}{56}$, or $\dfrac{8x-7}{6x+2}=\dfrac{9}{14}$.

17. $\dfrac{3x-2}{2x-5}+\dfrac{3x-21}{5}=\dfrac{6x-22}{10}$.

18. $\dfrac{4x+3}{9}=\dfrac{8x+19}{18}-\dfrac{7x-29}{5x-12}$.

19. $\dfrac{6p^2+p}{15p}-\dfrac{2p-4}{7p-13}=\dfrac{2p-1}{5}$.

20. Solve the equation $\dfrac{x-1}{x-2}+\dfrac{x-6}{x-7}=\dfrac{x-5}{x-6}+\dfrac{x-2}{x-3}$.

SOLUTION. — It will be observed that if the fractions in each member were connected by the sign −, and if the terms of each member were united, the numerators of the resulting fractions would be simple. The fractions can be made to meet this condition by transposing one fraction in each member before clearing of fractions.

Transposing, $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$.

Uniting terms, $\dfrac{-1}{x^2-5x+6}=\dfrac{-1}{x^2-13x+42}$.

Since the fractions are equal and their numerators are equal, their denominators must be equal.

Then, $x^2-5x+6=x^2-13x+42$.

$$\therefore x=4\tfrac{1}{2}.$$

21. $\dfrac{x-1}{x-2}+\dfrac{x-7}{x-8}=\dfrac{x-5}{x-6}+\dfrac{x-3}{x-4}$.

22. $\dfrac{x-3}{x-4}+\dfrac{x-7}{x-8}=\dfrac{x-6}{x-7}+\dfrac{x-4}{x-5}$.

23. $\dfrac{v+2}{v+1}-\dfrac{v+3}{v+2}=\dfrac{v+5}{v+4}-\dfrac{v+6}{v+5}$.

Algebraic Representation

204. 1. What part of $m - n$ is p?

2. Indicate the sum of l and m divided by 2, and that result multiplied by n.

3. Indicate the product of s and $(r-1)$ divided by the nth power of the sum of t and v.

4. A boy who had m marbles lost $\frac{1}{a}$ of them. How many marbles had he left?

5. By what number must x be multiplied that the product shall be z?

6. Indicate the result when the sum of a, b, and $-c$ is to be divided by the square of the sum of a and b.

7. It is t miles from Albany to Utica. The Empire State Express runs s miles an hour. How long does it take this train to go from Albany to Utica?

8. A cabinetmaker worked x days on two pieces of work. For one he received v dollars, and for the other w dollars. What were his average earnings per day for that time?

9. A train runs x miles an hour and an automobile $x-y$ miles an hour. How much longer will it take the automobile to run s miles than the train?

10. Indicate the result when b is added to the numerator and subtracted from the denominator of the fraction $\frac{a}{c}$.

11. A farmer had $\frac{1}{x}$ of his crop in one field, $\frac{1}{y}$ in a second, and $\frac{1}{z}$ in a third. What part of his crop had he in these three fields?

12. A student spends $\frac{1}{m}$ of his income for room rent, $\frac{1}{n}$ for board, $\frac{1}{s}$ for books, and $\frac{1}{r}$ for clothing. If his income is x dollars, how much has he left?

Problems

205. Solve the following problems and verify each solution:

1. If a large lemon grown in Mexico had weighed $2\frac{1}{2}$ pounds less, its weight would have been $\frac{2}{3}$ of its actual weight. What was its actual weight?

2. The crew of the *Lusitania* numbers 800. If this is 200 less than $\frac{1}{3}$ the number of passengers and crew that may be accommodated, what is the passenger capacity?

3. The box of a Chinese sedan chair is $1\frac{1}{2}$ feet higher than it is long and the area of ~~the~~ floor is 4 square feet. Find its three dimensions, if the capacity of the box is 14 cubic feet.

4. The sum of the heaviest loads that can be carried by a man, a horse, and an elephant is 2900 pounds. The elephant can carry 10 times as much as the horse, and the horse $1\frac{2}{3}$ times as much as the man. What load can each carry?

5. The first issue of *The Sun* devoted $\frac{1}{3}$ of its columns to advertisements and $\frac{1}{4}$ to miscellaneous news. The rest of the paper, 5 columns, was devoted to poetry, finance, and shipping news. How many columns did it contain?

6. Of the world's supply of rubber one year, South America produced $\frac{1}{3}$ and Africa $\frac{1}{5}$. How much was produced by each, if the rest of the world produced 26,600 tons?

7. A large ear of seed corn exhibited at the Iowa Experiment Station sold for $150. At the same rate, if it had weighed 8 ounces less, it would have sold for $90. How much did it weigh?

8. The feathers that a Toulouse goose yields in a year are valued at $2.80. If it yielded 4 ounces more, they would be worth $3.50. Find the weight of the yield of feathers.

9. The number of vessels entering at New York in one year was 11,399. If $\frac{1}{4}$ of the number of steamships was 449 more than $\frac{1}{5}$ of the number of sailing vessels, how many of each were there?

10. Find the world's production of nickel in a year when the United States and Canada together produced $\frac{1}{2}$ of it, England $\frac{11}{60}$ of it, and the rest of the world 3800 tons.

11. In a recent year, the United States produced 2 times as much aluminium as Germany, $1\frac{1}{3}$ times as much as France, and 6200 tons more than England. If all these countries produced 19,800 tons, how much did the United States produce?

12. The largest log in a shipment of mahogany sent to New Orleans weighed 14,000 pounds. If the weight of the rest of the shipment had been 1 ton more, the weight of this log would have been $\frac{1}{13}$ of the total weight of the shipment. Find the weight of the shipment.

13. In constructing the Hall of Records building in New York City, 600,000 pounds of copper were used. The dome lacks 5250 pounds of having $\frac{1}{8}$ as much as the rest of the building. How many pounds of copper are there in the dome?

14. The freight charges on a car load of hay were $\frac{6}{7}$ as much as on a car load of apples. If there were 10 tons of hay and the charges on each ton were \$2 less than $\frac{1}{7}$ of the charges on all the apples, find the charges on each car load.

15. The number of pound cans of salmon in a case is 4 more than $\frac{1}{18}$ of the number of cans that can be packed in a minute in a Washington cannery. If 1000 cases can be packed in an hour, how many cans are there in a case?

16. A boy sold from his garden a certain number of bunches of beets. If he had sold 7 bunches more he would have received \$11 for them. If he had sold 5 bunches less he would have received \$9.80. How many bunches did he sell and at what price?

17. If the number of pounds of alligator teeth sold in a given year had been 50 less, the approximate number of teeth would have been 14,000; if 200 less, the number of teeth would have been 3500. Find the number of pounds sold and the average number of teeth in a pound.

REVIEW

206. 1. What three signs are to be considered in connection with a fraction? What is the sign of a fraction?

2. Under what conditions may the sign of the numerator or of the denominator of a fraction be changed?

3. Show that $\frac{5}{9-x^2}=\frac{-5}{x^2-9}$.

4. What is the effect of changing the sign of an odd number of factors in either term of a fraction? an even number?

5. Show that $\frac{(x-y)(x-y)}{(z-w)(u-z)}=\frac{-x^2+2xy-y^2}{(z-w)(z-u)}$.

6. When is a fraction in its lowest terms? What principle applies to the reduction of fractions to higher or lower terms?

7. Reduce to lowest terms:

$$\frac{x(4a^2-9b^2)}{4a^2x^2+12abx^2+9b^2x^2} \text{ and } \frac{6lx+6sx-14ly-14sy}{36x^2-196y^2}.$$

8. In reducing a fraction to an equal fraction having a given denominator, how is the number found by which both terms are to be multiplied?

9. Reduce $\frac{3b}{a-2b}$ to a fraction whose denominator is $a^2-4ab+4b^2$.

10. Define highest common factor; lowest common multiple. Illustrate by finding the highest common factor and the lowest common multiple of $ax+ay$, x^2-y^2, and $x^2+2xy+y^2$.

11. What is the reciprocal of a fraction? of any number? Give the reciprocals of $\frac{x}{y}$, $\frac{a+b}{x-y}$, and x.

12. Define complex fraction and illustrate by writing one. Simplify the one you have written.

Reduce to an integral or a mixed expression:

13. $\frac{x^2 - 9y^2 + 7}{x - 3y}$.

14. $\frac{4a^3 + 20a^2b + 27ab^2 + 9b^3}{2a + 3b}$.

Reduce the following mixed expressions to fractions:

15. $x^2 - xy + y^2 - \frac{2y}{x + y}$.

16. $\frac{m^2 + n^2}{m + n} - n - m$.

Simplify:

17. $\frac{s + t}{2rl} + \frac{s - t}{4rl} - \frac{s^2 + t^2}{4rl(s + t)}$.

18. $\frac{a + x}{x + 2} + \frac{a - x}{2 - x} - \frac{2(x^2 - 2a)}{x^2 - 4}$.

19. $\left(\frac{x^3 - y^3}{xy + y^2} \times \frac{x^2 + xy}{x - y}\right) \div \left(\frac{x^2y + xy^2}{x^2 + 2xy + y^2} \times \frac{x + y}{y^2}\right)$.

20. $\left(\frac{a^2 + 2ab + b^2}{a^2 - 2ab + b^2} - \frac{a + b}{a - b}\right) \times \left(\frac{3a - 3b}{2b} \div \frac{6a + 6b}{2a - 2b}\right)$.

21. What is meant by clearing an equation of fractions? State the axiom upon which it is based.

22. What precautions must be taken to secure correct results in solving equations that involve fractions?

Solve, and verify each result:

23. $\frac{5x}{2} - \frac{7x}{4} = 6$.

24. $20x + \frac{10x}{3} = \frac{700}{6}$.

25. $\frac{x + 7}{4} = \frac{x - 3}{2} + 1$.

26. $\frac{3x - 2}{10} - \frac{7 - 2x}{5} = \frac{x - 5}{15}$.

27. $\frac{x + 3}{x - 3} - \frac{x}{4} = \frac{12}{24} - \frac{x - 3}{4}$.

28. $\frac{x + 7}{x + 2} + \frac{x + 8}{3} = \frac{3x - 15}{9} - 3$.

29. $2.04x - 3.1 - 2.95x = 8.12 - 5x + 1.05$.

SIMPLE EQUATIONS

ONE UNKNOWN NUMBER

207. The student already knows what an equation is; he has solved several different kinds; and he knows some of the kinds by name. In this chapter and the next he will meet some of the same kinds with the treatment extended to a few new forms and some additional methods of solution.

208. An equation that does not involve an unknown number in any denominator is called an **integral equation.**

$x + 5 = 8$ and $\frac{2x}{3} + 5 = 8$ are integral equations. Though the second equation contains a fraction, the unknown number x does not appear in the denominator.

209. An equation that involves an unknown number in any denominator is called a **fractional equation.**

$x + 5 = \frac{8}{x}$ and $\frac{2x}{x-1} = 7$ are fractional equations.

210. Any number that satisfies an equation is called a **root** of the equation.

2 is a root of the equation $3x + 4 = 10$.

211. Finding the roots of an equation is called **solving the equation**.

212. Two equations that have the same roots, each equation having all the roots of the other, are called **equivalent equations.**

$x + 3 = 7$ and $2x = 8$ are equivalent equations, each being satisfied for $x = 4$ and for no other value of x.

Numerical Equations

213. By applying axioms to the solution of equations, the endeavor is made to change to *equivalent* equations, each simpler than the preceding, until an equation is obtained having the unknown number in one member and the known numbers in the other.

Solve, and verify each result:

1. $8v = 24.$
2. $9r = 54.$
3. $\frac{1}{4}r = 1.5.$
4. $\frac{1}{2}x = 2.5.$
5. $11 + x = 15.$
6. $20 + x = 30.$
7. $7y - 5 = 2.$
8. $2z + 3 = 9.$
9. $4h + 3 = 7.$
10. $6r - 7 = 5.$
11. $\frac{1}{2}b + 3 = 8.$
12. $\frac{1}{5}x + 2 = 6.$
13. $8x - 7 = 3 + 6x.$
14. $7x + 6 = 6x + 8.$
15. $5x - 10 = 2x + 20.$
16. $4r - 18 = 20 + \frac{1}{5}r.$
17. $5n - (2n + 3) = 12.$
18. $17t + 5(2 - 3t) = 18.$
19. $5x - (4 - \overline{6x - 3}) = 26.$
20. $(2w - 1)^2 = 4(w - 3)^2.$
21. $21x + (x - 4)^2 = (5 + x)^2.$
22. $(12x + 6) \div 3 = 9 - 3x.$
23. $(x + 1)(x + 2) = 11 + x^2.$
24. $\frac{1}{4}x - 4 + \frac{3}{4}x = 16 + \frac{1}{4}x - 10.$
25. $x(x + 5) - 6 = x(x - 1) + 12.$
26. $3(2 - x) - 2(x + 3) = 6 - 2x.$
27. $x - (2 + 4x) = 13 - 5(x + 5).$
28. $2\{x - \overline{2x - 2}\} = 3\{x - (3x - 3)\}.$
29. $6x - 13 - 9x + x = 4x - 12 + 3x - 6x - 13.$
30. $36 + 5x - 22 - (7x - 16) = 5x + 17 - (2x + 22).$
31. $2(r - 5)(r - 4) = (r - 4)(r - 3) + (r - 2)(r - 5).$
32. $12x - (6x - 17x - 15 - x) = 15 - (2 - 17x + 6x - 4 - 12x).$

33. $3x - \frac{x}{5} = 14.$

34. $\frac{2x}{3} - \frac{5}{6} = \frac{x}{4}.$

35. $\frac{2x}{3} - \frac{7x}{8} + \frac{5x}{18} + \frac{x}{24} = \frac{4}{9}.$

36. $\frac{3t-5}{4} - \frac{7t-13}{6} = 3 - \frac{t+3}{2}.$

37. $\frac{r}{2}(2-r) - \frac{r}{4}(3-2r) = \frac{r+10}{6}.$

38. $\frac{6r+3}{15} - \frac{3r-1}{5r-25} = \frac{2r-9}{5}.$

39. $\frac{s+1}{s+2} + \frac{s+6}{s+7} = \frac{s+2}{s+3} + \frac{s+5}{s+6}.$

40. $\frac{x^3+1}{x-1} - \frac{x^3-1}{x+1} = \frac{8}{x^2-1} + 2x.$

41. $\frac{5x+2}{3} - \left(x - \frac{3x-1}{2}\right) = \frac{3x+19}{2} - \left(\frac{x+1}{6} + 5\right).$

Literal Equations

214. 1. Solve the equation $\frac{x-b^2}{a} = \frac{x-a^2}{b}$ for x.

Solution

$$\frac{x-b^2}{a} = \frac{x-a^2}{b}.$$

Clearing of fractions, $bx - b^3 = ax - a^3.$

Transposing, etc., $ax - bx = a^3 - b^3.$

$$(a-b)x = a^3 - b^3.$$

Dividing by $(a-b)$, $x = a^2 + ab + b^2.$

Verification. — Since a and b may have any numerical value, let $a = 2$ and $b = 1$; then $x = a^2 + ab + b^2 = 4 + 2 + 1 = 7$, and the given equation becomes $\frac{7-1}{2} = \frac{7-4}{1}$, or $3 = 3$; consequently, the equation is satisfied for $x = a^2 + ab + b^2$.

Solve for x, and verify each result:

2. $\frac{c^2-x}{nx}+\frac{n^2}{cx}=\frac{1}{c}$.

3. $1-\frac{ab}{x}=\frac{7}{ab}-\frac{49}{abx}$.

4. $rx+s^2=r^2-sx$.

5. $a^2x-b^4=b^2x-a^4$.

6. $\frac{a^2+b^2}{2\,bx}-\frac{a-b}{2\,bx^2}=\frac{b}{x}$.

7. $\frac{2\,x-a}{x-a}-\frac{x-a}{x+a}=1$.

8. $\frac{x}{b}-\frac{x+2\,b}{a}=\frac{a}{b}-3$.

9. $\frac{x-a}{b}+\frac{2\,x}{a}=5+\frac{6\,b}{a}$.

10. $6+l-2\,x=l(x-2)$.

11. $c^5x+d^{10}=c^{10}+d^5x$.

12. $\frac{x-2\,a}{a}+\frac{x}{b}=\frac{a^2+b^2}{ab}$.

13. $\frac{a^2}{bx}+\frac{b^2}{ax}=\frac{a+b}{ab}-\frac{3(a+b)}{x}$.

14. $6\,x+18(1-\frac{1}{2}\,a)=a(x-a)$.

15. $a^2(a-x)=abx+b^2(b+x)$.

16. $b(2\,x-9\,c-14\,b)=c(c-x)$.

17. $a(x-a-2\,b)+b(x-b)+c(x+c)=0$.

18. $(a-x)(x-b)+(a+x)(x-b)=(a-b)^2$.

19. $(m+x)^2+(m+x)(n-x)=(m+n)^2$.

20. $(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-b)$.

21. $\frac{a-b+c}{x+a}=\frac{b-a+c}{x-a}$.

22. $\frac{x-1}{a-1}-\frac{a-1}{x-1}=\frac{x^2-a^2}{(a-1)(x-1)}$.

23. $\frac{1}{m+n}-\frac{2\,mn}{(m+n)^3}-\frac{m}{(m+n)^2}=\frac{x-n}{(m+n)^2}$.

24. $\frac{a+x}{a}-\frac{2\,x}{a+x}+\frac{x^2(x-a)}{a(a^2-x^2)}=\frac{1}{3}$.

Suggestion. — Simplify as much as possible before clearing the equation of fractions.

25. $\frac{x^2(b-x)}{b^2(b^2-x^2)}+\frac{5\,(b-x)}{b(x-b)}-\frac{x-b}{b^2}=0$.

Problems

215. Review the general directions for solving problems given on page 45.

1. What is the weight of a turtle, from which $6\frac{3}{4}$ pounds of tortoise shell is taken, if this is $\frac{1}{20}$ of the turtle's whole weight?

2. The powder and the shell used in a twelve-inch gun weigh 1265 pounds. The powder weighs 15 pounds more than $\frac{1}{4}$ as much as the shell. Find the weight of each.

3. One day three lace makers earned 80 cents. The beginner earned $\frac{1}{4}$ as much as the expert maker, and the average worker earned 3 times as much as the beginner. How much did each earn?

4. One ton of coal will make 8.7 tons of steam. If the *Lusitania* requires 1200 tons of coal a day for this purpose, how many tons of steam are required an hour?

5. A grocer paid $8.50 for a molasses pump and 5 feet of tubing. He paid 12 times as much for the pump as for each foot of tubing. How much did the pump cost? the tubing?

6. In lighting a hall a certain number of 16-candle power electric lamps and twice as many 20-candle power lamps were used. The total illumination amounted to 224 candle power. Find the number of lamps of each kind used.

7. At the waterworks 2 large pumps and 4 small ones delivered 4800 gallons of water per minute. Each of the large pumps delivered 4 times as much water as each small pump. How many gallons per minute did each pump deliver?

8. The crew of a United States battleship in target practice made 11 hits in less than a minute. If $\frac{3}{4}$ of the number of shots fired was 9 times the number of misses, how many shots were fired?

9. The courtyard of a palace is 101 feet longer than it is wide. If its width were decreased 25 feet, its length would be twice its width. Find the dimensions of the courtyard.

10. In making 5000 pounds of brass there were used $8\frac{1}{2}$ times as much copper as tin, and twice as much tin as zinc. How many pounds of each metal were used?

11. A merchant bought 62 barrels of flour, part at $\$4\frac{3}{4}$ per barrel, the rest at $\$5\frac{1}{2}$ per barrel. If he paid \$320 for the flour, how many barrels of each grade did he buy?

12. A dealer paid \$185 for 25 boxes of candles. If he paid \$9 a box for part of them and \$6.50 a box for the rest, how many did he buy at each price?

13. A merchant purchased an assortment of bath robes for \$480. By selling $\frac{1}{4}$ of them at \$6 each, $\frac{1}{6}$ of them at \$7 each, $\frac{1}{3}$ of them at \$5 each, and the rest, or $\frac{1}{4}$ of them, at \$8 each, he gained \$128. How many did he sell at each price?

14. In a certain balloon race, the sum of the distances covered by the *Lotus II* and the *United States* was 1025 miles. The distance covered by the former was 50 miles more than $\frac{1}{2}$ of that covered by the latter. How far did each travel?

15. A newspaper reporter saved $\frac{1}{5}$ of his weekly salary, or \$1 more than was saved by an artist on the same paper, whose salary was \$5 greater but who saved only $\frac{1}{7}$ of it. How much did the reporter earn per week? the artist?

16. During a year of 365 days one locality had 6 days less of 'clear' weather than of 'cloudy' weather, and 4 days more of 'clear' than of 'partly cloudy' weather. Find the number of days of each kind of weather during the year.

17. The bark from a cork tree lost $\frac{1}{5}$ of its weight by being boiled. The boiled cork was then scraped, its weight thus being reduced $\frac{1}{4}$. How much did the cork weigh before and after these two operations, if the entire loss was 16 pounds?

18. At a certain depth a diver saw the sun as a reddish disk. At a depth 25 feet more than twice this depth it could still be faintly seen. If darkness occurred on descending 100 feet more, or at a total depth of 325 feet, at what depth did the sun appear as a reddish disk?

19. Find a fraction whose value is $\frac{4}{7}$ and whose denominator is 15 greater than its numerator.

20. Find a fraction whose value is $\frac{2}{3}$ and whose numerator is 3 greater than half of its denominator.

21. The numerator of a certain fraction is 8 less than the denominator. If each term of the fraction is decreased by 5, the resulting fraction equals $\frac{1}{3}$. What is the fraction?

22. An acre of wheat yielded 2000 pounds more of straw than of grain. The weight of the grain was .3 of the total weight of grain and straw. How many 60-pound bushels of wheat were produced?

23. The total diameter of a large wooden fly wheel is 30 feet. The number of inches in the thickness of the rim is 2 less than the number of feet from the center to the rim. How thick is the rim?

24. A shipment of 83,000 postal cards in two sizes weighed 472 pounds. The smaller cards weighed 5 pounds per 1000 and the larger ones weighed 6 pounds 3 ounces per 1000. Find the number of cards of each size in the shipment.

25. I paid 18¢ more for a screen door, 7 feet by 3 feet, than for 3 window screens, each $2\frac{2}{3}$ feet by 3 feet. Find the price per square foot in each case, if it was 3¢ less for window screens.

26. A grocer bought a box of soap containing 72 cakes for $4.50. Some of the soap he sold at 3 cakes for 25¢, and the rest at 10¢ a cake. This gave a profit of $1.90. How many cakes did he sell at each price?

27. It costs 3.6¢ less to travel 100 miles on the Swiss railroads under public management than it did to travel 90 miles when they were under private management. The average fare per mile under private management was 1.9¢ more than it is under public management. Find the rate per mile under each.

28. Two orange pickers together earned $4.50 a day, and one of them picked 20 boxes more than the other. If the slower one had picked twice as many as he did, they would have earned $6.50. How much did each receive a box?

29. A can do a piece of work in 8 days. If B can do it in 10 days, in how many days can both working together do it?

SOLUTION

Let $x =$ the required number of days.

Then, $\frac{1}{x} =$ the part of the work both can do in 1 day,

$\frac{1}{8} =$ the part of the work A can do in 1 day,

$\frac{1}{10} =$ the part of the work B can do in 1 day;

$$\therefore \frac{1}{x} = \frac{1}{8} + \frac{1}{10}.$$

Solving, $x = 4\frac{4}{9}$, the required number of days.

30. A can do a piece of work in 10 days, B in 12 days, and C in 8 days. In how many days can all together do it?

31. It takes a man 6 days to make a Panama hat, and a boy 7 days. How long would it take them, if they could work together?

32. The average amount of coal blasted out by a keg of powder can be mined by one man in 2 days and by another in 3 days. How long would it take them to mine it if they worked together?

33. A and B can dig a ditch in 10 days, B and C can dig it in 6 days, and A and C in $7\frac{1}{2}$ days. In what time can each man do the work?

SUGGESTION. — Since A and B can dig $\frac{1}{10}$ of the ditch in 1 day, B and C $\frac{1}{6}$ of it in 1 day, and A and C $\frac{2}{15}$ of it in 1 day, $\frac{1}{10} + \frac{1}{6} + \frac{2}{15}$ is twice the part they can all dig in 1 day.

34. A and B can load a car in $1\frac{3}{4}$ hours, B and C in $2\frac{1}{2}$ hours, and A and C in $2\frac{1}{3}$ hours. How long will it take each alone to load it?

35. In a certain year, New York State furnished 153.9 million pens, or 54 % of all that were made in the United States. How many pens were made in the United States?

36. The per cent of copper contained in an ancient die found in Egypt was $2\frac{1}{2}$ % more than 3 times the per cent of tin. If these metals formed $92\frac{1}{2}$ % of the die, what per cent of each did it contain?

37. Of the population of Mexico at one time the per cent of whites was $\frac{1}{2}$ that of Indians. Mixed races formed 5 % more than the per cent of Indians. Find the per cent of each.

38. Crude oil when refined produces $2\frac{1}{2}$ times as much kerosene as it does gasoline, and the remainder, which is 65 %, is fuel oil. If a certain refinery produces 2250 barrels of kerosene a day, what is its daily capacity of crude oil?

39. The units' digit of a two-digit number exceeds the tens' digit by 5. If the number increased by 63 is divided by the sum of its digits, the quotient is 10. Find the number.

SOLUTION

Let $x =$ the digit in tens' place.

Then, $x + 5 =$ the digit in units' place,

and $10x + (x + 5) =$ the number;

$$\therefore \frac{10x + (x + 5) + 63}{2x + 5} = 10;$$

whence, $x = 2$,

and $x + 5 = 7$.

Therefore, the number is 27.

40. The tens' digit of a two-digit number is 3 times the units' digit. If the number diminished by 33 is divided by the difference of the digits, the quotient is 10. Find the number.

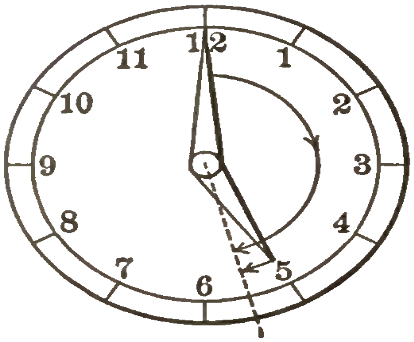
41. The tens' digit of a two-digit number is $\frac{1}{2}$ of the units' digit. If the number increased by 27 is divided by the sum of its digits, the quotient is $6\frac{1}{4}$. Find the number.

42. An officer, attempting to arrange his men in a solid square, found that with a certain number of men on a side he had 34 men over, but with one man more on a side he needed 35 men to complete the square. How many men had he?

SUGGESTION. — With x men on a side, the square contained x^2 men; with $(x + 1)$ men on a side, there were places for $(x + 1)^2$ men.

43. A regiment drawn up in the form of a solid square was reënforced by 240 men. When the regiment was formed again in a solid square, there were four more men on a side. How many men were there in the regiment at first?

44. At what time between 5 and 6 o'clock will the hands of a clock be together?



SOLUTION

Starting with the hands in the position shown, at 5 o'clock, let x represent the number of minute spaces passed over by the minute hand after 5 o'clock until the hands come together. In the same time the hour hand will pass over $\frac{1}{12}$ of x minute spaces.

Since they are 25 minute spaces apart at 5 o'clock,

$$x - \frac{x}{12} = 25;$$

$\therefore x = 27\frac{3}{11}$, the number of minutes after 5 o'clock.

45. At what time between 1 and 2 o'clock will the hands of a clock be together?

46. At what time between 10 and 11 o'clock will the hands of a clock point in opposite directions?

47. At what two different times between 4 and 5 o'clock will the hands of a clock be 15 minute spaces apart?

48. Mr. Reynolds invested \$800, a part at 6%, the rest at 5%. The total annual interest was \$45. Find how much money he invested at each rate.

SUGGESTION. — Let $x =$ the number of dollars invested at 6%.
Then, $800 - x =$ the number of dollars invested at 5%;
$\therefore \frac{6}{100}x + \frac{5}{100}(800 - x) = 45.$

49. A man put out \$4330 in two investments. On one of them he gained 12%, and on the other he lost 5%. If his net gain was \$251, what was the amount of each investment?

50. Mr. Bailey loaned some money at 4% interest, but received \$48 less interest on it annually than Mr. Day, who had loaned $\frac{4}{5}$ as much at 6%. How much did each man loan?

51. A man paid \$80 for insuring two houses for \$6000 and \$4000, respectively. The rate for the second house was $\frac{1}{8}$% greater than that for the first. What were the two rates?

52. United States silver coins are $\frac{9}{10}$ pure silver, or '$\frac{9}{10}$ fine.' How much pure silver must be melted with 250 ounces of silver $\frac{4}{5}$ fine to render it of the standard fineness for coinage?

SUGGESTION. — Let $x =$ the number of ounces of pure silver to be added.
Then, $\frac{4}{5}(250) + x =$ the number of ounces of pure silver after the addition.
Also, $\frac{9}{10}(250 + x) =$ the number of ounces of pure silver after the addition.

53. In an alloy of 90 ounces of silver and copper there are 6 ounces of silver. How much copper must be added that 10 ounces of the new alloy may contain $\frac{2}{5}$ of an ounce of silver?

54. If 80 pounds of sea water contain 4 pounds of salt, how much fresh water must be added that 49 pounds of the new solution may contain $1\frac{3}{4}$ pounds of salt?

55. Four gallons of alcohol 90 % pure is to be made 50 % pure. What quantity of water must be added?

56. Of 24 pounds of salt water, 12 % is salt. In order to have a solution that shall contain 4 % salt, how many pounds of pure water should be added?

57. A man rows downstream at the rate of 6 miles an hour and returns at the rate of 3 miles an hour. How far downstream can he go and return within 9 hours?

58. An airship traveled 11 miles with the wind in the same time as 1 mile against it. If it traveled 55 miles and returned in 12 hours, what was its rate against the wind? with the wind?

59. A train went 905.4 miles in a certain length of time. Another train with a speed 3 miles greater per hour covered 54 miles more in the same length of time. What was the speed of each train?

60. An express train whose rate is 40 miles an hour starts 1 hour and 4 minutes after a freight train and overtakes it in 1 hour and 36 minutes. How many miles does the freight train run per hour?

Solution of Formulæ

216. A **formula** expresses a principle or a rule in symbols. The solution of problems in commercial life, and in mensuration, mechanics, heat, light, sound, electricity, etc., often depends upon the ability to solve formulæ.

EXERCISES

217. 1. The circumference of a circle is equal to π (= 3.1416) times the diameter, or

$$C = \pi D.$$

Solve the formula for D and find, to the nearest inch, the diameter of the wheel of a locomotive, if the circumference of the wheel is 194.78 inches.

SOLUTION

From $C = \pi D$, $\pi D = C$.

$$\therefore D = \frac{C}{\pi} = \frac{194.78}{3.1416} = 62.0+.$$

Hence, to the nearest inch, the diameter is 62 inches.

2. Area of a triangle $= \frac{1}{2}$(base × altitude), or

$$A = \frac{1}{2}bh.$$

Solve for b, then find the base of a triangle whose area is 600 square feet and altitude 40 feet.

3. The area of a trapezoid is equal to the product of the altitude and half the sum of the bases; that is,

$$A = h \cdot \frac{1}{2}(b + b').$$

The bases are b and b'. b' is read 'b-prime.'

Solve for h, then find the altitude of a trapezoid whose area is 96 square inches and whose bases are 14 inches and 10 inches, respectively.

4. The volume of a pyramid $= \frac{1}{3}$(base × altitude), or

$$V = \frac{1}{3}Bh.$$

Solve for B, then find the area of the base of a pyramid whose volume is 252 cubic feet and altitude 9 feet.

5. The charge (c) for a telegram from New York to Chicago, 40¢ for 10 words and 3¢ for each additional word, may be found by the formula,

$$c = 40 + 3(n - 10),$$

in which n stands for the number of words.

Find the cost of a 16-word message.

Solve for n, then find how many words can be sent for $1.

6. In the formula, $$i = p \cdot \frac{r}{100} \cdot t,$$

i denotes the interest on a principal of p dollars at simple interest at r% for t years.

Solve for t, then find the time $300 must be on interest at 5% to yield $60 interest.

Solve for r. At what rate of interest will $4500 yield $900 interest in 5 years?

Solve for p. What principal at $3\frac{1}{2}$% will yield $210 annually?

7. The formula for the space (s) passed over by a body that moves with uniform velocity (v) during a given time (t) is

$$s = vt.$$

Solve for v, then find the velocity of sound when the conditions are such that it travels 8640 feet in 8 seconds.

8. The formula for converting a temperature of F degrees Fahrenheit into its equivalent temperature of C degrees Centigrade is

$$C = \frac{5}{9}(F - 32).$$

Solve for F and express 25° Centigrade (the mean annual temperature in Havana) in degrees Fahrenheit.

Solve:

9. $s = \frac{1}{2}at^2$, for a.

10. $F = Ma$, for a.

11. $W = Fs$, for s.

12. $P = I^2R$, for R.

13. $Mv_1 = mv_2$, for m.

14. $E = \frac{1}{2}Mv^2$, for M.

15. $s = v_0t + \frac{1}{2}at^2$, for a.

16. $s = \frac{1}{2}a(2t - 1)$, for t.

17. Any sort of a bar resting on a fixed point or edge is called a **lever**; the point or edge is called the **fulcrum**.

A lever will just balance when the numerical product of the power (p) and its distance (d) from the fulcrum (F) is equal to the numerical product of the weight (W) and its distance (D) from the fulcrum; that is, when

$$pd = WD.$$

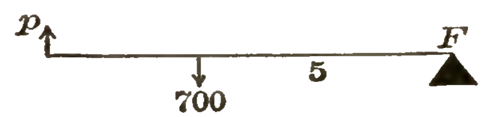
Solve for W and find what weight a power of 150 (pounds) will support by means of the lever shown, if $d = 7$ (feet) and $D = 3$ (feet).

Find for what values of p, d, W, or D the following levers will balance, each lever being 8 feet long:

18. 

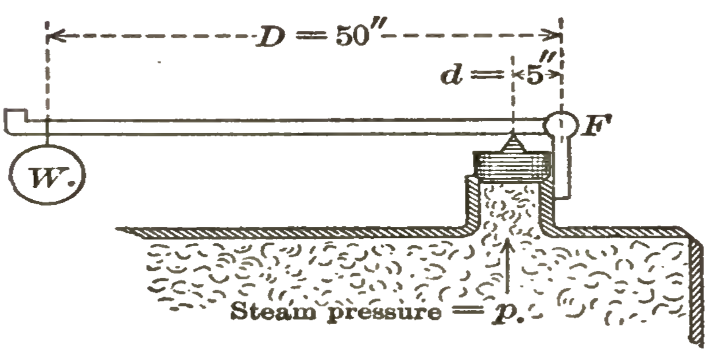
20. 

19. 

21. 

22. Philip, who weighs 114 pounds, and William, who weighs 102 pounds, are balanced on the ends of a 9-foot plank. Neglecting the weight of the plank, how far is Philip from the fulcrum?

23. The figure illustrates the lever of a safety valve, the power being the steam pressure (p) acting on the end of the piston above. The area of the end of the piston is 16 square inches. What weight (W) must be hung on the end of the lever so that when the steam pressure rises to 100 pounds per square inch the piston will rise and allow steam to escape?



24. The number of pounds pressure (P) on A square feet of surface of any body submerged to a depth of h feet in a liquid that weighs w pounds per cubic foot is given by the formula

$$P = wAh.$$

Fresh water weighs about $62\frac{1}{2}$ pounds per cubic foot, and ordinary sea water about 64 pounds per cubic foot.

Find the pressure on 1 square foot of surface at the bottom of a standpipe in which the water is 30 feet high; at the bottom of the ocean at a depth of 3000 feet.

25. Solve $P = wAh$ for h and find the value of h when $P = 5000$, $w = 62\frac{1}{2}$, and $A = 8$.

26. At what depth in fresh water will the pressure on an object having a total area of 4 square feet be 2000 pounds?

27. How deep in the ocean can a diver go, without danger, in a suit of armor that can sustain safely a pressure of 140 pounds per square inch (20,160 pounds per square foot)?

28. If the pressure per square foot on the bottom of a tank holding 18 feet of petroleum is 990 pounds, what is the weight of the petroleum per cubic foot?

29. The side of a chest lying in 25 feet of water was 5 square feet in area and sustained a pressure of 8000 pounds. Was the chest submerged in fresh water or in salt water?

Solve:

30. $\frac{E}{E'} = \frac{R}{R'}$, for R.

31. $I = \frac{E}{R + r}$, for r.

32. $a = \frac{v_1 - v_0}{t}$, for v_1.

33. $V = V_0\left(1 + \frac{t}{273}\right)$, for t.

34. Solve $\frac{E}{e} = \frac{R + r}{r}$, for R; for r.

35. Solve $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ for f_1; for f_2.

36. Solve $\frac{1}{8}Wl = \frac{SI}{c}$, for W; for S; for $\frac{I}{c}$.

SIMULTANEOUS SIMPLE EQUATIONS

TWO UNKNOWN NUMBERS

218. In the equation $x + y = 12$,
x and y may have an unlimited number of pairs of values,

as $\qquad x = 1$ and $y = 11$;

or $\qquad x = 2$ and $y = 10$; etc.

For since $\qquad y = 12 - x$,

if any value is assigned to x, a corresponding value of y may be obtained.

An equation that is satisfied by an unlimited number of sets of values of its unknown numbers is called an **indeterminate equation.**

219. Principle. — *Any single equation involving two or more unknown numbers is indeterminate.*

220. The equations

$$\left.\begin{aligned} 2x + 2y &= 10 \\ \text{and} \quad 3x + 3y &= 15 \end{aligned}\right\}$$

express but one relation between x and y; namely, that their sum is 5. In fact, the equations are *equivalent* to

$$x + y = 5$$

and to each other. Such equations are often called **dependent equations,** for either may be *derived* from the other.

221. The equations

$$\left.\begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}\right\}$$

express two distinct relations between x and y, namely, that their sum is 5 and their difference is 1. The equations cannot be reduced to the same equation; that is, they are not *equivalent.*

Equations that express different relations between the unknown numbers involved, and so cannot be reduced to the same equation, are called **independent equations.**

222. Each of the equations

$$\left.\begin{aligned} x+y&=5 \\ x-y&=1 \end{aligned}\right\}$$

is satisfied separately by an unlimited number of sets of values of x and y, but these letters have only one set of values in both equations, namely,

$$x=3 \text{ and } y=2.$$

Two or more equations that are satisfied by the same set or sets of values of the unknown numbers form a **system of simultaneous, or consistent, equations.**

223. The equations $\left.\begin{aligned} x+y&=5 \\ x+y&=7 \end{aligned}\right\}$

have no set of values of x and y in common.

Such equations are called **inconsistent equations.**

224. The student is familiar with the methods of solving simple equations involving *one* unknown number. The general method of solving a system of two independent simultaneous simple equations in *two* unknown numbers, as

$$\left.\begin{aligned} x+y&=5 \\ x-y&=3 \end{aligned}\right\}$$

is to combine the equations, using axioms **1–5** (§§ 68, 74) in such a way as to obtain an equation involving x alone, and another involving y alone, which may be solved separately by previous methods.

The process of deriving from a system of simultaneous equations another system involving fewer unknown numbers is called **elimination.**

Elimination by Addition or Subtraction

225. Elimination by addition or subtraction has been discussed and applied to the solution of simultaneous equations in §§ 99–103.

EXERCISES

226. 1. Solve the equations $2x + 3y = 7$ and $3x + 4y = 10$.

SOLUTION

$$\begin{cases} 2x + 3y = 7, & (1) \\ 3x + 4y = 10. & (2) \end{cases}$$

(1) × 4, $8x + 12y = 28.$ (3)

(2) × 3, $9x + 12y = 30.$ (4)

(4) − (3), $x = 2.$ (5)

Substituting (5) in (1), $4 + 3y = 7.$

$\therefore y = 1.$

To *verify*, substitute 2 for x and 1 for y in the given equations.

RULE. — *If necessary, multiply or divide the equations by such numbers as will make the coefficients of the quantity to be eliminated numerically equal.*

Eliminate by addition if the resulting coefficients have unlike signs, or by subtraction if they have like signs.

Solve by addition or subtraction, and verify results:

2. $\begin{cases} 7x - 5y = 52, \\ 2x + 5y = 47. \end{cases}$

3. $\begin{cases} 3x + 2y = 23, \\ x + y = 8. \end{cases}$

4. $\begin{cases} 3x - 4y = 7, \\ x + 10y = 25. \end{cases}$

5. $\begin{cases} 2x - 10y = 15, \\ 2x - 4y = 18. \end{cases}$

6. $\begin{cases} 3d + 4y = 25, \\ 4d + 3y = 31. \end{cases}$

7. $\begin{cases} 5p + 6q = 32, \\ 7p - 3q = 22. \end{cases}$

8. $\begin{cases} 3a + 6z = 39, \\ 9a - 4z = 51. \end{cases}$

9. $\begin{cases} 6x - 5y = 33, \\ 4x + 4y = 44. \end{cases}$

10. $\begin{cases} 2a + 3b = 17, \\ 3a + 2b = 18. \end{cases}$ 11. $\begin{cases} 3m + 11n = 67, \\ 5m - 3n = 5. \end{cases}$

Elimination by Comparison

227. If $x = 8 - y,$ (1)

and also $x = 2 + y,$ (2)

by axiom 5, the two expressions for x must be equal.

$$\therefore 8 - y = 2 + y.$$

By *comparing* the values of x in the given equations, (1) and (2), we have eliminated x and obtained an equation involving y alone.

This method is called **elimination by comparison.**

EXERCISES

228. 1. Solve the equations $2x - 3y = 10$ and $5x + 2y = 6$.

SOLUTION

$$\begin{cases} 2x - 3y = 10, & (1) \\ 5x + 2y = 6. & (2) \end{cases}$$

From (1), $x = \frac{10 + 3y}{2}.$ (3)

From (2), $x = \frac{6 - 2y}{5}.$ (4)

Comparing the values of x in (3) and (4),

$$\frac{10 + 3y}{2} = \frac{6 - 2y}{5}.$$

Solving, $y = -2.$

Substituting -2 for y in either (3) or (4),

$$x = 2.$$

To *verify*, substitute 2 for x and -2 for y in the given equations.

RULE. — *Find an expression for the value of the same unknown number in each equation, equate the two expressions, and solve the equation thus formed.*

Solve by comparison, and verify results:

2. $\begin{cases} 3x - 2y = 10, \\ x + y = 70. \end{cases}$

3. $\begin{cases} 5x + y = 22, \\ x + 5y = 14. \end{cases}$

4. $\begin{cases} 2x + 3y = 24, \\ 5x - 3y = 18. \end{cases}$

5. $\begin{cases} 3x + 5y = 14, \\ 2x - 3y = 3. \end{cases}$

6. $\begin{cases} 3v + 2y = 36, \\ 5v - 9y = 23. \end{cases}$

7. $\begin{cases} 2y + 3x = 22, \\ 7x - 3y = 13. \end{cases}$

8. $\begin{cases} 2s + 7t = 8, \\ 3s + 9t = 9. \end{cases}$

9. $\begin{cases} 4u + 6v = 19, \\ 3u - 2v = \frac{9}{2}. \end{cases}$

10. $\begin{cases} 4v + 3w = 34, \\ 11v + 5w = 87. \end{cases}$

11. $\begin{cases} 4x - 13y = 5, \\ 3x + 11y = -17. \end{cases}$

12. $\begin{cases} 4x + 3y = 10, \\ 12x - 11y = -10. \end{cases}$

13. $\begin{cases} 18x - 3y = 4y, \\ 1 - 4x + 3y = 27. \end{cases}$

Elimination by Substitution

229. Given $$3x + 2y = 27, \qquad (1)$$
and $$x - y = 4. \qquad (2)$$

On solving (2) for x, its value is found to be $x = 4 + y$.

If $4 + y$ is *substituted* for x in (1), $3x$ will become $3(4 + y)$, and the resulting equation

$$3(4 + y) + 2y = 27 \qquad (3)$$

will involve y only, x having been eliminated.

Solving (3), $y = 3$.

Substituting 3 for y in (2), $x = 7$.

This method is called **elimination by substitution**.

RULE. — *Find an expression for the value of either of the unknown numbers in one of the equations.*

Substitute this value for that unknown number in the other equation, and solve the resulting equation.

EXERCISES

230. Solve by substitution, and verify results:

1. $\begin{cases} x - y = 4, \\ 4y - x = 14. \end{cases}$

2. $\begin{cases} x + y = 10, \\ 6x - 7y = 34. \end{cases}$

3. $\begin{cases} 3x - 4y = 26, \\ x - 8y = 22. \end{cases}$

4. $\begin{cases} 6y - 10x = 14, \\ y - x = 3. \end{cases}$

5. $\begin{cases} y + 1 = 3x, \\ 5x + 9 = 3y. \end{cases}$

6. $\begin{cases} 17 = 3x + z, \\ 7 = 3z - 2x. \end{cases}$

7. $\begin{cases} 4y = 10 - x, \\ y - x = 5. \end{cases}$

8. $\begin{cases} 7z - 3x = 18, \\ 2z - 5x = 1. \end{cases}$

9. $\begin{cases} 3 - 15y = -x, \\ 3 + 15y = 4x. \end{cases}$

10. $\begin{cases} 1 - x = 3y, \\ 3(1 - x) = 40 - y. \end{cases}$

231. Three standard methods of elimination have been given. Though each is applicable under all circumstances, in special cases each has its peculiar advantages. The student should endeavor to select the method best adapted or to invent a method of his own.

EXERCISES

232. Solve by any method, verifying all results:

1. $\begin{cases} x + z = 13, \\ x - z = 5. \end{cases}$

2. $\begin{cases} 3x + y = 10, \\ x + 3y = 6. \end{cases}$

3. $\begin{cases} 4x + 5y = -2, \\ 5x + 4y = 2. \end{cases}$

4. $\begin{cases} 5x - y = 28, \\ 3x + 5y = 28. \end{cases}$

5. $\begin{cases} x + 3 = y - 3, \\ 2(x + 3) = 6 - y. \end{cases}$

6. $\begin{cases} 5x - y = 12, \\ x + 3y = 12. \end{cases}$

7. $\begin{cases} 4(2 - x) = 3y, \\ 2(2 - x) = 2(y - 2). \end{cases}$

8. $\begin{cases} (x + 1) + (y - 2) = 7, \\ (x + 1) - (y - 2) = 5. \end{cases}$

Eliminate before or after clearing of fractions, as may be more advantageous:

9. $\begin{cases} x + \dfrac{y}{3} = 11, \\ \dfrac{x}{3} + 3y = 21. \end{cases}$

11. $\begin{cases} \dfrac{x}{2} - \dfrac{2y}{3} = -2, \\ \dfrac{5x}{2} + \dfrac{y}{3} = 12. \end{cases}$

10. $\begin{cases} \dfrac{3x}{4} + \dfrac{4y}{5} = 21, \\ \dfrac{2x}{3} + \dfrac{3y}{5} = 17. \end{cases}$

12. $\begin{cases} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 8, \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11. \end{cases}$

13. $\begin{cases} x + \frac{1}{2}(3x - y - 1) = \frac{1}{4} + \frac{3}{4}(y - 1), \\ \frac{1}{5}(4x + 3y) = \frac{1}{10}(7y + 24). \end{cases}$

Equations of the form $\dfrac{a}{x} + \dfrac{b}{y} = c$, though not simple equations, may be solved as simple equations for some of their roots by first regarding $\dfrac{1}{x}$ and $\dfrac{1}{y}$ as the unknown numbers.

14. Solve the equations $\begin{cases} \dfrac{4}{x} - \dfrac{3}{y} = \dfrac{14}{5}, & (1) \\ \dfrac{4}{x} + \dfrac{10}{y} = \dfrac{50}{3}. & (2) \end{cases}$

SOLUTION. (2) — (1), $\dfrac{13}{y} = \dfrac{208}{15}.$

$\therefore \dfrac{1}{y} = \dfrac{16}{15}.$ (3)

Substituting (3) in (1), $\dfrac{4}{x} - \dfrac{48}{15} = \dfrac{14}{5}.$

$\therefore \dfrac{1}{x} = \dfrac{3}{2}.$ (4)

From (4) and (3), $x = \dfrac{2}{3}$ and $y = \dfrac{15}{16}.$

Solve, and verify results:

15. $\begin{cases} \dfrac{5}{x} - \dfrac{3}{y} = -2, \\ \dfrac{25}{x} + \dfrac{1}{y} = 6. \end{cases}$

16. $\begin{cases} \dfrac{7}{x} - \dfrac{8}{y} = -1, \\ \dfrac{1}{x} + \dfrac{3}{y} = \dfrac{25}{28}. \end{cases}$

17. $\begin{cases} \dfrac{3}{2x} - \dfrac{1}{y} = -3, \\ \dfrac{5}{2x} + \dfrac{3}{y} = 23. \end{cases}$

18. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 5, \\ \dfrac{5}{x} - \dfrac{2}{y} = 7. \end{cases}$

19. $\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} = \dfrac{9}{8}, \\ \dfrac{3}{x} + \dfrac{4}{y} = \dfrac{11}{12}. \end{cases}$

20. $\begin{cases} \dfrac{5}{3x} + \dfrac{4}{y} = 3, \\ \dfrac{7}{y} - \dfrac{1}{6x} = 2\frac{1}{6}. \end{cases}$

Literal Simultaneous Equations

233. 1. Solve the equations $\begin{cases} ax + by = m, \\ cx + dy = n. \end{cases}$

SOLUTION

$$ax + by = m \quad (1)$$

$$cx + dy = n \quad (2)$$

(1) × d, $$adx + bdy = dm \quad (3)$$

(2) × b, $$bcx + bdy = bn \quad (4)$$

(3) − (4), $$(ad - bc)x = dm - bn$$

$$\therefore\ x = \frac{dm - bn}{ad - bc}. \quad (5)$$

(1) × c, $$acx + bcy = cm \quad (6)$$

(2) × a, $$acx + ady = an \quad (7)$$

(7) − (6), $$(ad - bc)y = an - cm$$

$$\therefore\ y = \frac{an - cm}{ad - bc}. \quad (8)$$

In solving literal simultaneous equations, elimination is usually performed most easily by addition or subtraction.

Solve for x and y in exercises 2–15, and test results by assigning suitable values to the other letters:

2. $\begin{cases} ax + by = m, \\ bx - ay = c. \end{cases}$

3. $\begin{cases} ax - by = m, \\ cx - dy = r. \end{cases}$

4. $\begin{cases} ax = by, \\ x + y = ab. \end{cases}$

5. $\begin{cases} m(x + y) = a, \\ n(x - y) = 2a. \end{cases}$

6. $\begin{cases} a(x - y) = 5, \\ bx - cy = n. \end{cases}$

7. $\begin{cases} a(a - x) = b(y - b), \\ ax = by. \end{cases}$

8. $\begin{cases} x + y = b - a, \\ bx - ay + 2ab = 0. \end{cases}$

9. $\begin{cases} x - y = a - b, \\ ax + by = a^2 - b^2. \end{cases}$

Eliminate before or after clearing of fractions as seems best:

10. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} - 2 = 0, \\ bx - ay = 0. \end{cases}$

11. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a}, \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{b}. \end{cases}$

12. $\begin{cases} \dfrac{a}{x} - \dfrac{b}{y} = -1, \\ \dfrac{b}{x} - \dfrac{a}{y} = -1. \end{cases}$

13. $\begin{cases} \dfrac{x + 1}{y + 1} = \dfrac{a + b + 1}{a - b + 1}, \\ x - y = 2b. \end{cases}$

14. $\begin{cases} \dfrac{1}{x - a} = \dfrac{1}{a - y}, \\ \dfrac{x + y}{x - y} = a. \end{cases}$

15. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c, \\ \dfrac{x}{b} + \dfrac{y}{c} = d. \end{cases}$

Problems

234. Solve the following problems and verify each solution:

1. The length of a lot is 20 yards more than its width, and its perimeter is 360 yards. Find its dimensions.

2. The best Panama hats bought in Colombia cost \$5 each, and the cheapest cost 50 cents each. If 18 hats cost \$45, how many hats of each kind were bought?

3. A grocer sold 2 boxes of raspberries and 3 of cherries to one customer for 54 ¢, and 3 boxes of raspberries and 2 of cherries to another for 56 ¢. Find the price of each per box.

4. A druggist wishes to put 500 grains of quinine into 3-grain and 2-grain capsules. He fills 220 capsules. How many capsules of each size does he fill?

5. On the Fourth of July, 850 glasses of soda water were sold at a fountain, some at 5 ¢ each, the others at 10 ¢ each. The receipts were $ 55. How many were sold at each price?

6. A fruit dealer bought 36 pineapples for $ 2.50. He sold some at 12 ¢ each and the rest at 10 ¢ each, thereby gaining $ 1.50. How many did he sell at each price?

7. The receipts from 300 tickets for a musical recital were $ 125. Adults were charged 50 ¢ each and children 25 ¢ each. How many tickets of each kind were sold?

8. A dealer packed 1800 Christmas wreaths in 6 barrels and 6 cases. Later he packed 2250 wreaths in 9 barrels and 7 cases. Find the capacity of a barrel; of a case.

9. East African hemp is worth $ 25 more per ton than Mexican hemp. If 2 tons of African hemp are worth $ 25 less than $2\frac{1}{2}$ tons of Mexican hemp, find the value of each per ton.

10. A natural bridge in Utah has a span of 60 feet more than its height. If its height were 200 feet less, it would be $\frac{1}{5}$ of its span. Find the height and the span of the bridge.

11. A chimney at Bolton, England, is 100 feet lower than one at Glasgow, and $\frac{1}{3}$ of the height of the latter is 64 feet more than $\frac{1}{4}$ of the height of the former. Find the height of each.

12. An errand boy went to the bank to deposit some bills for his employer. Some of them were 1-dollar bills, and the rest 2-dollar bills. The number of bills was 38 and their value was $ 50. Find the number of each.

13. A grocer bought 1416 oranges of two sizes. Of one kind it took 360 oranges to fill a box and of the other 48. If there were 10 boxes in all, find the number of boxes of each kind.

14. The cost of firing 20 shots from a Japanese battleship was $ 4040. The shots from the large cannon cost $ 400 each and every shot from the small cannon cost $ 70. How many shots of each kind were fired?

15. In Berlin, Germany, a mason received 80¢ more in 5 days than a painter received in 6 days, each working 10 hours a day. The former earned the same in 9 days as the latter did in 12 days. What was the hourly wage of each man?

16. The champion National League baseball team one year won 62 games more than it lost. The team that came second played 154 games, winning 16 less than the first and losing 18 more than the first. How many games did each team win and how many did each lose?

17. During a rate war between rival steamship lines the passage for 2 immigrants from Bremen to New York cost $ 7.14 less than the normal rate for 1, and the passage for 7 immigrants $ 4.76 less than the normal cost for 3. Find the normal rate and the reduced rate.

18. A man noticed that a 15-word message by telegraph cost him 40¢ and a 22-word message 54¢, between the same two cities. Find the charge for the first 10 words and the charge for each additional word.

19. The United States imported 38 million bunches of bananas one year. The cost, at 30¢ each for the larger bunches and 20¢ each for the smaller ones, was 8.55 million dollars. How many bunches of each size were imported?

20. A steam pipe was inclosed in a wooden case. The diameter of the pipe was $\frac{2}{3}$ of the diameter of the case. The radius of the case was 2 inches less than the diameter of the pipe. What was the diameter of each?

21. A man invested \$4000, a part at 5 % and the rest at 4 %. If the annual income from both investments was \$175, what was the amount of each investment?

22. At a factory where 1000 men and women were employed, the average daily wage was \$2.50 for a man and \$1.50 for a woman. If labor cost \$2340 per day, how many men were employed? how many women?

23. It required 60 inches of tape to bind the four edges of a card on which a photograph was mounted. The length of the card was 6 inches greater than the width. How many inches long was the card? how many inches wide?

24. The German railroads carried 153 million first-class and second-class passengers one year. The number would have been 180 million if 4 times as many had traveled first class. How many traveled first class? second class?

25. In one hour 1375 vehicles passed a merchant's door on Broadway, New York City. The horse-drawn vehicles would have equaled the automobiles in number, had there been 50 more of the former and 25 less of the latter. How many of each passed his door?

26. Probably the highest dock in the world is on the Victoria Nyanza. Its height above sea level is 50 feet less than 15 times its length, and the sum of its height and length is 3950 feet. Find its height above sea level.

27. The American and British tourists to Japan during a recent year numbered 2794. If there had been twice as many Americans and 3 times as many British, the number would have been 6682. How many tourists were there from each country?

28. The number of students attending the University of Berlin at one time was 9277 more than the number attending at Munich, and $\frac{1}{2}$ the number at Berlin plus $\frac{1}{3}$ the number at Munich equaled 9591. How many attended each university?

29. Under the present contract, it costs $24.15 less a year per lamp to maintain electric lights in a certain city than it did under the previous one, and the expense of 6 lamps then was $46.35 more than that of 7 lamps now. Find the present yearly expense per lamp.

30. On the same day two seats in the New York Stock Exchange sold for $192,500. If one of them had sold for $1500 less, and the other for $1000 more, the prices of the two would have been equal. Find the price of each.

31. A man had 10 fox skins, some of which were silver fox, worth $300 a pelt, and some black fox, worth $750 a pelt. If he had had 3 less of the former and 3 more of the latter, the total value would have been $6150. How many had he of each?

32. The quantity of peanuts raised in the United States in a year is 135 million pounds more than the quantity of all nuts imported, and $\frac{1}{5}$ of the former equals $\frac{1}{2}$ of the latter. Find the number of pounds of nuts imported and of peanuts raised in the United States.

33. The receipts from a football game were $700. Admission tickets to the grounds were sold for 50¢, and to the grand stand for 25¢ in addition. If twice as many persons had purchased tickets for the grand stand, the receipts would have been $800. How many tickets of each kind were sold?

34. A train of 25 cars loaded with iron ore was run out on a dock and the ore emptied into pockets beneath the tracks. The ore filled 7 pockets and $\frac{1}{2}$ of another. To fill this last pocket, then, required 16 tons less than 2 extra car loads. What was the capacity of a car? of a pocket?

35. If 100 pounds of soft coal in burning can evaporate 50 pounds more water than 6 gallons of oil, and if 60 pounds of coal can evaporate 10 pounds less water than 4 gallons of oil, how many pounds of water can 1 pound of coal evaporate? 1 gallon of oil?

36. A proposed tunnel under Bering Strait would be in three sections, each of which would be $\frac{1}{4}$ of a mile longer, and two of which together would be $12\frac{3}{4}$ miles longer than the Simplon tunnel. Find the length of the proposed tunnel; of the Simplon tunnel.

37. If 1 is added to the numerator of a certain fraction, the value of the fraction becomes $\frac{3}{4}$; if 2 is added to the denominator, the value of the fraction becomes $\frac{1}{2}$. What is the fraction?

SUGGESTION. — Let $x =$ the numerator and $y =$ the denominator.

38. If the numerator of a certain fraction is decreased by 2, the value of the fraction is decreased by $\frac{1}{2}$; but if the denominator is increased by 4, the value of the fraction is decreased by $\frac{5}{8}$. What is the fraction?

39. A certain number expressed by two digits is equal to 7 times the sum of its digits; if 27 is subtracted from the number, the difference will be expressed by reversing the order of the digits. What is the number?

SUGGESTION. — The sum of x tens and y units is $(10x + y)$ units; of y tens and x units, $(10y + x)$ units.

40. Find a number that is 3 greater than 6 times the sum of its two digits, if the units' digit is 2 less than the tens' digit.

41. A crew can row 8 miles downstream and back, or 12 miles downstream and halfway back in $1\frac{1}{2}$ hours. What is their rate of rowing in still water and the velocity of the stream?

42. A man rows 12 miles downstream and back in 11 hours. The current is such that he can row 8 miles downstream in the same time as 3 miles upstream. What is his rate of rowing in still water, and what is the velocity of the stream?

43. A quantity of wheat could be thrashed by two machines in 6 days, but the larger machine worked alone for 8 days and was then replaced by the smaller, which finished in 3 days. How long would it have taken the larger machine to thrash all of the wheat? the smaller machine?

THREE UNKNOWN NUMBERS

235. The student has been solving systems of *two* independent simultaneous equations involving *two* unknown numbers. In general,

PRINCIPLE. — *Every system of independent simultaneous simple equations, involving the same number of unknown numbers as there are equations, can be solved, and is satisfied by one and only one set of values of its unknown numbers.*

EXERCISES

236. 1. Solve $\begin{cases} x + 2y + 3z = 14, & (1) \\ 2x + y + 2z = 10, & (2) \\ 3x + 4y - 3z = 2. & (3) \end{cases}$

SOLUTION. — Eliminating z by combining (1) and (3),

(1) + (3), $4x + 6y = 16.$ (4)

Eliminating z by combining (2) and (3),

(2) × 3, $6x + 3y + 6z = 30$

(3) × 2, $6x + 8y - 6z = 4$

Adding, $12x + 11y = 34.$ (5)

Eliminating x by combining (5) and (4),

(4) × 3, $12x + 18y = 48$ (6)

(6) − (5), $7y = 14$; $\therefore y = 2.$

Substituting the value of y in (4), $4x + 12 = 16$; $\therefore x = 1.$

Substituting the values of x and y in (1),

$$1 + 4 + 3z = 14; \therefore z = 3.$$

VERIFICATION. — Substituting $x = 1$, $y = 2$, and $z = 3$ in the given equations,

(1) becomes $1 + 4 + 9 = 14$, or $14 = 14$;

(2) becomes $2 + 2 + 6 = 10$, or $10 = 10$;

and (3) becomes $3 + 8 - 9 = 2$, or $2 = 2$;

that is, the given equations are satisfied for $x = 1$, $y = 2$, and $z = 3$.

Solve, and verify all results:

2. $\begin{cases} x+3y-z=10, \\ 2x+5y+4z=57, \\ 3x-y+2z=15. \end{cases}$

3. $\begin{cases} x+y+z=53, \\ x+2y+3z=105, \\ x+3y+4z=134. \end{cases}$

4. $\begin{cases} x-y+z=30, \\ 3y-x-z=12, \\ 7z-y+2x=141. \end{cases}$

5. $\begin{cases} 8x-5y+2z=53, \\ x+y-z=9, \\ 13x-9y+3z=71. \end{cases}$

6. $\begin{cases} x+3y+4z=83, \\ x+y+z=29, \\ 6x+8y+3z=156. \end{cases}$

7. $\begin{cases} 3x-2y+z=2, \\ 2x+5y+2z=27, \\ x+3y+3z=25. \end{cases}$

8. $\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=32, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12. \end{cases}$

9. $\begin{cases} \frac{1}{6}x-\frac{1}{5}y+\frac{1}{4}z=3, \\ \frac{1}{5}x-\frac{1}{4}y+\frac{1}{5}z=1, \\ \frac{1}{4}x-\frac{1}{3}y+\frac{1}{2}z=5. \end{cases}$

10. There are three numbers such that the sum of $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third is 12; of $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third is 9; and the sum of the numbers is 38. What are the numbers?

11. A, B, and C have certain sums of money. If A gives B \$100, they will have the same amount; if A gives C \$100, C will have twice as much as A; and if B gives C \$100, C will have 4 times as much as B. What sum has each?

12. A quantity of water sufficient to fill three jars of different sizes will fill the smallest jar 4 times; the largest jar twice with 4 gallons to spare; or the second jar 3 times with 2 gallons to spare. What is the capacity of each jar?

13. A contractor used 3 scows to convey sand from his dredge to the dumping ground. He was credited by the inspector for:

April 20, scows	a, b, c, a, b, c, a, and b,	8 loads, 3230 cu. yd.
April 21, scows	c, a, b, c, a, b, and c,	7 loads, 2820 cu. yd.
April 22, scows	a, b, c, a, b, c, and a,	7 loads, 2870 cu. yd.

Find the capacity of each scow.

GRAPHIC SOLUTIONS

SIMPLE EQUATIONS

237. When related quantities in a series are to be compared, as, for instance, the population of a town in successive years, recourse is often had to a method of representing quantities by *lines*. This is called the **graphic** method.

By this method, quantity is photographed in the process of change. The whole range of the variation of a quantity, presented in this vivid pictorial way, is easily comprehended at a glance; it stamps itself on the memory.

238. In Fig. 1 is shown the population of a town throughout its variations during the first 13 years of the town's existence.

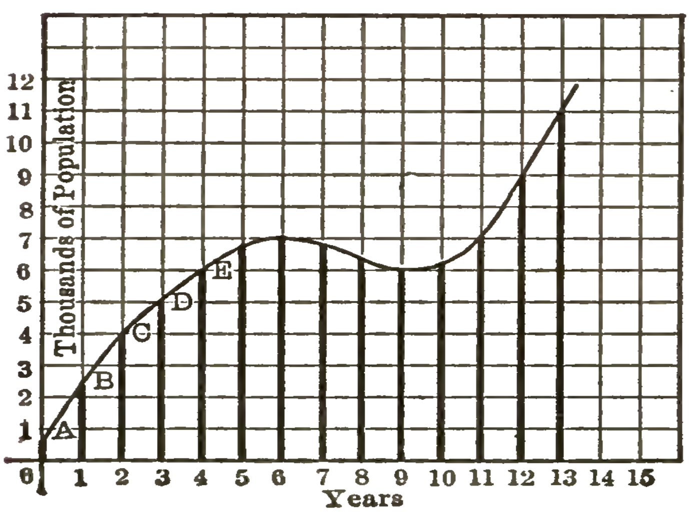


FIG. 1.

The population at the end of 2 years, for example, is represented by the length of the heavy black line drawn upward from 2, and is 4000; the population at the end of 6 years is 7000; at the end of 10 years, 6300 approximately; and so on.

239. Every point of the curved line shown in Fig. 1 exhibits a pair of corresponding values of two related quantities, years and population. For instance, the position of E shows that the population at the end of 4 years was 6000.

Such a line is called a **graph**.

Graphs are useful in numberless ways. The statistician uses them to present information in a telling way. The broker or merchant uses them to compare the rise and fall of prices. The physician uses them to record the progress of diseases. The engineer uses them in testing materials and in computing. The scientist uses them in his investigations of the laws of nature. In short, graphs may be used whenever two related quantities are to be compared throughout a series of values.

240. The graph in Fig. 2 represents the rate in gallons per day per person at which water was used in New York City during a certain day of 24 hours.

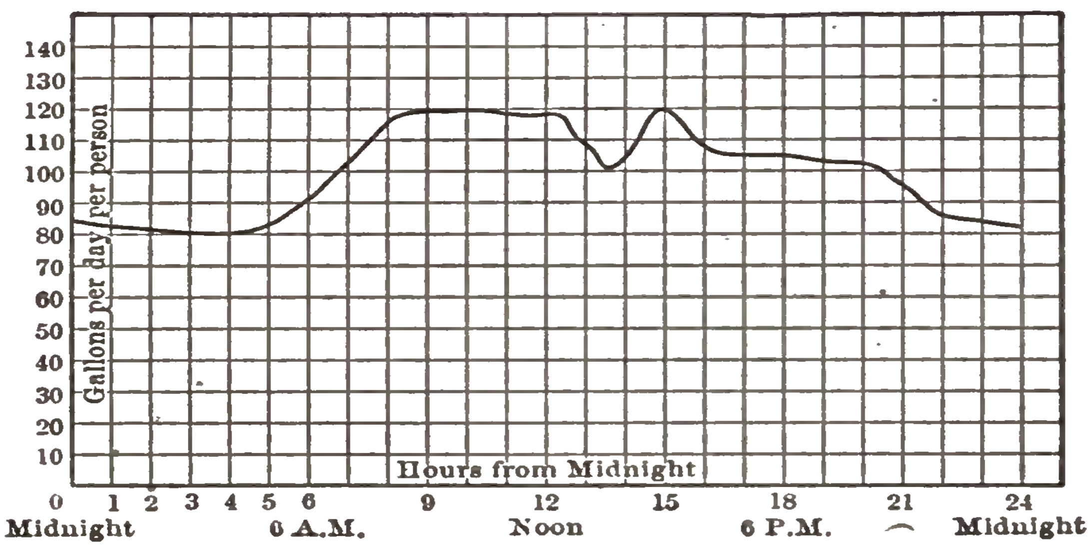


FIG. 2.

Thus, if each horizontal space represents 1 hour (from midnight) and each vertical space 10 gallons, at midnight water was being used at the rate of about 84 gallons per day per person; at 6 A.M., about 91 gallons; at 1 P.M., the 13th hour, about 108 gallons; etc.

1. What was the approximate consumption of water at 2 A.M.? at noon? at 1:30 P.M.? at 3 P.M.? at 6 P.M.?

2. What was the maximum rate during the day? the minimum rate? at what time did each occur?

3. During what hours was the rate most uniform? What was the rate at the middle of each hour?

4. What was the average increase per hour between 6 A.M. and 8 A.M.? the average decrease between 4 P.M. and 8 P.M.?

241. The graphs in Fig. 3 present to the eye in a forceful way the remarkable contrasts in the months of July of the years 1901 and 1904 by showing the *average daily maximum temperature* for ten cities that cover the part of the United States east of the Rocky Mountains.

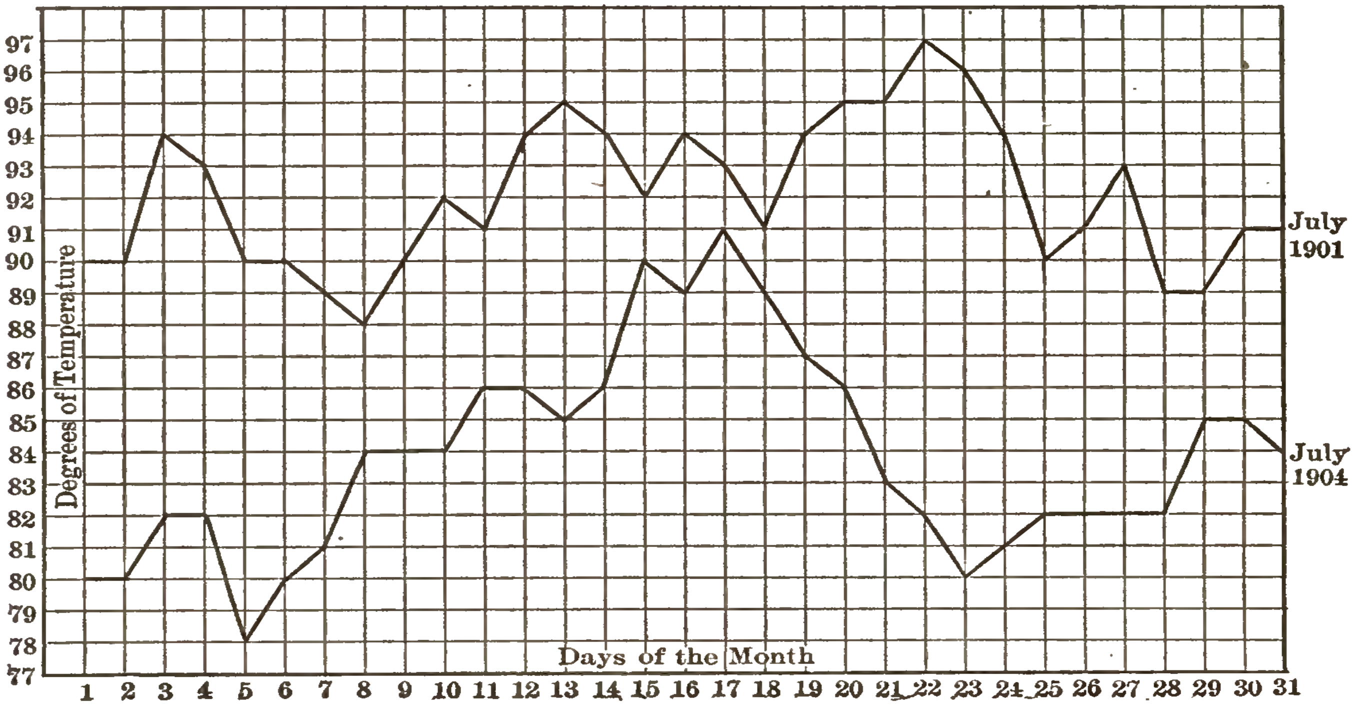


FIG. 3.

The vertical spaces for 0° to 76° inclusive are omitted. In the following 'temperature' means 'average maximum temperature.'

1. In which year was the month of July the hotter?

2. On how many days of July, 1901, was the temperature *below* 90°? How many days of July, 1904, had a temperature *above* 90°?

3. What was the *highest* temperature for July, 1901, and on what day did it occur? for July, 1904? Give the *lowest* temperature for July of each of these years and the date of its occurrence.

4. Which year had the smaller range of temperature for July? How many degrees hotter was the Fourth of July, 1901, than the Fourth of July, 1904?

5. Find the difference between the highest temperature of July, 1901, and the lowest temperature of July, 1904.

242. Fig. 4 gives two graphs, — one showing the height of water above zero of the gage in the Cumberland River at Nashville, the other showing the same thing for the Arkansas River at Little Rock, from daily observations taken in September of the year 1908.

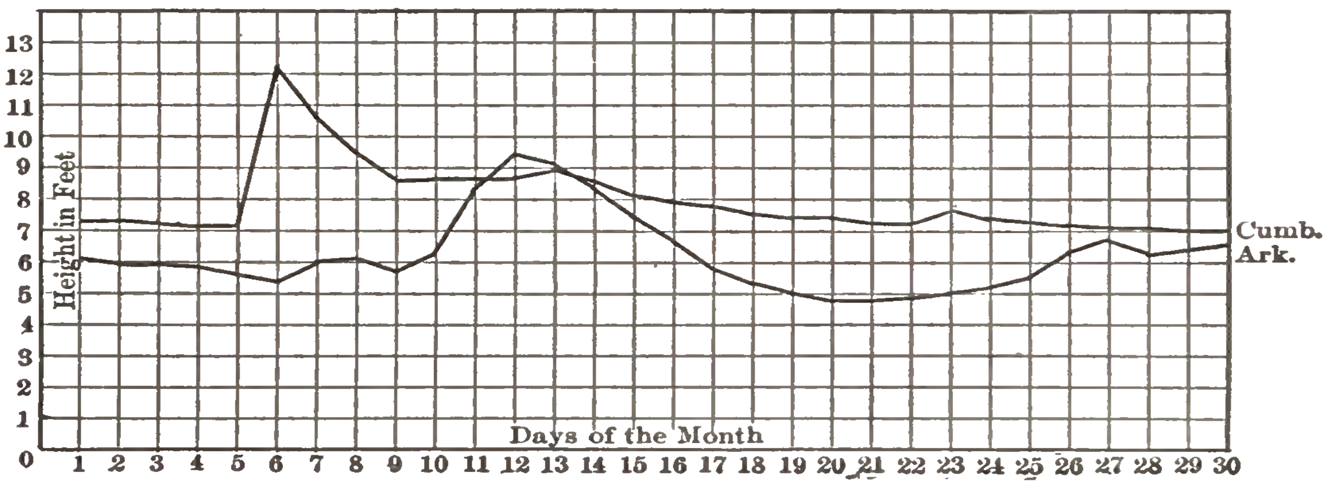


FIG. 4.

In giving heights read the graphs to the nearest tenth of a foot.

1. When was the water in the Cumberland highest? lowest? What was the maximum height? the minimum? the range between them?

2. How many days *later* than the Cumberland was the Arkansas at its maximum? How many days *earlier* than the Cumberland was the Arkansas at its minimum?

3. What was the range for the Arkansas during the month?

4. State the difference between the maximum readings for the two rivers; between their minimum readings.

5. On what days were the readings higher for the Arkansas than for the Cumberland?

6. What part of the month shows the greatest and most rapid changes in the height of the Cumberland? the Arkansas?

7. Which river had the least variation in height during the last half of the month?

8. Give the time of the greatest change in a single day for either river. How much was this change?

EXERCISES

243. 1. Letting each horizontal space represent 10 years and each vertical space 1 million of population, locate points from the pairs of corresponding values (years and millions of population given below) and connect these points with a line, thus constructing a population graph of the United States:

1810, 7.2; 1820, 9.6; 1830, 12.8; 1840, 17.1; 1850, 23.2; 1860, 31.4; 1870, 38.6; 1880, 50.2; 1890, 62.6; 1900, 76.3.

2. From the graph of exercise 1, tell the period during which the increase in the population was greatest; least.

3. The average price of tin in cents per pound for the months of a certain year was: Jan., 23.4; Feb., 24.7; Mar., 26.2; Apr., 27.3; May, 29.3; June, 29.3; July, 28.3; Aug., 28.1; Sept., 26.6; Oct., 25.8; Nov., 25.4; Dec., 25.3.

Draw a graph to show the variation in the price of tin during the year with each horizontal space representing 1 month and each vertical space 1 cent.

4. Construct the graph of exercise 3, letting each horizontal space represent 1 cent and each vertical space 1 month.

244. Let x and y be two *algebraic* quantities so related that $y = 2x - 3$. It is evident that we may give x a series of values, and obtain a corresponding series of values of y; and that *the number of such pairs of values of x and y is unlimited.* All of these values are represented in the graph of $y = 2x - 3$. Just as in the preceding illustrations, so in the graph of $y = 2x - 3$, Fig. 5, values of x are represented by lines laid off on or parallel to an **x-axis**, X′X, and values of y by lines laid off on or parallel to a **y-axis**, Y′Y, usually drawn perpendicular to the x-axis.

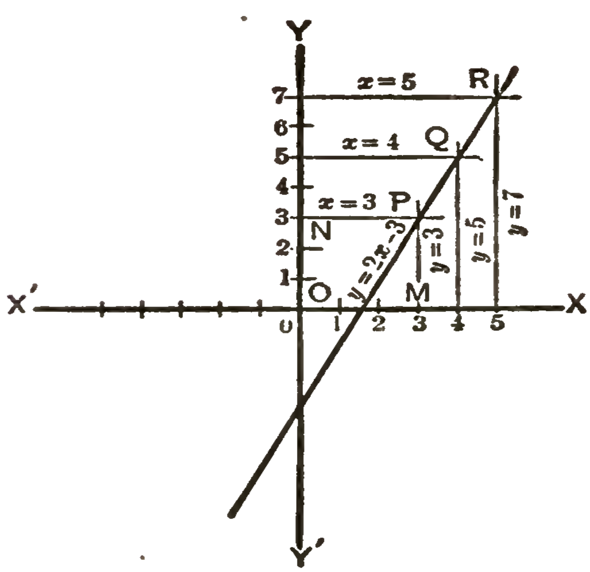


FIG. 5.

For example, the position of P shows that $y = 3$ when $x = 3$; the position of Q shows that $y = 5$ when $x = 4$; the position of R shows that $y = 7$ when $x = 5$; etc.

Evidently every point of the graph gives a pair of corresponding values of x and y.

245. Conversely, to locate any point with reference to two axes for the purpose of representing a pair of corresponding values of x and y, the value of x may be laid off on the x-axis as an *x-distance*, or **abscissa**, and that of y on the y-axis as a *y-distance*, or **ordinate**. If from each of the points on the axes obtained by these measurements, a line parallel to the other axis is drawn, the intersection of these two lines locates the point.

Thus, in Fig. 5, to represent the corresponding values $x = 3$, $y = 3$, a point P may be located by measuring 3 units from O to M on the x-axis and 3 units from O to N on the y-axis, and then drawing a line from M parallel to OY, and one from N parallel to OX, producing these lines until they intersect.

246. The abscissa and ordinate of a point referred to two perpendicular axes are called the **rectangular coördinates,** or simply the **coördinates,** of the point.

Thus, in Fig. 5, the coördinates of P are $OM (= NP)$ and $MP (= ON)$.

247. By universal custom *positive* values of x are laid off from O as a zero-point, or **origin,** toward the *right*, and *negative* values toward the *left*. Also *positive* values of y are laid off *upward* and *negative* values *downward*.

The point A in Fig. 6 may be designated as 'the point (2, 3),' or by the equation $A = (2, 3)$.

Similarly,

$B = (-2, 4)$, $C = (-3, -1)$, and $D = (1, -2)$.

The abscissa is always written first.

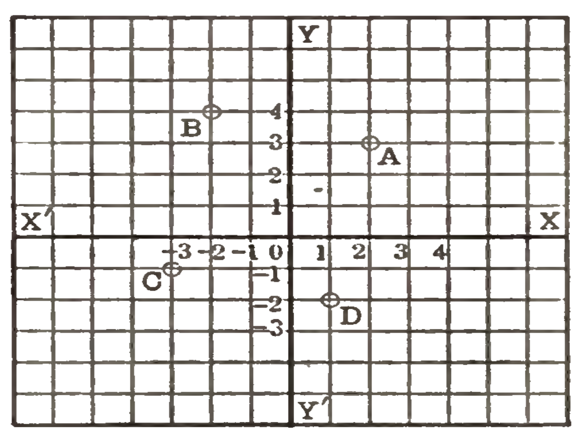


FIG. 6.

248. Plotting points and constructing graphs.

EXERCISES

NOTE. — The use of paper ruled in small squares, called coördinate paper, is advised in plotting graphs.

Draw two axes at right angles to each other and locate:

1. $A = (3, 2)$.
2. $B = (3, -2)$.
3. $C = (4, 3)$.
4. $D = (4, -3)$.
5. $E = (5, 5)$.
6. $F = (-5, 5)$.
7. $G = (-2, 5)$.
8. $H = (-3, -4)$.
9. $L = (0, 4)$.
10. $M = (0, -5)$.
11. $N = (3, 0)$.
12. $P = (-6, 0)$.
13. Where do all points having the abscissa 0 lie? the ordinate 0?
14. What are the coördinates of the origin?
15. Construct the graph of the equation $2y - x = 2$.

SOLUTION

Solving for y, $y = \frac{1}{2}(x + 2)$.

Values are now given to x and corresponding values are computed for y by means of this equation. The numbers substituted for x need not be large. Convenient numbers to be substituted for x in this instance are the even integers from -6 to $+6$.

When $x = -6$, $y = -2$. These values locate the point $A = (-6, -2)$.
When $x = -4$, $y = -1$. These values serve to locate $B = (-4, -1)$.
Other points may be located in the same way.

A record of the work should be kept as follows:

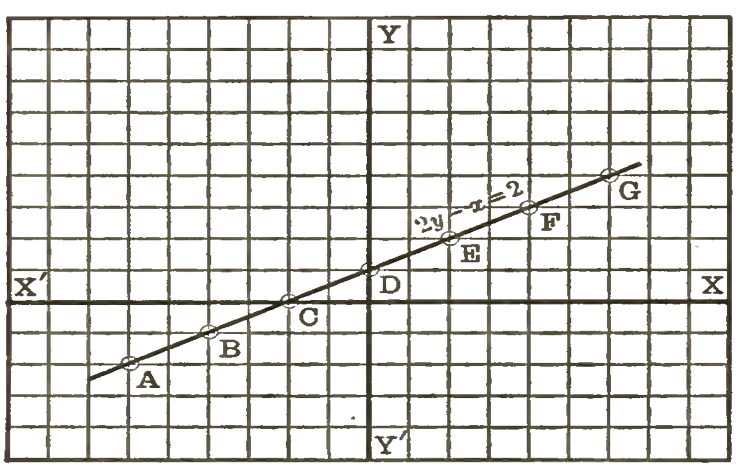


FIG. 7.

$y = \frac{1}{2}(x + 2)$

x	y	POINT
-6	-2	A
-4	-1	B
-2	0	C
0	1	D
2	2	E
4	3	F
6	4	G

A line drawn through A, B, C, D, etc., is the graph of $2y - x = 2$.

Construct the graph of each of the following:

16. $y = 3x - 7$.
17. $y = 2x + 1$.
18. $y = 2x - 1$.
19. $3x - y = 4$.
20. $4x - y = 10$.
21. $x - 2y = 2$.
22. $3x = 2y$.
23. $2x + y = 1$.
24. $2x + 3y = 6$.

249. It can be proved by the principle of the similarity of triangles that:

PRINCIPLE. — *The graph of a simple equation is a straight line.*

For this reason simple equations are sometimes called **linear equations.**

250. Since a straight line is determined by two points, to plot the graph of a linear equation, *plot two points and draw a straight line through them.*

It is often convenient to plot the points where the graph intersects the axes. To find where it intersects the x-axis, let $y = 0$; to find where it intersects the y-axis, let $x = 0$.

Thus, in $y = \frac{1}{2}(x + 2)$, when $y = 0$, $x = -2$, locating C, Fig. 7; when $x = 0$, $y = 1$, locating D.

Draw a straight line through C and D.

If the equation has no absolute term, $x = 0$ when $y = 0$, and this method gives only one point. In any case it is desirable, *for the sake of accuracy*, to plot points some distance apart, as A and G, in Fig. 7.

EXERCISES

251. Construct the graph of each of the following:

1. $y = x - 2$.
2. $y = 2 - x$.
3. $y = 9 - 4x$.
4. $y = 4x - 9$.
5. $y = 10 - 2x$.
6. $y = 2x - 10$.
7. $y = 2x - 4$.
8. $2x - 3y = 6$.
9. $2x + 3y = 0$.
10. $x - 4y = 3$.
11. $7x - y = 14$.
12. $4 - x = 2y$.
13. $3x + 4y = 12$.
14. $5x - 2y = 10$.
15. $8x - 3y = -6$.
16. $-2x + y = -3$.
17. $-3x + 4y = 8$.
18. $5x + 3y = 7\frac{1}{2}$.
19. $x - \frac{1}{2}y = 3$.
20. $\frac{1}{2}x + \frac{1}{3}y = 2$.
21. $.7x - .3y = .4$.

252. Graphic solution of simultaneous linear equations.

I. Let it be required to solve graphically the equations

$$\begin{cases} y = 2 + x, & (1) \\ y = 6 - x. & (2) \end{cases}$$

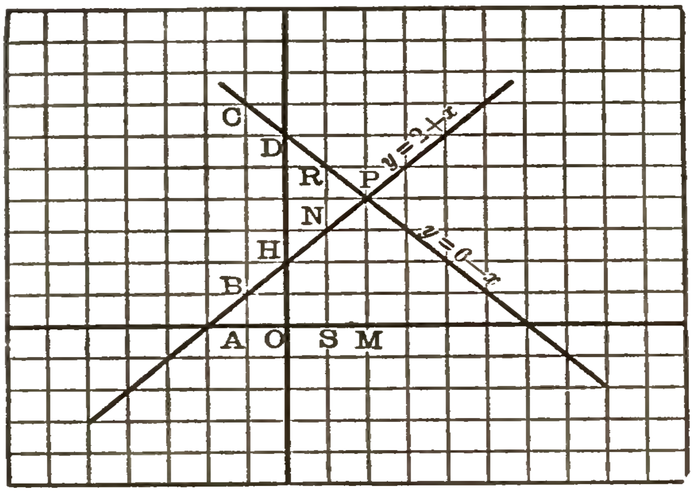


FIG. 8.

As in § 248, construct the graph of each equation, as shown in Fig. 8.

1. When $x = -1$, the value of y in (1) is represented by AB, and in (2) by AC.

Therefore, when $x = -1$, the equations are not satisfied by the same values of y.

2. Compare the values of y when $x = 0$; when $x = 1$; 2.

3. For what value of x are the values of y in the two equations equal, or coincident?

4. What values of x and y will satisfy *both* equations?

The required values of x and y, then, are represented graphically by the coördinates of P, *the intersection of the graphs.*

II. Let the given equations be

$$\begin{cases} x + y = 7, \\ 2x + 2y = 14. \end{cases}$$

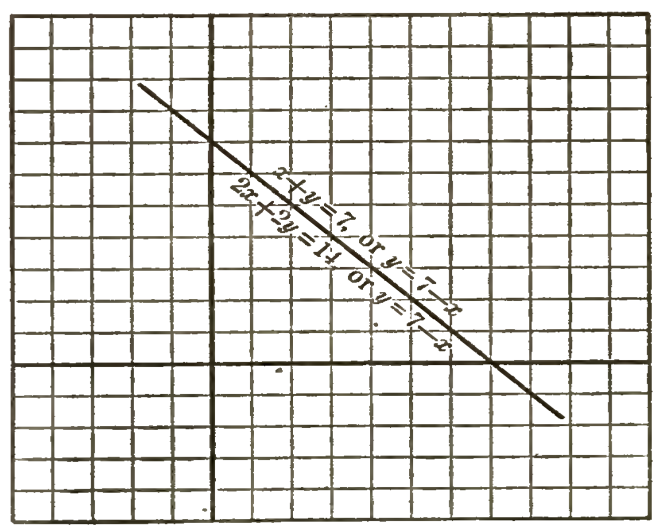


FIG. 9.

5. What happens if we try to eliminate either x or y?

6. Since $y = 7 - x$ in both equations, what will be the relative positions of any two points plotted for the same value of x? the relative positions of the two graphs?

7. The algebraic analysis shows that the equations are **indeterminate.**

The graphic analysis also shows that the equations are indeterminate, for *their graphs coincide.*

III. Let the given equations be

$$\begin{cases} y = 6 - x, & (1) \\ y = 4 - x. & (2) \end{cases}$$

8. When $x = -1$, how much greater is the value of y in (1) than in (2), as shown both by the equations and their graphs?

9. Compare the y's for other values of x.

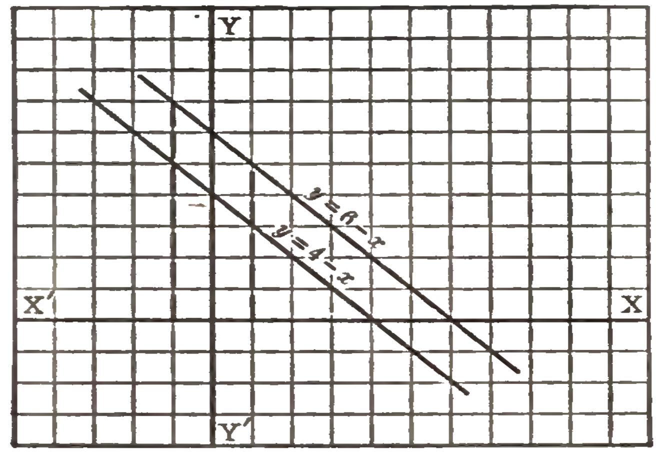


FIG. 10.

10. For every value of x the values of y in the two equations differ by 2, and the graphs are 2 units apart, vertically.

In algebraic language, the equations cannot be simultaneous; that is, they are **inconsistent**.

In graphical language, their graphs *cannot intersect*, being parallel straight lines.

253. PRINCIPLES. — 1. *A single linear equation involving two unknown numbers is indeterminate.*

2. *Two linear equations involving two unknown numbers are determinate, provided the equations are independent and simultaneous.*

They are satisfied by one, and only one, pair of common values.

3. *The pair of common values is represented graphically by the coördinates of the intersection of their graphs.*

EXERCISES

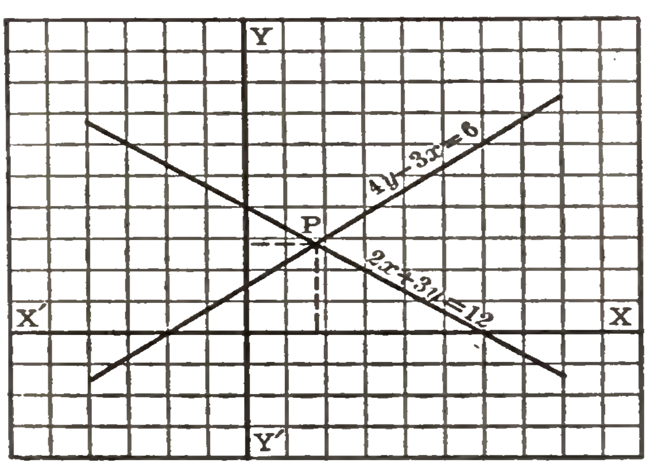


FIG. 11.

254. 1. Solve graphically the equations

$$\begin{cases} 4y - 3x = 6, \\ 2x + 3y = 12. \end{cases}$$

SOLUTION. — On plotting the graphs of both equations, as in § 248, it is found that they intersect at a point P, whose coördinates are 1.8 and 2.8, approximately.

Hence, $x = 1.8$ and $y = 2.8$.

The coördinates of P are estimated to the nearest tenth.

NOTE. — In solving simultaneous equations by the graphic method the same axes must be used for the graphs of both equations.

Construct the graphs of each of the following systems of equations. Solve, if possible. If there is no solution, tell why.

2. $\begin{cases} x - y = 1, \\ x + y = 9. \end{cases}$

3. $\begin{cases} x + y = 3, \\ x + 2y = 4. \end{cases}$

4. $\begin{cases} x = 4 + y, \\ y = 3 + x. \end{cases}$

5. $\begin{cases} 2x - y = 5, \\ 4x + y = 16. \end{cases}$

6. $\begin{cases} 3x = y + 9, \\ 2y = 6x - 18. \end{cases}$

7. $\begin{cases} y = 4x, \\ x - y = 3. \end{cases}$

8. $\begin{cases} x = \frac{1}{2}(y + 4), \\ y = 2(x - 2). \end{cases}$

9. $\begin{cases} x + y = -3, \\ x - 2y = -12. \end{cases}$

10. $\begin{cases} x + y = 4, \\ y = 2 - x. \end{cases}$

11. $\begin{cases} x = 2(y + 1), \\ 21 = 2(2x + y). \end{cases}$

12. $\begin{cases} x + y = 8, \\ 2x - 6y = -9. \end{cases}$

13. $\begin{cases} 2x - 5y = 5, \\ 10y = 2x + 1. \end{cases}$

14. $\begin{cases} 3y = 2x - 7, \\ 2x = 6 + 3y. \end{cases}$

15. $\begin{cases} 3(x - 4) = 2y, \\ 6(y + 6) = 9x. \end{cases}$

16. $\begin{cases} 10x + y = 14, \\ 8x - 5y = -2. \end{cases}$

17. $\begin{cases} 2x + 3y = 8, \\ 3x + 2y = 8. \end{cases}$

18. $\begin{cases} 4y + 3x = 5, \\ 4x - 3y = 3. \end{cases}$

19. $\begin{cases} x + 3y = -6, \\ 2x - 4y = -12. \end{cases}$

20. $\begin{cases} 4x - 10y = 0, \\ 2x + y = 12. \end{cases}$

21. $\begin{cases} x - 2y = 2, \\ 2y - 6x = 3. \end{cases}$

22. $\begin{cases} 3x + 4y = 10, \\ 6x + 8y = 20. \end{cases}$

23. $\begin{cases} \frac{3}{2}x + \frac{5}{2}y = 3\frac{1}{2}, \\ 10x - 2y = 14. \end{cases}$

REVIEW

255. 1. Distinguish between integral and fractional equations; between dependent and independent equations.

2. What is meant by the root of an equation? by solving an equation?

3. Define and illustrate equivalent equations.

4. What is a formula? Give a simple formula that has been used in solving some problem.

5. Find three values for x and y in $x + y = 15$. What kind of an equation is this, and why?

6. Define simultaneous equations; elimination. State the axiom upon which elimination by addition is based; elimination by comparison.

7. Outline the method of elimination by addition or subtraction; by substitution.

8. State what is meant by a graph. Of what practical use are graphs?

9. Define abscissa; ordinate; coördinates; origin.

10. In making graphs, where are positive values of x and y laid off? negative values? Interpret the equation $A = (-4, 3)$.

11. What is the abscissa of any point of the y-axis? the ordinate of any point of the x-axis? What are the coördinates of the origin?

12. Why are simple equations sometimes called linear equations?

13. Construct the graph of $2y = 3x - 4$.

14. How many points is it necessary to plot in drawing the graph of a simple equation? Why?

15. Tell how to determine where a graph crosses the x-axis; the y-axis.

16. In drawing the graph of the equation $3y = 2x$, what is the result, if the only points plotted are those where the graph intersects the axes? What must be done in a case like this?

17. Of what does the graphical solution of two simultaneous simple equations consist?

18. Solve graphically and algebraically:

$$\begin{cases} 2x - 3y = 10, \\ 5x + 2y = \ \ 6. \end{cases}$$

Compare the results obtained.

19. If a system of two linear equations is indeterminate, how will this fact be shown by the graphs of the equations referred to the same axes?

20. Draw the graphs of the two equations

$$\begin{cases} x + y = 6, \\ x = 13 - y, \end{cases}$$

and tell the algebraic meaning of the fact that the two graphs do not intersect.

21. From the following select the integral equations; the fractional equations; the numerical equations; the literal equations; the indeterminate equations:

(1) $3x + 5y = 19.$

(2) $\dfrac{7}{2x} + \dfrac{15}{3x} = 51.$

(3) $ax + bx = c.$

(4) $\dfrac{2x}{3} + \dfrac{25x}{9} = 31.$

(5) $5x + 2 = 3x - 10.$

(6) $\dfrac{a}{bx} + \dfrac{c}{dy} = a^2b^2.$

22. Classify the following sets of equations as equivalent equations, dependent equations, independent equations, simultaneous equations, or inconsistent equations:

(1) $\begin{cases} 3x = 18, \\ x + 5 = 11. \end{cases}$

(2) $\begin{cases} x + y = \ \ 7, \\ x + y = 11. \end{cases}$

(3) $\begin{cases} x + \ \ y = 6, \\ 3x - \ \ y = 2. \end{cases}$

(4) $\begin{cases} x - \ \ y = 2, \\ 3x - 3y = 6. \end{cases}$

INVOLUTION

256. The process of finding any required power of an expression is called **involution.**

257. The following illustrate powers of positive numbers, of negative numbers, of powers, of products, and of quotients, and show that every case of involution is an example of multiplication of *equal* factors.

POWERS OF A POSITIVE NUMBER	POWERS OF A NEGATIVE NUMBER	POWERS OF A POWER
$2 = 2^1$	$-2 = (-2)^1$	$4 = 2^2$
2	-2	4
$4 = 2^2$	$4 = (-2)^2$	$16 = (2^2)^2 = 2^4$
2	-2	4
$8 = 2^3$	$-8 = (-2)^3$	$64 = (2^2)^3 = 2^6$
2	-2	4
$16 = 2^4$	$16 = (-2)^4$	$256 = (2^2)^4 = 2^8$

POWER OF A PRODUCT

$$(2 \cdot 3)^2 = (2 \cdot 3) \times (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2.$$

POWER OF A QUOTIENT

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2^2}{3^2}.$$

258. From these examples and § 78, it is seen that, for involution:

Law of Signs. — *All powers of a positive number are positive; even powers of a negative number are positive, and odd powers are negative.*

259. From the examples in § 257 observe that, for involution:

Law of Exponents. — *The exponent of a power of a number is equal to the exponent of the number multiplied by the exponent of the power to which the number is to be raised.*

260. The last two examples in § 257 illustrate the following:

PRINCIPLES. — 1. *Any power of a product is equal to the product of its factors each raised to that power.*

2. *Any power of the quotient of two numbers is equal to the quotient of the numbers each raised to that power.*

261. AXIOM 6. — *The same powers of equal numbers are equal.*

Thus, if $x = 3$, $x^2 = 3^2$, or 9; also $x^4 = 3^4$, or 81; etc.

262. Involution of monomials.

EXERCISES

1. What is the third power of $4a^3b$?

SOLUTION

$$(4a^3b)^3 = 4a^3b \times 4a^3b \times 4a^3b = 64a^9b^3.$$

2. What is the fifth power of $-2ab^2$?

SOLUTION

$$(-2ab^2)^5 = -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 = -32a^5b^{10}.$$

To raise an integral term to any power:

RULE. — *Raise the numerical coefficient to the required power and annex to it each letter with an exponent equal to the product of its exponent by the exponent of the required power.*

Make the power positive or negative according to the law of signs.

Raise to the power indicated:

3. $(ab^2c^3)^2$.
4. $(a^3b^2c)^4$.
5. $(2a^2c)^3$.
6. $(7a^2m^5)^2$.
7. $(-4c^2y^5)^3$.
8. $(-2a^3n^5)^4$.
9. $(abcx)^m$.
10. $(2e^2x^c)^6$.
11. $(-1)^{99}$.
12. $(-1)^{200}$.
13. $(3bc)^n$.
14. $(2a^2x^3)^n$.

15. What is the square of $-\frac{5a^3x^2}{7b^2c}$?

SOLUTION

$$\left(-\frac{5a^3x^2}{7b^2c}\right)^2 = -\frac{5a^3x^2}{7b^2c} \times -\frac{5a^3x^2}{7b^2c} = \frac{25a^6x^4}{49b^4c^2}.$$

To raise a fraction to any power:

RULE. — *Raise both numerator and denominator to the required power and prefix the proper sign to the result.*

Raise to the power indicated:

16. $\left(\frac{1}{4b}\right)^2$.
17. $\left(\frac{2x}{7y}\right)^2$.
18. $\left(\frac{3x^2}{10y^3}\right)^2$.
19. $\left(-\frac{5}{ab}\right)^2$.
20. $\left(-\frac{2}{3x}\right)^4$.
21. $\left(-\frac{3x}{2y}\right)^3$.
22. $\left(-\frac{2a}{b}\right)^6$.
23. $\left(-\frac{b^3c^2}{a^2x}\right)^2$.
24. $\left(\frac{a^2b^3}{xy^4}\right)^n$.

263. Involution of polynomials.

The following are type forms of *squares* of polynomials:

§ 85, $(a+x)^2 = a^2 + 2ax + x^2$.

§ 88, $(a-x)^2 = a^2 - 2ax + x^2$.

§ 91, $(a-x+y)^2 = a^2 + x^2 + y^2 - 2ax + 2ay - 2xy$.

EXERCISES

264. Raise to the second power:

1. $2a+b$.
2. $2a-b$.
3. $x+3y$.
4. $3x-4y^3$.
5. $5m^4-11$.
6. $4rs^2+t^3$.
7. $2a+3b-4c$.
8. $5a^2-1+4n^3$.
9. $3r^2+2s+t^4$.

Raise to the required power by multiplication:

10. $(x+y)^3$.
11. $(x-y)^3$.
12. $(x+y)^4$.
13. $(x-y)^4$.
14. $(x+y)^5$.
15. $(x-y)^5$.

THE BINOMIAL FORMULA

265. By actual multiplication,

$$(a + x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3.$$

$$(a - x)^3 = a^3 - 3\,a^2x + 3\,ax^2 - x^3.$$

$$(a + x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4.$$

$$(a - x)^4 = a^4 - 4\,a^3x + 6\,a^2x^2 - 4\,ax^3 + x^4.$$

From the *expansions* just given, and as many others as the student may wish to obtain by multiplication, the following *observations* may be made in regard to any *positive integral power* of any *binomial*, the letter a standing for the first term and x for the second:

1. *The number of terms is one greater than the index of the required power.*

2. *The first term contains a only; the last term x only; all other terms contain both a and x.*

3. *The exponent of a in the first term is the same as the index of the required power and it decreases 1 in each succeeding term; the exponent of x in the second term is 1, and it increases 1 in each succeeding term.*

4. *In each term the sum of the exponents of a and x is equal to the index of the required power.*

5. *The coefficient of the first term is 1; the coefficient of the second term is the same as the index of the required power.*

6. *The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing this product by the number of the term.*

7. *All the terms are positive, if both terms of the binomial are positive.*

8. *The terms are alternately positive and negative, if the second term of the binomial is negative.*

EXERCISES

266. 1. Write by inspection the fifth power of $(b-y)$.

SOLUTION

Substituting b for a and y for x and applying the observations of § 265, (2 and 3 for the letters and exponents, 5 and 6 for the coefficients, and 8 for the signs) we have

$$(b-y)^5 = b^5 - 5b^4y + 10b^3y^2 - 10b^2y^3 + 5by^4 - y^5.$$

NOTE. — Observe that *the coefficients of the latter half of the expansion are the same as those of the first half, written in the reverse order.*

Expand:

2. $(m+n)^5$.
3. $(m-n)^5$.
4. $(a-c)^3$.
5. $(a+b)^3$.
6. $(b+d)^4$.
7. $(x-y)^4$.
8. $(c-n)^6$.
9. $(x-a)^7$.
10. $(d-y)^8$.
11. $(b+y)^6$.
12. $(x+4)^3$.
13. $(x+5)^3$.
14. $(x-2)^3$.
15. $(x+2)^4$.
16. $(a-3)^4$.

17. Write the expansion of $(2c^2-5)^4$.

SOLUTION

§ 265, $\quad (a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4.$

Substituting $2c^2$ for a and 5 for x, we have

$$(2c^2-5)^4 = (2c^2)^4 - 4(2c^2)^3 5 + 6(2c^2)^2 5^2 - 4(2c^2)5^3 + 5^4$$
$$= 16c^8 - 160c^6 + 600c^4 - 1000c^2 + 625.$$

18. Expand $(1+x^2)^3$, and test the result.

SOLUTION

§ 265, $\quad (a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$

Substituting 1 for a and x^2 for x, we have

$$(1+x^2)^3 = 1^3 + 3(1)^2(x^2) + 3(1)(x^2)^2 + (x^2)^3$$
$$= 1 + 3x^2 + 3x^4 + x^6.$$

TEST. — When $x=1$, $(1+x^2)^3 = 8$, and $1+3x^2+3x^4+x^6 = 8$; hence, $(1+x^2)^3 = 1+3x^2+3x^4+x^6$, and the expansion is correct.

Expand, and test results:

19. $(x+2y)^4$.
20. $(2x-y)^3$.
21. $(2x-5)^3$.
22. $(x^2-10)^4$.
23. $(1-3x^2)^4$.
24. $(5x^2-ab)^3$.
25. $(1+a^2b^2)^4$.
26. $(2ax-b)^5$.
27. $(1-x)^7$.
28. $(1-2x)^6$.
29. $(x-\frac{1}{2})^6$.
30. $(\frac{1}{2}x-\frac{1}{3}y)^4$.

Expand:

31. $\left(2a+\frac{1}{2}\right)^5$.
32. $\left(\frac{x}{y}-\frac{y}{x}\right)^4$.
33. $\left(\frac{x}{y}-\frac{y}{x}\right)^6$.
34. $\left(3a^2+\frac{b}{6}\right)^3$.
35. $\left(1+\frac{3x}{2}\right)^5$.
36. $\left(\frac{3}{5}+\frac{5x}{3}\right)^4$.
37. $\left(\frac{1}{2x}-2x\right)^5$.
38. $\left(\frac{1}{a}-a\right)^6$.
39. $\left(x+\frac{1}{x}\right)^7$.

40. Expand $(r-s-t)^3$.

SOLUTION

Since $(r-s-t)^3$ may be written in the binomial form, $(\overline{r-s}-t)^3$, we may substitute $(r-s)$ for a and t for x in

§ 265, $$(a-x)^3 = a^3-3a^2x+3ax^2-x^3.$$

Then, we have

$$\begin{aligned}(r-s-t)^3 &= (\overline{r-s}-t)^3\\ &= (r-s)^3-3(r-s)^2t+3(r-s)t^2-t^3\\ &= r^3-3r^2s+3rs^2-s^3-3t(r^2-2rs+s^2)+3rt^2-3st^2-t^3\\ &= r^3-3r^2s+3rs^2-s^3-3r^2t+6rst-3s^2t+3rt^2-3st^2-t^3.\end{aligned}$$

41. Expand $(a+b-c-d)^3$.

SUGGESTION. $(a+b-c-d)^3 = (\overline{a+b}-\overline{c+d})^3$, a binomial form.

Expand:

42. $(a+x-y)^3$.
43. $(a-m-n)^3$.
44. $(a-x+y)^3$.
45. $(a-x-y)^3$.
46. $(a+x+2)^3$.
47. $(a-x-2)^3$.
48. $(a+2b-3c)^3$.
49. $(a+b+x+y)^3$.

EVOLUTION

267. Just as (§ 132) one of the *two* equal factors of a number is its **second, or square, root**; so one of the *three* equal factors of a number is its **third, or cube, root**; one of the *four* equal factors, the **fourth root**; etc.

The second root of a number, as a, is indicated by $\sqrt{a}$; the third root by $\sqrt[3]{a}$; the fourth root by $\sqrt[4]{a}$; the fifth root by $\sqrt[5]{a}$; etc.

The sign $\sqrt{\ }$ is called the **root sign**, or the **radical sign**; the small figure in its opening is called the **index** of the root.

When no index is written, the second, or square, root is meant.

268. The process $2^3 = 2 \cdot 2 \cdot 2 = 8$ illustrates *involution*.

The process $\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$ illustrates **evolution**, which will be defined here as the process of finding a root of a number, or as the *inverse of involution*.

269. You have learned (§ 132) that *every number has two square roots, one positive and the other negative.*

For example, $\sqrt{25} = +5$ or -5.

The roots may be written together, thus: ± 5, read '*plus or minus five*'; or ∓ 5, read '*minus or plus five*.'

270. The square root of -16 is not 4, for $4^2 = +16$; nor -4, for $(-4)^2 = +16$. No number so far included in our number system can be a square root of -16 or of any other negative number.

It would be inconvenient and confusing to regard $\sqrt{a}$ as a number only when a is positive. In order to preserve the generality of the discussion of number, it is necessary, therefore, to admit square roots of negative numbers into our number system. The square roots of -16 are written

$$\sqrt{-16} \text{ and } -\sqrt{-16}.$$

Such numbers are called **imaginary numbers** and, in contrast, numbers that do not involve a square root of a negative number are called **real numbers**.

271. Just as every number has two square roots, so every number has three cube roots, four fourth roots, etc.

For example, the cube roots of 8 are the roots of the equation $x^3 = 8$, which later will be found to be

$$2, \ -1 + \sqrt{-3}, \text{ and } -1 - \sqrt{-3}.$$

The present discussion is concerned only with *real* roots.

272. Since $2^3 = 8$, $\sqrt[3]{8} = 2$.

Since $(-2)^3 = -8$, $\sqrt[3]{-8} = -2$.

Since $2^4 = 16$ and $(-2)^4 = 16$, $\sqrt[4]{16} = \pm 2$.

Since $2^5 = 32$, $\sqrt[5]{32} = 2$.

Since $(-2)^5 = -32$, $\sqrt[5]{-32} = -2$.

A root is **odd** or **even** according as its index is odd or even.

273. It follows from the preceding illustrations and from the law of signs for involution (§ 258) that, for real roots:

Law of Signs. — *An odd root of a number has the same sign as the number.*

An even root of a number may have either sign.

274. A real root of a number, if it has the same sign as the number itself, is called a **principal root** of the number.

The principal square root of 25 is 5, but not -5. The principal cube root of 8 is 2; of -8 is -2.

275. Axiom 7. — *The same roots of equal numbers are equal.*

Thus, if $x = 16$, $\sqrt{x} = 4$; if $x = 8$, $\sqrt[3]{x} = 2$; etc.

276. Since $(2^2)^3 = 2^{2\times 3} = 2^6$, the principal cube root of 2^6 is

$$\sqrt[3]{2^6} = 2^{6\div 3} = 2^2.$$

Hence, for evolution:

Law of Exponents. — *The exponent of any root of a number is equal to the exponent of the number divided by the index of the root.*

277. Since $(5\,a)^2 = 5^2a^2 = 25\,a^2$, the principal square root of $25\,a^2$ is

$$\sqrt{25\,a^2} = \sqrt{25}\cdot\sqrt{a^2} = 5\,a.$$

Hence, for principal roots:

PRINCIPLE. — *Any root of a product may be obtained by taking that root of each of the factors and finding the product of the results.*

278. Since $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$, the principal fourth root of $\frac{16}{81}$ is

$$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{\sqrt[4]{2^4}}{\sqrt[4]{3^4}} = \frac{2}{3}.$$

Hence, for principal roots:

PRINCIPLE. — *Any root of the quotient of two numbers is equal to that root of the dividend divided by that root of the divisor.*

279. Evolution of monomials.

EXERCISES

1. Find the square root of $36\,a^6b^2$.

SOLUTION

Since, in squaring a monomial, § 262, the coefficient is squared and the exponents of the letters are multiplied by 2, to find the square root, the *square root* of the coefficient must be found, and to it must be annexed the letters each with its exponent *divided* by 2.

The square root of 36 is 6, and the square root of the literal factors is a^3b. Therefore, the *principal* square root of $36\,a^6b^2$ is $6\,a^3b$.

The square root may also be $-6\,a^3b$, since $-6\,a^3b \times -6\,a^3b = 36\,a^6b^2$.

$$\therefore \sqrt{36\,a^6b^2} = \pm 6\,a^3b.$$

2. Find the cube root of $-125\,x^6y^{21}$.

SOLUTION

$$\sqrt[3]{-125\,x^6y^{21}} = -5\,x^2y^7, \text{ the real root.}$$

To find the root of an integral term:

RULE. — *Find the required root of the numerical coefficient, annex to it the letters each with its exponent divided by the index of the root sought, and prefix the proper sign to the result.*

Find real roots:

3. $\sqrt[3]{a^3b^9c^{15}}$.
4. $\sqrt{a^6b^{16}c^{14}}$.
5. $\sqrt[5]{a^{10}x^5y^{30}}$.
6. $\sqrt[4]{a^{4n}b^8c^{12}}$.
7. $\sqrt{x^{4n}y^8z^{2m}}$.
8. $\sqrt[3]{-8\,a^6b^{15}}$.
9. $\sqrt[5]{-32\,x^{10}y^{10}}$.
10. $\sqrt{16\,x^4y^2}$.
11. $\sqrt[7]{-a^{21}b^{35}x^{14}}$.
12. $\sqrt[5]{-243\,y^{5n}}$.
13. $\sqrt{(-mb^3)^2}$.
14. $\sqrt[3]{(-a^2b)^9}$.
15. $-\sqrt[9]{a^{18}b^9c^{27}}$.
16. $-\sqrt[3]{-27\,p^9r^3}$.
17. $-\sqrt[7]{-128\,a^{14}n^{28}}$.

18. Find the cube root of $\dfrac{-8\,x^9y^6}{27\,m^3n^{12}}$.

SOLUTION

$$\sqrt[3]{\frac{-8\,x^9y^6}{27\,m^3n^{12}}} = \frac{\sqrt[3]{-8\,x^9y^6}}{\sqrt[3]{27\,m^3n^{12}}} = \frac{-2\,x^3y^2}{3\,mn^4} = -\frac{2\,x^3y^2}{3\,mn^4}.$$

To find the root of a fractional term:

RULE. — *Find the required root of both numerator and denominator, and prefix the proper sign to the resulting fraction.*

Find real roots:

19. $\sqrt{\dfrac{64\,a^4b^6}{81\,m^2n^8}}$.
20. $\sqrt[7]{\dfrac{(x-y)^{14}}{128\,x^{14}}}$.
21. $\sqrt[3]{-125\dfrac{x^3}{y^9}}$.
22. $\sqrt[5]{\dfrac{-32\,a^5x^{10}}{243\,y^{15}}}$.
23. $\sqrt[3]{\left(-\dfrac{27\,a^3}{64\,b^6}\right)^2}$.
24. $\sqrt[3]{-\dfrac{125\,x^{12}y^{12}}{1728\,c^3}}$.

280. To find the square root of a polynomial.

EXERCISES

1. Derive the process for finding the square root of $a^2 + 2ab + b^2$.

PROCESS

$$
\begin{array}{lll}
 & a^2 + 2ab + b^2 & \underline{|a + b} \\
 & a^2 & \\
\text{Trial divisor, } 2a & \quad |2ab + b^2 & \\
\text{Complete divisor, } 2a + b & \quad |2ab + b^2 &
\end{array}
$$

EXPLANATION. — Since $a^2 + 2ab + b^2$ is the square of $(a + b)$, we know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

Since the first term of the root is a, it may be found by taking the square root of a^2, the first term of the power. On subtracting a^2, there is a remainder of $2ab + b^2$.

The second term of the root is known to be b, and that may be found by dividing the first term of the remainder by twice the part of the root already found. This divisor is called a *trial* divisor.

Since $2ab + b^2$ is equal to $b(2a + b)$, the complete divisor which multiplied by b produces the remainder $2ab + b^2$ is $2a + b$; that is, the complete divisor is found by adding the second term of the root to twice the root already found.

On multiplying the complete divisor by the second term of the root and subtracting, there is no remainder; then, $a + b$ is the required root.

2. Find the square root of $9x^2 - 30xy + 25y^2$.

PROCESS

$$
\begin{array}{lll}
 & 9x^2 - 30xy + 25y^2 & \underline{|3x - 5y} \\
 & 9x^2 & \\
\text{Trial divisor, } 6x & \quad |-30xy + 25y^2 & \\
\text{Complete divisor, } 6x - 5y & \quad |-30xy + 25y^2 &
\end{array}
$$

Find the square root of:

3. $4x^2 + 12x + 9$.

4. $x^2 + 2x + 1$.

5. $1 - 4m + 4m^2$.

6. $c^2 - 12c + 36$.

7. $4x^2 + 4x + 1$.

8. $16 + 24x + 9x^2$.

Since, in squaring $a + b + c$, $a + b$ may be represented by x, and the square of the number by $x^2 + 2xc + c^2$, the square root of a number whose root consists of *more than the two terms* may be obtained in the same way as in exercise 1, *by considering the terms already found as one term.*

9. Find the square root of $4x^4 + 12x^3 - 3x^2 - 18x + 9$.

PROCESS

$$\begin{array}{l|l|l}
 & 4x^4 + 12x^3 - 3x^2 - 18x + 9 & 2x^2 + 3x - 3 \\
 & 4x^4 & \\
\hline
4x^2 & 12x^3 - 3x^2 & \\
4x^2 + 3x & 12x^3 + 9x^2 & \\
\hline
4x^2 + 6x & -12x^2 - 18x + 9 & \\
4x^2 + 6x - 3 & -12x^2 - 18x + 9 & \\
\hline
\end{array}$$

EXPLANATION.—Proceeding as in exercise 2, we find that the first two terms of the root are $2x^2 + 3x$.

Considering $(2x^2 + 3x)$ as the first term of the root, we find the next term of the root as we found the second term, by dividing the remainder by twice the part of the root already found. Hence, the trial divisor is $4x^2 + 6x$, and the next term of the root is -3. Annexing this, as before, to the trial divisor already found, we find that the complete divisor is $4x^2 + 6x - 3$. Multiplying this by -3 and subtracting the product from $-12x^2 - 18x + 9$, we have no remainder.

Hence, the square root of the number is $2x^2 + 3x - 3$.

RULE.—*Arrange the terms of the polynomial with reference to the consecutive powers of some letter.*

Find the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, used as a trial divisor, and the quotient will be the next term of the root. Write this result in the root, and annex it to the trial divisor to form the complete divisor.

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before and continue in this manner until all the terms of the root have been found.

Find the square root of:

10. $25a^2 - 40a + 16$.

11. $900x^2 + 60x + 1$.

12. $x^2 + xy + \frac{1}{4}y^2$.

13. $4x^4 - 52x^2 + 169$.

14. $9x^4 - 12x^3 + 10x^2 - 4x + 1$.

15. $x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4$.

16. $x^8 + 2a^6x^2 - a^4x^4 - 2a^2x^6 + a^8$.

17. $25x^4 + 4 - 12x - 30x^3 + 29x^2$.

18. $1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6$.

19. $\frac{a^4}{9} - \frac{4a^3}{3} + 4a^2 + \frac{a^2b}{3} - 2ab + \frac{b^2}{4}$.

20. Find *four* terms of the square root of $1 + x$.

SQUARE ROOT OF ARITHMETICAL NUMBERS

281. Compare the number of digits in the square root of each of the following numbers with the number of digits in the number itself:

NUMBER	ROOT	NUMBER	ROOT	NUMBER	ROOT
1	1	1'00	10	1'00'00	100
25	5	10'24	32	56'25'00	750
81	9	98'01	99	99'80'01	999

From the preceding comparison it may be observed that:

PRINCIPLE. — *If a number is separated into periods of two digits each, beginning at units, its square root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one digit.

282. If the number of units expressed by the tens' digit is represented by t and the number of units expressed by the units' digit by u, any number consisting of tens and units will be represented by $t+u$, and its square by $(t+u)^2$, or $t^2+2tu+u^2$.

Since $25 = 20 + 5$, $25^2 = (20 + 5)^2 = 20^2 + 2(20 \times 5) + 5^2 = 625$.

EXERCISES

283. 1. Find the square root of 3844.

FIRST PROCESS

$$\begin{array}{rr|l} & 38'44 & \underline{60+2} \\ t^2 = & 36\,00 & \\ \hline 2\,t = 120 & 2\,44 & \\ u = 2 & & \\ \hline 2\,t + u = 122 & \underline{2\,44} & \end{array}$$

EXPLANATION. — Separating the number into periods of two digits each (Prin., § 281), we find that the root is composed of two digits, tens and units. Since the largest square in 38 is 6, the tens of the root cannot be greater than 6 tens, or 60. Writing 6 tens in the root, squaring, and subtracting from 3844, we have a remainder of 244.

Since the square of a number composed of tens and units is equal to (*the square of the tens*) + (*twice the product of the tens and the units*) + (*the square of the units*), when the square of the tens has been subtracted, the remainder, 244, is twice the product of the tens and the units, plus the square of the units, or only a little more than twice the product of the tens and the units.

Therefore, 244 divided by twice the tens is approximately equal to the units. 2×6 tens, or 120, then, is a *trial*, or *partial divisor*. On dividing 244 by the trial divisor, the units' figure is found to be 2.

Since twice the tens are to be multiplied by the units, and the units also are to be multiplied by the units to obtain the square of the units, in order to abridge the process the tens and units are first added, forming the *complete divisor* 122, which is then multiplied by the units. Thus, $(120 + 2)$ multiplied by $2 = 244$.

Therefore, the square root of 3844 is 62.

SECOND PROCESS

$$\begin{array}{rr|l} & 38'44 & \underline{62} \\ t^2 = & 36 & \\ \hline 2\,t = 120 & 2\,44 & \\ u = 2 & & \\ \hline 2\,t + u = 122 & \underline{2\,44} & \end{array}$$

EXPLANATION. — In practice it is usual to place the figures of the same order in the same column, and to disregard the ciphers on the right of the products.

Since any number may be regarded as composed of tens and units, the foregoing processes have a general application.

Thus, $346 = 34$ tens $+ 6$ units; $2377 = 237$ tens $+ 7$ units.

2. Find the square root of 104976.

SOLUTION

		10′49′76 | 324
		9
Trial divisor	$= 2 \times 30 = 60$	1 49
Complete divisor	$= 60 + 2 = 62$	1 24
Trial divisor	$= 2 \times 320 = 640$	25 76
Complete divisor	$= 640 + 4 = 644$	25 76

RULE. — *Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period and write its root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root last found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

1. When the number is not a perfect square, annex periods of decimal ciphers and continue the process.
2. Decimals are pointed off from the decimal point toward the right.
3. The square root of a common fraction may be obtained by finding the square root of both numerator and denominator separately or by reducing the fraction to a decimal and then finding the root.

Find the square root of:

3. 529.
4. 2209.
5. 4761.
6. 57121.
7. 42025.
8. 95481.
9. 2480.04.
10. 10.9561.
11. .001225.

Find the square root of:

12. $\frac{625}{729}$. 14. $\frac{169}{225}$. 16. $\frac{289}{324}$. 18. $\frac{576}{784}$.

13. $\frac{576}{841}$. 15. $\frac{196}{1156}$. 17. $\frac{361}{400}$. 19. $\frac{289}{961}$.

Find the square root to two decimal places:

20. $\frac{3}{4}$. 22. $\frac{5}{8}$. 24. $\frac{5}{6}$. 26. $\frac{7}{8}$.

21. $\frac{4}{5}$. 23. .6. 25. $\frac{2}{9}$. 27. $\frac{5}{16}$.

ROOTS BY FACTORING

284. The method of finding the cube root of polynomials and of arithmetical numbers, analogous to the one just given for square root, is beyond the scope of this text; but a method of finding the cube root, or any other root, of a number that is a perfect power of the same degree as the index of the required root is here mentioned because of its simplicity.

This method consists in factoring, grouping the factors, and taking the required root of each group.

Thus, $\sqrt[3]{42875} = \sqrt[3]{5 \cdot 5 \cdot 5 \times 7 \cdot 7 \cdot 7} = \sqrt[3]{5^3 \times 7^3} = 5 \times 7 = 35$;

also,
$$\begin{aligned}\sqrt{x^4 + 2x^3 - 3x^2 - 4x + 4} &= \sqrt{(x-1)^2(x+2)^2}\\ &= (x-1)(x+2)\\ &= x^2 + x - 2.\end{aligned}$$

EXERCISES

285. Find, by the method of factoring:

1. Square root of $a^6 - 12a^3 + 36$.
2. Cube root of $x^3 - 15x^2 + 75x - 125$.
3. Fourth root of $x^4 - 8x^3 + 24x^2 - 32x + 16$.
4. Fifth root of $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$.

Find the indicated root:

5. $\sqrt[3]{3375}$. 7. $\sqrt[6]{262144}$. 9. $\sqrt[5]{4084101}$.

6. $\sqrt[4]{1296}$. 8. $\sqrt[5]{759375}$. 10. $\sqrt[8]{16777216}$.

RADICALS

286. Thus far the exponents used have been *positive integers* only, and the **laws of exponents** have been based on this idea; but since zero, fractional, and negative exponents may occur in algebraic processes, they must follow the same laws as are given for positive integral exponents; hence, it becomes necessary to discover *meanings* for these *new kinds* of exponents, because, for example, in a^0, a^{-2}, and $a^{\frac{2}{3}}$, the exponents $0, -2$, and $\frac{2}{3}$ cannot show how many times a is used as a factor (§ 9).

287. Meaning of zero and negative exponents.

By notation, §§ 9, 10, $$a^2 = 1 \cdot a \cdot a. \quad (1)$$

Dividing both members of this equation by a, the first member by subtracting exponents (§ 32) and the second by taking out the factor a, we have

$$a^1 = 1 \cdot a. \quad (2)$$

Dividing (2) by a, $$a^0 = 1. \quad (3)$$

Dividing (3) by a, $$a^{-1} = \frac{1}{a}. \quad (4)$$

Dividing (4) by a, $$a^{-2} = \frac{1}{a^2}. \quad (5)$$

The meaning of a zero exponent, illustrated in (3), and of a negative exponent, in (4) and (5), may be stated as follows:

Any number with a zero exponent is equal to 1.

Any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

288. The meaning of a negative exponent shows that:

PRINCIPLE. — *Any factor may be transferred from one term of a fraction to the other without changing the value of the fraction, provided the sign of the exponent is changed.*

289. Meaning of a fractional exponent.

Just as, § 276, $\sqrt{a^2} = a^{2 \div 2} = a;$

so $\sqrt[3]{a^2} = a^{2 \div 3} = a^{\frac{2}{3}}.$ That is,

The numerator of a fractional exponent with positive integral terms indicates a power and the denominator a root.

Since the operations may be performed in either order:

The fractional exponent as a whole indicates a root of a power or a power of a root.

EXERCISES

290. Find a simple value for:

1. $5^0.$ 3. $2^{-5}.$ 5. $(-3)^0.$ 7. $(a^n b^2 q)^0.$

2. $4^{-2}.$ 4. $3^{-3}.$ 6. $(-6)^{-2}.$ 8. $(-\frac{1}{3})^{-2}.$

9. Which is the greater, $(\frac{1}{5})^2$ or $(\frac{1}{5})^3$? $(\frac{1}{5})^{-2}$ or $(\frac{1}{5})^{-3}$?

10. Write $5x^{-3}y^2$ with positive exponents.

SOLUTION. — By § 287, $5x^{-3}y^2 = 5y^2\frac{1}{x^3} = \frac{5y^2}{x^3}.$

Write with positive exponents:

11. $2x^{-1}.$ 13. $a^{-1}b^{-1}.$ 15. $4a^2c^{-2}.$

12. $5a^{-5}.$ 14. $x^{-3}y^{-2}.$ 16. $3ax^{-2}$

17. Write $\frac{3a^2y}{bx^2}$ without a denominator.

SOLUTION. — By § 288, $\frac{3a^2y}{bx^2} = 3a^2b^{-1}x^{-2}y.$

Write without a denominator:

18. $\frac{ax}{by}.$ 19. $\frac{mn}{a^2}.$ 20. $\frac{1}{a^{-2}b^2}.$ 21. $\left(\frac{x}{y}\right)^2.$

22. Find the value of $16^{\frac{3}{4}}$.

First Solution. $16^{\frac{3}{4}} = \sqrt[4]{16^3} = \sqrt[4]{16 \cdot 16 \cdot 16}$
$= \sqrt[4]{(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)}$
$= \sqrt[4]{(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)}$
$= 2 \cdot 2 \cdot 2 = 8.$

Second Solution. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8.$

In numerical exercises it is usually best to find the root first.

Simplify, taking only principal roots:

23. $8^{\frac{1}{3}}$.
24. $8^{\frac{2}{3}}$.
25. $64^{\frac{2}{3}}$.
26. $32^{\frac{3}{5}}$.
27. $64^{-\frac{2}{3}}$.
28. $(-8)^{-\frac{4}{3}}$.

29. Which is the greater, $27^{\frac{2}{3}}$ or $(-27)^{-\frac{2}{3}}$? $(\frac{1}{4})^{\frac{3}{2}}$ or $(\frac{1}{4})^{-\frac{3}{2}}$?

30. Express $\sqrt[3]{a^2bc^{-4}}$ with positive fractional exponents.

Solution.
$$\sqrt[3]{a^2bc^{-4}} = a^{\frac{2}{3}}b^{\frac{1}{3}}c^{-\frac{4}{3}} = \frac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{c^{\frac{4}{3}}}.$$

Express with positive fractional exponents:

31. $\sqrt{ab^3}$.
32. $\sqrt{xy}$.
33. $(\sqrt{x})^3$.
34. $(\sqrt[5]{y})^4$.
35. $(\sqrt[3]{xy})^{-2}$.
36. $5\sqrt{x^{-1}y^{-1}}$.

Express roots with radical signs and powers with positive exponents:

37. $a^{\frac{2}{5}}$.
38. $x^{\frac{4}{5}}$.
39. $x^{\frac{5}{4}}$.
40. $a^{\frac{1}{3}}b^{\frac{2}{3}}$.
41. $x^{\frac{5}{6}}y^{\frac{1}{6}}$.
42. $a^{\frac{1}{4}}b^{-\frac{3}{4}}$.
43. $a^{\frac{1}{2}} \div x^{\frac{1}{2}}$.
44. $x^{\frac{2}{3}} \div y^{\frac{4}{3}}$.

Multiply:

45. a^3 by a^{-2}.
46. a^2 by a^{-1}.
47. a^4 by a^{-4}.
48. a by a^{-3}.
49. a^2 by a^0.
50. $x^{\frac{1}{2}}$ by $x^{\frac{1}{2}}$.

Divide:

51. a^5 by a^6.
52. a^3 by a^0.
53. a^2 by a^{-2}.
54. $x^{\frac{5}{2}}$ by $x^{-\frac{1}{2}}$.
55. $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$.
56. $x^{n-\frac{3}{2}}$ by x^{n-2}.

Solve for values of x corresponding to principal roots by applying axioms 6 and 7 (§§ 261, 275), and test each result:

57. $x^{\frac{1}{2}} = 7.$

58. $x^{\frac{3}{4}} = 8.$

59. $x^{\frac{4}{3}} = 81.$

60. $\frac{1}{3} x^{\frac{3}{2}} = 72.$

61. $x^{-\frac{1}{2}} = 6.$

62. $x^{-\frac{2}{3}} = 144.$

63. $25 x^{-\frac{2}{3}} = 1.$

64. $x^{\frac{5}{3}} + 32 = 0.$

291. An indicated root of a number is called a **radical**; the number whose root is required is called the **radicand.**

$\sqrt{5a}$, $(x^6)^{\frac{1}{3}}$, $\sqrt[3]{a^2+2}$, and $(x+y)^{\frac{1}{4}}$ are radicals whose radicands are, respectively, $5a$, x^6, a^2+2, and $x+y$.

292. The **order** of a radical is shown by the index of the root or by the denominator of the fractional exponent.

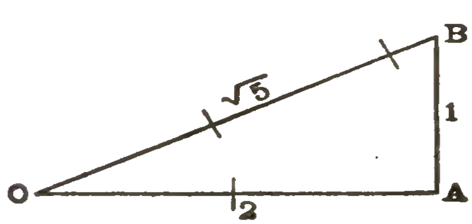
$\sqrt{a+x}$ and $(b-x)^{\frac{1}{2}}$ are radicals of the second order.

293. In the discussion and treatment of radicals only *principal roots* will be considered.

Thus, $\sqrt{16}$ will be taken to represent only the principal square root of 16, or 4. The other square root will be denoted by $-\sqrt{16}$.

294. Graphical representation of a radical of the second order. In geometry it is shown that the hypotenuse of a right triangle is equal to the *square root of the sum of the squares of the other two sides;* consequently, a radical of the second order may be represented graphically by the *hypotenuse* of a right triangle whose other two sides are such that the sum of their squares is equal to the radicand.

Thus, to represent $\sqrt{5}$ graphically, since it may be observed that $5 = 2^2 + 1^2$, draw OA 2 units in length, then draw AB 1 unit in length in a direction perpendicular to OA. Draw OB, completing the right-angled triangle OAB. Then, the length of OB represents $\sqrt{5}$ in its relation to the unit length.



EXERCISES

295. Represent graphically:

1. $\sqrt{2}$.
2. $\sqrt{10}$.
3. $\sqrt{13}$.
4. $\sqrt{17}$.
5. $\sqrt{34}$.
6. $\sqrt{25}$.
7. $\sqrt{\frac{5}{4}}$.
8. $\sqrt{\frac{13}{36}}$.

296. A number that is, or may be, expressed as an integer or as a fraction with integral terms, is called a **rational number.**

3, $\frac{1}{2}$, $\sqrt{25}$, and .333 are rational numbers.

297. A number that cannot be expressed as an integer or as a fraction with integral terms is called an **irrational number.**

$\sqrt{2}$, $4^{\frac{1}{3}}$, $1 + \sqrt{3}$, and $\sqrt{1 + \sqrt{3}}$ are irrational numbers.

From § 294, it will be observed that the irrational number $\sqrt{5}$ can be represented graphically by a line of *exact* length, though it cannot be represented exactly by decimal figures, for $\sqrt{5} = 2.236\ldots$, which is an endless decimal.

298. When the indicated root of a *rational* number cannot be obtained exactly, the expression is called a **surd.**

$\sqrt{2}$ is a surd, since 2 is rational but has no rational square root.

$\sqrt{1 + \sqrt{3}}$ is not a surd, because $1 + \sqrt{3}$ is not rational.

Radicals may be either rational or irrational, but surds are always irrational.

Both $\sqrt{4}$ and $\sqrt{3}$ are radicals, but only $\sqrt{3}$ is a surd.

299. A surd may contain a *rational factor*, that is, a factor whose radicand is a perfect power of a degree corresponding to the order of the surd.

The rational factor may be removed and written as the coefficient of the irrational factor.

In $\sqrt{8} = \sqrt{4 \times 2}$ and $\sqrt[3]{54} = \sqrt[3]{27 \times 2}$, the rational factors are $\sqrt{4}$ and $\sqrt[3]{27}$, respectively; that is, $\sqrt{8} = 2\sqrt{2}$ and $\sqrt[3]{54} = 3\sqrt[3]{2}$.

300. In the following pages it will be assumed that irrational numbers obey the same law as rational numbers. For proofs of the generality of these laws, the reader is referred to the author's *Advanced Algebra.*

REDUCTION OF RADICALS

301. To reduce a radical to its simplest form.

As the work progresses the student will discover the meaning of *simplest form.*

EXERCISES

302. 1. Reduce $\sqrt{20\,a^6}$ to its simplest form by writing the rational factor as the coefficient of the irrational factor.

PROCESS

$$\sqrt{20\,a^6} = \sqrt{4\,a^6 \times 5} = \sqrt{4\,a^6} \times \sqrt{5} = 2\,a^3\sqrt{5}$$

Explanation. — Since the highest factor of $20\,a^6$ that is a perfect square is $4\,a^6$, $\sqrt{20\,a^6}$ is separated into two factors, a rational factor $\sqrt{4\,a^6}$, and an irrational factor $\sqrt{5}$; that is, § 277, $\sqrt{20\,a^6} = \sqrt{4\,a^6} \times \sqrt{5}$.

On finding the square root of $4\,a^6$ and prefixing the root to the irrational factor as a coefficient, the result is $2\,a^3\sqrt{5}$.

2. Reduce $\sqrt[3]{-864}$ to its simplest form.

PROCESS

$$\sqrt[3]{-864} = \sqrt[3]{-216 \times 4} = \sqrt[3]{-216} \times \sqrt[3]{4} = -6\sqrt[3]{4}$$

Rule. — *Separate the radical into two factors one of which is its highest rational factor.*

Find the required root of the rational factor, multiply the result by the coefficient, if any, of the given radical, and place the product as the coefficient of the irrational factor.

Reduce to simplest form:

3. $\sqrt{12}$.
4. $\sqrt{75}$.
5. $\sqrt[3]{16}$.
6. $\sqrt{128}$.
7. $\sqrt[3]{250}$.
8. $\sqrt[4]{32}$.
9. $\sqrt{18\,a^2}$.
10. $\sqrt{25\,b}$.
11. $\sqrt{98\,c^3}$.
12. $\sqrt{50\,a}$.
13. $\sqrt{243\,a^5x^{10}}$.
14. $\sqrt[3]{128\,a^6b^4}$.
15. $(a^3 + 5\,a^2)^{\frac{1}{2}}$.
16. $\sqrt{18\,x - 9}$.
17. $\sqrt[3]{x^6 - 2\,x^3}$.

18. Reduce $\sqrt{\frac{a^2}{2y^3}}$ to its simplest form; that is, to a radical having an integral radicand.

PROCESS

$$\sqrt{\frac{a^2}{2y^3}} = \sqrt{\frac{a^2 \times 2y}{2y^3 \times 2y}} = \sqrt{\frac{a^2}{4y^4}} \times \sqrt{2y} = \frac{a}{2y^2}\sqrt{2y}$$

EXPLANATION. — Since the denominator must be removed from the radical, and since the radical is of the second order, the denominator must be made a perfect square. The smallest factor that will do this is $2y$.

On multiplying both terms of the fraction by this factor, the largest rational factor of the resulting radical is found to be $\sqrt{\frac{a^2}{4y^4}}$, or $\frac{a}{2y^2}$.

Therefore, the irrational factor is $\sqrt{2y}$ and its coefficient is $\frac{a}{2y^2}$.

Reduce to simplest form:

19. $\sqrt{\frac{1}{2}}$.

20. $\sqrt{\frac{2}{3}}$.

21. $\sqrt{\frac{3}{8}}$.

22. $\sqrt[3]{\frac{4}{5}}$.

23. $\sqrt[3]{\frac{5}{12}}$.

24. $\sqrt{\frac{2a^3}{b}}$.

25. $\sqrt{\frac{5x^4y^2}{2a^2}}$.

26. $\sqrt[4]{\frac{x}{y}}$.

27. $\sqrt{\frac{2}{3y^5}}$.

28. $\sqrt{\frac{4a}{3x^2}}$.

29. $\sqrt{\frac{3x}{50a^3y}}$.

303. Although $\frac{3}{6} = \frac{1}{2}$, it does not follow that $64^{\frac{3}{6}} = 64^{\frac{1}{2}}$, for each fractional exponent denotes a power of a root of 64, and the roots and powers taken are not the same for $64^{\frac{1}{2}}$ as for $64^{\frac{3}{6}}$. By trial, however, it is found that each number is equal to 8; and in general it may be proved that

A number having a fractional exponent is not changed in value by reducing the fractional exponent to higher or lower terms.

EXERCISES

304. 1. Reduce $\sqrt[6]{9a^2}$ to its simplest form; that is, to a radical having the smallest index possible.

PROCESS

$$\sqrt[6]{9a^2} = \sqrt[6]{(3a)^2} = (3a)^{\frac{2}{6}} = (3a)^{\frac{1}{3}} = \sqrt[3]{3a}$$

2. Reduce $\sqrt[9]{64\,a^6b^{15}}$ to its simplest form.

PROCESS

$$\sqrt[9]{64\,a^6b^{15}} = \sqrt[9]{2^6a^6b^6b^9} = b(2\,ab)^{\frac{6}{9}} = b(2\,ab)^{\frac{2}{3}} = b\sqrt[3]{4\,a^2b^2}$$

Simplify:

3. $\sqrt[4]{36}$.
4. $\sqrt[4]{25}$.
5. $\sqrt[4]{1600}$.
6. $\sqrt[6]{27\,a^3}$.
7. $\sqrt[4]{9\,a^2b^2c^6}$.
8. $\sqrt[4]{121\,a^6x^4}$.

305. The student has doubtless discovered that:

A radical is in its **simplest form** when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power whose exponent corresponds with the index of the root.

$\sqrt{7}$ is in its simplest form; but $\sqrt{\frac{3}{4}}$ is not in its simplest form, because $\frac{3}{4}$ is not integral in form; $\sqrt{8}$ is not in its simplest form, because 4, a factor of 8, has an exact square root; $\sqrt[6]{25}$, or $25^{\frac{1}{6}}$, is not in its simplest form, because $25^{\frac{1}{6}} = (5^2)^{\frac{1}{6}} = 5^{\frac{2}{6}} = 5^{\frac{1}{3}}$, or $\sqrt[3]{5}$.

MISCELLANEOUS EXERCISES

306. Reduce to simplest form:

1. $\sqrt{600}$.
2. $\sqrt{500}$.
3. $\sqrt[5]{160}$.
4. $\sqrt[3]{3000}$.
5. $\sqrt[3]{189}$.
6. $\sqrt{84}$.
7. $\sqrt[3]{72}$.
8. $\sqrt[3]{192}$.
9. $\sqrt[4]{144}$.
10. $\sqrt[6]{81}$.
11. $\sqrt[6]{343}$.
12. $\sqrt[4]{289}$.
13. $\sqrt{\frac{1}{3}}$.
14. $\sqrt{\frac{1}{x^3}}$.
15. $\sqrt[3]{\frac{a}{3\,b^2}}$.
16. $\sqrt{405\,a^5y^2}$.
17. $(135\,x^4y^5)^{\frac{1}{3}}$.
18. $\sqrt{8 - 20\,b^2}$.
19. $5\sqrt{4\,a^2 + 4}$.
20. $\sqrt[6]{a^4b^2c^4d^8}$.
21. $(16\,x - 16)^{\frac{1}{2}}$.

307. A surd that has a rational coefficient is called a **mixed surd.**

$2\sqrt{2}$, $a\sqrt[3]{x^2}$, and $(a - b)\sqrt{a + b}$ are mixed surds.

308. A surd that has no rational coefficient except unity is called an **entire surd.**

$\sqrt{5}$, $\sqrt[3]{11}$, and $\sqrt{a^2 + x^2}$ are entire surds.

309. To reduce a mixed surd to an entire surd.

EXERCISES

1. Express $2a\sqrt{5b}$ as an entire surd.

PROCESS

$$2a\sqrt{5b} = \sqrt{4a^2}\sqrt{5b} = \sqrt{4a^2 \times 5b} = \sqrt{20a^2b}$$

RULE. — *Raise the coefficient to a power corresponding to the index of the given radical, and introduce the result under the radical sign as a factor.*

Express as entire surds:

2. $2\sqrt{2}$.	5. $3\sqrt[3]{3}$.	8. $\frac{1}{2}\sqrt{2}$.	11. $\frac{4}{5}\sqrt{4\frac{3}{8}}$.
3. $3\sqrt{5}$.	6. $4\sqrt{5}$.	9. $\frac{3}{4}\sqrt{x^5}$.	12. $\frac{3}{2}\sqrt{\frac{33}{27}a^2}$.
4. $5\sqrt{2}$.	7. $\frac{1}{2}\sqrt{8}$.	10. $\frac{1}{2}\sqrt{bc}$.	13. $\frac{2}{3}\sqrt[3]{1\frac{1}{8}}$.

310. To reduce radicals to the same order.

EXERCISES

1. Reduce $\sqrt[4]{3}$, $\sqrt{2}$, and $\sqrt[3]{4}$ to radicals of the same order.

PROCESS

$$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27}$$
$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64}$$
$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256}$$

RULE. — *Express the given radicals with fractional exponents having a common denominator.*

Raise each number to the power indicated by the numerator of its fractional exponent, and indicate the root expressed by the common denominator.

Reduce to radicals of the same order:

2. $\sqrt{2}$ and $\sqrt[4]{3}$.

3. $\sqrt{5}$ and $\sqrt[3]{6}$.

4. $\sqrt[4]{7}$ and $\sqrt{10}$.

5. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$.

6. $\sqrt[6]{4}$, $\sqrt[4]{2}$, and $\sqrt{3}$.

7. $\sqrt[10]{13}$, $\sqrt{5}$, and $\sqrt[5]{4}$.

8. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{\frac{1}{27}}$.

9. $\sqrt{ab}$, $\sqrt[3]{ab^2}$, and $\sqrt[4]{2}$.

10. $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{x}$, and $\sqrt[6]{y}$.

11. $\sqrt[3]{a+b}$ and $\sqrt{x+y}$.

12. Which is greater, $\sqrt[5]{5}$ or $\sqrt{2}$? $\sqrt[3]{4}$ or $\sqrt{3}$?

13. Which is greater, $\sqrt[3]{3}$ or $\sqrt[4]{4}$? $3\sqrt{2}$ or $2\sqrt[3]{4}$?

Arrange in order of ascending value:

14. $\sqrt[3]{3}$, $\sqrt{2}$, and $\sqrt[6]{7}$.

15. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$.

16. $\sqrt[3]{2}$, $\sqrt[5]{3}$, and $\sqrt[15]{30}$.

17. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt{2\frac{1}{2}}$, and $\sqrt[3]{4}$.

18. $\sqrt{7}$, $\sqrt[4]{48}$, $\sqrt[3]{4}$, and $\sqrt[6]{63}$.

19. $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt[6]{13}$, $\sqrt[12]{150}$.

ADDITION AND SUBTRACTION OF RADICALS

311. Radicals that in their simplest form are of the same order and have the same radicand are called **similar radicals.**

Thus, $2\sqrt{3}$ and $7\sqrt{3}$ are similar radicals.

312. PRINCIPLE. — *Only similar radicals can be united into one term by addition or subtraction.*

EXERCISES

313. 1. Find the sum of $\sqrt{50}$, $2\sqrt[6]{8}$, and $6\sqrt{\frac{1}{2}}$.

PROCESS

$$\begin{aligned} \sqrt{50} &= 5\sqrt{2} \\ 2\sqrt[6]{8} &= 2\sqrt{2} \\ 6\sqrt{\tfrac{1}{2}} &= 3\sqrt{2} \\ \hline \text{Sum} &= 10\sqrt{2} \end{aligned}$$

EXPLANATION. — To ascertain whether the given expressions are similar radicals, each may be reduced to its simplest form. Since, in their simplest form, they are of the same order and have the same radicand, they are similar, and their sum is obtained by prefixing the sum of the coefficients to the common radical factor.

Find the sum of:

2. $\sqrt{50}$, $\sqrt{18}$, and $\sqrt{98}$.

3. $\sqrt{27}$, $\sqrt{12}$, and $\sqrt{75}$.

4. $\sqrt{20}$, $\sqrt{80}$, and $\sqrt{45}$.

5. $\sqrt{28}$, $\sqrt{63}$, and $\sqrt{700}$.

6. $\sqrt[3]{250}$, $\sqrt[3]{16}$, and $\sqrt[3]{54}$.

7. $\sqrt[3]{128}$, $\sqrt[3]{686}$, and $\sqrt[3]{\frac{1}{4}}$.

8. $\sqrt[3]{135}$, $\sqrt[3]{320}$, and $\sqrt[3]{625}$.

9. $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{-32}$.

10. $\sqrt{\frac{1}{2}}$, $\sqrt{12\frac{1}{2}}$, $\sqrt{\frac{1}{8}}$, and $\sqrt{1\frac{1}{8}}$.

11. $\sqrt{\frac{1}{3}}$, $\sqrt{75}$, $\frac{2}{3}\sqrt{3}$, and $\sqrt{12}$.

12. $\sqrt{\frac{3}{4}}$, $\frac{1}{3}\sqrt{3}$, $\frac{7}{6}\sqrt[4]{9}$, and $\sqrt{147}$.

13. $\sqrt[3]{40}$, $\sqrt{28}$, $\sqrt[6]{25}$, and $\sqrt{175}$.

14. $\sqrt{147}$, $4\sqrt{20}$, $\sqrt{75}$, and $\sqrt{605}$.

15. $\sqrt[3]{192}$, $\sqrt{80}$, $4\sqrt{45}$, and $5\sqrt[3]{24}$.

Simplify:

16. $\sqrt{245} - \sqrt{405} + \sqrt{45}$.

17. $\sqrt{12} + 3\sqrt{75} - 2\sqrt{27}$.

18. $5\sqrt{72} + 3\sqrt{18} - \sqrt{50}$.

19. $\sqrt[3]{128} + \sqrt[3]{686} - \sqrt[3]{54}$.

20. $\sqrt{112} - \sqrt{343} + \sqrt{448}$.

21. $\sqrt{\frac{a}{x^2}} + \sqrt{\frac{a}{y^2}} - \sqrt{\frac{a}{z^2}}$.

22. $\sqrt{\frac{ax^4}{by^2}} - \sqrt{\frac{16\,ax^2}{by^2}}$.

23. $\sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ac}} + \sqrt{\frac{c}{ab}}$.

24. $\sqrt{(a+b)^2c} - \sqrt{(a-b)^2c}$.

25. $\sqrt[3]{abx} - \sqrt[6]{a^2b^2x^2} + \sqrt[9]{8\,a^3b^3x^3}$.

26. $\sqrt{3\,x^3 + 30\,x^2 + 75\,x} - \sqrt{3\,x^3 - 6\,x^2 + 3\,x}$.

27. $\sqrt{5\,a^5 + 30\,a^4 + 45\,a^3} - \sqrt{5\,a^5 - 40\,a^4 + 80\,a^3}$.

28. $\sqrt{50} + \sqrt[6]{9} - 4\sqrt{\frac{1}{2}} + \sqrt[3]{-24} + \sqrt[9]{27} - \sqrt[4]{64}$.

29. $\sqrt{\frac{2}{3}} + 6\sqrt{\frac{5}{4}} - \frac{1}{5}\sqrt{18} + \sqrt[4]{36} - \sqrt[8]{\frac{16}{81}} + \sqrt[6]{125} - 2\sqrt{\frac{2}{25}}$.

MULTIPLICATION OF RADICALS

314. $$a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}.$$

That is, $$\sqrt{a} \times \sqrt[3]{a} = \sqrt[6]{a^3} \times \sqrt[6]{a^2} = \sqrt[6]{a^5}.$$

Since fractional exponents to be united by addition must be expressed with a common denominator, radicals to be united by multiplication must be expressed with a common root index.

EXERCISES

315. 1. Multiply $\sqrt{7}$ by $\sqrt{5}$; $5\sqrt{3}$ by $2\sqrt{15}$.

PROCESSES

$$\sqrt{7} \times \sqrt{5} = \sqrt{7 \times 5} = \sqrt{35}$$

$$5\sqrt{3} \times 2\sqrt{15} = 5 \times 2\sqrt{3 \times 15} = 10\sqrt{45} = 10\sqrt{9}\sqrt{5} = 30\sqrt{5}$$

2. Multiply $2\sqrt{3}$ by $3\sqrt[3]{2}$.

PROCESS

$$2\sqrt{3} = 2 \cdot 3^{\frac{1}{2}} = 2 \cdot 3^{\frac{3}{6}} = 2\sqrt[6]{27}$$

$$3\sqrt[3]{2} = 3 \cdot 2^{\frac{1}{3}} = 3 \cdot 2^{\frac{2}{6}} = 3\sqrt[6]{4}$$

$$2\sqrt{3} \times 3\sqrt[3]{2} = 2\sqrt[6]{27} \times 3\sqrt[6]{4} = 2 \times 3\sqrt[6]{27 \times 4} = 6\sqrt[6]{108}$$

RULE. — *If the radicals are not of the same order, reduce them to the same order.*

Multiply the coefficients for the coefficient of the product and the radicands for the radical factor of the product; simplify the result, if necessary.

Multiply:

3. $\sqrt{2}$ by $\sqrt{8}$.

4. $\sqrt{2}$ by $\sqrt{6}$.

5. $\sqrt{3}$ by $\sqrt{15}$.

6. $2\sqrt{5}$ by $3\sqrt{10}$.

7. $3\sqrt{20}$ by $2\sqrt{2}$.

8. $2\sqrt[3]{3}$ by $3\sqrt[3]{45}$.

9. $2\sqrt[4]{6}$ by $3\sqrt{6}$.

10. $3\sqrt{3}$ by $2\sqrt[3]{5}$.

11. $2\sqrt[3]{24}$ by $\sqrt[3]{18}$.

12. $\sqrt{2xy}$ by $3\sqrt[3]{x^2y^3}$.

Find the value of:

13. $\sqrt{mn} \times \sqrt[4]{m^2n} \times \sqrt[8]{mn^4}$.

14. $\sqrt{2\,axy} \times \sqrt[3]{xy} \times \sqrt[4]{a^2xy}$.

15. $\sqrt{a-b} \times \sqrt[4]{a^2b^2} \times \sqrt[4]{(a-b)^{-2}}$.

16. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{3}{4}}$.

17. $\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}}$.

18. $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.

19. $27^{\frac{1}{4}} \times 9^{\frac{1}{3}} \times 81^{\frac{1}{3}}$.

20. Multiply $2\sqrt{2} + 3\sqrt{3}$ by $5\sqrt{2} - 2\sqrt{3}$.

Solution

$$\begin{array}{l} 2\sqrt{2} + 3\sqrt{3} \\ \underline{5\sqrt{2} - 2\sqrt{3}} \\ 20 \quad + 15\sqrt{6} \\ \underline{\qquad\;\; - \;\;4\sqrt{6} - 18} \\ 20 \quad + 11\sqrt{6} - 18 = 2 + 11\sqrt{6}. \end{array}$$

Multiply:

21. $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$.

22. $\sqrt{7} + \sqrt{2}$ by $\sqrt{7} - \sqrt{2}$.

23. $\sqrt{6} - \sqrt{5}$ by $\sqrt{6} - \sqrt{5}$.

24. $5 - \sqrt{5}$ by $1 + \sqrt{5}$.

25. $4\sqrt{7} + 1$ by $4\sqrt{7} - 1$.

26. $2\sqrt{2} + \sqrt{3}$ by $4\sqrt{2} + \sqrt{3}$.

27. $a^2 - ab\sqrt{2} + b^2$ by $a^2 + ab\sqrt{2} + b^2$.

28. $x\sqrt{x} - x\sqrt{y} + y\sqrt{x} - y\sqrt{y}$ by $\sqrt{x} + \sqrt{y}$.

Expand:

29. $(\sqrt{3 + \sqrt{5}})(\sqrt{3 - \sqrt{5}})$.

30. $(\sqrt{9 + \sqrt{6}})(\sqrt{9 - \sqrt{6}})$.

31. $(\sqrt{6 + \sqrt{11}})(\sqrt{6 - \sqrt{11}})$.

32. $(\sqrt{5\,a + a\sqrt{5}})(\sqrt{5\,a - a\sqrt{5}})$.

DIVISION OF RADICALS

316. $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{3}{6}-\frac{2}{6}} = a^{\frac{1}{6}}.$

That is, $\sqrt{a} \div \sqrt[3]{a} = \sqrt[6]{a^3} \div \sqrt[6]{a^2} = \sqrt[6]{a^3 \div a^2} = \sqrt[6]{a}.$

In division, when one fractional exponent is subtracted from another, the exponents must be expressed with a common denominator. When one radical is divided by another, the radicals must be expressed with a common root index.

EXERCISES

317. 1. Divide $\sqrt{60}$ by $\sqrt{12}$.

PROCESS

$$\sqrt{60} \div \sqrt{12} = \sqrt{60 \div 12} = \sqrt{5}$$

2. Divide $\sqrt[3]{2}$ by $\sqrt{2}$.

PROCESS

$$\sqrt[3]{2} \div \sqrt{2} = \sqrt[6]{4} \div \sqrt[6]{8} = \sqrt[6]{\tfrac{4}{8}} = \sqrt[6]{\tfrac{32}{64}} = \tfrac{1}{2}\sqrt[6]{32}$$

RULE. — *If necessary, reduce the radicals to the same order.*

To the quotient of the coefficients annex the quotient of the radicands written under the common radical sign, and reduce the result to its simplest form.

Find quotients:

3. $\sqrt{50} \div \sqrt{8}$.
4. $\sqrt{72} \div 2\sqrt{6}$.
5. $4\sqrt{5} \div \sqrt{40}$.
6. $6\sqrt{7} \div \sqrt{126}$.
7. $\sqrt[3]{4} \div \sqrt{2}$.
8. $7\sqrt{75} \div 5\sqrt{28}$.
9. $\sqrt[3]{16} \div \sqrt[6]{32}$.
10. $\sqrt{2ab^3} \div \sqrt[4]{a^4b^4}$.
11. $\sqrt[3]{a^2x^2} \div \sqrt{2ax}$.
12. $\sqrt[3]{9a^2b^2} \div \sqrt{3ab}$.
13. $\sqrt[4]{4x^2y^2} \div \sqrt[3]{2xy}$.
14. $\sqrt{a-b} \div \sqrt{a+b}$.

15. Divide $\sqrt{15} - \sqrt{3}$ by $\sqrt{3}$.

16. Divide $\sqrt{6} - 2\sqrt{3} + 4$ by $\sqrt{2}$.

17. Divide $\sqrt{2} + 2 + \frac{1}{3}\sqrt{42}$ by $\frac{1}{3}\sqrt{6}$.

18. Divide $5\sqrt{2} + 5\sqrt{3}$ by $\sqrt{10} + \sqrt{15}$.

19. Divide $5 + 5\sqrt{30} + 36$ by $\sqrt{5} + 2\sqrt{6}$.

INVOLUTION AND EVOLUTION OF RADICALS

318. In finding powers and roots of radicals, it is frequently convenient to use fractional exponents.

EXERCISES

319. 1. Find the cube of $2\sqrt{ax^5}$.

SOLUTION. $(2\sqrt{ax^5})^3 = 2^3(a^{\frac{1}{2}}x^{\frac{5}{2}})^3 = 8a^{\frac{3}{2}}x^{\frac{15}{2}} = 8\sqrt{a^3x^{15}} = 8ax^7\sqrt{ax}.$

2. Find the square of $3\sqrt[6]{x^5}$.

SOLUTION. $(3\sqrt[6]{x^5})^2 = 9(x^{\frac{5}{6}})^2 = 9x^{\frac{5}{3}} = 9\sqrt[3]{x^5} = 9x\sqrt[3]{x^2}.$

Square:

3. $3\sqrt{ab}$.

4. $2\sqrt[3]{3x}$.

5. $x\sqrt[3]{2x^3}$.

6. $n^2\sqrt{4b}$.

Cube:

7. $2\sqrt{5}$.

8. $3\sqrt{2}$.

9. $2\sqrt[3]{a^2}$.

10. $\sqrt[4]{a^2b^3}$.

Involve as indicated:

11. $(-2\sqrt{2ab})$.

12. $(-\sqrt{2}\sqrt[6]{x})^3$.

13. $(-\sqrt{2}\sqrt[3]{ax^2})^4$.

14. $(-2\sqrt{x}\sqrt[3]{y})^5$.

15. Obtain by the binomial formula the cube of $\sqrt{2} + 1$.

SOLUTION.
$$(\sqrt{2}+1)^3 = (\sqrt{2})^3 + 3(\sqrt{2})^2 \cdot 1 + 3\sqrt{2} \cdot 1^2 + 1^3$$
$$= 2\sqrt{2} + 6 + 3\sqrt{2} + 1$$
$$= 7 + 5\sqrt{2}.$$

Expand:

16. $(2+\sqrt{6})^2$.

17. $(2+\sqrt{2})^2$.

18. $(2+\sqrt{5})^3$.

19. $(2-\sqrt{3})^3$.

20. $(\sqrt{7}-\sqrt{6})^2$.

21. $(2\sqrt{2}-\sqrt{3})^2$.

22. What is the fourth root of $\sqrt{2x}$?

SOLUTION. $\sqrt[4]{\sqrt{2x}} = [(2x)^{\frac{1}{2}}]^{\frac{1}{4}} = (2x)^{\frac{1}{8}} = \sqrt[8]{2x}.$

Find the square root of:

23. $\sqrt{2}$.

24. $\sqrt[3]{5}$.

25. $\sqrt[6]{x^2}$.

26. $\sqrt[5]{x^{12}}$.

Find the cube root of:

27. $\sqrt{2x}$.

28. $\sqrt[4]{8m^3x^3}$.

29. $-27\sqrt{x^6}$.

30. $-64\sqrt[5]{a^3y^3}$.

Simplify the following indicated roots:

31. $\sqrt{\sqrt[3]{4a^2x^4}}$.

32. $\sqrt[3]{\sqrt{a^{12}x^4}}$.

33. $(\sqrt{8a^3x^3})^{\frac{1}{3}}$.

320. A binomial, one or both of whose terms are surds, is called a **binomial surd.**

$\sqrt{2}+\sqrt{5}$, $2+\sqrt{5}$, $\sqrt[3]{2}+1$, and $\sqrt{3}-\sqrt[3]{2}$ are binomial surds.

321. A binomial surd whose surd or surds are of the second order is called a **binomial quadratic surd.**

$\sqrt{2}+\sqrt{5}$ and $2+\sqrt{5}$ are binomial quadratic surds.

322. Two binomial quadratic surds that differ only in the sign of one of the terms are called **conjugate surds.**

$3+\sqrt{5}$ and $3-\sqrt{5}$ are conjugate surds; also $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$.

323. The square root of a binomial quadratic surd by inspection.

The square of a binomial may be written in the form

$$(a+b)^2 = (a^2+b^2)+2ab.$$

Thus, $(\sqrt{2}+\sqrt{6})^2 = (2+6)+2\sqrt{12} = 8+2\sqrt{12}.$

Therefore, the terms of the square root of $8+2\sqrt{12}$ may be obtained by separating $\sqrt{12}$ into two factors such that the sum of their squares is 8. They are $\sqrt{2}$ and $\sqrt{6}$.

That is, $\sqrt{8+2\sqrt{12}} = \sqrt{2}+\sqrt{6}.$

PRINCIPLE. — *The terms of the square root of a binomial quadratic surd that is a perfect square may be obtained by dividing the irrational term by 2 and then separating the quotient into two factors, the sum of whose squares is the rational term.*

EXERCISES

324. 1. Find the square root of $14 + 8\sqrt{3}$.

SOLUTION

$$14 + 8\sqrt{3} = 14 + 2(4\sqrt{3}) = 14 + 2\sqrt{48}.$$

Since $\sqrt{48} = \sqrt{6} \times \sqrt{8}$ and $14 = 6 + 8$,

$$\sqrt{14 + 8\sqrt{3}} = \sqrt{6} + \sqrt{8} = \sqrt{6} + 2\sqrt{2}.$$

2. Find the square root of $11 - 6\sqrt{2}$.

SOLUTION

$$\sqrt{11 - 6\sqrt{2}} = \sqrt{11 - 2\sqrt{18}} = \sqrt{9} - \sqrt{2} = 3 - \sqrt{2}.$$

Find the square root of:

3. $12 + 2\sqrt{35}$.
4. $16 - 2\sqrt{60}$.
5. $15 + 2\sqrt{26}$.
6. $16 - 2\sqrt{55}$.
7. $11 + 2\sqrt{30}$.
8. $7 - 2\sqrt{10}$.
9. $12 - 6\sqrt{3}$.
10. $17 + 12\sqrt{2}$.
11. $12 + 4\sqrt{5}$.
12. $11 + 4\sqrt{7}$.
13. $15 - 6\sqrt{6}$.
14. $18 + 6\sqrt{5}$.

RATIONALIZATION

325. Suppose that it is required to find the approximate value of $\frac{1}{\sqrt{3}}$, having given $\sqrt{3} = 1.732 \cdots$.

$$1.732\cdots \overline{)1.000000}(.577\cdots \qquad\qquad 3\overline{)1.732\cdots}$$
$$8660 \qquad\qquad .577\cdots$$
$$\cdots$$

We may obtain a decimal approximately equal to $\frac{1}{\sqrt{3}}$, as in the first process (incomplete), by dividing 1 by $1.732 \cdots$; but a great saving of labor may be effected by first changing the fraction to an equal fraction having a *rational* denominator, thus:

$$\frac{1}{\sqrt{3}} = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3},$$

and employing the second process.

326. The process of multiplying an expression containing a surd by any number that will make the product rational is called **rationalization.**

327. The factor by which a surd expression is multiplied to render the product rational is called the **rationalizing factor.**

328. The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator is called **rationalizing the denominator.**

EXERCISES

329. Find the value of each of the following to the nearest third decimal place, taking $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, and $\sqrt{5}=2.23607$:

1. $\dfrac{5}{\sqrt{2}}$.
2. $\sqrt{\dfrac{2}{3}}$.
3. $\dfrac{2}{\sqrt{5}}$.
4. $\dfrac{3}{\sqrt{6}}$.
5. $\dfrac{6}{\sqrt{8}}$.
6. $\dfrac{4}{\sqrt{12}}$.
7. $\dfrac{10}{\sqrt{45}}$.
8. $\dfrac{15}{\sqrt{50}}$.
9. $\dfrac{1}{\sqrt{125}}$.

Rationalize the denominator of each of the following, using the smallest, or lowest, rationalizing factor possible:

10. $\dfrac{\sqrt{a}}{\sqrt{b}}$.
11. $\dfrac{1}{\sqrt{x^7}}$.
12. $\dfrac{ax}{\sqrt{2a^3x}}$.
13. $\dfrac{\sqrt{63x}}{\sqrt{8b^2}}$.
14. $\dfrac{2\sqrt{a}}{\sqrt{by}}$.
15. $\dfrac{\sqrt{6}}{\sqrt[3]{12}}$.
16. $\dfrac{\sqrt{a}}{\sqrt[3]{ax^2}}$.
17. $\sqrt{\dfrac{x^2y}{3xy^2}}$.
18. $\dfrac{\sqrt{r-1}}{\sqrt{r+1}}$.
19. $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$.
20. $\sqrt{\dfrac{x-2}{x+2}}$.
21. $\sqrt[3]{\dfrac{a+b}{(a-b)^2}}$.

330. *The product of any two conjugate surds is rational.*

For, by § 94, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$.

Hence, *a binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

EXERCISES

331. 1. Rationalize the denominator of $\dfrac{2}{3-\sqrt{5}}$.

SOLUTION

$$\frac{2}{3-\sqrt{5}}=\frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{2(3+\sqrt{5})}{9-5}=\frac{3+\sqrt{5}}{2}.$$

2. Rationalize the denominator of $\dfrac{6}{\sqrt{7}+\sqrt{3}}$.

SOLUTION

$$\frac{6}{\sqrt{7}+\sqrt{3}}=\frac{6(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}=\frac{6(\sqrt{7}-\sqrt{3})}{7-3}=\frac{3(\sqrt{7}-\sqrt{3})}{2}.$$

Rationalize the denominator of:

3. $\dfrac{3}{2+\sqrt{3}}$.

4. $\dfrac{5}{\sqrt{5}-\sqrt{3}}$.

5. $\dfrac{8}{\sqrt{3}+\sqrt{5}}$.

6. $\dfrac{1}{\sqrt{3}-\sqrt{2}}$.

7. $\dfrac{3}{2-\sqrt{2}}$.

8. $\dfrac{6}{\sqrt{6}-2\sqrt{3}}$.

9. $\dfrac{b}{a+2\sqrt{b}}$.

10. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

11. $\dfrac{3-\sqrt{ab}}{3+\sqrt{ab}}$.

12. $\dfrac{19}{3\sqrt{3}-2\sqrt{2}}$.

13. $\dfrac{x-3}{\sqrt{x+1}+2}$.

14. $\dfrac{x}{x+\sqrt{x^2-1}}$.

15. $\dfrac{x+y}{\sqrt{x+y}+\sqrt{x-y}}$.

Reduce to a decimal, to the nearest hundredth:

16. $\dfrac{3}{2-\sqrt{3}}$.

17. $\dfrac{4}{3+\sqrt{5}}$.

18. $\dfrac{5}{\sqrt{3}-\sqrt{2}}$.

RADICAL EQUATIONS

332. An equation involving an irrational root of an unknown number is called an **irrational**, or **radical, equation.**

$x^{\frac{1}{2}}=3$, $\sqrt{x+1}=\sqrt{x-4}+1$, and $\sqrt[3]{x-1}=2$ are radical equations.

333. A radical equation may be freed of radicals, wholly or in part, by raising both members, suitably prepared, to the same power. If the given equation contains more than one radical, involution may have to be repeated.

When the following equations have been freed of radicals, the resulting equations will be found to be simple equations. Other varieties of radical equations are treated subsequently.

EXERCISES

334. 1. Given $\sqrt{2x}+4=10$, to find the value of x.

SOLUTION

$$\sqrt{2x}+4=10.$$

Transposing, $\sqrt{2x}=6.$

Squaring, $2x=36.$

$\therefore x=18.$

VERIFICATION. — Substituting 18 for x in the given equation and (§ 293) considering only the *positive* value of $\sqrt{2x}$, we have $\sqrt{36}+4=10$; that is, $10=10$; hence, the equation is satisfied for $x=18$.

2. Given $\sqrt{x-7}+\sqrt{x}=7$, to find the value of x.

SOLUTION

$$\sqrt{x-7}+\sqrt{x}=7.$$

Transposing, $\sqrt{x-7}=7-\sqrt{x}.$

Squaring, $x-7=49-14\sqrt{x}+x.$

Transposing and combining, $14\sqrt{x}=56.$

Dividing by 14, $\sqrt{x}=4.$

Squaring, $x=16.$

VERIFICATION. $\sqrt{16-7}+\sqrt{16}=\sqrt{9}+\sqrt{16}=3+4=7$; that is, $7=7$.

3. Solve, if possible, the equation $\sqrt{x-7}-\sqrt{x}=7$.

SOLUTION

Transposing, squaring, simplifying, etc.,

$$\sqrt{x}=-4.$$

Squaring, $x=16.$

VERIFICATION. — Substituting 16 for x in the given equation and (§ 293) considering only the *positive* value of $\sqrt{x-7}$ and of $\sqrt{x}$, the first member becomes

$$\sqrt{16-7}-\sqrt{16}=\sqrt{9}-\sqrt{16}=3-4=-1;$$

but the second member of the given equation is 7; hence, $x=16$ does not satisfy the equation.

That is, the equation has no root, or is *impossible.*

General Directions. — *Transpose so that the radical term, if there is but one, or the most complex radical term, if there is more than one, may constitute one member of the equation.*

Then raise each member to a power corresponding to the order of that radical and simplify.

If the equation is not freed of radicals by the first involution, proceed again as at first.

Solve, and verify results, denoting impossible equations:

4. $\sqrt{x+1}=3$.

5. $\sqrt{x+5}=4$.

6. $\sqrt{x-7}=1$.

7. $\sqrt{x-a^2}=b$.

8. $\sqrt[3]{x-1}=2$.

9. $\sqrt[3]{x-a^3}=a$.

10. $\sqrt{x}+b=a$.

11. $1+\sqrt{x}=5$.

12. $2\sqrt{x}=6-\sqrt{x}$.

13. $\sqrt{x+16}-\sqrt{x}=2$.

14. $\sqrt{2x}-\sqrt{2x-3}=1$.

15. $\sqrt{2x}+\sqrt{2x-3}=1$.

16. $\sqrt{x^2+x+1}=2-x$.

17. $3\sqrt{x^2-9}=3x-3$.

18. $\sqrt{3x+7}+\sqrt{3x}=7$.

19. $2\sqrt{x}+\sqrt{4x-11}=1$.

20. $5-\sqrt{x+5}=\sqrt{x}$.

21. $\sqrt{x^2-5x+7}+2=x$.

Solve, and verify results, denoting impossible equations:

22. $4 - \sqrt{4 - 8x + 9x^2} = 3x.$

23. $\sqrt{3x - 5} + \sqrt{3x + 7} = 6.$

24. $\sqrt{4x + 5} - 2\sqrt{x - 1} = 9.$

25. $\sqrt{2x - 1} + \sqrt{2x + 4} = 5.$

26. $\sqrt{5x - 1} - 1 = \sqrt{5x + 16}.$

27. $\sqrt{7 + 3\sqrt{5x - 16}} - 4 = 0.$

28. $2x - \sqrt{4x^2 - \sqrt{16x^2 - 7}} = 1.$

29. $\sqrt{4x} - \sqrt{x} = \sqrt{9x - 32}.$

30. $\sqrt{2(x^2 + 3x - 5)} = (x + 2)\sqrt{2}.$

31. $\sqrt{2(x + 1)} + \sqrt{2x - 1} = \sqrt{8x + 1}.$

32. Solve the equation $\sqrt{2x} - \sqrt{2x - 7} = \dfrac{3}{\sqrt{2x - 7}}.$

SUGGESTION.—Clear the equation of fractions.

33. Solve the equation $\dfrac{\sqrt{3x} + 15}{\sqrt{3x} + 5} = \dfrac{\sqrt{3x} + 6}{\sqrt{3x} + 1}.$

SUGGESTION. — Some labor may be saved by reducing each fraction to a mixed number and *simplifying before clearing of fractions.*

Thus,
$$1 + \frac{10}{\sqrt{3x} + 5} = 1 + \frac{5}{\sqrt{3x} + 1}.$$

Canceling, and dividing both members by 5,
$$\frac{2}{\sqrt{3x} + 5} = \frac{1}{\sqrt{3x} + 1}.$$

Solve and verify:

34. $\dfrac{\sqrt{s} - 1}{\sqrt{s} + 5} = \dfrac{\sqrt{s} - 3}{\sqrt{s} - 1}.$

35. $\dfrac{\sqrt{t} - 3}{\sqrt{t} + 1} = \dfrac{\sqrt{t} - 4}{\sqrt{t} - 2}.$

36. $\dfrac{\sqrt{2x} + 9}{\sqrt{2x} - 7} = \dfrac{\sqrt{2x} + 20}{\sqrt{2x} - 12}.$

37. $\dfrac{\sqrt{x} + 18}{\sqrt{x} + 2} = \dfrac{32}{\sqrt{x} + 6} + 1.$

REVIEW

335. 1. Distinguish between involution and evolution.

2. Give the law of exponents for involution; for evolution.

3. For what values of n between 1 and 12 is $(-2)^n$ positive? negative? What is the sign for any power of a positive number?

4. How is a fraction raised to a power? How is the root of a fraction found? Raise $\frac{9}{16}$ to the second power. $\sqrt{\frac{9}{16}} = ?$

5. How is the power of a product found? the root of a product?

6. What operation is indicated by a radical sign? In what other way may this operation be indicated? Illustrate.

7. How many values has $\sqrt{25}$? What is the principal square root of 25? What is the principal cube root of -8?

8. What is the index of a root? What index is meant when none is expressed?

9. Distinguish between real and imaginary numbers and illustrate each.

10. What is the sign of an odd root of a number? of an even root of a number?

11. What is the meaning of x^0?

12. When a number has a fractional exponent, what does the numerator of the exponent show? the denominator?

13. What is the meaning of x^{-2}? How may any factor be transferred from one term of a fraction to the other?

Illustrate by writing without a denominator: $\frac{x^2}{a^{-3}b}$.

14. Expand $(x - y)^6$ by the binomial formula. How does the number of terms correspond with the exponent of the power? What is the coefficient of the first and last terms? of the second term? How are the coefficients of the other terms obtained?

15. Define and illustrate radical, radicand, entire surd, and mixed surd.

16. What is meant by the order of a radical? Illustrate by giving radicals of different orders.

17. How may a radical of the second order be represented graphically?

Illustrate by representing graphically $\sqrt{26}$.

18. What is a rational number? an irrational number?

From the following select the rational numbers:

$$8;\ \tfrac{2}{9};\ \sqrt{3};\ \sqrt[3]{8};\ -7\tfrac{2}{3};\ \sqrt{25};\ 5^{\frac{1}{2}}.$$

19. Are $\sqrt[3]{15}$ and $\sqrt{1+\sqrt{2}}$ radicals? surds? Are all radicals surds? Are all surds radicals?

20. How may the coefficient of a radical be placed under the radical sign?

Express as entire surds: $\frac{1}{3}\sqrt{2}$; $9\sqrt{bc}$; $\frac{2}{7}\sqrt[3]{x^2y}$.

21. When is a radical in its simplest form?

Illustrate by reducing $\sqrt{40\,b^2c}$, $\sqrt{\frac{2}{3}}$, and $\sqrt[6]{4}$, each to its simplest form.

22. What are similar radicals? When numbers have fractional exponents with different denominators, what must be done to the fractional exponents before the numbers can be multiplied? Find the value of $5^{\frac{1}{3}} \times 10^{\frac{1}{2}} \times 6^{\frac{1}{4}}$.

23. What is a binomial surd? a binomial quadratic surd? What are conjugate surds?

24. Define rationalization; rationalizing factor.

How may a binomial quadratic surd be rationalized?

Rationalize the denominator of $\dfrac{7\,x}{\sqrt{a}+\sqrt{b}}$.

25. What is a radical equation? Give general directions for solving a radical equation.

Solve and verify: $\sqrt{x}+\sqrt{3+x}=3$.

QUADRATIC EQUATIONS

336. The equation $x - 2 = 0$ is of the *first* degree and has *one* root, $x = 2$. Similarly, $x - 3 = 0$ is of the first degree and has one root, $x = 3$. Consequently, the product of these two simple equations, which is

$$(x - 2)(x - 3) = 0, \text{ or } x^2 - 5x + 6 = 0,$$

is of the *second degree* and has *two* roots, 2 and 3.

337. An equation that, when simplified, contains the *square* of the unknown number, but no higher power, is called an equation of the *second degree*, or a **quadratic equation.**

It is evident, therefore, that quadratic equations may be of two kinds — those which contain only the second power of the unknown number, and those which contain both the second and first powers.

$x^2 = 15$ and $3x^2 + 2x = 4$ are quadratic equations.

PURE QUADRATIC EQUATIONS

338. An equation that contains only the second power of the unknown number is called a **pure quadratic.**

$2x^2 = 8$ and $4x^2 - 2x^2 = 16$ are pure quadratic equations.

339. The equation $x^2 = 16$ has two roots, for it may be reduced to the form $(x - 4)(x + 4) = 0$, which is equivalent to the two simple equations,

$$x - 4 = 0 \text{ and } x + 4 = 0,$$

each of which has one root, namely, $+4$ and -4. That is,

PRINCIPLE. — *Every pure quadratic equation has two roots, numerically equal but opposite in sign.*

EXERCISES

340. 1. Given $10x^2 = 99 - x^2$, to find the value of x.

SOLUTION

$$10x^2 = 99 - x^2.$$

Transposing, etc., $11x^2 = 99.$

Dividing by 11, $x^2 = 9.$

Taking the square root of each member, § 275,

$$x = \pm 3.$$

NOTE. — Strictly speaking, the last equation should be $\pm x = \pm 3$, which stands for the equations, $+x = +3$, $+x = -3$, and $-x = -3$, and $-x = +3$. But since the last two equations may be derived from the first two, by changing signs, the first two express *all* the values of x.

For convenience, then, the two expressions, $x = +3$ and $x = -3$, are written $x = \pm 3$.

Consequently, in finding the square roots of the members of an equation, it will be sufficient to write the double sign before the root of *one* member.

2. Find the roots of the equation $3x^2 = 24$.

SOLUTION

$$3x^2 = 24.$$

Dividing by 3, $x^2 = 8.$

Taking the square root, $x = \pm 2\sqrt{2}.$

VERIFICATION. — The given equation becomes $24 = 24$ and is therefore satisfied when either $+2\sqrt{2}$ or $-2\sqrt{2}$ is substituted for x.

Solve, and verify each result:

3. $3x^2 - 5 = 22.$
4. $2x^2 + 3x^2 = 80.$
5. $4x^2 = \frac{1}{9}.$
6. $\frac{3}{4}x^2 - 5 = 22.$
7. $5x^2 - 75 = 2x^2.$
8. $2x^2 - 25 = 73.$
9. $7x^2 = 4x^2 + 24.$
10. $4n^2 + 9 = 5n^2 - 7.$
11. $(x+2)^2 - 4(x+2) = 4.$
12. $(3u - 8)(3u + 8) = 17.$
13. $(x+1)^3 - (x-1)^3 = 38.$
14. $(x+1)^2 = x(3x+2) - 3.$
15. $5(s+2) = 3s^2 + s(5-s).$
16. $(2r+1)(2r+3) = 8(r+3).$

Problems

341. 1. The length of a 10-acre field is 4 times its width. What are its dimensions?

2. How many rods of fence will inclose a square garden whose area is $2\frac{1}{2}$ acres?

3. A 4-foot length of stone curbing contains 3840 cubic inches. If its width is 5 times its thickness, find these dimensions.

4. The width of the largest American flag is $\frac{3}{5}$ of its length. The area of the flag is 1500 square feet. What are the dimensions of the flag?

5. A farmer sold his pumpkins for $50. The number of tons was 8 times the number of dollars he received per ton. Find the number of tons sold and the price per ton.

6. A lace worker in Switzerland received $12 for a piece of work. The number of days he worked on it was $\frac{1}{12}$ of the number of cents he earned per day. What were his daily earnings?

7. The area of the face of a square drawing board is 5 square feet. Find the dimensions of a rectangular board 3 inches longer and 3 inches narrower, if its face has the same area.

8. A shipment of railroad ties measuring 400,000 board feet contained as many car loads as there were board feet in a tie. If each car held 250 ties, find the total number of ties and the number of board feet in one tie.

Formulæ

342. Solve the following formulæ from *physics*:

1. $s = \frac{1}{2}gt^2$, for t.

2. $E = \frac{1}{2}Mv^2$, for v.

3. $P = I^2R$, for I.

4. $F = \dfrac{mv^2}{R}$, for v.

5. $G = \dfrac{mm'}{d^2}$, for d.

6. When $g = 32.16$, formula 1 gives the number of feet (s) through which a body will fall in t seconds, starting from rest. How long will it take a brick to fall to the sidewalk from the top of a building 100.5 feet high?

7. To lighten a balloon at the height of 2500 feet, a bag of sand was let fall. Find the time, to the nearest tenth of a second, required for it to reach the earth.

Solve the following *geometrical* formulæ:

8. $c^2 = a^2 + b^2$, for b.

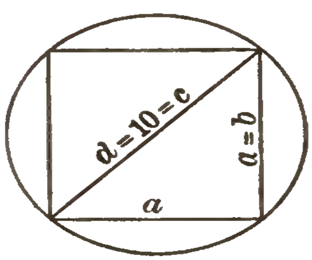
9. $4\,m^2 = 2(a^2 + b^2) - c^2$, for m.

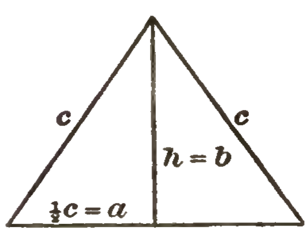
10. $A = .7854\,d^2$, for d.

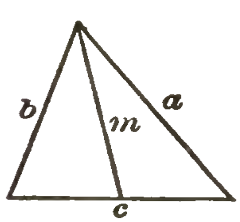
11. $V = \frac{1}{3}\pi r^2 h$, for r.

12. Using formula 8, find the hypotenuse (c) of a right triangle whose other two sides are $a = 8$ and $b = 6$.

13. By means of formula 8, find the side (a) of a right triangle whose hypotenuse (c) is 5 and whose side (b) is 3.







14. From formula 8 and the accompanying figure find, to the nearest tenth, the side (a) of a square inscribed in a circle whose diameter (d) is 10.

15. Using formula 8 and the accompanying figure, deduce a formula for the altitude (h) of an equilateral triangle in terms of its side (c).

16. From formula 9, find the length of the median (m) to the side (c) of the triangle in the accompanying figure, if $a = 11$, $b = 8$, and $c = 9$.

17. Substituting in formula 10, find, to the nearest tenth of a foot, the diameter (d) of a circle whose area (A) is 1000 square feet.

18. Using formula 11, find, to the nearest centimeter, the radius (r) of the base of a conical vessel 20 centimeters high ($h = 20$) that will hold a liter of water ($V = 1$ liter $= 1000$ cu. cm.; $\pi = 3.1416$).

AFFECTED QUADRATIC EQUATIONS

343. A quadratic equation that contains both the second and the first powers of one unknown number is called an **affected quadratic.**

$x^2 + 3x = 10$ and $4x^2 - x = 3$ are affected quadratics.

344. To solve affected quadratics by factoring.

Reduce the equation to the form $ax^2 + bx + c = 0$, factor the first member, and equate each factor to zero, as in § 163, thus obtaining two simple equations together equivalent to the given quadratic.

Thus, $3x^2 = 10x - 3.$

Transposing, $3x^2 - 10x + 3 = 0.$

Factoring, $(x - 3)(3x - 1) = 0.$

$\therefore x - 3 = 0$ or $3x - 1 = 0$;

whence, $x = 3$ or $\frac{1}{3}$.

EXERCISES

345. Solve by factoring, and verify results:

1. $x^2 - 5x + 6 = 0.$
2. $x^2 + 10x + 21 = 0.$
3. $x^2 + 12x - 28 = 0.$
4. $x^2 - 20x + 51 = 0.$
5. $x^2 - 5x = 24.$
6. $x^2 - 1 = 3(x + 1).$
7. $x^2 + 10x = 39.$
8. $60 + x^2 = 17x.$
9. $x(x - 1) = 42.$
10. $x^2 - 3 = 2(x + 6).$
11. $x^2 - 11(x + 3) = 9.$
12. $55 + x(x + 16) = 0.$
13. $2x^2 - 7x + 3 = 0.$
14. $2z^2 - z - 3 = 0.$
15. $3v^2 - 2v - 8 = 0.$
16. $10r^2 - 27r + 5 = 0.$
17. $6(s^2 + 1) = 13s.$
18. $2x^2 + 7x = 4.$
19. $4x^2 + 6 = 11x.$
20. $3w^2 + 3w = 6.$
21. $2t(t + 3) + 4 = 0.$
22. $3(x^2 - 2) - 7x = 0.$
23. $4y^2 + 8y + 3 = 0.$
24. $10(2 - 3x + x^2) = 0.$

346. First method of completing the square.

Since $$(x + a)^2 = x^2 + 2\,ax + a^2,$$

the general form of the perfect square of a binomial is

$$x^2 + 2\,ax + a^2.$$

Consequently, *an expression like* $x^2 + 2ax$ *may be made a perfect square by adding the term* a^2, which it will be observed *is the square of half the coefficient of* x.

Thus, to solve $$x^2 + 6\,x = -5$$

by the method of taking the square root of both members (the method used in solving pure quadratics), we must **complete the square** in the first member.

The number to be added is the square of half the coefficient of x; that is, $(\frac{6}{2})^2$, or 9. *The same number must be added to the second member to preserve the equality.*

Therefore, Ax. 1, $$x^2 + 6\,x + 9 = -5 + 9;$$

that is, $$x^2 + 6\,x + 9 = 4.$$

Taking the square root, § 275, $x + 3 = \pm 2$;

whence, $$x = -3 + 2 \text{ or } -3 - 2.$$

$$\therefore x = -1 \text{ or } -5.$$

EXERCISES

347. **1.** Solve the equation $x^2 - 5\,x - 14 = 0$.

SOLUTION

$$x^2 - 5\,x - 14 = 0.$$

Transposing, $$x^2 - 5\,x = 14.$$

Completing the square, $$x^2 - 5\,x + \tfrac{25}{4} = 14 + \tfrac{25}{4} = \tfrac{81}{4}.$$

Taking the square root, $$x - \tfrac{5}{2} = \pm \tfrac{9}{2};$$

whence, $$x = \tfrac{5}{2} + \tfrac{9}{2} \text{ or } \tfrac{5}{2} - \tfrac{9}{2}.$$

$$\therefore x = 7 \text{ or } -2.$$

VERIFICATION. — Either 7 or -2 substituted for x in the given equation reduces it to $0 = 0$; that is, the given equation is satisfied by these values of x.

2. Solve the equation $4x^2 + 4x - 2 = 0$.

SOLUTION

$$4x^2 + 4x - 2 = 0.$$

Transposing, $4x^2 + 4x = 2$.

Dividing by 4, $x^2 + x = \frac{1}{2}$.

Completing the square, $x^2 + x + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Taking the square root, $x + \frac{1}{2} = \pm \frac{1}{2}\sqrt{3}$.

$$\therefore x = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3},$$

which would usually be written, $x = \frac{1}{2}(-1 \pm \sqrt{3})$.

Steps in the solution of an affected quadratic equation by the first method of completing the square are:

1. *Transpose so that the terms containing x^2 and x are in one member and the known terms in the other.*
2. *Make the coefficient of x^2 positive unity by dividing both members by the coefficient of x^2.*
3. *Complete the square by adding to each member the square of half the coefficient of x.*
4. *Find the square root of both members.*
5. *Solve the two simple equations thus obtained.*

Solve, and verify all results:

3. $x^2 - 4x = 5$.
4. $x^2 - 6x = 7$.
5. $8x = x^2 - 9$.
6. $x^2 + 2x = 15$.
7. $x^2 - 2x = 24$.
8. $x^2 + 8x = -15$.
9. $63 = x^2 + 2x$.
10. $x^2 - 12x = -11$.
11. $x^2 - 40 = 6x$.
12. $y^2 = 10 - 3y$.
13. $v^2 + 5v = 14$.
14. $n(n - 1) = 2$.
15. $v^2 + 3v = 1$.
16. $2x(x - 2) = 8$.
17. $r^2 + 4r - 7 = 0$.
18. $l^2 - 11l + 28 = 0$.
19. $5x^2 - 3x - 2 = 0$.
20. $3x^2 - 6x = -2$.

348. Hindoo method of completing the square.

EXERCISES

1. Solve the general quadratic equation $ax^2 + bx + c = 0$.

SOLUTION

	$ax^2 + bx + c = 0.$	(1)
Transposing c,	$ax^2 + bx = -c.$	(2)
Multiplying by a,	$a^2x^2 + abx = -ac.$	(3)
Completing the square,	$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac.$	(4)
Multiplying by 4,	$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$	(5)
Taking the square root,	$2ax + b = \pm\sqrt{b^2 - 4ac}.$	(6)
	$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	(7)

It is evident that (5) can be obtained by multiplying (2) by $4a$ and adding b^2 to both members. Hence, when a quadratic has the general form of (1), if the absolute term is transposed to the second member, as in (2), the square may be completed and **fractions avoided** by

Multiplying by 4 times the coefficient of x^2 and adding to each member the square of the coefficient of x in the given equation.

This is called the **Hindoo method** of completing the square.

NOTE.—The formula for the values of x, given in (7) and known as the **quadratic formula**, may be used in obtaining the roots of any quadratic by substituting the numerical values of a, b, and c found in the given equation after it is reduced to the form $ax^2 + bx + c = 0$.

Solve by the Hindoo method, then by the quadratic formula:

2. $3x^2 + 4x = 4$.
3. $2x^2 - 11x + 12 = 0$.
4. $5x^2 - 14x = -8$.
5. $2x^2 + 5x = 7$.
6. $2x^2 + 7x = -6$.
7. $3x^2 - 7x = -2$.
8. $4x^2 - x - 3 = 0$.
9. $5x^2 - 7x = -2$.
10. $6x^2 + 5x = -1$.
11. $2 + 5x + 2x^2 = 0$.
12. $6x^2 + 2 = 7x$.
13. $4x^2 + 4x = 3$.
14. $3x + 2x^2 = 9$.
15. $15x^2 - 7x - 2 = 0$.

349. Miscellaneous equations to be solved by any method.

EXERCISES

General Directions. — 1. *Reduce the equation to the general form* $ax^2 + bx + c = 0$.

2. *If the factors are readily seen, solve by factoring.*

3. *If the factors are not readily seen, solve by completing the square or by formula.*

NOTE. — In reducing fractional equations to the general form, observe the cautions given on page 146.

Solve, and verify each result:

1. $x^2 + 5 = 6x$.
2. $x^2 = 3x + 10$.
3. $5x + 3x^2 = 2$.
4. $2x^2 - 7x = 2$.
5. $x^2 - 12x = 28$.
6. $x^4 - 16 = x^2(x^2 - 1)$.
7. $x^2 - 13x - 30 = 0$.
8. $x^2 + 4(x - 3) = 0$.
9. $6 + 11x + 3x^2 = 0$.
10. $(x + 5)^2 = 10x + 74$.
11. $4x^2 - 3x - 2 = 0$.
12. $(x + 7)(x - 9) = 1 - 2x$.
13. $x + \frac{1}{x} - \frac{5}{2} = 0$.
14. $\frac{x}{9(x - 1)} = \frac{x - 2}{6}$.
15. $\frac{x}{12} + \frac{x^2 - 15}{5x} = \frac{x}{5}$.
16. $\frac{x}{x - 5} - \frac{x - 5}{x} = \frac{3}{2}$.
17. $\frac{x + 4}{x - 2} + 3 = \frac{(x + 3)^2}{x^2 - 9}$.
18. $\frac{x^2}{x - 2} = \frac{4}{x - 2} + 5$.
19. $\frac{x}{x + 2} + \frac{1}{2} = \frac{x + 2}{2x}$.
20. $\frac{5x}{x + 7} + \frac{x + 6}{x + 3} = 3$.
21. $\frac{x + 2}{x - 7} - \frac{x + 5}{x - 5} = 1$.
22. $\frac{2x - 3}{x^2 - 3x} = 2 - \frac{3}{x^2 - 3x}$.
23. $\frac{x - 2}{x + 2} - \frac{x + 2}{2 - x} = \frac{40}{x^2 - 4}$.

Find roots to two decimal places:

24. $x^2 - 4x - 1 = 0$.
25. $v^2 + 6v + 7 = 0$.
26. $u^2 + 5u + 5.5 = 0$.
27. $t^2 - 12t + 16.5 = 0$.

Literal Equations

350. The methods of solution for literal quadratic equations are the same as for numerical quadratics. The method by factoring (§ 344) is recommended when the factors can be seen readily. If it is necessary to complete the square, the first method (§ 346) is usually more advantageous, provided the coefficient of x^2 is $+1$, otherwise the Hindoo method (§ 348) is better, because by its use fractions are avoided. Results may be tested by substituting simple numerical values for the literal known numbers.

EXERCISES

351. Solve for x by the method best adapted:

1. $x^2 - ax = ab - bx.$
2. $x^2 + ax = ac + cx.$
3. $x^2 = (m - n)x + mn.$
4. $5x - 2ax = x^2 - 10a.$
5. $x^2 + 3bx = 5cx + 15bc.$
6. $6x^2 + 3ax = 2bx + ab.$
7. $acx^2 - bcx - bd + adx = 0.$
8. $x^2 + 4mx + 3nx + 12mn = 0.$
9. $x^2 = 4ax - 2a^2.$
10. $x^2 - ax - a^2 = 0.$
11. $4ax - x^2 = 3a^2.$
12. $5ax + 6a^2 = 6x^2.$
13. $21b^2 - 4bx = x^2.$
14. $\frac{7m^2}{12} - mx = \frac{x^2}{3}.$
15. $\frac{x^2}{3b} = \frac{5x}{4} + \frac{b}{3}.$
16. $\frac{x}{x-1} - \frac{x}{x+1} = m.$
17. $x + \frac{a^2}{x} = \frac{a^2}{b} + b.$
18. $2x - \frac{3x^2}{a} = a - 2x.$
19. $\frac{1}{ax+4} = 1 - \frac{ax-4}{16}.$
20. $x^2 + \frac{a}{b}x = \frac{a+b}{b}.$
21. $x^2 + 2 = \left(\frac{2a^2+1}{a}\right)x.$
22. $x^2 - \frac{2x}{ab} = \frac{4(ab-1)}{ab}.$

23. $x^2 - 2(a - b)x = 4\,ab.$

24. $x^2 - 2\,x(m - n) = 2\,mn.$

25. $x^2 + 2(a + 8)x = -32\,a.$

26. $x^2 + x + bx + b = a(x + 1).$

27. $a(2x - 1) + 2\,bx - b = x(2\,x - 1).$

28. $x^2 + 4(a - 1)x = 8\,a - 4\,a^2.$

29. $\dfrac{1}{a + b + x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$

30. $\dfrac{2\,a + x}{2\,a - x} + \dfrac{a - 2\,x}{a + 2\,x} = \dfrac{8}{3}.$

31. $a(x - 2\,a + b) + a(x + a - b) = x^2 - (a - b)^2.$

RADICAL EQUATIONS

352. In §§ 333, 334, the student learned how to free radical equations of radicals, the cases treated there being such as lead to simple equations. The radical equations in this chapter lead to quadratic equations, but the methods of freeing them of radicals are the same as in the cases already discussed.

EXERCISES

353. 1. Solve the equation $2\sqrt{x} - x = x - 8\sqrt{x}.$

SOLUTION

$$2\sqrt{x} - x = x - 8\sqrt{x}.$$

Dividing by $\sqrt{x}$, $\quad 2 - \sqrt{x} = \sqrt{x} - 8.$

Transposing, etc., $\quad \sqrt{x} = 5.$

Squaring, $\quad x = 25.$

VERIFICATION. — When $x = 25$,

$$\text{1st member} = 2\sqrt{25} - 25 = 10 - 25 = -15;$$

$$\text{2d member} = 25 - 8\sqrt{25} = 25 - 40 = -15.$$

Hence, $x = 25$ is a root of the equation; $x = 0$, the root of the equation $\sqrt{x} = 0$, also is a root of the given equation, removed by dividing both members by $\sqrt{x}$.

2. Solve and verify $\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0$.

SOLUTION

$$\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0.$$

Transposing, $\sqrt{x+1}+\sqrt{x-2}=\sqrt{2x-5}$.

Squaring, $x+1+2\sqrt{x^2-x-2}+x-2=2x-5$.

Simplifying, $\sqrt{x^2-x-2}=-2$.

Squaring, $x^2-x-2=4$.

Solving, $x=-2$ or 3.

VERIFICATION. — Substituting -2 for x in the given equation,

$$\sqrt{-1}+\sqrt{-4}-\sqrt{-9}=0;$$

that is, $\sqrt{-1}+2\sqrt{-1}-3\sqrt{-1}=0$.

Therefore, -2 is a root of the given equation.

Substituting 3 for x in the given equation,

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=0,$$

which is not true according to the convention adopted in § 293.

Hence, 3 is not to be regarded as a root of the given equation.

NOTE. — The equation could be verified for $x=3$, if the negative square root of 1 were taken in the second term and the positive square root in the third, thus:

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=2+(-1)-(+1)=0.$$

This is an improper method of verification, however, for it has been agreed previously that the square root sign shall denote only the *positive* square root.

Solve, and verify each result:

3. $8\sqrt{x}-8x=\frac{3}{2}$.

4. $3x+\sqrt{x}=5\sqrt{4x}$.

5. $x-1+\sqrt{x+5}=0$.

6. $x-5-\sqrt{x-3}=0$.

7. $\sqrt{4x+17}+\sqrt{x+1}-4=0$.

8. $1+\sqrt{(3-5x)^2+16}=2(3-x)$.

9. $\sqrt{1+x\sqrt{x^2+12}}=1+x$.

10. $\sqrt{x-1}+\sqrt{2x-1}-\sqrt{5x}=0$.

11. $\sqrt{2x-7}-\sqrt{2x}+\sqrt{x-7}=0.$

12. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{2x}.$

13. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

14. $\sqrt{2x+\sqrt{10x+1}}=\sqrt{2x}+1.$

15. $\sqrt{6+x}+\sqrt{x}-\sqrt{10-4x}=0.$

16. $\sqrt{4x-3}-\sqrt{2x+2}=\sqrt{x-6}.$

17. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{5x-14}.$

18. $\sqrt{x^2+8}-\dfrac{6}{\sqrt{x^2+8}}=x.$

19. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

Find roots to two decimal places:

20. $x-\sqrt{6x}=6.$

21. $3\sqrt{x}=2(x-2).$

22. $2+2\sqrt{x+3}+x=0.$

23. $\sqrt{x-2}+\sqrt{2x}=3.$

Problems

354. 1. The sum of two numbers is 8, and their product is 15. Find the numbers.

Solution. — Let $x=$ one number.
Then, $8-x=$ the other.
Since their product is 15, $(8-x)x=15.$
Solving, $x=3$ or 5,
and $8-x=5$ or 3.

Therefore, the numbers are 3 and 5.

2. Divide 20 into two parts whose product is 96.

3. Divide 14 into two parts whose product is 45.

4. Find two consecutive positive integers the sum of whose squares is 61.

Solve the following problems and verify each solution:

5. A plumber received $24 for some work. The number of hours that he worked was 20 less than the number of cents per hour that he earned. Find his hourly wage.

6. A man's life insurance for a certain time cost $20.80. If the number of weeks was 12 more than the number of cents he paid per week, what was the weekly premium?

7. The area of a revolving floor in a hall in Paris is 2650 square feet. The width is 3 feet less than the length. What are the dimensions of the floor?

8. The base of the tower of the Metropolitan Life Building in New York City is 6375 square feet in area. The length of the base is 10 feet greater than the width. What are the dimensions of the base?

9. The 1860 bunches of asparagus from an acre of land were sold in boxes each holding 1 less than $\frac{1}{2}$ as many bunches as there were boxes. Find the number of bunches in a box.

10. One of the largest electric signs in the world is 14,560 square feet in area. The length of the sign lacks 22 feet of being twice the width. Find the dimensions of the sign.

11. The area of the plate glass floor of the highest bridge in the world, built across Royal Gorge in Colorado, is 5060 square feet. The length is 10 feet more than 10 times its width. What is its length?

12. If the average amount deposited in the postal savings banks of Canada by each depositor one year had been $70 less, and if the number of depositors had been equal to the average number of dollars each deposited, the total deposit would have been $40,000. Find the average amount each deposited.

13. The length of a steel barge used for coal on the Ohio River is 5 feet more than 5 times its width. If the depth is 8 feet and the capacity of the barge is 28,080 cubic feet, what is the width? the length?

14. The area of the plate used in a giant camera is $37\frac{1}{3}$ square feet. The width of the plate is 8 inches more than $\frac{1}{2}$ the length. Find the dimensions of the plate.

15. The *Chester*, a United States scouting cruiser, steamed 106.12 knots on its trial voyage. The number of knots that it went in one hour was 1.47 less than 7 times the number of hours spent on its trial voyage. Find its speed per hour.

16. A piece of silk made from spiders' web and exhibited at the Paris Exposition was 36 times as long as it was wide. If its width had been increased 9 inches, it would have contained $13\frac{1}{2}$ square yards. Find its length and width.

17. The area of one side wall of a square reservoir, cut in the solid rock at Bowling Green, Ohio, is 2200 square feet, and the depth of the reservoir is 2 feet more than $\frac{1}{5}$ of its length. Find its three dimensions.

18. Some boys laid out basket-ball grounds 30 feet greater in length than in width, but to change the area to the prescribed limit of 3500 square feet, they reduced the length 10 feet. How much too large had they laid out the grounds?

19. A party hired a coach for $12. In consequence of the failure of 3 of them to pay, each of the others had to pay 20 cents more. How many persons were in the party?

SOLUTION

Let $x =$ the number of persons.

Then, $x - 3 =$ the number that paid.

$\frac{12}{x} =$ the number of dollars each should have paid,

and $\frac{12}{x-3} =$ the number of dollars each paid.

Therefore, $\frac{12}{x-3} - \frac{1}{5} = \frac{12}{x}.$

Solving, $x = 15$ or -12.

The second value of x is evidently inadmissible, since there could not be a negative number of persons.

Hence, the number of persons in the party was 15.

20. A club had a dinner that cost \$ 60. If there had been 5 persons more, the share of each would have been \$ 1 less. How many persons were there in the club?

21. A party of young people agreed to pay \$ 8 for a sleigh ride. As 4 were obliged to be absent, the cost for each of the rest was 10 cents greater. How many went on the ride?

22. Find two consecutive integers the sum of whose reciprocals is $\frac{9}{20}$.

23. The dry gum used per day in gumming United States postage stamps costs \$ 48. If 400 pounds more were used at the same total cost, the price per pound would be 2 cents less. How much gum is used daily?

24. A man earned \$ 48 by shearing sheep. The number of cents he earned per fleece was 2 more than the number of days he worked. How many sheep did he shear, if he averaged 100 a day?

25. The weight of 80 four-inch spikes was 3 pounds less than the weight of 80 five-inch spikes. If 1 pound of the former contained 6 spikes more than 1 pound of the latter, how many of each kind weighed 1 pound?

26. A tub of dairy butter weighed 20 pounds less than a tub of creamery butter, and 360 pounds of dairy butter required 3 tubs more than the same amount of creamery butter. What weight of butter was there in a tub of each kind?

27. Mr. Field paid \$ 8 for one mile of No. 9 steel wire and \$ 2.88 for one mile of No. 14 wire. The No. 9 wire weighed 224 pounds more, and cost $\frac{1}{2}$ cent per pound less, than the No. 14 wire. Find the cost of each per pound.

28. A boy earned \$ 420 by delivering bills. If he had received 50 cents more per thousand, he would have earned as much by delivering 70 thousand less than he did. How much was he paid per thousand?

29. A merchant sold a hunting coat for $11, and gained a per cent equal to the number of dollars the coat cost him. What was his per cent of gain?

30. A moving picture film 150 feet long is made up of a certain number of individual pictures. If these pictures were $\frac{1}{4}$ of an inch longer, there would be 600 less for the same length of film. How long is each separate picture?

31. In Detroit, a machine for making pills turned out 250,000 pills per hour. If, in boxing these, 5 pills more were put into each box, the number of boxes would be 2500 less. How many pills does each box contain?

32. Two coats of paint applied to the sides of a barn having an area of 195 square yards required 69 pounds of paint. One pound covered $1\frac{1}{2}$ square yards more for the second coat than for the first. What area did 1 pound of paint cover for each coat?

33. A train started 16 minutes late, but finished its run of 120 miles on time by going 5 miles per hour faster than usual. What was the usual rate per hour?

34. Two automobiles went a distance of 60 miles, one making 6 miles per hour faster time than the other and completing the journey $\frac{5}{6}$ of an hour sooner. How long was each on the way?

35. The distance covered by an aeroplane on one occasion was 45 miles. If its rate per minute had been $\frac{1}{4}$ of a mile more, the distance would have been covered in 9 minutes less time. Find the speed of the aeroplane per minute.

36. If each cable of the Manhattan Bridge contained 9 strands more and each strand 20 wires more, the number of wires in a strand would be 6 times as many as the number of strands in a cable. How many strands are there in a cable, if there are 9472 wires in a cable?

37. To run around a track 1320 feet in circumference took one man 5 seconds less time than it took another who ran 2 feet per second slower. How long did it take each man?

38. Some rugs made in India have 400 knots to the square inch. A fast weaver ties 40 knots more per minute than a boy. If the former weaves a square inch in $13\frac{1}{3}$ minutes less time than the latter, how many knots does each tie per minute?

39. A cistern can be filled by two pipes in 24 minutes. If it takes the smaller pipe 20 minutes longer to fill the cistern than the larger pipe, in what time can the cistern be filled by each pipe?

Solution

Let $x =$ the number of minutes required by the larger pipe.

Then, $x + 20 =$ the number of minutes required by the smaller pipe.

Since $\frac{1}{x} =$ the part that the larger pipe fills in one minute,

$\frac{1}{x+20} =$ the part that the smaller pipe fills in one minute,

and $\frac{1}{24} =$ the part that both pipes fill in one minute.

Then, $$\frac{1}{x} + \frac{1}{x+20} = \frac{1}{24}.$$

Solving, $$x = 40 \text{ or } -12.$$

The negative value is inadmissible. Hence, the larger pipe can fill the cistern in 40 minutes, and the smaller pipe in 60 minutes.

40. A city reservoir can be filled by two of its pumps in 3 days. The larger pump alone would take $1\frac{3}{4}$ days less time than the smaller. In what time can each fill the reservoir?

41. A company owned two plants that together made 25,200 concrete building blocks in 12 days. Working alone, one plant would have required 7 days more time than the other. What was the daily capacity of each plant?

42. The number of strawberry baskets made by a machine was 12 more per minute than the number of peach baskets made by another machine. One day the former machine started 45 minutes after the latter, but each finished 2400 baskets at the same instant. Find the rate of each per minute.

Formulæ

355. 1. In any right-angled triangle (Fig. 1), $c^2 = a^2 + b^2$. Find all sides when $a = c - 2$ and $b = c - 4$.

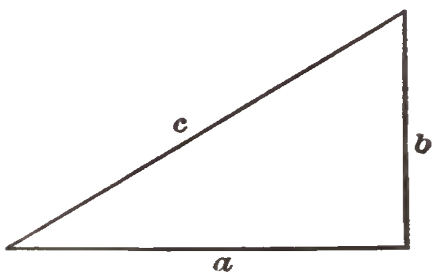


FIG. 1.

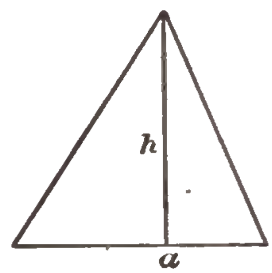


FIG. 2.

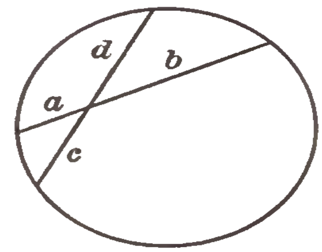


FIG. 3.

2. The area (A) of a triangle (Fig. 2) is expressed by the formula $A = \frac{1}{2}ah$. If the altitude (h) of a triangle is 2 inches greater than the base (a) and the area is 60 square inches, what is the length of the base?

3. If two chords intersect in a circle, as shown in Fig. 3, $a \times b$ is always equal to $c \times d$. Compute a and b when $c = 4$, $d = 6$, and $b = a + 5$.

4. The formula $h = a + vt - 16t^2$ gives, approximately, the height (h) of a body at the end of t seconds, if it is thrown vertically *upward*, starting with a velocity of v feet per second from a position a feet high.

Solve for t, and find how long it will take a skyrocket to reach a height of 796 feet, if it starts from a platform 12 feet high with an initial velocity of 224 feet per second.

5. How long will it take a bullet to reach a height of 25,600 feet, if it is fired vertically upward from the level of the ground with an initial velocity of 1280 feet per second?

6. When a body is thrown vertically *downward*, an approximate formula for its height is $h = a - vt - 16t^2$, in which h, a, v, and t stand for the same elements as in exercise 4.

Solve for t, and find when a ball thrown vertically downward from the Eiffel tower, height 984 feet, with an initial velocity of 24 feet per second, will be 368 feet above ground.

Find, to the nearest second, when it will reach the ground.

GRAPHIC SOLUTIONS

QUADRATIC EQUATIONS — ONE UNKNOWN NUMBER

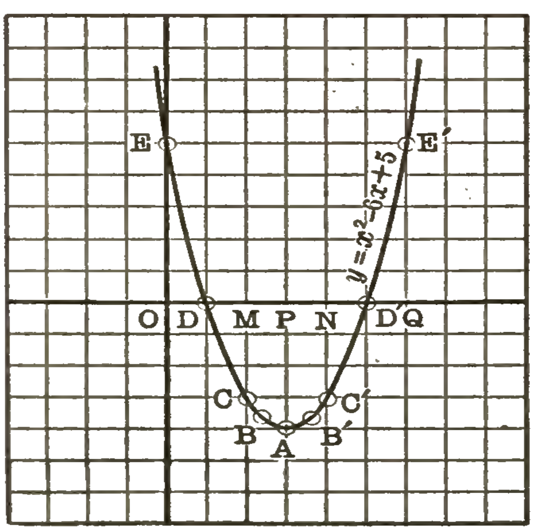
356. Let it be required to solve graphically, $x^2 - 6x + 5 = 0$.

To solve the equation graphically, we must first draw the graph of $x^2 - 6x + 5$. To do this, let $y = x^2 - 6x + 5$.

The graph of $y = x^2 - 6x + 5$ will represent *all* the corresponding real values of x and of $x^2 - 6x + 5$, and among them will be the values of x that make $x^2 - 6x + 5$ equal to zero, that is, the *roots* of the equation $x^2 - 6x + 5 = 0$.

When the coefficient of x^2 is $+1$, as in this instance, it is convenient to take for the first value of x a number equal to half the coefficient of x with its sign changed. Next, values of x differing from this value *by equal amounts* may be taken.

Thus, first substituting $x = 3$, it is found that $y = -4$, locating the point $A = (3, -4)$. Next give values to x differing from 3 by equal amounts, as $2\frac{1}{2}$ and $3\frac{1}{2}$, 2 and 4, 1 and 5, 0 and 6. It will be found that y has the same value for $x = 3\frac{1}{2}$ as for $x = 2\frac{1}{2}$, for $x = 4$ as for $x = 2$, etc.



The table below gives a record of the points and their coördinates:

$y = x^2 - 6x + 5$

x	y	POINTS
3	-4	A
$2\frac{1}{2}$, $3\frac{1}{2}$	$-3\frac{3}{4}$	B, B'
2, 4	-3	C, C'
1, 5	0	D, D'
0, 6	5	E, E'

Plotting the points A; B, B'; C, C'; etc., whose coördinates are given in the preceding table, and drawing a smooth curve through them, we obtain the graph of $y = x^2 - 6x + 5$ as shown in the figure.

It will be observed from the work of the preceding page that:

When $x = 3$, $x^2 - 6x + 5 = -4$, which is represented by the *negative* ordinate PA.

When $x = 2$ and also when $x = 4$, $x^2 - 6x + 5 = -3$, which is represented by the equal *negative* ordinates MC and NC'.

When $x = 0$ and also when $x = 6$, $x^2 - 6x + 5 = 5$, represented by the equal *positive* ordinates OE and QE'.

Thus, it is seen that the ordinates change sign as the curve crosses the x-axis.

At D and at D', where the ordinates are equal to 0, the value of $x^2 - 6x + 5$ is 0, and the abscissas are $x = 1$ and $x = 5$.

Hence, the roots of the given equation are 1 and 5.

The curve obtained by plotting the graph of $x^2 - 6x + 5$, or of any quadratic expression of the form $ax^2 + bx + c$, is a **parabola**.

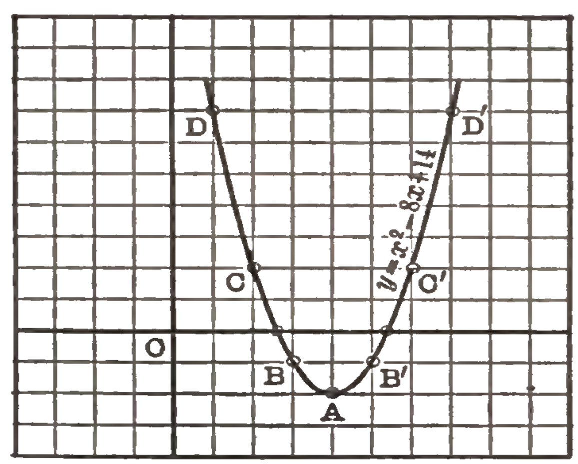
EXERCISES

357. 1. Solve graphically the equation $x^2 - 8x + 14 = 0$.

Solution. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:

$y = x^2 - 8x + 14.$

x	y	Points
4	-2	A
3, 5	-1	B, B'
2, 6	2	C, C'
1, 7	7	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 14$, which *crosses* the x-axis approximately at $x = 2.6$ and $x = 5.4$.

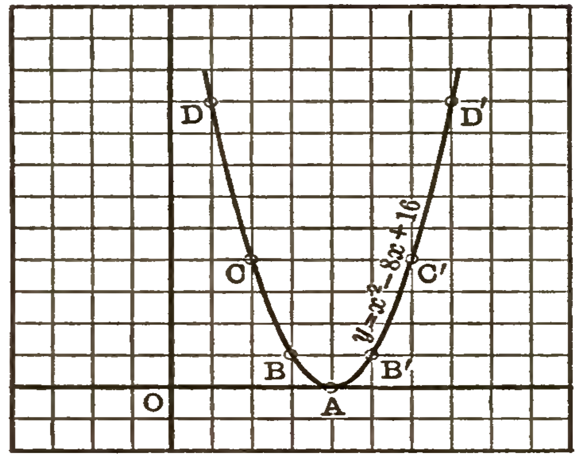
Hence, to the nearest tenth, the roots of $x^2 - 8x + 14 = 0$ are 2.6 and 5.4.

2. Solve graphically the equation $x^2 - 8x + 16 = 0$.

Solution. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:

$y = x^2 - 8x + 16$

x	y	Points
4	0	A
3, 5	1	B, B'
2, 6	4	C, C'
1, 7	9	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 16$, which *touches* the x-axis at $x = 4$.

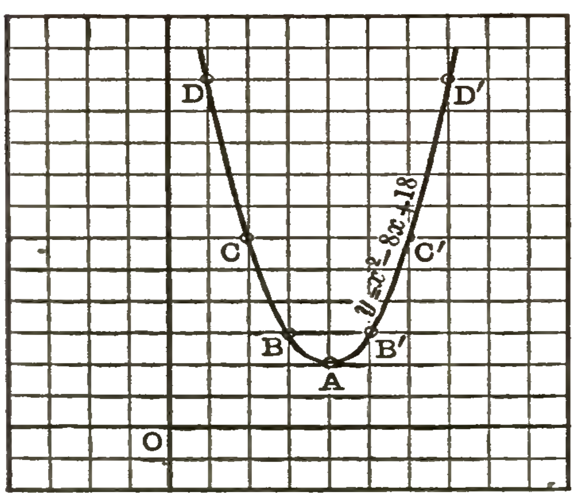
This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 16 = 0$ are *equal*, both being represented by the abscissa of the point of contact.

Hence, the roots are 4 and 4.

Note. — The student may show that 4 and 4 are the roots by solving the equation algebraically.

3. Solve graphically the equation $x^2 - 8x + 18 = 0$.

Solution. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:



$y = x^2 - 8x + 18$

x	y	Points
4	2	A
3, 5	3	B, B'
2, 6	6	C, C'
1, 7	11	D, D'

Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 18$, which neither crosses nor touches the x-axis. This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 18 = 0$ are *imaginary*.

NOTE. — The student may show that the roots are imaginary by solving the equation algebraically.

In each of the preceding graphs, the point A, whose ordinate is the least algebraically that any point in the graph has, is called the **minimum point**.

When the coefficient of x^2 is $+1$, it is evident that:

PRINCIPLES. — 1. *If the minimum point lies below the x-axis, the roots are real and unequal.*

2. *If the minimum point lies on the x-axis, the roots are real and equal.*

3. *If the minimum point lies above the x-axis, the roots are imaginary.*

Solve graphically, giving real roots to the nearest tenth:

4. $x^2 - 4x + 3 = 0$.
5. $x^2 - 6x + 7 = 0$.
6. $x^2 - 4x = -2$.
7. $x^2 + 2(x + 1) = 0$.
8. $x^2 - 4x + 6 = 0$.
9. $x^2 - 2x - 2 = 0$.
10. $x^2 = 6x - 9$.
11. $x^2 + 4x + 2 = 0$.
12. $x^2 - 2x + 6 = 0$.
13. $x^2 - 4x - 1 = 0$.
14. Solve graphically $4x - 2x^2 + 1 = 0$.

SUGGESTION. — On dividing both members of the given equation by -2, the coefficient of x^2, the equation becomes

$$x^2 - 2x - \tfrac{1}{2} = 0.$$

The roots may be found by plotting the graph of $y = x^2 - 2x - \frac{1}{2}$.

Solve graphically, giving real roots to the nearest tenth:

15. $2x^2 + 8x + 7 = 0$.
16. $2x^2 - 12x + 15 = 0$.
17. $12x - 4x^2 - 1 = 0$.
18. $11 + 8x + 2x^2 = 0$.

NOTE. — Another method of solving quadratic equations graphically is given in § 379.

EQUATIONS IN QUADRATIC FORM

358. An equation that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other, as $ax^{2n} + bx^n + c = 0$, in which n represents any number, is in the **quadratic form.**

EXERCISES

359. 1. Given $x^4 - 10x^2 + 24 = 0$, to find the value of x.

SOLUTION

$$x^4 - 10x^2 + 24 = 0.$$

Factoring, $\quad (x^2 - 4)(x^2 - 6) = 0.$

Hence, $\quad x^2 - 4 = 0$ or $x^2 - 6 = 0$;

whence, $\quad x = \pm 2$ or $x = \pm\sqrt{6}$.

Each of these values substituted in the given equation is found to verify; hence, ± 2 and $\pm\sqrt{6}$ are roots of the equation.

2. Given $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$, to find the values of x.

FIRST SOLUTION

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$$

Let $x^{\frac{1}{4}} = m$, then $x^{\frac{1}{2}} = m^2$ and the equation becomes

$$m^2 - m - 6 = 0.$$

Solving by factoring, $\quad m = 3$ or -2;

that is, $\quad x^{\frac{1}{4}} = 3$ or -2.

Raising to the fourth power, $\quad x = 81$ or 16.

SECOND SOLUTION

Transposing, $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6.$

Completing the square, $x^{\frac{1}{2}} - x^{\frac{1}{4}} + (\frac{1}{2})^2 = \frac{25}{4}.$

Taking the square root, $x^{\frac{1}{4}} - \frac{1}{2} = \pm \frac{5}{2}.$

$\therefore x^{\frac{1}{4}} = 3 \text{ or } -2.$

Raising to the fourth power, $x = 81 \text{ or } 16.$

Since $x = 16$ does not satisfy the given equation, 16 is not a root and should be rejected.

Solve, and verify each result:

3. $x^4 - 13x^2 + 36 = 0.$
4. $x^4 - 18x^2 + 32 = 0.$
5. $x^4 - 14x^2 + 45 = 0.$
6. $2x^4 - 10x^2 + 8 = 0.$
7. $3x^4 - 11x^2 + 8 = 0.$
8. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0.$
9. $x^{\frac{2}{3}} - x^{\frac{1}{3}} = 6.$
10. $x + 2\sqrt{x} = 3.$
11. $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 12 = 0.$
12. $(x-3)^2 + 2(x-3) = 3.$
13. $(x^2-1)^2 - 4(x^2-1) = 5.$
14. $(x^2-4)^2 - 3(x^2-4) = 10.$

15. Solve the equation $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24.$

SOLUTION

$$x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24. \quad (1)$$

Adding 18 to both members, we have

$$x^2 - 7x + 18 + \sqrt{x^2 - 7x + 18} = 42. \quad (2)$$

Put m for $\sqrt{x^2 - 7x + 18}$ and m^2 for $x^2 - 7x + 18$.

Then, transposing, $m^2 + m - 42 = 0. \quad (3)$

Solving by factoring, $m = 6 \text{ or } -7. \quad (4)$

That is, $\sqrt{x^2 - 7x + 18} = 6, \quad (5)$

or $\sqrt{x^2 - 7x + 18} = -7. \quad (6)$

Squaring (5), $x^2 - 7x + 18 = 36.$

Solving, $x = 9 \text{ or } -2.$

Since, in accordance with § 293, the radical in (6) cannot equal a negative number, $\sqrt{x^2 - 7x + 18} = -7$ is an impossible equation.

Hence, the only roots of (1) are 9 and -2.

16. Solve the equation $x + 2\sqrt{x+3} = 21$, and verify results.

17. Solve $x^2 - 3x + 2\sqrt{x^2 - 3x + 6} = 18$, and verify results.

18. Solve the equation $x^6 - 9x^3 + 8 = 0$.

SOLUTION

$$x^6 - 9x^3 + 8 = 0. \qquad (1)$$

Factoring, $$(x^3 - 1)(x^3 - 8) = 0. \qquad (2)$$

Therefore, $$x^3 - 1 = 0, \qquad (3)$$

or $$x^3 - 8 = 0. \qquad (4)$$

If the values of x are found by transposing the known terms in (3) and (4) and then taking the cube root of each member, only *one* value of x will be obtained from each equation. But if the equations are factored, *three* values of x are obtained for each.

Factoring (3), $$(x - 1)(x^2 + x + 1) = 0, \qquad (5)$$

and (4), $$(x - 2)(x^2 + 2x + 4) = 0. \qquad (6)$$

Writing each factor equal to zero, and solving, we have:

From (5), $x = 1,\ \frac{1}{2}(-1 + \sqrt{-3}),\ \frac{1}{2}(-1 - \sqrt{-3})$. (7)

From (6), $x = 2,\ -1 + \sqrt{-3},\ -1 - \sqrt{-3}$. (8)

NOTE. — Since the values of x in (7) are obtained by factoring $x^3 - 1 = 0$, they may be regarded as the *three cube roots of the number* 1. Also, the values of x in (8) may be regarded as the *three cube roots of the number* 8 (§ 271).

Solve:

19. $x^6 - 28x^3 + 27 = 0$.

20. $x^4 - 16 = 0$.

21. Find the three cube roots of -1.

22. Find the three cube roots of -8.

23. Solve the equation $x^4 + 4x^3 - 8x + 3 = 0$.

SOLUTION

By applying the factor theorem (§ 146), the factors of the first member are found to be $x - 1$, $x + 3$, and $x^2 + 2x - 1$; that is,

$$(x - 1)(x + 3)(x^2 + 2x - 1) = 0.$$

Solving, $$x = 1,\ -3,\ -1 \pm \sqrt{2}.$$

Solve, and verify each result:

24. $x^3 + x^2 - 4x = -2.$

25. $x^4 - 4x^3 + 8x = -3.$

26. $x^4 - 8x^2 + 5x + 6 = 0.$

27. $x^4 - 6x^3 + 27x = 10.$

28. $x^4 + 6x^3 + 7x^2 - 6x - 8 = 0.$

29. $x^4 + 2x^3 - 10x^2 - 11x + 30 = 0.$

MISCELLANEOUS EXERCISES

360. Solve, and verify each result:

1. $x^4 - 25x^2 + 144 = 0.$

2. $x^4 - 45x^2 + 500 = 0.$

3. $2x^4 - 11x^2 + 12 = 0.$

4. $5x^4 - 24x^2 + 16 = 0.$

5. $x^{\frac{1}{2}} - 2x^{\frac{1}{4}} - 3 = 0.$

6. $x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = -2.$

7. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 3.$

8. $x^{\frac{4}{3}} - 10x^{\frac{2}{3}} + 9 = 0.$

9. $(x^2 - 1)^2 - 4(x^2 - 1) + 3 = 0.$

10. $(x^2 - 6)^2 - 7(x^2 - 6) - 30 = 0.$

11. $(x^2 - 2x)^2 - 2(x^2 - 2x) = 3.$

12. $x - 6x^{\frac{1}{2}} + 8 = 0.$

13. $x + 20 - 9\sqrt{x} = 0.$

14. $2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 1 = 0.$

15. $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 10 = 0.$

16. $x - 5 + 2\sqrt{x - 5} = 8.$

17. $2x - 3\sqrt{2x + 5} = -5.$

18. $2x - 6\sqrt{2x - 1} = 8.$

19. $x - 3 = 21 - 4\sqrt{x - 3}.$

20. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

21. $4x^4 - 4x^3 - 7x^2 + 4x + 3 = 0.$

22. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$

23. $x^2 - x - \sqrt{x^2 - x + 4} - 8 = 0.$

24. $x^2 - 5x + 2\sqrt{x^2 - 5x - 2} = 10.$

SIMULTANEOUS QUADRATIC EQUATIONS

361. Two simultaneous *quadratic* equations involving two unknown numbers generally lead to equations of the fourth degree, and therefore they cannot usually be solved by quadratic methods.

However, there are some simultaneous equations involving quadratics that may be solved by quadratic methods, as shown in the following cases.

362. When one equation is simple and the other of higher degree.

Equations of this class may be solved by finding the value of one unknown number in terms of the other in the simple equation, and then substituting that value in the other equation.

EXERCISES

363. 1. Solve the equations $\begin{cases} x + y = 7, \\ 3x^2 + y^2 = 43. \end{cases}$

SOLUTION

$$x + y = 7. \qquad (1)$$

$$3x^2 + y^2 = 43. \qquad (2)$$

From (1), $y = 7 - x. \qquad (3)$

Substituting in (2), $3x^2 + (7 - x)^2 = 43. \qquad (4)$

Solving, $x = 3 \text{ or } \frac{1}{2}. \qquad (5)$

Substituting 3 for x in (3), $y = 4. \qquad (6)$

Substituting $\frac{1}{2}$ for x in (3), $y = \frac{13}{2}. \qquad (7)$

That is, x and y each have two values $\begin{cases} \text{when } x = 3, y = 4, \\ \text{when } x = \frac{1}{2}, y = \frac{13}{2}. \end{cases}$

Solve, and verify results:

2. $\begin{cases} x^2 + y^2 = 20, \\ x = 2y. \end{cases}$

3. $\begin{cases} 10x + y = 3xy, \\ y - x = 2. \end{cases}$

4. $\begin{cases} x = 6 - y, \\ x^3 + y^3 = 72. \end{cases}$

5. $\begin{cases} xy(x - 2y) = 10, \\ xy = 10. \end{cases}$

6. $\begin{cases} x^2 + xy = 12, \\ x - y = 2. \end{cases}$

7. $\begin{cases} m^2 - 3n^2 = 13, \\ m - 2n = 1. \end{cases}$

8. $\begin{cases} 3x(y + 1) = 12, \\ 3x = 2y. \end{cases}$

9. $\begin{cases} 3rs - 10r = s, \\ 2 - s = -r. \end{cases}$

364. An equation that is not affected by interchanging the unknown numbers involved is called a **symmetrical equation.**

$2x^2 + xy + 2y^2 = 4$ and $x^2 + y^2 = 9$ are symmetrical equations.

365. When both equations are symmetrical.

Though equations of this class may usually be solved by substitution, as in §§ 362, 363, it is preferable to find values for $x + y$ and $x - y$ and then solve these simple equations for x and y.

EXERCISES

366. 1. Solve the equations $\begin{cases} x + y = 7, \\ xy = 10. \end{cases}$

SOLUTION

$x + y = 7.$ (1)

$xy = 10.$ (2)

Squaring (1), $x^2 + 2xy + y^2 = 49.$ (3)

Multiplying (2) by 4, $4xy = 40.$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 9.$ (5)

Taking the square root, $x - y = \pm 3.$ (6)

From (1) + (6), $x = 5$ or $2.$

From (1) − (6), $y = 2$ or $5.$

2. Solve the equations $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$

SOLUTION

$$x^2 + y^2 = 25. \quad (1)$$

$$x + y = 7. \quad (2)$$

Squaring (2), $x^2 + 2xy + y^2 = 49. \quad (3)$

Subtracting (1) from (3), $2xy = 24. \quad (4)$

Subtracting (4) from (1), $x^2 - 2xy + y^2 = 1. \quad (5)$

Taking the square root, $x - y = \pm 1. \quad (6)$

From (2) + (6), $x = 4$ or 3.

From (2) − (6), $y = 3$ or 4.

Solve, and verify each result:

3. $\begin{cases} x^2 + y^2 = 50, \\ xy = 7. \end{cases}$

4. $\begin{cases} x + y = 8, \\ x^2 + y^2 = 34. \end{cases}$

5. $\begin{cases} x + y = 9, \\ x^3 + y^3 = 243. \end{cases}$

6. $\begin{cases} x + y = 8, \\ x^2 + xy + y^2 = 49. \end{cases}$

7. $\begin{cases} x^2 + xy + y^2 = 31, \\ x^2 + y^2 = 26. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 8, \\ x^2 - xy + y^2 = 4. \end{cases}$

9. $\begin{cases} xy = 12, \\ x - xy + y = -5. \end{cases}$

10. $\begin{cases} x^2 + 3xy + y^2 = 31, \\ xy = 6. \end{cases}$

11. $\begin{cases} x^2 + y^2 = 100, \\ (x + y)^2 = 196. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 13, \\ x^2 + y^2 + xy = 19. \end{cases}$

367. An equation *all* of whose terms are of the same degree with respect to the unknown numbers is called a **homogeneous equation.**

$x^2 - xy = y^2$ and $3x^3 + y^3 = 0$ are homogeneous equations.

An equation like $x^2 - xy + y^2 = 21$ is said to be **homogeneous in the unknown terms.**

368. When both equations are quadratic and homogeneous in the unknown terms.

Substitute vy for x, solve for y^2 in each equation, and compare the values of y^2 thus found, forming a quadratic in v.

EXERCISES

369. 1. Solve the equations $\begin{cases} x^2 - xy + y^2 = 21, \\ y^2 - 2\,xy = -15. \end{cases}$

SOLUTION

$$x^2 - xy + y^2 = 21. \qquad (1)$$

$$y^2 - 2\,xy = -15. \qquad (2)$$

Assume $x = vy$. (3)

Substituting in (1), $v^2y^2 - vy^2 + y^2 = 21.$ (4)

Substituting in (2), $y^2 - 2\,vy^2 = -15.$ (5)

Solving (4) for y^2, $y^2 = \dfrac{21}{v^2 - v + 1}.$ (6)

Solving (5) for y^2, $y^2 = \dfrac{15}{2\,v - 1}.$ (7)

Comparing the values of y^2, $\dfrac{15}{2\,v - 1} = \dfrac{21}{v^2 - v + 1}.$ (8)

Clearing, etc., $5\,v^2 - 19\,v + 12 = 0.$ (9)

Factoring, $(v - 3)(5\,v - 4) = 0.$ (10)

$\therefore\ v = 3 \text{ or } \frac{4}{5}.$ (11)

Substituting 3 for v in (7) or in (6), $y = \pm\sqrt{3}$
and since $x = vy$, $x = \pm 3\sqrt{3}.$ (12)

Substituting $\frac{4}{5}$ for v in (7) or in (6), $y = \pm 5$
and since $x = vy$, $x = \pm 4.$ (13)

When the double sign is used, as in (12) and in (13), it is understood that the roots shall be associated by taking the *upper* signs together and the *lower* signs together.

Hence, $\begin{cases} x = 3\sqrt{3}; & -3\sqrt{3}; & 4; & -4; \\ y = \sqrt{3}; & -\sqrt{3}; & 5; & -5. \end{cases}$

Solve, and verify results:

2. $\begin{cases} x^2 + 3\,y^2 = 84, \\ xy - y^2 = 8. \end{cases}$

3. $\begin{cases} 2\,x^2 - 3\,xy + 2\,y^2 = 100, \\ x^2 + y^2 = 125. \end{cases}$

4. $\begin{cases} x^2 - xy + y^2 = 21, \\ x^2 + 2y^2 = 27. \end{cases}$

5. $\begin{cases} x(x - y) = 6, \\ x^2 + y^2 = 5. \end{cases}$

6. $\begin{cases} xy + 3y^2 = 20, \\ x^2 - 3xy = -8. \end{cases}$

7. $\begin{cases} x^2 + xy = 12, \\ xy + 2y^2 = 5. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 44, \\ xy - y^2 = 8. \end{cases}$

9. $\begin{cases} x^2 + xy = 77, \\ xy - y^2 = 12. \end{cases}$

370. Special devices.

Many systems of simultaneous equations that belong to one or more of the preceding classes, and many that belong to none of them, may be solved readily by *special devices.* It is impossible to lay down any fixed line of procedure, but the object often aimed at is to find values for *any two* of the expressions, $x + y$, $x - y$, and xy, from which the values of x and y may be obtained. Various manipulations are resorted to in attaining this object, according to the forms of the given equations.

EXERCISES

371. 1. Solve the equations $\begin{cases} x^2 + xy = 12, \\ xy + y^2 = 4. \end{cases}$

SOLUTION

$$x^2 + xy = 12. \quad (1)$$

$$xy + y^2 = 4. \quad (2)$$

Adding (1) and (2), $$x^2 + 2xy + y^2 = 16. \quad (3)$$

$$\therefore x + y = +4 \text{ or } -4. \quad (4)$$

Subtracting (2) from (1), $$x^2 - y^2 = 8. \quad (5)$$

Dividing (5) by (4), $$x - y = +2 \text{ or } -2. \quad (6)$$

Combining (4) and (6), $$x = 3 \text{ or } -3;\ y = 1 \text{ or } -1.$$

NOTE. — The first value of $x - y$ corresponds only to the first value of $x + y$, and the second value only to the second value.

Consequently, there are only two pairs of values of x and y.

Observe that the given equations belong to the class treated in § 368. The special device adopted here, however, gives a much neater and simpler solution than the method presented in that case.

2. Solve the equations $\begin{cases} x^2+y^2+x+y=14, \\ xy=3. \end{cases}$

SOLUTION

$$x^2 + y^2 + x + y = 14. \quad (1)$$

$$xy = 3. \quad (2)$$

Adding twice the second equation to the first,

$$x^2 + 2\,xy + y^2 + x + y = 20.$$

Completing the square, $(x+y)^2 + (x+y) + (\frac{1}{2})^2 = 20\frac{1}{4}$.

Taking the square root, $x + y + \frac{1}{2} = \pm \frac{9}{2}$.

$$\therefore x + y = 4 \text{ or } -5. \quad (3)$$

Equations (2) and (3) give two pairs of simultaneous equations,

$$\begin{cases} x+y=4 \\ xy=3 \end{cases} \text{ and } \begin{cases} x+y=-5 \\ xy=3 \end{cases}$$

Solving, the corresponding values of x and y are found to be

$$\begin{cases} x = 3;\ 1;\ \frac{1}{2}(-5+\sqrt{13});\ \frac{1}{2}(-5-\sqrt{13}); \\ y = 1;\ 3;\ \frac{1}{2}(-5-\sqrt{13});\ \frac{1}{2}(-5+\sqrt{13}). \end{cases}$$

Symmetrical except as to sign. — When one of the equations is symmetrical and the other would be symmetrical if one or more of its signs were changed, or when both equations are of the latter type, the system may be solved by the methods used for symmetrical equations (§ 365).

3. Solve the equations $\begin{cases} x^2+y^2=53, \\ x-y=5. \end{cases}$

SOLUTION

$$x^2 + y^2 = 53. \quad (1)$$

$$x - y = 5. \quad (2)$$

Squaring (2), $x^2 - 2\,xy + y^2 = 25. \quad (3)$

Subtracting (3) from (1), $2\,xy = 28. \quad (4)$

Adding (4) and (1), $x^2 + 2\,xy + y^2 = 81. \quad (5)$

Taking the square root, $x + y = \pm 9. \quad (6)$

From (6) and (2), $x = 7 \text{ or } -2;$

and $y = 2 \text{ or } -7.$

Division of one equation by the other. — The reduction of equations of higher degree to quadratics is often effected by dividing one of the given equations by the other, *member by member*.

4. Solve the equations $\begin{cases} x^3 - y^3 = 26, \\ x - y = 2. \end{cases}$

SUGGESTION. — Dividing the first equation by the second,

$$x^2 + xy + y^2 = 13.$$

Therefore, solve the system,

$$\begin{cases} x^2 + xy + y^2 = 13, \\ x - y = 2, \end{cases}$$

instead of the given system.

Elimination of similar terms. — It is often advantageous to eliminate similar terms by addition or subtraction, just as in simultaneous simple equations.

5. Solve the equations $\begin{cases} xy + x = 35, \\ xy + y = 32. \end{cases}$

SOLUTION

$$xy + x = 35. \quad (1)$$

$$xy + y = 32. \quad (2)$$

Subtracting (2) from (1), $x - y = 3$;

whence, $y = x - 3$. (3)

Substituting (3) in (1), $x(x - 3) + x = 35$,

or $x^2 - 2x = 35$.

Solving, $x = 7$ or -5. (4)

Substituting (4) in (3), $y = 4$ or -8.

Solve, using the methods illustrated in exercises 1–5; verify:

6. $\begin{cases} m^2 + mn = 2, \\ mn + n^2 = -1. \end{cases}$

7. $\begin{cases} p^2 + q^2 + p + q = 14, \\ pq = -6. \end{cases}$

8. $\begin{cases} a^2 + b^2 = 130, \\ a - b = 2. \end{cases}$

9. $\begin{cases} r^3 - s^3 = 54, \\ r - s = 6. \end{cases}$

10. $\begin{cases} xy + x = 32, \\ xy + y = 27. \end{cases}$

11. $\begin{cases} 2x^2 - 3y^2 = 5, \\ 3x^2 - 2y^2 = 30. \end{cases}$

372. All the solutions in §§ 362–371 are but illustrations of methods that are important because they are often applicable. The student is urged to use his ingenuity in devising other methods or modifications of these whenever the given system does not yield readily to the devices illustrated, or whenever a simpler solution would result.

MISCELLANEOUS EXERCISES

373. Solve, and verify each result:

1. $\begin{cases} x + y = 3, \\ xy = 2. \end{cases}$

2. $\begin{cases} 5x^2 - 4y^2 = 44, \\ 4x^2 - 5y^2 = 19. \end{cases}$

3. $\begin{cases} 1 + x = y, \\ x^2 + y^2 = 61. \end{cases}$

4. $\begin{cases} x^2 - xy = 48, \\ xy - y^2 = 12. \end{cases}$

5. $\begin{cases} a + ab + 28 = 0, \\ b + ab + 40 = 0. \end{cases}$

6. $\begin{cases} x^2 + y^2 + x + y = 26, \\ xy = -12. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 40, \\ xy = 12. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 28, \\ x + y = 4. \end{cases}$

9. $\begin{cases} xy + x^2 = 44, \\ xy + y^2 = -28. \end{cases}$

10. $\begin{cases} x^2 + 3xy - y^2 = 9, \\ x + 2y = 4. \end{cases}$

11. $\begin{cases} 4xy = 96 - x^2y^2, \\ x + y = 6. \end{cases}$

12. $\begin{cases} x^2 - xy = 8, \\ xy + y^2 = 12. \end{cases}$

13. $\begin{cases} x(x + y) = x, \\ y(x - y) = -1. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 151, \\ x^2 + y^2 = 106. \end{cases}$

15. $\begin{cases} 1 + x = y, \\ 4 + 4x^3 = y^3. \end{cases}$

16. $\begin{cases} x^4 - y^4 = 369, \\ x^2 - y^2 = 9. \end{cases}$

17. $\begin{cases} x^2 - xy = 6, \\ x^2 + y^2 = 61. \end{cases}$

18. $\begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$

19. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^3 - y^3 = 19. \end{cases}$

20. $\begin{cases} x^2 + 3xy = y^2 + 9, \\ x + 3y = 5. \end{cases}$

Problems

374. 1. The sum of two numbers is 12, and their product is 32. What are the numbers?

2. The sum of two numbers is 17, and the sum of their squares is 157. What are the numbers?

3. The difference of two numbers is 1, and the difference of their cubes is 91. What are the numbers?

4. The area of a rectangular mirror is 88 square feet and its perimeter is 38 feet. Find its dimensions.

5. The perimeter of a rectangular ginseng bed is 18 rods and its area is $20\frac{1}{4}$ square rods. What are its demensions?

6. It takes 52 rods of fence to inclose a rectangular garden containing 1 acre. How long and how wide is the garden?

7. The product of two numbers is 59 greater than their sum, and the sum of their squares is 170. What are the numbers?

8. If 63 is subtracted from a certain number expressed by two digits, its digits will be transposed; and if the number is multiplied by the sum of its digits, the product will be 729. What is the number?

9. The smallest of the printed muslin flags made in this country has an area of $5\frac{1}{4}$ square inches. If the width were $\frac{1}{4}$ of an inch less, the length would be twice the width. Find the length and width.

10. A man's ice bill was $18. If ice had cost $1 less per ton, he would have received $\frac{3}{5}$ of a ton more for the same money. How many tons did he use and what was the price per ton?

11. The car of the airship *America* is 107 feet longer than it is wide. If its length were $\frac{1}{5}$ as great, the area of the floor would be 184 square feet. Find its dimensions.

12. A Chinaman received $1.60 for rolling joss sticks. Had he been paid 8 cents more for each lot of ten thousand, he would have had to roll one lot less for the same amount of money. How many such lots did he roll and how much was he paid per lot?

13. The area of a wall painting in a restaurant in Philadelphia is 340 square feet. If its length were 12 feet less and its width 12 feet greater, it would be square. Find its length and width.

14. Before the reduction in letter postage between the United States and Great Britain, it cost 10 ¢ to send a certain letter that now would just go for 4 ¢. What is the weight of the letter, if the postage was reduced 3 ¢ an ounce?

15. If each of a farmer's maple trees had yielded 2 pounds more of sugar, he could have made 750 pounds. If he had had 50 trees more, he could have made 600 pounds. Find the number of trees he had and the yield per tree.

16. A woman in Saxony received 1 ¢ an hour more for making chiffon hats than for weaving straw hats. If she received 21 ¢ for her work on the former and 20 ¢ for her work on the latter, working 14 hours in all, how much did she receive per hour for each?

17. An Illinois farmer raised broom corn and pressed the 6120 pounds of brush into bales. If he had made each bale 20 pounds heavier, he would have had 1 bale less. How many bales did he press and what was the weight of each?

18. If the length of the platform of an elevator were 9 feet less and its width 9 feet more, its area would be 361 square feet. If its length were 9 feet more and its width 9 feet less, its area would be 37 square feet. Find its dimensions.

19. A man bought 4 more loads of sand than of gravel, paying $.50 less per load for sand than for gravel. The sand cost him $ 9 and the gravel $ 10. What quantities of each did he buy? What prices did he pay?

20. The capital stock of a creamery company is $ 8850. If there had been 10 times as many stockholders, each man's share would have been $ 45 less. How many stockholders were there and what was each man's share?

21. Mr. Fuller paid \$2.25 for some Italian olive oil, and \$2 for $\frac{1}{2}$ gallon less of French olive oil, which cost \$.50 more per gallon. How much of each kind did he buy and at what price?

22. In papering a room, 18 yards of border were required, while 40 yards of paper $\frac{1}{2}$ yard wide were needed to cover the ceiling exactly. Find the length and breadth of the room.

23. The water surface area of a tank at Washington, D. C., used in testing ship models is 20,210 square feet. If the length were 3 feet greater, it would be 11 times the width. What are the dimensions of the water surface?

24. If the elm beetle had killed 500 trees less one year in Albany, New York, the total estimated loss would have been \$10,000; if the value of each tree had been $\frac{1}{4}$ as much, the total loss would have been \$3750. How many elm trees were killed?

25. The total area of a window screen whose length is 4 inches greater than its width is 10 square feet. The area inside the wooden frame is 8 square feet. Find the width of the frame.

26. A rectangular skating rink together with a platform around it 25 feet wide covered 37,500 square feet of ground. The area of the platform was $\frac{7}{8}$ of the area of the rink. What were the dimensions of the rink?

27. The course for a 36-mile yacht race is the perimeter of a right triangle, one leg of which is 3 miles longer than the other. How long is each side of the course?

28. At simple interest a sum of money amounted to \$2472 in 9 months and to \$2528 in 16 months. Find the amount of money at interest and the rate.

29. Two men working together can complete a piece of work in $6\frac{2}{3}$ days. If it would take one man 3 days longer than the other to do the work alone, in how many days can each man do the work alone?

GRAPHIC SOLUTIONS

QUADRATIC EQUATIONS — TWO UNKNOWN NUMBERS

EXERCISES

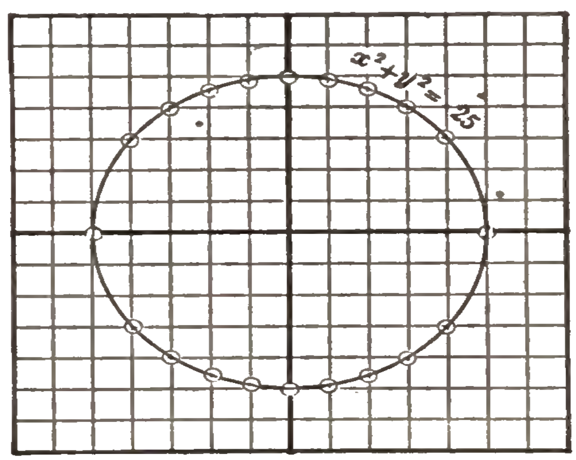
375. 1. Construct the graph of the equation $x^2 + y^2 = 25$.

Solution. — Solving for y, $y = \pm\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, we substitute only values of x between and including -5 and $+5$. The corresponding values of y, or of $\pm\sqrt{25-x^2}$, are recorded in the table below.

It will be observed that each value substituted for x, except ± 5, gives two values of y, and that values of x numerically equal give the same values of y; thus, when $x = 2$, $y = \pm 4.6$, and also when $x = -2$, $y = \pm 4.6$.

x	y
0	± 5
± 1	± 4.9
± 2	± 4.6
± 3	± 4
± 4	± 3
± 5	0



The values given in the table serve to locate twenty points of the graph of $x^2 + y^2 = 25$. Plotting these points and drawing a smooth curve through them, the graph is apparently a *circle*. It may be proved by geometry that this graph is a circle whose radius is 5.

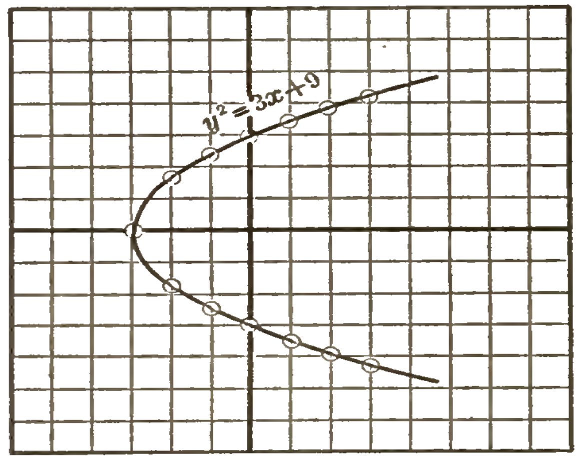
The graph of any equation of the form $x^2 + y^2 = r^2$ is a **circle** whose radius is r and whose center is at the origin.

2. Construct the graph of the equation $y^2 = 3x + 9$.

SOLUTION. — Solving for y, $y = \pm\sqrt{3x+9}$.

It will be observed that any value smaller than -3 substituted for x will make y imaginary; consequently, no point of the graph lies to the left of $x = -3$. Beginning with $x = -3$, we substitute values for x and determine the corresponding values of y, as recorded in the table:

x	y
-3	0
-2	± 1.7
-1	± 2.4
0	± 3
1	± 3.5
2	± 3.9
3	± 4.2



Plotting these points and drawing a smooth curve through them, the graph obtained is apparently a *parabola* (§ 356).

The graph of any equation of the form $y^2 = ax + c$ is a parabola.

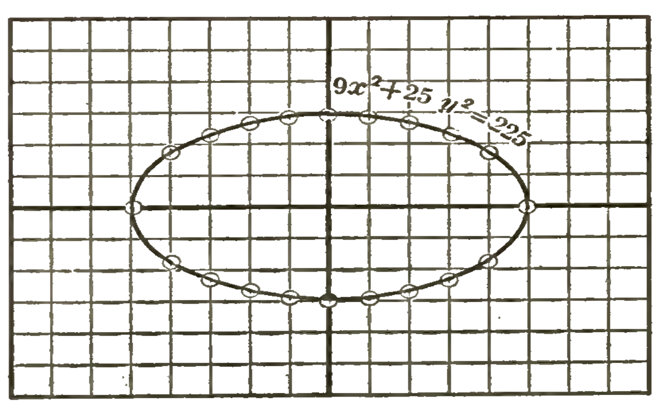
3. Construct the graph of the equation $9x^2 + 25y^2 = 225$.

SOLUTION. — Solving for y, $y = \pm\frac{3}{5}\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, no point of the graph lies farther to the right or to the left of the origin than 5 units; consequently, we substitute for x only values between and including -5 and $+5$.

Corresponding values of x and y are given in the table:

x	y
0	± 3
± 1	± 2.9
± 2	± 2.7
± 3	± 2.4
± 4	± 1.8
± 5	0



Plotting the twenty points tabulated on the preceding page, and drawing a smooth curve through them, we have the graph of $9x^2 + 25y^2 = 225$, which is called an *ellipse*.

The graph of any equation of the form $b^2x^2 + a^2y^2 = a^2b^2$ is an **ellipse.**

4. Construct the graph of the equation $4x^2 - 9y^2 = 36$.

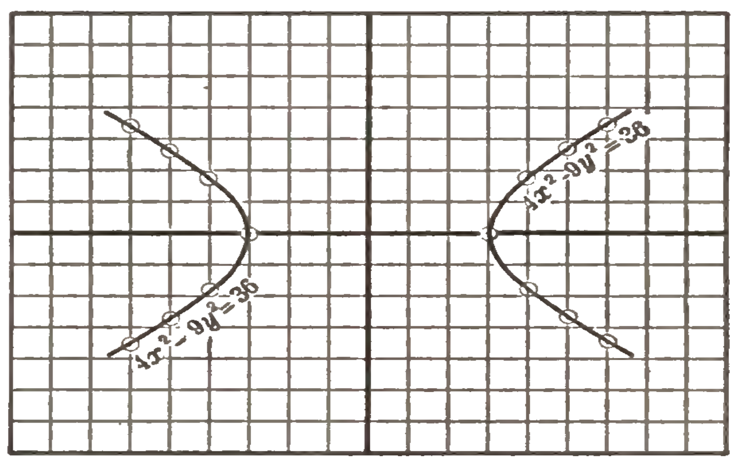
SOLUTION

Solving for y, $$y = \pm \tfrac{2}{3}\sqrt{x^2 - 9}.$$

Since any value numerically less than 3 substituted for x will make the value of y imaginary, no point of the graph lies between $x = +3$ and $x = -3$; consequently, we substitute for x only ± 3 and values numerically greater than 3.

Corresponding values of x and y are given in the table:

x	y
± 3	0
± 4	± 1.8
± 5	± 2.7
± 6	± 3.5



Plotting these fourteen points, it is found that half of them are on one side of the y-axis and half on the other side, and since there are no points of the curve between $x = +3$ and $x = -3$, the graph has two separate *branches*, that is, it is *discontinuous*.

Drawing a smooth curve through each group of points, the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an *hyperbola*.

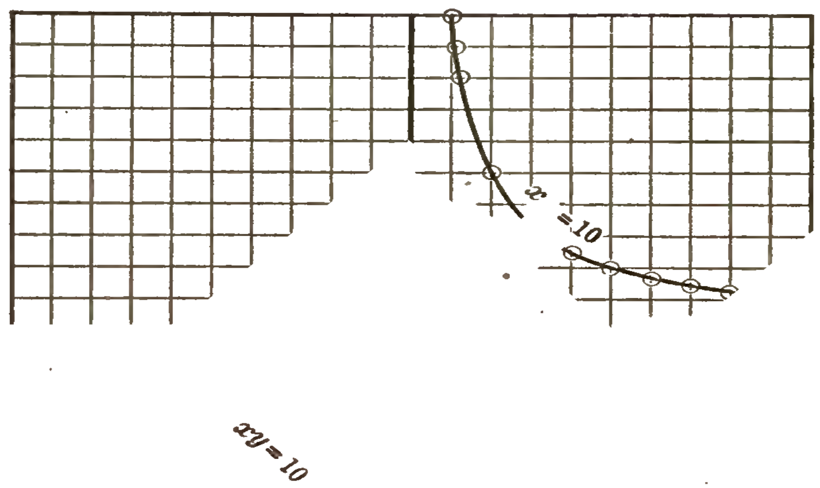
The graph of any equation of the form $b^2x^2 - a^2y^2 = a^2b^2$ is an **hyperbola.**

An hyperbola has two **branches** and is called a **discontinuous** curve.

5. Construct the graph of the equation $xy = 10$.

Solution. — Substituting values for x and solving for y, the corresponding values found are as given in the table:

x	y	x	y
1	10	-1	-10
2	5	-2	-5
3	$3\frac{1}{3}$	-3	$-3\frac{1}{3}$
4	$2\frac{1}{2}$	-4	$-2\frac{1}{2}$
5	2	-5	-2
6	$1\frac{2}{3}$	-6	$-1\frac{2}{3}$
7	$1\frac{3}{7}$	-7	$-1\frac{3}{7}$
8	$1\frac{1}{4}$	-8	$-1\frac{1}{4}$
9	$1\frac{1}{9}$	-9	$-1\frac{1}{9}$
10	1	-10	-1



Plotting these points and drawing a smooth curve through each group of points, the two branches of the curve found constitute the graph of the equation $xy = 10$, which is an *hyperbola*.

The graph of any equation of the form $xy = c$ is an **hyperbola**.

Construct the graph of:

6. $x^2 + y^2 = 9$.
7. $y^2 = 5x + 8$.
8. $9x^2 + 16y^2 = 144$.
9. $9x^2 - 16y^2 = 144$.
10. $xy = 12$.
11. $xy = -6$.

376. Summary. — The types of equations and their respective graphs, here given, will aid the student in plotting graphs, but he will meet other forms of equations that will have some of the same kinds of graphs, the varieties in equations giving rise to varieties in form, size, or location of the graphs.

For example, § 375, exercises 4 and 5 are both equations of the hyperbola, but they are differently located and of different size and shape.

It is possible to determine many characteristics of the various graphs from their equations alone, but a discussion of this is beyond the province of algebra. In the study of graphs, therefore, the student will rely principally on plotting a sufficient number of points to determine their form accurately. The following *types* have been studied:

I.	$ax + by = c$ (§ 248)	**Straight line**
II.	$x^2 + y^2 = r^2$	**Circle**
III.	$\left\{\begin{array}{l} ax^2 + bx + c = 0, \text{ or} \\ y = ax^2 + bx + c \end{array}\right\}$ (§ 356)	**Parabola**
IV.	$y^2 = ax + c$	**Parabola**
V.	$b^2x^2 + a^2y^2 = a^2b^2$	**Ellipse**
VI.	$b^2x^2 - a^2y^2 = a^2b^2$	**Hyperbola**
VII.	$xy = c$	**Hyperbola**

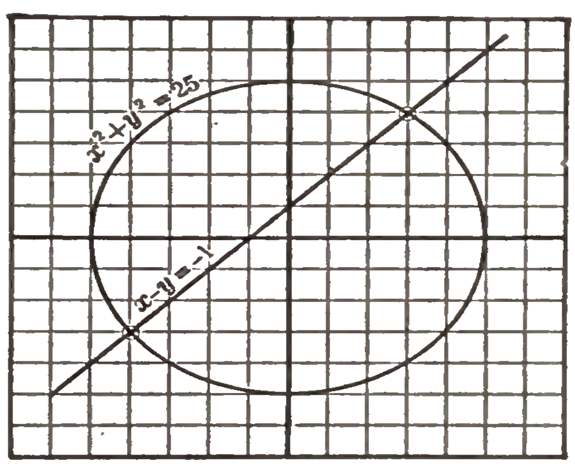
SIMULTANEOUS QUADRATIC EQUATIONS

377. The graphic method of solving simultaneous equations that involve quadratics is precisely the same as for simultaneous linear equations (§ 252). Construct the graph of each equation, both graphs being referred to the same axes, and determine the coördinates of the points where the graphs intersect. If they do not intersect, interpret this fact.

NOTE. — The student should construct the following graphs for himself. Roots are expected to the nearest tenth of a unit. To obtain this degree of accuracy, numerous points should be plotted and a scale of about $\frac{1}{2}$ inch to 1 unit should be used.

EXERCISES

378. 1. Solve graphically $\left\{\begin{array}{l} x^2 + y^2 = 25, \\ x - y = -1. \end{array}\right.$



SOLUTION. — Constructing the graphs of these equations, we find the first, as in § 375, exercise 1, to be a circle; and the second, as in § 248, a straight line.

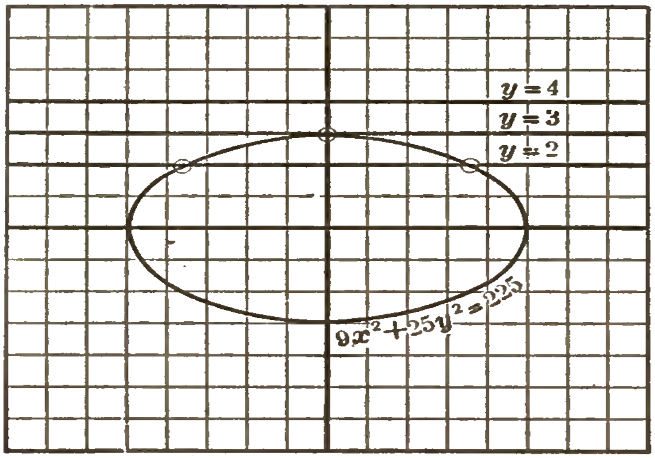
The line intersects the circle in two points $(-4, -3)$ and $(3, 4)$.

Hence, there are two solutions

$x = -4, y = -3$ and $x = 3, y = 4$.

TEST. — The student may test the roots found graphically by performing the algebraic solution.

2. Solve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 2. \end{cases}$



SOLUTION. — On constructing the graphs (for the first, see exercise 3, § 375), it is found that they intersect at the points $x = 3.7$, $y = 2$ and $x = -3.7$, $y = 2$.

Since the graphs have these two points in common, and no others, their coördinates are the only values of x and y that satisfy both equations and are the roots sought.

Observe that the pairs of values $x = 3.7$, $y = 2$ and $x = -3.7$, $y = 2$ are *real*, and different, or *unequal*.

NOTE. — The roots are estimated to the nearest tenth; their accuracy may be tested by performing the algebraic solution.

3. Solve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 3. \end{cases}$

SOLUTION. — Imagine the straight line $y = 2$ in the figure for exercise 2 to move upward until it coincides with the line $y = 3$. The real unequal roots represented by the coördinates of the points of intersection approach equality, and when the line becomes the tangent line $y = 3$, they coincide.

Hence, the given system of equations has *two real equal roots* $x = 0$, $y = 3$ and $x = 0$, $y = 3$.

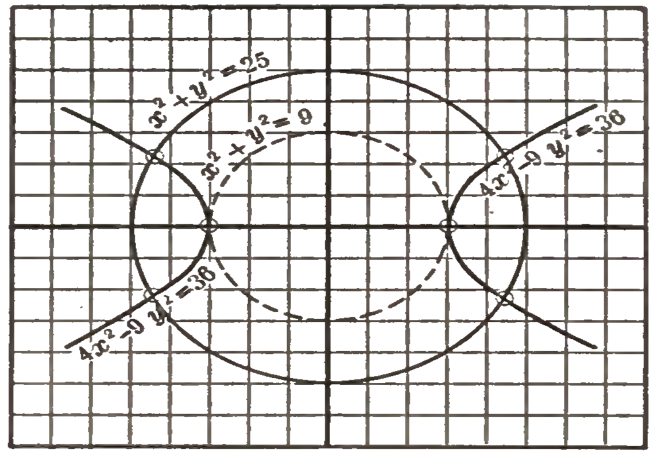
4. Find the nature of the roots of $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 4. \end{cases}$

SOLUTION. — Imagine the straight line $y = 2$ in the figure for exercise 2 to move upward until it coincides with the line $y = 4$. The graphs will cease to have any points in common, showing that the given equations have *no common real values* of x and y.

It is shown by the algebraic solution of the equations that there are two roots and that both are *imaginary*.

A system of two independent simultaneous equations in x and y, one simple and the other quadratic, has two roots.

The roots are real and unequal if the graphs intersect, real and equal if the graphs are tangent to each other, and imaginary if the graphs have no points in common.



5. Solve graphically

$$\begin{cases} 4x^2 - 9y^2 = 36, \\ x^2 + y^2 = 25. \end{cases}$$

Solution. — The graphs (see exercises 4 and 1, § 375) show that both of the given equations are satisfied by *four* different pairs of real values of x and y:

$$\begin{cases} x = 4.5; & 4.5; & -4.5; & -4.5; \\ y = 2.2; & -2.2; & -2.2; & 2.2. \end{cases}$$

6. What would be the nature of the roots in exercise 5, if the second equation were $x^2 + y^2 = 9$?

Solution. — Imagine the circle $x^2 + y^2 = 25$ in exercise 5 to become smaller and smaller until it coincides with the circle $x^2 + y^2 = 9$ (see dotted circle in the cut). The four real unequal roots represented by the coördinates of the points of intersection of the graphs come together in pairs at the points (3, 0) and (−3, 0) where the circle $x^2 + y^2 = 9$ is tangent to the hyperbola $4x^2 - 9y^2 = 36$; consequently, the equations $4x^2 - 9y^2 = 36$ and $x^2 + y^2 = 9$ have two pairs of *equal real roots*, namely:

$$\begin{cases} x = 3, & 3, & -3, & -3; \\ y = 0, & 0, & 0, & 0. \end{cases}$$

A system of two independent simultaneous quadratic equations in x and y has four roots.

An intersection of the graphs represents a real root, and a point of tangency, a pair of equal real roots. If there are less than four real roots, the other roots are imaginary.

It is not possible to solve *any* two simultaneous equations in x and y, that involve quadratics, *by quadratic methods*, but approximate values of the real roots may *always* be found *by the graphic method*.

Find by graphic methods, to the nearest tenth, the real roots of the following, and the number of imaginary roots, if there are any. Discuss the graphs and the roots.

7. $\begin{cases} 4x^2 - 9y^2 = 36, \\ x - 3y = 1. \end{cases}$

8. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4x^2 + 9y^2 = 36. \end{cases}$

9. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4y = x^2 - 16. \end{cases}$

10. $\begin{cases} 9x^2 + 16y^2 = 144, \\ 3x + 4y = 12. \end{cases}$

11. $\begin{cases} 9x^2 + 16y^2 = 144, \\ x^2 + y^2 = \frac{49}{4}. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 4, \\ x = y - 5. \end{cases}$

13. $\begin{cases} x^2 - 4y^2 = 4, \\ x^2 + y^2 = 4. \end{cases}$

14. $\begin{cases} x - y = 2, \\ xy = -1. \end{cases}$

15. $\begin{cases} x^2 + y^2 = 9, \\ y = x^2 - 5x + 6. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 9, \\ x = y^2 + 5y + 6. \end{cases}$

17. $\begin{cases} y = x^2 - 4, \\ x = (y + 1)(y + 4). \end{cases}$

18. $\begin{cases} y = x^2 - 5x + 4, \\ x = y^2 - 4y + 3. \end{cases}$

379. Another graphic method of solving quadratic equations in one unknown number (§ 356).

It has been seen that the real roots of *simultaneous equations* are the coördinates of the points where their graphs intersect or are tangent to each other, and that when there is no point in common, the roots are imaginary.

In §§ 356, 357, it was found that the real roots of a *quadratic equation* were the abscissas of the points where the graph of the quadratic expression crossed or touched the x-axis, and that when it had no point in common with the x-axis, the roots were imaginary.

In other words, the solution of a quadratic equation in x was made to depend upon the solution of the simultaneous system,

(I). $\begin{cases} y = ax^2 + bx + c, & \text{(a parabola)} \\ y = 0, & \text{(a straight line)} \end{cases}$

the second being the equation of the x-axis.

In the following method, by substituting y for x^2 in the given equation,

$$ax^2 + bx + c = 0,$$

the equation is divided into the simultaneous system,

(II). $\begin{cases} ay + bx + c = 0, & \text{(a straight line)} \\ y = x^2. & \text{(a parabola)} \end{cases}$

The solution of this system for x gives the required roots of $ax^2 + bx + c = 0$.

It will be observed that whether system (I) or (II) is used, the solution requires the construction of a parabola and a straight line, but the advantage of using (II) instead of (I) lies in the fact that the parabola $y = x^2$ is the same for all quadratic equations in x and when once constructed can be used for solving any number of equations, while with (I) a different parabola must be constructed for each equation solved.

EXERCISES

380. 1. Solve graphically the equation $x^2 - 2x - 8 = 0$.

SOLUTION

Substituting y for x^2, we have

$$y - 2x - 8 = 0.$$

Consequently, the values of x that satisfy the system,

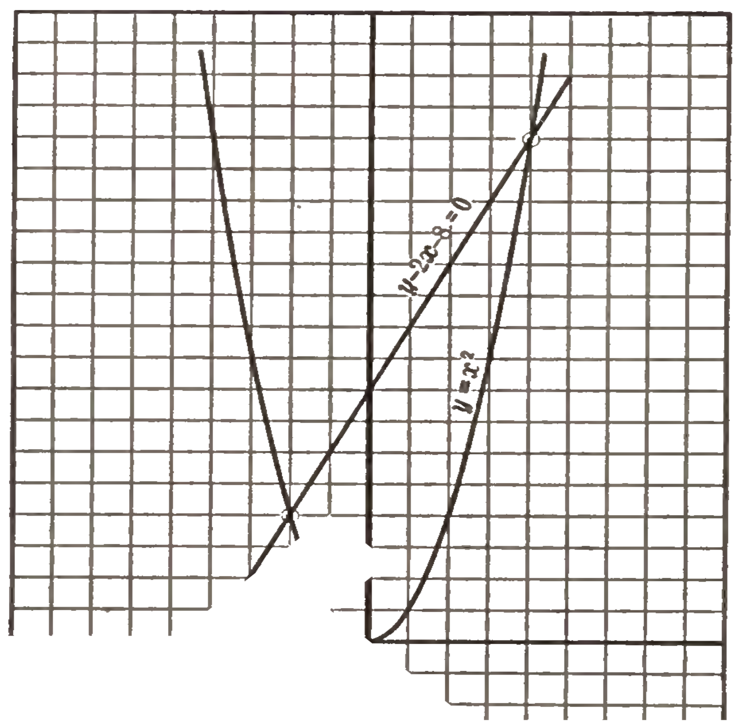
$$\begin{cases} y - 2x - 8 = 0, \\ y = x^2, \end{cases}$$

are the same as those that satisfy the given equation.

Constructing the graph of $y = x^2$, we have the parabola shown in the figure.

Constructing the graph of $y - 2x - 8 = 0$, a straight line, we find that it intersects the parabola at $x = -2$ and $x = 4$.

Hence, the roots of the equation $x^2 - 2x - 8 = 0$ are -2 and 4.



Solve graphically, giving roots to the nearest tenth:

2. $x^2 + x - 2 = 0$.
3. $x^2 - x - 6 = 0$.
4. $x^2 - 3x - 4 = 0$.
5. $x^2 - 2x - 15 = 0$.
6. $x^2 + 5x + 14 = 0$.
7. $x^2 - 7x + 18 = 0$.
8. $2x^2 - x = 6$.
9. $2x^2 - x - 15 = 0$.
10. $3x^2 + 5x - 28 = 0$.
11. $6x^2 - 7x - 20 = 0$.
12. $8x^2 + 14x - 15 = 0$.
13. $15x^2 + 2x - 20 = 0$.

RATIO AND PROPORTION

RATIO

381. The relation of two numbers that is expressed by the quotient of the first divided by the second is called their **ratio.**

382. The **sign of ratio** is a colon (:).

A ratio is expressed also in the form of a fraction.

The ratio of a to b is written $a : b$ or $\frac{a}{b}$.

The colon is sometimes regarded as derived from the sign of division by omitting the line.

383. To compare two quantities they must be expressed in *terms of a common unit.*

Thus, to indicate the ratio of 20¢ to \$1, both quantities must be expressed either in cents or in dollars, as 20¢ : 100¢ or \$ $\frac{1}{5}$: \$1.

There can be no ratio between 2 pounds and 3 feet.

The ratio of two quantities is the *ratio* of their *numerical measures.*

Thus, the ratio of 4 rods to 5 rods is the ratio of 4 to 5.

384. The first term of a ratio is called the **antecedent,** and the second, the **consequent.** Both terms form a **couplet.**

The antecedent corresponds to a dividend or numerator; the consequent, to a divisor or denominator.

In the ratio $a : b$, or $\frac{a}{b}$, a is the antecedent, b the consequent, and the terms a and b form a couplet.

385. The ratio of the reciprocals of two numbers is called the **reciprocal, or inverse, ratio** of the numbers.

It may be expressed by interchanging the terms of the couplet.

The inverse ratio of a to b is $\frac{1}{a}:\frac{1}{b}$. Since $\frac{1}{a}\div\frac{1}{b}=\frac{b}{a}$, the inverse ratio of a to b may be written $\frac{b}{a}$ or $b:a$.

Properties of Ratios

386. It is evident from the definition of a ratio that ratios have the same properties as fractions; that is, they may be *reduced to higher or lower terms, added, subtracted,* etc. Hence,

PRINCIPLES. — 1. *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

2. *Multiplying the antecedent or dividing the consequent of a ratio by any number multiplies the ratio by that number.*

3. *Dividing the antecedent or multiplying the consequent by any number divides the ratio by that number.*

EXERCISES

387. 1. What is the ratio of $8m$ to $4m$? of $4m$ to $8m$?

2. Express the ratio of $6:9$ in its lowest terms; the ratio $12x:16y$; $am:bm$; $20ab:10bc$; $(m+n):(m^2-n^2)$.

3. Which is the greater ratio, $2:3$ or $3:4$? $4:9$ or $2:5$?

4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? $\frac{1}{2}$ to $\frac{1}{3}$? $\frac{2}{3}$ to $\frac{3}{4}$?

SUGGESTION. — When fractions have a common denominator, they have the ratio of their numerators.

5. What is the inverse ratio of $3:10$? of $12:7$?

Reduce to lowest terms the ratios expressed by:

6. $10:2$.	8. $3:27$.	10. $\frac{12}{24}$.	12. $75\div100$.
7. $12:6$.	9. $4:40$.	11. $\frac{16}{48}$.	13. $60\div120$.

14. What is the ratio of 15 days to 30 days? of 21 days to 1 week? of 1 rod to 1 mile?

Find the value of each of the following ratios:

15. $\frac{4}{5}x : \frac{2}{5}x^2$. 17. $2\frac{1}{2} : 7\frac{1}{2}$. 19. $a^2b^3x^4 : a^4b^2x^2$.

16. $\frac{3}{4}ab : \frac{1}{2}ac$. 18. $.7m : .8n$. 20. $(x^2 - y^2) : (x - y)^2$.

21. If 9 is subtracted from 4 and then from 5, find the ratio of the first remainder to the second.

22. Change each to a ratio whose antecedent shall be 1:

$$5:20;\quad 3x:12x;\quad \tfrac{3}{2}:\tfrac{5}{6};\quad .4:1.2.$$

23. When the antecedent is $6x$ and the ratio is $\frac{1}{3}$, what is the consequent?

PROPORTION

388. An equality of ratios is called a **proportion.**

$3:10 = 6:20$ and $a:x = b:y$ are proportions.

The double colon (: :) is often used instead of the sign of equality.

The double colon has been supposed to represent the extremities of the lines that form the sign of equality.

The proportion $a:b = c:d$, or $a:b::c:d$, is read, 'the ratio of a to b is equal to the ratio of c to d,' or 'a is to b as c is to d.'

389. In a proportion, the first and fourth terms are called the **extremes,** and the second and third terms, the **means.**

In $a:b = c:d$, a and d are the extremes, b and c are the means.

390. Since a proportion is an equality of ratios each of which may be expressed as a fraction, a proportion may be expressed as an equation each member of which is a fraction.

Hence, it follows that:

General Principle. — *The changes that may be made in a proportion without destroying the equality of its ratios correspond to the changes that may be made in the members of an equation without destroying their equality and in the terms of a fraction without altering the value of the fraction.*

Properties of Proportions

391. PRINCIPLE 1.—*In any proportion the product of the extremes is equal to the product of the means.*

For, given $a:b = c:d,$

or $\frac{a}{b} = \frac{c}{d}.$

Clearing of fractions, $ad = bc.$

Test the following by principle 1 to find whether they are true proportions:

1. $6:16 = 3:8.$ 2. $\frac{12}{15} = \frac{16}{20}.$ 3. $7:8 = 10:12$

392. In the proportion $a:m = m:b$, m is called a **mean proportional** between a and b.

By Prin. 1, $m^2 = ab;$

$\therefore\ m = \sqrt{ab}.$

Hence, *a mean proportional between two numbers is equal to the square root of their product.*

1. Show that the mean proportional between 3 and 12 is either 6 or -6. Write both proportions.

2. Find two mean proportionals between 4 and 25.

393. PRINCIPLE 2.—*Either extreme of a proportion is equal to the product of the means divided by the other extreme.*

Either mean is equal to the product of the extremes divided by the other mean.

For, given $a:b = c:d.$

By Prin. 1, $ad = bc.$

Solving for a, d, b, and c, in succession, Ax. 4,

$$a = \frac{bc}{d},\ d = \frac{bc}{a},\ b = \frac{ad}{c},\ c = \frac{ad}{b}.$$

1. Solve the proportion $3:4 = x:20$, for x.

2. Solve the proportion $x:a = 2m:n$, for x.

3. If $a : b = b : c$, the term c is called a **third proportional** to a and b. Find a third proportional to 6 and 2.

4. In the proportion $a : b = c : d$, the term d is called a **fourth proportional** to a, b, and c. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

394. PRINCIPLE 3. — *If the product of two numbers is equal to the product of two other numbers, one pair of them may be made the extremes and the other pair the means of a proportion.*

For, given $ad = bc$.

Dividing by bd, Ax. 4, $\frac{a}{b} = \frac{c}{d}$;

that is, $a : b = c : d$.

By dividing both members of the given equation, or of $bc = ad$, by the proper numbers, various proportions may be obtained; but in all of them a and d will be the extremes and b and c the means, or *vice versa*, as illustrated in the proofs of principles 4 and 5.

1. If a men can do a piece of work in x days, and if b men can do the same work in y days, the number of days' work for one man may be expressed by either ax or by. Form a proportion between a, b, x, and y.

2. The formula $pd = WD$ (See p. 166)

expresses the physical law that, when a lever just balances, the product of the numerical measures of the power and its distance from the fulcrum is equal to the product of the numerical measures of the weight and its distance from the fulcrum. Express this law by means of a proportion.

395. PRINCIPLE 4. — *If four numbers are in proportion, the ratio of the antecedents is equal to the ratio of the consequents; that is, the numbers are in proportion by* **alternation.**

For, given $a : b = c : d$.

Then, Prin. 1, $ad = bc$.

Dividing by cd, Ax. 4, $\frac{a}{c} = \frac{b}{d}$;

that is, $a : c = b : d$.

396. Principle 5. — *If four numbers are in proportion, the ratio of the second to the first is equal to the ratio of the fourth to the third;* that is, *the numbers are in proportion by* **inversion.**

For, given $a : b = c : d.$

Then, Prin. 1, $ad = bc.$

$\therefore bc = ad.$

Dividing by ac, Ax. 4, $\frac{b}{a} = \frac{d}{c};$

that is, $b : a = d : c.$

397. Principle 6. — *If four numbers are in proportion, the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **composition.**

For, given $a : b = c : d,$

or $\frac{a}{b} = \frac{c}{d}.$

Then, Ax. 1, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$

or $\frac{a+b}{b} = \frac{c+d}{d};$

that is, $a + b : b = c + d : d.$

Similarly, taking the given proportion by inversion (Prin. 5), and adding 1 to both members, we obtain

$$a + b : a = c + d : c.$$

398. Principle 7. — *If four numbers are in proportion, the difference between the terms of the first ratio is to either term of the first ratio as the difference between the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **division.**

For, in the proof of Prin. 6, if 1 is *subtracted* instead of added, the following proportions are obtained:

$$a - b : b = c - d : d,$$

and $a - b : a = c - d : c.$

399. PRINCIPLE 8. — *If four numbers are in proportion, the sum of the terms of the first ratio is to their difference as the sum of the terms of the second ratio is to their difference;* that is, *the numbers are in proportion by* **composition and division.**

For, given $a : b = c : d.$

By Prin. 6, $\frac{a+b}{b} = \frac{c+d}{d}.$ (1)

By Prin. 7, $\frac{a-b}{b} = \frac{c-d}{d}.$ (2)

Dividing (1) by (2), Ax. 4, $\frac{a+b}{a-b} = \frac{c+d}{c-d};$

that is, $a + b : a - b = c + d : c - d.$

400. PRINCIPLE 9. — *In a proportion, if both terms of a couplet or both antecedents, or both consequents are multiplied or divided by the same number, the resulting four numbers form a proportion.*

For, given $a : b = c : d,$

or $\frac{a}{b} = \frac{c}{d}.$

Then, § 181, $\frac{ma}{mb} = \frac{nc}{nd}$, or $ma : mb = nc : nd.$

Also, Ax. 3, $\frac{a}{b} \cdot \frac{m}{n} = \frac{c}{d} \cdot \frac{m}{n}$, or $ma : nb = mc : nd.$

401. PRINCIPLE 10. — *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

For, given $a : b = c : d = e : f,$

or $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$, the value of each ratio.

Then, Ax. 3, $a = br,\ c = dr,\ e = fr;$

whence, Ax. 1, $a + c + e = (b + d + f)r,$

$$\therefore \frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f};$$

that is, $a + c + e : b + d + f = a : b$ or $c : d$ or $e : f.$

EXERCISES

402. 1. Find the value of x in the proportion $3:5=x:55$.

SOLUTION. $3:5=x:55$.

Prin. 2, $x=\frac{3 \cdot 55}{5}=33$.

Find the value of x in each of the following proportions:

2. $2:3=6:x$.
3. $5:x=4:3$.
4. $1:x=x:9$.
5. $x+2:x=10:6$.
6. $x:x-1=15:12$.
7. $x+2:x-2=3:1$.

8. Show that a mean proportional between any two numbers having like signs has the sign $\pm$.

9. Find two mean proportionals between $\sqrt{2}$ and $\sqrt{8}$.

10. Find a third proportional to 4 and 6.

11. Find a fourth proportional to 3, 8, and $7\frac{1}{2}$.

Test to see whether the following are true proportions:

12. $5\frac{1}{2}:3=4:1\frac{1}{2}$.
13. $4:13=2:6\frac{1}{2}$.
14. $5:7x=10:14x$.
15. $2.4a:.8a=6a:2a$.

16. Given $a:b=c:d$,
to prove that $2a+3c:2a-3c=8b+12d:8b-12d$.

PROOF. — Given $a:b=c:d$.

By alternation, Prin. 4, $a:c=b:d$.

Expressing as a fractional equation, $\frac{a}{c}=\frac{b}{d}$.

Multiplying the first member by $\frac{2}{3}$ and the second by $\frac{8}{12}$, the equal of $\frac{2}{3}$,

$$\frac{2a}{3c}=\frac{8b}{12d};$$

that is, $2a:3c=8b:12d$.

By composition and division, Prin. 8,

$$2a+3c:2a-3c=8b+12d:8b-12d.$$

When $a:b=c:d$, prove that:

17. $d:b=c:a$.
18. $c:d=\frac{1}{b}:\frac{1}{a}$.
19. $b^3:d^3=a^3:c^3$.
20. $a^2:b^2c^2=1:d^2$.
21. $ma:\frac{b}{2}=mc:\frac{d}{2}$.
22. $ac:bd=c^2:d^2$.

When $a:b = c:d$, prove that:

23. $a+b:c+d = a-b:c-d$.

24. $2a+5b:2a = 2c+5d:2c$.

25. $4a-3b:4c-3d = a:c$.

26. $a:a+b = a+c:a+b+c+d$.

27. $a+b:c+d = \sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

Problems

403. 1. The ratio of the rate of a local train in the New York subway to an express train is 1 : 2. If the local train runs 15 miles an hour, find the rate of the express train.

2. The consumption of gas in New York City one year was to that in Chicago as 7 : 4. If 12 billion cubic feet were consumed in Chicago, what was the consumption in New York?

3. Michigan produces yearly 25 % of the iron ore of the United States. The ratio of Michigan's output to Minnesota's is 5 : 8. What per cent of the country's output does Minnesota produce?

4. The United States manufactured 285 million pens one year. The ratio of the steel pens to the whole number of pens was 18 : 19. How many steel pens were manufactured?

5. A diver descended 210 feet into a lake. The ratio of this distance to the distance that is usually considered the limit for divers is 7 : 5. Find the usual limit for divers.

6. How many pounds of tea are made from 4200 pounds of the green leaf, if the ratio of the weight of the manufactured tea to that of the green leaf is 5 : 21?

7. Two machines, one old and one modern, turn out 960 pins per minute. The ratio of the number turned out by the old machine to the number turned out by the modern one is 1 : 15. How many were turned out by each machine?

8. Find a number that added to each of the numbers 1, 2, 4, and 7 will give four numbers in proportion.

9. The United States published 20,000 newspapers recently. The relation of this number to those published in the whole world was 2 : 5. How many were published in the world?

10. The ratio of the greatest length of Lake Erie to the greatest length of Lake Michigan is 5 : 6. What is the length of each, if Lake Michigan is 50 miles longer than Lake Erie?

11. The ratio of the loss of life in the Lisbon earthquake to that in the Messina earthquake is 12 : 23. If 55,000 more lives were lost in the latter than in the former, find the loss of life in each earthquake.

12. The length of a giant candle was to that of a Christmas candle as 40 : 1. If 8 times the length of the latter was 96 inches less than that of the former, find the length of each.

13. The wool sales for one week in New York amounted to 555,000 pounds. The ratio of the domestic sales to the foreign was 14 : 23. What were the foreign sales?

14. Out of a lot of shell caps, 100 times the number rejected by the government inspector for imperfections was to the total number as 3 : 11. If 1097 were accepted, how many were rejected?

15. In one year Egypt and Russia together sent $9\frac{1}{4}$ million pounds of eggs to Paris. If Egypt had sent twice as many, the ratio of this number to those sent by Russia would have been 1 : 18. How many pounds were sent by each country?

16. The sum of the three dimensions of a block of ice is 77 inches, and the width, 22 inches, is a mean proportional between the other two dimensions. Find the length and thickness.

17. The ratio of the length of a gold nugget to its width was 11 : 6, but if its length had been $\frac{1}{2}$ of an inch more, the ratio would have been 2 : 1. Find its length and width.

18. The area of the right triangle shown in Fig. 1 may be expressed either as $\frac{1}{2}ab$ or as $\frac{1}{2}ch$. Form a proportion whose terms shall be a, b, c, h.

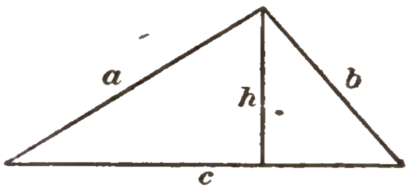


FIG. 1.

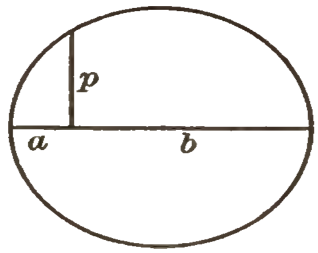


FIG. 2.

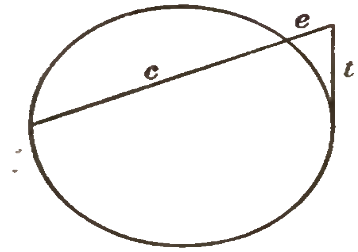


FIG. 3.

19. In Fig. 2, the perpendicular p, which is 20 feet long, is a mean proportional between a and b, the parts of the diameter, which is 50 feet long. Find the length of each part.

20. In Fig. 3, the tangent t is a mean proportional between the whole secant $c+e$, and its external part e. Find the length of t, if $e = 9\frac{3}{5}$ and $c = 50\frac{2}{5}$.

21. The strings of a musical instrument produce sound by vibrating. The relation between the number of vibrations N and N' of two strings, different only in their lengths l and l', is expressed by the proportion

$$N : N' = l' : l.$$

A c string and a g string, exactly alike except in length, vibrate 256 and 384 times per second, respectively. If the c string is 42 inches long, find the length of the g string.

22. If L and l are the lengths of two pendulums and T and t the times they take for an oscillation, then

$$T^2 : t^2 = L : l.$$

A pendulum that makes one oscillation per second is approximately 39.1 inches long. How often does a pendulum 156.4 inches long oscillate?

23. Using the proportion of exercise 22, find how many feet long a pendulum would have to be to oscillate once a minute.

GENERAL REVIEW

404. 1. Distinguish between known and unknown numbers.

2. When $\times$, $\div$, or both occur in connection with $+$, $-$, or both in an expression, what is the sequence of operations?

Illustrate by finding the value of: $7 - 3 \times 2 + 6 \div 2$.

3. Name and illustrate three ways of indicating multiplication; two ways of indicating division.

4. When is $x^n - y^n$ exactly divisible by $x + y$? by $x - y$?

5. When is a trinomial a perfect square? When is a fraction in its lowest terms? What are similar fractions?

6. By what principle may cancellation be used in reducing fractions to lowest terms?

7. Factor the following by three different methods:

$$(a^2 - 2)^2 - a^2.$$

8. Define power; root; like terms; transposition; simultaneous equations; surd.

9. Express the following without parentheses:

$$(a^2x^m)^n; \quad -[-(a^2)^2]^2; \quad (a^3)^4; \quad (a^4)^3; \quad (-a^2)^5.$$

10. What is the sign of any power of a positive number? of any even power of a negative number? of any odd power of a negative number?

11. How may the involution of a trinomial be performed by the use of the binomial formula?

Illustrate by raising $x + 2y - z$ to the third power.

12. Explain the meaning of a negative integral exponent; of a positive fractional exponent; of a zero exponent.

13. Define evolution; binomial surd; similar surds; conjugate surds; symmetrical equation.

14. Is $\sqrt{2+\sqrt{4}}$ a surd? State reasons for your answer.

15. Represent $\sqrt{10}$ inches by a line.

16. Why is it specially important to test the values of the unknown number found in the solution of radical equations?

17. Define coördinate axes; imaginary number; axiom; coefficient; elimination.

18. How is the degree of an equation determined? What is the degree of $x+b=c$? of $x^2+3x=7$? of $5x+xy=11$?

19. What name is given to an equation of the first degree? of the second degree? of any higher degree?

20. What is a pure quadratic equation? a complete quadratic equation? Illustrate each.

21. What is the root of an equation? What is the principle relating to the roots of a pure quadratic equation?

Illustrate by solving the following:

$$7x^2-5=23.$$

22. Give two methods of completing the square in the solution of affected quadratic equations. When is it advantageous to use the Hindoo method?

Solve the following equation by each method:

$$3x^2+5x=22.$$

23. Outline the method of solving quadratic equations by factoring.

Illustrate by solving the following:

$$2x^2-5x=12.$$

24. When is an equation in the quadratic form? Illustrate.

25. What roots should be associated when the roots of a system of equations are given thus: $x = \pm 2$, $y = \mp 3$?

26. Explain how, in the solution of problems, negative roots of quadratic equations, while mathematically correct, are often inadmissible.

27. What is the advantage of letting $x^2 = y$ in the graphic solution of a quadratic equation of the form $ax^2 + bx + c = 0$?

28. How does the graph of a quadratic equation show the fact, if the roots are real and equal? real and unequal? imaginary?

29. What is the form of the graph of a simple equation? of two inconsistent equations? of two indeterminate equations?

30. What is the form of the graph of an equation like $ax^2 + bx + c = 0$? like $x^2 + y^2 = r^2$?

31. What is meant by the minimum point of a graph?

Solve graphically the following equation and indicate the minimum point of the graph:

$$x^2 + 6x + 5 = 0.$$

32. How many roots has a simple equation? a quadratic equation? a system of two independent simultaneous equations, one simple and the other quadratic? a system of two independent simultaneous quadratic equations?

33. Define homogeneous equation; antecedent; consequent; inverse ratio; proportion.

34. Give and illustrate two principles relating to ratios. Upon what do these principles depend?

35. Find the ratio of $(x - y)^3$ to $x^2 - 2xy + y^2$.

36. In the following proportion indicate the means; the extremes; the mean proportional; the third proportional:

$$x : y = y : z.$$

37. Find a fourth proportional to $3a$, $9a$, and $5a$.

38. Add $x\sqrt{y}+y\sqrt{x}+\sqrt{xy}$, $x^{\frac{1}{2}}y^{\frac{1}{2}}-\sqrt{x^2y}-\sqrt{xy^2}$, $\sqrt{x^2y}-\sqrt{xy^2}-\sqrt{xy}$, and $y\sqrt{x}-x\sqrt{4y}-\sqrt{9xy}$.

39. Simplify $a-\{b-a-[a-b-(2a+b)+(2a-b)-a]-b\}$.

40. Divide x^4-y^4 by $x-y$ by inspection. Test.

41. Separate $a^{12}-1$ into six rational factors.

42. Reduce $\dfrac{a^2-b^2-c^2-2bc}{a^2-b^2+c^2+2ac}$ to its lowest terms.

43. Simplify $\dfrac{x}{x+1}-\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}$.

44. Simplify $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(c-b)(c-a)}+\dfrac{1}{(c-a)(a-b)}$.

45. Reduce to the simplest form: $\sqrt{\frac{2}{3}}$; $\sqrt[4]{25a^4}$; $\dfrac{\sqrt{3}+3\sqrt{2}}{\sqrt{6}+2}$.

Solve the following equations for x:

46. $3x^2-2x=65$.

47. $x^4+\dfrac{1}{2}=\dfrac{3x^2}{2}$.

48. $\sqrt{x-9}=\sqrt{x}-1$.

49. $x^2+\sqrt{x^2+16}=14$.

Solve the following for the letters involved:

50. $\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}=10,\\ \dfrac{3}{x}+\dfrac{2}{y}=10.\end{cases}$

51. $\begin{cases}x^2+y^2=25,\\ x+y=7.\end{cases}$

52. $\begin{cases}2x+3y+z=9,\\ x+2y+3z=13,\\ 3x+y+2z=11.\end{cases}$

53. $\begin{cases}x^2+xy=24,\\ y^2+xy=12.\end{cases}$

Solve graphically:

54. $\begin{cases}x-y=1,\\ x^2+y^2=16.\end{cases}$

55. $\begin{cases}4x^2+9y^2=36,\\ y=3.\end{cases}$

Find the value of x in the following proportions:

56. $7\frac{2}{3}:6=x:4\frac{1}{2}$.

57. $x:7.2=3.9:117$.

58. The railways of the United States use annually 150 million tons of coal. If the amount used in drawing trains is $\frac{1}{19}$ as much as goes up the smokestacks, how much is used to draw trains?

59. In one year about 30,000 vessels passed a lighthouse in Massachusetts. The number that used steam was to the number of the remainder as 1 : 5. How many used steam?

60. Out of 63 bakeries inspected in a certain city, the number of 'absolutely clean' ones was 3 more than that of the 'fairly clean' ones, and the number of 'unsanitary' ones was 2 less than twice that of the 'absolutely clean' ones. Find the number of bakeries in each class.

61. The number of parts in a certain manufacturer's mower is twice that in his horse rake and $\frac{3}{19}$ that in his binder. If the binder has 3500 parts more than the rake, how many parts has each machine?

62. Two men earned $3.50 one day for picking pine needles. They were paid 25 cents per 100 pounds. How many pounds did each pick, if one picked $\frac{3}{4}$ as many as the other?

63. One of the largest rugs ever made in this country contains 3180 square feet. Its length is 7 feet greater than its width. What are its dimensions?

64. Alfred the Great measured time by candles lighted in succession. The number used in a day was $\frac{1}{2}$ the number of inches in the length of each candle, and each burned at the rate of 3 inches per hour. How many candles were used per day and how long was each?

65. A target used in practice by the United States fleet was 1 foot longer than it was wide and 18 feet longer than the square bull's-eye. The area of the target exclusive of the bull's-eye was 411 square feet. Find the dimensions of each.

66. A good operator usually earns $1.80 a day by binding derby hats. If she bound 1 dozen more hats and received 5¢ less per dozen, she would earn 5¢ less a day. How many hats does she bind a day and how much does she receive per dozen?

67. How far down a river whose current runs 3 miles an hour can a steamboat go and return in 8 hours, if its rate of sailing in still water is 12 miles an hour?

68. A person who can walk n miles an hour has a hours at his disposal. How far may he ride in a coach that travels m miles an hour and return on foot within the allotted time?

69. The first copy of *The Sun* was printed on a sheet $5\frac{1}{2}$ inches longer than it was wide. If the length lacked 6 inches of being twice the width, find the dimensions of the sheet.

70. The *Lusitania* is 26 feet less than 6 times as long as the *Clermont*, and $\frac{1}{10}$ of the length of the *Lusitania* is 11 feet more than $\frac{1}{2}$ of the length of the *Clermont*. Find the length of each.

71. A woman has 13 square feet to add to the area of the rug she is weaving. She therefore increases the length $\frac{1}{8}$ and the width $\frac{1}{4}$, which makes the perimeter 4 feet greater. Find the dimensions of the finished rug.

72. The inventor of toothpicks sold 16,250,000 during his first year of business. Had there been 75,000 more toothpicks in each box, the number of boxes sold would have been 15 fewer. How many boxes did he sell?

73. Two passengers together have 400 pounds of baggage and are charged, for the excess above the weight allowed free, 40 cents and 60 cents, respectively. If the baggage had belonged to one of them, he would have been charged $1.50. How much baggage is one passenger allowed without charge?

74. A railway train, after traveling 2 hours at its usual rate, was detained 1 hour by an accident. It then proceeded at $\frac{3}{5}$ of its former rate, and arrived $7\frac{2}{3}$ hours behind time. If the accident had occurred 50 miles farther on, the train would have arrived $6\frac{1}{3}$ hours behind time. What was the whole distance traveled by the train?

405. This page contains the questions given in the Elementary Algebra examination of the Regents of the University of the State of New York for June, 1909.

References show where the text provides instruction necessary to answer these questions.

1. Divide $6x^3 + 11x^2 - 1$ by $3x - 1 + 2x^2$ (§§ 38, 108).

2. Find the prime factors of $1 - \frac{x^2}{4}$ (§§ 134, 155), $9a^4 - 90a^3 + 189a^2$ (§§ 133, 142, 155), $a^3 + b^3$ (§ 136), $4x^4 + 3x^2y^2 + 9y^4$ (§ 154), $ax + 4a - 4x - 16$ (§ 145).

3. Solve $\begin{cases} 3x + 8 = 4y + 2, \\ \frac{4x}{3} + \frac{y}{9} = 3 \end{cases}$ (§§ 231, 232).

Give an axiom justifying each step in the solution (§ 224).

4. Find a number such that if it is added to 1, 4, 9, 16, respectively, the results will form a proportion (§ 403).

5. Solve $\sqrt{x+1} + \sqrt{x-2} = \sqrt{2x+3}$ (§§ 352, 353).

6. Find the square root of $\frac{a^2}{9} + \frac{2ab}{15} - \frac{2a}{3} + \frac{b^2}{25} - \frac{2b}{5} + 1$ (§ 280).

7. Simplify *three* of the following: $\sqrt[3]{-125x^6}$, $\sqrt[5]{\frac{-y^{10}}{32x^{15m}}}$, $\sqrt[3]{108r^4}$ (§ 301), $\sqrt[4]{\frac{4}{9}} \times \sqrt[4]{\frac{16}{27}}$ (§§ 314, 315), $\sqrt{75} - 4\sqrt{243} + 2\sqrt{108}$ (§§ 311–313).

8. Solve $\begin{cases} x^2 + 2xy = 55, \\ 2x^2 - xy = 35. \end{cases}$ (Elimination of similar terms, page 270.)

9. If the speed of a railway train should be lessened 4 miles an hour, the train would be half an hour longer in going 180 miles. Find the rate of the train (§§ 205, 215).

10. If the greater of two numbers is divided by the less, the quotient is 2 and the remainder is 3. The square of the greater number exceeds 6 times the square of the less by 25. Find the numbers (§ 374).

State of New York Regents' examination for June, 1910:

1. Simplify $\dfrac{1+\dfrac{a}{a+1}}{a+\dfrac{1}{1+a}} \div \dfrac{(a+1)^2-a^2}{a^3-1}$ (§ 200).

2. Find the prime factors of *four* of the following: a^4-16 (§§ 134, 149); $20x^2-60xy+45y^2$ (§§ 133, 138); $6ax+10ay-21bx-35by$ (§ 145); $y^2-y+\frac{1}{4}$ (§ 138); $1-\dfrac{x^3}{8}$ (§ 136).

3. Solve $\begin{cases} ax+by=m, \\ bx+ay=n \end{cases}$ (§ 233).

4. Reduce *each* of the following to its simplest form: $(\sqrt{3}+5\sqrt{2})(2\sqrt{3}+3\sqrt{2})$ [§ 315]; $\sqrt{108}\div\sqrt{432}$ (§ 317); $\sqrt{\frac{2}{3}}$ (§ 302); $\sqrt{18}+\sqrt{50}-\sqrt{72}$ (§ 313); $\sqrt[3]{2}\times\sqrt[3]{4}$ (§ 315).

5. It required as many days for a number of men to dig a trench as there were men; if there had been 6 men more the work would have been done in 8 days. Find the number of men (§§ 166, 354).

6. Solve $x^2+5ax=14a^2$ (§§ 350, 351).

7. Solve $\begin{cases} 2x^2+xy=24, \\ x^2-y^2=5 \end{cases}$ (§ 368).

8. The length of a picture at the inner edge of the frame is twice its width; the frame is 4 inches wide and has an area of 328 square inches. Find the dimensions of the picture (§§ 125 and 215 or § 234).

9. Solve $\sqrt{x+7}+\sqrt{x}=7$ (§§ 333, 334).

10. The area of a certain square is $\frac{4}{9}$ of the area of a certain other square and its side is $\frac{1}{4}$ of a yard less. Find the side of each square (§ 374).

11. Two quarts of alcohol are mixed with 5 quarts of water. Find the number of quarts of alcohol that must be added to make the mixture three fourths alcohol (§ 215).

12. Define exponent, surd, term, reciprocal, elimination (§§ 9, 298, 3, 196, 224, respectively, and the glossary).

406. The following questions were asked in the Minnesota High School Board examination for Elementary Algebra in June, 1910.

1. Simplify $a-[-\{-(-3a-\overline{2a-b})\}]$ (§ 116).

2. A's age exceeds B's by 20 years. Ten years ago A was twice as old as B. Find the age of each (§§ 125, 215; also § 234).

3. Factor:

(1) $6bx-15ab-4dx+10ad$ (§ 145).
(2) x^4-16y^4 (§§ 134, 149).
(3) $1-18x-63x^2$ (§ 144).
(4) $64a^6+8$ (§§ 133, 136).
(5) x^3-27 (§ 136).

4. Find the algebraic sum of $\frac{x-y}{xy}-\frac{x-a}{ax+bx}+\frac{y-b}{ay+by}$ (§§ 191, 192).

5. Find the product of $\frac{a^3+27}{a^2-16}\times\frac{a+4}{a^2-3a+9}$ (§§ 193, 194).

6. Find the quotient of $\frac{b-a}{b}\div(a-b)$ (§§ 172, 195–198).

7. Seven men and 5 boys earn \$11.25 per day and at the same wages 4 men and 12 boys earn \$11 per day. Find the wages of each per day (§§ 104, 234).

8. Simplify $\sqrt{16a}+\sqrt{81a}+\sqrt{144a^2b^2}$ (§ 313).

9. Multiply $4\sqrt{8}-\sqrt{32}+2\sqrt{50}$ by $\sqrt{2}$ (§ 315).

10. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour, the running time is decreased 2 hours. Find the speed of the train (§ 354).

11. Find two numbers in the proportion of 3 to 5 whose sum is 160 (§ 403).

12. Simplify $\frac{2m-3+\frac{1}{m}}{\frac{2m-1}{m}}$ (§ 200).

Minnesota High School Board examination for May, 1907:

1. Define any five: Algebra, term, root, power, binomial, reciprocal (glossary).

2. From the sum of $a^3 + 2a^2b + 5ab^2 + 3b^3$ and $2a^3 - 3a^2b + 3ab^2 - b^3$, take their difference (§§ 62, 63, 66).

3. Express in simplest form:

$$-(4x - \overline{7y+4}) - \{-2x - (3y - \overline{4y - x + 4})\} \text{ (§ 116).}$$

4. Divide $3x^{2n} + 13x^{2n-1} + 15x^{2n-2} + 9x^{2n-3}$ by $x^n + 3x^{n-1}$ (§ 108).

5. Factor any five: (*a*) $x^{2n} - 10x^n + 16$ (§§ 142, 159); (*b*) $x^{12} - y^{12}$ (§§ 134, 136, 149); (*c*) $12x^2 + 2xy - 2y^2$ (§§ 133, 144); (*d*) $x^3 + 15x^2 - x - 15$ (§§ 145, 134); (*e*) $a^2 - 16 + b^2 + 2ab$ (§ 151); (*f*) $x^4 + 4y^4$ (§§ 153, 154).

6. Expand by the binomial formula, showing all the steps:

$$(2a - x)^5 \text{ [§§ 265, 266].}$$

7. Solve for x:

$$\begin{cases} \dfrac{x+8}{2} = 2 - \dfrac{3y - x}{6}, \\ \dfrac{2x+y}{2} - \dfrac{9x-7}{8} = \dfrac{7y - 4x + 36}{16} \end{cases} \text{ (§§ 231, 232).}$$

8. One half of A's money is equal to B's, and five eighths of B's is equal to C's; together they have \$1450. How much has each? (§§ 47, 75, 125, 205, 215).

9. Simplify $\dfrac{\dfrac{1}{1-a} + \dfrac{1}{1+x}}{\dfrac{1}{1-a} - \dfrac{1}{1+x}}$ (§ 200).

10. A man bought a suit of clothes for \$24 and paid for it in two-dollar bills and fifty-cent pieces, giving twice as many coins as bills. How many bills did he give? (§§ 125, 215, 234).

11. Five years ago the sum of the ages of A and B was 40 years. B is now four times as old as A. What is the present age of each? (§§ 125, 215; also § 234).

FACTORS AND MULTIPLES

407. This chapter gives a brief treatment of highest common factor (§ 183) and lowest common multiple (§ 189) for the benefit of any who may desire a little more work in these topics than their application affords in fractions, the only place in elementary algebra where they are applied.

HIGHEST COMMON FACTOR

408. An expression that is a factor of each of two or more expressions is called a **common factor** of them.

409. The common factor of two or more expressions that has the largest numerical coefficient and is of the highest degree is called their **highest common factor**.

The common factors of $4a^3b^2$ and $6a^2b$ are 2, a, b, a^2, $2a$, $2b$, $2a^2$, ab, $2ab$, a^2b, and $2a^2b$ with sign + or −. Of these, $2a^2b$ (or $-2a^2b$) has the largest numerical coefficient and is of the highest degree, and is therefore the highest common factor.

The highest common factor may be positive or negative, but usually only the positive sign is taken.

The highest common factor of two or more expressions is equal to the product of all their common prime factors.

410. Expressions that have no common prime factor, except 1, are said to be **prime to each other.**

EXERCISES

411. 1. Find the h. c. f. of $12a^4b^2c$ and $32a^2b^3c^3$.

SOLUTION

The arithmetical greatest common divisor or highest common factor of 12 and 32 is 4. The highest common factor of a^4b^2c and of $a^2b^3c^3$ is a^2b^2c. Hence, h. c. f. $= 4a^2b^2c$.

RULE. — *To the greatest common divisor of the numerical coefficients annex each common literal factor with the least exponent it has in any of the expressions.*

Find the highest common factor of:

2. $10\,x^3y^2$, $10\,x^2y^3$, and $15\,xy^4z$.

3. $70\,a^6b^3$, $21\,a^4b^4$, and $35\,a^4b^5$.

4. $8\,m^7n^3$, $28\,m^6n^4$, and $56\,m^5n^2$.

5. $4\,b^3cd$, $6\,b^2c^2$, and $24\,abc^3$.

6. $3(a+b)^2$ and $6(a+b)^3$.

7. $6(a+b)^2$ and $4(a+b)(a-b)$.

8. $12(a-x)^3$, $6(a-x)^2$, and $(a-x)^4$.

9. $10(x-y)^4z^3$ and $15(z-y)(x-y)^3$.

10. What is the h. c. f. of $3\,x^3 - 3\,xy^2$ and $6\,x^3 - 12\,x^2y + 6\,xy^2$?

PROCESS

$$3\,x^3 - 3\,xy^2 = 3\,x(x+y)(x-y)$$
$$6\,x^3 - 12\,x^2y + 6\,xy^2 = 2 \cdot 3\,x(x-y)(x-y)$$
$$\therefore \text{h. c. f.} = 3\,x(x-y)$$

EXPLANATION. — For convenience in selecting the common factors, the expressions are resolved into their simplest factors.

Since the only common prime factors are 3, x, and $(x-y)$, the highest common factor sought (§ 409) is their product $3\,x(x-y)$.

Find the highest common factor of:

11. $a^2 - x^2$ and $a^2 - 2\,ax + x^2$.

12. $a^2 - b^2$ and $a^2 + 2\,ab + b^2$.

13. $x^3 + y^3$ and $x^2 + 2\,xy + y^2$.

14. $x^2 - 2\,x - 15$ and $x^2 - x - 20$.

15. $a^2 + 7\,a + 12$ and $a^2 + 5\,a + 6$.

Find the highest common factor of:

16. $x^4 + x^2y^2 + y^4$ and $x^2 + xy + y^2$.

17. $x^3 + y^3$, $x^5 + y^5$, and $x^2y + xy^2$.

18. $a^4 + a^2b^4 + b^8$ and $3a^2 - 3ab^2 + 3b^4$.

19. $ax - y + xy - a$ and $ax^2 + x^2y - a - y$.

20. $a^2b - b - a^2c + c$ and $ab - ac - b + c$.

21. $1 - 4x^2$, $1 + 2x$, and $4a - 16ax^2$.

22. $24x^3y^3 + 8x^2y^3$ and $8x^3y^3 - 8x^2y^3$.

23. $6x^2 + x - 2$ and $2x^2 - 11x + 5$.

24. $17abc^3d^5 - 51a^3bc^4d^4$ and $abc^2d^2 - 3a^3bc^3d$.

25. $38xyz - 95x^3yz^2$ and $34xy^2z - 85x^2yz^2$.

26. $6r^7 + 10r^6s - 4r^5s^2$ and $2r^7 + 2r^6s - 4r^5s^2$.

27. $x^4 - x^3 - 2x^2$, $x^4 - 2x^3 - 3x^2$, and $x^4 - 3x^3 - 4x^2$.

28. $7l^3t^3 + 35l^2t^3 + 42lt^3$ and $7l^4t^3 + 21l^3t^3 - 28l^2t^3 - 84lt^3$.

29. $x^2 + a^2 - b^2 + 2ax$, $x^2 - a^2 + b^2 + 2bx$, and $x^2 - a^2 - b^2 - 2ab$.

30. $x^2 - 6x + 5$ and $x^3 - 5x^2 + 7x - 3$.

SUGGESTION. — Apply the factor theorem to the second expression.

31. $x^3 - 4x + 3$ and $x^3 + x^2 - 37x + 35$.

32. $9 - n^2$ and $n^2 - n - 6$.

SUGGESTION. — Change $9 - n^2$ to $-(n^2 - 9) = -(n + 3)(n - 3)$.

33. $1 - x^2$ and $x^3 - 6x^2 - 9x + 14$.

34. $(9 - x^2)^2$ and $x^3 + 2x^2 - 9x - 18$.

SUGGESTION. $(9 - x^2)^2 = (x^2 - 9)^2$.

35. $(4 - c^2)^2$ and $c^3 + 9c^2 + 26c + 24$.

36. $xy - y^2$, $-(y^3 - x^2y)$, and $x^2y - xy^2$.

37. $16 - s^4$, $2s - s^2$, and $s^2 - 4s + 4$.

38. $y^4 - x^4$, $x^5 + y^5$, and $y^2 + 2yx + x^2$.

39. $(y - x)^2(n - m)^3$ and $(x^2y - y^3)(m^2n - 2mn^2 + n^3)$.

LOWEST COMMON MULTIPLE

412. An expression that exactly contains each of two or more given expressions is called a **common multiple** of them.

$6abx$ is a common multiple of a, $3b$, $2x$, and $6abx$. These numbers may have other common multiples, as $12abx$, $6a^2b^2x$, $18a^3bx^2$, etc.

413. The expression having the smallest numerical coefficient and of *lowest degree* that will exactly contain each of two or more given expressions is called their **lowest common multiple.**

$6abx$ is the lowest common multiple of a, $3b$, $2x$, and $6abx$.

The lowest common multiple may have either sign + or −, though usually only the positive sign is taken.

The lowest common multiple of two or more expressions is equal to the product of all their different prime factors, each factor being used the greatest number of times it occurs in any of the expressions.

EXERCISES

414. 1. What is the l. c. m. of $12x^2yz^4$, $6a^2xy^2$, and $8axyz^2$?

Solution

The lowest common multiple of the numerical coefficients is found as in arithmetic. It is 24.

The literal factors of the lowest common multiple are each letter with the highest exponent it has in any of the given expressions. They are, therefore, a^2, x^2, y^2, and z^4.

The product of the numerical and literal factors, $24a^2x^2y^2z^4$, is the lowest common multiple of the given expressions.

Find the lowest common multiple of:

2. a^3x^2y, a^2xy^3, and ax^2y.

3. $10a^2b^2c^2$, $5ab^2c$, and $25b^3c^3d^3$.

4. $16a^2b^3c$, $24c^3de$, and $36a^4b^2d^2e^3$.

5. $18a^2br^2$, $12p^2q^2r$, and $54ab^2p^3q$.

6. What is the l. c. m. of $x^2 - 2xy + y^2$, $y^2 - x^2$, and $x^3 + y^3$?

PROCESS

$$x^2 - 2xy + y^2 = (x - y)(x - y)$$
$$y^2 - x^2 = -(x^2 - y^2) = -(x + y)(x - y)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$\therefore \text{l. c. m.} = (x - y)^2(x + y)(x^2 - xy + y^2)$$
$$= (x - y)^2(x^3 + y^3)$$

RULE. — *Factor the expressions into their prime factors.*

Find the product of all their different prime factors, using each factor the greatest number of times it occurs in any of the given expressions.

The factors of the l. c. m. may often be selected without separating the expressions into their prime factors.

Find the lowest common multiple of:

7. $x^2 - y^2$ and $x^2 + 2xy + y^2$.

8. $x^2 - y^2$ and $x^2 - 2xy + y^2$.

9. $x^2 - y^2$, $x^2 + 2xy + y^2$, and $x^2 - 2xy + y^2$.

10. $a^2 - n^2$ and $3a^3 + 6a^2n + 3an^2$.

11. $x^4 - 1$ and $a^2x^2 + a^2 - b^2x^2 - b^2$.

12. $a^2 + 1$, $ab - b$, $a^2 + a$, and $1 - a^2$.

13. $2x + y$, $2xy - y^2$, and $4x^2 - y^2$.

14. $1 + x$, $x - x^2$, $1 + x^2$, and $x^2(1 - x)$.

15. $2x + 2$, $5x - 5$, $3x - 3$, and $x^2 - 1$.

16. $16b^2 - 1$, $12b^2 + 3b$, $20b - 5$, and $2b$.

17. $1 - 2x^2 + x^4$, $(1 - x)^2$, and $1 + 2x + x^2$.

18. $b^2 - 5b + 6$, $b^2 - 7b + 10$, and $b^2 - 10b + 16$.

Find the lowest common multiple of:

19. $x^2 + 7x - 8$, $x^2 - 1$, $x + x^2$, and $3ax^2 - 6ax + 3a$.

20. $x^2 - a^2$, $a - 2x$, $a^2 + 2ax$, and $a^3 - 3a^2x + 2ax^2$.

21. $m^3 - x^3$, $m^2 + mx$, $m^2 + mx + x^2$, and $(m + x)x^2$.

22. $2 - 3x + x^2$, $x^2 + 4x + 4$, $x^2 + 3x + 2$, and $1 - x^2$.

23. $x^2 - y^2$, $x^4 + x^2y^2 + y^4$, $x^3 + y^3$, and $x^2 + xy + y^2$.

24. $x^3 + x^2y + xy^2 + y^3$ and $x^3 - x^2y + xy^2 - y^3$.

25. $a^2 + 4a + 4$, $a^2 - 4$, $4 - a^2$, and $a^4 - 16$.

26. $a^2 - (b + c)^2$, $b^2 - (c + a)^2$, and $c^2 - (a + b)^2$.

27. $2(ax^2 - x^3)^2$, $3x(a^2x - x^3)^3$, and $6(a^2x^2 - a^4)$.

28. $(yz^2 - xyz)^2$, $y^2(xz^2 - x^3)$, and $x^2z^2 + 2xz^3 + z^4$.

Suggestion. — In solving the following, use the factor theorem.

29. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.

30. $x^3 - 5x^2 - 4x + 20$ and $x^3 + 2x^2 - 25x - 50$.

31. $x^3 - 4x^2 + 5x - 2$ and $x^3 - 8x^2 + 21x - 18$.

32. $x^3 + 5x^2 + 7x + 3$ and $x^3 - 7x^2 - 5x + 75$.

33. $x^3 + 2x^2 - 4x - 8$, $x^3 - x^2 - 8x + 12$, $x^3 + 4x^2 - 3x - 18$.

GLOSSARY

Abscissa. A distance measured along or parallel to the x-axis.

Absolute Term. A term that does not contain an unknown number.

Absolute Value. The value of a number without regard to its sign.

Addends. Numbers to be added.

Addition. The process of finding a simple expression for the algebraic sum of two or more numbers.

Affected Quadratic. A quadratic equation that contains both the second and first powers of one unknown number.

Algebra. That branch of mathematics which treats of general numbers and the nature and use of equations. It is an extension of arithmetic and it uses both figures and letters to express numbers.

Algebraic Expression. A number represented by algebraic symbols.

Algebraic Numbers. Positive and negative numbers, whether integers or fractions.

Algebraic Sum. The result of adding two or more algebraic numbers.

Antecedent. The first term of a ratio.

Arrangement. When a polynomial is arranged so that in passing from left to right the several powers of some letter are successively *higher* or *lower*, the polynomial is said to be **arranged** according to the **ascending** or **descending powers**, respectively, of that letter.

Axes of Reference. Two straight lines that intersect, usually at right angles, used to locate a point or points in a plane.

Axiom. A principle so simple as to be self-evident.

Binomial. An algebraic expression of two terms.

Binomial Formula. The formula or principle by means of which any indicated power of a binomial may be expanded.

Binomial Quadratic Surd. A binomial surd whose surd or surds are of the second order.

Binomial Surd. A binomial, one or both of whose terms are surds.

Clearing an Equation of Fractions. The process of changing an equation containing fractions to an equation without fractions.

Coefficient. When one of the two factors into which a number can be resolved is a *known* number, it is usually written first and called the coefficient of the other factor.

In a broader sense, either one of the two factors into which a number can be resolved may be considered the *coefficient* of the other.

Co-factor. Same as *Coefficient.*

Common Factor. A factor of each of two or more numbers.

Common Multiple. An expression that exactly contains each of two or more given expressions.

Complete Quadratic. Same as *Affected Quadratic.*

Complex Fraction. A fraction one or both of whose terms contains a fraction.

Compound Expression. Same as *Polynomial.*

Conditional Equation. An equation that is true for only certain values of its letters.

Conjugate Surds. Two binomial quadratic surds that differ only in the sign of one of the terms.

Consequent. The second term of a ratio.

Consistent Equations. Same as *Simultaneous Equations.*

Coördinates. See *Rectangular Coördinates.*

Couplet. The two terms of a ratio.

Cube. Same as *Third Power.*

Cube Root. One of the three equal factors of a number.

Cubic Surd. A surd of the third order.

Degree of an Expression. The term of highest degree in any rational integral expression determines the **degree of the expression.**

Degree of a Term. The sum of the exponents of the literal factors of a rational integral term determines the **degree of the term.**

Denominator. The *divisor* in an algebraic fraction.

Dependent Equations. Two or more equations that express the same relation between the unknown numbers involved are often called **dependent equations**, for each may be *derived* from any one of the others.

Derived Equations. Same as *Dependent Equations.*

Difference. The result of subtracting one number from another. That is, the **difference** is the algebraic number that added to the subtrahend gives the minuend.

Dissimilar Fractions. Fractions that have different denominators.

Dissimilar Terms. Terms that contain different letters or the same letters with different exponents.

Dividend. In division, the number that is divided.

Division. The process of finding one of two factors when their product and one of the factors is given.

Divisor. In division, the number by which the dividend is divided.

Elimination. The process of deriving from a system of simultaneous equations another system involving fewer unknown numbers.

Entire Surd. A surd that has no rational coefficient except unity.

Equation. A statement of the equality of two numbers or expressions.

Equation of the First Degree. Same as *Simple Equation.*

Equation of the Second Degree. Same as *Quadratic Equation.*

Equivalent Equations. Two equations that have the same roots, each equation having all the roots of the other.

Even Root. A root whose index is an even number.

Evolution. The process of finding any required root of a number.

Exponent. A small figure or letter placed at the right and a little above a number to indicate how many times the number is to be used as a factor.

Extremes. The first and fourth terms of a proportion.

Factor. Each of two or more numbers whose product is a given number.

Factoring. The process of separating a number into its factors.

Formula. An expression of a principle or a rule in symbols.

Fourth Proportional. The fourth number of four different numbers that form a proportion.

Fourth Root. One of the four equal factors of a number.

Fraction. In algebra, an indicated division.

Fractional Equation. An equation that involves an unknown number in any denominator.

Fractional Expression. An expression, any term of which is a fraction.

Fulcrum. The point or edge upon which a lever rests.

General Number. A literal number to which any value may be assigned.

Graph. A picture (line or lines) every point of which exhibits a pair of corresponding values of two related quantities.

Graph of an Equation. The line or lines containing all the points, and only those, whose coördinates satisfy a given equation.

Higher Equation. An equation that contains a higher power of the unknown number than the second.

Highest Common Factor. The common factor of two or more expressions that has the largest numerical coefficient and is of the highest degree.

It is equal to the product of all the common factors of the expressions.

Homogeneous Equation. An equation *all* of whose terms are of the same degree with respect to the unknown numbers.

Identical Equation. An equation whose members are identical, or such that they may be reduced to the same form.

Identity. Same as *Identical Equation.*

Imaginary Number. A number that involves an indicated even root of a negative number.

Incomplete Quadratic. Same as *Pure Quadratic.*

Inconsistent Equations. Two or more equations that are not satisfied in common by any set of values of the unknown numbers.

Independent Equations. Two or more equations that express different relations between the unknown numbers involved, and so cannot be reduced to the same equation.

Indeterminate Equation. An equation that is satisfied by an unlimited number of sets of values of its unknown numbers.

Index of a Power. Same as *Exponent.*

Index of a Root. A small figure or letter written in the opening of a radical sign to indicate what root of a number is sought.

Integer. Same as *Whole Number.*

Integral Equation. An equation that does not involve an unknown number in any denominator.

Integral Expression. An expression that contains no fraction.

Inverse Ratio. Same as *Reciprocal Ratio.*

Involution. The process of finding any required power of an expression.

Irrational Equation. An equation involving an irrational root of an unknown number.

Irrational Expression. An expression that contains an irrational number.

Irrational Number. A number that cannot be expressed as an integer or as a fraction with integral terms.

Known Number. A general number or a number whose value is known.

Lever. Any sort of a bar resting on a fixed point or edge.

Like Terms. Same as *Similar Terms.*

Linear Equation. Same as *Simple Equation.*

Literal Coefficient. A coefficient composed of letters.

Literal Equation. An equation one or more of whose known numbers is expressed by letters.

Literal Numbers. Letters that are used for numbers.

Lowest Common Denominator. The denominator of lowest degree, having the least numerical coefficient, to which two or more fractions can be reduced.

It is equal to the lowest common multiple of the given denominators.

Lowest Common Multiple. The expression having the smallest numerical coefficient and of lowest degree that will exactly contain each of two or more given expressions.

Lowest Terms. When the terms of a fraction have no common factor, the fraction is said to be in its lowest terms.

Mean Proportional. A number that serves as both means of a proportion.

Means. The second and third terms of a proportion.

Members of an Equation. In an equation, the number on the left of the sign of equality is called the **first member** of the equation, and the number on the right is called the **second member.**

Minimum Point of a graph. The point of a graph that has the algebraically least ordinate.

Minuend. In subtraction, the number from which the subtraction is made.

Mixed Coefficient. A coefficient composed of both figures and letters.

Mixed Expression. An expression some of whose terms are integral and some fractional.

Mixed Number. Same as *Mixed Expression.*

Mixed Surd. A surd that has a rational coefficient.

Monomial. An algebraic expression of one term only.

Multiplicand. In multiplication, the number multiplied.

Multiplication. When the multiplier is a positive integer, the process of taking the multiplicand as many times as there are units in the multiplier.

In general, the process of finding a number that is obtained from the multiplicand just as the multiplier is obtained from unity.

Multiplier. In multiplication, the number by which the multiplicand is multiplied.

Negative Number. A number less than zero.

Negative Term. A term preceded by $-$.

Numerator. The *dividend* in an algebraic fraction.

Numerical Coefficient. A coefficient composed of figures.

Numerical Equation. An equation all of whose known numbers are expressed by figures.

Odd Root. A root whose index is odd.

Order of a radical or of a surd is indicated by the index of the root or by the denominator of the fractional exponent.

Ordinate. A distance measured along or parallel to the y-axis.

Origin. The intersection of the axes of reference.

Perfect Square. An expression that may be separated into two equal factors.

Polynomial. An algebraic expression of more than one term.

Positive Number. A number greater than zero.

Positive Term. A term preceded by +, expressed or understood.

Power of a Number. The product obtained when the number is used a certain number of times as a factor.

Prime Number. A number that has no factors except itself and 1.

Prime to Each Other. Expressions that have no common prime factor except 1 are said to be **prime to each other.**

Principal Root. A real root of a number that has the same sign as the number itself.

Product. The result of multiplying one number by another.

Proportion. An equality of ratios.

Pure Quadratic. An equation that contains only the second power of the unknown number.

Quadratic Equation. An equation that, when simplified, contains the *square* of the unknown number, but no higher power.

Quadratic Form. An expression that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other.

Quadratic Surd. A surd of the second order.

Quotient. The result of dividing one number by another.

Radical. An indicated root of a number.

Radical Equation. Same as *Irrational Equation.*

Radical Sign. Same as *Root Sign.*

Radicand. A number whose root is required.

Ratio. The relation of two numbers that is expressed by the quotient of the first divided by the second.

Rational Expression. An expression that contains no irrational number.

Rationalization. The process of multiplying an expression containing a surd by any number that will make the product rational.

Rationalizing Factor. The factor by which a surd expression is multiplied to render the product rational.

Rationalizing the Denominator. The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator.

Rational Number. A number that is, or may be, expressed as an integer or as a fraction with integral terms.

Real Number. A number that does not involve the even root of a negative number.

Reciprocal of a number is 1 divided by the number.

Reciprocal of a Fraction is the fraction *inverted* or 1 divided by the fraction.

Reciprocal Ratio. The ratio of the reciprocals of two numbers is called the **reciprocal ratio** of the numbers.

Rectangular Coördinates. The abscissa and ordinate of a point referred to two perpendicular axes are called the **rectangular coördinates** of the point.

Reduction. The process of changing the form of an expression without changing its value.

Remainder in subtraction. Same as *Difference.*

Root of an Equation. Any number that satisfies the equation.

Root of a Number. When the factors of a number are all equal, one of the factors is called a **root** of the number.

Root Sign. The symbol $\sqrt{\ }$ written before a number denotes that a root of the number is sought.

Satisfied. When an equation is reduced to an identity by the substitution of certain known numbers for the unknown numbers, the equation is said to be **satisfied.**

Second Power. When a number is used *twice* as a factor, the product is called the **second power** of the number.

Second Root. Same as *Square Root.*

Sign of Addition is $+$, read '*plus.*'

Sign of a Fraction. The sign written before the dividing line of a fraction.

Sign of Continuation is $\cdots$, read '*and so on*' or '*and so on to.*'

Sign of Deduction is $\therefore$, read '*therefore*' or '*hence.*'

Sign of Division is $\div$, read '*divided by.*'
Division is also indicated by a fraction, the numerator being the dividend and the denominator the divisor.

Sign of Equality is $=$, read '*is equal to*' or '*equals.*'

Sign of Multiplication is $\times$ or the dot $(\cdot)$, read '*multiplied by.*'
Multiplication is also indicated by the absence of sign.

Sign of Ratio is a colon (:), read '*is to.*'

Sign of Subtraction is $-$, read '*minus.*'

Signs of Aggregation. Signs used to group numbers that are to be regarded as a single number.
They are *parentheses*, ( ) ; *brackets*, [ ] ; *braces*, { } ; the *vinculum*, $\overline{\quad}$; and the *vertical bar*, |.

Signs of Direction. Same as *Signs of Quality.*

Signs of Opposition. Same as *Signs of Quality.*

Signs of Quality. The signs $+$ and $-$ when used to denote positive and negative numbers.

Similar Fractions. Fractions that have the same denominator.

Similar Radicals. Radicals that in their simplest form are of the same order and have the same radicand.

Similar Terms. Terms that contain the same letters with the same exponents.

Simple Equation. An integral equation that involves only the first power of one unknown number in any term when similar terms have been united.

Simple Expression. Same as *Monomial.*

Simplest Form of a Radical. A radical is in its simplest form when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power whose exponent corresponds with the index of the root.

Simultaneous Equations. Two or more equations that are satisfied by the same set or sets of values of the unknown numbers form a **system of simultaneous equations.**

Solving an Equation. Finding the roots of an equation.

Square. Same as *Second Power.*

Square Root. One of the two equal factors of a number.

Substitution. When a particular number takes the place of a letter, or general number, the process is called **substitution**.

Subtraction. The process of finding one of two numbers when their sum and the other number are given.

Subtraction is the *inverse* of addition.

Subtrahend. In subtraction, the number that is subtracted.

Sum. See *Algebraic Sum.*

Surd. The indicated root of a *rational* number that cannot be obtained exactly.

Symmetrical Equation. An equation that is not affected by interchanging the unknown numbers involved.

Term. An algebraic expression whose parts are not separated by the signs + or −.

Terms of a Fraction. The numerator and denominator of a fraction.

Third Power. When a number is used *three* times as a factor, the product is called the **third power** of the number.

Third Proportional. The consequent of the second ratio when the means of a proportion are identical.

Third Root. Same as *Cube Root.*

Transposition. The process of removing a term from one member of an equation to the other.

Trinomial. An algebraic expression of three terms.

Trinomial Square. A trinomial that is a perfect square.

Unknown Number. A number whose value is to be found.

Unlike Terms. Same as *Dissimilar Terms.*

Whole Number. A unit or an aggregate of units.

X-axis. The horizontal *axis of reference* is usually called the **x-axis.**

Y-axis. The vertical *axis of reference* is usually called the **y-axis.**

The End.