C 40188 **(Pages : 3)** **Name..**

Reg. No.....................................

SXTH SEMESTER (CUCBCSS–UG) DEGREE EXAMINATION, MARCH 2023

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

(2017–2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the twelve questions.
Each question 1 mark.

1. Define greatest common divisor of a and b.
2. State Euclid's lemma.
3. Examine whether the Diophantine equation $6x + 51y = 22$ has an integer solution.
4. If p is a prime and $p \mid ab$, show that $p \mid a$ or $p \mid b$.
5. Define Euclidean number. List the first five Euclidean numbers.
6. State Wilson's Theorem.
7. Find $\phi(360)$.
8. Define Vector Space.
9. Give an example to show that union of two subspaces need not be subspace.
10. Check whether the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (2x, y - 2, 4y)$ is linear. Justify your claim.
11. State the Dimension theorem.
12. When we say that a linear map is an isomorphism. Give an example for an isomorphism ?

(12 × 1 =12 marks)

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 marks.

13. Prove that $3a^2 - 1$ is never a perfect square.
14. If $\gcd(a, b) = d$, prove that $\gcd\left(\frac{a}{d} \frac{b}{d}\right) = 1$.

Turn over

15. Use the Euclidean Algorithm to find $\gcd(12378, 3054)$.

16. Prove that every integer $n > 1$ can be expressed as a product of primes.

17. Prove that the number $\sqrt{3}$ is irrational.

18. Using Sieve of Eratosthenes find all primes not exceeding 100.

19. Show that 41 divides $2^{20} - 1$.

20. Let $N = a_m 10^m + a_{m-1} 10^{m-1} + \ldots + a_1 10 + a_0$ be the decimal expansion of the positive integer $N, 0 \le a_k < 10$, and let $S = a_0 + a_1 + \ldots + a_m$. Prove that $9 \mid N$ if and only if $9 \mid S$.

21. Find the remainder when 15 ! is divided by 17.

22. Prove that every line through the origin is a subspace of $\mathbb{R}^2$.

23. If the vector space V has a finite basis B then show that every basis of V is finite and has the same number of elements as B.

24. Let $f : V \to W$ be linear. Prove that if X is subspace of V then $f^{\to}(X)$ is a subspace of W.

25. Find *Im f* and *Ker f* when $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $f(a,b,c) = (a+b, b+c, a+c)$.

26. Let $f : V \to W$ be a linear map. Prove that f is injective if and only if $ker\, f = \{0\}$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 *marks.*

27. If a and b are given integers, not both zero, prove that the set $T = \{ax + by : x, y \text{ are integers}\}$ is precisely the set of all multiples of $d = \gcd(a,b)$.

28. Prove that $\gcd(a,b)\, lcm(a,b) = ab$, where a and b are positive integers.

29. Find the complete solution of the linear Diophantine equation $172x + 20y = 1000$. Also find solutions in positive integers if they exist.

30. Using Chinese Remainder Theorem, solve the system of congruences $x \equiv 1 (\text{mod } 3)$, $x \equiv 2 (\text{mod } 5)$, $x \equiv 3 (\text{mod } 7)$.

31. Let p be a prime and suppose that $p \mid a$. Prove that $a^{p-1} \equiv 1(\text{mod } p)$.

32. (a) Prove that $\langle S \rangle = \text{span } S$. (4 marks)

(b) Show that $\{(1,1,1), (1,2,3), (2,-1,1), (2,-1,1)\}$ is a basis of $\mathbb{R}^3$. (3 marks)

33. If the vector space V has a finite basis B then show that every basis of V is finite and has the same number of elements as B.

34. Let V and W be vector spaces over a field F. If $\{v_1, v_2, \ldots, v_n\}$ is a basis of V and $w_1, w_2, \ldots, w_n$ are elements of W then show that there is a unique linear mapping $f : V \to W$ such that $f(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

35. If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that $f(1,1,1) = (1,1,1)$, $f(1,2,3) = (-1,-2,-3)$, $f(1,1,2) = (2,2,4)$, then find $f(x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. (a) Prove that the quadratic congruence $x^2 + 1 \equiv 0(\text{mod } p)$, where p is an odd prime, has a solution if and only if $p \equiv 1(\text{mod} 4)$.

(10 marks)

(b) Prove that If $n > 1$, then the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n\phi(n)$.

(3 marks)

37. (a) State and Prove Division Algorithm. (10 marks)

(b) Prove that square of any integer is either $3k$ or $3k + 1$. (3 marks)

38. Show that the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x,y,z) = (x+z, x+y+2z, 2x+y+3z)$ is neither surjective nor injective.

(3 × 13 = 39 marks)

C 20210 **(Pages : 3)** **Name..................................**

Reg. No..............................

SIXTH SEMESTER (CUCBCSS–UG) DEGREE EXAMINATION, MARCH 2022

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

(2014 to 2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *questions.*
Each question carries 1 *mark*.

1. State general version of the Division Algorithm.
2. Find lcm (306, 657).
3. State the Fundamental Theorem of Arithmetic.
4. Explain the method of Sieve of Eratosthenes.
5. Define Pseudoprime. Give an example of a Pseudoprime number.
6. Find $\sigma(12)$.
7. Define Euler's Phi Function.
8. Prove that the set $X = \{(x, 0) : x \in \mathbb{R}\}$ is a subspace of the vector space $\mathbb{R}^2$.
9. Define basis of a vector space.
10. Check whether the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (x + y, z, 0)$ is linear. Justify your claim.
11. Define the rank and nullity of the linear map.
12. For $(x, y) \in \mathbb{R}^2$, let $S = \{(x, y)\}$. Find $\langle S \rangle$.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *questions.*
Each question carries 4 *marks*.

13. Show that square of any odd integer is of the form $8k + 1$.
14. If $a \mid c$ and $b \mid c$, with $gcd(a, b) = 1$, prove that $ab \mid c$.
15. Use the Euclidean Algorithm to obtain integers x and y satisfying $gcd(24, 138) = 24x + 138y$.
16. Determine all solutions in the positive integers of the Diophantine equation $18x + 5y = 48$.

Turn over

17. Prove that the number $\sqrt{2}$ is irrational.

18. For arbitrary integers a and b, prove that $a \equiv b(\text{mod}\, n)$ if and only if a and b leave the same non-negative remainder when divided by n.

19. Use the binary exponential algorithm to compute 5^{110} (mod 131).

20. Solve the linear congruence $18x \equiv (\text{mod}\, 42)$.

21. If p and q are distinct primes with $a^p \equiv a(\text{mod}\, q)$ and $a^q \equiv a(\text{mod}\, p)$, then prove that $a^{pq} \equiv a(\text{mod}\, pq)$.

22. Prove that intersection of two subspaces of vector space V over a field is again a subspace.

23. Let A = Span {(1, 2, 0, 1), (– 1, 1, 1, 1)} and B = Span {(0, 0, 1, 1), (2, 2, 2, 2)} be two subspaces of $\mathbb{R}^4$. Determine $A \cap B$ and compute its dimension.

24. If V is a vector space over the set $\mathbb{C}$ of complex numbers of dimension n, prove that V can be regarded as a vector space over $\mathbb{R}$ of dimension $2n$.

25. Let f : V → W be linear. Prove that if Y is subspace of W then $f^{\leftarrow}(Y)$ is a subspace of V.

26. Let V be a vector space of dimension $n \geq 1$ over a field F. Prove that V is isomorphic to the vector space F^n.

(10 × 4 = 40 marks)

Section C

Answer any **six** *questions.*
Each question carries 7 *marks.*

27. Given integers a and b, not both of which are zero, prove that there exists integers x and y such that $gcd\,(a, b) = ax + by$.

28. Prove that the linear Diophantine equation $ax + by = c$ has an integer solution if and only if $d \mid c$ where $d = \gcd(a, b)$.

29. Prove that there are infinite number of primes.

30. Using Chinese Remainder Theorem, solve the system of congruences $x \equiv 5(\text{mod}\, 11), x \equiv 14(\text{mod}\, 29), x \equiv 15(\text{mod}\, 31)$.

31. If p is a prime, then prove that $(p-1)! \equiv -1(\text{mod}\, p)$.

32. Prove that a non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

33. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be the mappings given by $f(x) = \cos^2 x, g(x) = \sin^2 x, h(x) = \cos 2x$. Consider the subspace of $\text{Diff}\,(\mathbb{R}, \mathbb{R})$ given by W = Span {f, g, h}. Find a basis for W.

34. Show that the linear mapping $f:\mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x,y,z)=(x+y+z,\, 2x-y-z, x+2y-z)$ is both surjective and injective.

35. State and prove Dimension Theorem.

(6 × 7 = 42 marks)

Section D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. (a) If $a=qb+r$, prove that $\gcd(a,b)=\gcd(b,r)$. (4 marks)

(b) If a cock is worth 5 coins, a hen 3 coins, and three chicks together 1 coin, how many cocks, hens, and chicks, totalling 100, can be bought for 100 coins.

(9 marks)

37. Prove that the lienar congruence $ax \equiv b \pmod n$ has a solution if and only if $d \mid b$, where $d=\gcd(a,n)$. Moreover if $d \mid b$, prove that it has d mutually incongruent solutions modulo n.

38. Prove that every linearly independent subset I of a finite dimensional vector space V can be extended to form a basis.

[2 × 13 = 26 marks]

C 1249 (Pages : 3) Name...

Reg. No...

SIXTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION, MARCH 2021

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. State Division algorithm.
2. Find $gcd(272,1479)$.
3. Show that the Diophantine equation $14x + 35y = 651$ has an integer solution.
4. Define e-prime. Give an example of a prime number.
5. Find the last two digits of the number 9^{9^9}.
6. State Fermat's theorem.
7. Find $\tau(180)$.
8. Define subspace of a vector space.
9. Give a basis for $Mat_{2\times 2}(\mathbb{R})$.
10. What do you mean by a Linear Transformation ?
11. Give an injective linear map which is not surjective.
12. State the Dimension theorem.

(12 × 1 = 12 marks)

Section B

Answer at least **eight** *questions.*
Each question carries 6 *marks.*
All questions can be attended.
Overall Ceiling 48.

13. Show that the expression $\dfrac{a\left(a^2+2\right)}{3}$ is an integer for all $a \geq 1$.
14. If $a|bc$, with $gcd(a, b) = 1$, prove that $a|c$.

Turn over

15. Use the Euclidean Algorithm to obtain integers x and y satisfying $gcd(119, 272) = 119x + 272y$.

16. Prove that the number $\sqrt{5}$ is irrational.

17. Find the canonical form of 2093.

18. If p_n is the n^{th} prime number, prove that $p_n \leq 2^{2n-1}$.

19. Find the remainder obtained upon dividing the sum $1! + 2! + 3! + \ldots + 99! + 100!$ by 12.

20. Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a polynomial function of x with integral coefficients c_k. If $a \equiv b (\text{mod}\, n)$, prove that $P(a) \equiv P(b)(\text{mod}\, n)$.

21. If $(n-1)! \equiv -1(\text{mod } n)$, then prove that n must be prime.

22. Prove that every plane through the origin is a subspace of $\mathbb{R}^3$.

23. Show that $\{(1, 1, 0, 0), (-1, -1, 1, 2), (1, -1, 1, 3), (0, 1, -1, -3)\}$ is a basis of $\mathbb{R}^4$.

24. Let S be a subset of the vector space V. Prove that S is basis if and only if S is a minimal spanning set.

25. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map given by $f(a, b, c, d) = (a + b, b - c, a + d)$. Find $Im\, f$ and a basis for $Im\, f$.

26. Prove that if the linear mapping $f : V \longrightarrow W$ is injective and $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent subset of V then $\{f(v_1), f(v_2), \ldots, f(v_n)\}$ is a linearly independent subset of W.

(8 × 6 = 48 marks)

Section C

Answer at least **five** *questions.*

Each question carries 9 *marks.*

All questions can be attended.

Overall Ceiling 45.

27. Let a and b be integers, not both zero. Prove that a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.

28. Prove that if $k > 0$, then $gcd(ka, kb) = k\,gcd(a, b)$.

29. State and Prove Fundamental Theorem of Arithmetic.

30. Prove that the system of linear congruences $ax + by \equiv r \pmod n$; $cx + dy \equiv s \pmod n$ has a unique solution modulo n whenever $gcd(ad - bc, n) = 1$.

31. If n is an odd pseudoprime, prove that $M_n = 2^n - 1$ is a larger one.

32. Let V be a vector space that is spanned by the finite set G = $\{v_1, v_2, v_3, \dots, v_n\}$. If I = $\{w_1, w_2, \dots, w_m\}$ is linearly independent subset of V then show that $m \leq n$.

33. Let V be finite dimensional vector space. If G is a finite spanning set of V and if I is a linearly independent subset of V such that $I \subseteq G$ then prove that there is a basis B of V such that $I \subseteq B \subseteq G$.

34. Prove that a linear mapping is completely and uniquely determined by its action on basis.

35. If $f: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map such that $f(1, 1, 0) = (1, 2), f(1, 0, 1) = (0, 0), f(0, 1, 1) = (2, 1)$, then find $f(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$.

(5 × 9 = 45 marks)

Section D

Answer any **one** *question.*
The question carries 15 *marks.*

36. a) Let a and b be integers, not both zero. For positive integer d, prove that $d = gcd(a, b)$ if and only if

(i) $d|a$ and $d|b$

(ii) Whenever $c|a$ and $c|b$, then $c|d$.

b) A customer bought a dozen pieces of fruit, apples and oranges, for $1.32. If an apple costs 3 cents more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought.

37. State and prove Chinese Remainder Theorem.

38. Let V and W be vector spaces each of dimension n over a field F. If $f : V \longrightarrow W$ is linear then prove that the following statements are equivalent :

(i) f is injective ;

(ii) f is surjective ;

(iii) f is bijective ;

(iv) f carries bases to bases.

(1 × 15 = 15 marks)

C 60051 (Pages : 4) Name.................................

Reg. No.................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2019

(CUCBCSS)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. State the division algorithm.
2. State the Fundamental Theorem of Arithmetic.
3. Give an example to show that $a^2 \equiv b^2 \pmod{n}$ does not imply $a \equiv b \pmod{n}$.
4. Define multiplicative functions.
5. If x is a real number, what are the possible values of $[x]+[-x]$?
6. Define Euler phi function.
7. Find the highest power of 5 dividing 1000 !.
8. Define subspace of a vector space V.
9. Find Span S where $S = \{(1,0,0)\} \subseteq \mathbb{R}^3$.
10. Define a linear transformation.
11. Give two different bases for $\mathbb{R}^2$.
12. Define null space of a linear transformation.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 *marks.*

13. Prove that the square of any odd integer is of the form $8k+1$ where k is an integer.
14. Prove that if $\gcd(a,b) = d$, then $\gcd\left(a/d, b/d\right) = 1$ where a, b are integers.

Turn over

15. Let $\gcd(a, b) = 1$. Prove that $\gcd(a+b, a-b) = 1$ or 2.

16. Determine all solutions of the Diophantine equation $56x + 72y = 40$.

17. Prove that if $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$ and $ac \equiv bc \pmod{n}$.

18. Solve the congruence $18x \equiv 30 \pmod{42}$.

19. Prove that if p is a prime, then $a^p \equiv a \pmod{p}$ for any integer n.

20. Prove that τ is a multiplicative function.

21. Prove that the intersection of two subspaces of a vector space V is again a subspace of V.

22. Check whether the vectors $(1, 1, 0)$ and $(2, 5, 3)$ and $(0, 1, 1)$ in $\mathbb{R}^3$ are linearly independent.

23. Let W be a subspace of a vector space V. Prove that dim W = dim V if and only if V = W.

24. Prove that $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(a, b) = (a+b, a-b, b)$ is a linear transformation.

25. Let V and W be vector spaces. Prove that if the linear mapping $f : V \to W$ is injective and $\{v_1, v_2, ..., v_n\}$ is a linearly independent subset of V, then $\{f(v_1), f(v_2), ..., f(v_n)\}$ is a linearly independent subset of W.

26. Let V and W be vector spaces. Prove that the linear mapping $f : V \to W$ is injective if and only if Ker $f = \{0\}$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Prove that $\sqrt{2}$ is irrational.

28. Prove that the sequence of primes is infinite.

29. Prove that the linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d \backslash b$ where $d = \gcd(a, n)$. If $d \backslash b$ prove that the congruence has d mutually incongruent solutions modulo n.

30. Use Chinese Remainder Theorem to find the smallest non-negative solution of the given system of congruences :

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7.$$

31. If n and r are positive integers with $1 \leq r < n$, then the binomial co-efficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is an integer.

32. Let V be a vector space over a field F. Prove the following :

(a) $\lambda 0_V = 0_V \ \forall \lambda \in F$.

(b) $0_F \, x = 0_V \ \forall_x \in V$.

(c) If $\lambda x = 0_V$, then either $\lambda = 0_F$ or $x = 0_V$.

33. Let S and T be two non-empty finite subsets of a vector space V such that $S \subseteq T$. Prove the following :

(a) If T is linearly independent, then so is S. (b) If S is linearly dependent, then so is T.

34. Let V be a finite dimensional vector space. If G is a finite spanning set of V and if I is a linearly independent subset of V such that $I \subseteq G$, prove that there is a basis B of V such that $I \subseteq B \subseteq G$.

35. Let V and W be vector spaces of finite dimension over a field F. If $f : V \rightarrow W$ be linear, prove that $\dim V = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. Let $N = a_m 10^m + a_{m-1} 10^{m-1} + \ldots + a_1 10 + a_0$ be the decimal expansion of the positive integer N; $0 \leq a_k < 10$ and let $S = a_0 + a_1 + \ldots + a_m$,

$T = a_0 - a_1 + a_2 - ... + (-1)^m a_m$. Prove the following :

(a) $9 \mid N$ if and only if $9 \mid S$.

(b) $11 \mid N$ if and only if $11 \mid T$.

(c) Use the results in (a) and (b) to show that 1571724 is divisible by both 9 and 11.

37. (a) If n is a positive integer and $\gcd(a, n) = 1$, prove that $a^{\phi(n)} \equiv 1 \pmod{n}$. Deduce that if p is a prime and p / a, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) For $n > 1$, prove that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n \phi(n)$.

38. Prove the following :

(a) Let V be a vector space of dimension $n \geq 1$ over a field F. Then V is isomorphic to the vector space F^n.

(b) If V and W are vector spaces of the same dimension n over a field F, then V and W are isomorphic.

(c) A linear mapping is completely and uniquely determined by its action on a basis.

(2 × 13 = 26 marks)

D 40044 (Pages : 3) Name.....................................

Reg. No.................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2018

(CUCBCSS—UG)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Define gcd of two integers.
2. Find lcm (– 15, 20).
3. Define a Diophantine equation in two variables.
4. Write the canonical form of 180.
5. State Wilson's theorem.
6. Define a pseudoprime.
7. Find $\phi(9)$.
8. Define subspace of a vector space.
9. Give a spanning subset of the vector space of all polynomial functions over $\mathbb{R}$.
10. Show that any set of vectors which contains the zero vector is linearly dependent.
11. Define a linear map.
12. Define kernel of a linear map.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of* **fourteen** *questions.*
Each question carries 4 *marks.*

13. Show that $\dfrac{a\left(a^2+2\right)}{3}$ is a positive integer for any positive integer a.
14. Prove that two non-zero integers a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.
15. Find gcd (1769, 2378).

Turn over

16. Is $\sqrt{2}$ a rational number ? Justify your answer.

17. Find the remainder when 41^{65} is divided by 7.

18. Define an absolute pseudoprime. Illustrate with an example.

19. If p and q are primes, show that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

20. If n is a squarefree positive integer, prove that the positive divisors of n is 2^r, where r is the number of positive divisors of n.

21. Show that for a positive integer r, the product of any r consecutive positive integers is divisible by $r!$.

22. Define a vector space.

23. Prove that a non-empty subset W of a vector space V over a field F is a subspace of V if and only if $c\alpha + \beta \in W$ for all $\alpha, \beta \in W$ and for all $c \in F$.

24. Show that the set $\{e_1, e_2, e_3, e_4\}$, where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$, is a basis of $\mathbb{R}^4$.

25. Show that the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(a, b) = (a + b, a - b)$ is linear.

26. If V is a vector space of dimension $n \geq 1$ over a field F, show that V is isomorphic to the vector space F^n.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of* **nine** *questions.*
Each question carries 7 marks.

27. Given integers a and b, not both of which are zero, prove that there exist integers x and y such that $\gcd(a, b) = ax + by$.

28. Prove that the linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$ where $d = \gcd(a, b)$. Verify whether the Diophantine equation $14x + 35y = 93$ can be solved.

29. Solve the system of congruences $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$.

30. State and prove Fermat's Little Theorem.

31. Obtain the number and sum of positive divisors of a positive integer n.

32. Are the intersection and union of two subspaces of a vector space V again subspaces of V? Justify your answer.

33. A non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

34. Show that the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$ is not surjective.

35. Let V and W be vector spaces over a field F. If the set $\{v_1, v_2, v_3, \ldots, v_n\}$ is a basis of V and if $w_1, w_2, w_3, \ldots, w_n$ are elements of W, prove that there is a unique linear mapping $f : V \to W$ such that $f(v_i) = w_i$ $(i = 1, 2, 3, \ldots, n)$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of* **three** *questions.*
Each question carries 13 *marks.*

36. Prove that if a and b are integers with $b \neq 0$, then there exist unique integers q and r such that $a = qh + r,\ 0 \leq r \leq |b|$.

37. Show that Euler's phi-function is multiplicative.

38. If V and W are vector spaces over a field F and if $f : V \to W$ is a linear map, prove that :

    (a) $f(v_1 - v_2) = f(v_1) - f(v_2)$

    (b) the set Ker $f = \{v \in V : f(v) = 0\}$ is a subspace of V.

    (c) for any subspace X of V, $f(X)$ is a subspace of W.

    (d) f is an isomorphism if and only if Ker $f = \{0\}$.

(2 × 13 = 26 marks)

C 21565 (Pages : 2) Name..............................

Reg. No..............................

SIXTH SEMESTER B.Sc. DEGREE (SUPPLEMENTARY/IMPROVEMENT) EXAMINATION, MARCH 2017

(UG–CCSS)

Mathematics

MM 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 30 Weightage

I. Answer *all* twelve questions :

1 Define a pseudoprime to the base a.

2 $\tau(12) =$ ————.

3 $\left[-\frac{3}{2}\right] =$ ————.

4 State Euclid's lemma.

5 Write the set of least non-negative residues modulo n.

6 State Chinese Remainder Theorem.

7 Without performing the divisions determine whether the integer 1, 571, 724 is divisible by 11.

8 The rank of every n-rowed non-singular matrix A is ————.

9 Rank of matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is ————.

10 The characteristic roots of a skew-Hermitian matrix are either ———— or ————.

11 If two matrices A and B have the same size and the same rank, they are ————.

12 The system AX = O in n unknowns has a non-trivial solution if ————.

(12 × ¼ = 3 weightage)

II. Answer *all nine* questions :

13 If $g.c.d.\ (a, b) = d$ then prove that $g.c.d.\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

14 Check whether the Diophantine equation $6x + 51y = 22$ has a solution.

15 Establish the relation $2^{117} = 44 \bmod 117$.

16 Calculate ϕ (360).

17 If p is a prime then prove that $a^p \equiv a \pmod{p}$ for any integer a.

Turn over

18 If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that $\tau(n) = (k_1 + 1)(k_2 + 1) \dots (k_r + 1)$.

19 Prove that rank of a non-singular matrix is equal to the rank of its reciprocal matrix.

20 Show that if $\lambda_1, \lambda_2, \dots \lambda_n$ are n eigen values of a square matrix A of order n then the eigen values of the matrix A^2 be $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$.

21 State Cayley-Hamilton theorem.

(9 × 1 = 9 weightage)

III. Answer any *five* questions from seven :

22 For positive integers a and b, prove that $\gcd(a, b)\ l.c.m.(a, b) = ab$.

23 Find the g.c.d. of 12378 and 3054 and express it as the linear combination of 12378 and 3054.

24 Prove that there is an infinite number of primes.

25 If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_r}\right)$.

26 Find the number of zeros with which the decimal representation of 50! terminates.

27 Compute the inverse of $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 1 & 0 \end{bmatrix}$.

28 Show that equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve the same.

(5 × 2 = 10 weightage)

IV. Answer any *two* questions from three :

29 State and prove Chinese Remainder theorem.

30 Prove that the function ϕ is a multiplicative function.

31 Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. Hence obtain the value of A^{-1}.

(2 × 4 = 8 weightage)

C 1745 (Pages : 3) Name..............................

Reg. No..............................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2016

(UG—CCSS)

Core Course—Mathematics

MM 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 30 Weightage

Section A

Answer **all** *questions.*

1. Find gcd (306, 657).
2. Find the canonical form of 360.
3. Express 105 in binary system.
4. Find J (180), the number of divisors of 180.
5. Give an example of a multiplicative function.
6. Find ϕ (400).
7. What is the rank of a non-singular matrix of order n ?
8. State True or False. 'Elementary transformations change the rank of a matrix'.
9. Fill in the blanks :

   The row nullity and the column nullity of a square matrix are ————————.
10. Fill in the blank :

    Inconsistent equations have ———————— solutions.
11. State Cayley-Hamilton theorem.
12. The characteristic roots of a unitary matrix are ————————. Fill up the blank.

(12 × ¼ = 3 weightage)

Section B

Answer all questions.

13. Write down the relation connecting gcd and lcm of two positive integers a and b.
14. Check whether the following Diophantine equation can be solved $6x + 51y = 22$.
15. Show $a \equiv a \pmod{n}$ if a is an integer.
16. State Fermat's Little Theorem.
17. If p is a prime number then find $\phi(p^k)$.
18. Find the number of zeroes with which 50! terminates.

Turn over

19. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

20. Express the system of equations in matrix poem

$x + 2y + z = 10.$

$2x + y + 5z = 9.$

21. Find characteristic roots of the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

(9 × 1 = 9 weightage)

Section C

Answer any **five** *questions.*

22. Carry out the Euclidean Algorithm for the numbers 119 and 272 to find gcd and express gcd as a linear combination of 119 and 272.

23. Using sieve of Eratosthenes find the number of prime not exceeding 50.

24. Prove Fermat's Little theorem.

25. Show 18! + 1 = M(437) ; a multiple of 437.

26. By reducing to normal form find the rank of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$.

27. Find non-singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}.$$

28. Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

(5 × 2 = 10 weightage)

Section D

Answer any **two** *questions.*

29. (i) Prove the fundamental theorem of Arithmetic.

    (ii) Prove if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$ where a, b, c are integers.

30. (i) Show if $n = p_1^{k_1} p_2^{k_2} .... p_r^{k_r}$ then $J(n) = (k_1+1)(k_2+1).....(k_r+1)$ and

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \cdot \frac{p_2^{k_2+1}-1}{p_2-1} ...... \frac{p_r^{k_r+1}-1}{p_r-1}.$$

    (ii) Verify J (mn) = J (m) J(n) using suitable example.

31. Check for consistency and solve the system

$$x+2y+3z=14$$
$$3x+y+2z=11$$
$$2x+3y+z=11.$$

(2 × 4 = 8 weightage)

C 80028 **(Pages : 3)** Name.................................

Reg. No.................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2015

(U.G.–CCSS)

Core Course—Mathematics

MM 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 30 Weightage

Section A

Answer all **twelve** *questions.*

1. Find g.c.d. (143, 227).
2. State fundamental theorem of Arithmetic.
3. Express 4725 in canonical form.
4. State Feromat's little theorem.
5. Find the sum of divisions of 180.
6. When will you say a number theoretic function f is multiplicative.
7. Find ϕ (360).
8. Define rank of a matrix.
9. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.
10. Find characteristic root of the matrix $A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.
11. State the nature of the characteristic roots of Hermitian matrices.
12. State Cayley Hamilton theorem.

(12 × ¼ = 3 weightage)

Section B

Answer all **nine** *questions.*

13. Prove if g.c.d. $(a, b) = d$ then g.c.d. $\left(a/d, b/d\right) = 1$.
14. Use of Euclidean algorithm to find x and y which satisfies g.c.d. (56, 72) = 56 x + 72y.

Turn over

d/a d/b 1 = ax + by d | ax + by

15. Check whether the following Diophantine equation can be solved $6x + 51y = 22$.

16. Show $a^7 \equiv a \pmod{42}$ for all a.

17. Determine the highest power of 3 dividing 80!

18. Reduce to the normal form to find rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$.

19. State the Sylvester's law of nullity.

20. Show that the characteristic roots of a triangular matrix are just the diagonal elements of that matrix.

21. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(9 × 1 = 9 weightage)

Section C

Anwer any **five** *questions.*

22. Prove there is an infinite number of primes.

23. If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then prove $a \equiv c \pmod{n}$.

24. Show $181 + 1 \equiv 0 \pmod{437}$.

25. Show $\sigma(n) = \sigma(n+1)$ if $n = 14$ where $\sigma(n)$ = sum of divisors of n.

26. Prove Euler's theorem, $a^{\phi(n)} \equiv 1 \pmod{n}$ if $n \geq 1$ and g.c.d. $(a, n) = 1$.

27. Find non-singular matrices P and Q such that PAQ is in the normal form where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$.

28. Solve the system of equations :

$$x + 3y - 2z = 0$$

$$2x - y + 4z = 0$$

$$x - 11y + 14z = 0.$$

(5 × 2 = 10 weightage)

Section D

Anwer any **two** *questions.*

29 (a) Prove the fundamental theorem of arithmetic.

(b) Solve the linear congruence equation $6x \equiv 15 \pmod{21}$.

30 (a) State and prove Wilson's theorem.

(b) If n is an odd integer then prove $\phi(2n) = \phi(n)$.

31 Show that the equations :

$$x + 2y - z = 3$$

$$3x - y + 2z = 1$$

$$2x - 2y + 3z = 2$$

$$x - y + z = -1.$$

are consistent and solve the same.

(2 × 4 = 8 weightage)

C 60111

(Pages : 3)

Name.....................................

Reg. No..................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2014

(U.G.—CCSS)

Mathematics

MM 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours

Maximum : 30 Weightage

I. Answer all *twelve* questions :

1 $\sigma(5) =$ ____________.

2 The value of $\sum_{n=1}^{6} \sigma(n) =$ ____________.

3 If p is a prime and $k > 0$ then $\phi\left(p^k\right) =$ ____________.

4 If a, b, c are integers and $gcd\,(a, b, c) = 1$ then $gcd\,(a, c) =$ ____________.

5 If A is a non-singular matrix of order n then the rank of A is ____________.

6 The system AX = 0 in n unknows has a trivial solution if :

(a) $\rho(A) > n$.

(b) $\rho(A) = n$.

(c) $\rho(A) < n$.

(d) None of these.

7 State Cayley-Hamilton theorem.

8 Define Nullity of a matrix.

9 If A is a matrix of order $m \times n$ and R is a non-singular matrix of order m, then $\rho(RA) =$ ____________.

10 If N is a positive integer then $\sum_{n=1}^{N} \sigma(n) = \sum_{n=1}^{N}$ ____________.

11 State Chinese Remainder theorem.

12 If $ca \equiv cb \pmod{n}$ and $gcd\,(c, n) = 1$ then $a \equiv$ ____________ (mod n).

(12 × ¼ = 3 weightage)

Turn over

II. Answer all *nine* questions :

13 When two integers a and b are said to be relatively prime ?

14 Find *lcm* (306, 657).

15 State Fermat's theorem.

16 Prove that the function μ is a multiplicative function.

17 Show that no skew-symmetric matrix can be of rank 1.

18 If $A = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$ find $\rho(A)$.

19 Define Null space of a matrix.

20 Find the remainder when $1! + 2! + 3! + \dots + 99! + 100!$ is divided by 12.

21 Prove that if $gcd\,(a, b) = d$ then $gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

(9 × 1 = 9 weightage)

III. Answer any *five* questions from seven :

22 Prove that the fourth power of any integer is of the form $5k$ or $5k + 1$.

23 Let a and b be integers, not both zero prove that a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.

24 Prove that for positive integers a and b $gcd\,(a, b)\, lcm\,(a, b) = ab$.

25 Show that 8 divides $7^{2n+1} + 1$.

26 Using Wilson's theorem prove that $18! + 1 \equiv 0 \pmod{23}$.

27 Find non-singular matrices P and Q such that PAQ is in the normal form given

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}.$$

28 Show that the characteristic roots of an idempotent matrix are either zero or unity.

(5 × 2 = 10 weightage)

IV. Answer *two* questions from three :

29 State and prove Chinese Remainder theorem.

30 State and prove Fundamental theorem of Arithmetic.

31 Using Cayley-Hamilton theorem show that $A^3 - 6A^2 + 11A - 6I = 0$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$ and hence find A^{-1}.

(2 × 4 = 8 weightage)

C 40397 (Pages : 3) Name..............................

Reg. No..............................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2013

(CCSS)

Mathematics

MM 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 30 Weightage

I. Answer all *twelve* questions :

1 If Rank (A) = Ran $(c) < n$ then the system :

(a) Is consistent. (b) Has infinite solutions.

(c) Is inconsistent. (d) None of these.

2 If A is a non-singular matrix then $\left(A^{-1}\right)^{-1}$ = ———.

3 If $n > 1$ the sum of the positive integers less than n and relatively prime to n is ———.

4 Define the rank of a matrix.

5 The value of $\sum_{n=1}^{6}\left[\frac{6}{n}\right]$ is ———.

6 Find $\phi(9)$.

7 The value of $\sigma(12)$ = ———.

8 If $ca \equiv cb \pmod p$ and $p \times c$, where p is a prime number then ——— $\equiv b \pmod p$.

9 If $p, q_1, q_2, \ldots q_n$ are all primes and $p / q_1 q_2 \ldots q_n$ then prove that $p = q_k$ for same k.

10 The linear congruence $ax \equiv b(\bmod n)$ has a unique solution modulo n if gcd (a, n) = ———.

11 State Wilson's theorem.

12 If n is a positive integer then :

$$\sum_{n=1}^{N} \mathcal{T}(n) = \sum_{n=1}^{N} \text{———}.$$

(12 × ¼ = 3 weightage)

II. Answer all *nine* questions :

13 State Cayley-Hamilton theorem.

14 Define Nullity of a matrix.

Turn over

15 What is the value of $\begin{vmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{vmatrix}$.

16 If $A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$ find $\rho(AB)$.

17 Under what condition the rank of the matrix $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$ is 3.

18 Given integers a, b, c. If g c d $(a, b) = 1$ and $g\ c\ d\ (a, c) = 1$ what is $g\ c\ d\ (a, bc)$.

19 Prove that $\sigma(mn) = \sigma(m)\,\sigma(n)$.

20 If $ca \equiv cb \pmod{n}$ then prove that $a \equiv b \pmod{n/d}$ where $d = \gcd(c, n)$.

21 If p is a prime and $p\,/\,ab$ then prove that p/a or p/b.

(9 × 1 = 9 weightage)

III. Answer any *five* questions from seven :

22 Prove that the cube of any integer has one of the forms $7k$ or $7k \pm 1$.

23 Find the g.c.d. of (595, 252).

24 Prove that there is an infinite number of primes.

25 Prove that 13 divides $4^{2n+1} + 3^{n+2}$.

26 Prove that if $n > 1$ the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n \phi(n)$.

27 Reducing to the normal form find the rank of the matrix $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$.

28 Find the inverse of $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 1 & 0 \end{bmatrix}$.

(5 × 2 = 10 weightage)

IV. Answer any *two* questions from three :

29 Find non-singular matrices P and Q such that PAQ is in the normal form where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$.

30 State and prove Fermat's theorem.

31 Use the Euclidean Algorithm to obtain integers x and y satisfying :

$$\gcd(858, 325) = 858x + 325y$$

(2 × 4 = 8 weightage)