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A new perspective of ground band energy formulae

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Abstract. A host of alternative energy formulae for the ground bands of even Z even N nuclei are available in the literature. The usual approach is to compare the relative numerical accuracy of the predictions of the level energies by these formulae, for varying deformations of the nuclear core and for high spins. The soft rotor formula and variable moment of inertia model, the *ab* and *pq* formulae, the rotation vibration interaction and power index formulae are illustrated. Here, a new perspective is presented, with emphasis on the limitation of the region of their physical validity and on deriving useful meaning of their parameters.

Keywords. Nuclear structure; ground band; moment of inertia; softness parameter.

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1. Introduction

The γ-ray energies E_γ and intensities I_γ provide primary data on the spectral properties of atomic nuclei. The patterns of the spectra and the large $E2$ transition rates (much larger than the single-particle estimates) led Bohr and Mottelson to suggest the existence of collective motion in some nuclei arising from the coherent motion of the nucleons associated with their fast motion in single-particle orbits in the central field of the nucleus [1]. The central potential field may be spherical or deformed, leading to equispaced energy levels or rotor-like pattern respectively. The present work deals only with nuclear level energy patterns, observed in the spectra of even Z even N nuclei, which display collective motion. The prediction of the transition intensity I_γ is not dealt with here, which requires a different framework. Each of these aspects of the nuclear structure can also be dealt with in the microscopic treatments with different degrees of single-particle aspects, for example, the spherical shell model, the deformed shell model, random phase approximation (RPA), the generator coordinate method (GCM), the mean-field theories and algebraic model called interacting boson model and their variations based on symmetries and group structures. The empirical work in the unified collective model serves to provide a visual view of the problem and set up a task for the microscopic theories to give a deeper understanding. It is also known that we still do not know the exact form of nuclear force. While the shell model deals with static potential field, in the collective model, the central field is dynamic, capable of rotation and vibration. Further comments on this subject will be given in the summary section.

In the analytically solvable models, the level energies are given in terms of related quantum numbers. In the unified collective model of Bohr and Mottelson, the level energies are given by simple formulae based on spherical and rotational symmetries. The level energies of the ground-state band (gsb) of a well-deformed even Z even N nucleus ($R_{4/2} = E(4^+)/E(2^+) \sim 10/3$) are given by the Bohr–Mottelson (BM) rotation formula

$$E(I) = (\hbar^2/2\theta)I(I+1), \tag{1}$$

where θ is the moment of inertia (MoI) of the nucleus. The nuclear moment of inertia (MoI) differs from the rigid body motion. The rotation of the nucleus arises from the surface wave in the nuclear core. In the microscopic view, this is caused by the pairing interaction between the nucleons.

For higher spin states, and/or for the softer nuclei, an extension of eq. (1) is given by BM extended expression (2)

$$E(I) = AX - BX^2 + CX^3 \ldots, \tag{2}$$

with $X = I(I+1)$ and $A = \hbar^2/2\theta$, due to the perturbation of the rotational symmetry. The negative B-term accounts for the centrifugal stretching effect of

the rotation of the nuclear core, the band-mixing and the Coriolis antipairing (CAP) effects [1]. The third term compensates for the too large effect of the B-term. Even with the three terms, eq. (2) converges only for the well-deformed nuclei ($R_{4/2} \geq 3.2$). For the even–even nuclei with energy ratio $R_{4/2} > 3.2$, the convergence of series (2) is illustrated in table 1, where the radius of convergence is also given, which depends on the ratio B/A in eq. (2).

The convergence of series (2) gets progressively poorer at higher spins and for shape transitional (in between the spherical and well deformed) nuclei, with energy ratio $R_{4/2} < 3.2$, with reducing $R_{4/2}$ [1]. Over the last four decades, numerous alternative expressions have been suggested [2–10] to faithfully reproduce the spectral patterns of the given nuclei. Broadly speaking, these relate to the following:

(i) (a) Expansion of the moment of inertia (MoI) θ_I in terms of spin I, as in the soft rotor formula (SRF) [2], also called (earlier) the nuclear softness model (NSM) [3],

$$E(I) = I(I+1)/2\theta_0(1+\alpha I). \tag{3}$$

(b) The variable moment of inertia (VMI) model of Mariscotti *et al* [4] which includes, a potential term:

$$E(I) = I(I+1)/2\theta_I + 1/2C(\theta_I - \theta_0)^2 \tag{4}$$

leading to the cubic form in θ_I

$$\theta_I^3 - \theta_I^2\theta_0 - I(I+1)/2C = 0. \tag{5}$$

(c) A combination of NSM/SRF and VMI models, called VMINS3 model [5], uses $\theta_I = \theta_0(1+\alpha I)$ in eq. (4), instead of the condition $\partial E/\partial\theta = 0$ used in the original formulation of VMI model [4].

These models seek to account for the spin-dependent centrifugal stretching and Coriolis antipairing (CAP) and band mixing effects. The VMI model was meant to predict the moment of inertia (MoI) of high spin states produced in heavy ion reactions.

(ii) Alternative expression dealing with the expansion of MoI with spin I or excitation energy E, to improve upon eq. (1), but in closed form, was given as *ab* formula by Holmberg and Lipas [6]

$$E(I) = a[(1+bI(I+1))^{1/2} - 1]. \tag{6}$$

It leads to the quadratic form: $E^2 + 2aE = a^2bI(I+1)$ for its solution.

Its modified form called *pq* formula was also given by Zeng *et al* [7].

$$[(1+2E/p)^{1/2} + 1] = 2pq\theta. \tag{7}$$

It leads to the cubic expression

$$E^3 + p^2qI(I+1)E - 1/2p^2q^2[I(I+1)]^2 = 0. \tag{8}$$

(iii) (a) A completely different approach, viz. change of $X = I(I+1)$ to $X = I$, or a mix of the two, with 2 or 3 terms, was suggested in anharmonic vibrator model (AHV) by Das *et al* [8].

$$E(I) = a'I + b'I(I-2) + c'I(I-2)(I-4). \tag{9}$$

An equivalent empirical expression (for two terms) was suggested by Ejiri *et al* [9].

$$E(I) = aI(I+1) + bI. \tag{10}$$

It could be extended to three terms, including $cI^2(I+1)$. In a previous study by Gupta and Kavathekar [10], empirically derived eq. (10) was interpreted to represent a combination of rotation energy and vibration energy (both) in each state of the ground-state band. It was a new view, but necessarily holding for the shape transitional nuclei, away from the limiting symmetries of a spherical anharmonic vibrator or a soft deformed rotor, even at low spins. It enabled an explicit view of the change in rotation–vibration structure of the nuclei with a change in deformation.

It was shown (Harris (1965) see p. 25 and figure 4.11 on p. 70 in [1]) that an expansion in powers of frequency ω^2 converges faster than series (2) in powers of quantum mechanical spin I. It is explained in terms of the response of the nucleonic motion to the rotating potential in the cranking model.

(b) Gupta *et al* [11] also proposed a single-term compact formula, called the power index formula (11)

$$E(I) = aI^b. \tag{11}$$

Table 1. Convergence of eq. (2) (BM-2, p.66 T-2) [1].

	^{156}Gd	^{158}Gd	^{162}Dy	^{164}Dy	^{168}Er	^{178}Hf
A	15.03	13.33	13.52	12.28	13.34	15.62
$B/A \times 10^3$	2.362	1.069	0.934	0.738	0.547	1.005
Radius of convergence (p. 71 of [1]) is given by $I(I+1) = 4/27\|A/B\|$						
$= 150/(B/A)$	8	10	12	14	16	10
$= R_{4/2}$	3.24	3.29	3.294	3.301	3.309	3.247

Table 2. Level energies, rms deviation (keV) and relative error σ, SRF parameters MoI θ_0 in $(\text{MeV})^{-1}$ and softness parameter α (last two columns). For VMI θ_0 and C and for ab formula a and b, see the last two columns. Blank spaces in fixed mode denote original values.

Model	$E(2)$	$E(4)$	$E(6)$	$E(8)$	$E(10)$	$E(12)$	(rmsd, σ)	θ_0	(α or C) or a and b
^{156}Gd									
Expt.	88.97	288.18	584.7	965.1	1416.0	1924.4			
SRFix (θ_0, α)			588.6	982.0	1461.2	2019.7	(533, 3.0)	32.7	0.015
SRFit	89.7	287.5	580.9	959.5	1414.4	1937.8	(7.5, 0.73)	32.0	0.021
VMI (θ_0, C)	89.76	290.8	588.6	969.3	1421.4	1936.1	(6.3)	33.0	2.825
ab (a, b)	88.0	287.1			1412.7	1913.7	(5.7, 0.66)	4634	0.0064
^{154}Gd									
Expt.	123.07	371.01	717.74	1144.5	1637.2	2184.7			
SRFix (θ_0, α)			711.1	1121.2	1585.5	2093.1	(54, 2.8)	21.8	0.059
SRFit	122.0	372.4	721.7	1148.3	1636.9	2175.7	(5.4, 0.63)	22.3	0.050
VMI (θ_0, C)	128.77	388.1	739.7	1162.4	1643.1	2173.3	(14.3)	21.7	1.70
ab (a, b)	114.7	364.4			1621.7	2134.4	(27, 3.8)	2486	0.0157
^{152}Gd									
Expt.	344.28	755.40	1227.3	1746.7	2300.4	2883.7			
SRFix (θ_0, α)			1182	1615	2051	2489	(243, 9.7)	4.2	0.54
SRFit	320.7	770.0	1263.5	1777.4	2302.4	2834.1	(37, 3.0)	5.7	0.32
VMI (θ_0, C)	No solution								
ab (a, b)	279.7	727.7			2275.4	2809	(53, 10)	656	0.157

A major change is the use of the non-integer power index b, as distinguished from all the other formulae cited above, which use only integer powers of spin I, $(1.0 < b < 2.0)$.

One may note that, in quantum mechanics, one deals with integers (e.g. integer powers of spin I), but in classical mechanics, continuous variation is allowed (as in frequency ω). Thus, the continuous variation allowed for index b which depends on the deformation, is akin to the frequency ω. The highly successful expression (11) in reproducing the level energies of the ground band in all shape transitional nuclei [11] (without needing an expansion of the MoI or the series expansion in powers of spin I), points to its physical relevance. The index b may be obtained from the ratio of any two level energies.

In the present work, we seek to further explore the physics of the above energy expressions. In §2, a brief discussion is given on the above models, along with their relative accuracy of energy and parameter predictions. Discussion and summary are given in §3, where we also discuss the importance of the parameters of these various formulae and the links to the microscopic theories.

2. Review of the energy formulae

(i) (a) The soft rotor formula (SRF) [2] eq. (3), also called the nuclear softness model (NSM) [3], employs the concept of increase of MoI with spin I, $\theta_I = \theta_0(1 + \alpha I)$. It is solved for the ground-state MoI (θ_0) and the softness parameter (α), either in the predictive mode, from the energy of 2^+, 4^+ states, or in the reproduction mode, from all the level energies via the root mean square (rms) fit method [2]. It is a simple formula with transparent meaning of the parameters. With increasing $R_{4/2}$ (deformation), θ_0 increases and the softness parameter α goes towards zero. It yields a reasonable calculated energy spectrum up to high spins and for all nuclei $(2.0 < R_{4/2} < 10/3)$ [2].

The predictions of SRF (level energies and variation of parameters with N, Z) are illustrated here for the three isotopes $^{152-156}$Gd, ranging from the anharmonic vibrator in ^{152}Gd to the well-deformed rotor in ^{156}Gd (table 2). For ^{156}Gd, the ground-state MoI $\theta_0 = 32.0\,(\text{MeV})^{-1}$ by the root mean square (rms) fit method, and 32.7 by the prediction method from $E(2_1^+)$ and $E(4_1^+)$. At $I^\pi = 2^+$, θ increases to 33.4 and at 12^+ to 40.3 $(\text{MeV})^{-1}$ (an increase of ∼25%). The softness parameter α is 0.015 in the predictive mode (upper row in table 2), for the $I^\pi = 2^+$ and 4^+ levels. In the rms fit mode for spin I upto 12^+, it is 0.021 (lower row). Higher spins yield larger α in the rms fit mode.

For ^{154}Gd, $\theta_0 = 22.3\,(\text{MeV})^{-1}$ by the rms fit method (lower row) and 21.8 by the prediction method (upper row) from $E(2_1^+)$ and $E(4_1^+)$. At $I^\pi = 2_1^+$, it increases to 24.6 and at 12^+ to 35.9 $(\text{MeV})^{-1}$ (an increase of

~60%). Here the softness parameter $\alpha = 0.050$ in the rms fit mode.

For ^{152}Gd, $\theta_0 = 5.7$ (MeV)$^{-1}$ by the rms fit method and 4.2 by the prediction method from $E(2_1^+)$ and $E(4_1^+)$. At $I = 2$, it increases to 9.4 and at 12^+ to 27.5 (MeV)$^{-1}$ (an increase of 300% with $\alpha = 0.317$ in the rms fit mode. These results agree with those from ref. [2] (see figures 2 and 4 of [2]).

These illustrations show that for the SRF, its physical validity is limited to softness parameter $\alpha < 0.1$, because for larger α, the increase in the MoI with spin I is rather too large. Even for $\alpha = 0.1$, the MoI at spin $I = 10$ would be doubled, which is not probable due to the centrifugal stretching or Coriolis antipairing (CAP) effects, so that the prediction of the MoI and the softness parameter are model-dependent, even if the level energy reproduction is quite good (relative error $\sigma = 3\%$ even for ^{152}Gd). This should provide a useful guide for the microscopic theories.

(b) In the VMI model, one uses the rotational kinetic moment of inertia expression $\theta_I = (2I - 1)/E_\gamma$ for spins $I = 2$ and 4 and solves the quadratic equation in θ_0, for the ground-state MoI θ_0 and stretching constant C, which are then used for solving the cubic eq. (5) in θ_I

$$\theta_I^3 - \theta_I^2\theta_0 - I(I+1)/2C = 0. \tag{5}$$

For all spins I, an iteration procedure is used to get the best-fit values of θ_0, C and θ_I [4].

The energy fits in the VMI model were found good for almost all nuclei by Mariscotti *et al* [4], with $R_{4/2} > 2.2$. The VMI moments of inertia are often used by the experimentalists to characterize a nucleus. But even the values obtained by iteration are based on the rotor model expression (1). For a test of these features, we have solved the VMI eqs (4) and (5) for the three isotopes $^{152-156}$Gd. In table 2, we cite the VMI results for ^{154,156}Gd (no solution is obtainable for ^{152}Gd, with $R_{4/2} = 2.19$).

For ^{156}Gd, the rotor model MoI $= 3/E_2 = 33.7$ (MeV)$^{-1}$. The ground-state MoI $\theta_0 = 32.9$ (MeV)$^{-1}$. At 2_1^+ it is 33.9 from VMI model, equal to $\theta_{\rm RM}$ and at $I^\pi = 12^+$, it rises to 46.0 (MeV)$^{-1}$, i.e. a rise of 36% in a well-deformed nucleus with $R_{4/2} = 3.24$. Also the rms deviation falls from 39.4 to 6.3 keV after 21 iterations. The increase of MoI with spin, almost corresponds to the values from the SRF. Thus, the effect of the potential term on MoI is small. The values of θ_I tabulated in [4] agree with our values.

The $N = 90$ isotope ^{154}Gd lies at the onset of the deformed region. It is one of the four $N = 90$ isotopes which are good examples of the analytically solvable collective Hamiltonian and are called the $X(5)$ nuclei [12,13]. In the $X(5)$ approximation, one separates the β and γ degrees of freedom and assumes simple forms for them. For ^{154}Gd, the rotor model MoI $= 3/E_2 = 24.4$ (MeV)$^{-1}$. The MoI $\theta_0 = 21.7$ (MeV)$^{-1}$. At $I^\pi = 2^+$, $\theta = 24.7 = \theta_{\rm RM}$ and at $I^\pi = 12^+$ it rises to 45.2, i.e. a rise of almost 100% in a $X(5)$ soft deformed rotor nucleus with $R_{4/2} = 3.01$ [12,13]. Note that it is almost equal to the value for ^{156}Gd (at 12^+). Also, the rms deviation error falls from 37.7 to 14.3 keV after four iterations. The iteration process does not significantly affect the rise of MoI with spin. It does affect the values of the coefficient θ_0 and stretching constant C, so as to improve the energy fits. For ^{152}Gd, no solution is obtained. Earlier, Gupta and Hamilton [14] have illustrated at length the anomaly of using a rotor model-based MoI expression for the shape transitional nuclei, which leads to the above stated features of the SRF and the VMI models. Again, the above stated features of the VMI model should serve for the microscopic theories.

(ii) The *ab* formula [6] represents an attempt to obtain a closed form expression, improving on the BM series expression (2). But its validity is also limited to only well-deformed nuclei ($R_{4/2} > 3.0$). Holmberg and Lipas [6] noted that the solutions from spin $I^\pi = 6^+$ and 8^+ were better than from $I^\pi = 2^+$ and 4^+. This is illustrated here for $^{152-156}$Gd for both ways of the solution in table 3.

The values for $E(2_1^+)$ for $A = 156$, 154, 152 are 89, 123 and 344 keV respectively. Thus, from $(2^+, 4^+)$ pair, a/E_2 in eq. (6) varies from 38 to 9 and to less than 1.0 respectively, and b varies through a factor of 100.

The rms deviation X' (from $6^+, 8^+$) for spin up to 12^+ varies from about 6 keV to 53 keV in the above examples (table 3). Since X and X' are energy-dependent, we also calculated the root mean square of the sum of the relative energy deviations, called σ. The σ' value rises from 0.7% to 10% for rotational to the vibrational nucleus. Hence, the *ab* formula may be useful, only for the well-deformed nuclei ($R_{4/2} > 3.0$). Further, from table 3 we note that the parameters a and b vary in a large range and too fast with a slight change in $R_{4/2}$. The value of a and a' are too big and the value of b, b' are too small. This explains the limitations of the formula, as noted in [6]. The energy values predicted from the *pq* formula (eqs (7) and (8)) are slightly better, but the solution process is quite involved [7]. The results quoted in [7] for the well-deformed nuclei, derived from the rms fit to all the energy levels were reasonably good.

(iii) (a) The expressions (9) and (10) are basically similar. So we discuss only eq. (10).

Since a vast number of atomic nuclei lie in between the vibrational and the rotational limits, Gupta and Kavathekar [10] justified the view, that the energy of

Table 3. Parameters a, b from ab formula, rms deviation X and σ (relative error). The primed numbers belong to $(6^+, 8^+)$ option.

Z	A	$R_{4/2}$	a	a'	$b \times 10$	$b' \times 10$	X	X'	σ (%)	σ' (%)
64	156	3.24	3378	4634	0.089	0.064	47	5.7	2.8	0.66
64	154	3.014	1108	2486	0.39	0.157	202	27	10.7	3.8
64	152	2.19	224	656	9.08	1.725	270	53	10.8	9.7

each state of a transitional nucleus is a mixture of rotational and vibrational energy in eq. (10). For 2_1^+ states, these were called ROTE and VIBE respectively.

The 2-term expression (10) leads to a linear relation of energy ratios

$$R(I) = R_{4/2} I(I-2)/8 - I(I-4)/4. \quad (12)$$

This relation, in effect, explains the Mallmann plots [15] for ground band energies. Long ago, Mallmann [15] had noted that for any given even Z even N nucleus, its energy level pattern is determined by the energy ratio $R_{4/2}$. In figure 1, the linear plot of $R_{12/2}$ vs. $R_{4/2}$ is illustrated, along with the predictions from the SRF (dashed curve) and the power index (power law) formula.

In table 4, the predicted level energies and the associated parameters are listed for the RVI formula in eq. (10) and the power index formula (11), for the three Gd isotopes, representing the well-deformed nucleus ^{156}Gd, the $X(5)$ nucleus ^{154}Gd [12,13] and the vibrational nucleus ^{152}Gd. The rotation–vibration interaction (RVI) expression, if solved in fixed mode, yields results in reasonable agreement with data ($\sigma = 1.6\%$ in ^{152}Gd). But with coefficients a, b, c determined from rms fit method give significantly good predictions. In deformed ^{156}Gd, the vibration energy contribution bI (with negative b) is part of RVI energy and the nucleus is a good rotor. The relative error (σ) is only 0.05%. In ^{154}Gd, with $R_{4/2} = 3.01$, σ is 1.4% and at 2^+ the vibrational component is about 20%. In ^{152}Gd ($R_{4/2} = 2.19$) the rotational component is only about 20% and σ is only 0.13%.

In the RVI expression, first we get the degree of mixture of ROTE and VIBE in the 2_1^+ state (and any other state). Note that vibration coefficient b is negative for ^{156}Gd, indicating that it serves as RV interaction coefficient, rather than contributing to vibration. Thus, RVI coefficients yield useful information on the nature of the spectrum. Also it is valid for the whole region ($2.0 < R_{4/2}10/3$) from vibrational to the deformed limit.

(b) Next we look at the power index formula (11). Here, one obtains the values of the power index b for all the ground band states, say up to $I^\pi = 12^+$, and using average b, one deduces the values of the scaling coefficient a for all the states (table 5). Then, using the average b and average a one derives all the state energies.

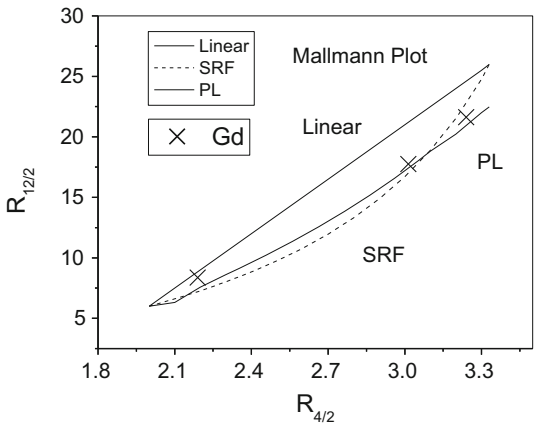

Figure 1. The Mallmann plot for $R_{12/2}$. Linear expression (12) is indicated by the straight line. The dashed curve indicates the SRF values and the continuous line marked PL is for power law. The experimental values for the Gd isotopes are indicated by cross (×).

The deduced relative rms deviation σ values for all the three isotopes $^{152-156}$Gd are small (table 4).

The energy ratio $R_{12/2}$ for $^{152-156}$Gd are displayed in figure 1 (by cross symbol) on the Mallmann plot. The data lie on the PL curve. Thus, the formula is valid for the whole region of $R_{4/2}$ between 2.0 and 3.3 (figure 1). Also there is no assumption of rotation or vibration. The variation of the index b and scaling coefficient a with spin for $^{152-156}$Gd is shown in table 5. Note the almost constant values of b and a with spin I, less than 5%, even for the vibrational nucleus ^{152}Gd. The power index formula can also be solved in the rms fit method by using $\log_{10}(E) = \log_{10}(a) + b \log_{10}(\text{spin})$ for all spins. This yields slightly improved energies (table 4).

3. Discussion and summary

The first two categories of the two-term energy formulae SRF, VMI, ab and pq are based on the assumption that the moment of inertia (θ_I) increases with spin I. In all of them, one uses the quantum mechanical form $I(I+1)$ in the numerator for square of angular momentum operator. For well-deformed nuclei, beyond the $X(5)$ limit ($R_{4/2} > 3.0$), it is a valid assumption. But for the nuclei far from the rotor limit ($2.0 < R_{4/2} < 3.0$), it leads

Table 4. Level energies (keV) and model parameters. For RVI the last three columns list the coefficient a, b and c. For power index formula, last two columns list a (keV) and b.

	$E(2)$	$E(4)$	$E(6)$	$E(8)$	$E(10)$	$E(12)$	(rmsd, σ)	a	b	c
^{156}Gd										
Exp.	88.97	288.18	584.7	965.1	1416.0	1924.4				
RVIfix				965.6	1418	1929	(2.8, 0.16)	15.7	−0.9	−0.027
RVIfit	88.9	288.3	584.7	965.1	1416.1	1924.5	(0.06, 0.05)	15.7	−1.07	−0.28
PLav (a, b)	89.2	292.4	585.5	958.3	1404.4	1919.2	(6.1, 0.75)	27.2	1.713	
PLfit (a, b)	88.5	291.5	585.5	960.4	1409.8	1929.3	(4.0, 0.60)	26.8	1.720	
^{154}Gd										
Exp.	123.07	371.01	717.74	1144.5	1637.2	2184.7				
RVIfix				1137	1603	2091	(57, 2.7)	19.4	6.6	−0.054
RVI fit	126.0	370.3	715.5	1144.3	1639.6	2183.6	(2.6, 1.4)	17.3	13.2	−0.36
PLav (a, b)	123.3	374.7	717.7	1138.8	1628.8	2182.0	(4.6, 0.5)	48.5	1.604	
PLfit (a, b)	122.6	373.9	717.9	1140.4	1632.9	2189.4	(3.3, 0.4)	40.2	1.609	
^{152}Gd										
Exp.	344.28	755.40	1227.3	1746.7	2300.4	2883.7				
RVIfix				1754	2331	2950	(42, 1.6)	9.2	145	−0.012
RVIfit	343.7	756.3	1227.3	1745.8	2301.3	2883	(1.03, 0.13)	10.2	142.7	−0.22
PLav (a, b)	346	777	1247	1743	2262	2797	(40, 2.0)	154	1.166	
PLfit (a, b)	338.0	770.1	1246.6	1754.5	2287	2840	(21, 1.3)	148	1.168	

Table 5. Variation of power index b and scaling index a (keV).

Spin	2^+	4^+	6^+	8^+	10^+	12^+
^{156}Gd b		1.696	1.714	1.720	1.720	1.716
^{154}Gd b		1.592	1.605	1.609	1.608	1.606
^{152}Gd b		1.134	1.157	1.172	1.180	1.186
^{156}Gd a	27.1	26.8	27.2	27.4	27.4	27.3
^{154}Gd a	40.5	40.2	40.5	40.8	40.8	40.6
^{152}Gd a	153.5	150.1	152.0	154.7	157.1	159.2

to (model-dependent) arithmetically modified values of the MoI θ_0 and the softness parameter α, and θ_I, even if the energy fits are good (see [14] for the anomalous moment of inertia).

Casten *et al* [16], while studying the relation of energy $E(4^+)$ to $E(2^+)$, defined the anharmonicity regime for the nuclei with $R_{4/2} < 3.1$. For this region, the third category of energy expressions has a greater validity. The RVI expression (10) assumes a combination of rotation and vibration symmetry, with or without RV interaction. The linear relation (12) of the energy ratios $R_{I/2}$, based on the two term expression, is useful for studying the shape transition with changing $R_{4/2}$ for the full range of $2.0 < R_{4/2} < 10/3$ (linear curve in figure 1). But the three-term expression yields realistic energy values for all the nuclei. The two-term expression (10) is also supported by the microscopic theory of collective motion and the two-term interacting boson model expression [17]

$$H_{\mathrm{IBM}} = \varepsilon n_d + kQ \cdot Q. \tag{13}$$

Here, the first term is the boson energy term of the $U(5)$ symmetry and the second term represents the $SU(3)$ symmetry. The ratio ε/k is often used for studying the shape transition from $U(5)$ to $SU(3)$ limit. Also, in the quadrupole operator

$$Q - (d^+s + s^+d) + \chi(d^+d) \tag{14}$$

the coefficient χ may be varied to induce symmetry mixing. This corresponds to the mixing term of RVI interaction.

In some microscopic theories, using the Hartree–Fock–Bogoliubov (HFB) approximation, one invokes the quadrupole interaction for generating the deformation and the pairing interaction for inducing sphericity. An example of this approach is in the pairing plus quadupole model of Kumar and Baranger [18]. In other mean-field theories (spherical shell model or deformed shell model) also, one uses the potential well with minimum at $|\beta| > 0$, to reproduce the requisite deformation. The width and shape of the potential well explain the other spectral features.

Lastly, the single-term power index formula does not involve the notion of rotation and vibration. While the index b expresses the degree of deformation, the inverse of scaling coefficient a corresponds to the MoI. The variation of power index b_I gives a transparent indication of any change in the structure at any spin I. The non-integer power index faithfully sums up the dependence of collective motion on (quantum mechanical) spin I. While spin I accounts for the quantum mechanical solution of the Hamiltonian, the variation of the non-integer power index with spin accounts for the continuing variation in the deformation energy. Angular momentum projection of the wave functions of the mean-field theory is often used for deriving spectral features. Even in the 'beyond mean-field theories' (for example, see ref. [19]), the angular momentum projection is used for restoring the broken symmetries in Hartree–Fock–Bogoliubov (HFB) treatment. So, spin specification is an essential part of the microscopic theories. The HFB process treats the quadrupole interaction and the pairing at equal footing. The intimate mixing of rotational and vibrational symmetries in the power index b of the power index formula may be considered to correspond to this aspect at the macroscopic level.

The energy ratio $R_{I/2} = (I/2)^b$, where the index b itself is $\log_{10}(R_{4/2})/\log_{10}(2)$, gives the compact form of the linear relation eq. (12) in the more realistic form, as apparent in the PL curve of figure 1, including the effect of the RVI interaction term as well. The three Gd isotopes illustrate the basic features of all the cited models. Both the SRF and PL curves in figure 1 illustrate their usefulness for almost all nuclei, spherical, deformed or shape transitional. The individual nucleus is characterized by its scaling coefficient a (inverse MoI).

Our focus in the above is on the physical content of the formulae, rather than on the slight differences in the accuracy (rms deviation) of level energy predictions. The SRF model is simple. Its use for deducing parameters θ_0 and α of $K^\pi = 2_1^+$ γ-bands of the deformed nuclei was illustrated by Gupta *et al* in ref. [20]. In the study of β-bands of nuclei in A =140–180 region, the softness parameter α for β-bands was found to be almost equal to that of ground bands (figure 8 in [21]), which supported the interpretation of $K^\pi = 0_2^+$ bands, as axially-symmetric β-vibration. The correspondence of the stiffness parameter C (gsb) from VMI model, to the energy ratio $R_{02} = E(0_2^+)/E(2_1^+)$ was also illustrated (see figure 7 in [21]). The RVI coefficient c displayed a valley at $N = 88-90$ corresponding to the valley of band head energy $E(0_2^+)$ (see figure 3 in [21]). Thus, the model parameters of the cited models provide useful information on the nuclear structure of the ground band and the excited bands, in their region of validity, for the predictions in microscopic theories.

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