THE MATHEMATICS OF INVESTMENT

(With answers in a separate pamp

BY

WILLIAM L. HART, Ph.D.

ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MINNESOTA

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PREFACE

This book provides an elementary course in the theory and the application of annuities certain and in the mathematical aspects of life insurance. The book is particularly adapted to the needs of students in colleges of business administration, but it is also fitted for study by college students of mathematics who are not specializing in business. Annuities certain and their applications are considered in Part I, life insurance is treated in Part II, and a treatment of logarithms and of progressions is given in Part III. The prerequisites for the study of the book are three semesters of high school algebra and an acquaintance with progressions and logarithmic computation. Very complete preparation would be furnished by three semesters of high school algebra and a course in college algebra including logarithms.

The material in the book has been thoroughly tested by the author through the teaching of it, in mimeographed form, for two years in classes at the University of Minnesota. It has been aimed in this book to present the subject in such a way that its beautiful simplicity and great usefulness will be thoroughly appreciated by all of the students to whom it is taught. Features of the book which will appeal to teachers of the subject are as follows:

1. **Illustrative examples are consistently used throughout** to illuminate new theory, to illustrate new methods, and to supply models for the solution of problems by the students.

2. **Large groups of problems are supplied** to illustrate each topic, and, in addition, sets of miscellaneous problems are given at the close of each important chapter, while a review set is placed at the end of each of the major parts of the book.

3. **Flexibility in the length of the course is provided for**; the teacher can conveniently choose from this book either a one or a two semester, three-hour course, on account of the latitude afforded by (*a*) the large number of problems, (*b*) the segregation of optional methods and difficult topics into **Supplementary** Sections whose omission does not break the continuity of the remainder of the book, and (*c*) the possibility of the omission of all of Part II, where life insurance is considered.

4. **The concept of an equation of value is emphasized** as a unifying principle throughout.

5. **Formulas are simplified and reduced** to as small a number as the author considers possible, if the classical notation of the subject is to be preserved. In Part I, a simplification is effected by the use of the interest period instead of the year as a time unit in a final pair of formulas for the amount and for the present value of an annuity certain. By use of these two formulas, the present values and the amounts of most annuities met in practice can be conveniently computed with the aid of the standard tables. In the applications of annuities certain, very few new formulas are introduced. The student is called upon to recognize all usual problems involving amortization, sinking funds, bonds, etc., as merely different instances of a single algebraic problem; that is, the finding of one unknown quantity by the solution of one of the fundamental pair of annuity formulas.

6. **Interpolation methods are used to a very great extent** and their logical and practical completeness is emphasized. Some problems solved by interpolation are treated by other methods, as well, and such optional methods are found segregated into Supplementary Sections.

7. **Practical aspects** of the subject are emphasized throughout.

8. **Very complete tables are provided**, including a five-place table of logarithms, the values of the interest and annuity functions for twenty-five interest rates, tables of the most essential insurance functions, and a table of squares, square roots, and reciprocals. The tables may be obtained either bound with the book or bound separately.

9. **In the discussion of life annuities and life insurance, the emphasis is placed on methods and on principles** rather than on manipulative proficiency. It is aimed to give the student a clear conception of the mathematical foundations of the subject. No attempt is made to prepare the student as an insurance actuary, but the treatment in this book is an excellent introduction to more advanced courses in actuarial science.

The interest and annuity tables prepared in connection with this book make possible the solution of most problems accurately to cents, if ordinary arithmetic is used. Results can be obtained with sufficient accuracy for most class purposes if the five-place table of logarithms is used in the computations. If the teacher considers it desirable to use seven-place logarithms, the author recommends the use of Glover's *Tables from Applied Mathematics*. These excellent tables contain the values, and the seven-place logarithms of the values, of the interest and annuity functions, a standard seven-place table of logarithms, and a variety of other useful tables dealing with insurance and statistics.

The author acknowledges his indebtedness to Professor James Glover for his permission to publish in the tables of this book certain extracts from Glover's *Tables* which were published, for the first time, in that book.

University of Minnesota,
January 1, 1924.

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MATHEMATICS OF INVESTMENT

PART I — ANNUITIES CERTAIN

CHAPTER I

SIMPLE INTEREST AND SIMPLE DISCOUNT

1. Definition of interest. — Interest is the income received from invested capital. The invested capital is called the **principal**; at any time after the investment of the principal, the sum of the principal plus the interest due is called the **amount**. The interest charge is usually stated as a **rate per cent** of the principal per year. If $\$P$ is the principal, r the rate expressed as a decimal, and i the interest for 1 year, then by definition $i = Pr$, or

$$r = \frac{i}{P}. \tag{1}$$

Thus, if \$1000 earns \$36.60 interest in one year, $r = \frac{36.60}{1000} = .0366$, or the rate is 3.66%. Also, if $P = \$1$ in equation 1, then $r = i$, or the rate r equals the interest on \$1 for 1 year.

2. Simple interest. — If interest is computed on the original principal during the whole life of a transaction, **simple interest** is being charged. The simple interest on a principal P is proportional to the time P is invested. Thus, if the simple interest for 1 year is \$1000, the interest for 5.7 years is \$5700.

Let I be the interest earned by P in t years, and let A be the amount due at the end of t years; since **amount = principal plus interest,**

$$A = P + I. \tag{2}$$

Since the interest earned by P in 1 year is Pr, the interest earned in t years is $t(Pr)$ or

$$I = Prt. \tag{3}$$

Hence $P + I = P + Prt$, so that, from equation 2,

$$A = P(1 + rt). \tag{4}$$

It is important to realize that equation 4 relates two sums, P and A, which are equally desirable if money can be invested at, or is worth, the simple interest rate r. The possession of P at any instant is as desirable as the possession of A at a time t years later, because if P is invested at the rate r, it will grow to the amount A in t years. We shall call P the **present value** or **present worth** of the amount A, due at the end of t years.

3. Ordinary and exact interest. — Simple interest is computed by equation 3, where t is the time in years. If the time is given in days, there are two varieties of interest used, **ordinary** and **exact** simple interest. In computing ordinary interest we assume one year to have 360 days, while for exact interest we assume 365 days.

Example 1. — Find the ordinary and the exact interest at 5% on \$5000 for 59 days.

Solution. — For the ordinary interest I_o use equation 3 with $t = \frac{59}{360}$, and for the exact interest I_e use $t = \frac{59}{365}$.

$$I_o = 5000(.05)\tfrac{59}{360} = \$40.97; \qquad I_e = 5000(.05)\tfrac{59}{365} = \$40.41.$$

A relation exists between the ordinary interest I_o and the exact interest I_e on a principal P for D days at the rate r.

$$I_o = Pr\frac{D}{360} = \frac{1}{360}PrD; \qquad I_e = Pr\frac{D}{365} = \frac{1}{365}PrD. \tag{5}$$

From equations 5 it is seen that $PrD = 360\ I_o = 365\ I_e$ or

$$\frac{I_o}{I_e} = \frac{73}{72}. \tag{6}$$

which shows that ordinary interest is greater than exact interest. From equation 6,

$$I_o = \frac{73}{72}I_e = I_e + \frac{I_e}{72}, \tag{7}$$

$$I_e = \frac{72}{73}I_o = I_o - \frac{I_o}{73}. \tag{8}$$

Thus, if the exact interest $I_e = \$40.41$, we obtain from equation 7,

$$I_o = 40.41 + \frac{40.41}{72} = \$40.97.$$

EXERCISE I

In the first five problems find the interest by use of equation 3.

Prob.	Principal	Rate	Time	Interest
1.	$ 5,570.	3.5%	75 days	exact
2.	8,000.	.045	93 days	ordinary
3.	115,380.	.0625	80 days	exact
4.	4,838.70	7.5%	35 days	ordinary
5.	2,500.	.055	27 days	exact

6. The exact interest on a certain principal for a certain number of days is $50.45. Find the ordinary interest for the same period of time.

7. The ordinary interest on a certain principal for a certain number of days is $35.67. Find the exact interest for the same period.

8. In problem 1 find the ordinary interest by use of the result of problem 1.

When the rate is 6%, the ordinary interest for 60 days is $(.01)P$, which is obtained by moving the decimal point in P two places to the left, while the interest for 6 days is $(.001)P$. These facts are the basis of the 6% method for computing ordinary interest at 6% or at rates conveniently related to 6%.

Example 2. — By the 6% method find the ordinary interest on $1389.20 for 83 days at 6%, and at 4.5%.

Solution. — $13.892 is interest at 6% for 60 days
4.168 is interest at 6% for 18 days (3 times 6 days)
1.158 is interest at 6% for 5 days ($\frac{1}{12}$ of 60 days)
$19.218 is interest at 6% for 83 days
4.805 is interest for 83 days at 1.5% ($\frac{1}{4}$ of 6%)
$14.413 is interest for 83 days at 4.5%

The extensive interest tables used in banks make it unnecessary to perform multiplications or divisions in computing simple interest. Table IV in this book makes it unnecessary to perform divisions.

Example 3. — Find the exact interest at 5% on $8578 for 96 days.

Solution. — From Table IV the interest at 1% on $10,000 for 96 days is $26.3013699. The interest on $8578 is (.8578) (5) (26.3014) = $112.81.

To find the time between two dates, it is sometimes assumed that each month has 30 days. For example,

$$\begin{array}{llr} \text{February 23, 1922, is } 1922:2:23 \text{ or} & 1921:14:23 & \\ \text{June 3, 1921, is} & \underline{1921:\ 6:\ 3} & \\ \text{Elapsed time} = & 0:\ 8:20 = 260 \text{ days} & \end{array} \tag{9}$$

The exact time can be found from Table III. February 23, 1922, is the 54th day of 1922 or the 419th day from January 1, 1921. June 3, 1921, is the 154th day from January 1, 1921. The elapsed time is $(419 - 154) = 265$ days.

NOTE. — In this book, for the sake of uniformity, proceed as follows, unless otherwise directed, in problems involving simple interest: (*a*) use ordinary interest if the time interval is given in days; (*b*) in computing the number of days between dates, find the exact number of days; (*c*) if the time is given in months, reduce it to a fraction of a year on the basis of 12 months to the year, without changing to days. Methods in the business world are lacking in uniformity in these respects, and, in any practical application, explicit information should be obtained as to the procedure to be followed.

EXERCISE II

Find the ordinary interest in the first five problems by use of the 6% method.

1. $P =$ \$3957.50, $t = 170$ days, $r = .06$.
2. $P =$ \$3957.50, $t = 170$ days, $r = .07$.
3. $P =$ \$4893.75, $t = 53$ days, $r = .04$.
4. $P =$ \$13,468.60, $t = 41$ days, $r = .03$.
5. $P =$ \$9836.80, $t = 134$ days, $r = .05$.
6. Find the exact interest in problem 4 by use of Table IV.
7. Find the ordinary interest on \$8500 at 6% from August 11, 1921, to March 13, 1922. Use the approximate number of days, as in expression 9 above.
8. Find the ordinary interest in problem 7, but use the exact number of days.
9. Find the ordinary and exact interest on \$1750 at 5% from April 3, 1921, to October 13, 1921, using the exact number of days.
10. (*a*) Find the ordinary and exact interest in problem 9, using the approximate number of days. (*b*) Which of the four methods of problems 9 and 10 is the most favorable to a creditor?

4. Algebraic problems.—If a sufficient number of the quantities (A, P, I, r, t) are given, the others can be determined by equations 2, 3, and 4. When the rate r, or the time t, is unknown, equation 3 should be used. When the present value P is unknown, equation 4 is most useful.

Example 1. — If a $1000 principal increases to $1250 when invested at simple interest for 3 years, what is the interest rate?

Solution. — $P = \$1000$, $A = \$1250$, $t = 3$. The interest $I = \$250$. From $I = Prt$, $250 = 3000\,r$, or $r = .0833$.

Example 2. — What principal invested at 5.5% simple interest will amount to $1150 after 2 years, 6 months?

Solution. — Use equation 4. $1150 = P[1 + 2.5(.055)] = 1.1375\,P$; $P = \dfrac{1150}{1.1375} = \$1010.99$. An equivalent statement of Example 2 would be, "Find the present value of $1150, due at the end of 2 years, 6 months, if money is worth 5.5% simple interest."

EXERCISE III

Find the missing quantities in the table below:

Prob.	A	P	I	Rate	Time
1.		$ 750.		.04	3 yr., 6 mo.
2.	$3500.			.055	2 yr.
3.		3500.		.058	2 yr.
4.			$150	.075	6 mo.
5.	2500.			.035	2 yr., 3 mo.
6.	1200.			.06	2 yr., 6 mo.
7.		1800.50	300	.055	
8.			650	.03	3 yr., 9 mo.
9.		1680.		.0375	11 mo.
10.	9850.50			.0725	1 yr., 6 mo.

11. Find the present value of $6000, due after 8 months, if money is worth 9%.

12. W borrowed $360 from B and agreed to repay it at the end of 8 months, with simple interest at the rate 5.25%. What did W pay at the end of 8 months?

13. Find the present worth of \$1350, due at the end of 2 months, if money is worth 5% simple interest.

√ 14. At the end of 3 months I must pay \$1800 to B. To cancel this obligation immediately, what should I pay B if he is willing to accept payment and is able to reinvest money at 6% simple interest?

Example 3. — A merchant is offered a \$50 discount for cash payment of a \$1200 bill due after 60 days. If he pays cash, at what rate may he consider his money to be earning interest for the next 60 days?

Solution. — He would pay \$1150 now in place of \$1200 at the end of 60 days. To find the interest rate under which \$1150 is the present value of \$1200, due in 60 days, use $I = Prt$; $50 = \frac{1150r}{6}$, or $r = .26087$. His money earns interest at the rate 26.087%; he could afford to borrow at any smaller rate in order to be able to pay cash.

15. A merchant is offered a \$30 discount for cash payment of a \$1000 bill due at the end of 30 days. What is the largest rate at which he could afford to borrow money in order to pay cash?

16. A 3% discount is offered for cash payment of a \$2500 bill, due at the end of 90 days. At what rate is interest earned over the 90 days if cash payment is made?

17. The terms of payment for a certain debt are: net cash in 90 days or 2% discount for cash in 30 days. At what rate is interest earned if the discount is taken advantage of?

HINT. — For a \$100 bill, \$100 paid after 90 days, or \$98 at the end of 30 days, would cancel the debt.

5. Simple discount. — The process of finding the present value P of an amount A, due at the end of t years, is called **discounting** A. The difference between A and its present value P, or $A - P$, is called the **discount** on A. From $A = P + I$, $I = A - P$; thus I, which is the interest on P, *also is the discount on* A. If \$1150 is the discounted value of \$1250, due at the end of 7 months, the discount on the \$1250 is \$100; the interest on \$1150 for 7 months is \$100.

In considering I as the interest on a known principal P we computed I at a certain rate per cent of P. In considering I as the discount on a known amount A, it is convenient to compute I at a certain rate per cent per year, of A. If i is the discount on

A, due at the end of 1 year, and if d is the discount rate expressed as a decimal, then by definition $i = Ad$, or

$$d = \frac{i}{A}. \tag{10}$$

Simple discount, like simple interest, is discount which is proportional to the time. If Ad is the discount on A, due in 1 year, the discount on an amount A due at the end of t years is $t(Ad)$, or

$$I = Adt. \tag{11}$$

From $P = A - I$, $P = A - Adt$, or

$$P = A(1 - dt). \tag{12}$$

If the time is given in days, we may use either **exact** or **ordinary** simple discount, according as we use one year as equal to 365 or to 360 days, in finding the value of t.

NOTE. — In simple discount problems in this book, for the sake of uniformity, proceed as follows unless otherwise directed: (*a*) if the time is given in days, use ordinary discount; (*b*) in computing the number of days between dates, find the exact number of days; (*c*) if the time is given in months, reduce it to a fraction of a year on the basis of 12 months to the year. Business practices are not uniform, and hence in any practical application of discount one should obtain explicit information as to the procedure to follow.

Example 1. — Find the discount rate if \$340 is the present value of \$350, due after 60 days.

Solution. — From $I = Adt$, $10 = \frac{350\,d}{6}$; $d = .17143$, or 17.143%.

Example 2. — If the discount rate is 6%, find the present value of \$300, due at the end of 3 months.

Solution. — From $I = Adt$, $I = \frac{300(.06)}{4} = 4.50$. $P = A - I = 300 - 4.50 = \$295.50$.

NOTE. — If A is known and P is unknown, it is easier to find P when the discount rate is given than when the interest rate is known. To appreciate this fact compare the solution of Example 2 above with that of Example 2 of Section 4, where a quotient had to be computed. This simplifying property of discount rates is responsible for their wide use in banking and business.

The use of a discount rate in finding the present value P of a known amount A is equivalent to the use of some interest rate, which is always different from the discount rate.

Example 3. — (*a*) If a 6% discount rate is charged in discounting amounts due after one year, what equivalent interest rate could be used? (*b*) What would be the interest rate if amounts due after 3 months were being discounted?

Solution. — (*a*) Suppose $A = \$100$, due after 1 year. Then $I = 100(.06) = \$6$, and $P = \$94$. Let r be the equivalent interest rate, and use $I = Prt$. $6 = 94r$; $r = .063830$. (*b*) If $A = \$100$, due after 3 months, $I = \frac{100(.06)}{4} = \$1.50$, and $P = \$98.50$. From $I = Prt$, $1.50 = \frac{98.50r}{4}$; $r = .060914$.

Compare the results of Example 3. When a discount rate is being used, the equivalent interest rate is larger for long-term than for short-term transactions and in both cases is larger than the discount rate.

The brief methods available for computing simple interest apply as well to the computation of simple discount because both operations involve multiplication by a small decimal. Thus, we may use the 6% method for computing discount, and simple interest tables may be used as simple discount tables.

EXERCISE IV

1. Find the discount rate if the discount on $1500, due after one year, is $82.50.

2. Find the discount rate if the present value of $1250, due after 8 months, is $1193.75.

Find the missing quantities in the table by use of equations 11 and 12.

PROB.	DISCOUNTED VALUE, P	AMOUNT, A	A IS DUE AFTER	I, DISCOUNT ON A OR INT. ON P	DISCOUNT RATE, d
3.		$1200	1 yr.		.05
4.	$145.50	150	6 mo.		
5.			3 mo.	$250	.07
6.		2000	90 da.		.045
7.		800	120 da.		.06
8.	357.75	375	9 mo.		
9.			5 mo.	300	.08
10.	750.		72 da.		.0625
11.		1500	6 mo.	35	
12.		7500		100	.06

13. Write in words problems equivalent to problems 3, 4, 10, and 12 above.

14. (*a*) What simple interest rate would be equivalent to the charge made in problem 3; (*b*) in problem 7?

15. If $d = .045$, (*a*) what is the equivalent interest rate for a 1-year transaction; (*b*) for one whose term is 4 months?

16. What discount rate would be equivalent to the use of a 6% interest rate in a 1-year transaction? HINT.—Let $P = \$100$; find A and I, and use $I = Adt$.

17. What discount rate would be equivalent to the use of a 6% interest rate in a 6-month transaction?

SUPPLEMENTARY NOTE. — Formulas can be obtained relating the discount rate d and the equivalent interest rate r on 1-year transactions. Suppose that \$1 is due at the end of 1 year. Then, in equations 11 and 12, $A = \$1$, $I = d$, and $P = 1 - d$. From $I = Prt$ with $t = 1$, $d = (1 - d)r$, or

$$r = \frac{d}{1 - d}. \tag{13}$$

From equation 13, $r - rd = d$, or $r = d(1 + r)$, so that

$$d = \frac{r}{1 + r}. \tag{14}$$

It must be remembered that equations 13 and 14 connect the discount and the interest rates only in the case of 1-year transactions.

SUPPLEMENTARY EXERCISE V

1. By use of equations 13 and 14, solve problems 14, 15, 16, and 17 of Exercise IV.

6. Banking use of simple discount in lending money. — If X asks for a \$1000 loan for six months from a bank B which is charging 6% discount, B will cause X to sign a note promising to pay B \$1000 at the end of 6 months. Then, B will give X the present value of the \$1000 which he has promised to pay. The bank computes this present value by use of its discount rate. $P = 1000 - 30 = \$970$, which X receives. The transaction is equivalent to B lending X \$970 for 6 months. The interest rate which X is paying is that which is equivalent to the 6% discount rate. The banker would tell X that he is paying 6% **interest in advance**, but this would merely be a colloquial manner of stating that the discount rate is 6%. In this book the phrase *interest in advance* is always used in this customary colloquial sense.

Example 1. — X requests a loan of \$9000 for 3 months from a bank B charging 5% discount. Find the immediate proceeds of the loan and the interest rate which X is paying.

Solution. — X promises to pay \$9000 at the end of 3 months. Discount on \$9000 for 3 months at 5% is \$112.50. Immediate proceeds, which X receives, are \$9000 − \$112.50 = \$8887.50. To determine the interest rate, use $I = Prt$. $112.5 = 8887.5\,r(\frac{1}{4})$, or $r = .050633$.

Example 2. — X wishes to receive \$9000 as the immediate proceeds of a 90-day loan from a bank B which is charging 5% interest payable in advance. For what sum will X draw the note which he will give to B?

Solution. — $P = \$9000$, $t = 90$ days, $d = .05$, and A is unknown. From equation 12, $9000 = A(1 - .0125)$. $A = \$9113.92$, for which X will draw the note.

EXERCISE VI

Determine how much X receives from the bank B. In the first three problems, also determine the interest rate which X is paying.

Prob.	Loan Requested by X	For	Discount Rate of B
1.	\$5,000	6 months	.065
2.	1,750	75 days	.07
3.	3,570	90 days	.06
4.	190	30 days	.05
5.	7,500	45 days	.055
6.	3,800	3 months	.0625

Determine the size of the loan which X would request from B if X desired the immediate proceeds given in the table.

Prob.	Immediate Proceeds	Term of Loan	Discount Rate of B
7.	\$ 3,500	30 days	.06
8.	8,000	4 months	.05
9.	1,300	3 months	.07
10.	150,000	60 days	.055
11.	4,300	90 days	.045
12.	9,350	6 months	.05

7. Discounting notes. — The discounting of promissory notes gives rise to problems similar to those of Exercise VI. Consider the following notes:

NOTE (*a*)

> *Minneapolis, June 1, 1922.*
> *Six months after date I promise to pay to* Y *or order* \$5000 *without interest. Value received.* Signed X.

NOTE (*b*)

> *Chicago, June 1, 1922.*
> *One hundred and eighty days after date I promise to pay to* Y *or order* \$5000 *together with simple interest from date at the rate* 7%. *Value received.* Signed X.

On August 1, Y sells note (*a*) to a bank B. The sale is accomplished by Y indorsing the note, transferring his rights to B, who will receive the \$5000 on the maturity date. The transaction is called *discounting the note* because B pays Y the present or discounted value of \$5000, due on December 1, 1922.

Example 1. — If B discounts notes at 5%, what will Y receive on August 1?

Solution. — *B* is using the discount rate 5% in computing present values. The discount on \$5000, due after 4 months, is \$83.33; B will pay Y \$4916.67.

Example 2. — On July 31, Y discounts note (*b*) at the bank B of Example 1. What are Y's proceeds from the sale of the note?

Solution. — B first computes the maturity value of the note, or what X will pay on the maturity date, which is November 28. The transaction is equivalent to discounting this maturity value. Term of the discount is 120 days (July 31 to November 28). From equation 4, the maturity value of the note is $5000(1 + .035) = \$5175.00$. Discount for 120 days at 5% on \$5175 is \$86.25. Proceeds $= \$5175 - \$86.25 = \$5088.75$.

EXERCISE VII

1. X paid Y for an order of goods with the following note:

> *Chicago, June 1, 1922.*
> *Sixty days after date I promise to pay to* Y *or order* \$375 *at the Continental Trust Company. Value received.* Signed X.

Thirty days later, Y discounted this note at a bank charging 5.5% discount. Find Y's proceeds from the sale of the note.

2. Find the proceeds in problem 1, if the discount rate is 6%.

3. The bank B of problem 1, after buying the note from Y, immediately rediscounted it at a Federal Reserve Bank[1] whose rediscount rate was .035. What did B receive for the note?

4. The holder of a non-interest-bearing note dated October 1, 1911, payable 4 months after date, discounted it at a bank on October 1, at the rate 4%. The bank's discount on the note was $20. What was the face of the note?

5. X owes a firm Y $800, due immediately. In payment X draws a 90-day non-interest-bearing note for such a sum that, if Y immediately discounts it at a bank charging 6% discount, the proceeds will be $800. What is the face value of the note?

HINT. — In equation 12, $P = \$800$ and A is unknown.

6. X draws a 60-day non-interest-bearing note in payment of a bill for $875, due now. What should be the face of the note so that the immediate proceeds to the creditor will be $875 if he discounts it immediately at a bank whose discount rate is 6.5%?

Find the proceeds from the sale of the following notes:

PROB.	FACE OF NOTE	DATE OF NOTE	TERM	NOTE BEARS INT. AT	SOLD ON	DISC. RATE OF BUYER
7.	$ 450	6/10/17	30 days	.06	6/20/17	.07
8.	1200	5/12/18	120 days	.05	6/26/18	.06
9.	375	3/25/20	90 days	.07	4/24/20	.04
10.	470	11/20/21	60 days	.00	12/ 5/21	.065
11.[2]	325	4/30/20	3 months	.06	6/ 1/20	.08
12.[3]	3000	8/14/19	6 months	.08	12/16/19	.06

[1] A Federal Reserve Bank discounts commercial notes brought to it by banks belonging to the Federal Reserve System. The rate of the Federal Reserve Bank is called a rediscount rate because all notes discounted by it have been discounted previously by some other bank. This previous discounting has no effect so far as the computation of the present value by the Federal Reserve Bank is concerned.

[2] The note is due on 7/31/20, the last day of the third month from April. Find the exact number of days between 6/1/20 and 7/31/20.

[3] Find the exact number of days between 12/16/19 and 2/14/20.

13. Y owes W $5000 due now. In payment Y draws a 45-day non-interest-bearing note, which W discounts immediately at a bank charging 6% interest in advance. What is the face of the note if W's proceeds are $5000?

14. W desires $2500 as the immediate proceeds of a 6-month loan from a bank which charges 7% interest in advance. What loan will W request?

15. A bank B used the rate 6% in discounting a 90-day note for $1000. The note was immediately rediscounted by B at a Federal Reserve Bank whose rate was 4%. Find B's profit on the transaction.

CHAPTER II

COMPOUND INTEREST

8. Definition of compound interest. — If, at stated intervals during the term of an investment, the interest due is added to the principal and thereafter earns interest, the sum by which the original principal has increased by the end of the term of the investment is called **compound interest**. At the end of the term, the total amount due, which consists of the original principal plus the compound interest, is called the **compound amount**.

We speak of interest being **compounded**, or **payable**, or **converted** into principal. The time between successive conversions of interest into principal is called the **conversion period**. In a compound interest transaction we must know (*a*) the conversion period and (*b*) the rate at which interest is earned during a conversion period. Thus, if the rate is 6%, compounded quarterly, the conversion period is 3 months and interest is earned at the rate 6% per year during each period, or at the rate 1.5% per conversion period.

Example 1. — Find the compound amount after 1 year if $100 is invested at the rate 8%, compounded quarterly.

Solution. — The rate per conversion period is .02. Original principal is $100.

At end of 3 mo. $2.000 interest is due; new principal is $102.000.
At end of 6 mo. $2.040 interest is due; new principal is $104.040.
At end of 9 mo. $2.081 interest is due; new principal is $106.121.
At end of 1 yr. $2.122 interest is due; new principal is $108.243.

The compound interest earned in 1 year is $8.243. The rate of increase of principal per year is $\frac{8.243}{100} = .08243$, or 8.243%.

EXERCISE VIII

1. By the method of Example 1 find the compound amount after 1 year if $100 is invested at the rate 6%, payable quarterly. What was the compound amount after 6 months? At what rate per year does principal increase in this case?

2. Find the annual rate of growth of principal under the rate .04, converted quarterly.

NOTE. — Hereafter, the unqualified word *interest* will always refer to *compound interest*. If a transaction extends over more than 1 year, compound interest should be used. If the time involved is less than 1 year, simple interest generally is used.

9. The compound interest formula. — Let the interest rate per conversion period be r, expressed as a decimal. Let P be the original principal and let A be the compound amount to which P accumulates by the end of k conversion periods. Then, we shall prove that

$$A = P(1 + r)^k. \qquad (15)$$

The method of Example 1, Section 8, applies in establishing equation 15.

Original principal invested is P.
Interest due at end of 1st period is Pr.
New principal at end of 1st period is $P + Pr = P(1 + r)$.
Interest due at end of 2d period is $P(1 + r)r$.
New principal at end of 2d period is $P(1 + r) + P(1 + r)r = P(1 + r)^2$.

By the end of each period, the principal on hand at the beginning of the period has been multiplied by $(1 + r)$. Hence, by the end of k periods, the original principal P has been multiplied k successive times by $(1 + r)$ or by $(1 + r)^k$. Therefore, the compound amount after k periods is $P(1 + r)^k$.

If money can be invested at the rate r per period, the sums P and A, connected by equation 15, are equally desirable, because if P is invested now it will grow to the value A by the end of k periods. We shall call P the **present value** of A, due at the end of k periods.

The fundamental problems under compound interest are the following:

(*a*) The **accumulation** problem, or the determination of the amount A when we know the principal P, the interest rate, and the time for which P is invested. To accumulate P, means to find the compound amount A resulting from the investment of P.

(*b*) The **discount** problem, or the determination of the present value P of a known amount A, when we know the interest rate and the date on which A is due. To discount A means to find its present value P. The discount on A is $(A - P)$.

The accumulation problem is solved by equation 15.

Example 1. — Find the compound amount after 9 years and 3 months on a principal $P = \$3000$, if the rate is 6%, compounded quarterly.

Solution. — The rate per period is $r = .015$; the number of periods is $k = 4(9.25) = 37$.

$A = 3000(1.015)^{37} = 3000(1.73477663) = \$5204.33$. (Using Table V)

The compound interest earned is $\$5204.33 - \$3000 = \$2204.33$.

NOTE. — If interest is converted m times per year, find the number k of conversion periods in n years from the equation $k = mn$.

To solve the discount problem we first solve equation 15 for P, obtaining

$$P = \frac{A}{(1+r)^k} = A\left(\frac{1}{(1+r)^k}\right) = A(1+r)^{-k}. \qquad (16)$$

Example 2. — Find the present value of \$5000, due at the end of 4 years and 6 months, if money earns 4%, converted semi-annually.

Solution. — Rate per period is $r = .02$; number of periods is $k = 2(4.5) = 9$.

$P = 5000(1.02)^{-9} = 5000(.83675527) = \$4183.78$. (Using Table VI)

The discount on A is $\$5000 - \$4183.78 = \$816.22$.

NOTE. — Recognize that Example 2 involves the formation of a product when solved by Table VI. A problem is solved incorrectly if available tables are not used to simplify the work. Since products are easier to compute than quotients, the following solution of Example 2 should be considered incorrect, although mathematically flawless, because a quotient is computed.

$$P = \frac{5000}{(1.02)^9} = \frac{5000}{1.19509257} = \$4183.78. \text{ (Using Table V)}$$

In describing interest rates in the future, a standard abbreviation will be used. When we state the rate to be $(.05, m = 2)$, the "$m = 2$" signifies that interest is compounded twice per year, or semi-annually. The rate $(.08, m = 1)$ means 8%, compounded annually; $(.07, m = 12)$ means 7%, converted monthly; $(.06, m = 4)$ means 6%, compounded quarterly.

NOTE. — The quantity $(1 + r)$ in equation 15 is sometimes called the accumulation factor, while the quantity $\frac{1}{1+r}$ or $(1+r)^{-1}$ in equation 16 is called the discount factor. In many books the letter v is used to denote the discount factor, or $v = (1+r)^{-1}$. Thus, at the rate 7%, $v^4 = (1.07)^{-4}$.

EXERCISE IX

1. By use of the binomial theorem verify all digits of the entry for $(1.02)^4$ in Table V.

2. In Table VI verify all digits of the entry for $(1.02)^{-6}$ by using the entry for $(1.02)^6$ in Table V and by completing the long division involved.

HINT. — $(1.02)^{-6} = \frac{1}{(1.02)^6} = \frac{1}{1.12616242}$.

3. Find the compound amount on $3,000,000 after 16 years and 3 months, if the rate is (.06, $m = 4$).

4. Accumulate a $40,000 principal for 15 years under the rate (.05, $m = 4$). What compound interest is earned?

5. Find the present value of $6000, due after $4\frac{1}{2}$ years, if money can earn interest at the rate (.08, $m = 4$). What is the discount on the $6000?

6. Discount $5000 for 19 years and 6 months, at the rate (.05, $m = 2$).

In the table below, find that quantity, P or A, which is not given. In the first four problems, before doing the numerical work, write equivalent problems in words.

PROB.	PRINCIPAL, P	AMOUNT, A	P ACCUMULATES FOR, OR A IS DUE AFTER	RATE
7.		$4000	5 yr., 6 mo.	.04, $m = 2$
8.	$1000.		10 yr., 3 mo.	.07, $m = 4$
9.	3000.		12 yr.	.06, $m = 1$
10.		6000	8 yr., 6 mo.	.03, $m = 4$
11.	2500.		13 yr., 9 mo.	.08, $m = 4$
12.		1500	7 yr., 6 mo.	.06, $m = 2$
13.	576.50		3 yr., 6 mo.	.06, $m = 12$
14.	1398.50		15 yr., 3 mo.	.05, $m = 4$
15.		8300	14 yr., 6 mo.	.055, $m = 2$
16.		9500	5 yr.	.045, $m = 1$
17.	1300.		2 yr., 9 mo.	.03, $m = 4$
18.	1.		75 yr.	.05, $m = 1$
19.		100	100 yr.	.035, $m = 1$
20.[1]	100.		175 yr.	.045, $m = 1$
21.[1]		100	173 yr.	.065, $m = 1$
22.		1	30 int. periods	.04, per period
23.	1.		35 int. periods	.05, per period

[1] In problem 20 use $(1.045)^{175} = (1.045)^{100}(1.045)^{75}$. In problem 21 use $(1.065)^{-100}(1.065)^{-73}$.

24. (*a*) If the rate is i, compounded annually, and if the original principal is P, derive the formula for the compound amount after 10 years. (*b*) After n years.

25. If $100 had been invested in the year 1800 A.D. at the rate (.03, $m = 1$), what would be the compound amount now?

26. (*a*) If the rate is j, compounded m times per year, derive a formula for the compound amount of a principal P after 10 years. (*b*) After n years.

10. Nominal and effective rates. — Under a given type of compound interest the rate per year at which principal grows is called the **effective rate**. The per cent quoted in stating a type of compound interest is called the **nominal rate**; it is the rate per year at which money earns interest *during a conversion period.* In the illustrative Example 1 of Section 8 it was seen that, when the nominal rate was 8%, converted quarterly, the effective rate was 8.24%. We shall say " the rate is (j, m) " to abbreviate " the nominal rate is j, converted m times per year." Let i represent the effective rate.

The effective rate i corresponding to the nominal rate j, converted m times per year, satisfies the equation

$$1 + i = \left(1 + \frac{j}{m}\right)^m. \tag{17}$$

To prove this, consider investing $P = \$1$ for 1 year at the rate (j, m). The rate per period is $\frac{j}{m}$ and the number of periods in 1 year is m. The amount A after 1 year and the interest I earned in that time are

$$A = \left(1 + \frac{j}{m}\right)^m; \quad I = \left(1 + \frac{j}{m}\right)^m - 1.$$

The rate of increase of principal per year is $i = \frac{I}{P} = I$, because $P = \$1$. Hence

$$i = \left(1 + \frac{j}{m}\right)^m - 1. \tag{18}$$

Transpose the 1 in equation 18 and equation 17 is obtained.

Example 1. — What is the effective rate corresponding to the rate (.05, $m = 4$)?

Solution. — Use equation 17. $1 + i = (1.0125)^4 = 1.05094534.$
$i = .05094534.$

Example 2. — What nominal rate, if converted 4 times per year, will yield the effective rate 6%?

Solution. — From equation 17, $1.06 = \left(1 + \frac{j}{4}\right)^4.$

$$1 + \frac{j}{4} = (1.06)^{\frac{1}{4}} = 1.01467385, \text{ from Table X.} \qquad (19)$$

$$j = 4(.01467385) = .05869540.$$

Table XI furnishes an easier solution. From equation 19,

$$\frac{j}{4} = (1.06)^{\frac{1}{4}} - 1; \; j = 4[(1.06)^{\frac{1}{4}} - 1] = .05869538. \qquad \text{(Table XI)}$$

Example 3. — What nominal rate, converted quarterly, will give the same yield as $(.05, m = 2)$?

Solution. — Let j be the unknown nominal rate. The effective rate i corresponding to $(.05, m = 2)$ must equal the effective rate corresponding to the nominal rate j, compounded quarterly. From equation 17,

$$1 + i = (1.025)^2; \; 1 + i = \left(1 + \frac{j}{4}\right)^4. \; \therefore (1.025)^2 = \left(1 + \frac{j}{4}\right)^4.$$

$$1 + \frac{j}{4} = (1.025)^{\frac{1}{2}} = 1.01242284. \; j = 4(.01242284) = .04969136.$$

EXERCISE X

1. (*a*) In Table X verify the entry for $(1.05)^{\frac{1}{2}}$ by use of Table II. (*b*) In Table XI verify the entry for $p = 4$ and $i = .05$, by using Table X.

2. Find the effective rates corresponding to the rates $(.06, m = 2)$ and $(.06, m = 4)$.

3. Find the nominal rate which, if converted semi-annually, yields the effective rate .05. (*a*) Solve by Table X. (*b*) Read the result out of Table XI.

Solve the problems in the table orally by the aid of Tables V and XI. State equivalent problems in words.

PROB.	j	m	i	PROB.	j	m	i
4.	.07	2		10.		12	.04
5.		2	.07	11.	.03	2	
6.	.10	4		12.		2	.0275
7.		2	.035	13.		4	.025
8.	.09	3		14.	.05	1	
9.	.09	4		15.		1	.06

16. Derive a formula for the nominal rate which, if converted p times per year, gives the effective rate i. NOTE. — The resulting value of j is denoted by the notation j_p, as in Table XI.

17. Which interest rate is the better, 5% compounded monthly or 5.5% compounded semi-annually?

18. Which rate is the better, $(.062, m = 1)$ or $(.06, m = 2)$?

19. Determine the nominal rate which, if converted semi-annually, may be used in place of the rate 5%, compounded quarterly.

20. What nominal rate compounded quarterly could equitably replace the rate $(.04, m = 2)$?

NOTE. — When interest is compounded annually, the nominal and the effective rates are equal because in this case both represent the rate of increase of principal per year. This equality is seen also by placing $m = 1$ in equation 17. Hence, to say that *money is worth the effective rate* 5% is equivalent to saying that *money is worth the nominal rate* 5%, *compounded annually*.

If m, the number of conversion periods per year of a nominal rate, is increased, the corresponding effective rate is also increased. If $j = .06$, the effective rates i for different values of m are:

$m =$	1	2	4	12	52	365
$i =$	.06000	.06090	.06136	.06168	.06180	.06183

We could consider interest converted every day or every moment or every second, or, as a limiting case, converted continuously (m = infinity). The more frequent the conversions, the more just is the interest method from the standpoint of a lender, so that interest, converted continuously, is theoretically the most ideal. The effective rate does *not* increase enormously as we increase the frequency of compounding. When $j = .06$, in the extreme case of continuous conversion (see Section 18, below), $i = .06184$, only slightly in excess of .06183, which is the effective rate when $m = 365$.

11. Interest for fractional parts of a period. — In deriving equation 15 we assumed k to be an integer. Let us agree as a new definition that the compound amount A shall be given by equation 15 also when k is not an integer.

Example 1. — Accumulate \$1000 for 2 years and 2 months, at the rate (.08, $m = 4$).

Solution. — The rate per period is $r = .02$, and $k = 4(2\frac{1}{6}) = 8\frac{2}{3}$.

$$A = 1000(1.02)^{8\frac{2}{3}} = 1000(1.02)^{8}(1.02)^{\frac{2}{3}} = 1000(1.1716594)(1.02)^{\frac{2}{3}}$$

$$\tfrac{2}{3}\log 1.02 = 0.005734.$$
$$\log 1171.66 = 3.068801.$$
$$\log A = 3.074535; \quad A = \$1187.23.$$

Example 2. — Find the present value of \$3500, due at the end of 2 years and 10 months, if money is worth (.07, $m = 2$).

Solution. — The rate per period is $r = .035$, and $k = 2(2\frac{5}{6}) = 5\frac{2}{3}$.

$$P = 3500(1.035)^{-5\frac{2}{3}} = 3500(1.035)^{-6}(1.035)^{\frac{1}{3}}.$$
$$P = 3500(.81350064)(1.01153314) = \$2880.09.^{1} \quad \text{(Tables VI and X)}$$

The methods of Examples 1 and 2 are complicated unless k is a convenient number. Approximate practical methods are described below.

Rule 1. — To find the compound amount after k periods when k is not an integer, (*a*) compute the compound amount after the largest number of whole conversion periods contained in the given time. (*b*) Accumulate this amount for the remaining time at simple interest at the given nominal rate.

Example 3. — Find the amount in Example 1 by use of Rule 1.

Solution. — Compound amount after 2 years is $1000(1.02)^8 = \$1171.66$. Simple interest at the rate 8% for 2 months on \$1171.66 is \$15.62. Amount at end of 2 years and 2 months is $1171.66 + 15.62 = \$1187.28$, slightly greater than the result of Example 1. The use of Rule 1 is always slightly in favor of the creditor.

Rule 2. — To find the present value of A, due at the end of k periods, when k is not an integer, (*a*) discount the amount A for the smallest number of whole periods *containing* the given time. This gives the discounted value of A at a time *before* the present.

[1] If five-place logarithms are used in multiplications or divisions, the results will be accurate to only four significant figures. Hence, in Example 2, if we desire P accurately to cents, ordinary multiplication must be used (unless logarithm tables with seven or more places are available). In performing the ordinary multiplication, as in finding P in Example 2, the student is advised to use the abridged method described in the Appendix, Note 4.

(*b*) Accumulate this result up to the present time at simple interest at the given nominal rate.

Example 4. — Solve Example 2 by use of Rule 2.

Solution. — The smallest number of conversion periods containing 2 years and 10 months is 6 periods, or 3 years. Discounted value of \$3500, 3 years before due, or 2 months before the present, is $3500(1.035)^{-6} = \$2847.25$. Simple interest on \$2847.25 at 7% for 2 months is \$33.22. Present value is $2847.25 + 33.22 = \$2880.47$, which is greater than the result of Example 2. Results computed by use of Rule 2 are always slightly larger than the true present values as found from equation 16.

NOTE. — Unless otherwise directed, use the methods of Examples 1 and 2 when k is not an integer. Compute the time between dates approximately, as in expression 9, of Chapter I, and reduce to years on the basis of 360 days to the year.

EXERCISE XI

Find P or A, whichever is not given. Use Table X whenever possible.

PROB.	PRESENT VALUE, P	AMOUNT, A	P ACCUMULATES FOR, OR A IS DUE AFTER	INTEREST RATE
1.	\$2000		3 years, 3 mo.	.06, $m = 2$
2.		1000	5 years, 1 mo.	.07, $m = 4$
3.	8000		16 years, 8 mo.	.05, $m = 1$
4.	4000		13 years, 7 mo.	.08, $m = 4$
5.		5000	11 years, 5 mo.	.04, $m = 2$
6.	1000		6 years, 4 mo.	.05, $m = 2$
7.		1500	7 years, 10 mo.	.06, $m = 4$
8.	1500		7 years, 10 mo.	.06, $m = 4$

9. Find the amount in problem 1 by use of Rule 1.

10. Find the present value in problem 5 by use of Rule 2.

11. Find the amount in problem 4 by use of Rule 1.

12. Find the present value in problem 7 by use of Rule 2.

13. On June 1, 1920, X borrows \$2000 from Y and agrees to pay the compound amount on whatever date he settles his account. By use of Rule 1, determine what X should pay on August 1, 1922, if interest is at the rate 6%, compounded quarterly? 2175.52

14. At the end of 5 years and 3 months, \$10,000 is due. Discount it to the present time if money is worth (.05, $m = 2$). Use Rule 2.

12. Graphical comparison of simple and compound interest. — In Figure 1 the straight line EF is the graph of the equation

$$A = 1 + .06\,t,$$

the amount after t years if \$1 is invested at simple interest at the rate .06. The curved line EH is the graph of the equation

$$A = (1.06)^t,$$

the amount after t years if \$1 is invested at the rate .06 compounded annually. This curve was sketched through the points corresponding to the following table of values:

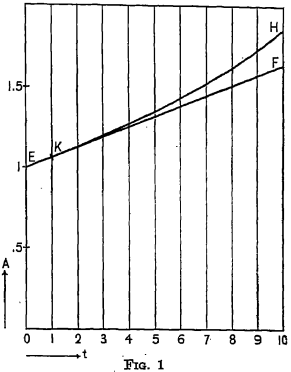

FIG. 1

A	1	1.0147	1.0296	1.06	1.124	1.338	1.791
t	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	5	10

The entries for $t = \frac{1}{2}$ and $t = \frac{1}{4}$ are from Table X. In Figure 2, that part of the curves for which $t = 0$ to $t = 1$ has been magnified (and distorted vertically, for emphasis). Figure 2 shows that, when the time is less than one conversion period, the amount at simple interest is greater than the amount at compound interest. The two amounts are the same when $t = 1$, and, thereafter, the compound amount rapidly grows greater than the amount at simple interest. The ratio, (compound amount) ÷ (amount at simple interest), approaches infinity as t approaches infinity.

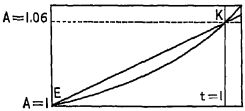

FIG. 2

EXERCISE XII

1. (*a*) Draw graphs on the same coördinate system, of the amount at simple interest, rate 5%, and of the amount at $(.05, m = 1)$, for a principal of \$1, from $t = 0$ to $t = 10$ years. (*b*) Draw a second graph of that part of the curves for which $t = 0$ to $t = 1$ with your original scales magnified 10 times.

13. Values of obligations. — A financial obligation is a promise to pay, or, an obligation is equivalent to a promissory note. Consider the following obligations or notes:

(*a*) *Three years and* 9 *months after date,* X *promises to pay* \$1000 *to* Y *or order.*

(*b*) *Three years and* 9 *months after date,* X *promises to pay* \$1000 *together with all accumulated interest at the rate* 6%, *compounded quarterly, to* Y *or order.*

Example 1. — One year after date of note (*a*), what does Y receive on discounting it with a banker B to whom money is worth $(.05, m = 4)$?

Solution. — B pays the present value of \$1000, due after 2 years and 9 months, or $1000(1.0125)^{-11} = \$872.28$.

Example 2. — One year after date of note (*b*), what is its value to a man W to whom money is worth $(.07, m = 4)$?

Solution. — Maturity value of obligation (*b*) is $1000(1.015)^{15} = \$1250.23$. Its value to W, 2 years and 9 months before due, is $1250.23(1.0175)^{-11} = \$1033.03$.

Under a stipulated rate of interest, the value of an obligation, n years after its maturity date, is the compound amount which would be on hand if the maturity value had been invested for n years at the stipulated interest rate.

Example 3. — Note (*b*) was not paid when due. What should X pay at the end of 5 years to cancel the obligation if money is worth $(.07, m = 4)$ to Y?

Solution. — Maturity value of note is $1000(1.015)^{15} = \$1250.23$. Value at the rate $(.07, m = 4)$ to be paid by X, 1 year and 3 months after maturity date, is $1250.23(1.0175)^{5} = \$1363.52$.

NOTE. — In all succeeding problems in compound interest, reckon elapsed time between dates approximately, as in expression 9 of Chapter I. If it is stated that a sum is *due* on a certain date, the sum is understood to be *due* *without interest.* If a sum is *due with accumulated interest,* this fact will be mentioned explicitly.

EXERCISE XIII

1. If money is worth (.07, $m = 2$) to W, what would he pay to Y for note (*a*), above, 3 months after date of the note?

2. If money is worth (.06, $m = 2$) to W, what should he pay to Y for note (*b*), above, 3 months after date of the note?

3. X borrows $1500 from Y and gives him the following note:

> BOSTON, *July 15, 1922.*
>
> *Three years and* 6 *months after date, I promise to pay to* Y *or order at the First National Bank,* $1500 *together with accumulated interest at the rate* (.07, $m = 2$)
>
> *Value received.* Signed X.

On January 15, 1923, what does Y receive on selling this note to a bank which uses the rate (.06, $m = 2$) in discounting?

4. What would Y receive for the note in problem 3 if he discounted it on July 15, 1922, at a bank using the rate (.055, $m = 2$)?

5. X owes $300, due with accumulated interest at the rate (.04, $m = 4$) at the end of 5 years and 3 months. What is the value of this obligation two years before it is due to a man to whom money is worth (.06, $m = 1$)?

6. At the end of $4\frac{1}{2}$ years, $7000 is due, together with accumulated interest at the rate (.045, $m = 2$). (*a*) Find the value of this obligation $2\frac{1}{2}$ years before it is due if money is worth (.05, $m = 2$). (*b*) What is its value then under the rate (.045, $m = 2$)?

7. On May 15, 1918, $10,000 was borrowed. It was to be repaid on August 15, 1921, with accumulated interest at the rate (.08, $m = 4$). No payment was made until August 15, 1923. What was due then if money was considered worth (.07, $m = 2$) after August 15, 1921?

8. On May 15, 1922, what was the value of the obligation of problem 7 if money was worth (.07, $m = 4$) after August 15, 1921?

9. Find the value of the obligation of problem 7 on November 15, 1923, if money is worth (.05, $m = 4$), commencing on August 15, 1921.

10. X owes Y (*a*) $2000, due in 2 years, and (*b*) $1000, due in $3\frac{1}{2}$ years with accumulated interest at the rate (.05, $m = 2$). At the end of one year what should X pay to cancel the obligations if money is worth (.04, $m = 2$) to Y?

HINT. — X should pay the sum of the values of his obligations.

11. At the end of 3 years and 3 months, \$10,000 is due with accumulated interest at $(.05, m = 4)$. (*a*) What is the value of this obligation at the end of 5 years if money is worth $(.07, m = 4)$? (*b*) What is its value then if money is worth $(.04, m = 4)$?

12. The note of problem 3 is sold by Y on October 15, 1924, to a banker to whom money is worth 6%, effective. By use of Rule 2 of Section 11 find the amount the banker will pay.

The value of an obligation depends on when it is due. Hence, to compare two obligations, due on different dates, **the values of the obligations must be compared on some common date.**

Example 4. — If money is worth $(.05, m = 1)$, which is the more valuable obligation, (*a*) \$1200 due at the end of 2 years, or (*b*) \$1000 due at the end of 4 years with accumulated interest at $(.06, m = 2)$?

Solution. — Compare values at the end of 4 years under the rate $(.05, m = 1)$. The value of (*a*) after 4 years is $1200(1.05)^2 = \$1323.00$. The value of (*b*) after 4 years is $1000(1.03)^8 = \$1266.77$. Hence, (*a*) is the more valuable.

Note. — The value of an obligation on any date, the present for example, is the sum of money which if possessed to-day is as desirable as the payment promised in the obligation. If the present values of two obligations are the same, their values at any future time must likewise be equal, because these future values are the compound amounts of the two equal present values. Similarly, if the present values are equal, the values at any previous date must have been equal, because these former values would be the results obtained on discounting the two equal present values to the previous date. Hence, any comparison date may be used in comparing the values of two obligations, because if their values are equal on one date they are equal on all other dates, both past and future. If the value of one obligation is greater than that of another on one date, it will be the greater on all dates. For instance, in Example 4 above, on comparing values at the end of 3 years, the value of (*a*) is $1200(1.05) = \$1260.00$; the value of (*b*) is $1000(1.03)^8(1.05)^{-1} = \$1206.45$. Hence, as in the original solution, (*a*) is seen to be the more valuable. The comparison date should be selected so as to minimize the computation required. Therefore, the original solution of Example 4 was the most desirable.

EXERCISE XIV

1. If money is worth $(.04, m = 2)$, which obligation is the more valuable: (*a*) \$1400 due after 2 years, or (*b*) \$1500 due after 3 years?

2. If money is worth $(.05, m = 2)$, which obligation is the more valuable: (*a*) \$1400 due after 5 years, or (*b*) \$1000 due after 4 years with accumulated interest at (.07, $m = 2$)? Use 4 years from now as the comparison date.

3. Solve problem 2 with 6 years from now as the comparison date.

4. If money is worth (.06, $m = 2$), compare the value of (*a*) \$5000 due after 4 years with (*b*) an obligation to pay \$4000 after 3 years with accumulated interest at (.05, $m = 1$).

5. Compare the set of obligations (*a*) with set (*b*) if money is worth (.06, $m = 2$):

(*a*) \$1600 due after 3 years; \$1000 due after 2 years with accumulated interest at the rate (.04, $m = 2$).

(*b*) \$1200 due after 2 years; \$1400 due after $2\frac{1}{2}$ years with accumulated interest at the rate (.05, $m = 2$).

6. Which obligation is the more valuable if money is worth (.06, $m = 4$): (*a*) \$8000 due after $3\frac{1}{2}$ years with accumulated interest at (.05, $m = 4$), or (*b*) \$8500 due after 3 years?

14. Equations of value. — An **equation of value** is an equation stating that the sum of the values, on a certain **comparison date,** of one set of obligations equals the sum of the values on this date of another set. Equations of value are the most powerful tools available for solving problems throughout the mathematics of investment.

NOTE. — In writing an equation of value, the comparison date must be explicitly mentioned, and every term in the equation must represent the value of some obligation on this date. To avoid errors, make preliminary lists of the sets of obligations being compared.

Example 1. — W owes Y (*a*) \$1000 due after 10 years, (*b*) \$2000 due after 5 years with accumulated interest at (.05, $m = 2$), and (*c*) \$3000 due after 4 years with accumulated interest at (.04, $m = 1$). W wishes to pay in full by making two equal payments at the ends of the 3d and 4th years. If money is worth (.06, $m = 2$) to Y, find the size of W's payments.

Solution. — Let \$$x$ be the payment. W wishes to replace his old obligations by two new ones. Let 4 years from now be the comparison date.

OLD OBLIGATIONS	NEW OBLIGATIONS
(*a*) \$1000 due in 10 years.	\$$x$ due in 3 years.
(*b*) $2000(1.025)^{10}$ due in 5 years.	\$$x$ due in 4 years.
(*c*) $3000(1.04)^4$ due in 4 years.	

In the following equation of value the left member is the sum of the values of the old obligations on the comparison date. This sum must equal the sum of the values of the new obligations given in the right member.

$$1000(1.03)^{-12} + 2000(1.025)^{10}(1.03)^{-2} + 3000(1.04)^{4} = x(1.03)^{2} + x. \quad (20)$$
$$6624.16 = x(1.0609) + x = 2.0609x.$$
$$x = \$3214.21.$$

If 5 years from the present were used as the comparison date, the equation would be

$$1000(1.03)^{-10} + 2000(1.025)^{10} + 3000(1.04)^{4}(1.03)^{2} = x(1.03)^{4} + x(1.03)^{2}, \quad (21)$$

from which, of course, the same value of x is obtained because equation 21 could be obtained by multiplying both sides of equation 20 by $(1.03)^2$. All obligations were accumulated for one more year in writing equation 21 as compared with equation 20.

EXERCISE XV

Solve each problem by writing an equation of value. List the obligations being compared.

1. W owes Y \$1000 due after 4 years and \$2000 due after 3 years and 3 months. What sum paid now will discharge these debts if money is worth (.08, $m = 4$) to Y?

2. W desires to discharge his obligations in problem 1 by two equal payments made at the ends of 1 year and of 1 year and 6 months, respectively. Find the payments if money is worth (.06, $m = 4$) to Y.

3. W desires to pay his obligations in problem 1 by three equal payments made after 1, 2, and 3 years. Find the payments if money is worth (.06, $m = 4$) to Y.

4. What payment made at the end of 2 years will discharge the following obligations if money is worth (.05, $m = 2$): (*a*) \$10,000 due after 4 years, and (*b*) \$2000 due after $3\frac{1}{2}$ years with accumulated interest at (.07, $m = 2$)?

5. If money is worth (.06, $m = 2$), determine the size of the equal payments which, if made at the ends of the 1st and 2d years, will discharge the obligations of problem 4.

6. What sum, paid at the end of 2 years, will complete payment of the obligations of problem 4 if twice that sum was previously paid at the end of the first year? Money is worth (.08, $m = 2$).

7. W owed Y \$1000 due after 3 years, and \$3000 due with accumulated interest at (.06, $m = 2$) after $4\frac{1}{2}$ years. W paid \$1500 after 2 years. What should he pay at the end of $3\frac{1}{2}$ years to cancel his debts if money is worth 7%, compounded semi-annually, to Y? \$3025.98 ✓

8. A man, owing the obligations (*a*) and (*b*) of problem 4, paid \$8000 at the end of 3 years. What single additional payment should he make at the end of 5 years to cancel his obligations if money is worth $(.04, m = 2)$ to his creditor?

9. Determine whether it would be to the creditor's advantage in problem 8 to stipulate that money is worth $(.05, m = 2)$ to him.

15. Interpolation methods. — The usual problem in compound interest, where the rate or the time is the only unknown quantity, may be solved approximately by interpolating in Table V. The method is the same as that used in finding a number N from a logarithm table when $\log N$ is known.

Example 1. — Find the nominal rate under which \$2350 will accumulate to \$3500 by the end of 4 years and 9 months, if interest is compounded quarterly.

Solution. — Let r be the unknown rate per period. The nominal rate will be $4r$. From equation 15,

$$3500 = 2350(1 + r)^{19}; \qquad (1 + r)^{19} = \frac{3500}{2350} = 1.4894.$$

i	$(1 + i)^{19}$
.02	1.4568
$i = r$	1.4894
.0225	1.5262

The first and third entries in the table are from the row in Table V for $n = 19$. In finding r by interpolation we assume that r is the same proportion of the way from .02 to .0225 as 1.4894 is of the way from 1.4568 to 1.5262. Since $1.5262 - 1.4568 = .0694$, and $1.4894 - 1.4568 = .0326$, then 1.4894 is $\frac{326}{694}$ of the way from 1.4568 to 1.5262. The distance from .02 to .0225 is $.0225 - .02 = .0025$. Hence $r = .02 + \frac{326}{694}(.0025) = .0212$. The nominal rate is $4r = .0848$.

Interest rates per period determined as above are usually in error by not more than[1] $\frac{1}{10}$ of the difference between the table rates used. Thus, the value of r above is probably in error by not more than $\frac{1}{10}(.0025)$ or about .0001. The error happens to be much less, because a solution by exact methods gives $r = .02119$. Results obtained by interpolation should be computed to one more than the number of decimal places which are expected to be accurate.

NOTE. — When interpolating, it is sufficient to use only four decimal places of the entries in Table V. Use of more places does not increase the accuracy of the final results and causes unnecessary computation.

[1] The author gives no theoretical justification for this statement. He has verified its truth for numerous examples distributed over the complete range of Table V.

Example 2. — How long will it take \$5250 to accumulate to \$7375 if invested at $(.06, m = 4)$?

Solution. — Let k be the necessary number of interest periods.

$$7375 = 5250(1.015)^k; \qquad (1.015)^k = \frac{7375}{5250} = 1.4048.$$

n	$(1.015)^n$
22	1.3876
$n = k$	1.4048
23	1.4084

The first and third entries in the table are from Table V. Since $1.4084 - 1.3876 = .0208$, and $1.4048 - 1.3876 = .0172$, then k is $\frac{172}{208}$ of the way from 22 to 23, or $k = 22 + \frac{172}{208} = 22.83$ periods of 3 months. The time is $\frac{22.83}{4} = 5.71$ years. A value of k obtained as above is in error by not more than $\frac{1}{2}$ of the interest rate[1] per period. The error in Example 2 is much less, because an exact solution of the problem gives $k = 22.831$.

Example 3. — X owes Y \$1000 due after 1 year, and \$2000 due after 3 years with accumulated interest at $(.05, m = 2)$. When would the payment of \$4000 balance X's account if money is worth $(.06, m = 4)$ to Y?

Solution. — Let k be the number of conversion periods of the rate $(.06, m = 4)$ between the present and the date when \$4000 should be paid. With the present as a comparison date, the equation of value for the obligations is

$$4000(1.015)^{-k} = 1000(1.015)^{-4} + 2000(1.025)^{6}(1.015)^{-12} = 2882.09.$$
$$(1.015)^{-k} = .72052.$$

From interpolation in Table VI, $k = 22 + \frac{17}{1065} = 22.02$. X should pay \$4000 after $\frac{22.02}{4} = 5.50$ years.

Example 4. — How long will it take for money to double itself if left to accumulate at $(.06, m = 2)$?

Solution. — Let $P = \$1$ and $A = \$2$. If k represents the necessary number of conversion periods, a solution by interpolation gives $k = 23.44$; the time is 11.72 years. Another approximate method is furnished by the following rule.

Rule 1.[2] — To determine the time necessary for money to double itself at compound interest: (*a*) Divide .693 by the rate per period. (*b*) Add .35 to this result. The sum is the time in conversion periods. The error of this approximate result generally is less than a few hundredths of a period.

On solving Example 4 by this rule, $k = \frac{.693}{.03} + .35 = 23.45$.

[1] For justification of this statement see Appendix, Note 5. A knowledge of the calculus is necessary in reading this note.

[2] For a proof of this rule see Appendix, Note 1.

EXERCISE XVI[1]

Solve all problems by interpolation unless otherwise directed. In each problem in the table, find the missing quantity.

Prob.	Amount, A	Principal, P	P Accumulates for, or A is Due after	Nominal Rate	Conversions per Year
1.	$2735	$1500		.05	1
2.	2500	2000		.06	2
3.	2	1	15 years		1
4.	1000	750	3 years, 9 mo.		4
5.	5010	4250		.07	2
6.	6575	4270	7 years, 6 mo.		2
7.	3000	1000		.05	2

8. Find the nominal rate under which $3500 is the present value of $5000, due at the end of $12\frac{1}{2}$ years. Interest is compounded semi-annually.

9. How long will it take for money to quadruple itself if invested at $(.06, m = 2)$?

10. (*a*) At what nominal rate compounded annually will money double in 14 years? (*b*) Solve by use of Rule 1.

11. If money is worth $(.07, m = 1)$, when will the payment of $4000 cancel the obligations (*a*) $2000 due after 3 years, and (*b*) $2000 due after 7 years?

12. If money is worth $(.05, m = 2)$, when will the payment of $3000 cancel the obligations (*a*) $1500 due after 3 years, and (*b*) $1000 due at the end of $2\frac{1}{2}$ years with accumulated interest at $(.06, m = 4)$?

13. By use of Rule 1, determine how long it takes for money to double itself under each of the following rates: (*a*) $(.06, m = 4)$; (*b*) $(.04, m = 2)$; (*c*) $(.05, m = 2)$; (*d*) $(.03, m = 1)$.

14. By use of the results of problem 13, determine how long it takes for money to quadruple itself under each of the four rates in problem 13.

15. If money is worth $(.04, m = 2)$, when will the payment of $3500 cancel the liabilities (*a*) $1000 due after 18 months, and (*b*) $2000 due after $2\frac{1}{2}$ years?

[1] The Miscellaneous Problems at the end of the chapter may be taken up immediately after the completion of Exercise XVI.

SUPPLEMENTARY MATERIAL

16. Logarithmic methods. — Problems may arise to which the tables at hand do not apply, or in which more accuracy is desired than is obtainable by interpolation methods. Logarithmic methods are available in such cases.

Example 1. — Find the present value of \$350.75, due at the end of 6 years and 6 months, if interest is at the rate (.0374, $m = 2$).

Solution. — $P = 350.75(1.0187)^{-13} = \dfrac{350.75}{(1.0187)^{13}}$.

$$\begin{aligned} \log 350.75 &= 2.54500 \\ 13 \log 1.0187 = 13(.0080463) &= 0.10460 \quad \text{(Using Table II)} \\ \text{(subtract) } \log P &= 2.44040. \quad P = \$275.68. \end{aligned}$$

If Table I were used in obtaining $\log (1.0187)^{13}$, $13 \log 1.0187 = 13(.00804) = 0.10452$, in error by .00008.

Example 2. — If interest is converted quarterly, find the nominal rate under which \$2350 is the present value of \$2750, due after 4 years and 9 months.

Solution. — Let r be the unknown rate per period; the nominal rate is $4r$.

$$2750 = 2350(1+r)^{19}; \quad 1 + r = \left(\frac{2750}{2350}\right)^{\frac{1}{19}}.$$

$$\begin{aligned} \log 2750 &= 2.43933 \\ \log 2350 &= 2.37107 \\ \text{log quot.} &= 0.06826. \quad \tfrac{1}{19} \text{ log quot.} = 0.00359. \end{aligned}$$

$\therefore 1 + r = 1.0083$, $r = .0083$. The nominal rate is $4r = .0332$, converted quarterly.

Example 3. — How long will it take for \$3500 to accumulate to \$4708 if interest is at the rate (.08, $m = 4$)?

Solution. — Let k be the necessary number of conversion periods.

$$4708 = 3500(1.02)^k; \quad (1.02)^k = \frac{4708}{3500}. \quad \therefore k \log 1.02 = \log \frac{4708}{3500}.$$

$$\begin{aligned} \log 4708 &= 3.67284 & \log 1.02 &= 0.0086002 \\ \log 3500 &= 3.54407 & & \\ \text{log quot.} &= 0.12877. & \therefore k(0.0086002) &= .12877. \\ k &= \frac{12877}{860.02}. & \log 12877 &= 4.10982 \\ & & \log 860.02 &= 2.93451 \\ & & \text{(subtract) } \log k &= 1.17531 \end{aligned}$$

The time is $k = 14.973$ periods of 3 months, or 3.743 years.

EXERCISE XVII

Use exact logarithmic methods in all problems on this page. Use Table II whenever advisable.

1. Find the compound amount after 3 years and 3 months, if \$3500 is invested at the rate (.063, $m = 4$).

2. Find the present value of \$3500 which is due at the end of 8 years and 6 months, if money is worth (.078, $m = 2$).

3. At what nominal rate, converted quarterly, is \$5000 the present value of \$7300, due at the end of 2 years and 9 months?

4. Find the length of time necessary for a principal of \$2000 to accumulate to \$3600, if interest is at the rate (.05, $m = 1$).

5. Solve problems 2 and 5 of Exercise XVI by exact methods.

6. Solve problems 3 and 4 of Exercise XVI by exact methods.

7. Find the nominal rate which, if converted semi-annually, yields the effective rate .0725.

8. Find the nominal rate which, if converted semi-annually, is equivalent to the rate .068, compounded quarterly.

9. (*a*) Determine how long it will take for money to double itself at the rate (.06, $m = 1$). (*b*) Compare your answer with the result you obtain on using Rule 1 of Section 15.

10. One dollar is allowed to accumulate at (.03, $m = 2$). A second dollar accumulates at (.06, $m = 1$). When will the compound amount on the second dollar be three times that on the first?

HINT. — Take the logarithm of both sides of the equation obtained.

17. The equated time. — The **equated date** for a set of obligations is the date on which they could be discharged by a single payment equal to the sum of the maturity values of the obligations. The time between the present and the equated date is called the **equated time**, and it is found by solving an equation of value.

Example 1. — If money is worth (.05, $m = 2$), find the equated time for the payment of the obligations (*a*) \$2000 due after 3 years, and (*b*) \$1000 due after 2 years with accumulated interest at the rate (.04, $m = 2$).

Solution. — The sum of the maturity values of (*a*) and (*b*) is $2000 + 1000(1.02)^4 = \$3082.43$. Let the equated time be k conversion periods of the rate (.05, $m = 2$). The value of the obligation \$3082.43, due after k periods (on the equated date), must be equal to the sum of the values of the given obligations. With the present as the comparison date, the corresponding equation of value is

$$3082.4(1.025)^{-k} = 2000(1.025)^{-6} + 1000(1.02)^{4}(1.025)^{-4} = 2705.2.$$

$$(1.025)^{k} = \frac{3082.4}{2705.2} = 1.1394.$$

By the method of Section 16, $k = 5.286$ six-month periods or, the equated time is 2.643 years. By the interpolation method, $k = 5.28$. The present was used as the comparison date above to avoid having k appear on both sides of the equation.

To obtain the equated time approximately, the following rule is usually used.

Rule 1.[1] — Multiply the maturity value of each obligation by the time in years (or months, or days) to elapse before it is due. Add these products and divide by the sum of the maturity values to obtain the equated time.

On using this rule in Example 1 above, we obtain

$$\text{equated time} = \frac{3(2000) + 2(1082.4)}{3082.4} = 2.65 \text{ years.}$$

Rule 1 is always used in finding the equated date for short-term commercial accounts. The equated date for an account is also called the **average date** and the process of finding the average date is called **averaging the account.** Since Rule 1 does not involve the interest rate, it is unnecessary to state the rate when asking for the equated date for an account.

NOTE. — Results obtained by use of Rule 1 are always a little too large, so that a debtor is favored by its use. The accuracy of the rule is greater when the interest rate is low than when it is high. The accuracy is greater for short-term than for long-term obligations.

EXERCISE XVIII

1. If money is worth (.05, $m = 1$), find the equated time for the payment of (*a*) \$1000 due after 3 years, and (*b*) \$2000 due after 4 years. Solve by Rule 1.

2. Solve problem 1 by the exact method of Example 1 above.

[1] For derivation of the rule see Appendix, Note 2.

3. (I) If money is worth (.07, $m = 2$), find the equated time for the payment of (*a*) \$1000 due after 3 years and (*b*) \$2000 due after 4 years with accumulated interest at (.05, $m = 2$). Solve by the exact method. (II) Solve by Rule 1.

4. Find the equated time for an account requiring the payment of \$55 after 3 months, \$170 after 9 months, and \$135 after 7 months. Use Rule 1. $7\frac{1}{3}$

5. (*a*) A man owes four 180-day, non-interest-bearing notes dated as follows: March 9, for \$400; May 24, for \$250; August 13, for \$525; August 30, for \$500. By use of Rule 1 find the equated time and the equated date for the payment of the notes, considering for convenience that March 9 is the present. (*b*) How much must be paid on the equated date to cancel these obligations, if no other payment is made?

6. If money is worth 6%, simple interest, what should be paid 30 days after the equated date in problem 5 in order to balance the account, if no other payment is made?

18. Continuously convertible interest; the force of interest.[1] — The compound amount on \$1 at the end of one year, if interest is at the nominal rate j, converted m times per year, is $A = \left(1 + \frac{j}{m}\right)^m$. It was seen at the end of Section 10 that, as m increases, the amount A increases. As m increases without bound, or in other words, as m *approaches infinity*, the amount A does *not* increase without bound but approaches a limiting value e^j, where $e = 2.7182818^+$ is the base of the Naperian, or natural, system of logarithms. To prove this we use the theory of limits.

$$\lim_{m=\infty} A = \lim_{m=\infty}\left(1 + \frac{j}{m}\right)^m = \lim_{m=\infty}\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{j}}\right]^j = \left[\lim_{m=\infty}\left(1 + \frac{j}{m}\right)^{\frac{m}{j}}\right]^j.$$

It is known that[2] $\lim_{m=\infty}\left(1 + \frac{j}{m}\right)^{\frac{m}{j}} = e.$ Therefore,

$$\lim_{m=\infty}\left(1 + \frac{j}{m}\right)^m = e^j. \qquad (22)$$

It is customary to say that this limiting value e^j is the compound amount on \$1 at the end of one year in the ideal case where

[1] A knowledge of the theory of limits is advisable in reading this section.
[2] See Granville's *Calculus*, Revised Edition, page 22.

interest is **converted continuously**. For every value of m, the effective rate i corresponding to the nominal rate j is given by $i = \left[\left(1 + \frac{j}{m}\right)^m - 1\right]$. Hence, in the limiting case where interest is converted continuously, it follows from equation 22 that

$$i = \lim_{m=\infty}\left(1 + \frac{j}{m}\right)^m - 1 = e^j - 1.$$

$$1 + i = e^j. \qquad (23)$$

Example 1. — Find the effective rate if the nominal rate is .05, converted continuously.

Solution. — $1 + i = e^{.05}$. $\log(1 + i) = .05 \log e$, where $e = 2.71828$.

$$.05 \log e = .05(0.43429) = 0.02171 = \log(1 + i).$$

$$\therefore 1 + i = 1.0513;\ i = .0513.$$

The **force of interest**, corresponding to a given effective rate i, is the nominal rate which, if converted *continuously*, will yield the effective rate i. Hence, if δ represents the force of interest, the value $j = \delta$ must satisfy equation 23, or

$$1 + i = e^{\delta}. \qquad (24)$$

Example 2. — Find the force of interest if the effective rate is .06.

Solution. — $1.06 = e^{\delta}$. $\therefore \delta \log e = \log 1.06$.

$$\delta = \frac{\log 1.06}{\log e} = \frac{.0253059}{.43429} = .058269.$$

Under the effective rate i, the compound amount of a principal P at the end of n years is $A = P(1 + i)^n$. If the nominal rate is j, converted continuously, $(1 + i) = e^j$; hence $A = P(e^j)^n$, or

$$A = Pe^{nj}.$$

To compute A we use logarithms; $\log A = \log P + nj \log e$.

EXERCISE XIX

1. Find the effective rate if the nominal rate is .06, converted continuously.

2. Find the force of interest if the effective rate is .05. 4.879%

3. (*a*) Find the amount after 20 years if \$2000 is invested at the rate .07, converted continuously. (*b*) Compare your answer with the compound amount in case the rate is (.07, $m = 4$).

MISCELLANEOUS PROBLEMS

1. A man, in buying a house, is offered the option of paying \$1000 cash and \$1000 annually for the next 4 years, or \$650 cash and \$1100 annually for the next 4 years. If money is worth (.06, $m = 1$), which method is the better from the purchaser's standpoint?

2. A merchant desires to obtain \$6000 from his banker. (*a*) If the loan is to be for 90 days and if the banker charges 6% interest in advance, for what sum will the merchant make out the note which he will give to the banker? (*b*) What simple interest rate is the man paying?

3. A merchant who originally invested \$6000 has \$8000 capital at the end of 6 years. What has been the annual rate of growth of his capital if the rate is assumed to have been uniform through the 6 years?

4. If gasoline consumption is to increase at the rate of 5% per year, when will the consumption be double what it is now? Solve by two methods.

5. When will the payment of \$5000 cancel the obligations \$2000 due after 3 years, and \$2500 due after 6 years? Money is worth (.05, $m = 2$).

6. A certain life insurance company lends money to policy holders at 6% interest, payable in advance, and allows repayment of all or part of the loan at any time. Six months before the maturity of a \$2000 loan, the policy holder A sends a check for \$800 to apply on his loan. What additional sum will A pay at maturity?

HINT. — First find the sum, due in 6 months, of which \$800 is the present value.

7. A must pay B \$2000 after 2 years, and \$1000 after 3 years and 6 months. At the end of 1 year A paid B \$1500. If money is worth (.05, $m = 2$), what additional equal payments at the ends of 2 years and 6 months and of 3 years will cancel A's liability?

8. (*a*) If you were a creditor, would you specify that money is worth a high or a low rate of interest to you, if one of your debtors desired to pay the value of an obligation on a date before it is due? Justify your answer in one sentence. (*b*) If a debtor desires to discharge an obligation by making payment on a date after it was due, what rate, high or low, should the creditor specify as the worth of money?

9. At the end of 4 years and 7 months, \$3000 is due. Find its present value by the practical rule if money is worth (.08, $m = 4$).

10. A man has his money invested in bonds which yield 5%, payable semi-annually. If he desires to reinvest his money, what is the lowest rate, payable quarterly, which his new securities should yield?

11. One dollar is invested at simple interest, rate 5%. A second dollar is invested at (.05, $m = 1$). When will the compound amount on the second dollar be double the simple interest amount on the first?

HINT. — Solve the equation by interpolation; see illustrative Example 1 in Appendix, Note 3.

12. After how long a time will the compound amount on $1 at the rate (.06, $m = 2$) be double the amount on a second $1 at the rate (.035, $m = 1$)?

HINT. — Use either interpolation or logarithmic methods. See problem **11.**

13. If $100 is invested now, what will be the compound amount after 20 years if the effective rate of interest for the first 5 years will be 6%, whereas interest will be at the rate (.04, $m = 2$) for the last 15 years?

14. A man owes $2000, due at the end of 10 years. Find its present value if it is assumed money will be worth 4% effective, for the first 5 years, and 6% effective, for the last 5 years.

15. If $100 is due at the end of 5 years, discount it to the present time, (*a*) under the rate 5%, compounded annually; (*b*) under the simple interest rate 5%; (*c*) under the simple discount rate 5%.

CHAPTER III

ANNUITIES CERTAIN

19. Definitions. — An **annuity** is a sequence of periodic payments. An **annuity certain** is one whose payments extend over a fixed term of years. For instance, the monthly payments made in purchasing a house on the instalment plan, form an annuity certain. A **contingent annuity** is one whose payments last for a period of time which depends on events whose dates of occurrence cannot be accurately foretold. For instance, a sequence of payments (such as the premiums on an insurance policy) which ends at the death of some individual form a contingent annuity. In Part I of this book we consider only annuities certain.

The sum of the payments of an annuity made in one year is called the **annual rent.** The time between successive payment dates is the **payment interval.** The time between the beginning of the first payment interval and the end of the last, is called the **term** of the annuity. Unless otherwise stated, *all payments of an annuity are equal, and they are due at the ends of the payment intervals;* the first payment is due at the end of the first interval, and the last is due at the end of the term. Thus, for an annuity of $50 per month for 15 years, the payment interval is 1 month, the annual rent is $600, and the term is 15 years; the first payment is due after 1 month, and the last, after 15 years.

Under a specified rate of interest, the **present value** of an annuity is the sum of the present values of all payments of the annuity. The **amount** of an annuity is the sum of the compound amounts that would be on hand at the end of the term if all payments should accumulate at interest until then from the dates on which they are due.

NOTE 1. — Consider an annuity of $100, payable annually for 5 years, with interest at the rate 4%, effective. We obtain the present value A of this annuity by adding the 2d column in the table below, and the amount S by adding the 3d column.

Payment of $100 Due at End of	Present Value of Payment	Compound Amount at End of Term if Payment is Left to Accumulate at Interest
1 year	$100(1.04)^{-1}$ = 96.15385	$100(1.04)^4$ = 116.98586
2 years	$100(1.04)^{-2}$ = 92.45562	$100(1.04)^3$ = 112.48640
3 years	$100(1.04)^{-3}$ = 88.89964	$100(1.04)^2$ = 108.16000
4 years	$100(1.04)^{-4}$ = 85.48042	$100(1.04)$ = 104.00000
5 years	$100(1.04)^{-5}$ = 82.19271	100 = 100.00000
	(add) A = $445.18224	(add) S = $541.63226

The present value $A = \$445.18$ is as desirable as the future possession of all payments of the annuity. The amount $S = \$541.63$, possessed at the end of 5 years, is as desirable as all of the payments. Hence, A should be the present value of S, due at the end of the term, or we should have $S = A(1.04)^5$. This relation is verified to hold;

$$A(1.04)^5 = 445.182 \times 1.21665290 = \$541.632 = S. \tag{25}$$

Note 2. — In the table below it is verified that, if a fund is formed by investing $445.182 at 4% effective, this fund will provide for all payments of the annuity of Note 1 and, in so doing, will become exactly exhausted at the time of the last payment. This result can be foreseen theoretically because $445.182 is the sum of the present values of all of the payments.

Year	In Fund at Beginning of Year	Int. at 4% Due at End of Year	In Fund at End of Year before Payment is Made	Payment at End of Year
1	$445.182	$17.807	$462.989	$100
2	362.989	14.520	377.509	100
3	277.509	11.100	288.609	100
4	188.609	7.544	196.153	100
5	96.153	3.846	99.999	100

EXERCISE XX

1. (*a*) Form a table as in Note 1 above in order to find the present value and the amount of an annuity which pays $1000 at the end of each 6 months for 3 years. Money is worth 6%, compounded semi-annually. (*b*) Verify as in equation 25, that A is the present value of S, due at the end of the term. (*c*) Form a table as in Note 2, to verify that the present value A, if invested at $(.06, m = 2)$, creates a fund exactly sufficient to provide the payments of the annuity.

20. The examples below[1] illustrate methods used later to obtain fundamental annuity formulas.

Example 1. — If money is worth (.06, $m = 4$), find the present value A and the amount S of an annuity whose annual rent is $200, payable semi-annually for 15 years.

Solution. — Each payment is $100. The entries in the 2d and 4th columns below are verified by the principles of compound interest.

Payment of $100 Due at the End of	Present Value of Payment	Time from Date of Payment to End of Term	Comp. Amt. at End of Term if Payt. is Left to Accumulate at Int.
6 months	$100(1.015)^{-2}$	14 yr., 6 mo.	$100(1.015)^{58}$
1 year	$100(1.015)^{-4}$	14 years	$100(1.015)^{56}$
etc.	etc.	etc.	etc.
14 yr., 6 mo.	$100(1.015)^{-58}$	6 months	$100(1.015)^{2}$
15 years	$100(1.015)^{-60}$	0 months	100
	Sum $= A$		Sum $= S$

Hence, $S = 100[1 + (1.015)^2 + \cdots \text{etc.} \cdots + (1.015)^{56} + (1.015)^{58}]$.

The bracket contains a geometrical progression of 30 terms for which the ratio is $w = (1.015)^2$, the first term $a = 1$, and the last term $L = (1.015)^{58}$. By the formula for the sum of a geometrical progression,[2]

$$S = 100\frac{wL - a}{w - 1} = 100\frac{(1.015)^{60} - 1}{(1.015)^2 - 1} = 100\frac{2.44321978 - 1}{1.03022500 - 1} = \$4774.918.$$

On adding the 2d column in the table we obtain the present value

$$A = 100[(1.015)^{-60} + (1.015)^{-58} + \cdots \text{etc.} \cdots + (1.015)^{-4} + (1.015)^{-2}].$$

The geometrical progression in the bracket has the ratio $w = (1.015)^2$, while $a = (1.015)^{-60}$ and $L = (1.015)^{-2}$. Since $wL = 1$,

$$A = 100\frac{1 - (1.015)^{-60}}{(1.015)^2 - 1} = 100\frac{1 - .40929597}{1.03022500 - 1} = \$1954.356.$$

The present value of $4774.918, due at the end of 15 years, should equal A, or $S = A(1.015)^{60}$. We verify that

$$A(1.015)^{60} = (1954.356)(2.44321978) = \$4774.920. \qquad (26)$$

[1] Geometrical Progressions in Part III, Chapter XII, should be studied if the student has not met them previously. Section 20 may be omitted without disturbing the continuity of the succeeding sections, but geometrical progressions are needed in Sections 21 and 22.

[2] See Part III, Section 90.

Example 2. — Find the present value A of an annuity of \$100 pe month for 3 years and 6 months, if money is worth $(.05, m = 2)$.

Solution. — A is the sum of the entries in the 2d row below.

Payment of \$100 due after	1 month	2 months	etc.	3 yr., 5 mo.	3 yr., 6 mo.
Present value of payment	$100(1.025)^{-\frac{1}{6}}$	$100(1.025)^{-\frac{2}{6}}$	etc.	$100(1.025)^{-\frac{41}{6}}$	$100(1.025)^{-\frac{4}{6}}$

$$A = 100\,[(1.025)^{-\frac{42}{6}} + (1.025)^{-\frac{41}{6}} + \cdots \text{ etc.} \cdots + (1.025)^{-\frac{2}{6}} + (1.025)^{-\frac{1}{6}}].$$

The ratio of the geometrical progression is $w = (1.025)^{\frac{1}{6}}$; $a = (1.025)^{-7}$ an $L = (1.025)^{-\frac{1}{6}}$. Since $wL - a = 1 - (1.025)^{-7}$,

$$A = 100\,\frac{1 - (1.025)^{-7}}{(1.025)^{\frac{1}{6}} - 1} = 100\,\frac{1 - .84126524}{1.00412392 - 1} = \$3849.14. \text{ (Tables VI and X}$$

EXERCISE XXI

In each problem derive formulas for A and S for the annuity describe using the method of Examples 1 and 2 above.

1. An annuity whose annual rent is \$200, payable quarterly for 1 years. Money is worth $(.08, m = 4)$. Compute the formulas for A and and verify as in equation 26 that A is the present value of S, due at th end of the term. $A = 50 \times \frac{1-(1.02)^{-48}}{.02} = \$1533.66$, $S = 50 \times$

2. Fifteen successive annual payments of \$1000, the first due aft 1 year. Money is worth $(.05, m = 2)$. Compute A and S and veri that A is the present value of S, due at the end of the term.

3. Payments of \$100, made at the end of each 3 months for 15 year Money is worth $(.05, m = 4)$.

4. (*a*) The annual rent of the annuity is \$2000, the payment interv is 3 months, and the term is $12\frac{1}{2}$ years. Money is worth $(.06, m = 1$ (*b*) Solve the problem if money is worth $(.06, m = 2)$.

5. An annuity which pays \$100 at the end of each interest period f 10 interest periods. Money is worth .045, per interest period.

21. Formulas for A and S in the most simple case. — Co sider the annuity paying \$1 at the end of each year for n year Let $(a_{\overline{n}|}\ at\ i)$ be the present value, and $(s_{\overline{n}|}\ at\ i)$ be the amount this annuity when money is worth the rate i compounded annuall The entries in the table below are easily verified.

Payment of \$1 Due at the End of	Present Value of the Payment	Time from Date of Payment to End of Term	Comp. Amt. at End of Term if Payt. is Left to Accumulate at Int.
1 year	$(1+i)^{-1}$	$(n-1)$ yr.	$(1+i)^{n-1}$
2 years	$(1+i)^{-2}$	$(n-2)$ yr.	$(1+i)^{n-2}$
etc.	etc.	etc.	etc.
$(n-1)$ yr.	$(1+i)^{-(n-1)}$	1 year	$(1+i)$
n years	$(1+i)^{-n}$	0 years	1
	Sum $= (a_{\overline{n}\vert}\, at\ i)$		Sum $= (s_{\overline{n}\vert}\, at\ i)$

Hence,

$$(s_{\overline{n}|}\ at\ i) = 1 + (1+i) + \cdots \text{etc.} \cdots + (1+i)^{n-2} + (1+i)^{n-1}.$$

This is a geometrical progression where the ratio $w = (1+i)$, the first term $a = 1$, and the last term $L = (1+i)^{n-1}$. Since $(wL - a) = (1+i)^n - 1$, and $(w-1) = i$, the formula $\dfrac{wL - a}{w - 1}$ gives

$$(s_{\overline{n}|}\ at\ i) = \frac{(1+i)^n - 1}{i}. \qquad (27)$$

On adding the 2d column of the table we obtain

$$(a_{\overline{n}|}\ at\ i) = (1+i)^{-n} + (1+i)^{-n+1} + \cdots \text{etc.} \cdots + (1+i)^{-2} + (1+i)^{-1},$$

which is a geometrical progression with the ratio $w = (1+i)$, $a = (1+i)^{-n}$, and $L = (1+i)^{-1}$. Since

$$(wL - a) = [(1+i)(1+i)^{-1} - (1+i)^{-n}], \text{ and } (w-1) = i,$$

$$(a_{\overline{n}|}\ at\ i) = \frac{1 - (1+i)^{-n}}{i}. \qquad (28)$$

If each payment of the annuity had been $\$R$ instead of \$1, the present value A and the amount S would have been $A = R(a_{\overline{n}|}\ at\ i)$ and $S = R(s_{\overline{n}|}\ at\ i)$.

It is important to realize that formulas 27 and 28 may be used **whenever the payment interval of the annuity equals the conversion period of the interest rate.** In deriving the formulas, the interest period was called 1 year, merely for concreteness. Hence, if i is the interest rate per period, then $R(s_{\overline{n}|}\ at\ i)$ represents the amount and $R(a_{\overline{n}|}\ at\ i)$ the present value of an annuity which pays $\$R$ at the end of each interest period for n periods. Thus,

$100(a_{\overline{18}|}$ *at* .025) is the present value of an annuity paying \$100 at the end of each interest period for 18 periods if money is worth the rate .025 per period.

Example 1. — Find the amount and the present value of an annuity paying \$150 at the end of each 3 months for 15 years and 6 months, if money is worth 6%, compounded quarterly.

Solution. — Since the payment interval equals the interest period, formulas 27 and 28 apply with the number of payments $n = 62$, and with $i = .015$.

$$\text{Amount} = 150(s_{\overline{62}|} \textit{ at } .015) = 150(101.13773956) = \$15170.66.$$
$$\text{Pr. val.} = 150(a_{\overline{62}|} \textit{ at } .015) = 150(\ 40.18080408) = \$6027.12.$$

The value of $s_{\overline{62}|}$ is from Table VII and that of $a_{\overline{62}|}$ is from Table VIII.

NOTE. — Recognize that the solution above makes no use of the explicit expressions for $a_{\overline{62}|}$ and $s_{\overline{62}|}$ because their values are tabulated. The use of the explicit formulas for $a_{\overline{n}|}$ or $s_{\overline{n}|}$ in such a case would be a complicated, and therefore an incorrect method.

EXERCISE XXII

1. (*a*) In Table VII verify the entry for $(s_{\overline{37}|} \textit{ at } .02) = \dfrac{(1.02)^{37} - 1}{.02}$ by use of Table V. (*b*) Verify the entry for $(a_{\overline{38}|} \textit{ at } .04)$ in Table VIII by use of Table VI.

2. Find the present value and the amount of an annuity which pays \$500 at the end of each year for 20 years, if money is worth $(.05, m = 1)$.

3. If money is worth $(.05, m = 2)$, find the present value and the amount of an annuity whose annual rent is \$240, payable semi-annually for 13 years and 6 months.

Find the present values and the amounts of the annuities below.

PROB.	EACH PAYMENT	PAYMENT INTERVAL	TERM	ANNUAL RENT	INTEREST RATE
4.	\$ 50	3 mo.	14 yr., 9 mo.		.06, $m = 4$
5.	10,000	1 yr.	18 yr.		.055, $m = 1$
6.	500	6 mo.	19 yr., 6 mo.		.07, $m = 2$
7.		6 mo.	15 yr.	\$1000	.055, $m = 2$
8.	300	1 yr.	25 yr.		.04, $m = 1$
9.		6 mo.	23 yr.	2000	.03, $m = 2$
10.		1 mo.	7 yr.	2400	.06, $m = 12$

√(11) In purchasing a house a man agrees to pay \$1000 cash and \$1000 at the end of each 6 months for the next 6 years. If money is worth (.07, $m = 2$), what would be an equivalent cash valuation for the house?

HINT. — The cash price is the sum of the present values of all payments. The present value of the first payment is \$1000. The remaining 12 payments come at the ends of the payment intervals and hence form a standard annuity whose present value is $1000(a_{\overline{12}|}$ *at* .035).

12. The man of problem 11 has just paid the installment due at the end of 4 years and 6 months. What additional payment, if made immediately, would cancel his remaining indebtedness if money is worth (.08, $m = 2$)?

HINT. — His remaining indebtedness at any time, or the principal outstanding, is the present value of all remaining payments.

13. If you deposit \$50 at the end of each 3 months in a savings bank which pays interest quarterly at the rate 3%, how much will be to your credit after 20 years and 6 months, if you make no withdrawals?

14. A man in buying a house has agreed to pay \$1000 at the beginning of each 6 months until 29 installments have been paid. If money is worth 6%, compounded semi-annually, what is an equivalent cash price for the house?

15. At the end of each year a corporation places \$5000 in a depreciation fund which is to provide for plant replacement at the end of 12 years. (*a*) What sum will be in the fund at the end of 12 years if it accumulates at the effective rate 7%? (*b*) What sum is in the fund at the beginning of the seventh year?

16. A man desires to deposit with a trust company a sufficient sum to provide his family with \$500 at the end of each 3 months for the next 15 years. If the trust company credits interest at the rate 6%, quarterly, on all funds, what should the man deposit?

HINT. — See the table of Note 2 of Section 19.

22. Further annuity formulas. — Consider the annuity whose annual rent is \$1, payable p times per year for n years. Each of the np payments is $\frac{\$1}{p}$; the first is due at the end of $\frac{1}{p}$ years, and the others are due at intervals of $\frac{1}{p}$ years for the rest of the term. If money is worth the rate i, compounded annually, let $(s_{\overline{n}|}^{(p)}$ *at* $i)$ represent the amount of the annuity and $(a_{\overline{n}|}^{(p)}$ *at* $i)$ its present value. To derive formulas for $a_{\overline{n}|}^{(p)}$ and $s_{\overline{n}|}^{(p)}$ we form the table below.

PAYMENT OF $\frac{\$1}{p}$ DUE AT THE END OF	PRESENT VALUE OF THE PAYMENT	TIME FROM DATE OF PAYMENT TO END OF TERM	COMP. AMT. AT END OF TERM IF PAYT. IS LEFT TO ACCUMULATE AT INTEREST
$\frac{1}{p}$ years	$\frac{\$1}{p}(1+i)^{-\frac{1}{p}}$	$\left(n-\frac{1}{p}\right)$ yr.	$\frac{\$1}{p}(1+i)^{n-\frac{1}{p}}$
$\frac{2}{p}$ years	$\frac{\$1}{p}(1+i)^{-\frac{2}{p}}$	$\left(n-\frac{2}{p}\right)$ yr.	$\frac{\$1}{p}(1+i)^{n-\frac{2}{p}}$
etc.	etc.	etc.	etc.
$\left(n-\frac{1}{p}\right)$ yr.	$\frac{\$1}{p}(1+i)^{-\left(n-\frac{1}{p}\right)}$	$\frac{1}{p}$ years	$\frac{\$1}{p}(1+i)^{\frac{1}{p}}$
n years	$\frac{\$1}{p}(1+i)^{-n}$	0 years	$\frac{\$1}{p}$

Hence, on adding the fourth column we obtain

$$(s_{\overline{n}|}^{(p)} \text{ at } i) = \frac{\$1}{p}[1 + (1+i)^{\frac{1}{p}} + \cdots \text{etc.} \cdots + (1+i)^{n-\frac{2}{p}} + (1+i)^{n-\frac{1}{p}}].$$

The progression in the bracket has the ratio $w = (1+i)^{\frac{1}{p}}$, the first term $a = 1$, and the last term $L = (1+i)^{n-\frac{1}{p}}$. Since $wL - a = (1+i)^n - 1$, and $w - 1 = (1+i)^{\frac{1}{p}} - 1$,

$$(s_{\overline{n}|}^{(p)} \text{ at } i) = \frac{1}{p}\left(\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1}\right) = \frac{(1+i)^n - 1}{p[(1+i)^{\frac{1}{p}} - 1]}. \qquad (29)$$

The denominator of the last fraction is the expression we have previously called[1] (j_p *at* i). On multiplying numerator and denominator of the last fraction by i, we obtain

$$(s_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i}{i}\,\frac{(1+i)^n - 1}{j_p} = \frac{i}{j_p}\,\frac{(1+i)^n - 1}{i}.$$

Since, by formula 27, the last fraction is ($s_{\overline{n}|}$ *at* i),

$$(s_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i}{j_p}(s_{\overline{n}|} \text{ at } i). \qquad (30)$$

[1] See Exercise X, Problem 16. Also see heading of Table XI. The fact that (j_p *at* i) is the nominal rate which, if converted p times per year, yields the effective rate i, is of importance in the applications of equation 29. We use j_p merely as a convenient abbreviation for its complicated algebraic expression.

From the second column of the table we obtain

$$(a_{\overline{n}|}^{(p)} \text{ at } i) = \frac{\$1}{p}[(1+i)^{-n} + (1+i)^{-n+\frac{1}{p}} + \cdots \text{etc.} \cdots + (1+i)^{-\frac{1}{p}}].$$

The ratio of the geometrical progression in the bracket is $w = (1+i)^{\frac{1}{p}}$, the first term $a = (1+i)^{-n}$, and $L = (1+i)^{-\frac{1}{p}}$. Since $wL - a = 1 - (1+i)^{-n}$,

$$(a_{\overline{n}|}^{(p)} \text{ at } i) = \frac{1}{p}\left(\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1}\right) = \frac{1-(1+i)^{-n}}{p[(1+i)^{\frac{1}{p}} - 1]}. \tag{31}$$

From this expression we derive, as in formula 30,

$$(a_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i}{j_p}(a_{\overline{n}|} \text{ at } i). \tag{32}$$

If the sum of the payments made in 1 year, or the annual rent, had been $\$R$ instead of \$1, the present value of the annuity would have been $R(a_{\overline{n}|}^{(p)} \text{ at } i)$ and the amount, $R(s_{\overline{n}|}^{(p)} \text{ at } i)$.

In the discussion above, money was worth the rate i, compounded once per year, while the annuity was payable p times per year for n years and the sum of the payments made in 1 year was \$1. The word *year* was used in this statement and in the proof of formulas 29 to 32 for the sake of concreteness. All of the reasoning remains valid if the word *year* is changed throughout to *interest period.* Thus, when money is worth the rate i, **per interest period, if an annuity is payable p times per interest period for a term of n interest periods, and if the sum of the payments made in one interest period is $\$R$,** the present value A and the amount S are given by

$$\begin{aligned} A &= R(a_{\overline{n}|}^{(p)} \text{ at } i) = R\frac{i}{j_p}(a_{\overline{n}|} \text{ at } i); \\ S &= R(s_{\overline{n}|}^{(p)} \text{ at } i) = R\frac{i}{j_p}(s_{\overline{n}|} \text{ at } i). \end{aligned} \tag{33}$$

Example 1.—If money is worth $(.05, m = 2)$, find A and S for an annuity of fifty quarterly payments of \$100 each, the first due at the end of three months.

Solution. — Payments occur twice in each interest period. Hence, use formulas 33 with the data listed below. Tables XII, VIII, and VII are used in computing.

$n = 2(12.5) = 25$ int. periods,
$p = 2$, $R = \$200$, $i = .025$.

$$A = 200(a^{(2)}_{\overline{25}|} \text{ at } .025) = 200\frac{.025}{j_2}(a_{\overline{25}|} \text{ at } .025),$$

$$A = 200(1.00621142)(18.42437642),$$

$$A = \$3707.76.$$

$$S = 200(s^{(2)}_{\overline{25}|} \text{ at } .025) = 200\frac{.025}{j_2}(s_{\overline{25}|} \text{ at } .025) = 200(1.00621142)(34.15776393),$$

$$S = \$6873.99.$$

Example 2. — If money is worth $(.06, m = 4)$, find A and S for an annuity whose annual rent is \$1000, payable monthly for 12 years and 3 months.

Solution. — Use formulas 33, because the payments occur three times in each interest period.

$n = 4(12\frac{1}{4}) = 49$ int. periods,
$p = 3$, $R = \$250$, $i = .015$.

$$A = 250(a^{(3)}_{\overline{49}|} \text{ at } .015) = 250\frac{.015}{j_3}(a_{\overline{49}|} \text{ at } .015),$$

$$A = 250(1.00498346)(34.52468339).$$

$$S = 250(s^{(3)}_{\overline{49}|} \text{ at } .015) = 250\frac{.015}{j_3}(s_{\overline{49}|} \text{ at } .015) = 250(1.00498346)(71.60869758).$$

NOTE. — When $p = 1$, formulas 29 and 31 reduce to formulas 27 and 28, or $(s^{(1)}_{\overline{n}|} \text{ at } i) = (s_{\overline{n}|} \text{ at } i)$ and $(a^{(1)}_{\overline{n}|} \text{ at } i) = (a_{\overline{n}|} \text{ at } i)$. We may obtain the same results on placing $p = 1$ in formulas 30 and 32, because $(j_1 \text{ at } i) = 1[(1 + i) - 1] = i$ and $\frac{i}{j_1} = 1$. These results could have been foretold because, when $p = 1$, the payment interval equals the interest period, and hence formulas 27 and 28 apply as well as formulas 29 and 31. In the future think of $(s_{\overline{n}|} \text{ at } i)$ as $(s^{(1)}_{\overline{n}|} \text{ at } i)$, with the value of p left off and understood to be $p = 1$ (just as we omit the exponent 1 in algebra when we write x instead of x^1). Thus, formulas 30 and 32 express the present value and the amount of an annuity payable p times per interest period in terms of the present value and the amount of an annuity payable *once* per interest period.

EXERCISE XXIII

1. Verify the entry in Table XII for $i = .06$ and $p = 2$.

HINT. — $\frac{.06}{j_2} = \frac{.06}{.05912603}$, from Table XI. Complete the division.

2. If money is worth $(.06, m = 2)$, find the present value and the amount of an annuity whose term is 9 years and 6 months, and whose annual rent is \$1200, payable monthly.

Compute the present values and the amounts of the annuities below.

PROB.	ANNUAL RENT	EACH PAYMENT	PAYMENT INTERVAL	TERM	INTEREST RATE
3.	$1000		6 mo.	15 yr.	.05, $m = 1$
4.	5000		1 mo.	12 yr.	.06, $m = 1$
5.		$500	6 mo.	9 yr., 6 mo.	.07, $m = 2$
6.		225	3 mo.	19 yr.	.05, $m = 1$
7.		200	3 mo.	8 yr., 6 mo.	.08, $m = 2$
8.	2000		3 mo.	10 yr., 6 mo.	.055, $m = 2$
9.	500		1 mo.	6 yr., 3 mo.	.06, $m = 4$
10.		750	3 mo.	8 yr.	.04, $m = 1$

11. In buying a farm it has been agreed to pay $100 at the end of each month for the next 25 years. If money is worth the effective rate 7%, what would be an equivalent cash valuation for the farm? 14,427.5

12. If $50 is deposited in a bank at the end of every month for the next 15 years and is left to accumulate, what will be on hand at the end of 15 years if the bank pays 6%, compounded annually on deposits? 14,345.6

13. A sinking fund is being accumulated by payments of $1000, made at the end of each 3 months. Just after the 48th payment to the fund has been made, how much is in the fund if it accumulates at (.045, $m = 1$)?

14. An investment yields $50 at the end of each 3 months, and payments will continue for 25¼ years. What is a fair valuation for the project if money is worth (.05, $m = 2$)?

15. How much could a railroad company afford to pay to eliminate a dangerous crossing requiring the attention of two watchmen, each receiving $75 per month, if money is worth (.04, $m = 1$)? Assume that the crossing will be used for 50 years.

16. Prove from formula 29, that $(s_{\overline{1}|}^{(p)} \text{ at } i) = \frac{i}{j_p}$. Thus $\frac{i}{j_p}$ is the sum which, if paid at the end of 1 year, is equivalent to p payments of $\frac{\$1}{p}$ made at equal intervals during the year.

23. The most general annuity formulas. — Consider the annuity whose annual rent is $1, payable p times per year for n years. To find the present value and the amount of this annuity when money is worth the nominal rate j, compounded m times per year, we might first compute the corresponding effective rate i and then use formulas 29 and 31. It is better to use equation 17 to obtain entirely new formulas in terms of the given quantities j and m. From equation 17,

$$(1+i)^n = \left[\left(1+\frac{j}{m}\right)^m\right]^n = \left(1+\frac{j}{m}\right)^{mn};$$

$$(1+i)^{\frac{1}{p}} = \left(1+\frac{j}{m}\right)^{\frac{m}{p}};\ (1+i)^{-n} = \left(1+\frac{j}{m}\right)^{-mn}.$$

On substituting these expressions in formulas 29 and 31 we obtain

$$\left.\begin{aligned}(s_{\overline{n}|}^{(p)}\ at\ j,\ m) &= \frac{\left(1+\frac{j}{m}\right)^{mn}-1}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]};\\ (a_{\overline{n}|}^{(p)}\ at\ j,\ m) &= \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]}.\end{aligned}\right\} \qquad (34)$$

If the annual rent of the annuity above were $\$R$ instead of $\$1$, the present value would be $R(a_{\overline{n}|}^{(p)}\ at\ j,\ m)$ and the amount would be $R(s_{\overline{n}|}^{(p)}\ at\ j,\ m)$.

NOTE. — Formulas 34 include all previous formulas as special cases, because, when $m = 1$ and $j = i$, formulas 34 reduce to formulas 29 and 31, from which we started. Thus, think of $(s_{\overline{n}|}^{(p)}\ at\ i)$ as being $(s_{\overline{n}|}^{(p)}\ at\ j = i,\ m = 1)$ with the value of m left out and understood to be $m = 1$. Likewise, $(a_{\overline{n}|}\ at\ i) = (a_{\overline{n}|}^{(1)}\ at\ j = i,\ m = 1)$.

24. Summary. — For an annuity under **Case 1** below we usually may compute the present value A and the amount S by means of our tables. For an annuity under **Case 2**, the explicit formulas for A and S must be computed with much less aid from the tables.

Case 1. — The annuity is **payable p times per interest period**, where p is an integer. The method of Section 22 applies, with additional simplification when $p = 1$. If

p = the number of payments per interest period,
n = the term, expressed in interest periods,
i = the rate, per interest period, and
$\$R$ = the sum of the payments made in one interest period, then

$$A = R(a_{\overline{n}|}^{(p)}\ at\ i) = R\frac{i}{j_p}(a_{\overline{n}|}\ at\ i);\ S = R(s_{\overline{n}|}^{(p)}\ at\ i) = R\frac{i}{j_p}(s_{\overline{n}|}\ at\ i). \quad \text{(I)}$$

When $p = 1$, $\$R$ is the **annuity payment**, n is the **number of payments**, and

$$A = R(a_{\overline{n}|} \text{ at } i); \qquad S = R(s_{\overline{n}|} \text{ at } i). \qquad \text{(II)}$$

The values of A and S in I and II can usually be computed by Tables VII, VIII, and XII.

NOTE. — One or more of Tables VII, VIII, and XII will not apply if i is not a table interest rate, or if n is not an integer. In that case the explicit formulas 29 and 31 for $(a^{(p)}_{\overline{n}|} \text{ at } i)$ and $(s^{(p)}_{\overline{n}|} \text{ at } i)$ must be computed.

Case 2. — The annuity is **not payable an integral number of times per interest period.** The general formulas 34 must be used, and if

n = the term in years, p = the number of payments per year,
$\$R$ = the annual rent, j = the nominal rate, and
m = the number of conversion periods per year, then

$$A = R(a^{(p)}_{\overline{n}|} \text{ at } j, m) = R\,\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{p\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right]};$$

$$S = R(s^{(p)}_{\overline{n}|} \text{ at } j, m) = R\,\frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{p\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right]}. \qquad \text{(III)}$$

SUPPLEMENTARY NOTE. — From formulas II and III it can be proved that

$$(a^{(1)}_{\overline{n}|} \text{ at } j, m) = \left(a_{\overline{nm}|} \text{ at } \frac{j}{m}\right)\frac{1}{\left(s_{\overline{m}|} \text{ at } \frac{j}{m}\right)}; \quad (s^{(1)}_{\overline{n}|} \text{ at } j, m) = \left(s_{\overline{nm}|} \text{ at } \frac{j}{m}\right)\frac{1}{\left(s_{\overline{m}|} \text{ at } \frac{j}{m}\right)}.$$

These formulas can be used to simplify the computation of the present values and the amounts of many annuities coming under Case 2. Other simplifying formulas could be derived but they would not be of sufficiently general application to justify their consideration.

Example 1. — An annuity will pay \$500 semi-annually for 8 years. Find the present value A if money is worth $(.06, m = 4)$.

Solution. — The annuity comes under Case 2.

Case 2
$n = 8$ years, $p = 2$,
$j = .06$, $m = 4$, $R = \$1000$.

$$A = 1000(a^{(2)}_{\overline{8}|} \text{ at } .06, m = 4),$$

$$A = 1000\,\frac{1 - (1.015)^{-32}}{2[(1.015)^2 - 1]},$$

$$A = \frac{1000(1 - .6209929)}{2(1.03022500 - 1)} = \$6269.76.$$

Example 2. — In buying a house a man has agreed to pay \$1000 cash, and \$200 at the end of each month for 4 years and 3 months. If money is worth (.06, $m = 2$), what would be an equivalent cash price for the property?

Solution. — First disregard the cash payment. The other payments form an annuity under Case 1 whose present value is

Case 1
$n = 8.5$ int. periods,
$p = 6, i = .03, R = \$1200.$

$$A = 1200(a_{\overline{8.5}|}^{(6)} \text{ at } .03) = 1200\frac{.03}{j_6}(a_{\overline{8.5}|} \text{ at } .03),$$

$$A = 1200(1.01242816)\frac{1 - (1.03)^{-8.5}}{.03}.$$

$$(1.03)^{-8.5} = (1.03)^{-9}(1.03)^{\frac{1}{2}} = (.76641673)(1.01488916) = .7778280.$$

$$A = \frac{1200(1.01242816)(1 - .7778280)}{.03} = \$8997.33.$$

The equivalent cash price is \$1000 + \$8997.30 = \$9997.33.

Example 3. — At the end of each 3 months a man deposits \$50 with a building and loan association. What sum is to his credit at the end of 4 years if interest is accumulating at the rate (.075, $m = 2$), from the date of each deposit?

Solution. — The amount on hand is the amount of an annuity which comes under Case 1.

Case 1
$n = 8$ int. periods,
$p = 2, i = .0375, R = \$100.$

$$S = 100(s_{\overline{8}|}^{(2)} \text{ at } .0375),$$

$$S = 100\frac{(1.0375)^8 - 1}{2[(1.0375)^{\frac{1}{2}} - 1]}. \quad \text{(Formula 29)}$$

$\frac{1}{2}\log(1.0375) = 0.0079940$, from Table II.

$(1.0375)^{\frac{1}{2}} = 1.018577$, from Table II.

$8 \log(1.0375) = 8(.0159881) = 0.12790.$

$(1.0375)^8 = 1.3425$, from Table I.

$$S = \frac{100(1.3425 - 1)}{2(.018577)}.$$

$$S = \frac{34.25}{.037154} = \$921.8.$$

The answer is not stated to five digits because the numerator 34.25 was obtainable only to four digits from Table I.

NOTE. — In every problem where the present value or amount of an annuity is to be computed, first list the case and the elements of the annuity as in the examples above.

NOTE. — To find A and S for an annuity we could always proceed as under Case 2, even though the annuity comes under Case 1. Thus, for the annuity of Example 3 above, the term is $n = 4$ years, the annual rent is $R = \$200$, payable $p = 4$ times per year, $j = .075$, and $m = 2$. Hence, from formulas III of Case 2,

$$S = 200(s_{\overline{4}|}^{(4)} \text{ at } .075, m = 2) = 200\frac{(1.0375)^8 - 1}{4[(1.0375)^{\frac{1}{2}} - 1]},$$

which is the same as obtained above. The only difference in method is that, under Case 2, the fundamental time unit is the year, whereas under Case 1 it is the interest period. The classification of annuity computations under two cases would not be advisable if we were always to compute A and S by the explicit formulas, as is necessary in Example 3. But, if we used the general formulas of Case 2, with the year as a time unit, in problems under Case 1 to which Tables VII, VIII, and XII apply, unnecessary computational confusion would result and other inconvenient auxiliary formulas would have to be derived. Hence, use the method of Case 1 whenever possible.

EXERCISE XXIV

Compute A and S for each annuity in the table. Use Table II when it is an aid to accuracy.

Prob.	Annual Rent	Each Payment	Payment Interval	Term	Interest Rate
1.	$10,000		1 month	15 years	.05, $m = 4$
2.		$ 400	1 month	12 years	.06, $m = 1$
3.		2500	6 months	19 yr., 6 mo.	.05, $m = 4$
4.	500		6 months	7 yr., 6 mo.	.05, $m = 2$
5.	240		3 months	11 yr., 6 mo.	.04, $m = 4$
6.		150	1 year	18 years	.09, $m = 4$
7.	100		6 months	28 years	.05, $m = 1$
8.	5,000		3 months	6 yr., 9 mo.	.07, $m = 2$
9.		125	6 months	10 years	.0625, $m = 1$
10.	2,000		1 year	15 years	.05, $m = 2$
11.	900		3 months	9 yr., 3 mo.	.08, $m = 4$
12.		700	6 months	20 years	.005, $m = 1$
13.		50	3 months	30 years	.048, $m = 2$
14.		100	4 months	9 years	.04, $m = 2$
15.	3,000		3 months	12 years	.04, $m = 2$
16.	500		1 year	35 years	.07, $m = 2$
17.		200	6 months	12 years	.055, $m = 2$
18.	1,200		6 months	15 yr., 6 mo.	.03, $m = 4$
19.		150	3 months	6 yr., 3 mo.	.06, $m = 2$
20.		250	4 months	9 years	.04, $m = 2$
21.	500		6 months	10 years	.045, $m = 2$
22.		25	1 month	17 years	.06, $m = 1$

23. To provide for the retirement of a bond issue at the end of 20 years, a city will place $100,000 in a sinking fund at the end of each 6 months. (*a*) If the fund accumulates at the rate (.05, $m = 2$), what sum will be available at the end of 20 years? (*b*) What sum is in the fund at the beginning of the 12th year?

24. An investment will yield \$50 at the end of each month for the next 15 years. If money is worth (.05, $m = 4$), what would be a fair present valuation for the project?

25. A depreciation fund is being accumulated by semi-annual deposits of \$250 in a bank which pays 5%, compounded quarterly. How much will be in the fund just after the 30th deposit?

26. A will decrees that X shall receive \$1000 at the beginning of each 6 months until 10 payments have been made. If money is worth (.06, $m = 2$), on what sum should X's inheritance tax be computed, assuming that the payments will certainly be made?

27. A certain bond has attached coupons for \$5 each, payable at the end of each year for the next 25 years. If money is worth 5% effective, find the present value of the coupons.

28. The bond of problem 27 will be redeemed for \$100 by the issuing corporation at the end of 25 years. What should an investor pay for the bond if he desires 5% effective on his investment?

HINT. — He should pay the present value of the coupons plus the present value of the redemption price.

29. A farm is to be paid for by 10 successive annual installments of \$5000 in addition to a cash payment of \$15,000. What is an equivalent cash price for the farm if money is worth (.05, $m = 2$)?

30. (*a*) At the end of the 5th year in problem 29, after the payment due has been made, the debtor wishes to make an additional payment immediately which will cancel his remaining liability. The creditor is willing to accept payment if money is considered worth 4% effective. What does the debtor pay? (*b*) Why should the creditor specify the rate 4% effective instead of a higher rate, (.05, $m = 1$) for instance?

HINT. — Find the present value of the remaining payments.

31. A man has been placing \$100 in a bank at the end of each month for the last 12 years. What is to his credit if his savings have been accumulating at the rate 6%, compounded semi-annually from their dates of deposit?

32. A man wishes to donate immediately to a university sufficient money to provide for the erection and the maintenance, for the next 50 years, of a building which will cost \$500,000 to erect and will require \$1000 at the end of each month to maintain. How much should he donate if the university is able to invest its funds at 5%, converted semi-annually?

\$721,962

33. A certain bond has attached coupons for \$5 each, payable semi-annually for the next 10 years. At the end of 10 years the bond will be redeemed for \$125. What should an investor pay for the bond if he desires 6%, compounded semi-annually, on his investment?

HINT. — See problem 28.

34. A man, who borrowed a sum of money, is to discharge the liability by paying \$500 at the end of each 3 months for the next 8 years. What sum did he borrow if the creditor's interest rate is $(.055, m = 2)$?

35. (*a*) In problem 34, at the end of 4 years, just after the installment due has been paid, what additional payment would cancel the remaining liability if money is still worth $(.055, m = 2)$ to the creditor? (*b*) What would be the payment if money is worth $(.04, m = 4)$ to the creditor?

36. A man X agreed to pay \$1000 to his creditor at the end of each 6 months for 15 years, but defaulted on his first 7 payments. (*a*) What should X pay at the end of 4 years, if money is worth $(.06, m = 2)$ to his creditor? (*b*) What should he pay if money is worth $(.05, m = 2)$?

37. In problem 36, at the end of 4 years, X desires to make a single payment which will cancel his liability due to his previous failure to pay, and also will discharge the liability of the payments due in the future. (*a*) What should he pay if money is worth $(.06, m = 2)$ to his creditor? (*b*) Find the payment if the rate is $(.05, m = 2)$.

38. A certain bond has attached coupons of \$2 each, payable quarterly for the next 20 years, and at the end of that time the bond itself will be redeemed for \$110. What should a man pay for the bond if he considers money worth 6%, effective?

39. Prove by use of formulas 34 that the present value of an annuity, accumulated at the rate (j, m) for n years, will equal the amount of the annuity; that is, prove algebraically that

$$A\left(1 + \frac{j}{m}\right)^{mn} = R(a_{\overline{n}|}^{(p)} \text{ at } j, m)\left(1 + \frac{j}{m}\right)^{mn} = R(s_{\overline{n}|}^{(p)} \text{ at } j, m) = S.$$

Another statement of this result would be that "A is the present value of S, due at the end of the term of the annuity."

40. What is the amount of an annuity whose term is 14 years, and whose present value is \$1575, if interest is at the rate $(.06, m = 2)$?

HINT. — Use the result of problem 39.

41. What is the effective rate of interest in use if the present value of an annuity is \$2500, the amount \$3750, and the term 10 years?

42. A will bequeaths to a boy who is now 10 years old, $20,000 worth of bonds which pay 6% interest semi-annually. The will requires that half of the interest shall be deposited in a savings bank which pays 4%, compounded quarterly. The accumulation of the savings account, and the bonds themselves, are to be given to the boy on his 25th birthday. Find the value of the property received by him on that date.

43. A man desires to deposit with a trust company a sufficient sum to provide his family an annuity of $200 per month for 10 years. What should he deposit if the trust company will credit interest at the rate 5% compounded quarterly, on the unexpended balance of the fund?

44. If you can invest money at $(.03, m = 2)$, what is the least sum you would take at the present time in return for a contract on your part to pay $100 at the end of each 6 months for the next 15 years?

45. If money is worth $(.04, m = 1)$, is it more profitable to pay $100 at the end of each month for 3 years as rent on a motor truck, or to buy one for $3000, assuming that the truck will be useless after 3 years? Assume in both cases that you would have to pay the upkeep.

25. Annuities due. — The payments of the standard annuities considered previously were made at the ends of the payment intervals. An **annuity due** is one whose payments occur at the **beginning of each interval**, so that the first payment is due immediately. The definitions of the amount and of the present value of an annuity as given in Section 19 apply without change of wording to an annuity due. It must be noticed, however, that the last payment of an annuity due occurs at the beginning of the last interval, whereas the end of the term is the end of this interval. Hence, the amount of an annuity due is the **sum of the compound amounts of the payments one interval after the last payment is made.** For an annuity due whose annual rent is $100, payable quarterly for 6 years, the last $25 payment is made at the end of 5 years and 9 months. The amount of this annuity is the sum of the compound amounts of the payments at the end of 6 years, the end of the term.

For the treatment of annuities due and for other purposes in the future, it is essential to recognize that, regardless of when a sequence of periodic payments start, **they will form an ordinary annuity if judged from a date one payment interval before the first payment.** Hence, one interval before the first payment, the sum of the discounted values of the payments is the present value of the ordinary annuity they form. Moreover, the sum of the accumulated values of the payments on the last payment date is the amount of this ordinary annuity.

Example 1. — If money is worth (.05, $m = 2$), find the present value A and the amount S of an annuity due whose annual rent is \$100, payable quarterly for 6 years.

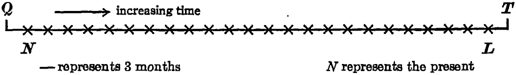

FIG. 3

Solution.— Consider the time scale in Figure 3, where × represents a payment date, T is the end of the term, 6 years from the present, and L is the last payment date, 3 months before T. Q is 3 months before the present. Considered from Q the payments form an ordinary annuity whose term ends at L and whose present value A' and amount S' are

Case 1
$n = 12$ int. periods,
$p = 2$, $i = .025$, $R = \$50$.

$$A' = 50(a^{(2)}_{\overline{12}|} \text{ at } .025),$$
$$S' = 50(s^{(2)}_{\overline{12}|} \text{ at } .025).$$

Since A' is the sum of the discounted values of the payments at Q, 3 months before the present, we accumulate A' for 3 months to find A, the present value of the annuity due.

$$A = A'(1.025)^{\frac{1}{2}} = 50(a^{(2)}_{\overline{12}|} \text{ at } .025)(1.025)^{\frac{1}{2}}.$$

Since S' is the sum of the accumulated values of the payments at L, we accumulate S' for 3 months to find S, which is the sum of the values at time T.

$$S = S'(1.025)^{\frac{1}{2}} = 50(s^{(2)}_{\overline{12}|} \text{ at } .025)(1.025)^{\frac{1}{2}}.$$

Tables VII, VIII, X, and XII would be used to compute A and S.

Second solution. — The first \$25 payment is cash and the remaining payments form an ordinary annuity, as judged from the present. Its present value A' is

Case 1
$n = 11.5$ int. periods,
$p = 2$, $i = .025$, $R = \$50$.

$$A' = 50(a^{(2)}_{\overline{11.5}|} \text{ at } .025).$$
$$\text{Hence, } A = 25 + 50(a^{(2)}_{\overline{11.5}|} \text{ at } .025).$$

To find S, first consider a new annuity consisting of all payments of the annuity due, with an additional \$25 due at time T. Since T is the last payment date of the new sequence of payments, the sum of their values at time T is the amount S' of an ordinary annuity, or

Case 1
$n = 12.5$ int. periods,
$p = 2$, $i = .025$, $R = \$50$.

$$S' = 50(s^{(2)}_{\overline{12.5}|} \text{ at } .025).$$

The value of the additional \$25 payment at time T is included in S', or $S' = S + 25$. $S = 50(s^{(2)}_{\overline{12.5}|} \text{ at } .025) - 25$. To find the numerical values of A and S, $a_{\overline{11.5}|}$ and $s_{\overline{12.5}|}$ must be computed from formulas 29 and 31. Hence, the first solution was less complicated numerically. In some problems, however, the second solution would be the least complicated.

Two rules may be stated corresponding, respectively, to the two methods of solution considered above.

Rule 1. — To find A and S for an annuity due, first find the present value A' and the amount S' of an ordinary annuity having the same term, annual rent, and payment interval. Then:

(*a*) A is the compound amount on A' after one payment interval.
(*b*) S is the compound amount on S' after one payment interval.

Rule 2. — To find A for an annuity due, first find A', the present value of all payments, omitting the first. Then, if W is the annuity payment, $A = A' + W$. To find S first obtain S', the amount of the ordinary annuity having a payment at the end of the term in addition to the payments of the annuity due. Then, $S = S' - W$.

NOTE. — It is customary in actuarial textbooks to use black roman type to indicate amounts and present values of annuities due. Thus ($\mathbf{s}^{(p)}_{\overline{n}|}$ *at j, m*) represents the amount, and ($\mathbf{a}^{(p)}_{\overline{n}|}$ *at j, m*) the present value of an annuity due whose annual rent is \$1, payable p times per year for n years, if money is worth (j, m).

EXERCISE XXV

In each problem draw a figure similar to Figure 3. Find A and S for each annuity due in the table, by use of the specified rule.

PROB.	TERM	PAYMENT INTERVAL	ANNUAL RENT	INTEREST RATE	RULE
1.	10 yr.	3 mo.	\$ 300	.06, $m = 4$	2
2.	7 yr., 6 mo.	6 mo.	500	.05, $m = 2$	2
3.	12 yr., 6 mo.	6 mo.	3600	.03, $m = 1$	2
4.	12 yr.	3 mo.	1000	.05, $m = 4$	1

5. Carry through the solution of problem 2 by Rule 1 far enough to be able to state why it is inconvenient.

6. A man deposited \$100 in a bank at the beginning of each 3 months for 10 years. (*a*) What sum is to his credit at the end of 10 years if the bank credits 6% interest quarterly from the date of deposit? (*b*) What sum is to his credit at the end of 9 years and 9 months, after the deposit has been made at that time?

7. In purchasing a house, a man has agreed to pay \$100 at the beginning of each month for the next 5 years. (*a*) If money is worth 6% effective, find the present value of the payments. (*b*) If money is worth (.06, m = 12), find their present value.

8. A man was loaned \$75 on the 1st of each month, for 12 months each year, during the four years of his college course. (*a*) If his creditor considers money worth 3% effective, what is the liability of the debt at the end of the 4 years? (*b*) If the debtor makes no payment until four years after he graduates, what should he pay then to settle in full?

9. If money is worth the effective rate i, prove that $(\mathbf{a}_{\overline{n}|}\ at\ i)$, the present value, and $(\mathbf{s}_{\overline{n}|}\ at\ i)$, the amount of an annuity due of \$1 payable annually for n years, are given by

$$(\mathbf{a}_{\overline{n}|}\ at\ i) = 1 + (a_{\overline{n-1}|}\ at\ i), \qquad (\mathbf{s}_{\overline{n}|}\ at\ i) = (s_{\overline{n+1}|}\ at\ i) - 1.$$

10. Prove that

$$(\mathbf{s}^{(p)}_{\overline{n}|}\ at\ j, m) = (1+i)^{\frac{1}{p}}(s^{(p)}_{\overline{n}|}\ at\ j, m);\ (\mathbf{a}^{(p)}_{\overline{n}|}\ at\ j, m) = (1+i)^{\frac{1}{p}}(a^{(p)}_{\overline{n}|}\ at\ j, m).$$

26. Deferred annuities. — A **deferred annuity** is one whose term does not begin until the expiration of a certain length of time. Thus, an annuity whose term is 6 years, deferred 4 years, and whose annual rent is \$1000, payable semi-annually, consists of 12 payments of \$500, the first due after (4 years + 6 months) and the last, after (4 years + 6 years).

Example 1. — If money is worth (.05, m = 1), find A and S for the deferred annuity of the last paragraph.

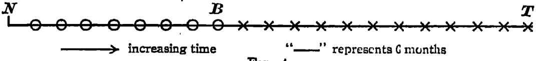

FIG. 4

In Fig. 4, × represents a payment date of the deferred annuity, N, the present, T, the end of 10 years, the end of the term, and B, the beginning of the term.

Solution. — Consider the time scale in Figure 4. The payments form an ordinary annuity when judged from B, 6 months before the first payment. S', the amount, and A', the present value (at B), of this ordinary annuity are

Case 1
$n = 6$ int. periods,
$p = 2, i = .05, R = \$1000.$

$$S' = 1000(s_{\overline{6}|}^{(2)} \text{ at } .05).$$
$$A' = 1000(a_{\overline{6}|}^{(2)} \text{ at } .05).$$

S' is the sum of accumulated values at time T. Since S, the amount of the deferred annuity, is also equal to the sum of values at T,

$$S = S' = 1000(s_{\overline{6}|}^{(2)} \text{ at } .05).$$

Since A' is the sum of the discounted values at B, we must discount A' for 4 years to obtain the present value A.

$$A = A'(1.05)^{-4} = 1000(a_{\overline{6}|}^{(2)} \text{ at } .05)(1.05)^{-4}.$$

Second solution for A. — Consider a new annuity having payments of $500 at the end of each 6 months for the first 4 years as well as for the last 6. The new payment dates are indicated by circles in Figure 4. The present value A of the deferred annuity equals the present value A' of the new annuity over the whole 10 years minus the present value A'' of the payments over the first 4 years, which are not to be received. Both A' and A'' are the present values of ordinary annuities.

Case 1
$n = 10$, and 4, int. periods,
$p = 2, i = .05, R = \$1000.$

$$A' = 1000(a_{\overline{10}|}^{(2)} \text{ at } .05).$$
$$A'' = 1000(a_{\overline{4}|}^{(2)} \text{ at } .05).$$

$$A = A' - A'' = 1000\left[(a_{\overline{10}|}^{(2)} \text{ at } .05) - (a_{\overline{4}|}^{(2)} \text{ at } .05)\right],$$
$$A = 1000\frac{.05}{j_2}\left[(a_{\overline{10}|} \text{ at } .05) - (a_{\overline{4}|} \text{ at } .05)\right].$$

From Example 1 it is clear that the **amount of a deferred annuity equals the amount of an ordinary annuity having the same term.** Corresponding to the two methods used above in obtaining A, we state the two rules below.

Rule 1. — To obtain A for an annuity whose term is deferred w years, first find A', the present value of the ordinary annuity having the same term. Then, A equals the value of A' discounted for w years.

Rule 2. — If the term of the annuity is n years, deferred w years, then

A = [*present value of an ordinary annuity with term* $(w + n)$ *years*]
− (*present value of an ordinary annuity with term* w *years*),

where these new annuities have the same annual rent and payment interval as the deferred annuity.

NOTE. — The present values and the amounts of deferred annuities are indicated in actuarial writings by the symbols for ordinary annuities with a number prefixed showing the time for which the term is deferred. Thus

$$({}_{w|}a_{\overline{n}|}^{(p)} \text{ at } j, m) \text{ and } ({}_{w|}s_{\overline{n}|}^{(p)} \text{ at } j, m)$$

represent the present value and the amount when the term is deferred w years.

EXERCISE XXVI[1]

In each problem draw a figure similar to Figure 4. Find the present value of each deferred annuity in the table, by use of the specified rule.

PROB.	TERM	TERM DEFERRED	PAYMENT INTERVAL	ANNUAL RENT	INTEREST RATE	RULE
1.	6 yr.	10 yr., 6 mo.	3 mo.	\$1000	.05, $m = 4$	2
2.	7 yr.	8 yr., 6 mo.	1 mo.	200	.06, $m = 2$	1
3.	9 yr.	12 yr.	1 yr.	300	.07, $m = 1$	2
4.	13 yr.	10 yr., 6 mo.	1 mo.	1200	.05, $m = 1$	1

5. Carry through the solution of problem 4 by Rule 2 until you are able to state why it is inconvenient.

6. Solve problem 3 if money is worth (.07, $m = 2$).

7. A man will receive a pension of \$50 at the end of each month for 10 years, first payment to occur 1 month after he is 65 years old. Assuming that he will live to receive all payments, find the present value of his expectation if money is worth (.04, $m = 1$), and if he is now 50 years old.

8. A certain mine will yield a semi-annual profit of \$50,000, the first payment to come at the end of 7 years, and the last after 42 years, at which time the mine will become worthless. What is a fair valuation for the mine if money is worth 5%, effective?

9. A recently paved road will require no upkeep until the end of 3 years, at which time \$3000 will be needed for repairs. After that, \$3000 will be used for repairs at the end of each 6 months for 15 years. Find the present value of all future upkeep if money is worth (.05, $m = 2$).

[1] The Miscellaneous Problems at the end of the chapter may be taken up immediately after the completion of Exercise XXVI.

10. By use of Rules 1 and 2 prove the relations below, for an annuity whose annual rent is \$1, payable p times per year, and whose term is n years, deferred w years.

$$({}_{w|}a_{\overline{n}|}^{(p)} \text{ at } i) = (1+i)^{-w}(a_{\overline{n}|}^{(p)} \text{ at } i).$$
$$({}_{w|}a_{\overline{n}|}^{(p)} \text{ at } i) = (a_{\overline{w+n}|}^{(p)} \text{ at } i) - (a_{\overline{w}|}^{(p)} \text{ at } i).$$

SUPPLEMENTARY MATERIAL

27.[1] **Continuous annuities.** — If money is worth the effective rate i, the present value of an annuity whose annual rent is \$1, payable p times per year, is

$$(a_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i}{j_p}(a_{\overline{n}|} \text{ at } i).$$

We may consider an annuity payable weekly, $p = 52$, or daily, $p = 365$, and we may ask what value does $(a_{\overline{n}|}^{(p)} \text{ at } i)$ approach as p becomes large without bound? We obtain

$$\lim_{p=\infty}(a_{\overline{n}|}^{(p)} \text{ at } i) = i(a_{\overline{n}|} \text{ at } i)\lim_{p=\infty}\frac{1}{(j_p \text{ at } i)}.$$

But, from Section 18 we have $\lim_{p=\infty} j_p = \delta$, the force of interest corresponding to the effective rate i. Hence

$$\lim_{p=\infty}(a_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i(a_{\overline{n}|} \text{ at } i)}{\delta}.$$

In the same way it can be shown that

$$\lim_{p=\infty}(s_{\overline{n}|}^{(p)} \text{ at } i) = \frac{i(s_{\overline{n}|} \text{ at } i)}{\delta}.$$

As $p \doteq \infty$, the annuity approaches the ideal case of an annuity whose annual rent is payable continuously. If we let $(\bar{a}_{\overline{n}|} \text{ at } i)$ represent the present value and $(\bar{s}_{\overline{n}|} \text{ at } i)$ the amount of a continuous annuity, the results above show that

$$(\bar{a}_{\overline{n}|} \text{ at } i) = \frac{i(a_{\overline{n}|} \text{ at } i)}{\delta}, \quad (\bar{s}_{\overline{n}|} \text{ at } i) = \frac{i(s_{\overline{n}|} \text{ at } i)}{\delta}. \tag{35}$$

Recall that $1 + i = e^{\delta}$ so that $\delta = \dfrac{\log(1+i)}{\log e}$. Since $\log e = \log 2.7182818 = 0.4342945$, we obtain from equation 35,

[1] Section 18 is a prerequisite for the reading of this section.

$$(\bar{a}_{\overline{n}|} \; at \; i) = \frac{i(a_{\overline{n}|} \; at \; i)(.4342945)}{\log\,(1 + i)}, \quad (\bar{s}_{\overline{n}|} \; at \; i) = \frac{i(s_{\overline{n}|} \; at \; i)(.4342945)}{\log(1 + i)}.$$

The present value and the amount of an annuity, which is payable continuously, differ but slightly from the corresponding quantities for an annuity which is payable a very large number of times per year (see problem 1 below). Hence we may use $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ as approximations for $a^{(p)}_{\overline{n}|}$ and $s^{(p)}_{\overline{n}|}$ if p is very large.

EXERCISE XXVII

1. (*a*) If money is worth 6%, effective, find the present value and the amount of an annuity whose annual rent is \$100, and whose term is 10 years, if the annuity is payable continuously. (*b*) Solve the problem if the annuity is payable monthly.

2. If one year equals 360 days, find approximately the amount of an annuity of \$1 per day for 20 years if money is worth 4%, effective.
HINT. — Use a continuous annuity as an appropriation.

3. A member of a labor union has agreed to contribute \$.20 per day to a benefit fund for 3 years. Under the rate (.05, $m = 1$), find approximately the present value of his agreement if a year has 365 days.

4. An industrial insurance policy for \$100 calls for a premium of 10 cents at the end of each week. Find approximately, by use of a continuous annuity, the equivalent premium which could be paid at the end of the year if money is worth $3\frac{1}{2}$%.

NOTE. — If the conversion period of an interest rate is not stated, assume it to be 1 year.

28.[1] **Computations of high accuracy.** — The binomial theorem can be used in interest computations to which the tables do not apply. As a special case of the binomial theorem,[2] we have

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{n(n-1)\cdots(n-r+1)x^r}{r!} + \cdots \qquad (36)$$

When n is a positive integer, series 36 contains $(n + 1)$ terms, the last of which is x^n. If n is a negative integer or a fraction, the series contains infinitely many terms. In this case, if x lies

[1] A knowledge of the binomial theorem is needed in this section.
[2] See page 93 in Rietz and Crathorne's *College Algebra*.

between -1 and $+1$, the infinite series converges, and, if x is very small, the sum of the first few terms gives a good approximation to the value of $(1+x)^n$. The proof of this statement is too difficult for an elementary treatment.

Example 1. — Find the value of $(a^{(4)}_{\overline{6|}}\ at\ .033)$ accurately to six significant figures.

Solution. — From formula 31,

$$(a^{(4)}_{\overline{6|}}\ at\ .033) = \frac{1-(1.033)^{-6}}{4\left[(1.033)^{\frac{1}{4}}-1\right]}. \tag{37}$$

From equation (36) with $n = \frac{1}{4}$ and $x = .033$,

$$(1.033)^{\frac{1}{4}} = 1 + \tfrac{1}{4}(.033) - \tfrac{3}{32}(.033)^2 + \tfrac{7}{128}(.033)^3 - \tfrac{77}{16(128)}(.033)^4 + \cdots \tag{38}$$

$$4\left[(1.033)^{\frac{1}{4}} - 1\right] = .03300000 - .00040838 + .00000786 - .00000018 = .03259930.$$

The next term in series 38, beyond the last one computed, is negligible in the 8th decimal place, and hence our result is accurate to the 7th decimal place with only a slight doubt as to the 8th. To compute $(1.033)^{-6}$, first compute $(1.033)^6$ and then take the reciprocal; $(1.033)^6 = 1.2150718$; $(1.033)^{-6} = .8229966$. Hence,

$$(a^{(4)}_{\overline{6|}}\ at\ .033) = \frac{.1770034}{.03259930} = 5.429669.$$

The final 9 is not dependable because the final 0 in the denominator was doubtful.

EXERCISE XXVIII

1. Compute $(s^{(2)}_{\overline{10|}}\ at\ .0325)$ and $(a^{(2)}_{\overline{10|}}\ at\ .0325)$ accurately to five significant figures.

2. Compute $(j_{52}\ at\ .02)$ accurately to the 7th decimal place.

3. A man pays $50 to a building and loan association at the end of each week. If the deposits accumulate at the rate $(.06,\ m = 2)$, how much will be to his credit at the end of 3 years? Obtain the result correct to four significant figures.

MISCELLANEOUS PROBLEMS

1. If $1750 is the present value of an annuity whose term is 12 years, what is its amount if money is worth $(.05,\ m = 4)$?

2. If $100 is deposited in a bank at the beginning of each month for 10 years, what is the accumulated amount at the time of the last payment if interest is at the rate 6% compounded semi-annually on all money from the date of deposit?

3. In problem 2, what is the amount at the end of 10 years?

4. A man W has occupied a farm for 5 years and, pending decision of a case in court, has paid no rent to B to whom the farm is finally awarded. What should W pay at the end of 5 years if rent of $100 should have been paid monthly, in advance, and if money is worth $(.06, m = 12)$?

5. What should W pay in problem 4 if the rent is considered due at the end of each month and if money is worth 6%, compounded annually?

6. A building will cost $500,000. It will require, at the beginning of each year, $5000 for heat and light, $5000 for janitor service, and, at the end of each year, $3000 for small repairs. It is to be completely renovated, at a cost of $20,000 at the end of each 15 years. If the cost of the annual repairs is included in the cost of renovation at the end of each 15 years, and if the building is to be renovated at the end of 90 years, what present sum will provide for the erection of the building and for its upkeep for the next 90 years, if money is worth 6% effective?

7. A young man, just starting a four-year college course, estimates his future earning power, in excess of living expenses, at $100 per month for the first 3 years after graduation from college, $200 per month for the next 7 years, and $300 per month for the next 20 years. What is the present value of this earning power, if money is worth 4%, effective?

8. If the man in problem 7 should place his surplus earnings in a bank, what will he have at the end of his working life if his savings earn interest at the rate $(.05, m = 2)$, for the first 10 years, and at the rate $(.04, m = 1)$ for the balance of the time?

9. In purchasing a homestead from the government, a war veteran has agreed to pay $100 at the end of 5 years, and monthly thereafter until the last payment occurs at the end of 9 years. What is the present value of his agreement if money is worth $(.045, m = 1)$?

10. At the end of 3 years, the man of problem 9 decides to pay off his obligation to the government immediately. What should he pay if money is worth $(.045, m = 1)$?

11. A savings bank accepts deposits of $1 at the beginning of each week during the year from small depositors who are creating Christmas funds. Just after the 52d payment, what will each fund amount to if the bank accumulates the savings at 6%, effective?

12. A contract provides for the payment of $1000 at the end of each 6 months for the next 25 years. What is the present value of the contract if the future liabilities are discounted at $(.06, m = 2)$ over the last 15 years of the life of the transaction and at $(.05, m = 2)$ over the first 10 years?

13. In purchasing a house it was agreed to pay \$50 at the end of each month for a certain time. The purchaser desires to change to annual payments. What should he pay at the end of each year if money is considered worth $(.05, m = 2)$?

14. At what rate of interest compounded semi-annually will \$1650 be the present value and \$3500 the amount of an annuity whose term is 14 years, if the annuity is payable weekly? Would the result be any different if the annuity were payable annually?

15. If money is worth 5%, effective, what is the least sum which you would accept now in return for a contract on your part to pay \$50 at the end of each month for 20 years, first payment to occur at the end of 10 years and 1 month?

CHAPTER IV

PROBLEMS IN ANNUITIES

29. In every problem below, the payment interval of the annuity and the conversion period of the interest rate will be given or, in other words, the p and m of equations I and III of Section 24 will always be known. For an annuity under Case 1 there remain for consideration the five quantities (A, S, R, i, n). If three of (S, R, i, n) are known, we use $S = R(s_{\overline{n}|}^{(p)} \text{ at } i)$ to find the fourth; while, if three of (A, R, i, n) are known, we use $A = R(a_{\overline{n}|}^{(p)} \text{ at } i)$. If an annuity comes under Case 2, similar remarks apply to the four quantities (S, R, j, n) *and to* (A, R, j, n). Problems in which A and S were unknown were treated in Chapter III.

30. Determination of the payment. — For future convenience it is essential to know that

$$\frac{1}{(s_{\overline{n}|} \text{ at } i)} = \frac{1}{(a_{\overline{n}|} \text{ at } i)} - i, \text{ or } \frac{1}{(s_{\overline{n}|} \text{ at } i)} + i = \frac{1}{(a_{\overline{n}|} \text{ at } i)}. \tag{39}$$

From formulas 27 and 28 we obtain

$$\frac{1}{(s_{\overline{n}|} \text{ at } i)} + i = \frac{i}{(1+i)^n - 1} + i = \frac{i + i(1+i)^n - i}{(1+i)^n - 1}.$$

$$i + \frac{1}{(s_{\overline{n}|} \text{ at } i)} = \frac{i(1+i)^n}{(1+i)^n - 1}.$$

$$\frac{1}{(a_{\overline{n}|} \text{ at } i)} = \frac{i}{1-(1+i)^{-n}} = \frac{(1+i)^n}{(1+i)^n}\left(\frac{i}{1-(1+i)^{-n}}\right).$$

$$\frac{1}{(a_{\overline{n}|} \text{ at } i)} = \frac{i(1+i)^n}{(1+i)^n - 1},$$

and hence relation 39 is true.

Example 1. — Find the value of $\dfrac{1}{(s_{\overline{15}|} \text{ at } .05)}$.

Solution. — From Table IX, $\frac{1}{(a_{\overline{15}|} \text{ at } .05)} = .09634229$. Therefore, from relation 39, $\frac{1}{(s_{\overline{15}|} \text{ at } .05)} = .09634229 - .05 = .04634229$. On doing this subtraction mentally, we are able to read the result .04634229 directly from Table IX.

NOTE. — From Example 1 we see that, because of relation 39, Table IX gives the values of $\frac{1}{(s_{\overline{n}|} \text{ at } i)}$ as well as those of $\frac{1}{(a_{\overline{n}|} \text{ at } i)}$. It would be equally convenient to have a table of the values of $\frac{1}{(s_{\overline{n}|} \text{ at } i)}$ from which we could obtain those of $\frac{1}{(a_{\overline{n}|} \text{ at } i)}$ by adding the interest rate i.

Example 2. — What annuity, payable quarterly for 20 years and 6 months, could be purchased for $5000, if money is worth (.05, $m = 4$)?

Solution. — The present value of the annuity is $5000. Let $x be the quarterly payment.

Case 1
$n = 82$ int. periods,
$p = 1$, $i = .0125$,
$A = \$5000$, $R = \$x$.

$$5000 = x(a_{\overline{82}|} \text{ at } .0125), \quad x = \frac{5000}{(a_{\overline{82}|} \text{ at } .0125)},$$

$$x = 5000 \frac{1}{(a_{\overline{82}|} \text{ at } .0125)} = 5000(.01956437)$$

$x = \$97.822$. The annual rent is $4x = \$391.29$.

Example 3. — If money is worth (.06, $m = 2$), find the annual rent of an annuity, payable quarterly for $11\frac{1}{2}$ years, if its amount is $10,000.

Solution. — Let $x be the sum of the payments in one interest period.

Case 1
$n = 23$ int. periods,
$p = 2$, $i = .03$,
$S = \$10{,}000$, $R = \$x$.

$$10000 = x(s^{(2)}_{\overline{23}|} \text{ at } .03) = x\frac{.03}{j_2}(s_{\overline{23}|} \text{ at } .03),$$

$$x = \frac{10000(j_2 \text{ at } .03)}{.03(s_{\overline{23}|} \text{ at } .03)} = \frac{10000(j_2 \text{ at } .03)}{.03} \frac{1}{(s_{\overline{23}|} \text{ at } .03)}.$$

$$x = \frac{10000(.02977831)}{.03}(.03081390) = \$305.86.$$

Tables IX and XI were used. The annual rent is $2x = \$611.72$.

NOTE. — Recognize that the solutions above were arranged so as to avoid computing quotients, except for the easy division by .03.

Example 4. — Find the annual rent if $3500 is the present value of an annuity which is payable semi-annually for 8 years. Interest is at the rate 5%, compounded quarterly.

Solution. — Let \$x be the annual rent.

Case 2
$n = 8$ yr., $p = 2$,
$j = .05$, $m = 4$,
$R = \$x$, $A = \$3500$.

$$3500 = x(a_{\overline{8|}}^{(2)} \text{ at } .05,\ m = 4) = x\frac{1 - (1.0125)^{-32}}{2[(1.0125)^2 - 1]}.$$

$$3500 = x\frac{.32801593}{2(.02515625)}. \qquad \text{(Tables V and VI)}$$

$$x = \frac{7000(.02515625)}{.32801593} = \$536.84.$$

EXERCISE XXIX

1. Compute $\dfrac{1}{(a_{\overline{15|}} \text{ at } .05)} = \dfrac{1}{10.379658}$ to verify the entry in Table IX.

Find the annual rents of the annuities below.

Prob.	Payment Interval	Interest Rate	Term	Amount	Present Value
2.	3 mo.	.06, $m = 4$	12 yr., 3 mo.		\$ 6,500
3.	6 mo.	.05, $m = 2$	17 yr., 6 mo.	\$ 8,500	
4.	1 yr.	.04, $m = 1$	15 yr.		3,000
5.	6 mo.	.05, $m = 1$	8 yr.		15,000
6.	3 mo.	.05, $m = 2$	5 yr., 6 mo.	3,750	
7.	3 mo.	.06, $m = 1$	15 yr.		4,000
8.	1 yr.	.05, $m = 1$	17 yr.		7,000
9.	6 mo.	.05, $m = 4$	12 yr., 6 mo.	10,000	
10.	1 yr.	.07, $m = 2$	9 yr.		2,500
11.	6 mo.	.07, $m = 2$	9 yr.	2,500	

12. If money is worth 6%, effective, find the annuity, payable annually for 25 years, which may be purchased for \$1000.

13. If money is worth 6%, effective, find the annuity, payable annually for 10 years, whose amount is \$1.

14. If money is worth 4%, compounded annually, what annuity, payable annually for 15 years, may be purchased for \$1?

15. If money is worth the effective rate i, derive a formula for the payment of the annuity, payable annually for n years, which may be purchased for \$1.

16. If money is worth the effective rate i, derive a formula for the payment of the annuity, payable annually for n years, whose amount is \$1.

17. In order to create a fund of \$2000 by the end of 10 years, what must a man deposit at the end of each 6 months in a bank which credits interest semi-annually at the rate 3%?

18. The present liability of a debt is \$12,000. If money is worth 5.5%, compounded semi-annually, what should be paid at the end of each year for 10 years to discharge the liability in full?

31. Determination of the term. — If the term of an annuity is unknown, interpolation methods furnish the solution of the problem with sufficient accuracy for practical purposes.

Example 1. — For how long must a man deposit \$175 at the end of each 3 months in a bank in order to accumulate a fund of \$7500, if the bank credits interest quarterly at the rate 6%?

Solution. — The deposits form an annuity whose amount is \$7500. Let the unknown term be k interest periods.

Case 1
$n = k$ int. periods,
$p = 1,\ i = .015,$
$R = \$175,\ S = \$7500.$

$$7500 = 175(s_{\overline{k}|}\ at\ .015),$$

$$(s_{\overline{k}|}\ at\ .015) = \frac{7500}{175} = 42.857.$$

n	$(s_{\overline{n}|}\ at\ .015)$
33	42.299
k	42.857
34	43.933

From the column in Table VII for $i = .015$, we obtain the first and third entries at the left. By interpolation,

$$k = 33 + \frac{558}{1634} = 33.341 \text{ int. periods.}$$

The term is $\frac{k}{4} = 8.34$ years. However, since an annuity whose term is not an integral number of payment intervals has not been defined, this result is useful only because it permits us to make the following statement: The \$175 payments must continue for 8.5 years to create a fund of at least \$7500; when the 33d payment occurs at the end of 8.25 years, the fund amounts to less than \$7500.

Example 2. — Find the term of an annuity whose present value is \$8500 and whose annual rent is \$2000, payable quarterly. Interest is at the rate (.06, m = 2).

Solution. — Let the unknown term be k interest periods.

Case 1
$n = k$ int. periods,
$p = 2,\ i = .03,$
$R = \$1000,\ A = \$8500.$

$$8500 = 1000(a^{(2)}_{\overline{k}|}\ at\ .03) = 1000\,\frac{.03}{j_2}(a_{\overline{k}|}\ at\ .03),$$

$$(a_{\overline{k}|}\ at\ .03) = \frac{8500(j_2\ at\ .03)}{30},$$

$$(a_{\overline{k}|}\ at\ .03) = \frac{8500(.02977831)}{30} = 8.437.$$

n	($a_{\overline{n}\rceil}$ at .03)
9	7.786
k	8.437
10	8.530

The first and third entries at the left are from the column in Table VIII for $i = .03$. $k = 10 - \frac{93}{744} = 9.88$ interest periods. The term is $\frac{k}{2} = 4.94$ years. Hence, for an annuity whose term is 4.75 years, the present value is less than \$8500, while the present value would be greater than \$8500 if the term were 5 years.

NOTE. — Values of k found by interpolation in Tables VII and VIII are in error by less than half of the interest rate per period.[1] In interpolating, use three decimal places of the table entries and compute the value of k to three decimal places.

EXERCISE XXX[2]

Find the terms of the annuities below.

PROB.	PAYMENT INTERVAL	INTEREST RATE	PRESENT VALUE	AMOUNT	ANNUAL RENT
1.	1 yr.	.05, $m = 1$		\$5000	\$ 500
2.	6 mo.	.06, $m = 2$		7500	250
3.	1 yr.	.03, $m = 1$	\$8000		400
4.	3 mo.	.08, $m = 4$	9000		1000
5.	6 mo.	.05, $m = 1$	6500		1300
6.	1 mo.	.045, $m = 2$		3500	600
7.	6 mo.	.05, $m = 2$	8500		1000
8.	3 mo.	.05, $m = 2$		8500	1000
9.	1 mo.	.03, $m = 1$	4600		2500
10.	1 yr.	.07, $m = 1$		7450	700

11. For how many full years will it be necessary to deposit \$250 at the end of each year to accumulate a fund of at least \$3500, if the deposits earn 5%, compounded annually?

12. The cash value of a house is \$15,000. In buying it on the installment plan a purchaser has agreed to pay \$1000 at the end of each 6 months as long as necessary. For how long must he pay if money is worth 6%, compounded semi-annually?

32. Determination of the interest rate. —

Example 1. — Under what nominal rate, converted quarterly, is \$7150 the present value of an annuity whose annual rent is \$880, payable quarterly for 12 years and 6 months?

[1] For justification of this statement see Appendix, Note 6.

[2] See supplementary Section 33 for other problems in which the term is unknown.

Solution. — Let r be the unknown rate per period.

Case 1
n = 50 int. periods,
$p = 1, i = r,$
R = \$220, A = \$7150.

$$7150 = 220(a_{\overline{50}|} \text{ at } r),$$
$$(a_{\overline{50}|} \text{ at } r) = \tfrac{7150}{220} = 32.500.$$

i	$(a_{\overline{50}\|} \text{ at } i)$
.0175	33.141
r	32.500
.0200	31.424

The first and third entries at the left were obtained from the row in Table VIII for $n = 50$. Since

$$.0200 - .0175 = .0025,$$
$$r = .0175 + \tfrac{641}{1717}(.0025) = .0175 + .00093 = .01843.$$

The nominal rate is $j = 4r = 4(.01843) = .07372$, converted quarterly.

NOTE. — A value of r obtained as above usually is in error by not more than [1] $\frac{1}{25}$th of the difference of the table rates used in the interpolation. Hence, $r = .01843$ probably is in error by not more than $\frac{1}{25}(.0025) = .0001$, and the nominal rate $j = .07372$ is in error by not more than .0004. We are justified only in saying that the rate is approximately .0737, with doubt as to the last digit.

Supplementary Example 2. — Determine the nominal rate in Example 1 accurately to hundredths of 1%.

Solution. — From Example 1, $(a_{\overline{50}|} \text{ at } r) = 32.500$, and $r = .0184$, approximately. It is probable that r is between .0184 and .0185, or else between .0184 and .0183. Since our tables do not use the rate .0184, we compute

$$(a_{\overline{50}|} \text{ at } .0184) = \frac{1 - (1.0184)^{-50}}{.0184}. \qquad 50 \log 1.0184 = 50(.0079184) = .39592.$$

$$(a_{\overline{50}|} \text{ at } .0184) = \frac{1 - .40186}{.0184} = 32.507. \qquad \log (1.0184)^{-50} = 9.60408 - 10. \quad (1.0184)^{-50} = .40186.$$

Since 32.507 is greater than 32.500, r must be greater than .0184, and probably is between .0184 and .0185. By logarithms, $(a_{\overline{50}|} \text{ at } .0185) = 32.438$.

i	$(a_{\overline{50}\|} \text{ at } i)$
.0184	32.507
r	32.500
.0185	32.438

From interpolation in the table at the left,

$$r = .0184 + \tfrac{7}{69}(.0001) = .018410.$$

The nominal rate is $j = 4r = .073640$, which is certainly accurate to hundredths of 1%, and is probably accurate to thousandths of 1%.

[1] The author gives no theoretical justification for this statement. He has verified its truth for numerous examples scattered over the range of Tables VII and VIII.

NOTE. — We could obtain the solution of Example 1 with any desired degree of accuracy by successive computations as in Example 2. Our accuracy would be limited only by the extent of the logarithm tables at our disposal.

In Example 1, the most simple formulas (Case 1, with $p = 1$) applied because the conversion period equaled the payment interval. In more complicated examples, the solution may be obtained by first considering a new problem of the simple type met in Example 1.

Example 3. — Under what nominal rate, converted semi-annually, is \$7150 the present value of an annuity whose annual rent is \$880, payable quarterly for 12 years and 6 months?

Solution. — Let the unknown nominal rate be j. We could use the formulas of Case 1, with $p = 2$, in the solution, but the work would be slightly complicated. Instead, we first solve the following new problem: "*Determine the nominal rate, w, converted quarterly, under which the present value of the annuity will be* \$7150." We choose quarterly conversions here because the annuity is payable quarterly. This new problem is the Example 1 solved above, so that $w = .0737$. The rate j, compounded semi-annually, must be equivalent to the rate .0737, compounded quarterly, because the present value of the annuity is \$7150 under both of these rates. Hence, the effective rates corresponding to these two rates must be the same.[1] From equation 17, if i represents the effective rate,

$$1 + i = \left(1 + \frac{j}{2}\right)^2, \quad 1 + i = \left(1 + \frac{.0737}{4}\right)^4 = (1.0184)^4.$$

$$\left(1 + \frac{j}{2}\right)^2 = (1.0184)^4; \quad 1 + \frac{j}{2} = (1.0184)^2 = 1.03714.$$

Table II was used in computing 1.03714. The desired nominal rate is $j = 2(.03714) = .07428$, or approximately .0743, with doubt as to the last digit.

EXERCISE XXXI[2]

In the first ten problems find the nominal rates as closely[3] as is possible by interpolation in the tables.

[1] For a similar problem see Section 10, illustrative Example 3.

[2] The Miscellaneous Problems at the end of the chapter may be taken up immediately after the completion of Exercise XXXI.

[3] If the instructor desires, the students may be requested to obtain accuracy to hundredths of 1%, as in Example 2 above.

Prob.	Annual Rent	Payment Interval	Interest Period	Amount	Present Value	Term
1.	$1000	1 year	1 year	$15,700		12 yr.
2.	100	1 year	1 year		$ 1,785	25 yr.
3.	500	1 year	1 year		5,390	15 yr.
4.	100	6 mo.	6 mo.		1,110	17 yr., 6 mo.
5.	400	3 mo.	3 mo.	2,500		5 yr., 3 mo.
6.	1000	1 year	1 year	53,000		26 yr.
7.[1]	200	3 mo.	1 year	2,750		9 yr.
8.[1]	200	3 mo.	6 mo.	2,750		9 yr.
9.	2400	1 mo.	1 year		14,500	8 yr.
10.	500	6 mo.	3 mo.	17,500		24 yr., 6 mo.

11. A man has paid $100 to a building and loan association at the end of each 3 months for the last 10 years. If he now has $5500 to his credit, at what nominal rate, converted quarterly, does the association compute interest?

12. By use of the result of problem 11, find the effective rate of interest paid by the association of problem 11.

13. It has been agreed to pay $1100 at the end of each 6 months for 8 years. Under what nominal rate, compounded semi-annually, would this agreement be equivalent to a cash payment of $14,000?

14. A fund of $12,000 has been deposited with a trust company in order to provide an income of $400 at the end of each 3 months, for 10 years, at which time the fund will be exhausted. At what effective rate does the trust company credit interest on the fund?

Hint. — First find the equivalent nominal rate, payable quarterly.

SUPPLEMENTARY MATERIAL

33. Difficult cases and exact methods in finding the term. — When the formulas of Case 2 apply to an annuity, it is necessary to use the explicit formulas III in finding the term if it is unknown.

Example 1. — The amount of an annuity is $8375, and the annual rent is $1700, payable semi-annually. What is the term if money is worth $(.06, m = 4)$?

[1] See illustrative Example 3. The same preliminary work should be used for both of problems 7 and 8. First determine the nominal rate, converted quarterly, under which $2750 is the amount.

Solution. — Case 2 applies. Hence, let the unknown term be k years.

Case 2
$n = k$ yr., $p = 2$,
$j = .06$, $m = 4$,
$R = \$1700$, $A = \$8375$.

$$8375 = 1700(s_{\overline{k}|}^{(2)} \text{ at } .06, m = 4).$$

$$8375 = 1700\frac{(1.015)^{4k} - 1}{2[(1.015)^2 - 1]}.$$

From Table V, the denominator is 2(.030225).

$$(1.015)^{4k} - 1 = \frac{8375(2)(.030225)}{1700} = .29781.$$

$$(1.015)^{4k} = 1 + .29781 = 1.29781. \quad (40)$$

(*a*) To solve equation 40 by interpolation, we use entries from the column in Table V for $i = .015$. We obtain

n	$(1.015)^n$
17	1.28802
$4k$	1.29781
18	1.30734

$$4k = 17 + \frac{979}{1932} = 17.507, \text{ or } k = 4.377.$$

(*b*) To obtain the exact value of k from equation 40, take the logarithm of both sides of the equation, using Table II for log 1.015.

$$4k \log 1.015 = \log 1.29781; \quad 4k(.0064660) = .11321.$$

$$k = \frac{11321}{4(646.60)} = \frac{11321}{2586.4}. \qquad \begin{aligned} \log 11321 &= 4.05389 \\ \log 2568.4 &= 3.41270 \\ \hline \log k &= 0.64119 \end{aligned}$$

$$k = 4.3771.$$

The solutions of problems, treated by interpolation in Section 31, may be obtained by solving exponential equations, as in solution (*b*) above. In these exact solutions it is always necessary to use the explicit algebraic expressions for the present values and the amounts of the annuities concerned.

Example 2. — If the rate is (.06, $m = 2$), find the term of an annuity whose present value is \$8500 and whose annual rent is \$2000, payable quarterly.

Solution. — Let the unknown term be k interest periods.

Case 1
$n = k$ int. periods,
$p = 2$, $i = .03$,
$R = \$1000$, $A = \$8500$.

$$8500 = 1000(a_{\overline{k}|}^{(2)} \text{ at } .03) = 1000\frac{1 - (1.03)^{-k}}{2[(1.03)^{\frac{1}{2}} - 1]}.$$

$$8500 = 1000\frac{1 - (1.03)^{-k}}{.02977831}. \quad \text{(Table XI)}$$

$$1 - (1.03)^{-k} = \frac{8500(.02977831)}{1000} = .25312, \quad \text{(By Table I)}$$

$$(1.03)^{-k} = 1 - .25312 = .74688. \quad \therefore \; -k \log 1.03 = \log .74688.$$

$$-k(.0128372) = 9.87325 - 10 = -.12675. \qquad \text{(log 1.03 from Table II)}$$

$$k = \frac{12675}{1283.7} = 9.8738 \text{ periods of 6 months.}$$

The term is $\frac{k}{2} = 4.9369$ years. Compare this with the result by interpolation in Example 2 of Section 31.

EXERCISE XXXII

1. At the end of each 6 months a man deposits $200 in a bank which credits interest quarterly at the rate 3%. For how many years must the deposits continue in order to create a fund of $3000? Use the exact method (*b*) of Example 1 above.

2. If money is worth 6%, compounded monthly, for how long must payments of $2000 be made at the end of each 6 months in order to discharge a debt whose present liability is $30,000? Solve by both an interpolation and an exact method.

3. Solve illustrative Example 1 of Section 31 by the exact method.

4. Solve problem 9 of Exercise XXX by the exact method.

5. To create an educational fund for a daughter, a father decides to deposit $500 at the end of each 6 months in a bank which credits interest annually, from the date of deposit, at the rate 4%. When will the fund amount to at least $6000? Solve by the exact method.

MISCELLANEOUS PROBLEMS

1. If money is worth 5%, effective, what equal payments should be made at the end of each year for 10 years in purchasing a house whose equivalent cash price is $5000?

2. If a man saves $200 at the end of each month, when will he be able to buy an automobile, worth $3000, if his deposits accumulate at the rate 5%, compounded semi-annually?

3. A depreciation fund is being accumulated by equal deposits at the end of each month in a bank which credits 6% interest monthly on deposits from date of deposit. What is the monthly deposit if the fund contains $7000 at the end of 5 years?

4. In purchasing a house, worth $20,000 cash, a man has agreed to pay $5000 cash and $1000 semi-annually for 9 years. What interest rate, compounded semi-annually, is being used in the transaction?

5. An insurance policy, on maturing, gives the policy holder the option of an immediate endowment of $15,000 or an annuity, payable quarterly for 10 years. Under the rate 3.5%, effective, what will be the quarterly payment of the annuity?

6. A fund of $50,000 has been deposited with a trust company which credits interest quarterly on all funds at the rate 5%. For how long will this fund furnish a man payments of $1000 at the end of each 3 months?

7. On the death of her husband, a widow deposited her inherited estate of $25,000 with a trust company. If interest is credited semi-annually on the fund at the nominal rate 4%, for how long will the widow be able to withdraw $800 at the end of each 6 months?

8. A certain loan bureau lends money to heads of families on the following plan: In return for a $100 loan, $9 must be paid at the end of each month for 1 year. What effective rate of interest is being charged?

HINT. — First find the nominal rate, compounded monthly.

9. A certain homestead is worth $5000 cash. The government sold this to an ex-soldier under the agreement that he should pay $1000 at the end of each 6 months until the liability is discharged. If interest is at the rate (.04, $m = 2$), for how long must the payments continue?

10. The annual rent of an annuity is $50, payable annually. The present value of the annuity is $400 and the amount is $600. Find the effective rate of interest by use of the relation 39 of Section 30.

11. A certain farm has a cash value of $20,000. If money is worth (.05, $m = 2$), what equal payments, made at the beginning of each 6 months for 6 years, would complete the purchase of the farm?

12. A man borrowed $2000 under the agreement that interest should be at the rate (.06, $m = 2$) during the life of the transaction. He made no payments of either interest or principal for 4 years. At that time, he agreed to discharge all liability in connection with the debt by making equal payments at the end of each 3 months for 3 years. Find the quarterly payment.

CHAPTER V

THE PAYMENT OF DEBTS BY PERIODIC INSTALLMENTS

34. Amortization of a debt. — A debt, whose present value is A, is said to be **amortized** under a given rate of interest, if **all liabilities as to principal and interest are discharged** by a sequence of periodic payments. When the payments are equal, as is usually the case, they form an annuity whose present value must equal A, the original liability. Hence, most problems in the amortization of debts involve the present value formulas for annuities. Many amortization problems have been solved in previous chapters.

Example 1. — A man borrows $15,000, with interest payable annually at the rate 5%. The debt is to be paid, interest as due and original principal included, by equal installments at the end of each year for 5 years. (*a*) Find the annual payment. (*b*) Form a schedule showing the progress of repayment (or **amortization**) of the principal.

Case 1
$n = 5$ int. periods,
$p = 1$, $i = .05$,
$R = \$x$, $A = \$15,000$.

Solution. — Let $\$x$ be the payment. The present value of the payment annuity, at the rate (.05, $m = 1$), must equal $15,000. $15000 = x(a_{\overline{5}|}\ at\ .05)$.

$$x = 15000 \frac{1}{(a_{\overline{5}|}\ at\ .05)} = \$3464.622.$$

AMORTIZATION SCHEDULE

YEAR	OUTSTANDING PRINCIPAL AT BEGINNING OF YEAR	INTEREST AT 5% DUE AT END OF YEAR	ANNUAL PAYMENT AT END OF YEAR	FOR REPAYMENT OF PRINCIPAL AT END OF YEAR
1	$15,000.000	$ 750.000	$ 3,464.622	$ 2,714.622
2	12,285.378	614.269	3,464.622	2,850.353
3	9,435.025	471.751	3,464.622	2,992.871
4	6,442.154	322.108	3,464.622	3,142.514
5	3,299.640	164.982	3,464.622	3,299.640
Totals	$46,462.197	$2323.110	$17,323.110	$15,000.000

NOTE 1. — The schedule shows that the payments satisfy the creditor's demands for interest and likewise return his principal in installments. If x = \$3464.622 was computed correctly, we know, without forming the schedule, that these facts must be true because the present value of the five payments is \$15,000. The checks on the arithmetic done in the table are that the last total should be \$15,000, the sum of the second and the last should equal the third, and the second should be interest on the first total for one year at 5%. Notice that the repayments of principal increase from year to year, while the interest payments decrease. Amortization schedules are very useful in the bookkeeping of both debtor and creditor because the exact outstanding liability at every interest date is clearly shown. The outstanding principal, or liability at any date, is sometimes called the book value of the debt at that time.

NOTE 2. — Since money is worth 5%, in Example 1, we may assume that the debtor invests the \$15,000 at 5% immediately after borrowing it. The accumulation of this fund should provide for all the annual payments, to be made to the creditor, because their present value is \$15,000. A numerical verification of this fact is obtained in the amortization table above if we merely alter the titles of the columns, as below, leaving the rest of the table unchanged.

YEAR	IN FUND AT BEGINNING OF YEAR	INTEREST RECEIVED AT END OF YEAR	PAYMENT TO CREDITOR AT END OF YEAR	TAKEN FROM FUND AT END OF YEAR
1	\$15,000	\$750	\$3464.622	\$2714.622
—	—	—	—	—

Thus, at the end of the first year, the debtor receives \$750 from his invested fund and, in order to make the payment of \$3464.622 to his creditor, he takes \$2714.62 from the principal. By the end of 5 years, the fund reduces to zero.

Example 2. — In Example 1, without using the amortization schedule, determine the principal outstanding at the beginning of the third year.

Solution. — The outstanding principal, or liability, is the present value of all payments remaining to be made. These form an annuity whose term is three years. The outstanding principal is

Case 1
$n = 3$ int. periods,
$p = 1, i = .05, R = \$3464.62.$

$$3464.62(a_{\overline{3}|} \; at \; .05) = \$9435.03.$$

This is the third entry of the first column of the amortization schedule.

Example 3. — A debt whose present value is \$30,000, bearing interest at the rate 4.5%, compounded semi-annually, is to be amortized in 10 years by equal payments at the end of each 3 months. (*a*) Find the quarterly payment. (*b*) Find the principal outstanding at the end of 5 years, after the payment due has been made.

Solution. — (a) Let $x be the quarterly payment. The present value of the payment annuity must equal $30,000.

Case 1
$n = 20$ int. periods,
$p = 2, i = .0225,$
$R = 2x, A = \$30{,}000.$

$$30000 = 2x(a_{\overline{20}|}^{(2)} \text{ at } .0225) = 2x\frac{.0225}{j_2}(a_{\overline{20}|} \text{ at } .0225).$$

$$x = \frac{15000\,j_2}{.0225}\,\frac{1}{(a_{\overline{20}|} \text{ at } .0225)}.$$

$$x = \frac{15000(.02237484)}{.0225}(.06264207) = \$934.403.$$

(b) At the end of the 5th year, or the beginning of the 6th, the outstanding liability, L, is the present value of payments extending over 5 years.

Case 1
$n = 10$ int. periods,
$i = .0225, p = 2, R = \$1868.81.$

$$L = 1868.81(a_{\overline{10}|}^{(2)} \text{ at } .0225).$$

$$L = 1868.81\frac{.0225}{j_2}(a_{\overline{10}|} \text{ at } .0225).$$

$$L = \$16{,}661.95.$$

EXERCISE XXXIII

√ 1. A loan of $5000, with interest at 6%, payable semi-annually, is to be amortized by six semi-annual payments, the first due after 6 months. (a) Find the payment, to three decimal places. (b) Form the amortization schedule for the debt.

2. In problem 1, without using the amortization table, find the principal unpaid at the end of 1 year and 6 months, just after the payment due has been made.

3. A man deposits $10,000 with a trust company which credits 5% interest annually. The fund is to provide equal payments at the end of each year for 5 years, at the end of which time the fund is to be exhausted. (a) Find the annual payment to three decimal places. (b) Form a table showing the amortization of the fund.

HINT. — See Note 2, Section 34; think of the trust company as the debtor.

4. In problem 3, without using the table, find the amount remaining in the fund at the end of 2 years, after the payment due has been made.

5. A purchaser of a house owes $7500, and interest at 6% is payable semi-annually on all amounts remaining due. He wishes to discharge his debt, principal and interest included, by twelve equal semi-annual installments, the first due after 6 months. Find the necessary semi-annual payment.

6. A street assessment of $500 against a certain piece of real estate is to be amortized, with interest at 6%, by six equal annual payments, the first due after 1 year. What part of the assessment will remain unpaid at the beginning of the 4th year, after the payment due has been made?

7. A house is worth \$25,000 and the owner, on selling, desires the equivalent of interest at the rate 5%, payable semi-annually. (*a*) What quarterly installment, for 8 years, in addition to a cash payment of \$5000, would satisfy the owner? (*b*) How much of the principal of the debt remains unpaid at the end of 3 years and 6 months, after the payment due has been made?

8. A debt of \$12,000, with interest payable semi-annually at the rate 5%, is to be amortized in 10 years by equal semi-annual installments, the first due after 6 months. What part of the debt will remain unpaid at the beginning of the 6th year, after the payment due has been made?

9. In problem 8, what part of the 11th payment is interest and what part is repayment of principal?

10. A debt will be discharged, principal and interest, at 6% effective, included, by payments of \$1200 at the end of each year for 12 years. (*a*) What is the original principal of the debt? (*b*) What principal will remain outstanding at the beginning of the 5th year? (*c*) What part of the 5th payment will be interest and what part repayment of principal?

11. A trust fund of \$100,000 was created to provide a regular income at the end of each month for 20 years. If the trust company uses the interest rate 4%, converted semi-annually, what is the monthly payment, if the fund is to be exhausted by the end of 20 years?

12. It was agreed to amortize a debt of \$20,000 with interest at 5%, by 12 equal annual payments, the first due in one year. The debtor failed to make the first four payments. What payment at the end of 5 years would bring the debtor up to date on his contract?

13. A debt of \$38,000 is to be amortized by payments of \$2000 at the end of each 3 months for 6 years. (*a*) Find the nominal rate, compounded quarterly, at which interest is being paid. (*b*) What is the effective rate of interest?

14. A certain insurance policy on maturing gives the option of \$10,000 cash or \$345 at the end of each 6 months for 20 years. What rate of interest is being used by the insurance company?

35. Amortization of a bonded debt. — In amortizing a debt which is in the form of a bond issue, the periodic payments cannot be exactly equal. If the bonds are of \$1000 denomination, for example, the principal repayments must be multiples of \$1000, because any individual bond must be retired in one installment.

Example 1. — Construct a schedule for the amortization, by 10 annual payments as nearly equal as possible, of a \$10,000 debt which is outstanding in bonds of \$100 denomination, and which bears 4% interest payable annually. The first payment is due at the end of 1 year.

Solution. — Let \$$x$ be the annual payment, which would be made if the payments were to be equal.

Case 1
n = 10 int. periods,
$p = 1, i = .04,$
$R = \$x, A = \$10{,}000.$

$$10000 = x(a_{\overline{10|}} \text{ at } .04).$$

$$x = 10000 \frac{1}{(a_{\overline{10|}} \text{ at } .04)} = \$1232.91.$$

The annual payments should be as close as possible to \$1232.91. Thus, at the end of the 1st year, the interest due is \$400, leaving 1232.91 − 400.00 = \$832.91 available for repayment of principal. Hence, retire 8 bonds, or \$800 of the principal on this date, making a total payment of \$1200. At the end of the next year 1232.91 − 368.00 = \$864.91 is available for retiring bonds. Therefore, pay 9 bonds or \$900 of the principal.

Amortization Schedule for a Bonded Debt

Year	Principal Outstanding at Beginning of Year	Interest Due at End of Year	Bonds Retired at End of Year	Total Payment at End of Year
1	\$10,000	\$ 400	8	\$ 1,200
2	9,200	368	9	1,268
3	8,300	332	9	1,232
4	7,400	296	9	1,196
5	6,500	260	10	1,260
6	5,500	220	10	1,220
7	4,500	180	11	1,280
8	3,400	136	11	1,236
9	2,300	92	11	1,192
10	1,200	48	12	1,248
Totals	\$58,300	\$2,332	100	\$12,332

EXERCISE XXXIV

1. A \$1,000,000 debt is outstanding in the form of \$1000 bonds which pay 6% interest annually. Construct a schedule for the retirement of the debt, principal and interest included, by five annual payments as nearly equal as possible, the first payment due at the end of 1 year.

2. A \$1,000,000 issue of bonds, paying 5% interest annually, consists of 500 bonds of \$100, 200 bonds of \$500, 200 of \$1000, and 130 of \$5000

denomination. Construct a schedule for the amortization of the debt by 10 annual payments as nearly equal as possible.

HINT. — In the schedule, make a separate column for each class of bonds.

36. Problems in which the periodic payment is known. — If the present liability of a debt, the interest rate, and the size and frequency of the amortization payments are known, the term of the payment annuity can be found as in Section 31.

Example 1. — A house is valued at \$10,000 cash. It is agreed to pay \$1200 cash and \$1200 at the end of each 6 months as long as necessary to amortize the given cash value with interest at 5%, payable semi-annually. (*a*) For how long must the payments continue? (*b*) Construct an amortization schedule.

Solution. — (*a*) After the cash payment of \$1200, \$8800 remains due. Let k be the time in interest periods necessary to amortize it with interest at the rate (.05, $m = 2$).

Case 1
$n = k$ int. periods,
$p = 1, i = .025,$
$R = \$1200,\ A = \$8800.$

$$8800 = 1200(a_{\overline{k}|}\ at\ .025);$$
$$(a_{\overline{k}|}\ at\ .025) = 7.333.$$

By interpolation in Table VIII, $k = 8.20$. Hence, 8 full payments of \$1200 must be made in addition to the first cash payment. After the \$1200 payment is made, at the end of 4 years, some principal is still outstanding because k is greater than 8. A partial payment will be necessary at the next payment date. These conclusions are verified in the schedule below.

(*b*) AMORTIZATION SCHEDULE

PAYMENT INTERVAL	OUTSTANDING PRINCIPAL AT BEGINNING OF INTERVAL	INTEREST DUE AT END OF INTERVAL	TOTAL PAYMENT AT END OF INTERVAL	PRINCIPAL REPAID AT END OF INTERVAL
1	\$8800.000	\$220.000	\$1200.	\$ 980.000
2	7820.000	195.500	1200.	1004.500
3	6815.500	170.388	1200.	1029.612
4	5785.888	144.647	1200.	1055.353
5	4730.535	118.263	1200.	1081.737
6	3648.798	91.220	1200.	1108.780
7	2540.018	63.500	1200.	1136.500
8	1403.518	35.088	1200.	1164.912
9	238.606	5.965	244.571	238.606
Totals	\$41,782.863	\$1044.571	\$9844.571	\$8800.000

Example 2. — Without using the amortization table, find the principal still unpaid in Example 1 at the end of $2\frac{1}{2}$ years, after the payment due has been made.

Solution. — Let $\$M$ be the amount remaining due. The payment of $\$M$ at the end of $2\frac{1}{2}$ years, in addition to the payments already made, would complete the payment of the debt whose original principal was \$8800. Hence, this "Old Obligation" must have the same value as the "New Obligations" listed below.

OLD OBLIGATION	NEW OBLIGATIONS
\$8800 due at the beginning of the transaction.	(*a*) $\$M$ due at the end of $2\frac{1}{2}$ years. (*b*) Payments of \$1200 due at the end of each 6 months for $2\frac{1}{2}$ years.

To find M, write an equation of value, under the rate $(.05, m = 2)$, with the end of $2\frac{1}{2}$ years as the comparison date. The sum of the values of obligations (*b*) is the amount of the annuity they form.

$$8800(1.025)^5 = M + 1200(s_{\overline{5|}} \text{ at } .025).$$

$$M = 8800(1.025)^5 - 1200(s_{\overline{5|}} \text{ at } .025) = 9956.39 - 6307.59 = \$3648.80, \quad (41)$$

which checks with the proper entry in the table of Example 1. The debtor could close the transaction at the end of $2\frac{1}{2}$ years by paying the regular installment plus \$3648.80 or $(1200 + 3648.80) = \$4848.80$.

NOTE 1. — Recognize that $8800(1.025)^5$ is the amount the creditor should have at the end of $2\frac{1}{2}$ years if he invested \$8800 at $(.05, m = 2)$, whereas he actually has possession of only $1200(s_{\overline{5|}} \text{ at } .025)$ as a consequence of the payments received from the debtor. Hence, equation 41 shows that M is the difference between what the creditor should have and what he actually has.

NOTE 2. — By the method of Example 2 we can find the final installment in Example 1 without computing the amortization table. Let $\$N$ be the amount remaining due just after the last full payment, at the end of 4 years. Then,

$$N = 8800(1.025)^8 - 1200(s_{\overline{8|}} \text{ at } .025) = 10721.946 - 10483.339 = \$238.607.$$

To close the transaction at the end of $4\frac{1}{2}$ years, the necessary payment is \$238.607 plus interest for 6 months at 5%, or $238.607 + 5.965 = \$244.57$.

EXERCISE XXXV

1. (*a*) How long will it take to amortize a debt whose present value is \$10,000 if payments of \$2000 are made at the end of each year and if these payments include interest at the rate 5%, payable annually. (*b*) Form an amortization schedule for this debt.

2. (*a*) Without using the table in problem 1, find the principal outstanding at the beginning of the 3d year. (*b*) Find the size of the final payment.

3. A debt of \$50,000, with interest payable quarterly at the rate 8%, is being amortized by payments of \$1500 at the end of each 3 months. (*a*) What is the outstanding liability just after the 10th payment? (*b*) Find the final installment.

4. A trust fund of \$100,000 is invested at the rate 6%, compounded semi-annually. Principal and interest are to provide payments of \$5000 at the end of each 6 months until the fund is exhausted. (*a*) How many full payments of \$5000 will be made? (*b*) What will be the size of the final partial payment?

5. The purchaser of a farm has agreed to pay \$1000 at the end of each 3 months for 5 years. (*a*) If these payments include interest at the rate 6%, payable quarterly, what is the outstanding principal at the beginning of the transaction? (*b*) Find the outstanding liability at the beginning of the 3d year. Notice that, since the exact number of the remaining payments is known, part (*b*) should be done like illustrative Example 2 of Section 34; it would be clumsy to use the method of illustrative Example 2 of the present section.

37. Sinking fund method. — A **sinking fund** is a fund formed in order to pay an obligation falling due at some future date. In the following section, unless otherwise stated, it is assumed that the sinking funds involved are created by investing equal periodic payments. Then, the amount in a sinking fund at any time is the amount of the annuity formed by the payments, and examples involving sinking funds can be solved by use of the formulas for the amount of an annuity. Thus, to create a fund of \$10,000 at the end of 10 years by investing $\$x$ at the end of each 6 months for 10 years, at the rate (.06, $m = 2$), x must satisfy

$$10000 = x(s_{\overline{20}|}\ at\ .03); \quad x = 10000 \frac{1}{(s_{\overline{20}|}\ at\ .03)} = \$372.157.$$

Suppose that $\$A$ is borrowed under the agreement that interest shall be paid when due and that the principal shall be paid in one installment at the end of n years. If the debtor provides for the future payment of $\$A$ at the end of n years by the creation of a sinking fund, invested under his own control, his debt is said to be retired by the **sinking fund method**. Under this method, the expense of the debt to the debtor is the sum of (*a*) and (*b*) below:

(*a*) Interest on $\$A$, paid periodically to the creditor when due.

(*b*) Periodic deposits, necessary to create a sinking fund of $\$A$, to pay the principal when due.

NOTE 1. — Recognize that the sinking fund is a private affair of the debtor. The rate of interest paid by the debtor on $\$A$ bears no relation to the rate of interest at which the debtor is able to invest his sinking fund. Usually, the desire for absolute safety for the fund would compel the debtor to invest it at a lower rate than he himself pays on his debt.

Example 1. — A debt of $10,000 is contracted under the agreement that interest shall be paid semi-annually at the rate 6%, and that the principal shall be paid in one installment at the end of 2½ years. (*a*) Under the sinking fund method, what is the semi-annual expense of the debt if the debtor invests his fund at $(.04, m = 2)$? (*b*) Form a table showing the accumulation of the fund.

Solution. — (*a*) Let $\$x$ be the semi-annual deposit to the sinking fund, whose amount at the end of 2½ years is $10,000.

Case 1
$n = 5$ int. periods,
$p = 1, i = .02,$
$R = \$x, S = \$10{,}000.$

$$10000 = x(s_{\overline{5}|} \text{ at } .02);$$

$$x = 10000 \frac{1}{(s_{\overline{5}|} \text{ at } .02)} = \$1921.584.$$

Interest due semi-annually on the principal of the debt is $(.03)(10000) = \$300$. Semi-annual expense is $300 + 1921.58 = \$2221.58$.

(*b*) TABLE SHOWING GROWTH OF SINKING FUND

PAYMENT INTERVAL	IN FUND AT BEGINNING OF INTERVAL	INTEREST AT 4% RECEIVED ON FUND AT END OF INTERVAL	PAYMENT TO FUND AT END OF INTERVAL	IN FUND AT END OF INTERVAL
1	0	0	$1921.584	$1921.584
2	$1921.584	$ 38.432	1921.584	3881.600
3	3881.600	77.632	1921.584	5880.816
4	5880.816	117.616	1921.584	7920.016
5	7920.016	158.400	1921.584	10000.000

NOTE 2. — The **book value** of the debtor's indebtedness, or his net indebtedness, at any time may be defined as the difference between what he owes and what he has in his sinking fund. Thus, at the end of 2 years, the book value of the debt is $10000 - 7920.016 = \$2079.984$.

NOTE 3. — The amount in the sinking fund at any time is the amount of the payment annuity up to that date and can be found without forming the table (*b*). Thus, the amount in the fund at the end of 2 years is

$$1921.584(s_{\overline{4}|} \text{ at } .02) = \$7920.02.$$

EXERCISE XXXVI

1. A debt of $50,000, with interest payable semi-annually at the rate 6%, is to be paid at the end of 3 years by the accumulation of a sinking fund. (*a*) If payments to the fund are made at the end of each 6 months and accumulate at the rate 3%, compounded semi-annually, what is the total semi-annual expense of the debt? (*b*) Form a table showing the accumulation of the fund.

2. (*a*) In problem 1, without using the table, determine the amount in the sinking fund at the end of 2 years. (*b*) What is the book value of the debtor's indebtedness at this time?

3. A loan of $10,000 bears 5% interest, payable semi-annually, and a sinking fund is created by payments at the end of each 6 months in order to repay the principal at the end of 4 years. Find the expense of the debt if the fund accumulates at the rate (.04, $m = 2$).

4. A city issues $100,000 worth of bonds bearing 6% interest, payable annually, and is compelled by law to create a sinking fund to retire the bonds at the end of 10 years. If payments to the fund occur at the end of each year and are invested at 6%, effective, what is the annual expense of the debt?

5. How much is in the sinking fund in problem 4 at the beginning of the 7th year?

6. A loan of $5000 is made under the agreement that interest shall be paid semi-annually at the rate 5.5% on all principal remaining due and that the principal shall be paid in full on or before the end of 6 years. (*a*) Find the semi-annual expense if the debt is amortized by equal payments at the end of each 6 months for 6 years. (*b*) Find the semi-annual expense to retire the debt by the sinking fund method at the end of 6 years, if payments to the fund are made at the end of each 6 months and are invested at (.04, $m = 2$). (*c*) Find the semi-annual expense under the sinking fund method if the fund earns (.06, $m = 2$). (*d*) Which method is most advantageous to the debtor?

7. A sinking fund is established by payments at the end of each 3 months in order to accumulate a fund of $300,000 at the end of 15 years. Find the quarterly payment if interest is at the rate (.06, $m = 2$).

8. To accumulate a fund of $100,000, payments of $5000 are invested at the end of each 6 months at the rate 5%, compounded semi-annually. (*a*) How many full payments must be made? (*b*) How much must be paid on the last payment date?

38. Comparison of the amortization and the sinking fund methods. — To amortize a debt of $\$A$, in n years, with interest payable annually at the rate i, by payments of $\$R$ at the end of each year, we have $A = R(a_{\overline{n}|} \text{ at } i)$ or

$$R = A \frac{1}{(a_{\overline{n}|} \text{ at } i)}. \tag{42}$$

Since $\dfrac{1}{(a_{\overline{n}|} \text{ at } i)} = i + \dfrac{1}{(s_{\overline{n}|} \text{ at } i)}$,

$$R = Ai + A \frac{1}{(s_{\overline{n}|} \text{ at } i)}. \tag{43}$$

If the sinking fund method is used to retire this debt, the amount in the fund at the end of n years is $\$A$. If payments of $\$W$ are made to the fund at the end of each year and accumulate at the effective rate r, then $A = W(s_{\overline{n}|} \text{ at } r)$ or

$$W = A \frac{1}{(s_{\overline{n}|} \text{ at } r)}. \tag{44}$$

The annual interest on $\$A$ at the rate i is Ai, so that the total annual expense E under the sinking fund plan is

$$E = Ai + W = Ai + A \frac{1}{(s_{\overline{n}|} \text{ at } r)}. \tag{45}$$

When $r = i$ in equation 45, E equals R, as given in equation 43. Thus, **the amortization payment $\$R$ is sufficient to pay interest on $\$A$ at the rate i, and to create a sinking fund which amounts to $\$A$ at the end of n years, if the fund also earns interest at the rate i.** Hence, the amortization method may be considered as a special case of the sinking fund method, where the creditor is custodian of the sinking fund and invests it at the rate i.

When r is less than i, $(s_{\overline{n}|} \text{ at } r)$ is less than $(s_{\overline{n}|} \text{ at } i)$, so that $A \dfrac{1}{(s_{\overline{n}|} \text{ at } r)}$ is greater than $A \dfrac{1}{(s_{\overline{n}|} \text{ at } i)}$ and E is greater than R. Similarly, when r is greater than i, the sinking fund expense is less than the amortization expense.

NOTE. — The conclusions of the last paragraph are obvious without the use of any formulas. If the debtor is able to invest his fund at the rate r, greater than i, his expense will be less than under the amortization method because, under the latter, he is investing a sinking fund with his creditor at the smaller rate i.

NOTE. — Equation 42 is sometimes called the **amortization equation**, and equation 44 is called the **sinking fund equation**. Table IX for $\frac{1}{a_{\overline{n}|}}$ may be called a *table of amortization charges* for a debt of \$1 ($A = 1$ in equation 42), and a table of the values of $\frac{1}{s_{\overline{n}|}}$ would be a *table of sinking fund charges* for a debt of \$1.

MISCELLANEOUS PROBLEMS

In solving the more difficult problems of the set below, the student should recall that the writing of an equation of value, for a conveniently selected comparison date, furnishes a systematic method of solution, as in illustrative Example 2 of Section 36.

1. At the end of 5 years, a man will pay \$15,000 cash for a house. (*a*) What equal amounts should he save at the end of each year to accumulate the money if his savings earn 6%, effective? (*b*) What should he save at the *beginning* of each year in order to accumulate the money if the savings earn 6%, effective?

2. A loan of \$5000 is made, with interest at 6%, payable semi-annually. Is it better to amortize the debt in 6 years by equal semi-annual installments, or to pay interest when due and to retire the principal in one installment at the end of 6 years by the accumulation of a sinking fund by semi-annual payments, invested at $(.04, m = 2)$?

3. A man, purchasing a farm worth \$20,000 cash, agrees to pay \$5000 cash and \$1500 at the end of each 6 months. (*a*) If the payments include annual interest at the rate 5%, effective, how many full payments of \$1500 will be necessary? (*b*) What is the purchaser's equity in the house at the beginning of the 3d year?

4. A county has an assessed valuation of \$50,000,000. The county borrows \$500,000 at 5%, payable annually, and is to retire the principal at the end of 20 years through the accumulation of a sinking fund by annual payments invested at 4%, effective. By how much, per dollar of assessed valuation, will the annual taxes of the county be raised on account of the expense of the debt?

5. A debt of \$25,000, with interest payable semi-annually at the rate 6%, is to be amortized by equal payments, at the beginning of each 6 months for 12 years. (*a*) Determine the payment. (*b*) At the beginning of the 4th year, after the payment due has been made, what principal remains outstanding?

6. A debt of \$12,000, with interest at 5%, compounded quarterly, is to be amortized by equal payments at the end of each 3 months for 8 years. At the end of 4 years, what payment, in addition to the one due, would cancel the remaining liability if the creditor should permit the future payments to be discounted, under the rate $(.04, m = 4)$?

7. A debt of \$100,000, with interest at 5% payable annually, will be retired at the end of 10 years by the accumulation of a sinking fund by annual payments invested at 4%, effective. Considering the total annual expense of the debtor as an annuity, under what rate of interest is the present value of this annuity equal to \$100,000? The answer obtained is the rate at which the debtor could afford to amortize his debt, instead of using the sinking fund method described in the problem.

8. In order to accumulate a fund of \$155,000, a corporation invests \$20,000 at the end of each 3 months at the rate $(.08, m = 4)$. (*a*) How many full payments of \$20,000 must be made? (*b*) Three months after the last full payment of \$20,000 is made, what partial payment will complete the fund?

9. The original liability of a debt was \$30,000, and interest is at the rate 5%, effective. Payments of \$2000 were made at the end of each year for 7 years. At the end of that time it was decided to amortize the remaining indebtedness by equal payments at the end of each year for 8 years. Find the annual payment.

HINT. — Equations of value furnish a systematic method for solving problems of this type. Let $\$x$ be the annual payment. Then, the value of the "Old Obligation" below must equal the sum of the values of the "New Obligations" on whatever comparison date is selected.

OLD OBLIGATION	NEW OBLIGATIONS
\$30,000 due at the beginning of the transaction.	(*a*) \$2000 due at the end of each year for 7 years. (*b*) Eight annual payments of $\$x$, the first due at the end of 8 years.

The end of 7 years is the most convenient comparison date.

10. At the end of 5 years, \$10,000 must be paid. (*a*) What equal amounts should the debtor deposit in a savings bank at the end of each 6 months in order to provide for the payment, if his savings accumulate at the rate (.04, $m = 2$)? (*b*) How much must he deposit semi-annually if his first deposit occurs immediately and the last at the end of 5 years?

11. A debtor borrows \$20,000, which is to be repaid, together with all accumulated interest at the rate (.05, $m = 4$), at the end of 6 years. (*a*) In order to pay the debt when due, what equal deposits must be made in a sinking fund at the end of each 6 months if the fund accumulates at the rate (.05, $m = 2$)? (*b*) At what rate, compounded semi-annually, could the debtor just as well have borrowed the \$20,000, under the agreement that it be amortized in 6 years by equal payments at the end of each 6 months?

12. A certain state provided for the sale of farms to war veterans under the agreement that (*a*) interest should be computed at the rate (.04, $m = 2$), and (*b*) the total liability should be discharged by 10 equal semi-annual installments, the first due at the end of $3\frac{1}{2}$ years. Find the necessary installment if the farm is worth \$6000 cash.

13. Under what rate of interest will 25 semi-annual payments of \$500, the first due immediately, amortize a debt of \$9700? Determine both the nominal rate, compounded semi-annually, and the effective rate.

14. A sinking fund is being accumulated by payments of \$1000 at the end of each year. For the first 10 years the fund earns 6%, effective, and, for the next 6 years, 4%, effective. What is the size of the fund at the end of 16 years? It is advisable, first, to find the amount at the end of 16 years due to the payments during the first 10 years.

15. In order to create a fund of \$50,000 by the end of 20 years, what equal payments should be made at the end of each 6 months if the fund accumulates at the rate (.04, $m = 2$) for the first 10 years and at 6%, effective, for the last 10 years?

16. A fund of \$250,000 is given to a university. The principal and interest of this fund are to provide payments of \$2000 at the beginning of each month until the fund is exhausted. If the university succeeds in investing the fund at 5%, compounded semi-annually, how many full payments of \$2000 will be made?

17. Under what nominal rate, compounded semi-annually, would it be just as economical to amortize a debt in 10 years by equal payments at the end of each 6 months, as to pay interest semi-annually at the rate

6%, on the principal, and to repay the principal at the end of 10 years by the accumulation of a sinking fund by equal payments at the end of each 6 months, invested at the rate (.04, $m = 2$).

18. A factory is worth \$100,000 cash. At the time of purchase \$25,000 was paid. Payments of \$8000, including interest, were made at the end of each year for 6 years. The liability was completely discharged by six more annual payments of \$9000, including interest, the first occurring at the end of the 7th year. What effective rate of interest did the debtor pay?

HINT. — Write an equation of value at the end of 6 years. Transpose all quantities in the equation to one side and solve by interpolation. See Appendix, Note 3, Example 2.

SUPPLEMENTARY MATERIAL

39. Funds invested with building and loan associations. — A **building and loan association** is a coöperative enterprise whose main purpose is to provide funds from which loans may be made to members of the association desiring to build homes. Some members are investors only, and do not borrow from the association. Others are simultaneously borrowers and investors. Shares of stock are sold to members generally in units of \$100. Each share is paid for by equal periodic installments called **dues**, payable at the beginning of each month. Profits of the association arise from investing the money received as dues. Members share in the profits in proportion to the amount they have paid on their shares of stock, and their profits are credited as payments on their stock. When the periodic payments on a \$100 share, plus the credited earnings, have reached \$100, the share is said to **mature**. The owner may then withdraw its value or may allow it to remain invested with the association.

Over moderate periods of time, the interest rate received by an association on its invested funds is approximately constant. The amount to the credit of a member, who has been making periodic payments on a share, is the amount of the annuity formed by his payments, with interest at the rate being earned by the association.

Example 1. — A member pays \$2 at the beginning of each month on a share in an association whose funds earn 6%, compounded monthly. What is to the credit of the member just after the 20th payment?

Solution. — The payments form an ordinary annuity of 20 payment intervals, if the term is considered to begin 1 month before the first payment is made. The amount on the 20th payment date is the amount of this ordinary annuity, or $2(s_{\overline{20}|}\ at\ .005) = \$41.96$.

Example 2. — A member pays $1 at the beginning of each month on a $100 share. If the association is earning at the rate (.06, $m = 12$), when will the share mature?

Solution. — Let k be the number of installments necessary to bring the amount to the member's credit up to $100.

Case 1
$n = k$ int. periods,
$p = 1,\ i = .005,$
$R = \$1,\ S = \$100$

$$100 = (s_{\overline{k}|}\ at\ .005).$$

By interpolation in Table VII, $k = 81.3$. Just after the 81st payment, at the beginning of the 81st month, the member is credited with $(s_{\overline{81}|}\ at\ .005) = \$99.558$. By the beginning of the 82d month, this book value has earned $(.005)(99.558) = \$.498$, and the new book value is $99.558 + .498 = \$100.056$. Hence, no payment is necessary at the beginning of the 82d month. Properly speaking, the share matured to the value $100 at a time during the 81st month, but the member ordinarily would be informed of the maturity at the beginning of the 82d month.

Example 3. — A $100 share matures just after the 83d monthly payment of $1. (*a*) What rate, compounded monthly, is the association earning on its funds? (*b*) What is the effective rate?

Solution. — (*a*) Let r be the rate per period of 1 month. The amount of the payment annuity is $100,

Case 1
$n = 83$ int. periods,
$p = 1,\ i = r,$
$R = \$1,\ S = \$100.$

$$100 = (s_{\overline{83}|}\ at\ r).$$

By interpolation in Table VII, $r = \frac{5}{12}\% + \frac{108}{364}(\frac{1}{12}\%) = .441\%$. The nominal rate is $12(.00441) = .0529$, compounded monthly. (*b*) The effective rate i is obtained from

$$1 + i = (1.00441)^{12} = 1.05422. \qquad \text{(By Table II)}$$

Hence $i = .0542$, approximately.

EXERCISE XXXVII

1. A member pays $1 at the beginning of each month on a share in an association which earns 5%, compounded monthly. What is to the member's credit just after the 50th payment?

2. When will the share of stock in problem 1 mature to the value $100, and what payment, if any, will be necessary on the maturity date?

3. If an association earns 6%, compounded monthly, when will a $100 share mature if the dues are $2 per month per share?

4. The monthly due on a \$100 share is \$.50, and the share will mature, approximately, just after the 131st monthly payment. (*a*) What rate, compounded monthly, does the association earn on its funds? (*b*) What is the effective rate?

5. An association earns approximately 6%, compounded quarterly, on its funds. (*a*) Find the date, to the nearest month, on which a \$100 share will mature if the monthly due is \$1, making your computation under the assumption that \$3 is paid at the beginning of each 3 months. (*b*) Without any computation, tell why your result, as computed in (*a*), is smaller (or larger) than the actual result.

6. The monthly due on each \$100 share is \$1, in an association where the shares mature just after the 80th payment. What is the effective rate earned on the shares?

7. A man paid \$30 per month for 80 months as dues on thirty \$100 shares in an association, and, to mature his stock at the beginning of the 81st month, a partial payment of \$6 was necessary. At what rate, compounded monthly, did his money increase during the 80 months?

HINT. — If he should pay \$30 at the beginning of the 81st month, the book value of his shares would be \$3024.

40. Retirement of loans made by building and loan associations. — If a man borrows \$$A$ from a building and loan association, he is usually caused, at that time, to become a member of the association by subscribing for \$$A$ worth of stock. He must pay monthly interest (usually in advance) on the principal of his loan and dues on his stock. When his stock matures, with the value \$$A$, the association appropriates it as repayment of the principal of the loan. Thus, if a man borrows \$2000 from an association which charges borrowers 7% interest, payable monthly in advance, the interest due at the beginning of each month is $\frac{1}{12}(.07)(2000) = \$11.67$. The borrower subscribes for twenty \$100 shares of stock on which he pays \$20 as dues at the beginning of each month if the due per share is \$1. Thus, the monthly expense of the debt is \$31.67. Payments continue until the twenty shares mature with the value \$2000, at which time the association takes them as repayment of the principal of the debt, and closes the transaction. This method of retiring a debt is essentially a sinking fund plan, where the debtor's fund is invested in stock of the association. The debtor is benefited by this method because the rate received on his stock investment is higher than could safely be obtained in the outside market.

Example 1. — A man borrows \$2000 under the conditions of the preceding paragraph. If shares in the association mature at the end of 82 months, at what nominal rate, compounded monthly, may the borrower consider that he is amortizing his debt?

Solution. — The debtor pays \$31.67 at the beginning of each month for 82 months. We wish the rate under which \$2000 is the present value of this annuity due. Let r be the unknown rate per period of 1 month. The first \$31.67 is paid cash and the remaining 81 payments form an ordinary annuity, under Case 1, with $p = 1$. Hence,

$$2000 = 31.67 + 31.67(a_{\overline{81|}}\ at\ r),$$

$$(a_{\overline{81|}}\ at\ r) = \frac{1968.33}{31.67} = 62.151.$$

By interpolation in Table VIII, $r = \frac{7}{12}\% + \frac{226}{387}(\frac{1}{6}\%) = .681\%$. The nominal rate is $12(.00681) = .082$, approximately, compounded monthly. The effective rate i, if desired, can be obtained, with the aid of Table II, from

$$1 + i = (1.00681)^{12}.$$

EXERCISE XXXVIII

1. An association charges borrowers 6% interest payable monthly in advance and issues \$100 shares on which the monthly dues are \$1 per share. (*a*) At what rate of interest, compounded monthly, may a borrower consider his loan to be amortized, if shares in the association mature at the end of the 84th month, without a payment at that time? (*b*) What is the effective rate of interest?

2. Which would be more profitable, to borrow from the association of problem 1, or to pay 5% interest monthly in advance to some other lending source, and to repay the principal of the loan at the end of 84 months by the accumulation of a sinking fund, at the rate 6%, compounded monthly, if payments to the fund are made at the beginning of each month for 84 months?

3. An association charges borrowers 7% interest, payable monthly in advance, and issues \$100 shares on which the monthly dues are \$1 per share. If the shares mature at the end of 80 months, without a payment at that time, at what effective rate does a borrower amortize his debt?

4. An association charges 6% interest payable monthly in advance, and issues \$100 shares on which the monthly due per share is \$.50. If the shares mature at the end of 130 months, without a payment at that time, what is the effective rate paid by a borrower?

CHAPTER VI

DEPRECIATION, PERPETUITIES, AND CAPITALIZED COST

41. Depreciation; sinking fund plan. — Fixed assets, such as buildings and machinery, diminish in value through use. **Depreciation** is defined as that part of their loss in value which cannot be remedied by current repairs. In every business enterprise, the effects of depreciation should be foreseen and funds should be accumulated whose object is to supply the money needed for the replacement of assets when worn out. The deposits in these depreciation funds are called **depreciation charges**,[1] and are deducted periodically, under the heading of expense, from the current revenues of the business.

NOTE 1. — The replacement cost of an asset equals its cost when new minus its salvage or scrap value when worn out. Thus, if a machine costs $1000, and has a scrap value of $50 when worn out at the end of 10 years, its replacement cost is $950, the amount needed in addition to the scrap value in order to buy a new machine worth $1000. The replacement cost is also called the **wearing value**; it is the value which is lost through wear during the life of the asset.

A depreciation fund is essentially a sinking fund whose amount at the end of the life of the asset equals the replacement cost. Many different methods are in use for estimating the proper depreciation charge. Under the **sinking fund method**, the periodic depreciation charges are equal and are invested at compound interest at a specified rate. Under this plan, (*a*) the depreciation charges form an annuity whose amount equals the replacement cost, and (*b*) the depreciation fund is a sinking fund to which we may apply the methods for sinking funds used in Section 37.

Example 1. — A machine costs $1000 and it will wear out in 10 years. When worn out, its scrap value is $50. Under the sinking fund plan,

[1] In this book, unless otherwise specified, we shall always assume that the charge for depreciation during each year is made at the end of the year.

determine the depreciation charge which should be made at the end of each year for 10 years, if the fund is invested at 5%, effective.

Solution. — Let $x be the annual charge. The replacement cost S = $950.

Case 1
$n = 10$ int. periods,
$p = 1, i = .05$,
$R = \$x, S = \$950$.

$$950 = x(s_{\overline{10|}} \text{ at } .05),$$

$$x = 950 \frac{1}{(s_{\overline{10|}} \text{ at } .05)} = \$75.529.$$

DEPRECIATION TABLE

YEAR	INT. DUE ON FUND AT END OF YEAR	PAYMENT TO FUND AT END OF YEAR	IN FUND AT END OF YEAR	BOOK VALUE AT END OF YEAR
0	$ 0	$ 0	$ 0	$1000.00
1	0	75.529	75.529	924.47
2	3.777	75.529	154.835	845.16
3	7.742	75.529	238.106	761.89
4	11.905	75.529	325.540	674.46
5	16.277	75.529	417.346	582.65
6	20.867	75.529	513.742	486.26
7	25.687	75.529	614.958	385.04
8	30.748	75.529	721.235	278.76
9	36.062	75.529	832.826	167.17
10	41.641	75.529	949.996	50.00

In Figure 5 the growth of the fund and the decrease in the book value are represented graphically.

NOTE 2. — A good depreciation plan is in harmony with the fundamental principle of economics that capital invested in an enterprise should be kept intact. Thus, at the end of 2 years in Example 1, the $154.84 in the fund should be considered as capital, originally invested in the machine, which has been returned through the revenues of the business. The book value of the machine, 1000 − 154.84 = $845.16, is the amount of capital still invested in the machine.

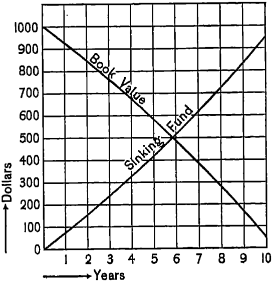

FIG. 5

The **condition per cent** of an asset at any time is the ratio of its remaining wearing value to its wearing value when new.

Example 2. — In Example 1, find the condition per cent at the end of 6 years.

Solution. — Original wearing value is \$950. Book value at end of 6 years is \$486.26, and the remaining wearing value is $486.26 - 50 = \$436.26$. The condition per cent is $\frac{436.26}{950} = .45923$ or 45.923%.

EXERCISE XXXIX

1. In illustrative Example 1 above, find the amount in the depreciation fund at the end of 4 years, without using the depreciation table.

2. A machine costs \$3100 when new, it wears out in 12 years, and its final scrap value is \$100. Under the sinking fund plan, determine the depreciation charge which should be made at the end of each 6 months if the fund accumulates at $(.05, m = 2)$.

3. (*a*) In problem 2, without forming a depreciation table, find the amount in the depreciation fund, and the book value of the asset, at the end of $5\frac{1}{2}$ years. (*b*) What is the condition per cent at this time?

4. A building costs \$100,000 and it will last 20 years, at the end of which time its salvage value is \$5000. Under the sinking fund plan, when the fund earns 5%, effective, determine the size of the depreciation fund at the end of 6 years, if deposits in the fund are made at the end of each year.

5. A motor truck has an original value of \$2500, a probable life of 6 years, and a final salvage value of \$200. Its depreciation is to be covered by deposits in a fund at the end of each 3 months. Find the quarterly deposit if the fund earns $(.055, m = 2)$.

6. A manufacturing plant is composed of part (*a*) whose cost is \$90,000, life is 15 years, and salvage value is \$6000, and part (*b*) whose cost is \$50,000, life is 20 years, and salvage value is \$5000. If depreciation charges are made at the end of each year and accumulate at $4\frac{1}{2}\%$, effective, what is the total annual charge for the plant?

42. Straight line method. — Consider an asset whose life is n years and whose replacement cost (wearing value) is $\$S$. Suppose that annual depreciation charges are placed in a fund which does not earn interest. Then, in order to have $\$S$ at the end of n years, the annual charge must be $\frac{S}{n}$. The fund increases each year by $\frac{S}{n}$, and hence the book value decreases each year by $\frac{S}{n}$. This method is called the **straight line method**, because we obtain straight lines as the graphs of the book value and of the amount in the fund (see problem 1 below).

NOTE. — An essential characteristic of the straight line method is that, under it, all of the annual decreases in book value are equal to $\frac{1}{n}$th of the total wearing value S. It is usually stated, under this method, that $\frac{S}{n}$ *is written off the book value each year* for depreciation. Recall, from the table in Example 1 of Section 41, that the annual decreases in book value are not equal under the sinking fund plan; they increase as the asset grows old.

NOTE. — The straight line method is the special case of the sinking fund method, where the rate earned on the depreciation fund is 0%. Hereafter, any general reference to the use of the sinking fund method should be understood to include the straight line plan as one possibility. In supplementary Section 48 below, another depreciation plan is considered which applies well to assets whose depreciation in early life is large compared to that in old age. The student is referred to textbooks on valuation and on accounting for many other special methods which are in use. Usually, each of these is particularly desirable for a certain type of assets.

EXERCISE XL

1. A building costs $50,000 and has a salvage value of $5000 when worn out at the end of 15 years. (*a*) Under the straight line method, form a table showing the amount in the depreciation fund and the book value at the end of each year. (*b*) Draw separate graphs of the book value and of the amount in the fund, using the years as abscissas.

2. In problem 1, find the annual depreciation charge under the sinking fund method, where the fund earns 4%, effective, and compare with the charge in problem 1.

43. Composite life. — If the plant of an enterprise consists of several parts whose lives are of different lengths, it is useful to have a definition for the average or composite life of the plant as a whole. Let $\$S$ be the sum of the wearing values (replacement costs) and $\$D$ the sum of the annual depreciation charges for all parts. Under the sinking fund method let i be the effective rate earned on the depreciation fund. Then, the **composite life** is defined as the term of years, necessary for an annuity of $\$D$, paid annually, to have the amount \$$S$. If k is the composite life in years, then

$$S = D(s_{\overline{k}|} \text{ at } i), \tag{46}$$

which can be solved for k by interpolation.

Under the straight line method of depreciation, k is the number of times \$$D$ must be paid into a fund in order that the fund, without earning interest, should equal \$$S$. Hence

$$S = kD, \quad k = \frac{S}{D}. \tag{47}$$

NOTE. — In equation 46 place $i = 0\%$. Then $(s_{\overline{k}|} \text{ at } 0) = k$ and equation 46 becomes $S = kD$, as found above. This is an obvious consequence of the fact that the straight line method is the sinking fund method with $i = 0\%$.

Example 1. — A plant consists of part A, with life 20 years, original cost \$55,000, and scrap value \$5000; part B with life 15 years, original cost \$23,000, and scrap value \$3000; part C, with life 10 years, original cost \$16,000 and scrap value \$1000. Determine the composite life, (*a*) under the sinking fund method, with interest at 4%, effective, and (*b*) under the straight line method.

Solution. — (*a*) The total wearing value is 50,000 + 20,000 + 15,000 = \$85,000. Under the sinking fund method, the annual charge for part A is $50000 \dfrac{1}{(s_{\overline{20}|} \text{ at } .04)} = \$1679.09$. Similarly, the charges for B and C are \$998.82 and \$1249.36, respectively. The total annual charge is \$3927.27. Let k be the composite life. Then, the annuity of \$3927.27, paid annually for k years, should have the amount \$85,000, or

Case 1
$n = k$ int. periods,
$p = 1, i = .04,$
$R = \$3927.27, S = \$85,000.$

$$85000 = 3927.27(s_{\overline{k}|} \text{ at } .04);$$
$$(s_{\overline{k}|} \text{ at } .04) = 21.644.$$

By interpolation in Table VII, $k = 15.90$ years.

(*b*) Under the straight line method, the annual depreciation charges for the parts A, B, and C are, respectively, \$2500, \$1333.33, and \$1500. The total annual charge is \$5333.33. The composite life k is

$$k = \frac{85000}{5333.33} = 15.938 \text{ years.}$$

NOTE. — Compare the results above. When the rate on the sinking fund is 0% (the straight line method), the composite life differs from the life when $i = 4\%$, by only $(15.94 - 15.90) = .04$ year. Thus, under the sinking fund method, regardless of the rate earned on the fund, the composite life may be obtained approximately by finding the life under straight line depreciation.

EXERCISE XLI

1. Find the composite life for the plant with parts A, B, and C below, under the sinking fund method (a) at 3%, effective, and (b) at 6%, effective, and under (c) the straight line method.

Part	Life	Cost	Scrap Value
A	10	\$20,500	\$ 500
B	20	35,750	750
C	16	19,000	1000

44. Valuation of a mine. — A mine, or any similar property, is a depreciable asset which becomes valueless when all of the ore is removed. Part of the net revenue of the mine should be used to accumulate a depreciation, or **redemption fund**, which will return the original invested capital when the mine is exhausted. The revenue remaining, after the depreciation charge, is the owner's net return on his investment.

NOTE. — We shall assume, in this book, that the annual revenue, or royalty, from a mine, or similar enterprise, is payable in one installment at the end of the year.

Example 1. — A mine, whose life is 20 years, costs \$200,000 cash. What should be the net annual revenue in order to pay 6% interest, annually, on the invested capital, and to provide an annual deposit for a redemption fund which accumulates at 4%, effective?

Case 1
$n = 20$ int. periods,
$p = 1$, $i = .04$,
$R = \$x$, $S = \$200{,}000$.

Solution. — Let \$$x$ be the annual deposit in the sinking fund to provide \$200,000 at the end of 20 years.

$$200{,}000 = x(s_{\overline{20|}}\ at\ .04); \quad x = \$6{,}716.35.$$

Annual interest at 6% on \$200,000 is \$12,000. Annual revenue required is 12,000 + 6,716.35 = \$18,716.35.

Mining engineers furnish accurate estimates, for any given mine, of the life, k years, and of the annual revenue, \$$R$. Suppose that a purchaser pays \$$P$ for a mine. If the annual revenue payments of \$$R$ are exactly sufficient to amortize the original invested principal P at the effective rate i, then P is the present value of the annuity formed by the revenue installments, or

$$P = R(a_{\overline{k|}}\ at\ i). \qquad (48)$$

Recall[1] that the amortization payments are exactly sufficient to pay interest at the rate i on P and to accumulate a sinking (redemption) fund at the rate i to repay P at the end of k years. Therefore, the price paid under the assumption that P is amortized is the price we should obtain on assuming that the investor receives the rate i on his investment and places the surplus revenue in a redemption fund which accumulates at the rate i.

Consider determining the purchase price P if the buyer desires the effective rate i on his investment and is able to invest his redemption fund at the effective rate r. Let \$$D$ be the depreciation charge deposited annually in the redemption fund, which accumulates to the amount P at the end of k years. Then

$$P = D(s_{\overline{k}|} \text{ at } r)\,; \qquad D = P\frac{1}{(s_{\overline{k}|} \text{ at } r)}.$$

Annual interest on \$$P$ at the rate i is Pi. Since

$$\textit{revenue} = (\textit{interest on capital}) + (\textit{depreciation charge}), \tag{49}$$

$$R = Pi + D = Pi + P\frac{1}{(s_{\overline{k}|} \text{ at } r)} = P\left(i + \frac{1}{(s_{\overline{k}|} \text{ at } r)}\right). \tag{50}$$

$$P = \frac{R}{i + \dfrac{1}{(s_{\overline{k}|} \text{ at } r)}}. \tag{51}$$

Example 2. — The annual revenue from a mine will be \$30,000 until it becomes exhausted at the end of 25 years. What should be paid for the mine if 8% is to be earned on the invested capital while a redemption fund accumulates at 5%?

Solution. — From formula 51

$$P = \frac{30000}{.08 + \dfrac{1}{(s_{\overline{25}|} \text{ at } .05)}} = \frac{30000}{.10095246} = \$297{,}170. \qquad \text{(Table IX)}$$

NOTE. — In equation 51 place $r = i$, and use formula 39. One obtains $P = R(a_{\overline{k}|} \text{ at } i)$, as obtained previously for this case in equation 48.

EXERCISE XLII

1. A purchaser paid \$300,000 for a mine which will be exhausted at the end of 50 years. What annual revenue from the mine will be required to pay 7% on the investment and to provide an annual deposit in a redemption fund which accumulates at 5%, effective?

[1] See Section 38.

2. The annual revenue from a mine will be \$50,000 until the ore is exhausted at the end of 50 years. A purchaser desires 7% on his investment. What should he pay for the mine if his redemption fund accumulates (*a*) at 7%; (*b*) at 5%; (*c*) at 4%?

3. The privileges of a certain patent last for 10 years and the annual royalties from it will be \$75,000. If a redemption fund can be accumulated at 5%, what should an investor pay for the patent rights if he demands 6% on his investment?

4. An investor in an oil property desires 10% on his investment and assumes that he can accumulate a redemption fund at 5%. What should he pay for an oil field whose net annual revenue for its 10 years of life is estimated at \$100,000?

5. A purchaser of a wooden ship estimates that the boat will be practically valueless at the end of 6 years. If the net earnings for each of these years will be \$50,000, what should the purchaser pay if he desires 8% on his investment and accumulates a depreciation fund at 4%?

45. Perpetuities. — A **perpetuity** is an annuity whose payments continue forever. Present values[1] of perpetuities are useful in capitalization problems.

Suppose that \$1000 is invested at 6%, effective. Then, \$60 interest is received at the end of each year, forever. That is, at 6%, the present value of a perpetuity of \$60, paid annually, is \$1000. Similarly, if \$$A$ is invested at the rate i, payable annually, it will yield $R = Ai$ interest annually, forever. Hence, the present value \$$A$ of a perpetuity of \$$R$ paid at the end of each year is obtained from $Ai = R$; or,

$$A = \frac{R}{i}. \tag{52}$$

NOTE 1. — If, in the paragraph above, we change the word *year* to *interest period*, it is seen that, when the interest rate per period is i, the present value of a perpetuity of \$$R$, paid at the end of each interest period, is $\frac{R}{i}$.

Example 1. — At the end of each 6 months, \$50 is required to clean a statue. If money is worth (.04, $m = 2$), what is the present value of all future renovation?

Solution. — The future renovation costs form a perpetuity whose present value, by formula 52, is $\frac{50}{.02} = \$2500$.

[1] The notion of the *amount of a perpetuity* is meaningless and useless, since the end of the term of a perpetuity does not exist.

Consider a perpetuity of \$1 paid at the end of each k years, and let $(a_{\infty,k}\ at\ i)$ be its present value when money is worth the effective rate i. In order to find a formula for $a_{\infty,k}$, first let us determine the installment \$$x$ which, if paid into a fund at the end of each year for k years, will accumulate to \$1 at the end of k years. Then, \$1 is the amount of the annuity of \$$x$ per annum and $1 = x(s_{\overline{k}|}\ at\ i)$; $x = \dfrac{1}{(s_{\overline{k}|}\ at\ i)}$. Therefore, a perpetuity of \$$x$ per annum will create a fund from which \$1 can be paid at the end of each k years, forever. Hence, the present value of the perpetuity of \$1 at the end of each k years is equal to the present value of the perpetuity of \$$x$, per annum. Therefore, from formula 52 with $R = x = \dfrac{1}{(s_{\overline{k}|}\ at\ i)}$,

$$(a_{\infty,k}\ at\ i) = \frac{x}{i} = \frac{1}{i}x = \frac{1}{i}\cdot\frac{1}{(s_{\overline{k}|}\ at\ i)}.$$

The present value \$$A$ of a perpetuity of \$$R$ paid at the end of each k years is

$$A = R(a_{\infty,k}\ at\ i) = \frac{R}{i}\cdot\frac{1}{(s_{\overline{k}|}\ at\ i)}. \tag{53}$$

NOTE 2. — Thus, if money is worth (.05, $m = 1$), the present value of a perpetuity of \$90,000 paid at the end of each 20 years is

$$A = \frac{90000}{.05}\,\frac{1}{(s_{\overline{20}|}\ at\ .05)} = \frac{90000}{.05}(.03024259) = \$54{,}436.5. \quad \text{(Table IX)}$$

If i is not a table rate, formula 53 must be computed by inserting the explicit formula for $(s_{\overline{k}|}\ at\ i)$.

NOTE 3. — Recognize that formula 52, or formula 53, applies to a perpetuity whose first payment comes at the end of the first payment interval. The present value of a *perpetuity due*, or of a *deferred perpetuity*, can be obtained by the methods used for the corresponding type of annuity.

EXERCISE XLIII

1. (*a*) Find the present value of a perpetuity which pays \$100 at the end of each 3 months, if money is worth (.08, $m = 4$). (*b*) What is the present value if the payments occur at the beginning of each 3 months?

√2. An enterprise will yield \$5000 net profit at the end of each year. At 4%, find the capitalized value of the enterprise, where the "capitalized value at 4%" is the present value of all future earnings.

3. A bridge must be repainted each 5 years at a cost of \$8000. If money is worth 5%, find the present value of all future repainting.

4. A certain depreciable asset must be replaced at the end of each 25 years at a cost of \$50,000. At 6%, find the present value of all future replacements.

5. Find the present value of an annuity of \$1000 paid at the end of each year for 75 years, if money is worth (.04, $m = 1$). Compare the result with the present value of a perpetuity of \$1000, paid annually.

6. To repair a certain road, \$1000 will be needed at the beginning of the 4th year and annually thereafter. Find the present value of all future repairs if money is worth 5%, effective.

46. Capitalized cost. — The **capitalized cost** of an asset is defined as the *first cost plus the present value of all future replacements,* which it is assumed will continue forever. Let \$$C$ be the first cost and \$$R$ the replacement cost of an asset which must be renewed at the end of each k years. Then, the capitalized cost \$$K$ equals \$$C$ plus the present value of a perpetuity of \$$R$ paid at the end of each k years. When money is worth the effective rate i, we obtain, on using formula 53,

$$K = C + \frac{R}{i}\frac{1}{(s_{\overline{k}|} \text{ at } i)}. \tag{54}$$

If the replacement cost equals the first cost, $R = C$. Then, on changing C to $\frac{Ci}{i}$ and on placing $R = C$ in formula 54,

$$K = \frac{Ci}{i} + \frac{C}{i}\frac{1}{(s_{\overline{k}|} \text{ at } i)} = \frac{C}{i}\left(i + \frac{1}{(s_{\overline{k}|} \text{ at } i)}\right). \quad \text{(See formula 39)}$$

$$K = \frac{C}{i}\frac{1}{(a_{\overline{k}|} \text{ at } i)}. \tag{55}$$

Example 1. — A machine costs \$3000 new and must be renewed at the end of each 15 years. (*a*) Find the capitalized cost when money is worth (.05, $m = 1$), if the final scrap value of the machine is \$500; (*b*) if the scrap value is zero.

Solution. — (*a*) Use formula 54 with $C =$ \$3000 and $R =$ \$2500.

$$K = 3000 + \frac{2500}{.05}\frac{1}{(s_{\overline{15}|} \text{ at } .05)} = \$5317.12.$$

(b) When the scrap value is zero, $C = R = \$3000$. From formula 55,

$$K = \frac{3000}{.05} \frac{1}{(a_{\overline{15}|} \; at \; .05)} = \$5780.54.$$

NOTE 1. — If the renewal cost of an asset is $\$R$ and its life k years, the annual depreciation charge $\$D$ is given by $R = D(s_{\overline{k}|} \; at \; i)$, or,

$$D = R \frac{1}{(s_{\overline{k}|} \; at \; i)}. \tag{56}$$

These future depreciation charges form a perpetuity of $\$D$, paid annually, whose present value is $\frac{D}{i}$, or

$$\frac{1}{i} D = \frac{1}{i}\left(R \frac{1}{(s_{\overline{k}|} \; at \; i)}\right) = \frac{R}{i} \frac{1}{(s_{\overline{k}|} \; at \; i)}.$$

This result is the same (see formula 54) as the present value of all future renewal costs. Hence, the definition of capitalized cost may be restated to be "the first cost plus the present value of all future depreciation charges."

NOTE 2. — In formula 54, multiply both sides by i. Then

$$Ki = Ci + R \frac{1}{(s_{\overline{k}|} \; at \; i)} = Ci + D. \qquad \text{(See equation 56)}$$

Thus, if an enterprise earns interest at the rate i on the capitalized cost K, the revenue Ki provides for the interest Ci at the rate i on the invested capital C and likewise for the annual depreciation charge D.

If two assets are available for serving the same purpose, that one should be used whose capitalized cost is least. If their capitalized costs are the same, both assets are equally economical.

Example 2. — A certain type of pavement costs \$12 per square yard, laid in place, and must be renewed at the same cost every 10 years. How much could a highway commission afford to pay to improve the pavement so that it would last 15 years, if money is worth 4%, effective?

Solution. — Let $\$x$ be the cost per square yard of the improved type of pavement, whose life is 15 years. If this type is just as economical as the old, its capitalized cost must be the same. The capitalized costs of the two types, as given by equation 55, are equated below:

$$\frac{12}{.04} \frac{1}{(a_{\overline{10}|} \; at \; .04)} = \frac{x}{.04} \frac{1}{(a_{\overline{15}|} \; at \; .04)}.$$

$$x = \frac{12(a_{\overline{15}|} \; at \; .04)}{(a_{\overline{10}|} \; at \; .04)} = \frac{12(11.1183874)}{8.1108958} = \$16.450.$$

The commission could afford to pay anything less than (16.450 − 12) or \$4.450 to improve the old pavement.

EXERCISE[1] XLIV

1. Find the capitalized cost of a plant whose original cost is $200,000 and whose life is 25 years, if its final salvage value is $15,000. Money is worth 4%.

2. A section of pavement costing $50,000 has a life of 25 years. Find its capitalized cost if the renewal cost is $50,000, and if money is worth 3%.

3. If it costs $2000 at the end of each year to maintain a section of railroad, how much would it pay to spend, immediately, to improve the section so that the annual maintenance would be reduced to $500? Money is worth 5½%.

4. A bridge must be rebuilt every 50 years at a cost of $45,000. Find the capitalized cost if the first cost is $75,000, and if money is worth 3%.

5. One machine costs $15,000, lasts 25 years, and has a final salvage value of $1000. Another machine for the same purpose costs $18,000, lasts 28 years, and has a salvage value of $2000. If money is worth 5%, which machine should be used?

6. Would it be better to use tile costing $18 per thousand and lasting 15 years, or to use other material costing $22 per thousand and lasting 20 years, if money is worth 5% and if neither material has a scrap value?

7. A corporation is considering the use of motor trucks worth $5000 each, whose life is 4 years and salvage value is zero. How much would it pay to spend, per truck, to obtain other trucks whose life would be 6 years, and final salvage value zero? Money is worth 5%.

8. The interior of a room can be painted at a cost of $10 and the painting must be repeated every 2 years. If money is worth 6%, how much could one afford to pay for papering the room if the paper would need renewal every 3 years?

9. A certain manufacturing plant involves one part worth $100,000 new and needing replacement every 10 years at a cost of $90,000, and a second part costing $52,000 new and needing renewal every 12 years at a cost of $50,000. What should be the net operating revenue in order to yield 7% on the capitalized cost?

10. A certain dam will cost $100,000 and will need renewal at a cost of $50,000 every 10 years. If money is worth 4½%, how much could one afford to pay in addition to $100,000 to make the dam of permanent type?

[1] After this exercise the student may proceed immediately to the Miscellaneous Problems at the end of the chapter.

SUPPLEMENTARY MATERIAL

47. Difficult cases under perpetuities. — Perpetuities are met to which formulas 52 and 53 do not apply. A systematic means for finding the present values of all perpetuities is furnished by infinite geometrical progressions.

Example 1. — If money is worth (.05, $m = 2$), find the present value of a perpetuity of \$6 paid at the end of each 3 months.

Solution. — The present value \$$A$ of the perpetuity is the sum of the present values of all of the payments as listed below:

Payment of \$6 due at end of	3 mo.	6 mo.	9 mo.	etc.	to infinity.
Present value of payment	$6(1.025)^{-\frac{1}{2}}$	$6(1.025)^{-1}$	$6(1.025)^{-\frac{3}{2}}$	etc.	to infinitely many terms.

$A = 6[(1.025)^{-\frac{1}{2}} + (1.025)^{-1} + (1.025)^{-\frac{3}{2}} + \cdots$ etc. $\cdots$ *to infinitely many terms*].

The bracket contains an infinite geometrical progression whose first term $a = (1.025)^{-\frac{1}{2}}$, and whose ratio $w = (1.025)^{-\frac{1}{2}}$. The sum [1] of the series is $\frac{a}{1-w}$;

$$A = \frac{6(1.025)^{-\frac{1}{2}}}{1-(1.025)^{-\frac{1}{2}}} = \frac{6(1.025)^{-\frac{1}{2}}}{1-(1.025)^{-\frac{1}{2}}} \cdot \frac{(1.025)^{\frac{1}{2}}}{(1.025)^{\frac{1}{2}}}.$$

$$A = \frac{6}{(1.025)^{\frac{1}{2}} - 1} = \frac{6}{.0124228} = \$482.98. \qquad \text{(Table X)}$$

NOTE. — The formulas 52 and 53 of Section 45 can be obtained by the method of Example 1 (see problems 3 and 5 below).

EXERCISE XLV

Use geometrical progressions unless otherwise directed.

1. The annual rent of a perpetuity is \$1000, payable in semi-annual installments. Find the present value when money is worth (.06, $m = 1$).

2. Find the present value of the perpetuity in problem 1 if money is worth (.06, $m = 4$).

3. Derive the formula 52 for the present value of a perpetuity of \$1 paid annually, when money is worth the effective rate i. This present value is generally denoted by the symbol a_∞; that is, (a_∞ *at* i) $= \frac{1}{i}$.

[1] See Formula 22, Section 91.

4. (a) By use of a geometrical progression, find the present value of a perpetuity of \$100 paid at the end of each 10 years, if money is worth 5%, effective. (b) Compare with the result obtained by use of formula 53.

5. Derive formula 53 for R ($a_{\infty,k}$ *at* i), the present value of a perpetuity of \$$R$ paid at the end of each k years, with money worth the effective rate i.

6. At 6%, effective, find the capitalized value of an enterprise which yields a net revenue of \$500 at the end of each month.

7. Let ($a_{\infty}^{(p)}$*at* i) represent the present value, when money is worth i, effective, of a perpetuity whose annual rent is \$1, paid in p installments per year. Prove that ($a_{\infty}^{(p)}$*at* i) $= \frac{1}{j_p} = \frac{i}{j_p}$ (a_{∞} *at* i).

8. If money is worth (.06, $m = 4$), find the present value of a perpetuity of \$1000, paid semi-annually, by use of formula 53.

9. If money is worth (.05, $m = 2$), find the present value of a perpetuity of \$100 paid monthly, by use of the result of problem 7.

10. An irrigation system has just been completed. There will be no repair expense until the end of two years, after which \$50 will be needed at the end of each 6 months. If money is worth 4%, effective, find the present value of the future upkeep. Solve by any method.

48. Constant percentage method of depreciation. — Under the **constant percentage method**, the book value decreases each year by a fixed percentage of the value at the beginning of the year. If the life of the asset is n years, the constant percentage r, expressed as a decimal, must be chosen so that the original cost \$$C$ is reduced to the residual book value \$$R$ at the end of n years. The decrease in the first year is Cr, and the value at the end of 1 year is $C - Cr = C(1 - r)$. Similarly, the book value at the end of each year is $(1 - r)$ times the value at the beginning of the year. By the end of n years, the original value C has been multiplied n times by $(1 - r)$, or by $(1 - r)^n$, and the residual scrap value R is $C(1 - r)^n$. Therefore, we may obtain r from

$$C(1 - r)^n = R, \text{ or, } 1 - r = \sqrt[n]{\frac{R}{C}}. \qquad (57)$$

Each annual reduction in book value is the depreciation charge for that year, and, if we consider all of these reductions in book value placed in a depreciation fund which **does not earn interest**, the fund will contain the replacement cost at the end of n years.

Example 1. — For a certain asset, the original cost is $3000, the life is 6 years, and the scrap value is $500. Find the annual percentage of depreciation under the constant percentage method and form a table showing the changes in book value.

Solution. — From equation 57,

$$1 - r = \sqrt[6]{\frac{500}{3000}} = \sqrt[6]{\frac{1}{6}}. \qquad \log \tfrac{1}{6} = 9.22185 - 10.$$

$$1 - r = .74183. \qquad \log(1 - r) = \tfrac{1}{6}\log\tfrac{1}{6} = 9.87031 - 10.$$

$$r = .25817.$$

The book values in the table below were computed by 5-place logarithms. Thus, at the end of 3 years, the book value is $B = 3000(1 - r)^3$;

$$\log(1 - r)^3 = 3\log(1 - r) = 3(9.87031 - 10) = 9.61093 - 10$$

$$\log 3000 = 3.47712$$

$$\log B = 3.08805; \ B = \$1224.7.$$

DEPRECIATION TABLE

YEAR	BOOK VALUE AT END OF YEAR	DEPRECIATION DURING YEAR	IN DEPR. FUND AT END OF YEAR
1	$2225.5	$774.5	$ 774.5
2	1651.0	574.5	1349.0
3	1224.7	426.3	1775.3
4	908.6	316.1	2091.4
5	674.0	234.6	2326.0
6	500.0	174.0	2500.0

NOTE. — When the scrap value R is relatively small, the method above gives ridiculously high depreciation charges in the early years. The method breaks down completely when $R = 0$.

EXERCISE XLVI

1. (*a*) Find the annual percentage of depreciation under the constant percentage method, for a machine whose original cost is $10,000, life is 5 years, and scrap value is $1000. (*b*) Form a depreciation table and draw a graph of the changes in book value.

2. In problem 1, find the annual depreciation charge under the sinking fund plan, where the fund earns 4%, effective, and compare with the result of problem 1.

3. A machine, whose life is 20 years, costs \$50,000 when new and has a scrap value of \$5000 when worn out. Find the annual rate of depreciation under the constant percentage method.

4. For a certain asset, the depreciation in value during the early years of its life is known to be very great, as compared with the later years. Which of the three methods, straight line, sinking fund, or constant percentage, would give a series of book values most in harmony with the actual values during the life of the asset? Justify your answer.

MISCELLANEOUS PROBLEMS

Depreciation, in problems below, is under the sinking fund method.

1. An automobile costs \$3500 when new, and its salvage value at the end of 6 years is \$400. (*a*) If the depreciation fund earns 4%, by how much is the book value decreased during the 4th year? (*b*) By how much is the book value decreased during the 4th year, under the straight line method?

2. A hotel has been built at a cost of \$1,000,000 in an oil-boom city which will die at the end of 25 years. Assuming that the assets can be sold for \$100,000 at that time, what must be the net annual revenue during the 25 years to earn 7% on the investment and to cover depreciation, where the depreciation fund earns 4%?

3. A syndicate will build a theater in a boom city which will die at the end of 30 years. For each \$10,000 unit of net annual profit expected, how much can the syndicate afford to spend on the theater if 8% is desired on the investment while a redemption fund to cover the initial investment is accumulated at 5%? Assume that the theater will be valueless at the end of 30 years.

4. A machine, worth \$100,000 new, will yield 12% net annual operating profit on its original cost, if no depreciation charges are made. If the life of the machine is 20 years, what annual profit will it yield if annual depreciation charges are made, where the depreciation fund accumulates at 4%, effective? Assume that the final salvage value of the machine is zero.

5. The life of a mine is 30 years, and its net annual revenue is \$50,000. Find the purchase price to yield an investor 7%, if the redemption fund accumulates at 4%.

6. A mine will yield a net annual revenue of \$25,000 for 20 years. It was purchased for \$200,000. If, at this price, the investor considers

that he obtains 10% on his investment, at what rate does he accumulate his redemption fund?

7. A certain railroad will cost $60,000 per mile to build. To maintain the roadbed in good condition will cost $500 per mile, payable at the beginning of each year. At the end of each 30 years, the tracks must be relaid at a cost of $30,000. What is the present value of the construction and of all future maintenance and renewals, if money is worth 5%?

8. Find the capitalized value at 6%, effective, of a farm whose net annual revenue is $3000.

9. An automobile with a wearing value of $1200 has a life of 5 years. Upkeep and repairs cost the equivalent of $450 at the end of each year. What is the annual maintenance expense if the owner accumulates a depreciation fund by annual charges invested at 5%?

10. A certain piece of forest land will yield a net annual revenue of $25,000 for 15 years, at the end of which time the cut-over land will be sold for $15,000. (*a*) If money is worth 6% to an investor, what should he pay for the property? (*b*) If the investor desires 9% on his invested capital, and assumes that he can accumulate a redemption fund at 5% to return his original capital at the end of 15 years, find the price he should pay for the land, by use of the method which was used in deriving formula 51 for the valuation of a mine.

CHAPTER VII

BONDS

49. Terminology. — A **bond** is a written contract to pay a definite **redemption price** $\$C$ on a specified **redemption date** and to pay equal **dividends** $\$D$ periodically until after the redemption date. The dividends are usually payable semi-annually, but may be paid annually or in any other regular fashion. The principal $\$F$ mentioned in the face of the bond is called the **face value** or **par value**. A bond is said to be **redeemed at par** if $C = F$ (as is usually the case), and at a **premium** if C is greater than F. The interest rate named in a bond is called the dividend rate. The dividend $\$D$ is described in a bond by saying that it is the interest, semi-annual or otherwise, on the par value F at the dividend rate.

NOTE. — The following is an extract from an ordinary bond:

The Kansas Improvement Corporation acknowledges itself to owe and, for value received, promises to pay to bearer FIVE HUNDRED DOLLARS *on January* 1*st*, 1926, *with interest on said sum from and after January* 1*st*, 1920, *at the rate* 6% *per annum, payable semi-annually, until the said principal sum is paid. Furthermore, an additional* 10% *of the said principal shall be paid to bearer on the date of redemption.*

For this bond, $F = \$500$, $C = \$550$, and the semi-annual dividend $D = \$15$ is semi-annual interest at 6% on \$500. A bond is named after its face F and dividend rate, so that the extract is from a \$500, 6% bond. Corresponding to each dividend D there usually would be attached to the bond an individual coupon containing a written contract to pay $\$D$ on the proper date.

50. When an investor purchases a bond, the interest rate i which he receives on his investment is computed assuming that he will hold the bond until it is redeemed. It is important to recognize that the investment rate i is *not the same as the dividend rate of the bond*, except in very special cases, because i depends on all of the following: $\$P$, the price paid for the bond; $\$C$, the redemption price; the time to elapse before the redemption date; the number of times per year dividends are paid, and the size of $\$D$, the periodic dividend.

In this chapter we shall solve two principal problems. First, the determination of the price $\$P$ which should be paid for a specified bond, if we know the investment rate demanded by the buyer. Second, the determination of the investment rate if we know the price which the investor had to pay.

51. Purchase price to yield a given rate. — The essential features of a bond contract are the promises (*a*) **to pay $\$C$ on the redemption date** and (*b*) **to pay the annuity formed by the periodic dividends of $\$D$.**[1] If an investor desires a specified investment rate, the price $\$P$ he is willing to pay on purchasing the bond is

$$P = \textbf{(present value of \$C due on the redemption date)} \qquad (58)$$
$$+ \textbf{(present value of the annuity formed by the dividends)},$$

where present values are computed under the investor's rate.

Example 1. — A \$1000, 6% bond, with dividends payable semi-annually, will be redeemed at 105% at the end of 15 years. Find the price to yield an investor (.05, $m = 1$).

Solution. — "At 105%" means at a premium of 5% over the par value. $F = \$1000$, $C = 1000 + 50 = \$1050$. The semi-annual dividend $D = (.03)1000 = \$30$. The redemption price \$1050 is due at the end of 15 years. Hence, at the rate (.05, $m = 1$),

Div. annuity, Case 1 $n = 15$ int. periods, $p = 2$, $i = .05$, $R = \$60$.

$$P = 1050(1.05)^{-15} + 60(a_{\overline{15|}}^{(2)} \; at \; .05).$$

$$P = 505.07 + 60 \frac{.05}{j_2} (a_{\overline{15|}} \; at \; .05) = \$1135.54.$$

EXERCISE XLVII

1. A \$1000, 5% bond, with dividends payable semi-annually, will be redeemed at 108% at the end of 7 years. Find the price to yield an investor 6%, compounded semi-annually.

The bonds in the table are redeemable at par. Find the purchase prices. The *life* is the time to the redemption date.

[1] When a bond is sold on a dividend date, the seller takes the dividend $\$D$ which is due. The purchaser will receive the *future* dividends, which form an ordinary annuity whose first payment is due at the end of one dividend interval, and whose last payment is due on the redemption date.

Prob.	Par Value	Life	Dividend Rate	Dividends Payable	Investment Rate
2.	$ 1,000	10 yr., 6 mo.	5%	semi-ann.	(.06, $m = 2$)
3.	100	17 yr.	6%	annually	(.07, $m = 1$)
4.	1,000	14 yr.	7%	semi-ann.	(.08, $m = 1$)
5.	500	9 yr.	8%	semi-ann.	(.04, $m = 2$)
6.	2,000	7 yr.	4%	quarterly	(.05, $m = 4$)
7.	1,000	8 yr., 6 mo.	3%	semi-ann.	(.06, $m = 2$)
8.	1,000,000	13 yr., 6 mo.	5½%	semi-ann.	(.04, $m = 2$)
9.	100,000	19 yr.	5%	semi-ann.	(.06, $m = 1$)

10. A $10,000, 5% bond, whose dividends are payable annually, will be redeemed at par at the end of 30 years. Find the purchase prices to yield (*a*) 5%, effective; (*b*) 7%, effective; (*c*) 4%, effective. Compare your results.

11. A $1000, 6% bond, whose dividends are payable semi-annually, is purchased to yield 5%, effective. Find the price if the bond is to be redeemed at the end of (*a*) 5 years; (*b*) 20 years; (*c*) 75 years. Compare your results.

√ **12.** A $100,000, 5% bond is redeemable at 110% at the end of 15 years, and dividends are payable annually. Find the price to yield (.06, $m = 2$).

If a bond is redeemable at par ($C = F$) and if the investor's interest period equals the interval between successive dividends, it is easy to compute the premium ($P - F$), the excess of the price P over par value F. Let k be the number of dividend periods to elapse before the bond matures, r the dividend rate per dividend interval, and i the investor's rate per interest period. Then, a dividend $D = Fr$ is due at the end of each interest period and the redemption price F is due at the end of k periods. The equations below are easily verified.

Div. annuity, Case 1 $n = k$ int. periods, $p = 1, i = i, R = Fr.$

$$P = Fr(a_{\overline{k}|} \text{ at } i) + F(1 + i)^{-k}.$$
$$P - F = Fr(a_{\overline{k}|} \text{ at } i) + F(1 + i)^{-k} - F.$$

From formula 28, $-Fi\,(a_{\overline{k}|} \text{ at } i) = -Fi\dfrac{1 - (1 + i)^{-k}}{i} = F(1 + i)^{-k} - F.$

Therefore, $P - F = Fr(a_{\overline{k}|} \text{ at } i) - Fi(a_{\overline{k}|} \text{ at } i) = (Fr - Fi)(a_{\overline{k}|} \text{ at } i).$

$$\textit{Premium} = P - F = F(r - i)(a_{\overline{k}|} \text{ at } i). \tag{59}$$

NOTE 1. — Formula 59 shows that, when r is greater than i, $P - F$ is positive, or the bond is purchased at a positive premium over par value F. When r is less than i, $P - F$ is negative or the bond is purchased at a negative premium, that is, at a discount from the par value F.

Example 2. — A $1000, 6% bond, with dividends payable semi-annually, is redeemable at par at the end of 20 years. (*a*) Find the price to yield an investor (.05, $m = 2$). (*b*) To yield (.07, $m = 2$).

Solution. — (*a*) From formula 59 with $F = \$1000$, the premium is $P - F = 1000(.03 - .025)(a_{\overline{40}|}\ at\ .025) = 5(a_{\overline{40}|}\ at\ .025) = \$125.51$. $P = F + 125.51 = \$1125.51$. (*b*) Premium $= P - F = 1000(.03 - .035)(a_{\overline{40}|}\ at\ .035) = -5(a_{\overline{40}|}\ at\ .035) = -\$111.78$. $P = F - 111.78 = \$882.22$. In this case, we say the discount is $111.78.

NOTE 2. — Equation 59 could have been proved by direct reasoning. Suppose r is greater than i. Then, if an investor should pay $\$F$ for the bond, he would desire Fi as interest on each dividend date. Since each dividend is Fr, he would be receiving $(Fr - Fi) = F(r - i)$ excess income at the end of each interest period for k periods. Hence, he should pay, in addition to $\$F$, a premium equal to the present value of the annuity formed by the excess income or $F(r - i)(a_{\overline{k}|}\ at\ i)$. Similarly, when r is less than i, if the investor should pay $\$F$ for the bond, there would be a deficiency in income of $Fi - Fr = F(i - r)$ at the end of each interest period. Hence, the present value of the deficiency or $F(i - r)(a_{\overline{k}|}\ at\ i)$ should be returned to the investor as a discount from the price F we supposed paid.

VALUES TO THE NEAREST CENT OF A $100,000, 5% BOND WITH SEMI-ANNUAL DIVIDENDS

INVEST. RATE WITH $m = 2$	TIME TO REDEMPTION DATE			
	10½ YEARS	11 YEARS	11½ YEARS	12 YEARS
.0400	108505.60	108829.02	109146.10	109456.96
.0405	108060.01	108365.61	108665.14	108958.72
.0410	107616.62	107904.58	108186.75	108463.25
.0415	107175.43	107445.93	107710.93	107970.54
.0420	106736.43	106989.64	107237.65	107480.56
.0425	106299.59	106535.71	106766.91	106993.30
.0430	105864.92	106084.11	106298.69	106508.75
.0435	105432.40	105634.84	105832.97	106026.89
.0440	105002.01	105187.88	105369.74	105547.69
.0445	104573.75	104743.21	104908.99	105071.16
.0450	104147.61	104300.84	104450.70	104597.26

NOTE 3. — To facilitate practical work with bonds, extensive tables have been computed showing the purchase prices of bonds redeemable at par.[1] The table on page 116 illustrates those found in bond tables.

EXERCISE XLVIII

In the future use formula 59 to find P whenever $F = C$ and the investor's interest period equals the dividend interval. Otherwise use the fundamental method involving formula 58.

1. Find the price to yield 4%, compounded semi-annually, of a $1000, 5% bond, with dividends payable semi-annually, redeemable at par at the end of $15\frac{1}{2}$ years.

2. Verify all entries in the bond table on page 116, corresponding to the investment yields .04 and .045.

3. A $5000 bond, paying a $100 dividend semi-annually, is redeemable at par at the end of 11 years. Find the price to yield (.06, $m = 2$).

4. A man W signs a note promising to pay $2000 to M at the end of 5 years, and to pay interest semi-annually on the $2000 at the rate $6\frac{1}{2}$%. (*a*) What will M receive on discounting this note immediately at a bank which uses the interest rate 7%, compounded semi-annually? (*b*) What will M receive if the bank uses the rate 7% effective?

5. Find the price to yield 5%, effective, of a $10,000, 7% bond, with dividends payable annually, which is redeemable at par at the end of (*a*) 10 years; (*b*) 15 years; (*c*) 40 years. (*d*) Explain in a brief sentence how and why the price of a bond changes as the time to maturity increases, if the investor's rate is less than the dividend rate.

6. Find the price to yield 6%, effective, of a $10,000, 4% bond with annual dividends, which is redeemable at par at the end of (*a*) 5 years; (*b*) 10 years; (*c*) 80 years. (*d*) Explain in one brief sentence how and why the price of a bond changes as the time to maturity is increased, if the investor's rate is greater than the dividend rate.

52. Changes in book value. — On a dividend date, it is convenient to use the term **book value** for the price $\$P$ at which a bond would sell under a given investment rate i. Recall that this price $\$P$, at which a purchaser could buy the bond, is the sum of the present values, under the rate i, of all payments promised in the bond. Hence, the dividends $\$D$ together with the redemption payment $\$C$ are sufficient to pay interest at rate i on the invested

[1] Sprague's *Complete Bond Tables* contain the purchase prices to the nearest cent for a bond of $1,000,000 par value, corresponding to a wide range of investment rates.

principal $P, and to return the principal intact. If a bond is purchased at a premium over the redemption price $C, only $C of the original principal $P is returned at redemption. Therefore, the remaining principal, which equals the premium $(P - C)$ originally paid for the bond, is returned in installments, or is *amortized*, through the dividend payments. Thus, each dividend $D, in addition to paying interest due on principal, provides a partial payment of principal. These payments reduce the invested principal, or book value, from $P on the date of purchase to $C on the redemption date.

Example 1. — A $1000, 6% bond pays dividends semi-annually and will be redeemed at 110% on July 1, 1925. It is bought on July 1, 1922, to yield (.04, $m = 2$). Find the price paid and form a table showing the change in book value and the payment for amortization of the premium on each interest date.

Solution.— $C = \$1100$, $D = \$30$. $P = 1100(1.02)^{-6} + 30(a_{\overline{6}|}\ at\ .02) = \$1144.811$.

TABLE OF BOOK VALUES FOR A BOND BOUGHT AT A PREMIUM

DATE	INT. AT 4% DUE ON BOOK VALUE	DIVIDEND RECEIVED	FOR AMORTIZATION OF PREMIUM	FINAL BOOK VALUE
July 1, 1922				$1144.811
Jan. 1, 1923	$22.896	$30.000	$7.104	1137.707
July 1, 1923	22.754	30.000	7.246	1130.461
Jan. 1, 1924	22.609	30.000	7.391	1123.070
July 1, 1924	22.461	30.000	7.539	1115.531
Jan. 1, 1925	22.311	30.000	7.689	1107.842
July 1, 1925	22.157	30.000	7.843	1099.999

On Jan. 1, 1923, for example, interest due on book value is .02(1144.81) = $22.896. Hence, the $30 dividend pays the interest due and leaves (30 − 22.896) = $7.104 for repayment, or amortization, of the premium; the new book value is 1144.811 − 7.104 = $1137.707. The check on the computation is that the final book value should be $1100, the redemption price.

If a bond is purchased at a discount from the redemption price, that is, if P is less than C, the redemption payment C exceeds the original investment P by $(C - P)$. Hence, this excess must be the accumulated value on the redemption date of that part of the interest on the investment which the payments of D on the dividend dates were insufficient to meet. Therefore, on each dividend date, the payment D is less than the interest due on invested principal; the interest which is not paid represents a new investment in the bond, whose book value is thereby increased. This writing up of the book value on dividend dates is called **accumulating the discount** because the book value increases from P on the date of purchase, to C on the redemption date, the total increase amounting to the original discount $(C - P)$.

Example 2. — A \$1000, 4% bond pays dividends semi-annually and will be redeemed at 105% on January 1, 1924. It is purchased on January 1, 1921, to yield (.06, $m = 2$). Find the price and form a table showing the accumulation of the discount.

Solution. — $C = \$1050$, and $D = \$20$.

$$P = 1050(1.03)^{-6} + 20(a_{\overline{6}|} \text{ at } .03) = \$987.702.$$

TABLE OF BOOK VALUES FOR A BOND BOUGHT AT A DISCOUNT

DATE	INT. AT 6% DUE ON BOOK VALUE	DIVIDEND RECEIVED	FOR ACCUMULATION OF DISCOUNT	FINAL BOOK VALUE
Jan. 1, 1921				\$ 987.702
July 1, 1921	\$29.631	\$20.000	\$ 9.631	997.333
Jan. 1, 1922	29.920	20.000	9.920	1007.253
July 1, 1922	30.218	20.000	10.218	1017.471
Jan. 1, 1923	30.524	20.000	10.524	1027.995
July 1, 1923	30.840	20.000	10.840	1038.835
Jan. 1, 1924	31.165	20.000	11.165	1050.000

In forming the row for July 1, 1921, for example, interest due at 6% is .03(987.702) = 29.631. Of this, only \$20 is paid. The balance, 29.631 − 20 = \$9.631, is considered as a new investment, raising the book value of the bond to 987.702 + 9.631 = \$997.333. In his bookkeeping on July 1, 1921, the investor records the receipt of \$29.631 interest although only \$20 actually came into his hands. Also, his books show a new investment of \$9.631 in the bond.

Recognize that, when a bond is purchased at a premium, the dividend D is the sum of the interest I on the investment plus a payment for amortization of the premium, or

$$I = D - (\text{amortization payment}). \qquad (60)$$

Thus, in illustrative Example 1 above on Jan. 1, 1923, the interest is 22.896 = 30 − 7.104. In accounting problems this fact is of importance. For instance, if a trust company purchases the bond of Example 1 for a trust fund, \$1144.81 of the capital is invested. Suppose that the trust company considers all of each \$30 dividend as interest and expends it for the beneficiary of the fund. Then, on July 1, 1925, the company faces an illegal loss of \$44.81 in the capital of the fund, because \$1100 is received at redemption in place of \$1144.81 invested. The company should consider only the entries in the 2d column of the table of Example 1 as income for the beneficiary.

Similarly, when a bond is bought at a discount,

$$I = D + (\textbf{payment for accumulation of the discount}). \quad (61)$$

Thus, in illustrative Example 2 above on July 1, 1921, the interest is $29.631 = 20 + 9.631$.

From equations 60 and 61 we obtain, respectively,

$$(\textit{amortization payment}) = D - I,$$
$$(\textit{payt. for accumulation of discount}) = I - D.$$

Let P_0 and P_1 be the book values on two successive dividend dates, at the same yield. Then, if the bond is at a premium, P_1 equals P_0 minus the amortization payment, or $P_1 = P_0 - (D - I)$;

$$P_1 = P_0 + (I - D). \quad (62)$$

If the bond is at a discount, P_1 equals P_0 plus the payment for the accumulation of the discount or $P_1 = P_0 + (I - D)$, the same as found in equation 62. Hence, equation 62 holds true for all bonds.

EXERCISE XLIX

1. A \$1000, 8% bond pays dividends semi-annually on February 1 and August 1, and is redeemable at par on August 1, 1925. It is purchased on February 1, 1923, to yield (.06, $m = 2$). Form a table showing the amortization of the premium.

2. A \$1000, 5% bond pays dividends annually on March 1, and is redeemable at 110% on March 1, 1931. It is purchased on March 1, 1925, to yield (.07, $m = 1$). Form a table showing the accumulation of the discount.

3. By use of formula 58, find the book value of the bond of problem 2 on March 1, 1927, to yield (.07, $m = 1$) and thus verify the proper entry in the table of problem 2. Any book value in the tables of problems 1 and 2 could be computed in this way without forming the tables.

4. Under the investment rate (.04, $m = 1$), the book value of a \$100, 5% bond on January 1, 1921, is \$113.55, and dividends are payable annually on January 1. Find the amount of the interest on the investment, and of the payment for amortization on January 1, 1922.

5. A \$1000, 4% bond pays dividends annually on August 15 and is redeemable at par on August 15, 1935. An investor purchased it on August 15, 1923, to yield (.06, $m = 1$). (*a*) Without forming a table, find how much interest on invested capital should be recorded as received, in the accounts of the investor, on August 15, 1928. (*b*) How much new principal does the investor invest in the bond on August 15, 1928?

53. Price at a given yield between interest dates. — The price of a bond on any date is the sum of the present values of all future payments promised in the bond. Let the investment rate be (i, $m = 1$), and suppose the last dividend was paid $\frac{1}{k}$ th year ago. At that time, the price P_0 was the sum of the present values, at the rate (i, $m = 1$) of all future bond payments. To-day, the price P is the sum of the present values of these same payments because no more dividends have as yet been paid. Hence, P equals P_0 accumulated at the rate (i, $m = 1$) for $\frac{1}{k}$ years, or

$$P = P_0(1 + i)^{\frac{1}{k}}. \qquad (63)$$

This price is on a strict compound interest yield basis. In practice, P is defined as P_0, accumulated for $\frac{1}{k}$ years at the rate i, *simple interest*; $P = P_0\left(1 + \frac{1}{k}i\right)$,

$$P = P_0 + P_0 i \frac{1}{k}. \qquad (64)$$

That is, P equals P_0 plus simple interest on P_0 from the last dividend date at the investment rate i.

NOTE 1. — The use of equation 64 favors the seller because it gives a slightly larger value of P than equation 63. The difference in price is negligible except in large transactions. Use equation 64 in all problems in Exercise L on page 123.

Example 1. — A \$1000, 6% bond, with dividends payable July 1 and January 1, is redeemable at 110% on July 1, 1925. Find the price to yield (.05, $m = 2$) on August 16, 1922.

Solution. — July 1, 1922, was the last dividend date. The price P_0 then was $P_0 = 1100(1.025)^{-6} + 30(a_{\overline{6}|}\ at\ .025) = \$1113.77$. Simple interest on

\$1113.77 from July 1 to Aug. 16 at the investment rate 5%, is \$6.96. The price on Aug. 16 is $P = 1113.77 + 6.96 = \$1120.73$.

It is proper to consider that the dividend on a bond accrues (or is earned) continuously during each dividend interval. Thus, d days after a dividend date, the

(accrued dividend) = (simple int. for d days on the face F at dividend rate). (65)

Example 2. — In Example 1, find the accrued dividend on August 16, 1922.

Solution. — From July 1 to Aug. 16 is 45 days. Accrued dividend is $\frac{45}{360}(.06)(1000) = \$7.50$.

NOTE 2. — In using equation 65, take 360 days as 1 year, and find the approximate number of days between dates, as in expression 9, Chapter I.

When a bond is purchased at a given yield between interest dates, part of the price P is a payment to the seller because of the dividend accrued since the last dividend date. The remainder of P is the present value of future dividend accruals and of the future redemption payment. This remainder of P corresponds to what was defined as the book value of a bond in Section 52. Hence, between dividend dates, the price is

$$P = \text{(Accrued Dividend to Date)} + \text{(Book Value)}; \quad (66)$$

$$\text{(Book Value)} = P - \text{(Accrued Dividend)}. \quad (67)$$

Equation 67 is also true on dividend dates; the accrued dividend is zero because the seller appropriates the dividend which is due, and hence the book value and the purchase price are the same, as they were previously defined to be in Section 52.

Example 3. — For the bond of Example 1 above, find the book value on August 16, 1922, to yield $(.05, m = 2)$.

Solution. — On Aug. 16, $P = \$1120.73$, from Example 1. The accrued dividend to Aug. 16 is \$7.50, from Example 2. Book value on Aug. 16 is $1120.73 - 7.50 = \$1113.23$, from equation 67.

NOTE 3. — The accrued dividend, although earned, is not due until the next dividend date. Hence, theoretically, in equation 66 we should use, instead of the accrued dividend, its value discounted to date from the next dividend date. Thus, in Example 3 we should theoretically subtract the present value at $(.05, m = 2)$ on Aug. 16, 1922, of \$7.50 due on Jan. 1, 1923, or $7.50(1.025)^{-\frac{5}{4}} = \$7.35$. The difference (in this case \$.15) always is small unless a large transaction is involved and, hence, it is the practice to use equation 67 as it stands.

EXERCISE L

1. A $1000, 8% bond, with dividends payable January 16 and July 16, is redeemable at 110% on July 16, 1928. Find the purchase price and the book value on September 16, 1921, to yield (.04, $m = 2$).

Find the purchase prices and the book values of the bonds below on the specified dates. All bonds are redeemable at par.

Prob.	Par Value	Div. Rate	Dividend Dates	Redemp. Date	Date of Purchase	Invest. Rate
2.	$1000	5%	June 1	6/1/1932	5/16/1922	(.07, $m = 1$)
3.	100	4%	Jan. 1, July 1	7/1/1940	8/13/1931	(.04, $m = 2$)
4.	5000	$4\frac{1}{2}$%	May 1	5/1/1952	9/1 /1924	(.06, $m = 1$)
5.	1000	6%	June 1, Dec. 1	6/1/1937	8/16/1923	(.05, $m = 2$)
6.	100	$6\frac{1}{2}$%	May 1, Nov. 1	5/1/1934	3/1 /1927	(.07, $m = 2$)

The book value between dividend dates may be found very easily by interpolation between the book values at the last and at the next dividend dates. This method is especially easy if a bond table is available.

Example 4. — A $100, 6% bond pays dividends on July 1 and January 1, and is redeemable at par on January 1, 1940. Find the book value and the purchase price on September 1, 1924, to yield (.04, $m = 2$).

Date	Book Value
7/1/1924	$122.938
9/1/1924	B
1/1/1925	$122.396

Solution. — In the table below the book values to yield (.04, $m = 2$) on 7/1/1924 and 1/1/1925 were computed from equation 59. Let B be the book value on Sept. 1, which is $\frac{1}{3}$ of the way from July 1 to Jan. 1. Hence, since $122.938 - 122.396 = .542$, $B = 122.938 - \frac{1}{3}(.542) = \$122.757$. From equation 66, the purchase price $P = 122.757 + 1 = \$123.757$.

EXERCISE LI

1. A $1000, 4% bond pays dividends annually on July 1, and is redeemable at par on July 1, 1937. (*a*) By interpolation find the book value on November 1, 1928, to yield (.06, $m = 1$). (*b*) Find the purchase price on November 1, 1928.

2. Find the book value in problem 1 by the method of illustrative example 3, Section 53, and compare with the result of problem 1.

3. A $5000, 6% bond pays dividends semi-annually on May 1 and November 1, and is redeemable at par on November 1, 1947. By use of interpolation find the book value and the purchase price to yield (.04, $m = 2$) on July 1, 1930.

4. A \$1000, 5% bond, with dividends payable March 1 and September 1, is redeemable at par on March 1, 1935. By use of the bond table of Section 51, find by interpolation the book value to yield (.045, $m = 2$) on April 1, 1924.

SUPPLEMENTARY NOTE. — The interpolation method of illustrative Example 4, page 123, gives the same book value as is obtained by the method of Example 3, which uses equation 67. To prove this, let the time to the present from the last dividend date be $\frac{1}{k}$ th part of a dividend interval. Let P_0 be the book value on the last, and P_1 that on the next dividend date, P the purchase price to-day, D the periodic dividend, I the interest on P_0 for a whole dividend interval at the investment rate, and B the book value of the bond to-day. First use the method of Example 3. Interest to date on P_0 at the investment rate is $\frac{1}{k}(I)$, and the accrued dividend to date is $\frac{1}{k}(D)$. From equation 64, $P = P_0 + \frac{I}{k}$, and from equation 67, $B = P - \frac{D}{k}$, or

$$B = P_0 + \frac{I}{k} - \frac{D}{k} = P_0 + \frac{I - D}{k}. \tag{68}$$

By interpolation, as in Example 4, since the present is $\frac{1}{k}$th interval from the last dividend date, B is $\frac{1}{k}$th part of the way from P_0 to P_1, or $B = P_0 + \frac{1}{k}(P_1 - P_0)$. From equation 62, $P_1 - P_0 = I - D$, and hence $B = P_0 + \frac{I - D}{k}$, the same as in equation 68.

DATE	BOOK VALUE
Last div. date	P_0
Present	B
Next div. date	P_1

Equation 68 shows that, when a bond is selling at a discount, the accumulation of the discount in $\frac{1}{k}$ th interval is $\frac{1}{k}$ th of the total accumulation for the interval, for, in equation 61 it is seen that $I - D$ is the accumulation for the whole interval. Similarly, if we write equation 68 as $B = P_0 - \frac{D - I}{k}$, it is seen that the amortization of the premium on a bond in $\frac{1}{k}$th interval is $\frac{1}{k}$th of the amortization for the whole interval.

54. Professional practices in bond transactions. — An investor buying a particular bond cannot usually demand a specified yield from his investment. He must pay whatever price is asked for that particular bond on the financial market. On bond exchanges, and in most private transactions, the purchase price of a bond is described to a purchaser as a certain **quoted price** plus the **accrued dividend.**[1] That is, the market quotation on a bond is what we have previously called the book value of the bond, in equation 67.

NOTE 1. — The quotation for a bond is given as a percentage of its par value. That is, a $10,000 bond, quoted at $93\frac{1}{4}$, has a book value of $9325.

Example 1. — A $10,000, 6% bond, with dividends payable June 1 and December 1, is quoted at $93\frac{1}{4}$ on May 1. Find the purchase price.

Solution. — Quotation = book value = $9325.00. Accrued dividend since December 1 is $250. From equation 66, the price is $9325 + 250 = \$9575$.

NOTE 2. — Bond exchange methods are simplified by the quotation of book values instead of actual purchase[2] prices. If the yield at which a bond sells remains constant, the book value changes very slowly through the accumulation of the discount, or amortization of the premium, as the case may be. Hence, when the practice is to quote book values, the market quotations of bonds change very slowly and any violent fluctuation in them is due to a distinct change in the yields at which the bonds are selling. On the other hand, if the actual purchase price of a bond were the market quotation, the quotation would increase as the dividend accrued and then, at each dividend date, a violent decrease would occur when the dividend was paid. Thus, even though the yield at which a bond were selling should remain constant, large fluctuations in its market quotation would occur.

EXERCISE LII

1. (*a*) A $1000, 5% bond, with dividends payable February 1 and August 1, is quoted at 98.75 on May 1; find the purchase price. (*b*) If the bond is purchased for $993.30 on April 1, find the market quotation then.

2. The interest dates for the 2d $4\frac{1}{4}$% Liberty Loan bonds are May 15 and November 15. Take their closing quotation on the New York Stock Exchange from the morning newspaper and determine the purchase price for a $10,000 bond of this issue.

3. A $1000, 6% bond whose dividend dates are January 1 and July 1 is quoted at $103\frac{1}{2}$ on October 16. Find the total price paid by a purchaser if he pays a brokerage commission of $\frac{1}{8}$% of the par value.

[1] In bond market parlance, it is called accrued *interest*. The more proper word *dividend* has been consistently used in this book to avoid pitfalls which confront the beginner. As seen in Section 52, equations 60 and 61, the dividend is not the same as the interest on the investment. The terminology *accrued interest* in bond dealings must be learned by the student and appreciated to mean *accrued dividend* in the sense of this chapter.

[2] The purchase price is called the *flat price* in bond parlance, as contrasted with the *price, and accrued interest* quotation customarily used.

55. Approximate bond yields. — On a given date the book value of a bond is quoted on the market and the problem is met of determining the yield obtained by an investor on purchasing the bond and holding it to maturity. We first consider an approximate method of solution, using mere arithmetic.

NOTE. — A bond salesman, in speaking of *the yield* on a bond, usually refers to an investment rate compounded the *same number of times per year as dividends are paid.* Thus, by the yield on a quarterly bond, he means the investment rate, compounded quarterly. We shall follow this customary usage in the future. Moreover, in computing yields it is usual to neglect the accrued dividend and brokerage fee paid at the time of purchase in addition to the book value. A yield is computed with reference to the book value of the bond.

The justification of the following rules is apparent on reading them. Let $\$B$ be the quoted book value of a bond, t the time in years before its maturity, and $\$C$ its redemption price. The invested principal changes from $\$B$ at purchase to $\$C$ at redemption, so that the average book value $\$B_0$ is given by $B_0 = \frac{1}{2}(B + C)$. Even though a bond pays dividends quarterly or semi-annually, in using the rules below *proceed as if the dividends were payable annually* at the dividend rate and let $\$D$ be this annual dividend.

Rule 1. — When the quoted book value B is at a premium over C. — Compute $\$A$, the average annual amortization of the premium from $A = \frac{\text{Premium}}{t}$. Compute $\$I$, the average annual interest on the investment from $I = D - A$.[1] Then, the approximate yield r equals the average annual interest divided by the average invested capital or $r = \frac{I}{B_0}$.

Rule 2. — When B is at a discount from C. — Compute $\$T$, the average annual accumulation of the discount from $T = \frac{\text{Discount}}{t}$. Compute I from $I = D + T$.[2] Then, the approximate yield r equals the average annual interest divided by the average invested capital, or $r = \frac{I}{B_0}$.

[1] See equation 60. [2] See equation 61.

Example 1. — A \$1000, 5% bond pays dividends semi-annually and is redeemable at 110%. Eleven years before its maturity, the book value is quoted on the market at 93. Estimate the yield.

Solution. — Considering its dividends annual, $D = \$50$, $C = \$1100$, and the book value $B = \$930$. Using Rule 2, the average accumulation of the discount is[1] $\frac{170}{11} = \$15.5$, and $I = 50 + 15.5 = \$65.5$. The average invested capital is $\frac{1}{2}(930 + 1100) = \$1015$. The approximate yield $r = \frac{65.5}{1015} = .065$, or 6.5%.

Example 2. — A \$1000, 5% bond pays dividends on July 1 and January 1 and is redeemable at par on January 1, 1961. Its quoted book value on May 1, 1922, is 113. Estimate the yield.

Solution. — Use Rule 1 with $B = \$1130$, $t = 38\frac{2}{3}$ years, $D = \$50$, and $C = \$1000$. We find $B_0 = \$1065$. To find the average amortization of the premium we take $t = 39$, the nearest whole number, because the inaccuracy of our rule when t is large makes refinements in computation useless. $A = \frac{130}{39} = \$3.3$, $I = 50 - 3.3 = \$46.7$, and therefore $r = \frac{46.7}{1065} = .044$, or 4.4%.

NOTE. — The author has experimentally verified that Rules 1 and 2 give estimated yields within .2% of the truth if: (*a*) the yield is between 4% and 8%, (*b*) the time to maturity is less than 40 years, and (*c*) the difference between the dividend rate and the yield is less than 3%. Greater accuracy is obtained under favorable circumstances. For a bond whose term is more than 30 years, as in Example 2 above, take t as the whole number nearest to the time to maturity in years. In all other cases use the exact time to the nearest month.

EXERCISE LIII

Estimate the yields of the following bonds, by use of Rules 1 and 2.

PROB.	FACE	TO BE REDEEMED AT	MARKET QUOT.	DIVIDEND		TIME TO MATURITY
				RATE	PAID	
1.[2]	\$1000	par	107.24	5%	semi-ann.	$11\frac{1}{2}$ years
2.[2]	100	par	160.30	7%	semi-ann.	25 years
3.[2]	1000	par	84.28	4%	semi-ann.	40 years
4.	100	110%	96.50	6%	annually	$23\frac{1}{2}$ years
5.	100	105%	115.00	5%	annually	8 years
6.	100	115%	98.75	$3\frac{1}{2}$%	annually	20 years

[1] Inspect the table of illustrative Example 2, Section 52. In that example, on computing the average semi-annual accumulation as in Rule 2, we obtain $\frac{1}{6}(62.30) = \$10.4$, a result very close to all of the semi-annual accumulations.

[2] The yields in the first three problems, determined by accurate means, are as follows: (1) 4.2%; (2) 3.4%; (3) 4.9%. Compare your results as found from Rules 1 and 2 in order to form an opinion of their accuracy.

7. A \$10,000, 5% bond, with dividends payable June 1 and December 1, is redeemable at par on December 1, 1950. On May 23, 1925, it is quoted at 89. Estimate the yield.

8. A Kingdom of Belgium $7\frac{1}{2}$% bond, whose dividends are semi-annual, may be redeemed at 115% at the end of 8 years. Estimate its yield under the assumption that it will be redeemed then, if it is now quoted at 94.

56. Yield on a dividend date by interpolation. — When the quoted value of a bond is given on a dividend date, the yield may be determined by interpolation. When annuity tables, but no bond tables, are available, proceed as follows:

(*a*) Find the estimated yield r as in Section 55.

(*b*) Compute the book value of the bond at the rate r_1 nearest to r for which the annuity tables may be used.

(*c*) Inspect the result of (*b*) and then compute the book value for another rate r_2, chosen so that the true yield is probably between r_1 and r_2. Select r_2 as near as possible to r_1.

(*d*) Find the yield i by interpolation between the results in (*b*) and (*c*).

Example 1. — A \$100, 6% bond, with semi-annual dividends, is redeemable at par. The quoted book value, $10\frac{1}{2}$ years before maturity, is \$111.98. Find the yield.

Solution. — (*a*) Average annual interest $I = 6 - \frac{11.98}{10.5} = \$4.9$; estimated yield $r = \frac{4.9}{106} = 4.6\%$. (*b*) Book value $10\frac{1}{2}$ years before maturity to yield $(.045, m = 2)$ is \$112.44 (by equation 59). (*c*) Since \$112.44 is greater than \$111.98, the yield is greater than .045, and is probably between .045 and .05. The book value at $(.05, m = 2)$ is \$108.09. Let $(i, m = 2)$ be the yield. In the table, $112.44 - 108.09 = 4.35$; $112.44 - 111.98 = .46$, and $.05 - .045 = .005$. Hence, $i = .045 + \frac{46}{435}(.005) = .0455$, or the yield is approximately 4.55%, compounded semi-annually.

Invest. Rate	Book Value
$.045, m = 2$	112.44
$i, m = 2$	111.98
$.05, m = 2$	108.09

Note 1. — A more exact solution[1] may be obtained as follows: At the yield $(.0455, m = 2)$, found above, compute the book value P, using logarithms in equation 59 because the annuity tables do not apply; $P = 100 + .725$

[1] A solution as in Example 1 gives a result which is in error by not more than $\frac{1}{20}$th of the difference between the table rates used in the interpolation. We are, essentially, interpolating in Table VIII, and hence our result is subject only to the error we meet in using that table.

($a_{\overline{21}|}$ *at* .02275) = 111.998. Since \$111.998 is greater than \$111.980, the yield i is greater than .0455, and is probably between .0455 and .0456. By logarithms, the book value at (.0456, m = 2) is \$111.910. From interpolation as in Example 1, $i = .0455 + \frac{18}{88}(.0001) = .045520$. The yield is 4.5520%, compounded semi-annually, with a possible error in the last decimal place.

NOTE 2. — The method of Example 1 is very easy if the desired book values can be read directly from a bond table (see problem 2 below). If the bond table uses interest rates differing by $\frac{1}{10}$%, results obtained by interpolation in the table are in error by not more than a few .001%. Extension of the accuracy of a solution as in Note 1 is limited only by the extent of the logarithm tables at our disposal.

NOTE 3. — If the book value B is given on a day between dividend dates, the yield may be accurately obtained by the method of Section 58 below. An approximate result can be found by assuming B as the book value on the nearest dividend date and computing the corresponding yield.

EXERCISE LIV

Find the yield in each problem as in Example 1, page 128. If the instructor so directs, extend the accuracy as in Note 1 above.

1. A \$100, 4% bond pays dividends on January 1 and July 1 and is redeemable at par on January 1, 1932. (*a*) Find the yield if the quoted value on July 1, 1919, is 89.32. (*b*) Find the effective rate of interest yielded by investing in the bond.

2. A \$100, 5% bond pays semi-annual dividends and is redeemable at par. By use of the bond table of Section 51, find the yield if the quoted value 11 years before maturity is 107.56.

For each bond in the table, par value is \$100. Find the yields.

PROB.	TO BE REDEEMED AT	DIVIDEND		TIME TO MATURITY	BOOK VALUE
		RATE	PAYABLE		
3.	110%	5%	annually	30 years	\$ 78.50
4.	par	4½%	semi-ann.	15½ years	110.75
5.	par	3%	semi-ann.	19 years	83.30
6.	par	6%	annually	12 years	121.00
7.	105%	5%	semi-ann.	24 years	88.00
8.	par	6%	quarterly	10¼ years	107.00

9. On January 1, 1923, a purchaser paid \$87.22, exclusive of brokerage, for a \$100, 4% bond whose dividends are payable July 1 and January 1

and which is redeemable at par on January 1, 1932. Find the yield obtained if the investor holds the bond to maturity.

10. A \$100, 5% bond pays semi-annual dividends and is redeemable at par at the end of 9 years. If it is quoted at 83.20, find the effective rate of interest yielded by the investment.

57. Special types of bond issues. — On issuing a set of bonds, a corporation, instead of desiring to redeem all bonds on one date, may prefer to redeem the issue in installments. The bonds are then said to form a **serial issue**. The price of the whole issue to net an investor a specified yield is the sum of the prices he should pay for the bonds entering in each redemption installment.

Example 1. — A \$1,000,000 issue of 6% bonds was made on January 1, 1920, with dividends payable semi-annually, and the issue is redeemable serially in 10 equal annual installments. Find the price at which all bonds outstanding on January 1, 1927, could be purchased to yield an investor $(.04, m = 2)$.

Solution. — There is \$300,000 outstanding. The price of the bonds for \$100,000, which are redeemable at the end of 1 year, is $1000(a_{\overline{2}|}\ at\ .02) + 100000 =$ \$101,941.56; the prices of the bonds redeemable in the installments paid at the end of 2 years and of 3 years are \$103,807.73 and \$105,601.43, respectively. The total price of outstanding bonds is \$311,350.72.

An **annuity bond**, with face value \$$F$, is a bond promising the payment of an annuity. The periodic payment \$$S$ of the annuity is described as the installment which, if paid periodically during the life of the bond, is sufficient to redeem the face \$$F$ in installments and to pay interest as due at the dividend rate on all of the face \$$F$ not yet redeemed. That is, the payments of \$$S$ amortize the face \$$F$ with interest at the dividend rate. When F and the dividend rate are known, S can be found by the methods of the amortization chapter. At a given investment yield, the price of an annuity bond is the present value of the annuity it promises. The annuity is always paid the same number of times per year as dividends are payable on the bond.

Example 2. — A certain ten-year, \$10,000 annuity bond with the dividend rate 5% is redeemable in semi-annual installments of \$$S$ each. (*a*) Find S. (*b*) Find the purchase price of the bond, 5 years before maturity, to yield 6%, effective.

Solution. — (a) The payments of $S amortize $10,000 at (.05, $m = 2$).

Bond annuity, Case 1
$n = 20$ int. per., $R = \$S$,
$A = \$10,000$, $p = 1$, $i = .025$.

$$10{,}000 = S(a_{\overline{20}|} \text{ at } .025)\,;\ S = \$641.47.$$

(b) The price A at the yield (.06, $m = 1$) is the present value of semi-annual payments of S made for 5 years.

Case 1
$n = 5$ int. per., $p = 2$,
$i = 06$, $R = \$1282.94$.

$$A = 1282.94(a^{(2)}_{\overline{5}|} \text{ at } .06) = \$5484.09.$$

EXERCISE LV[1]

1. A $100,000 serial issue of 5% bonds, with dividends payable semi-annually, is redeemable in 5 equal annual installments. The issue was made July 1, 1927. On July 1, 1930, find the price of all outstanding bonds to net the investor (.06, $m = 2$).

2. For the bonds purchased in problem 1, form a table showing, on each dividend date, the dividend received, the installment (if any) which is paid, the interest due on the book value, and the final book value.

3. A house worth $12,000 cash is purchased under the following agreement: $2000 of the principal is to be paid at beginning of each year for six years; interest at 6% is to be paid semi-annually on all principal outstanding. Two years later, the written contract embodying this agreement was sold to a banker, who purchased the remaining rights of the creditor to yield 7%, effective. What did the banker pay?

4. A 5-year annuity bond for $20,000, with the dividend rate 6%, payable semi-annually, is issued on June 1, 1921. (a) Find the price on June 1, 1922, to yield (.03, $m = 2$). (b) Find the price on September 1, 1922, to yield (.03, $m = 2$).

5. On June 1, 1924, find the price of the bond of problem 4 to yield (.06, $m = 1$).

SUPPLEMENTARY MATERIAL

58. Yield of a bond between dividend dates. — If the quoted value of a bond is given on a day between dividend dates, the yield may be found by interpolation by essentially the same procedure, with steps (a), (b), (c) and (d), as used in Section 56 on a dividend date.

[1] After the completion of Exercise LV, the student may immediately proceed to the consideration of the Miscellaneous Problems at the end of the chapter.

Example 1. — A \$100, 4% bond pays dividends annually on December 1 and is redeemable at par on December 1, 1931. Find the yield on February 1, 1926, if the book value is quoted at 95.926.

Solution. — (*a*) As in Section 55, the average annual interest on the investment is $I = 4 + .71 = \$4.71$; the estimated yield is $\frac{4.71}{97.96} = .048$. (*b*) The nearest table rate is 5%. To find the book value at 5% on Feb. 1, 1926, first compute the values at 5% on Dec. 1, 1925, and Dec. 1, 1926, the last and the next interest dates. The results, 94.924 and 95.671, are placed in the first row of the table below, and from them we find by interpolation the book value

TABLE OF BOOK VALUES

YIELD	DEC. 1, 1925	FEB. 1, 1926	DEC. 1, 1926
$.05, m = 1$	\$94.924	\$95.048	\$95.671
$i, m = 1$		95.926	
$.045, m = 1$	97.421	97.485	97.805

on Feb. 1, 1926. Since $95.671 - 94.924 = .747$, the book value on Feb. 1 at 5% is $94.924 + \frac{1}{6}(.747) = 95.048$. (*c*) Since 95.048 is less than 95.926 (the given book value), the yield is less than .05 and is probably between .045 and .05. Prices at .045 on Dec. 1, 1925, and on Dec. 1, 1926, were computed and from them the book value on Feb. 1, 1926, at .045 was obtained by interpolation. (*d*) The yield i is obtained by interpolation in the column of the table for Feb. 1. $97.485 - 95.048 = 2.437$; $97.485 - 95.926 = 1.559$; $.05 - .045 = .005$; hence $i = .045 + \frac{1559}{2437}.005 = .0482$. The yield is approximately 4.82%, compounded annually, with a possible small error in the last digit.

NOTE. — The method above is extremely simple if the desired book values can be read from a bond table. The accuracy of the solution can be extended by the method of Note 1, Section 56.

EXERCISE LVI

1. By use of the bond table of Section 51, find the yield of a \$100, 5% bond, with dividends payable on September 1 and March 1, and redeemable at par on September 1, 1928, if the quoted book value December 1, 1917, is \$106.78.

2. The interest dates of a \$100, 4% bond are July 1 and January 1, and it is redeemable at par on January 1, 1930. (*a*) Find the yield if it is quoted at 83.25 on September 1, 1923. (*b*) Find the effective rate of interest yielded by the bond.

3. A man pays $87.22, exclusive of the brokerage commission, on September 1, 1923, for a $100, 4% bond whose dividends are payable July 1 and January 1, and which is redeemable at par on January 1, 1930. Determine the yield, if the bond is held to maturity.

4. A $1000, 5% bond pays dividends annually on June 16 and is redeemable at 110% on June 16, 1937. Find the yield if it is quoted at 112.06 on November 16, 1932.

5. The 3d Liberty Loan $4\frac{1}{4}$% bonds are redeemable at par on September 15, 1928. Interest dates are September 15 and March 15. If the bonds were quoted at 85 on May 15, 1921, what was the investment yield?

MISCELLANEOUS PROBLEMS

In the following problems, the word **interest** is used in the colloquial sense in connection with bonds in place of the word **dividend** previously used.

1. A certain $1000, 5% bond pays interest annually. It is stipulated that, at the option of the debtor corporation, it may be redeemed at par on any interest date after the end of 10 years. The bond certainly will be redeemed at par by the end of 20 years. At what purchase price would a purchaser be certain to obtain 6% or more on his investment?

2. What is the proper price for the bond in problem 1 to yield 4%, or more?

3. In return for a loan of $5000, W gives his creditor H the following note:

> *Norfolk, June* 1, 1915.
>
> *For value received, I promise to pay, to* H *or order*, $5000 *at the end of* 6 *years and to pay interest on this sum semi-annually at the rate* 6%. Signed, W.

On December 1, 1916, H sold this note to an investor desiring $(.07, m = 2)$ on his investment. What did H receive?

4. What would H have received if he had sold the note to the same investor as in problem 3 on February 1, 1917?

5. Two $1000 bonds are redeemable at par and pay 4% interest semi-annually. Their quoted prices on a certain date to yield $(.05, m = 2)$ are $973 and $941.11, respectively. Without using annuity tables, and without computation, state which bond has the longer term to run and justify your answer.

6. Determine the term of the bond in problem 5 quoted at \$941.11.

7. A \$100, 4% bond, redeemable at par in 20 years, pays interest semi-annually. If it is quoted at \$92.10, what is the effective rate of interest obtained by an investor?

8. On June 1, 1921, a corporation has its surplus invested in bonds which are redeemable at par on June 1, 1928, and which pay interest semi-annually at the rate 5%. If the bonds are quoted at 102.74, would it pay the corporation to sell the bonds and reinvest the proceeds in Government bonds which net 4.65%, effective?

9. A \$1,000,000 issue of 5½% bonds, paying interest annually, is to be redeemed at 110% in twenty annual installments. The first installment is to be paid at the end of 5 years and the last at the end of 24 years. It is desired that the annual payments (dividends on unpaid bonds and the redemption installment included) at the end of each year for the last 20 years shall be equal. Determine the payment.

10. A house worth \$12,000 is purchased under the following agreement \$2,000 is to be paid cash and the balance of the principal is to be paid in four equal installments due at the ends of the 2d, 4th, 6th, and 8th years Interest at 6% is to be paid semi-annually on all sums remaining due The note signed by the purchaser is sold after 3 years by the original owner of the house. If the purchaser of the note demands 7%, compounded semi-annually, on his investment, what does he pay for the note?

11. A trust fund of \$20,000 is invested in bonds which yield 5% annually. The trust agreement states that ¾ of the income shall be given to the beneficiary each year and that the balance shall be re-invested in savings bank which pays 5%, compounded annually. The whole fund shall be turned over to the beneficiary after 10 years. If money is worth 6%, effective, to the beneficiary, what sum would he take now in place of his interest in the trust fund? Assume that he will live 10 years.

12. A corporation can sell at par a \$1,000,000 issue of 5½% bonds redeemable at par in 20 years and paying interest annually. To pay them at maturity the corporation would accumulate a sinking fund by annual deposits invested at 4½%, effective. Would it be better for the corporation to sell at par a \$1,000,000 issue of 5% bonds, if these are redeemable in such annual installments during the 20 years that the total annual payments, dividends and redemption payments included, will be equal

REVIEW PROBLEMS ON PART I

1. In purchasing a farm, \$5000 will be paid at the end of each year for 10 years. (*a*) What is the equivalent cash price if money is worth 5%, effective? (*b*) What must be paid at the end of the 6th year to complete the purchase of the farm?

2. A depreciation fund is being accumulated by semi-annual deposits of \$250 in a bank paying interest semi-annually at the rate 5%. What is in the fund just after the 30th payment?

3. A man wishes to donate to a university sufficient money to provide for the erection and the maintenance, for the next 50 years, of a building which will cost \$500,000 to erect and will require \$2000 at the end of each 3 months to maintain. What should he donate if the university is able to invest its funds at 5%, compounded semi-annually?

4. A debt of \$100,000 bears interest at 6%, payable semi-annually. A sinking fund is being accumulated by payments at the end of each 6 months to repay the principal in one installment at the end of 10 years. If the sinking fund earns 4% interest, compounded semi-annually, what is the total semi-annual expense of the debt?

5. A debt of \$100,000 is contracted under the agreement that interest at 6% shall be paid semi-annually on all sums remaining due. What payment at the end of each 6 months for 10 years will amortize this debt?

6. By use of a geometrical progression derive the expression for the amount of an annuity whose annual rent is \$2000, payable in semi-annual installments for 10 years, if money is worth 6%, compounded quarterly.

7. Find the present value of an annuity whose annual rent is \$3000, payable semi-annually for $20\frac{1}{2}$ years, if money is worth $(.05, m = 4)$.

8. A merchant owes \$6000 due immediately. For what sum should he make out a 90-day, non-interest-bearing note, so that his creditor may realize \$6000 on it if he discounts it immediately at a bank whose discount rate is 8%?

9. If money is worth 5%, effective, find the equal payments which if made at the ends of the first and of the third years would discharge the liability of the following debts: (1) \$1000 due without interest at the end of 3 years; (2) \$2000 due, with accumulated interest at the rate $(06, m = 2)$, at the end of 4 years.

10. A trust fund of \$100,000 is invested at 6%, effective. Payments of \$10,000 will be made from the fund at the end of each year as long as possible. (*a*) Find how many full payments of $10,000 will be made. (*b*) How much will be left in the fund just after the last full payment of $10,000?

11. Find the nominal rate of interest, compounded quarterly, under which payments of $1000 at the end of each 3 months for 20 years will be sufficient to accumulate a fund of $200,000.

12. Find the purchase price, to yield 6%, effective, of a $100, 5% bond with interest payable semi-annually, which is to be redeemed at 110% at the end of 10 years.

13. Estimate the yield on a bond which is quoted at 78, 10 years before it is due, if it is to be redeemed at par and if its dividend rate is 6%, payable annually.

14. Find the capitalized cost of a machine, whose original cost is $200,000, which must be renewed at a cost of $150,000 every 20 years. Money is worth 5%, effective.

15. Find the yield of a $100, 6% bond, with semi-annual dividends, which is quoted at 93.70, $10\frac{1}{2}$ years before it is due, and is redeemable at par.

16. A $100, 5% bond, quoted at 86.33 on September 1, 1926, yields 6% if held to maturity. The last coupon date was July 1. What is the purchase price on September 1?

17. A man deposited $100 in a bank at the beginning of each 3 months for 10 years. What is to his credit at the end of 10 years if the bank pays 8%, compounded quarterly?

18. A man deposited $50 in a fund at the end of each month for 20 years, at which time deposits ceased. What will be in the fund 10 years later if it accumulated for the first 20 years at the rate 6%, effective, and at the rate 4%, effective, for the remainder of the time?

19. The cash price of a farm is $5000 and money is worth $(.06, m = 2)$. What equal payments made quarterly will have an equivalent value, if the first payment is due at the end of 3 years and 9 months, and the last at the end of $13\frac{1}{2}$ years?

20. A corporation issues $200,000 worth of 6% bonds, redeemable at par at the end of 15 years, with interest payable semi-annually. The corporation is compelled by the terms of the issue to accumulate a sinking fund, to pay the bonds at maturity, by payments at the end of each 6 months, which are invested at $(.04, m = 2)$. The bonds are sold by the corporation at 95 (95% of their par value). Considering the total semi-annual expense as an annuity, under what rate of interest is the corporation amortizing the loan it realizes from the bond issue?

21. The present liability of a debt is $100,000. It is agreed that payments of $5000 shall be made at the end of each 6 months for 10 years, and that, during this time, the payments include interest at the rate 6%, payable semi-annually. Then, commencing with a first payment at the end of $10\frac{1}{2}$ years, semi-annual installments of $10,000 shall be paid as long as necessary to discharge the debt. (*a*) After the end of 10 years, if the payments include interest at the rate 5%, payable semi-annually, how many full payments of $10,000 must be made? (*b*) What part of the payment at the end of $10\frac{1}{2}$ years is interest on outstanding principal and what part is principal repayment?

22. From whose standpoint, that of the debtor or of the creditor, is compound interest more desirable than simple interest? Tell why in one sentence.

23. A 90-day note whose face value is $2000 bears interest at 6%. It is discounted at a bank 30 days before due. What are the proceeds if the banker's discount rate is 8%?

24. A man borrows a sum of money for 72 days from a bank, charging 5% interest payable in advance. (*a*) What interest rate is he paying? (*b*) What interest rate would he be paying if he borrowed money for 1 year from this bank?

25. Estimate the yield of a bond whose redemption value is $135, whose dividends are each $10, and are paid annually, and whose purchase price 6 years before due is $147.

26. (*a*) Find the yield[1] of a $100, 6% bond *bought* for $103.53 on October 1, 1921. Coupons are payable semi-annually on February 1 and August 1 and the bond will be redeemed at par on February 1, 1928. (*b*) Find the effective rate of interest yielded by the bond.

27. The principal of a debt of $200,000 is to be paid after 20 years by the accumulation of a sinking fund into which 79 quarterly payments will be made, starting with the first payment in 6 months. Find the quarterly payment if the fund grows at 6%, compounded quarterly.

28. To amortize a certain debt at 6%, effective, 40 semi-annual payments of $587.50 must be made. Just after the 26th payment, what principal will be outstanding?

29. (*a*) Find the capitalized worth at (.06, m = 12) of an enterprise which will yield a monthly income of $100, forever, first payment due now. (*b*) What is the present worth in (*a*) if the first monthly payment is due at the end of 6 months?

[1] Find the yield as in Section 58, or, if that section has not been studied, use the method of Section 55.

30. A bridge will need renewal at a cost of \$100,000 every 25 years. Under 5% interest, what is the present equivalent of all future renewals?

31. (*a*) By use of a geometrical progression determine a formula for the amount of an annuity whose annual rent is \$20,000, which is paid quarterly for 30 years, if money is worth 7%, compounded annually. (*b*) Without a geometrical progression find the present value of the annuity.

32. A house is worth \$50,000. In purchasing it \$20,000 is paid cash and the remainder is to be paid, principal and interest at $(.05,\ m = 2)$ included, by semi-annual installments of \$2000, first payment to be made at the end of 2 years. (*a*) Determine by interpolation how many whole payments of \$2000 will be necessary. (*b*) What liability will be outstanding just before the last full payment of \$2000?

33. A debt of \$50,000 is contracted and interest is at the rate 5%, compounded annually. The only payments (including interest) made were \$5000 at the end of 2 years, and six annual payments of \$3000, starting with one at the end of 5 years. At the end of 10 years what additional payment would complete payment of the debt?

34. If money is worth $(.05,\ m = 2)$, find the equal payments which must be made at the ends of the 3d and 4th years in order to discharge the following liabilities: (1) \$5000 due at the end of 6 years, without interest; (2) \$4000 due at the end of 5 years with all accumulated interest at $(.06,\ m = 1)$.

35. A debt of \$100,000 is contracted and it is agreed that it shall be paid, principal and interest included, by equal payments at the end of each 6 months for 20 years. Interest is at the rate $(4\%,\ m = 2)$ for the first 10 years and at $(5\%,\ m = 2)$ for the next 10 years. What single rate of interest over the whole 20 years would have resulted in the same payments?

36. A man invested \$100,000 in a certain enterprise. At the ends of each of the next 10 years he was paid \$4000 and, in addition, he received a payment of \$25,000 at the end of 6 years. At the end of 10 years he sold his investment holdings for \$80,000. Considering the whole period of 10 years, what was the effective rate of interest yielded by the investment?

HINT. — Write an equation of value; solve by interpolation as in Note 3 in the Appendix.

37. Mr. A borrows \$5000 from B to finance his college course and gives B a note, promising to pay \$5000 at the end of 10 years, together with all accumulations at 3%, compounded semi-annually. (*a*) What will A pay at the end of 10 years? (*b*) At the end of 5 years, B sells A's promissory note to a bank, which discounts it, considering money as worth (.05, $m = 1$). What does B realize from the sale?

38. Find the price at which a $100, 5% bond would be quoted on the market on September 1, 1922, to yield the investor (.06, $m = 2$). The bond is to be redeemed at par on August 1, 1928, and interest dates of the bond are August 1 and February 1.

39. An industrial commission awards $10,000 damages to the wife of a workman killed in an accident, but suggests that this sum be paid out by a trust company in quarterly installments of $200, the first payment due immediately. (*a*) If the trust company pays (.04, $m = 4$) on money, for how long will payments continue? (*b*) At the end of 10 years, the wife takes the balance of her fund. What amount does she receive?

40. Determine the capitalized cost of a machine worth $5000 new, due to wear out in 20 years, and renewable with a scrap value of $1000. Money is worth .05, effective.

41. Find the purchase price on December 1, 1920, of a $100, 6% bond with annual dividends, to yield at least 5%, if the bond may at the option of the issuing company be redeemed at 110% on any December 1 from 1930 to 1935, inclusive, or at par on any December 1 from 1943 to 1950. Justify your price.

42. A father wills to his son, who is just 20 years old, $20,000 of stock which pays dividends annually at the rate 6%. The will directs that the earnings shall be held to his son's credit in a bank paying $3\frac{1}{2}$%, effective, and that all accumulations as well as the original property shall become the direct possession of his son on his 30th birthday. Assuming that the market value of the stock on the 30th birthday will be $20,000, what is the present value of the estate for the son on his 20th birthday, assuming that money is worth $4\frac{1}{2}$% and that the son will certainly live to age 30?

43. If money is worth (.06, $m = 2$), what equal installments paid at the ends of the 2d and 3d years will cancel the liability of the following obligations: (*a*) $1000 due without interest at the end of 5 years, and (*b*) $2000 due with accumulated interest at the rate 4%, compounded annually, at the end of 6 years?

44. Two years and 9 months ago X borrowed $2000 from Y, and has paid nothing since then. (*a*) If interest is at the rate 6%, payable semi-annually, determine the theoretical compound amount which X should pay to settle his debt immediately. (*b*) Determine the amount by the practical rule.

45. At the end of each 6 months, \$200,000 is placed in a fund which accumulates at the rate (.06, $m = 2$). (*a*) How many full payments of \$200,000 will be necessary to accumulate a fund of \$1,000,000? (*b*) What smaller payment will be needed to complete the fund on the next date of deposit after the last \$200,000 payment?

46. Find the annual expense of a bond issue for \$500,000 paying 5% annually, if it is to be retired at the end of 20 years by the accumulation of a sinking fund by annual payments invested at 4%, effective.

47. In problem 46, at what effective rate of interest could the borrower just as well borrow \$500,000 if it is agreed to amortize the debt by equal payments made at the ends of the next 20 years?

48. How much is necessary for the endowment of a research fellowship paying \$3000 annually, at the beginning of each year, to the fellow and supplying a research plant, whose original cost is \$10,000, which requires \$2000 at the beginning of each year for repairs and supplies? Money is worth 4%, effective.

49. A banker employs his money in 90-day loans at 6% interest, payable in advance. At what effective rate is he investing his resources?

50. Find the present value and the amount of an annuity of \$50 per year for 20 years if money is worth 4%, payable annually. Use no tables and do entirely by arithmetic, knowing that $(1.04)^{20} = 2.191123$.

51. \$100,000 falls due at the end of 10 years. The debtor put \$8000 into a sinking fund at the end of each of the first 3 years. He then decided to make equal annual deposits in his sinking fund for the remainder of the time in order to accumulate the necessary \$100,000. If the fund earns (.04, $m = 1$), what was the annual deposit?

52. A corporation is to retire, by payments at the end of each of the next 10 years, a debt of \$105,000 bearing 5% interest, payable annually. The tenth annual payment, including interest, is to be \$15,000. The other nine are to be equal in amount and are to include interest. Determine the size of these nine payments.

53. Compute the purchase price to yield (.05, $m = 4$) of a \$1000, 6% bond redeemable at 110% in $12\frac{1}{2}$ years, if it pays interest semi-annually.

54. Compute the present value of an annuity whose annual rent is \$3000, payable quarterly for 6 years, if interest is at the rate 5.2%, effective.

55. The maximum sum insured under the War Risk Insurance Act pays \$57.50 at the beginning of each month for 20 years certain after death or disability. What would be the equivalent cash sum payable at death, or disability, at $3\frac{1}{2}$% interest?

56. A company issues $100,000 worth of 4%, 20-year bonds, which it wishes to pay at maturity by the accumulation of a sinking fund into which equal deposits will be made at the end of each year. The fund will earn 5% during the first five years, 4½% for the next 5 years, and 4% for the last 10 years. Determine the annual deposit.

57. The amount of a certain annuity, whose term is 7 years, is $3595 and the present value of the annuity is $2600. (*a*) Determine the effective rate of interest. (*b*) Determine the nominal rate, if it is compounded quarterly.

58. How long will it take to pay for a house worth $20,000 if interest is at 5%, effective, and if payments of $4000, including interest, are made at the beginning of each year? Find the last annual payment which will be made, assuming that the debtor never pays more than $4000 at one time.

59. A sum of $1000 is due at the end of two years. (*a*) Discount it to the present time under the simple interest rate 6%. (*b*) Discount it under the simple discount rate 6%. (*c*) Discount it under the compound interest rate (.06, $m = 1$).

60. A concern issues $200,000 worth of serial bonds, paying 5% interest annually. It is provided that $30,000 shall be used at the end of each year to retire bonds at par and to pay interest. How long will it take to retire the issue? Disregard the denomination of the bonds.

61. Find the value of a mine which will net $18,000 per year for 30 years if the investment yield is to be 6% and if the redemption fund is to be accumulated at 3½%, compounded annually.

62. A man expects to go into business when he has saved $5000. He now has $2000 and can invest his savings at (5%, $m = 1$). How much must he save at the end of each year to obtain the necessary amount by the end of 5 years?

63. Find by interpolation the composite life on a 4% basis of a plant consisting of: Part (A), with life 10 years, cost new, $13,000, scrap value, $2000; Part (B), with life 15 years, cost new, $20,000, and scrap value, $3000.

64. How much could a telephone company afford to pay per $10 unit cost in improving the material in its poles in order to increase the length of life from 15 to 25 years? The poles have no scrap value when worn out, and money is worth (.05, $m = 1$).

65. What are the net proceeds if a 90-day note for $1000, bearing 6% interest, is discounted at 8%?

66. X requests a 60-day loan of \$1000 from a bank charging 6% interest in advance. How much money does the bank give him and what interest rate is X paying on the loan?

67. A woman has funds on deposit in a bank paying (.04, $m = 2$). Should she reinvest in bonds yielding .0415, effective?

68. How long will it take for a fund of \$3500 to grow to \$4750 if invested at the rate 6%, compounded quarterly?

69. The sums \$200, \$500, and \$1000 are due without interest in 1, 2, and 3 years respectively. When would the payment of \$1700 equitably discharge these debts if money is worth (.06, $m = 1$)?

70. A father has 3 children aged 4, 7, and 9. He wishes to present each one with \$1000 at age 21. In order to do so he decides to deposit equal sums in a bank at the end of each year for 10 years. If it is assumed that the children will certainly live and that the bank pays (5%, $m = 1$), how much must the father deposit annually?

71. Which is worth more, if money is worth 6%, effective: (*a*) an income of 12 annual payments of \$500, first payment to be made at the end of 2 years, or (*b*) 120 monthly payments of \$50, first payment due at the end of 3 years and 1 month?

72. A \$100, 5% bond pays interest quarterly and is redeemable at 110% at the end of 10 years. Find its price to yield 6%, effective.

73. Find the present value and the amount of an annuity of \$3000 payable at the end of each 3 years for 21 years if interest is at the rate (.05, $m = 2$).

74. Find the nominal rate, converted quarterly, under which money will treble in 20 years.

75. (*a*) What effective rate is yielded by purchasing at par a \$100, 4% bond, redeemable at par, which pays interest quarterly? (*b*) What rate, compounded semi-annually, does the investment yield?

76. (*a*) In order to retire a \$10,000 debt at the end of 8 years a sinking fund will be accumulated by equal semi-annual deposits, the first due immediately and the last at the end of $7\frac{1}{2}$ years. Find the semi-annual payment if the fund is invested at the rate (.04, $m = 2$). (*b*) Find the size of the payments, under the same rate, if the first is made immediately and the last at the end of 8 years.

77. X lends \$600 to B, who promises to repay it at the end of 6 years with all accumulated interest at (.06, $m = 2$). At the end of 3 years, B desires to pay in full. If X is now able to invest funds at only 4%, effective, what should the debtor pay?

78. Find the nominal rate, converted quarterly, which yields the effective rate .0635.

79. (*a*) A house costs \$23,000 cash. If interest is at the rate (.05, $m = 1$), what equal payments made at the beginning of each 6 months for $6\frac{1}{2}$ years will amortize the debt? (*b*) What liability is outstanding at the beginning of the 3d year before the payment due is made?

80. How much must a man provide to purchase and maintain forever an ambulance costing \$6000 new, renewable every 4 years at a cost of \$4500 and requiring annual upkeep of \$1500 payable at the beginning of each year? Money is worth 4%, effective.

81. A corporation was loaned \$200,000 and, in return, made annual payments of \$12,000 for 8 years in addition to making a final payment of \$200,000 at the end of 9 years. What rate of interest did the corporation pay?

82. A loan of \$100,000 is to be amortized by equal payments at the end of each year for 20 years. During the first 10 years the payments are to include interest at 5%, effective, and, during the last 10 years, interest at 6%, effective. Determine the annual payment.

83. \$10,000 is invested at 6%, effective. Principal and interest are to yield a fixed income at the end of each 6 months for 10 years, at the end of which time the principal is to be exhausted. Determine the semi-annual income.

84. A house worth \$10,000 cash is purchased by B. A cash payment of \$2000 is made and it is agreed in the contract to pay \$500 of principal at the end of each 6 months until the principal is repaid and, in addition, to pay interest at the rate 6% semi-annually on all unpaid principal. Just after the payments are made at the end of two years, an investor buys the contract to yield 7%, compounded semi-annually, on the investment. What does the investor pay?

85. A farm worth \$15,000 cash is purchased by B, who contracts to pay \$2000 at the beginning of each 6 months, these payments including semi-annual interest at 6%, until the liability is discharged. At the end of 4 years, just after the payments due are made, the contract signed by B is sold to an investor to yield him (.07, $m = 2$) on the investment. What does he pay?

86. A state, in making farm loans to ex-soldiers, grants them the following terms: (*a*) interest shall be computed at the rate (.04, $m = 2$) throughout the life of the loan; (*b*) no interest shall be paid, but it shall accumulate as a liability, during the first 4 years; (*c*) the total indebted-

ness shall be discharged by equal monthly payments, the first due at the end of 4 years and 1 month and the last at the end of 10 years. Determine the monthly payment on a loan of $2000.

87. (*a*) A boy aged 15 years will receive the accumulations at 5%, effective, of an estate now worth $30,000, when he reaches the age 21. What is the present value of his inheritance at $3\frac{1}{2}$%, effective, assuming that he will certainly live to age 21? (*b*) Suppose that the boy is to receive, annually, the income at 5% from the estate and to receive the principal at age 21. Find the present value of the inheritance at $3\frac{1}{2}$%, effective.

88. The quotation of a certain $100, 5% bond to-day (an interest date) is 88.37 and it yields 7% to an investor. Find the purchase price and market quotation 2 months later at the same yield.

89. A note signed by Y promises to pay $1000 at the end of 90 days with interest at 5%. (*a*) What would the holder X obtain on selling the note 30 days later to a banker whose discount rate is 6%? (*b*) What would he obtain if the note were discounted under the simple interest rate 6%?

90. A certain man invests $1500 at the rate $(.04, m = 1)$ on each of his birthdays, starting at age 35 and ending at age 65. (*a*) At age 65, what does he have on hand? (*b*) Suppose that at age 65 he decides to save no more and to spend all of his savings by taking from them an equal amount at the end of each month for 15 years, and suppose that he will certainly live that long. What can he take per month if the savings remain invested at $(4\%, m = 1)$? (*c*) If he desires to have $5000 left at the end of the 15 years, what will be his monthly allowance?

91. A depreciation fund is being formed by semi-annual deposits, to replace an article worth $10,000 new, when it becomes worn out after 6 years. (*a*) If money is worth $(5\%, m = 1)$, what is the semi-annual charge if the scrap value of the article is $1000? (*b*) How much is in the depreciation fund just after the third deposit? (*c*) Find the condition per cent of the article at the end of 3 years.

92. A debt of $50,000 is being amortized with interest at $(.06, m = 2)$ by 24 equal semi-annual payments, the first payment cash. Find the payment and determine how much principal is outstanding just after the 12th payment.

93. Find the present value of a perpetuity of $1000, payable semi-annually, if interest is at the rate 6%, effective.

94. A man borrowed $10,000, which he agreed to amortize with interest at the rate 5%, payable annually, by equal payments, at the end of each year for 12 years. Immediately after borrowing the money he invested it at 7%, payable semi-annually. In balancing his books at the end of 12 years, what is his accumulated profit on the transaction?

95. A loan agency offers loans to salaried workers under the following plan. In return for a $100 loan, payments of $8.70 must be made at the end of each month for 1 year. Determine the nominal rate, compounded quarterly, under which the transaction is executed.

96. A corporation can raise money by selling 6% bonds, with semi-annual dividends, at 95% of par value. To provide for their redemption at par at the end of 15 years, a sinking fund would be accumulated by investing equal semi-annual deposits at (.04, $m = 2$). The corporation also can raise money by issuing, at par, 15-year, 7% annuity bonds redeemable in semi-annual installments. (*a*) Which method would entail the least semi-annual expense in raising $100,000 by a bond issue? (*b*) If money can be invested at (.04, $m = 2$) by the corporation, what would be the equivalent profit, in values at the end of 15 years, from choosing the best method?

97. A corporation will issue $1,000,000 worth of 5% bonds, paying interest semi-annually and redeemable at par in the following amounts: $200,000 at the end of 5 years; $300,000 at the end of 10 years; $500,000 at the end of 15 years. A banking syndicate bids $945,000 for the issue. Under what interest rate is the corporation borrowing on the proceeds of the bond issue?

98. An investor paid $300,000 for a mine and spent $30,000 additional at the beginning of each year for the first 3 years for running expenses. Equal annual operating profits were received beginning at the end of the 3d year and ceasing with a profit at the end of 25 years, when the mine became exhausted. The investor reinvested all revenue from the mine at 5%, effective. What was the net operating profit for the last 23 years if, at the end of 25 years, he has as much as if he had received, and reinvested at 5%, effective, 8% interest annually on all capital invested in the mine and likewise had received back his capital intact at the end of 25 years?

99. A man who borrowed $100,000 under the rate 6%, payable semi-annually, is to discharge all principal and interest obligations by equal payments at the end of each quarter for 8 years. At the end of 2 years, his creditor agrees to permit him to discharge his future obligations by 4 equal semi-annual payments, the first due immediately. (*a*) What will be the semi-annual payment if the creditor, in computing it, uses the rate

5%, compounded semi-annually? (*b*) What will be the semi-annual payment if the rate (.07, $m = 2$) is used in the computation?

100. A *contract for deed* is the name assigned to the following type of agreement in real estate transactions: In purchasing a piece of property worth \$2000 cash B agrees to pay \$500 cash and to pay \$25 at the end of each month, these payments to include interest at the rate 6%, payable monthly, until the property is paid for. The owner A agrees on his part to deliver the deed for the property to B when payment is completed.

Six months after the contract above was made, A sells it to an investor, who obtains the rate 7%, compounded monthly, on his investment. What does he pay, if A has already received the \$25 due on the contract on this date?

PART II — LIFE INSURANCE

CHAPTER VIII

LIFE ANNUITIES

59. Probability. — The mathematical definition of probability makes precise the meaning customarily assigned to the words *chance* or *probability* as used, for example, in regard to the winning of a game. Thus, if a bag contains 7 black and 3 white balls and if a ball is drawn at random, the *chance* of a white ball being obtained is $\frac{3}{10}$ because, out of 10 balls in the bag, 3 are white.

Definition. — If an event E can happen in h ways and fail in u ways, *all of which are equally likely*, the **probability** p of the event happening is

$$p = \frac{h}{u + h}, \tag{1}$$

and the probability q the event failing is

$$q = \frac{u}{u + h}. \tag{2}$$

NOTE 1.— In the ball problem above, the event E was the drawing of a white ball; $h = 3$, $h + u = 10$, $p = \frac{3}{10}$. The probability of failure $q = \frac{7}{10}$. The denominator $(u + h)$ in the formulas should be remembered as the total number of ways in which E can happen or fail.

From formulas 1 and 2, it is seen that p and q are both less than 1. Moreover,

$$p + q = \frac{h}{h + u} + \frac{u}{h + u} = \frac{u + h}{u + h} = 1,$$

or the sum of the probabilities of failure and of success is 1. If an event is *certain to happen*, $u = 0$ and $p = \frac{h}{h} = 1$.

EXERCISE LVII

1. An urn contains 10 white and 33 black balls. What is the probability that a ball drawn at random will be white?

2. A deck of 52 cards contains 4 aces. On drawing a card at random from a deck, what is the probability that it will be an ace?

3. Out of a class of 50 containing 20 girls and 30 boys, one member is chosen by lot. What is the probability that a girl will be picked?

4. A cubical die with six faces, numbered from 1 to 6, is tossed. What is the probability that it will fall with the number 4 up?

5. A coin is tossed. What is the probability that it will fall head up?

6. If the probability of a man living for at least 10 years is .8, find the probability of him dying within 10 years.

7. If the probability of winning a game is $\frac{3}{5}$, what is the probability of losing?

NOTE 2. — It is important to recognize that when we say, as in problem 7, above, "the probability of winning is $\frac{3}{5}$," we mean: (*a*) if a *very large number of games are played*, it is to be expected that approximately $\frac{3}{5}$ of them will be won, and (*b*) if the number of games played becomes larger and larger without bound, it is to be expected that the quotient, of the number of them which are won divided by the total played, will approach $\frac{3}{5}$ as a limiting value. We *do not imply*, for instance, that out of 45 games played exactly $\frac{3}{5}$ of 45, or 27 games will be won. We must recognize that, if only a few games are played, it may happen that more, or equally well less, than $\frac{3}{5}$ of the total will be won.

8. As a coöperative class exercise, toss a coin 400 times and record at each trial whether or not the coin falls head up. How many were heads out of (*a*) the first 10 trials; (*b*) 50 trials; (*c*) 400 trials? Compare in each case the number of heads with $\frac{1}{2}$ of the number of trials so as to appreciate Note 2, above.

NOTE 3. — The assumption in the definition of probability that all ways of happening or failing are *equally likely*, is a very important qualification. For example, we might reason as follows: A man selected at random will either live one day or else he will die before to-morrow. Hence, there are only two possibilities to consider, and the probability of dying before to-morrow is $\frac{1}{2}$. This ridiculous conclusion would neglect the fact that he is *more* likely to live than to die, and hence our definition of probability should not be applied.

60. Mortality Table. — Table XIII was formed from the accumulated experience of many American life insurance companies. This table should be considered as showing the observed deaths among a group[1] of 100,000 people of the same age, all of whom

[1] The actual construction of a mortality table is a very difficult matter and cannot be considered here. It is, of course, impossible to obtain for observation 100,000 children of the same age, 10 years, and to keep a record of the deaths until all have died. However, data obtained by insurance companies, or census records of births and deaths, can be used to create a table equivalent to a death record of a representative group of 100,000 people of the same age, all of whom were alive at age 10.

were alive at age 10. In the mortality table, l_x represents the number of the group still alive at age x, and d_x the number of the group dying between ages x and $x + 1$. Thus, $l_{25} = 89{,}032$, and $d_{25} = 718$ (= 89032 − 88314). In general, $d_x = l_x - l_{x+1}$. Out of l_x alive at age x, l_{x+n} remain alive at age $x + n$, and hence $l_x - l_{x+n}$ die between ages x and $x + n$. Thus, $l_{25} - l_{28}$ or 2154 die between ages 25 and 28.

When the exact probability of the happening of an event is unknown, the probability may sometimes be determined by observation and statistical analysis. Suppose that an event has been observed to happen h times out of m trials in the past. Then, as an approximation to the probability of occurrence we may take $p = \frac{h}{m}$. This estimated value of p becomes increasingly reliable as the number of observed cases increases. The statistical method is used in determining all probabilities in regard to the death or survival of an individual selected at random; our *observed data* is the tabulated record given in the mortality table.

Example 1. — A man is alive at age 25. (*a*) Find the probability that he will live at least 13 years. (*b*) Find the probability that he will die in the year after he is 42.

Solution. — (*a*) We observe $l_{25} = 89{,}032$ men alive at age 25. Of these, 79,611($= l_{38}$) remain alive at age 38. The probability of living to age 38 is $\frac{l_{38}}{l_{25}} = \frac{79611}{89032}$. (*b*) Out of 89,032 alive at age 25, 785 (= 76567 − 75782 $= d_{42}$) die in their 43d year. The probability of dying is $\frac{d_{42}}{l_{25}} = \frac{785}{89032}$.

EXERCISE LVIII

In the first nine problems find the probability:

1. That a boy aged 10 will live to graduate from college at age 22.

2. That a man aged 33 will live to receive an inheritance payable at age 45.

3. That a boy aged 15 will reach age 80.

4. (*a*) That a man aged 56 will die within 5 years. (*b*) That he will die during the 5th year.

5. That a man aged 24 will live to age 25.

6. That a man aged 28 will die in his 38th year.

7. That a man aged 28 will die in the year after he is 38.

8. That a man aged 40 will live at least 12 years.

9. That a man aged 35 will live at least 20 years.

10. If a man is alive at age 22, between what ages is he most likely to die and what is his probability of dying in that year?

NOTE. — The problems in probability solved in the future in the theory of life annuities and of life insurance, so far as it is presented in this book, will be like those of Exercise LVIII. None of the well known theorems on probability are needed in solving such problems; the mere definition of probability is sufficient. Hence, no further theorems on probability are discussed in this text. The student is referred for their consideration to books on college algebra.

61. Formulas used with Table XIII. — In Table XIII, we verify that the number dying between ages 25 and 28 is $l_{25} - l_{28} = d_{25} + d_{26} + d_{27}$, the sum of those dying in their 25th, 26th, or 27th years. Similarly, those dying between ages x and $x + n$ are

$$l_x - l_{x+n} = d_x + d_{x+1} + \cdots + d_{x+n-1}. \quad (3)$$

From Table XIII, $l_{96} = 0$, and hence l_{97}, l_{98}, etc., are zero because all are dead before reaching age 96. The group of l_x alive at age x are those who die in the future years, so that

$$l_x = d_x + d_{x+1} + \cdots + d_{95}. \quad (4)$$

It is convenient to use "(x)" to abbreviate *a man aged x*. Let ${}_np_x$ represent the probability that (x) will live at least n years, or that (x) will still be alive at age $x + n$. Since l_{x+n} remain alive at age $(x + n)$, out of l_x alive at age x,

$${}_np_x = \frac{l_{x+n}}{l_x}. \quad (5)$$

When $n = 1$, we omit the $n = 1$ on ${}_np_x$ and write p_x for the probability that (x) will live 1 year;

$$p_x = \frac{l_{x+1}}{l_x}. \quad (6)$$

The values of p_x are tabulated in Table XIII. Let q_x represent the probability that (x) will die before age $(x + 1)$. Since d_x of the group of l_x die in the first year,

$$q_x = \frac{d_x}{l_x}. \quad (7)$$

7. That a man aged 28 will die in the year after he is 38.

8. That a man aged 40 will live at least 12 years.

9. That a man aged 35 will live at least 20 years.

10. If a man is alive at age 22, between what ages is he most likely to die and what is his probability of dying in that year?

NOTE. — The problems in probability solved in the future in the theory of life annuities and of life insurance, so far as it is presented in this book, will be like those of Exercise LVIII. None of the well known theorems on probability are needed in solving such problems; the mere definition of probability is sufficient. Hence, no further theorems on probability are discussed in this text. The student is referred for their consideration to books on college algebra.

61. Formulas used with Table XIII. — In Table XIII, we verify that the number dying between ages 25 and 28 is $l_{25} - l_{28} = d_{25} + d_{26} + d_{27}$, the sum of those dying in their 25th, 26th, or 27th years. Similarly, those dying between ages x and $x + n$ are

$$l_x - l_{x+n} = d_x + d_{x+1} + \cdots + d_{x+n-1}. \tag{3}$$

From Table XIII, $l_{96} = 0$, and hence l_{97}, l_{98}, etc., are zero because all are dead before reaching age 96. The group of l_x alive at age x are those who die in the future years, so that

$$l_x = d_x + d_{x+1} + \cdots + d_{95}. \tag{4}$$

It is convenient to use "(x)" to abbreviate *a man aged x*. Let ${}_np_x$ represent the probability that (x) will live at least n years, or, that (x) will still be alive at age $x + n$. Since l_{x+n} remain alive at age $(x + n)$, out of l_x alive at age x,

$${}_np_x = \frac{l_{x+n}}{l_x}. \tag{5}$$

When $n = 1$, we omit the $n = 1$ on ${}_np_x$ and write p_x for the probability that (x) will live 1 year;

$$p_x = \frac{l_{x+1}}{l_x}. \tag{6}$$

The values of p_x are tabulated in Table XIII. Let q_x represent the probability that (x) will die before age $(x + 1)$. Since d_x of the group of l_x die in the first year,

$$q_x = \frac{d_x}{l_x}. \tag{7}$$

The values of q_x are tabulated in Table XIII. Let $_{n|}q_x$ represent the probability that x will die in the year after reaching age $(x+n)$, between the ages $(x+n)$ and $(x+n+1)$. Of the original group of l_x alive at age x, d_{x+n} die in the year after reaching age $(x+n)$. Hence

$$_{n|}q_x = \frac{d_{x+n}}{l_x}. \tag{8}$$

Let $|_nq_x$ represent the probability that x will die before reaching age $x+n$. Of the group of l_x alive at age x, $(l_x - l_{x+n})$ will die before reaching age $x+n$, and hence the probability of dying is

$$|_nq_x = \frac{l_{x+n} - l_x}{l_x}. \tag{9}$$

Example 1. — State in words the probabilities denoted by the following symbols and find their values by the formulas above:

$$(a)\ {}_{17}p_{35};\ (b)\ {}_{15|}q_{22};\ (c)\ |_{15}q_{22}.$$

Solution. — (*a*) $_{17}p_{25}$ is the probability that a man aged 25 will be alive at age 42; $_{17}p_{25} = \frac{l_{42}}{l_{25}} = \frac{76567}{89032}$. (*b*) $_{15|}q_{22}$ is the probability that a man aged 22 will die in the year after he reaches age 37; $_{15|}q_{22} = \frac{d_{37}}{l_{22}} = \frac{742}{91192}$. (*c*) $|_{15}q_{22}$ is the probability that a man aged 22 will die before he is 15 years older (before reaching age 22 + 15 = 37); $|_{15}q_{22} = \frac{l_{22} - l_{37}}{l_{22}}$.

EXERCISE LIX

1. Find the probability that a man aged 25 will live at least (*a*) 30 years; (*b*) 40 years; (*c*) 70 years.

2. Find the probability that a man aged 30 will die in the year after reaching age 40.

3. From formula 7 find the probability of a man aged 23 dying within 1 year, and verify the entry in Table XIII.

4. From formula 6 find the probability of a man aged 37 being alive at age 38, and verify the table entry.

5. Find the probability that a man aged 33 will die in the year after reaching age 55.

6. State in words the probabilities represented by the following symbols and express them as quotients by the formulas above: $_{16}p_{42}$; $_{15|}q_{42}$; $_{17|}q_{25}$; $|_{17}q_{25}$; q_{70}; $_{10}p_{23}$; p_{43}; $_{18|}q_{52}$.

7. From formulas 5 and 9 prove that ${}_np_x = 1 - {}_{|n}q_x$.

NOTE. — If we consider the event of (x) living for at least n years, the failure of the event means that (x) dies within n years. Hence, the result of problem 7 should be true because, from Section 59 the sum of the probabilities of the success and of the failure of an event is 1, and $p = 1 - q$.

8. What is the probability of a man aged 26 dying some time after he reaches age 45?

9. Verify formula 4 for $x = 90$.

10. Verify formula 3 for $x = 53$ and $n = 5$.

62. Mathematical expectation; present value of an expectation. — If a man gambles in a game where the stake is $100, and where his probability of winning is .6, his *chances* are worth .6 (100) = $60. Such a statement is made precise in the following

Definition. — *If p is the probability of a person receiving a sum $S, the* **mathematical expectation** *of the person is pS.*

If the sum $S is due at the end of n years, the mathematical expectation *at the end of n years* is pS. If money is worth the effective rate i, the **present value $A of the expectation** is given by

$$A = pS(1 + i)^{-n}. \qquad (10)$$

NOTE. — In the future, the arithmetical work in all examples will be performed by 5-place logarithms.

Example 1. — If money is worth $3\frac{1}{2}\%$, find the present value of the expectation of a man aged 25 who is promised a payment of $5000 at the end of 12 years if he is still alive.

Solution. — The probability p of receiving the payment is the probability of the man living to age 37, or $p = {}_{12}p_{25}$. From equation 10, the present value of the expectation is

$$A = {}_{12}p_{25}5000(1.035)^{-12} = \frac{5000(1.035)^{-12}l_{37}}{l_{25}}. \qquad \text{(Formula 5)}$$

$$A = \$2986.4. \qquad \text{(Tables VI and XIII)}$$

NOTE. — In Example 1 it would be said that the payment of $5000 at the end of 12 years is contingent (or dependent) on the survival of the man. For brevity, in using formula 10, we shall speak of the *present value of a contingent payment* instead of, more completely, the present value of the *expectation of this payment.*

EXERCISE LX

1. In playing a game for a stake of $50, what is the mathematical expectation of a player whose probability of winning is .3?

NOTE. — Suppose that a professional gambler should operate the game of problem 1 and charge each player the value of his mathematical expectation as a fee for entering the game. Then, if a *very large number of players enter the game*, the gambler may expect to win, or lose, approximately nothing. This follows from the facts pointed out in Note 2, Section 59, because, if a large number play, approximately .3 of them may be expected to win the stake, and the money won would, in this case, be approximately equal to the total fees collected by the gambler. If, however, the gambler should admit only a *few players* to the game, he might happen to win, or equally well lose, a large sum, because out of a few games he has no right to expect that exactly 3 of them will be won. The principle involved in this note is fundamental in the theory of insurance, and finds immediate application in problem 4, below. In any financial operation which is essentially similar to that of the professional gambler above, the safety of the operator depends on his obtaining a large number of players for his game.

2. At the end of 10 years a man will receive $10,000 if he is alive. At 5% interest, find the present worth of his expectation if his probability of living is .8.

3. A young man, aged 20, on entering college is promised $1000 at the end of 4 years, if he graduates with honors. At 5% interest, find the present value of his expectation.

4. Out of 1,000,000 buildings of a certain type, assume that the equivalent of 2500 total losses, payable at the end of the year, will be suffered through fire in the course of one year. (*a*) If an owner insures his building for $20,000 for one year, what is the present value of his expectation, at 3% interest? (*b*) What is the least price that a fire insurance company could be expected to charge for insuring his building?

5. A certain estate will be turned over to the heir on his 23d birthday. If the estate will then be worth $50,000, what is the present worth of the inheritance if money is worth 4½% and if the heir is now 14 years old?

6. A boy aged 15 has been willed an estate worth $10,000 now. The will directs that the estate shall be allowed to accumulate at the rate (.04, $m = 2$) until the heir is 21. If money is worth 3½% to the boy, find the present value of his expectation.

63. Present value of a pure endowment. — If $1 is to be paid to (x) when he reaches age $(x + n)$, we shall say he has (or is promised) an n-**year pure endowment** of $1. Let ${}_nE_x$ be the present value of this endowment when money is worth the effective rate i. The probability p of the endowment being paid equals the probability of (x) living to age $x + n$, or $p = {}_np_x$. Hence, from formula 10, with $S = 1$,

$$ {}_nE_x = {}_np_x(1+i)^{-n} = \frac{l_{x+n}(1+i)^{-n}}{l_x}. \qquad \text{(Formula 5)}$$

In the future we shall use v as an abbreviation for the discount factor $(1 + i)^{-1}$. Thus, $v = (1 + i)^{-1}$, $v^2 = (1 + i)^{-2}$, etc., $v^n = (1 + i)^{-n}$. Hence,

$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x}. \qquad (11)$$

The present value $\$A$ of an n-year pure endowment of $\$R$ to a man aged x is given by

$$A = R({}_nE_x) = \frac{Rv^n l_{x+n}}{l_x}. \qquad (12)$$

NOTE 1. — Remember the subscript $(x + n)$ on l_{x+n} in formula 12 as the age at which (x) receives the endowment.

Example 1. — A man aged 35 is promised a \$3000 payment at age 39. Find the present value of this promise if money is worth 6%, effective.

Solution. — The man aged 35 has a 4-year pure endowment of \$3000. From formula 12, its present value is

$$A = 3000({}_4E_{35}) = \frac{3000\,v^4 l_{39}}{l_{35}} = \frac{3000(1.06)^{-4} l_{39}}{l_{35}} = \$2290.3. \quad \text{(Tables VI, XIII)}$$

NOTE 2. — Formula 11 may be derived by the following method. The present value ${}_nE_x$ is the sum which, if contributed now by a man aged x, will make possible the payment of \$1 to him at the end of n years, if money can be invested at the rate i, effective. Suppose that l_x men of age x make equal contributions to a common fund with the object of providing all survivors of the group with \$1 payments at the end of n years. Since l_{x+n} men will survive, the necessary payments at the end of n years total $\$l_{x+n}$. The present value of this amount at the rate i is $l_{x+n}(1 + i)^{-n} = l_{x+n}v^n$, which is the sum needed in the common fund. Hence, the share which each of the l_x people must contribute is

$$\frac{v^n l_{x+n}}{l_x},$$

the same as obtained in formula 11.

EXERCISE LXI

1. A man aged 31 is promised a gift of \$10,000 when he reaches age 41. Find the present value of the promise at $3\frac{1}{2}\%$ interest.

2. State in words what is represented by \$2000 $({}_{17}E_{38})$ and find its value at 5% interest.

3. A will specifies that the estate shall be turned over to the heir, now aged 23, when he reaches 30 years of age. If the estate will then amount to \$150,000 find the present value of the inheritance at 4% interest.

4. A man aged 25 has \$1000 cash. What pure endowment, payable at the end of 20 years, could he purchase from an insurance company which will compute the endowment at a 4% rate? Make use of equation 12, to determine the unknown quantity R.

5. If money is worth $3\frac{1}{2}\%$, what endowment payable at age 45 could a man aged 30 purchase for \$7500?

64. Whole life annuity. — A **whole life annuity** is an annuity whose periodic payments continue as long as a certain individual (or individuals) survives. We shall deal only with the case where *one* individual is concerned. In speaking of a *life annuity* we shall always mean a *whole* life annuity unless otherwise specified.

NOTE 1. — The periodic payments of all annuities will be supposed equal and will be due at the ends of the payment intervals unless otherwise stated. When no rate of interest is specified, it will be understood as the rate i, effective.

Example 1. — If money is worth $3\frac{1}{2}\%$, find the present value of a life annuity of \$1000 payable annually to a man aged 92.

Solution. — He is promised a payment, or endowment, of \$1000 at age 93, another at 94, and a third at 95, which he will receive if he is alive when they are due. No payment is possible after he is of age 95 because he is certainly dead at age 96. The present value A of his expectation from the annuity is the sum of the present values of the equivalent three endowments, due in 1, 2, and 3 years. From formula 12, the present values of these endowments are $1000{}_1E_{92}$, $1000{}_2E_{92}$, and $1000{}_3E_{92}$. Hence, by use of formula 12 and tables VI and XIII, we obtain

$$A = 1000({}_1E_{92} + {}_2E_{92} + {}_3E_{92});$$

$$A = 1000\left(\frac{vl_{93}}{l_{92}} + \frac{v^2l_{94}}{l_{92}} + \frac{v^3l_{95}}{l_{92}}\right);$$

$$A = 1000\left(\frac{(1.035)^{-1}l_{93} + (1.035)^{-2}l_{94} + (1.035)^{-3}l_{95}}{l_{92}}\right) = \$456.65.$$

Let a_x be the present value of a ife annuity of \$1 payable at the end of each year to a man now aged x. This annuity is equivalent to pure endowments of \$1 payable at ages $(x + 1)$, $(x + 2)$, $\cdots$, to age 95. The present values of these endowments are tabulated below, and a_x equals their sum.

Age at Which \$1 Endowment is Payable	Time from Now until Endowment is Payable	Present Value of the Endowment
$x+1$	1 yr.	${}_1E_x = \frac{vl_{x+1}}{l_x}$
$x+2$	2 yr.	${}_2E_x = \frac{v^2l_{x+2}}{l_x}$
.	.	.
.	.	.
.	.	.
95	$95-x$	${}_{95-x}E_x = \frac{v^{95-x}l_{95}}{l_x}$

On adding the last column we obtain

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + \cdots + v^{95-x}l_{95}}{l_x}. \tag{13}$$

The present value \$$A$ of a life annuity of \$$R$ paid at the end of each year is given by $A = Ra_x$.

Note 2. — In contrast to the annuities *certain* considered in Part I, life annuities are called *contingent* annuities, because their payments are contingent (or dependent) on the survival of (x). The life annuity was interpreted as being paid to (x). Recognize that a_x is the present value of payments made at the end of each year during the life of (x), *regardless of who receives the payments.*

EXERCISE LXII

1. By the process of Example 1 above, find the present value of a life annuity of \$2000 payable at the end of each year to a man now aged 91, if money is worth 5%.

2. If money is worth 6%, find the present value of a whole life pension of \$1000 payable at the end of each year to a man now aged 92. Use formula 13.

3. (*a*) By use of formula 13, write the explicit expression which would be computed in finding the present value of a life annuity of \$500 paid at the end of each year to a man aged 65, if money is worth 6%. (*b*) How many multiplications would be necessary in computing the numerator?

4. By the method used in deriving the formula for a_x, find the present value of a life annuity of \$1000 payable at the end of each 3 years to a man aged 35, if money is worth (.04, m = 1). Do not compute the expression obtained.

65. Commutation symbols. — Auxiliary symbols (such as D_k and N_k below), called **commutation symbols**, are used in life annuity and insurance formulas. From formula 11 for ${}_nE_x$, we obtain

$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^x}{v^x}\frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x}. $$

Let D_k be an abbreviation for $v^k l_k$, or

$$ D_k = v^k l_k. \tag{14} $$

Thus, $D_{50} = v^{50} l_{50}$. Hence, $v^x l_x = D_x$, $v^{x+n} l_{x+n} = D_{x+n}$, and

$$ {}_nE_x = \frac{D_{x+n}}{D_x}. \tag{15} $$

The present value $\$A$ of an n-year pure endowment of $\$R$ is

$$ A = R({}_nE_x) = \frac{RD_{x+n}}{D_x}. \tag{16} $$

Example 1. — Compute D_{25} if money is worth $3\frac{1}{2}\%$.

Solution. — $D_{25} = v^{25} l_{25} = (1.035)^{-25}(89032) = 37674.$

NOTE. — It is very customary for insurance companies to use $3\frac{1}{2}\%$ as the rate in annuity computations. The values of D_{10}, D_{11}, to D_{95} at $3\frac{1}{2}\%$ are tabulated in Table XIV, and the result of Example 1 above is seen to check the proper table entry. Formulas 15 and 16 may be used, in connection with Table XIV, only when the rate is $3\frac{1}{2}\%$. Tables of the values of D_k at other rates [1] are found in collections of actuarial tables. In problems in this book, when the rate is not $3\frac{1}{2}\%$, formulas 11 and 12 must be used.

To simplify formula 13 for a_x, multiply numerator and denominator by v^x. We obtain

$$ a_x = \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \cdots + v^{95} l_{95}}{v^x l_x}. $$

Since $v^x l_x = D_x$, $v^{x+1} l_{x+1} = D_{x+1}$, etc., $v^{95} l_{95} = D_{95}$,

$$ a_x = \frac{D_{x+1} + D_{x+2} + \cdots + D_{95}}{D_x}. \tag{17} $$

Introduce N_k as an abbreviation for the sum of all D's from D_k to D_{95}:

$$ N_k = D_k + D_{k+1} + \cdots + D_{95}. \tag{18} $$

Thus, $N_{90} = D_{90} + D_{91} + D_{92} + D_{93} + D_{94} + D_{95}$. Since the numerator in formula 17 is N_{x+1},

$$ a_x = \frac{N_{x+1}}{D_x}. \tag{19} $$

[1] See *Tables of Applied Mathematics*, by Glover.

The present value $\$A$ of a life annuity of $\$R$ per year to (x) is

$$A = Ra_x = \frac{RN_{x+1}}{D_x}. \tag{20}$$

NOTE. — The values of N_k are tabulated in Table XIV for the rate $3\frac{1}{2}\%$. For this rate, formulas 19 and 20 may be used in connection with Table XIV. For all other rates the previous formula 13 must be used. The values of N_k for a few other interest rates are found in actuarial tables.

Example 2. — If money is worth $3\frac{1}{2}\%$, what life annuity, payable at the end of each year, can a man aged 50 purchase for $10,000?

Solution. — Let R be the payment of the annuity. From formula 20,

$$10000 = R(a_{50}) = R\frac{N_{51}}{D_{50}},$$

$$R = \frac{10000D_{50}}{N_{51}} = \frac{10000(12498.6)}{169165} = \$738.83.$$

EXERCISE LXIII

1. Compute the value of D_{38} for $i = .035$ and verify the entry in Table XIV.

2. By use of formula 18 and Table XIV for the D's, find the value of (*a*) N_{95}; (*b*) N_{94}; (*c*) N_{93}.

3. Find the present value of a life annuity of $1000 at the end of each year for a man aged 24, at $3\frac{1}{2}\%$.

4. (*a*) Find the present value of a pure endowment of $3500 at the end of 12 years for a man aged 33, at $3\frac{1}{2}\%$. (*b*) Find the present value of the endowment at the rate 4%.

5. A man aged 65 is promised a pension of $2000 at the end of each year as long as he lives. (*a*) If money is worth $3\frac{1}{2}\%$, find the present value of his pension. (*b*) What is the present value if $2000 is to be paid at the beginning of each year?

6. An estate is worth $100,000 and is invested at 5%, effective. The annual income is willed to a woman, aged 30, for the rest of her life. Find the present value of her inheritance if money is worth $3\frac{1}{2}\%$.

7. A man aged 45 has agreed to pay a $75 insurance premium at the end of each year as long as he lives. At $3\frac{1}{2}\%$ interest, what is the present value of his premiums from the standpoint of the insurance company?

8. A man aged 26 has agreed to pay $50 insurance premiums at the end of each year for the rest of his life. At $3\frac{1}{2}\%$, what is the present value of his premiums?

9. A man aged 60 gives \$10,000 to an insurance company in return for an annuity contract promising him payments at the end of each year as long as he lives. If money is worth $3\frac{1}{2}\%$ to the company, what annual payment does he receive?

10. From the formulas previously developed, prove that

$$a_x = vp_x(1 + a_{x+1}).$$

NOTE. — Recognize that this formula would make the computation of a table of the values of a_x very simple. First, we should compute a_{95}, which is zero; then, $a_{94} = vp_{94}(1 + a_{95})$ gives the value of a_{94}, etc., for $a_{93}, a_{92}, \cdots$, down to a_{10}.

66. Temporary and deferred life annuities. — A **temporary life annuity** of $\$R$ per year for n years to (x) furnishes payments of $\$R$ at the end of 1 year, 2 years, etc., to the end of n years, if (x) continues to live. The payments cease at the end of n years, even though (x) remains alive. Let $a_{x\overline{n}|}$ represent the present value of a temporary life annuity of \$1 paid annually for n years to (x). This annuity promises n pure endowments whose present values are tabulated below.

Age at Which \$1 Endowment is Payable	Time from Now until Endowment is Payable	Present Value of the Endowment
$x+1$	1 yr.	${}_1E_x = \frac{vl_{x+1}}{l_x}$
$x+2$	2 yr.	${}_2E_x = \frac{v^2l_{x+2}}{l_x}$
.	.	.
.	.	.
.	.	.
$x+n$	n yr.	${}_nE_x = \frac{v^nl_{x+n}}{l_x}$

The sum of the present values is $a_{x\overline{n}|}$, or

$$a_{x\overline{n}|} = \frac{vl_{x+1} + v^2l_{x+2} + \cdots + v^nl_{x+n}}{l_x}. \tag{21}$$

Formula 21 applies for all interest rates. To obtain a formula in terms of N and D (which, with our tables, will be useful only when the rate is $3\frac{1}{2}\%$), multiply numerator and denominator in equation 21 by v^x.

$$a_{x\overline{n}|} = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + \cdots + v^{x+n}l_{x+n}}{v^x l_x},$$

$$a_{x\overline{n}|} = \frac{D_{x+1} + D_{x+2} + \cdots + D_{x+n}}{D_x}.$$

From formula 18,

$$N_{x+1} = D_{x+1} + D_{x+2} + \cdots + D_{x+n} + D_{x+n+1} + \cdots + D_{95},$$

$$N_{x+n+1} = D_{x+n+1} + \cdots + D_{95}.$$

Hence, $N_{x+1} - N_{x+n+1} = D_{x+1} + D_{x+2} + \cdots + D_{x+n}$, and therefore

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \tag{22}$$

If the annual payment of the temporary annuity is $\$R$, the present value $\$A$ is given by

$$A = R(a_{x\overline{n}|}) = \frac{R(N_{x+1} - N_{x+n+1})}{D_x}. \tag{23}$$

The definition of a **deferred life annuity** is similar to that for a deferred annuity certain (see Section 26, Part I). A life annuity of $1 per year, whose term is deferred 10 years, to a man aged 30, promises the first $1 payment at the end of (10 + 1) or 11 years, and $1 annually thereafter. Let $_{n|}a_x$ be the present value of a life annuity of $1 per year, whose term is deferred n years, to a man aged x. The first payment of the deferred annuity is due at the end of $(n + 1)$ years. It is clear that a whole life annuity of $1 per year to a man aged x pays him $1 at the end of each year for the first n years, and also at the end of each year after that, provided that he lives. The payments during the first n years form a temporary life annuity whose present value is $a_{x\overline{n}|}$. The payments after the nth. year are those of the deferred annuity, whose present value we are representing by $_{n|}a_x$. Hence, the present value a_x of the whole life annuity is the sum of the other two present values or

$$a_x = {}_{n|}a_x + a_{x\overline{n}|}; \tag{24}$$

$$_{n|}a_x = a_x - a_{x\overline{n}|}. \tag{25}$$

On using formulas 19 and 22 in formula 25,

$$_{n|}a_x = \frac{N_{x+1}}{D_x} - \frac{N_{x+1} - N_{x+n+1}}{D_x}, \tag{26}$$

$$_{n|}a_x = \frac{N_{x+n+1}}{D_x}. \tag{27}$$

The present value $\$A$ of a life annuity of $\$R$ per year, deferred n years, for a man aged x, is

$$A = R(_{n|}a_x) = \frac{RN_{x+n+1}}{D_x}. \tag{28}$$

EXERCISE LXIV

1. At $3\frac{1}{2}\%$, find the present value of a life annuity of \$1000 per annum, deferred 20 years, to a man aged 23.

2. At $3\frac{1}{2}\%$, find the present value of a life annuity of \$2000 paid annually for 25 years to a man aged 45.

3. If money is worth 5%, find the present value of a life annuity of \$1000 paid annually for 3 years to a man aged 27.

4. A man aged 25 will pay 20 annual premiums of \$50 each on a life insurance policy, if the man remains alive. If the first premium is cash, find their present value, at $3\frac{1}{2}\%$.

5. A man aged 50 gives an insurance company \$10,000 in return for a contract to pay him a fixed income at the end of each year for 20 years, if he lives. If money is worth $3\frac{1}{2}\%$ to the company, what is the annual income? Use formula 23.

√ **6.** A man aged 40 pays an insurance company \$20,000 in return for a contract to pay him a life annuity whose first annual payment will be made when his age is 65. Find the annual payment, if money is worth $3\frac{1}{2}\%$ to the insurance company, by use of formula 28.

7. A corporation has promised to pay an employee, now aged 48, a pension of \$1000 at the end of each year, starting with a payment on his 61st birthday. At $3\frac{1}{2}\%$, what is the present value of this obligation?

NOTE. — Any pension system instituted by a company constitutes a definite present obligation whose value can be determined by finding, as in the problem above, the present value of the pension promised to each employee.

√ **8.** A man aged 43 estimates his future earnings at \$5000 at the end of each year for the next 25 years. At $3\frac{1}{2}\%$, find the capitalized (present) value of his earning power.

67. Annuities due.—A life annuity due is one whose payments occur at the *beginnings* of the payment intervals, so that the first payment is cash. Let $\mathbf{a}_x$ be the present value of a life annuity due of \$1 paid annually to (x). A cash payment of \$1 is due and the remaining payments of \$1 at the end of each year form an ordinary life annuity whose present value is a_x. Hence, the present value of the annuity due is given by

$$\mathbf{a}_x = 1 + a_x. \tag{29}$$

From formula 19,

$$\mathbf{a}_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x} = \frac{D_x + (D_{x+1} + D_{x+2} + \cdots + D_{95})}{D_x}. \tag{30}$$

$$\mathbf{a}_x = \frac{N_x}{D_x}. \tag{31}$$

The present value \$$A$ of a life annuity due of \$$R$ paid annually is

$$A = R(\mathbf{a}_x) = \frac{RN_x}{D_x}. \tag{32}$$

Let $\mathbf{a}_{x\overline{n}|}$ be the present value of a temporary life annuity due, whose term is n years, paying \$1 annually, to a man aged x. The first \$1 is paid cash and the remaining payments form an ordinary temporary life annuity whose term is $(n - 1)$ years. Hence,

$$\mathbf{a}_{x\overline{n}|} = 1 + a_{x\,\overline{n-1}|}. \tag{33}$$

From formula 22, $a_{x\,\overline{n-1}|} = \dfrac{N_{x+1} - N_{x+n-1+1}}{D_x} = \dfrac{N_{x+1} - N_{x+n}}{D_x}.$

Hence, $\mathbf{a}_{x\overline{n}|} = 1 + \dfrac{N_{x+1} - N_{x+n}}{D_x} = \dfrac{D_x + N_{x+1} - N_{x+n}}{D_x}.$

Since $D_x + N_{x+1} = N_x$,

$$\mathbf{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}. \tag{34}$$

The present value \$$A$ of a temporary annuity due of \$$R$ payable annually for n years to (x) is

$$A = R(\mathbf{a}_{x\overline{n}|}) = \frac{R(N_x - N_{x+n})}{D_x}. \tag{35}$$

Example 1. — In a certain insurance policy, the present value of the benefits promised to the policyholder is \$3500. If the policyholder is of age 27, what equal premiums should he pay to the insurance company at the beginning of each year for 10 years, in payment of the policy, if money is worth $3\frac{1}{2}\%$ to the company?

Solution. — Let $\$R$ be the annual premium. The premiums form a tem-
rary life annuity due whose present value equals $3500. From formula
, with A = \$3500, x = 27, and n = 10,

$$3500 = \frac{R(N_{27} - N_{37})}{D_{27}}; \qquad R = \frac{3500\, D_{27}}{N_{27} - N_{37}}.$$

$$R = \frac{3500(34601)}{287510} = \$421.21. \qquad \text{(Table XIV)}$$

NOTE 1. — In discussing premiums on life insurance policies, formulas 32
d 35 are of great use. Formulas 16, 28, 32, and 35 of this chapter are the
es we shall use most frequently in the future.

Summary of present value formulas

Pure endowment: $A = R({}_nE_x) = R\dfrac{D_{x+n}}{D_x}$. **(16)**

Whole life annuity: $A = R(a_x) = \dfrac{N_{x+1}}{D_x}$. **(20)**

Temporary life annuity: $A = R(a_{x\overline{n}|}) = R\dfrac{N_{x+1} - N_{x+n+1}}{D_x}$. **(23)**

Deferred life annuity: $A = R({}_n|\,a_x) = R\dfrac{N_{x+n+1}}{D_x}$. **(28)**

Whole life annuity due: $A = R(\mathrm{a}_x) = R\dfrac{N_x}{D_x}$. **(32)**

Temp. life annuity due: $A = R(\mathrm{a}_{x\overline{n}|}) = R\dfrac{N_x - N_{x+n}}{D_x}$. **(35)**

MISCELLANEOUS PROBLEMS

1. A man aged 40 pays \$10,000 to an insurance company in return for ontract to pay him a fixed annual income for life, starting with a pay- nt on his 60th birthday. Find the annual income if money is worth % to the company.

2. At age 65 a man considers whether he should (*a*) pay his total ings of \$20,000 to an insurance company for a life annuity whose first ıual payment would occur in 1 year, or (*b*) invest his savings at 6%, ›ctive. Find the difference in his annual income under the two methods, uming that money is worth $3\frac{1}{2}\%$ to the insurance company.

3. In problem 2, what will be received by the heirs of the man at his .th if he adopts plan (*a*)? What will they receive under plan (*b*)?

4. A certain insurance policy taken out by a man aged 28 calls for premiums of \$200 at the beginning of each year as long as he lives. Find the present value of these premiums at $3\frac{1}{2}\%$.

5. A certain insurance policy matures when the policyholder is of age 35 and gives him \$2000 cash or the option of equal payments at the beginning of each year for 10 years as long as he lives. If money is worth $3\frac{1}{2}\%$, find the annual payment under the optional plan.

6. A boy of 16 has been left an estate of \$100,000, which is invested at 5%, effective. If he lives, he will receive the income annually for the next 10 years and the principal of the estate when he reaches age 26. If money is worth $3\frac{1}{2}\%$, find the present value of his inheritance.

7. Derive formula 13 for a_x by the mutual benefit fund reasoning used in Note 2, Section 63. Thus, at the end of 1 year, $\$l_{x+1}$ will be needed for payments; $\$l_{x+2}$ at the end of 2 years, etc., $\$l_{95}$ at the end of $(95 - x)$ years. Discount all of these payments and divide by l_x.

8. Derive formula 21 for $a_{x\overline{n}|}$ by the mutual fund method of reasoning.

9. A man aged 22 agrees to pay \$50 as the premium on an insurance policy at the beginning of each year for 10 years if he lives. Find the present value of his premiums at $3\frac{1}{2}\%$ interest.

10. The present value of the benefits promised in a certain insurance policy is \$8000. If the policyholder is aged 30, what equal premiums should he agree to pay at the beginning of each year for 15 years, provided he lives, if money is worth $3\frac{1}{2}\%$ to the insurance company?

11. A man is to receive a life annuity of \$2000 per year, the first payment occurring on his 55th birthday. If he postpones the annuity so that the first annual payment will occur on his 65th birthday, what will be the annual income, if the new annuity has the same present value as the former one, under $3\frac{1}{2}\%$ interest?

12. A certain professor at age 65 enters upon a pension of the Carnegie Foundation which will pay \$2000 at the end of each year for life. In order to have, at age 65, an amount equal to the present value at $3\frac{1}{2}\%$ of the pension he is to receive, what equal sums would the professor have had to have invested annually at 5%, assuming that his first investment would have occurred at age 41 and his last at age 65?

CHAPTER IX

LIFE INSURANCE

68. Terminology. — **Insurance** is an indemnity or protection against loss. The business of insuring people against any variety of disaster is on a scientific basis only when a large number of individuals are insured under one organization, so that individual losses may be distributed over the whole group according to some scientific principle of mutuality. That is, each of the insured should pay in proportion to what he is promised as an insurance benefit. In this chapter we shall discuss the principles and most simple aspects of the scientific type of life insurance furnished by **old line**, or **legal reserve** companies.

When an individual is insured by a company, he and the company sign a written contract, called a **policy**. The individual is called a **policyholder**, or the **insured**. In the contract the company promises to pay certain sums of money, called **benefits**, if certain events occur. The person to whom the benefits are to be paid is called the **beneficiary**. The insured agrees to pay certain sums called **gross** or **office premiums** in return for the contracted benefits. The **policy date** is the day the contract was entered into. The successive years after this date are called **policy years**.

The fundamental problem of a company is to determine the premiums which should be charged a policyholder in return for specified benefits. Every insurance company adopts a certain mortality table and an assumed rate of earnings on invested funds as the basis for its computations. We shall use the American Experience Table and $3\frac{1}{2}\%$, as is the custom among many companies. The **net premiums** for a policy are those **whose present value is equal to the present value of the policy benefits** under the following assumptions: (*a*) *the benefits from the policy will be paid at the ends of the years in which they fall due;* (*b*) *the company's funds will earn interest at exactly the specified rate* ($3\frac{1}{2}\%$ in our case);

(c) *the deaths among the policyholders will occur at exactly the rate given by the mortality table* (Table XIII in our case). Under these assumptions, if a company were run without profit or administrative expense, it could afford to issue policies in return for these net premiums. The actual gross premiums for a policy are the net premiums plus certain amounts which provide for the administrative expense of the company and for added expense due to violations of the theoretical conditions (a), (b), and (c) assumed above. In computing gross premiums, insurance companies use their own individual methods. Our discussion is concerned entirely with net premiums and related questions.

NOTE. — In the future, if the interest rate in a problem is not given, it is understood to be 3½%.

69. Net single premium; whole life insurance. — If a policyholder agrees to pay all premium obligations in one installment, it is payable immediately on the policy date and is called the **single premium** for the policy. The **net single premium** is the present value on the policy date of all benefits of the policy.

A whole life insurance of $\$R$ on the life of (x) is an agreement by the company to pay $\$R$ to the beneficiary at the end of the year in which (x) dies. A policy containing this contract is called a **whole life policy.**

Example 1. — Find the net single premium for a whole life policy for $1000 for a man aged 91.

Solution. — Suppose that the company issues whole life policies for $1000 to l_{91}, or 462 men of age 91. During the first year, $d_{91} = 246$ men will die; $246,000 in death claims will be payable to beneficiaries at the end of 1 year. The present value of this payment is $246{,}000(1.035)^{-1} = 246{,}000\,v$. The other entries below are easily verified.

POLICY YEAR	DEATHS DURING YEAR	BENEFITS DUE AT END OF YEAR	PRESENT VALUE OF BENEFITS
1	$d_{91} = 246$	$246,000	$246{,}000\,v$
2	$d_{92} = 137$	137,000	$137{,}000\,v^2$
3	$d_{93} = 58$	58,000	$58{,}000\,v^3$
4	$d_{94} = 18$	18,000	$18{,}000\,v^4$
5	$d_{95} = 3$	3,000	$3{,}000\,v^5$.

Hence, on the policy date, the insurance company should obtain through the net single premiums from the l_{91} men, a fund equal to the sum of the last column. This sum, divided by 462, is the share or net single premium paid by each of the l_{91} men. By use of Table VI, we find that each pays

$$\frac{246000\,v + 137000\,v^2 + 58000\,v^3 + 18000\,v^4 + 3000\,v^5}{462} = \$943.93.$$

Let $\$A_x$ be the net single premium for a whole life insurance of \$1 on the life of (x). To obtain A_x by the method of Example 1 above, suppose that a company issues whole life policies for \$1 insurance to each of l_x men of age x. During the first policy year, d_x will die; $\$d_x$ in death benefits is payable to beneficiaries at the end of 1 year. The present value of these benefits at the rate i is $d_x(1 + i)^{-1} = vd_x$. The other entries below are easily verified.

Policy Year	Deaths during Year	Benefits Due at End of Year	Present Value of Benefits
1	d_x	$\$d_x$	vd_x
2	d_{x+1}	$\$d_{x+1}$	v^2d_{x+1}
3	d_{x+2}	$\$d_{x+2}$	v^3d_{x+2}
.	.	.	.
.	.	.	.
.	.	.	.
$96 - x$	d_{95}	$\$d_{95}$	$v^{96-x}d_{95}$

In the last row, notice that when the group reaches age 95, the policies have been in force $(95 - x)$ years, or the $(96 - x)$th year is just entered on. Hence, $\$d_{95}$ is due at the end of $(96 - x)$ years. From the net single premiums paid on the policy date, the company must obtain a fund equal to the sum of the values in the last column. The share of each of the l_x men, or his net single premium A_x, is

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots + v^{96-x}d_{95}}{l_x}. \tag{36}$$

On multiplying numerator and denominator above by v^x,

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + \cdots + v^{96}d_{95}}{v^xl_x}.$$

Introduce a new symbol $C_k = v^{k+1}d_k$. Thus, $C_{53} = v^{54}d_{53}$, $C_{93} = v^{94}d_{93}$, etc., $v^{x+1}d_x = C_x$, $v^{x+2}d_{x+1} = C_{x+1}$, and $v^{96}d_{95} = C_{95}$. Hence

$$A_x = \frac{C_x + C_{x+1} + \cdots + C_{95}}{D_x}.$$

Introduce a new symbol

$$M_k = C_k + C_{k+1} + \cdots + C_{95}. \tag{37}$$

Thus, $M_{92} = C_{92} + C_{93} + C_{94} + C_{95}$; $M_x = C_x + \cdots + C_{95}$.

Hence
$$A_x = \frac{M_x}{D_x}. \tag{38}$$

The net single premium A for a whole life policy of $\$R$ for (x) is

$$A = R(A_x) = \frac{RM_x}{D_x}. \tag{39}$$

NOTE. — The values of M_k for the rate $3\frac{1}{2}\%$, $v = (1.035)^{-1}$, are given in Table XIV.

EXERCISE LXV

Use formulas 38 and 39 unless otherwise directed.

1. Compute the values of C_{94} and of C_{95} at $3\frac{1}{2}\%$; verify the entries for M_{94} and for M_{95} in Table XIV.

2. By the method of illustrative Example 1, page 166, find the net single premium for a whole life insurance for \$1000 for a man aged 93, if interest is at the rate 5%.

3. Find the net single premium for a whole life insurance of \$1000 on the life of a man, (a) aged 90; (b) aged 50; (c) aged 30; (d) aged 10.

4. How much whole life insurance can a man aged 50 purchase from a company for \$1500 cash?

5. How much whole life insurance can a man aged 35 purchase from a company for \$2000 cash?

70. Term insurance. — An n-year term insurance for $\$R$ on the life of (x) promises the payment of $\$R$ at the end of the year in which (x) dies, only on condition that his death occurs within n years. Thus, a 5-year term insurance gives no benefit unless (x) dies within 5 years. Let $A^{1}_{x\overline{n|}}$ represent the present value of an n-year term insurance for \$1 on the life of (x). To obtain the value of $A^{1}_{x\overline{n|}}$, assume that the company issues n-year term insurance policies for $1 to each of l_x men aged x. The present values of the benefits which will be paid are tabulated below; the policy has no force after n years.

POLICY YEAR	DEATHS DURING YEAR	BENEFITS DUE AT END OF YEAR	PRESENT VALUE OF BENEFITS
1	d_x	$\$d_x$	vd_x
2	d_{x+1}	$\$d_{x+1}$	v^2d_{x+1}
.	.	.	.
.	.	.	.
.	.	.	.
n	d_{x+n-1}	$\$d_{x+n-1}$	v^nd_{x+n-1}

The net single premium paid by each man is the sum of the last column, divided by l_x, or

$$A^1_{x\overline{n}|} = \frac{vd_x + v^2d_{x+1} + \cdots + v^nd_{x+n-1}}{l_x}. \tag{40}$$

On multiplying numerator and denominator above by v^x and on using the symbol $C_k = v^{k+1}d_k$,

$$A^1_{x\overline{n}|} = \frac{C_x + C_{x+1} + \cdots + C_{x+n-1}}{D_x}. \tag{41}$$

Since $\quad M_x = C_x + C_{x+1} + \cdots + C_{x+n-1} + C_{x+n} + \cdots + C_{95},$

and $\quad M_{x+n} = \qquad C_{x+n} + \cdots + C_{95},$

it is seen that the numerator in equation 41 is $M_x - M_{x+n}$; hence

$$A^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}. \tag{42}$$

The net single premium $\$A$ for n-year term insurance of $\$R$ on the life of (x) is

$$A = RA^1_{x\overline{n}|} = \frac{R(M_x - M_{x+n})}{D_x}. \tag{43}$$

The net single premium for a 1-year term insurance for (x) is called the **natural premium** at age x. The natural premium for $1 insurance is obtained from equation 42, with $n = 1$:

$$\textit{Natural Premium} = A^1_{x\overline{1}|} = \frac{M_x - M_{x+1}}{D_x} = \frac{C_x}{D_x}, \tag{44}$$

where $M_x - M_{x+1} = C_x$, because of formula 37.

EXERCISE LXVI

Use formulas 42 and 43 unless otherwise specified.

1. By use of the method used in deriving formula 40, find the expression for the net single premium for a 3-year term insurance for $1000 on the life of a man aged 25 and compute its value at 5% interest.

2. Find the net single premium for a 10-year insurance for $2000 on the life of a man aged 31.

3. Find the natural premium for $1 insurance at age 22; at age 90.

4. (*a*) Find the net single premium for a whole life insurance of $1000 at age 50. (*b*) Find the net single premium at age 50 for a 10-year term insurance for $1000.

5. How much term insurance for 10 years can be purchased for $2000 cash by a man aged 35?

6. How much term insurance for 10 years can be purchased for $2000 cash by a man aged 55?

71. Endowment insurance. — An n-year **endowment insurance** of $\$R$ on the life of a man aged x furnishes

(a) *a payment of* $\$R$ *at the end of the year in which* (x) *dies, if he dies within* n *years, and* (b) *a pure endowment of* $\$R$ *to* (x) *at the end of* n *years if* (x) *is alive at that time.*

Thus, a 20-year endowment insurance of $1000 pays $1000 at death, if it occurs within 20 years; or, if (x) is alive at the end of 20 years, he receives the endowment of $1000. Let $A_{x\overline{n}|}$ represent the net single premium (or present value) of an n-year endowment insurance of $1 on the life of (x). The present value $A_{x\overline{n}|}$ is the sum of the present values of (*a*) the n-year term insurance for $1 on the life of (x), and of (*b*) the n-year pure endowment of $1 to (x). Hence, on using formulas 16 and 42,

$$A_{x\overline{n}|} = A^{1}_{x\overline{n}|} + {}_nE_x = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x},$$

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \qquad (45)$$

If the endowment insurance is for $\$R$, the net single premium $\$A$ is given by

$$A = RA_{x\overline{n}|} = \frac{R(M_x - M_{x+n} + D_{x+n})}{D_x}. \qquad (46)$$

EXERCISE LXVII

1. (*a*) Compute the net single premium for a $1000, 20-year endowment insurance on the life of a man aged 23. (*b*) Compute the present value of a pure endowment of $1000 payable to the man at age 43. (*c*) Find the net single premium for a 20-year term insurance for $1000 on the life of the man aged 23, by using (*a*) and (*b*).

2. Find the net single premium for a 10-year endowment insurance for $5000 on the life of a man aged 30.

3. (*a*) Find the net single premium for a 10-year endowment insurance for $3000 on the life of a man aged 26. (*b*) Find the net single premium for a 10-year term insurance for $3000 on his life. (*c*) From the results of (*a*) and (*b*), find the present value of a 10-year pure endowment of $3000 for the man.

4. How much 20-year endowment insurance can a man aged 33 purchase for $3000 cash?

5. How much 10-year endowment insurance can a man aged 45 purchase for $2500 cash?

72. Annual premiums. — If the net premiums for a policy are payable annually, instead of in one installment (the net single premium), they must satisfy the condition that the

$$\textbf{(pr. val. of annual premiums)} = \textbf{(net single premium)}, \quad (47)$$

because the net single premium is the present value of the policy benefits. When paid annually, the premiums for a policy are always equal and are payable at the beginnings of the years, as long as the policyholder lives.

Example 1. — (*a*) Find the net single premium for a 10-year term insurance for $10,000 on the life of a man aged 46. (*b*) Find the equivalent annual premium which the man might agree to pay for 10 years, if he lives.

Solution. — (*a*) From formula 43, the net single premium is

$$10000(A^{\ 1}_{46\overline{10}|}) = \frac{10000(M_{46} - M_{56})}{D_{46}} = \$1119.30. \quad \text{(Table XIV)}$$

That is, the present value of the insurance benefits is $1119.30. (*b*) Let P be the annual premium. The 10 premiums form a 10-year life annuity due paid by a man aged 46. Their present value is $P(a_{46\overline{10}|})$, and it must equal $1119.30. Hence,

$$1119.30 = P(a_{46\overline{10}|}) = P\frac{N_{46} - N_{56}}{D_{46}}. \quad \text{(Formula 35)}$$

$$P = \frac{1119.30\ D_{46}}{N_{46} - N_{56}} = \$137.33. \quad \text{(Table XIV)}$$

The annual payments of $137.33 have a value equivalent to $1119.30 paid cash.

NOTE 1. — The solution of (a) was not necessary in order to solve (b) above. Thus, we may write, immediately, from equation 47,

$$P(a_{46\overline{10|}}) = 10000(A^1_{46\overline{10|}}).$$

$$P\frac{N_{46} - N_{56}}{D_{46}} = 10000\frac{M_{46} - M_{56}}{D_{46}}; \qquad P = \frac{10000(M_{46} - M_{56})}{N_{46} - N_{56}} = \$137.33.$$

In insurance practice the most simple forms of insurance policies are those tabulated below. Their names, policy benefits, and manner of premium payment should be memorized. All premiums are payable in advance, at the beginning of the year. The numbers assigned are merely for later convenience in this book.

NUMBER	NAME OF POLICY	POLICY BENEFITS	PREMIUMS PAID
I	Ordinary life	Whole life insurance	Annually for life
II	n-payment life	Whole life insurance	Annually for n years
III	n-year term	n-year term insurance	Annually for n years
IV	n-year endowment	(a) n-year pure endowment (b) n-year term insurance	Annually for n years

To determine the net annual premiums for these policies, we use the fundamental equation 47, and the method [1] of Note 1 above. Consider an ordinary life policy for $1 for a man aged x. Let P_x be the net annual premium. The premiums paid by the man aged x form a whole life annuity due whose present value is $P_x(a_x)$. The net single premium for the policy is A_x. Hence, from equation 47,

$$P_x(a_x) = A_x; \quad P_x\frac{N_x}{D_x} = \frac{M_x}{D_x}. \qquad \text{(Formulas 32, 38)}$$

$$P_x = \frac{M_x}{N_x}. \tag{48}$$

[1] The student is advised to solve problem 1 of Exercise LXVIII below before reading the rest of the section.

Let ${}_nP_x$ be the net annual premium for an n-payment life policy for \$1 for a man aged x. The premium payments by (x) form an n-year life annuity due whose present value is ${}_nP_x(a_{x\overline{n}|})$. Hence, from equation 47,

$${}_nP_x(a_{x\overline{n}|}) = A_x;\ {}_nP_x\frac{N_x - N_{x+n}}{D_x} = \frac{M_x}{D_x}. \quad \text{(Formulas 35, 38)}$$

$${}_nP_x = \frac{M_x}{N_x - N_{x+n}}. \quad (49)$$

It is left as an exercise (problem 3, below) for the student to prove that the net annual premium $P^1_{x\overline{n}|}$ for an n-year term insurance policy for \$1 for (x) is given by

$$P^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \quad (50)$$

Let $P_{x\overline{n}|}$ be the net annual premium for an n-year endowment policy for \$1 for (x). The premiums paid by the man aged x form an n-year life annuity due whose present value is $P_{x\overline{n}|}(a_{x\overline{n}|})$. The net single premium for the policy is $A_{x\overline{n}|}$. From equation 47,

$$P_{x\overline{n}|}(a_{x\overline{n}|}) = A_{x\overline{n}|};$$

$$P_{x\overline{n}|}\frac{N_x - N_{x+n}}{D_x} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \quad \text{(Formulas 35, 45)}$$

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \quad (51)$$

If the policies I to IV are for \$$R$ instead of \$1, the annual premiums are found from formulas 48 to 51 by multiplying by R.

EXERCISE LXVIII

1. By the method of Note 1, page 172, find the net annual premium for a 5-payment life policy for \$1000 for a man aged 45.

2. A man aged 29 has agreed to pay 15 annual premiums of \$100 for a certain policy. Find the net single premium for the policy.

3. Establish formula 50 for the net annual premium for an n-year term insurance policy for \$1 for a man aged x.

4. The net single premium for a certain policy for a man aged 26 is \$3500. (a) What is the net annual premium if he agrees to pay premiums

annually for life? (*b*) What is the net annual premium if he agrees to pay annually for 12 years?

5. For a man aged 25, find the net annual premium (*a*) for an ordinary life policy for $2000; (*b*) for a 20-payment life policy for $2000.

6. Find the net annual premium for a 10-year term policy for $1000 for a man (*a*) aged 32; (*b*) aged 42; (*c*) aged 62.

7. Find the net annual premium for a 20-year endowment policy for $1000 for a person of your own age.

8. (*a*) Find the net annual premium for a 5-year term policy for $1000 for a man aged 40. (*b*) Find the natural premiums for a $1000 insurance at each of the ages 40, 41, 42, 43, and 44. (*c*) Compare the result of (*a*) with the five results in (*b*).

9. A whole life insurance policy for $1000 taken at age 30 states that the annual premiums were computed as if: (*a*) term insurance of $1000 were given for the first year, and (*b*) an ordinary whole life policy were then written when the man reaches age 31. Find the net premium (*a*) for the first year, and (*b*) for the subsequent years.

NOTE. — Such a policy is very common and is said to be written on the **1-year term plan.** The advantages from an insurance company's standpoint are apparent after reading the next chapter.

10. A certain endowment policy for $1000 taken at age 23 provides that the net premium for the 1st year is that for 1-year term insurance and that the net premiums for the remaining 19 years are those for a 19-year endowment policy for $1000, taken at age 24. Find the net annual premium (*a*) for the first year; (*b*) for subsequent years. This policy is another example of the 1-year term plan.

11. How much insurance on the 20-payment life plan can a man aged 32 purchase for a net annual premium of $75? It is advisable, first, to find the equivalent net single premium.

12. How large a 20-year endowment policy can a man aged 23 purchase for net annual premiums of $100?

13. (*a*) For a boy aged 16, find the net annual premium for a $1000 endowment policy, which matures at age 85. (*b*) Find the net annual premium for an ordinary life policy for $1000 taken at age 16. (*c*) Explain the small difference between the results.

14. A man aged 30 takes out a policy which provides him with $10,000 insurance for the first 10 years, $8000 for the next 10 years, and $5000 for the remainder of his life. He is to pay premiums annually for 10 years. Find the net annual premium.

15. A certain policy on maturing at age 55 offers the option of a pure endowment of \$2000, or an equivalent amount of *paid up whole life insurance*, that is, as much insurance as the \$2000, considered as a net single premium, will buy. Find the amount of paid up insurance.

NOTE ON GROSS PREMIUMS. — Premiums previously discussed were net premiums, or present values of the benefits to be paid under the policy. In conducting an insurance company there is expense due to the salaries paid to administrative officials, the commissions paid to agents for obtaining new policyholders, the expense of the medical examination of policyholders, bookkeeping expense, etc. To provide for these items and for unforeseen contingencies, it is necessary for the company to add to the net premium an amount called the **loading**. The net premium plus the loading is the gross or actual premium paid by the policyholder. Sometimes the loading is determined as a certain percentage of the net premium plus a constant charge independent of the nature of the policy. Sometimes the loading may be determined simply as a percentage of the net premium, the percentage either being independent of the policyholder's age, or varying with it. Each company uses its own method for loading, but the resulting gross premiums of all large, well-managed companies are essentially the same.

SUPPLEMENTARY MATERIAL

73. Net single premiums as present values of expectations. — A whole life policy on a life aged x promises only one payment, due at the end of the year in which (x) dies. However, we may think of the policy as promising payments at the end of each year up to the man's 96th birthday, the payment at each date being *contingent on his death during the preceding year*. Then, the method used in deriving formulas for life annuities may be used to obtain the present value, or net single premium, for the policy.

Consider obtaining the net single premium A_x for a whole life insurance of \$1 on the life of (x). At the end of 1 year, \$1 will be paid if (x) dies during the preceding year. The probability of (x) dying in this year is $\frac{d_x}{l_x}$; from formula 10 with $S = 1$, the present value of the expectation of this payment is $\frac{d_x}{l_x}(1+i)^{-1}$ or $\frac{vd_x}{l_x}$. The other present values below are verified similarly.

Payment of $1 Will be Made at End of	If Man Dies between Ages	Probability of Payment Being Made	Present Value of the Payment
1 yr.	x and $(x+1)$	$\frac{d_x}{l_x}$	$\frac{vd_x}{l_x}$
2 yr.	$(x+1)$ and $(x+2)$	$\frac{d_{x+1}}{l_x}$	$\frac{v^2d_x}{l_x}$
.	.	.	.
.	.	.	.
.	.	.	.
$96-x$ yr.	95 and 96	$\frac{d_{95}}{l_x}$	$\frac{v^{96-x}d_{95}}{l_x}$

The expression obtained for A_x on adding the present values in the last column is the same as previously obtained in formula 36.

Note. — From the present point of view, an insurance company may be likened to a gambler who plays against all of the beneficiaries of the policies. The net single premiums are the present values of the expectations of the beneficiaries. So many players are involved as opponents of the company that the probabilities of winning and of losing as given by the mortality table will be practically certain to operate. Hence, the company will neither lose nor win in the long run.

EXERCISE LXIX

1. By the method which was used above to obtain A_x, find the expression for the present value of a 10-year term insurance policy for $1000 on a life aged 29.

2. By the method above, derive the formula 40 for $A^1_{x\overline{n|}}$.

74. Policies of irregular type. — Equation 47 enables us to find the premiums for any policy for which the present value of the benefits is known.

Example 1. — A policy written for a man aged 32 promises the following benefits: (*a*) Term insurance for $5000 for 28 years; (*b*) a life annuity of $1000 paid annually, first payment due at age 60. It is agreed that premiums shall be paid annually for 28 years. Find P, the net annual premium.

Solution. — The present value of benefit (*a*) is $5000(A^1_{32\overline{28|}})$; benefit (*b*) is a life annuity, term deferred 27 years, whose present value is $1000({}_{27}|a_{32})$.

The annual premiums form a 28-year temporary annuity due, whose present value is $P(a_{32\overline{28}|})$. Hence, from equation 47,

$$P(a_{32\overline{28}|}) = 5000(A^{1}_{32\overline{28}|}) + 1000({}_{27}|a_{32}),$$

$$P\frac{N_{32} - N_{60}}{D_{32}} = 5000\frac{M_{32} - M_{60}}{D_{32}} + 1000\frac{N_{60}}{D_{32}}. \qquad \text{(Formulas 35, 42, 28)}$$

$$P = \frac{1000\ N_{60} + 5000(M_{32} - M_{60})}{N_{32} - N_{60}} = \$234.43. \qquad \text{(Table XIV)}$$

EXERCISE LXX

Find the periodic premium payment for each policy described. The *age of policyholder* is the age at the time the policy was written.

Prob.	Benefits of Policy	Age of Policy-holder	Method of Paying Premiums
1.	(*a*) 10-year term insurance for $1000. (*b*) A pure endowment of $2000 at the end of 10 years.	27	10 annual premiums
2.	(*a*) Term insurance of $10,000 for 20 years. (*b*) Life annuity of $1000 paid annually, first payment at age 66.	46	20 annual premiums
3.	Life annuity of $1000 paid annually, first payment at age 65.	30	25 annual premiums
4.	Life annuity of $1000 paid annually, first payment at age 70.	45	10 annual premiums
5.	(*a*) Term insurance of $10,000 for first 10 years. (*b*) Term insurance for $5000 for next 20 years. (*c*) Life annuity of $2000, paid annually with first payment at age 65.	45	20 annual premiums

6. In what way does the policy in problem 1 differ from a 10-year endowment policy?

Note. — The policy of problem 4 is called an *annuity policy* and is a familiar form for those wishing protection in their old age. This same feature of protection is present in the policy of problem 2.

CHAPTER X

POLICY RESERVES

75. Policy reserve. — At age 30, the natural premium for $1000 insurance, that is, the net single premium for 1-year term insurance for $1000, is found to be $8.14. This is the sum which each of l_{30} men of age 30 should pay in order to provide benefits of $1000 in the case of all of the group who will die within 1 year. The $8.14 premium is the actual expense of an insurance company in insuring a man aged 30 for $1000 for 1 year. The expense of insurance for 1 year increases continually during life, after an early age, being $17.94 per $1000 insurance at age 55 and $139.58 at age 80.

Consider a man aged 30 who takes out a $1000, ordinary life policy. Throughout life he pays a **net level premium** (that is, a constant premium) of $17.19, as obtained from formula 48, and is insured for $1000 all during life. The expense of the company in insuring him during the first year is the natural premium, $8.14. Hence, in the first year the man pays (17.19 − 8.14) = $9.05 more than the expense. The insurance company may be considered to place this unused $9.05 in a reserve fund which will accumulate at interest for future needs. Up to age 54, each annual premium of $17.19 is more than the expense of insurance and the company places the excess over expense in the reserve fund. At age 55 the $17.19 premium is less than the insurance expense, which is $17.94. The deficiency, (17.94 − 17.19) = $.75, is taken from the reserve fund. From then on until the end of life, the expense of insurance is met more and more largely from the reserve fund. Thus, at age 80, the expense is $139.58 (the natural premium, as given above) so that (139.58 − 17.19) = $122.39 comes from the reserve.

For every insurance policy (except a 1-year term policy) where a net level premium P is paid, the annual expense of insurance during the early policy years is less than the premium P. Hence, the insurance company should place the unused parts of the premiums in a **reserve fund** and accumulate it at interest to answer the future needs of the policy. When the expense of insurance, in later years, becomes greater than the level premium, the deficiency is made up by contributions from the reserve fund. The reserve funds should be regarded as a possession of the policyholders, merely held and invested by the insurance company.

The reserve on a policy at the end of any policy year, before the next premium due is paid, is called the **terminal reserve** for that year. In this chapter we consider the determination of the terminal reserve for a given year.

Example 1. — Form a table showing the terminal reserves for the first 6 years for a 5-payment life policy for \$1000 written at age 40.

Solution. — From formula 49, the net annual premium is \$89.4574. Assume that the company issues the same policy to each of l_{40} = 78,106 men. The following table shows the disposition of the funds received as premiums.

Policy Year	Premiums Paid at Start of Year	Reserve Fund at Start of Year	Accum. Fund at 3½%, at End of Year	Reserve Fund after Death Benefits at End of Year	Reserve per Survivor at End of Year
1	\$6,987,160	\$ 6,987,160	\$ 7,231,710	\$ 6,466,710	\$ 84
2	6,918,725	13,385,435	13,848,925	13,074,925	171
3	6,849,485	19,924,410	20,621,764	19,836,764	262
4	6,779,261	26,616,025	27,547,586	26,750,586	357
5	6,707,963	33,458,549	34,629,598	33,817,598	456
6	0	33,817,598	35,001,224	34,173,224	466
.		.	.	.	.
.		.	.	.	.
.		.	.	.	.

For example, at the beginning of the first year, (78,106)(89.4574) is received in premiums. At the end of the year, interest at 3½% added to the premium fund gives \$7,231,710. During the year, d_{40} = 765 deaths occurred so that \$765,000 is payable to beneficiaries, leaving \$6,466,710. There are l_{41} = 77,341 survivors; the total fund \$6,466,710, divided by 77,341, gives \$84 as the share, or reserve, per policy. At the beginning of the 2d year, 77,341 men pay premiums, etc. After the 5th year, no more premiums will be received, so that all death benefits in the future come from the fund, \$33,817,598, on hand at the end of 5 years, and its future accumulations at interest.

EXERCISE LXXI

1. Assume that an insurance company issues $1000 ordinary life policies to each of l_{92} men of age 92. Compute a table showing the disposition of funds received as premiums and the total reserve per policy at the end of each year.

76. Remaining benefits of a policy; computation of the reserve. — At any time after a policy is written, the **remaining benefits** of a policy are the promised payments of the policy as they affect the policyholder at his attained age.

Example 1. — A certain insurance policy, written for a man aged 32, promises him (*a*) temporary life insurance for $1000 for 25 years; (*b*) a pure endowment of $1000 payable at the end of 25 years; (*c*) a life annuity of $1000 payable annually, first payment at age 60. (1) Describe the remaining benefits, 8 years later. (2) Find the present value of the remaining benefits, 8 years later.

Solution. — (1) Eight years later, the *attained age* of the man is 40 years. The policy promises (*a*) term insurance for $1000 for 17 years to a man aged 40; (*b*) a pure endowment of $1000 payable at the end of 17 years to a man now aged 40; (*c*) a deferred life annuity, for a man aged 40, of $1000 paid annually. Since the first payment of the annuity is due at age 60, which is 20 years later, the annuity is deferred 19 years.

(2) The present value of the remaining benefits at age 40 is the sum of the present values or net single premiums for the three individual benefits or

$$1000\, A^{1}_{40\overline{17}|} + 1000\,{}_{17}E_{40} + 1000\,{}_{19|}a_{40},$$

which can be computed by use of the proper formulas.

Consider the conditions in regard to a policy, written for a man aged x, n years after the policy date. The attained age of the policyholder is $x + n$, and the reserve fund for the policy contains a certain amount $\$V$, the terminal reserve at the end of n years. The company is liable for the remaining benefits of the policy, and the policyholder is liable for the future premiums. Since all future benefits must be paid from the reserve and from the future premiums, the following equation is satisfied:

$$\begin{pmatrix}\text{Single premium for}\\ \text{(pr. val. of) remaining}\\ \text{benefits at age } x+n\end{pmatrix} = \begin{pmatrix}\text{Pr. val. at age } x+n\\ \text{of net premiums}\\ \text{due in the future}\end{pmatrix} + \begin{pmatrix}\text{Terminal}\\ \text{reserve at}\\ \text{age } x+n\end{pmatrix} \quad (52)$$

To find the terminal reserve on a policy, first find the net annual premium and then use equation 52.

Example 2. — Find the terminal reserve at the end of 6 years on a 20-year endowment policy for $1000 written at age 24.

Solution. — (*a*) The net annual premium is $1000(P_{24\overline{20}|}) = \$39.085$, from formula 51. (*b*) The remaining benefits at the attained age of 30 years are a pure endowment of $1000 payable at the end of 14 years to a man now aged 30 and term insurance for $1000 for 14 years on a life aged 30; in other words, the benefits form a 14-year endowment insurance for $1000 for a man aged 30. The remaining premiums form a 14-year temporary annuity due. Let V be the reserve at the end of 6 years. From equation 52,

$$1000(A_{30\overline{14}|}) = 39.085(a_{30\overline{14}|}) + V,$$

$$V = \frac{1000(M_{30} - M_{44} + D_{44})}{D_{30}} - \frac{39.085(N_{30} - N_{44})}{D_{30}}. \quad \text{(Formulas 46, 35)}$$

$$V = \frac{1000(M_{30} - M_{44} + D_{44}) - 39.085(N_{30} - N_{44})}{D_{30}}$$

$$V = \frac{6634000}{30441} = \$217.9. \quad \text{(Table XIV)}$$

The method used in Example 2 may be applied in the case of any standard policy to obtain a general formula for the reserve at the end of a given number of years. For example, consider an ordinary life policy for $1 written for a man aged x. Let ${}_nV_x$ represent the terminal reserve at the end of n years. The net annual premium is $P_x = \frac{M_x}{N_x}$. The remaining benefit at the attained age $(x + n)$ is whole life insurance for $1 for a man aged $(x + n)$. The remaining premiums form a whole life annuity due of P_x payable annually by a man now aged $(x + n)$. From equation 52,

$${}_nV_x = A_{x+n} - P_x(a_{x+n}).$$

$${}_nV_x = \frac{M_{x+n}}{D_{x+n}} - \frac{M_x}{N_x} \cdot \frac{N_{x+n}}{D_{x+n}}. \quad (53)$$

NOTE 1. — For the advantage of the insurance actuary, who has occasion to compute the reserves on numerous policies, it is advisable to develop general formulas and convenient numerical methods for the computation of reserves. In the case of a student meeting the subject for the first time, it is more important to appreciate thoroughly the truth of equation 52. Such appreciation is attained only by direct application of the equation. The problems of Exercise LXXII below should be solved by direct application of equation 52, as was done in illustrative Example 2 above.

EXERCISE LXXII[1]

1. If the net single premium for the remaining benefits of a policy is \$745, and if the present value of the future premiums is \$530, what is the reserve?

2. At an attained age of 42, the net single premium for the remaining benefits of a policy is \$750. There are six annual premiums of \$50 remaining to be paid, the first due immediately. Find the policy reserve.

3. At the attained age of 44, the reserve on a certain policy is \$500. Annual premiums of \$25, the first due immediately, must be paid for the remainder of life. Find the present value of the remaining policy benefits.

4. A \$1000, 10-payment life policy is written at age 34. (*a*) Find the reserve on the policy at the end of 6 years. (*b*) Find the reserve at the end of 10 years.

5. A \$1000, 5-payment life policy is written at age 40. (*a*) Take the premium as computed in illustrative Example 1, Section 75; compute the reserve at the end of 3 years and compare with the result given in the table of that example. (*b*) Find the terminal reserve at the end of 5 years and compare with the table.

6. A \$2000, 20-year endowment policy is written at age 33. (*a*) Find the terminal reserve at the end of 15 years. (*b*) What is the terminal reserve at the end of 20 years, before the endowment is paid?

7. In the case of a 1-year term policy, why is the reserve zero at the end of the year?

8. An ordinary life policy for \$5000 is written at age 25. Find the terminal reserve at the end of 15 years.

9. Derive a formula for the terminal reserve at the end of n years for an n-payment life policy written at age x. (*b*) Derive a formula for the reserve at the end of m years, where m is greater than n.

10. Find the reserve at the end of 5 years for a 10-year term policy for \$10,000 written at age 35.

11. (*a*) Find the reserve at the end of 5 years for an ordinary life policy for \$10,000 written at age 35. (*b*) Compare your answer with that in problem 10 and give a brief explanation of the difference.

12. A man aged 25 pays the net single premium for a 10-year term insurance for \$1000. What is the policy reserve, 5 years later?

13. A man aged 30 pays the net single premium for a whole life insurance for \$1000. Ten years later, what is the policy reserve?

[1] After the completion of Exercise LXXII, the student may proceed immediately to the Miscellaneous Problems at the end of the chapter.

14. Derive a general formula, as in equation 53, for the reserve at the end of m years for an n-year endowment policy for \$1 written at age x.

NOTE 2. — The method for computing reserves, furnished by equation 52, is called the *prospective method* because the future history of the policy is the basis for the equation. *Retrospective* methods also are used.

NOTE 3. — Insurance companies are subject to legal regulation. It is usually specified by state law that, at periodic times, an insurance company must show net assets equal to the sum of the reserves for all of its outstanding policies. The law specifies a standard mortality table and interest rate to be used. A company is insolvent if it cannot show net assets equal to the necessary reserve. It is likewise recognized by law that a company's reserve belongs to its policyholders as a whole. Hence, the reserve on a policy is the basis for its cash **surrender value**, the amount which the company must pay to a policyholder if he decides to withdraw from the company and surrender his policy. The cash surrender value equals the reserve, minus a surrender charge. The surrender charge in most states is specified by law and may be considered as a charge by the insurance company on account of the expense entailed in finding a new policyholder to take the place of the one withdrawing. This charge is legitimate because the theoretical reserve was computed by the company on the assumption that it had so many policyholders that the laws of averages, as dealt with in using the mortality table, would hold. Hence, the number of policyholders must be maintained and any one withdrawing should pay for the expense of obtaining a new policyholder in his place.

NOTE 4. — It should be recognized that the discussion in the preceding three chapters is merely an introduction to the subject of life annuities and of life insurance. We have not considered **joint life**, or **survivorship annuities** and **insurance**. Moreover, the subject of reserves requires a thorough treatment, beyond what we have given, from both the theoretical and the computational standpoint. The surplus of a company, its manner of declaring dividends to policyholders, and many other practical questions connected with the accounting and business methods of insurance companies have not even been mentioned. The student who wishes to pursue the subject farther is referred to the *Text Book of the Institute of Actuaries*, and to the courses of study described by the Educational Committees of the Actuarial Society of America and of the American Institute of Actuaries.

SUPPLEMENTARY EXERCISE LXXIII

Students working the problems below should have previously completed Supplementary Section 74 of Chapter IX.

1. A policy written for a person aged 38 promises whole life insurance for \$10,000, and a life annuity of \$1000 payable annually, with the first payment at age 65. Premiums are payable annually for 20 years. (*a*) Find the reserve at the end of 10 years. (*b*) Find the reserve at the end of 20 years.

2. A policy written at age 27 promises $1000 term insurance for 20 years and a pure endowment of $5000 at the end of 20 years. Premiums are payable annually for 10 years. Find the reserve at the end of 6 years.

3. A policy written at age 15 promises 20-year endowment insurance for $1000, and the premiums are payable annually for 10 years. (*a*) Determine the reserve at the end of 10 years. (*b*) Determine the reserve at the end of 9 years.

4. A certain pure annuity policy written at age 40 promises a life annuity of $1000 with the first payment at age 61. The premiums are payable annually for 21 years. Find the reserve (*a*) at the end of 5 years; (*b*) at the end of 20 years.

NOTE. — When a corporation or association promises a pension to a person, its act is equivalent to writing a pure annuity policy for the person involved. Hence, a pension association should be considered solvent only when its reserve fund is equal to the sum of the reserves on each of its pension contracts. As judged by this standard, there are an unfortunately large number of insolvent pension associations in operation. Their insolvency does not become apparent until after they have been operating long enough so that the theoretical reserve (which they do not possess) becomes necessary in order to meet liabilities falling due.

5. A group of workers of the same age entered a pension association which promised $500 annual payments for life, starting with payments at age 61. At age 55, 10,000 workers remain alive. They are required to pay $50 at the beginning of each year up to and including their 60th birthdays. How much should the association have on hand as a reserve before the $50 payments due at age 55 have been made?

MISCELLANEOUS PROBLEMS ON INSURANCE[1]

1. Write a sample of each of the following types of insurance policies, stating the age of the policyholder, the benefits he will receive, and how he is required to pay premiums: (*a*) 20-payment life; (*b*) 10-year endowment; (*c*) ordinary whole life; (*d*) 10-year term.

[1] Insurance companies mentioned in these problems are assumed to operate under assumptions (*a*), (*b*), and (*c*) of Section 68.

2. (*a*) A man aged 47 desires to set aside a sufficient sum which he can invest at 5%, effective, to pay him an annual income of $1000 for 10 years, starting with a payment on his 61st birthday. Find the amount set aside, *assuming that he will certainly live* to age 70. (*b*) At age 47 what would he have to pay to an insurance company for a contract to pay him $1000 at the end of each year for life, with the first payment at age 61, with the understanding that the company would compute the charge in accordance with the principles of scientific life insurance, at $3\frac{1}{2}\%$?

3. A woman offers $3000 to a benevolent organization on condition that the organization pay her 5% interest thereon at the end of each year for life. If the organization can purchase the required annuity for her from an insurance company, which uses the rate $3\frac{1}{2}\%$, will it pay to accept her offer if she is 55 years old?

4. According to a will, a trust fund of $200,000 will go to a charity at the death of a girl who is now aged 19, and she is to receive the income at 4% for the remainder of her life. On a $3\frac{1}{2}\%$ basis, find the present value of (*a*) her inheritance and of (*b*) the bequest to the charity.

5. A man borrows $200,000, on which he pays 5% interest annually. The principal is due at the end of 8 years. To protect his creditor he is compelled to take out an 8-year term insurance policy for $200,000. Assume that the man will certainly live to the end of 8 years, and find the present value at 6%, effective, of all of his payments on account of the debt, assuming that he pays merely the net premiums for his insurance as computed by a company which uses the rate $3\frac{1}{2}\%$. His age is 40 years.

6. A man aged 35 pays the net single premium on a whole life insurance for $1000. What is the policy reserve 10 years later?

7. A man aged 30 took out a 10-payment life policy. At the end of 10 years he desires to convert it into a 20-year endowment insurance as of that date. How much paid up endowment insurance will he obtain if the company permits all of his reserve to be used for that purpose? Notice that his reserve is the net single premium for the new insurance.

8. (*a*) Find the net annual premium at age 43 for an ordinary life policy for $2000. (*b*) Suppose that the man is alive at the end of 25 years. Find the reserve on his policy and compare it with the sum he would have on hand if he had invested all of his annual premiums at 5%, effective.

9. A man aged 42 borrows $100,000 and agrees to pay $4\frac{1}{2}\%$ interest annually. He agrees to provide for the payment of the principal at his death, or at the end of 10 years if he lives, by taking out a 10-year endowment policy for $100,000, with the creditor as beneficiary. The debtor

considers his future payments, assuming (1) that he will pay merely the net premiums at $3\frac{1}{2}\%$ for his policy, (2) that he will certainly live to the end of 10 years, and (3) that he is able to invest his money at 7%, effective. He asks if it would pay to borrow $100,000 elsewhere at 6%, payable annually, with the agreement that the principal may be repaid at the end of 10 years through the accumulation of a sinking fund. (*a*) Which method is best? (*b*) In terms of present values, how much could the debtor save by selecting the best method?

10. Compare the net single premiums for whole life insurance for $1000 (*a*) at ages 25 and 26; (*b*) at ages 75 and 76. (*c*) For which pair is the change in cost greatest?

PART III—AUXILIARY SUBJECTS

CHAPTER XI

LOGARITHMS

77. Definition of logarithms. — Logarithms are exponents. The **logarithm** of a number N with respect to a base a, where a is $> 0, \neq 1$, is the exponent of the power to which a must be raised to obtain N. That is, by definition, if

$$N = a^x, \tag{1}$$

then, the logarithm of N with respect to the base a is x; or, in abbreviated form,

$$\log_a N = x. \tag{2}$$

Thus, since $49 = 7^2$, then $\log_7 49 = 2$; since $1000 = 10^3$, then $\log_{10} 1000 = 3$. Also, if $\log_5 N = 2$, then, from equation 1, $N = 5^2$; if $\log_6 N = 4$, then $N = 6^4 = 1296$.

In the future, whenever we talk of the *logarithm of a number* we shall be referring to a positive number N. This is necessary because, in the definition of a logarithm, the base a is positive, and hence only positive numbers N have logarithms as long as the x in equation 1 is a real number.

EXERCISE LXXIV

1. Since $3^2 = 9$, what is $\log_3 9$?
2. Since $5^4 = 625$, what is $\log_5 625$?
3. Since $100 = 10^2$, what is $\log_{10} 100$?
4. Since $2^3 = 8$, what is $\log_2 8$?
5. Since $10^0 = 1$, what is $\log_{10} 1$? Since $17^0 = 1$, what is $\log_{17} 1$? Since every number to the power zero is 1, what is the logarithm of 1 with respect to every base; that is, since $a^0 = 1$, what is $\log_a 1$?
6. What is $\log_6 36$?
7. What is $\log_{10} 10{,}000$?
8. What is $\log_2 16$?
9. What is $\log_7 7$?
10. What is $\log_5 25$?
11. What is $\log_a a$?

12. If $\log_4 N = 2$, find N.
13. If $\log_3 N = 4$, find N.
14. If $\log_{10} N = 5$, find N.
15. If $\log_4 N = \frac{1}{2}$, find N.
16. If $\log_2 N = \frac{1}{2}$, find N.
17. If $\log_a N = 6.5$, find N.
18. Find $\log_9 81$.
19. Find $\log_{10} 1$.
20. Find $\log_{16} 4$.
21. Find $\log_{100} 10$.
22. If $10^{1.5} = 31.62$, find $\log_{10} 31.62$.
23. If $10^{.699} = 5$, find $\log_{10} 5$.
24. If $10^{2.4814} = 303$, find $\log_{10} 303$.
25. If $10^{.4771} = 3$, find $\log_{10} 3$.

Express in another way the fact that:

26. $\log_{10} 86.5 = 1.9370$.
27. $\log_{10} 684 = 2.8351$.
28. $\log_{10} 6.6 = .8195$.
29. $\log_{10} 4730 = 3.6749$.
30. $343 = 7^3$.
31. $\sqrt{3} = 1.732$.
32. If $N = \frac{1}{4}$, find $\log_4 N$. HINT.— $\frac{1}{4} = (4)^{-1}$.
33. If $N = \frac{1}{16}$, find $\log_2 N$. HINT.— $\frac{1}{16} = \frac{1}{2^4} = 2^{-4}$.
34. If $N = .1$, find $\log_{10} N$. HINT.— $.1 = \frac{1}{10}$.
35. Find $\log_{10} .001$. Find $\log_{10} .00001$. Find $\log_{10} .0000001$.

78. Properties of logarithms. — Logarithms have properties which make them valuable tools for simplifying arithmetical computation.

Property I. — *The logarithm of the product of two numbers M and N is equal to the sum of the logarithms of M and N:*

$$\log_a MN = \log_a M + \log_a N. \tag{3}$$

Proof. — Let $\log_a M = x$ and $\log_a N = y$. Then,

since	$\log_a M = x$, then $M = a^x$,	(Def. of logarithms)
and since	$\log_a N = y$, then $N = a^y$.	(Def. of logarithms)
Hence,	$MN = a^x a^y = a^{x+y}$.	(Law of exponents)
Since	$MN = a^{x+y}$, then $\log_a MN = x + y$.	(Def. of logarithms)
Hence,	$\log_a MN = \log_a M + \log_a N$.	(Subst. $x = \log_a M$; $y = \log_a N$)

Property II. — *The logarithm of the quotient of two numbers, M divided by N, is equal to the logarithm of the numerator minus the logarithm of the denominator:*

$$\log_a \frac{M}{N} = \log_a M - \log_a N. \tag{4}$$

Proof. — Let $\log_a M = x$ and $\log_a N = y$. Then,

since	$\log_a M = x$, then $M = a^x$,	(Def. of logarithms)
and since	$\log_a N = y$, then $N = a^y$.	(Def. of logarithms)

Hence, $\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$. (Law of exponents)

Since $\frac{M}{N} = a^{x-y}$, then $\log_a \frac{M}{N} = x - y$. (Def. of logarithms)

Hence, $\log_a \frac{M}{N} = \log_a M - \log_a N$. (Subst. $x = \log_a M$; $y = \log_a N$)

Property III. — *The logarithm of a number N, raised to a power k, is k times the logarithm of N:*

$$\log_a N^k = k \log_a N. \quad (5)$$

Proof. — Let $\log_a N = x$; then $N = a^x$, by the definition of logarithms.
Hence, $N^k = (a^x)^k = a^{kx}$. (Law of exponents)
Since $N^k = a^{kx}$, then $\log_a N^k = kx$. (Def. of logarithms)
Hence, $\log_a N^k = k \log_a N$. (Subst. $x = \log_a N$)

NOTE. — In the future we shall deal entirely with logarithms to the base 10. Hence, for convenience, instead of writing $\log_{10} N$ we shall write merely $\log N$, understanding that the base always is 10. Logarithms to the base 10 are called **Common Logarithms**; the name Briggs' logarithms is also used, in honor of an Englishman named Henry Briggs (1556–1630), who computed the first table of Common Logarithms.

Example 1. — Given that: $\log 2 = .3010$, $\log 5 = .6990$, $\log 17 = 1.2305$, find the logarithms of each of the following numbers: 34, 85, $\frac{17}{5}$, $\sqrt{17}$, 25.

Solution. — $\log 34 = \log 2(17) = \log 2 + \log 17 = .3010 + 1.2305 = 1.5315.$
$\log 85 = \log 5(17) = \log 5 + \log 17 = .6990 + 1.2305 = 2.2295.$ (Prop. I)
$\log \frac{17}{5} = \log 17 - \log 5 = 1.2305 - .6990 = .5315.$ (Prop. II)
$\log \sqrt{17} = \log 17^{\frac{1}{2}} = \frac{1}{2} \log 17 = \frac{1}{2}(1.2305) = .61525.$ (Prop. III with $k = \frac{1}{2}$)
$\log 25 = \log 5^2 = 2 \log 5 = 2(.6990) = 1.3980.$ (Prop. III with $k = 2$)

EXERCISE LXXV

In the problems below find the logarithms of the given numbers, given that:

$\log 2 = .3010$	$\log 3 = .4771$	$\log 5 = .6990$
$\log 7 = .8451$	$\log 11 = 1.0414$	$\log 13 = 1.1139$
$\log 17 = 1.2305$	$\log 23 = 1.3617$	$\log 29 = 1.4624$

1. 6	**2.** 9	**3.** 46	**4.** 51	**5.** $\frac{17}{8}$	**6.** $\frac{5}{13}$
7. 20	**8.** $\frac{34}{5}$	**9.** $\sqrt[3]{5}$	**10.** $\sqrt{7}$	**11.** 49	**12.** 16
13. 5^3	**14.** $\sqrt[4]{17}$	**15.** $\frac{17}{7}$	**16.** 50	**17.** 55	**18.** 154
19. $\frac{77}{15}$	**20.** $\frac{23}{29}$	**21.** 10	**22.** 100	**23.** 1000	**24.** 10,000

25. 230 **26.** 2300 **27.** 23,000 **28.** 230,000 **29.** $.1 = \frac{1}{10}$ **30.** $.01 = \frac{1}{100}$
31. .001 **32.** .0001 **33.** $.5 = \frac{5}{10}$ **34.** .05 **35.** .005 **36.** .0005

79. Common logarithms. — If one number $N = 10^x$ is larger than another number $M = 10^y$, then x must be larger than y. Since $x = \log N$ and $y = \log M$, it follows that, if N is larger than M, then $\log N$ is larger than $\log M$. Thus, since 9 is larger than 7, log 9 must be larger than log 7.

The table below gives the logarithms of certain powers of 10.

Since:	Then:
$10000 = 10^4$	$\log 10000 = 4$
$1000 = 10^3$	$\log 1000 = 3$
$100 = 10^2$	$\log 100 = 2$
$10 = 10^1$	$\log 10 = 1$
$.1 = \frac{1}{10} = 10^{-1}$	$\log .1 = -1$
$.01 = \frac{1}{100} = 10^{-2}$	$\log .01 = -2$
$.001 = \frac{1}{1000} = 10^{-3}$	$\log .001 = -3$

Consider the number 7, or any other number between 1 and 10. Since 7 is greater than 1 and less than 10, log 7 is greater than log 1, which is 0, and is less than log 10, which is 1. That is, since 7 is between 1 and 10, log 7 lies between 0 and 1. Hence, log 7 = 0 + (*a proper fraction*). From a table of logarithms, as described later, log 7 = .84510, approximately, so that the fraction mentioned above is .84510. Similarly, since 750 is between 100 and 1000, log 750 lies between 2 and 3; therefore, log 750 = 2 + (*a proper fraction*); since 5473 is between 1000 and 10,000, log 5473 = 3 + (*a proper fraction*). In the same manner, since .15 lies between .1 and 1, log .15 lies between − 1 and 0, and hence[1] log .15 = − 1 + (*a proper fraction*). In general, *the logarithm of every positive number can be expressed as an integer, either positive or negative, plus a positive proper fraction.*

The integral part of a logarithm is called its **characteristic.** When a number N is greater than 1, the characteristic of $\log N$ is positive; when N is less than 1, the characteristic of $\log N$ is negative.

[1] Any number between − 1 and 0 can be expressed as − 1 + (a proper fraction). Thus, − .57 = − 1 + .43; − .88 = − 1 + .12, etc.

The fractional part of a logarithm is called its **mantissa.**

Thus, given that $\log 700 = 2.84510$, the characteristic of log 700 is 2, and the mantissa is .84510; given that $\log .27 = -1 + .43136$, the characteristic of log .27 is -1 and the mantissa is .43136.

80. Properties of the mantissa and the characteristic. — Given that $\log 3.8137 = .58134$, then, by use of Properties I and II of Section 78, and from the logarithms of powers of 10 given in Section 79, we prove the following results:

$$\log 3813.7 = \log 1000(3.8137) = \log 1000 + \log 3.8137 = 3 + .58134 = 3.58134.$$
$$\log 381.37 = \log 100(3.8137) = \log 100 + \log 3.8137 = 2 + .58134 = 2.58134.$$
$$\log 38.137 = \log 10(3.8137) = \log 10 + \log 3.8137 = 1 + .58134 = 1.58134.$$
$$\log 3.8137 = 0.58134.$$

$$\log .38137 = \log \frac{3.8137}{10} = \log 3.8137 - \log 10 = .58134 - 1 = -1 + .58134.$$

$$\log .038137 = \log \frac{3.8137}{100} = \log 3.8137 - \log 100 = .58134 - 2 = -2 + .58134.$$

$$\log .0038137 = \log \frac{3.8137}{1000} = \log 3.8137 - \log 1000 = .58134 - 3 = -3 + .58134.$$

NOTE. — The characteristics of the logarithms above could have been obtained as in Section 79. Thus, since 3813.7 lies between 1000 and 10,000, log 3813.7 lies between 3 and 4; $\log 3813.7 = 3 +$ (a proper fraction). Therefore, the characteristic of log 3813.7 is 3, as found above.

From inspection above, we see that .58134 is the mantissa of all of the logarithms. This result, which obviously would hold for any succession of digits as well as it does for the digits 3, 8, 1, 3, 7, may be summarized as follows:

Rule 1. — The mantissa of the logarithm of a number *N* depends only on the succession of digits in *N*. If two numbers have the same succession of digits, that is, if they differ only in the position of the decimal point, their logarithms have the same mantissa.

The logarithms above also illustrate facts about the characteristic.

Rule 2. — The characteristic of the logarithm of a number greater than 1 is positive and is 1 less than the number of digits in the number to the left of the decimal point.

NOTE. — Thus, in accordance with Rule 2, 3 is the characteristic of log 3813.7; 2 is the characteristic of log 381.37, etc. Rule 2 is justified in general by recognizing that, if a number N has $(k + 1)$ digits to the left of the decimal point, then N is between 10^k and 10^{k+1}; hence $\log N$ is between k and $(k + 1)$ and $\log N = k +$ (a proper fraction). That is, k is the characteristic of $\log N$.

Rule 3. — If a number N is less than 1, the characteristic of $\log N$ is a negative integer; if the first significant figure of N appears in the kth decimal place, then the characteristic of $\log N$ is $-k$.

Thus, the first significant figure of .38137 is 3 and appears in the first decimal place, and, in accordance with Rule 3, the characteristic of log .38137 is -1. The first significant figure of .038137 appears in the 2d decimal place, while the characteristic of log .038137 is -2, etc.

NOTE. — It may appear strange to the student that we write, for example, $\log .0038137 = -3 + .58134$, instead of performing the subtraction. For every number N which is less than 1, $\log N$ is a negative number; thus, $\log .0038137 = -3 + .58134 = -2.41866$. Written in this way, the mantissa .58134 and the characteristic -3 are lost sight of. We write the logarithm in the form $-3 + .58134$ to keep the characteristic and the mantissa in a prominent position.

Since the mantissa depends merely on the succession of digits in the number, it is customary to speak of a mantissa as corresponding to a given succession of digits without thinking of any decimal point being associated with the digits. Thus, above, we would say that the mantissa for the digits 38137 is .58134.

Example 1. — Given that the mantissa for the digits 5843 is .76664, find log 5843; log 584.3; log 58,430,000; log 5.843; log .0005843.

Solution. — The characteristic of log 5843 is 3; hence, log 5843 = 3.76664. Similarly, log 584.3 = 2.76664; log 58,430,000 = 7.76664; log 5.843 = 0.76664; log .0005843 = $-4 + .76664$.

EXERCISE LXXVI

1. Given that .75101 is the mantissa for the digits 56365, find log 5636.5; log 56365; log 563.65; log 56,365,000; log .0056365; log .56365.

2. Given that .93046 is the mantissa for 85204, find log 85.204; log 852,040,000; log 8.5204; log 85204; log .085204; log .0000085204.

3. Given that .39863 is the mantissa for 2504, find log 2504; log 2.504; log 25,040; log .2504; log .00000000002504.

4. Given that log 273.7 = 2.43727, find log 2.737; log 27.37; log 27,370; log .02737; log .002737. Make use of Rule 1.

5. Given that log 68,025 = 4.83267, find log 68.025; log 6.8025; log .68025; log 6802.5; log .00068025.

6. What is the mantissa of log 1; of log 10; of log 10,000; of log .1; of log .00001?

81. Tables of mantissas. — The mantissa for a given succession of digits can be computed by the methods of advanced mathematics. The computed mantissas are then gathered in *tables of logarithms* which, more correctly, should be called tables of mantissas. Except in special cases, mantissas are infinite decimal fractions. Thus the mantissa for 10705 is .02958667163045713486 to 20 decimal places. In a 5-place table of logarithms, this mantissa would be recorded *correct to* 5 *decimal places*, giving .02959. In an 8-place table, it would be recorded as .02958667, correct to 8 decimal places.

NOTE. — Table I in this book is a 5-place table of logarithms. A decimal point is understood in front of each tabulated mantissa. To find the mantissa for $N = 3553$, for example, go to the sixth page of Table I. Find the digits 355 in column headed N; the mantissa for 3553 is entered in the corresponding row under the column headed 3. The entry is "060," but the first two digits of the mantissa are understood to be "55," the same as for the first entry in the row. Thus, the mantissa for 3553 is .55060. From Table I the student should now verify that:

FOR THE DIGITS BELOW	THE MANTISSA IS
3630	.55991
3947	.59627
4589	.66172
9331	.96993
9332	.96997
9333[1]	.97002

Example 1. — Find log 38570; log .008432.

Solution. — By inspection, the characteristic of log 38570 is 4; the mantissa as found in Table I is .58625. Hence, log 38570 = 4.58625. The characteristic of log .008432 is -3; log .008432 = $-3 + .92593$.

[1] In Table I, for 9333, we find the entry "*002." The asterisk (*) on the "002" means that the first two digits are to be changed from 96, as at the beginning of the row, to 97.

In order to obtain greater convenience in computation, it is customary to write negative characteristics in a different manner than heretofore. Thus, in log .008432 = − 3 + .92593, change the − 3 to (7 − 10). Then log .008432 = − 3 + .92593 = 7 − 10 + .92593 = 7.92593 − 10. Recognize clearly that log .008432 = − 3 + .92593 = − 2.07407. We verify that 7.92593 − 10 = − 2.07407. The two ways introduced for writing log .008432 are merely two different ways of writing the negative number − 2.07407, which is the actual logarithm involved. Similarly, log .8432 = − 1 + .92593 = 9.92593 − 10; log .000'000'000'008432 = − 12 + .92593 = 8 − 20 + .92593 = 8.92593 − 20, etc.

NOTE. — The change from the new form to the old or vice versa is easy. Thus, given that log .05383 = 8.73102 − 10, we see that the characteristic is (8 − 10) or − 2; given that log .005849 = − 3 + .76708, then log .005849 = 7.76708 − 10.

EXERCISE LXXVII

1. What are the characteristics of the following logarithms: 9.8542 − 10; 7.7325 − 10; 6.5839 − 10; 4.3786 − 10?

2. Write the following logarithms in the other form: − 3 + .5678; − 5 + 7654; − 7 + .8724; − 1 + .9675.

3. Write the following logarithms as pure negative numbers: − 3 + .5674; − 1 + .7235; 9.7536 − 10; 7.2539 − 10.

4. By use of Table I verify the logarithms given below:

N	Log N	N	Log N
3515.	3.54593	35.88	1.55485
.01832	8.26293 − 10	1.170	0.06819
889,900	5.94934	.0008141	6.91068 − 10
.6761	9.83001 − 10	27,770	4.44358
621.8	2.79365	.00004788	5.68015 − 10

NOTE. — When the characteristic of log N is 0, log N is equal to its mantissa. Thus, log 1.578 = 0.19811. Hence, a table of mantissas is a table of the actual logarithms of all numbers between 1 and 10.

82. Logarithms of numbers with five significant figures. — If a number N has five significant digits, log N cannot be read directly from the table. We must use the process of *interpolation* as described in the following examples.

Example 1. — Find log 25.637.

Solution. — The characteristic is 1. To find the mantissa, recognize that 25.637 is between 25.630 and 25.640; the mantissas for 2563 and for 2564 were read from Table I and the logarithms of 25.630 and 25.640 are given in the table below. Since 25.637 is .7 of the way from 25.630 toward 25.640, we assume [1] that log 25.637 is .7 of the way from 1.40875 toward 1.40892. The total way, or difference, is .40892 − .40875 = .00017; .7 of the way is .7(.00017) = .000119. We reduce this to .00012, the nearest number of five decimal places. Hence,

Number	Logarithm
25.630	1.40875
25.637	? ?
25.640	1.40892

$$\log 25.637 = 1.40875 + .00012 = 1.40887.$$

Note. — At first, the student should do all interpolation in detail as in Example 1 above. Afterward, he should aim to gain speed by doing the arithmetic mentally. The small tables in the column in Table I headed PP, an abbreviation for *proportional parts*, are given to reduce the arithmetical work.

Example 2. — Find log .0017797.

Solution. — The characteristic is − 3 or (7 − 10). The digits 17797 form a number between 17790 and 17800. The tabular difference between the corresponding mantissas is (.25042 − .25018) = .00024, or 24 units in the 5th decimal place. Since 17797 is .7 of the way from 17790 to 17800, we wish .7(24). By multiplication, .7(24) = 16.8. This should be found *without multiplication* from the small table headed 24 under the column PP. From this table we read .1(24) = 2.4, .2(24) = 4.8, etc., .7(24) = 16.8. Hence, the mantissa for 17797 is .25018 + .17 = .25035, and

Number	Mantissa
17790	.25018
17797	? ?
17800	.25042

$$\log .0017797 = 7.25035 - 10.$$

Note. — The following situation is sometimes met in interpolating. Suppose that .5(15) = 7.5 is the part of the tabular difference which we must add. We may, with equal justification, call 7.5 either 7 or 8. As a definite rule in this book, whenever such an ambiguity is met, we agree to choose the even number. Hence, we choose 8 above. Similarly, in using .7(15), or 10.5, we should call it 10, because we have a choice between 10 and 11.

[1] This assumption is justified by the first paragraph of Section 79. Since 25.637 is between 25.630 and 25.640, log 25.637 must be between log 25.630 and log 25.640. In interpolating as in Example 1, we merely go one step farther than this admitted fact when we assume that the change in the logarithm is proportional to the change in the number. This assumption, although not exactly true, is sufficiently accurate for all practical purposes.

EXERCISE LXXVIII

1. Verify the following logarithms:

log 256.32 = 2.40878	log 8956.1 = 3.95211
log 13.798 = 1.13982	log 931.42 = 2.96915
log .073563 = 8.86666 − 10	log 33.581 = 1.52609
log .59834 = 9.77695 − 10	log .00047178 = 6.67374 − 10
log 1.1675 = 0.06725	log 676.93 = 2.83054

2. Find the logarithms of the following numbers:

18.156	.31463	.061931	151.11
5321.7	83196	48.568	6319.1
67.589	113.42	384.22	9.3393
.031562	.92156	.52793	.000031579
.009567	5.6319	1.1678	83.462

83. To find the number when the logarithm is given.

Example 1. — Find N if $\log N = 7.67062 - 10$.

Solution. — Since the characteristic is $(7 - 10) = -3$, the first significant figure of N will appear in the 3d decimal place; $N = .00\ldots.$ To find the digits of N, we must obtain the number whose mantissa is .67062. We search for this mantissa, or those nearest to it, in Table I; we find .67062 as the mantissa of 4684. Hence, $N = .004684$.

Example 2. — Find N if $\log N = 5.41152$.

Solution. — We wish the 5-figure number whose mantissa is .41152. On inspecting Table I we find the tabular mantissas .41145 and .41162 between which .41152 lies. The total way, or *tabular difference*, between .41145 and .41162 is .00017, or 17 units in the 5th decimal place. The *partial difference* $.41152 - .41145 = .00007$, or 7 units in the 5th decimal place. Hence, .41152 is $\frac{7}{17}$ of the way from .41145 to .41162. We then assume that the number x, whose mantissa is .41152, is $\frac{7}{17}$ of the way from 25790 to 25800. The total way, or difference, is 10 units in the 5th place; $\frac{7}{17}(10) = 4.1$; the nearest unit is 4. Hence, .41152 is the mantissa of $25790 + 4 = 25794$. Since the characteristic of $\log N$ is 5, $N = 257{,}940$.

NUMBER	MANTISSA
25790	.41145
x	.41152
25800	.41162

NOTE. — The arithmetic in Example 2 above is simplified by use of the table headed 17 under the column of proportional parts. In Example 2 we desire (10), which we can easily obtain if we know $\frac{7}{17}$ correct to the nearest tenth. m the table headed 17, we read $.4(17) = 6.8$, or $\frac{6.8}{17} = .4$; $.5(17) = 8.5$, $\frac{8.5}{17} = .5$. Since 7 is between 6.8 and 8.5, $\frac{7}{17}$ is between .4 and .5, but is rest to .4. Thus, $\frac{7}{17}(10) = 4$, to the nearest unit. With practice, this result should be obtained almost instantaneously. Thus, *look under the table headed* 17 *for the number nearest to* 7; *we find* 6.8; *at the left it is shown that this is* .4 *of* 17; *hence* $\frac{1}{17}(10) = 4$.

EXERCISE LXXIX

1. Find the numbers corresponding to the given logarithms and verify the answers given:

$\log N = 3.21388$;	$N = 1636.4$.	$\log N = 3.75097$;	$N = 5636$.
$\log N = 8.40415 - 10$;	$N = .02536$.	$\log N = 0.46839$;	$N = 2.9403$.
$\log N = 2.15931$;	$N = 144.31$.	$\log N = 3.33590$;	$N = 2167.2$.
$\log N = 9.52163 - 10$;	$N = .33238$.	$\log N = 8.65267 - 10$;	$N = .044944$.
$\log N = 0.89651$;	$N = 7.8797$.	$\log N = 0.35217$;	$N = 2.2499$.

2. Find the numbers corresponding to the following logarithms:

$\log N = 5.21631$	$\log N = 3.19008$	$\log N = 9.64397 - 10$
$\log N = 1.39875$	$\log N = 7.55642 - 10$	$\log N = 2.57938 - 10$
$\log N = 8.95321 - 10$	$\log N = 0.89577$	$\log N = 1.77871$
$\log N = 4.32111 - 10$	$\log N = 1.21352$	$\log N = 7.77853$
$\log N = 2.15678$	$\log N = 8.45673 - 10$	$\log N = 3.15698$

84. Computation of products and of quotients.

Example 1. — Compute $P = 787.97 \times .0033238 \times 14.431$.

Solution. — From Property I of Section 78, $\log P$ is the sum of the logarithms of the factors. From Table I,

$$\begin{aligned} \log 787.97 &= 2.89651 \\ \log .0033238 &= 7.52163 - 10 \\ \log 14.431 &= 1.15931 \\ \hline \text{(add) } \log P &= 11.57745 - 10 = 1.57745 \end{aligned}$$

From Table I, $P = 37.796$.

Example 2. — Compute $Q = \dfrac{4.8031 \times 269.97 \times 1.6364}{78797 \times 253.6}$.

Solution. — From Property II of Section 78, $\log Q$ equals the logarithm of the numerator minus the logarithm of the denominator. Both numerator and denominator are products whose logarithms are determined by Property I.

$$\begin{aligned} \log 4.8031 &= 0.68152 \\ \log 269.97 &= 2.43131 \\ \log 1.6364 &= 0.21389 \\ \hline \text{(add) } \log \text{Numer.} &= 3.32672 \\ \log \text{Denom.} &= 7.30066 \\ \hline \text{(subtract) } \log Q &= \ ? \end{aligned} \qquad \begin{aligned} \log 78797 &= 4.89651 \\ \log 253.6 &= 2.40415 \\ \hline \text{(add) } \log \text{Denom.} &= 7.30066 \end{aligned}$$

We recognize that the result on subtracting will be negative. To obtain $\log Q$ in standard form, we add and also subtract 10 from the log numerator.

$$\begin{array}{rl} \log \text{Numer.} = 3.32672 & = 13.32672 - 10 \\ \log \text{Denom.} & = 7.30066 \\ \hline (\text{subtract}) \log Q & = 6.02606 - 10; \quad Q = .00010618. \end{array}$$

NOTE. — Before computing any expression by logarithms, a *computing form* should be made. Thus, the first operation in solving Example 2 above was to write down the following form:

$$\begin{array}{rl} \log 4.8031 & = \\ \log 269.97 & = \\ \log 1.6364 & = \\ \hline (\text{add}) \log \text{Numer.} & = \\ \log \text{Denom.} & = \\ \hline (\text{subtract}) \log Q & = \end{array} \qquad \begin{array}{rl} \log 78797 & = \\ \log 253.6 & = \\ \hline (\text{add}) \log \text{Denom.} & = \end{array}$$

A systematic form prevents errors and makes it easy to repeat the work if it is desired to check the computation.

EXERCISE LXXX

Compute by logarithms:

1. $563.7 \times 8.2156 \times .00565$.
2. $4.321 \times 21.98 \times .99315$.
3. $\frac{675.31}{13.215}$.
4. $\frac{56.854}{2356.7}$.
5. $\frac{.008315}{.0003156}$.
6. $\frac{783.12 \times 11.325}{8932}$.
7. $\frac{85 \times 73 \times 139.58}{3215.7 \times .4563}$.
8. $\frac{9.325 \times 531.75}{.8319 \times .5685}$.
9. $\frac{.42173 \times .21567}{.3852 \times .956}$.
10. $\frac{5.3172 \times .4256}{18.11 \times 31.581}$.

85. Computation of powers and of roots.

Example 1. — Find $(.3156)^4$.

Solution. — From Property III of Section 78 with $k = 4$,

$\log (.3156)^4 = 4 \log .3156 = 4(9.49914 - 10) = 37.99656 - 40 = 7.99656 - 10$. From Table I, $(.3156)^4 = .009921$.

Example 2. — Find $\sqrt[3]{856.31}$.

Solution. — $\sqrt[3]{856.31} = (856.31)^{\frac{1}{3}}$. From Property III with $k = \frac{1}{3}$, $\log \sqrt[3]{856.31} = \frac{1}{3} \log 856.31 = \frac{2.93263}{3} = 0.97754$; hence, $\sqrt[3]{856.31} = 9.4960$.

Example 3. — Find $\sqrt[6]{.08351}$; $\sqrt[3]{.08351}$.

Solution. — Since $\sqrt[6]{.08351} = (.08351)^{\frac{1}{6}}$, we obtain from Property III, $\log \sqrt[6]{.08351} = \frac{1}{6}\log .08351 = \frac{8.92174 - 10}{6}$. If we divide this as it stands, we obtain $1.48696 - \frac{10}{6}$, a most inconvenient form. Hence, we add and, at the same time, subtract 50 from log .08351 in order that, after the division by 6, the result will be in the standard form for logarithms with negative characteristics. Hence,

$$\log \sqrt[6]{.08351} = \frac{8.92174 - 10}{6} = \frac{50 + 8.92174 - 10 - 50}{6} = \frac{58.92174 - 60}{6}.$$

$\log \sqrt[6]{.08351} = 9.82029 - 10$; hence, from Table I, $\sqrt[6]{.08351} = .66113$.

From Property III, $\log \sqrt[3]{.08351} = \frac{1}{3}\log .08351 = \frac{8.92174 - 10}{3}$.

$$\log \sqrt[3]{.08351} = \frac{28.92174 - 30}{3} = 9.64058 - 10; \text{ hence, } \sqrt[3]{.08351} = .43710.$$

EXERCISE LXXXI

Compute by logarithms:

1. $(175)^2$.
2. $(56.73)^3$.
3. $(.013821)^4$.
4. $\sqrt{531.2}$.
5. $\sqrt[3]{.079677}$.
6. $(.38956)^{\frac{1}{4}}$.
7. $(353.3 \times 1.6888)^2$.
8. $\sqrt[5]{199.62}$.
9. $(1.05)^7$.
10. $(1.06)^{\frac{1}{4}}$.
11. $(1.03)^{14}$.
12. $(1.06)^{29}$.
13. $\frac{1}{85.75}$.
14. $\frac{1}{(45.6)^2}$.
15. $(1.03)^{-6} = \frac{1}{(1.03)^6}$.
16. $\sqrt{\frac{56.35 \times 4.3157}{21.36 \times \sqrt{521.9}}}$.

Hint. — For this problem, the computing form is:

log 56.35 =	log 521.9 =
log 4.3157 =	½ log 521.9 =
(add) log Numer. =	log 21.36 =
log Denom.	(add) log Denom. =
(subtract) log fract. =	
½ log fract. =	Result =

17. $\frac{535 \times 831.5 \times (1.03)^6}{475 \times 938}$.
18. $(189.5)^4$.
19. $\sqrt[4]{.00356}$.
20. $\sqrt{896.33}$.
21. $\frac{(153.2)^2 \times 257.3}{1893.2 \times 35830}$.
22. $\frac{.03156 \times 75.31}{221.38 \times (.3561)^2}$.
23. $(1.04)^{25}$.
24. $(1.035)^{-6}$.

86. Problems in computation. — It is very important to realize that the properties I, II, and III of logarithms may be used in computing products, quotients, and powers, but that they may *not* be used in computing differences or sums except in the auxiliary manner illustrated below.

Example 1. — Compute $Q = \dfrac{\sqrt{896} + (.567)(35.3)}{532 - (15.31)^2}$.

Solution. — By logarithms, we perform each of the three computations below.

$$\log \sqrt{896} = \tfrac{1}{2}\log 896 = \frac{2.95231}{2} = 1.47616;\ \sqrt{896} = 29.934.$$

$$\begin{aligned} \log .567 &= 9.75358 - 10 \\ \log 35.3 &= 1.54777 \\ \hline \log \text{prod.} &= 11.30135 - 10;\ (.567)(35.3) = 20.015. \end{aligned}$$

$$\log (15.31)^2 = 2 \log 15.31 = 2(1.18498) = 2.36996;\ \text{hence, } (15.31)^2 = 234.40.$$

Therefore, using the results above, $Q = \dfrac{29.934 + 20.015}{532 - 234.40} = \dfrac{49.949}{297.60}$.

$$\begin{aligned} \log 49.949 &= 1.69853 = 11.69853 - 10 \\ \log 297.60 &= 2.47363 = \ \ 2.47363 \\ \hline (\text{subtract}) \log Q &= \ \ 9.22490 - 10;\ \text{hence, } Q = .16784. \end{aligned}$$

NOTE. — A computation done with a 5-place table of logarithms will give results which are accurate to 4 significant figures, but the 5th figure always will be open to question. Each mantissa in the table, and each of those we determine by interpolation is subject to an error of part of 1 unit in the 5th decimal place, even though all of our interpolation is done correctly. During a long computation, these accumulated errors in the logarithms, together with the allowable error due to our final interpolation, cause an unavoidable error in the 5th significant figure of our final result. Therefore, if a number with more than 5 significant figures, such as 2,986,533, is met in a computation with a 5-place table, we should reduce this number to 2,986,500, the nearest number having 5 significant figures, before finding its logarithm. To retain more than 5 significant digits is fictitious accuracy, since our final results will be accurate to only 4 digits. For the same reason, in looking up the number corresponding to a given logarithm, the interpolation should not be carried beyond the nearest unit in the 5th significant place.

NOTE. — Logarithmic computation of products, quotients, and powers must deal entirely with positive numbers, according to the statements of Section 77. Hence, if negative numbers are involved, we first compute the expression by logarithms as if all numbers were positive, and then by inspection determine the proper sign to be assigned to the result. Thus, to compute $(-75.3) \times (-8.392) \times (-32.15)$ we first find $75.3 \times 8.392 \times 32.15 = 20316$; then, we note that a negative sign must be attached, giving -20316 as the result.

EXERCISE LXXXII

Compute by logarithms:

1. $\dfrac{(35.6)^2 + 89.53}{\sqrt{111.39} - 2.513}$.

2. $\dfrac{1.931 \times 5.622 - \sqrt[3]{.3921}}{5.923}$.

3. $\dfrac{(1.03)^5 - 1}{(1.03)^{\frac{1}{2}} - 1}$.

4. $\dfrac{1 - (1.04)^{-4}}{.04}$.

5. $\dfrac{(1.07)^6 - 1}{.07}$.

6. $\dfrac{251 + 63.95 \times 41.27}{787}$.

7. 395×856.

8. $\dfrac{3852}{5321}$.

9. $(\log 395)(\log 856)$. That is, compute $(2.59660)(2.93247)$.

10. $\dfrac{\log 3852}{\log 5321}$. That is, compute $\dfrac{3.58569}{3.72599}$.

11. $\dfrac{\log 88 - 2}{654}$.

12. $\dfrac{3 \log (1.04)}{\log 2}$.

13. $\dfrac{\log 6.532}{\log 1.04}$.

14. $\dfrac{\log 8.957}{\log 1.06}$.

15. $153.5(1.025)^{10}$.

16. $35.285(1.04)^{-5}$.

17. $(1.05)^{\frac{3}{4}}$.

18. $(1.035)^{\frac{5}{12}}$.

19. $12[(1.02)^{\frac{1}{12}} - 1]$.

20. $\dfrac{\log .85 + 3}{(.235)^2}$.

87. Exponential equations. — An equation in which the unknown is involved in an exponent is called an **exponential equation.** Thus, $3^z - 7 = 27$ is an exponential equation for z. In this section we shall treat exponential equations of the type that can be solved by use of the following rule:

Rule. — To solve a simple exponential equation, take the logarithm of both sides of the equation and solve the resulting equation.

Example 1. — Solve the equation $13^{2x+2} = (356)5^x$.

Solution. — Take the logarithm of both sides of the equation, making use of Property I. Then $(2x + 2) \log 13 = \log 356 + x \log 5$, or

$$(2x+2)(1.11394) = 2.55145 + x(.69897). \qquad \text{(Table I)}$$
$$2.22788\,x + 2.22788 = 2.55145 + .69897\,x.$$
$$1.52891\,x = .32357;$$
$$x = \frac{.32357}{1.52891} = .21164. \qquad \begin{array}{r} \log .32357 = 9.50997 - 10 \\ \log 1.5289 = 0.18438 \\ \hline \text{(subtract)} \log x = 9.32559 - 10 \end{array}$$

The exponential equations met in applications to the mathematics of investment are of the form

$$A^v = B, \tag{6}$$

where A and B are constants, and where v is a function of the unknown quantity.

Example 2. — Solve $(1.07)^{2n} = 4.57$.

Solution. — Taking the logarithm of both sides, we obtain

$$2\,n \log 1.07 = \log 4.57; \quad n = \frac{\log 4.57}{2 \log 1.07} = \frac{.65992}{2(.02938)} = \frac{.65992}{.05876}.$$
$$\begin{array}{r} \log .65992 = 9.81949 - 10 \\ \log .05876 = 8.76908 - 10 \\ \hline \text{(subtract)} \log n = 1.05041; \quad n = 11.231. \end{array}$$

EXERCISE LXXXIII

Solve the following equations:

1. $(1.05)^n = 6.325$.
2. $15^x = 95$.
3. $12^{x+1} = 38$.
4. $(1.025)^{2n} = 3.8261$.
5. $53^x = 569$.
6. $2^n = 31$.
7. $5^x = 27(2^x)$.
8. $25(6^x) = 282$.
9. $\dfrac{(1.035)^n - 1}{.035} = 2.75$.

HINT. — Clear the equation of fractions and reduce to the form of equation 6, obtaining $(1.035)^n = 1.06875$.

10. $(1.045)^{-n} = .753$.

HINT. — The equation becomes $-n \log 1.045 = 9.87679 - 10 = -.12321$.

11. $(1.03)^{-n} = .8321$.
12. $850(1.05)^n = 1638$.
13. $65.30(1.025)^{-n} = 52.67$.
14. $750\,\dfrac{(1.02)^n - 1}{.02} = 3500$.

SUPPLEMENTARY MATERIAL

88. Logarithms to bases different from 10. — To avoid confusion we shall explicitly denote the bases for all logarithms met in this section. From Section 77, $x = \log_a N$ satisfies the equation $a^x = N$. By solving this exponential equation, we can find x when N and a are given. Thus, taking the logarithm to the base 10 of both sides of $a^x = N$, we obtain

$$x \log_{10} a = \log_{10} N; \quad x = \frac{\log_{10} N}{\log_{10} a},$$

or
$$x = \log_a N = \frac{\log_{10} N}{\log_{10} a} = \frac{1}{\log_{10} a} \log_{10} N. \qquad (7)$$

NOTE. — Equation 7 enables us to find the logarithm of any number with respect to a given base a, provided that we have a table of logarithms to the base 10. The quantity $\log_{10} a$ is called the modulus of the system of logarithms to the base 10 with respect to the system to the base a.

The **natural system** of logarithms is that system where the base is the number $e = 2.718281828\ldots$. The number e is a very important mathematical constant and logarithms to the base e are useful in advanced mathematics. From an 8-place table, we find

$$\log_{10} e = 0.43429448; \ \log .43429448 = 9.63778431 - 10.$$

Example 1. — Find $\log_e 35$.

Solution. — Let $x = \log_e 35$. Then, $e^x = 35$; taking the logarithm of both sides to the base 10, $x \log_{10} e = \log_{10} 35$;

$$x = \frac{\log_{10} 35}{\log_{10} e} = \frac{1.54407}{0.43429}. \qquad \begin{aligned} \log 1.5441 &= 10.18868 - 10 \\ \log .43429 &= 9.63778 - 10 \end{aligned}$$

$$x = 3.5555. \qquad \text{(subtract) } \log x = 0.55090$$

EXERCISE LXXXIV

1. Find $\log_e 75$; $\log_e 10$; $\log_e 830$; $\log_e 657$.
2. Find the natural logarithm of 4368.
3. Find $\log_6 353$; $\log_5 10$; $\log_9 895$; $\log_{15} 33$.
4. If a and b are any two positive numbers, prove that

$$\log_b N = \log_a N \cdot \log_b a.$$

HINT. — Let $x = \log_a N$ and $y = \log_b N$. Then $N = a^x = b^y$. Take the logarithm with respect to the base b of both sides of the equation $b^y = a^x$.

CHAPTER XII

PROGRESSIONS

89. Arithmetical progressions. — A **progression** is a sequence of numbers formed according to some law. An **arithmetical progression** is a progression in which each term is obtained from the next preceding term by the addition of a fixed constant called the **common difference.** Thus, 3, 6, 9, 12, · · · , etc., is an arithmetical progression in which the common difference is 3. Similarly, 3, $\frac{5}{2}$, 2, $\frac{3}{2}$, · · · , etc., is an arithmetical progression in which the common difference is $(-\frac{1}{2})$.

Let a represent the first term of an arithmetical progression, d the common difference, and n the number of terms in the progression. Then, in the progression,

$$\left\{\begin{aligned} a &= \text{1st term,} \\ a + d &= \text{2d term,} \\ a + 2d &= \text{3d term,} \end{aligned}\right. \qquad \left\{\begin{array}{l} a + 3d = \text{4th term,} \\ \cdots \text{etc.} \cdots \\ a + (n-1)d = n\text{th term.} \end{array}\right. \tag{8}$$

If we let l represent the last, or the nth, term, we have proved that

$$\boldsymbol{l = a + (n - 1)\, d.} \tag{9}$$

If we start with the last term, the next to the last term is formed by subtracting d, the second from the last by subtracting $2\,d$, etc. That is, in going backward, we meet an arithmetical progression with the common difference $(-d)$. Thus,

$$\left\{\begin{aligned} l &= \text{last term,} \\ l - d &= \text{1st from last term,} \\ l - 2\,d &= \text{2d from last term,} \end{aligned}\right. \qquad \left\{\begin{array}{l} l - 3\,d = \text{3d from last term,} \\ \cdots \text{etc.} \cdots \\ a = l - (n-1)d = (n-1)\text{st from} \\ \quad \text{last.} \end{array}\right. \tag{10}$$

Let s represent the sum of the terms of the progression. Then, we obtain equation 11 below by using the terms as given in equations 8, and equation 12 by using equations 10.

$$s = a + (a + d) + (a + 2\,d) + \cdots \text{etc.} \cdots + [a + (n-1)d]. \tag{11}$$
$$s = l + (l - d) + (l - 2\,d) + \cdots \text{etc.} \cdots + [l - (n-1)d]. \tag{12}$$

On adding equations 11 and 12, we obtain

$$2s = (a + l) + (a + l) + (a + l) + \cdots \text{etc.} \cdots + (a + l). \quad (13)$$

There are n terms in equation 13, one corresponding to each term of the progression. Hence, $2s = n(a + l)$, or

$$s = \frac{n}{2}(a + l). \quad (14)$$

If any three of the quantities (a, d, n, l, s) are given, the equations 9 and 14 enable us to find the other two. We call (a, d, n, l, s) the **elements** of the progression.

Example 1. — In an arithmetical progression with the first term 3, the 6th term is 28. Find the common difference and the intermediate terms.

Solution. — We have $a = 3$, $n = 6$, and $l = 28$. Hence, from equation 9, $28 = 3 + 5d$; $5d = 25$; $d = 5$. The terms of the progression are 3, 8, 13, 18, 23, 28.

EXERCISE LXXXV

1. Find the last term and the sum of the progression
3, 5, 7, 9, ... to twelve terms.

2. Find the sum of the progression 5, 4, 3, 2, ..., to eighteen terms.

3. Find the last term and the sum of the progression
1000(.05), 950(.05), 900(.05), ... etc., to twenty terms.

4. If 10 is the first term and 33 is the 20th term of an arithmetical progression, find the common difference and the sum of the progression.

5. If 15 is the 4th term and 32 is the 10th term of an arithmetical progression, find the intermediate terms.

90. Geometrical progressions. — A **geometrical progression** is a progression in which each term is formed by multiplying the preceding term by a fixed constant r. The number r is called the **common ratio** of the progression because the ratio of any term to the preceding term is equal to r. Thus, 4, 12, 36, 108, $\cdots$ etc., is a geometrical progression with the common ratio $r = 3$. The sequence

$$(1.05),\ (1.05)^2,\ (1.05)^3,\ (1.05)^4,\ \cdots \text{etc.},$$

is a geometrical progression with the ratio $r = (1.05)$.

Let a represent the first term, r the common ratio, and n the number of terms in a geometrical progression. Then,

$$\begin{cases} a = \text{1st term}, \\ ar = \text{2d term}, \\ ar^2 = \text{3d term}, \\ ar^3 = \text{4th term}, \end{cases} \qquad \begin{cases} ar^5 = \text{5th term}, \\ \ldots \text{etc.}, \\ ar^{n-2} = (n-1)\text{st term}, \\ ar^{n-1} = n\text{th term}. \end{cases}$$

If we let l represent the last, or nth, term, we have proved that

$$l = ar^{n-1}. \tag{15}$$

Let s represent the sum of the terms of the progression. Then

$$s = a + ar + ar^2 + \cdots \text{etc.} \cdots + ar^{n-2} + ar^{n-1}, \tag{16}$$

$$rs = ar + ar^2 + ar^3 + \cdots \text{etc.} \cdots + ar^{n-1} + ar^n. \tag{17}$$

On subtracting equation 17 from equation 16, all terms will cancel except a from equation 16 and $-ar^n$ from equation 17. Thus,

$$s - rs = s(1 - r) = a - ar^n.$$

Hence,

$$s = \frac{a - ar^n}{1 - r} = \frac{a(1 - r^n)}{1 - r}. \tag{18}$$

Since $l = ar^{n-1}$, then $rl = ar^n$; on substituting this in the first fraction of equation 18,

$$s = \frac{rl - a}{r - 1}. \tag{19}$$

Example 1. — Find the sum of $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ etc.... to six terms.

Solution. — Use formula 18 with $a = 1$, $r = \frac{1}{3}$, and $n = 6$.

$$s = \frac{[1 - (\frac{1}{3})^6]}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{729}}{\frac{2}{3}} = \frac{364}{243}.$$

Example 2. — Find an expression for the sum of

$$1 + (1.05)^{.5} + (1.05) + (1.05)^{1.5} + \ldots \text{etc.} \ldots + (1.05)^{23.5}.$$

Solution. — The terms form a geometrical progression for which $a = 1$, $r = (1.05)^{.5}$, and $l = (1.05)^{23.5}$. From formula 19,

$$s = \frac{(1.05)^{.5}(1.05)^{23.5} - 1}{(1.05)^{.5} - 1} = \frac{(1.05)^{24} - 1}{(1.05)^{.5} - 1}.$$

EXERCISE LXXXVI

1. Find the last term and the sum of 25, 5, 1, $\frac{1}{5}$, $\frac{1}{25}$, ... etc. to seven terms.

2. Find the last term and the sum of 2, 4, 8, ... etc. to eighteen terms.

3. Find the ratio, the number of terms, and the sum for the progression 3, 9, 27, ... etc. ... to 729.

4. Find the sum of $2, -1, \frac{1}{2}, \cdots$ etc. to eight terms.

5. Find the sum of $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{128}$.

Find expressions for the following sums:

6. $(1.05) + (1.05)^2 + (1.05)^3 + \cdots \text{etc.} \cdots + (1.05)^{28}$.

7. $(1.04)^2 + (1.04)^4 + (1.04)^6 + \cdots \text{etc.} \cdots + (1.04)^{38}$.

8. $(1.06)^{-25} + (1.06)^{-24} + (1.06)^{-23} + \cdots \text{etc.} \cdots + (1.06)^{-1}$.

9. $(1.03)^{-1} + (1.03)^{-2} + (1.03)^{-3} + \cdots \text{etc.} \cdots + (1.03)^{-17}$.

10. $(1.02) + (1.02)^2 + (1.02)^3 + \cdots \text{etc.} \cdots + (1.02)^{60}$.

91. Infinite geometrical progressions. — Consider the following hypothetical example. A certain jar contains two quarts of water. One quart is poured out; then, $\frac{1}{2}$ of the remainder, or $\frac{1}{2}$ quart, is poured out; then, $\frac{1}{2}$ of the remainder, or $\frac{1}{4}$ quart, is poured out, etc., without ceasing. The amounts poured out are

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \cdots \text{etc.} \cdots \textit{to infinitely many terms.}$$

The sum of the amounts poured out up to and including the nth pouring is

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \text{etc.} \cdots + \frac{1}{2^{n-1}}.$$

Since the amount originally in the jar was 2 quarts, s_n can never exceed 2. Also, it is clear intuitionally that, as n increases without bound, s_n must approach the value 2 because the amount of water left in the jar approaches 0 as the process continues. We can prove this fact mathematically; from formula 19,

$$s_n = \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} = \left(\frac{1}{2^n} - 1\right)(-2) = 2 - \frac{1}{2^{n-1}}. \tag{20}$$

As n grows large without bound, $\frac{1}{2^{n-1}}$ continually decreases and approaches zero. Thus, from equation 20 we prove that, as n increases without bound, s_n approaches the limit 2, as was seen intuitionally above. Hence, we may agree, by definition, to call this value 2 *the sum of the infinite geometrical progression*, or to say

$$2 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots \text{etc.} \cdots \textit{to infinitely many terms.}$$

This example shows that a sensible definition, in accordance with our intuitions, may be given for the sum of an infinite geometrical progression.

In general, consider any infinite geometrical progression for which the ratio r is numerically less than 1, that is, for which r lies between -1 and $+1$. The terms of the progression are

$$a, ar, ar^2, ar^3, \cdots \text{etc.} \cdots \textit{to infinitely many terms.}$$

Let s_n represent the sum of the first n terms of the progression:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

The statement *as n approaches infinity* will be used as an abbreviation for the statement *as n increases without bound.*

The sum S of an infinite geometrical progression is defined as the limiting value, if any exists, approached by s_n as n approaches infinity.

From formula 18,

$$s_n = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \tag{21}$$

As n approaches infinity, it is evident that r^n approaches zero [1] because r is numerically less than 1. Hence, from equation 21 it is seen that, as n approaches infinity, s_n approaches $\frac{a}{1-r}$ as a limiting value, because the other term in equation 21 approaches zero. Since, by definition, this limiting value of s_n is the value we assign to the sum

$$S = a + ar + ar^2 + \cdots \text{etc.} \cdots \textit{to infinitely many terms,}$$

we have proved that

$$S = \frac{a}{1 - r}. \tag{22}$$

Example 1. — Find the sum of the progression

$$(1.04)^{-2} + (1.04)^{-4} + (1.04)^{-6} + \cdots \text{etc.} \cdots \textit{to infinitely many terms.}$$

Solution. — The ratio of the infinite geometrical progression is $r = (1.04)^{-2}$; $a = (1.04)^{-2}$. From formula 22, the sum is

$$S = \frac{(1.04)^{-2}}{1 - (1.04)^{-2}}.$$

Example 2. — Express the infinite repeating decimal .08333 ⋯ as a fraction.

Solution. — We verify that .08333 ⋯ equals .08 plus

$$.003 + .0003 + .00003 + \cdots \text{etc.} \cdots \textit{to infinitely many terms.}$$

[1] For a rigorous proof of this intuitional fact the student is referred to the theory of limits as presented, for example, in books on the Calculus.

These terms form an infinite geometrical progression with $a = .003$, and $r = .1$. Their sum is $\frac{.003}{1 - .1} = \frac{.003}{.9}$. Hence,

$$.08333 \cdots = .08 + \frac{.003}{.9} = \frac{8}{100} + \frac{3}{900} = \frac{25}{300} = \frac{1}{12}.$$

NOTE. — By the method of Example 2 above, any infinite repeating decimal can be shown to represent a fraction whose numerator and denominator are integers.

EXERCISE LXXXVII

Find the sums of the following progressions:

1. $2 + 1 + \frac{1}{2} + \cdots$ to infinitely many terms.

2. $5 + 1 + \frac{1}{5} + \frac{1}{25} + \cdots$ to infinitely many terms.

3. $\frac{1}{(1.05)} + \frac{1}{(1.05)^2} + \frac{1}{(1.05)^3} + \cdots$ to infinitely many terms.

4. $(1.04)^{-1} + (1.04)^{-2} + (1.04)^{-3} + \cdots$ to infinitely many terms.

5. $(1.03)^{-2} + (1.03)^{-4} + (1.03)^{-6} + \cdots$ to infinitely many terms.

6. $(1.01)^{-1} + (1.01)^{-2} + (1.01)^{-3} + \cdots$ to infinitely many terms.

Express the following infinite decimals as fractions:

7. $.333333 \cdots$.
8. $.66666 \cdots$.

9. $.11111 \cdots$.
10. $.41111 \cdots$.

11. $.5636363 \cdots$.
12. $.24222222 \cdots$.

APPENDIX

Note 1

Proof of Rule 1, Section 15, Part I. — Consider the equation

$$2 = (1 + r)^n.$$

The solution of this equation for n is the time required for money to double itself if r is the rate per period. On taking the logarithms, with respect to the base $e = 2.71828 \ldots$, of both sides of the equation, we obtain

$$n = \frac{\log 2}{\log (1 + r)},$$

where "log" means "$\log_e$." From textbooks on the Calculus, we find that $\log (1 + r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \cdots = r(1 - \frac{r}{2} + \frac{r^2}{3} - \cdots)$,

and from a table of natural logarithms we obtain $\log 2 = .693$. Hence

$$n = \frac{.693}{r(1 - \frac{r}{2} + \frac{r^2}{3} - \cdots)} = \frac{.693}{r}(1 + \frac{r}{2} - \frac{r^2}{12} + \cdots).$$

On neglecting the powers of r in the parenthesis from r^2 on, we obtain as an approximate solution

$$n = \frac{.693}{r} + \frac{.693}{2} = \frac{.693}{r} + .35.$$

Note 2

Proof of Rule 1, Section 17, Part I. — Consider three obligations whose maturity values are S_1, S_2, and S_3, which are due, respectively, at the ends of n_1, n_2, and n_3 years. We shall prove Rule 1 for this special case. The reasoning and the details of proof are the same for the case of any number of obligations. Let i be the effective rate of interest, and let n be the equated time. By definition, n satisfies the equation

$$(1 + i)^{-n}(S_1 + S_2 + S_3) = S_1(1 + i)^{-n_1} + S_2(1 + i)^{-n_2} + S_3(1 + i)^{-n_3}. \quad (1)$$

By use of the binomial theorem, we obtain the following infinite series:

$$(1+i)^{-n} = 1 - ni + \frac{n(1+n)}{2}i^2 - \cdots;$$

$$(1+i)^{-n_1} = 1 - n_1 i + \frac{n_1(1+n_1)}{2}i^2 - \cdots;$$

$$(1+i)^{-n_2} = 1 - n_2 i + \frac{n_2(1+n_2)}{2}i^2 - \cdots;$$

$$(1+i)^{-n_3} = 1 - n_3 i + \frac{n_3(1+n_3)}{2}i^2 - \cdots.$$

Since i is small, we make only a slight error if we use only the first two terms of each infinite series as an approximate value for the corresponding power of $(1 + i)$. On using these approximations in equation 1, we obtain

$$(1 - ni)(S_1 + S_2 + S_3) = S_1(1 - n_1 i) + S_2(1 - n_2 i) + S_3(1 - n_3 i).$$

On expanding both sides and on solving for n, we obtain

$$n = \frac{n_1 S_1 + n_2 S_2 + n_3 S_3}{n_1 + n_2 + n_3},$$

which establishes Rule 1 for the present case.

Note 3

Solution of equations by interpolation. — The method of interpolation which the student has used in connection with logarithm and compound interest tables can be used in solving equations whose solution would otherwise present very great difficulties.

Example 1. — Solve for n in the equation

$$5(1.06)^n = 7.5 + 7.5(.06)n. \tag{1}$$

Solution. — On rewriting the equation and on using the abbreviation $F(n)$ for the left member, we obtain

$$F(n) = 5(1.06)^n - 7.5 - .45\,n = 0.$$

We desire a value $n = k$ such that $F(k) = 0$. If we find a value $n = n_1$ such that $F(n_1)$ is negative, and another value $n = n_2$ such that $F(n_2)$ is positive, then it will follow that there is a value $n = k$, between n_1 and n_2, such that $F(k) = 0$. That is, there must be a solution $n = k$ between n_1 and n_2. From a rough inspection of Table V we guess that the solution is greater than $n = 19$. With the aid of Table V we compute $F(n)$ for $n = 19$, 20, and 21. $F(19) = -.922$; $F(20) = -.464$; $F(21) = +.048$. Hence, there is a solution $n = k$ of the equation between $n = 20$ and $n = 21$. We find k by interpolation in the table below where we use the fact that $F(k) = 0$. The total difference in the tabular entries is $.048 - (-.464) = .512$. The partial difference is $0 - (-.464) = .464$. Hence, since 0 is $\frac{464}{512} = .91$ of the way from $-.464$ to $+.048$, we assume that the solution k is .91 of the way from 20 to 21, or that $k = 20 + .91 = 20.91$. Of course, this is only an approximate solution of the equation, but such a one is extremely useful in practical applications. An inspection of the equation shows that there cannot be any other solution because $5(1.06)^n$ increases much more rapidly than $.45\,n$, and hence $F(n)$ will be positive for all values of n greater than 21.

n	$F(n)$
20	$-.464$
$n = k$	0
21	.048

Example 2. — A man invests \$6000 in the stock of a corporation. He receives a \$400 dividend at the end of each year for 10 years. At the end of that time he sells his holdings for \$5000. Considering the whole 10-year period, at what effective rate may the man consider his investment to have been made?

Solution. — Let r be the effective rate. With the end of 10 years as a comparison date, we write the following equation of value:

$$6000(1 + r)^{10} = 5000 + 400(s_{\overline{10|}}\ at\ r),$$

$$F(r) = 6000(1 + r)^{10} - 5000 - 400(s_{\overline{10|}}\ at\ r) = 0. \qquad (1)$$

We shall solve equation 1 by interpolation.

If the \$1000 loss in capital had been uniformly distributed over the 10 years, the loss per year would have been \$100. Hence, under this false (but approximately true) condition, the net annual income would have been \$300. The average invested capital would have been $\frac{1}{2}(6000 + 5000) = \$5500$. Hence, since $\frac{300}{5500} = .055$, we guess[1] .055 as an approximation to the solution of the equation. When $r = .055$, $F(.055) = +98.73$. Since this is positive, the solution must be less than .055. We find $F(.05) = -257.80$. Hence, the solution $r = k$ of equation 1, for which $F(k) = 0$ is between $r = .055$ and $r = .05$. We interpolate in the table below. $98.7 - (-257.8) = 356.5$; $0 - (-257.8) = 257.8$; $.055 - .05 = .005$. Hence,

$$k = .05 + \frac{257.8}{356.5}(.005) = .05 + .0036,$$ or, approximately, $k = .054$. The solution could be obtained accurately to hundredths (or to thousandths, or less) of 1%, if desired, by the method used in Example 2, Section 32, Part I.

r	$F(r)$
.05	-257.8
$r = k$	0
.055	98.7

Note 4

Abridged multiplication. — Consider forming the product $(11.132157) \times (893.214)$. We decide in advance that we desire the result accurately

[1] Notice the similarity between this reasoning and that employed in Section 55 of Part I.

to the nearest digit in the second decimal place. The ordinary multiplication would proceed as at the left below, while the abridged method proceeds as at the right.

Ordinary Method	Abridged Method	Multiply by
11.132157	××××× 11.132157	
893.214	893.214	
44528628	8905.7256	800
11132157	1001.8935	90
22264314	33.3963	3
33396471	2.2264	.2
100189413	.1113	.01
89057256	.0444	.004
9943.398481598	9943.3975	Add
Result = 9943.40	*Result* = 9943.40	

In multiplying by the abridged method we proceed as follows:

Since we desire the result to be accurate in the 2d decimal place, we carry two extra places, or four decimal places, for safety. To multiply by 893.214 we multiply in succession by 800, 90, 3, .2, .01, and .004 and then add the results (this is the same as is done in the ordinary method of multiplication, except that the multiplications are performed in the reverse order). We first multiply by 800, that is, we multiply by 8 and then move the decimal point. All digits of 11.132157 are used in this operation in order to obtain four significant decimal places in the result. To obtain four decimal places when multiplying by 90 we need one less digit of 11.132157; we put × over the "7" to indicate that we multiply 11.13215 at this time. We put × over the "5" and then multiply 11.1321 by 3; we put × over the last "1" and then multiply 11.132 by .2; etc. The advantages of this method are obvious. Less labor is involved, the decimal point is accurately located, and fewer mistakes will occur in the final addition.

Note 5

Accuracy of the interpolation method in solving for the time in the compound interest equation. — Consider the equation

$$A = (1 + r)^n, \qquad (1)$$

where A and r are known. To determine the value of n by interpolation, we first find from our interest table (Table V if $A > 1$, Table VI if $A < 1$) two integers n_1 and n_2, $n_2 - n_1 = 1$, such that the corresponding values $A_1 = (1 + r)^{n_1}$ and $A_2 = (1 + r)^{n_2}$ include A between them. That is, $A_1 < A < A_2$. Then, as obtained by interpolation, the solution of equation 1 is

$$N = n_1 + \frac{A - A_1}{A_2 - A_1}.$$

The exact solution of equation 1 is obtained by taking the logarithm of both sides; $\log A = n \log (1 + r)$, where *log* means log_e.

$$n = \frac{\log A}{\log (1 + r)}. \qquad (2)$$

From equation 2 we obtain

$$\frac{dn}{dA} = \frac{1}{A \log (1 + r)}; \quad \frac{d^2n}{dA^2} = \frac{-1}{A^2 \log (1 + r)}. \qquad (3)$$

Hence since dn/dA is positive, n is an increasing function of A. Moreover, since d^2n/dA^2 is negative, the graph of n as a function of A, with the A-axis horizontal, will be concave downward, as in Figure 6, distorted for illustration. It is seen graphically that the difference between n, as given in equation 2, and N is given by the line EF in the figure. This error is less than DH, where H is the point in which the tangent (drawn at P) intersects the ordinate at A_2. Since $CD = 1$,

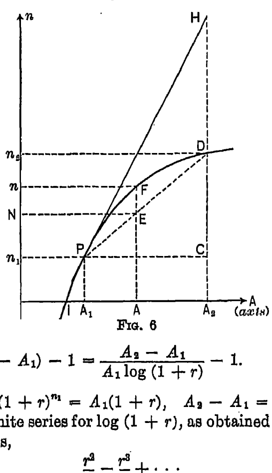

FIG. 6

$$DH = CH - CD = \frac{dn}{dA_1}(A_2 - A_1) - 1 = \frac{A_2 - A_1}{A_1 \log (1 + r)} - 1.$$

Since $A_2 = (1 + r)^{n_2} = (1 + r)(1 + r)^{n_1} = A_1(1 + r)$, $A_2 - A_1 = rA_1$. Hence, on inserting the infinite series for $\log (1 + r)$, as obtained from any textbook on the Calculus,

$$DH = \frac{r}{r - \frac{r^2}{2} + \frac{r^3}{3} - \cdots} - 1 = \frac{\frac{r^2}{2} - \frac{r^3}{3} + \cdots}{r - \frac{r^2}{2} + \frac{r^3}{3} - \cdots}.$$

$$DH \leqq \frac{r}{2}\frac{1}{1-\frac{r}{2}}.$$

If $r \leqq .10$, as is the case in the tables of this book, $DH \leqq \frac{r}{2}\left(\frac{1}{.95}\right)$, which is approximately $\frac{1}{2}r$, if computed to only two decimal places. Hence, if we are computing results to only two decimal places, a solution of equation 1, obtained by interpolation, is in error by at most $\frac{1}{2}r$.

Note 6

Accuracy of the interpolation method in solving for the time in the annuity equations. — Consider the equation

$$(s_{\overline{n}|} \text{ at } r) = S, \tag{1}$$

where S and r are known. From equation 1, on inserting the explicit algebraic expression for $(s_{\overline{n}|}$ at $r)$, we obtain

$$\frac{(1+r)^n - 1}{r} = S; \ (1+r)^n = Sr + 1.$$

If we solve equation 1 for n by interpolation in Table VII, our solution is the same as we should obtain in solving the equivalent equation

$$(1+r)^n = A \tag{2}$$

(where $A = Sr + 1$) for n, by interpolation in Table V. For, the solution of equation 1 by interpolation would be

$$n = n_1 + \frac{S - S_1}{S_2 - S_1},$$

while that for equation 2 would be

$$n = n_1 + \frac{A - A_1}{A_2 - A_1} = n_1 + \frac{Sr + 1 - S_1r - 1}{S_2r + 1 - S_1r - 1},$$

$$n = n_1 + \frac{S - S_1}{S_2 - S_1},$$

which is the same as the result for equation 1. Hence, it follows from Note 5 of the Appendix that the error in the solution of equation 1 obtained by interpolation in Table VII is at most $\frac{1}{2}$ of the interest rate r. Similarly, it follows that, if we should solve for n in the equation $(a_{\overline{n}|} \text{ at } r) = A$, by interpolating in Table VIII, the error of the result would be at most $\frac{1}{2}r$.

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Numbers refer to pages

TABLE I

COMMON LOGARITHMS OF NUMBERS

TO FIVE DECIMAL PLACES

Pages 2 to 19

TABLE II

COMMON LOGARITHMS OF NUMBERS

FROM 1.00000 *to* 1.10000

TO SEVEN DECIMAL PLACES

Pages 20 to 21

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	340	380
01	432	475	518	561	604	647	689	732	775	817
02	860	903	945	988	*030	*072	*115	*157	*190	*242
03	01 284	326	368	410	452	494	536	578	620	662
04	703	745	787	828	870	912	953	905	*036	*078
05	02 119	160	202	243	284	325	366	407	440	400
06	531	572	612	653	694	735	776	816	857	898
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302
08	03 342	883	423	463	503	543	583	623	663	703
09	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
11	532	571	610	650	689	727	766	805	844	883
12	922	961	999	*038	*077	*115	*154	*192	*231	*269
13	05 308	346	385	423	461	500	538	576	614	652
14	690	729	767	805	843	881	918	956	994	*032
15	06 070	108	145	183	221	258	296	333	371	408
16	446	483	521	558	595	633	670	707	744	781
17	819	856	893	930	967	*004	*041	*078	*115	*151
18	07 188	225	262	298	335	372	408	445	482	518
19	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
21	08 279	314	350	386	422	458	493	529	565	600
22	636	672	707	743	778	814	849	884	920	955
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
24	09 342	377	412	447	482	517	552	587	621	656
25	691	726	760	795	830	864	899	934	968	*003
26	10 037	072	106	140	175	209	243	278	312	346
27	380	415	449	483	517	551	585	619	653	687
28	721	755	789	823	857	890	924	958	992	*025
29	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
31	727	760	793	826	860	893	926	959	992	*024
32	12 057	090	123	156	189	222	254	287	320	352
33	385	418	450	483	516	548	581	613	646	678
34	710	743	775	808	840	872	905	937	969	*001
35	13 033	066	098	130	162	194	226	258	290	322
36	354	386	418	450	481	513	545	577	609	640
37	672	704	735	767	799	830	862	893	925	956
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
39	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
41	922	953	983	*014	*045	*076	*106	*137	*168	*198
42	15 229	259	290	320	351	381	412	442	473	503
43	534	564	594	625	655	685	715	746	776	806
44	836	866	897	927	957	987	*017	*047	*077	*107
45	16 137	167	197	227	256	286	316	346	376	406
46	435	465	495	524	554	584	613	643	673	702
47	732	761	791	820	850	879	909	938	967	997
48	17 026	056	085	114	143	173	202	231	260	289
49	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	0	1	2	3	4	5	6	7	8	9

PP

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

PP

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	0	1	2	3	4	5	6	7	8	9

P P

	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	**23**
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	**21**
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

P P

N	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
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PP

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9	19.8	18.9

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PP

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250	39 794	811	829	846	863	881	898	915	933	950
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123
52	40 140	157	175	192	209	226	243	261	278	295
53	312	329	346	364	381	398	415	432	449	466
54	483	500	518	535	552	569	586	603	620	637
55	654	671	688	705	722	739	756	773	790	807
56	824	841	858	875	892	909	926	943	960	976
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
58	41 162	179	196	212	229	246	263	280	296	313
59	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
61	664	681	697	714	731	747	764	780	797	814
62	830	847	863	880	896	913	929	946	963	979
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
64	42 160	177	193	210	226	243	259	275	292	308
65	325	341	357	374	390	406	423	439	455	472
66	488	504	521	537	553	570	586	602	619	635
67	651	667	684	700	716	732	749	765	781	797
68	813	830	846	862	878	894	911	927	943	959
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
71	297	313	329	345	361	377	393	409	425	441
72	457	473	489	505	521	537	553	569	584	600
73	616	632	648	664	680	696	712	727	743	759
74	775	791	807	823	838	854	870	886	902	917
75	933	949	965	981	996	*012	*028	*044	*059	*075
76	44 091	107	122	138	154	170	185	201	217	232
77	248	264	279	295	311	326	342	358	373	389
78	404	420	436	451	467	483	498	514	529	545
79	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
81	871	886	902	917	932	948	963	979	994	*010
82	45 025	040	056	071	086	102	117	133	148	163
83	179	194	209	225	240	255	271	286	301	317
84	332	347	362	378	393	408	423	439	454	469
85	484	500	515	530	545	561	576	591	606	621
86	637	652	667	682	697	712	728	743	758	773
87	788	803	818	834	849	864	879	894	909	924
88	939	954	969	984	*000	*015	*030	*045	*060	*075
89	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
91	389	404	419	434	449	464	479	494	509	523
92	538	553	568	583	598	613	627	642	657	672
93	687	702	716	731	746	761	776	790	805	820
94	835	850	864	879	894	909	923	938	953	967
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114
96	47 129	144	159	173	188	202	217	232	246	261
97	276	290	305	319	334	349	363	378	392	407
98	422	436	451	465	480	494	509	524	538	553
99	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
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P P

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6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

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300	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	615	629	643	657	671	686	700
07	714	728	742	756	770	785	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	845	859	872	886	900	914	927	941	955
16	969	982	996	*010	*024	*037	*051	*065	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	325	338	352	365
19	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
21	651	664	678	691	705	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 055	068	081	095	108	121	135	148	162	175
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	415	428	441
27	455	468	481	495	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*075	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	375	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	815	827	840	853	866	879
38	892	905	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	555	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	845	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
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350	54 407	419	432	444	456	469	481	494	506	518
51	531	543	555	568	580	593	605	617	630	642
52	654	667	679	691	704	716	728	741	753	765
53	777	790	802	814	827	839	851	864	876	888
54	900	913	925	937	949	962	974	986	998	*011
55	55 023	035	047	060	072	084	096	108	121	133
56	145	157	169	182	194	206	218	230	242	255
57	267	279	291	303	315	328	340	352	364	376
58	388	400	413	425	437	449	461	473	485	497
59	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
61	751	763	775	787	799	811	823	835	847	859
62	871	883	895	907	919	931	943	955	967	979
63	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
64	56 110	122	134	146	158	170	182	194	205	217
65	229	241	253	265	277	289	301	312	324	336
66	348	360	372	384	396	407	419	431	443	455
67	467	478	490	502	514	526	538	549	561	573
68	585	597	608	620	632	644	656	667	679	691
69	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
71	937	949	961	972	984	996	*008	*019	*031	*043
72	57 054	066	078	089	101	113	124	136	148	159
73	171	183	194	206	217	229	241	252	264	276
74	287	299	310	322	334	345	357	368	380	392
75	403	415	426	438	449	461	473	484	496	507
76	519	530	542	553	565	576	588	600	611	623
77	634	646	657	669	680	692	703	715	726	738
78	749	761	772	784	795	807	818	830	841	852
79	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
81	58 092	104	115	127	138	149	161	172	184	195
82	206	218	229	240	252	263	274	286	297	309
83	320	331	343	354	365	377	388	399	410	422
84	433	444	456	467	478	490	501	512	524	535
85	546	557	569	580	591	602	614	625	636	647
86	659	670	681	692	704	715	726	737	749	760
87	771	782	794	805	816	827	838	850	861	872
88	883	894	906	917	928	939	950	961	973	984
89	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
91	218	229	240	251	262	273	284	295	306	318
92	329	340	351	362	373	384	395	406	417	428
93	439	450	461	472	483	494	506	517	528	539
94	550	561	572	583	594	605	616	627	638	649
95	660	671	682	693	704	715	726	737	748	759
96	770	780	791	802	813	824	835	846	857	868
97	879	890	901	912	923	934	945	956	966	977
98	988	999	*010	*021	*032	*043	*054	*065	*076	*086
99	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
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P P

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4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

P P

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400	60 206	217	228	239	249	260	271	282	293	304	
01	314	325	336	347	358	369	379	390	401	412	
02	423	433	444	455	466	477	487	498	509	520	
03	531	541	552	563	574	584	595	606	617	627	
04	638	649	660	670	681	692	703	713	724	735	
05	746	756	767	778	788	799	810	821	831	842	
06	853	863	874	885	895	906	917	927	938	949	
07	959	970	981	991	*002	*013	*023	*034	*045	*055	
08	61 066	077	087	098	109	119	130	140	151	162	
09	172	183	194	204	215	225	236	247	257	268	
410	278	289	300	310	321	331	342	352	363	374	
11	384	395	405	416	426	437	448	458	469	479	
12	490	500	511	521	532	542	553	563	574	584	
13	595	606	616	627	637	648	658	669	679	690	
14	700	711	721	731	742	752	763	773	784	794	
15	805	815	826	836	847	857	868	878	888	899	
16	909	920	930	941	951	962	972	982	993	*003	
17	62 014	024	034	045	055	066	076	086	097	107	
18	118	128	138	149	159	170	180	190	201	211	
19	221	232	242	252	263	273	284	294	304	315	
420	325	335	346	356	366	377	387	397	408	418	
21	428	439	449	459	469	480	490	500	511	521	
22	531	542	552	562	572	583	593	603	613	624	
23	634	644	655	665	675	685	696	706	716	726	
24	737	747	757	767	778	788	798	808	818	829	
25	839	849	859	870	880	890	900	910	921	931	
26	941	951	961	972	982	992	*002	*012	*022	*033	
27	63 043	053	063	073	083	094	104	114	124	134	
28	144	155	165	175	185	195	205	215	225	236	
29	246	256	266	276	286	296	306	317	327	337	
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31	448	458	468	478	488	498	508	518	528	538	
32	548	558	568	579	589	599	609	619	629	639	
33	649	659	669	679	689	699	709	719	729	739	
34	749	759	769	779	789	799	809	819	829	839	
35	849	859	869	879	889	899	909	919	929	939	
36	949	959	969	979	988	998	*008	*018	*028	*038	
37	64 048	058	068	078	088	098	108	118	128	137	
38	147	157	167	177	187	197	207	217	227	237	
39	246	256	266	276	286	296	306	316	326	335	
440	345	355	365	375	385	395	404	414	424	434	
41	444	454	464	473	483	493	503	513	523	532	
42	542	552	562	572	582	591	601	611	621	631	
43	640	650	660	670	680	689	699	709	719	729	
44	738	748	758	768	777	787	797	807	816	826	
45	836	846	856	865	875	885	895	904	914	924	
46	933	943	953	963	972	982	992	*002	*011	*021	
47	65 031	040	050	060	070	079	089	099	108	118	
48	128	137	147	157	167	176	186	196	205	215	
49	225	234	244	254	263	273	283	292	302	312	
450	321	331	341	350	360	369	379	389	398	408	
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P P

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450	65 321	331	341	350	360	369	379	389	398	408
51	418	427	437	447	456	466	475	485	495	504
52	514	523	533	543	552	562	571	581	591	600
53	610	619	629	639	648	658	667	677	686	696
54	706	715	725	734	744	753	763	772	782	792
55	801	811	820	830	839	849	858	868	877	887
56	896	906	916	925	935	944	954	963	973	982
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
58	66 087	096	106	115	124	134	143	153	162	172
59	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
61	370	380	389	398	408	417	427	436	445	455
62	464	474	483	492	502	511	521	530	539	549
63	558	567	577	586	596	605	614	624	633	642
64	652	661	671	680	689	699	708	717	727	736
65	745	755	764	773	783	792	801	811	820	829
66	839	848	857	867	876	885	894	904	913	922
67	932	941	950	960	969	978	987	997	*006	*015
68	67 025	034	043	052	062	071	080	089	099	108
69	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
71	302	311	321	330	339	348	357	367	376	385
72	394	403	413	422	431	440	449	459	468	477
73	486	495	504	514	523	532	541	550	560	569
74	578	587	596	605	614	624	633	642	651	660
75	669	679	688	697	706	715	724	733	742	752
76	761	770	779	788	797	806	815	825	834	843
77	852	861	870	879	888	897	906	916	925	934
78	943	952	961	970	979	988	997	*006	*015	*024
79	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
81	215	224	233	242	251	260	269	278	287	296
82	305	314	323	332	341	350	359	368	377	386
83	395	404	413	422	431	440	449	458	467	476
84	485	494	502	511	520	529	538	547	556	565
85	574	583	592	601	610	619	628	637	646	655
86	664	673	681	690	699	708	717	726	735	744
87	753	762	771	780	789	797	806	815	824	833
88	842	851	860	869	878	886	895	904	913	922
89	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
91	108	117	126	135	144	152	161	170	179	188
92	197	205	214	223	232	241	249	258	267	276
93	285	294	302	311	320	329	338	346	355	364
94	373	381	390	399	408	417	425	434	443	452
95	461	469	478	487	496	504	513	522	531	539
96	548	557	566	574	583	592	601	609	618	627
97	636	644	653	662	671	679	688	697	705	714
98	723	732	740	749	758	767	775	784	793	801
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06	415	424	432	441	449	458	467	475	484	492
07	501	509	518	526	535	544	552	561	569	578
08	586	595	603	612	621	629	638	646	655	663
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12	927	935	944	952	961	969	978	986	995	*003
13	71 012	020	029	037	046	054	063	071	079	088
14	096	105	113	122	130	139	147	155	164	172
15	181	189	198	206	214	223	231	240	248	257
16	265	273	282	290	299	307	315	324	332	341
17	349	357	366	374	383	391	399	408	416	425
18	433	441	450	458	466	475	483	492	500	508
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21	684	692	700	709	717	725	734	742	750	759
22	767	775	784	792	800	809	817	825	834	842
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24	933	941	950	958	966	975	983	991	999	*008
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36	916	925	933	941	949	957	965	973	981	989
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
38	73 078	086	094	102	111	119	127	135	143	151
39	159	167	175	183	191	199	207	215	223	231
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41	320	328	336	344	352	360	368	376	384	392
42	400	408	416	424	432	440	448	456	464	472
43	480	488	496	504	512	520	528	536	544	552
44	560	568	576	584	592	600	608	616	624	632
45	640	648	656	664	672	679	687	695	703	711
46	719	727	735	743	751	759	767	775	783	791
47	799	807	815	823	830	838	846	854	862	870
48	878	886	894	902	910	918	926	933	941	949
49	957	965	973	981	989	997	*005	*013	*020	*028
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62	974	981	989	997	*005	*012	*020	*028	*035	*043
63	75 051	059	066	074	082	089	097	105	113	120
64	128	136	143	151	159	166	174	182	189	197
65	205	213	220	228	236	243	251	259	266	274
66	282	289	297	305	312	320	328	335	343	351
67	358	366	374	381	389	397	404	412	420	427
68	435	442	450	458	465	473	481	488	496	504
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72	740	747	755	762	770	778	785	793	800	808
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76	76 042	050	057	065	072	080	087	095	103	110
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84	641	649	656	664	671	678	686	693	701	708
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86	790	797	805	812	819	827	834	842	849	856
87	864	871	879	886	893	901	908	916	923	930
88	938	945	953	960	967	975	982	989	997	*004
89	77 012	019	026	034	041	048	056	063	070	078
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91	159	166	173	181	188	195	203	210	217	225
92	232	240	247	254	262	269	276	283	291	298
93	305	313	320	327	335	342	349	357	364	371
94	379	386	393	401	408	415	422	430	437	444
95	452	459	466	474	481	488	495	503	510	517
96	525	532	539	546	554	561	568	576	583	590
97	597	605	612	619	627	634	641	648	656	663
98	670	677	685	692	699	706	714	721	728	735
99	743	750	757	764	772	779	786	793	801	808
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17	79 029	036	043	050	057	064	071	078	085	092
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21	309	316	323	330	337	344	351	358	365	372
22	379	386	393	400	407	414	421	428	435	442
23	449	456	463	470	477	484	491	498	505	511
24	518	525	532	539	546	553	560	567	574	581
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26	657	664	671	678	685	692	699	706	713	720
27	727	734	741	748	754	761	768	775	782	789
28	796	803	810	817	824	831	837	844	851	858
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31	80 003	010	017	024	030	037	044	051	058	065
32	072	079	085	092	099	106	113	120	127	134
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36	346	353	359	366	373	380	387	393	400	407
37	414	421	428	434	441	448	455	462	468	475
38	482	489	496	502	509	516	523	530	536	543
39	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
41	686	693	699	706	713	720	726	733	740	747
42	754	760	767	774	781	787	794	801	808	814
43	821	828	835	841	848	855	862	868	875	882
44	889	895	902	909	916	922	929	936	943	949
45	956	963	969	976	983	990	996	*003	*010	*017
46	81 023	030	037	043	050	057	064	070	077	084
47	090	097	104	111	117	124	131	137	144	151
48	158	164	171	178	184	191	198	204	211	218
49	224	231	238	245	251	258	265	271	278	285
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57	757	763	770	776	783	790	796	803	809	816
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62	086	092	099	105	112	119	125	132	138	145
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66	347	354	360	367	373	380	387	393	400	406
67	413	419	426	432	439	445	452	458	465	471
68	478	484	491	497	504	510	517	523	530	536
69	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
71	672	679	685	692	698	705	711	718	724	730
72	737	743	750	756	763	769	776	782	789	795
73	802	808	814	821	827	834	840	847	853	860
74	866	872	879	885	892	898	905	911	918	924
75	930	937	943	950	956	963	969	975	982	988
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
77	83 059	065	072	078	085	091	097	104	110	117
78	123	129	136	142	149	155	161	168	174	181
79	187	193	200	206	213	219	225	232	238	245
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82	378	385	391	398	404	410	417	423	429	436
83	442	448	455	461	467	474	480	487	493	499
84	506	512	518	525	531	537	544	550	556	563
85	569	575	582	588	594	601	607	613	620	626
86	632	639	645	651	658	664	670	677	683	689
87	696	702	708	715	721	727	734	740	746	753
88	759	765	771	778	784	790	797	803	809	816
89	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
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92	84 011	017	023	029	036	042	048	055	061	067
93	073	080	086	092	098	105	111	117	123	130
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02	634	640	646	652	658	665	671	677	683	689
03	696	702	708	714	720	726	733	739	745	751
04	757	763	770	776	782	788	794	800	807	813
05	819	825	831	837	844	850	856	862	868	874
06	880	887	893	899	905	911	917	924	930	936
07	942	948	954	960	967	973	979	985	991	997
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09	065	071	077	083	089	095	101	107	114	120
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15	431	437	443	449	455	461	467	473	479	485
16	491	497	503	509	516	522	528	534	540	546
17	552	558	564	570	576	582	588	594	600	606
18	612	618	625	631	637	643	649	655	661	667
19	673	679	685	691	697	703	709	715	721	727
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21	794	800	806	812	818	824	830	836	842	848
22	854	860	866	872	878	884	890	896	902	908
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24	974	980	986	992	998	*004	*010	*016	*022	*028
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26	094	100	106	112	118	124	130	136	141	147
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31	392	398	404	410	415	421	427	433	439	445
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34	570	576	581	587	593	599	605	611	617	623
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36	688	694	700	705	711	717	723	729	735	741
37	747	753	759	764	770	776	782	788	794	800
38	806	812	817	823	829	835	841	847	853	859
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41	982	988	994	999	*005	*011	*017	*023	*029	*035
42	87 040	046	052	058	064	070	075	081	087	093
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45	216	221	227	233	239	245	251	256	262	268
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47	332	338	344	349	355	361	367	373	379	384
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52	622	628	633	630	645	651	656	662	008	574
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1025	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014 1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034	5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015 3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039	6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018 2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044	7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019 1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047	9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049	7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051	6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023 2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063	5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064	9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027 3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071	7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075	4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083	6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035 0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036 2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N	0	1	2	3	4	5	6	7	8	9

TABLE III—The Number of Each Day of the Year

Day of Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29		88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

Note. — In leap years, after February 28, add 1 to the tabulated number.

TABLE IV — Ordinary and Exact Interest at 1% on $10,000

Exact Interest for 1 to 365 Days						Ordinary Interest for 1 to 360 Days					
Days	Interest	For days below add to interest column				Days	Interest	For days below add to interest column			
		$20	$40	$60	$80			$20	$40	$60	$80
1	$ .2739726	74	147	220	293	1	$ .2777778	73	145	217	289
2	.5479452	75	148	221	294	2	.5555556	74	146	218	290
3	.8219178	76	149	222	295	3	.8333333	75	147	219	291
4	1.0958904	77	150	223	296	4	1.1111111	76	148	220	292
5	1.3698630	78	151	224	297	5	1.3888889	77	149	221	293
6	1.6438356	79	152	225	298	6	1.6666667	78	150	222	294
7	1.9178082	80	153	226	299	7	1.9444444	79	151	223	295
8	2.1917808	81	154	227	300	8	2.2222222	80	152	224	296
9	2.4657534	82	155	228	301	9	2.5000000	81	153	225	297
10	2.7397260	83	156	229	302	10	2.7777778	82	154	226	298
11	3.0136986	84	157	230	303	11	3.0555556	83	155	227	299
12	3.2876712	85	158	231	304	12	3.3333333	84	156	228	300
13	3.5616438	86	159	232	305	13	3.6111111	85	157	229	301
14	3.8356164	87	160	233	306	14	3.8888889	86	158	230	302
15	4.1095890	88	161	234	307	15	4.1666667	87	159	231	303
16	4.3835616	89	162	235	308	16	4.4444444	88	160	232	304
17	4.6575342	90	163	236	309	17	4.7222222	89	161	233	305
18	4.9315068	91	164	237	310	18	5.0000000	90	162	234	306
19	5.2054795	92	165	238	311	19	5.2777778	91	163	235	307
20	5.4794521	93	166	239	312	20	5.5555556	92	164	236	308
21	5.7534247	94	167	240	313	21	5.8333333	93	165	237	309
22	6.0273973	95	168	241	314	22	6.1111111	94	166	238	310
23	6.3013699	96	169	242	315	23	6.3888889	95	167	239	311
24	6.5753425	97	170	243	316	24	6.6666667	96	168	240	312
25	6.8493151	98	171	244	317	25	6.9444444	97	169	241	313
26	7.1232877	99	172	245	318	26	7.2222222	98	170	242	314
27	7.3972603	100	173	246	319	27	7.5000000	99	171	243	315
28	7.6712329	101	174	247	320	28	7.7777778	100	172	244	316
29	7.9452055	102	175	248	321	29	8.0555556	101	173	245	317
30	8.2191781	103	176	249	322	30	8.3333333	102	174	246	318
31	8.4931507	104	177	250	323	31	8.6111111	103	175	247	319
32	8.7671233	105	178	251	324	32	8.8888889	104	176	248	320
33	9.0410959	106	179	252	325	33	9.1666667	105	177	249	321
34	9.3150685	107	180	253	326	34	9.4444444	106	178	250	322
35	9.5890411	108	181	254	327	35	9.7222222	107	179	251	323
36	9.8630137	109	182	255	328	36	10.0000000	108	180	252	324
37	10.1369863	110	183	256	329	37	10.2777778	109	181	253	325
38	10.4109589	111	184	257	330	38	10.5555556	110	182	254	326
39	10.6849315	112	185	258	331	39	10.8333333	111	183	255	327
40	10.9589041	113	186	259	332	40	11.1111111	112	184	256	328
41	11.2328767	114	187	260	333	41	11.3888889	113	185	257	329
42	11.5068493	115	188	261	334	42	11.6666667	114	186	258	330
43	11.7808219	116	189	262	335	43	11.9444444	115	187	259	331
44	12.0547945	117	190	263	336	44	12.2222222	116	188	260	332
45	12.3287671	118	191	264	337	45	12.5000000	117	189	261	333
46	12.6027397	119	192	265	338	46	12.7777778	118	190	262	334
47	12.8767123	120	193	266	339	47	13.0555556	119	191	263	335
48	13.1506849	121	194	267	340	48	13.3333333	120	192	264	336
49	13.4246575	122	195	268	341	49	13.6111111	121	193	265	337
50	13.6986301	123	196	269	342	50	13.8888889	122	194	266	338
51	13.9726027	124	197	270	343	51	14.1666667	123	195	267	339
52	14.2465753	125	198	271	344	52	14.4444444	124	196	268	340
53	14.5205479	126	199	272	345	53	14.7222222	125	197	269	341
54	14.7945206	127	200	273	346	54	15.0000000	126	198	270	342
55	15.0684932	128	201	274	347	55	15.2777778	127	199	271	343
56	15.3424658	129	202	275	348	56	15.5555556	128	200	272	344
57	15.6164384	130	203	276	349	57	15.8333333	129	201	273	345
58	15.8904111	131	204	277	350	58	16.1111111	130	202	274	346
59	16.1643836	132	205	278	351	59	16.3888889	131	203	275	347
60	16.4383562	133	206	279	352	60	16.6666667	132	204	276	348
61	16.7123288	134	207	280	353	61	16.9444444	133	205	277	349
62	16.9863014	135	208	281	354	62	17.2222222	134	206	278	350
63	17.2602740	136	209	282	355	63	17.5000000	135	207	279	351
64	17.5342466	137	210	283	356	64	17.7777778	136	208	280	352
65	17.8082192	138	211	284	357	65	18.0555556	137	209	281	353
66	18.0821918	139	212	285	358	66	18.3333333	138	210	282	354
67	18.3561644	140	213	286	359	67	18.6111111	139	211	283	355
68	18.6301370	141	214	287	360	68	18.8888889	140	212	284	356
69	18.9041096	142	215	288	361	69	19.1666667	141	213	285	357
70	19.1780822	143	216	289	362	70	19.4444444	142	214	286	358
71	19.4520548	144	217	290	363	71	19.7222222	143	215	287	359
72	19.7260274	145	218	291	364	72	20.0000000	144	216	288	360
73	20.0000000	146	219	292	365						

TABLE V — COMPOUND AMOUNT OF 1

$(1 + i)^n$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0041 6667	1.0050 0000	1.0058 3333	1.0075 0000	1.0100 0000
2	1.0083 5069	1.0100 2500	1.0117 0069	1.0150 5625	1.0201 0000
3	1.0125 5210	1.0150 7513	1.0176 0228	1.0226 6917	1.0303 0100
4	1.0167 7112	1.0201 5050	1.0235 3830	1.0303 3919	1.0406 0401
5	1.0210 0767	1.0252 5125	1.0295 0894	1.0380 6673	1.0510 1005
6	1.0252 6187	1.0303 7751	1.0355 1440	1.0458 5224	1.0615 2015
7	1.0295 3379	1.0355 2940	1.0415 5490	1.0536 9613	1.0721 3535
8	1.0338 2352	1.0407 0704	1.0476 3064	1.0615 9885	1.0828 5671
9	1.0381 3111	1.0459 1058	1.0537 4182	1.0695 6084	1.0936 8527
10	1.0424 5666	1.0511 4013	1.0598 8805	1.0775 8255	1.1046 2213
11	1.0468 0023	1.0563 9583	1.0660 7133	1.0856 6441	1.1156 6835
12	1.0511 6190	1.0616 7781	1.0722 9008	1.0938 0690	1.1268 2503
13	1.0555 4174	1.0669 8620	1.0785 4511	1.1020 1045	1.1380 9328
14	1.0599 3983	1.0723 2113	1.0848 3662	1.1102 7553	1.1494 7421
15	1.0643 5625	1.0776 8274	1.0911 6483	1.1186 0259	1.1609 6896
16	1.0687 9106	1.0830 7115	1.0975 2996	1.1269 9211	1.1725 7864
17	1.0732 4436	1.0884 8651	1.1039 3222	1.1354 4455	1.1843 0443
18	1.0777 1621	1.0939 2894	1.1103 7182	1.1439 6039	1.1961 4748
19	1.0822 0670	1.0993 9858	1.1168 4890	1.1525 4009	1.2081 0895
20	1.0867 1589	1.1048 9558	1.1233 6395	1.1611 8414	1.2201 9004
21	1.0912 4387	1.1104 2006	1.1299 1690	1.1698 9302	1.2323 9194
22	1.0957 9072	1.1159 7216	1.1365 0808	1.1786 6722	1.2447 1586
23	1.1003 5652	1.1215 5202	1.1431 3771	1.1875 0723	1.2571 6302
24	1.1049 4134	1.1271 5978	1.1498 0602	1.1964 1353	1.2697 3465
25	1.1095 4526	1.1327 9558	1.1565 1322	1.2053 8663	1.2824 3200
26	1.1141 6836	1.1384 5955	1.1632 5955	1.2144 2703	1.2952 5631
27	1.1188 1073	1.1441 5185	1.1700 4523	1.2235 3523	1.3082 0888
28	1.1234 7244	1.1498 7261	1.1768 7049	1.2327 1175	1.3212 9097
29	1.1281 5358	1.1556 2197	1.1837 3557	1.2419 5709	1.3345 0388
30	1.1328 5422	1.1614 0008	1.1906 4069	1.2512 7176	1.3478 4892
31	1.1375 7444	1.1672 0708	1.1975 8610	1.2606 5630	1.3613 2740
32	1.1423 1434	1.1730 4312	1.2045 7202	1.2701 1122	1.3749 4068
33	1.1470 7398	1.1789 0833	1.2115 9869	1.2796 3706	1.3886 9009
34	1.1518 5346	1.1848 0288	1.2186 6634	1.2892 3434	1.4025 7699
35	1.1566 5284	1.1907 2689	1.2257 7523	1.2989 0359	1.4166 0276
36	1.1614 7223	1.1966 8052	1.2329 2559	1.3086 4537	1.4307 6878
37	1.1663 1170	1.2026 6393	1.2401 1765	1.3184 6021	1.4450 7647
38	1.1711 7133	1.2086 7725	1.2473 5167	1.3283 4866	1.4595 2724
39	1.1760 5121	1.2147 2063	1.2546 2789	1.3383 1128	1.4741 2251
40	1.1809 5142	1.2207 9424	1.2619 4655	1.3483 4861	1.4888 6373
41	1.1858 7206	1.2268 9821	1.2693 0791	1.3584 6123	1.5037 5237
42	1.1908 1319	1.2330 3270	1.2767 1220	1.3686 4969	1.5187 8989
43	1.1957 7491	1.2391 9786	1.2841 5969	1.3789 1456	1.5339 7779
44	1.2007 5731	1.2453 9385	1.2916 5062	1.3892 5642	1.5493 1757
45	1.2057 6046	1.2516 2082	1.2991 8525	1.3996 7584	1.5648 1075
46	1.2107 8446	1.2578 7892	1.3067 6383	1.4101 7341	1.5804 5885
47	1.2158 2940	1.2641 6832	1.3143 8662	1.4207 4971	1.5962 6344
48	1.2208 9536	1.2704 8916	1.3220 5388	1.4314 0533	1.6122 2608
49	1.2259 8242	1.2768 4161	1.3297 6586	1.4421 4087	1.6283 4834
50	1.2310 9068	1.2832 2581	1.3375 2283	1.4529 5693	1.6446 3182

TABLE V — COMPOUND AMOUNT OF 1

$(1+i)^n$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	1.2362 2002	1.2896 4194	1.3453 2504	1.4638 5411	1.6610 7814
52	1.2413 7114	1.2960 9015	1.3531 7277	1.4748 3301	1.6776 8892
53	1.2465 4352	1.3025 7060	1.3610 6628	1.4858 9426	1.6944 6581
54	1.2517 3745	1.3090 8346	1.3690 0583	1.4970 3847	1.7114 1047
55	1.2569 5302	1.3156 2887	1.3769 9170	1.5082 6626	1.7285 2457
56	1.2621 9033	1.3222 0702	1.3850 2415	1.5195 7825	1.7458 0982
57	1.2674 4946	1.3288 1805	1.3931 0340	1.5309 7509	1.7632 6792
58	1.2727 3050	1.3354 6214	1.4012 2990	1.5424 5740	1.7809 0060
59	1.2780 3354	1.3421 3946	1.4094 0374	1.5540 2583	1.7987 0960
60	1.2833 5868	1.3488 5015	1.4176 2526	1.5656 8103	1.8166 9670
61	1.2887 0601	1.3555 9440	1.4258 9474	1.5774 2363	1.8348 6367
62	1.2940 7561	1.3623 7238	1.4342 1246	1.5892 5431	1.8532 1230
63	1.2994 6760	1.3691 8424	1.4425 7870	1.6011 7372	1.8717 4443
64	1.3048 8204	1.3760 3016	1.4509 9374	1.6131 8252	1.8904 6187
65	1.3103 1905	1.3829 1031	1.4594 5787	1.6252 8139	1.9093 6649
66	1.3157 7872	1.3898 2486	1.4679 7138	1.6374 7100	1.9284 6015
67	1.3212 6113	1.3967 7399	1.4765 3454	1.6497 5203	1.9477 4475
68	1.3267 6638	1.4037 5785	1.4851 4766	1.6621 2517	1.9672 2220
69	1.3322 9458	1.4107 7664	1.4938 1102	1.6745 9111	1.9868 9442
70	1.3378 4580	1.4178 3053	1.5025 2492	1.6871 5055	2.0067 6337
71	1.3434 2016	1.4249 1968	1.5112 8965	1.6998 0418	2.0268 3100
72	1.3490 1774	1.4320 4428	1.5201 0550	1.7125 5271	2.0470 9931
73	1.3546 3865	1.4392 0450	1.5289 7279	1.7253 9685	2.0675 7031
74	1.3602 8298	1.4464 0052	1.5378 9170	1.7383 3733	2.0882 4601
75	1.3659 5082	1.4536 3252	1.5468 6283	1.7513 7486	2.1001 2847
76	1.3716 4229	1.4609 0069	1.5558 8620	1.7645 1017	2.1302 1975
77	1.3773 5746	1.4682 0519	1.5649 6220	1.7777 4400	2.1515 2195
78	1.3830 9645	1.4755 4622	1.5740 9115	1.7910 7708	2.1730 3717
79	1.3888 5935	1.4829 2395	1.5832 7334	1.8045 1015	2.1947 6754
80	1.3946 4627	1.4903 3857	1.5925 0910	1.8180 4398	2.2167 1522
81	1.4004 5729	1.4977 9026	1.6017 9874	1.8316 7931	2.2388 8237
82	1.4062 9253	1.5052 7921	1.6111 4257	1.8454 1691	2.2612 7119
83	1.4121 5209	1.5128 0561	1.6205 4090	1.8592 5753	2.2838 8390
84	1.4180 3605	1.5203 6964	1.6299 9405	1.8732 0196	2.3067 2274
85	1.4239 4454	1.5279 7148	1.6395 0235	1.8872 5098	2.3297 8997
86	1.4298 7764	1.5356 1134	1.6490 6612	1.9014 0536	2.3530 8787
87	1.4358 3546	1.5432 8940	1.6586 8567	1.9156 6590	2.3766 1875
88	1.4418 1811	1.5510 0585	1.6683 6134	1.9300 3339	2.4003 8494
89	1.4478 2568	1.5587 6087	1.6780 9344	1.9445 0865	2.4243 8879
90	1.4538 5829	1.5665 5468	1.6878 8232	1.9590 9246	2.4486 3267
91	1.4599 1603	1.5743 8745	1.6977 2830	1.9737 8565	2.4731 1900
92	1.4659 9902	1.5822 5939	1.7076 3172	1.9885 8905	2.4978 5019
93	1.4721 0735	1.5901 7069	1.7175 9290	2.0035 0346	2.5228 2869
94	1.4782 4113	1.5981 2154	1.7276 1210	2.0185 2974	2.5480 5698
95	1.4844 0047	1.6061 1215	1.7376 8993	2.0336 6871	2.5735 3755
96	1.4905 8547	1.6141 4271	1.7478 2646	2.0489 2123	2.5992 7293
97	1.4967 9624	1.6222 1342	1.7580 2211	2.0642 8814	2.6252 6565
98	1.5030 3289	1.6303 2449	1.7682 7724	2.0797 7030	2.6515 1831
99	1.5092 9553	1.6384 7611	1.7785 9219	2.0953 6858	2.6780 3349
100	1.5155 8426	1.6466 6849	1.7889 6731	2.1110 8384	2.7048 1383

TABLE V — COMPOUND AMOUNT OF 1

$$(1 + i)^n$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	1.5218 9919	1.6549 0183	1.7994 0295	2.1269 1697	2.7318 6197
102	1.5282 4044	1.6631 7634	1.8098 9947	2.1428 6885	2.7591 8059
103	1.5346 0811	1.6714 9223	1.8204 5722	2.1589 4036	2.7867 7239
104	1.5410 0231	1.6798 4969	1.8310 7655	2.1751 3242	2.8146 4012
105	1.5474 2315	1.6882 4894	1.8417 5783	2.1914 4591	2.8427 8652
106	1.5538 7075	1.6966 9018	1.8525 0142	2.2078 8175	2.8712 1438
107	1.5603 4521	1.7051 7363	1.8633 0768	2.2244 4087	2.8999 2653
108	1.5668 4665	1.7136 9950	1.8741 7697	2.2411 2417	2.9289 2579
109	1.5733 7518	1.7222 6800	1.8851 0967	2.2579 3260	2.9582 1505
110	1.5799 3091	1.7308 7934	1.8961 0614	2.2748 6710	2.9877 9720
111	1.5865 1395	1.7395 3373	1.9071 6676	2.2919 2860	3.0176 7517
112	1.5931 2443	1.7482 3140	1.9182 9190	2.3091 1807	3.0478 5192
113	1.5997 6245	1.7569 7256	1.9294 8194	2.3264 3645	3.0783 3044
114	1.6064 2812	1.7657 5742	1.9407 3725	2.3438 8472	3.1091 1375
115	1.6131 2157	1.7745 8621	1.9520 5822	2.3614 6386	3.1402 0489
116	1.6198 4291	1.7834 5914	1.9634 4522	2.3791 7484	3.1716 0693
117	1.6265 9226	1.7923 7644	1.9748 9865	2.3970 1865	3.2033 2300
118	1.6333 6973	1.8013 3832	1.9864 1889	2.4149 9629	3.2353 5623
119	1.6401 7543	1.8103 4501	1.9980 0634	2.4331 0876	3.2677 0980
120	1.6470 0950	1.8193 9673	2.0096 6138	2.4513 5708	3.3003 8689
121	1.6538 7204	1.8284 9372	2.0213 8440	2.4697 4226	3.3333 9076
122	1.6607 6317	1.8376 3619	2.0331 7581	2.4882 6532	3.3667 2467
123	1.6676 8302	1.8468 2437	2.0450 3600	2.5069 2731	3.4003 9192
124	1.6746 3170	1.8560 5849	2.0569 6538	2.5257 2927	3.4343 9584
125	1.6816 0933	1.8653 3878	2.0689 6434	2.5446 7224	3.4687 3980
126	1.6886 1603	1.8746 6548	2.0810 3330	2.5637 5728	3.5034 2719
127	1.6956 5193	1.8840 3880	2.0931 7266	2.5829 8546	3.5384 6147
128	1.7027 1715	1.8934 5900	2.1053 8284	2.6023 5785	3.5738 4608
129	1.7098 1181	1.9029 2629	2.1176 6424	2.6218 7553	3.6095 8454
130	1.7169 3602	1.9124 4092	2.1300 1728	2.6415 3960	3.6456 8039
131	1.7240 8992	1.9220 0313	2.1424 4238	2.6613 5115	3.6821 3719
132	1.7312 7363	1.9316 1314	2.1549 3996	2.6813 1128	3.7189 5856
133	1.7384 8727	1.9412 7121	2.1675 1044	2.7014 2112	3.7561 4815
134	1.7457 3097	1.9509 7757	2.1801 5425	2.7216 8177	3.7937 0963
135	1.7530 0485	1.9607 3245	2.1928 7182	2.7420 9439	3.8316 4673
136	1.7603 0903	1.9705 3612	2.2056 6357	2.7626 6009	3.8699 6319
137	1.7676 4365	1.9803 8880	2.2185 2994	2.7833 8005	3.9086 6282
138	1.7750 0884	1.9902 9074	2.2314 7137	2.8042 5540	3.9477 4945
139	1.7824 0471	2.0002 4219	2.2444 8828	2.8252 8731	3.9872 2695
140	1.7898 3139	2.0102 4340	2.2575 8113	2.8464 7697	4.0270 9922
141	1.7972 8902	2.0202 9462	2.2707 5036	2.8678 2554	4.0673 7021
142	1.8047 7773	2.0303 9609	2.2839 9640	2.8893 3424	4.1080 4391
143	1.8122 9763	2.0405 4808	2.2973 1971	2.9110 0424	4.1491 2435
144	1.8198 4887	2.0507 5082	2.3107 2074	2.9328 3677	4.1906 1559
145	1.8274 3158	2.0610 0457	2.3241 9995	2.9548 3305	4.2325 2175
146	1.8350 4588	2.0713 0959	2.3377 5778	2.9769 9430	4.2748 4697
147	1.8426 9190	2.0816 6614	2.3513 9470	2.9993 2175	4.3175 9544
148	1.8503 6978	2.0920 7447	2.3651 1117	3.0218 1667	4.3607 7139
149	1.8580 7966	2.1025 3484	2.3789 0765	3.0444 8029	4.4043 7910
150	1.8658 2106	2.1130 4752	2.3927 8461	3.0673 1389	4.4484 2290

TABLE V — COMPOUND AMOUNT OF 1

$$(1+i)^n$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0112 5000	1.0125 0000	1.0150 0000	1.0175 0000	1.0200 0000
2	1.0226 2656	1.0251 5625	1.0302 2500	1.0353 0625	1.0404 0000
3	1.0341 3111	1.0379 7070	1.0456 7838	1.0534 2411	1.0612 0800
4	1.0457 6509	1.0509 4534	1.0613 6355	1.0718 6903	1.0824 3216
5	1.0575 2994	1.0640 8215	1.0772 8400	1.0906 1656	1.1040 8080
6	1.0694 2716	1.0773 8318	1.0934 4326	1.1097 0235	1.1261 6242
7	1.0814 5821	1.0908 5047	1.1098 4491	1.1291 2215	1.1486 8567
8	1.0936 2462	1.1044 8610	1.1264 9259	1.1488 8178	1.1716 5938
9	1.1059 2789	1.1182 9218	1.1433 8998	1.1689 8721	1.1950 9257
10	1.1183 6958	1.1322 7083	1.1605 4083	1.1894 4449	1.2189 9442
11	1.1309 5124	1.1464 2422	1.1779 4894	1.2102 6977	1.2433 7431
12	1.1436 7444	1.1607 5452	1.1956 1817	1.2314 3931	1.2682 4179
13	1.1565 4078	1.1752 6395	1.2135 5244	1.2529 8950	1.2936 0663
14	1.1695 5186	1.1899 5475	1.2317 5573	1.2749 1682	1.3194 7876
15	1.1827 0932	1.2048 2918	1.2502 3207	1.2972 2786	1.3458 6834
16	1.1960 1480	1.2198 8955	1.2689 8555	1.3199 2935	1.3727 8571
17	1.2094 6997	1.2351 3817	1.2880 2033	1.3430 2811	1.4002 4142
18	1.2230 7650	1.2505 7739	1.3073 4064	1.3665 3111	1.4282 4625
19	1.2368 3611	1.2662 0961	1.3269 5075	1.3904 4540	1.4568 1117
20	1.2507 5052	1.2820 3723	1.3468 5501	1.4147 7820	1.4859 4740
21	1.2648 2146	1.2980 6270	1.3670 5783	1.4395 3681	1.5156 6634
22	1.2790 5071	1.3142 8848	1.3875 6370	1.4647 2871	1.5459 7967
23	1.2934 4003	1.3307 1709	1.4083 7715	1.4903 6146	1.5768 9926
24	1.3079 9123	1.3473 5105	1.4295 0281	1.5164 4279	1.6084 3725
25	1.3227 0613	1.3641 9294	1.4509 4535	1.5429 8054	1.6406 0599
26	1.3375 8657	1.3812 4535	1.4727 0953	1.5699 8269	1.6734 1811
27	1.3526 3442	1.3985 1092	1.4948 0018	1.5974 5739	1.7068 8648
28	1.3678 5156	1.4159 9230	1.5172 2218	1.6254 1290	1.7410 2421
29	1.3832 3989	1.4336 9221	1.5399 8051	1.6538 5762	1.7758 4469
30	1.3988 0134	1.4516 1336	1.5630 8022	1.6828 0013	1.8113 6158
31	1.4145 3785	1.4697 5853	1.5865 2642	1.7122 4913	1.8475 8882
32	1.4304 5140	1.4881 3051	1.6103 2432	1.7422 1349	1.8845 4059
33	1.4465 4398	1.5067 3214	1.6344 7918	1.7727 0223	1.9222 3140
34	1.4628 1760	1.5255 6629	1.6589 9637	1.8037 2452	1.9606 7603
35	1.4792 7430	1.5446 3587	1.6838 8132	1.8352 8970	1.9998 8955
36	1.4959 1613	1.5639 4382	1.7091 3954	1.8674 0727	2.0398 8734
37	1.5127 4519	1.5834 9312	1.7347 7663	1.9000 8689	2.0806 8509
38	1.5297 6357	1.6032 8678	1.7607 9828	1.9333 3841	2.1222 9879
39	1.5469 7341	1.6233 2787	1.7872 1025	1.9671 7184	2.1647 4477
40	1.5643 7687	1.6436 1946	1.8140 1841	2.0015 9734	2.2080 3966
41	1.5819 7611	1.6641 6471	1.8412 2868	2.0366 2530	2.2522 0046
42	1.5997 7334	1.6849 6677	1.8688 4712	2.0722 6624	2.2972 4447
43	1.6177 7079	1.7060 2885	1.8968 7982	2.1085 3090	2.3431 8936
44	1.6359 7071	1.7273 5421	1.9253 3302	2.1454 3019	2.3900 5314
45	1.6543 7538	1.7489 4614	1.9542 1301	2.1829 7522	2.4378 5421
46	1.6729 8710	1.7708 0797	1.9835 2621	2.2211 7728	2.4866 1129
47	1.6918 0821	1.7929 4306	2.0132 7910	2.2600 4789	2.5363 4351
48	1.7108 4105	1.8153 5485	2.0434 7829	2.2995 9872	2.5870 7039
49	1.7300 8801	1.8380 4679	2.0741 3046	2.3398 4170	2.6388 1179
50	1.7495 5150	1.8610 2237	2.1052 4242	2.3807 8893	2.6915 8803

TABLE V — COMPOUND AMOUNT OF 1

$$(1 + i)^n$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	1.7692 3395	1.8842 8515	2.1368 2106	2.4224 5274	2.7454 1979
52	1.7891 3784	1.9078 3872	2.1688 7337	2.4648 4566	2.8003 2819
53	1.8092 6564	1.9316 8670	2.2014 0647	2.5079 8046	2.8563 3475
54	1.8296 1988	1.9558 3279	2.2344 2757	2.5518 7012	2.9134 6144
55	1.8502 0310	1.9802 8070	2.2679 4398	2.5965 2785	2.9717 3067
56	1.8710 1788	2.0050 3420	2.3019 6314	2.6419 6708	3.0311 6529
57	1.8920 6084	2.0300 9713	2.3364 9259	2.6882 0151	3.0917 8859
58	1.9133 5259	2.0554 7335	2.3715 3998	2.7352 4503	3.1536 2436
59	1.9348 7780	2.0811 6676	2.4071 1308	2.7831 1182	3.2166 9685
60	1.9566 4518	2.1071 8135	2.4432 1978	2.8318 1628	3.2810 3079
61	1.9786 5744	2.1335 2111	2.4798 6807	2.8813 7306	3.3466 5140
62	2.0009 1733	2.1601 9013	2.5170 6609	2.9317 9709	3.4135 8443
63	2.0234 2765	2.1871 9250	2.5548 2208	2.9831 0354	3.4818 5612
64	2.0461 9121	2.2145 3241	2.5931 4442	3.0343 0785	3.5514 9324
65	2.0692 1087	2.2422 1407	2.6320 4158	3.0884 2574	3.6225 2311
66	2.0924 8940	2.2702 4174	2.6715 2221	3.1424 7319	3.6949 7357
67	2.1160 2999	2.2986 1976	2.7115 9504	3.1974 6647	3.7688 7304
68	2.1398 3533	2.3273 5251	2.7522 6896	3.2534 2213	3.8442 5050
69	2.1639 0848	2.3564 4442	2.7935 5300	3.3103 5702	3.9211 3551
70	2.1882 5245	2.3858 9997	2.8354 5629	3.3682 8827	3.9995 5822
71	2.2128 7020	2.4157 2372	2.8779 8814	3.4272 3331	4.0795 4939
72	2.2377 6508	2.4459 2027	2.9211 5796	3.4872 0990	4.1611 4038
73	2.2629 3994	2.4764 9427	2.9649 7533	3.5482 3607	4.2443 6318
74	2.2883 9801	2.5074 5045	3.0094 4996	3.6103 8020	4.3292 5045
75	2.3141 4249	2.5387 9358	3.0545 9171	3.6735 1098	4.4158 3546
76	2.3401 7659	2.5705 2850	3.1004 1059	3.7377 9742	4.5041 5216
77	2.3665 0358	2.6026 6011	3.1469 1674	3.8032 0888	4.5942 3521
78	2.3931 2675	2.6351 9336	3.1941 2050	3.8697 6503	4.6861 1991
79	2.4200 4942	2.6681 3327	3.2420 3230	3.9374 8592	4.7798 4231
80	2.4472 7498	2.7014 8494	3.2906 6279	4.0063 9192	4.8754 3916
81	2.4748 0682	2.7352 5350	3.3400 2273	4.0765 0378	4.9729 4794
82	2.5026 4840	2.7694 4417	3.3901 2307	4.1478 4260	5.0724 0690
83	2.5308 0319	2.8040 6222	3.4409 7492	4.2204 2984	5.1738 5504
84	2.5592 7473	2.8391 1300	3.4925 8954	4.2942 8737	5.2773 3214
85	2.5880 6657	2.8746 0191	3.5449 7838	4.3694 3740	5.3828 7878
86	2.6171 8232	2.9105 3444	3.5981 5306	4.4459 0255	5.4905 3636
87	2.6466 2562	2.9469 1612	3.6521 2535	4.5237 0584	5.6003 4708
88	2.6764 0016	2.9837 5257	3.7069 0723	4.6028 7070	5.7123 5402
89	2.7065 0966	3.0210 4948	3.7625 1084	4.6834 2003	5.8266 0110
90	2.7369 5789	3.0588 1260	3.8189 4851	4.7653 8080	5.9431 3313
91	2.7677 4867	3.0970 4775	3.8762 3273	4.8487 7496	6.0619 9570
92	2.7988 8584	3.1357 6085	3.9343 7622	4.9336 2853	6.1832 3570
93	2.8303 7331	3.1749 5786	3.9933 9187	5.0199 6703	6.3069 0042
94	2.8622 1501	3.2146 4483	4.0532 9275	5.1078 1645	6.4330 3843
95	2.8944 1402	3.2548 2780	4.1140 9214	5.1972 0324	6.5616 9920
96	2.9269 7700	3.2955 1324	4.1758 0352	5.2881 5429	6.6929 3318
97	2.9599 0550	3.3367 0716	4.2384 4057	5.3806 9699	6.8267 9184
98	2.9932 0452	3.3784 1600	4.3020 1718	5.4748 5019	6.9633 2768
99	3.0268 7807	3.4206 4620	4.3665 4744	5.5706 6923	7.1025 9423
100	3.0609 3045	3.4634 0427	4.4320 4565	5.6681 5594	7.2446 4612

TABLE V — COMPOUND AMOUNT OF 1

$(1 + i)^n$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000	1.0350 0000
2	1.0455 0625	1.0506 2500	1.0557 5625	1.0609 0000	1.0712 2500
3	1.0690 3014	1.0768 0063	1.0847 8955	1.0027 2700	1.1087 1788
4	1.0930 8332	1.1038 1289	1.1146 2126	1.1255 0881	1.1475 2300
5	1.1176 7769	1.1314 0821	1.1452 7334	1.1502 7407	1.1876 8631
6	1.1428 2544	1.1506 9342	1.1767 6836	1.1040 5230	1.2292 5533
7	1.1685 3901	1.1886 8575	1.2091 2940	1.2298 7387	1.2722 7926
8	1.1948 3114	1.2184 0290	1.2423 8055	1.2667 7008	1.3168 0004
9	1.2217 1484	1.2488 6297	1.2765 4602	1.3047 7318	1.3028 9735
10	1.2492 0343	1.2800 8454	1.3116 5103	1.3439 1038	1.4105 9876
11	1.2773 1050	1.3120 8666	1.3477 2144	1.3842 3387	1.4590 6972
12	1.3060 4990	1.3448 8882	1.3847 8378	1.4257 6080	1.5110 0866
13	1.3354 3611	1.3785 1104	1.4228 0533	1.4085 3371	1.5639 5606
14	1.3654 8343	1.4129 7382	1.4619 9413	1.5125 8972	1.6186 9452
15	1.3962 0680	1.4482 9817	1.5021 9896	1.5579 6742	1.6753 4883
16	1.4276 2146	1.4845 0562	1.5435 0944	1.6047 0644	1.7339 8604
17	1.4597 4294	1.5216 1826	1.5859 5505	1.6528 4763	1.7946 7555
18	1.4925 8716	1.5596 5872	1.6295 6973	1.7024 3306	1.8574 8920
19	1.5261 7037	1.5986 5019	1.6743 8290	1.7535 0605	1.9225 0132
20	1.5605 0920	1.6386 1644	1.7204 2843	1.8061 1123	1.9897 8886
21	1.5956 2066	1.6795 8185	1.7677 4021	1.8602 9457	2.0594 3147
22	1.6315 2212	1.7215 7140	1.8163 5307	1.9161 0341	2.1315 1158
23	1.6682 3137	1.7646 1068	1.8663 0278	1.9735 8651	2.2061 1448
24	1.7057 6658	1.8087 2595	1.9176 2610	2.0327 9411	2.2833 2849
25	1.7441 4632	1.8539 4410	1.9703 6082	2.0937 7793	2.3632 4498
26	1.7833 8962	1.9002 9270	2.0245 4575	2.1565 9127	2.4459 5856
27	1.8235 1588	1.9478 0002	2.0802 2075	2.2212 8901	2.5315 6711
28	1.8645 4499	1.9964 9502	2.1374 2682	2.2879 2768	2.6201 7196
29	1.9064 9725	2.0464 0739	2.1962 0606	2.3565 6551	2.7118 7798
30	1.9493 9344	2.0975 6758	2.2566 0173	2.4272 6247	2.8067 9370
31	1.9932 5479	2.1500 0677	2.3186 5828	2.5000 8035	2.9050 3148
32	2.0381 0303	2.2037 5694	2.3824 2138	2.5750 8276	3.0067 0750
33	2.0839 6034	2.2588 5086	2.4479 3797	2.6523 3524	3.1119 4235
34	2.1308 4945	2.3153 2213	2.5152 5626	2.7319 0530	3.2208 6033
35	2.1787 9356	2.3732 0519	2.5844 2581	2.8138 6245	3.3335 9045
36	2.2278 1642	2.4325 3532	2.6554 0752	2.8982 7833	3.4502 6611
37	2.2779 4229	2.4933 4870	2.7285 2370	2.9852 2608	3.5710 2543
38	2.3291 9599	2.5556 8242	2.8035 5810	3.0747 8348	3.6960 1132
39	2.3816 0290	2.6195 7448	2.8806 5595	3.1670 2008	3.8253 7171
40	2.4351 8897	2.6850 6384	2.9598 7399	3.2620 3779	3.9592 5972
41	2.4899 8072	2.7521 9043	3.0412 7052	3.3598 9893	4.0978 3381
42	2.5460 0528	2.8209 9520	3.1249 0546	3.4606 9589	4.2412 5799
43	2.6032 9040	2.8915 2008	3.2108 4036	3.5645 1677	4.3897 0202
44	2.6618 6444	2.9638 0808	3.2991 3847	3.6714 5227	4.5433 4160
45	2.7217 5639	3.0379 0328	3.3898 6478	3.7815 9584	4.7023 5855
46	2.7829 9590	3.1138 5086	3.4830 8606	3.8950 4372	4.8669 4110
47	2.8456 1331	3.1916 9713	3.5788 7093	4.0118 9503	5.0372 8404
48	2.9096 3961	3.2714 8956	3.6772 8988	4.1322 5188	5.2135 8898
49	2.9751 0650	3.3532 7680	3.7784 1535	4.2562 1944	5.3960 6459
50	3.0420 4640	3.4371 0872	3.8823 2177	4.3839 0602	5.5849 2686

TABLE V — COMPOUND AMOUNT OF 1

$(1 + i)^n$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	3.1104 9244	3.5230 3644	3.9890 8562	4.5154 2320	5.7803 0030
52	3.1804 7852	3.6111 1235	4.0987 8547	4.6508 8500	5.9827 1327
53	3.2520 3029	3.7013 9016	4.2115 0208	4.7904 1247	6.1921 0824
54	3.3252 1017	3.7939 2491	4.3273 1838	4.9341 2485	6.4088 3202
55	3.4000 2740	3.8887 7303	4.4463 1964	5.0821 4859	6.6331 4114
56	3.4765 2802	3.9859 0230	4.5685 9343	5.2346 1305	6.8653 0108
57	3.5547 4990	4.0856 4217	4.6942 2075	5.3916 5144	7.1055 8062
58	3.6347 3177	4.1877 8322	4.8233 2107	5.5534 0098	7.3542 8215
59	3.7165 1324	4.2924 7780	4.9559 6239	5.7200 0301	7.6116 8203
60	3.8001 3470	4.3997 8975	5.0922 5136	5.8916 0310	7.8780 9090
61	3.8856 3782	4.5097 8440	5.2322 8827	6.0683 5120	8.1538 2408
62	3.9730 6467	4.6225 2010	5.3761 7020	6.2504 0173	8.4392 0703
63	4.0624 5802	4.7380 9233	5.5240 2105	6.4379 1370	8.7345 8020
64	4.1538 6394	4.8565 4464	5.6759 3162	6.6310 5120	9.0402 9051
65	4.2473 2588	4.9779 5826	5.8320 1074	6.8299 8273	9.3567 0008
66	4.3428 9071	5.1024 0721	5.9924 0020	7.0348 8222	9.6841 8520
67	4.4406 0576	5.2299 6739	6.1571 9130	7.2459 2808	10.0231 3168
68	4.5405 1939	5.3607 1658	6.3265 1400	7.4633 0654	10.3739 4129
69	4.6426 8107	5.4947 3440	6.5004 9319	7.6872 0574	10.7370 2024
70	4.7471 4140	5.6321 0286	6.6792 5676	7.9178 2191	11.1128 2526
71	4.8539 5208	5.7729 0543	6.8629 3632	8.1553 5657	11.5017 7414
72	4.9631 6600	5.9172 2806	7.0516 6700	8.4000 1727	11.9043 3624
73	5.0748 3723	6.0651 5876	7.2455 8791	8.6520 1778	12.3209 8801
74	5.1890 2107	6.2167 8773	7.4448 4158	8.9115 7832	12.7522 2259
75	5.3057 7405	6.3722 0743	7.6495 7472	9.1789 2567	13.1985 5038
76	5.4251 5396	6.5315 1261	7.8599 3802	9.4542 9344	13.6604 9064
77	5.5472 1093	6.6948 0043	8.0760 8032	9.7379 2224	14.1386 1713
78	5.6720 3237	6.8621 7044	8.2981 7809	10.0300 5091	14.6334 6873
79	5.7996 5310	7.0337 2470	8.5263 7801	10.3309 6171	15.1456 4013
80	5.9301 4530	7.2095 6782	8.7608 5402	10.6408 9056	15.6757 3754
81	6.0635 7357	7.3898 0701	9.0017 7751	10.9601 1727	16.2243 8835
82	6.2000 0397	7.5745 5219	9.2493 2030	11.2889 2079	16.7922 4195
83	6.3395 0406	7.7639 1599	9.5036 8286	11.6275 8842	17.3799 7041
84	6.4821 4290	7.9580 1380	9.7650 3414	11.9764 1607	17.9882 6938
85	6.6279 9112	8.1569 6424	10.0335 7258	12.3357 0855	18.6178 5881
86	6.7771 2002	8.3608 8834	10.3094 0583	12.7057 7981	19.2694 8387
87	6.9296 0614	8.5699 1055	10.5929 0999	13.0869 5320	19.9439 1580
88	7.0855 2228	8.7841 5832	10.8843 1405	13.4795 6180	20.6419 5285
89	7.2449 4653	9.0037 6228	11.1836 3331	13.8839 4865	21.3644 2120
90	7.4079 5782	9.2288 5033	11.4911 8322	14.3004 6711	22.1121 7595
91	7.5746 3088	9.4595 7774	11.8071 9076	14.7294 8112	22.8861 0210
92	7.7450 6621	9.6960 6718	12.1318 8851	15.1713 6556	23.6871 1508
93	7.9193 3020	9.9384 6886	12.4655 1544	15.6265 0652	24.5161 6473
94	8.0975 1512	10.1869 3058	12.8083 1711	16.0953 0172	25.3742 3040
95	8.2797 0921	10.4416 0385	13.1605 4584	16.5781 6077	26.2623 2856
96	8.4660 0267	10.7026 4305	13.5224 6085	17.0755 0550	27.1815 1006
97	8.6564 8773	10.9702 1004	13.8943 2852	17.5877 7070	28.1328 6201
98	8.8512 5871	11.2444 6530	14.2764 2255	18.1154 0388	29.1175 1311
99	9.0504 1203	11.5255 7693	14.6690 2417	18.6588 6600	30.1366 2607
100	9.2540 4630	11.8137 1635	15.0724 2234	19.2186 3198	31.1914 0798

TABLE V — COMPOUND AMOUNT OF 1

$(1 + i)^n$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000	1.0600 0000
2	1.0816 0000	1.0920 2500	1.1025 0000	1.1130 2500	1.1236 0000
3	1.1248 6400	1.1411 6613	1.1576 2500	1.1742 4138	1.1910 1600
4	1.1698 5856	1.1925 1860	1.2155 0625	1.2388 2465	1.2624 7696
5	1.2166 5290	1.2461 8194	1.2762 8156	1.3069 6001	1.3382 2558
6	1.2653 1902	1.3022 6012	1.3400 9564	1.3788 4281	1.4185 1911
7	1.3159 3178	1.3608 6183	1.4071 0042	1.4546 7916	1.5036 3026
8	1.3685 6905	1.4221 0061	1.4774 5544	1.5346 8651	1.5938 4807
9	1.4233 1181	1.4860 9514	1.5513 2822	1.6190 9427	1.6894 7896
10	1.4802 4428	1.5529 6942	1.6288 9463	1.7081 4446	1.7908 4770
11	1.5394 5406	1.6228 5305	1.7103 3936	1.8020 9240	1.8982 9856
12	1.6010 3222	1.6958 8143	1.7958 5633	1.9012 0749	2.0121 9647
13	1.6650 7351	1.7721 9610	1.8856 4914	2.0057 7390	2.1329 2826
14	1.7316 7645	1.8519 4492	1.9799 3160	2.1160 9146	2.2609 0396
15	1.8009 4351	1.9352 8244	2.0789 2818	2.2324 7649	2.3965 5819
16	1.8729 8125	2.0223 7015	2.1828 7459	2.3552 6270	2.5403 5168
17	1.9479 0050	2.1133 7681	2.2920 1832	2.4848 0215	2.6927 2779
18	2.0258 1652	2.2084 7877	2.4066 1923	2.6214 6627	2.8543 3915
19	2.1068 4918	2.3078 6031	2.5269 5020	2.7656 4691	3.0255 9950
20	2.1911 2314	2.4117 1402	2.6532 9771	2.9177 5749	3.2071 3547
21	2.2787 6807	2.5202 4116	2.7859 6259	3.0782 3415	3.3995 6360
22	2.3699 1879	2.6336 5201	2.9252 6072	3.2475 3703	3.6035 3742
23	2.4647 1554	2.7521 6635	3.0715 2376	3.4261 5157	3.8197 4966
24	2.5633 0416	2.8760 1383	3.2250 9994	3.6145 8990	4.0489 3464
25	2.6658 3633	3.0054 3446	3.3863 5494	3.8133 9235	4.2918 7072
26	2.7724 6978	3.1406 7901	3.5556 7269	4.0231 2893	4.5493 8296
27	2.8833 6858	3.2820 0956	3.7334 5632	4.2444 0102	4.8223 4594
28	2.9987 0332	3.4296 9999	3.9201 2914	4.4778 4307	5.1116 8670
29	3.1186 5145	3.5840 3649	4.1161 3560	4.7241 2444	5.4183 8790
30	3.2433 9751	3.7453 1813	4.3219 4238	4.9839 5129	5.7434 9117
31	3.3731 3341	3.9138 5745	4.5380 3949	5.2580 6861	6.0881 0064
32	3.5080 5875	4.0899 8104	4.7649 4147	5.5472 6238	6.4533 8668
33	3.6483 8110	4.2740 3018	5.0031 8854	5.8523 6181	6.8405 8988
34	3.7943 1634	4.4663 6154	5.2533 4797	6.1742 4171	7.2510 2528
35	3.9460 8890	4.6673 4781	5.5160 1537	6.5138 2501	7.6860 8679
36	4.1039 3255	4.8773 7846	5.7918 1614	6.8720 8538	8.1472 5200
37	4.2680 8986	5.0968 6049	6.0814 0694	7.2500 5008	8.6360 8712
38	4.4388 1345	5.3262 1921	6.3854 7729	7.6488 0283	9.1542 5235
39	4.6163 6599	5.5658 9908	6.7047 5115	8.0694 8699	9.7035 0749
40	4.8010 2063	5.8163 6454	7.0399 8871	8.5133 0877	10.2857 1794
41	4.9930 6145	6.0781 0094	7.3919 8815	8.9815 4076	10.9028 6101
42	5.1927 8391	6.3516 1548	7.7615 8756	9.4755 2550	11.5570 3267
43	5.4004 9527	6.6374 3818	8.1496 6693	9.9966 7940	12.2504 5463
44	5.6165 1508	6.9361 2290	8.5571 5028	10.5464 9677	12.9854 8191
45	5.8411 7568	7.2482 4843	8.9850 0779	11.1265 5409	13.7646 1083
46	6.0748 2271	7.5744 1961	9.4342 5818	11.7385 1456	14.5904 8748
47	6.3178 1562	7.9152 6849	9.9059 7109	12.3841 3287	15.4659 1673
48	6.5705 2824	8.2714 5557	10.4012 6965	13.0652 6017	16.3938 7173
49	6.8333 4937	8.6436 7107	10.9213 3313	13.7838 4948	17.3775 0403
50	7.1066 8335	9.0326 3627	11.4673 9979	14.5419 6120	18.4201 5427

TABLE V — COMPOUND AMOUNT OF 1

$(1 + i)^n$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	7.3909 5068	9.4391 0490	12.0407 6978	15.3417 6007	19.5253 6353
52	7.6865 8871	9.8638 6403	12.6428 0826	16.1855 6037	20.6968 8534
53	7.9940 5226	10.3077 3853	13.2749 4868	17.0757 7252	21.9386 9846
54	8.3138 1435	10.7715 8077	13.9386 9611	18.0149 4001	23.2550 2037
55	8.6463 6692	11.2563 0817	14.6356 3092	19.0057 6171	24.6503 2159
56	8.9922 2160	11.7628 4204	15.3674 1246	20.0510 7860	26.1293 4089
57	9.3519 1046	12.2921 6993	16.1357 8309	21.1538 8793	27.6971 0134
58	9.7259 8688	12.8453 1758	16.9425 7224	22.3173 5176	29.3589 2742
59	10.1150 2635	13.4233 5687	17.7897 0085	23.5448 0611	31.1204 6307
60	10.5196 2741	14.0274 0793	18.6791 8589	24.8397 7045	32.9876 9085
61	10.9404 1250	14.6586 4129	19.6131 4519	26.2059 5782	34.9669 5230
62	11.3780 2900	15.3182 8014	20.5938 0245	27.6472 8550	37.0649 6944
63	11.8331 5016	16.0076 0275	21.6234 9257	29.1678 8620	39.2888 6761
64	12.3064 7617	16.7279 4487	22.7046 6720	30.7721 1994	41.6461 9967
65	12.7987 3522	17.4807 0239	23.8399 0056	32.4645 8654	44.1449 7165
66	13.3106 8463	18.2673 3400	25.0318 9559	34.2501 3880	46.7936 6994
67	13.8431 1201	19.0893 6403	26.2834 9037	36.1338 9643	49.6012 9014
68	14.3968 3649	19.9483 8541	27.5976 6488	38.1212 6074	52.5773 6755
69	14.9727 0995	20.8460 6276	28.9775 4813	40.2179 3008	55.7320 0960
70	15.5716 1835	21.7841 3558	30.4264 2554	42.4299 1623	59.0759 3018
71	16.1944 8308	22.7644 2168	31.9477 4681	44.7635 6163	62.6204 8599
72	16.8422 6241	23.7888 2066	33.5451 3415	47.2255 5751	66.3777 1515
73	17.5159 5290	24.8593 1759	35.2223 9086	49.8229 6318	70.3603 7806
74	18.2165 9102	25.9779 8688	36.9835 1040	52.5632 2615	74.5820 0074
75	18.9452 5460	27.1469 9629	38.8326 8592	55.4542 0359	79.0569 2079
76	19.7030 6485	28.3686 1112	40.7743 2022	58.5041 8479	83.8003 3603
77	20.4911 8744	29.6451 9862	42.8130 3623	61.7219 1495	88.8283 5620
78	21.3108 3494	30.9792 3256	44.9536 8804	65.1166 2027	94.1580 5757
79	22.1632 6834	32.3732 9802	47.2013 7244	68.6980 3439	99.8075 4102
80	23.0497 9907	33.8300 9643	49.5614 4107	72.4764 2628	105.7959 9348
81	23.9717 9103	35.3524 5077	52.0395 1312	76.4626 2973	112.1437 5309
82	24.9306 6267	36.9433 1106	54.6414 8878	80.6680 7436	118.8723 7828
83	25.9278 8918	38.6057 6006	57.3735 6322	85.1048 1845	126.0047 2097
84	26.9650 0475	40.3430 1926	60.2422 4138	89.7855 8347	133.5650 0423
85	28.0436 0494	42.1584 5513	63.2543 5344	94.7237 9056	141.5789 0449
86	29.1653 4914	44.0555 8561	66.4170 7112	99.9335 9904	150.0736 3875
87	30.3319 6310	46.0380 8696	69.7379 2467	105.4299 4698	159.0780 5708
88	31.5452 4163	48.1098 0087	73.2248 2091	111.2285 9407	168.6227 4050
89	32.8070 5129	50.2747 4191	76.8860 6195	117.3461 6674	178.7401 0493
90	34.1193 3334	52.5371 0530	80.7303 6505	123.8002 0591	189.4645 1123
91	35.4841 0668	54.9012 7503	84.7668 8330	130.6092 1724	200.8323 8190
92	36.9034 7094	57.3718 3241	89.0052 2747	137.7927 2419	212.8823 2482
93	38.3796 0978	59.9535 6487	93.4554 8884	145.3713 2402	225.6552 6431
94	39.9147 9417	62.6514 7529	98.1282 6328	153.3667 4684	239.1945 8017
95	41.5113 8594	65.4707 9168	103.0346 7645	161.8019 1791	253.5462 5498
96	43.1718 4138	68.4169 7730	108.1864 1027	170.7010 2340	268.7590 3028
97	44.8987 1503	71.4957 4128	113.5957 3078	180.0895 7969	284.8845 7209
98	46.6946 6363	74.7130 4964	119.2755 1732	189.9945 0657	301.9776 4642
99	48.5624 5018	78.0751 3687	125.2392 9319	200.4442 0443	320.0963 0520
100	50.5049 4818	81.5885 1803	131.5012 5785	211.4686 3567	339.3020 8351

TABLE V — COMPOUND AMOUNT OF 1

$(1+i)^n$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000	1.0850 0000
2	1.1342 2500	1.1449 0000	1.1556 2500	1.1664 0000	1.1772 2500
3	1.2079 4963	1.2250 4300	1.2422 9688	1.2597 1200	1.2772 8913
4	1.2864 6635	1.3107 9601	1.3354 6914	1.3604 8896	1.3858 5870
5	1.3700 8666	1.4025 5173	1.4356 2933	1.4693 2808	1.5036 5669
6	1.4591 4230	1.5007 3035	1.5433 0153	1.5868 7432	1.6314 6751
7	1.5539 8655	1.6057 8148	1.6590 4914	1.7138 2427	1.7701 4225
8	1.6549 9567	1.7181 8618	1.7834 7783	1.8509 3021	1.9206 0434
9	1.7625 7039	1.8384 5921	1.9172 3866	1.9990 0463	2.0838 5571
10	1.8771 3747	1.9671 5136	2.0610 3156	2.1589 2500	2.2609 8344
11	1.9991 5140	2.1048 5195	2.2156 0893	2.3316 3900	2.4531 6703
12	2.1290 9624	2.2521 9159	2.3817 7960	2.5181 7012	2.6616 8623
13	2.2674 8750	2.4098 4500	2.5604 1307	2.7196 2373	2.8879 2956
14	2.4148 7418	2.5785 3415	2.7524 4405	2.9371 9362	3.1334 0357
15	2.5718 4101	2.7590 3154	2.9588 7735	3.1721 6911	3.3997 4288
16	2.7390 1067	2.9521 6375	3.1807 9315	3.4259 4264	3.6887 2102
17	2.9170 4637	3.1588 1521	3.4193 5264	3.7000 1805	4.0022 6231
18	3.1066 5438	3.3799 3228	3.6758 0409	3.9960 1950	4.3424 5461
19	3.3085 8691	3.6165 2754	3.9514 8940	4.3157 0106	4.7115 6325
20	3.5236 4506	3.8696 8446	4.2478 5110	4.6609 5714	5.1120 4612
21	3.7526 8199	4.1405 6237	4.5664 3993	5.0338 3372	5.5465 7005
22	3.9966 0632	4.4304 0174	4.9089 2293	5.4365 4041	6.0180 2850
23	4.2563 8573	4.7405 2986	5.2770 9215	5.8714 6365	6.5295 6002
24	4.5330 5081	5.0723 6695	5.6728 7406	6.3411 8074	7.0845 7360
25	4.8276 9911	5.4274 3264	6.0983 3961	6.8484 7520	7.6867 6236
26	5.1414 9955	5.8073 5292	6.5557 1508	7.3963 5321	8.3401 3716
27	5.4756 9702	6.2138 6763	7.0473 9371	7.9880 6147	9.0490 4881
28	5.8316 1733	6.6488 3836	7.5759 4824	8.6271 0639	9.8182 1796
29	6.2106 7245	7.1142 5705	8.1441 4436	9.3172 7490	10.6527 6649
30	6.6143 6616	7.6122 5504	8.7549 5519	10.0626 5689	11.5582 5164
31	7.0442 9996	8.1451 1290	9.4115 7683	10.8676 6944	12.5407 0303
32	7.5021 7946	8.7152 7080	10.1174 4509	11.7370 8300	13.6066 6279
33	7.9898 2118	9.3253 3975	10.8762 5347	12.6760 4964	14.7632 2913
34	8.5091 5950	9.9781 1354	11.6919 7248	13.6901 3361	16.0181 0360
35	9.0622 5487	10.6765 8148	12.5688 7042	14.7853 4429	17.3796 4241
36	9.6513 0143	11.4239 4219	13.5115 3570	15.9681 7184	18.8569 1201
37	10.2786 3603	12.2236 1814	14.5249 0088	17.2456 2558	20.4597 4053
38	10.9467 4737	13.0792 7141	15.6142 6844	18.6252 7563	22.1988 2824
39	11.6582 8595	13.9948 2041	16.7853 3858	20.1152 9768	24.0857 2865
40	12.4160 7453	14.9744 5784	18.0442 3897	21.7245 2150	26.1330 1558
41	13.2231 1938	16.0226 6989	19.3975 5689	23.4624 8322	28.3543 2190
42	14.0826 2214	17.1442 5678	20.8523 7366	25.3394 8187	30.7644 3927
43	14.9979 9258	18.3443 5475	22.4163 0168	27.3666 4042	33.3794 1660
44	15.9728 6209	19.6284 5959	24.0975 2431	29.5559 7166	36.2166 6702
45	17.0110 9813	21.0024 5176	25.9048 3863	31.9204 4939	39.2950 8371
46	18.1168 1951	22.4726 2338	27.8477 0153	34.4740 8534	42.6351 6583
47	19.2944 1278	24.0457 0702	29.9362 7915	37.2320 1217	46.2591 5402
48	20.5485 4961	25.7289 0651	32.1815 0008	40.2105 7314	50.1911 8309
49	21.8842 0533	27.5299 2997	34.5951 1259	43.4274 1899	54.4574 3365
50	23.3066 7868	29.4570 2506	37.1897 4603	46.9016 1251	59.0863 1551

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.9958 5062	0.9950 2488	0.9942 0050	0.9925 5583	0.9900 9901
2	0.9917 1846	0.9900 7450	0.9884 3463	0.9851 6708	0.9802 9605
3	0.9876 0345	0.9851 4876	0.9827 0220	0.9778 3333	0.9705 9015
4	0.9835 0551	0.9802 4752	0.9770 0302	0.9705 5417	0.9609 8034
5	0.9794 2457	0.9753 7067	0.9713 3688	0.9633 2920	0.9514 6569
6	0.9753 6057	0.9705 1808	0.9657 0301	0.9561 5802	0.9420 4524
7	0.9713 1343	0.9656 8963	0.9601 0301	0.9490 4022	0.9327 1805
8	0.9672 8308	0.9608 8520	0.9545 3480	0.9419 7540	0.9234 8322
9	0.9632 6946	0.9561 0468	0.9489 0007	0.9349 6318	0.9143 3982
10	0.9592 7249	0.9513 4794	0.9434 0534	0.9280 0315	0.9052 8695
11	0.9552 9211	0.9466 1489	0.9380 2354	0.9210 9494	0.8963 2372
12	0.9513 2824	0.9419 0534	0.9325 8347	0.9142 3815	0.8874 4923
13	0.9473 8082	0.9372 1924	0.9271 7405	0.9074 3241	0.8786 6260
14	0.9434 4978	0.9325 5646	0.9217 9780	0.9006 7733	0.8699 6297
15	0.9395 3505	0.9279 1688	0.9164 5183	0.8939 7254	0.8613 4947
16	0.9356 3656	0.9233 0037	0.9111 3086	0.8873 1766	0.8528 2126
17	0.9317 5425	0.9187 0684	0.9058 6272	0.8807 1231	0.8443 7749
18	0.9278 8805	0.9141 3616	0.9005 9023	0.8741 5014	0.8360 1731
19	0.9240 3789	0.9095 8822	0.8953 7620	0.8676 4878	0.8277 3992
20	0.9202 0371	0.9050 6290	0.8901 8346	0.8611 8985	0.8195 4447
21	0.9163 8544	0.9005 6010	0.8850 2084	0.8547 7901	0.8114 3017
22	0.9125 8301	0.8960 7971	0.8798 8810	0.8484 1589	0.8033 9621
23	0.9087 9636	0.8916 2160	0.8747 8525	0.8421 0014	0.7954 4179
24	0.9050 2542	0.8871 8567	0.8697 1103	0.8358 3140	0.7875 6613
25	0.9012 7012	0.8827 7181	0.8646 6803	0.8296 0933	0.7797 6844
26	0.8975 3041	0.8783 7991	0.8596 5339	0.8234 3358	0.7720 4796
27	0.8938 0622	0.8740 0986	0.8546 6782	0.8173 0380	0.7644 0392
28	0.8900 9748	0.8696 6155	0.8497 1118	0.8112 1966	0.7568 3557
29	0.8864 0413	0.8653 3488	0.8447 8327	0.8051 8080	0.7493 4215
30	0.8827 2610	0.8610 2973	0.8398 8305	0.7991 8690	0.7419 2292
31	0.8790 6334	0.8567 4600	0.8350 1304	0.7932 3762	0.7345 7715
32	0.8754 1577	0.8524 8358	0.8301 7038	0.7873 3202	0.7273 0411
33	0.8717 8334	0.8482 4237	0.8253 5581	0.7814 7158	0.7201 0307
34	0.8681 6599	0.8440 2226	0.8205 6915	0.7756 5418	0.7129 7334
35	0.8645 6364	0.8398 2314	0.8158 1026	0.7698 8008	0.7059 1420
36	0.8609 7024	0.8356 4492	0.8110 7807	0.7641 4896	0.6989 2495
37	0.8574 0372	0.8314 8748	0.8063 7511	0.7584 6061	0.6920 0490
38	0.8538 4603	0.8273 5073	0.8016 9854	0.7528 1440	0.6851 5337
39	0.8503 0310	0.8232 3455	0.7970 4908	0.7472 1032	0.6783 6967
40	0.8467 7487	0.8191 3886	0.7924 2600	0.7416 4706	0.6716 5314
41	0.8432 6128	0.8150 6354	0.7878 3002	0.7361 2701	0.6650 0311
42	0.8397 6227	0.8110 0850	0.7832 6189	0.7306 4716	0.6584 1892
43	0.8362 7778	0.8069 7363	0.7787 1936	0.7252 0809	0.6518 9992
44	0.8328 0775	0.8029 5884	0.7742 0317	0.7198 0952	0.6454 4546
45	0.8293 5211	0.7989 6402	0.7697 1318	0.7144 5114	0.6390 5492
46	0.8259 1082	0.7949 8907	0.7652 4023	0.7091 3264	0.6327 2764
47	0.8224 8380	0.7910 3390	0.7608 1116	0.7038 5374	0.6264 6301
48	0.8190 7100	0.7870 9841	0.7563 9884	0.6986 1414	0.6202 6041
49	0.8156 7237	0.7831 8250	0.7520 1210	0.6934 1353	0.6141 1921
50	0.8122 8784	0.7792 8607	0.7476 5080	0.6882 5165	0.6080 3882

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1+i)^{-n}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	0.8089 1735	0.7754 0902	0.7433 1480	0.6831 2819	0.6020 1864
52	0.8055 6084	0.7715 5127	0.7390 0394	0.6780 4286	0.5960 5800
53	0.8022 1827	0.7677 1270	0.7347 1809	0.6720 9540	0.5001 5649
54	0.7988 8956	0.7638 9324	0.7304 5700	0.6070 8551	0.5843 1336
55	0.7955 7467	0.7600 9277	0.7262 2080	0.6630 1291	0.5785 2808
56	0.7922 7353	0.7563 1122	0.7220 0008	0.6580 7733	0.5728 0008
57	0.7889 8608	0.7525 4847	0.7178 2179	0.6531 7849	0.5071 2879
58	0.7857 1228	0.7488 0445	0.7136 5878	0.6483 1012	0.5615 1305
59	0.7824 5207	0.7450 7906	0.7005 1991	0.6434 8095	0.5550 5411
60	0.7792 0538	0.7413 7220	0.7054 0505	0.6386 9070	0.5504 4962
61	0.7759 7216	0.7376 8378	0.7013 1405	0.6339 4511	0.5440 0962
62	0.7727 5236	0.7340 1371	0.6972 4678	0.6292 2592	0.5396 0358
63	0.7695 4591	0.7303 6190	0.6932 0310	0.6245 4185	0.5342 6097
64	0·7063 5278	0.7267 2826	0.6801 8286	0.6198 0266	0.5289 7120
65	0.7631 7289	0.7231 1209	0.6851 8594	0.6152 7807	0.5237 3302
66	0.7600 0620	0.7195 1512	0.6812 1221	0.6106 9784	0.5185 4844
67	0.7568 5265	0.7159 3544	0.6772 6151	0.6061 5170	0.5134 1429
68	0.7537 1218	0.7123 7357	0.6733 3373	0.6016 3940	0.5083 3009
69	0.7505 8474	0.7088 2943	0.6694 2873	0.5971 6070	0.5032 0801
70	0.7474 7028	0.7053 0291	0.6655 4638	0.5927 1533	0.4983 1480
71	0.7443 6874	0.7017 9394	0.6616 8654	0.5883 0306	0.4933 8105
72	0.7412 8008	0.6983 0243	0.6578 4909	0.5839 2363	0.4884 0609
73	0.7382 0423	0.6948 2829	0.6540 8389	0.5795 7681	0.4836 5940
74	0.7351 4114	0.6913 7143	0.6502 4082	0.5752 6234	0.4788 7078
75	0.7320 9076	0.6879 3177	0.6464 6975	0.5709 7999	0.4741 2949
76	0.7290 5304	0.6845 0923	0.6427 2054	0.5667 2952	0.4694 3514
77	0.7260 2792	0.6811 0371	0.6389 9308	0.5025 1069	0.4647 8726
78	0.7230 1536	0.6777 1518	0.6352 8724	0.5583 2326	0.4601 8541
79	0.7200 1529	0.6743 4342	0.6316 0289	0.5541 6701	0.4556 2912
80	0.7170 2768	0.6709 8847	0.6279 3991	0.5500 4170	0.4511 1794
81	0.7140 5246	0.6676 5022	0.6242 9817	0.5459 4710	0.4466 5142
82	0.7110 8959	0.6643 2858	0.6206 7755	0.5418 8207	0.4422 2913
83	0.7081 3901	0.6610 2346	0.6170 7793	0.5378 4911	0.4378 5063
84	0.7052 0067	0.6577 3479	0.6134 9919	0.5338 4527	0.4335 1547
85	0.7022 7453	0.6544 6248	0.6099 4120	0.5298 7123	0.4202 2324
86	0.6993 6052	0.6512 0644	0.6064 0384	0.5259 2678	0.4249 7350
87	0.6964 5861	0.6479 6661	0.6028 8700	0.5220 1160	0.4207 6585
88	0.6935 6874	0.6447 4290	0.5993 9056	0.5181 2575	0.4165 9085
89	0.6906 9086	0.6415 3522	0.5959 1439	0.5142 6873	0.4124 7510
90	0.6878 2493	0.6383 4350	0.5924 5838	0.5104 4043	0.4083 0110
91	0.6849 7088	0.6351 6766	0.5890 2242	0.5066 4083	0.4043 4771
92	0.6821 2868	0.6320 0763	0.5856 0638	0.5028 6911	0.4003 4427
93	0.6792 9827	0.6288 6331	0.5822 1015	0.4991 2567	0.3963 8046
94	0.6764 7960	0.6257 3464	0.5788 3363	0.4954 1009	0.3024 5590
95	0.6736 7263	0.6226 2153	0.5754 7668	0.4917 2217	0.3885 7020
96	0.6708 7731	0.6195 2391	0.5721 3920	0.4880 6171	0.3847 2207
97	0.6680 9359	0.6164 4170	0.5688 2108	0.4844 2850	0.3809 1383
98	0.6653 2141	0.6133 7483	0.5655 2220	0.4808 2233	0.3771 4241
99	0.6625 6074	0.6103 2321	0.5622 4245	0.4772 4301	0.3734 0832
100	0.6598 1153	0.6072 8678	0.5589 8172	0.4736 9033	0.3697 1121

TABLE VI — PRESENT VALUE OF 1

$v^n = (1 + i)^{-n}$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	0.6570 7372	0.6042 6545	0.5557 3991	0.4701 6410	0.3660 5071
102	0.6543 4727	0.6012 5015	0.5525 1689	0.4666 6412	0.3624 2644
103	0.6516 3214	0.5982 6781	0.5493 1257	0.4631 9019	0.3588 3806
104	0.6489 2827	0.5952 9130	0.5461 2683	0.4597 4213	0.3552 8521
105	0.6402 3562	0.5023 2971	0.5429 5957	0.4563 1973	0.3517 6753
106	0.6435 5415	0.5893 8279	0.5398 1067	0.4529 2281	0.3482 8469
107	0.6408 8380	0.5864 5054	0.5366 8004	0.4495 5117	0.3448 3632
108	0.6382 2453	0.5835 3288	0.5335 6750	0.4462 0464	0.3414 2210
109	0.6355 7630	0.5806 2973	0.5304 7313	0.4428 8302	0.3380 4168
110	0.6329 3905	0.5777 4102	0.5273 9665	0.4395 8012	0.3346 9474
111	0.6303 1275	0.5748 6669	0.5243 3801	0.4363 1377	0.3213 8093
112	0.6276 9734	0.5720 0666	0.5212 9711	0.4330 6577	0.3280 9993
113	0.6250 9279	0.5691 6085	0.5182 7385	0.4298 4196	0.3248 5141
114	0.6224 9904	0.5663 2921	0.5152 6812	0.4266 4124	0.3216 3506
115	0.6199 1606	0.5635 1165	0.5122 7982	0.4234 6615	0.3184 5056
116	0.6173 4379	0.5607 0811	0.5093 0885	0.4203 1379	0.3152 9758
117	0.6147 8220	0.5579 1852	0.5063 5512	0.4171 8491	0.3121 7582
118	0.6122 3123	0.5551 4280	0.5034 1851	0.4140 7931	0.3090 8497
119	0.6096 9086	0.5523 8090	0.5004 9893	0.4109 9683	0.3060 2473
120	0.6071 6102	0.5496 3273	0.4975 9629	0.4079 3780	0.3029 9478
121	0.6046 4168	0.5468 9824	0.4947 1047	0.4049 0055	0.2999 9483
122	0.6021 3279	0.5441 7736	0.4918 4140	0.4018 8640	0.2970 2459
123	0.5996 3431	0.5414 7001	0.4889 8896	0.3988 9409	0.2940 8375
124	0.5971 4620	0.5387 7612	0.4861 5307	0.3959 2525	0.2911 7203
125	0.5946 6842	0.5360 9565	0.4833 3363	0.3920 7792	0.2882 8914
126	0.5922 0091	0.5334 2850	0.4805 3058	0.3900 5252	0.2854 3479
127	0.5897 4365	0.5307 7463	0.4777 4369	0.3871 4891	0.2826 0870
128	0.5872 9658	0.5281 3396	0.4749 7302	0.3842 6691	0.2798 1060
129	0.5848 5966	0.5255 0643	0.4722 1841	0.3814 0636	0.2770 4019
130	0.5824 3286	0.5228 9197	0.4694 7978	0.3785 6711	0.2742 9722
131	0.5800 1613	0.5202 9052	0.4667 5703	0.3757 4899	0.2715 8141
132	0.5776 0942	0.5177 0201	0.4640 5007	0.3729 5185	0.2688 9248
133	0.5752 1270	0.5151 2637	0.4613 5881	0.3701 7553	0.2662 3018
134	0.5728 2593	0.5125 6356	0.4586 8316	0.3674 1988	0.2635 9424
135	0.5704 4906	0.5100 1349	0.4560 2303	0.3646 8475	0.2609 8439
136	0.5680 8205	0.5074 7611	0.4533 7832	0.3619 6997	0.2584 0030
137	0.5657 2486	0.5049 5135	0.4507 4895	0.3592 7541	0.2558 4197
138	0.5633 7745	0.5024 3916	0.4481 3483	0.3566 0090	0.2533 0888
139	0.5610 3979	0.4999 3946	0.4455 3587	0.3539 4630	0.2508 0087
140	0.5587 1182	0.4974 5220	0.4429 5198	0.3513 1147	0.2483 1770
141	0.5563 9351	0.4949 7731	0.4403 8308	0.3486 9625	0.2458 5911
142	0.5540 8483	0.4925 1474	0.4378 2908	0.3461 0049	0.2434 2486
143	0.5517 8572	0.4900 6442	0.4352 8989	0.3435 2406	0.2410 1471
144	0.5494 9615	0.4876 2628	0.4327 6542	0.3409 6681	0.2386 2843
145	0.5472 1609	0.4852 0028	0.4302 5560	0.3384 2860	0.2362 6577
146	0.5449 4548	0.4827 8635	0.4277 6033	0.3359 0928	0.2339 2650
147	0.5426 8429	0.4803 8443	0.4252 7953	0.3334 0871	0.2316 1040
148	0.5404 3249	0.4779 9446	0.4228 1312	0.3309 2676	0.2293 1723
149	0.5381 9003	0.4756 1637	0.4203 6102	0.3284 6329	0.2270 4676
150	0.5359 5688	0.4732 5012	0.4179 2313	0.3260 1815	0.2247 9877

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.9888 7515	0.9876 5432	0.9852 2167	0.9828 0098	0.9803 9216
2	0.9778 7407	0.9754 0106	0.9706 6175	0.9658 9777	0.9611 6878
3	0.9669 9537	0.9634 1833	0.9563 1090	0.9492 8528	0.9423 2233
4	0.9562 3770	0.9515 2428	0.9421 8423	0.9329 5851	0.9238 4543
5	0.9455 9970	0.9397 7706	0.9282 6033	0.9169 1254	0.9057 3081
6	0.9350 8005	0.9281 7488	0.9145 4219	0.9011 4254	0.8879 7138
7	0.9246 7743	0.9167 1593	0.9010 2679	0.8856 4378	0.8705 6018
8	0.9143 9054	0.9053 9845	0.8877 1112	0.8704 1157	0.8534 9037
9	0.9042 1808	0.8942 2069	0.8745 9224	0.8554 4135	0.8367 5527
10	0.8941 5881	0.8831 8093	0.8616 6723	0.8407 2860	0.8203 4830
11	0.8842 1142	0.8722 7746	0.8489 3323	0.8262 0889	0.8042 6304
12	0.8743 7470	0.8615 0860	0.8363 8742	0.8120 5788	0.7884 9318
13	0.8646 4742	0.8508 7269	0.8240 2702	0.7980 9128	0.7730 3253
14	0.8550 2835	0.8403 6809	0.8118 4928	0.7843 6490	0.7578 7502
15	0.8455 1629	0.8299 9318	0.7998 5150	0.7708 7459	0.7430 1473
16	0.8361 1005	0.8197 4635	0.7880 3104	0.7576 1631	0.7284 4581
17	0.8268 0846	0.8096 2602	0.7763 8526	0.7445 8605	0.7141 6256
18	0.8176 1034	0.7996 3064	0.7649 1159	0.7317 7990	0.7001 5937
19	0.8085 1455	0.7897 5866	0.7536 0747	0.7191 9401	0.6864 3076
20	0.7995 1995	0.7800 0855	0.7424 7042	0.7068 2458	0.6729 7133
21	0.7906 2542	0.7703 7881	0.7314 9795	0.6946 6789	0.6597 7582
22	0.7818 2983	0.7608 6796	0.7206 8763	0.6827 2028	0.6468 3904
23	0.7731 3210	0.7514 7453	0.7100 3708	0.6709 7817	0.6341 5592
24	0.7645 3112	0.7421 9707	0.6995 4392	0.6594 3800	0.6217 2149
25	0.7560 2583	0.7330 3414	0.6892 0583	0.6480 9632	0.6095 3087
26	0.7476 1516	0.7239 8434	0.6790 2052	0.6369 4970	0.5975 7928
27	0.7392 9806	0.7150 4626	0.6689 8574	0.6259 9479	0.5858 6204
28	0.7310 7348	0.7062 1853	0.6590 9925	0.6152 2829	0.5743 7455
29	0.7229 4040	0.6974 9978	0.6493 5887	0.6046 4697	0.5631 1231
30	0.7148 9780	0.6888 8867	0.6397 6243	0.5942 4764	0.5520 7089
31	0.7069 4467	0.6803 8387	0.6303 0781	0.5840 2716	0.5412 4597
32	0.6990 8002	0.6719 8407	0.6209 9292	0.5739 8247	0.5306 3330
33	0.6913 0287	0.6636 8797	0.6118 1568	0.5641 1053	0.5202 2873
34	0.6836 1223	0.6554 9429	0.6027 7407	0.5544 0839	0.5100 2817
35	0.6760 0715	0.6474 0177	0.5938 6608	0.5448 7311	0.5000 2761
36	0.6684 8667	0.6394 0916	0.5850 8974	0.5355 0183	0.4902 2315
37	0.6610 4986	0.6315 1522	0.5764 4309	0.5262 9172	0.4806 1093
38	0.6536 9578	0.6237 1873	0.5679 2423	0.5172 4002	0.4711 8719
39	0.6464 2352	0.6160 1850	0.5595 3126	0.5083 4400	0.4619 4822
40	0.6392 3216	0.6084 1334	0.5512 6232	0.4996 0098	0.4528 9042
41	0.6321 2080	0.6009 0206	0.5431 1559	0.4910 0834	0.4440 1021
42	0.6250 8855	0.5934 8352	0.5350 8925	0.4825 6348	0.4353 0413
43	0.6181 3454	0.5861 5656	0.5271 8153	0.4742 6386	0.4267 6875
44	0.6112 5789	0.5789 2006	0.5193 9067	0.4661 0699	0.4184 0074
45	0.6044 5774	0.5717 7290	0.5117 1494	0.4580 9040	0.4101 9680
46	0.5977 3324	0.5647 1397	0.5041 5265	0.4502 1170	0.4021 5373
47	0.5910 8355	0.5577 4219	0.4967 0212	0.4424 6850	0.3942 6836
48	0.5845 0784	0.5508 5649	0.4893 6170	0.4348 5848	0.3865 3761
49	0.5780 0528	0.5440 5579	0.4821 2975	0.4273 7934	0.3789 5844
50	0.5715 7506	0.5373 3905	0.4750 0468	0.4200 2888	0.3715 2788

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	0.5652 1637	0.5307 0524	0.4679 8491	0.4128 0475	0.3642 4302
52	0.5589 2843	0.5241 5332	0.4610 6887	0.4057 0492	0.3571 0100
53	0.5527 1044	0.5176 8229	0.4542 5505	0.3987 2719	0.3500 9902
54	0.5465 6162	0.5112 9115	0.4475 4192	0.3918 6947	0.3432 3433
55	0.5404 8120	0.5049 7892	0.4409 2800	0.3851 2970	0.3365 0425
56	0.5344 6843	0.4087 4401	0.4344 1182	0.3785 0585	0.3299 0613
57	0.5285 2256	0.4925 8727	0.4279 9194	0.3719 9592	0.3234 3738
58	0.5226 4282	0.4865 0594	0.4216 6694	0.3655 9796	0.3170 9547
59	0.5168 2850	0.4804 9970	0.4154 3541	0.3593 1003	0.3108 7791
60	0.5110 7887	0.4745 6760	0.4092 9597	0.3531 3025	0.3047 8227
61	0.5053 9319	0.4687 0874	0.4032 4726	0.3470 5676	0.2988 0614
62	0.4997 7077	0.4629 2222	0.3972 8794	0.3410 8772	0.2929 4720
63	0.4942 1090	0.4572 0713	0.3914 1669	0.3352 2135	0.2872 0314
64	0.4887 1288	0.4515 6259	0.3856 3221	0.3294 5587	0.2815 7170
65	0.4832 7602	0.4459 8775	0.3799 3321	0.3237 8956	0.2760 5069
66	0.4778 9965	0.4404 8173	0.3743 1843	0.3182 2069	0.2706 3793
67	0.4725 8309	0.4350 4368	0.3687 8663	0.3127 4761	0.2653 3130
68	0.4673 2568	0.4296 7277	0.3633 3658	0.3073 6866	0.2601 2873
69	0.4621 2675	0.4243 6817	0.3579 6708	0.3020 8222	0.2550 2817
70	0.4569 8566	0.4191 2905	0.3526 7692	0.2968 8670	0.2500 2761
71	0.4519 0177	0.4139 5462	0.3474 6495	0.2917 8054	0.2451 2511
72	0.4468 7443	0.4088 4407	0.3423 3000	0.2867 6221	0.2403 1874
73	0.4419 0302	0.4037 9661	0.3372 7093	0.2818 3018	0.2356 0661
74	0.4369 8692	0.3988 1147	0.3322 8663	0.2769 8298	0.2309 8687
75	0.4321 2551	0.3938 8787	0.3273 7599	0.2722 1914	0.2264 5771
76	0.4273 1818	0.3890 2506	0.3225 3793	0.2675 3724	0.2220 1737
77	0.4225 6433	0.3842 2228	0.3177 7136	0.2629 3586	0.2176 6408
78	0.4178 6337	0.3794 7870	0.3130 7523	0.2584 1362	0.2133 9616
79	0.4132 1470	0.3747 9387	0.3084 4850	0.2539 6916	0.2092 1192
80	0.4086 1775	0.3701 6679	0.3038 9015	0.2496 0114	0.2051 0973
81	0.4040 7194	0.3655 9683	0.2993 9916	0.2453 0825	0.2010 8797
82	0.3995 7670	0.3610 8329	0.2949 7454	0.2410 8919	0.1971 4507
83	0.3951 3148	0.3566 2547	0.2906 1531	0.2369 4269	0.1932 7948
84	0.3907 3570	0.3522 2268	0.2863 2050	0.2328 6751	0.1894 8968
85	0.3863 8882	0.3478 7426	0.2820 8917	0.2288 6242	0.1857 7420
86	0.3820 9031	0.3435 7951	0.2779 2036	0.2249 2621	0.1821 3157
87	0.3778 3961	0.3393 3779	0.2738 1316	0.2210 5770	0.1785 6036
88	0.3736 3621	0.3351 4843	0.2697 6666	0.2172 5572	0.1750 5918
89	0.3694 7956	0.3310 1080	0.2657 7997	0.2135 1914	0.1716 2665
90	0.3653 6916	0.3269 2425	0.2618 5218	0.2098 4682	0.1682 6142
91	0.3613 0448	0.3228 8814	0.2579 8245	0.2062 3766	0.1649 6217
92	0.3572 8503	0.3189 0187	0.2541 6990	0.2026 9057	0.1617 2762
93	0.3533 1029	0.3149 6481	0.2504 1369	0.1992 0450	0.1585 5649
94	0.3493 7976	0.3110 7636	0.2467 1300	0.1957 7837	0.1554 4754
95	0.3454 9297	0.3072 3591	0.2430 6699	0.1924 1118	0.1523 9955
96	0.3416 4941	0.3034 4287	0.2394 7487	0.1891 0190	0.1494 1132
97	0.3378 4861	0.2996 9666	0.2359 3583	0.1858 4953	0.1464 8169
98	0.3340 9010	0.2959 9670	0.2324 4909	0.1826 5310	0.1436 0950
99	0.3303 7340	0.2923 4242	0.2290 1389	0.1795 1165	0.1407 9363
100	0.3266 9805	0.2887 3326	0.2256 2944	0.1764 2422	0.1380 3297

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379	0.9661 8357
2	0.9564 7444	0.9518 1440	0.9471 8833	0.9425 9591	0.9335 1070
3	0.9354 2732	0.9285 9941	0.9218 3779	0.9151 4166	0.9019 4271
4	0.9148 4335	0.9059 5064	0.8971 6573	0.8884 8705	0.8714 4223
5	0.8947 1232	0.8838 5429	0.8731 5400	0.8626 0878	0.8419 7317
6	0.8750 2427	0.8622 9687	0.8497 8491	0.8374 8426	0.8135 0064
7	0.8557 6940	0.8412 6524	0.8270 4128	0.8130 9151	0.7859 9096
8	0.8369 3835	0.8207 4657	0.8049 0635	0.7894 0923	0.7594 1156
9	0.8185 2161	0.8007 2836	0.7833 6385	0.7664 1673	0.7337 3097
10	0.8005 1013	0.7811 9840	0.7623 9791	0.7440 9391	0.7089 1881
11	0.7828 9499	0.7621 4478	0.7419 9310	0.7224 2128	0.6849 4571
12	0.7656 6748	0.7435 5589	0.7221 3440	0.7013 7988	0.6617 8330
13	0.7488 1905	0.7254 2038	0.7028 0720	0.6809 5134	0.6394 0415
14	0.7323 4137	0.7077 2720	0.6839 9728	0.6611 1781	0.6177 8179
15	0.7162 2628	0.6904 6556	0.6656 9078	0.6418 6195	0.5968 9062
16	0.7004 6580	0.6736 2493	0.6478 7424	0.6231 6694	0.5767 0591
17	0.6850 5212	0.6571 9506	0.6305 3454	0.6050 1645	0.5572 0378
18	0.6699 7763	0.6411 6591	0.6136 5892	0.5873 9461	0.5383 6114
19	0.6552 3484	0.6255 2772	0.5972 3496	0.5702 8603	0.5201 5569
20	0.6408 1647	0.6102 7094	0.5812 5057	0.5536 7575	0.5025 6588
21	0.6267 1538	0.5953 8629	0.5656 9398	0.5375 4928	0.4855 7090
22	0.6129 2457	0.5808 6467	0.5505 5375	0.5218 9250	0.4691 5063
23	0.5994 3724	0.5666 9724	0.5358 1874	0.5066 9175	0.4532 8563
24	0.5862 4668	0.5528 7535	0.5214 7809	0.4919 3374	0.4379 5713
25	0.5733 4639	0.5393 9059	0.5075 2126	0.4776 0557	0.4231 4699
26	0.5607 2997	0.5262 3472	0.4939 3796	0.4636 9473	0.4088 3767
27	0.5483 9117	0.5133 9973	0.4807 1821	0.4501 8906	0.3950 1224
28	0.5363 2388	0.5008 7778	0.4678 5227	0.4370 7675	0.3816 5434
29	0.5245 2213	0.4886 6125	0.4553 3068	0.4243 4636	0.3687 4815
30	0.5129 8008	0.4767 4269	0.4431 4421	0.4119 8676	0.3562 7841
31	0.5016 9201	0.4651 1481	0.4312 8391	0.3999 8715	0.3442 3035
32	0.4906 5233	0.4537 7055	0.4197 4103	0.3883 3703	0.3325 8971
33	0.4798 5558	0.4427 0298	0.4085 0708	0.3770 2625	0.3213 4271
34	0.4692 9641	0.4319 0534	0.3975 7380	0.3660 4490	0.3104 7605
35	0.4589 6960	0.4213 7107	0.3869 3314	0.3553 8340	0.2999 7686
36	0.4488 7002	0.4110 9372	0.3765 7727	0.3450 3243	0.2898 3272
37	0.4389 9268	0.4010 6705	0.3664 9856	0.3349 8294	0.2800 3161
38	0.4293 3270	0.3912 8492	0.3566 8959	0.3252 2615	0.2705 6194
39	0.4198 8528	0.3817 4139	0.3471 4316	0.3157 5355	0.2614 1250
40	0.4106 4575	0.3724 3062	0.3378 5222	0.3065 5684	0.2525 7247
41	0.4016 0954	0.3633 4695	0.3288 0995	0.2976 2800	0.2440 3137
42	0.3927 7216	0.3544 8483	0.3200 0968	0.2889 5922	0.2357 7910
43	0.3841 2925	0.3458 3886	0.3114 4495	0.2805 4294	0.2278 0590
44	0.3756 7653	0.3374 0376	0.3031 0944	0.2723 7178	0.2201 0231
45	0.3674 0981	0.3291 7440	0.2949 9702	0.2644 3862	0.2126 5924
46	0.3593 2500	0.3211 4576	0.2871 0172	0.2567 3653	0.2054 6787
47	0.3514 1809	0.3133 1294	0.2794 1773	0.2492 5876	0.1985 1968
48	0.3436 8518	0.3056 7116	0.2719 3940	0.2419 9880	0.1918 0645
49	0.3361 2242	0.2982 1576	0.2646 6122	0.2349 5029	0.1853 2024
50	0.3287 2608	0.2909 4221	0.2575 7783	0.2281 0708	0.1790 5337

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	0.3214 9250	0.2838 4006	0.2506 8402	0.2214 6318	0.1729 9843
52	0.3144 1810	0.2769 2208	0.2439 7471	0.2150 1280	0.1671 4824
53	0.3074 9936	0.2701 0870	0.2374 4497	0.2087 5029	0.1614 9589
54	0.3007 3287	0.2635 7928	0.2310 9000	0.2026 7019	0.1560 3467
55	0.2941 1528	0.2571 5052	0.2249 0511	0.1967 6717	0.1507 5814
56	0.2876 4330	0.2508 7855	0.2188 8575	0.1910 3609	0.1456 6004
57	0.2813 1374	0.2447 5956	0.2130 2749	0.1854 7193	0.1407 3433
58	0.2751 2347	0.2387 8982	0.2073 2603	0.1800 6984	0.1359 7520
59	0.2690 6940	0.2329 6568	0.2017 7716	0.1748 2508	0.1313 7701
60	0.2631 4856	0.2272 8359	0.1963 7679	0.1697 3309	0.1269 3431
61	0.2573 5801	0.2217 4009	0.1911 2097	0.1647 8941	0.1226 4184
62	0.2516 9487	0.2163 3179	0.1860 0581	0.1599 8972	0.1184 9453
63	0.2461 5635	0.2110 5541	0.1810 2755	0.1553 2982	0.1144 8747
64	0.2407 3971	0.2059 0771	0.1761 8253	0.1508 0565	0.1106 1591
65	0.2354 4226	0.2008 8557	0.1714 6718	0.1464 1325	0.1068 7528
66	0.2302 6138	0.1959 8593	0.1668 7804	0.1421 4879	0.1032 6114
67	0.2251 9450	0.1912 0578	0.1624 1172	0.1380 0853	0.0997 6922
68	0.2202 3912	0.1865 4223	0.1580 6493	0.1339 8887	0.0963 9538
69	0.2153 9278	0.1819 9241	0.1538 3448	0.1300 8628	0.0931 3563
70	0.2106 5309	0.1775 5358	0.1497 1726	0.1262 9736	0.0899 8612
71	0.2060 1769	0.1732 2300	0.1457 1023	0.1226 1880	0.0869 4311
72	0.2014 8429	0.1689 9805	0.1418 1044	0.1190 4737	0.0840 0300
73	0.1970 5065	0.1648 7615	0.1380 1503	0.1155 7998	0.0811 6232
74	0.1927 1458	0.1608 5478	0.1343 2119	0.1122 1357	0.0784 1770
75	0.1884 7391	0.1569 3149	0.1307 2622	0.1089 4521	0.0757 6590
76	0.1843 2657	0.1531 0389	0.1272 2747	0.1057 7205	0.0732 0376
77	0.1802 7048	0.1493 6965	0.1238 2235	0.1026 9131	0.0707 2827
78	0.1763 0365	0.1457 2649	0.1205 0837	0.0997 0030	0.0683 3650
79	0.1724 2411	0.1421 7218	0.1172 8309	0.0967 9641	0.0660 2560
80	0.1686 2993	0.1387 0457	0.1141 4412	0.0939 7710	0.0637 9285
81	0.1649 1925	0.1353 2153	0.1110 8917	0.0912 3990	0.0616 3561
82	0.1612 9022	0.1320 2101	0.1081 1598	0.0885 8243	0.0595 5131
83	0.1577 4105	0.1288 0098	0.1052 2237	0.0860 0236	0.0575 3750
84	0.1542 6997	0.1256 5949	0.1024 0620	0.0834 9743	0.0555 9178
85	0.1508 7528	0.1225 9463	0.0996 6540	0.0810 6547	0.0537 1187
86	0.1475 5528	0.1196 0452	0.0969 9795	0.0787 0434	0.0518 9553
87	0.1443 0835	0.1166 8733	0.0944 0190	0.0764 1198	0.0501 4060
88	0.1411 3286	0.1138 4130	0.0918 7533	0.0741 8639	0.0484 4503
89	0.1380 2724	0.1110 6468	0.0894 1638	0.0720 2562	0.0468 0679
90	0.1349 8997	0.1083 5579	0.0870 2324	0.0699 2779	0.0452 2395
91	0.1320 1953	0.1057 1296	0.0846 9415	0.0678 9105	0.0436 9464
92	0.1291 1445	0.1031 3460	0.0824 2740	0.0659 1364	0.0422 1704
93	0.1262 7331	0.1006 1912	0.0802 2131	0.0639 9383	0.0407 8941
94	0.1234 9468	0.0981 6500	0.0780 7427	0.0621 2993	0.0394 1006
95	0.1207 7719	0.0957 7073	0.0759 8469	0.0603 2032	0.0380 7735
96	0.1181 1950	0.0934 3486	0.0739 5104	0.0585 6342	0.0367 8971
97	0.1155 2029	0.0911 5596	0.0719 7181	0.0568 5769	0.0355 4562
98	0.1129 7828	0.0889 3264	0.0700 4556	0.0552 0164	0.0343 4359
99	0.1104 9221	0.0867 6355	0.0681 7086	0.0535 9383	0.0331 8221
100	0.1080 6084	0.0846 4737	0.0663 4634	0.0520 3284	0.0320 6011

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730	0.9433 9623
2	0.9245 5621	0.9157 2995	0.9070 2948	0.8984 5242	0.8899 9644
3	0.8889 9636	0.8762 9660	0.8638 3760	0.8516 1366	0.8396 1928
4	0.8548 0419	0.8385 6134	0.8227 0247	0.8072 1674	0.7920 9366
5	0.8219 2711	0.8024 5105	0.7835 2617	0.7651 3435	0.7472 5817
6	0.7903 1453	0.7678 9574	0.7462 1540	0.7252 4583	0.7049 6054
7	0.7599 1781	0.7348 2846	0.7106 8133	0.6874 3681	0.6650 5711
8	0.7306 9021	0.7031 8513	0.6768 3936	0.6515 9887	0.6274 1237
9	0.7025 8674	0.6729 0443	0.6446 0892	0.6176 2926	0.5918 9846
10	0.6755 6417	0.6439 2768	0.6139 1325	0.5854 3058	0.5583 9478
11	0.6495 8093	0.6161 9874	0.5846 7929	0.5549 1050	0.5267 8753
12	0.6245 9705	0.5896 6386	0.5568 3742	0.5259 8152	0.4969 6936
13	0.6005 7409	0.5642 7164	0.5303 2135	0.4985 6068	0.4688 3902
14	0.5774 7508	0.5399 7286	0.5050 6795	0.4725 6937	0.4423 0096
15	0.5552 6450	0.5167 2044	0.4810 1710	0.4479 3305	0.4172 6506
16	0.5339 0818	0.4944 6932	0.4581 1152	0.4245 8109	0.3936 4628
17	0.5133 7325	0.4731 7639	0.4362 9669	0.4024 4653	0.3713 6442
18	0.4936 2812	0.4528 0037	0.4155 2065	0.3814 6590	0.3503 4379
19	0.4746 4242	0.4333 0179	0.3957 3396	0.3615 7906	0.3305 1301
20	0.4563 8695	0.4146 4286	0.3768 8948	0.3427 2896	0.3118 0473
21	0.4388 3360	0.3967 8743	0.3589 4236	0.3248 6158	0.2941 5540
22	0.4219 5539	0.3797 0089	0.3418 4987	0.3079 2567	0.2775 0510
23	0.4057 2633	0.3633 5013	0.3255 7131	0.2918 7267	0.2617 9726
24	0.3901 2147	0.3477 0347	0.3100 6791	0.2766 5656	0.2469 7855
25	0.3751 1680	0.3327 3060	0.2953 0277	0.2622 3370	0.2329 9863
26	0.3606 8923	0.3184 0248	0.2812 4073	0.2485 6275	0.2198 1003
27	0.3468 1657	0.3046 9137	0.2678 4832	0.2356 0450	0.2073 6795
28	0.3334 7747	0.2915 7069	0.2550 9364	0.2233 2181	0.1956 3014
29	0.3206 5141	0.2790 1502	0.2429 4632	0.2116 7944	0.1845 5674
30	0.3083 1867	0.2670 0002	0.2313 7745	0.2006 4402	0.1741 1013
31	0.2964 6026	0.2555 0241	0.2203 5947	0.1901 8390	0.1642 5484
32	0.2850 5794	0.2444 9991	0.2098 6617	0.1802 6910	0.1549 5740
33	0.2740 9417	0.2339 7121	0.1998 7254	0.1708 7119	0.1461 8622
34	0.2635 5209	0.2238 9589	0.1903 5480	0.1619 6321	0.1379 1153
35	0.2534 1547	0.2142 5444	0.1812 9029	0.1535 1963	0.1301 0522
36	0.2436 6872	0.2050 2817	0.1726 5741	0.1455 1624	0.1227 4077
37	0.2342 9685	0.1961 9921	0.1644 3563	0.1379 3008	0.1157 9318
38	0.2252 8543	0.1877 5044	0.1566 0536	0.1307 3941	0.1092 3885
39	0.2166 2061	0.1796 6549	0.1491 4797	0.1239 2362	0.1030 5552
40	0.2082 8904	0.1719 2870	0.1420 4568	0.1174 6314	0.0972 2219
41	0.2002 7793	0.1645 2507	0.1352 8160	0.1113 3947	0.0917 1905
42	0.1925 7493	0.1574 4026	0.1288 3962	0.1055 3504	0.0865 2740
43	0.1851 6820	0.1506 6054	0.1227 0440	0.1000 3322	0.0816 2962
44	0.1780 4635	0.1441 7276	0.1168 6133	0.0948 1822	0.0770 0908
45	0.1711 9841	0.1379 6437	0.1112 9651	0.0898 7509	0.0726 5007
46	0.1646 1386	0.1320 2332	0.1059 9668	0.0851 8965	0.0685 3781
47	0.1582 8256	0.1263 3810	0.1009 4921	0.0807 4849	0.0646 5831
48	0.1521 9476	0.1208 9771	0.0961 4211	0.0765 3885	0.0609 9840
49	0.1463 4112	0.1156 9158	0.0915 6391	0.0725 4867	0.0575 4566
50	0.1407 1262	0.1107 0965	0.0872 0373	0.0687 6652	0.0542 8836

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	0.1353 0059	0.1059 4225	0.0830 5117	0.0651 8153	0.0512 1544
52	0.1300 9672	0.1013 8014	0.0790 9635	0.0617 8344	0.0483 1645
53	0.1250 9300	0.0970 1449	0.0753 2986	0.0585 6250	0.0455 8156
54	0.1202 8173	0.0928 3683	0.0717 4272	0.0555 0948	0.0430 0147
55	0.1156 5551	0.0888 3907	0.0683 2640	0.0526 1562	0.0405 6742
56	0.1112 0722	0.0850 1347	0.0650 7276	0.0498 7263	0.0382 7115
57	0.1069 3002	0.0813 5260	0.0619 7406	0.0472 7263	0.0361 0486
58	0.1028 1733	0.0778 4938	0.0590 2291	0.0448 0818	0.0340 6119
59	0.0988 6282	0.0744 9701	0.0562 1230	0.0424 7221	0.0321 3320
60	0.0950 6040	0.0712 8901	0.0535 3552	0.0402 5802	0.0303 1434
61	0.0914 0423	0.0682 1915	0.0509 8621	0.0381 5926	0.0285 9843
62	0.0878 8868	0.0652 8148	0.0485 5830	0.0361 6992	0.0269 7965
63	0.0845 0835	0.0624 7032	0.0462 4600	0.0342 8428	0.0254 5250
64	0.0812 5803	0.0597 8021	0.0440 4381	0.0324 9695	0.0240 1179
65	0.0781 3272	0.0572 0594	0.0419 4648	0.0308 0279	0.0226 5264
66	0.0751 2762	0.0547 4253	0.0399 4903	0.0291 9696	0.0213 7041
67	0.0722 3809	0.0523 8519	0.0380 4670	0.0276 7485	0.0201 6077
68	0.0694 5970	0.0501 2937	0.0362 3495	0.0262 3208	0.0190 1959
69	0.0667 8818	0.0479 7069	0.0345 0948	0.0248 6453	0.0179 4301
70	0.0642 1940	0.0459 0497	0.0328 6617	0.0235 6828	0.0169 2737
71	0.0617 4942	0.0439 2820	0.0313 0111	0.0223 3960	0.0159 6921
72	0.0593 7445	0.0420 3655	0.0298 1058	0.0211 7498	0.0150 6530
73	0.0570 9081	0.0402 2637	0.0283 9103	0.0200 7107	0.0142 1254
74	0.0548 9501	0.0384 9413	0.0270 3908	0.0190 2471	0.0134 0806
75	0.0527 8367	0.0368 3640	0.0257 5150	0.0180 3290	0.0126 4911
76	0.0507 5353	0.0352 5023	0.0245 2524	0.0170 9279	0.0119 3313
77	0.0488 0147	0.0337 3228	0.0233 5737	0.0162 0170	0.0112 5767
78	0.0469 2449	0.0322 7969	0.0222 4512	0.0153 5706	0.0106 2044
79	0.0451 1970	0.0308 8965	0.0211 8582	0.0145 5646	0.0100 1928
80	0.0433 8433	0.0295 5948	0.0201 7698	0.0137 9759	0.0094 5215
81	0.0417 1570	0.0282 8658	0.0192 1617	0.0130 7828	0.0089 1713
82	0.0401 1125	0.0270 6850	0.0183 0111	0.0123 9648	0.0084 1238
83	0.0385 6851	0.0259 0287	0.0174 2963	0.0117 5022	0.0079 3621
84	0.0370 8510	0.0247 8744	0.0165 9965	0.0111 3765	0.0074 8699
85	0.0356 5875	0.0237 2003	0.0158 0919	0.0105 5701	0.0070 6320
86	0.0342 8726	0.0226 9860	0.0150 5637	0.0100 0664	0.0066 6340
87	0.0329 6852	0.0217 2115	0.0143 3940	0.0094 8497	0.0062 8622
88	0.0317 0050	0.0207 8579	0.0136 5657	0.0089 9049	0.0059 3040
89	0.0304 8125	0.0198 9070	0.0130 0626	0.0085 2180	0.0055 9472
90	0.0293 0890	0.0190 3417	0.0123 8691	0.0080 7753	0.0052 7803
91	0.0281 8163	0.0182 1451	0.0117 9706	0.0076 5643	0.0049 7928
92	0.0270 9772	0.0174 3016	0.0112 3530	0.0072 5728	0.0046 9743
93	0.0260 5550	0.0166 7958	0.0107 0028	0.0068 7894	0.0044 3154
94	0.0250 5337	0.0159 6132	0.0101 9074	0.0065 2032	0.0041 8070
95	0.0240 8978	0.0152 7399	0.0097 0547	0.0061 8040	0.0039 4405
96	0.0231 6325	0.0146 1626	0.0092 4331	0.0058 5820	0.0037 2081
97	0.0222 7235	0.0139 8685	0.0088 0315	0.0055 5279	0.0035 1019
98	0.0214 1572	0.0133 8454	0.0083 8395	0.0052 6331	0.0033 1150
99	0.0205 9204	0.0128 0817	0.0079 8471	0.0049 8892	0.0031 2406
100	0.0198 0004	0.0122 5663	0.0076 0449	0.0047 2883	0.0029 4723

TABLE VI — PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593	0.9216 5899
2	0.8816 5928	0.8734 3873	0.8653 3261	0.8573 3882	0.8494 5529
3	0.8278 4909	0.8162 9788	0.8049 6057	0.7938 3224	0.7829 0810
4	0.7773 2309	0.7628 9521	0.7488 0053	0.7350 2985	0.7215 7428
5	0.7298 8084	0.7129 8618	0.6965 5863	0.6805 8320	0.6650 4542
6	0.6853 3412	0.6663 4222	0.6479 6152	0.6301 6963	0.6129 4509
7	0.6435 0621	0.6227 4974	0.6027 5490	0.5834 9040	0.5649 2635
8	0.6042 3119	0.5820 0910	0.5607 0223	0.5402 6888	0.5206 6945
9	0.5673 5323	0.5439 3374	0.5215 8347	0.5002 4897	0.4798 7968
10	0.5327 2604	0.5083 4929	0.4851 9393	0.4631 9349	0.4422 8542
11	0.5002 1224	0.4750 9280	0.4513 4319	0.4288 8286	0.4076 3633
12	0.4696 8285	0.4440 1196	0.4198 5413	0.3971 1376	0.3757 0168
13	0.4410 1676	0.4149 6445	0.3905 6198	0.3676 9792	0.3462 6883
14	0.4141 0025	0.3878 1724	0.3633 1347	0.3404 6104	0.3191 4178
15	0.3888 2652	0.3624 4602	0.3379 6602	0.3152 4170	0.2941 3989
16	0.3650 9533	0.3387 3460	0.3143 8699	0.2918 9047	0.2710 9667
17	0.3428 1251	0.3165 7439	0.2924 5302	0.2702 6895	0.2498 5869
18	0.3218 8969	0.2958 6392	0.2720 4932	0.2502 4903	0.2302 8450
19	0.3022 4384	0.2765 0832	0.2530 6913	0.2317 1206	0.2122 4378
20	0.2837 9703	0.2584 1900	0.2354 1315	0.2145 4821	0.1956 1639
21	0.2664 7608	0.2415 1309	0.2189 8897	0.1986 5575	0.1802 9160
22	0.2502 1228	0.2257 1317	0.2037 1997	0.1839 4051	0.1661 6738
23	0.2349 4111	0.2109 4688	0.1894 9830	0.1703 1528	0.1531 4965
24	0.2206 0198	0.1971 4662	0.1762 7749	0.1576 9934	0.1411 5176
25	0.2071 3801	0.1842 4918	0.1639 7906	0.1460 1790	0.1300 9378
26	0.1944 9579	0.1721 9549	0.1525 3866	0.1352 0176	0.1199 0210
27	0.1826 2515	0.1609 3037	0.1418 9643	0.1251 8682	0.1105 0885
28	0.1714 7902	0.1504 0221	0.1319 9668	0.1159 1372	0.1018 5148
29	0.1610 1316	0.1405 6282	0.1227 8761	0.1073 2752	0.0938 7233
30	0.1511 8607	0.1313 6712	0.1142 2103	0.0993 7733	0.0865 1828
31	0.1419 5875	0.1227 7301	0.1062 5212	0.0920 1605	0.0797 4035
32	0.1332 9460	0.1147 4113	0.0988 3918	0.0852 0005	0.0734 9341
33	0.1251 5925	0.1072 3470	0.0919 4343	0.0788 8893	0.0677 3586
34	0.1175 2042	0.1002 1934	0.0855 2877	0.0730 4531	0.0624 2936
35	0.1103 4781	0.0936 6294	0.0795 6164	0.0676 3454	0.0575 3858
36	0.1036 1297	0.0875 3546	0.0740 1083	0.0626 2458	0.0530 3095
37	0.0972 8917	0.0818 0884	0.0688 4729	0.0579 8572	0.0488 7645
38	0.0913 5134	0.0764 5686	0.0640 4399	0.0536 9048	0.0450 4742
39	0.0857 7590	0.0714 5501	0.0595 7580	0.0497 1341	0.0415 1836
40	0.0805 4075	0.0667 8038	0.0554 1935	0.0460 3093	0.0382 6577
41	0.0756 2512	0.0624 1157	0.0515 5288	0.0426 2123	0.0352 6799
42	0.0710 0950	0.0583 2857	0.0479 5617	0.0394 6411	0.0325 0506
43	0.0666 7559	0.0545 1268	0.0446 1039	0.0365 4084	0.0299 5858
44	0.0626 0619	0.0509 4643	0.0414 9804	0.0338 3411	0.0276 1160
45	0.0587 8515	0.0476 1349	0.0386 0283	0.0313 2788	0.0254 4848
46	0.0551 9733	0.0444 9859	0.0359 0961	0.0290 0730	0.0234 5482
47	0.0518 2848	0.0415 8747	0.0334 0428	0.0268 5861	0.0216 1734
48	0.0486 6524	0.0388 6679	0.0310 7375	0.0248 6908	0.0199 2382
49	0.0456 9506	0.0363 2410	0.0289 0582	0.0230 2693	0.0183 6297
50	0.0429 0616	0.0339 4776	0.0268 8913	0.0213 2123	0.0169 2439

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0041 6667	2.0050 0000	2.0058 3333	2.0075 0000	2.0100 0000
3	3.0125 1736	3.0150 2500	3.0175 3403	3.0225 5625	3.0301 0000
4	4.0250 6952	4.0301 0013	4.0351 3631	4.0452 2542	4.0604 0100
5	5.0418 4064	5.0502 5063	5.0586 7460	5.0755 6461	5.1010 0501
6	6.0628 4831	6.0755 0188	6.0881 8354	6.1136 3135	6.1520 1506
7	7.0881 1018	7.1058 7039	7.1236 9794	7.1594 8358	7.2135 3521
8	8.1176 4397	8.1414 0870	8.1652 5284	8.2131 7071	8.2856 7056
9	9.1514 6740	9.1821 1583	9.2128 8349	9.2747 7856	9.3685 2727
10	10.1895 9860	10.2280 2641	10.2666 2531	10.3443 3940	10.4622 1254
11	11.2320 5526	11.2791 6654	11.3265 1398	11.4219 2194	11.5668 3467
12	12.2788 5549	12.3355 6237	12.3925 8529	12.5075 8036	12.6825 0301
13	13.3300 1739	13.3972 4018	13.4648 7537	13.6013 9325	13.8093 2804
14	14.3855 5013	14.4642 2039	14.5434 2048	14.7034 0370	14.9474 2132
15	15.4454 9896	15.5365 4752	15.6282 5710	15.8136 7023	16.0968 9554
16	16.5098 5520	16.6142 3026	16.7194 2193	16.9322 8183	17.2578 6449
17	17.5786 4027	17.6973 0141	17.8169 5189	18.0592 7304	18.4304 4314
18	18.6518 9063	18.7857 8791	18.9208 8411	19.1947 1849	19.6147 4757
19	19.7296 0684	19.8797 1685	20.0312 5593	20.3386 7888	20.8108 9504
20	20.8118 1353	20.9791 1544	21.1481 0493	21.4912 1897	22.0190 0399
21	21.8985 2942	22.0840 1101	22.2714 6887	22.6524 0312	23.2391 9403
22	22.9897 7330	23.1944 3107	23.4013 8577	23.8222 9614	24.4715 8598
23	24.0855 6402	24.3104 0322	24.5378 9386	25.0009 6336	25.7163 0183
24	25.1859 2054	25.4319 5524	25.6810 3157	26.1884 7059	26.9734 6485
25	26.2908 6187	26.5591 1502	26.8308 3759	27.3848 8412	28.2431 9950
26	27.4004 0713	27.6919 1059	27.9873 5081	28.5902 7075	29.5256 3150
27	28.5145 7540	28.8303 7015	29.1506 1035	29.8046 9778	30.8208 8781
28	29.6333 8622	29.9745 2200	30.3206 5558	31.0282 3301	32.1290 9669
29	30.7568 5867	31.1243 9461	31.4975 2607	32.2609 4476	33.4503 8766
30	31.8850 1224	32.2800 1658	32.6812 0164	33.5029 0184	34.7848 9153
31	33.0178 6646	33.4414 1666	33.8719 0233	34.7541 7361	36.1327 4045
32	34.1554 4090	34.6086 2375	35.0694 8843	36.0148 2991	37.4940 6785
33	35.2977 5524	35.7816 6686	36.2740 6045	37.2849 4113	38.8690 0853
34	36.4448 2022	36.9605 7520	37.4856 5013	38.5645 7819	40.2576 9862
35	37.5966 8268	38.1453 7807	38.7043 2548	39.8538 1253	41.6602 7560
36	38.7533 3552	39.3361 0496	39.9301 0071	41.1527 1612	43.0768 7836
37	39.9148 0775	40.5327 8549	41.1630 2030	42.4613 6140	44.5076 4714
38	41.0811 1945	41.7354 4942	42.4031 4305	43.7798 2170	45.9527 2361
39	42.2522 9078	42.9441 2666	43.6504 9562	45.1081 7037	47.4122 5085
40	43.4283 4199	44.1588 4730	44.9051 2352	46.4464 8164	48.8863 7336
41	44.6092 9342	45.3796 4153	46.1670 7007	47.7948 3026	50.3752 3709
42	45.7951 6548	46.6065 3974	47.4363 7798	49.1532 9148	51.8789 8046
43	46.9859 7866	47.8395 7244	48.7130 9018	50.5219 4117	53.3977 7036
44	48.1817 5358	49.0787 7030	49.9972 4988	51.9008 5573	54.9317 5715
45	49.3825 1088	50.3241 6415	51.2889 0050	53.2901 1215	56.4810 7472
46	50.5882 7134	51.5757 8497	52.5880 8575	54.6897 8790	58.0458 8547
47	51.7990 5581	52.8336 6390	53.8948 4059	56.0999 6140	59.6263 4432
48	53.0148 8521	54.0978 3222	55.2092 3621	57.5207 1111	61.2226 0777
49	54.2357 8056	55.3683 2138	56.5312 9009	58.9521 1644	62.8348 3385
50	55.4617 6298	56.6451 6299	57.8610 5595	60.3942 5732	64.4631 8218

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	56.6928 5366	57.9283 8880	59.1985 7877	61.8472 1424	66.1078 1401
52	57.9290 7388	59.2180 3075	60.5439 0381	63.3110 6835	67.7688 0215
53	59.1704 4503	60.5141 2090	61.8970 7659	64.7859 0136	69.4465 8107
54	60.4169 8855	61.8166 9150	63.2581 4287	66.2717 9562	71.1410 4088
55	61.6687 2600	63.1257 7496	64.6271 4870	67.7688 3400	72.8524 5735
56	62.9256 7902	64.4414 0384	66.0041 4040	69.2771 0035	74.5809 8102
57	64.1878 6935	65.7636 1086	67.3891 6455	70.7966 7800	76.3267 9174
58	65.4553 1881	67.0924 2891	68.7822 6801	72.3276 5360	78.0900 5066
59	66.7280 4930	68.4278 9105	70.1834 9701	73.8701 1109	79.8709 6025
60	68.0060 8284	69.7700 3051	71.5929 0165	75.4241 3603	81.6696 6086
61	69.2894 4152	71.1188 8066	73.0105 2091	76.9898 1705	83.4863 6655
62	70.5781 4753	72.4744 7507	74.4364 2165	78.5672 4150	85.3212 3022
63	71.8722 2314	73.8368 4744	75.8706 3411	80.1564 9590	87.1744 4252
64	73.1716 9074	75.2060 3168	77.3132 1281	81.7576 6062	89.0461 8695
65	74.4765 7278	76.5820 6184	78.7642 0655	83.3708 5214	90.9366 4882
66	75.7868 9184	77.9649 7215	80.2236 6442	84.9961 3353	92.8460 1531
67	77.1026 7055	79.3547 9701	81.6916 3579	86.6336 0453	94.7744 7546
68	78.4239 3168	80.7515 7099	83.1681 7034	88.2833 5657	96.7222 2021
69	79.7506 9806	82.1553 2885	84.6533 1800	89.9454 8174	98.6894 4242
70	81.0829 9264	83.5661 0549	86.1471 2902	91.6200 7285	100.6763 3684
71	82.4208 3844	84.9839 3602	87.6496 5394	93.3072 2340	102.6831 0021
72	83.7642 5860	86.4088 5570	89.1609 4359	95.0070 2758	104.7099 3121
73	85.1132 7634	87.8408 9998	90.6810 4900	96.7195 8028	106.7570 3052
74	86.4679 1500	89.2801 0448	92.2100 2188	98.4449 7714	108.8246 0083
75	87.8281 9797	90.7265 0500	93.7479 1367	100.1833 1446	110.9128 4684
76	89.1941 4880	92.1801 3752	95.2947 7650	101.9346 8932	113.0219 7530
77	90.5657 9109	93.6410 3821	96.8506 6270	103.6991 9940	115.1521 9506
78	91.9431 4855	95.1092 4340	98.4156 2490	105.4769 4349	117.3037 1701
79	93.3262 4500	96.5847 8962	99.9897 1604	107.2680 2056	119.4767 5418
80	94.7151 0436	98.0677 1357	101.5729 8038	109.0725 3072	121.6715 2172
81	96.1097 5062	99.5580 5214	103.1654 9849	110.8905 7470	123.8882 3604
82	97.5102 0792	101.0558 4240	104.7672 9723	112.7222 5401	126.1271 1931
83	98.9165 0045	102.5611 2161	106.3784 3980	114.5676 7091	128.3883 9050
84	100.3286 5254	104.0739 2722	107.9989 8070	116.4269 2845	130.6722 7440
85	101.7466 8859	105.5942 9685	109.6289 7475	118.3001 3041	132.9789 9715
86	103.1706 3312	107.1222 6834	111.2684 7710	120.1873 8139	135.3087 8712
87	104.6005 1076	108.6578 7968	112.9175 4322	122.0887 8675	137.6618 7499
88	106.0363 4622	110.2011 6908	114.5762 2889	124.0044 5265	140.0384 9374
89	107.4781 6433	111.7521 7402	116.2445 9022	125.9344 8604	142.4388 7868
90	108.9259 9002	113.3109 3580	117.9226 8367	127.8789 9460	144.8632 6746
91	110.3798 4831	114.8774 9048	119.6105 6599	129.8380 8715	147.3119 0014
92	111.8397 6434	116.4518 7793	121.3082 9429	131.8118 7280	149.7850 1914
93	113.3057 6336	118.0341 3732	123.0159 2601	133.8004 6185	152.2828 6933
94	114.7778 7071	119.6243 0800	124.7335 1891	135.8039 6531	154.8056 9803
95	116.2561 1184	121.2224 2954	126.4611 3110	137.8224 9505	157.3537 5501
96	117.7405 1230	122.8285 4169	128.1988 2103	139.8561 6377	159.9272 9256
97	119.2310 9777	124.4426 8440	129.9466 4749	141.9050 8499	162.5265 6548
98	120.7278 9401	126.0648 9782	131.7046 6960	143.9693 7313	165.1518 3114
99	122.2309 2690	127.6952 2231	133.4729 4684	146.0491 4343	167.8033 4945
100	123.7402 2243	129.3336 9842	135.2515 3903	148.1445 1201	170.4813 8294

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	125.2558 0009	130.0803 6692	137.0405 0034	150.2555 0585	173.1861 0677
102	126.7777 0580	132.6352 0875	138.8390 0020	152.3825 1281	175.9180 5874
103	128.3050 4633	134.2084 4500	140.6408 0876	154.5253 8106	178.0772 3033
104	120.8405 5444	135.9609 3732	142.4702 6508	150.6843 2202	181.4640 1172
105	131.3815 5675	137.6407 8701	144.3013 4253	158.8504 5444	184.2786 5184
106	132.9280 7090	130.3380 3594	146.1431 0030	161.0500 0035	187.1214 3830
107	134.4828 5065	141.0347 2612	147.9056 0178	163.2587 8210	189.0026 5274
108	136.0431 9586	142.7308 9075	140.8580 0946	165.4832 2290	102.8925 7927
109	137.6100 4251	144.4535 9925	151.7330 8043	167.7243 4714	105.8215 0506
110	130.1834 1760	140.1758 0725	153.6181 9610	169.9822 7074	108.7707 2011
111	140.7633 4860	147.9007 4658	155.5143 0225	172.2571 4684	201.7675 1731
112	142.3408 6255	140.6402 8032	157.4214 6001	174.5400 7544	204.7851 0248
113	143.9420 8008	151.3945 1172	159.3307 6001	176.8581 9351	207.8330 4441
114	145.5427 4942	153.1514 8428	161.2692 4285	179.1846 2906	210.0113 7485
115	147.1491 7754	154.9172 4170	163.2090 8010	181.5285 1408	214.0204 8800
116	148.7622 9012	156.6918 2791	165.1620 3832	183.8800 7854	217.1606 0349
117	150.3821 4203	158.4752 8704	167.1254 8354	186.2691 5338	220.3323 0042
118	152.0087 3420	160.2676 0348	169.1003 8219	188.6661 7203	223.5356 2343
119	153.6421 0401	162.0690 0180	171.0868 0109	191.0811 6832	226.7700 7966
120	155.2822 7045	163.8703 4081	173.0848 0743	193.5142 7708	230.0386 8046
121	156.9202 8805	165.6987 4354	175.0044 0881	195.9656 3416	233.3390 7635
122	158.5831 6098	167.5272 3726	177.1158 5321	198.4353 7642	236.6724 6712
123	160.2430 2415	169.3648 7344	179.1490 2002	200.9236 4174	240.0391 0179
124	161.9116 0717	171.2116 9781	181.1940 6502	203.4305 6005	243.4395 8370
125	163.5862 3887	173.0677 5030	183.2510 3040	205.9562 9832	246.8739 7054
126	165.2678 4810	174.9330 9508	185.3199 9474	208.5009 7056	250.3427 1934
127	166.9564 6423	176.8077 6056	187.4010 2805	211.0647 2784	253.8461 4653
128	168.6521 1016	178.6917 9936	189.4942 0071	213.6477 1330	257.3846 0800
129	170.3548 3331	180.5852 5830	191.5995 8355	216.2500 7115	260.9584 5408
130	172.0646 4512	182.4881 8465	193.7172 4778	218.8719 4668	264.5680 3862
131	173.7815 8114	184.4006 2557	195.8472 6506	221.5134 8628	268.2137 1000
132	175.5056 7106	186.3226 2870	197.9897 0744	224.1748 3743	271.8958 5619
133	177.2369 4469	188.2542 4184	200.1446 4740	226.8561 4871	275.6148 1475
134	178.9754 3196	190.1955 1305	202.3121 5785	229.5575 6982	279.3709 6290
135	180.7211 6293	192.1464 9062	204.4923 1210	232.2792 5160	283.1046 7253
136	182.4741 6777	194.1072 2307	206.6851 8302	235.0213 4598	286.9963 1926
137	184.2344 7681	196.0777 5910	208.8908 4740	237.7840 0608	290.8662 8245
138	186.0021 2046	198.0581 4798	211.1093 7744	240.5673 8612	294.7749 4527
139	187.7771 2920	200.0484 3872	213.3408 4881	243.3716 4152	298.7226 9473
140	189.5595 3400	202.0486 8092	215.5853 3709	246.1960 2883	302.7099 2167
141	191.3493 6539	204.0589 2432	217.8420 1822	249.0434 0580	306.7370 2089
142	193.1466 5441	206.0792 1804	220.1136 0858	251.9112 3134	310.8043 9110
143	194.9514 3214	208.1096 1504	222.3976 0408	254.8005 6558	314.9124 3501
144	196.7637 2077	210.1501 6311	224.6949 8469	257.7115 6982	319.0615 5936
145	198.5835 7865	212.2009 1303	227.0057 0544	260.6444 0659	323.2521 7495
146	200.4110 1023	214.2619 1850	229.3299 0538	263.5992 3964	327.4846 9670
147	202.2460 5610	216.3332 2809	231.6676 6317	266.5762 3394	331.7595 4367
148	204.0887 4800	218.4148 9423	234.0190 5787	269.5755 5569	336.0771 3911
149	205.9391 1779	220.5069 6870	236.3841 6004	272.5973 7236	340.4379 1050
150	207.7971 0744	222.6095 0354	238.7630 7060	275.6418 5265	344.8422 8960

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0112 5000	2.0125 0000	2.0150 0000	2.0175 0000	2.0200 0000
3	3.0338 7656	3.0376 5625	3.0452 2500	3.0528 0625	3.0004 0000
4	4.0680 0767	4.0756 2695	4.0900 0338	4.1062 3036	4.1216 0800
5	5.1137 7276	5.1265 7229	5.1522 6693	5.1780 8038	5.2040 4016
6	6.1713 0270	6.1906 5444	6.2295 5093	6.2087 0596	6.3081 2096
7	7.2407 2980	7.2680 3762	7.3220 0410	7.3784 0831	7.4342 8338
8	8.3221 8807	8.3588 8809	8.4328 3011	8.5075 3045	8.5829 6905
9	9.4158 1269	9.4633 7420	9.5593 3169	9.6504 1224	9.7546 2843
10	10.5217 4058	10.5816 6637	10.7027 2167	10.8253 9945	10.9497 2100
11	11.6401 1016	11.7139 3720	11.8632 6249	12.0148 4304	12.1687 1542
12	12.7710 6140	12.8603 6142	13.0412 1143	13.2251 0371	13.4120 8973
13	13.9147 3584	14.0211 1504	14.2368 2060	14.4565 4303	14.6803 3152
14	15.0712 7662	15.1603 7088	15.4503 8205	15.7005 3253	15.9739 3815
15	16.2408 2848	16.3863 3463	16.6821 3778	16.9844 4935	17.2934 1602
16	17.4235 3780	17.5911 6382	17.9323 6984	18.2816 7721	18.6302 8525
17	18.6195 5260	18.8110 5336	19.2013 5539	19.6016 0656	20.0120 7096
18	19.8290 2257	20.0461 9153	20.4893 7572	20.9446 3468	21.4123 1238
19	21.0520 9907	21.2067 6893	21.7967 1636	22.3111 6578	22.8405 5863
20	22.2889 3519	22.5629 7854	23.1236 6710	23.7016 1119	24.2973 6980
21	23.5396 8571	23.8450 1577	24.4705 2211	25.1163 8038	25.7833 1719
22	24.8045 0717	25.1430 7847	25.8375 7904	26.5559 2620	27.2080 8354
23	26.0835 5788	26.4573 6695	27.2251 4364	28.0206 5400	28.8449 6321
24	27.3769 9700	27.7880 8403	28.6335 2080	29.5110 1637	30.4218 6247
25	28.6849 8913	29.1354 3508	30.0630 2361	31.0274 5915	32.0302 9072
26	30.0076 9526	30.4996 2802	31.5139 6896	32.5704 3909	33.6709 0572
27	31.3452 8183	31.8808 7337	32.9866 7850	34.1404 2238	35.3443 2383
28	32.6979 1625	33.2793 8429	34.4814 7867	35.7378 7977	37.0512 1031
29	34.0657 6781	34.6953 7659	35.9987 0085	37.3032 9267	38.7022 3451
30	35.4490 0769	36.1290 6880	37.5386 8137	39.0171 5020	40.5680 7921
31	36.8478 0903	37.5806 8216	39.1017 6159	40.6999 5042	42.3794 4079
32	38.2623 4688	39.0504 4069	40.6882 8801	42.4121 9955	44.2270 2961
33	39.6927 9829	40.5385 7120	42.2986 1233	44.1544 1305	46.1115 7020
34	41.1393 4227	42.0453 0334	43.9330 9152	45.9271 1527	48.0338 0160
35	42.6021 5987	43.5708 6963	45.5920 8789	47.7308 3979	49.9944 7763
36	44.0814 3417	45.1155 0550	47.2759 6921	49.5661 2049	51.9943 6719
37	45.5773 5030	46.6794 4932	48.9851 0874	51.4335 3675	54.0342 5453
38	47.0900 9549	48.2626 4243	50.7198 8538	53.3336 2365	56.1149 3962
39	48.6198 5906	49.8862 2921	52.4806 8366	55.2669 6206	58.2372 3841
40	50.1668 3248	51.4895 5708	54.2678 9301	57.2341 3390	60.4019 8318
41	51.7312 0934	53.1331 7654	56.0819 1232	59.2357 3124	62.6100 2284
42	53.3131 8545	54.7973 4125	57.9231 4100	61.2723 5654	64.8622 2330
43	54.9129 5879	56.4823 0801	59.7919 8812	63.3446 2278	67.1594 6777
44	56.5307 2957	58.1883 3687	61.6888 6794	65.4531 5367	69.5026 5712
45	58.1667 0028	59.9156 9108	63.6142 0096	67.5985 8386	71.8927 1027
46	59.8210 7566	61.6646 3721	65.5684 1398	69.7815 5908	74.3305 6447
47	61.4940 6276	63.4354 4518	67.5519 4018	72.0027 3637	76.8171 7576
48	63.1858 7097	65.2283 8824	69.5652 1920	74.2627 8425	79.3535 1027
49	64.8967 1201	67.0437 4810	71.6086 9758	76.5623 8298	81.9405 8066
50	66.6268 0002	68.8817 8989	73.6828 2804	78.9022 2468	84.5794 0145

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	68.3703 5152	70.7428 1226	75.7880 7046	81.2830 1361	87.2709 8948
52	70.1455 8548	72.6270 9741	77.9248 9152	83.7054 6635	90.0164 0927
53	71.9347 2332	74.5340 3013	80.0937 6489	86.1703 1201	92.8167 3746
54	73.7439 8895	76.4066 2283	82.2951 7136	88.6782 9247	95.6730 7221
55	75.5736 0883	78.4224 5502	84.5295 9893	91.2301 6259	98.5865 3365
56	77.4238 1193	80.4027 3031	86.7975 4292	93.8266 9043	101.5582 6432
57	79.2948 2981	82.4077 7052	89.0995 0606	96.4686 5752	104.5894 2961
58	81.1868 9665	84.4378 6765	91.4359 9865	99.1568 5902	107.6812 1820
59	83.1002 4923	86.4933 4099	93.8075 3863	101.8921 0405	110.8348 4257
60	85.0351 2704	88.5745 0776	96.2146 5171	104.6752 1588	114.0515 3042
61	86.9917 7222	90.6816 8910	98.6578 7149	107.5070 3215	117.3325 7021
62	88.9704 2966	92.8152 1022	101.1377 3956	110.3884 0522	120.6792 2161
63	90.9713 4699	94.9754 0034	103.6548 0565	113.3202 0231	124.0928 0604
64	92.9947 7464	97.1625 9285	106.2096 2774	116.3033 0585	127.5746 6216
65	95.0409 6586	99.3771 2526	108.8027 7215	119.3386 1370	131.1261 5541
66	97.1101 7672	101.6193 3933	111.4348 1374	122.4270 3944	134.7486 7852
67	99.2026 0021	103.8895 8107	114.1063 3594	125.5695 1263	138.4436 5209
68	101.3186 9021	106.1882 0083	116.8179 3098	128.7669 7910	142.2125 2513
69	103.4585 3154	108.5155 5334	119.5701 9995	132.0204 0124	146.0567 7563
70	105.6224 4002	110.8719 9776	122.3687 5295	135.3307 5826	149.9779 1114
71	107.8106 9247	113.2578 9773	125.1992 0024	138.6990 4653	153.9774 6937
72	110.0235 6276	115.6736 2145	128.0771 9738	142.1262 7984	158.0570 1875
73	112.2613 2784	118.1195 4172	130.9983 5534	145.6134 8974	162.2181 5913
74	114.5242 6778	120.5960 3599	133.9633 3067	149.1617 2581	166.4625 2231
75	116.8126 6579	123.1034 8644	136.9727 8063	152.7720 5601	170.7917 7276
76	119.1268 0828	125.6422 8002	140.0273 7234	156.4455 6699	175.2076 0821
77	121.4669 8487	128.2128 0852	143.1277 8292	160.1833 6441	179.7117 6038
78	123.8334 8845	130.8154 6863	146.2746 9967	163.9865 7329	184.3059 9558
79	126.2266 1520	133.4506 6199	149.4688 2010	167.8563 3832	188.9921 1549
80	128.6466 6462	136.1187 9526	152.7108 5247	171.7988 2424	193.7719 5780
81	131.0939 3960	138.8202 8020	156.0015 1525	175.8002 1617	198.6473 9696
82	133.5687 4642	141.5555 3370	159.3415 3798	179.8767 1995	203.6203 4490
83	136.0713 9481	144.3249 7787	162.7316 6105	184.0245 6255	208.6927 5180
84	138.6021 9801	147.1290 4010	166.1726 3597	188.2449 9239	213.8666 0683
85	141.1614 7273	149.9681 5310	169.6652 2551	192.5392 7976	219.1439 3897
86	143.7495 3930	152.8427 5501	173.2102 0389	196.9087 1716	224.5268 1775
87	146.3667 2162	155.7532 8045	176.8083 5695	201.3546 1971	230.0173 5411
88	149.0133 4724	158.7002 0557	180.4604 8230	205.8783 2555	235.6177 0119
89	151.6897 4739	161.6839 5814	184.1673 8954	210.4811 9625	241.3300 5521
90	154.3962 5705	164.7050 0702	187.9299 0038	215.1646 1718	247.1566 5632
91	157.1332 1494	167.7638 2021	191.7488 4880	219.9299 9798	253.0997 8944
92	159.9009 6361	170.8608 6796	195.6250 8102	224.7787 7295	259.1617 8523
93	162.6998 4945	173.9966 2881	199.5594 5784	229.7124 0148	265.3450 2094
94	165.5302 2276	177.1715 8667	203.5528 4971	234.7323 6850	271.6519 2135
95	168.3924 3776	180.3862 3151	207.6061 4246	239.8401 8495	278.0849 5078
96	171.2868 5269	183.6410 5940	211.7202 3459	245.0373 8619	284.6466 5898
97	174.2138 2978	186.9365 7264	215.8960 3811	250.3255 4248	291.3395 9216
98	177.1737 3537	190.2732 7980	220.1344 7868	255.7062 3947	298.1663 8400
99	180.1669 3989	193.6516 9580	224.4364 9586	261.1810 9866	305.1297 1168
100	183.1938 1796	197.0723 4200	228.8030 4330	266.7517 6789	312.2323 0591

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0225 0000	2.0250 0000	2.0275 0000	2.0300 0000	2.0350 0000
3	3.0680 0625	3.0756 2500	3.0832 5625	3.0909 0000	3.1062 2500
4	4.1370 3639	4.1525 1563	4.1680 4580	4.1836 2700	4.2149 4288
5	5.2301 1971	5.2563 2852	5.2826 0706	5.3091 3581	5.3624 6588
6	6.3477 9740	6.3877 3673	6.4270 4040	6.4684 0988	6.5501 5218
7	7.4906 2284	7.5474 3015	7.6047 0876	7.6624 6218	7.7794 0751
8	8.6591 6180	8.7361 1590	8.8138 3825	8.8923 3605	9.0516 8677
9	9.8539 9300	9.9545 1880	10.0562 1880	10.1591 0613	10.3684 9581
10	11.0757 0784	11.2033 8177	11.3327 6482	11.4638 7931	11.7313 9316
11	12.3249 1127	12.4834 6631	12.6444 1585	12.8077 9569	13.1419 9192
12	13.6022 2177	13.7955 5297	13.9921 3729	14.1920 2956	14.6019 6164
13	14.9082 7176	15.1404 4170	15.3769 2107	15.6177 9045	16.1130 3030
14	16.2437 0788	16.5189 5284	16.7997 8639	17.0863 2416	17.6769 8636
15	17.6091 9130	17.9319 2666	18.2617 8052	18.5989 1380	19.2956 8088
16	19.0053 9811	19.3802 2483	19.7639 7948	20.1568 8130	20.9710 2971
17	20.4330 1957	20.8647 3045	21.3074 8892	21.7615 8774	22.7050 1575
18	21.8927 6251	22.3863 4871	22.8934 4487	23.4144 3537	24.4996 9130
19	23.3853 4966	23.9460 0743	24.5230 1460	25.1168 6844	26.3571 8050
20	24.9115 2003	25.5446 5761	26.1973 9750	26.8703 7449	28.2796 8181
21	26.4720 2923	27.1832 7405	27.9178 2593	28.6764 8572	30.2694 7068
22	28.0676 4989	28.8628 5590	29.6855 6615	30.5367 8030	32.3289 0215
23	29.6991 7201	30.5844 2730	31.5019 1921	32.4528 8370	34.4604 1373
24	31.3674 0338	32.3490 3798	33.3682 2199	34.4264 7022	36.6665 2821
25	33.0731 6996	34.1577 6393	35.2858 4810	36.4592 6432	38.9498 5669
26	34.8173 1628	36.0117 0803	37.2562 0802	38.5530 4225	41.3131 0168
27	36.6007 0590	37.9120 0073	39.2807 5467	40.7096 3352	43.7590 6024
28	38.4242 2178	39.8598 0075	41.3609 7542	42.9309 2252	46.2906 2734
29	40.2887 6677	41.8562 9577	43.4984 0224	45.2188 5020	48.9107 9930
30	42.1952 6402	43.9027 0316	45.6946 0830	47.5754 1571	51.6226 7728
31	44.1446 5746	46.0002 7074	47.9512 1003	50.0026 7818	54.4294 7098
32	46.1379 1226	48.1502 7751	50.2698 6831	52.5027 5852	57.3345 0247
33	48.1760 1528	50.3540 3445	52.6522 9099	55.0778 4128	60.3412 1005
34	50.2599 7563	52.6128 8531	55.1002 2765	57.7301 7652	63.4531 5240
35	52.3908 2508	54.9282 0744	57.6154 8391	60.4620 8181	66.6740 1274
36	54.5696 1864	57.3014 1263	60.1999 0972	63.2759 4427	70.0076 0318
37	56.7974 3506	59.7339 4794	62.8554 0724	66.1742 2250	73.4578 6930
38	59.0753 7735	62.2272 9664	65.5839 3094	69.1594 4927	77.0288 0472
39	61.4045 7334	64.7829 7906	68.3874 8904	72.2342 3275	80.7249 0604
40	63.7861 7624	67.4025 5354	71.2681 4499	75.4012 5973	84.5502 7775
41	66.2213 6521	70.0876 1737	74.2280 1898	78.6632 9753	88.5095 3747
42	68.7113 4592	72.8398 0781	77.2692 8950	82.0231 9645	92.6073 7128
43	71.2573 5121	75.6608 0300	80.3941 9496	85.4838 9234	96.8486 2928
44	73.8606 4161	78.5523 2308	83.6050 3532	89.0484 0911	101.2383 3130
45	76.5225 0605	81.5161 3116	86.9041 7370	92.7198 6139	105.7816 7290
46	79.2442 6243	84.5540 3443	90.2940 3857	96.5014 5723	110.4840 3145
47	82.0272 5834	87.6678 8530	93.7771 2463	100.3965 0095	115.3509 7255
48	84.8728 7165	90.8595 8243	97.3559 9556	104.4083 9598	120.3882 5659
49	87.7825 1126	94.1310 7199	101.0332 8544	108.5406 4785	125.6018 4557
50	90.7576 1776	97.4843 4879	104.8117 0079	112.7968 6729	130.9979 1016

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	93.7996 6416	100.9214 5751	108.6940 2256	117.1807 7331	136.5828 3702
52	96.9101 5661	104.4444 9305	112.6831 0818	121.6961 9651	142.3632 3631
53	100.0906 8513	108.0556 0629	116.7818 9365	126.3470 8240	148.3459 4958
54	103.3426 7442	111.7569 9645	120.9933 0573	131.1374 9488	154.5380 5782
55	106.6678 8460	115.5509 2136	125.3207 1411	136.0716 1072	160.9468 8084
56	110.0679 1200	119.4396 9440	129.7670 3375	141.1537 6831	167.5800 3099
57	113.5444 4002	123.4256 8676	134.3356 2718	146.3883 8136	174.4453 3207
58	117.0991 8902	127.5113 2893	139.0298 5602	151.7800 3280	181.5509 1869
59	120.7339 2169	131.6991 1215	143.8531 7790	157.3334 3379	188.9052 0085
60	124.4504 3493	135.9915 8995	148.8091 4038	163.0534 3680	196.5168 8288
61	128.2505 6972	140.3913 7970	153.9013 9174	168.9450 3091	204.3949 7378
62	132.1362 0754	144.9011 6419	159.1336 8002	175.0133 9110	212.5487 9786
63	136.1092 7221	149.5236 9330	164.5098 5622	181.2637 9284	220.9880 0570
64	140.1717 3083	154.2617 8563	170.0338 7726	187.7017 0662	229.7225 8599
65	144.3255 9477	159.1183 3027	175.7098 0889	194.3327 5782	238.7628 7650
66	148.5729 2066	164.0962 8853	181.5418 2803	201.1627 4055	248.1195 7718
67	152.9158 1137	169.1986 9574	187.5342 2892	208.1976 2277	257.8037 6238
68	157.3564 1713	174.4286 6314	193.6914 2021	215.4435 5145	267.8268 9400
69	161.8969 3651	179.7893 7971	200.0179 3427	222.9068 5800	278.2008 3535
70	166.5396 1758	185.2841 1421	206.5184 2746	230.5940 6374	288.9378 6450
71	171.2867 5898	190.9162 1706	213.1976 8422	238.5118 8565	300.0506 8085
72	176.1407 1100	196.6891 2249	220.0606 2054	246.6672 4222	311.5524 6400
73	181.1038 7705	202.6063 5055	227.1122 8760	255.0672 5949	323.4568 0024
74	186.1787 1429	208.6715 0931	234.3578 7551	263.7192 7727	335.7777 8824
75	191.3677 3536	214.8882 9705	241.8027 1709	272.6308 5559	348.5300 1083
76	196.6735 0041	221.2605 0447	249.4522 9181	281.8097 8126	361.7285 6121
77	202.0986 6337	227.7920 1709	257.3122 2083	291.2640 7469	375.3890 6085
78	207.6458 8329	234.4868 1751	265.3883 1615	301.0019 9693	389.5276 7708
79	213.3179 1567	241.3489 8705	273.6864 0485	311.0320 5084	404.1611 4671
80	219.1175 6877	248.3827 1265	282.2128 7345	321.3630 1855	419.3067 8685
81	225.0477 1407	255.5922 8047	290.9737 2747	332.0039 0910	434.9825 2430
82	231.1112 8763	262.9820 8748	299.9755 0498	342.9640 2638	451.2069 1274
83	237.3112 9160	270.5566 3966	309.2248 3137	354.2529 4717	467.9991 5469
84	243.6507 9567	278.3205 5566	318.7285 1423	365.8805 3558	485.3791 2610
85	250.1329 3857	286.2785 6955	328.4935 4837	377.8569 5165	503.3673 9448
86	256.7609 2969	294.4355 3370	338.5271 2095	390.1926 6020	521.9852 5329
87	263.5380 5060	302.7964 2213	348.8366 1678	402.8984 4001	541.2547 3715
88	270.4676 5674	311.3663 3268	359.4296 2374	415.9853 9321	561.1986 5295
89	277.5531 7002	320.1504 9100	370.3139 3839	429.4649 5500	581.8406 0581
90	284.7981 2555	329.1542 5328	381.4975 7170	443.3489 0365	603.2050 2701
91	292.2060 8337	338.3831 0061	392.9887 5492	457.6493 7076	625.3172 0295
92	299.7807 2025	347.8426 8735	404.7959 4568	472.3788 5189	648.2033 0506
93	307.5257 8645	357.5387 5453	416.9278 3418	487.5502 1744	671.8904 2073
94	315.4451 1065	367.4772 2330	429.3933 4962	503.1767 2397	696.4065 8540
95	323.5426 3177	377.6641 5308	442.2016 6674	519.2720 2569	721.7808 1595
96	331.8223 4099	388.1057 5783	455.3622 1257	535.8501 8645	748.0431 4451
97	340.2883 4366	398.8084 0177	468.8846 7342	552.9256 9205	775.2246 5457
98	348.9448 3139	409.7786 1182	482.7790 0194	570.5134 6281	803.3575 1748
99	357.7960 9010	421.0230 7711	497.0554 2449	588.6288 6609	832.4750 3059
100	366.8465 0213	432.5486 5404	511.7244 4867	607.2877 3270	862.6116 5666

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n|}} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0400 0000	2.0450 0000	2.0500 0000	2.0550 0000	2.0600 0000
3	3.1216 0000	3.1370 2500	3.1525 0000	3.1680 2500	3.1836 0000
4	4.2464 6400	4.2781 9113	4.3101 2500	4.3422 6638	4.3746 1600
5	5.4163 2256	5.4707 0973	5.5256 3125	5.5810 9103	5.6370 9296
6	6.6329 7546	6.7168 9166	6.8019 1281	6.8880 5103	6.9753 1854
7	7.8982 9448	8.0191 5179	8.1420 0845	8.2668 9384	8.3938 3765
8	9.2142 2626	9.3800 1362	9.5491 0888	9.7215 7300	9.8974 6791
9	10.5827 9531	10.8021 1423	11.0265 6432	11.2562 5951	11.4913 1598
10	12.0061 0712	12.2882 0937	12.5778 9254	12.8753 5379	13.1807 9494
11	13.4863 5141	13.8411 7879	14.2067 8716	14.5834 9825	14.9716 4264
12	15.0258 0546	15.4650 3184	15.9171 2652	16.3855 9065	16.8699 4120
13	16.6268 3768	17.1599 1327	17.7129 8285	18.2867 9814	18.8821 3767
14	18.2919 1119	18.9321 0937	19.5986 3199	20.2925 7203	21.0150 6593
15	20.0235 8764	20.7840 5429	21.5785 6359	22.4086 6350	23.2759 6988
16	21.8245 3114	22.7193 3673	23.6574 9177	24.6411 3999	25.6725 2808
17	23.6975 1239	24.7417 0689	25.8403 6636	26.9964 0269	28.2128 7976
18	25.6454 1288	26.8550 8370	28.1323 8467	29.4812 0483	30.9056 5255
19	27.6712 2940	29.0635 6246	30.5390 0391	32.1026 7110	33.7599 9170
20	29.7780 7858	31.3714 2277	33.0659 5410	34.8683 1801	36.7855 9120
21	31.9692 0172	33.7831 3680	35.7192 5181	37.7860 7550	39.9927 2668
22	34.2479 6979	36.3033 7795	38.5052 1440	40.8643 0965	43.3922 9028
23	36.6178 8858	38.9370 2996	41.4304 7512	44.1118 4669	46.9958 2760
24	39.0826 0412	41.6891 9631	44.5019 9887	47.5379 9825	50.8155 7735
25	41.6459 0829	44.5652 1015	47.7270 9882	51.1525 8816	54.8645 1200
26	44.3117 4462	47.5706 4460	51.1134 5376	54.9659 8051	59.1563 8272
27	47.0842 1440	50.7113 2361	54.6691 2645	58.9891 0943	63.7057 6568
28	49.9675 8298	53.9933 3317	58.4025 8277	63.2335 1045	68.5281 1162
29	52.9662 8630	57.4230 3316	62.3227 1191	67.7113 5353	73.6397 9832
30	56.0849 3775	61.0070 6966	66.4388 4750	72.4354 7797	79.0581 8622
31	59.3283 3526	64.7523 8779	70.7607 8988	77.4194 2926	84.8016 7739
32	62.7014 6867	68.6662 4524	75.2988 2937	82.6774 9787	90.8897 7803
33	66.2095 2742	72.7562 2628	80.0637 7084	88.2247 6025	97.3431 6471
34	69.8579 0851	77.0302 5646	85.0669 5938	94.0771 2207	104.1837 5460
35	73.6522 2486	81.4966 1800	90.3203 0735	100.2513 6378	111.4347 7087
36	77.5983 1385	86.1639 6581	95.8363 2272	106.7651 8870	119.1208 6666
37	81.7022 4640	91.0413 4427	101.6281 3886	113.6372 7417	127.2681 1866
38	85.9703 3626	96.1382 0476	107.7095 4580	120.8873 2425	135.9042 0578
39	90.4091 4971	101.4644 2398	114.0950 2309	128.5361 2708	145.0584 5813
40	95.0255 1570	107.0303 2306	120.7997 7424	136.6056 1407	154.7619 6562
41	99.8265 3633	112.8466 8760	127.8397 6295	145.1189 2285	165.0476 8356
42	104.8195 9778	118.9247 8854	135.2317 5110	154.1004 6360	175.9505 4457
43	110.0123 8160	125.2764 0402	142.9933 3866	163.5759 8910	187.5075 7724
44	115.4128 7696	131.9138 4220	151.1430 0559	173.5726 6850	199.7580 3188
45	121.0293 9204	138.8499 6510	159.7001 5587	184.1191 6527	212.7435 1379
46	126.8705 6772	146.0982 1353	168.6851 6366	195.2457 1936	226.5081 2462
47	132.9453 9043	153.6726 3314	178.1194 2185	206.9842 3392	241.0986 1210
48	139.2632 0604	161.5879 0168	188.0253 9294	219.3683 6679	256.5645 2882
49	145.8337 3429	169.8593 5720	198.4266 6259	232.4336 2696	272.9584 0055
50	152.6670 8366	178.5030 2828	209.3479 9572	246.2174 7645	290.3359 0458

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	159.7737 6700	187.5356 0455	220.8153 9550	260.7594 3765	308.7560 5886
52	167.1647 1708	196.9747 0946	232.8561 6528	276.1012 0072	328.2814 2239
53	174.8513 0039	206.8386 3408	245.4989 7354	292.2867 7309	348.9783 0773
54	182.8453 5805	217.1463 7202	258.7739 2222	309.3625 4561	370.9170 0620
55	191.1591 7299	227.9179 5038	272.7126 1833	327.3774 8562	394.1720 2657
56	199.8055 3901	239.1742 0756	287.3482 4924	346.3832 4733	418.8223 4816
57	208.7977 0151	250.9371 0960	302.7156 6171	366.4343 2593	444.9516 8905
58	218.1496 7197	263.2292 7953	318.8514 4479	387.5882 1386	472.6487 9040
59	227.8756 5885	276.0745 9711	335.7940 1703	409.9055 6562	502.0077 1782
60	237.9906 8520	289.4979 5398	353.5837 1788	433.4503 7173	533.1281 8089
61	248.5103 1201	303.5253 0190	372.2629 0378	458.2901 4217	566.1158 7174
62	259.4507 2511	318.1840 0319	391.8760 4897	484.4960 9999	601.0828 2405
63	270.8287 5412	333.5022 8333	412.4698 5141	512.1433 8549	638.1477 9349
64	282.6019 0428	349.5098 8608	434.0933 4398	541.3112 7170	677.4366 6110
65	294.9683 8045	366.2378 3096	456.7980 1118	572.0833 9164	719.0828 6076
66	307.7671 1507	383.7185 3335	480.6379 1174	604.5479 7818	763.2278 3241
67	321.0778 0030	401.9858 6735	505.6698 0733	638.7981 1098	810.0215 0236
68	334.9209 1231	421.0752 3138	531.9532 9770	674.9320 1341	859.6227 9250
69	349.3177 4880	441.0236 1670	559.5509 6258	713.0532 7415	912.2001 6005
70	364.2904 5870	461.8696 7955	588.5285 1071	753.2712 0423	967.9321 6965
71	379.8620 7711	483.0538 1513	618.9549 3025	795.7011 2046	1027.0080 9983
72	396.0505 0019	506.4182 3081	650.9026 8306	840.4646 8209	1089.6285 8582
73	412.8988 2200	530.2070 5747	684.4478 1721	887.6902 3960	1156.0063 0097
74	430.4147 7550	555.0663 7505	719.6702 0807	937.5132 0278	1226.3666 7903
75	448.0313 0652	581.0443 0193	756.6537 1848	990.0764 2893	1300.9486 7977
76	467.5766 2118	608.1913 5822	795.4864 0440	1045.5306 3252	1380.0056 0055
77	487.2796 8603	636.5599 0934	836.2607 2462	1104.0348 1731	1463.8059 3659
78	507.7708 7347	666.2051 6796	879.0737 6085	1165.7567 3226	1552.6342 9278
79	529.0817 0841	697.1844 0052	924.0274 4889	1230.8733 5254	1646.7923 5035
80	551.2449 7075	729.5576 9854	971.2288 2134	1299.5713 8693	1746.5998 9137
81	574.2947 7582	763.3877 9497	1020.7902 6240	1372.0478 1321	1852.3958 8485
82	598.2605 0085	798.7402 4575	1072.8297 7552	1448.5104 4294	1964.5396 3794
83	623.1972 2952	835.0835 5080	1127.4712 6430	1529.1785 1730	2083.4120 1022
84	649.1251 1870	874.2893 1680	1184.8448 2752	1614.2833 3575	2209.4167 3719
85	676.0901 2345	914.0323 3612	1245.0870 6889	1704.0689 1921	2342.9817 4142
86	704.1337 2839	956.7907 9125	1308.3414 2234	1798.7927 0977	2484.5606 4591
87	733.2090 7753	1000.8463 7685	1374.7584 9345	1898.7263 0881	2634.6342 8466
88	763.0310 4063	1046.8844 6381	1444.4964 1812	2004.1562 5579	2793.7123 4174
89	795.1762 8225	1094.9942 6468	1517.7212 3903	2115.3848 4986	2962.3350 8225
90	827.9833 3354	1145.2690 0659	1594.6073 0098	2232.7310 1660	3141.0751 8718
91	862.1026 0688	1197.8061 1189	1675.3376 6003	2356.5312 2252	3330.5396 9841
92	897.5867 7356	1252.7073 8692	1760.1045 4033	2487.1404 3976	3531.3720 8032
93	934.4902 4450	1310.0792 1933	1849.1097 7080	2624.9331 6394	3744.2544 0514
94	972.8098 5428	1370.0327 8420	1942.5652 6564	2770.3044 8796	3969.9096 6944
95	1012.7846 4845	1432.6842 5049	2040.6935 2892	2923.6712 3480	4209.1042 4961
96	1054.2960 3439	1498.1550 5117	2143.7282 0537	3085.4731 5271	4462.6505 0459
97	1097.4678 7577	1566.5720 2847	2251.9146 1564	3256.1741 7611	4731.4095 3486
98	1142.3665 9080	1638.0677 6976	2365.5103 4642	3436.2637 5580	5016.2941 0696
99	1189.0612 5443	1712.7808 1939	2484.7858 6374	3626.2582 6237	5318.2717 5337
100	1237.0237 0461	1790.8559 5627	2610.0251 5093	3826.7024 6680	5638.3680 5857

TABLE VII — AMOUNT OF ANNUITY OF 1 PER PERIOD

$$(s_{\overline{n}|} \text{ at } i) = \frac{(1+i)^n - 1}{i}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0650 0000	2.0700 0000	2.0750 0000	2.0800 0000	2.0850 0000
3	3.1992 2500	3.2149 0000	3.2306 2500	3.2464 0000	3.2622 2500
4	4.4071 7463	4.4309 4300	4.4729 2188	4.5061 1200	4.5395 1413
5	5.0936 4098	5.7507 3901	5.8083 9102	5.8666 0000	5.0253 7283
6	7.0637 2764	7.1532 9074	7.2440 2034	7.3359 2004	7.4290 2052
7	8.5228 6994	8.6540 2109	8.7873 2187	8.9228 0336	0.0604 9702
8	10.0768 5648	10.2598 0257	10.4463 7101	10.6366 2763	10.8306 3027
9	11.7318 5215	11.9779 8875	12.2298 4883	12.4875 5784	12.7512 4301
10	13.4944 2254	13.8164 4706	14.1470 8750	14.4865 6247	14.8350 0032
11	15.3715 6001	15.7835 9932	10.2081 1006	10.6454 8746	17.0000 8270
12	17.3707 1141	17.8884 5127	18.4237 2799	18.9771 2646	10.5402 4979
13	19.4998 0765	20.1406 4286	20.8055 0759	21.4952 9658	22.2109 3003
14	21.7672 9515	22.5504 8780	23.3659 2066	24.2149 2030	25.0988 0550
15	24.1821 6933	25.1290 2201	26.1183 6470	27.1521 1393	28.2322 0010
16	26.7540 1034	27.8880 5355	29.0772 4206	30.3242 8304	31.0320 1204
17	29.4930 2101	30.8402 1730	32.2580 3521	33.7502 2509	35.3207 3306
18	32.4100 6738	33.9990 3251	35.6773 8785	37.4502 4374	39.3229 9538
19	35.5167 2176	37.3789 6479	39.3531 9194	41.4462 6324	43.6654 4998
20	38.8253 0807	40.9954 9232	43.3046 8134	45.7619 6430	48.3770 1323
21	42.3489 5373	44.8651 7678	47.5525 3244	50.4229 2144	53.4890 5930
22	46.1016 3573	49.0057 3910	52.1189 7237	55.4567 5510	59.0350 2940
23	50.0982 4205	53.4361 4090	57.0278 9530	60.8932 9557	65.0530 5790
24	54.3546 2778	58.1766 7076	62.3049 8744	66.7647 5922	71.5832 1882
25	58.8876 7859	63.2490 3772	67.9778 6150	73.1059 3995	78.6677 0242
26	63.7153 7769	68.6764 7036	74.0762 0112	79.9544 1515	86.3546 5478
27	68.8568 7725	74.4838 2328	80.6319 1620	87.3507 6836	94.6940 0103
28	74.3325 7427	80.6976 9091	87.6793 0001	95.3388 2083	103.7437 4075
29	80.1641 9159	87.3465 2927	95.2552 5816	103.9659 3022	113.5619 5871
30	86.3748 6405	94.4607 8032	103.3994 0252	113.2832 1111	124.2147 2520
31	92.9892 3021	102.0730 4137	112.1543 5771	123.3458 0800	135.7720 7084
32	100.0335 3017	110.2181 5426	121.5659 3454	134.2135 3744	148.3130 7987
33	107.5357 0963	118.9334 2506	131.6833 7003	145.9506 2044	161.9203 4260
34	115.5255 3076	128.2587 6481	142.5596 3310	158.6266 7007	176.6835 7170
35	124.0346 9026	138.2368 7835	154.2516 0558	172.3168 0368	192.7016 7530
36	133.0960 4513	148.9134 5984	166.8204 7000	187.1021 4797	210.0813 1780
37	142.7482 4056	160.3374 0202	180.3320 1170	203.0703 1081	228.9382 2981
38	153.0268 8259	172.5610 2017	194.8560 1258	220.3159 4540	249.3079 7935
39	163.9736 2995	185.6402 9158	210.4711 8102	238.9412 2103	271.5968 0759
40	175.6319 1590	199.6351 1199	227.2565 1960	259.0565 1871	295.6825 3024
41	188.0479 9044	214.6095 6983	245.3007 5857	280.7810 4021	321.8155 5182
42	201.2711 0981	230.6322 3972	264.6983 1546	304.2435 2342	350.1608 7372
43	215.3537 3195	247.7764 9650	285.5506 8012	329.5830 0530	380.9343 1209
44	230.3517 2453	266.1208 5125	307.9669 0080	356.9496 4572	414.3137 2959
45	246.3245 8602	285.7493 1084	332.0645 1511	386.5056 1738	450.5303 9061
46	263.3356 8475	306.7517 0260	357.9693 5375	418.4260 6677	489.8254 8032
47	281.4525 0426	329.2243 8508	385.8170 5528	452.9001 5211	532.4606 4615
48	300.7469 1704	353.2700 9300	415.7533 3442	490.1321 6428	578.7198 0107
49	321.2954 6665	378.9989 9951	447.9348 3451	530.3427 3742	628.9109 8416
50	343.1796 7198	406.5289 2947	482.5299 4709	573.7701 5642	683.3684 1782

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1 + i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.9958 5062	0.9950 2488	0.9942 0050	0.9925 5583	0.9900 9901
2	1.9875 6908	1.9850 9938	1.9826 3513	1.9777 2291	1.9703 9506
3	2.9751 7253	2.9702 4814	2.9653 3733	2.9555 6624	2.9409 8521
4	3.9586 7804	3.9504 9566	3.9423 4034	3.9261 1041	3.9019 6555
5	4.9381 0261	4.9258 6633	4.9136 7723	4.8894 3961	4.8534 3124
6	5.9134 6318	5.8963 8441	5.8793 8084	5.8455 9763	5.7954 7647
7	6.8847 7661	6.8620 7404	6.8394 8385	6.7946 3785	6.7281 9453
8	7.8520 5969	7.8229 5924	7.7940 1875	7.7366 1325	7.6516 7775
9	8.8153 2915	8.7790 6392	8.7430 1781	8.6715 7642	8.5660 1758
10	9.7746 0164	9.7304 1186	9.6865 1315	9.5995 7958	9.4713 0453
11	10.7298 9374	10.6770 2673	10.6245 3669	10.5206 7452	10.3676 2825
12	11.6812 2198	11.6189 3207	11.5571 2016	11.4349 1267	11.2550 7747
13	12.6286 0280	12.5561 5131	12.4842 9511	12.3423 4508	12.1337 4007
14	13.5720 5257	13.4887 0777	13.4060 9201	13.2430 2242	13.0037 0304
15	14.5115 8762	14.4166 2465	14.3225 4473	14.1369 9495	13.8650 5252
16	15.4472 2418	15.3399 2502	15.2336 8160	15.0243 1261	14.7178 7378
17	16.3789 7843	16.2586 3186	16.1395 3432	15.9050 2492	15.5622 5127
18	17.3068 6048	17.1727 6802	17.0401 3354	16.7791 8107	16.3982 6858
19	18.2309 0438	18.0823 5624	17.9355 0974	17.6468 2984	17.2260 0850
20	19.1511 0809	18.9874 1915	18.8256 9320	18.5080 1969	18.0455 5297
21	20.0674 9352	19.8879 7925	19.7107 1404	19.3627 9870	18.8569 8313
22	20.9800 7653	20.7840 5896	20.5906 0220	20.2112 1459	19.6603 7934
23	21.8888 7289	21.6756 8055	21.4653 8745	21.0533 1473	20.4558 2113
24	22.7938 9831	22.5628 6622	22.3350 9938	21.8891 4614	21.2433 8726
25	23.6951 6843	23.4456 3803	23.1997 0741	22.7187 5547	22.0231 5570
26	24.5926 9884	24.3240 1794	24.0594 2079	23.5421 8905	22.7952 0366
27	25.4865 0566	25.1980 2780	24.9140 8862	24.3594 9286	23.5596 0759
28	26.3766 0254	26.0676 8936	25.7637 9979	25.1707 1251	24.3164 4316
29	27.2630 0868	26.9330 2423	26.6085 8307	25.9758 9331	25.0657 8530
30	28.1457 3278	27.7940 5397	27.4484 6702	26.7750 8021	25.8077 0822
31	29.0247 9012	28.6507 9997	28.2834 8066	27.5683 1783	26.5422 8537
32	29.9002 1189	29.5032 8355	29.1136 5044	28.3556 5045	27.2695 8947
33	30.7719 9524	30.3515 2592	29.9390 0025	29.1371 2203	27.9896 9255
34	31.6401 6122	31.1955 4818	30.7595 7540	29.9127 7621	28.7026 6589
35	32.5047 2480	32.0353 7132	31.5753 8566	30.6826 5629	29.4085 8009
36	33.3657 0109	32.8710 1624	32.3864 6403	31.4468 0525	30.1075 0504
37	34.2231 0481	33.7025 0372	33.1928 3074	32.2052 6576	30.7995 0994
38	35.0769 5084	34.5298 5445	33.9945 3828	32.9580 8016	31.4846 6330
39	35.9272 5304	35.3530 8900	34.7915 8736	33.7052 9048	32.1630 3298
40	36.7740 2881	36.1722 2786	35.5840 1306	34.4469 3844	32.8346 8611
41	37.6172 9000	36.9872 9141	36.3718 4487	35.1830 6545	33.4996 8922
42	38.4570 5236	37.7982 0091	37.1551 0870	35.9137 1260	34.1581 0814
43	39.2933 3013	38.6052 7354	37.9338 2012	36.6389 2070	34.8100 0806
44	40.1261 3788	39.4082 3238	38.7080 2029	37.3587 3022	35.4554 5352
45	40.9554 8099	40.2071 9640	39.4777 4248	38.0731 8136	36.0945 0844
46	41.7814 0081	41.0021 8547	40.2429 9170	38.7823 1401	36.7272 3608
47	42.6038 8401	41.7932 1937	41.0038 0287	39.4861 6774	37.3536 9909
48	43.4229 5562	42.5803 1778	41.7602 0170	40.1847 8189	37.9739 5949
49	44.2386 2799	43.3635 0028	42.5122 1380	40.8781 9542	38.5880 7871
50	45.0509 1582	44.1427 8635	43.2598 0460	41.5664 4707	39.1961 1753

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1+i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	45.8598 3317	44.9181 9537	44.0031 7940	42.2495 7525	39.7081 3017
52	46.6653 9401	45.6897 4664	44.7421 8335	42.9276 1812	40.3941 9423
53	47.4676 1228	46.4574 5934	45.4769 0144	43.6006 1351	40.9843 5072
54	48.2665 0184	47.2213 5258	46.2073 5853	44.2685 9902	41.5686 0408
55	49.0620 7651	47.9814 4535	46.9335 7933	44.9316 1193	42.1471 9210
56	49.8543 5003	48.7377 5657	47.6555 8841	45.5896 8926	42.7199 9224
57	50.6433 3612	49.4903 0505	48.3734 1020	46.2428 6776	43.2871 2102
58	51.4290 4840	50.2391 0950	49.0870 6898	46.8911 8388	43.8486 3468
59	52.2115 0046	50.9841 8855	49.7965 8889	47.5346 7382	44.4045 8870
60	52.9907 0584	51.7255 6075	50.5019 9394	48.1733 7352	44.9550 3841
61	53.7666 7800	52.4632 4453	51.2033 0800	48.8073 1863	45.5000 3803
62	54.5394 3035	53.1972 5824	51.9005 5478	49.4365 4455	46.0396 4161
63	55.3089 7627	53.9276 2014	52.5937 5787	50.0610 8640	46.5739 0258
64	56.0753 2905	54.6543 4839	53.2829 4073	50.6809 7906	47.1028 7385
65	56.8385 0194	55.3774 6109	53.9681 2068	51.2962 5713	47.6266 0777
66	57.5985 0814	56.0969 7621	54.6493 3888	51.9060 5497	48.1451 5621
67	58.3553 6078	56.8129 1165	55.3266 0040	52.5131 0667	48.6585 7050
68	59.1090 7296	57.5252 8522	55.9999 3413	53.1147 4607	49.1669 0149
69	59.8596 5770	58.2341 1465	56.6693 6287	53.7110 0677	49.6701 9940
70	60.6071 2798	58.9394 1756	57.3349 0925	54.3046 2210	50.1685 1435
71	61.3514 9672	59.6412 1151	57.9965 9579	54.8929 2516	50.6618 9539
72	62.0927 7680	60.3395 1394	58.6544 4488	55.4768 4880	51.1503 9148
73	62.8309 8103	61.0343 4222	59.3084 7877	56.0564 2561	51.6340 5097
74	63.5661 2216	61.7257 1366	59.9587 1959	56.6316 8795	52.1129 2175
75	64.2982 1292	62.4136 4543	60.6051 8934	57.2026 6794	52.5870 5124
76	65.0272 6596	63.0981 5466	61.2479 0088	57.7693 9746	53.0564 8637
77	65.7532 9388	63.7792 5836	61.8869 0207	58.3319 0815	53.5212 7364
78	66.4763 0924	64.4569 7350	62.5221 9021	58.8902 3141	53.9814 5905
79	67.1963 2453	65.1313 1691	63.1537 9310	59.4443 9842	54.4370 8817
80	67.9133 5221	65.8023 0538	63.7817 3301	59.9944 4012	54.8882 0611
81	68.6274 0467	66.4699 5561	64.4060 3118	60.5403 8722	55.3348 5753
82	69.3384 9426	67.1342 8410	65.0267 0874	61.0822 7019	55.7770 8666
83	70.0466 3326	67.7958 0705	65.6437 8667	61.6201 1930	56.2149 3729
84	70.7518 8393	68.4530 4244	66.2572 8585	62.1539 6456	56.6484 5276
85	71.4541 0846	69.1075 0491	66.8672 2705	62.6838 3579	57.0776 7600
86	72.1534 6898	69.7587 1135	67.4736 3089	63.2097 6257	57.5026 4951
87	72.8499 2759	70.4066 7796	68.0765 1789	63.7317 7427	57.9234 1535
88	73.5434 9633	71.0514 2086	68.6759 0845	64.2499 0002	58.3400 1520
89	74.2341 8720	71.6929 5608	69.2718 2283	64.7641 6875	58.7524 9030
90	74.9220 1212	72.3312 9058	69.8642 8121	65.2746 0018	59.1608 8148
91	75.6069 8300	72.9664 6725	70.4533 0363	65.7812 4981	59.5652 2919
92	76.2891 1168	73.5984 7487	71.0389 1001	66.2841 1892	59.9655 7346
93	76.9684 0995	74.2273 3818	71.6211 2017	66.7832 4458	60.3619 5392
94	77.6448 8955	74.8530 7282	72.1999 5379	67.2786 5467	60.7544 0982
95	78.3185 6218	75.4756 9434	72.7754 3047	67.7703 7685	61.1429 8002
96	78.9894 3950	76.0952 1825	73.3475 6967	68.2584 3856	61.5277 0299
97	79.6575 3308	76.7116 5995	73.9163 9075	68.7428 6705	61.9086 1682
98	80.3228 5450	77.3250 3478	74.4819 1294	69.2236 8938	62.2857 5923
99	80.9854 1524	77.9353 5799	75.0441 5539	69.7009 3239	62.6591 6755
100	81.6452 2677	78.5426 4477	75.6031 3712	70.1746 2272	63.0288 7877

TABLE VIII—PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1+i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	82.3023 0049	79.1469 1021	76.1588 7702	70.6447 8682	63.3949 2947
102	82.9566 4777	79.7481 6937	76.7113 9392	71.1114 5094	63.7573 5591
103	83.6082 7991	80.3464 3718	77.2607 0648	71.5746 4113	64.1161 9397
104	84.2572 0818	80.9417 2854	77.8068 3331	72.0343 8325	64.4714 7918
105	84.9034 4381	81.5340 5825	78.3497 9288	72.4907 0298	64.8232 4671
106	85.5469 9795	82.1234 4104	78.8896 0355	72.9436 2579	65.1715 3140
107	86.1878 8175	82.7098 9158	79.4262 8359	73.3931 7696	65.5163 6772
108	86.8261 0628	83.2934 2446	79.9598 5115	73.8393 8160	65.8577 8983
109	87.4616 8258	83.8740 5419	80.4903 2428	74.2822 6461	66.1958 3151
110	88.0946 2163	84.4517 9522	81.0177 2093	74.7218 5073	66.5305 2625
111	88.7249 3437	85.0266 6101	81.5420 5895	75.1581 6450	66.8619 0718
112	89.3526 3171	85.5986 0850	82.0633 5600	75.5912 3027	67.1900 0710
113	89.9777 2450	86.1678 2942	82.5816 2991	76.0210 7223	67.5148 5852
114	90.6002 2354	86.7341 5802	83.0968 9803	76.4477 1437	67.8364 9358
115	91.2201 3050	87.2976 7027	83.6091 7785	76.8711 8052	68.1549 4414
116	91.8374 8338	87.8583 7838	84.1184 8671	77.2914 9431	68.4702 4172
117	92.4522 6558	88.4162 9090	84.6248 4182	77.7086 7922	68.7824 1755
118	93.0644 9081	88.9714 3070	85.1282 6033	78.1227 5853	69.0915 0252
119	93.6741 8767	89.5238 2059	85.6287 5926	78.5337 5536	69.3975 2725
120	94.2813 4869	90.0734 5333	86.1263 5554	78.9416 9207	69.7005 2203
121	94.8859 9036	90.6203 5157	86.6210 6602	79.3465 9322	70.0005 1686
122	95.4881 2315	91.1645 2892	87.1129 0742	79.7484 7062	70.2975 4145
123	96.0877 5747	91.7059 9893	87.6018 9638	80.1473 7432	70.5916 2520
124	96.6849 0367	92.2447 7505	88.0880 4946	80.5432 9957	70.8827 9722
125	97.2795 7209	92.7808 7070	88.5713 8308	80.9362 7749	71.1710 8636
126	97.8717 7301	93.3142 9920	89.0519 1361	81.3263 3001	71.4565 2115
127	98.4615 1666	93.8450 7384	89.5296 5731	81.7134 7892	71.7391 2985
128	99.0488 1324	94.3732 0780	90.0046 3032	82.0977 4583	72.0189 4045
129	99.6336 7290	94.8987 1422	90.4768 4873	82.4791 5219	72.2959 8064
130	100.2161 0570	95.4216 0619	90.9463 2851	82.8577 1929	72.5702 7786
131	100.7961 2180	95.9418 9071	91.4130 8554	83.2334 6828	72.8418 5927
132	101.3737 3131	96.4595 9872	91.8771 3561	83.6064 2013	73.1107 5175
133	101.9489 4401	96.9747 2500	92.3384 9442	83.9765 9506	73.3769 8193
134	102.5217 0994	97.4872 8865	92.7971 7758	84.3440 1554	73.6405 7617
135	103.0922 1800	97.9973 0214	93.2532 0060	84.7087 0029	73.9015 6056
136	103.6603 0104	98.5047 7825	93.7065 7892	85.0706 7026	74.1599 6095
137	104.2260 2590	99.0097 2900	94.1573 2787	85.4299 4567	74.4158 0293
138	104.7894 0335	99.5121 6875	94.6054 6270	85.7865 4657	74.6691 1181
139	105.3504 4314	100.0121 0821	95.0509 9857	86.1404 9288	74.9199 1268
140	105.9091 5496	100.5095 0041	95.4939 5050	86.4918 0484	75.1682 3038
141	106.4655 4847	101.0045 3772	95.9343 3364	86.8405 0059	75.4140 8948
142	107.0196 3330	101.4970 5240	96.3721 6272	87.1866 0108	75.6575 1434
143	107.5714 1002	101.9871 1088	96.8074 5201	87.5301 2514	75.8985 2005
144	108.1209 1517	102.4747 4310	97.2402 1804	87.8710 9105	76.1371 5747
145	108.6681 3126	102.9590 4344	97.6704 7304	88.2095 2055	76.3734 2324
146	109.2130 7674	103.4427 2079	98.0082 3307	88.5454 2982	76.6073 4974
147	109.7557 0103	103.9231 1422	98.5235 1350	88.8788 3854	76.8389 6014
148	110.2961 9353	104.4011 0808	98.9403 2663	89.2097 6530	77.0682 7737
149	110.8343 8356	104.8767 2505	99.3666 8765	89.5382 2858	77.2953 2413
150	111.3703 4044	105.3499 7518	99.7846 1078	89.8642 4673	77.5201 2290

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1-(1+i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.9888 7515	0.9876 5432	0.9852 2167	0.9828 0098	0.9803 9216
2	1.9667 4923	1.9631 1538	1.9558 8342	1.9486 9875	1.9415 6094
3	2.9337 4460	2.9265 3371	2.9122 0042	2.8979 8403	2.8838 8327
4	3.8899 8230	3.8780 5798	3.8543 8465	3.8300 4254	3.8077 2870
5	4.8355 8200	4.8178 3504	4.7826 4497	4.7478 5508	4.7134 5951
6	5.7706 6205	5.7460 0992	5.6971 8717	5.6489 9702	5.6014 3089
7	6.6953 3948	6.6627 2585	6.5982 1396	6.5346 4139	6.4719 9107
8	7.6097 3002	7.5681 2429	7.4859 2508	7.4050 5297	7.3254 8144
9	8.5139 4810	8.4623 4498	8.3605 1732	8.2604 9432	8.1622 3671
10	9.4081 0690	9.3455 2591	9.2221 8455	9.1012 2291	8.9825 8501
11	10.2923 1832	10.2178 0337	10.0711 1779	9.9274 9181	9.7868 4805
12	11.1666 9302	11.0793 1197	10.9075 0521	10.7395 4969	10.5753 4122
13	12.0313 4044	11.9301 8466	11.7315 3222	11.5376 4097	11.3483 7375
14	12.8863 6880	12.7705 5275	12.5433 8150	12.3220 0587	12.1062 4877
15	13.7318 8509	13.6005 4592	13.3432 3301	13.0928 8046	12.8492 6350
16	14.5679 9514	14.4202 9227	14.1312 6405	13.8504 9677	13.5777 0931
17	15.3948 0360	15.2299 1829	14.9076 4931	14.5950 8282	14.2918 7188
18	16.2124 1395	16.0295 4893	15.6725 6089	15.3268 0272	14.9920 3125
19	17.0209 2850	16.8193 0759	16.4261 6837	16.0460 5673	15.6784 6201
20	17.8204 4845	17.5993 1613	17.1686 3879	16.7528 8130	16.3514 3334
21	18.6110 7387	18.3696 9495	17.9001 3673	17.4475 4919	17.0112 0916
22	19.3929 0371	19.1305 6291	18.6208 2437	18.1302 0948	17.6580 4820
23	20.1660 3580	19.8820 3744	19.3308 6145	18.8012 4764	18.2922 0412
24	20.9305 6693	20.6242 3451	20.0304 0537	19.4606 8565	18.9139 2560
25	21.6865 9276	21.3572 6865	20.7196 1120	20.1087 8196	19.5234 5647
26	22.4342 0792	22.0812 5299	21.3986 3172	20.7457 3166	20.1210 3576
27	23.1735 0598	22.7962 9925	22.0676 1746	21.3717 2644	20.7068 9780
28	23.9045 7940	23.5025 1778	22.7267 1071	21.9869 5474	21.2812 7236
29	24.6275 1986	24.2000 1756	23.3760 7558	22.5916 0171	21.8443 8466
30	25.3424 1766	24.8889 0623	24.0158 3801	23.1858 4934	22.3964 5555
31	26.0493 6233	25.5692 9010	24.6461 4582	23.7698 7650	22.9377 0152
32	26.7484 4236	26.2412 7418	25.2671 3874	24.3438 5897	23.4683 3482
33	27.4397 4522	26.9049 6215	25.8789 5442	24.9079 6951	23.9885 6355
34	28.1233 5745	27.5604 5644	26.4817 2849	25.4623 7789	24.4985 9172
35	28.7993 6460	28.2078 5822	27.0755 9458	26.0072 5100	24.9986 1933
36	29.4678 5127	28.8472 6737	27.6606 8431	26.5427 5283	25.4888 4248
37	30.1289 0114	29.4787 8259	28.2371 2740	27.0690 4455	25.9694 5341
38	30.7825 9692	30.1025 0133	28.8050 5103	27.5862 8457	26.4406 4060
39	31.4290 2044	30.7185 1983	29.3645 8288	28.0946 2857	26.9025 8883
40	32.0682 5260	31.3269 3316	29.9158 4520	28.5942 2955	27.3554 7924
41	32.7003 7340	31.9278 3522	30.4589 6079	29.0852 8789	27.7994 8945
42	33.3254 6195	32.5213 1874	30.9940 5004	29.5678 0135	28.2347 9358
43	33.9435 9649	33.1074 7530	31.5212 3157	30.0420 6522	28.6615 6233
44	34.5548 5438	33.6863 9536	32.0406 2223	30.5081 7221	29.0799 6307
45	35.1593 1212	34.2581 6825	32.5523 3718	30.9662 6261	29.4901 5987
46	35.7570 4536	34.8228 8222	33.0564 8983	31.4164 7431	29.8923 1360
47	36.3481 2891	35.3806 2442	33.5531 9195	31.8589 4281	30.2865 8196
48	36.9326 3674	35.9314 8091	34.0425 5365	32.2938 0129	30.6731 1957
49	37.5106 4202	36.4755 3670	34.5246 8339	32.7211 8063	31.0520 7801
50	38.0822 1708	37.0128 7574	34.9996 8807	33.1412 0946	31.4236 0589

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1+i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	38.6474 3345	37.5435 8009	35.4676 7298	33.5540 1421	31.7878 4892
52	39.2063 0188	38.0677 3431	35.9287 4185	33.9597 1913	32.1449 4992
53	39.7590 7232	38.5854 1660	36.3829 9090	34.3584 4033	32.4950 4894
54	40.3056 3304	39.0967 0770	36.8305 3882	34.7503 1579	32.8382 8327
55	40.8461 1514	39.6016 8607	37.2714 0081	35.1354 4550	33.1747 8752
56	41.3805 8358	40.1004 3128	37.7058 7803	35.5139 5135	33.5046 9365
57	41.9091 0613	40.5930 1855	38.1338 7058	35.8859 4727	33.8281 3103
58	42.4317 4896	41.0795 2449	38.5555 3751	36.2515 4523	34.1452 2650
59	42.9485 7740	41.5600 2419	38.9709 7292	36.6108 5520	34.4561 0441
60	43.4596 5638	42.0345 0179	39.3802 0880	36.9639 8552	34.7608 8668
61	43.9650 4052	42.5033 0054	39.7835 1614	37.3110 4228	35.0596 9282
62	44.4648 2029	42.9662 2275	40.1808 0408	37.6521 3000	35.3526 4002
63	44.9590 3119	43.4234 2088	40.5722 2077	37.9873 5135	35.6398 4316
64	45.4477 4407	43.8749 9247	40.9578 5298	38.3168 0723	35.9214 1486
65	45.9310 2009	44.3209 8022	41.3377 8018	38.6405 9678	36.1974 6555
66	46.4089 1975	44.7614 0195	41.7121 0461	38.9588 1748	36.4681 0348
67	46.8815 0284	45.1965 0563	42.0808 0125	39.2715 0509	36.7334 3478
68	47.3488 2852	45.6261 7840	42.4442 2783	39.5789 3375	36.9935 6351
69	47.8109 5527	46.0505 4056	42.8021 9490	39.8810 1597	37.2485 9108
70	48.2679 4094	46.4696 7562	43.1548 7183	40.1779 0207	37.4986 1929
71	48.7198 4270	46.8836 3024	43.5023 3078	40.4696 8321	37.7437 4441
72	49.1667 1714	47.2924 7431	43.8446 0077	40.7564 4542	37.9840 6314
73	49.6086 2016	47.6962 7093	44.1819 3771	41.0382 7500	38.2196 6075
74	50.0456 0708	48.0950 8240	44.5142 2434	41.3152 5857	38.4506 5602
75	50.4777 3259	48.4889 7027	44.8416 0034	41.5874 7771	38.6771 1433
76	50.9050 5077	48.8779 9533	45.1641 3826	41.8550 1495	38.8991 3170
77	51.3276 1510	49.2622 1761	45.4819 0902	42.1179 5081	39.1167 9578
78	51.7454 7847	49.6419 9640	45.7949 8485	42.3763 0443	39.3301 9104
79	52.1586 9317	50.0164 9027	46.1034 3335	42.6303 3359	39.5394 0380
80	52.5673 1092	50.3866 5706	46.4073 2349	42.8799 3474	39.7445 1359
81	52.9713 8280	50.7522 5389	46.7067 2265	43.1252 4298	39.9456 0156
82	53.3709 5957	51.1133 3717	47.0016 0720	43.3663 3217	40.1427 4663
83	53.7660 9104	51.4699 6204	47.2923 1251	43.6032 7486	40.3360 2611
84	54.1568 2074	51.8221 8532	47.5786 3301	43.8361 4237	40.5255 1579
85	54.5432 1557	52.1700 5958	47.8607 2218	44.0650 0479	40.7112 8990
86	54.9253 0588	52.5136 3009	48.1386 4254	44.2899 3099	40.8934 2156
87	55.3031 4540	52.8529 7088	48.4124 5571	44.5109 8869	41.0719 8192
88	55.6767 8100	53.1881 2531	48.6822 2237	44.7282 4441	41.2470 4110
89	56.0462 0120	53.5191 3011	48.9480 0234	44.9417 6355	41.4186 0774
90	56.4116 3041	53.8460 0035	49.2098 5452	45.1516 1037	41.5869 2916
91	56.7729 3490	54.1689 4850	49.4678 3090	45.3578 4803	41.7518 9133
92	57.1302 1092	54.4878 5037	49.7220 0086	45.5605 3860	41.9136 1805
93	57.4835 3021	54.8028 1518	49.9724 2055	45.7597 4310	42.0721 7545
94	57.8329 0007	55.1138 9154	50.2191 3355	45.9555 2147	42.2276 2200
95	58.1784 0204	55.4211 2744	50.4622 0054	46.1479 3265	42.3800 2254
96	58.5200 5235	55.7245 7031	50.7016 7541	46.3370 3455	42.5294 3386
97	58.8579 0090	56.0242 0608	50.9376 1124	46.5228 8408	42.6759 1555
98	59.1919 9106	56.3202 0308	51.1700 0034	46.7055 3718	42.8195 2505
99	59.5223 0446	56.6126 0610	51.3990 7422	46.8850 4882	42.9603 1867
100	59.8490 0251	56.9013 3936	51.6247 0307	47.0614 7304	43.0983 5164

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1+i)^{-n}}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379	0.9661 8357
2	1.9344 6955	1.9274 2415	1.9204 2434	1.9134 6970	1.8996 9428
3	2.8698 9687	2.8560 2356	2.8422 6213	2.8286 1135	2.8016 3698
4	3.7847 4021	3.7619 7421	3.7394 2787	3.7170 9840	3.6730 7921
5	4.6794 5253	4.6458 2850	4.6125 8186	4.5797 0719	4.5150 5238
6	5.5544 7680	5.5081 2536	5.4623 6078	5.4171 9144	5.3285 5302
7	6.4102 4626	6.3493 9060	6.2894 0806	6.2302 8296	6.1145 4398
8	7.2471 8461	7.1701 3717	7.0943 1441	7.0196 9219	6.8739 5554
9	8.0657 0622	7.9708 6553	7.8776 7820	7.7861 0892	7.6076 8651
10	8.8662 1635	8.7520 6393	8.6400 7616	8.5302 0284	8.3166 0532
11	9.6491 1134	9.5142 0871	9.3820 6920	9.2526 2411	9.0015 5104
12	10.4147 7882	10.2577 6460	10.1042 0360	9.9540 0399	9.6633 3433
13	11.1635 9787	10.9831 8497	10.8070 1086	10.6349 5533	10.3027 3849
14	11.8959 8924	11.6909 1217	11.4910 0814	11.2960 7314	10.9205 2028
15	12.6121 6551	12.3813 7773	12.1566 9892	11.9379 3509	11.5174 1090
16	13.3126 8131	13.0550 0266	12.8045 7315	12.5611 0203	12.0941 1681
17	13.9976 8343	13.7121 9772	13.4351 0769	13.1661 1847	12.6513 2059
18	14.6676 6106	14.3533 6363	14.0487 6661	13.7535 1308	13.1896 8173
19	15.3228 9590	14.9788 9134	14.6460 0157	14.3237 9911	13.7098 3742
20	15.9637 1237	15.5891 6229	15.2272 5213	14.8774 7486	14.2124 0330
21	16.5904 2775	16.1845 4857	15.7929 4612	15.4150 2414	14.6979 7420
22	17.2033 5232	16.7654 1324	16.3434 9987	15.9369 1664	15.1671 2484
23	17.8027 8955	17.3321 1048	16.8793 1861	16.4436 0839	15.6204 1047
24	18.3890 3624	17.8849 8583	17.4007 9670	16.9355 4212	16.0583 0760
25	18.9623 8263	18.4243 7642	17.9083 1795	17.4131 4769	16.4815 1459
26	19.5231 1260	18.9506 1114	18.4022 5592	17.8768 4242	16.8903 5226
27	20.0715 0376	19.4640 1087	18.8829 7413	18.3270 3147	17.2853 6451
28	20.6078 2764	19.9648 8866	19.3508 2040	18.7641 0823	17.6670 1885
29	21.1323 4977	20.4535 4991	19.8061 5708	19.1884 5459	18.0357 6700
30	21.6453 2985	20.9302 9259	20.2493 0130	19.6004 4135	18.3920 4541
31	22.1470 2186	21.3954 0741	20.6805 8520	20.0004 2840	18.7362 7576
32	22.6376 7419	21.8491 7796	21.1003 2023	20.3887 6553	19.0688 6547
33	23.1175 2977	22.2918 8094	21.5088 3332	20.7657 9178	19.3902 0818
34	23.5868 2618	22.7237 8628	21.9064 0712	21.1318 3668	19.7006 8423
35	24.0457 9577	23.1451 5734	22.2933 4026	21.4872 2007	20.0006 6110
36	24.4946 6579	23.5562 5107	22.6699 1753	21.8322 5250	20.2904 9381
37	24.9336 5848	23.9573 1812	23.0364 1609	22.1672 3544	20.5705 2542
38	25.3629 9118	24.3486 0304	23.3931 0568	22.4924 6159	20.8410 8736
39	25.7828 7646	24.7303 4443	23.7402 4884	22.8082 1513	21.1024 9087
40	26.1935 2221	25.1027 7505	24.0781 0106	23.1147 7197	21.3550 7234
41	26.5951 3174	25.4661 2200	24.4069 1101	23.4123 9997	21.5991 0371
42	26.9879 0390	25.8206 0683	24.7269 2069	23.7013 5920	21.8348 8281
43	27.3720 3316	26.1664 4569	25.0383 6563	23.9819 0213	22.0626 8870
44	27.7477 0969	26.5038 4945	25.3414 7507	24.2542 7392	22.2827 9102
45	28.1151 1950	26.8330 2386	25.6364 7209	24.5187 1254	22.4954 5026
46	28.4744 4450	27.1541 6962	25.9235 7381	24.7754 4907	22.7009 1813
47	28.8258 6259	27.4674 8255	26.2029 9154	25.0247 0783	22.8994 3780
48	29.1695 4777	27.7731 5371	26.4749 3094	25.2667 0664	23.0912 4425
49	29.5056 7019	28.0713 6947	26.7395 9215	25.5016 5693	23.2765 0450
50	29.8343 9627	28.3623 1168	26.9971 6998	25.7297 6401	23.4556 1757

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1-(1+i)^{-n}}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	30.1558 8877	28.6461 5774	27.2478 5400	25.9512 2719	23.6286 1030
52	30.4703 0687	28.9230 8072	27.4918 2871	26.1662 3999	23.7957 6454
53	30.7778 0623	29.1932 4948	27.7202 7368	26.3749 9028	23.9572 6043
54	31.0785 3910	29.4568 2876	27.9603 6308	26.5776 6047	24.1132 9510
55	31.3726 5438	29.7139 7928	28.1852 6879	26.7744 2764	24.2640 5323
56	31.6602 9768	29.9648 5784	28.4041 5454	26.9654 6373	24.4097 1327
57	31.9416 1142	30.2096 1740	28.6171 8203	27.1509 3566	24.5504 4760
58	32.2167 3489	30.4484 0722	28.8245 0806	27.3310 0549	24.6864 2281
59	32.4858 0429	30.6813 7290	29.0262 8522	27.5058 3058	24.8177 9981
60	32.7489 5285	30.9086 5649	29.2226 6201	27.6755 6367	24.9447 3412
61	33.0063 1086	31.1303 9657	29.4137 8298	27.8403 5307	25.0673 7596
62	33.2580 0573	31.3467 2836	29.5997 8879	28.0003 4279	25.1858 7049
63	33.5041 6208	31.5577 8377	29.7808 1634	28.1556 7261	25.3003 5706
64	33.7449 0179	31.7636 9148	29.9569 9887	28.3064 7826	25.4109 7388
65	33.9803 4405	31.9645 7705	30.1284 6605	28.4528 9152	25.5178 4016
66	34.2106 0543	32.1605 6298	30.2953 4409	28.5950 4031	25.6211 1030
67	34.4357 9993	32.3517 6876	30.4577 5581	28.7330 4884	25.7208 7951
68	34.6560 3905	32.5383 1099	30.6158 2074	28.8670 3771	25.8172 7480
69	34.8714 3183	32.7203 0340	30.7696 5522	28.9971 2399	25.9104 1052
70	35.0820 8492	32.8978 5698	30.9193 7247	29.1234 2135	26.0003 9664
71	35.2881 0261	33.0710 7998	31.0650 8270	29.2460 4015	26.0873 3975
72	35.4895 8691	33.2400 7803	31.2068 9314	29.3650 8752	26.1713 4275
73	35.6866 8756	33.4049 5417	31.3440 0816	29.4806 6750	26.2525 0508
74	35.8793 5214	33.5658 0895	31.4792 2936	29.5928 8106	26.3309 2278
75	36.0678 2605	33.7227 4044	31.6099 5558	29.7018 2628	26.4066 8808
76	36.2521 5262	33.8758 4433	31.7371 8304	29.8075 9833	26.4798 9244
77	36.4324 2310	34.0252 1398	31.8610 0540	29.9102 8064	26.5506 2072
78	36.6087 2675	34.1709 4047	31.9815 1377	30.0099 8094	26.6189 5721
79	36.7811 5085	34.3131 1265	32.0987 9685	30.1067 8035	26.6849 8281
80	36.9497 8079	34.4518 1722	32.2129 4098	30.2007 6345	26.7487 7567
81	37.1147 0004	34.5871 3875	32.3240 3015	30.2920 0335	26.8104 1127
82	37.2759 9026	34.7191 5970	32.4321 4613	30.3805 8577	26.8699 6258
83	37.4337 3130	34.8479 6074	32.5373 6850	30.4665 8813	26.9275 0008
84	37.5880 0127	34.9736 2023	32.6397 7469	30.5500 8556	26.9830 9186
85	37.7388 7655	35.0962 1486	32.7394 4009	30.6311 5103	27.0368 0373
86	37.8864 3183	35.2158 1938	32.8364 3804	30.7098 5537	27.0886 9026
87	38.0307 4018	35.3325 0671	32.9308 3994	30.7862 6735	27.1388 3986
88	38.1718 7304	35.4463 4801	33.0227 1527	30.8604 5374	27.1872 8480
89	38.3099 0028	35.5574 1269	33.1121 3165	30.9324 7936	27.2340 9168
90	38.4448 0025	35.6657 6848	33.1991 5489	31.0024 0714	27.2793 1564
91	38.5769 0978	35.7714 8144	33.2838 4005	31.0702 9820	27.3230 1028
92	38.7060 2423	35.8746 1604	33.3662 7644	31.1362 1184	27.3652 2732
93	38.8322 9754	35.9752 3510	33.4464 9776	31.2002 0567	27.4060 1673
94	38.9557 0221	36.0734 0016	33.5245 7202	31.2623 3560	27.4454 2680
95	39.0765 6040	36.1691 7089	33.6005 5671	31.3226 5592	27.4835 0415
96	39.1946 8800	36.2626 0574	33.6745 0775	31.3812 1034	27.5202 9387
97	39.3102 0920	36.3537 6170	33.7464 7056	31.4380 7703	27.5558 3948
98	39.4231 8748	36.4426 9434	33.8165 2512	31.4932 7867	27.5901 8308
99	39.5336 7968	36.5294 5700	33.8846 9508	31.5468 7250	27.6233 6529
100	39.6417 4052	36.6141 0520	33.9510 4232	31.5989 0534	27.6554 2840

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1 - (1 + i)^{-n}}{i}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 0730	0.9433 9623
2	1.8860 9467	1.8726 6775	1.8594 1043	1.8463 1971	1.8333 9207
3	2.7750 9103	2.7489 6435	2.7232 4803	2.6970 3338	2.0730 1105
4	3.6298 9522	3.5875 2570	3.5450 5050	3.5051 5012	3.4051 0561
5	4.4518 2233	4.3899 7674	4.3294 7607	4.2702 8448	4.2123 6370
6	5.2421 3686	5.1578 7248	5.0756 9200	4.9955 3031	4.9173 2433
7	6.0020 5467	5.8927 0094	5.7863 7340	5.6820 0712	5.5823 8144
8	6.7327 4487	6.5958 8007	6.4632 1276	6.3345 0500	0.2007 9381
9	7.4353 3161	7.2687 9050	7.1078 2168	6.9521 9525	6.8016 9227
10	8.1108 9578	7.9127 1818	7.7217 3493	7.5376 2583	7.3600 8705
11	8.7604 7671	8.5289 1692	8.3064 1422	8.0925 3633	7.8868 7458
12	9.3850 7376	9.1185 8078	8.8632 5164	8.6185 1785	8.3838 4304
13	9.9856 4785	9.6828 5242	9.3935 7299	9.1170 7853	8.8520 8200
14	10.5631 2293	10.2228 2528	9.8986 4094	0.5806 4790	9.2049 8393
15	11.1183 8743	10.7395 4573	10.3796 5804	10.0375 8094	9.7122 4800
16	11.6522 9561	11.2340 1505	10.8377 6950	10.4621 6203	10.1058 9527
17	12.1656 6885	11.7071 9143	11.2740 6625	10.8646 0860	10.4772 5969
18	12.6592 9697	12.1599 9180	11.6895 8690	11.2460 7447	10.8276 0348
19	13.1339 3940	12.5932 9359	12.0853 2086	11.6076 5352	11.1581 1649
20	13.5903 2634	13.0079 3645	12.4622 1034	11.9503 8249	11.4699 2122
21	14.0291 5995	13.4047 2388	12.8211 5271	12.2752 4406	11.7640 7662
22	14.4511 1533	13.7844 2476	13.1630 0258	12.5831 6073	12.0415 8172
23	14.8568 4167	14.1477 7489	13.4885 7388	12.8750 4240	12.3033 7808
24	15.2469 6314	14.4954 7837	13.7986 4179	13.1516 9805	12.5503 5753
25	15.6220 7994	14.8282 0896	14.0939 4457	13.4139 3266	12.7833 5616
26	15.9827 6918	15.1466 1145	14.3751 8530	13.6624 9541	13.0031 6619
27	16.3295 8575	15.4513 0282	14.6430 3362	13.8980 9991	13.2105 3414
28	16.6630 6822	15.7428 7351	14.8981 2726	14.1214 2172	13.4061 6428
29	16.9837 1463	16.0218 8853	15.1410 7358	14.3331 0116	13.5907 2102
30	17.2920 3330	16.2888 8854	15.3724 5103	14.5337 4617	13.7648 3115
31	17.5884 9356	16.5443 9095	15.5928 1050	14.7239 2907	13.9290 8599
32	17.8735 5150	16.7888 9086	15.8026 7667	14.9041 9817	14.0840 4339
33	18.1476 4567	17.0228 6207	16.0025 4921	15.0750 6936	14.2302 2961
34	18.4111 9776	17.2467 5796	16.1929 0401	15.2370 3257	14.3681 4114
35	18.6646 1323	17.4610 1240	16.3741 9429	15.3905 5220	14.4982 4636
36	18.9082 8195	17.6660 4058	16.5468 5171	15.5360 6843	14.6209 8713
37	19.1425 7880	17.8622 3979	16.7112 8734	15.6739 9851	14.7367 8031
38	10.3678 6423	18.0499 0023	16.8678 9271	15.8047 3703	14.8460 1916
39	19.5844 8484	18.2296 5572	17.0170 4067	15.9286 6154	14.9490 7468
40	19.7927 7388	18.4015 8442	17.1590 8635	16.0461 2469	15.0462 9687
41	19.9930 5181	18.5661 0049	17.2943 6796	16.1574 6416	15.1380 1592
42	20.1856 2674	18.7235 4975	17.4232 0758	16.2629 9920	15.2245 4332
43	20.3707 9494	18.8742 1029	17.5459 1198	16.3630 8242	15.3061 7294
44	20.5488 4129	10.0183 8305	17.6627 7331	16.4578 5003	15.3831 8202
45	20.7200 3970	19.1563 4742	17.7740 6982	16.5477 2572	15.4558 3209
46	20.8846 5356	19.2883 7074	17.8800 6650	16.6329 1537	15.5243 6990
47	21.0429 3612	19.4147 0884	17.9810 1571	16.7136 6386	15.5890 2821
48	21.1951 3088	19.5356 0654	18.0771 5782	10.7902 0271	15.6500 2661
49	21.3414 7200	19.6512 9813	18.1687 2173	16.8627 5139	15.7075 7227
50	21.4821 8462	19.7620 0778	18.2559 2546	16.9315 1790	15.7618 6004

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1-(1+i)^{-n}}{i}$$

n	4%	4½%	5%	5½%	6%
51	21.6174 8521	19.8679 5003	18.3390 7663	16.9966 9943	15.8130 7607
52	21.7475 8193	19.9693 3017	18.4180 7298	17.0584 8287	15.8613 9252
53	21.8726 7403	20.0663 4466	18.4934 0284	17.1170 4538	15.9069 7408
54	21.9929 5667	20.1591 8140	18.5651 4556	17.1725 5486	15.9499 7554
55	22.1086 1218	20.2480 2057	18.6334 7106	17.2251 7048	15.9905 4297
56	22.2198 1940	20.3330 3404	18.6985 4473	17.2750 4311	16.0288 1412
57	22.3267 4943	20.4143 8664	18.7605 1879	17.3223 1575	16.0649 1898
58	22.4295 6676	20.4922 3002	18.8195 4170	17.3671 2393	16.0989 8017
59	22.5284 2957	20.5667 3303	18.8757 5400	17.4095 9614	16.1311 1337
60	22.6234 8997	20.6380 2204	18.9292 8952	17.4498 5416	16.1614 2771
61	22.7148 0421	20.7062 4118	18.9802 7574	17.4880 1343	16.1900 2614
62	22.8027 8280	20.7715 2266	19.0288 3404	17.5241 8334	16.2170 0579
63	22.8872 0124	20.8339 9298	19.0750 8003	17.5584 0702	16.2424 5829
64	22.9685 4927	20.8937 7310	19.1191 2384	17.5909 6457	16.2664 7009
65	23.0466 8109	20.9509 7013	19.1610 7033	17.6217 0737	16.2891 2272
66	23.1218 0061	21.0057 2165	19.2010 1936	17.6509 6433	16.3104 9314
67	23.1940 4770	21.0581 0684	19.2390 6606	17.6786 3017	16.3306 5300
68	23.2635 0740	21.1082 3621	19.2753 0101	17.7048 7125	16.3496 7840
69	23.3302 9558	21.1562 0090	19.3098 1048	17.7297 3570	16.3676 1650
70	23.3945 1408	21.2021 1187	19.3426 7665	17.7533 0406	16.3845 4387
71	23.4562 6440	21.2460 4007	19.3739 7776	17.7756 4366	16.4005 1308
72	23.5156 3885	21.2880 7662	19.4037 8834	17.7968 1804	16.4155 7838
73	23.5727 2060	21.3283 0298	19.4321 7937	17.8168 8970	16.4297 9093
74	23.6276 2408	21.3667 0711	19.4592 1845	17.8359 1441	16.4431 9899
75	23.6804 0834	21.4036 3360	19.4849 6995	17.8539 4731	16.4558 4810
76	23.7311 6187	21.4388 8383	19.5094 0510	17.8710 4010	16.4677 8123
77	23.7799 6333	21.4726 1611	19.5328 5257	17.8872 4180	16.4790 3889
78	23.8268 8782	21.5048 9579	19.5550 9768	17.9025 9887	16.4896 5933
79	23.8720 0752	21.5357 8545	19.5762 8351	17.9171 5532	16.4996 7802
80	23.9153 9185	21.5653 4493	19.5964 6048	17.9309 5201	16.5091 3077
81	23.9571 0754	21.5936 3161	19.6156 7665	17.9440 3120	16.5180 4700
82	23.9972 1870	21.6207 0001	19.6339 7776	17.9564 2768	16.5264 0028
83	24.0357 8730	21.6466 0288	19.6514 0739	17.9681 7789	16.5343 0640
84	24.0728 7240	21.6713 9032	19.6680 0704	17.9793 1554	16.5418 8348
85	24.1085 3116	21.6951 1035	19.6838 1623	17.9898 7255	16.5489 4068
86	24.1428 1842	21.7178 0805	19.6988 7200	17.9998 7910	16.5556 1008
87	24.1757 8694	21.7395 3000	19.7132 1200	18.0093 6410	16.5618 9630
88	24.2074 8745	21.7603 1588	19.7268 6857	18.0183 5466	16.5678 2670
89	24.2379 6870	21.7802 0658	19.7398 7483	18.0268 7645	16.5734 2141
90	24.2672 7750	21.7992 4076	19.7522 6174	18.0349 5308	16.5786 9944
91	24.2954 5023	21.8174 5526	19.7640 5880	18.0426 1041	16.5836 7872
92	24.3225 5605	21.8348 8542	19.7752 9410	18.0498 0709	16.5883 7615
93	24.3486 1245	21.8515 0499	19.7859 9438	18.0567 4662	16.5928 0769
94	24.3736 6582	21.8675 2631	19.7961 8512	18.0632 6694	16.5969 8830
95	24.3977 5559	21.8828 0030	19.8058 9059	18.0694 4734	16.6009 3244
96	24.4209 1884	21.8974 1055	19.8151 3300	18.0753 0553	16.6046 5325
97	24.4431 9110	21.9114 0340	19.8239 3705	18.0808 5833	16.6081 6344
98	24.4646 0002	21.9247 8794	19.8323 2100	18.0861 2104	16.6114 7494
99	24.4851 9896	21.9375 0612	19.8403 0671	18.0911 1055	16.6145 9900
100	24.5049 9900	21.9498 5274	19.8479 1020	18.0958 3039	16.6175 4623

TABLE VIII — PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$(a_{\overline{n}|} \text{ at } i) = \frac{1-(1+i)^{-n}}{i}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593	0.9216 5899
2	1.8206 2642	1.8080 1817	1.7955 6517	1.7832 6475	1.7711 1427
3	2.6484 7551	2.6243 1604	2.6005 2574	2.5770 9699	2.5540 2237
4	3.4257 9860	3.3872 1126	3.3493 2627	3.3121 2684	3.2755 9666
5	4.1556 7944	4.1001 9744	4.0458 8490	3.9927 1004	3.9406 4208
6	4.8410 1356	4.7665 3966	4.6938 4642	4.6228 7966	4.5535 8717
7	5.4845 1977	5.3892 8940	5.2966 0132	5.2063 7006	5.1185 1352
8	6.0887 5096	5.9712 9851	5.8573 0355	5.7466 3894	5.6391 8297
9	6.6561 0419	6.5152 3225	6.3788 8703	6.2468 8791	6.1190 6264
10	7.1888 3022	7.0235 8154	6.8640 8096	6.7100 8140	6.5613 4806
11	7.6890 4246	7.4986 7434	7.3154 2415	7.1389 6426	6.9689 8439
12	8.1587 2532	7.9426 8630	7.7352 7827	7.5360 7802	7.3446 8607
13	8.5997 4208	8.3576 5074	8.1258 4026	7.9037 7594	7.6909 5490
14	9.0138 4233	8.7454 6799	8.4891 5373	8.2442 3698	8.0100 0668
15	9.4026 6885	9.1079 1401	8.8271 1974	8.5594 7869	8.3042 3658
16	9.7677 6418	9.4466 4860	9.1415 0674	8.8513 6916	8.5753 3325
17	10.1105 7670	9.7632 2299	9.4339 5976	9.1216 3811	8.8251 9194
18	10.4324 6638	10.0590 8691	9.7060 0908	9.3718 8714	9.0554 7644
19	10.7347 1022	10.3355 9524	9.9590 7821	9.6035 9920	9.2677 2022
20	11.0185 0725	10.5940 1425	10.1944 9136	9.8181 4741	9.4633 3661
21	11.2849 8333	10.8355 2733	10.4134 8033	10.0168 0316	9.6436 2821
22	11.5351 0562	11.0612 4050	10.6171 9101	10.2007 4366	9.8097 0559
23	11.7701 3673	11.2721 8738	10.8066 8931	10.3710 5895	9.9629 4524
24	11.9907 3871	11.4693 3400	10.9829 6680	10.5287 5828	10.1040 9700
25	12.1978 7672	11.6535 8318	11.1469 4586	10.6747 7619	10.2341 9078
26	12.3923 7251	11.8257 7867	11.2994 8452	10.8099 7795	10.3540 9288
27	12.5749 9766	11.9867 0904	11.4413 8095	10.9351 6477	10.4646 0174
28	12.7464 7668	12.1371 1125	11.5733 7763	11.0510 7849	10.5664 5321
29	12.9074 8984	12.2776 7407	11.6961 6524	11.1584 0601	10.6603 2554
30	13.0586 7591	12.4090 4118	11.8103 8627	11.2577 8334	10.7468 4382
31	13.2006 3465	12.5318 1410	11.9166 3839	11.3497 9939	10.8265 8416
32	13.3339 2925	12.6465 5532	12.0154 7757	11.4349 9944	10.9000 7757
33	13.4590 8850	12.7537 9002	12.1074 2099	11.5138 8837	10.9678 1343
34	13.5766 0892	12.8540 0936	12.1929 4976	11.5869 3367	11.0302 4279
35	13.6869 5673	12.9476 7230	12.2725 1141	11.6545 6822	11.0877 8137
36	13.7905 6970	13.0352 0776	12.3465 2224	11.7171 9279	11.1408 1233
37	13.8878 5887	13.1170 1660	12.4153 6953	11.7751 7851	11.1896 8878
38	13.9792 1021	13.1934 7345	12.4794 1351	11.8288 6899	11.2347 3620
39	14.0649 8611	13.2649 2846	12.5389 8931	11.8785 8240	11.2762 5457
40	14.1455 2687	13.3317 0884	12.5944 0866	11.9246 1333	11.3145 2034
41	14.2211 5190	13.3941 2041	12.6459 6155	11.9672 3457	11.3497 8833
42	14.2921 6140	13.4524 4898	12.6939 1772	12.0066 9867	11.3822 0339
43	14.3588 3708	13.5069 6167	12.7385 2811	12.0432 3951	11.4122 5197
44	14.4214 4327	13.5579 0810	12.7800 2615	12.0770 7362	11.4398 6357
45	14.4802 2842	13.6055 2159	12.8186 2898	12.1084 0150	11.4653 1205
46	14.5354 2575	13.6500 2018	12.8545 3858	12.1374 0880	11.4887 6686
47	14.5872 5422	13.6916 0764	12.8879 4287	12.1642 6741	11.5103 8420
48	14.6359 1946	13.7304 7443	12.9190 1662	12.1891 3649	11.5303 0802
49	14.6816 1451	13.7667 9853	12.9479 2244	12.2121 6341	11.5486 7099
50	14.7245 2067	13.8007 4629	12.9748 1157	12.2334 8464	11.5655 0538

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n|}}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n|}}\ at\ i)}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0041 6667	1.0050 0000	1.0058 3333	1.0075 0000	1.0100 0000
2	0.5031 2717	0.5037 5312	0.5043 7024	0.5056 3200	0.5075 1244
3	0.3361 1496	0.3366 7221	0.3372 2976	0.3383 4579	0.3400 2211
4	0.2520 0958	0.2531 3279	0.2536 5044	0.2547 0501	0.2562 8109
5	0.2025 0693	0.2030 0997	0.2035 1357	0.2045 2242	0.2060 3980
6	0.1601 0504	0.1695 9546	0.1700 8594	0.1710 6891	0.1725 4837
7	0.1452 4800	0.1457 2854	0.1462 0086	0.1471 7488	0.1486 2828
8	0.1273 5512	0.1278 2886	0.1283 0351	0.1202 5552	0.1306 9029
9	0.1134 3876	0.1139 0736	0.1143 7698	0.1153 1029	0.1167 4037
10	0.1023 0506	0.1027 7057	0.1032 3632	0.1041 7123	0.1055 8208
11	0.0031 9757	0.0936 5903	0.0941 2175	0.0950 5094	0.0964 5408
12	0.0856 0748	0.0860 6043	0.0865 2675	0.0874 5148	0.0888 4879
13	0.0791 8532	0.0706 4224	0.0801 0004	0.0810 2188	0.0824 1482
14	0.0736 8082	0.0741 3600	0.0745 9205	0.0755 1146	0.0769 0117
15	0.0689 1045	0.0603 6436	0.0608 1999	0.0707 3639	0.0721 2378
16	0.0647 3055	0.0651 8937	0.0656 4401	0.0665 5879	0.0679 4460
17	0.0610 5387	0.0615 0579	0.0619 5966	0.0628 7321	0.0642 5806
18	0.0577 8053	0.0582 3173	0.0586 8499	0.0595 9766	0.0609 8205
19	0.0548 5191	0.0553 0253	0.0557 5532	0.0566 6740	0.0580 5175
20	0.0522 1030	0.0526 6645	0.0531 1889	0.0540 3063	0.0554 1532
21	0.0498 3183	0.0502 8163	0.0507 3383	0.0516 4543	0.0530 3075
22	0.0476 6427	0.0481 1380	0.0485 6585	0.0494 7748	0.0508 6371
23	0.0456 8531	0.0461 3465	0.0465 8663	0.0474 9846	0.0488 8584
24	0.0438 7130	0.0443 2061	0.0447 7258	0.0456 8474	0.0470 7347
25	0.0422 0270	0.0426 5186	0.0431 0388	0.0440 1650	0.0454 0675
26	0.0406 6247	0.0411 1163	0.0415 6376	0.0424 7003	0.0438 6888
27	0.0392 3645	0.0396 8565	0.0401 3793	0.0410 5176	0.0424 4553
28	0.0379 1230	0.0383 6167	0.0388 1415	0.0397 2871	0.0411 2444
29	0.0366 7974	0.0371 2014	0.0375 8186	0.0384 0723	0.0398 9502
30	0.0355 2036	0.0359 7892	0.0304 3191	0.0373 4816	0.0387 4811
31	0.0344 5330	0.0349 0304	0.0353 5633	0.0362 7352	0.0376 7573
32	0.0334 4458	0.0338 0453	0.0343 4815	0.0352 6634	0.0366 7080
33	0.0324 9708	0.0329 4727	0.0334 0124	0.0343 2048	0.0357 2744
34	0.0316 0540	0.0320 5586	0.0325 1020	0.0334 3053	0.0348 8907
35	0.0307 6476	0.0312 1550	0.0316 7024	0.0325 9170	0.0340 0368
36	0.0209 7000	0.0304 2104	0.0308 7710	0.0317 9973	0.0332 1431
37	0.0202 2003	0.0296 7139	0.0301 2698	0.0310 5082	0.0324 6805
38	0.0285 0875	0.0289 6045	0.0204 1640	0.0303 4157	0.0317 6150
39	0.0278 3402	0.0282 8607	0.0287 4258	0.0296 6893	0.0310 9160
40	0.0271 9810	0.0276 4552	0.0281 0251	0.0290 3016	0.0304 5560
41	0.0263 8352	0.0270 3631	0.0274 9379	0.0284 2276	0.0298 5102
42	0.0260 0303	0.0264 5622	0.0269 1420	0.0278 4452	0.0202 7563
43	0.0254 4961	0.0259 0320	0.0263 6170	0.0272 9338	0.0287 2737
44	0.0249 2141	0.0253 7541	0.0258 3443	0.0267 0751	0.0282 0441
45	0.0244 1675	0.0248 7117	0.0253 3073	0.0262 6521	0.0277 0505
46	0.0239 3409	0.0243 8804	0.0248 4905	0.0257 8405	0.0272 2775
47	0.0234 7204	0.0239 2733	0.0243 8708	0.0253 2532	0.0267 7111
48	0.0230 2920	0.0234 8503	0.0239 4624	0.0248 8504	0.0263 3384
49	0.0226 0468	0.0230 6087	0.0235 2265	0.0244 6292	0.0259 1474
50	0.0221 9711	0.0226 5376	0.0231 1611	0.0240 5787	0.0255 1273

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|} \text{ at } i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|} \text{ at } i)}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	0.0218 0557	0.0222 6269	0.0227 2563	0.0230 6888	0.0251 2680
52	0.0214 2916	0.0218 8075	0.0223 5027	0.0232 0503	0.0247 5603
53	0.0210 6700	0.0215 2507	0.0219 8019	0.0220 3540	0.0243 9956
54	0.0207 1830	0.0211 7086	0.0216 4157	0.0225 8938	0.0240 5658
55	0.0203 8234	0.0208 4139	0.0213 0671	0.0222 5005	0.0237 2637
56	0.0200 5843	0.0205 1797	0.0209 8390	0.0219 3478	0.0234 0823
57	0.0197 4593	0.0202 0598	0.0206 7251	0.0216 2496	0.0231 0156
58	0.0194 4426	0.0199 0481	0.0203 7190	0.0213 2597	0.0228 0573
59	0.0191 5287	0.0196 1392	0.0200 8170	0.0210 3727	0.0225 2020
60	0.0188 7123	0.0193 3280	0.0198 0120	0.0207 5836	0.0222 4445
61	0.0185 9888	0.0190 6096	0.0195 2999	0.0204 8873	0.0219 7800
62	0.0183 3536	0.0187 9796	0.0192 6762	0.0202 2795	0.0217 2041
63	0.0180 8025	0.0185 4337	0.0190 1366	0.0199 7560	0.0214 7125
64	0.0178 3315	0.0182 9681	0.0187 6773	0.0197 3127	0.0212 3013
65	0.0175 9371	0.0180 5789	0.0185 2946	0.0194 9460	0.0200 9667
66	0.0173 6156	0.0178 2627	0.0182 9848	0.0192 6524	0.0207 7052
67	0.0171 8639	0.0176 0163	0.0180 7440	0.0190 4286	0.0205 5136
68	0.0169 1788	0.0173 8366	0.0178 5716	0.0188 2716	0.0203 3888
69	0.0167 0574	0.0171 7206	0.0176 4622	0.0186 1785	0.0201 3280
70	0.0164 9971	0.0169 6657	0.0174 4138	0.0184 1464	0.0199 3282
71	0.0162 9952	0.0167 6693	0.0172 4239	0.0182 1728	0.0197 3870
72	0.0161 0493	0.0165 7289	0.0170 4901	0.0180 2554	0.0195 5019
73	0.0159 1572	0.0163 8422	0.0168 6100	0.0178 3917	0.0193 6706
74	0.0157 3165	0.0162 0070	0.0166 7814	0.0176 5796	0.0191 8910
75	0.0155 5253	0.0160 2214	0.0165 0024	0.0174 8170	0.0190 1609
76	0.0153 7816	0.0158 4832	0.0163 2709	0.0173 1020	0.0188 4784
77	0.0152 0836	0.0156 7908	0.0161 5851	0.0171 4328	0.0186 8416
78	0.0150 4295	0.0155 1423	0.0159 9432	0.0169 8074	0.0185 2488
79	0.0148 8177	0.0153 5360	0.0158 3436	0.0168 2244	0.0183 6984
80	0.0147 2464	0.0151 9704	0.0156 7847	0.0166 6821	0.0182 1885
81	0.0145 7144	0.0150 4439	0.0155 2650	0.0165 1790	0.0180 7180
82	0.0144 2200	0.0148 9552	0.0153 7830	0.0163 7136	0.0170 2851
83	0.0142 7620	0.0147 5028	0.0152 3373	0.0162 2847	0.0177 8880
84	0.0141 3391	0.0146 0855	0.0150 9268	0.0100 8908	0.0176 5273
85	0.0139 9500	0.0144 7021	0.0149 5501	0.0159 5308	0.0175 1008
86	0.0138 5935	0.0143 3513	0.0148 2060	0.0158 2034	0.0173 9050
87	0.0137 2685	0.0142 0320	0.0146 8935	0.0156 9076	0.0172 6417
89	0.0135 9740	0.0140 7431	0.0145 6115	0.0155 6423	0.0171 4080
89	0.0134 7088	0.0139 4837	0.0144 3588	0.0154 4064	0.0170 2056
90	0.0133 4721	0.0138 2527	0.0143 1347	0.0153 1989	0.0169 0306
91	0.0132 2629	0.0137 0493	0.0141 9380	0.0152 0190	0.0167 8882
92	0.0131 0803	0.0135 8724	0.0140 7679	0.0150 8657	0.0166 7624
93	0.0129 9234	0.0134 7213	0.0139 6236	0.0149 7382	0.0165 6673
94	0.0128 7915	0.0133 5950	0.0138 5042	0.0148 6356	0.0164 5071
95	0.0127 6837	0.0132 4930	0.0137 4090	0.0147 5571	0.0163 5511
96	0.0126 5992	0.0131 4143	0.0136 3372	0.0146 5020	0.0162 5284
97	0.0125 5374	0.0130 3583	0.0135 2880	0.0145 4696	0.0161 5284
98	0.0124 4976	0.0129 3242	0.0134 2608	0.0144 4592	0.0160 5503
99	0.0123 4790	0.0128 3115	0.0133 2549	0.0143 4701	0.0159 5936
100	0.0122 4811	0.0127 3194	0.0132 2696	0.0142 5017	0.0158 6574

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|} \text{ at } i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|} \text{ at } i)}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	0.0121 5033	0.0126 3473	0.0131 3045	0.0141 5533	0.0157 7413
102	0.0120 5440	0.0125 3947	0.0130 3587	0.0140 6243	0.0156 8446
103	0.0119 6054	0.0124 4611	0.0129 4319	0.0139 7143	0.0155 9608
104	0.0118 6842	0.0123 5457	0.0128 5234	0.0138 8226	0.0155 1073
105	0.0117 7800	0.0122 6481	0.0127 6238	0.0137 9487	0.0154 2056
106	0.0116 8948	0.0121 7679	0.0126 7594	0.0137 0922	0.0153 4412
107	0.0116 0250	0.0120 9045	0.0125 9029	0.0136 2524	0.0152 6336
108	0.0115 1727	0.0120 0575	0.0125 0628	0.0135 4291	0.0151 8423
109	0.0114 3358	0.0119 2264	0.0124 2385	0.0134 6217	0.0151 0669
110	0.0113 5143	0.0118 4107	0.0123 4298	0.0133 8296	0.0150 3069
111	0.0112 7079	0.0117 6102	0.0122 6361	0.0133 0527	0.0149 5620
112	0.0111 9161	0.0116 8242	0.0121 8571	0.0132 2905	0.0148 8317
113	0.0111 1386	0.0116 0526	0.0121 0923	0.0131 5425	0·0148 1156
114	0.0110 3750	0.0115 2948	0.0120 3414	0.0130 8084	0.0147 4133
115	0.0109 6249	0.0114 5506	0.0119 6041	0.0130 0878	0.0146 7245
116	0.0108 8880	0.0113 8195	0.0118 8799	0.0129 3803	0.0146 0488
117	0.0108 1030	0.0113 1013	0.0118 1686	0.0128 6857	0.0145 3860
118	0.0107 4524	0.0112 3056	0.0117 4098	0.0128 0037	0.0144 7356
119	0.0106 7530	0.0111 7021	0.0116 7832	0.0127 3338	0.0144 0973
120	0.0106 0655	0.0111 0205	0.0116 1085	0.0126 6758	0.0143 4709
121	0.0105 3896	0.0110 3505	0.0115 4454	0.0126 0294	0.0142 8561
122	0.0104 7251	0.0109 6918	0.0114 7936	0.0125 3942	0.0142 2525
123	0.0104 0715	0.0109 0441	0.0114 1528	0.0124 7702	0.0141 6599
124	0.0103 4288	0.0108 4072	0.0113 5228	0.0124 1568	0.0141 0780
125	0.0102 7965	0.0107 7808	0.0112 9033	0.0123 5540	0.0140 5065
126	0.0102 1745	0.0107 1647	0.0112 2940	0.0122 9614	0.0139 9452
127	0.0101 5625	0.0106 5580	0.0111 6948	0.0122 3788	0.0139 3930
128	0.0100 9603	0.0105 9923	0.0111 1054	0.0121 8060	0.0138 8524
129	0.0100 3677	0.0105 3755	0.0110 5255	0.0121 2428	0.0138 3203
130	0.0099 7844	0.0104 7981	0.0109 9550	0.0120 6888	0.0137 7975
131	0.0099 2102	0.0104 2208	0.0109 3935	0.0120 1440	0.0137 2837
132	0.0098 6440	0.0103 6704	0.0108 8410	0.0119 6080	0.0136 7788
133	0.0098 0883	0.0103 1197	0.0108 2972	0.0119 0808	0.0136 2825
134	0.0097 5403	0.0102 5775	0.0107 7619	0.0118 5621	0.0135 7947
135	0.0097 0005	0.0102 0436	0.0107 2349	0.0118 0516	0.0135 3151
136	0.0096 4689	0.0101 5179	0.0106 7161	0.0117 5493	0.0134 8437
137	0.0095 9453	0.0101 0002	0.0106 2052	0.0117 0550	0.0134 3801
138	0.0095 4295	0.0100 4902	0.0105 7021	0.0116 5684	0.0133 9242
139	0.0094 9213	0.0099 9879	0.0105 2067	0·0116 0894	0.0133 4759
140	0.0094 4205	0.0099 4930	0.0104 7187	0.0115 6179	0.0133 0349
141	0.0093 9271	0·0099 0055	0.0104 2380	0.0115 1536	0.0132 6012
142	0.0093 4408	0.0098 5250	0.0103 7644	0.0114 6965	0.0132 1746
143	0.0092 9615	0.0098 0516	0.0103 2978	0.0114 2464	0.0131 7549
144	0.0092 4890	0.0097 5850	0.0102 8381	0.0113 8031	0.0131 3419
145	0.0092 0233	0.0097 1252	0.0102 3851	0.0113 3664	0.0130 9356
146	0.0091 5641	0.0096 6719	0.0101 9386	0.0112 9364	0.0130 5358
147	0.0091 1114	0.0096 2250	0.0101 4986	0.0112 5127	0.0130 1423
148	0.0090 6650	0.0095 7844	0.0101 0649	0.0112 0953	0.0129 7551
149	0.0090 2247	0.0095 3500	0.0100 6373	0.0111 6841	0.0129 3739
150	0.0089 7905	0.0094 9217	0.0100 2159	0.0111 2790	0.0128 9988

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|}\ at\ i)}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0112 5000	1.0125 0000	1.0150 0000	1.0175 0000	1.0200 0000
2	0.5084 5323	0.5093 9441	0.5112 7792	0.5131 6205	0.5150 4950
3	0.3408 6130	0.3417 0117	0.3433 8206	0.3450 0746	0.3467 5467
4	0.2570 7058	0.2578 6102	0.2594 4478	0.2610 3237	0.2626 2375
5	0.2068 0034	0.2075 6211	0.2090 8932	0.2106 2142	0.2121 5839
6	0.1732 9034	0.1740 3381	0.1755 2521	0.1770 2256	0.1785 2581
7	0.1403 5762	0.1500 8872	0.1515 5616	0.1530 3059	0.1545 1196
8	0.1314 1071	0.1321 3314	0.1335 8402	0.1350 4292	0.1365 0980
9	0.1174 5432	0.1181 7055	0.1196 0982	0.1210 5813	0.1225 1544
10	0.1062 9131	0.1070 0307	0.1084 3418	0.1098 7534	0.1113 2653
11	0.0971 5984	0.0978 6839	0.0992 9384	0.1007 3038	0.1021 7794
12	0.0895 5203	0.0002 5831	0.0916 7999	0.0931 1377	0.0945 5960
13	0.0831 1626	0.0838 2100	0.0852 4036	0.0866 7283	0.0881 1835
14	0.0776 0138	0.0783 0515	0.0797 2332	0.0811 5562	0.0826 0107
15	0.0728 2321	0.0735 2646	0.0749 4436	0.0763 7739	0.0778 2547
16	0.0686 4363	0.0693 4072	0.0707 6508	0.0721 9958	0.0736 5013
17	0.0649 5698	0.0656 6023	0.0670 7906	0.0685 1023	0.0699 6984
18	0.0616 8113	0.0623 8479	0.0638 0578	0.0652 4402	0.0667 0210
19	0.0587 5120	0.0594 5548	0.0608 7847	0.0623 2061	0.0637 8177
20	0.0561 1531	0.0568 2039	0.0582 4574	0.0596 0122	0.0611 5672
21	0.0537 3145	0.0544 3748	0.0558 6550	0.0573 1404	0.0587 8477
22	0.0515 6525	0.0522 7238	0.0537 0331	0.0551 5638	0.0566 3140
23	0.0495 8833	0.0502 9666	0.0517 3075	0.0531 8796	0.0546 6810
24	0.0477 7701	0.0484 8665	0.0499 2410	0.0513 8565	0.0528 7110
25	0.0461 1144	0.0468 2247	0.0482 6345	0.0497 2952	0.0512 2044
26	0.0445 7479	0.0452 8729	0.0467 3196	0.0482 0269	0.0496 9923
27	0.0431 5273	0.0438 6677	0.0453 1527	0.0467 9079	0.0482 9309
28	0.0418 3299	0.0425 4863	0.0440 0108	0.0454 8151	0.0469 8967
29	0.0406 0498	0.0413 2228	0.0427 7878	0.0442 6424	0.0457 7836
30	0.0394 5953	0.0401 7854	0.0416 3919	0.0431 2975	0.0446 4992
31	0.0383 8866	0.0391 0942	0.0405 7430	0.0420 7005	0.0435 9635
32	0.0373 8535	0.0381 0791	0.0395 7710	0.0410 7812	0.0426 1061
33	0.0364 4349	0.0371 6786	0.0386 4144	0.0401 4779	0.0416 8653
34	0.0355 5763	0.0362 8387	0.0377 6189	0.0392 7363	0.0408 1807
35	0.0347 2299	0.0354 5111	0.0369 3363	0.0384 5082	0.0400 0221
36	0.0339 3529	0.0346 6533	0.0361 5240	0.0376 7507	0.0392 3285
37	0.0331 9072	0.0339 2270	0.0354 1437	0.0369 4257	0.0385 0678
38	0.0324 8589	0.0332 1983	0.0347 1613	0.0362 4990	0.0378 2057
39	0.0318 1773	0.0325 5365	0.0340 5468	0.0355 9399	0.0371 7114
40	0.0311 8349	0.0319 2141	0.0334 2710	0.0349 7209	0.0365 5575
41	0.0305 8069	0.0313 2063	0.0328 3106	0.0343 8170	0.0359 7188
42	0.0300 0709	0.0307 4906	0.0322 6426	0.0338 2057	0.0354 1729
43	0.0294 6064	0.0302 0466	0.0317 2465	0.0332 8666	0.0348 8993
44	0.0289 3949	0.0296 8557	0.0312 1038	0.0327 7810	0.0343 8794
45	0.0284 4197	0.0291 9012	0.0307 1976	0.0322 9321	0.0339 0962
46	0.0279 6652	0.0287 1675	0.0302 5125	0.0318 3043	0.0334 5342
47	0.0275 1173	0.0282 6406	0.0298 0342	0.0313 8836	0.0330 1792
48	0.0270 7632	0.0278 3075	0.0293 7500	0.0309 6569	0.0326 0184
49	0.0266 5910	0.0274 1563	0.0289 6478	0.0305 6124	0.0322 0396
50	0.0262 5898	0.0270 1763	0.0285 7168	0.0301 7391	0.0318 2321

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n|}}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n|}}\ at\ i)}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	0.0258 7494	0.0266 3571	0.0281 9469	0.0298 0209	0.0314 5856
52	0.0255 0606	0.0202 6897	0.0278 3287	0.0294 4065	0.0311 0909
53	0.0251 5149	0.0259 1653	0.0274 8537	0.0291 0492	0.0307 7392
54	0.0248 1048	0.0256 7760	0.0271 5138	0.0287 7672	0.0304 5220
55	0.0244 8213	0.0252 5146	0.0268 3018	0.0284 6129	0.0301 4337
56	0.0241 6592	0.0249 3739	0.0265 2100	0.0281 5795	0.0298 4656
57	0.0238 6116	0.0246 3478	0.0262 2341	0.0278 6606	0.0295 6120
58	0.0235 6726	0.0243 4303	0.0259 3661	0.0275 8603	0.0292 8607
59	0.0232 8366	0.0240 6158	0.0256 6012	0.0273 1430	0.0290 2243
60	0.0230 0985	0.0237 8993	0.0253 9343	0.0270 5336	0.0287 6797
61	0.0227 4534	0.0235 2758	0.0251 3604	0.0268 0172	0.0285 2278
62	0.0224 8969	0.0232 7410	0.0248 8751	0.0265 5892	0.0282 8643
63	0.0222 4247	0.0230 2904	0.0246 4741	0.0263 2455	0.0280 5848
64	0.0220 0329	0.0227 9203	0.0244 1534	0.0260 9821	0.0278 3855
65	0.0217 7178	0.0225 6268	0.0241 9094	0.0258 7952	0.0276 2624
66	0.0215 4758	0.0223 4065	0.0239 7386	0.0266 6813	0.0274 2122
67	0.0218 3037	0.0221 2560	0.0237 6376	0.0254 6372	0.0272 2316
68	0.0211 1985	0.0219 1724	0.0235 6033	0.0252 6596	0.0270 3173
69	0.0209 1571	0.0217 1527	0.0233 6329	0.0250 7459	0.0268 4665
70	0.0207 1709	0.0215 1941	0.0231 7235	0.0248 8930	0.0266 6765
71	0.0205 2552	0.0213. 2941	0.0229 8727	0.0247 0985	0.0264 9440
72	0.0203 3896	0.0211 4501	0.0228 0779	0.0245 3600	0.0263 2683
73	0.0201 5779	0.0209 6600	0.0226 3368	0.0243 6750	0.0261 6454
74	0.0199 8177	0.0207 9215	0.0224 6473	0.0242 0413	0.0260 0736
75	0.0198 1072	0.0206 2325	0.0223 0072	0.0240 4570	0.0258 5508
76	0.0196 4442	0.0204 5910	0.0221 4146	0.0238 9200	0.0257 0751
77	0.0194 8269	0.0202 9953	0.0219 8076	0.0237 4284	0.0255 6447
78	0.0193 2536	0.0201 4435	0.0218 3045	0.0235 9806	0.0254 2576
79	0.0191 7226	0.0199 9341	0.0216 9036	0.0234 5748	0.0252 9123
80	0.0190 2323	0.0198 4652	0.0215 4832	0.0233 2093	0.0251 6071
81	0.0188 7812	0.0197 0356	0.0214 1019	0.0281 8828	0.0250 3405
82	0.0187 3678	0.0195 6437	0.0212 7583	0.0230 5936	0.0249 1110
83	0.0185 9908	0.0194 2881	0.0211 4509	0.0229 3406	0.0247 9173
84	0.0184 6489	0.0192 9675	0.0210 1784	0.0228 1223	0.0246 7581
85	0.0183 3409	0.0101 6808	0.0208 9396	0.0226 9375	0.0245 6321
86	0.0182 0654	0.0190 4207	0.0207 7333	0.0225 7850	0.0244 5381
87	0.0180 8215	0.0189 2041	0.0206 5584	0.0224 0636	0.0248 4750
88	0.0179 6081	0.0188 0119	0.0205 4138	0.0223 5724	0.0242 4416
89	0.0178 4240	0.0186 8490	0.0204 2984	0.0222 5102	0.0241 4370
90	0.0177 2684	0.0185 7146	0.0203 2113	0.0221 4760	0.0240 4602
91	0.0176 1403	0.0184 6076	0.0202 1516	0.0220 4690	0.0239 5101
92	0.0175 0387	0.0183 5271	0.0201 1182	0.0219 4882	0.0238 5859
93	0.0173 9620	0.0182 4724	0.0200 1104	0.0218 5327	0.0237 6868
94	0.0172 9119	0.0181 4425	0.0199 1273	0.0217 6017	0.0236 8118
95	0.0171 8851	0.0180 4366	0.0198 1681	0.0216 6944	0.0235 9602
96	0.0170 8816	0.0179 4540	0.0197 2321	0.0215 8101	0.0235 1313
97	0.0169 9007	0.0178 4941	0.0196 3186	0.0214 9480	0.0234 3242
98	0.0168 9418	0.0177 5560	0.0195 4268	0.0214 1074	0.0233 5383
99	0.0168 0041	0.0176 6391	0.0194 5560	0.0213 2876	0.0232 7729
100	0.0167 0870	0.0175 7428	0.0193 7057	0.0212 4880	0.0232 0274

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|}\ at\ i)}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000	1.0350 0000
2	0.5169 3758	0.5188 2716	0.5207 1825	0.5226 1084	0.5264 0040
3	0.3484 4458	0.3501 3717	0.3518 3243	0.3535 3030	0.3500 3418
4	0.2642 1893	0.2658 1788	0.2674 2059	0.2000 2705	0.2722 5114
5	0.2137 0021	0.2152 4686	0.2107 9832	0.2183 5457	0.2214 8137
6	0.1800 3496	0.1815 4997	0.1830 7083	0.1845 0750	0.1876 0821
7	0.1500 0025	0.1574 9543	0.1589 9747	0.1005 0635	0.1035 4449
8	0.1370 8462	0.1394 6735	0.1409 5795	0.1424 5630	0.1454 7005
9	0.1239 8170	0.1254 5089	0.1200 4095	0.1284 3380	0.1314 4601
10	0.1127 8768	0.1142 5876	0.1157 3972	0.1172 3051	0.1202 4137
11	0.1036 3649	0.1051 0596	0.1005 8029	0.1080 7745	0.1110 0107
12	0.0900 1740	0.0974 8713	0.0989 0871	0.1004 6200	0.1034 8305
13	0.0895 7686	0.0910 4827	0.0925 3252	0.0040 2054	0.0970 0157
14	0.0840 6230	0.0855 3653	0.0870 2457	0.0885 2634	0.0915 7073
15	0.0792 8852	0.0807 6646	0.0822 5017	0.0837 6658	0.0808 2507
16	0.0751 1663	0.0765 9899	0.0780 9710	0.0706 1085	0.0826 8483
17	0.0714 4039	0.0729 2777	0.0744 3186	0.0759 5253	0.0700 4313
18	0.0681 7720	0.0696 7008	0.0711 8063	0.0727 0870	0.0758 1084
19	0.0652 0182	0.0667 6062	0.0682 7802	0.0698 1388	0.0720 4033
20	0.0626 4207	0.0641 4713	0.0656 7173	0.0672 1571	0.0703 6108
21	0.0002 7572	0.0617 8733	0.0633 1041	0.0648 7178	0.0080 3650
22	0.0581 2821	0.0590 4661	0.0611 8040	0.0627 4730	0.0659 3207
23	0.0561 7097	0.0570 9638	0.0592 4410	0.0008 1300	0.0040 1880
24	0.0543 8023	0.0559 1282	0.0574 0863	0.0500 4742	0.0022 7283
25	0.0527 3599	0.0542 7592	0.0558 3907	0.0574 2787	0.0600 7404
26	0.0512 2134	0.0527 6875	0.0543 4116	0.0550 3829	0.0502 0540
27	0.0498 2188	0.0513 7687	0.0529 5776	0.0545 6421	0.0578 5241
28	0.0485 2525	0.0500 8793	0.0516 7738	0.0532 9323	0.0500 0265
29	0.0473 2081	0.0488 9127	0.0504 8935	0.0521 1467	0.0554 4538
30	0.0461 9934	0.0477 7764	0.0493 8442	0.0510 1926	0.0543 7133
31	0.0451 5280	0.0467 3900	0.0483 5453	0.0400 9803	0.0533 7240
32	0.0441 7415	0.0457 6831	0.0473 9263	0.0490 4602	0.0524 4150
33	0.0432 5722	0.0448 5938	0.0464 9253	0.0481 5612	0.0515 7242
34	0.0423 9655	0.0440 0675	0.0450 4875	0.0473 2106	0.0507 5006
35	0.0415 8731	0.0432 0558	0.0448 5645	0.0405 3929	0.0490 9835
36	0.0408 2522	0.0424 5158	0.0441 1132	0.0458 0379	0.0402 8416
37	0.0401 0643	0.0417 4090	0.0434 0953	0.0451 1162	0.0486 1325
38	0.0394 2753	0.0410 7012	0.0427 4704	0.0444 5934	0.0479 8214
39	0.0387 8543	0.0404 3615	0.0421 2250	0.0438 4385	0.0473 8775
40	0.0381 7738	0.0398 3623	0.0415 3151	0.0432 6238	0.0468 2728
41	0.0376 0087	0.0392 6786	0.0409 7200	0.0427 1241	0.0402 9822
42	0.0370 5364	0.0387 2876	0.0404 4175	0.0421 9167	0.0457 9828
43	0.0365 3364	0.0382 1688	0.0390 3871	0.0416 9811	0.0453 2539
44	0.0360 3901	0.0377 3037	0.0394 6100	0.0412 2985	0.0148 7768
45	0.0355 6805	0.0372 0752	0.0390 0693	0.0407 8518	0.0444 5343
46	0.0351 1921	0.0368 2676	0.0385 7403	0.0403 6254	0.0440 5108
47	0.0346 9107	0.0364 0669	0.0381 6358	0.0399 6051	0.0436 0019
48	0.0342 8233	0.0360 0599	0.0377 7158	0.0395 7777	0.0433 0046
49	0.0338 9179	0.0356 2348	0.0373 9773	0.0392 1314	0.0429 6167
50	0.0335 1836	0.0352 5806	0.0370 4092	0.0388 6550	0.0426 3371

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|}\ at\ i)}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	0.0331 6102	0.0349 0870	0.0367 0014	0.0385 3382	0.0423 2156
52	0.0328 1884	0.0345 7440	0.0363 7444	0.0382 1718	0.0420 2429
53	0.0324 9094	0.0342 5449	0.0360 6297	0.0379 1471	0.0417 4100
54	0.0321 7654	0.0339 4790	0.0357 6401	0.0376 2558	0.0414 7090
55	0.0318 7480	0.0336 5419	0.0354 7953	0.0373 4907	0.0412 1323
56	0.0315 8530	0.0333 7243	0.0352 0612	0.0370 8447	0.0409 6730
57	0.0313 0712	0.0331 0204	0.0349 4404	0.0368 3114	0.0407 3245
58	0.0310 3977	0.0328 4244	0.0346 9270	0.0365 8848	0.0405 0810
59	0.0307 8268	0.0325 9307	0.0344 5153	0.0363 5593	0.0402 9366
60	0.0305 3533	0.0323 5340	0.0342 2002	0.0361 3296	0.0400 8802
61	0.0302 9724	0.0321 2294	0.0339 9767	0.0359 1908	0.0398 9249
62	0.0300 6795	0.0319 0126	0.0337 8402	0.0357 1385	0.0397 0480
63	0.0298 4704	0.0316 8790	0.0335 7866	0.0355 1682	0.0395 2513
64	0.0296 3411	0.0314 8249	0.0333 8118	0.0353 2760	0.0393 5308
65	0.0294 2878	0.0312 8463	0.0331 9120	0.0351 4581	0.0391 8826
66	0.0292 3070	0.0310 9308	0.0330 0837	0.0349 7110	0.0390 3031
67	0.0290 3955	0.0309 1021	0.0328 3236	0.0348 0313	0.0388 7892
68	0.0288 5500	0.0307 3300	0.0326 6285	0.0346 4159	0.0387 3375
69	0.0286 7677	0.0305 6206	0.0324 9955	0.0344 8618	0.0385 9453
70	0.0285 0458	0.0303 9712	0.0323 4218	0.0343 3663	0.0384 6095
71	0.0283 3816	0.0302 3790	0.0321 9048	0.0341 9266	0.0383 3277
72	0.0281 7728	0.0300 8417	0.0320 4420	0.0340 5404	0.0382 0973
73	0.0280 2169	0.0299 3568	0.0319 0311	0.0339 2053	0.0380 9160
74	0.0278 7118	0.0297 9222	0.0317 6698	0.0337 9191	0.0379 7816
75	0.0277 2554	0.0296 5358	0.0316 3560	0.0336 6796	0.0378 6919
76	0.0275 8457	0.0295 1950	0.0315 0878	0.0335 4840	0.0377 6450
77	0.0274 4808	0.0293 8907	0.0313 8633	0.0334 3331	0.0376 6390
78	0.0273 1589	0.0292 6463	0.0312 6806	0.0333 2224	0.0375 6721
79	0.0271 8784	0.0291 4338	0.0311 5382	0.0332 1510	0.0374 7426
80	0.0270 6376	0.0290 2605	0.0310 4342	0.0331 1175	0.0373 8489
81	0.0269 4350	0.0289 1248	0.0309 3674	0.0330 1201	0.0372 9894
82	0.0268 2692	0.0288 0254	0.0308 3361	0.0329 1576	0.0372 1628
83	0.0267 1387	0.0286 9608	0.0307 3389	0.0328 2284	0.0371 3676
84	0.0266 0423	0.0285 9298	0.0306 3747	0.0327 3313	0.0370 6025
85	0.0264 9787	0.0284 9310	0.0305 4420	0.0326 4650	0.0369 8662
86	0.0263 9467	0.0283 9633	0.0304 5397	0.0325 6284	0.0369 1576
87	0.0262 9452	0.0283 0255	0.0303 6667	0.0324 8202	0.0368 4756
88	0.0261 9730	0.0282 1165	0.0302 8219	0.0324 0393	0.0367 8190
89	0.0261 0291	0.0281 2353	0.0302 0041	0.0323 2848	0.0367 1868
90	0.0260 1126	0.0280 3809	0.0301 2125	0.0322 5556	0.0366 5781
91	0.0259 2224	0.0279 5523	0.0300 4460	0.0321 8508	0.0365 9919
92	0.0258 3577	0.0278 7486	0.0299 7038	0.0321 1694	0.0365 4273
93	0.0257 5176	0.0277 9690	0.0298 9850	0.0320 5107	0.0364 8834
94	0.0256 7012	0.0277 2126	0.0298 2887	0.0319 8737	0.0364 3594
95	0.0255 9078	0.0276 4786	0.0297 6141	0.0319 2577	0.0363 8546
96	0.0255 1366	0.0275 7662	0.0296 9605	0.0318 6619	0.0363 3682
97	0.0254 3868	0.0275 0747	0.0296 3272	0.0318 0856	0.0362 8995
98	0.0253 6578	0.0274 4084	0.0295 7134	0.0317 5281	0.0362 4478
99	0.0252 9489	0.0273 7517	0.0295 1185	0.0316 9886	0.0362 0124
100	0.0252 2594	0.0273 1188	0.0294 5418	0.0316 4667	0.0361 5927

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n|}}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n|}}\ at\ i)}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000	1.0600 0000
2	0.5301 9608	0.5339 9756	0.5378 0488	0.5416 1800	0.5454 3689
3	0.3603 4854	0.3637 7336	0.3672 0856	0.3706 5407	0.3741 0981
4	0.2754 9005	0.2787 4365	0.2820 1183	0.2852 9449	0.2885 9149
5	0.2246 2711	0.2277 9164	0.2309 7480	0.2341 7644	0.2373 9640
6	0.1907 6190	0.1938 7839	0.1970 1747	0.2001 7895	0.2033 6263
7	0.1666 0961	0.1697 0147	0.1728 1982	0.1759 6442	0.1791 3502
8	0.1485 2783	0.1516 0965	0.1547 2181	0.1578 6401	0.1610 3504
9	0.1344 9299	0.1375 7447	0.1406 9008	0.1438 3946	0.1470 2224
10	0.1232 9094	0.1263 7882	0.1295 0458	0.1326 6777	0.1358 6796
11	0.1141 4904	0.1172 4818	0.1203 8889	0.1235 7065	0.1267 9294
12	0.1065 5217	0.1096 6619	0.1128 2541	0.1160 2923	0.1192 7703
13	0.1001 4373	0.1032 7535	0.1064 5577	0.1096 8426	0.1129 6011
14	0.0946 6807	0.0978 2032	0.1010 2397	0.1042 7912	0.1075 8491
15	0.0899 4110	0.0931 1381	0.0963 4229	0.0996 2560	0.1029 6276
16	0.0858 2000	0.0890 1537	0.0922 6991	0.0955 8254	0.0989 5214
17	0.0821 9852	0.0854 1758	0.0886 9914	0.0920 4197	0.0954 4480
18	0.0789 9333	0.0822 3690	0.0855 4622	0.0889 1992	0.0923 5654
19	0.0761 3862	0.0794 0734	0.0827 4501	0.0861 5006	0.0896 2086
20	0.0735 8175	0.0768 7614	0.0802 4259	0.0836 7933	0.0871 8456
21	0.0712 8011	0.0746 0057	0.0779 9611	0.0814 6478	0.0850 0455
22	0.0691 9881	0.0725 4565	0.0759 7051	0.0794 7123	0.0830 4557
23	0.0673 0906	0.0706 8249	0.0741 3682	0.0776 6965	0.0812 7848
24	0.0655 8683	0.0689 8703	0.0724 7090	0.0760 3580	0.0796 7900
25	0.0640 1196	0.0674 3903	0.0709 5246	0.0745 4935	0.0782 2672
26	0.0625 6738	0.0660 2137	0.0695 6432	0.0731 9307	0.0769 0435
27	0.0612 3854	0.0647 1946	0.0082 9186	0.0719 5228	0.0756 9717
28	0.0600 1298	0.0635 2081	0.0671 2253	0.0708 1440	0.0745 9255
29	0.0588 7993	0.0624 1461	0.0660 4551	0.0697 6857	0.0735 7961
30	0.0578 3010	0.0613 9154	0.0650 5144	0.0688 0539	0.0726 4891
31	0.0568 5535	0.0604 4345	0.0641 3212	0.0679 1065	0.0717 9222
32	0.0559 4859	0.0595 6320	0.0632 8042	0.0670 0519	0.0710 0234
33	0.0551 0357	0.0587 4453	0.0624 9004	0.0663 3469	0.0702 7293
34	0.0543 1477	0.0579 8191	0.0617 5545	0.0656 2958	0.0695 9843
35	0.0535 7732	0.0572 7045	0.0610 7171	0.0649 7493	0.0689 7386
36	0.0528 8688	0.0566 0578	0.0604 3446	0.0643 6635	0.0683 9483
37	0.0522 3957	0.0559 8402	0.0598 3979	0.0637 9993	0.0678 5743
38	0.0516 3192	0.0554 0169	0.0592 8423	0.0632 7217	0.0673 5812
39	0.0510 6083	0.0548 5567	0.0587 6462	0.0627 7991	0.0668 9377
40	0.0505 2349	0.0543 4315	0.0582 7816	0.0623 2034	0.0664 6154
41	0.0500 1738	0.0538 6158	0.0578 2229	0.0618 9090	0.0660 5886
42	0.0495 4020	0.0534 0868	0.0573 9471	0.0614 8927	0.0656 8342
43	0.0490 8989	0.0529 8235	0.0569 9333	0.0611 1337	0.0653 3312
44	0.0486 6454	0.0525 8071	0.0566 1625	0.0607 6128	0.0650 0606
45	0.0482 6246	0.0522 0202	0.0562 6173	0.0604 3127	0.0647 0050
46	0.0478 8205	0.0518 4471	0.0559 2820	0.0601 2175	0.0644 1485
47	0.0475 2189	0.0515 0734	0.0556 1421	0.0598 3129	0.0641 4768
48	0.0471 8065	0.0511 8858	0.0553 1843	0.0595 5854	0.0638 9766
49	0.0468 5712	0.0508 8722	0.0550 3965	0.0593 0230	0.0636 6356
50	0.0465 5020	0.0506 0215	0.0547 7674	0.0590 6145	0.0634 4429

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n|}}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n|}}\ at\ i)}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	0.0462 5885	0.0503 3232	0.0545 2867	0.0588 3405	0.0632 3880
52	0.0459 8212	0.0500 7679	0.0542 9450	0.0586 2186	0.0630 4017
53	0.0457 1915	0.0498 3469	0.0540 7334	0.0584 2136	0.0628 6551
54	0.0454 6010	0.0496 0519	0.0538 6438	0.0582 3245	0.0626 9602
55	0.0452 3124	0.0493 8754	0.0536 6080	0.0580 5458	0.0625 3696
56	0.0450 0487	0.0491 8105	0.0534 8010	0.0578 8698	0.0623 8705
57	0.0447 8032	0.0489 8506	0.0533 0343	0.0577 2966	0.0622 4744
58	0.0445 8401	0.0487 9897	0.0531 3626	0.0575 8006	0.0621 1574
59	0.0443 8836	0.0486 2221	0.0529 7802	0.0574 3959	0.0619 9200
60	0.0442 0185	0.0484 5426	0.0528 2818	0.0573 0707	0.0618 7572
61	0.0440 2398	0.0482 9462	0.0526 8627	0.0571 8203	0.0617 6642
62	0.0438 5430	0.0481 4284	0.0525 5183	0.0570 6400	0.0616 6366
63	0.0436 9237	0.0479 9848	0.0524 2442	0.0569 5258	0.0615 6704
64	0.0435 3780	0.0478 6115	0.0523 0365	0.0568 4737	0.0614 7615
65	0.0433 9019	0.0477 3047	0.0521 8915	0.0567 4800	0.0613 9066
66	0.0432 4921	0.0476 0608	0.0520 8057	0.0566 5413	0.0613 1022
67	0.0431 1451	0.0474 8765	0.0519 7757	0.0565 6544	0.0612 3454
68	0.0429 8578	0.0473 7487	0.0518 7986	0.0564 8163	0.0611 6330
69	0.0428 6272	0.0472 6745	0.0517 8715	0.0564 0242	0.0610 9625
70	0.0427 4506	0.0471 6511	0.0516 9915	0.0563 2754	0.0610 3313
71	0.0426 3253	0.0470 6759	0.0516 1563	0.0562 5675	0.0609 7370
72	0.0425 2489	0.0469 7465	0.0515 3633	0.0561 8982	0.0609 1774
73	0.0424 2190	0.0468 8606	0.0514 6103	0.0561 2652	0.0608 6505
74	0.0423 2334	0.0468 0159	0.0513 8953	0.0560 6665	0.0608 1542
75	0.0422 2900	0.0467 2104	0.0513 2161	0.0560 1002	0.0607 6867
76	0.0421 3869	0.0466 4422	0.0512 5709	0.0559 5645	0.0607 2463
77	0.0420 5221	0.0465 7094	0.0511 9580	0.0559 0577	0.0606 8315
78	0.0419 6939	0.0465 0104	0.0511 3756	0.0558 5781	0.0606 4407
79	0.0418 9007	0.0464 3434	0.0510 8222	0.0558 1243	0.0606 0724
80	0.0418 1408	0.0463 7069	0.0510 2962	0.0557 6948	0.0605 7254
81	0.0417 4127	0.0463 0995	0.0509 7963	0.0557 2884	0.0605 3984
82	0.0416 7150	0.0462 5197	0.0509 3211	0.0556 9036	0.0605 0903
83	0.0416 0463	0.0461 9663	0.0508 8694	0.0556 5395	0.0604 7998
84	0.0415 4054	0.0461 4379	0.0508 4399	0.0556 1947	0.0604 5261
85	0.0414 7909	0.0460 9334	0.0508 0316	0.0555 8683	0.0604 2681
86	0.0414 2018	0.0460 4516	0.0507 6433	0.0555 5593	0.0604 0249
87	0.0413 6370	0.0459 9915	0.0507 2740	0.0555 2667	0.0603 7956
88	0.0413 0953	0.0459 5522	0.0506 9228	0.0554 9896	0.0603 5795
89	0.0412 5758	0.0459 1325	0.0506 5888	0.0554 7273	0.0603 3757
90	0.0412 0775	0.0458 7316	0.0506 2711	0.0554 4788	0.0603 1836
91	0.0411 5995	0.0458 3486	0.0505 9689	0.0554 2435	0.0603 0025
92	0.0411 1410	0.0457 9827	0.0505 6815	0.0554 0207	0.0602 8318
93	0.0410 7010	0.0457 6331	0.0505 4080	0.0553 8096	0.0602 6708
94	0.0410 2789	0.0457 2991	0.0505 1478	0.0553 6097	0.0602 5190
95	0.0409 8738	0.0456 9799	0.0504 9003	0.0553 4204	0.0602 3758
96	0.0409 4850	0.0456 6749	0.0504 6648	0.0553 2410	0.0602 2408
97	0.0409 1119	0.0456 3834	0.0504 4407	0.0553 0711	0.0602 1135
98	0.0408 7538	0.0456 1048	0.0504 2274	0.0552 9101	0.0601 9935
99	0.0408 4100	0.0455 8385	0.0504 0245	0.0552 7577	0.0601 8803
100	0.0408 0800	0.0455 5839	0.0503 8314	0.0552 6132	0.0601 7736

TABLE IX — PERIODICAL PAYMENT OF ANNUITY WHOSE PRESENT VALUE IS 1

$$\frac{1}{(a_{\overline{n}|}\ at\ i)} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{(s_{\overline{n}|}\ at\ i)}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000	1.0850 0000
2	0.5492 6150	0.5530 9179	0.5569 2771	0.5607 6923	0.5640 1031
3	0.3775 7570	0.3810 5166	0.3845 3703	0.3880 3351	0.3015 3025
4	0.2919 0274	0.2952 2812	0.2985 6751	0.3019 2080	0.3052 8789
5	0.2406 3454	0.2438 9069	0.2471 6472	0.2504 5645	0.2537 6575
6	0.2065 6831	0.2097 9580	0.2130 4480	0.2163 1539	0.2196 0708
7	0.1823 3137	0.1855 5322	0.1888 0032	0.1920 7240	0.1953 6922
8	0.1642 3730	0.1674 6776	0.1707 2702	0.1740 1476	0.1773 3065
9	0.1502 3803	0.1534 8647	0.1567 0716	0.1600 7971	0.1634 2372
10	0.1391 0469	0.1423 7750	0.1456 8593	0.1490 2949	0.1524 0771
11	0.1300 5521	0.1333 5690	0.1366 9747	0.1400 7634	0.1434 9293
12	0.1225 6817	0.1259 0199	0.1292 7783	0.1326 9502	0.1361 5286
13	0.1162 8256	0.1196 5085	0.1230 6420	0.1265 2181	0.1300 2287
14	0.1109 4048	0.1143 4494	0.1177 0737	0.1212 9685	0.1248 4244
15	0.1063 5278	0.1097 9462	0.1132 8724	0.1168 2954	0.1204 2046
16	0.1023 7757	0.1058 5765	0.1093 9116	0.1129 7687	0.1166 1354
17	0.0989 0633	0.1024 2519	0.1060 0003	0.1096 2943	0.1133 1198
18	0.0958 5401	0.0994 1260	0.1030 2896	0.1067 0210	0.1104 3041
19	0.0931 5575	0.0967 5301	0.1004 1999	0.1041 2763	0.1079 0140
20	0.0907 5640	0.0943 9293	0.0980 9219	0.1018 5221	0.1056 7007
21	0.0886 1333	0.0922 8900	0.0960 2937	0.0998 3225	0.1036 9541
22	0.0866 9120	0.0904 0577	0.0941 8687	0.0980 3207	0.1019 3802
23	0.0849 6078	0.0887 1393	0.0925 3528	0.0964 2217	0.1003 7193
24	0.0833 9770	0.0871 8902	0.0910 5008	0.0949 7796	0.0989 0075
25	0.0819 8148	0.0858 1052	0.0897 1067	0.0936 7878	0.0977 1168
26	0.0806 9480	0.0845 6103	0.0884 9961	0.0925 0713	0.0965 8016
27	0.0795 2288	0.0834 2573	0.0874 0204	0.0914 4809	0.0955 6025
28	0.0784 5305	0.0823 9193	0.0864 0520	0.0904 8891	0.0946 3014
29	0.0774 7440	0.0814 4865	0.0854 9811	0.0896 1854	0.0938 0577
30	0.0765 7744	0.0805 8640	0.0846 7124	0.0888 2743	0.0930 5058
31	0.0757 5393	0.0797 9691	0.0839 1628	0.0881 0728	0.0923 6524
32	0.0749 9665	0.0790 7292	0.0832 2599	0.0874 5081	0.0917 4247
33	0.0742 9924	0.0784 0807	0.0825 9397	0.0868 5163	0.0911 7588
34	0.0736 5616	0.0777 9674	0.0820 1461	0.0863 0411	0.0906 5984
35	0.0730 6226	0.0772 3396	0.0814 8291	0.0858 0326	0.0901 8937
36	0.0725 1332	0.0767 1531	0.0809 9447	0.0853 4467	0.0897 6006
37	0.0720 0534	0.0762 3685	0.0805 4533	0.0849 2440	0.0893 6799
38	0.0715 3480	0.0757 9505	0.0801 3197	0.0845 3894	0.0890 0966
39	0.0710 9854	0.0753 8676	0.0797 5124	0.0841 8513	0.0886 8193
40	0.0706 9373	0.0750 0914	0.0794 0031	0.0838 6016	0.0883 8201
41	0.0703 1779	0.0746 5962	0.0790 7663	0.0835 6149	0.0881 0737
42	0.0699 6842	0.0743 3591	0.0787 7789	0.0832 8684	0.0878 5576
43	0.0696 4352	0.0740 3590	0.0785 0201	0.0830 3414	0.0876 2512
44	0.0693 4119	0.0737 5769	0.0782 4710	0.0828 0152	0.0874 1363
45	0.0690 5968	0.0734 9957	0.0780 1146	0.0825 8728	0.0872 1961
46	0.0687 9743	0.0732 5996	0.0777 9353	0.0823 8991	0.0870 4154
47	0.0685 5300	0.0730 3744	0.0775 9190	0.0822 0799	0.0868 7807
48	0.0683 2506	0.0728 3070	0.0774 0527	0.0820 4027	0.0867 2795
49	0.0681 1240	0.0726 3853	0.0772 3247	0.0818 8557	0.0865 9005
50	0.0679 1393	0.0724 5985	0.0770 7241	0.0817 4286	0.0864 6334

TABLE X — COMPOUND AMOUNT OF 1 FOR FRACTIONAL PERIODS

$$(1+i)^{\frac{1}{p}}$$

p	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
2	1.0020 8117	1.0024 0088	1.0029 1243	1.0037 4209	1.0049 8756
3	1.0013 8690	1.0016 6300	1.0019 4068	1.0024 9378	1.0033 2228
4	1.0010 4004	1.0012 4766	1.0014 5515	1.0018 6975	1.0024 9068
6	1.0006 9324	1.0008 3160	1.0009 6987	1.0012 4611	1.0016 5977
12	1.0003 4656	1.0004 1571	1.0004 8482	1.0006 2286	1.0008 2954
13	1.0003 1990	1.0003 8373	1.0004 4751	1.0005 7494	1.0007 6570
26	1.0001 5994	1.0001 0185	1.0002 2373	1.0002 8743	1.0003 8276

p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
2	1.0056 0027	1.0062 3050	1.0074 7208	1.0087 1205	1.0099 5050
3	1.0037 3602	1.0041 4943	1.0049 7521	1.0057 9963	1.0066 2271
4	1.0028 0081	1.0031 1046	1.0037 2909	1.0043 4658	1.0049 6293
6	1.0018 6627	1.0020 7257	1.0024 8452	1.0028 9562	1.0033 0589
12	1.0009 3270	1.0010 3575	1.0012 4149	1.0014 4677	1.0016 5158
13	1.0008 6092	1.0009 5604	1.0011 4594	1.0013 3540	1.0015 2444
26	1.0004 3037	1.0004 7790	1.0005 7280	1.0006 0748	1.0007 6193

p	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
2	1.0111 8742	1.0124 2284	1.0136 5076	1.0148 8916	1.0173 4050
3	1.0074 4444	1.0082 6484	1.0090 8306	1.0099 0163	1.0115 3314
4	1.0055 7815	1.0061 9225	1.0068 0522	1.0074 1707	1.0086 3745
6	1.0037 1532	1.0041 2392	1.0045 3168	1.0049 3862	1.0057 5004
12	1.0018 5594	1.0020 5984	1.0022 6328	1.0024 6627	1.0028 7090
26	1.0008 5016	1.0009 5017	1.0010 4396	1.0011 3752	1.0013 2401
52	1.0004 2709	1.0004 7497	1.0005 2184	1.0005 6860	1.0006 6170

p	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$	6%
2	1.0198 0390	1.0222 5242	1.0246 9508	1.0271 3103	1.0295 6302
3	1.0131 5941	1.0147 8046	1.0163 9636	1.0180 0713	1.0196 1282
4	1.0098 5341	1.0110 6499	1.0122 7224	1.0134 7518	1.0146 7385
6	1.0065 5820	1.0073 6312	1.0081 6485	1.0089 6340	1.0097 5880
12	1.0032 7374	1.0036 7481	1.0040 7412	1.0044 7170	1.0048 6755
26	1.0015 0903	1.0016 9439	1.0018 7831	1.0020 6138	1.0022 4303
52	1.0007 5453	1.0008 4684	1.0009 3871	1.0010 3016	1.0011 2118

p	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
2	1.0319 8837	1.0344 0804	1.0368 2207	1.0392 3048	1.0416 3333
3	1.0212 1347	1.0228 0012	1.0243 9081	1.0259 8557	1.0275 6644
4	1.0158 0828	1.0170 5853	1.0182 4460	1.0194 2655	1.0206 0440
6	1.0105 5107	1.0113 4026	1.0121 2638	1.0129 0946	1.0136 8952
12	1.0052 6169	1.0056 5415	1.0060 4492	1.0064 3403	1.0068 2149
26	1.0024 2504	1.0026 0564	1.0027 8544	1.0029 6443	1.0031 4262
52	1.0012 1179	1.0013 0197	1.0013 9175	1.0014 8112	1.0015 7008

TABLE XI — NOMINAL RATE j WHICH IF CONVERTED p TIMES PER YEAR GIVES EFFECTIVE RATE i

$$j_p = p[(1+i)^{\frac{1}{p}} - 1]$$

p	$\frac{5}{12}$%	$\frac{1}{2}$%	$\frac{7}{12}$%	$\frac{3}{4}$%	1%
2	.0041 6234	.0049 9377	.0058 2485	.0074 8599	.0099 7512
3	.0041 6089	.0049 9169	.0058 2203	.0074 8133	.0099 6685
4	.0041 6017	.0049 9065	.0058 2062	.0074 7900	.0099 6272
6	.0041 5945	.0049 8962	.0058 1921	.0074 7667	.0099 5859
12	.0041 5873	.0049 8858	.0058 1780	.0074 7434	.0099 5446
13	.0041 5868	.0049 8850	.0058 1769	.0074 7416	.0099 5414
26	.0041 5834	.0049 8802	.0058 1704	.0074 7309	.0099 5224
p	$1\frac{1}{8}$%	$1\frac{1}{4}$%	$1\frac{1}{2}$%	$1\frac{3}{4}$%	2%
2	.0112 1854	.0124 6118	.0149 4417	.0174 2410	.0199 0099
3	.0112 0807	.0124 4828	.0149 2562	.0173 9890	.0198 6813
4	.0112 0285	.0124 4183	.0149 1636	.0173 8631	.0198 5173
6	.0111 9763	.0124 3539	.0149 0710	.0173 7374	.0198 3534
12	.0111 9241	.0124 2895	.0148 9785	.0173 6119	.0198 1898
13	.0111 9200	.0124 2846	.0148 9714	.0173 6022	.0198 1772
26	.0111 8960	.0124 2549	.0148 9288	.0173 5443	.0198 1017
p	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%	$3\frac{1}{2}$%
2	.0223 7484	.0248 4567	.0273 1349	.0297 7831	.0346 9899
3	.0223 3333	.0247 9451	.0272 5170	.0297 0490	.0345 9943
4	.0223 1261	.0247 6899	.0272 2087	.0296 6829	.0345 4978
6	.0222 9192	.0247 4349	.0271 9009	.0296 3173	.0345 0024
12	.0222 7125	.0247 1804	.0271 5936	.0295 9524	.0344 5078
26	.0222 6013	.0247 0434	.0271 4283	.0295 7561	.0344 2420
52	.0222 5537	.0246 9848	.0271 3575	.0295 6721	.0344 1281
p	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
2	.0396 0781	.0445 0483	.0493 9015	.0542 3380	.0591 2003
3	.0394 7821	.0443 4138	.0491 8907	.0540 2139	.0588 3847
4	.0394 1363	.0442 5996	.0490 8894	.0539 0070	.0586 9538
6	.0393 4918	.0441 7874	.0489 8908	.0537 8036	.0585 5277
12	.0392 8488	.0440 9771	.0488 8949	.0536 6039	.0584 1061
26	.0392 5031	.0440 5417	.0488 3597	.0535 9593	.0583 3425
52	.0392 3551	.0440 3552	.0488 1306	.0535 6834	.0583 0157
p	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%	$8\frac{1}{2}$%
2	.0639 7674	.0688 1609	.0736 4414	.0784 6097	.0832 6667
3	.0636 4042	.0684 2737	.0731 9942	.0779 5670	.0826 9933
4	.0634 7314	.0682 3410	.0729 7840	.0777 0619	.0824 1758
6	.0633 0644	.0680 4156	.0727 5827	.0774 5674	.0821 3712
12	.0631 4033	.0678 4974	.0725 3903	.0772 0836	.0818 5792
26	.0630 5113	.0677 4676	.0724 2134	.0770 7506	.0817 0811
52	.0630 1295	.0677 0268	.0723 7098	.0770 1802	.0816 4401

TABLE XII — THE VALUE OF THE CONVERSION FACTOR

$$\frac{i}{j_p} = \frac{i}{p[(1+i)^{\frac{1}{p}} - 1]} = (s_{\overline{1}|}^{(p)} \text{ at } i)$$

p	$\frac{5}{12}$ %	$\frac{1}{2}$ %	$\frac{7}{12}$ %	$\frac{3}{4}$ %	1 %
2	1.0010 4058	1.0012 4844	1.0014 5621	1.0018 7150	1.0024 0378
3	1.0013 8761	1.0016 0482	1.0010 4103	1.0024 0585	1.0033 2506
4	1.0015 6115	1.0018 7305	1.0021 8485	1.0028 0812	1.0037 4223
6	1.0017 3471	1.0020 8131	1.0024 2781	1.0031 2046	1.0041 5861
12	1.0010 0820	1.0022 8900	1.0026 7080	1.0034 3286	1.0045 7510
13	1.0019 2104	1.0023 0563	1.0026 8950	1.0034 5600	1.0046 0714
26	1.0020 0170	1.0024 2182	1.0028 0166	1.0036 0111	1.0047 0041
p	$1\frac{1}{8}$ %	$1\frac{1}{4}$ %	$1\frac{1}{2}$ %	$1\frac{3}{4}$ %	2 %
2	1.0028 0463	1.0031 1520	1.0037 3004	1.0043 6176	1.0049 7525
3	1.0037 4068	1.0041 5516	1.0049 8340	1.0058 1084	1.0066 3733
4	1.0042 0802	1.0046 7537	1.0056 0755	1.0065 3878	1.0074 6850
6	1.0046 7730	1.0051 9575	1.0062 3101	1.0072 6707	1.0083 0125
12	1.0051 4583	1.0057 1632	1.0068 5652	1.0079 9571	1.0091 3389
13	1.0051 8188	1.0057 5637	1.0069 0458	1.0080 5177	1.0091 9796
26	1.0053 9818	1.0059 9669	1.0071 9296	1.0083 8820	1.0095 8243
p	$2\frac{1}{4}$ %	$2\frac{1}{2}$ %	$2\frac{3}{4}$ %	3 %	$3\frac{1}{2}$ %
2	1.0055 9371	1.0062 1142	1.0068 2837	1.0074 4458	1.0086 7475
3	1.0074 6292	1.0082 8761	1.0091 1141	1.0099 3431	1.0115 7748
4	1.0083 9839	1.0093 2677	1.0102 5422	1.0111 8072	1.0130 3004
6	1.0093 3444	1.0103 6665	1.0113 9780	1.0124 2810	1.0144 8578
12	1.0102 7107	1.0114 0725	1.0125 4243	1.0136 7602	1.0159 4203
26	1.0107 7565	1.0119 6786	1.0131 5908	1.0143 4920	1.0167 2674
52	1.0109 9195	1.0122 0819	1.0134 2343	1.0146 3757	1.0170 0316
p	4 %	$4\frac{1}{2}$ %	5 %	$5\frac{1}{2}$ %	6 %
2	1.0099 0195	1.0111 2021	1.0123 4754	1.0135 6596	1.0147 8151
3	1.0132 1713	1.0148 5328	1.0164 8507	1.0181 1522	1.0197 4104
4	1.0148 7744	1.0167 2026	1.0185 5942	1.0203 9495	1.0222 2688
6	1.0165 3957	1.0185 8053	1.0206 3570	1.0226 7810	1.0247 1676
12	1.0182 0351	1.0204 0109	1.0227 1479	1.0249 6465	1.0272 1070
26	1.0191 0023	1.0214 6980	1.0238 3548	1.0261 9729	1.0285 5526
52	1.0194 8470	1.0219 6231	1.0243 1602	1.0267 2586	1.0291 3186
p	$6\frac{1}{2}$ %	7 %	$7\frac{1}{2}$ %	8 %	$8\frac{1}{2}$ %
2	1.0159 9419	1.0172 0402	1.0184 1103	1.0196 1524	1.0208 1667
3	1.0213 6348	1.0229 8254	1.0245 9826	1.0262 1065	1.0278 1974
4	1.0240 5523	1.0258 8002	1.0277 0129	1.0295 1904	1.0313 3332
6	1.0267 5172	1.0287 8298	1.0308 1059	1.0328 3456	1.0348 5492
12	1.0294 5294	1.0316 9143	1.0339 2617	1.0361 5721	1.0383 8455
26	1.0309 0941	1.0332 5978	1.0356 0640	1.0379 4927	1.0402 8845
52	1.0315 3404	1.0339 3242	1.0363 2705	1.0387 1794	1.0411 0511

TABLE XIII — AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x	Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 766	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 532
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE XIV — COMMUTATION COLUMNS, AMERICAN EXPERIENCE TABLE, 3½%

Age x	D_x	N_x	M_x	Age x	D_x	N_x	M_x
10	70 801.0	1 575 535.3	17 612.01	53	10 787.4	145 915.7	5 853.005
11	67 081.5	1 504 643.4	17 009.80	54	10 252.4	135 128.2	5 682.801
12	65 180.0	1 430 661.9	16 606.20	55	9 733.40	124 875.8	5 510.544
13	62 600.4	1 371 472.0	16 131.12	56	9 220.60	115 142.4	5 335.808
14	59 938.4	1 308 963.5	15 673.06	57	8 740.17	105 912.8	5 158.573
15	54 471.6	1 249 025.0	15 234.05	58	8 264.44	97 172.04	4 978.405
16	55 104.2	1 191 553.4	14 810.17	59	7 801.83	88 908.20	4 795.266
17	52 832.0	1 136 449.2	14 402.30	60	7 351.65	81 106.38	4 608.926
18	50 653.0	1 083 616.2	14 009.83	61	6 913.44	73 754.73	4 419.322
19	48 562.8	1 032 962.4	13 631.68	62	6 486.75	66 841.28	4 226.413
20	46 556.2	984 399.6	13 267.32	63	6 071.27	60 354.54	4 030.290
21	44 630.8	937 843.4	12 916.25	64	5 666.85	54 283.27	3 831.187
22	42 782.8	893 212.6	12 577.53	65	5 273.33	48 616.41	3 629.300
23	41 009.2	850 420.0	12 250.71	66	4 890.55	43 343.08	3 424.843
24	39 307.1	809 420.6	11 935.38	67	4 518.65	38 452.53	3 218.321
25	37 673.6	770 113.6	11 631.14	68	4 157.82	33 933.88	3 010.290
26	36 106.1	732 439.9	11 337.59	69	3 808.32	29 776.06	2 801.396
27	34 601.5	696 333.8	11 053.97	70	3 470.67	25 967.74	2 592.538
28	33 157.4	661 732.4	10 779.04	71	3 145.43	22 497.07	2 384.657
29	31 771.3	628 575.0	10 515.18	72	2 833.42	19 351.64	2 170.018
30	30 440.8	596 803.6	10 259.02	73	2 535.75	16 518.22	1 977.167
31	29 163.5	566 362.9	10 011.17	74	2 253.57	13 982.47	1 780.731
32	27 937.5	537 199.3	9 771.375	75	1 987.87	11 728.90	1 591.240
33	26 760.5	509 261.8	9 539.044	76	1 739.39	9 741.028	1 409.988
34	25 630.1	482 501.3	9 313.638	77	1 508.63	8 001.633	1 238.047
35	24 544.7	456 871.2	9 094.955	78	1 295.73	6 492.999	1 076.158
36	23 502.5	432 326.5	8 882.798	79	1 100.65	5 197.271	924.893 7
37	22 501.4	408 824.0	8 676.415	80	923.338	4 096.624	784.894 6
38	21 539.7	386 322.6	8 475.658	81	763.234	3 173.286	655.924 5
39	20 615.5	364 782.9	8 279.860	82	620.465	2 410.052	538.965 7
40	19 727.4	344 167.4	8 088.915	83	494.995	1 789.587	434.477 6
41	18 873.6	324 440.0	7 902.231	84	386.641	1 294.592	342.862 4
42	18 052.9	305 566.3	7 719.738	85	294.610	907.951 3	263.005 0
43	17 263.6	287 513.4	7 540.910	86	217.598	613.341 7	196.850 0
44	16 504.4	270 249.8	7 365.480	87	154.383	395.743 8	141.000 3
45	15 773.6	253 745.5	7 192.800	88	103.963	241.360 0	95.801 07
46	15 070.0	237 971.9	7 022.682	89	65.623 1	137.397 8	60.076 82
47	14 392.1	222 901.9	6 854.337	90	38.304 7	71.774 70	35.877 52
48	13 738.5	208 509.8	6 687.406	91	20.186 0	33.470 01	19.055 09
49	13 107.9	194 771.3	6 521.419	92	9.118 80	13.283 99	8.699 695
50	12 498.6	181 663.4	6 355.436	93	3.222 36	4.164 21	3.081 545
51	11 909.6	169 164.7	6 189.012	94	0.827 611	0.941 84	.795 762
52	11 339.5	157 255.2	6 021.696	95	0.114 232	0.114 23	.110 369

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
1.00	1.0000	1.00000	3.16228	1.000000	**1.50**	2.2500	1.22474	3.87298	.666667
1.01	1.0201	1.00499	3.17805	.990099	1.51	2.2801	1.22882	3.88587	.662252
1.02	1.0404	1.00995	3.19374	.980392	1.52	2.3104	1.23288	3.89872	.657895
1.03	1.0609	1.01489	3.20936	.970874	1.53	2.3409	1.23693	3.91152	.653595
1.04	1.0816	1.01980	3.22490	.961538	1.54	2.3716	1.24097	3.92428	.649351
1.05	1.1025	1.02470	3.24037	.952381	1.55	2.4025	1.24499	3.93700	.645161
1.06	1.1236	1.02956	3.25576	.943396	1.56	2.4336	1.24900	3.94968	.641026
1.07	1.1449	1.03441	3.27109	.934579	1.57	2.4649	1.25300	3.96232	.636943
1.08	1.1664	1.03923	3.28634	.925926	1.58	2.4964	1.25698	3.97492	.632911
1.09	1.1881	1.04403	3.30151	.917431	1.59	2.5281	1.26095	3.98748	.628931
1.10	1.2100	1.04881	3.31662	.909091	**1.60**	2.5600	1.26491	4.00000	.625000
1.11	1.2321	1.05357	3.33167	.900901	1.61	2.5921	1.26886	4.01248	.621118
1.12	1.2544	1.05830	3.34664	.892857	1.62	2.6244	1.27279	4.02492	.617284
1.13	1.2769	1.06301	3.36155	.884956	1.63	2.6569	1.27671	4.03733	.613497
1.14	1.2996	1.06771	3.37639	.877193	1.64	2.6896	1.28062	4.04969	.609756
1.15	1.3225	1.07238	3.39116	.869565	1.65	2.7225	1.28452	4.06202	.606061
1.16	1.3456	1.07703	3.40588	.862069	1.66	2.7556	1.28841	4.07431	.602410
1.17	1.3689	1.08167	3.42053	.854701	1.67	2.7889	1.29228	4.08656	.598802
1.18	1.3924	1.08628	3.43511	.847458	1.68	2.8224	1.29615	4.09878	.595238
1.19	1.4161	1.09087	3.44964	.840336	1.69	2.8561	1.30000	4.11096	.591716
1.20	1.4400	1.09545	3.46410	.833333	**1.70**	2.8900	1.30384	4.12311	.588235
1.21	1.4641	1.10000	3.47851	.826446	1.71	2.9241	1.30767	4.13521	.584795
1.22	1.4884	1.10454	3.49285	.819672	1.72	2.9584	1.31149	4.14729	.581395
1.23	1.5129	1.10905	3.50714	.813008	1.73	2.9929	1.31529	4.15933	.578035
1.24	1.5376	1.11355	3.52136	.806452	1.74	3.0276	1.31909	4.17133	.574713
1.25	1.5625	1.11803	3.53553	.800000	1.75	3.0625	1.32288	4.18330	.571429
1.26	1.5876	1.12250	3.54965	.793651	1.76	3.0976	1.32665	4.19524	.568182
1.27	1.6129	1.12694	3.56371	.787402	1.77	3.1329	1.33041	4.20714	.564972
1.28	1.6384	1.13137	3.57771	.781250	1.78	3.1684	1.33417	4.21900	.561798
1.29	1.6641	1.13578	3.59166	.775194	1.79	3.2041	1.33791	4.23084	.558659
1.30	1.6900	1.14018	3.60555	.769231	**1.80**	3.2400	1.34164	4.24264	.555556
1.31	1.7161	1.14455	3.61939	.763359	1.81	3.2761	1.34536	4.25441	.552486
1.32	1.7424	1.14891	3.63318	.757576	1.82	3.3124	1.34907	4.26615	.549451
1.33	1.7689	1.15326	3.64692	.751880	1.83	3.3489	1.35277	4.27785	.546448
1.34	1.7956	1.15758	3.66060	.746269	1.84	3.3856	1.35647	4.28952	.543478
1.35	1.8225	1.16190	3.67423	.740741	1.85	3.4225	1.36015	4.30116	.540541
1.36	1.8496	1.16619	3.68782	.735294	1.86	3.4596	1.36382	4.31277	.537634
1.37	1.8769	1.17047	3.70135	.729927	1.87	3.4969	1.36748	4.32435	.534759
1.38	1.9044	1.17473	3.71484	.724638	1.88	3.5344	1.37113	4.33590	.531915
1.39	1.9321	1.17898	3.72827	.719424	1.89	3.5721	1.37477	4.34741	.529101
1.40	1.9600	1.18322	3.74166	.714286	**1.90**	3.6100	1.37840	4.35890	.526316
1.41	1.9881	1.18743	3.75500	.709220	1.91	3.6481	1.38203	4.37035	.523560
1.42	2.0164	1.19164	3.76829	.704225	1.92	3.6864	1.38564	4.38178	.520833
1.43	2.0449	1.19583	3.78153	.699301	1.93	3.7249	1.38924	4.39318	.518135
1.44	2.0736	1.20000	3.79473	.694444	1.94	3.7636	1.39284	4.40454	.515464
1.45	2.1025	1.20416	3.80789	.689655	1.95	3.8025	1.39642	4.41588	.512821
1.46	2.1316	1.20830	3.82099	.684932	1.96	3.8416	1.40000	4.42719	.510204
1.47	2.1609	1.21244	3.83406	.680272	1.97	3.8809	1.40357	4.43847	.507614
1.48	2.1904	1.21655	3.84708	.675676	1.98	3.9204	1.40712	4.44972	.505051
1.49	2.2201	1.22066	3.86005	.671141	1.99	3.9601	1.41067	4.46094	.502513
1.50	2.2500	1.22474	3.87298	.666667	**2.00**	4.0000	1.41421	4.47214	.500000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
2.00	4.0000	1.41421	4.47214	.500000	**2.50**	6.2500	1.58114	5.00000	.400000
2.01	4.0401	1.41774	4.48330	.497512	2.51	6.3001	1.58430	5.00999	.398406
2.02	4.0804	1.42127	4.49444	.495050	2.52	6.3504	1.58745	5.01996	.396825
2.03	4.1209	1.42478	4.50555	.492611	2.53	6.4009	1.59060	5.02991	.395257
2.04	4.1616	1.42829	4.51664	.490196	2.54	6.4516	1.59374	5.03984	.393701
2.05	4.2025	1.43178	4.52769	.487805	2.55	6.5025	1.59687	5.04975	.392157
2.06	4.2436	1.43527	4.53872	.485437	2.56	6.5536	1.60000	5.05964	.390625
2.07	4.2849	1.43875	4.54973	.483092	2.57	6.6049	1.60312	5.06952	.389105
2.08	4.3264	1.44222	4.56070	.480769	2.58	6.6564	1.60624	5.07937	.387597
2.09	4.3681	1.44568	4.57165	.478469	2.59	6.7081	1.60935	5.08920	.386100
2.10	4.4100	1.44914	4.58258	.476190	**2.60**	6.7600	1.61245	5.09902	.384615
2.11	4.4521	1.45258	4.59347	.473934	2.61	6.8121	1.61555	5.10882	.383142
2.12	4.4944	1.45602	4.60435	.471698	2.62	6.8644	1.61864	5.11859	.381679
2.13	4.5369	1.45945	4.61519	.469484	2.63	6.9169	1.62173	5.12835	.380228
2.14	4.5796	1.46287	4.62601	.467290	2.64	6.9696	1.62481	5.13809	.378788
2.15	4.6225	1.46629	4.63681	.465116	2.65	7.0225	1.62788	5.14782	.377358
2.16	4.6656	1.46969	4.64758	.462963	2.66	7.0756	1.63095	5.15752	.375940
2.17	4.7089	1.47309	4.65833	.460829	2.67	7.1289	1.63401	5.16720	.374532
2.18	4.7524	1.47648	4.66905	.458716	2.68	7.1824	1.63707	5.17687	.373134
2.19	4.7961	1.47986	4.67974	.456621	2.69	7.2361	1.64012	5.18652	.371747
2.20	4.8400	1.48324	4.69042	.454545	**2.70**	7.2900	1.64317	5.19615	.370370
2.21	4.8841	1.48661	4.70106	.452489	2.71	7.3441	1.64621	5.20577	.369004
2.22	4.9284	1.48997	4.71169	.450450	2.72	7.3984	1.64924	5.21536	.367647
2.23	4.9729	1.49332	4.72229	.448430	7.73	7.4529	1.65227	5.22494	.366300
2.24	5.0176	1.49666	4.73286	.446429	2.74	7.5076	1.65529	5.23450	.364964
2.25	5.0625	1.50000	4.74342	.444444	2.75	7.5625	1.65831	5.24404	.363636
2.26	5.1076	1.50333	4.75395	.442478	2.76	7.6176	1.66132	5.25357	.362319
2.27	5.1529	1.50665	4.76445	.440529	2.77	7.6729	1.66433	5.26308	.361011
2.28	5.1984	1.50997	4.77493	.438596	2.78	7.7284	1.66733	5.27257	.359712
2.29	5.2441	1.51327	4.78539	.436681	2.79	7.7841	1.67033	5.28205	.358423
2.30	5.2900	1.51658	4.79583	.434783	**2.80**	7.8400	1.67332	5.29150	.357143
2.31	5.3361	1.51987	4.80625	.432900	2.81	7.8961	1.67631	5.30094	.355872
2.32	5.3824	1.52315	4.81664	.431034	2.82	7.9524	1.67929	5.31037	.354610
2.33	5.4289	1.52643	4.82701	.429185	2.83	8.0089	1.68226	5.31977	.353357
2.34	5.4756	1.52971	4.83735	.427350	2.84	8.0656	1.68523	5.32917	.352113
2.35	5.5225	1.53297	4.84768	.425532	2.85	8.1225	1.68819	5.33854	.350877
2.36	5.5696	1.53623	4.85798	.423729	2.86	8.1796	1.69115	5.34790	.349650
2.37	5.6169	1.53948	4.86826	.421941	2.87	8.2369	1.69411	5.35724	.348432
2.38	5.6644	1.54272	4.87852	.420168	2.88	8.2944	1.69706	5.36656	.347222
2.39	5.7121	1.54596	4.88876	.418410	2.89	8.3521	1.70000	5.37587	.346021
2.40	5.7600	1.54919	4.89898	.416667	**2.90**	8.4100	1.70294	5.38516	.344828
2.41	5.8081	1.55242	4.90918	.414938	2.91	8.4681	1.70587	5.39444	.343643
2.42	5.8564	1.55563	4.91935	.413223	2.92	8.5264	1.70880	5.40370	.342466
2.43	5.9049	1.55885	4.92950	.411523	2.93	8.5849	1.71172	5.41295	.341297
2.44	5.9536	1.56205	4.93964	.409836	2.94	8.6436	1.71464	5.42218	.340136
2.45	6.0025	1.56525	4.94975	.408163	2.95	8.7025	1.71756	5.43139	.338983
2.46	6.0516	1.56844	4.95984	.406504	2.96	8.7616	1.72047	5.44059	.337838
2.47	6.1009	1.57162	4.96991	.404858	2.97	8.8209	1.72337	5.44977	.336700
2.48	6.1504	1.57480	4.97996	.403226	2.98	8.8804	1.72627	5.45894	.335570
2.49	6.2001	1.57797	4.98999	.401606	2.99	8.9401	1.72916	5.46809	.334448
2.50	6.2500	1.58114	5.00000	.400000	**3.00**	9.0000	1.73205	5.47723	.333333
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
3.00	9.0000	1.73205	5.47723	.333333	**3.50**	12.2500	1.87083	5.91608	.285714
3.01	9.0601	1.73494	5.48635	.332226	3.51	12.3201	1.87350	5.92453	.284900
3.02	9.1204	1.73781	5.49545	.331126	3.52	12.3904	1.87617	5.93296	.284091
3.03	9.1809	1.74069	5.50454	.330033	3.53	12.4609	1.87883	5.94138	.283286
3.04	9.2416	1.74356	5.51362	.328947	3.54	12.5316	1.88149	5.94979	.282486
3.05	9.3025	1.74642	5.52268	.327869	3.55	12.6025	1.88414	5.95819	.281690
3.06	9.3636	1.74929	5.53173	.326797	3.56	12.6736	1.88680	5.96657	.280899
3.07	9.4249	1.75214	5.54076	.325733	3.57	12.7449	1.88944	5.97495	.280112
3.08	9.4864	1.75499	5.54977	.324675	3.58	12.8164	1.89209	5.98331	.279330
3.09	9.5481	1.75784	5.55878	.323625	3.59	12.8881	1.89473	5.99166	.278552
3.10	9.6100	1.76068	5.56776	.322581	**3.60**	12.9600	1.89737	6.00000	.277778
3.11	9.6721	1.76352	5.57674	.321543	3.61	13.0321	1.90000	6.00833	.277008
3.12	9.7344	1.76635	5.58570	.320513	3.62	13.1044	1.90263	6.01664	.276243
3.13	9.7969	1.76918	5.59464	.319489	3.63	13.1769	1.90526	6.02495	.275482
3.14	9.8596	1.77200	5.60357	.318471	3.64	13.2496	1.90788	6.03324	.274725
3.15	9.9225	1.77482	5.61249	.317460	3.65	13.3225	1.91050	6.04152	.273973
3.16	9.9856	1.77764	5.62139	.316456	3.66	13.3956	1.91311	6.04979	.273224
3.17	10.0489	1.78045	5.63028	.315457	3.67	13.4689	1.91572	6.05805	.272480
3.18	10.1124	1.78326	5.63915	.314465	3.68	13.5424	1.91833	6.06630	.271739
3.19	10.1761	1.78606	5.64801	.313480	3.69	13.6161	1.92094	6.07454	.271003
3.20	10.2400	1.78885	5.65685	.312500	**3.70**	13.6900	1.92354	6.08276	.270270
3.21	10.3041	1.79165	5.66569	.311526	3.71	13.7641	1.92614	6.09098	.269542
3.22	10.3684	1.79444	5.67450	.310559	3.72	13.8384	1.92873	6.09918	.268817
3.23	10.4329	1.79722	5.68331	.309598	3.73	13.9129	1.93132	6.10737	.268097
3.24	10.4976	1.80000	5.69210	.308642	3.74	13.9876	1.93391	6.11555	.267380
3.25	10.5625	1.80278	5.70088	.307692	3.75	14.0625	1.93649	6.12372	.266667
3.26	10.6276	1.80555	5.70964	.306748	3.76	14.1376	1.93907	6.13188	.265957
3.27	10.6929	1.80831	5.71839	.305810	3.77	14.2129	1.94165	6.14003	.265252
3.28	10.7584	1.81108	5.72713	.304878	3.78	14.2884	1.94422	6.14817	.264550
3.29	10.8241	1.81384	5.73585	.303951	3.79	14.3641	1.94679	6.15630	.263852
3.30	10.8900	1.81659	5.74456	.303030	**3.80**	14.4400	1.94936	6.16441	.263158
3.31	10.9561	1.81934	5.75326	.302115	3.81	14.5161	1.95192	6.17252	.262467
3.32	11.0224	1.82209	5.76194	.301205	3.82	14.5924	1.95448	6.18061	.261780
3.33	11.0889	1.82483	5.77062	.300300	3.83	14.6689	1.95704	6.18870	.261097
3.34	11.1556	1.82757	5.77927	.299401	3.84	14.7456	1.95959	6.19677	.260417
3.35	11.2225	1.83030	5.78792	.298507	3.85	14.8225	1.96214	6.20484	.259740
3.36	11.2896	1.83303	5.79655	.297619	3.86	14.8996	1.96469	6.21289	.259067
3.37	11.3569	1.83576	5.80517	.296736	3.87	14.9769	1.96723	6.22093	.258398
3.38	11.4244	1.83848	5.81378	.295858	3.88	15.0544	1.96977	6.22896	.257732
3.39	11.4921	1.84120	5.82237	.294985	3.89	15.1321	1.97231	6.23699	.257069
3.40	11.5600	1.84391	5.83095	.294118	**3.90**	15.2100	1.97484	6.24500	.256410
3.41	11.6281	1.84662	5.83952	.293255	3.91	15.2881	1.97737	6.25300	.255754
3.42	11.6964	1.84932	5.84808	.292398	3.92	15.3664	1.97990	6.26099	.255102
4.43	11.7649	1.85203	5.85662	.291545	3.93	15.4449	1.98242	6.26897	.254453
3.44	11.8336	1.85472	5.86515	.290698	3.94	15.5236	1.98494	6.27694	.253807
3.45	11.9025	1.85742	5.87367	.289855	3.95	15.6025	1.98746	6.28490	.253165
3.46	11.9716	1.86011	5.88218	.289017	3.96	15.6816	1.98997	6.29285	.252525
3.47	12.0409	1.86279	5.89067	.288184	3.97	15.7609	1.99249	6.30079	.251889
3.48	12.1104	1.86548	5.89915	.287356	3.98	15.8404	1.99499	6.30872	.251256
3.49	12.1801	1.86815	5.90762	.286533	3.99	15.9201	1.99750	6.31664	.250627
3.50	12.2500	1.87083	5.91608	.285714	**4.00**	16.0000	2.00000	6.32456	.250000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
4.00	16.0000	2.00000	6.32456	.250000	**4.50**	20.2500	2.12132	6.70820	.222222
4.01	16.0801	2.00250	6.33246	.249377	4.51	20.3401	2.12368	6.71565	.221729
4.02	16.1604	2.00400	6.34035	.248756	4.52	20.4304	2.12603	6.72309	.221239
4.03	16.2409	2.00749	6.34823	.248139	4.53	20.5209	2.12838	6.73053	.220751
4.04	16.3216	2.00998	6.35610	.247525	4.54	20.6116	2.13073	6.73795	.220264
4.05	16.4025	2.01246	6.36396	.246914	4.55	20.7025	2.13307	6.74537	.219780
4.06	16.4836	2.01494	6.37181	.246305	4.56	20.7936	2.13542	6.75278	.219298
4.07	16.5649	2.01742	6.37966	.245700	4.57	20.8849	2.13776	6.76018	.218818
4.08	16.6464	2.01990	6.38749	.245098	4.58	20.9764	2.14009	6.76757	.218341
4.09	16.7281	2.02237	6.39531	.244499	4.59	21.0681	2.14243	6.77495	.217865
4.10	16.8100	2.02485	6.40312	.243902	**4.60**	21.1600	2.14476	6.78233	.217391
4.11	16.8921	2.02731	6.41093	.243309	4.61	21.2521	2.14709	6.78970	.216920
4.12	16.9744	2.02978	6.41872	.242718	4.62	21.3444	2.14942	6.79706	.216450
4.13	17.0569	2.03224	6.42651	.242131	4.63	21.4369	2.15174	6.80441	.215983
4.14	17.1396	2.03470	6.43428	.241546	4.64	21.5296	2.15407	6.81175	.215517
4.15	17.2225	2.03715	6.44205	.240964	4.65	21.6225	2.15639	6.81909	.215054
4.16	17.3056	2.03961	6.44981	.240385	4.66	21.7156	2.15870	6.82642	.214592
4.17	17.3889	2.04206	6.45755	.239808	4.67	21.8089	2.16102	6.83374	.214133
4.18	17.4724	2.04450	6.46529	.239234	4.68	21.9024	2.16333	6.84105	.213675
4.19	17.5561	2.04695	6.47302	.238663	4.69	21.9961	2.16564	6.84836	.213220
4.20	17.6400	2.04939	6.48074	.238095	**4.70**	22.0900	2.16795	6.85565	.212766
4.21	17.7241	2.05183	6.48845	.237530	4.71	22.1841	2.17025	6.86294	.212314
4.22	17.8084	2.05426	6.49615	.236967	4.72	22.2784	2.17256	6.87023	.211864
4.23	17.8929	2.05670	6.50384	.236407	4.73	22.3729	2.17486	6.87750	.211416
4.24	17.9776	2.05913	6.51153	.235849	4.74	22.4676	2.17715	6.88477	.210970
4.25	18.0625	2.06155	6.51920	.235294	4.75	22.5625	2.17945	6.89202	.210526
4.26	18.1476	2.06398	6.52687	.234742	4.76	22.6576	2.18174	6.89928	.210084
4.27	18.2329	2.06640	6.53452	.234192	4.77	22.7529	2.18403	6.90652	.209644
4.28	18.3184	2.06882	6.54217	.233645	4.78	22.8484	2.18632	6.91375	.209205
4.29	18.4041	2.07123	6.54981	.233100	4.79	22.9441	2.18861	6.92098	.208768
4.30	18.4900	2.07364	6.55744	.232558	**4.80**	23.0400	2.19089	6.92820	.208333
4.31	18.5761	2.07605	6.56506	.232019	4.81	23.1361	2.19317	6.93542	.207900
4.32	18.6624	2.07846	6.57267	.231481	4.82	23.2324	2.19545	6.94262	.207469
4.33	18.7489	2.08087	6.58027	.230947	4.83	23.3289	2.19773	6.94982	.207039
4.34	18.8356	2.08327	6.58787	.230415	4.84	23.4256	2.20000	6.95701	.206612
4.35	18.9225	2.08567	6.59545	.229885	4.85	23.5225	2.20227	6.96419	.206186
4.36	19.0096	2.08806	6.60303	.229358	4.86	23.6196	2.20454	6.97137	.205761
4.37	19.0969	2.09045	6.61060	.228833	4.87	23.7169	2.20681	6.97854	.205339
4.38	19.1844	2.09284	6.61816	.228311	4.88	23.8144	2.20907	6.98570	.204918
4.39	19.2721	2.09523	6.62571	.227790	4.89	23.9121	2.21133	6.99285	.204499
4.40	19.3600	2.09762	6.63325	.227273	**4.90**	24.0100	2.21359	7.00000	.204082
4.41	19.4481	2.10000	6.64078	.226757	4.91	24.1081	2.21585	7.00714	.203666
4.42	19.5364	2.10238	6.64831	.226244	4.92	24.2064	2.21811	7.01427	.203252
4.43	19.6249	2.10476	6.65582	.225734	4.93	24.3049	2.22036	7.02140	.202840
4.44	19.7136	2.10713	6.66333	.225225	4.94	24.4036	2.22261	7.02851	.202429
4.45	19.8025	2.10950	6.67083	.224719	4.95	24.5025	2.22486	7.03562	.202020
4.46	19.8916	2.11187	6.67832	.224215	4.96	24.6016	2.22711	7.04273	.201613
4.47	19.9809	2.11424	6.68581	.223714	4.97	24.7009	2.22935	7.04982	.201207
4.48	20.0704	2.11660	6.69328	.223214	4.98	24.8004	2.23159	7.05691	.200803
4.49	20.1601	2.11896	6.70075	.222717	4.99	24.9001	2.23383	7.06399	.200401
4.50	20.2500	2.12132	6.70820	.222222	**5.00**	25.0000	2.23607	7.07107	.200000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
5.00	25.0000	2.23607	7.07107	.200000	**5.50**	30.2500	2.34521	7.41620	.181818
5.01	25.1001	2.23830	7.07814	.199601	5.51	30.3601	2.34734	7.42294	.181488
5.02	25.2004	2.24054	7.08520	.199203	5.52	30.4704	2.34947	7.42967	.181159
5.03	25.3009	2.24277	7.09225	.198807	5.53	30.5809	2.35160	7.43640	.180832
5.04	25.4016	2.24499	7.09930	.198413	5.54	30.6916	2.35372	7.44312	.180505
5.05	25.5025	2.24722	7.10634	.198020	5.55	30.8025	2.35584	7.44983	.180180
5.06	25.6036	2.24944	7.11337	.197628	5.56	30.9136	2.35797	7.45654	.179856
5.07	25.7049	2.25167	7.12039	.197239	5.57	31.0249	2.36008	7.46324	.179533
5.08	25.8064	2.25389	7.12741	.196850	5.58	31.1364	2.36220	7.46994	.179211
5.09	25.9081	2.25610	7.13442	.196464	5.59	31.2481	2.36432	7.47663	.178891
5.10	26.0100	2.25832	7.14143	.196078	**5.60**	31.3600	2.36643	7.48331	.178571
5.11	26.1121	2.26053	7.14843	.195695	5.61	31.4721	2.36854	7.48999	.178253
5.12	26.2144	2.26274	7.15542	.195312	5.62	31.5844	2.37065	7.49667	.177936
5.13	26.3169	2.26495	7.16240	.194932	5.63	31.6969	2.37276	7.50333	.177620
5.14	26.4196	2.26716	7.16938	.194553	5.64	31.8096	2.37487	7.50999	.177305
5.15	26.5225	2.26936	7.17635	.194175	5.65	31.9225	2.37697	7.51665	.176991
5.16	26.6256	2.27156	7.18331	.193798	5.66	32.0356	2.37908	7.52330	.176678
5.17	26.7289	2.27376	7.19027	.193424	5.67	32.1489	2.38118	7.52994	.176367
5.18	26.8324	2.27596	7.19722	.193050	5.68	32.2624	2.38328	7.53658	.176056
5.19	26.9361	2.27816	7.20417	.192678	5.69	32.3761	2.38537	7.54321	.175747
5.20	27.0400	2.28035	7.21110	.192308	**5.70**	32.4900	2.38747	7.54983	.175439
5.21	27.1441	2.28254	7.21803	.191939	5.71	32.6041	2.38956	7.55645	.175131
5.22	27.2484	2.28473	7.22496	.191571	5.72	32.7184	2.39165	7.56307	.174825
5.23	17.3529	2.28692	7.23187	.191205	5.73	32.8329	2.39374	7.56968	.174520
5.24	27.4576	2.28910	7.23878	.190840	5.74	32.9476	2.39583	7.57628	.174216
5.25	27.5625	2.29129	7.24569	.190476	5.75	33.0625	2.39792	7.58288	.173913
5.26	27.6676	2.29347	7.25259	.190114	5.76	33.1776	2.40000	7.58947	.173611
5.27	27.7729	2.29565	7.25948	.189753	5.77	33.2929	2.40208	7.59605	.173310
5.28	27.8784	2.29783	7.26636	.189394	5.78	33.4084	2.40416	7.60263	.173010
5.29	27.9841	2.30000	7.27324	.189036	5.79	33.5241	2.40624	7.60920	.172712
5.30	28.0900	2.30217	7.28011	.188679	**5.80**	33.6400	2.40832	7.61577	.172414
5.31	28.1961	2.30434	7.28697	.188324	5.81	33.7561	2.41039	7.62234	.172117
5.32	28.3024	2.30651	7.29383	.187970	5.82	33.8724	2.41247	7.62889	.171821
5.33	28.4089	2.30868	7.30068	.187617	5.83	33.9889	2.41454	7.63544	.171527
5.34	28.5156	2.31084	7.30753	.187266	5.84	34.1056	2.41661	7.64199	.171233
5.35	28.6225	2.31301	7.31437	.186916	5.85	34.2225	2.41868	7.64853	.170940
5.36	28.7296	2.31517	7.32120	.186567	5.86	34.3396	2.42074	7.65506	.170649
5.37	28.8369	2.31733	7.32803	.186220	5.87	34.4569	2.42281	7.66159	.170358
5.38	28.9444	2.31948	7.33485	.185874	5.88	34.5744	2.42487	7.66812	.170068
5.39	29.0521	2.32164	7.34166	.185529	5.89	34.6921	2.42693	7.67463	.169779
5.40	29.1600	2.32379	7.34847	.185185	**5.90**	34.8100	2.42899	7.68115	.169492
5.41	29.2681	2.32594	7.35527	.184843	5.91	34.9281	2.43105	7.68765	.169205
5.42	29.3764	2.32809	7.36206	.184502	5.92	35.0464	2.43311	7.69415	.168919
5.43	29.4849	2.33024	7.36885	.184162	5.93	35.1649	2.43516	7.70065	.168634
5.44	29.5936	2.33238	7.37564	.183824	5.94	35.2836	2.43721	7.70714	.168350
5.45	29.7025	2.33452	7.38241	.183486	5.95	35.4025	2.43926	7.71362	.168067
5.46	29.8116	2.33666	7.38918	.183150	5.96	35.5216	2.44131	7.72010	.167785
5.47	29.9209	2.33880	7.39594	.182815	5.97	35.6409	2.44336	7.72658	.167504
5.48	30.0304	2.34094	7.40270	.182482	5.98	35.7604	2.44540	7.73305	.167224
5.49	30.1401	2.34307	7.40945	.182149	5.99	35.8801	2.44745	7.73951	.166945
5.50	30.2500	2.34521	7.41620	.181818	**6.00**	36.0000	2.44949	7.74597	.166667
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

ABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
36.0000	2.44949	7.74597	.166667	**6.50**	42.2500	2.54951	8.06226	.153846
36.1201	2.45153	7.75242	.166389	6.51	42.3801	2.55147	8.06846	.153610
36.2404	2.45357	7.75887	.166113	6.52	42.5104	2.55343	8.07465	.153374
36.3609	2.45561	7.76531	.165837	6.53	42.6409	2.55539	8.08084	.153139
36.4816	2.45764	7.77174	.165563	6.54	42.7716	2.55734	8.08703	.152905
36.6025	2.45967	7.77817	.165289	6.55	42.9025	2.55930	8.09321	.152672
36.7236	2.46171	7.78460	.165017	6.56	43.0336	2.56125	8.09938	.152439
36.8449	2.46374	7.79102	.164745	6.57	43.1649	2.56320	8.10555	.152207
36.9664	2.46577	7.79744	.164474	6.58	43.2964	2.56515	8.11172	.151976
37.0881	2.46779	7.80385	.164204	6.59	43.4281	2.56710	8.11788	.151745
37.2100	2.46982	7.81025	.163934	**6.60**	43.5600	2.56905	8.12404	.151515
37.3321	2.47184	7.81665	.163666	6.61	43.6921	2.57099	8.13019	.151286
37.4544	2.47386	7.82304	.163399	6.62	43.8244	2.57294	8.13634	.151057
37.5769	2.47588	7.82943	.163132	6.63	43.9569	2.57488	8.14248	.150830
37.6996	2.47790	7.83582	.162866	6.64	44.0896	2.57682	8.14862	.150602
37.8225	2.47992	7.84219	.162602	6.65	44.2225	2.57876	8.15475	.150376
37.9456	2.48193	7.84857	.162338	6.66	44.3556	2.58070	8.16088	.150150
38.0689	2.48395	7.85493	.162075	6.67	44.4889	2.58263	8.16701	.149925
38.1924	2.48596	7.86130	.161812	6.68	44.6224	2.58457	8.17313	.149701
38.3161	2.48797	7.86766	.161551	6.69	44.7561	2.58650	8.17924	.149477
38.4400	2.48998	7.87401	.161290	**6.70**	44.8900	2.58844	8.18535	.149254
38.5641	2.49199	7.88036	.161031	6.71	45.0241	2.59037	8.19146	.149031
38.6884	2.49399	7.88670	.160772	6.72	45.1584	2.59230	8.19756	.148810
38.8129	2.49600	7.89303	.160514	6.73	45.2929	2.59422	8.20366	.148588
38.9376	2.49800	7.89937	.160256	6.74	45.4276	2.59615	8.20975	.148368
39.0625	2.50000	7.90569	.160000	6.75	45.5625	2.59808	8.21584	.148148
39.1876	2.50200	7.91202	.159744	6.76	45.6976	2.60000	8.22192	.147929
39.3129	2.50400	7.91833	.159490	6.77	45.8329	2.60192	8.22800	.147710
39.4384	2.50599	7.92465	.159236	6.78	45.9684	2.60384	8.23408	.147493
39.5641	2.50799	7.93095	.158983	6.79	46.1041	2.60576	8.24015	.147275
39.6900	2.50998	7.93725	.158730	**6.80**	46.2400	2.60768	8.24621	.147059
39.8161	2.51197	7.94355	.158479	6.81	46.3761	2.60960	8.25227	.146843
39.9424	2.51396	7.94984	.158228	6.82	46.5124	2.61151	8.25833	.146628
40.0689	2.51595	7.95613	.157978	6.83	46.6489	2.61343	8.26438	.146413
40.1956	2.51794	7.96241	.157729	6.84	46.7856	2.61534	8.27043	.146199
40.3225	2.51992	7.96869	.157480	6.85	46.9225	2.61725	8.27647	.145985
40.4496	2.52190	7.97496	.157233	6.86	47.0596	2.61916	8.28251	.145773
40.5769	2.52389	7.98123	.156986	6.87	47.1969	2.62107	8.28855	.145560
40.7044	2.52587	7.98749	.156740	6.88	47.3344	2.62298	8.29458	.145349
40.8321	2.52784	7.99375	.156495	6.89	47.4721	2.62488	8.30060	.145138
40.9600	2.52982	8.00000	.156250	**6.90**	47.6100	2.62679	8.30662	.144928
41.0881	2.53180	8.00625	.156006	6.91	47.7481	2.62869	8.31264	.144718
41.2164	2.53377	8.01249	.155763	6.92	47.8864	2.63059	8.31865	.144509
41.3449	2.53574	8.01873	.155521	6.93	48.0249	2.63249	8.32466	.144300
41.4736	2.53772	8.02496	.155280	6.94	48.1636	2.63439	8.33067	.144092
41.6025	2.53969	8.03119	.155039	6.95	48.3025	2.63629	8.33667	.143885
41.7316	2.54165	8.03741	.154799	6.96	48.4416	2.63818	8.34266	.143678
41.8609	2.54362	8.04363	.154560	6.97	48.5809	2.64008	8.34865	.143472
41.9904	2.54558	8.04984	.154321	6.98	48.7204	2.64197	8.35464	.143266
42.1201	2.54755	8.05605	.154083	6.99	48.8601	2.64386	8.36062	.143062
42.2500	2.54951	8.06226	.153846	**7.00**	49.0000	2.64575	8.36660	.142857
n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

n	[illegible]	[illegible]	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
7.00	[illegible]	[illegible]	8.36660	.142857	**7.50**	56.2500	2.73861	8.66025	.133333
7.01	[illegible]	[illegible]	8.37257	.142653	7.51	56.4001	2.74044	8.66603	.133156
7.02	49.2804	[illegible]	8.37854	.142450	7.52	56.5504	2.74226	8.67179	.132979
7.03	49.4209	[illegible]	8.38451	.142248	7.53	56.7009	2.74408	8.67756	.132802
7.04	49.5616	2.65330	8.39047	.142045	7.54	56.8516	2.74591	8.68332	.132626
7.05	49.7025	[illegible]	8.39643	.141844	7.55	57.0025	2.74773	8.68907	.132450
7.06	49.8436	2.65707	8.40238	.141643	7.56	57.1536	2.74955	8.69483	.132275
7.07	49.9849	2.65895	8.40833	.141443	7.57	57.3049	2.75136	8.70057	.132100
7.08	50.1264	2.66083	8.41427	.141243	7.58	57.4564	2.75318	8.70632	.131926
7.09	50.2681	2.66271	8.42021	.141044	7.59	57.6081	2.75500	8.71206	.131752
7.10	50.4100	2.66458	8.42615	.140845	**7.60**	57.7600	2.75681	8.71780	.131579
7.11	50.5521	2.66646	8.43208	.140647	7.61	57.9121	2.75862	8.72353	.131406
7.12	50.6944	2.66833	8.43801	.140449	7.62	58.0644	2.76043	8.72926	.131234
7.13	50.8369	2.67021	8.44393	.140252	7.63	58.2169	2.76225	8.73499	.131062
7.14	50.9796	2.67208	8.44985	.140056	7.64	58.3696	2.76405	8.74071	.130890
7.15	51.1225	2.67395	8.45577	.139860	7.65	58.5225	2.76586	8.74643	.130719
7.16	51.2656	2.67582	8.46168	.139665	7.66	58.6756	2.76767	8.75214	.130548
7.17	51.4089	2.67769	8.46759	.139470	7.67	58.8289	2.76948	8.75785	.130378
7.18	51.5524	2.67955	8.47349	.139276	7.68	58.9824	2.77128	8.76356	.130208
7.19	51.6961	2.68142	8.47939	.139082	7.69	59.1361	2.77308	8.76926	.130039
7.20	51.8400	2.68328	8.48528	.138889	**7.70**	59.2900	2.77489	8.77496	.129870
7.21	51.9841	2.68514	8.49117	.138696	7.71	59.4441	2.77669	8.78066	.129702
7.22	52.1284	2.68701	8.49706	.138504	7.72	59.5984	2.77849	8.78635	.129534
7.23	52.2729	2.68887	8.50294	.138313	7.73	59.7529	2.78029	8.79204	.129366
7.24	52.4176	2.69072	8.50882	.138122	7.74	59.9076	2.78209	8.79773	.129199
7.25	52.5625	2.69258	8.51469	.137931	7.75	60.0625	2.78388	8.80341	.129032
7.26	52.7076	2.69444	8.52056	.137741	7.76	60.2176	2.78568	8.80909	.128866
7.27	52.8529	2.69629	8.52643	.137552	7.77	60.3729	2.78747	8.81476	.128700
7.28	52.9984	2.69815	8.53229	.137363	7.78	60.5284	2.78927	8.82043	.128535
7.29	53.1441	2.70000	8.53815	.137174	7.79	60.6841	2.79106	8.82610	.128370
7.30	53.2900	2.70185	8.54400	.136986	**7.80**	60.8400	2.79285	8.83176	.128205
7.31	53.4361	2.70370	8.54985	.136799	7.81	60.9961	2.79464	8.83742	.128041
7.32	53.5824	2.70555	8.55570	.136612	7.82	61.1524	2.79643	8.84308	.127877
7.33	53.7289	2.70740	8.56154	.136426	7.83	61.3089	2.79821	8.84873	.127714
7.34	53.8756	2.70924	8.56738	.136240	7.84	61.4656	2.80000	8.85438	.127551
7.35	54.0225	2.71109	8.57321	.136054	7.85	61.6225	2.80179	8.86002	.127389
7.36	54.1696	2.71293	8.57904	.135870	7.86	61.7796	2.80357	8.86566	.127226
7.37	54.3169	2.71477	8.58487	.135685	7.87	61.9369	2.80535	8.87130	.127065
7.38	54.4644	2.71662	8.59069	.135501	7.88	62.0944	2.80713	8.87694	.126904
7.39	54.6121	2.71846	8.59651	.135318	7.89	62.2521	2.80891	8.88257	.126743
7.40	54.7600	2.72029	8.60233	.135135	**7.90**	62.4100	2.81069	8.88819	.126582
7.41	54.9081	2.72213	8.60814	.134953	7.91	62.5681	2.81247	8.89382	.126422
7.42	55.0564	2.72397	8.61394	.134771	7.92	62.7264	2.81425	8.89944	.126263
7.43	55.2049	2.72580	8.61974	.134590	7.93	62.8849	2.81603	8.90505	.126103
7.44	55.3536	2.72764	8.62554	.134409	7.94	63.0436	2.81780	8.91067	.125945
7.45	55.5025	2.72947	8.63134	.134228	7.95	63.2025	2.81957	8.91628	.125786
7.46	55.6516	2.73130	8.63713	.134048	7.96	63.3616	2.82135	8.92188	.125628
7.47	55.8009	2.73313	8.64292	.133869	7.97	63.5209	2.82312	8.92749	.125471
7.48	55.9504	2.73496	8.64870	.133690	7.98	63.6804	2.82489	8.93308	.125313
7.49	56.1001	2.73679	8.65448	.133511	7.99	63.8401	2.82666	8.93868	.125156
7.50	56.2500	2.73861	8.66025	.133333	**8.00**	64.0000	2.82843	8.94427	.125000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

TABLE XV — SQUARES — SQUARE ROOTS — RECIPROCALS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
8.00	64.0000	2.82843	8.94427	.125000	**8.50**	72.2500	2.91548	9.21954	.117647
8.01	64.1601	2.83019	8.94986	.124844	8.51	72.4201	2.91719	9.22497	.117509
8.02	64.3204	2.83196	8.95545	.124688	8.52	72.5904	2.91890	9.23038	.117371
8.03	64.4809	2.83373	8.96103	.124533	8.53	72.7609	2.92062	9.23580	.117233
8.04	64.6416	2.83549	8.96660	.124378	8.54	72.9316	2.92233	9.24121	.117096
8.05	64.8025	2.83725	8.97218	.124224	8.55	73.1025	2.92404	9.24662	.116959
8.06	64.9636	2.83901	8.97775	.124069	8.56	73.2736	2.92575	9.25203	.116822
8.07	65.1249	2.84077	8.98332	.123916	8.57	73.4449	2.92746	9.25743	.116686
8.08	65.2864	2.84253	8.98888	.123762	8.58	73.6164	2.92916	9.26283	.116550
8.09	65.4481	2.84429	8.99444	.123609	8.59	73.7881	2.93087	9.26823	.116414
8.10	65.6100	2.84605	9.00000	.123457	**8.60**	73.9600	2.93258	9.27362	.116279
8.11	65.7721	2.84781	9.00555	.123305	8.61	74.1321	2.93428	9.27901	.116144
8.12	65.9344	2.84956	9.01110	.123153	8.62	74.3044	2.93598	9.28440	.116009
8.13	66.0969	2.85132	9.01665	.123001	8.63	74.4769	2.93769	9.28978	.115875
8.14	66.2596	2.85307	9.02219	.122850	8.64	74.6496	2.93939	9.29516	.115741
8.15	66.4225	2.85482	9.02774	.122699	8.65	74.8225	2.94109	9.30054	.115607
8.16	66.5856	2.85657	9.03327	.122549	8.66	74.9956	2.94279	9.30591	.115473
8.17	66.7489	2.85832	9.03881	.122399	8.67	75.1689	2.94449	9.31128	.115340
8.18	66.9124	2.86007	9.04434	.122249	8.68	75.3424	2.94618	9.31665	.115207
8.19	67.0761	2.86182	9.04986	.122100	8.69	75.5161	2.94788	9.32202	.115075
8.20	67.2400	2.86356	9.05539	.121951	**8.70**	75.6900	2.94958	9.32738	.114943
8.21	67.4041	2.86531	9.06091	.121803	8.71	75.8641	2.95127	9.33274	.114811
8.22	67.5684	2.86705	9.06642	.121655	8.72	76.0384	2.95296	9.33809	.114679
8.23	67.7329	2.86880	9.07193	.121507	8.73	76.2129	2.95466	9.34345	.114548
8.24	67.8976	2.87054	9.07744	.121359	8.74	76.3876	2.95635	9.34880	.114416
8.25	68.0625	2.87228	9.08295	.121212	8.75	76.5625	2.95804	9.35414	.114286
8.26	68.2276	2.87402	9.08845	.121065	8.76	76.7376	2.95973	9.35949	.114155
8.27	68.3929	2.87576	9.09395	.120919	8.77	76.9129	2.96142	9.36483	.114025
8.28	68.5584	2.87750	9.09945	.120773	8.78	77.0884	2.96311	9.37017	.113895
8.29	68.7241	2.87924	9.10494	.120627	8.79	77.2641	2.96479	9.37550	.113766
8.30	68.8900	2.88097	9.11043	.120482	**8.80**	77.4400	2.96648	9.38083	.113636
8.31	69.0561	2.88271	9.11592	.120337	8.81	77.6161	2.96816	9.38616	.113507
8.32	69.2224	2.88444	9.12140	.120192	8.82	77.7924	2.96985	9.39149	.113379
8.33	69.3889	2.88617	9.12688	.120048	8.83	77.9689	2.97153	9.39681	.113250
8.34	69.5556	2.88791	9.13236	.119904	8.84	78.1456	2.97321	9.40213	.113122
8.35	69.7225	2.88964	9.13783	.119760	8.85	78.3225	2.97489	9.40744	.112994
8.36	69.8896	2.89137	9.14330	.119617	8.86	78.4996	2.97658	9.41276	.112867
8.37	70.0569	2.89310	9.14877	.119474	8.87	78.6769	2.97825	9.41807	.112740
8.38	70.2244	2.89482	9.15423	.119332	8.88	78.8544	2.97993	9.42338	.112613
8.39	70.3921	2.89655	9.15969	.119190	8.89	79.0321	2.98161	9.42868	.112486
8.40	70.5600	2.89828	9.16515	.119048	**8.90**	79.2100	2.98329	9.43398	.112360
8.41	70.7281	2.90000	9.17061	.118906	8.91	79.3881	2.98496	9.43928	.112233
8.42	70.8964	2.90172	9.17606	.118765	8.92	79.5664	2.98664	9.44458	.112108
8.43	71.0649	2.90345	9.18150	.118624	8.93	79.7449	2.98831	9.44987	.111982
8.44	71.2336	2.90517	9.18695	.118483	8.94	79.9236	2.98998	9.45516	.111857
8.45	71.4025	2.90689	9.19239	.118343	8.95	80.1025	2.99166	9.46044	.111732
8.46	71.5716	2.90861	9.19783	.118203	8.96	80.2816	2.99333	9.46573	.111607
8.47	71.7409	2.91033	9.20326	.118064	8.97	80.4609	2.99500	9.47101	.111483
8.48	71.9104	2.91204	9.20869	.117925	8.98	80.6404	2.99666	9.47629	.111359
8.49	72.0801	2.91376	9.21412	.117786	8.99	80.8201	2.99833	9.48156	.111235
8.50	72.2500	2.91548	9.21954	.117647	**9.00**	81.0000	3.00000	9.48683	.111111
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$
9.00	81.0000	3.00000	9.48683	.111111	**9.50**	90.2500	3.08221	9.74679	.105263
9.01	81.1801	3.00167	9.49210	.110988	9.51	90.4401	3.08383	9.75192	.105152
9.02	81.3604	3.00333	9.49737	.110865	9.52	90.6304	3.08545	9.75705	.105042
9.03	81.5409	3.00500	9.50263	.110742	9.53	90.8209	3.08707	9.76217	.104932
9.04	81.7216	3.00666	9.50789	.110619	9.54	91.0116	3.08869	9.76729	.104822
9.05	81.9025	3.00832	9.51315	.110497	9.55	91.2025	3.09031	9.77241	.104712
9.06	82.0836	3.00998	9.51840	.110375	9.56	91.3936	3.09192	9.77753	.104603
9.07	82.2649	3.01164	9.52365	.110254	9.57	91.5849	3.09354	9.78264	.104493
9.08	82.4464	3.01330	9.52890	.110132	9.58	91.7764	3.09516	9.78775	.104384
9.09	82.6281	3.01496	9.53415	.110011	9.59	91.9681	3.09677	9.79285	.104275
9.10	82.8100	3.01662	9.53939	.109890	**9.60**	92.1600	3.09839	9.79796	.104167
9.11	82.9921	3.01828	9.54463	.109769	9.61	92.3521	3.10000	9.80306	.104058
9.12	83.1744	3.01993	9.54987	.109649	9.62	92.5444	3.10161	9.80816	.103950
9.13	83.3569	3.02159	9.55510	.109529	9.63	92.7369	3.10322	9.81326	.103842
9.14	83.5396	3.02324	9.56033	.109409	9.64	92.9296	3.10483	9.81835	.103734
9.15	83.7225	3.02490	9.56556	.109290	9.65	93.1225	3.10644	9.82344	.103627
9.16	83.9056	3.02655	9.57079	.109170	9.66	93.3156	3.10805	9.82853	.103520
9.17	84.0889	3.02820	9.57601	.109051	9.67	93.5089	3.10966	9.83362	.103413
9.18	84.2724	3.02985	9.58123	.108932	9.68	93.7024	3.11127	9.83870	.103306
9.19	84.4561	3.03150	9.58645	.108814	9.69	93.8961	3.11288	9.84378	.103199
9.20	84.6400	3.03315	9.59166	.108696	**9.70**	94.0900	3.11448	9.84886	.103093
9.21	84.8241	3.03480	9.59687	.108578	9.71	94.2841	3.11609	9.85393	.102987
9.22	85.0084	3.03645	9.60208	.108460	9.72	94.4784	3.11769	9.85901	.102881
9.23	85.1929	3.03809	9.60729	.108342	9.73	94.6729	3.11929	9.86408	.102775
9.24	85.3776	3.03974	9.61249	.108225	9.74	94.8676	3.12090	9.86914	.102669
9.25	85.5625	3.04138	9.61769	.108108	9.75	95.0625	3.12250	9.87421	.102564
9.26	85.7476	3.04302	9.62289	.107991	9.76	95.2576	3.12410	9.87927	.102459
9.27	85.9329	3.04467	9.62808	.107875	9.77	95.4529	3.12570	9.88433	.102354
9.28	86.1184	3.04631	9.63328	.107759	9.78	95.6484	3.12730	9.88939	.102249
9.29	86.3041	3.04795	9.63846	.107643	9.79	95.8441	3.12890	9.89444	.102145
9.30	86.4900	3.04959	9.64365	.107527	**9.80**	96.0400	3.13050	9.89949	.102041
9.31	86.6761	3.05123	9.64883	.107411	9.81	96.2361	3.13209	9.90454	.101937
9.32	86.8624	3.05287	9.65401	.107296	9.82	96.4324	3.13369	9.90959	.101833
9.33	87.0489	3.05450	9.65919	.107181	9.83	96.6289	3.13528	9.91464	.101729
9.34	87.2356	3.05614	9.66437	.107066	9.84	96.8256	3.13688	9.91968	.101626
9.35	87.4225	3.05778	9.66954	.106952	9.85	97.0225	3.13847	9.92472	.101523
9.36	87.6096	3.05941	9.67471	.106838	9.86	97.2196	3.14006	9.92975	.101420
9.37	87.7969	3.06105	9.67988	.106724	9.87	97.4169	3.14166	9.93479	.101317
9.38	87.9844	3.06268	9.68504	.106610	9.88	97.6144	3.14325	9.93982	.101215
9.39	88.1721	3.06431	9.69020	.106496	9.89	97.8121	3.14484	9.94485	.101112
9.40	88.3600	3.06594	9.69536	.106383	**9.90**	98.0100	3.14643	9.94987	.101010
9.41	88.5481	3.06757	9.70052	.106270	9.91	98.2081	3.14802	9.95490	.100908
9.42	88.7364	3.06920	9.70567	.106157	9.92	98.4064	3.14960	9.95992	.100806
9.43	88.9249	3.07083	9.71082	.106045	9.93	98.6049	3.15119	9.96494	.100705
9.44	89.1136	3.07246	9.71597	.105932	9.94	98.8036	3.15278	9.96995	.100604
9.45	89.3025	3.07409	9.72111	.105820	9.95	99.0025	3.15436	9.97497	.100503
9.46	89.4916	3.07571	9.72625	.105708	9.96	99.2016	3.15595	9.97998	.100402
9.47	89.6809	3.07734	9.73139	.105597	9.97	99.4009	3.15753	9.98499	.100301
9.48	89.8704	3.07896	9.73653	.105485	9.98	99.6004	3.15911	9.98999	.100200
9.49	90.0601	3.08058	9.74166	.105374	9.99	99.8001	3.16070	9.99500	.100100
9.50	90.2500	3.08221	9.74679	.105263	**10.00**	100.000	3.16228	10.0000	.100000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$1/n$