Linear Partial Differential Equations and Fourier Theory

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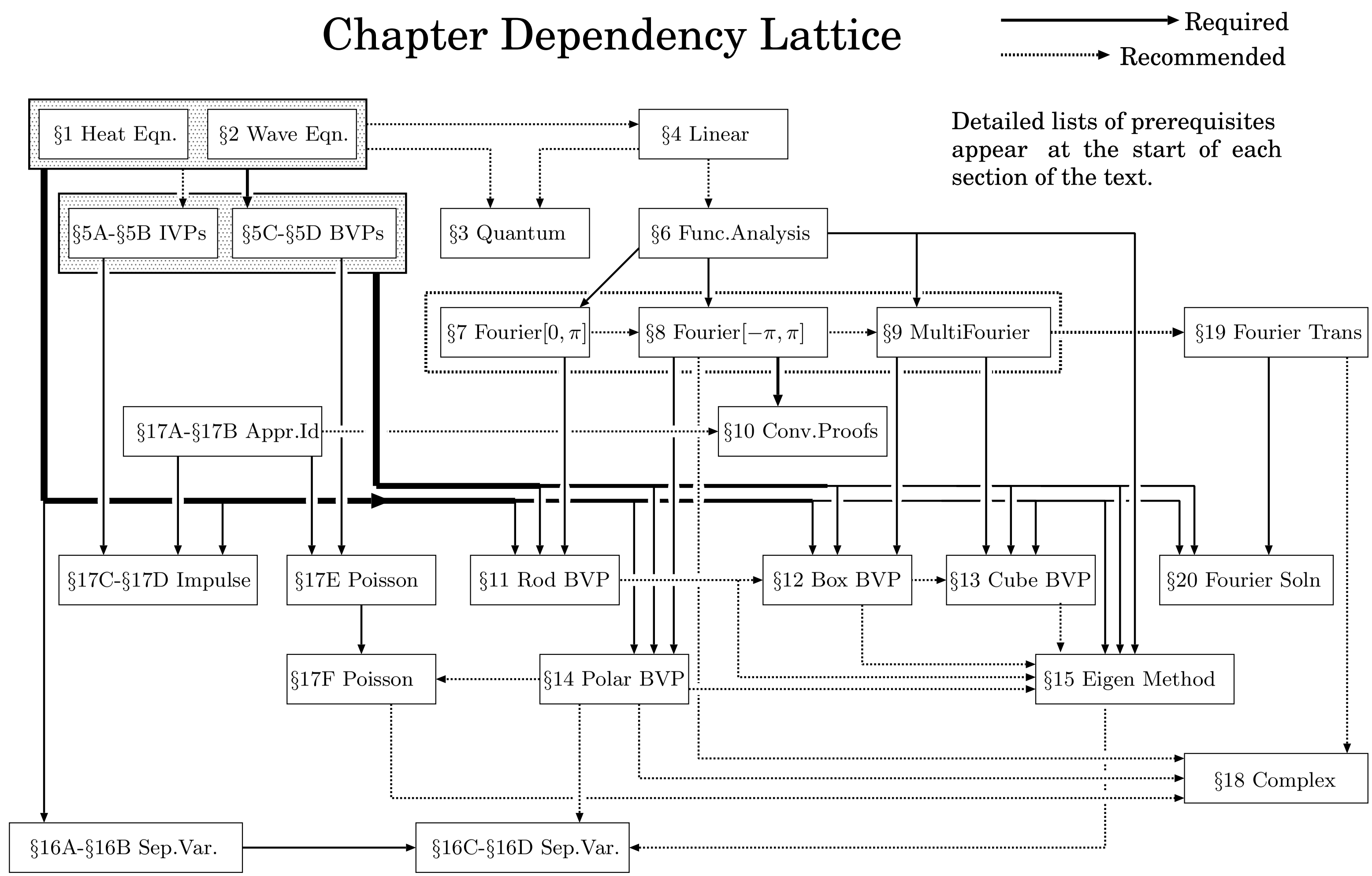
To Joseph and Emma Pivato
for their support, encouragement,
and inspiring example.

Chapter Dependency Lattice



Detailed lists of prerequisites appear at the start of each section of the text.

Contents

Preface

This is a textbook for an introductory course on linear partial differential equations (PDEs) and initial/boundary value problems (I/BVPs). It also provides a mathematically rigorous introduction to Fourier analysis (Chapters 7, 8, 9, 10, and 19), which is the main tool used to solve linear PDEs in Cartesian coordinates. Finally, it introduces basic functional analysis (Chapter 6) and complex analysis (Chapter 18). The first is necessary to rigorously characterize the convergence of Fourier series, and also to discuss eigenfunctions for linear differential operators. The second provides powerful techniques to transform domains and compute integrals, and also offers additional insight into Fourier series.

This book is not intended to be comprehensive or encyclopaedic. It is designed for a one-semester course (i.e. 36-40 hours of lectures), and it is therefore strictly limited in scope. First, it deals mainly with *linear* PDEs with constant coefficients. Thus, there is no discussion of characteristics, conservation laws, shocks, variational techniques, or perturbation methods, which would be germane to other types of PDEs. Second, the book focus mainly on concrete solution methods to specific PDEs (e.g. the Laplace, Poisson, Heat, Wave, and Schrödinger equations) on specific domains (e.g. line segments, boxes, disks, annuli, spheres), and spends rather little time on qualitative results about entire classes of PDEs (e.g elliptic, parabolic, hyperbolic) on general domains. Only after a thorough exposition of these special cases does the book sketch the general theory; experience shows that this is far more pedagogically effective then presenting the general theory first. Finally, the book does not deal at all with numerical solutions or Galerkin methods.

One slightly unusual feature of this book is that, from the very beginnning, it emphasizes the central role of eigenfunctions (of the Laplacian) in the solution methods for linear PDEs. Fourier series and Fourier-Bessel expansions are introduced as the orthogonal eigenfunction expansions which are most suitable in certain domains or coordinate systems. Separation of variables appears relatively late in the exposition (Chapter 16), as a convenient device to obtain such eigenfunctions. The only techniques in the book which are not either implicitly or explicitly based on eigenfunction expansions are impulse-response functions and Green's functions (Chapter 17) and complex-analytic methods (Chapter 18).

Prerequisites and intended audience. This book is written for third-year undergraduate students in mathematics, physics, engineering, and other mathematical sciences. The only prererequisites are (1) *multivariate calculus* (i.e. partial derivatives, multivariate integration, changes of coordinate system) and (2) *linear algebra* (i.e. linear operators and their eigenvectors).

It might also be helpful for students to be familiar with: (1) the basic theory of ordinary differential equations (specifically: Laplace transforms, Frobenius method); (2) some elementary vector calculus (specifically: divergence and gradient operators); and (3) elementary physics (to understand the physical motivation behind many of the problems). However, none of these three things are really required.

In addition to this background knowledge, the book requires some ability at abstract mathematical reasoning. Unlike some 'applied math' texts, we do not suppress or handwave the mathematical theory behind the solution methods. At suitable moments, the exposition introduces concepts like 'orthogonal basis', 'uniform convergence' vs. '$\mathbf{L}_2$-convergence', 'eigenfunction expansion', and 'self-adjoint operator'; thus, students must be intellectually capable of understanding abstract mathematical concepts of this nature. Likewise, the exposition is mainly organized in a 'definition → theorem → proof → example' format, rather than a 'problem → solution' format. Students must be able to understand abstract descriptions of general solution techniques, rather than simply learn by imitating worked solutions to special cases.

Acknowledgements. I would like to thank Xiaorang Li of Trent University, who read through an early draft of this book and made many helpful suggestions and corrections, and who also provided questions #6 and #7 on page 101, and also question # 8 on page 135. I also thank Peter Nalitolela, who proofread a penultimate draft and spotted many mistakes. I would like to thank several anonymous reviewers who made many useful suggestions, and I would also like to thank Peter Thompson of Cambridge University Press for recruiting these referees. I also thank Diana Gillooly of Cambridge University Press, who was very supportive and helpful throughout the entire publication process, especially concerning my desire to provide a free online version of the book, and to release the figures and problem sets under a Creative Commons license. I also thank the many students who used the early versions of this book, especially those who found mistakes or made good suggestions. Finally, I thank George Peschke of the University of Alberta, for being an inspiring example of good mathematical pedagogy.

None of these people are responsible for any remaining errors, omissions, or other flaws in the book (of which there are no doubt many). If you find an error or some other deficiency in the book, please contact me at

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This book would not have been possible without open source software. The book was prepared entirely on the LINUX operating system (initially REDHAT[1], and later UBUNTU[2]). All the text is written in Leslie Lamport's LaTeX2e typesetting language[3], and was authored using Richard Stallman's EMACS editor[4]. The illustrations were hand-drawn using William Chia-Wei Cheng's excellent TGIF

[1] `http://www.redhat.com`
[2] `http://www.ubuntu.com`
[3] `http://www.latex-project.org`
[4] `http://www.gnu.org/software/emacs/emacs.html`

object-oriented drawing program[5]. Additional image manipulation and post-processing was done with GNU Image Manipulation Program (GIMP)[6]. Many of the plots were created using GnuPlot[7].[8] I would like to take this opportunity to thank the many people in the open source community who have developed this software.

Finally and most importantly, I would like to thank my beloved wife and partner, Reem Yassawi, and our wonderful children, Leila and Aziza, for their support and for their patience with my many long absences.

[5] `http://bourbon.usc.edu:8001/tgif`

[6] `http://www.gimp.org`

[7] `http://www.gnuplot.info`

[8] Many other plots were generated using Waterloo Maple (`http://www.maplesoft.com`), which unfortunately is *not* open-source.

What's good about this book?

This text has many advantages over most other introductions to partial differential equations.

Illustrations. PDEs are physically motivated and geometrical objects; they describe curves, surfaces and scalar fields with special geometric properties, and the way these entities evolve over time under endogenous dynamics. To understand PDEs and their solutions, it is necessary to visualize them. Algebraic formulae are just a language used to communicate such visual ideas in lieu of pictures, and they generally make a poor substitute. This book has over 300 high-quality illustrations, many of which are rendered in three dimensions. In the online version of the book, most of these illustrations appear in full colour. Also, the website contains many animations which do not appear in the printed book.

Most importantly, on the book website, *all illustrations are freely available under a Creative Commons Attribution Noncommercial Share-Alike License*[9]. This means that you are free to download, modify, and utilize the illustrations to prepare your own course materials (e.g. printed lecture notes or beamer presentations), as long as you attribute the original author. Please visit

<http://xaravve.trentu.ca/pde>

Physical motivation. Connecting the math to physical reality is critical: it keeps students motivated, and helps them interpret the mathematical formalism in terms of their physical intuitions about diffusion, vibration, electrostatics, etc. Chapter 1 of this book discusses the physics behind the Heat, Laplace, and Poisson equations, and Chapter 2 discusses the wave equation. An unusual addition to this text is Chapter 3, which discusses quantum mechanics and the Schrödinger equation (one of the major applications of PDE theory in modern physics).

Detailed syllabus. Difficult choices must be made when turning a 600+ page textbook into a feasible twelve-week lesson plan. It is easy to run out of time or inadvertently miss something important. To facilitate this task, this book provides a lecture-by-lecture breakdown of how the author covers the material (page xxi). Of course, each instructor can diverge from this syllabus to suit the interests/background of her students, a longer/shorter teaching semester, or her personal taste. But the prefabricated syllabus provides a base to work from, and will save most instructors a lot of time and aggravation.

[9]See `http://creativecommons.org/licenses/by-nc-sa/3.0`.

Explicit prerequisites for each chapter and section. To save time, an instructor might want to skip a certain chapter or section, but she worries that it may end up being important later. We resolve this problem in two ways. First, page (iv) provides a *Chapter Dependency Lattice*, which summarises the large-scale structure of logical dependencies between the chapters of the book. Second, every section of every chapter begins with an explicit list of 'required' and 'recommended' prerequisite sections; this provides more detailed information about the small-scale structure of logical dependencies between sections. By tracing backward through this 'lattice of dependencies', you can figure out exactly what background material you must cover to reach a particular goal. This makes the book especially suitable for self-study.

Flat dependency lattice. There are many 'paths' through the twenty-chapter *Dependency Lattice* on page (iv), every one of which is only *seven* chapters or less in length. Thus, an instructor (or an autodidact) can design many possible syllabi, depending on her interests, and can quickly move to advanced material. The 'Recommended Syllabus' on page (xxi) describes a gentle progression through the material, covering most of the 'core' topics in a 12 week semester, emphasizing concrete examples and gradually escalating the abstraction level. The Chapter Dependency Lattice suggests some other possibilities for 'accelerated' syllabi focusing on different themes:

- *Solving PDEs with impulse response functions.* Chapters 1, 2, 5 and 17 only.
- *Solving PDEs with Fourier transforms.* Chapters 1, 2, 5, 19, and 20 only.

  (Pedagogically speaking, Chapters 8 and 9 will help the student understand Chapter 19, and Chapters 11-13 will help the student understand Chapter 20. Also, it is interesting to see how the 'impulse-response' methods of Chapter 17 yield the same solutions as the 'Fourier methods' of Chapter 20, using a totally different approach. However, strictly speaking, none of Chapters 8-13 or 17 is logically necessary.)
- *Solving PDEs with separation of variables.* Chapters 1, 2 and 16 only.

  (However, without at least Chapters 12 and 14, the ideas of Chapter 16 will seem somewhat artificial and pointless.)
- *Solving I/BVPs using eigenfunction expansions.* Chapters 1, 2, 4, 5, 6, and 15.

  (It would be pedagogically better to also cover Chapters 9 and 12, and probably Chapter 14. But strictly speaking, none of these is logically necessary.)
- *Tools for quantum mechanics.* Section 1B, then Chapters 3, 4, 6, 9, 13, 19, and 20 (skipping material on Laplace, Poisson, and wave equations in

Chapters 13 and 20, and adapting the solutions to the heat equation into solutions to the Schrödinger equation.)

- *Fourier theory.* Section 4A, then Chapters 6, 7, 8, 9, 10, and 19. Finally, Sections 18A, 18C, 18E and 18F provide a 'complex' perspective. (Section 18H also contains some useful computational tools).

- *Crash course in complex analysis.* Chapter 18 is logically independent of the rest of the book, and rigorously develops the main ideas in complex analysis from first principles. (However, the emphasis is on applications to PDEs and Fourier theory, so some of the material may seem esoteric or unmotivated if read in isolation from other chapters.)

Highly structured exposition, with clear motivation up front. The exposition is broken into small, semi-independent logical units, each of which is clearly labelled, and which has a clear purpose or meaning which is made immediately apparent. This simplifies the instructor's task; she doesn't need to spend time restructuring and summarizing the text material, because it is already structured in a manner which self-summarizes. Instead, instructors can focus more on explanation, motivation, and clarification.

Many 'practice problems' (with complete solutions and source code available on-line). Frequent evaluation is critical to reinforce material taught in class. This book provides an extensive supply of (generally simple) computational 'Practice Problems' at the end of each chapter. Completely worked solutions to virtually all of these problems are available on the book website. Also on the book website, *the LaTeX source code for all problems and solutions is freely available under a Creative Commons Attribution Noncommercial Share-Alike License*[10]. Thus, an instructor can download and edit this source code, and easily create quizzes, assignments, and matching solutions for her students.

Challenging exercises without solutions. Complex theoretical concepts cannot really be tested in quizzes, and do not lend themselves to canned 'practice problems'. For a more theoretical course with more mathematically sophisticated students, the instructor will want to assign some proof-related exercises for homework. This book has more than 420 such exercises scattered throughout the exposition; these are flagged by an "Ⓔ" symbol in the margin, as shown here. Many of these exercises ask the student to prove a major result from the text (or a component thereof). This is the best kind of exercise, because it reinforces the material taught in class, and gives students a sense of ownership of the mathematics. Also, students find it more fun and exciting to prove important theorems, rather than solving esoteric make-work problems. Ⓔ

[10]See `http://creativecommons.org/licenses/by-nc-sa/3.0`.

Appropriate rigour. The solutions of PDEs unfortunately involve many technicalities (e.g. different forms of convergence; derivatives of infinite function series, etc.). It is tempting to handwave and gloss over these technicalities, to avoid confusing students. But this kind of pedagogical dishonesty actually makes students *more* confused; they know something is fishy, but they can't tell quite what. Smarter students know they are being misled, and may lose respect for the book, the instructor, or even the whole subject.

In contrast, this book provides a rigorous mathematical foundation for all its solution methods. For example, Chapter 6 contains a careful explanation of L^2 spaces, the various forms of convergence for Fourier series, and the differences between them —including the 'pathologies' which can arise when one is careless about these issues. I adopt a 'triage' approach to proofs: The simplest proofs are left as exercises for the motivated student (often with a step-by-step breakdown of the best strategy). The most complex proofs I have omitted, but I provide multiple references to other recent texts. In between are those proofs which are challenging but still accessible; I provide detailed expositions of these proofs. Often, when the text contains several variants of the same theorem, I prove one variant in detail, and leave the other proofs as exercises.

Appropriate Abstraction. It is tempting to avoid abstractions (e.g. linear differential operators, eigenfunctions), and simply present *ad hoc* solutions to special cases. This cheats the student. The right abstractions provide simple yet powerful tools which help students understand a myriad of seemingly disparate special cases within a single unifying framework. This book provides students with the opportunity to learn an abstract perspective once they are ready for it. Some abstractions are introduced in the main exposition, others are in optional sections, or in the philosophical preambles which begin each major part of the book.

Gradual abstraction. Learning proceeds from the concrete to the abstract. Thus, the book begins each topic with a specific example or a low-dimensional formulation, and only later proceed to a more general/abstract idea. This introduces a lot of "redundancy" into the text, in the sense that later formulations subsume the earlier ones. So the exposition is not as "efficient" as it could be. This is a good thing. Efficiency makes for good reference books, but lousy texts.

For example, when introducing the heat equation, Laplace equation, and wave equation in Chapters 1 and 2, I first derive and explain the one-dimensional version of each equation, then the two-dimensional version, and finally, the general, D-dimensional version. Likewise, when developing the solution methods for BVPs in Cartesian coordinates (Chapters 11-13), I confine the exposition to the interval $[0, \pi]$, the square $[0, \pi]^2$ and the cube $[0, \pi]^3$, and assume all the coefficients in the differential equations are unity. Then the exercises ask the student to state and prove the appropriate generalization of each solution method for an interval/rectangle/box of arbitrary dimensions, and for equations with arbitrary coefficients. The general method for solving I/BVPs using eigenfunction expansions only appears in Chapter 15, after many special cases of this method have been thoroughly exposited in Cartesian and polar coordinates (Chapters 11-14).

Likewise, the development of Fourier theory proceeds in gradually escalating levels of abstraction. First we encounter Fourier (co)sine series on the interval $[0, \pi]$ (§7A); then on the interval $[0, L]$ for arbitrary $L > 0$ (§7B). Then Chapter 8 introduces 'real' Fourier series (i.e. with both sine and cosine terms) and then complex Fourier series (§8D). Then, in Chapter 9 introduce 2-dimensional (co)sine series, and finally, D-dimensional (co)sine series.

Expositional clarity. Computer scientists have long known that it is easy to write software that *works*, but it is much more difficult (and important) to write working software that *other people* can understand. Similarly, it is relatively easy to write formally correct mathematics; the real challenge is to make the mathematics easy to read. To achieve this, I use several techniques. I divide proofs into semi-independent modules ("claims"), each of which performs a simple, clearly-defined task. I integrate these modules together in an explicit hierarchical structure (with "subclaims" inside of "claims"), so that their functional interdependence is clear from visual inspection. I also explain formal steps with parenthetical heuristic remarks. For example, in a long string of (in)equalities, I often attach footnotes to each step, as follows:

"$A \underset{(*)}{=} B \underset{(\dagger)}{\leq} C \underset{(\ddagger)}{<} D$. Here, $(*)$ is because [...]; $(\dagger)$ follows from [...], and $(\ddagger)$ is because [...]."

Finally, I use letters from the same 'lexicographical family' to denote objects which 'belong' together. For example: If $\mathcal{S}$ and $\mathcal{T}$ are sets, then elements of $\mathcal{S}$ should be $s_1, s_2, s_3, \ldots$, while elements of $\mathcal{T}$ are $t_1, t_2, t_3, \ldots$. If $\mathbf{v}$ is a vector, then its entries should be $v_1, \ldots, v_N$. If $\mathbf{A}$ is a matrix, then its entries should be $a_{11}, \ldots, a_{NM}$. I reserve upper-case letters (e.g. $J, K, L, M, N, \ldots$) for the bounds of intervals or indexing sets, and then use the corresponding lower-case letters (e.g. $j, k, l, m, n, \ldots$) as indexes. For example, $\forall n \in \{1, 2, \ldots, N\}$, $A_n := \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk}^n$.

Clear and explicit statements of solution techniques. Many PDEs text contain very few theorems; instead they try to develop the theory through a long sequence of worked examples, hoping that students will 'learn by imitation', and somehow absorb the important ideas 'by osmosis'. However, less gifted students often just imitate these worked examples in a slavish and uncomprehending way. Meanwhile, the more gifted students do not want to learn 'by osmosis'; they want clear and precise statements of the main ideas.

The problem is that most solution methods in PDEs, if stated as theorems in full generality, are incomprehensible to many students (especially the non-math majors). My solution is this: I provide explicit and precise statements of the solution-method for almost every possible combination of (1) several major

PDEs, (2) several kinds of boundary conditions, and (3) several different domains. I state these solutions as *theorems*, not as 'worked examples'. I follow each of these theorems with several completely worked examples. Some theorems I prove, but most of the proofs are left as exercises (often with step-by-step hints).

Of course, this approach is highly redundant, because I end up stating more than twenty theorems which are all really special cases of three or four general results (for example, the general method for solving the heat equation on a compact domain with Dirichlet boundary conditions, using an eigenfunction expansion). However, this sort of redundancy is *good* in an elementary exposition. Highly 'efficient' expositions are pleasing to our aesthetic sensibilities, but they are dreadful for pedagogical purposes.

However, one must not leave the students with the impression that the theory of PDEs is a disjointed collection of special cases. To link together all the 'homogeneous Dirichlet heat equation' theorems, for example, I explicitly point out that they all utilize the same underlying strategy. Also, when a proof of one variant is left as an exercise, I encourage students to imitate the (provided) proofs of previous variants. When the students understand the underlying similarity between the various special cases, *then* it is appropriate to state the general solution. The students will almost feel they have figured it out for themselves, which is the best way to learn something.

Suggested Twelve-Week Syllabus

Week 1: *Heat and Diffusion-related PDEs*

- **Lecture 1:** §0A-§0E *Review of multivariate calculus; intro. to complex numbers*
- **Lecture 2:** §1A-§1B *Fourier's Law; The heat equation*
- **Lecture 3:** §1C-§1D *Laplace Equation; Poisson's Equation*

Week 2: *Wave-related PDEs; Quantum Mechanics*

- **Lecture 1:** §1E; §2A *Properties of harmonic functions; Spherical Means*
- **Lecture 2:** §2B-§2C *wave equation; telegraph equation*
- **Lecture 3:** Chap.3 *The Schrödinger equation in quantum mechanics*

Week 3: *General Theory*

- **Lecture 1:** §4A - §4C *Linear PDEs: homogeneous vs. nonhomogeneous*
- **Lecture 2:** §5A; §5B, *Evolution equations & Initial Value Problems*
- **Lecture 3:** §5C *Boundary conditions and boundary value problems*

Week 4: *Background to Fourier Theory*

- **Lecture 1:** §5D *Uniqueness of solutions to BVPs*; §6A *Inner products*
- **Lecture 2:** §6B-§6D L^2 *space; Orthogonality*
- **Lecture 3:** §6E(a,b,c) L^2 *convergence; Pointwise convergence; Uniform Convergence*

Week 5: *One-dimensional Fourier Series*

- **Lecture 1:** §6E(d) *Infinite Series*; §6F *Orthogonal bases*
  §7A *Fourier (co/sine) Series: Definition and examples*
- **Lecture 2:** §7C(a,b,c,d,e) *Computing Fourier series of polynomials, piecewise linear and step functions*
- **Lecture 3:** §11A-§11C *Solution to heat equation & Poisson equation on line segment.*

Week 6: *Fourier Solutions for BVPs in One and Two dimensions*

- **Lecture 1:** §11B- §12A; *wave equation on line segment & Laplace equation on a square.*
- **Lecture 2:** §9A-§9B *Multidimensional Fourier Series.*
- **Lecture 3:** §12B- §12C(i) *Solution to heat equation & Poisson equation on a square*

Week 7: *Fourier solutions for 2-dimensional BVPs in Cartesian & Polar Coordinates*

- **Lecture 1:** §12C(ii), §12D *Solution to Poisson equation & wave equation on a square*
- **Lecture 2:** §5C(iv); §8A-§8B *Periodic Boundary Conditions; Real Fourier Series.*

Lecture 3: §14A; §14B(a,b,c,d) *Laplacian in Polar coordinates; Laplace Equation on (co)Disk.*

Week 8: *BVP's in Polar Coordinates; Bessel functions*

Lecture 1: §14C *Bessel Functions.*

Lecture 2: §14D-§14F *Heat, Poisson, and wave equations in Polar coordinates.*

Lecture 3: §14G *Solving Bessel's equation with the Method of Frobenius.*

Week 9: *Eigenbases; Separation of variables.*

Lecture 1: §15A-§15B *Eigenfunction solutions to BVPs*

Lecture 2: §15B; §16A-§16B *Harmonic bases. Separation of Variables in Cartesian coordinates.*

Lecture 3: §16C-§16D *Separation of variables in polar and spherical coordinates. Legendre Polynomials.*

Week 10: *Impulse Response Methods.*

Lecture 1: §17A - §17C *Impulse response functions; convolution. Approximations of identity. Gaussian Convolution Solution for heat equation.*

Lecture 2: §17C-§17F, *Gaussian convolutions continued. Poisson's Solutions to Dirichlet problem on a half-plane and a disk.*

Lecture 3: §14B(v); §17D *Poisson solution on disk via polar coordinates; d'Alembert Solution to wave equation.*

Week 11: *Fourier Transforms.*

Lecture 1: §19A *One-dimensional Fourier Transforms.*

Lecture 2: §19B *Properties of one-dimensional Fourier transform.*

Lecture 3: §20A ; §20C *Fourier transform solution to heat equation; Dirchlet problem on Half-plane.*

Week 12: *Fourier Transform Solutions to PDEs.*

Lecture 1: §19D, §20B(i) *Multidimensional Fourier transforms; Solution to wave equation.*

Lecture 2: §20B(ii); §20E *Poisson's Spherical Mean Solution; Huygen's Principle. The General Method.*

Lecture 3: (Time permitting) §19G or §19H (*Heisenberg Uncertainty* or *Laplace transforms*).

In a longer semester or a faster paced course, one could also cover parts of Chapter 10 (*Proofs of Fourier Convergence*) and/or Chapter 18 (*Applications of Complex Analysis*)

I Motivating examples and major applications

A *dynamical system* is a mathematical model of a system evolving in time. Most models in mathematical physics are dynamical systems. If the system has only a finite number of 'state variables', then its dynamics can be encoded in an *ordinary differential equation* (ODE), which expresses the *time derivative* of each state variable (i.e. its rate of change over time) as a function of the other state variables. For example, *celestial mechanics* concerns the evolution of a system of gravitationally interacting objects (e.g. stars and planets). In this case, the 'state variables' are vectors encoding the position and momentum of each object, and the ODE describe how the objects move and accelerate as they gravitationally interact.

However, if the system has a very large number of state variables, then it is no longer feasible to represent it with an ODE. For example, consider the flow of heat or the propagation of compression waves through a steel bar containing 10^{24} iron atoms. We *could* model this using a 10^{24}-dimensional ODE, where we explicitly track the vibrational motion of each iron atom. However, such a 'microscopic' model would be totally intractable. Furthermore, it isn't necessary. The iron atoms are (mostly) immobile, and interact only with their immediate neighbours. Furthermore, nearby atoms generally have roughly the same temperature, and move in synchrony. Thus, it suffices to consider the macroscopic *temperature distribution* of the steel bar, or study the fluctuation of a macroscopic *density field*. This temperature distribution or density field can be mathematically represented as a smooth, real-valued function over some three-dimensional domain. The flow of heat or the propagation of sound can then be described as the *evolution* of this function over time.

We now have a dynamical system where the 'state variable' is not a finite system of vectors (as in celestial mechanics), but is instead a multivariate *function*. The evolution of this function is determined by its spatial geometry —e.g. the local 'steepness' and variation of the temperature gradients between warmer and cooler regions in the bar. In other words, the *time derivative* of the function (its rate of change over time) is determined by its *spatial derivatives* (which describe its slope and curvature at each point in space). An equation which relates the different derivatives of a multivariate function in this way is a *partial differential equation* (PDE). In particular, a PDE which describes a dynamical system is called an *evolution equation*. For example, the evolution equation which describes the flow of heat through a solid is called the *heat equation*. The equation which describes compression waves is the *wave equation*.

An *equilibrium* of a dynamical system is a state which is unchanging over time; mathematically, this means that the time-derivative is equal to zero. An equlibrium of an N-dimensional evolution equation satisfies an $(N-1)$-dimensional PDE called an *equilibrium equation*. For example, the equilibrium equations corresponding to the heat equation are the *Laplace equation* and the *Poisson equation* (depending on whether or not the system is subjected to external heat input).

PDEs are thus of central importance in the thermodynamics and acoustics of continuous media (e.g. steel bars). The heat equation also describes chemical diffusion in fluids, and also the evolving probability distribution of a particle performing a random walk called *Brownian motion*. It thus finds applications everywhere from mathematical biology to mathematical finance. When diffusion or Brownian motion is combined with deterministic drift (e.g. due to prevailing wind or ocean currents) it becomes a PDE called the *Fokker-Planck equation*.

The Laplace and Poisson equations describe the equilibria of such diffusion processes. They also arise in electrostatics, where they describe the shape of an electric field in a vacuum. Finally, solutions of the two-dimensional Laplace equation are good approximations of surfaces trying to minimize their elastic potential energy —that is, soap films.

The wave equation describes the resonance of a musical instrument, the spread of ripples on a pond, seismic waves propagating through the earth's crust, and shockwaves in solar plasma. (The motion of fluids themselves is described by yet another PDE, the *Navier-Stokes* equation). A version of the wave equation arises as a special case of Maxwell's equations of electrodynamics; this led to Maxwell's prediction of *electromagnetic waves*, which include radio, microwaves, X-rays, and visible light. When combined with a 'diffusion' term reminiscent of the heat equation, the wave equation becomes the *telegraph equation*, which describes the propagation and degradation of electrical signals travelling through a wire.

Finally, an odd-looking 'complex' version of the heat equation induces wave-like evolution in the complex-valued probability fields which describe the position and momentum of subatomic particles. This *Schrödinger equation* is the starting point of quantum mechanics, one of the two most revolutionary developments in physics in the twentieth century. The other revolutionary development was relativity theory. General relativity represents spacetime as a four-dimensional manifold, whose curvature interacts with the spatiotemporal flow of mass/energy through yet another PDE: the *Einstein equation*.

Except for the Einstein and Navier-Stokes equations, all the equations we have mentioned are *linear* PDEs. This means that a sum of two or more solutions to the PDE will also be a solution. This allows us to solve linear PDEs through the *method of superposition*: we build complex solutions by adding together many simple solutions. A particularly convenient class of simple solutions are *eigenfunctions*. Thus, an enormously powerful and general method for linear PDEs is to represent the solutions using *eigenfunction expansions*. The most natural eigenfunction expansion (in Cartesian coordinates) is the *Fourier series*.

Chapter 1

Heat and diffusion

"The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory." —Jean Joseph Fourier

1A Fourier's law

Prerequisites: §0A. **Recommended:** §0E.

1A(i) ...in one dimension

Figure 1A.1 depicts a material diffusing through a one-dimensional domain $\mathbb{X}$ (for example, $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0, L]$). Let $u(x, t)$ be the density of the material at the point $x \in \mathbb{X}$ at time $t > 0$. Intuitively, we expect the material to flow from regions of *greater* to *lesser* concentration. In other words, we expect the *flow* of the material at any point in space to be proportional to the *slope* of the curve $u(x, t)$ at that point. Thus, if $F(x, t)$ is the flow at the point x at time t, then

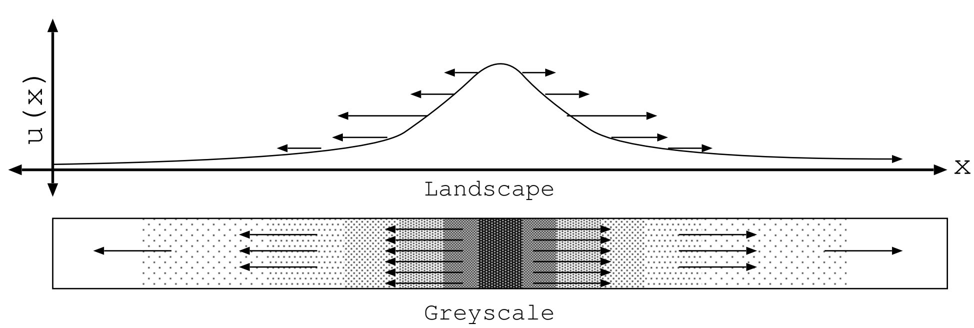
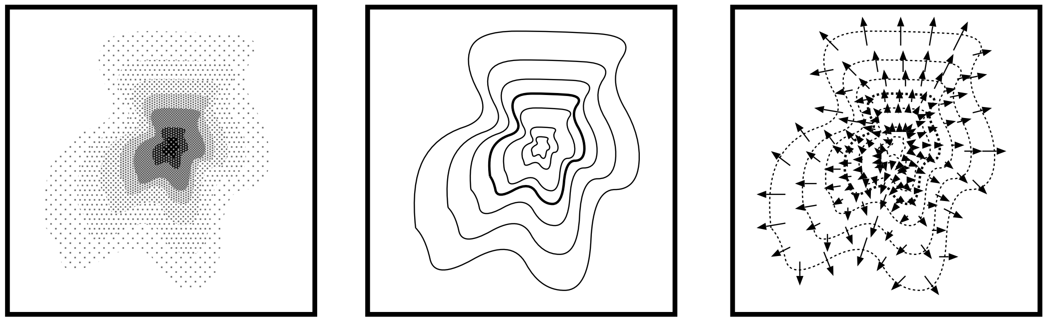


Figure 1A.1: Fourier's Law of Heat Flow in one dimension



Grayscale Temperature Distribution | Isothermal Contours | Heat Flow Vector Field

Figure 1A.2: Fourier's Law of Heat Flow in two dimensions

we expect:

$$F(x,t) \quad = \quad -\kappa \cdot \partial_x \, u(x,t)$$

where $\kappa > 0$ is a constant measuring the rate of diffusion. This is an example of **Fourier's Law.**

1A(ii) ...in many dimensions

Prerequisites: §0E.

Figure 1A.2 depicts a material diffusing through a two-dimensional domain $\mathbb{X} \subset \mathbb{R}^2$ (e.g. heat spreading through a region, ink diffusing in a bucket of water, etc.). We could just as easily suppose that $\mathbb{X} \subset \mathbb{R}^D$ is a D-dimensional domain. If $\mathbf{x} \in \mathbb{X}$ is a point in space, and $t \geq 0$ is a moment in time, let $u(\mathbf{x}, t)$ denote the concentration at $\mathbf{x}$ at time t. (This determines a function $u : \mathbb{X} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$, called a *time-varying scalar field.*)

Now let $\vec{\mathbf{F}}(\mathbf{x}, t)$ be a D-dimensional vector describing the *flow* of the material at the point $\mathbf{x} \in \mathbb{X}$. (This determines a *time-varying vector field* $\vec{\mathbf{F}} : \mathbb{R}^D \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}^D$.)

Again, we expect the material to flow from regions of high concentration to low concentration. In other words, material should flow *down the concentration gradient.* This is expressed by **Fourier's Law of Heat Flow** , which says:

$$\vec{\mathbf{F}} \quad = \quad -\kappa \cdot \nabla u,$$

where $\kappa > 0$ is is a constant measuring the rate of diffusion.

One can imagine u as describing a distribution of highly antisocial people; each person is always fleeing everyone around them and moving in the direction with the fewest people. The constant κ measures the average walking speed of these misanthropes.

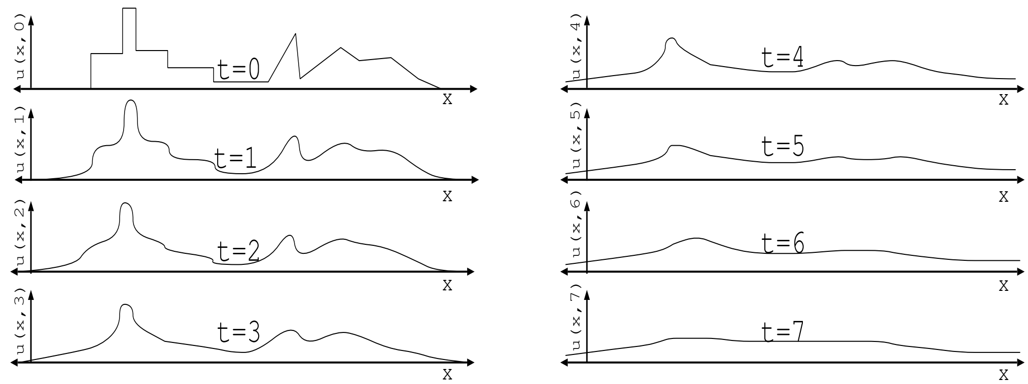


Figure 1B.1: The heat equation as "erosion".

1B The heat equation

Recommended: §1A.

1B(i) ...in one dimension

Prerequisites: §1A(i).

Consider a material diffusing through a one-dimensional domain $\mathbb{X}$ (for example, $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0, L]$). Let $u(x,t)$ be the density of the material at the location $x \in \mathbb{X}$ at time $t \in \mathbb{R}_{\not -}$, and let $F(x,t)$ be the flux of the material at the location x and time t. Consider the derivative $\partial_x F(x,t)$. If $\partial_x F(x,t) > 0$, this means that the flow is *diverging*[1] at this point in space, so the material there is spreading farther apart. Hence, we expect the concentration at this point to *decrease*. Conversely, if $\partial_x F(x,t) < 0$, then the flow is *converging* at this point in space, so the material there is crowding closer together, and we expect the concentration to *increase*. To be succinct: the concentration of material will *increase* in regions where F converges, and *decrease* in regions where F diverges. The equation describing this is:

$$\partial_t u(x,t) \quad = \quad -\partial_x F(x,t).$$

If we combine this with Fourier's Law, however, we get:

$$\partial_t u(x,t) \quad = \quad \kappa \cdot \partial_x \partial_x u(x,t),$$

which yields the **one-dimensional heat equation:**

$$\boxed{\partial_t u(x,t) \quad = \quad \kappa \cdot \partial_x^2 u(x,t).}$$

[1]See § 0E(ii) on page 558 for an explanation of why we say the flow is 'diverging' here.

Heuristically speaking, if we imagine $u(x,t)$ as the height of some one-dimensional "landscape", then the heat equation causes this landscape to be "eroded", as if it were subjected to thousands of years of wind and rain (see Figure 1B.1).

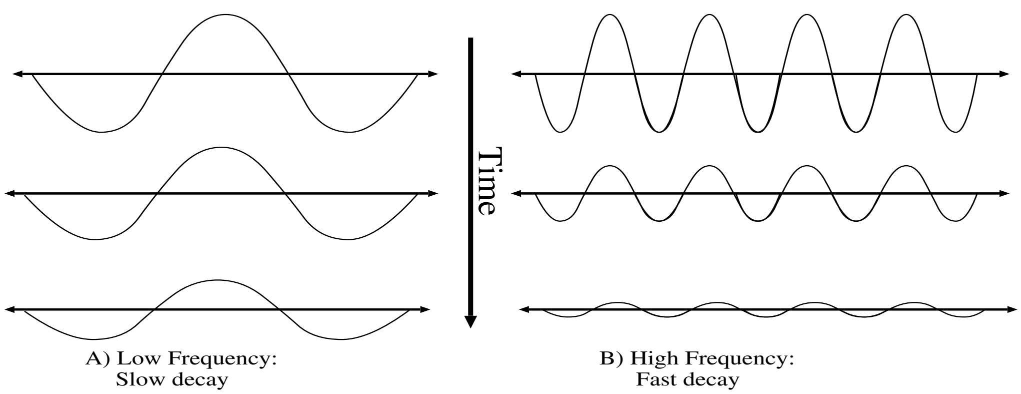


Figure 1B.2: Under the heat equation, the exponential decay of a periodic function is proportional to the square of its frequency.

Example 1B.1. For simplicity we suppose $\kappa = 1$.

(a) Let $u(x,t) = e^{-9t} \cdot \sin(3x)$. Thus, u describes a spatially sinusoidal function (with spatial frequency 3) whose magnitude decays exponentially over time.

(b) **The dissipating wave:** More generally, let $u(x,t) = e^{-\omega^2 \cdot t} \cdot \sin(\omega \cdot x)$. Then u is a solution to the one-dimensional heat equation, and looks like a standing wave whose amplitude decays exponentially over time (see Figure 1B.2). Notice that the decay rate of the function u is proportional to the square of its frequency.

(c) **The (one-dimensional) Gauss-Weierstrass Kernel:** Let

$$\mathcal{G}(x;t) \quad := \quad \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right).$$

Then $\mathcal{G}$ is a solution to the one-dimensional heat equation, and looks like a "bell curve", which starts out tall and narrow, and over time becomes broader and flatter (Figure 1B.3). ♢

Ⓔ **Exercise 1B.1.** Verify that the functions in Examples 1B.1(a,b,c) all satisfy the heat equation. ♦

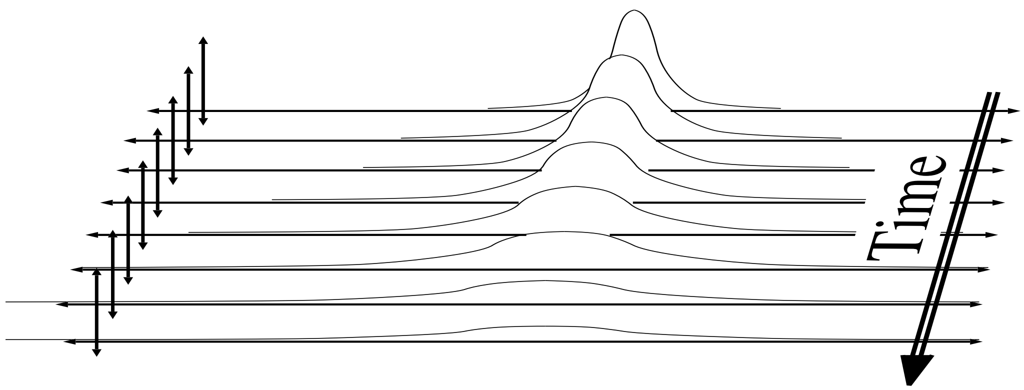


Figure 1B.3: The Gauss-Weierstrass kernel under the heat equation.

All three functions in Examples 1B.1 starts out very tall, narrow, and pointy, and gradually become shorter, broader, and flatter. This is generally what the heat equation does; it tends to flatten things out. If u describes a physical landscape, then the heat equation describes "erosion".

1B(ii) ...in many dimensions

Prerequisites: §1A(ii).

More generally, if $u : \mathbb{R}^D \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is the time-varying density of some material, and $\vec{\mathbf{F}} : \mathbb{R}^D \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is the flux of this material, then we would expect the material to *increase* in regions where $\vec{\mathbf{F}}$ converges, and to *decrease* in regions where $\vec{\mathbf{F}}$ diverges.[2] In other words, we have:

$$\partial_t u \;=\; -\mathrm{div}\,\vec{\mathbf{F}}.$$

If u is the density of some diffusing material (or heat), then $\vec{\mathbf{F}}$ is determined by **Fourier's Law**, so we get the **heat equation**

$$\partial_t u \;=\; \kappa \cdot \mathrm{div}\,\nabla u \;=\; \kappa \triangle u.$$

Here, $\triangle$ is the **Laplacian** operator[3], defined:

$$\boxed{\triangle u \;=\; \partial_1^2 u + \partial_2^2 u + \ldots \partial_D^2 u}$$

Exercise 1B.2. (a) If $D = 1$, and $u : \mathbb{R} \longrightarrow \mathbb{R}$, verify that $\mathrm{div}\,\nabla u(x) = u''(x) = \triangle u(x)$, for all $x \in \mathbb{R}$. Ⓔ

[2] See § 0E(ii) on page 558 for a review of the 'divergence' of a vector field.
[3] Sometimes the Laplacian is written as " ∇^2 ".

(b) If $D = 2$, and $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$, verify that $\operatorname{div} \nabla u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = \triangle u(x,y)$, for all $(x,y) \in \mathbb{R}^2$.

(c) For any $D \geq 2$ and $u : \mathbb{R}^D \longrightarrow \mathbb{R}$, verify that $\operatorname{div} \nabla u(\mathbf{x}) = \triangle u(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^D$.

♦

By changing to the appropriate time units, we can assume $\kappa = 1$, so the **heat equation** becomes:

$$\boxed{\partial_t u \;=\; \triangle u}.$$

For example,

- If $\mathbb{X} \subset \mathbb{R}$, and $x \in \mathbb{X}$, then $\triangle u(x;t) \;=\; \partial_x^2 u(x;t)$.
- If $\mathbb{X} \subset \mathbb{R}^2$, and $(x,y) \in \mathbb{X}$, then $\triangle u(x,y;t) \;=\; \partial_x^2 u(x,y;t) + \partial_y^2 u(x,y;t)$.

Thus, as we've already seen, the one-dimensional heat equation is

$$\partial_t u \quad = \quad \partial_x^2 u$$

and the the **two dimensional heat equation** is:

$$\boxed{\partial_t u(x,y;t) \quad = \quad \partial_x^2 u(x,y;t) \;+\; \partial_y^2 u(x,y;t)}$$

Example 1B.2.

(a) Let $u(x,y;t) \;=\; e^{-25\,t} \cdot \sin(3x)\sin(4y)$. Then u is a solution to the two-dimensional heat equation, and looks like a two-dimensional 'grid' of sinusoidal hills and valleys with horizontal spacing 1/3 and vertical spacing 1/4. As shown in Figure 1B.4, these hills rapidly subside into a gently undulating meadow, and then gradually sink into a perfectly flat landscape.

(b) **The (two-dimensional) Gauss-Weierstrass Kernel:** Let

$$\mathcal{G}(x,y;\; t) \quad := \quad \frac{1}{4\pi t} \exp\left(\frac{-x^2 - y^2}{4t}\right).$$

Then $\mathcal{G}$ is a solution to the two-dimensional heat equation, and looks like a mountain, which begins steep and pointy, and gradually "erodes" into a broad, flat, hill.

(c) **The D-dimensional Gauss-Weierstrass Kernel** is the function $\mathcal{G} : \mathbb{R}^D \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined

$$\boxed{\mathcal{G}(\mathbf{x};t) \;=\; \frac{1}{(4\pi t)^{D/2}} \exp\left(\frac{-\left\|\mathbf{x}\right\|^2}{4t}\right)}$$

Technically speaking, $\mathcal{G}(\mathbf{x};t)$ is a D-dimensional *symmetric normal probability distribution* with variance $\sigma = 2t$. ♢

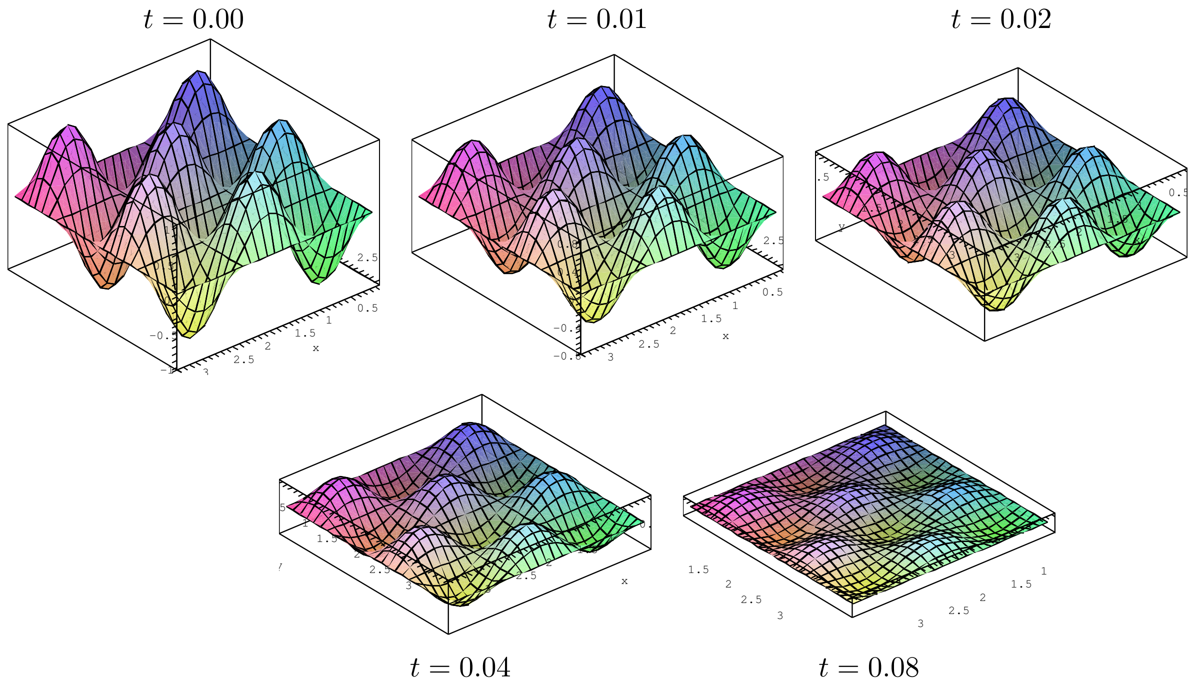


Figure 1B.4: Five snapshots of the function $u(x, y; t) = e^{-25\,t} \cdot \sin(3x)\sin(4y)$ from Example 1B.2.

Exercise 1B.3. Verify that the functions in Examples 1B.2(a,b,c) all satisfy the heat equation. ♦

Exercise 1B.4. Prove the *Leibniz rule* for Laplacians: if $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ are two scalar fields, and $(f \cdot g) : \mathbb{R}^D \longrightarrow \mathbb{R}$ is their product, then for all $\mathbf{x} \in \mathbb{R}^D$,

$$\triangle(f \cdot g)(\mathbf{x}) \quad = \quad g(\mathbf{x}) \cdot \Big(\triangle\, f(\mathbf{x}) \Big) \;+\; 2 \Big(\nabla f(\mathbf{x}) \Big) \bullet \Big(\nabla g(\mathbf{x}) \Big) \;+\; f(\mathbf{x}) \cdot \Big(\triangle\, g(\mathbf{x}) \Big).$$

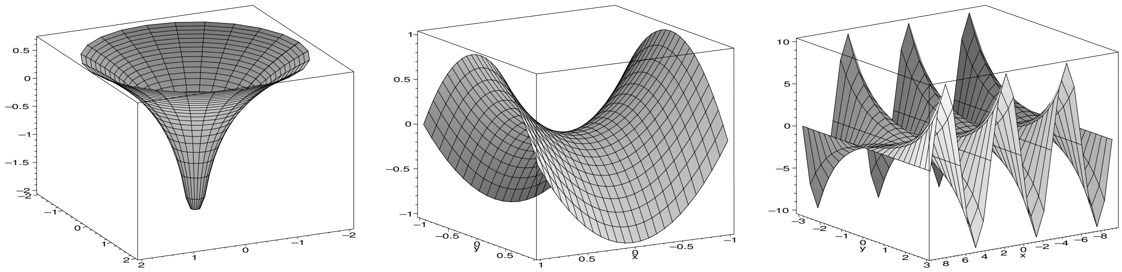
Hint: Combine the Leibniz rules for gradients and divergences (Propositions 0E.1(b) and 0E.2(b) on pages 558 and 560). ♦

1C Laplace's equation

Prerequisites: §1B.

If the heat equation describes the erosion/diffusion of some system, then an **equilibrium** or **steady-state** of the heat equation is a scalar field $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ satisfying **Laplace's Equation:**

$$\boxed{\triangle h \equiv 0.}$$



(A) **(B)** **(C)**

Figure 1C.1: Three harmonic functions: **(A)** $h(x,y) = \log(x^2 + y^2)$. **(B)** $h(x,y) = x^2 - y^2$. **(C)** $h(x,y) = \sin(x) \cdot \sinh(y)$. In all cases, note the telltale "saddle" shape.

A scalar field satisfying the Laplace equation is called a **harmonic function**.

Example 1C.1.

(a) If $D = 1$, then $\triangle h(x) = \partial_x^2 h(x) = h''(x)$; thus, the **one-dimensional Laplace equation** is just

$$\boxed{h''(x) = 0}$$

Suppose $h(x) = 3x + 4$. Then $h'(x) = 3$, and $h''(x) = 0$, so h is harmonic. More generally: the one-dimensional harmonic functions are just the *linear* functions of the form: $h(x) = ax + b$ for some constants $a, b \in \mathbb{R}$.

(b) If $D = 2$, then $\triangle h(x,y) = \partial_x^2 h(x,y) + \partial_y^2 h(x,y)$, so the **two-dimensional Laplace equation** reads:

$$\boxed{\partial_x^2 h + \partial_y^2 h \quad = \quad 0,}$$

or, equivalently, $\partial_x^2 h = -\partial_y^2 h$. For example:

- Figure 1C.1(B) shows the harmonic function $h(x,y) = x^2 - y^2$.
- Figure 1C.1(C) shows the harmonic function $h(x,y) = \sin(x) \cdot \sinh(y)$.

Ⓔ <u>**Exercise 1C.1**</u> Verify that these two functions are harmonic. ♢

The surfaces in Figure 1C.1 have a "saddle" shape, and this is typical of harmonic functions; in a sense, a harmonic function is one which is "saddle-shaped" at every point in space. In particular, notice that $h(x,y)$ has no maxima or minima anywhere; this is a universal property of harmonic functions (see Corollary 1E.2 on page 17). The next example seems to contradict this assertion, but in fact it doesn't...

Example 1C.2. Figure 1C.1(A) shows the harmonic function $h(x, y) = \log(x^2 + y^2)$ for all $(x, y) \neq (0, 0)$. This function is well-defined everywhere except at $(0, 0)$; hence, contrary to appearances, $(0, 0)$ is *not* an extremal point. [Verifying that h is harmonic is problem # 3 on page 20]. ♢

When $D \geq 3$, harmonic functions no longer define nice saddle-shaped *surfaces*, but they still have similar mathematical properties.

Example 1C.3.

(a) If $D = 3$, then $\triangle h(x, y, z) = \partial_x^2 h(x, y, z) + \partial_y^2 h(x, y, z) + \partial_z^2 h(x, y, z)$.

Thus, the **three-dimensional Laplace equation** reads:

$$\boxed{\partial_x^2 h + \partial_y^2 h + \partial_z^2 h \quad = \quad 0,}$$

For example, let $h(x, y, z) = \dfrac{1}{\|(x, y, z)\|} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ for all $(x, y, z) \neq (0, 0, 0)$. Then h is harmonic everywhere except at $(0, 0, 0)$.

[Verifying that h is harmonic is problem # 4 on page 21.]

(b) For any $D \geq 3$, the D**-dimensional Laplace equation** reads:

$$\boxed{\partial_1^2 h + \ldots + \partial_D^2 h \quad = \quad 0.}$$

For example, let $h(\mathbf{x}) = \dfrac{1}{\|\mathbf{x}\|^{D-2}} = \dfrac{1}{\left(x_1^2 + \cdots + x_D^2\right)^{\frac{D-2}{2}}}$ for all $\mathbf{x} \neq \mathbf{0}$.

Then h is harmonic everywhere everywhere in $\mathbb{R}^D \setminus \{\mathbf{0}\}$ (**Exercise 1C.2** Ⓔ Verify that h is harmonic on $\mathbb{R}^D \setminus \{\mathbf{0}\}$.) ♢

Harmonic functions have the convenient property that we can multiply together two lower-dimensional harmonic functions to get a higher dimensional one. For example:

- $h(x, y) = x \cdot y$ is a two-dimensional harmonic function (**Exercise 1C.3** Ⓔ Verify this).

- $h(x, y, z) = x \cdot (y^2 - z^2)$ is a three-dimensional harmonic function (**Exercise 1C.4** Verify this). Ⓔ

In general, we have the following:

Proposition 1C.4. *Suppose $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic and $v : \mathbb{R}^m \longrightarrow \mathbb{R}$ is harmonic, and define $w : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$ by $w(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) \cdot v(\mathbf{y})$ for $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Then w is also harmonic*

Proof. **Exercise 1C.5** Hint: First prove that w obeys a kind of Leibniz rule: $\triangle w(\mathbf{x}, \mathbf{y}) = v(\mathbf{y}) \cdot \triangle u(\mathbf{x}) + u(\mathbf{x}) \cdot \triangle v(\mathbf{y})$. □ Ⓔ

The function $w(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) \cdot v(\mathbf{y})$ is called a *separated solution*, and this theorem illustrates a technique called *separation of variables*. The next exercise also explores separation of variables. A full exposition of this technique appears in Chapter 16 on page 353.

Ⓔ **Exercise 1C.6.** (a) Let $\mu, \nu \in \mathbb{R}$ be constants, and let $f(x, y) = e^{\mu x} \cdot e^{\nu y}$. Suppose f is harmonic; what can you conclude about the relationship between μ and ν? (Justify your assertion).

(b) Suppose $f(x, y) = X(x) \cdot Y(y)$, where $X : \mathbb{R} \longrightarrow \mathbb{R}$ and $Y : \mathbb{R} \longrightarrow \mathbb{R}$ are two smooth functions. Suppose $f(x, y)$ is harmonic

[i] Prove that $\dfrac{X''(x)}{X(x)} = \dfrac{-Y''(y)}{Y(y)}$ for all $x, y \in \mathbb{R}$.

[ii] Conclude that the function $\dfrac{X''(x)}{X(x)}$ must equal a constant c independent of x. Hence $X(x)$ satisfies the ordinary differential equation $X''(x) = c \cdot X(x)$.

Likewise, the function $\dfrac{Y''(y)}{Y(y)}$ must equal $-c$, independent of y. Hence $Y(y)$ satisfies the ordinary differential equation $Y''(y) = -c \cdot Y(y)$.

[iii] Using this information, deduce the general form for the functions $X(x)$ and $Y(y)$, and use this to obtain a general form for $f(x, y)$. ♦

1D The Poisson equation

Prerequisites: §1C.

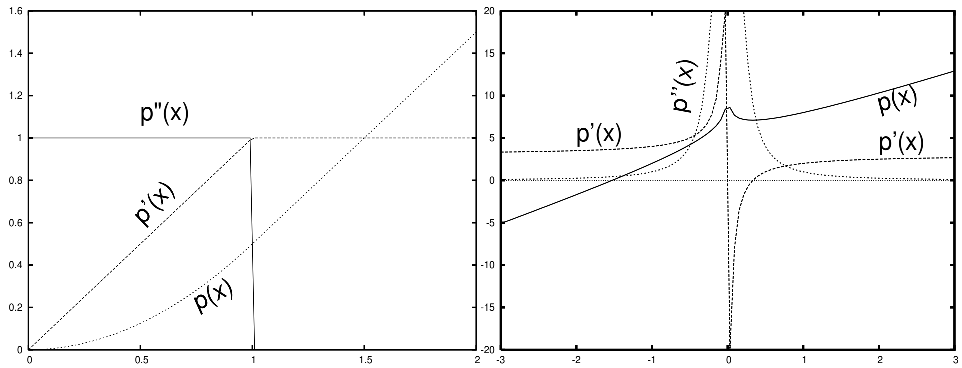
Imagine $p(\mathbf{x})$ is the concentration of a chemical at the point $\mathbf{x}$ in space. Suppose this chemical is being *generated* (or *depleted*) at different rates at different regions in space. Thus, in the absence of diffusion, we would have the **generation equation**

$$\partial_t\, p(\mathbf{x}, t) \;=\; q(\mathbf{x}),$$

where $q(\mathbf{x})$ is the rate at which the chemical is being created/destroyed at $\mathbf{x}$ (we assume that q is constant in time).

If we now included the effects of diffusion, we get the **generation-diffusion equation:**

$$\partial_t\, p \;=\; \kappa \triangle p \;+ q.$$



$$p''(x) = Q(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$p'(x) = \begin{cases} x & \text{if } 0 < x < 1; \\ 1 & \text{otherwise.} \end{cases}$$

$$p(x) = \begin{cases} x^2/2 & \text{if } 0 < x < 1; \\ x - \frac{1}{2} & \text{otherwise.} \end{cases}$$

(**A**)

$$p''(x) = Q(x) = 1/x^2;$$

$$p'(x) = -1/x + 3;$$

$$p(x) = -\log|x| + 3x + 5.$$

(**B**)

Figure 1D.1: Two one-dimensional potentials.

A *steady state* of this equation is a scalar field p satisfying **Poisson's Equation:**

$$\boxed{\triangle p = Q.}$$

where $Q(\mathbf{x}) \quad = \quad \dfrac{-q(\mathbf{x})}{\kappa}.$

Example 1D.1: One-Dimensional Poisson Equation

If $D = 1$, then $\triangle p(x) = \partial_x^2\, p(x) = p''(x)$; thus, the **one-dimensional Poisson equation** is just

$$\boxed{p''(x) = Q(x).}$$

We can solve this equation by twice-integrating the function $Q(x)$. If $p(x) = \int\int Q(x)$ is some double-antiderivative of G, then p clearly satisfies the Poisson equation. For example:

(a) Suppose $Q(x) = \begin{cases} 1 & \text{if } \ 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$. Then define

$$p(x) \quad = \quad \int_0^x \int_0^y q(z)\, dz\, dy \quad = \quad \begin{cases} 0 & \text{if } \ x < 0; \\ x^2/2 & \text{if } \ 0 < x < 1; \\ x - \frac{1}{2} & \text{if } \ 1 < x. \end{cases} \qquad \text{(Figure 1D.1A)}$$

(b) If $Q(x) = 1/x^2$ (for $x \neq 0$), then $p(x) \;=\; \int\int Q(x) \;=\; -\log|x| + ax + b$ (for $x \neq 0$), where $a, b \in \mathbb{R}$ are arbitrary constants. (see Figure 1D.1B) $\diamondsuit$

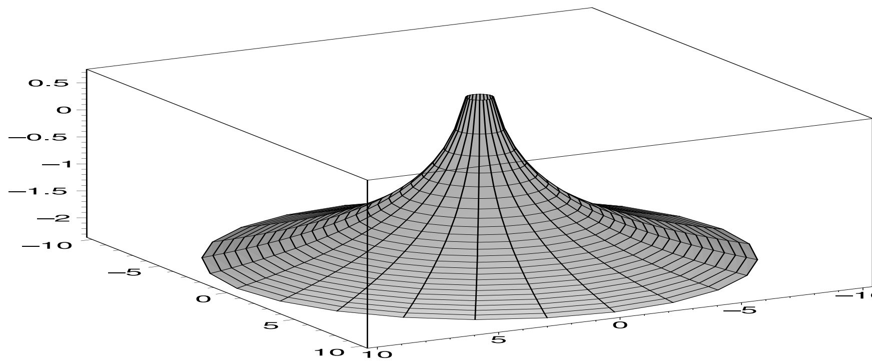


Figure 1D.2: The two-dimensional potential field generated by a concentration of charge at the origin.

Ⓔ **Exercise 1D.1.** Verify that the functions $p(x)$ in Examples (a) and (b) are both solutions to their respective Poisson equations. ♦

Example 1D.2: Electrical/Gravitational Fields

Poisson's equation also arises in classical field theory[4]. Suppose, for any point $\mathbf{x} = (x_1, x_2, x_3)$ in three-dimensional space, that $q(\mathbf{x})$ is charge density at $\mathbf{x}$, and that $p(\mathbf{x})$ is the the electric potential field at $\mathbf{x}$. Then we have:

$$\triangle\, p(\mathbf{x}) \;=\; \kappa\, q(\mathbf{x}) \qquad (\kappa \text{ some constant}) \tag{1D.1}$$

If $q(\mathbf{x})$ were the *mass* density at $\mathbf{x}$, and $p(\mathbf{x})$ were the *gravitational* potential energy, then we would get the same equation. (See Figure 1D.2 for an example of such a potential in two dimensions).

In particular, in a region where there is no charge/mass (i.e. where $q \equiv 0$), equation (1D.1) reduces to the Laplace equation $\triangle p \equiv 0$. Because of this, solutions to the Poisson equation (and especially the Laplace equation) are sometimes called **potentials**. ♢

Example 1D.3: The Coulomb Potential

Let $D = 3$, and let $p(x, y, z) \;=\; \dfrac{1}{\|(x, y, z)\|} \;=\; \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$. In Example 1C.3(a), we asserted that $p(x, y, z)$ was harmonic everywhere except at $(0, 0, 0)$, where it is not well-defined. For physical reasons, it is 'reasonable' to write the equation:

$$\triangle\, p(0, 0, 0) \quad = \quad \delta_0, \tag{1D.2}$$

[4] For a quick yet lucid introduction to electrostatics, see [Ste95, Chap.3].

where δ_0 is the 'Dirac delta function' (representing an infinite concentration of charge at zero)[5]. Then $p(x, y, z)$ describes the electric potential generated by a *point charge.* $\diamondsuit$

Exercise 1D.2. Check that $\nabla p(x, y, z) = \frac{-(x, y, z)}{\|(x, y, z)\|^3}$. This is the electric field generated by a point charge, as given by *Coulomb's Law* from classical electrostatics. $\blacklozenge$ Ⓔ

Exercise 1D.3. (a) Let $q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar field describing a charge density distribution. If $\vec{\mathbf{E}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the electric field generated by q, then *Gauss's law* saws $\operatorname{div} \vec{\mathbf{E}} = \kappa\, q$, where κ is a constant. If $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the electric potential field associated with $\vec{\mathbf{E}}$, then by definition, $\vec{\mathbf{E}} = \nabla p$. Use these facts to derive equation (1D.1). Ⓔ

(b) Suppose q is independent of the x_3 coordinate; that is, $q(x_1, x_2, x_3) = Q(x_1, x_2)$ for some function $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Show that p is also is independent of the x_3 coordinate; that is, $p(x_1, x_2, x_3) = P(x_1, x_2)$ for some function $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Show P and Q satisfy the *two*-dimensional version of the Poisson equation —that is that $\triangle P = \kappa\, Q$.

(This is significant because many physical problems have (approximate) translational symmetry along one dimension (e.g. an electric field generated by a long, uniformly charged wire or plate). Thus, we can reduce the problem to two dimensions, where powerful methods from complex analysis can be applied; see Section 18B on page 422.)
$\blacklozenge$

Notice that the electric/gravitational potential field is *not uniquely defined* by equation (1D.1). If $p(\mathbf{x})$ solves the Poisson equation (1D.1), then so does $\widetilde{p}(\mathbf{x}) = p(\mathbf{x}) + a$ for any constant $a \in \mathbb{R}$. Thus, we say that the potential field is well-defined *up to addition of a constant*; this is similar to the way in which the antiderivative $\int Q(x)$ of a function is only well-defined up to some constant.[6] This is an example of a more general phenomenon:

Proposition 1D.4. *Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let $p : \mathbb{X} \longrightarrow \mathbb{R}$ and $h : \mathbb{X} \longrightarrow \mathbb{R}$ be two functions on $\mathbb{X}$. Let $\widetilde{p}(\mathbf{x}) := p(\mathbf{x}) + h(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$. Suppose that h is harmonic —i.e. $\triangle h \equiv 0$. If p satisfies the Poisson Equation "$\triangle p \equiv q$", then $\widetilde{p}$ also satisfies this Poisson equation.*

Proof. **Exercise 1D.4** Hint: Notice that $\triangle\widetilde{p}(\mathbf{x}) = \triangle p(\mathbf{x}) + \triangle h(\mathbf{x})$. $\square$ Ⓔ

For example, if $Q(x) = 1/x^2$, as in Example 1D.1(b), then $p(x) = -\log(x)$ is a solution to the Poisson equation "$p''(x) = 1/x^2$". If $h(x)$ is a one-dimensional

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[5]Equation (1D.2) seems mathematically nonsensical, but it *can* be made mathematically meaningful, using *distribution theory.* However, this is far beyond the scope of this book, so for our purposes, we will interpret eqn. (1D.2) as purely metaphorical.

[6]For the purposes of the physical theory, this constant *does not matter*, because the field p is physically interpreted only by computing the *potential difference* between two points, and the constant a will always cancel out in this computation. Thus, the two potential fields $p(\mathbf{x})$ and $\widetilde{p}(\mathbf{x}) = p(\mathbf{x}) + a$ will generate identical physical predictions.

harmonic function, then $h(x) = ax + b$ for some constants a and b (see Example 1C.1(a) on page 10). Thus $\widetilde{p}(x) = -\log(x) + ax + b$, and we've already seen that these are also valid solutions to this Poisson equation.

1E Properties of harmonic functions

Prerequisites: §1C, §0H(ii). **Prerequisites** (for proofs): §2A, §17G, §0E(iii).

Recall that a function $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ is **harmonic** if $\triangle h \equiv 0$. Harmonic functions have nice geometric properties, which can be loosely summarized as 'smooth and gently curving'.

Theorem 1E.1. Mean Value Theorem
Let $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a scalar field. Then f is harmonic if and only if f is integrable, and:

$$\text{For any } \mathbf{x} \in \mathbb{R}^D, \text{ and any } R > 0, \quad f(\mathbf{x}) \;=\; \frac{1}{A(R)} \int_{\mathbb{S}(\mathbf{x};R)} f(\mathbf{s})\, d\mathbf{s}. \tag{1E.1}$$

Here, $\mathbb{S}(\mathbf{x};R) := \left\{\mathbf{s} \in \mathbb{R}^D \; ; \; \|\mathbf{s} - \mathbf{x}\| = R\right\}$ is the $(D-1)$-dimensional **sphere** *of radius R around $\mathbf{x}$, and $A(R)$ is the $(D-1)$-dimensional surface area of $\mathbb{S}(\mathbf{x};R)$.*

Ⓔ *Proof.* <u>**Exercise 1E.1**</u> (a) Suppose f is integrable and statement (1E.1) is true. Use the **Spherical Means** formula for the Laplacian (Theorem 2A.1) to show that f is harmonic.

(b) Now, suppose f is harmonic. Define $\phi : \mathbb{R}_{\neq} \longrightarrow \mathbb{R}$ by: $\phi(R) \;:=\; \frac{1}{A(R)} \int_{\mathbb{S}(\mathbf{x};R)} f(\mathbf{s})\, d\mathbf{s}$.

Show that $\phi'(R) \;=\; \frac{K}{A(R)} \int_{\mathbb{S}(\mathbf{x};R)} \partial_\perp f(\mathbf{s})\, d\mathbf{s}$, for some constant $K > 0$.

Here, $\partial_\perp f(\mathbf{s})$ is the *outward normal derivative* of f at the point $\mathbf{s}$ on the sphere (see page 564 for an abstract definition; see §5C(ii) on page 76 for more information).

(c) Let $\mathbb{B}(\mathbf{x};R) := \left\{\mathbf{b} \in \mathbb{R}^D \; ; \; \|\mathbf{b} - \mathbf{x}\| \leq R\right\}$ be the **ball** of radius R around $\mathbf{x}$. Apply *Green's Formula* (Theorem 0E.5(a) on page 564) to show that

$$\phi'(R) \;=\; \frac{K}{A(R)} \int_{\mathbb{B}(\mathbf{x};R)} \triangle f(\mathbf{b})\, d\mathbf{b}.$$

(d) Deduce that, if f is harmonic, then ϕ must be constant.

(e) Use the fact that f is continuous to show that $\lim_{r \to 0} \phi(r) = f(\mathbf{x})$. Deduce that $\phi(r) = f(\mathbf{x})$ for all $r \geq 0$. Conclude that, if f is harmonic, then statement (1E.1) must be true. □

Corollary 1E.2. Maximum Principle for harmonic functions

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain, and let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a nonconstant harmonic function. Then u has no local maximal or minimal points anywhere in the interior of $\mathbb{X}$.

If $\mathbb{X}$ is bounded (hence compact)*, then u **does** obtain a maximum and minimum, but only on the **boundary** of $\mathbb{X}$.*

Proof. (by contradiction). Suppose $\mathbf{x}$ *was* a local maximum of u somewhere in the interior of $\mathbb{X}$. Let $R > 0$ be small enough that $\mathbb{S}(\mathbf{x}; R) \subset \mathbb{X}$, and such that

$$u(\mathbf{x}) \quad \geq \quad u(\mathbf{s}) \qquad \text{for all } \mathbf{s} \in \mathbb{S}(\mathbf{x}; R), \tag{1E.2}$$

where this inequality is strict for at least one $\mathbf{s}_0 \in \mathbb{S}(\mathbf{x}; R)$.

Claim 1: *There is a nonempty open subset $\mathbb{Y} \subset \mathbb{S}(\mathbf{x}; R)$ such that $u(\mathbf{x}) > u(\mathbf{y})$ for all $\mathbf{y}$ in $\mathbb{Y}$.*

Proof. We know that $u(\mathbf{x}) > u(\mathbf{s}_0)$. But u is continuous, so there must be some open neighbourhood $\mathbb{Y}$ around $\mathbf{s}_0$ such that $u(\mathbf{x}) > u(\mathbf{y})$ for all $\mathbf{y}$ in $\mathbb{Y}$. $\diamond_{\text{Claim 1}}$

Equation (1E.2) and Claim 1 imply that

$$f(\mathbf{x}) \quad > \quad \frac{1}{A(R)} \int_{\mathbb{S}(\mathbf{x};R)} f(\mathbf{s})\, d\mathbf{s}.$$

But this contradicts the Mean Value Theorem. By contradiction, $\mathbf{x}$ cannot be a local maximum. (The proof for local minima is analogous). $\square$

A function $F : \mathbb{R}^D \longrightarrow \mathbb{R}$ is **spherically symmetric** if $F(\mathbf{x})$ depends only on the norm $\|\mathbf{x}\|$ (i.e. $F(\mathbf{x}) = f(\|\mathbf{x}\|)$ for some function $f : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$). For example, the function $F(\mathbf{x}) := \exp(-\|\mathbf{x}\|^2)$ is spherically symmetric.

If $h, F : \mathbb{R}^D \longrightarrow \mathbb{R}$ are two integrable functions, then their **convolution** is the function $h * F : \mathbb{R}^D \longrightarrow \mathbb{R}$ defined by

$$h * F(\mathbf{x}) \quad := \quad \int_{\mathbb{R}^D} h(\mathbf{y}) \cdot F(\mathbf{x} - \mathbf{y})\, d\mathbf{y}, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^D$$

(if this integral converges). We will encounter convolutions in § 10D(ii) on page 214 (where they will be used to prove the L^2 convergence of a Fourier series) and again in Chapter 17 (where they will be used to construct ‘impulse-response’ solutions for PDEs). For now, we state the following simple consequence of the Mean Value Theorem:

Lemma 1E.3. *If $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ is harmonic and $F : \mathbb{R}^D \longrightarrow \mathbb{R}$ is integrable and spherically symmetric, then $h * F = K \cdot h$, where $K \in \mathbb{R}$ is some constant.*

Proof. <u>**Exercise 1E.2**</u> $\square$ Ⓔ

Proposition 1E.4. Smoothness of harmonic functions
If $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a harmonic function, then h is infinitely differentiable.

Proof. Let $F : \mathbb{R}^D \longrightarrow \mathbb{R}$ be some infinitely differentiable, spherically symmetric, integrable function. For example, we could take $F(\mathbf{x}) := \exp(-\|\mathbf{x}\|^2)$. Then Proposition 17G.2(f) on page 410 says that $h * F$ is infinitely differentiable. But Lemma 1E.3 implies that $h * F = Kh$ for some constant $K \in \mathbb{R}$; thus, h is also infinitely differentiable.

(For another proof, see Theorem 6 on p. 28 of [Eva91, §2.2].) $\square$

Actually, we can go even further than this. A function $h : \mathbb{X} \longrightarrow \mathbb{R}$ is **analytic** if, for every $\mathbf{x} \in \mathbb{X}$, there is a multivariate Taylor series expansion for h around $\mathbf{x}$ with a nonzero radius of convergence.[7]

Proposition 1E.5. Harmonic functions are analytic
Let $\mathbb{X} \subseteq \mathbb{R}^D$ be an open set. If $h : \mathbb{X} \longrightarrow \mathbb{R}$ is a harmonic function, then h is analytic on $\mathbb{X}$.

Proof. For the case $D = 2$, see Corollary 18D.2 on page 451. For the general case $D \geq 2$, see Theorem 10 on p. 31 of [Eva91, §2.2]. $\square$

1F* Transport and diffusion

Prerequisites: §1B, §6A.

If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a "mountain", then recall that $\nabla u(\mathbf{x})$ points in the direction of *most rapid ascent* at $\mathbf{x}$. If $\vec{\mathbf{v}} \in \mathbb{R}^D$ is a vector, then the dot product $\vec{\mathbf{v}} \bullet \nabla u(\mathbf{x})$ measures how rapidly you would be ascending if you walked in direction $\vec{\mathbf{v}}$.

Suppose $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ describes a pile of leafs, and there is a steady wind blowing in the direction $\vec{\mathbf{v}} \in \mathbb{R}^D$. We would expect the pile to slowly move in the direction $\vec{\mathbf{v}}$. Suppose you were an observer fixed at location $\mathbf{x}$. The pile is moving past you in direction $\vec{\mathbf{v}}$, which is the same as you walking along the pile in direction $-\vec{\mathbf{v}}$; thus, you would expect the height of the pile at your location to increase/decrease at rate $-\vec{\mathbf{v}} \bullet \nabla u(\mathbf{x})$. The pile thus satisfies the **Transport Equation:**

$$\partial_t u \;=\; -\vec{\mathbf{v}} \bullet \nabla u.$$

Now, suppose that the wind does not blow in a *constant* direction, but instead has some complex spatial pattern. The wind velocity is therefore determined by a *vector field* $\vec{\mathbf{V}} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$. As the wind picks up leafs and carries them around, the **flux** of leafs at a point $\mathbf{x} \in \mathbb{X}$ is then the vector $\vec{\mathbf{F}}(\mathbf{x}) = u(\mathbf{x}) \cdot \vec{\mathbf{V}}(\mathbf{x})$.

[7] See Appendices 0H(ii) and 0H(v) on pages 569 and 576.

But the rate at which leafs are piling up at each location is the *divergence* of the flux. This results in **Liouville's Equation:**

$$\partial_t u \quad = \quad -\mathrm{div}\,\vec{\mathbf{F}} \quad = \quad -\mathrm{div}\,(u \cdot \vec{\mathbf{V}}) \underset{(*)}{=} \quad -\vec{\mathbf{V}} \bullet \nabla u \; - \; u \cdot \mathrm{div}\,\vec{\mathbf{V}}.$$

Here, $(*)$ is by the Leibniz rule for divergence (Proposition 0E.2(b) on page 560).

Liouville's equation describes the rate at which u-material accumulates when it is being pushed around by the $\vec{\mathbf{V}}$-vector field. Another example: $\vec{\mathbf{V}}(\mathbf{x})$ describes the flow of water at $\mathbf{x}$, and $u(\mathbf{x})$ is the buildup of some sediment at $\mathbf{x}$.

Now suppose that, in addition to the deterministic force $\vec{\mathbf{V}}$ acting on the leafs, there is also a "random" component. In other words, while being blown around by the wind, the leafs are also subject to some random diffusion. To describe this, we combine *Liouville's Equation* with the *heat equation*, to obtain the **Fokker-Plank** equation:

$$\partial_t u \quad = \quad \kappa \triangle u \; - \; \vec{\mathbf{V}} \bullet \nabla u \; - \; u \cdot \mathrm{div}\,\vec{\mathbf{V}}.$$

1G* Reaction and diffusion

Prerequisites: §1B.

Suppose A,B and C are three chemicals, satisfying the chemical reaction:

$$2A + B \quad \Longrightarrow \quad C$$

As this reaction proceeds, the A and B species are consumed, and C is produced. Thus, if a, b, c are the concentrations of the three chemicals, we have:

$$\partial_t c \quad = \quad R(t) \quad = \quad -\partial_t b \quad = \quad -\frac{1}{2}\partial_t a,$$

where $R(t)$ is the rate of the reaction at time t. The rate $R(t)$ is determined by the concentrations of A and B, and by a rate constant ρ. Each chemical reaction requires 2 molecules of A and one of B; thus, the reaction rate is given by

$$R(t) \quad = \quad \rho \cdot a(t)^2 \cdot b(t)$$

Hence, we get three ordinary differential equations, called the **reaction kinetic equations** of the system:

$$\left.\begin{array}{rcl} \partial_t a(t) & = & -2\rho \cdot a(t)^2 \cdot b(t) \\ \partial_t b(t) & = & -\rho \cdot a(t)^2 \cdot b(t) \\ \partial_t c(t) & = & \rho \cdot a(t)^2 \cdot b(t) \end{array}\right\} \qquad (1G.1)$$

Now, suppose that the chemicals A, B and C are in solution, but are not uniformly mixed. At any location $\mathbf{x} \in \mathbb{X}$ and time $t > 0$, let $a(\mathbf{x}, t)$ be the concentration of chemical A at location $\mathbf{x}$ at time t; likewise, let $b(\mathbf{x}, t)$ be the concentration of B and $c(\mathbf{x}, t)$ be the concentration of C. (This determines three *time-varying scalar fields*, $a, b, c : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$.) As the chemicals react, their concentrations at each point in space may change. Thus, the functions a, b, c will obey the equations (1G.1) at each point in space. That is, for every $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

$$\partial_t\, a(\mathbf{x}; t) \;\approx\; -2\rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t)$$

etc. However, the dissolved chemicals are also subject to *diffusion* forces. In other words, each of the functions a, b and c will also be obeying the heat equation. Thus, we get the system:

$$\begin{aligned}
\partial_t\, a &= \kappa_a \cdot \triangle a(\mathbf{x}; t) \;-\; 2\rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t) \\
\partial_t\, b &= \kappa_b \cdot \triangle b(\mathbf{x}; t) \;-\; \rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t) \\
\partial_t\, c &= \kappa_c \cdot \triangle c(\mathbf{x}; t) \;+\; \rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t)
\end{aligned}$$

where $\kappa_a, \kappa_b, \kappa_c > 0$ are three different diffusivity constants.

This is an example of a **reaction-diffusion system**. In general, in a reaction-diffusion system involving N distinct chemicals, the concentrations of the different species is described by a **concentration vector field u** $: \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^N$, and the chemical reaction is described by a **rate function** $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$. For example, in the previous example, $\mathbf{u}(\mathbf{x}, t) = \Big(a(\mathbf{x}, t),\ b(\mathbf{x}, t),\ c(\mathbf{x}, t)\Big)$, and

$$F(a, b, c) \;=\; \left[-2\rho a^2 b, \quad -\rho a^2 b, \quad \rho a^2 b\right].$$

The **reaction-diffusion equations** for the system then take the form

$$\partial_t\, u_n \;=\; \kappa_n \,\triangle\, u_n \;+\; F_n(\mathbf{u}),$$

for $n = 1, ..., N$

1H Practice problems

1. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be a differentiable scalar field. Show that $\operatorname{div} \nabla f(x_1, x_2, x_3, x_4) = \triangle f(x_1, x_2, x_3, x_4)$.
2. Let $f(x, y; t) \;=\; \exp(-34t) \cdot \sin(3x + 5y)$. Show that $f(x, y; t)$ satisfies the two-dimensional heat equation: $\partial_t\, f(x, y; t) \;=\; \triangle f(x, y; t)$.
3. Let $u(x, y) = \log(x^2 + y^2)$. Show that $u(x, y)$ satisfies the (two-dimensional) Laplace Equation, everywhere except at $(x, y) = (0, 0)$.

   Remark: If $(x, y) \in \mathbb{R}^2$, recall that $\|(x, y)\| := \sqrt{x^2 + y^2}$. Thus, $\log(x^2 + y^2) \;=\; 2 \log \|(x, y)\|$. This function is sometimes called the **logarithmic potential**.

4. If $(x, y, z) \in \mathbb{R}^3$, recall that $\|(x, y, z)\| := \sqrt{x^2 + y^2 + z^2}$. Define

$$u(x, y, z) \quad = \quad \frac{1}{\|(x, y, z)\|} \quad = \quad \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Show that u satisfies the (three-dimensional) Laplace equation, everywhere except at $(x, y, z) = (0, 0, 0)$.

Remark: Observe that $\nabla u(x, y, z) \ = \ \dfrac{-(x, y, z)}{\|(x, y, z)\|^3}$. What force field does this remind you of? Hint: $u(x, y, z)$ is sometimes called the **Coulomb potential**.

5. Let $u(x, y; t) \ = \ \dfrac{1}{4\pi t} \exp\left(\dfrac{-\|(x, y)\|^2}{4t}\right) \ = \ \dfrac{1}{4\pi t} \exp\left(\dfrac{-x^2 - y^2}{4t}\right)$ be the (two-dimensional) **Gauss-Weierstrass Kernel**. Show that u satisfies the (two-dimensional) heat equation, $\partial_t u \ = \ \triangle u$.

6. Let α and β be real numbers, and let $h(x, y) \ = \ \sinh(\alpha x) \cdot \sin(\beta y)$.

   (a) Compute $\triangle h(x, y)$.

   (b) Suppose h is **harmonic**. Write an equation describing the relationship between α and β.

Further reading

An analogy of the Laplacian can be defined on any Riemannian manifold, where it is sometimes called the **Laplace-Beltrami operator**. The study of harmonic functions on manifolds yields important geometric insights [War83, Cha93].

The reaction diffusion systems from §1G play an important role in modern mathematical biology [Mur93].

The heat equation also arises frequently in the theory of Brownian motion and other Gaussian stochastic processes on $\mathbb{R}^D$ [Str93].

Chapter 2

Waves and signals

"There is geometry in the humming of the strings." —Pythagoras

2A The Laplacian and spherical means

Prerequisites: §0A, §0B, §0H(v). **Recommended:** §1B.

Let $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a function of D variables. Recall that the **Laplacian** of u is defined:

$$\boxed{\triangle u \quad = \quad \partial_1^2 u + \partial_2^2 u + \ldots \partial_D^2 u.}$$

In this section, we will show that $\triangle u(\mathbf{x})$ measures the discrepancy between $u(\mathbf{x})$ and the 'average' of u in a small neighbourhood around $\mathbf{x}$.

Let $\mathbb{S}(\epsilon)$ be the D-dimensional "sphere" of radius ϵ around 0. For example:

- If $D = 1$, then $\mathbb{S}(\epsilon)$ is just a set with two points: $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$.
- If $D = 2$, then $\mathbb{S}(\epsilon)$ is the *circle* of radius ϵ: $\mathbb{S}(\epsilon) = \left\{(x, y) \in \mathbb{R}^2 \; ; \; x^2 + y^2 = \epsilon^2\right\}$
- If $D = 3$, then $\mathbb{S}(\epsilon)$ is the 3-dimensional spherical shell of radius ϵ:

$$\mathbb{S}(\epsilon) \quad = \quad \left\{(x, y, z) \in \mathbb{R}^3 \; ; \; x^2 + y^2 + z^2 = \epsilon^2\right\}.$$

- More generally, for any dimension D,

$$\mathbb{S}(\epsilon) \quad = \quad \left\{(x_1, x_2, \ldots, x_D) \in \mathbb{R}^D \; ; \; x_1^2 + x_2^2 + \ldots + x_D^2 = \epsilon^2\right\}.$$

Let A_ϵ be the "surface area" of the sphere. For example:

- If $D = 1$, then $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$ is a finite set, with two points, so we say $A_\epsilon = 2$.
- If $D = 2$, then $\mathbb{S}(\epsilon)$ is the circle of radius ϵ; the *perimeter* of this circle is $2\pi\epsilon$, so we say $A_\epsilon = 2\pi\epsilon$.

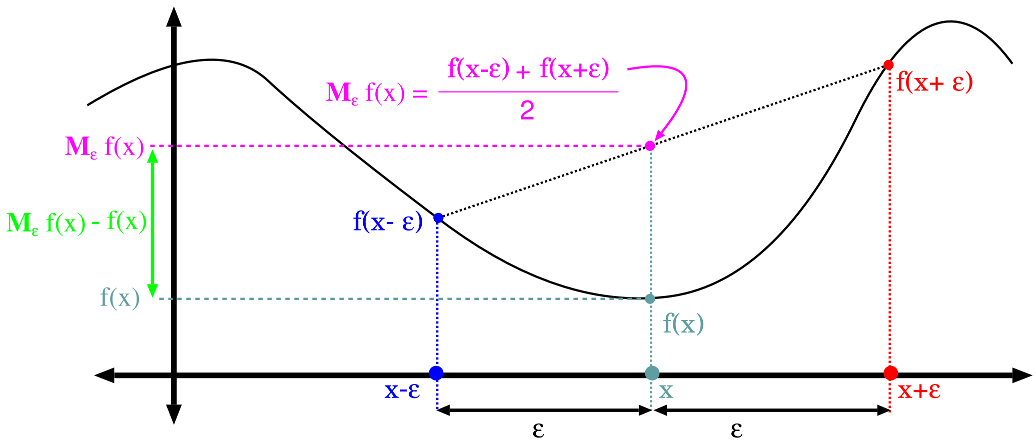


Figure 2A.1: Local averages: $f(x)$ vs. $\mathbf{M}_\epsilon f(x) := \frac{f(x-\epsilon)+f(x+\epsilon)}{2}$.

- If $D = 3$, then $\mathbb{S}(\epsilon)$ is the sphere of radius ϵ, which has *surface area* $4\pi\epsilon^2$.

Let $\mathbf{M}_\epsilon f(0) := \frac{1}{A_\epsilon} \int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\, d\mathbf{s}$; then $\mathbf{M}_\epsilon f(0)$ is the *average value* of $f(\mathbf{s})$ over all $\mathbf{s}$ on the surface of the ϵ-radius sphere around 0, which is called the **spherical mean** of f at 0. The interpretation of the integral sign "$\int$" depends on the dimension D of the space. For example, "$\int$" represents a *surface integral* if $D = 3$, a *line integral* if $D = 2$, and simple two-point sum if $D = 1$. Thus:

- If $D = 1$, then $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$, so that $\int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\ d\mathbf{s} = f(\epsilon) + f(-\epsilon)$; thus,
$$\mathbf{M}_\epsilon f \;=\; \frac{f(\epsilon) + f(-\epsilon)}{2}.$$

- If $D = 2$, then any point on the circle has the form $\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)$ for some angle $\theta \in [0, 2\pi)$. Thus, $\int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\, d\mathbf{s} \;=\; \int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ \epsilon\, d\theta$, so that
$$\mathbf{M}_\epsilon f \;=\; \frac{1}{2\pi\epsilon}\int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ \epsilon\, d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ d\theta,$$

Likewise, for any $\mathbf{x} \in \mathbb{R}^D$, we define $\mathbf{M}_\epsilon f(\mathbf{x}) := \frac{1}{A_\epsilon}\int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s}$ to be the average value of f over an ϵ-radius sphere around $\mathbf{x}$. Suppose $f : \mathbb{R}^D \longrightarrow \mathbb{R}$

is a smooth scalar field, and $\mathbf{x} \in \mathbb{R}^D$. One interpretation of the Laplacian is this: $\triangle f(\mathbf{x})$ measures the disparity between $f(\mathbf{x})$ and the *average value* of f in the immediate vicinity of $\mathbf{x}$. This is the meaning of the next theorem:

Theorem 2A.1.

(a) *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth scalar field, then* (as shown in Figure 2A.1), *for any $x \in \mathbb{R}$,*

$$\triangle f(x) \quad = \quad \lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \Big[\mathbf{M}_\epsilon\, f(\mathbf{x}) \quad - \quad f(\mathbf{x}) \Big] \quad = \quad \lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \left[\frac{f(x-\epsilon) + f(x+\epsilon)}{2} \quad - \quad f(x) \right].$$

(b) [1] *If $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a smooth scalar field, then for any $\mathbf{x} \in \mathbb{R}^D$,*

$$\triangle f(\mathbf{x}) \quad = \quad \lim_{\epsilon \to 0} \frac{C}{\epsilon^2} \Big[\mathbf{M}_\epsilon\, f(\mathbf{x}) \quad - \quad f(\mathbf{x}) \Big] \quad = \quad \lim_{\epsilon \to 0} \frac{C}{\epsilon^2} \left[\frac{1}{A_\epsilon} \int_{\mathbb{S}(\epsilon)} f(\mathbf{x} + \mathbf{s})\, d\mathbf{s} \quad - \quad f(\mathbf{x}) \right]$$

(Here C is a constant determined by the dimension D).

Proof. **(a)** Using *Taylor's theorem* (see § 0H(i) on page 568), we have:

$$\begin{aligned} f(x+\epsilon) &= f(x) + \epsilon f'(x) + \frac{\epsilon^2}{2} f''(x) + \mathcal{O}(\epsilon^3) \\ \text{and} \quad f(x-\epsilon) &= f(x) - \epsilon f'(x) + \frac{\epsilon^2}{2} f''(x) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Here, $f'(x) = \partial_x f(x)$ and $f''(x) = \partial_x^2 f(x)$. The expression "$\mathcal{O}(\epsilon)$" means "some function (we don't care which one) such that $\lim_{\epsilon \to 0} \mathcal{O}(\epsilon) = 0$".[2] Likewise, "$\mathcal{O}(\epsilon^3)$" means "some function (we don't care which one) such that $\lim_{\epsilon \to 0} \dfrac{\mathcal{O}(\epsilon^3)}{\epsilon^2} = 0$." Summing these two equations, we get

$$f(x+\epsilon) + f(x-\epsilon) \quad = \quad 2f(x) \; + \; \epsilon^2 \cdot f''(x) \; + \; \mathcal{O}(\epsilon^3).$$

Thus,

$$\frac{f(x-\epsilon) - 2f(x) + f(x+\epsilon)}{\epsilon^2} \quad = \quad f''(x) + \mathcal{O}(\epsilon).$$

[because $\mathcal{O}(\epsilon^3)/\epsilon^2 = \mathcal{O}(\epsilon)$.] Now take the limit as $\epsilon \to 0$, to get

$$\lim_{\epsilon \to 0} \frac{f(x-\epsilon) - 2f(x) + f(x+\epsilon)}{\epsilon^2} \quad = \quad \lim_{\epsilon \to 0} f''(x) + \mathcal{O}(\epsilon) \quad = \quad f''(x) \quad = \quad \triangle f(x),$$

[1] Part (b) of Theorem 2A.1 is not necessary for the physical derivation of the wave equation which appears later in this chapter. However, part (b) *is* required for to prove the Mean Value Theorem for harmonic functions (Theorem 1E.1 on page 16).

[2] Actually, "$\mathcal{O}(\epsilon)$" means slightly more than this —see §0H(i). However, for our purposes, this will be sufficient.

as desired.

(b) Define the **Hessian derivative matrix** of f at $\mathbf{x}$:

$$\mathsf{D}^2 f(\mathbf{x}) = \begin{bmatrix} \partial_1^2 f & \partial_1\partial_2 f & \ldots & \partial_1\partial_D f \\ \partial_2\partial_1 f & \partial_2^2 f & \ldots & \partial_2\partial_D f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_D\partial_1 f & \partial_D\partial_2 f & \ldots & \partial_D^2 f \end{bmatrix}$$

Then, for any $\mathbf{s} \in \mathbb{S}(\epsilon)$, the *Multivariate Taylor's theorem* (see § 0H(v) on page 576) says:

$$f(\mathbf{x}+\mathbf{s}) \quad = \quad f(\mathbf{x}) \quad + \quad \mathbf{s} \bullet \nabla f(\mathbf{x}) \quad + \quad \frac{1}{2}\mathbf{s} \bullet \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \quad + \quad \mathcal{O}(\epsilon^3).$$

Now, if $\mathbf{s} = (s_1, s_2, \ldots, s_D)$, then $\mathbf{s} \bullet \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \ = \ \sum_{c,d=1}^{D} s_c \cdot s_d \cdot \partial_c \partial_d\, f(\mathbf{x})$. Thus, for any $\epsilon > 0$, we have

$$\begin{aligned}
A_\epsilon \cdot \mathbf{M}_\epsilon\, f(\mathbf{x}) \quad &= \quad \int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s} \\
&= \int_{\mathbb{S}(\epsilon)} f(\mathbf{x})\ d\mathbf{s} \quad + \quad \int_{\mathbb{S}(\epsilon)} \mathbf{s} \bullet \nabla f(\mathbf{x})\ d\mathbf{s} \\
&\qquad + \quad \frac{1}{2}\int_{\mathbb{S}(\epsilon)} \mathbf{s} \bullet \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \quad + \quad \int_{\mathbb{S}(\epsilon)} \mathcal{O}(\epsilon^3)\ d\mathbf{s} \\
&= A_\epsilon f(\mathbf{x}) \quad + \quad \nabla f(\mathbf{x}) \bullet \int_{\mathbb{S}(\epsilon)} \mathbf{s}\ d\mathbf{s} \\
&\qquad + \quad \frac{1}{2}\int_{\mathbb{S}(\epsilon)} \left(\sum_{c,d=1}^{D} s_c s_d \cdot \partial_c \partial_d\, f(\mathbf{x}) \right) d\mathbf{s} \quad + \quad \mathcal{O}(\epsilon^{D+2}) \\
&= A_\epsilon f(\mathbf{x}) \quad + \quad \underbrace{\nabla f(\mathbf{x}) \bullet \mathbf{0}}_{(*)} \\
&\qquad + \quad \frac{1}{2}\sum_{c,d=1}^{D} \left(\partial_c\, \partial_d\, f(\mathbf{x}) \cdot \left(\int_{\mathbb{S}(\epsilon)} s_c s_d\ d\mathbf{s} \right) \right) \quad + \quad \mathcal{O}(\epsilon^{D+2}) \\
&= A_\epsilon f(\mathbf{x}) \quad + \quad \underbrace{\frac{1}{2}\sum_{d=1}^{D} \left(\partial_d^2\, f(\mathbf{x}) \cdot \left(\int_{\mathbb{S}(\epsilon)} s_d^2\ d\mathbf{s} \right) \right)}_{(\dagger)} \quad + \quad \mathcal{O}(\epsilon^{D+2}) \\
&= A_\epsilon f(\mathbf{x}) \quad + \quad \frac{1}{2} \triangle\, f(\mathbf{x}) \cdot \epsilon^{D+1} K \quad + \quad \mathcal{O}(\epsilon^{D+2}),
\end{aligned}$$

where $K := \int_{\mathbb{S}(1)} s_1^2\ d\mathbf{s}$. Here, $(*)$ is because $\int_{\mathbb{S}(\epsilon)} \mathbf{s}\ d\mathbf{s} = \mathbf{0}$, because the centre-of-mass of a sphere is at its centre, namely $\mathbf{0}$. $(\dagger)$ is because, if $c, d \in [1...D]$,

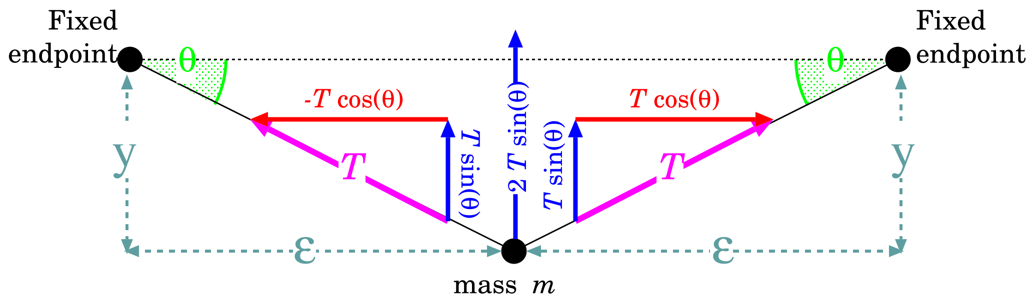


Figure 2B.1: A bead on a string

and $c \neq d$, then $\int_{\mathbb{S}(\epsilon)} s_c s_d \, d\mathbf{s} = 0$ (**Exercise 2A.1** *Hint:* Use symmetry). Thus, Ⓔ

$$\begin{aligned} A_\epsilon \cdot \mathbf{M}_\epsilon f(\mathbf{x}) \;-\; A_\epsilon f(\mathbf{x}) &= \frac{\epsilon^{D+1} K}{2} \triangle f(\mathbf{x}) \;+\; \mathcal{O}(\epsilon^{D+2}), \\ \text{so} \;\; \mathbf{M}_\epsilon f(\mathbf{x}) \;-\; f(\mathbf{x}) &= \frac{\epsilon^{D+1} K}{2A_\epsilon} \triangle f(\mathbf{x}) \;+\; \frac{1}{A_\epsilon} \mathcal{O}(\epsilon^{D+2}) \\ &\underset{(*)}{=} \frac{\epsilon^{D+1} K}{2A_1 \cdot \epsilon^{D-1}} \triangle f(\mathbf{x}) \;+\; \mathcal{O}\left(\frac{\epsilon^{D+2}}{\epsilon^{D-1}}\right) \\ &= \frac{\epsilon^2 K}{2A_1} \triangle f(\mathbf{x}) + \mathcal{O}(\epsilon^3), \end{aligned}$$

where $(*)$ is because $A_\epsilon = A_1 \cdot \epsilon^{D-1}$. Thus,

$$\frac{2A_1}{K\,\epsilon^2}\Big(\mathbf{M}_\epsilon f(\mathbf{x}) \;-\; f(\mathbf{x})\Big) \;=\; \triangle f(\mathbf{x}) + \mathcal{O}(\epsilon).$$

Now take the limit as $\epsilon \longrightarrow 0$, and set $C := \frac{2A_1}{K}$, to prove part (b). □

Exercise 2A.2. Let $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a smooth scalar field, such that $\mathbf{M}_\epsilon f(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$. Show that f is harmonic. ♦ Ⓔ

2B The wave equation

Prerequisites: §2A.

2B(i) ...in one dimension: the string

We want to mathematically describe vibrations propagating through a stretched elastic cord. We will represent the cord with a one-dimensional domain $\mathbb{X}$; either $\mathbb{X} = [0, L]$ or $\mathbb{X} = \mathbb{R}$. We will make several simplifying assumptions:

(W1) The cord has uniform thickness and density. Thus, there is a constant *linear density* $\rho > 0$, so that a cord-segment of length ℓ has mass $\rho\ell$.

(W2) The cord is *perfectly elastic*; meaning that it is infinitely flexible and does not resist bending in any way. Likewise, there is no air friction to resist the motion of the cord.

(W3) The only force acting on the cord is *tension*, which is force of magnitude T pulling the cord to the right, balanced by an equal but opposite force of magnitude $-T$ pulling the cord to the left. These two forces are in balance, so the cord exhibits no horizontal motion. The tension T must be constant along the whole length of the cord. Thus, the equilibrium state for the cord is to be perfectly straight. Vibrations are deviations from this straight position.[3]

(W4) The vibrational motion of the cord is entirely *vertical*; there is no horizontal component to the vibration. Thus, we can describe the motion using a scalar-valued function $u(x, t)$, where $u(x, t)$ is the vertical displacement of the cord (from its flat equilibrium) at point x at time t. We assume that $u(x, t)$ is relatively small relative to the length of the cord, so that the cord is not significantly stretched by the vibrations[4].

For simplicity, let's first imagine a single bead of mass m suspended at the midpoint of a (massless) elastic cord of length 2ϵ, stretched between two endpoints. Suppose we displace the bead by a distance y from its equilibrium, as shown in Figure 2B.1. The tension force T now pulls the bead diagonally towards each endpoint with force T. The horizontal components of the two tension forces are equal and opposite, so they cancel, so the bead experiences no net horizontal force. Suppose the cord makes an angle θ with the horizontal; then the vertical component of each tension force is $T\sin(\theta)$, so the total vertical force acting on the bead is $2T\sin(\theta)$. But $\theta = \arctan(\epsilon/y)$ by the geometry of the triangles in

[3] We could also incorporate the force of gravity as a constant downward force. In this case, the equilibrium position for the cord is to sag downwards in a 'catenary' curve. Vibrations are then deviations from this curve. This doesn't change the mathematics of this derivation, so we will assume for simplicity that gravity is absent and the cord is straight.

[4] If $u(x, t)$ was large, then the vibrations stretch the cord, and a *restoring force* acts against this stretching, as described by *Hooke's Law*. By assuming that the vibrations are small, we are assuming we can neglect Hooke's Law.

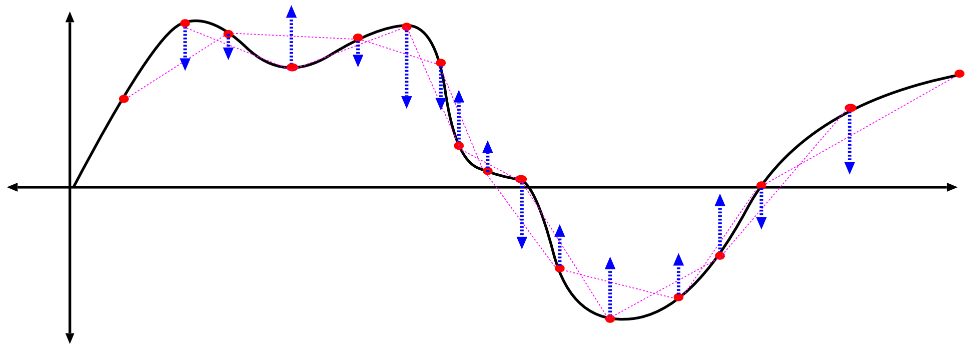


Figure 2B.2: Each bead feels a negative force proportional to its deviation from the local average.

Figure 2B.1, so $\sin(\theta) = \dfrac{y}{\sqrt{y^2+\epsilon^2}}$. Thus, the vertical force acting on the bead is

$$F \quad = \quad 2T\sin(\theta) \quad = \quad \frac{2Ty}{\sqrt{y^2+\epsilon^2}} \tag{2B.1}$$

Now we return to our original problem of the vibrating string. Imagine that we replace the string with a 'necklace' made up of small beads of mass m separated by massless elastic strings of length ϵ. Each of these beads, in isolation, behaves like the 'bead on a string' in Figure 2B.1. However, now, the vertical displacement of each bead is not computed relative to the horizontal, but instead relative to the *average height* of the two neighbouring beads. Thus, in eqn.(2B.1), we set $y := u(x) - \mathbf{M}_\epsilon\, u(x)$, where $u(x)$ is the height of bead x, and where $\mathbf{M}_\epsilon\, u := \frac{1}{2}[u(x-\epsilon)+u(x+\epsilon)]$ is the average of its neighbours. Substituting this into eqn.(2B.1), we get

$$F_\epsilon(x) \quad = \quad \frac{2T[u(x)-\mathbf{M}_\epsilon\, u(x)]}{\sqrt{[u(x)-\mathbf{M}_\epsilon\, u(x)]^2+\epsilon^2}} \tag{2B.2}$$

(Here, the "ϵ" subscript in "F_ϵ" is to remind us that this is just an ϵ-bead approximation of the real string). Each bead represents a length-ϵ segment of the original string, so if the string has linear density ρ, then each bead must have mass $m_\epsilon := \rho\epsilon$. Thus, by Newton's law, the vertical acceleration of bead x must be

$$\begin{aligned} a_\epsilon(x) \quad &= \quad \frac{F_\epsilon(x)}{m_\epsilon} \quad = \quad \frac{2T[u(x)-\mathbf{M}_\epsilon\, u(x)]}{\rho\,\epsilon\,\sqrt{[u(x)-\mathbf{M}_\epsilon\, u(x)]^2+\epsilon^2}} \\ &= \quad \frac{2T[u(x)-\mathbf{M}_\epsilon\, u(x)]}{\rho\,\epsilon^2\,\sqrt{[u(x)-\mathbf{M}_\epsilon\, u(x)]^2/\epsilon^2+1}} \end{aligned} \tag{2B.3}$$

Now, we take the limit as $\epsilon \to 0$, to get the vertical acceleration of the string at x:

$$a(x) \quad = \quad \lim_{\epsilon\to 0} a_\epsilon(x) \quad = \quad \frac{T}{\rho} \lim_{\epsilon\to 0} \frac{2}{\epsilon^2}\Big[u(x) - \mathbf{M}_\epsilon\, u(x)\Big] \cdot \lim_{\epsilon\to 0} \frac{1}{\sqrt{[u(x) - \mathbf{M}_\epsilon\, u(x)]^2/\epsilon^2 + 1}}$$
$$\underset{(*)}{=} \quad \frac{T}{\rho}\, \partial_x^2\, u(x) \frac{1}{\lim_{\epsilon\to 0} \sqrt{\epsilon^2 \cdot \partial_x^2\, u(x)^2 + 1}} \quad \underset{(\dagger)}{=} \quad \frac{T}{\rho}\, \partial_x^2\, u(x). \tag{2B.4}$$

Here, $(*)$ is because Theorem 2A.1(a) on page 25 says that $\lim_{\epsilon\to 0} \frac{2}{\epsilon^2}[u(x) - \mathbf{M}_\epsilon\, u(x)] = \partial_x^2\, u(x)$. Finally, $(\dagger)$ is because, for any value of $u'' \in \mathbb{R}$, we have $\lim_{\epsilon\to 0} \sqrt{\epsilon^2 u'' + 1} = 1$. We conclude that

$$a(x) \quad = \quad \frac{T}{\rho}\, \partial_x^2\, u(x) \quad = \quad \lambda^2\, \partial_x^2\, u(x),$$

where $\lambda := \sqrt{T/\rho}$. Now, the position (and hence, velocity and acceleration) of the cord is changing in time. Thus, a and u are functions of t as well as x. And of course, the acceleration $a(x,t)$ is nothing more than the second derivative of u with respect to t. Hence we have the **one-dimensional Wave Equation**:

$$\boxed{\partial_t^2\, u(x,t) \quad = \quad \lambda^2 \cdot \partial_x^2\, u(x,t).}$$

This equation describes the propagation of a transverse wave along an idealized string, or electrical pulses propagating in a wire.

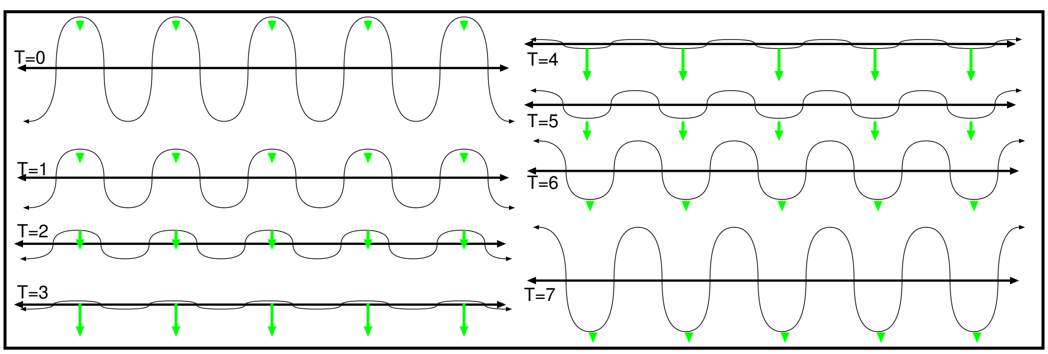


Figure 2B.3: A one-dimensional standing wave.

Example 2B.1. Standing Waves

(a) Suppose $\lambda^2 = 4$, and let $u(x;t) = \sin(3x) \cdot \cos(6t)$. Then u satisfies the Wave Equation and describes a *standing wave* with a *temporal frequency* of 6 and a *wave number* (or *spatial frequency*) of 3. (See Figure 2B.3)

(b) More generally, fix $\omega > 0$; if $u(x;t) = \sin(\omega \cdot x) \cdot \cos(\lambda \cdot \omega \cdot t)$, Then u satisfies the wave equation and describes a *standing wave* of *temporal frequency* $\lambda \cdot \omega$ and *wave number* ω. ◇

Exercise 2B.1. Verify examples (a) and (b) above. ♦ Ⓔ

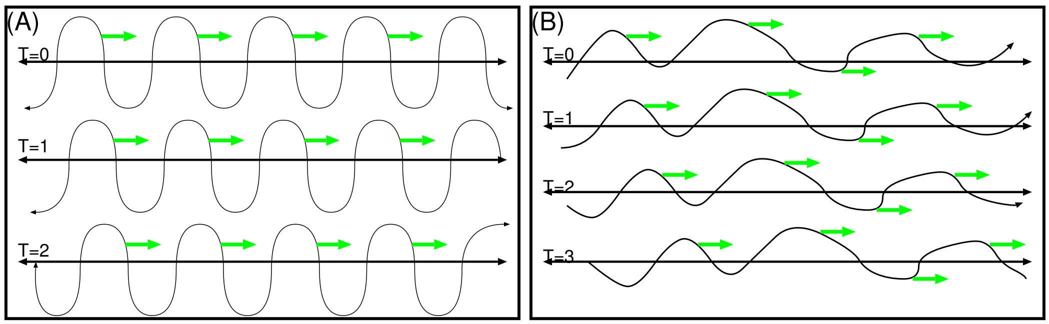


Figure 2B.4: **(A)** A one-dimensional sinusoidal travelling wave. **(B)** A general one-dimensional travelling wave.

Example 2B.2. Travelling Waves

(a) Suppose $\lambda^2 = 4$, and let $u(x;t) = \sin(3x - 6t)$. Then u satisfies the Wave Equation and describes a *sinusoidal travelling wave* with *temporal frequency* 6 and *wave number* 3. The wave crests move rightwards along the cord with velocity 2. (Figure 2B.4A).

(b) More generally, fix $\omega \in \mathbb{R}$ and let $u(x;t) = \sin(\omega \cdot x - \lambda \cdot \omega \cdot t)$. Then u satisfies the wave equation and describes a *sinusoidal travelling wave* of *wave number* ω. The wave crests move rightwards along the cord with velocity λ.

(c) More generally, suppose that f is any function of one variable, and define $u(x;t) = f(x - \lambda \cdot t)$. Then u satisfies the wave equation and describes a *travelling wave*, whose shape is given by f, and which moves rightwards along the cord with velocity λ (see Figure 2B.4B). ◇

Exercise 2B.2. Verify examples 2B.2(a,b,c) above. ♦ Ⓔ

Exercise 2B.3. According to Example 2B.2(c), you can turn any function into a travelling wave. Can you turn any function into a standing wave? Why or why not? ♦ Ⓔ

2B(ii) ...in two dimensions: the drum

Now, suppose $D = 2$, and imagine a two-dimensional "rubber sheet". Suppose $u(x, y; t)$ is the the vertical displacement of the rubber sheet at the point $(x, y) \in \mathbb{R}^2$ at time t. To derive the two-dimensional wave equation, we approximate this rubber sheet as a two-dimensional 'mesh' of tiny beads connected by massless, tense elastic strings of length ϵ. Each bead (x, y) feels a net vertical force $F = F_x + F_y$, where F_x is the vertical force arising from the tension in the x direction, and F_y is the vertical force from the tension in the y direction. Both of these are expressed by a formula similar to eqn.(2B.2). Thus, if bead (x, y) has mass m_ϵ, then it experiences acceleration $a = F/m_\epsilon = F_x/m_\epsilon + F_y/m_\epsilon = a_x + a_y$, where $a_x := F_x/m_\epsilon$ and $a_y := F_y/m_\epsilon$, and each of these is expressed by a formula similar to eqn.(2B.3). Taking the limit as $\epsilon \to 0$ as in eqn.(2B.4), we deduce that

$$a(x,y) \quad = \quad \lim_{\epsilon \to 0} a_{x,\epsilon}(x,y) \; + \; \lim_{\epsilon \to 0} a_{y,\epsilon}(x,y) \quad = \quad \lambda^2 \, \partial_x^2 \, u(x,y) + \lambda^2 \, \partial_y^2 \, u(x,y),$$

where λ is a constant determined by the density and tension of the rubber membrane. Again, we recall that u and a are also functions of time, and that $a(x, y; t) = \partial_t^2 \, u(x, y; t)$. Thus, we have the **two-dimensional Wave Equation:**

$$\boxed{\partial_t^2 \, u(x,y;t) \quad = \quad \lambda^2 \cdot \partial_x^2 \, u(x,y;t) \;\; + \;\; \lambda^2 \cdot \partial_y^2 \, u(x,y;t)} \tag{2B.5}$$

or, more abstractly:

$$\boxed{\partial_t^2 \, u \quad = \quad \lambda^2 \cdot \triangle u.}$$

This equation describes the propagation of wave energy through any medium with a linear restoring force. For example:

- Transverse waves on an idealized rubber sheet.
- Ripples on the surface of a pool of water.
- Acoustic vibrations on a drumskin.

Example 2B.3. Two-dimensional Standing Waves

(a) Suppose $\lambda^2 = 9$, and let $u(x, y; t) = \sin(3x) \cdot \sin(4y) \cdot \cos(15t)$. This describes a two-dimensional standing wave with temporal frequency 15.

(b) More generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\boldsymbol{\omega}\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$. Then the function

$$u(\mathbf{x};t) \quad := \quad \sin(\omega_1 x) \cdot \sin(\omega_2 y) \cdot \cos(\lambda \cdot \Omega t)$$

satisfies the 2-dimensional wave equation and describes a standing wave with temporal frequency $\lambda \cdot \Omega$.

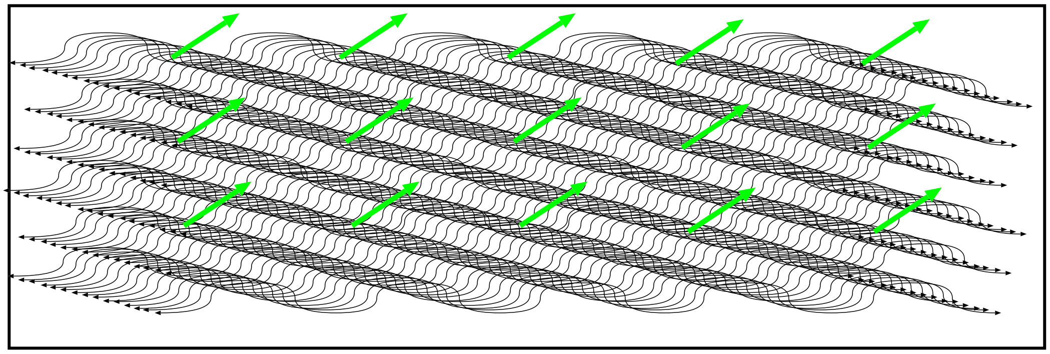


Figure 2B.5: A two-dimensional travelling wave.

(c) Even more generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\boldsymbol{\omega}\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$, as before.

$$\begin{aligned} \text{Let } SC_1(x) &= \text{either } \sin(x) \text{ or } \cos(x); \\ \text{let } SC_2(y) &= \text{either } \sin(y) \text{ or } \cos(y); \\ \text{and let } SC_t(t) &= \text{either } \sin(t) \text{ or } \cos(t). \end{aligned}$$

Then

$$u(\mathbf{x}; t) \quad = \quad SC_1(\omega_1 x) \cdot SC_2(\omega_2 y) \cdot SC_t(\lambda \cdot \Omega t)$$

satisfies the 2-dimensional wave equation and describes a standing wave with temporal frequency $\lambda \cdot \Omega$. ♢

Exercise 2B.4. Check examples (a), (b) and (c) above. ♦ Ⓔ

Example 2B.4. Two-dimensional Travelling Waves

(a) Suppose $\lambda^2 = 9$, and let $u(x, y; t) = \sin(3x + 4y + 15t)$. Then u satisfies the two-dimensional wave equation, and describes a sinusoidal travelling wave with **wave vector** $\boldsymbol{\omega} = (3, 4)$ and temporal frequency 15. (see Figure 2B.5).

(b) More generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\omega\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$. Then

$$u(\mathbf{x}; t) \;=\; \sin\Big(\omega_1 x + \omega_2 y + \lambda \cdot \Omega t\Big) \qquad \text{and} \qquad v(\mathbf{x}; t) \;=\; \cos\Big(\omega_1 x + \omega_2 y + \lambda \cdot \Omega t\Big)$$

both satisfy the two-dimensional wave equation, and describe sinusoidal travelling waves with **wave vector** $\boldsymbol{\omega}$ and temporal frequency $\lambda \cdot \Omega$. ♢

Exercise 2B.5. Check examples (a) and (b) above. ♦ Ⓔ

2B(iii) ...in higher dimensions:

The same reasoning applies for $D \geq 3$. For example, the 3-dimensional wave equation describes the propagation of (small amplitude[5]) sound-waves in air or water. In general, the wave equation takes the form

$$\partial_t^2 u \quad = \quad \lambda^2 \triangle u,$$

where λ is some constant (determined by the density, elasticity, pressure, etc. of the medium) which describes the speed-of-propagation of the waves.

By a suitable choice of time units, we can always assume that $\lambda = 1$. Hence, from now on, we will consider the simplest form of the **wave equation:**

$$\boxed{\partial_t^2 u \quad = \quad \triangle u.}$$

For example, fix $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_D) \in \mathbb{R}^D$ and let $\Omega = \|\boldsymbol{\omega}\|_2 = \sqrt{\omega_1^2 + \ldots + \omega_D^2}$. Then

$$u(\mathbf{x}; t) \quad = \quad \sin\Big(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_D x_D + \Omega t\Big) \quad = \quad \sin\Big(\boldsymbol{\omega} \bullet \mathbf{x} + \lambda \cdot \Omega \cdot t\Big)$$

satisfies the D-dimensional wave equation and describes a transverse wave of with **wave vector $\boldsymbol{\omega}$** propagating across D-dimensional space. (**Exercise 2B.6** Ⓔ Check this.)

2C The telegraph equation

Recommended: §2B(i), §1B(i).

Imagine a signal propagating through a medium with a linear restoring force (e.g. an electrical pulse in a wire, a vibration on a string). In an ideal universe, the signal obeys the Wave Equation. However, in the real universe, *damping effects* interfere. First, energy might "leak" out of the system. For example, if a wire is imperfectly insulated, then current can leak out into surrounding space. Also, the signal may get blurred by noise or frictional effects. For example, an electric wire will pick up radio waves ("crosstalk") from other nearby wires, while losing energy to electrical resistance. A guitar string will pick up vibrations from the air, while losing energy to friction.

Thus, intuitively, we expect the signal to propagate like a wave, but to be gradually smeared out and attenuated by noise and leakage (Figure 2C.6). The model for such a system is the **telegraph equation**:

$$\kappa_2 \partial_t^2 u + \kappa_1 \partial_t u + \kappa_0 u \quad = \quad \lambda \triangle u$$

[5] At large amplitudes, nonlinear effects become important and invalidate the physical argument used here.

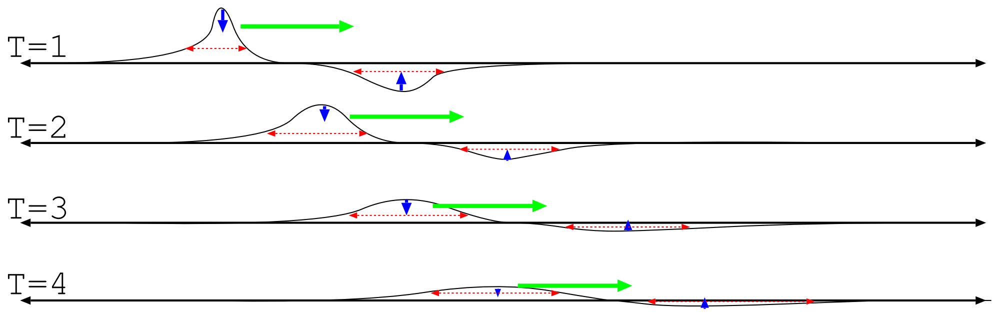


Figure 2C.6: A solution to the telegraph equation propagates like a wave, but it also diffuses over time due to noise, and decays exponentially in magnitude due to 'leakage'.

(where $\kappa_2, \kappa_1, \kappa_0, \lambda > 0$ are constants).

Heuristically speaking, this equation is a "sum" of two equations. The first,

$$\kappa_2 \partial_t^2 u \;=\; \lambda_1 \triangle u$$

is a version of the wave equation, and describes the "ideal" signal, while the second,

$$\kappa_1 \partial_t u \;=\; -\kappa_0 u + \lambda_2 \triangle u$$

describes energy lost due to leakage and frictional forces.

2D Practice problems

1. By explicitly computing derivatives, show that the following functions satisfy the (one-dimensional) wave equation $\partial_t^2 u \;=\; \partial_x^2 u$.

   (a) $u(x,t) \;=\; \sin(7x)\cos(7t)$.

   (b) $u(x,t) \;=\; \sin(3x)\cos(3t)$.

   (c) $u(x,t) \;=\; \frac{1}{(x-t)^2}$ (for $x \neq t$).

   (d) $u(x,t) \;=\; (x-t)^2 - 3(x-t) + 2$.

   (e) $v(x,t) \;=\; (x-t)^2$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any twice-differentiable function. Define $u : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ by $u(x,t) := f(x-t)$, for all $(x,t) \in \mathbb{R}\times\mathbb{R}$.

   Does u satisfies the (one-dimensional) wave equation $\partial_t^2 u \;=\; \triangle u$? Justify your answer.

3. Let $u(x,t)$ be as in 1(a) and let $v(x,t)$ be as in 1(e), and suppose $w(x,t) \;=\; 3u(x,t) - 2v(x,t)$. Conclude that w also satisfies the wave equation, without explicitly computing any derivatives of w.

4. Suppose $u(x,t)$ and $v(x,t)$ are both solutions to the wave equation, and $w(x,t) \;=\; 5u(x,t) + 2v(x,t)$. Conclude that w also satisfies the wave equation.

5. Let $u(x,t) \;=\; \displaystyle\int_{x-t}^{x+t} \cos(y)\ dy \;=\; \sin(x+t) - \sin(x-t)$. Show that u satisfies the (one-dimensional) wave equation $\partial_t^2 u \;=\; \triangle u$.

6. By explicitly computing derivatives, show that the following functions satisfy the (two-dimensional) wave equation $\partial_t^2 u \;=\; \triangle u$.

   (a) $u(x,y;t) \;=\; \sinh(3x) \cdot \cos(5y) \cdot \cos(4t)$.

   (b) $u(x,y;t) \;=\; \sin(x)\cos(2y)\sin(\sqrt{5}t)$.

   (c) $u(x,y;t) \;=\; \sin(3x - 4y)\cos(5t)$.

Chapter 3

Quantum mechanics

"[M]odern physics has definitely decided in favor of Plato. In fact the smallest units of matter are not physical objects in the ordinary sense; they are forms, ideas which can be expressed unambiguously only in mathematical language." —Werner Heisenberg

3A Basic framework

Prerequisites: §0C, §1B(ii).

Near the beginning of the twentieth century, physicists realized that electromagnetic waves sometimes exhibited particle-like properties, as if light was composed of discrete 'photons'. In 1923, Louis de Broglie proposed that, conversely, particles of matter might have wave-like properties. This was confirmed in 1927 by C.J. Davisson and L.H. Germer, and independently, by G.P. Thompson, who showed that an electron beam exhibited an unmistakable diffraction pattern when scattered off a metal plate, as if the beam was composed of 'electron waves'. Systems with many interacting particles exhibit even more curious phenomena. *Quantum mechanics* is a theory which explains these phenomena.

We will not attempt here to provide a physical justification for quantum mechanics. Historically, quantum theory developed through a combination of vaguely implausible physical analogies and wild guesses motivated by inexplicable empirical phenomena. By now, these analogies and guesses have been overwhelmingly vindicated by experimental evidence. The best justification for quantum mechanics is that it 'works', by which we mean that its theoretical predictions match all available empirical data with astonishing accuracy.

Unlike the heat equation in §1B and the Wave Equation in §2B, we cannot derive quantum theory from 'first principles', because the postulates of quantum mechanics *are* the first principles. Instead, we will simply state the main assumptions of the theory, which are far from self-evident, but which we hope you will accept because of the weight of empirical evidence in their favour. Quantum theory describes any physical system via a *probability distribution* on a certain *statespace*. This probability distribution evolves over time; the evolution is driven by a potential energy function, as described by a partial differential equation called the *Schrödinger equation.* We will now examine each of these concepts in turn.

Statespace: A system of N interacting particles moving in 3 dimensional space can be completely described using the $3N$-dimensional **state space** $\mathbb{X} := \mathbb{R}^{3N}$. An element of $\mathbb{X}$ consists of list of N ordered triples:

$$\mathbf{x} \quad = \quad (x_{11}, x_{12}, x_{13};\ x_{21}, x_{22}, x_{23};\ \ldots x_{N1}, x_{N2}, x_{N3}) \quad \in \quad \mathbb{R}^{3N},$$

where (x_{11}, x_{12}, x_{13}) is the spatial position of particle #1, (x_{21}, x_{22}, x_{23}) is the spatial position of particle #2, and so on.

Example 3A.1. (a) *Single electron* A single electron is a one-particle system, so it would be represented using a 3-dimensional statespace $\mathbb{X} = \mathbb{R}^3$. If the electron was confined to a two-dimensional space (e.g. a conducting plate), we would use $\mathbb{X} = \mathbb{R}^2$. If the electron was confined to a one-dimensional space (e.g. a conducting wire), we would use $\mathbb{X} = \mathbb{R}$.

(b) *Hydrogen Atom:* The common isotope of hydrogen contains a single proton and a single electron, so it is a two-particle system, and would be represented using a 6-dimensional state space $\mathbb{X} = \mathbb{R}^6$. An element of $\mathbb{X}$ has the form $\mathbf{x} = (x_1^p, x_2^p, x_3^p;\ x_1^e, x_2^e, x_3^e)$, where (x_1^p, x_2^p, x_3^p) are the coordinates of the proton, and (x_1^e, x_2^e, x_3^e) are those of the electron. $\diamondsuit$

Readers familiar with classical mechanics may be wondering how *momentum* is represented in this statespace. Why isn't the statespace $6N$-dimensional, with 3 'position' and 3 *momentum* coordinates for each particle? The answer, as we will see later, is that the *momentum* of a quantum system is implicitly encoded in the wavefunction which describes its position (see § 19G on page 511).

Potential Energy: We define a **potential energy** (or **voltage**) function $V : \mathbb{X} \longrightarrow \mathbb{R}$, which describes which states are 'prefered' by the quantum system. Loosely speaking, the system will 'avoid' states of high potential energy, and 'seek' states of low energy. The voltage function is usually defined using reasoning familiar from 'classical' physics.

Example 3A.2: Electron in ambient field

Imagine a single electron moving through an ambient electric field $\vec{\mathbf{E}}$. The statespace for this system is $\mathbb{X} = \mathbb{R}^3$, as in Example 3A.1(a). The potential function V is just the voltage of the electric field; in other words, V is any scalar function such that $-q_e \cdot \vec{\mathbf{E}} = \nabla V$, where q_e is the charge of the electron. For example:

(a) *Null field:* If $\vec{\mathbf{E}} \equiv 0$, then V will be a constant, which we can assume is zero: $V \equiv 0$.

(b) *Constant field:* If $\vec{\mathbf{E}} \equiv (E, 0, 0)$, for some constant $E \in \mathbb{R}$, then $V(x, y, z) = -q_e Ex + c$, where c is an arbitrary constant, which we normally set to zero.

(c) *Coulomb field:* Suppose the electric field $\vec{\mathbf{E}}$ is generated by a (stationary) point charge Q at the origin. Let ϵ_0 be the 'permittivity of free space'. Then Coulomb's law says that the electric voltage is given by

$$V(\mathbf{x}) \quad := \quad \frac{q_e \cdot Q}{4\pi\epsilon_0 \cdot |\mathbf{x}|}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

In SI units, $q_e \approx 1.60 \times 10^{-19}\,\mathtt{C}$, and $\epsilon_0 \approx 8.85 \times 10^{-12}\,\mathtt{C/N\,m^2}$. However, for simplicity, we will normally adopt 'atomic units' of charge and field strength, where $q_e = 1$ and $4\pi\epsilon_0 = 1$. Then the above expression becomes $V(\mathbf{x}) = Q/|\mathbf{x}|$.

(d) *Potential well:* Sometimes we confine the electron to some bounded region $\mathbb{B} \subset \mathbb{R}^3$, by setting the voltage equal to 'positive infinity' outside $\mathbb{B}$. For example, a low-energy electron in a cube made of conducting metal can move freely about the cube, but cannot leave[1] the cube. If the subset $\mathbb{B}$ represents the cube, then we define $V : \mathbb{X} \longrightarrow [0, \infty]$ by

$$V(\mathbf{x}) \quad = \quad \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \mathbb{B}; \\ +\infty & \text{if} \quad \mathbf{x} \notin \mathbb{B}. \end{cases}$$

(if '$+\infty$' makes you uncomfortable, then replace it with some 'really big' number). ♢

Example 3A.3: Hydrogen atom:

The system is an electron and a proton; the statespace of this system is $\mathbb{X} = \mathbb{R}^6$ as in Example 3A.1(b). Assuming there is no external electric field, the voltage function is defined

$$V(\mathbf{x}^p, \mathbf{x}^e) \quad := \quad \frac{q_e^2}{4\pi\epsilon_0 \cdot |\mathbf{x}^p - \mathbf{x}^e|}, \quad \text{for all } (\mathbf{x}^p, \mathbf{x}^e) \in \mathbb{R}^6.$$

where $\mathbf{x}^p$ is the position of the proton, $\mathbf{x}^e$ is the position of the electron, and q_e is the charge of the electron (which is also the charge of the proton, with reversed sign). If we adopt 'atomic' units where $q_e := 1$ and $4\pi\epsilon_0 = 1$, then this expression simplifies to

$$V(\mathbf{x}^p, \mathbf{x}^e) \quad := \quad \frac{1}{|\mathbf{x}^p - \mathbf{x}^e|}, \quad \text{for all} \quad (\mathbf{x}^p, \mathbf{x}^e) \in \mathbb{R}^6, \qquad ♢$$

[1] 'Cannot leave' of course really means 'is very highly unlikely to leave'.

Probability and Wavefunctions. Our knowledge of the classical properties of a quantum system is inherently incomplete. All we have is a time-varying probability distribution $\rho : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{R}_{\not-}$ which describes where the particles are likely or unlikely to be at a given moment in time.

As time passes, the probability distribution ρ evolves. However, ρ itself cannot exhibit the 'wavelike' properties of a quantum system (e.g. destructive interference), because ρ is a nonnegative function (and we need to add negative to positive values to get destructive interference). So, we introduce a complex-valued **wavefunction** $\omega : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{C}$. The wavefunction ω determines ρ via the equation:

$$\rho_t(\mathbf{x}) \quad := \quad |\omega_t(\mathbf{x})|^2, \qquad \text{for all } \mathbf{x} \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

(Here, as always in this book, we define $\rho_t(\mathbf{x}) := \rho(\mathbf{x}; t)$ and $\omega_t(\mathbf{x}) := \omega(\mathbf{x}; t)$; subscripts do *not* indicate derivatives). Now, ρ_t is supposed to be a probability density function, so ω_t must satisfy the condition

$$\int_{\mathbb{X}} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x} \quad = \quad 1, \qquad \text{for all } t \in \mathbb{R}. \tag{3A.1}$$

It is acceptable (and convenient) to relax condition (3A.1), and instead simply require

$$\int_{\mathbb{X}} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x} \quad = \quad W \; < \; \infty, \qquad \text{for all } t \in \mathbb{R}. \tag{3A.2}$$

where W is some finite constant, independent of t. In this case, we define $\rho_t(\mathbf{x}) := \frac{1}{W}|\omega_t(\mathbf{x})|^2$ for all $\mathbf{x} \in \mathbb{X}$. It follows that any physically meaningful solution to the Schrödinger equation must satisfy condition (3A.2). This excludes, for example, solutions where the magnitude of the wavefunction grows exponentially in the $\mathbf{x}$ or t variables.

For any fixed $t \in \mathbb{R}$, condition (3A.2) is usually expressed by saying that ω_t is *square-integrable*. Let $\mathbf{L}^2(\mathbb{X})$ denote the set of all square-integrable functions on $\mathbb{X}$. If $\omega_t \in \mathbf{L}^2(\mathbb{X})$, then the **$L^2$-norm** of ω is defined

$$\|\omega_t\|_2 \quad := \quad \sqrt{\int_{\mathbb{X}} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x}}.$$

Thus, a fundamental postulate of quantum theory is:

> *Let $\omega : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{C}$ be a wavefunction. To be physically meaningful, we must have $\omega_t \in \mathbf{L}^2(\mathbb{X})$ for all $t \in \mathbb{R}$. Furthermore, $\|\omega_t\|_2$ must be constant in time.*

We refer the reader to § 6B on page 105 for more information on L^2-norms and L^2-spaces.

3B The Schrödinger equation

Prerequisites: §3A. **Recommended:** §4B.

The wavefunction ω evolves over time in response to the potential field V. Let $\hbar$ be the 'rationalized' Planck constant

$$\hbar \quad := \quad \frac{h}{2\pi} \quad \approx \quad \frac{1}{2\pi} \times 6.6256 \times 10^{-34}\ \mathtt{J\,s} \quad \approx \quad 1.0545 \times 10^{-34}\ \mathtt{J\,s}.$$

Then the wavefunction's evolution is described by the **Schrödinger Equation:**

$$\mathbf{i}\hbar\,\partial_t\,\omega \quad = \quad \mathsf{H}\,\omega, \tag{3B.1}$$

where H is a linear differential operator called the **Hamiltonian** operator, defined by:

$$\mathsf{H}\,\omega_t(\mathbf{x}) \quad := \quad \frac{-\hbar^2}{2}\,\blacktriangle\,\omega_t(\mathbf{x}) + V(\mathbf{x})\cdot\omega_t(\mathbf{x}), \qquad \text{for all } \mathbf{x}\in\mathbb{X} \text{ and } t\in\mathbb{R}. \tag{3B.2}$$

Here, $\blacktriangle\,\omega_t$ is like the Laplacian of ω_t, except that the components for each particle are divided by the *rest mass* of that particle. The *potential function* $V : \mathbb{X} \longrightarrow \mathbb{R}$ encodes all the exogenous aspects of the system we are modelling (e.g. the presence of ambient electric fields). Substituting eqn.(3B.2) into eqn.(3B.1), we get

$$\mathbf{i}\hbar\,\partial_t\,\omega \quad = \quad \frac{-\hbar^2}{2}\,\blacktriangle\,\omega \;+\; V\cdot\omega, \tag{3B.3}$$

In 'atomic units', $\hbar = 1$, so the Schrödinger equation (3B.3) becomes

$$\mathbf{i}\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-1}{2}\,\blacktriangle\,\omega_t(\mathbf{x}) \;+\; V(\mathbf{x})\cdot\omega_t(\mathbf{x}), \qquad \text{for all } \mathbf{x}\in\mathbb{X} \text{ and } t\in\mathbb{R}.$$

Example 3B.1. (a) *Free Electron:* Let $m_e \approx 9.11\times 10^{-31}$ kg be the rest mass of an electron. A solitary electron in a null electric field (as in Example 3A.2(a)) satisfies the *free Schrödinger equation:*

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_t(\mathbf{x}). \tag{3B.4}$$

(In this case $\blacktriangle = \frac{1}{m_e}\triangle$, and $V \equiv 0$ because the ambient field is null). In atomic units, we set $m_e := 1$ and $\hbar := 1$, so eqn.(3B.4) becomes

$$\mathbf{i}\,\partial_t\,\omega \quad = \quad \frac{-1}{2}\,\triangle\,\omega \quad = \quad \frac{-1}{2}\left(\partial_1^2\omega + \partial_2^2\omega + \partial_3^2\omega\right). \tag{3B.5}$$

(b) *Electron vs. point charge:* Consider the *Coulomb* electric field, generated by a (stationary) point charge Q at the origin, as in Example 3A.2(c). A solitary electron in this electric field satisfies the Schrödinger equation

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_t(\mathbf{x}) \;+\; \frac{q_e\cdot Q}{4\pi\epsilon_0\cdot|\mathbf{x}|}\,\omega_t(\mathbf{x}).$$

In atomic units, we have $m_e := 1$, $q_e := 1$, etc. Let $\widetilde{Q} = Q/q_e$ be the charge Q converted in units of electron charge. Then the previous expression simplifies to

$$\mathbf{i}\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-1}{2}\,\triangle\,\omega_t(\mathbf{x}) \;+\; \frac{\widetilde{Q}}{|\mathbf{x}|}\,\omega_t(\mathbf{x}).$$

(c) *Hydrogen atom:* (see Example 3A.3) An interacting proton-electron pair (in the absence of an ambient field) satisfies the two-particle Schrödinger equation

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}^p,\mathbf{x}^e) \quad = \quad \frac{-\hbar^2}{2\,m_p}\triangle_p\,\omega_t(\mathbf{x}^p,\mathbf{x}^e) + \frac{-\hbar^2}{2\,m_e}\triangle_e\,\omega_t(\mathbf{x}^p,\mathbf{x}^e) + \frac{q_e^2\cdot\omega_t(\mathbf{x}^p,\mathbf{x}^e)}{4\pi\epsilon_0\cdot|\mathbf{x}^p-\mathbf{x}^e|}, \tag{3B.6}$$

where $\triangle_p\,\omega := \partial^2_{x_1^p}\omega + \partial^2_{x_2^p}\omega + \partial^2_{x_3^p}\omega$ is the Laplacian in the 'proton' position coordinates, and $m_p \approx 1.6727\times10^{-27}$ $\mathtt{kg}$ is the rest mass of a proton. Likewise, $\triangle_e\,\omega := \partial^2_{x_1^e}\omega + \partial^2_{x_2^e}\omega + \partial^2_{x_3^e}\omega$ is the Laplacian in the 'electron' position coordinates, and m_e is the rest mass of the electron. In atomic units, we have $4\pi\epsilon_0 = 1$, $q_e = 1$, and $m_e = 1$. If $\widetilde{m}_p \approx 1864$ is the ratio of proton mass to electron mass, then $2\widetilde{m}_p \approx 3728$, and eqn.(3B.6) becomes

$$\mathbf{i}\;\partial_t\,\omega_t(\mathbf{x}^p,\mathbf{x}^e) \quad = \quad \frac{-1}{3728}\,\triangle_p\;\omega_t(\mathbf{x}^p,\mathbf{x}^e) \;+\; \frac{-1}{2}\,\triangle_e\;\omega_t(\mathbf{x}^p,\mathbf{x}^e) \;+\; \frac{\omega_t(\mathbf{x}^p,\mathbf{x}^e)}{|\mathbf{x}^p-\mathbf{x}^e|}.$$

$\diamondsuit$

The major mathematical problems of quantum mechanics come down to finding solutions to the Schrödinger equations for various physical systems. In general it is very difficult to solve the Schrödinger equation for most 'realistic' potential functions. We will confine ourselves to a few 'toy models' to illustrate the essential ideas.

Example 3B.2: Free Electron with Known Velocity (Null Field)

Consider a single electron in a null electromagnetic field. Suppose an experiment has precisely measured the 'classical' velocity of the electron, and determined it to be $\mathbf{v} = (v_1, 0, 0)$. Then the wavefunction of the electron is given[2]

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,t\right)\cdot\exp\left(\frac{\mathbf{i}}{\hbar}\,m_e v_1\cdot x_1\right). \qquad \text{(see Figure 3B.1)} \tag{3B.7}$$

This ω satisfies the free Schrödinger equation (3B.4). [See practice problem # 1 on page 54 of §3D.]

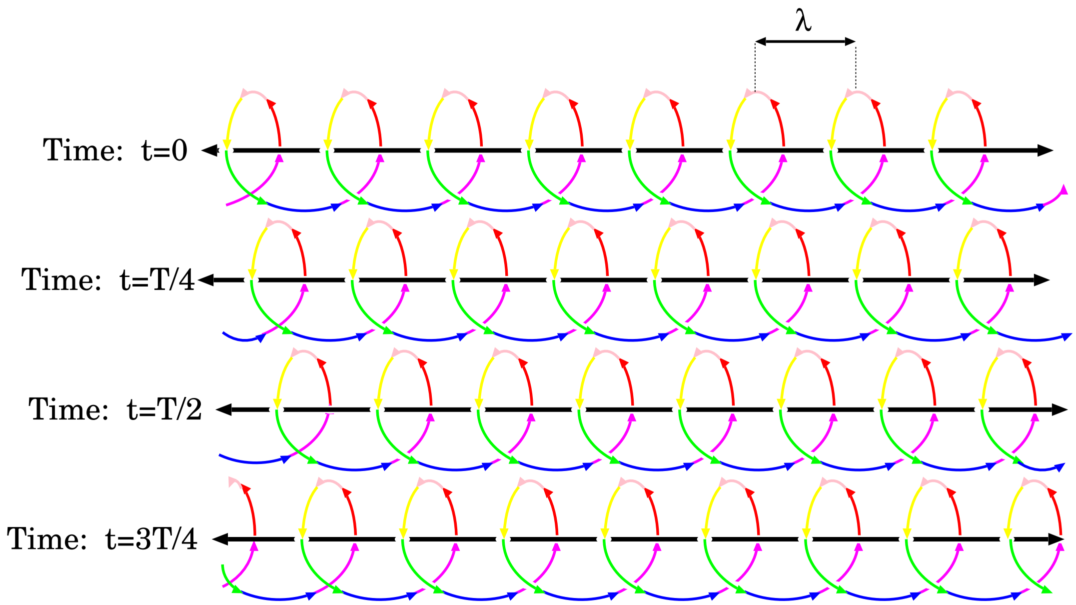


Figure 3B.1: Four successive 'snapshots' of the wavefunction of a single electron in a zero potential, with a precisely known velocity. Only one spatial dimension is shown. The angle of the spiral indicates complex phase.

Ⓔ **Exercise 3B.1.** (a) Check that the spatial wavelength λ of the function ω is given $\lambda = \dfrac{2\pi\hbar}{p_1} = \dfrac{h}{m_e v}$. This is the so-called *de Broglie wavelength* of an electron with velocity v.

(b) Check that the temporal period of ω is $T := \dfrac{2h}{m_e v^2}$.

(c) Conclude the *phase velocity* of ω (i.e. the speed at which the wavefronts propagate through space) is equal to v. ♦

More generally, suppose the electron has a precisely known velocity $\mathbf{v} = (v_1, v_2, v_3)$, with corresponding momentum vector $\mathbf{p} := m_e\,\mathbf{v}$. Then the wavefunction of the electron is given

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}}{\hbar}\,E_k\,t\right)\cdot\exp\left(\frac{\mathbf{i}}{\hbar}\,\mathbf{p}\bullet\mathbf{x}\right), \qquad (3B.8)$$

where $E_k := \frac{1}{2}m_e|\mathbf{v}|^2$ is kinetic energy, and $\mathbf{p}\bullet\mathbf{x} := p_1x_1 + p_2x_2 + p_3x_3$. If we convert to atomic units, then $E_k = \frac{1}{2}|\mathbf{v}|^2$ and $\mathbf{p} = \mathbf{v}$, and this function takes the simpler form

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}\,|\mathbf{v}|^2\,t}{2}\right)\cdot\exp\left(\mathbf{i}\,\mathbf{v}\bullet\mathbf{x}\right).$$

[2] We will not attempt here to justify *why* this is the correct wavefunction for a particle with this velocity. It is not obvious.

This ω satisfies the free Schrödinger equation (3B.5). [See practice problem # 2 on page 54 of §3D.]

The wavefunction (3B.8) represent a state of maximal uncertainty about the position of the electron. This is an extreme manifestation of the infamous *Heisenberg Uncertainty Principle*; by assuming that the electron's *velocity* was 'precisely determined', we have forced it's *position* to be entirely undetermined (see §19G for more information).

Indeed, the wavefunction (3B.8) violates our 'fundamental postulate' —the function ω_t is *not* square-integrable, because $|\omega_t(\mathbf{x})| = 1$ for all $\mathbf{x} \in \mathbb{R}$, so $\int_{\mathbb{R}^3} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x} = \infty$. Thus, wavefunction (3B.8) cannot be translated into a probability distribution, so it is not physically meaningful. This isn't too surprising, because wavefunction (3B.8) seems to suggest that the electron is equally likely to be anywhere in the (infinite) 'universe' $\mathbb{R}^3$! It is well known that the location of a quantum particle can be 'dispersed' over some region of space, but this seems a bit extreme. There are two solutions to this problem.

- Let $\mathbb{B}(R) \subset \mathbb{R}^3$ be a ball of radius R, where R is much larger than the physical system or laboratory apparatus we are modelling (e.g. $R = 1$ lightyear). Define the wavefunction $\omega_t^{(R)}(\mathbf{x})$ by (3B.8) for all $\mathbf{x} \in \mathbb{B}(R)$, and set $\omega_t^{(R)}(\mathbf{x}) = 0$ for all $\mathbf{x} \notin \mathbb{B}(R)$. This means that the position of the electron is still extremely dispersed (indeed, 'infinitely' dispersed for the purposes of any laboratory experiment), but the function $\omega_t^{(R)}$ is still square-integrable. Note that the function $\omega_t^{(R)}$ violates the Schrodinger equation at the boundary of $\mathbb{B}(R)$, but this boundary occurs very far from the physical system we are studying, so it doesn't matter. In a sense, the solution (3B.8) can be seen as the 'limit' of $\omega^{(R)}$ as $R \to \infty$.
- Reject the wavefunction (3B.8) as 'physically meaningless'. Our starting assumption —an electron with a precisely known velocity —has led to a contradiction. Our conclusion: a free quantum particle can *never* have a precisely known classical velocity. Any physically meaningful wavefunction in a vacuum must contain a 'mixture' of several velocities.

$\diamondsuit$

Remark. (*The meaning of phase*) At any point $\mathbf{x}$ in space and moment t in time, the wavefunction $\omega_t(\mathbf{x})$ can be described by its *amplitude* $A_t(\mathbf{x}) := |\omega_t(\mathbf{x})|$ and its *phase* $\phi_t(\mathbf{x}) := \omega_t(\mathbf{x})/A_t(\mathbf{x})$. We have already discussed the physical meaning of the amplitude: $|A_t(\mathbf{x})|^2$ is the *probability* that a classical measurement will produce the outcome $\mathbf{x}$ at time t. What is the meaning of phase?

The phase $\phi_t(\mathbf{x})$ is a complex number of modulus one —an element of the unit circle in the complex plane (hence $\phi_t(\mathbf{x})$ is sometimes called the *phase angle*). The 'oscillation' of the wavefunction ω over time can be imagined in terms of the 'rotation' of $\phi_t(\mathbf{x})$ around the circle. The 'wavelike' properties of quantum systems (e.g. interference patterns) occur because wavefunctions with different phases will partially cancel one another when they are superposed. In other words, it is because of *phase* that the Schrodinger Equation yields 'wave-like' phenomena, instead of yielding 'diffusive' phenomena like the heat equation.

However, like potential energy, phase is *not directly physically observable*. We can observe the phase *difference* between wavefunction α and wavefunction β (by observing cancelation between α and β), just as we can observe the potential energy *difference* between point A and point B (by measuring the energy released by a particle moving from point A to point B). However, it is not physically meaningful to speak of the 'absolute phase' of wavefunction α, just as it is not physically meaningful to speak of the 'absolute potential energy' of point A.

Indeed, inspection of the Schrödinger equation (3B.3) on page 41 will reveal that the speed of phase rotation of a wavefunction ω at point $\mathbf{x}$ is determined by the magnitude of the potential function V at $\mathbf{x}$. But we can arbitrarily increase V by a constant, without changing its physical meaning. Thus, we can arbitrarily 'accelerate' the phase rotation of the wavefunction without changing the physical meaning of the solution.

3C Stationary Schrödinger equation

Prerequisites: §3B. **Recommended:** §4B(iv).

A 'stationary' state of a quantum system is one where the probability density does not change with time. This represents a physical system which is in some kind of long-term equilibrium. Note that a stationary quantum state does *not* mean that the particles are 'not moving' (whatever 'moving' means for quanta). It instead means that they are moving in some kind of regular, confined pattern (i.e. an 'orbit') which remains qualitatively the same over time. For example, the orbital of an electron in a hydrogen atom should be a stationary state, because (unless the electron absorbs or emits energy) the orbital should stay the same over time.

Mathematically speaking, a stationary wavefunction ω yields a time-invariant probability density function $\rho : \mathbb{X} \longrightarrow \mathbb{R}$ such that, for any $t \in \mathbb{R}$,

$$|\omega_t(\mathbf{x})|^2 \quad = \quad \rho(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in \mathbb{X}.$$

The simplest way to achieve this is to assume that ω has the *separated* form

$$\omega_t(\mathbf{x}) \quad = \quad \phi(t) \cdot \omega_0(\mathbf{x}), \tag{3C.1}$$

where $\omega_0 : \mathbb{X} \longrightarrow \mathbb{C}$ and $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the conditions

$$|\phi(t)| \;=\; 1, \text{ for all } t \in \mathbb{R}, \quad \text{and} \quad |\omega_0(\mathbf{x})| \;=\; \sqrt{\rho(\mathbf{x})}, \text{ for all } x \in \mathbb{X}. \tag{3C.2}$$

Lemma 3C.1. *Suppose* $\omega_t(\mathbf{x}) = \phi(t) \cdot \omega_0(\mathbf{x})$ *is a separated solution to the Schrödinger equation, as in* eqn.(3C.1) *and* eqn.(3C.2). *Then there is some constant* $E \in \mathbb{R}$ *so that*

- $\phi(t) = \exp(-\mathbf{i}Et/\hbar)$, *for all* $t \in \mathbb{R}$.
- $\mathsf{H}\,\omega_0 = E \cdot \omega_0$; *in other words* ω_0 *is an* ***eigenfunction***[3] *of the Hamiltonian operator* H, *with* ***eigenvalue*** E.
- *Thus,* $\omega_t(\mathbf{x}) = e^{-\mathbf{i}Et/\hbar} \cdot \omega_0(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{X}$ *and* $t \in \mathbb{R}$.

Ⓔ *Proof.* **Exercise 3C.1** Hint: use separation of variables.[4] □

Physically speaking, E corresponds to the *total energy* (potential + kinetic) of the quantum system[5]. Thus, Lemma 3C.1 yields one of the key concepts of quantum theory:

> *Eigenfunctions of the Hamiltonian correspond to stationary quantum states. The eigenvalues of these eigenfunctions correspond to the energy level of these states.*

Thus, to get stationary states, we must solve the **stationary Schrödinger equation:**

$$\mathsf{H}\,\omega_0 \quad = \quad E \cdot \omega_0,$$

where $E \in \mathbb{R}$ is an unknown constant (the energy eigenvalue), and $\omega_0 : \mathbb{X} \longrightarrow \mathbb{C}$ is an unknown wavefunction.

Example 3C.2: The Free Electron

Recall 'free electron' of Example 3B.2. If the electron has velocity v, then the function ω in eqn.(3B.7) yields a solution to the stationary Schrödinger equation, with eigenvalue $E = \frac{1}{2} m_e\, v^2$. [See practice problem # 3 on page 54 of §3D]. Observe that E corresponds to the classical *kinetic energy* of an electron with velocity v. ♢

[3]See § 4B(iv) on page 63.

[4]See Chapter 16 on page 353.

[5]This is not obvious, but it's a consequence of the fact that the Hamiltonian $\mathsf{H}\,\omega$ measures the total energy of the wavefunction ω. Loosely speaking, the term $\frac{\hbar^2}{2} \blacktriangle \omega$ represents the 'kinetic energy' of ω, while the term $V \cdot \omega$ represents the 'potential energy'.

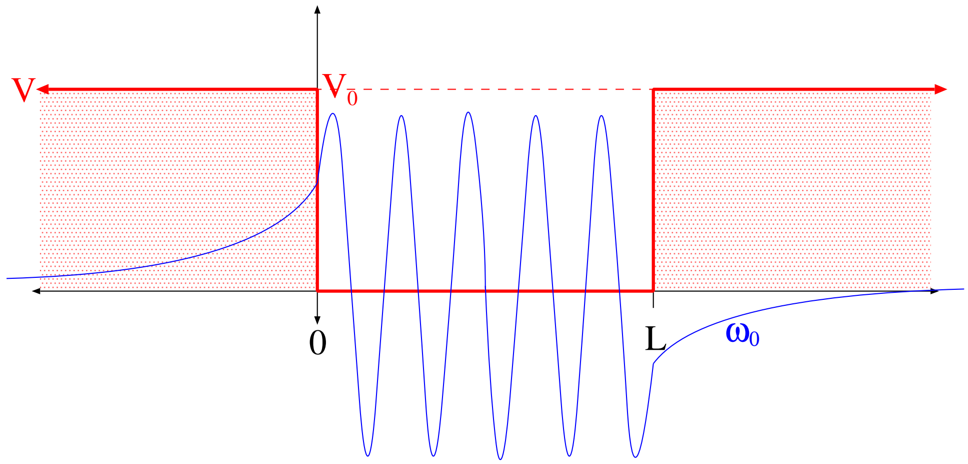


Figure 3C.1: The (stationary) wavefunction of an electron in a one-dimensional 'square' potential well, with finite voltage gaps.

Example 3C.3: One-dimensional square potential well; finite voltage

Consider an electron confined to a one-dimensional environment (e.g. a long conducting wire). Thus, $\mathbb{X} := \mathbb{R}$, and the wavefunction $\omega_0 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ obeys the one-dimensional Schrödinger equation

$$\mathbf{i}\partial_t \omega_0 \quad = \quad \frac{-1}{2}\partial_x^2 \omega_0 + V \cdot \omega_0,$$

where $V : \mathbb{R} \longrightarrow \mathbb{R}$ is the potential energy function, and we have adopted atomic units. Let $V_0 > 0$ be some constant, and suppose that

$$V(x) \quad = \quad \begin{cases} 0 & \text{if} \quad 0 \leq x \leq L; \\ V_0 & \text{if} \quad x < 0 \text{ or } L < x. \end{cases}$$

Physically, this means that V defines a 'potential energy well', which tries to confine the electron in the interval $[0, L]$, between two 'walls', which are voltage gaps of height V_0 (see Figure 3C.1). The corresponding stationary Schrödinger equation is:

$$\frac{-1}{2}\partial_x^2 \omega_0 + V \cdot \omega_0 \quad = \quad E \cdot \omega_0, \tag{3C.3}$$

where $E > 0$ is an (unknown) eigenvalue which corresponds to the energy of the electron. The function V only takes two values, so we can split eqn.(3C.3) into two equations, one inside the interval $[0, L]$, and one outside it:

$$\begin{array}{rcll} \frac{-1}{2}\partial_x^2 \omega_0(x) & = & E \cdot \omega_0(x), & \text{for } x \in [0, L]; \\ \frac{-1}{2}\partial_x^2 \omega_0(x) & = & (E - V_0) \cdot \omega_0(x), & \text{for } x \notin [0, L]. \end{array} \tag{3C.4}$$

Assume that $E < V_0$. This means that the electron's energy is less than the voltage gap, so the electron has insufficient energy to 'escape' the interval (at least in classical theory). The (physically meaningful) solutions to eqn.(3C.4) have the form

$$\omega_0(x) \quad = \quad \begin{cases} C\exp(\epsilon' x), & \text{if } x \in (-\infty, 0]; \\ A\sin(\epsilon x) + B\cos(\epsilon x), & \text{if } x \in [0, L]; \\ D\exp(-\epsilon' x), & \text{if } L \in [L, \infty). \end{cases} \tag{3C.5}$$

(See Figure 3C.1.) Here, $\epsilon := \sqrt{2E}$ and $\epsilon' := \sqrt{2E - 2V_0}$, and $A, B, C, D \in \mathbb{C}$ are constants. The corresponding solution to the full Schrödinger equation is:

$$\omega_t(x) \quad = \quad \begin{cases} Ce^{-\mathbf{i}(E-V_0)t} \cdot \exp(\epsilon' x), & \text{if } x \in (-\infty, 0]; \\ e^{-\mathbf{i}Et} \cdot (A\sin(\epsilon x) + B\cos(\epsilon x)), & \text{if } x \in [0, L]\ ; \\ De^{-\mathbf{i}(E-V_0)t} \cdot \exp(-\epsilon' x), & \text{if } L \in [L, \infty). \end{cases} \qquad \text{for all } t \in \mathbb{R}.$$

This has two consequences:

(a) With nonzero probability, the electron might be found *outside* the interval $[0, L]$. In other words, it is quantumly possible for the electron to 'escape' from the potential well, something which is classically impossible[6]. This phenomenon called *quantum tunnelling* (because the electron can 'tunnel' through the wall of the well).

(b) The system has a physically meaningful solution only for certain values of E. In other words, the electron is only 'allowed' to reside at certain discrete *energy levels*; this phenomenon is called *quantization of energy*.

To see **(a)**, recall that the electron has probability distribution

$$\rho(x) \;:=\; \frac{1}{W}|\omega_0(x)|^2, \quad \text{where } W \;:=\; \int_{-\infty}^{\infty} |\omega_0(x)|^2 \, dx.$$

Thus, if $C \neq 0$, then $\rho(x) \neq 0$ for $x < 0$, while if $D \neq 0$, then $\rho(x) \neq 0$ for $x > L$. Either way, the electron has nonzero probability of 'tunnelling' out of the well.

To see **(b)**, note that we must choose A, B, C, D so that ω_0 is continuously differentiable at the boundary points $x = 0$ and $x = L$. This means we must have

$$\begin{array}{rcccccccc} B & = & A\sin(0) + B\cos(0) & = & \omega_0(0) & = & C\exp(0) & = & C \\ \epsilon A & = & A\epsilon\cos(0) - B\epsilon\sin(0) & = & \omega_0'(0) & = & \epsilon' C\exp(0) & = & \epsilon' C \\ & & A\sin(\epsilon L) + B\cos(\epsilon L) & = & \omega_0(L) & = & D\exp(-\epsilon' L) & & \\ & & A\epsilon\cos(\epsilon L) - B\epsilon\sin(\epsilon L) & = & \omega_0'(L) & = & -\epsilon' D\exp(-\epsilon' L) & & \end{array} \tag{3C.6}$$

[6] Many older texts observe that the electron 'can penetrate the classically forbidden region', which has caused mirth to generations of physics students.

Clearly, we can satisfy the first two equations in (3C.6) by setting $B := C := \frac{\epsilon}{\epsilon'}A$. The third and fourth equations in (3C.6) then become

$$e^{\epsilon' L}\cdot\left(\sin(\epsilon L) + \frac{\epsilon}{\epsilon'}\cos(\epsilon L)\right)\cdot A \quad = \quad D \quad = \quad \frac{-\epsilon}{\epsilon'}e^{\epsilon' L}\cdot\left(\cos(\epsilon L) - \frac{\epsilon}{\epsilon'}\sin(\epsilon L)\right)A, \tag{3C.7}$$

Cancelling the factors $e^{\epsilon' L}$ and A from both sides and substituting $\epsilon := \sqrt{2E}$ and $\epsilon' := \sqrt{2E - 2V_0}$, we see that eqn.(3C.7) is satisfiable if and only if

$$\sin\left(\sqrt{2E}\cdot L\right) + \frac{\sqrt{E}\cdot\cos(\sqrt{2E}\cdot L)}{\sqrt{E - V_0}} \quad = \quad \frac{-\sqrt{E}\cdot\cos\left(\sqrt{2E}\cdot L\right)}{\sqrt{E - V_0}} + \frac{E\cdot\sin(\sqrt{2E}\cdot L)}{E - V_0}. \tag{3C.8}$$

Hence, eqn.(3C.4) has a physically meaningful solution only for those values of E which satisfy the transcendental equation (3C.8). The set of solutions to eqn.(3C.8) is an infinite discrete subset of $\mathbb{R}$; each solution for eqn.(3C.8) corresponds to an allowed 'energy level' for the physical system. ◊

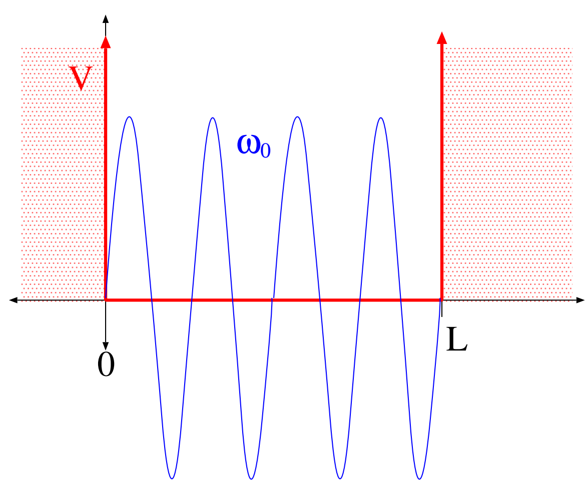


Figure 3C.2: The (stationary) wavefunction of an electron in an infinite potential well.

Example 3C.4: One-dimensional square potential well; infinite voltage

We can further simplify the model of Example 3C.3 by setting $V_0 := +\infty$, which physically represents a 'huge' voltage gap that totally confines the electron within the interval $[0, L]$ (see Figure 3C.2). In this case, $\epsilon' = \infty$, so $\exp(\epsilon' x) = 0$ for all $x < 0$ and $\exp(-\epsilon' x) = 0$ for all $x > L$. Hence, if ω_0 is as in eqn.(3C.5), then $\omega_0(x) \equiv 0$ for all $x \notin [0, L]$, and the constants C and D are no longer physically meaningful; we set $C = 0 = D$ for simplicity. Also, we must have $\omega_0(0) = 0 = \omega_0(L)$ to get a continuous solution; thus, we must set $B := 0$ in eqn.(3C.5). Thus, the stationary solution in eqn.(3C.5) becomes

$$\omega_0(x) \quad = \quad \begin{cases} 0 & \text{if } \ x \notin [0, L]; \\ A \cdot \sin(\sqrt{2E}\, x) & \text{if } \ x \in [0, L], \end{cases}$$

where A is a constant, and E satisfies the equation

$$\sin(\sqrt{2E}\, L) = 0. \qquad \text{(Figure 3C.2)} \tag{3C.9}$$

Assume for simplicity that $L := \pi$. Then eqn.(3C.9) is true if and only if $\sqrt{2E}$ is an integer, which means $2E \in \{0, 1, 4, 9, 16, 25, \ldots\}$, which means $E \in \{0,\ \frac{1}{2},\ 2,\ \frac{9}{2},\ 8,\ \frac{25}{2}, \ldots\}$. Here we see the phenomenon of *quantization of energy* in its simplest form. ♢

The set of eigenvalues of a linear operator is called the **spectrum** of that operator. For example, in Example 3C.4, the spectrum of the Hamiltonian operator H is the set $\{0,\ \frac{1}{2},\ 2,\ \frac{9}{2},\ 8,\ \frac{25}{2}, \ldots\}$. In quantum theory, the spectrum of the Hamiltonian is the set of allowed energy levels of the system.

Example 3C.5: Three-dimensional square potential well; infinite voltage

We can easily generalize Example 3C.4 to three dimensions. Let $\mathbb{X} := \mathbb{R}^3$, and let $\mathbb{B} := [0, \pi]^3$ be a cube with one corner at the origin, having sidelength $L = \pi$. We use the potential function $V : \mathbb{X} \longrightarrow \mathbb{R}$ defined

$$V(\mathbf{x}) \quad = \quad \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \mathbb{B}; \\ +\infty & \text{if} \quad \mathbf{x} \notin \mathbb{B}. \end{cases}$$

Physically, this represents an electron confined within a cube of perfectly conducting material with perfectly insulating boundaries[7]. Suppose the electron has energy E. The corresponding stationary Schrödinger equation is

$$\begin{array}{rcll} \frac{-1}{2} \triangle \omega_0(\mathbf{x}) & = & E \cdot \omega_0(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{B}; \\ \frac{-1}{2} \triangle \omega_0(\mathbf{x}) & = & -\infty \cdot \omega_0(\mathbf{x}) & \text{for } \mathbf{x} \notin \mathbb{B}; \end{array} \tag{3C.10}$$

(in atomic units). By reasoning similar Example 3C.4, we find that the physically meaningul solutions to eqn.(3C.10) have the form

$$\omega_0(\mathbf{x}) \quad = \quad \begin{cases} \frac{\sqrt{2}}{\pi^{3/2}} \sin(n_1 x_1) \cdot \sin(n_2 x_2) \cdot \sin(n_3 x_3) & \text{if } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{B}; \\ 0 & \text{if } \mathbf{x} \notin \mathbb{B}. \end{cases} \tag{3C.11}$$

where n_1, n_2, and n_3 are arbitrary integers (called the *quantum numbers* of the solution), and $E = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2)$ is the associated energy eigenvalue.

[7] Alternately, it could be any kind of particle, confined in a cubical cavity with impenetrable boundaries.

The corresponding solution to the full Schrödinger equation for all $t \in \mathbb{R}$ is

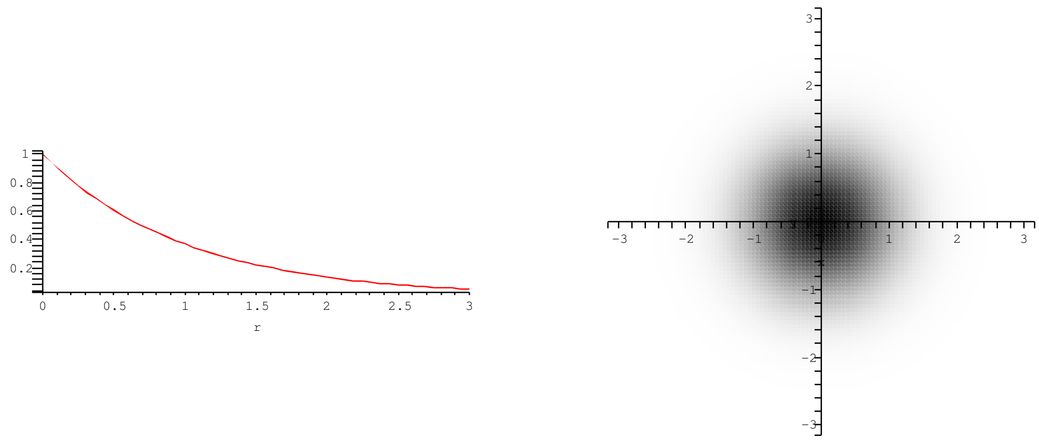
$$\omega_t(\mathbf{x}) = \begin{cases} \frac{\sqrt{2}}{\pi^{3/2}} e^{-\mathbf{i}(n_1^2+n_2^2+n_3^2)t/2} \cdot \sin(n_1 x_1)\sin(n_2 x_2)\sin(n_3 x_3) & \text{if } \mathbf{x} \in \mathbb{B}; \\ 0 & \text{if } \mathbf{x} \notin \mathbb{B}. \end{cases}$$

♢

Exercise 3C.2. (a) Check that eqn.(3C.11) is a solution for eqn.(3C.10). Ⓔ
(b) Check that $\rho := |\omega|^2$ is a probability density, by confirming that

$$\int_{\mathbb{X}} |\omega_0(\mathbf{x})|^2 \, d\mathbf{x} = \frac{2}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \sin(n_1 x_1)^2 \cdot \sin(n_2 x_2)^2 \cdot \sin(n_3 x_3)^2 \, dx_1 \, dx_2 \, dx_3 = 1,$$

(this is the reason for using the constant $\frac{\sqrt{2}}{\pi^{3/2}}$). ♦



(A) **(B)**

Figure 3C.3: The groundstate wavefunction for a hydrogen atom. **(A)** Probability density as a function of distance from the nucleus. **(B)** Probability density visualized in three dimensions.

Example 3C.6: Hydrogen Atom

In Example 3A.3 on page 39, we described the hydrogen atom as a two-particle system, with a six-dimensional state space. However, the corresponding Schrödinger equation (Example 3B.1(c)) is already too complicated for us to solve it here, so we will work with a simplified model.

Because the proton is 1864 times as massive as the electron, we can treat the proton as remaining effectively immobile while the electron moves around it. Thus, we can model the hydrogen atom as a *one*-particle system: a single electron moving in a Coulomb potential well, as described in Example 3B.1(b). The electron then satisfies the Schrödinger equation

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_t(\mathbf{x}) \;+\; \frac{q_e^2}{4\pi\epsilon_0\cdot|\mathbf{x}|}\cdot\omega_t(\mathbf{x}), \quad \forall\;\mathbf{x}\in\mathbb{R}^3. \qquad (3C.12)$$

(Recall that m_e is the mass of the electron, q_e is the charge of both electron and proton, ϵ_0 is the 'permittivity of free space', and $\hbar$ is the rationalized Plank constant.) Assuming the electron is in a stable orbital, we can replace eqn.(3C.12) with the *stationary* Schrödinger equation

$$\frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_0(\mathbf{x}) \;+\; \frac{q_e^2}{4\pi\epsilon_0\cdot|\mathbf{x}|}\cdot\omega_0(\mathbf{x}) \quad = \quad E\cdot\omega_0(\mathbf{x}), \quad \forall\;\mathbf{x}\in\mathbb{R}^3, \qquad (3C.13)$$

where E is the 'energy level' of the electron. One solution to this equation is

$$\omega(\mathbf{x}) \quad = \quad \frac{b^{3/2}}{\sqrt{\pi}}\exp(-b|\mathbf{x}|), \quad \text{where } b \;:=\; \frac{m\,q_e^2}{4\pi\epsilon_0\,\hbar^2}, \qquad (3C.14)$$

with corresponding energy eigenvalue

$$E \quad = \quad \frac{-\hbar^2}{2m}\cdot b^2 \quad = \quad \frac{-m\,q_e^4}{32\pi^2\epsilon_0^2\,\hbar^2} \qquad (3C.15)$$

Ⓔ **<u>Exercise 3C.3.</u>** (a) Verify that the function ω_0 in eqn.(3C.14) is a solution to eqn.(3C.13), with E given by eqn.(3C.15).

(b) Verify that the function ω_0 defines a probability density, by checking that $\int_{\mathbb{X}}|\omega|^2 = 1$. ♦

There are many other, more complicated solutions to eqn.(3C.13). However, eqn.(3C.14) is the simplest solution, and has the *lowest* energy eigenvalue E of any solution. In other words, the solution (3C.13) describes an electron in the *ground state*: the orbital of lowest potential energy, where the electron is 'closest' to the nucleus.

This solution immediately yields two experimentally testable predictions:

(a) The *ionization potential* for the hydrogen atom, which is the energy required to 'ionize' the atom, by stripping off the electron and removing it to an infinite distance from the nucleus.

(b) The *Bohr radius* of the hydrogen atom —that is, the 'most probable' distance of the electron from the nucleus.

To see **(a)**, recall that E is the sum of potential and kinetic energy for the electron. We assert (without proof) that there exist solutions to the stationary Schrödinger equation (3C.13) with energy eigenvalues arbitrarily close to zero (note that E is negative). These zero-energy solutions represent orbitals where the electron has been removed to some very large distance from the nucleus, and the atom is essentially ionized. Thus, the energy difference between these 'ionized' states and ω_0 is $E - 0 = E$, and this is the energy necessary to 'ionize' the atom when the electron is in the orbital described by ω_0.

By substituting in numerical values $q_e \approx 1.60 \times 10^{-19}$ C, $\epsilon_0 \approx 8.85 \times 10^{-12}$ C/N m^2, $m_e \approx 9.11 \times 10^{-31}$ kg, and $\hbar \approx 1.0545 \times 10^{-34}$ J s, the reader can verify that, in fact, $E \approx -2.1796 \times 10^{-18}$ J ≈ -13.605 eV, which is very close to -13.595 eV, the experimentally determined ionization potential for a hydrogen atom.[8]

To see **(b)**, observe that the probability density function for the distance r of the electron from the nucleus is given by

$$P(r) \quad = \quad 4\pi r^2 |\omega(r)|^2 \quad = \quad 4b^3 r^2 \exp(-2b|\mathbf{x}|).$$

(**<u>Exercise 3C.4</u>**). The *mode* of the radial probability distribution is the maximal point of $P(r)$; if we solve the equation $P'(r) = 0$, we find that the mode occurs at Ⓔ

$$r \quad := \quad \frac{1}{b} \quad = \quad \frac{4\pi\epsilon_0\hbar^2}{m_e\, q_e^2} \quad \approx \quad 5.29172 \times 10^{-11} \text{ m}. \qquad \diamondsuit$$

The Balmer Lines. Recall that the **spectrum** of the Hamiltonian operator H is the set of all eigenvalues of H. Let $\mathcal{E} = \{E_0 < E_1 < E_2 < \ldots\}$ be the spectrum of the Hamiltonian of the hydrogen atom from Example 3C.6, with the elements listed in increasing order. Thus, the smallest eigenvalue is $E_0 \approx -13.605$, the energy eigenvalue of the aforementioned ground state ω_0. The other, larger eigenvalues correspond to electron orbitals with higher potential energy.

When the electron 'falls' from a high energy orbital (with eigenvalue E_n, for some $n \in \mathbb{N}$) to a low energy orbital (with eigenvalue E_m, where $m < n$), it releases the energy difference, and emits a photon with energy $(E_n - E_m)$. Conversely, to 'jump' from a low E_m-energy orbital to a higher E_n-energy orbital, the electron must *absorb* a photon, and this photon must have exactly energy $(E_n - E_m)$.

Thus, the hydrogen atom can only emit or absorb photons of energy $|E_n - E_m|$, for some $n, m \in \mathbb{N}$. Let $\mathcal{E}' := \{|E_n - E_m| \,;\, n, m \in \mathbb{N}\}$. We call $\mathcal{E}'$ the *energy spectrum* of the hydrogen atom.

Planck's law says that a photon with energy E has frequency $f = E/h$, where $h \approx 6.626 \times 10^{-34}$ J s is Planck's constant. Thus, if $\mathcal{F} = \{E/h \,;\, E \in \mathcal{E}'\}$, then a hydrogen atom can only emit/absorb a photon whose frequency is in $\mathcal{F}$; we say $\mathcal{F}$ is the *frequency spectrum* of the hydrogen atom.

[8] The error of 0.01 eV is mainly due to our simplifying assumption of an 'immobile' proton.

Here lies the explanation for the empirical observations of 19th century physicists such as Balmer, Lyman, Rydberg, and Paschen, who found that an energized hydrogen gas has a distinct *emission spectrum* of frequencies at which it emits light, and an identical *absorption spectrum* of frequencies which the gas can absorb. Indeed, every chemical element has its own distinct spectrum; astronomers use these 'spectral signatures' to measure the concentrations of chemical elements in the stars of distant galaxies. Now we see that

> *The (frequency) spectrum of an atom is determined by the (eigenvalue) spectrum of the corresponding Hamiltonian.*

Further reading

Unfortunately, most other texts on partial differential equations do not discuss the Schrödinger equation; one of the few exceptions is the excellent text [Asm05]. For an lucid, fast, yet precise introduction to quantum mechanics in general, see [McW72]. For a more comprehensive textbook on quantum theory, see [Boh79]. A completely different approach to quantum theory uses Feynman's *path integrals*; for a good introduction to this approach, see [Ste95], which also contains excellent introductions to classical mechanics, electromagnetism, statistical physics, and special relativity. For a rigorous mathematical approach to quantum theory, an excellent introduction is [Pru81]; another source is [BEH94].

3D Practice problems

1. Let $v_1 \in \mathbb{R}$ be a constant. Consider the function $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\omega_t(x_1, x_2, x_3) \;=\; \exp\left(\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,t\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\,m_e v_1 \cdot x_1\right).$$

   Show that ω satisfies the (free) *Schrödinger equation:* $\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \;=\; \dfrac{-\hbar^2}{2\,m_e} \triangle\, \omega_t(\mathbf{x})$.

2. Let $\mathbf{v} := (v_1, v_2, v_3)$ be a three-dimensional velocity vector, and let $|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$. Consider the function $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\omega_t(x_1, x_2, x_3) \quad = \quad \exp\left(-\mathbf{i}\,|\mathbf{v}|^2\,t/2\right) \cdot \exp\left(\mathbf{i}\,\mathbf{v} \bullet \mathbf{x}\right).$$

   Show that ω satisfies the (free) *Schrödinger equation:* $\mathbf{i}\,\partial_t\,\omega \quad = \quad \dfrac{-1}{2}\triangle\omega$.

3. Consider the stationary Schrödinger equation for a null potential:

$$\mathsf{H}\,\omega_0 \quad = \quad E \cdot \omega_0, \qquad \text{where} \qquad \mathsf{H} = \frac{-\hbar^2}{2m_e}\triangle\,.$$

Let $v \in \mathbb{R}$ be a constant. Consider the function $\omega_0 : \mathbb{R}^3 \longrightarrow \mathbb{C}$ defined:

$$\omega_0(x_1, x_2, x_3) \;=\; \exp\left(\frac{\mathbf{i}}{\hbar}\, m_e v_1 \cdot x_1\right).$$

Show that ω_0 is a solution to the above stationary Schrödinger equation, with eigenvalue $E = \frac{1}{2} m_e\, v^2$.

4. Exercise 3C.2(a) (page 51).

5. Exercise 3C.3(a) (page 52).

II General theory

Chapter 4

Linear partial differential equations

"The Universe is a grand book which cannot be read until one first learns the language in which it is composed. It is written in the language of mathematics." —Galileo Galilei

4A Functions and vectors

Prerequisites: §0A.

Vectors: If $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1.5 \\ 3 \\ 1 \end{bmatrix}$, then we can add these two vectors *componentwise*:

$$\mathbf{v} + \mathbf{w} \;=\; \begin{bmatrix} 2-1.5 \\ 7+3 \\ -3+1 \end{bmatrix} \;=\; \begin{bmatrix} 0.5 \\ 10 \\ -2 \end{bmatrix}.$$

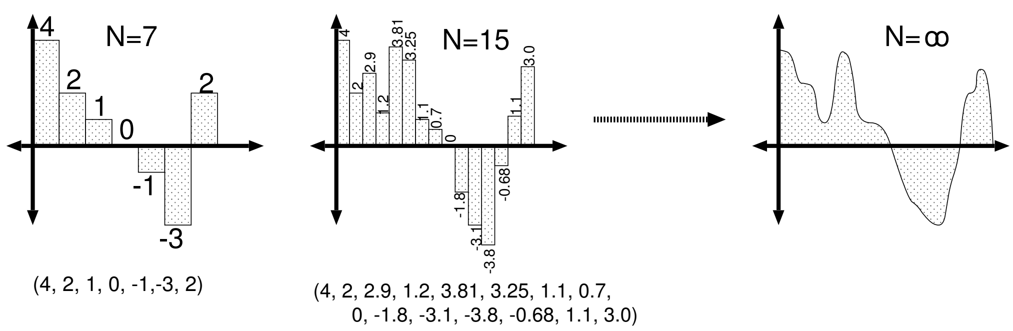


Figure 4A.1: We can think of a function as an "infinite-dimensional vector"

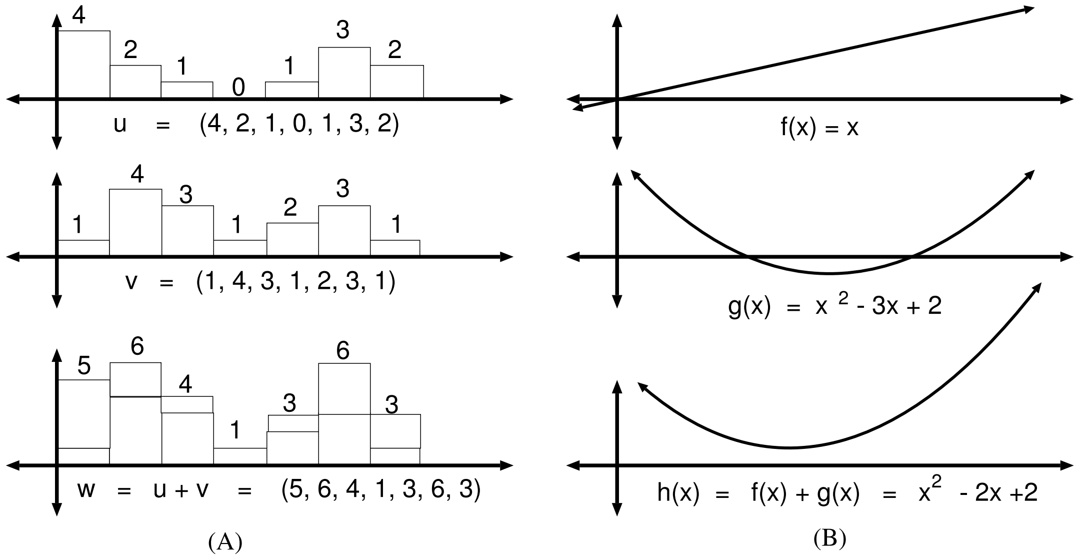


Figure 4A.2: **(A)** We add vectors *componentwise*: If $\mathbf{u} = (4,2,1,0,1,3,2)$ and $\mathbf{v} = (1,4,3,1,2,3,1)$, then the equation "$\mathbf{w} = \mathbf{v} + \mathbf{w}$" means that $\mathbf{w} = (5,6,4,1,3,6,3)$. **(B)** We add two functions *pointwise*: If $f(x) = x$, and $g(x) = x^2 - 3x + 2$, then the equation "$h = f + g$" means that $h(x) = f(x) + g(x) = x^2 - 2x + 2$ for every x.

In general, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is defined by:

$$u_n = v_n + w_n, \quad \text{for } n = 1, 2, 3 \tag{4A.1}$$

(see Figure 4A.2A) Think of $\mathbf{v}$ as a function $v : \{1, 2, 3\} \longrightarrow \mathbb{R}$, where $v(1) = 2$, $v(2) = 7$, and $v(3) = -3$. If we likewise represent $\mathbf{w}$ with $w : \{1, 2, 3\} \longrightarrow \mathbb{R}$ and $\mathbf{u}$ with $u : \{1, 2, 3\} \longrightarrow \mathbb{R}$, then we can rewrite eqn.(4A.1) as "$u(n) = v(n) + w(n)$ for $n = 1, 2, 3$". In a similar fashion, any N-dimensional vector $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ can be thought of as a function $u : [1...N] \longrightarrow \mathbb{R}$.

Functions as Vectors: Letting N go to infinity, we can imagine any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as a sort of "infinite-dimensional vector" (see Figure 4A.1). Indeed, if f and g are two functions, we can add them *pointwise*, to get a new function $h = f + g$, where

$$h(x) = f(x) + g(x), \quad \text{for all } x \in \mathbb{R} \tag{4A.2}$$

(see Figure 4A.2B) Notice the similarity between formulae (4A.2) and (4A.1), and the similarity between Figures 4A.2A and 4A.2B.

One of the most important ideas in the theory of PDEs is that *functions are infinite-dimensional vectors*. Just as with finite vectors, we can add them together, act on them with linear operators, or represent them in different *coordinate systems* on infinite-dimensional space. Also, the vector space $\mathbb{R}^D$ has a natural geometric structure; we can identify a similar geometry in infinite dimensions.

Let $\mathbb{X} \subseteq \mathbb{R}^D$ be some domain. The vector space of all continuous functions from $\mathbb{X}$ into $\mathbb{R}^m$ is denoted $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$. That is:

$$\mathcal{C}(\mathbb{X};\mathbb{R}^m) \quad := \quad \{f : \mathbb{X} \longrightarrow \mathbb{R}^m \ ; \ f \text{ is continuous}\}.$$

When $\mathbb{X}$ and $\mathbb{R}^m$ are obvious from context, we may just write "$\mathcal{C}$".

Exercise 4A.1. Show that $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$ is a vector space. ♦ Ⓔ

A scalar field $f : \mathbb{X} \longrightarrow \mathbb{R}$ is **infinitely differentiable** (or **smooth**) if, for every $N > 0$ and every $i_1, i_2, \ldots, i_N \in [1...D]$, the Nth derivative $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_N} f(\mathbf{x})$ exists at each $\mathbf{x} \in \mathbb{X}$. A vector field $f : \mathbb{X} \longrightarrow \mathbb{R}^m$ is **infinitely differentiable** (or **smooth**) if $f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where each of the scalar fields $f_1, \ldots, f_m : \mathbf{X} \longrightarrow \mathbb{R}$ is infinitely differentiable. The vector space of all smooth functions from $\mathbb{X}$ into $\mathbb{R}^m$ is denoted $\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m)$. That is:

$$\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m) \quad := \quad \{f : \mathbb{X} \longrightarrow \mathbb{R}^m \ ; \ f \text{ is infinitely differentiable}\}.$$

When $\mathbb{X}$ and $\mathbb{R}^m$ are obvious from context, we may just write "$\mathcal{C}^\infty$".

Example 4A.1.

(a) $\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ is the space of all smooth *scalar fields* on the *plane* (i.e. all functions $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$).

(b) $\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^3)$ is the space of all smooth *curves* in *three-dimensional space*. ♢

Exercise 4A.2. Show that $\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m)$ is a vector space, and thus, a linear subspace of $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$. ♦ Ⓔ

4B Linear operators

Prerequisites: §4A.

4B(i) ...on finite dimensional vector spaces

Let $\mathbf{v} := \begin{bmatrix}2\\7\end{bmatrix}$ and $\mathbf{w} := \begin{bmatrix}-1.5\\3\end{bmatrix}$, and let $\mathbf{u} := \mathbf{v}+\mathbf{w} = \begin{bmatrix}0.5\\10\end{bmatrix}$. If $\mathbf{A} := \begin{bmatrix}1 & -1\\4 & 0\end{bmatrix}$, then $\mathbf{A}\cdot\mathbf{u} = \mathbf{A}\cdot\mathbf{v} + \mathbf{A}\cdot\mathbf{w}$. That is:

$$\begin{bmatrix}1 & -1\\4 & 0\end{bmatrix}\cdot\begin{bmatrix}0.5\\10\end{bmatrix} = \begin{bmatrix}-9.5\\2\end{bmatrix} = \begin{bmatrix}-5\\8\end{bmatrix}+\begin{bmatrix}-4.5\\-6\end{bmatrix} = \begin{bmatrix}1 & -1\\4 & 0\end{bmatrix}\cdot\begin{bmatrix}2\\7\end{bmatrix}+\begin{bmatrix}1 & -1\\4 & 0\end{bmatrix}\cdot\begin{bmatrix}-1.5\\3\end{bmatrix};$$

Also, if $\mathbf{x} = 3\mathbf{v} = \left[\begin{smallmatrix} 6 \\ 21 \end{smallmatrix}\right]$, then $\mathbf{Ax} = 3\mathbf{Av}$. That is:

$$\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 21 \end{bmatrix} \quad = \quad \begin{bmatrix} -15 \\ 24 \end{bmatrix} \quad = \quad 3 \begin{bmatrix} -5 \\ 8 \end{bmatrix} \quad = \quad 3 \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

In other words, multiplication by the matrix $\mathbf{A}$ is a **linear operator** on the vector space $\mathbb{R}^2$. In general, a function $L : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is **linear** if:

- For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$, we have $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
- For all $\mathbf{v} \in \mathbb{R}^N$ and $r \in \mathbb{R}$, we have $L(r \cdot \mathbf{v}) = r \cdot L(\mathbf{v})$.

Every linear function from $\mathbb{R}^N$ to $\mathbb{R}^M$ corresponds to multiplication by some $N \times M$ matrix.

Example 4B.1.

(a) **Difference Operator:** Suppose $D : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is the function:

$$D \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad = \quad \begin{bmatrix} x_1 - x_0 \\ x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix}.$$

Then D corresponds to multiplication by the matrix $\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}$.

(b) **Summation operator:** Suppose $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^5$ is the function:

$$S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad = \quad \begin{bmatrix} 0 \\ x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

Then S corresponds to multiplication by the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

(c) **Multiplication operator:** Suppose $M : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ is the function

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \cdot x_1 \\ 2 \cdot x_2 \\ -5 \cdot x_3 \\ \frac{3}{4} \cdot x_4 \\ \sqrt{2} \cdot x_5 \end{bmatrix}$$

Then M corresponds to multiplication by the matrix $\begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & -5 & & \\ & & & \frac{3}{4} & \\ & & & & \sqrt{2} \end{bmatrix}$.

$\diamondsuit$

Remark Notice that the transformation D is a *left-inverse* to the transformation S. That is, $D \circ S = \mathbf{Id}$. (However, D is *not* a *right-inverse* to S, because if $\mathbf{x} = (x_0, x_1, \ldots, x_4)$, then $S \circ D(\mathbf{x}) = \mathbf{x} - (x_0, x_0, \ldots, x_0)$. $\diamondsuit$

4B(ii) ...on $\mathcal{C}^\infty$

Recommended: §1B, §1C, §2B.

A transformation $\mathsf{L} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is called a **linear operator** if, for any two differentiable functions $f, g \in \mathcal{C}^\infty$, we have $\mathsf{L}(f + g) = \mathsf{L}(f) + \mathsf{L}(g)$, and, for any real number $r \in \mathbb{R}$, we have $\mathsf{L}(r \cdot f) = r \cdot \mathsf{L}(f)$.

Example 4B.2.

(a) **Differentiation:** If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions, and $h = f + g$, then we know that, for any $x \in \mathbb{R}$,

$$h'(x) = f'(x) + g'(x).$$

Also, if $h = r \cdot f$, then $h'(x) = r \cdot f'(x)$. Thus, if we define the operation $\mathsf{D} : \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ by $\mathsf{D}[f] \ = \ f'$, then D is a linear transformation of $\mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$. For example, sin and cos are elements of $\mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$, and we have

$$\mathsf{D}[\sin] \ = \ \cos, \qquad \text{and} \qquad \mathsf{D}[\cos] \ = \ -\sin.$$

More generally, if $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ and $h = f + g$, then for any $i \in [1..D]$,

$$\partial_j h = \partial_j f + \partial_j g.$$

Also, if $h = r \cdot f$, then $\partial_j h = r \cdot \partial_j f$. In other words, the transformation $\partial_j : \mathcal{C}^\infty(\mathbb{R}^D; \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^D; \mathbb{R})$ is a linear operator.

(b) **Integration:** If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are integrable functions, and $h = f + g$, then we know that, for any $x \in \mathbb{R}$,

$$\int_0^x h(y)\ dy \ = \ \int_0^x f(y)\ dy \ + \ \int_0^x g(y)\ dy.$$

Also, if $h = r \cdot f$, then $\displaystyle\int_0^x h(y)\ dy \ = \ r \cdot \int_0^x f(y)\ dy.$

Thus, if we define the operation $\mathsf{S} : \mathcal{C}^\infty(\mathbb{R};\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ by

$$\mathsf{S}[f](x) \ = \ \int_0^x f(y)\ dy, \qquad \text{for all } x \in \mathbb{R}.$$

then S is a linear transformation. For example, sin and cos are elements of $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, and we have

$$\mathsf{S}[\sin] \ = \ 1 - \cos, \qquad \text{and} \qquad \mathsf{S}[\cos] \ = \ \sin.$$

(c) **Multiplication:** If $\gamma : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then define the operator $\Gamma : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ by: $\Gamma[f] = \gamma \cdot f$. In other words, for all $\mathbf{x} \in \mathbb{R}^D$, $\Gamma[f](\mathbf{x}) = \gamma(\mathbf{x}) \cdot f(\mathbf{x})$. Then Γ is a linear function, because, for any $f, g \in \mathcal{C}^\infty$, $\Gamma[f+g] = \gamma \cdot [f+g] = \gamma \cdot f + \gamma \cdot g = \Gamma[f] + \Gamma[g]$. ♢

Remark. Notice that the transformation D is a *left-inverse* for the transformation S, because the Fundamental Theorem of Calculus says that $\mathsf{D} \circ \mathsf{S}(f) = f$ for any $f \in \mathcal{C}^\infty(\mathbb{R})$. However, D is *not* a *right*-inverse for S, because in general $\mathsf{S} \circ \mathsf{D}(f) = f - c$, where $c = f(0)$ is a constant. ♢

Ⓔ **Exercise 4B.1.** Compare the three linear transformations in Example 4B.2 with those from Example 4B.1. Do you notice any similarities? ♦

Remark. Unlike linear transformations on $\mathbb{R}^N$, there is in general no way to express a linear transformation on $\mathcal{C}^\infty$ in terms of multiplication by some matrix. To convince yourself of this, try to express the three transformations from example 4B.2 in terms of “matrix multiplication”. ♢

Any combination of linear operations is also a linear operation. In particular, any combination of differentiation and multiplication operations is linear. Thus, for example, the second-derivative operator $\mathsf{D}^2[f] \ = \ \partial_x^2 f$ is linear, and the Laplacian operator

$$\triangle f = \partial_1^2 f + \ldots + \partial_D^2 f$$

is also linear; in other words, $\triangle[f+g] = \triangle f + \triangle g$.

A linear transformation that is formed by adding and/or composing multiplications and differentiations is called a **linear differential operator** . For example, the Laplacian $\triangle$ is a linear differential operator.

4B(iii) Kernels

If L is a linear function, then the **kernel** of L is the set of all vectors $\mathbf{v}$ such that $L(\mathbf{v}) = 0$.

Example 4B.3.

(a) Consider the differentiation operator ∂_x on the space $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$. The kernel of ∂_x is the set of all functions $u : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\partial_x\, u \equiv 0$ —in other words, the set of all **constant** functions.

(b) The kernel of ∂_x^2 is the set of all functions $u : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\partial_x^2\, u \equiv 0$ —in other words the set of all **flat** functions of the form $u(x) = ax + b$. $\diamondsuit$

Many partial differential equations are really equations for the kernel of some differential operator.

Example 4B.4.

(a) **Laplace's equation** "$\triangle u \equiv 0$" really just says: "u is in the kernel of $\triangle$."

(b) The **heat equation** "$\partial_t\, u = \triangle u$" really just says: "$u$ is in the kernel of the operator $\mathsf{L} = \partial_t - \triangle$." $\diamondsuit$

4B(iv) Eigenvalues, eigenvectors, and eigenfunctions

If L is a linear operator on some vector space, then an **eigenvector** of L is a vector $\mathbf{v}$ such that

$$\mathsf{L}(\mathbf{v}) = \lambda \cdot \mathbf{v},$$

for some constant $\lambda \in \mathbb{C}$, called the associated **eigenvalue**.

Example 4B.5. If $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then $L(\mathbf{v}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\mathbf{v}$, so $\mathbf{v}$ is an eigenvector for L, with eigenvalue $\lambda = -1$. $\diamondsuit$

If L is a linear operator on $\mathcal{C}^\infty$, then an eigenvector of L is sometimes called an **eigenfunction**.

Example 4B.6. Let $n, m \in \mathbb{N}$. Define $u(x, y) = \sin(n \cdot x) \cdot \sin(m \cdot y)$. Then

$$\triangle u(x, y) \quad = \quad -(n^2 + m^2) \cdot \sin(n \cdot x) \cdot \sin(m \cdot y) \quad = \quad \lambda \cdot u(x, y),$$

where $\lambda = -(n^2 + m^2)$. Thus, u is an eigenfunction of the linear operator $\triangle$, with eigenvalue λ. (**Exercise 4B.2** Verify the these claims.) $\diamondsuit$ Ⓔ

Eigenfunctions of linear differential operators (particularly, eigenfunctions of $\triangle$) play a central role in the solution of linear PDEs. This is implicit in Chapters 11-14 and 20, and is made explicit in Chapter 15.

4C Homogeneous vs. nonhomogeneous

Prerequisites: §4B.

If $\mathsf{L} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is a linear differential operator, then the equation "$\mathsf{L}u \equiv 0$" is called a **homogeneous linear** partial differential equation.

Example 4C.1. The following are linear homogeneous PDEs. Here $\mathbb{X} \subset \mathbb{R}^D$ is some domain.

(a) **Laplace's Equation**[1]**:** Here, $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{X};\ \mathbb{R})$, and $\mathsf{L} = \triangle$.

(b) **heat equation**[2]**:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{X} \times \mathbb{R};\ \mathbb{R})$, and $\mathsf{L} = \partial_t - \triangle$.

(c) **wave equation**[3]**:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{X} \times \mathbb{R};\ \mathbb{R})$, and $\mathsf{L} = \partial_t^2 - \triangle$.

(d) **Schrödinger Equation**[4]**:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^{3N} \times \mathbb{R};\ \mathbb{C})$, and, for any $\omega \in \mathcal{C}^\infty$ and $(\mathbf{x}; t) \in \mathbb{R}^{3N} \times \mathbb{R}$, $\mathsf{L}\,\omega(\mathbf{x}; t) \ := \ \frac{-\hbar^2}{2} \blacktriangle\,\omega(\mathbf{x}; t) \ + \ V(\mathbf{x}) \cdot \omega(\mathbf{x}; t) \ - \ \mathbf{i}\hbar\,\partial_t\,\omega(\mathbf{x}; t)$. (Here, $V : \mathbb{R}^{3N} \longrightarrow \mathbb{R}$ is some *potential function*, and $\blacktriangle$ is like a Laplacian operator, except that the components for each particle are divided by the rest mass of that particle.)

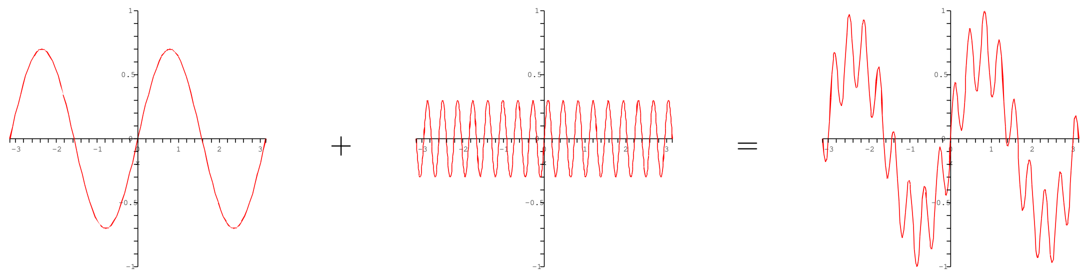
(e) **Fokker-Plank**[5]**:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{X} \times \mathbb{R};\ \mathbb{R})$, and, for any $u \in \mathcal{C}^\infty$, $\mathsf{L}(u) \ = \ \partial_t\,u \ - \ \triangle u \ + \ \vec{\mathbf{V}} \bullet \nabla u \ + \ u \cdot \operatorname{div} \vec{\mathbf{V}}$. ◊

Linear homogeneous PDEs are nice because we can combine two solutions together to obtain a third solution.

Example 4C.2.

(a) Let $u(x; t) = 7\sin[2t + 2x]$ and $v(x; t) = 3\sin[17t + 17x]$ be two travelling wave solutions to the wave equation. Then $w(x; t) = u(x; t) + v(x; t) = 7\sin(2t + 2x) + 3\sin(17t + 17x)$ is also a solution (see Figure 4C.1). To use a musical analogy: if we think of u and v as two "pure tones", then we can think of w as a "chord".

[1]See § 1C on page 9.
[2]See § 1B on page 5.
[3]See § 2B on page 27.
[4]See § 3B on page 41.
[5]See § 1F on page 18.



$u(x,t) = 7\sin(2t + 2x)$ $\quad$ $v(x;t) = 3\sin(17t + 17x)$ $\quad$ $w(x,t) = u(x;t) + v(x;t)$

Figure 4C.1: Example 4C.2(a).

(b) Let $f(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-x^2}{4t}\right]$, $\quad g(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-(x-3)^2}{4t}\right]$, and $h(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-(x-5)^2}{4t}\right]$ be one-dimensional Gauss-Weierstrass kernels, centered at 0, 3, and 5, respectively. Thus, f, g, and h are all solutions to the heat equation. Then, $F(x) = f(x) + 7 \cdot g(x) + h(x)$ is also a solution to the heat equation. If a Gauss-Weierstrass kernel models the erosion of a single "mountain", then the function F models the erosion of a little "mountain range", with peaks at 0, 3, and 5, and where the middle peak is seven times higher than the other two. $\diamondsuit$

These examples illustrate a general principle:

Theorem 4C.3. Superposition Principle for homogeneous Linear PDEs

Suppose L *is a linear differential operator, and* $u_1, u_2 \in \mathcal{C}^\infty$ *are solutions to the homogeneous linear PDE "*$\mathsf{L}u = 0$.*" Then, for any* $c_1, c_2 \in \mathbb{R}$, $u = c_1 \cdot u_1 + c_2 \cdot u_2$ *is also a solution.*

Proof. **<u>Exercise 4C.1</u>** $\square$ Ⓔ

If $q \in \mathcal{C}^\infty$ is some fixed nonzero function, then the equation "$\mathsf{L}p \equiv q$" is called a **nonhomogeneous** linear partial differential equation.

Example 4C.4. The following are linear *non*homogeneous PDEs

(a) The **antidifferentiation equation** $p' = q$ is familiar from first year calculus. The *Fundamental Theorem of Calculus* says that one solution to this equation is the integral function $p(x) = \displaystyle\int_0^x q(y)\, dy$.

(b) The **Poisson Equation**[6], "$\triangle p = q$", is a *nonhomogeneous* linear PDE. $\diamondsuit$

Recall Examples 1D.1 and 1D.2 on page 14, where we obtained *new* solutions to a nonhomogeneous equation by taking a single solution, and adding solutions of the *homogeneous* equation to this solution. These examples illustrates a general principle:

Theorem 4C.5. Subtraction Principle for nonhomogeneous linear PDEs

Suppose L *is a linear differential operator, and* $q \in \mathcal{C}^\infty$. *Let* $p_1 \in \mathcal{C}^\infty$ *be a solution to the nonhomogeneous linear PDE* "$\mathsf{L}p_1 = q$." *If* $h \in \mathcal{C}^\infty$ *is any solution to the homogeneous equation (i.e.* $\mathsf{L}h = 0$*), then* $p_2 = p_1 + h$ *is another solution to the* non*homogeneous equation. In summary:*

$$\left(\mathsf{L}p_1 = q; \quad \mathsf{L}h = 0; \quad \text{and } p_2 = p_1 + h. \right) \Longrightarrow \left(\mathsf{L}p_2 = q \right).$$

Ⓔ *Proof.* **Exercise 4C.2** □

If $\mathsf{P} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is *not* a linear operator, then a PDE of the form "$\mathsf{P}u \equiv 0$" or "$\mathsf{P}u \equiv g$" is called a **nonlinear** PDE. For example, if $F : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is some nonlinear 'rate function' describing chemical reactions, then the **reaction-diffusion equation**[7]

$$\partial_t \mathbf{u} = \triangle \mathbf{u} + F(\mathbf{u}),$$

is a *nonlinear* PDE, corresponding to the nonlinear differential operator $\mathsf{P}(\mathbf{u}) := \partial_t \mathbf{u} - \triangle \mathbf{u} - F(\mathbf{u})$.

The theory of linear partial differential equations is relatively simple, because solutions to linear PDEs interact in very nice ways, as shown by Theorems 4C.3 and 4C.5. The theory of *non*linear PDEs is much more complicated; furthermore, many of the methods which *do* exist for solving nonlinear PDEs involve somehow 'approximating' them with linear ones. In this book we shall concern ourselves only with linear PDEs.

4D Practice problems

1. For each of the following equations: u is an unknown function; q is always some fixed, predetermined function; and λ is always a constant.

   In each case, is the equation linear? If it is linear, is it homogeneous? Justify your answers.

[6] See § 1D on page 12
[7] See § 1G on page 19

(a) heat equation: $\partial_t u(\mathbf{x}) = \triangle u(\mathbf{x})$.

(b) Poisson Equation: $\triangle u(\mathbf{x}) = q(\mathbf{x})$.

(c) Laplace Equation: $\triangle u(\mathbf{x}) = 0$.

(d) Monge-Ampère Equation: $q(x,y) = \det \begin{bmatrix} \partial_x^2 u(x,y) & \partial_x \partial_y u(x,y) \\ \partial_x \partial_y u(x,y) & \partial_y^2 u(x,y) \end{bmatrix}$.

(e) Reaction-Diffusion $\partial_t u(\mathbf{x};t) = \triangle u(\mathbf{x};t) + q\Big(u(\mathbf{x};t)\Big)$.

(f) Scalar conservation Law $\partial_t u(x;t) = -\partial_x (q \circ u)(x;t)$.

(g) Helmholtz Equation: $\triangle u(\mathbf{x}) = \lambda \cdot u(\mathbf{x})$.

(h) Airy's Equation: $\partial_t u(x;t) = -\partial_x^3 u(x;t)$.

(i) Beam Equation: $\partial_t u(x;t) = -\partial_x^4 u(x;t)$.

(j) Schrödinger Equation: $\partial_t u(\mathbf{x};t) = \mathbf{i} \triangle u(\mathbf{x};t) + q(\mathbf{x};t) \cdot u(\mathbf{x};t)$.

(k) Burger's Equation: $\partial_t u(x;t) = -u(x;t) \cdot \partial_x u(x;t)$.

(l) Eikonal Equation: $|\partial_x u(x)| = 1$.

2. Which of the following are eigenfunctions for the 2-dimensional Laplacian $\triangle = \partial_x^2 + \partial_y^2$? In each case, if u is an eigenfunction, what is the eigenvalue?

(a) $u(x,y) = \sin(x)\sin(y)$ (Figure 5F.1(A) on page 100)

(b) $u(x,y) = \sin(x) + \sin(y)$ (Figure 5F.1(B) on page 100)

(c) $u(x,y) = \cos(2x) + \cos(y)$ (Figure 5F.1(C) on page 100)

(d) $u(x,y) = \sin(3x) \cdot \cos(4y)$.

(e) $u(x,y) = \sin(3x) + \cos(4y)$.

(f) $u(x,y) = \sin(3x) + \cos(3y)$.

(g) $u(x,y) = \sin(3x) \cdot \cosh(4y)$.

(h) $u(x,y) = \sinh(3x) \cdot \cosh(4y)$.

(i) $u(x,y) = \sinh(3x) + \cosh(4y)$.

(j) $u(x,y) = \sinh(3x) + \cosh(3y)$.

(k) $u(x,y) = \sin(3x + 4y)$.

(l) $u(x,y) = \sinh(3x + 4y)$.

(m) $u(x,y) = \sin^3(x) \cdot \cos^4(y)$.

(n) $u(x,y) = e^{3x} \cdot e^{4y}$.

(o) $u(x,y) = e^{3x} + e^{4y}$.

(p) $u(x,y) = e^{3x} + e^{3y}$.

Chapter 5

Classification of PDEs and problem types

"If one looks at the different problems of the integral calculus which arise naturally when one wishes to go deep into the different parts of physics, it is impossible not to be struck by the analogies existing. Whether it be electrostatics or electrodynamics, the propogation of heat, optics, elasticity, or hydrodynamics, we are led always to differential equations of the same family."

—Henri Poincaré

5A Evolution vs. nonevolution equations

Recommended: §1B, §1C, §2B, §4B.

An **evolution equation** is a PDE with a distinguished "time" coordinate, t. In other words, it describes functions of the form $u(\mathbf{x}; t)$, and the equation has the form:

$$\mathsf{D}_t\, u \;=\; \mathsf{D}_{\mathbf{x}}\, u$$

where D_t is some differential operator involving only derivatives in the t variable (e.g. ∂_t, ∂_t^2, etc.), while $\mathsf{D}_{\mathbf{x}}$ is some differential operator involving only derivatives in the $\mathbf{x}$ variables (e.g. ∂_x, ∂_y^2, $\triangle$, etc.)

Example 5A.1. The following are evolution equations:

(a) The heat equation "$\partial_t\, u = \triangle u$" of §1B.

(b) The wave equation "$\partial_t^2\, u = \triangle u$" of §2B.

(c) The telegraph equation "$\kappa_2 \partial_t^2\, u \;+\; \kappa_1 \partial_t\, u \;=\; -\kappa_0 u \;+\; \triangle u$" of §2C.

(d) The Schrödinger equation "$\partial_t\, \omega \;=\; \frac{1}{\mathbf{i}\hbar} \mathsf{H}\, \omega$" of §3B (here H is a Hamiltonian operator).

(e) Liouville's Equation, the Fokker-Plank equation, and Reaction-Diffusion Equations. ♢

Nonexample 5A.2. The following are *not* evolution equations:

(a) The Laplace Equation "$\triangle u = 0$" of §1C.

(b) The Poisson Equation "$\triangle u = q$" of §1D.

(c) The Helmholtz Equation "$\triangle u = \lambda u$" (where $\lambda \in \mathbb{C}$ is a constant —i.e. an eigenvalue of $\triangle$).

(d) The Stationary Schrödinger equation $\mathsf{H}\ \omega_0 \ = \ E \cdot \omega_0$ (where $E \in \mathbb{C}$ is a constant eigenvalue). ♢

In mathematical models of physical phenomena, most PDEs are evolution equations. Nonevolutionary PDEs generally arise as **stationary state** equations for evolution PDEs (e.g. Laplace's equation) or as **resonance states** (e.g. Sturm-Liouville, Helmholtz).

Order: The **order** of the differential operator $\partial_x^2 \partial_y^3$ is $2+3 = 5$. More generally, the **order** of the differential operator $\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_D^{k_D}$ is the sum $k_1 + \ldots + k_D$. The **order** of a general differential operator is the highest order of any of its terms. For example, the Laplacian is second order. The **order** of a PDE is the highest order of the differential operator that appears in it. Thus, the Transport Equation, Liouville's Equation, and the (nondiffusive) Reaction Equation is *first order*, but all the other equations we have looked at (the heat equation, the wave equation, etc.) are of *second order*.

5B Initial value problems

Prerequisites: §5A.

Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let L be a differential operator on $\mathcal{C}^\infty(\mathbb{X};\mathbb{R})$. Consider evolution equation

$$\partial_t u \ = \ \mathsf{L}\, u, \tag{5B.1}$$

for an unknown function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$. An **initial value problem** (IVP) for equation (5B.1) is the following problem:

> *Given some function* $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial conditions**), *find a continuous function* $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *which satisfies* (5B.1) *and also satisfies* $u(\mathbf{x}, 0) = f_0(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{X}$.

For example, suppose the domain $\mathbb{X}$ is an iron pan being heated on a gas flame stove. You turn off the flame (so there is no further heat entering the system) and then throw some vegetables into the pan. Thus, (5B.1) is the Heat Equation, and f_0 describes the initial distribution of heat: cold vegetables in a hot pan. The initial value problem asks: "How fast do the vegetables cook? How fast does the pan cool?"

Next, consider the second order-evolution equation

$$\partial_t^2 u \;=\; \mathsf{L}\, u, \tag{5B.2}$$

for a unknown function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$. An **initial value problem** (or **IVP**, or **Cauchy problem**) for (5B.2) is as follows:

> *Given a function* $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial position**), *and/or another function* $f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial velocity**), *find a continuously differentiable function* $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *which satisfies* (5B.2) *and also satisfies* $u(\mathbf{x}, 0) = f_0(\mathbf{x})$ *and* $\partial_t\, u(\mathbf{x}, 0) = f_1(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{X}$.

For example, suppose (5B.1) is the wave equation on $\mathbb{X} = [0, L]$. Imagine $[0, L]$ as a vibrating string. Thus, f_0 describes the initial displacement of the string, and f_1 its initial momentum.

If $f_0 \not\equiv 0$, and $f_1 \equiv 0$, then the string is initially at rest, but is released from a displaced state —in other words, it is *plucked* (e.g. in a guitar or a harp). Hence, the initial value problem asks: "How does a guitar string sound when it is plucked?"

On the other hand, if $f_0 \equiv 0$, and $f_1 \not\equiv 0$, then the string is initially flat, but is imparted with nonzero momentum —in other words, it is *struck* (e.g. by the hammer in the piano). Hence, the initial value problem asks: "How does a piano string sound when it is struck?"

5C Boundary value problems

Prerequisites: §0D, §1C. **Recommended:** §5B.

If $\mathbb{X} \subset \mathbb{R}^D$ is a finite domain, then $\partial\mathbb{X}$ denotes its **boundary**. The **interior** of $\mathbb{X}$ is the set $\mathsf{int}(\mathbb{X})$ of all points in $\mathbb{X}$ *not* on the boundary.

Example 5C.1.

(a) If $\mathbb{I} = [0, 1] \subset \mathbb{R}$ is the **unit interval**, then $\partial\mathbb{I} = \{0, 1\}$ is a two-point set, and $\mathsf{int}(\mathbb{I}) = (0, 1)$.

(b) If $\mathbb{X} = [0,1]^2 \subset \mathbb{R}^2$ is the **unit square**, then $\mathsf{int}(\mathbb{X}) = (0,1)^2$. and

$$\partial\mathbb{X} \quad = \quad \{(x,y) \in \mathbb{X}\ ;\ x=0 \text{ or } x=1 \text{ or } y=0 \text{ or } y=1\}.$$

(c) In polar coordinates on $\mathbb{R}^2$, let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1, \theta \in [-\pi,\pi)\}$ be the **unit disk**. Then $\partial\mathbb{D} = \{(1,\theta)\ ;\ \theta \in [-\pi,\pi)\}$ is the **unit circle**, and $\mathsf{int}(\mathbb{D}) = \{(r,\theta)\ ;\ r < 1,\ \theta \in [-\pi,\pi)\}$.

(d) In spherical coordinates on $\mathbb{R}^3$, let $\mathbb{B} = \left\{\mathbf{x} \in \mathbb{R}^3\ ;\ \|\mathbf{x}\| \leq 1\right\}$ be the 3-dimensional **unit ball** in $\mathbb{R}^3$. Then $\partial\mathbb{B} = \mathbb{S} := \{\left\{\mathbf{x} \in \mathbb{R}^D\ ;\ \|\mathbf{x}\| = 1\right\}$ is the **unit sphere**, and $\mathsf{int}(\mathbb{B}) = \left\{\mathbf{x} \in \mathbb{R}^D\ ;\ \|\mathbf{x}\| < 1\right\}$.

(e) In cylindrical coordinates on $\mathbb{R}^3$, let $\mathbb{X} = \{(r,\theta,z)\ ;\ r \leq R,\ -\pi \leq \theta \leq \pi,\ 0 \leq z \leq L\}$ be the **finite cylinder** in $\mathbb{R}^3$. Then $\partial\mathbb{X} = \{(r,\theta,z)\ ;\ r = R \text{ or } z = 0 \text{ or } z = L\}$.
♢

A **boundary value problem** (BVP) is a problem of the following kind:

> *Find a continuous function* $u : \mathbb{X} \longrightarrow \mathbb{R}$ *such that*
>
> 1. u *satisfies some PDE at all* $\mathbf{x}$ *in the* **interior** *of* $\mathbb{X}$.
> 2. u *also satisfies some other equation* (maybe a differential equation) *for all* $\mathbf{s}$ *on the* **boundary** *of* $\mathbb{X}$.

The condition u must satisfy on the boundary of $\mathbb{X}$ is called a **boundary condition**. Note that there is no 'time variable' in our formulation of a BVP; thus, typically the PDE in question is an 'equilibrium' equation, like the Laplace equation or the Poisson equation.

If we try to solve an evolution equation with specified initial conditions *and* specified boundary conditions, then we are confronted with an 'initial/boundary value problem'. Formally, an **initial/boundary value problem** (I/BVP) is a problem of the following kind:

> *Find a continuous function* $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *such that*
>
> 1. u *satisfies some* (evolution) *PDE at all* $\mathbf{x}$ *in the* **interior** *of* $\mathbb{X} \times \mathbb{R}_{\not-}$.
> 2. u *satisfies some boundary condition for all* $(\mathbf{s};t)$ *in* $(\partial\mathbb{X}) \times \mathbb{R}_{\not-}$.
> 3. $u(\mathbf{x};0)$ *also satisfies some initial condition* (as described in §5B) *for all* $\mathbf{x} \in \mathbb{X}$.

We will consider four kinds of boundary conditions: *Dirichlet*, *Neumann*, *Mixed*, and *Periodic*. Each of these boundary conditions has a particular physical interpretation, and yields particular kinds of solutions for a partial differential equation.

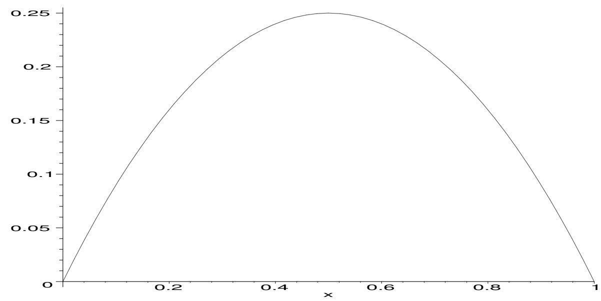


Figure 5C.1: $f(x) = x(1-x)$ satisfies homogeneous Dirichlet boundary conditions on the interval $[0,1]$.

5C(i) Dirichlet boundary conditions

Let $\mathbb{X}$ be a domain, and let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a function. We say that u satisfies **homogeneous Dirichlet boundary conditions (HDBC)** on $\mathbb{X}$ if:

$$\text{For all } \mathbf{s} \in \partial\mathbb{X}, \quad u(\mathbf{s}) \equiv 0.$$

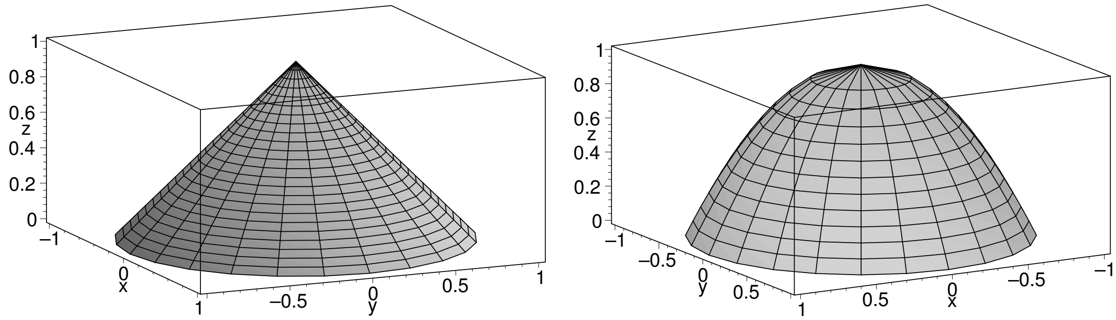
Physical interpretation.

Thermodynamic. (Heat equation, Laplace Equation, or Poisson Equation) In this case, u represents a temperature distribution. We imagine that the domain $\mathbb{X}$ represents some physical object, whose boundary $\partial\mathbb{X}$ is made out of metal or some other material which conducts heat almost perfectly. Hence, we can assume that *the temperature on the boundary is always equal to the temperature of the surrounding environment.*

We further assume that this environment has a constant temperature T_E (for example, $\mathbb{X}$ is immersed in a 'bath' of some uniformly mixed fluid), which remains constant during the experiment (for example, the fluid is present in large enough quantities that the heat flowing into/out of $\mathbb{X}$ does not measurably change it). We can then assume that the ambient temperature is $T_E \equiv 0$, by simply subtracting a constant temperature of T_E off the inside and the outside. (This is like changing from measuring temperature in degrees Kelvin to measuring in degrees Celsius; you're just adding 273^o to both sides, which makes no mathematical difference.)

Electrostatic. (Laplace equation or Poisson Equation) In this case, u represents an electrostatic potential. The domain $\mathbb{X}$ represents some compartment or region in space, whose boundary $\partial\mathbb{X}$ is made out of metal or some other perfect electrical conductor. Thus, the electrostatic potential within the metal boundary is a constant, which we can normalize to be zero.

Acoustic. (Wave equation) In this case, u represents the vibrations of some vibrating medium (e.g. a violin string or a drum skin). Homogeneous Dirich-



(A) **(B)**

Figure 5C.2: **(A)** $f(r, \theta) = 1 - r$ satisfies homogeneous Dirichlet boundary conditions on the disk $\mathbb{D} = \{(r, \theta) \ ; \ r \leq 1\}$, but is not smooth at zero. **(B)** $f(r, \theta) = 1 - r^2$ satisfies homogeneous Dirichlet boundary conditions on the disk $\mathbb{D} = \{(r, \theta) \ ; \ r \leq 1\}$, and is smooth everywhere.

let boundary conditions mean that the medium is *fixed* on the boundary $\partial\mathbb{X}$ (e.g. a violin string is clamped at its endpoints; a drumskin is pulled down tightly around the rim of the drum).

The set of *infinitely differentiable* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Dirichlet Boundary Conditions will be denoted $\mathcal{C}_0^\infty(\mathbb{X}; \mathbb{R})$ or $\mathcal{C}_0^\infty(\mathbb{X})$. Thus, for example

$$\mathcal{C}_0^\infty[0, L] \quad = \quad \left\{ f : [0, L] \longrightarrow \mathbb{R}; \quad f \text{ is smooth, and } f(0) \ = \ 0 \ = \ f(L) \right\}$$

The set of *continuous* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Dirichlet Boundary Conditions will be denoted $\mathcal{C}_0(\mathbb{X}; \mathbb{R})$ or $\mathcal{C}_0(\mathbb{X})$.

Example 5C.2.

(a) Suppose $\mathbb{X} = [0, 1]$, and $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined by $f(x) = x(1 - x)$. Then $f(0) = 0 = f(1)$, and f is smooth, so $f \in \mathcal{C}_0^\infty[0, 1]$. (See Figure 5C.1).

(b) Let $\mathbb{X} = [0, \pi]$.

1. For any $n \in \mathbb{N}$, let $\mathbf{S}_n(x) = \sin(n \cdot x)$ (see Figure 6D.1 on page 113). Then $\mathbf{S}_n \in \mathcal{C}_0^\infty[0, \pi]$.
2. If $f(x) = 5\sin(x) - 3\sin(2x) + 7\sin(3x)$, then $f \in \mathcal{C}_0^\infty[0, \pi]$. More generally, any finite sum $\displaystyle\sum_{n=1}^{N} B_n \mathbf{S}_n(x)$ (for some constants B_n) is in $\mathcal{C}_0^\infty[0, \pi]$.

3. If $f(x) = \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$ is a *uniformly convergent* Fourier sine series[1], then $f \in \mathcal{C}_0^\infty[0,\pi]$.

(c) Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ be the unit disk. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the 'cone' in Figure 5C.2(A), defined: $f(r,\theta) = (1-r)$. Then f is continuous, and $f \equiv 0$ on the boundary of the disk, so f satisfies Dirichlet boundary conditions. Thus, $f \in \mathcal{C}_0(\mathbb{D})$. However, f is not smooth (it is nondifferentiable at zero), so $f \notin \mathcal{C}_0^\infty(\mathbb{D})$.

(d) Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the 'dome' in Figure 5C.2(B), defined $f(r,\theta) = 1 - r^2$. Then $f \in \mathcal{C}_0^\infty(\mathbb{D})$.

(e) Let $\mathbb{X} = [0,\pi] \times [0,\pi]$ be the square of sidelength π.

1. For any $(n,m) \in \mathbb{N}^2$, let $\mathbf{S}_{n,m}(x,y) = \sin(n \cdot x) \cdot \sin(m \cdot y)$. Then $\mathbf{S}_{n,m} \in \mathcal{C}_0^\infty(\mathbb{X})$. (see Figure 9A.2 on page 181).
2. If $f(x) = 5\sin(x)\sin(2y) - 3\sin(2x)\sin(7y) + 7\sin(3x)\sin(y)$, then $f \in \mathcal{C}_0^\infty(\mathbb{X})$. More generally, any finite sum $\sum_{n=1}^{N}\sum_{m=1}^{M} B_{n,m}\mathbf{S}_{n,m}(x)$ is in $\mathcal{C}_0^\infty(\mathbb{X})$.
3. If $f = \sum_{n,m=1}^{\infty} B_{n,m}\mathbf{S}_{n,m}$ is a *uniformly convergent* two dimensional Fourier sine series[2], then $f \in \mathcal{C}_0^\infty(\mathbb{X})$.

◇

Exercise 5C.1. (i) Verify examples (b) to (e) above Ⓔ
(ii) Show that $\mathcal{C}_0^\infty(\mathbb{X})$ is a vector space.
(iii) Show that $\mathcal{C}_0(\mathbb{X})$ is a vector space. ♦

Arbitrary **nonhomogeneous Dirichlet boundary conditions** are imposed by fixing some function $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and then requiring:

$$u(\mathbf{s}) = b(\mathbf{s}), \quad \text{for all } \mathbf{s} \in \partial\mathbb{X}. \tag{5C.3}$$

For example, the **classical Dirichlet Problem** is to find a continuous function $u : \mathbb{X} \longrightarrow \mathbb{R}$ satisfying the Dirichlet condition (5C.3), such that u also satisfies Laplace's Equation: $\triangle u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathsf{int}(\mathbb{X})$.

[1] See § 7B on page 144.
[2] See § 9A on page 179.

Physical interpretations.

Thermodynamic. u describes a stationary temperature distribution on $\mathbb{X}$, where the temperature is *fixed* on the boundary. Different parts of the boundary may have different temperatures, so heat may be flowing *through* the region $\mathbb{X}$ from warmer boundary regions to cooler boundary regions. But the actual temperature distribution within $\mathbb{X}$ is in equilibrium.

Electrostatic. u describes an electrostatic potential field within the region $\mathbb{X}$. The voltage level on the boundaries is fixed (e.g. boundaries of $\mathbb{X}$ are wired up to batteries which maintain a constant voltage). However different parts of the boundary may have different voltages (the boundary is not a perfect conductor).

Minimal surface. u describes a minimal-energy surface (e.g. a soap film). The boundary of the surface is clamped in some position (e.g. the wire frame around the soap film); the interior of the surface must adapt to find the minimal energy configuration compatible with these boundary conditions. Minimal surfaces of low curvature are well-approximated by harmonic functions.

For example, if $\mathbb{X} = [0, L]$, and $b(0)$ and $b(L)$ are two constants, then the **Dirichlet Problem** is to find $u : [0, L] \longrightarrow \mathbb{R}$ such that

$$u(0) = b(0), \quad u(L) = b(L), \quad \text{and} \quad \partial_x^2 u(x) = 0, \quad \text{for } 0 < x < L. \qquad (5C.4)$$

That is, the temperature at the left-hand endpoint is fixed at $b(0)$, and at the right-hand endpoint is fixed at $b(L)$. The unique solution to this problem is the
Ⓔ function $u(x) = \Big(b(L) - b(0)\Big)\, x/L + b(0)$. (**Exercise 5C.2**).

5C(ii) Neumann boundary conditions

Suppose $\mathbb{X}$ is a domain with boundary $\partial\mathbb{X}$, and $u : \mathbb{X} \longrightarrow \mathbb{R}$ is some function. Then for any boundary point $\mathbf{s} \in \partial\mathbb{X}$, we use "$\partial_\perp u(\mathbf{s})$" to denote the **outward normal** derivative[3] of u on the boundary. Physically, $\partial_\perp u(\mathbf{s})$ is the *rate of change in u as you leave $\mathbb{X}$ by passing through $\partial\mathbb{X}$ in a perpendicular direction.*

Example 5C.3.

(a) If $\mathbb{X} = [0, 1]$, then $\partial_\perp u(0) = -\partial_x u(0)$ and $\partial_\perp u(1) = \partial_x u(1)$.

(b) Suppose $\mathbb{X} = [0, 1]^2 \subset \mathbb{R}^2$ is the unit square, and $(x, y) \in \partial\mathbb{X}$. There are four cases:

[3]This is sometimes indicated as $\dfrac{\partial u}{\partial \mathbf{n}}$ or $\dfrac{\partial u}{\partial \nu}$, or as "$\nabla u \bullet \vec{\mathbf{N}}$", or as "$\nabla u \bullet \vec{\mathbf{n}}$".

- If $x = 0$ (left edge), then $\partial_\perp u(0, y) = -\partial_x u(0, y)$.
- If $x = 1$ (right edge), then $\partial_\perp u(1, y) = \partial_x u(1, y)$.
- If $y = 0$ (top edge), then $\partial_\perp u(x, 0) = -\partial_y u(x, 0)$.
- If $y = 1$ (bottom edge), then $\partial_\perp u(x, 1) = \partial_y u(x, 1)$.

(If more than one of these conditions is true —for example, at $(0, 0)$ —then (x, y) is a corner, and $\partial_\perp u(x, y)$ is not well-defined).

(c) Let $\mathbb{D} = \{(r, \theta) \; ; \; r < 1\}$ be the unit disk in the plane. Then $\partial\mathbb{D}$ is the set $\{(1, \theta) \; ; \; \theta \in [-\pi, \pi)\}$, and for any $(1, \theta) \in \partial\mathbb{D}$, $\quad \partial_\perp u(1, \theta) \;=\; \partial_r u(1, \theta)$.

(d) Let $\mathbb{D} = \{(r, \theta) \; ; \; r < R\}$ be the disk of radius R. Then $\partial\mathbb{D} = \{(R, \theta) \; ; \; \theta \in [-\pi, \pi)\}$, and for any $(R, \theta) \in \partial\mathbb{D}$, $\quad \partial_\perp u(R, \theta) \;=\; \partial_r u(R, \theta)$.

(e) Let $\mathbb{B} = \{(r, \phi, \theta) \; ; \; r < 1\}$ be the unit ball in $\mathbb{R}^3$. Then $\partial\mathbb{B} = \{(r, \phi, \theta) \; ; \; r = 1\}$ is the unit sphere. If $u(r, \phi, \theta)$ is a function in polar coordinates, then for any boundary point $\mathbf{s} = (1, \phi, \theta)$, $\quad \partial_\perp u(\mathbf{s}) \;=\; \partial_r u(\mathbf{s})$.

(f) Suppose $\mathbb{X} = \{(r, \theta, z) \; ; \; r \le R, \; 0 \le z \le L, \; -\pi \le \theta < \pi\}$, is the **finite cylinder**, and $(r, \theta, z) \in \partial\mathbb{X}$. There are three cases:

- If $r = R$ (sides), then $\partial_\perp u(R, \theta, z) \;=\; \partial_r u(R, \theta, z)$.
- If $z = 0$ (bottom disk), then $\partial_\perp u(r, \theta, 0) \;=\; -\partial_z u(r, \theta, 0)$.
- If $z = L$ (top disk), then $\partial_\perp u(r, \theta, L) \;=\; \partial_z u(r, \theta, L)$.

♢

We say that u satisfies **homogeneous Neumann boundary conditions** if

$$\partial_\perp u(\mathbf{s}) \;=\; 0 \quad \text{for all } \mathbf{s} \in \partial\mathbb{X}. \tag{5C.5}$$

Physical Interpretations.

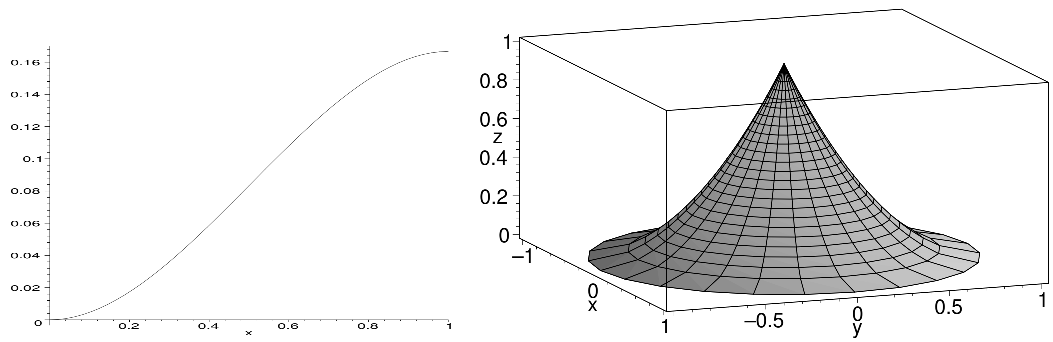
Thermodynamic. (Heat, Laplace, or Poisson equation) Suppose u represents a temperature distribution. Recall that Fourier's Law of Heat Flow (§ 1A on page 3) says that $\nabla u(\mathbf{s})$ is the speed and direction in which heat is flowing at $\mathbf{s}$. Recall that $\partial_\perp u(\mathbf{s})$ is the component of $\nabla u(\mathbf{s})$ which is perpendicular to $\partial\mathbb{X}$. Thus, homogeneous Neumann BC means that $\nabla u(\mathbf{s})$ is *parallel* to the boundary for all $\mathbf{s} \in \partial\mathbb{X}$. In other words *no heat is crossing the boundary*. This means that the boundary is a *perfect insulator*.

If u represents the concentration of a diffusing substance, then $\nabla u(\mathbf{s})$ is the flux of this substance at $\mathbf{s}$. Homogeneous Neumann Boundary conditions mean that the boundary is an *impermeable barrier* to this substance.

Electrostatic. (Laplace or Poisson equation) Suppose u represents an electric potential. Thus $\nabla u(\mathbf{s})$ is the *electric field* at $\mathbf{s}$. Homogeneous Neumann BC means that $\nabla u(\mathbf{s})$ is *parallel* to the boundary for all $\mathbf{s} \in \partial\mathbb{X}$; i.e. no field lines penetrate the boundary.

The set of *continuous* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Neumann boundary conditions will be denoted $\mathcal{C}_{\perp}(\mathbb{X})$. The set of *infinitely differentiable* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Neumann boundary conditions will be denoted $\mathcal{C}^{\infty}_{\perp}(\mathbb{X})$. Thus, for example

$$\mathcal{C}^{\infty}_{\perp}[0,L] \quad = \quad \Big\{ f : [0,L] \longrightarrow \mathbb{R}; \quad f \text{ is smooth, and } f'(0) \ = \ 0 \ = \ f'(L) \Big\}$$



(A) **(B)**

Figure 5C.3: **(A)** $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ satsfies homogeneous Neumann boundary conditions on the interval $[0,1]$. **(B)** $f(r,\theta) = (1-r)^2$ satisfies homogeneous Neumann boundary conditions on the disk $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$, but is not differentiable at zero.

Example 5C.4.

(a) Let $\mathbb{X} = [0,1]$, and let $f : [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ (See Figure 5C.3(A)). Then $f'(0) = 0 = f'(1)$, and f is smooth, so $f \in \mathcal{C}^{\infty}_{\perp}[0,1]$.

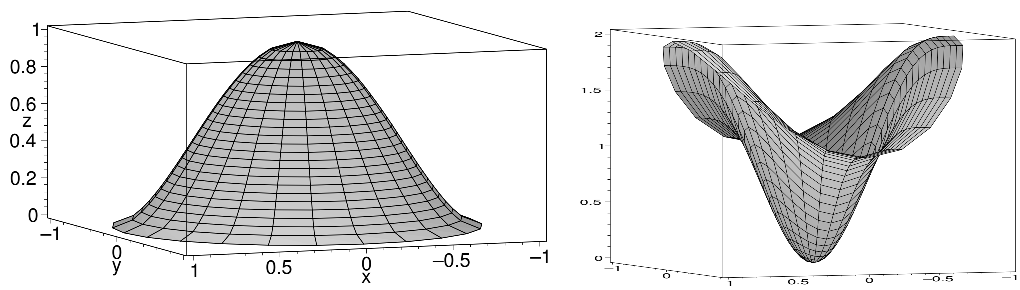
(b) Let $\mathbb{X} = [0,\pi]$.

1. For any $n \in \mathbb{N}$, let $\mathbf{C}_n(x) = \cos(n \cdot x)$ (see Figure 6D.1 on page 113). Then $\mathbf{C}_n \in \mathcal{C}^{\infty}_{\perp}[0,\pi]$.
2. If $f(x) = 5\cos(x) - 3\cos(2x) + 7\cos(3x)$, then $f \in \mathcal{C}^{\infty}_{\perp}[0,\pi]$. More generally, any finite sum $\displaystyle\sum_{n=1}^{N} A_n \mathbf{C}_n(x)$ (for some constants A_n) is in $\mathcal{C}^{\infty}_{\perp}[0,\pi]$.

3. If $f(x) = \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x)$ is a *uniformly convergent* Fourier cosine series[4], and the derivative series $f'(x) = -\sum_{n=1}^{\infty} nA_n\mathbf{S}_n(x)$ is *also* uniformly convergent, then $f \in \mathcal{C}_{\perp}^{\infty}[0,\pi]$.

(c) Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ be the unit disk.

1. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "witch's hat" of Figure 5C.3(B), defined: $f(r,\theta) := (1-r)^2$. Then $\partial_{\perp} f \equiv 0$ on the boundary of the disk, so f satisfies Neumann boundary conditions. Also, f is continuous on $\mathbb{D}$; hence $f \in \mathcal{C}_{\perp}(\mathbb{D})$. However, f is not smooth (it is nondifferentiable at zero), so $f \notin \mathcal{C}_{\perp}^{\infty}(\mathbb{D})$.



(A) **(B)**

Figure 5C.4: **(A)** $f(r,\theta) = (1-r^2)^2$ satisfies homogeneous Neumann boundary conditions on the disk, and is smooth everywhere. **(B)** $f(r,\theta) = (1+\cos(\theta)^2)\cdot(1-(1-r^2)^4)$ does *not* satisfy homogeneous Neumann boundary conditions on the disk, and is not constant on the boundary.

2. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "bell" of Figure 5C.4(A), defined: $f(r,\theta) := (1-r^2)^2$. Then $\partial_{\perp} f \equiv 0$ on the boundary of the disk, and f is smooth everywhere on $\mathbb{D}$, so $f \in \mathcal{C}_{\perp}^{\infty}(\mathbb{D})$.
3. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "flower vase" of Figure 5C.4(B), defined $f(r,\theta) := (1+\cos(\theta)^2)\cdot(1-(1-r^2)^4)$. Then $\partial_{\perp} f \equiv 0$ on the boundary of the disk, and f is smooth everywhere on $\mathbb{D}$, so $f \in \mathcal{C}_{\perp}^{\infty}(\mathbb{D})$. Note that, in this case, the angular derivative is nonzero, so f is not constant on the boundary of the disk.

(d) Let $\mathbb{X} = [0,\pi] \times [0,\pi]$ be the square of sidelength π.

1. For any $(n,m) \in \mathbb{N}^2$, let $\mathbf{C}_{n,m}(x,y) = \cos(nx)\cdot\cos(my)$ (see Figure 9A.2 on page 181). Then $\mathbf{C}_{n,m} \in \mathcal{C}_{\perp}^{\infty}(\mathbb{X})$.

[4]See § 7B on page 144.

2. If $f(x) = 5\cos(x)\cos(2y) - 3\cos(2x)\cos(7y) + 7\cos(3x)\cos(y)$, then $f \in \mathcal{C}^\infty_\perp(\mathbb{X})$. More generally, any finite sum $\sum_{n=1}^{N}\sum_{m=1}^{M} A_{n,m}\mathbf{C}_{n,m}(x)$ (for some constants $A_{n,m}$) is in $\mathcal{C}^\infty_\perp(\mathbb{X})$.

3. More generally, if $f = \sum_{n,m=0}^{\infty} A_{n,m}\mathbf{C}_{n,m}$ is a *uniformly convergent* two dimensional Fourier cosine series[5], and the derivative series

$$\begin{aligned} \partial_x f(x,y) &= -\sum_{n,m=0}^{\infty} nA_{n,m}\sin(nx)\cdot\cos(my) \\ \partial_y f(x,y) &= -\sum_{n,m=0}^{\infty} mA_{n,m}\cos(nx)\cdot\sin(my) \end{aligned}$$

are *also* uniformly convergent, then $f \in \mathcal{C}^\infty_\perp(\mathbb{X})$.

Ⓔ **Exercise 5C.3** Verify examples (b) to (d) ◊

Arbitrary **nonhomogeneous Neumann Boundary conditions** are imposed by fixing a function $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and then requiring

$$\partial_\perp u(\mathbf{s}) \;=\; b(\mathbf{s}) \quad \text{for all } \mathbf{s} \in \partial\mathbb{X}. \tag{5C.6}$$

For example, the **classical Neumann Problem** is to find a continuously differentiable function $u : \mathbb{X} \longrightarrow \mathbb{R}$ satisfying the Neumann condition (5C.6), such that u also satisfies Laplace's Equation: $\triangle u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathsf{int}(\mathbb{X})$.

Physical Interpretations.

Thermodynamic. Here u represents a temperature distribution, or the concentration of some diffusing material. Recall that Fourier's Law (§ 1A on page 3) says that $\nabla u(\mathbf{s})$ is the flux of heat (or material) at $\mathbf{s}$. Thus, for any $\mathbf{s} \in \partial\mathbb{X}$, the derivative $\partial_\perp u(\mathbf{s})$ is the flux of heat/material *across the boundary* at $\mathbf{s}$. The nonhomogeneous Neumann Boundary condition $\partial_\perp u(\mathbf{s}) = b(\mathbf{s})$ means that heat (or material) is being 'pumped' across the boundary at a constant rate described by the function $b(\mathbf{s})$.

Electrostatic. Here, u represents an electric potential. Thus $\nabla u(\mathbf{s})$ is the *electric field* at $\mathbf{s}$. Nonhomogeneous Neumann boundary conditions mean that the field vector perpendicular to the boundary is determined by the function $b(\mathbf{s})$.

[5]See § 9A on page 179.

5C(iii) Mixed (or Robin) boundary conditions

These are a combination of Dirichlet and Neumann-type conditions obtained as follows: Fix functions $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and $h, h_\perp : \partial\mathbb{X} \longrightarrow \mathbb{R}$. Then $(h, h_\perp, b)$-**mixed boundary conditions** are given:

$$h(\mathbf{s}) \cdot u(\mathbf{s}) \ + \ h_\perp(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \ = \ b(x) \quad \text{for all } \mathbf{s} \in \partial\mathbb{X}. \tag{5C.7}$$

For example:

- **Dirichlet Conditions** corresponds to $h \equiv 1$ and $h_\perp \equiv 0$.
- **Neumann Conditions** corresponds to $h \equiv 0$ and $h_\perp \equiv 1$.
- *No* boundary conditions corresponds to $h \equiv h_\perp \equiv 0$.
- **Newton's Law of Cooling** reads:

$$\partial_\perp u \ = \ c \cdot (u - T_E) \tag{5C.8}$$

  This describes a situation where the boundary is an *imperfect conductor* (with conductivity constant c), and is immersed in a bath with ambient temperature T_E. Thus, heat leaks in or out of the boundary at a rate proportional to c times the difference between the internal temperature u and the external temperature T_E. Equation (5C.8) can be rewritten:

$$c \cdot u \ - \ \partial_\perp u \ = \ b,$$

  where $b = c \cdot T_E$. This is the mixed boundary equation (5C.7), with $h \equiv c$ and $h_\perp \equiv -1$.

- **Homogeneous** mixed boundary conditions take the form:

$$h \cdot u \ + \ h_\perp \cdot \partial_\perp u \ \equiv \ 0.$$

  The set of functions in $\mathcal{C}^\infty(\mathbb{X})$ satisfying this property will be denoted $\mathcal{C}^\infty_{h,h_\perp}(\mathbb{X})$. Thus, for example, if $\mathbb{X} = [0, L]$, and $h(0)$, $h_\perp(0)$, $h(L)$ and $h_\perp(L)$ are four constants, then

$$\mathcal{C}^\infty_{h,h_\perp}[0, L] = \left\{ \begin{array}{ll} f : [0, L] \longrightarrow \mathbb{R}; & f \text{ is differentiable}, \quad h(0)f(0) \ - \ h_\perp(0)f'(0) = 0 \\ & \qquad \text{and } \ h(L)f(L) \ + \ h_\perp(L)f'(L) = 0. \end{array} \right\}$$

Remarks. (a) Note that there is some redundancy in this formulation. Equation (5C.7) is equivalent to

$$k \cdot h(\mathbf{s}) \cdot u(\mathbf{s}) \ + \ k \cdot h_\perp(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \ = \ k \cdot b(\mathbf{s}),$$

for any constant $k \neq 0$. Normally we chose k so that at least one of the coefficients h or $h_\perp$ is equal to 1.

(b) Some authors (e.g. [Pin98]) call this **general boundary conditions**, and, for mathematical convenience, write this as

$$\cos(\alpha)u \quad + \quad L \cdot \sin(\alpha)\partial_\perp u \quad = \quad T. \tag{5C.9}$$

where and α and T are parameters. Here, the "$\cos(\alpha)$, $\sin(\alpha)$" coefficients of (5C.9) are just a mathematical "gadget" to concisely express any weighted combination of Dirichlet and Neumann conditions. An expression of type (5C.7) can be transformed into one of type (5C.9) as follows: Let $\alpha := \arctan\left(\frac{h_\perp}{L \cdot h}\right)$ (if $h = 0$, then set $\alpha = \frac{\pi}{2}$) and let $T := b\frac{\cos(\alpha) + L\sin(\alpha)}{h + h_\perp}$. Going the other way is easier; simply define $h := \cos(\alpha)$, $h_\perp := L \cdot \sin(\alpha)$, and $T := b$.

5C(iv) Periodic boundary conditions

Periodic boundary conditions means that function u "looks the same" on opposite edges of the domain. For example, if we are solving a PDE on the **interval** $[-\pi, \pi]$, then periodic boundary conditions are imposed by requiring

$$u(-\pi) \;=\; u(\pi) \quad \text{and} \quad u'(-\pi) \;=\; u'(\pi).$$

Interpretation #1: Pretend that u is actually a small piece of an infinitely extended, periodic function $\widetilde{u} : \mathbb{R} \longrightarrow \mathbb{R}$, where, for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have:

$$\widetilde{u}(x + 2n\pi) \quad = \quad u(x).$$

Thus u must have the same value —and the same derivative —at x and $x+2n\pi$, for any $x \in \mathbb{R}$. In particular, u must have the same value and derivative at $-\pi$ and π. This explains the name "periodic boundary conditions".

Interpretation #2: Suppose you 'glue together' the left and right ends of the interval $[-\pi, \pi]$ (i.e. glue $-\pi$ to π). Then the interval looks like a a circle (where $-\pi$ and π actually become the 'same' point). Thus u must have the same value —and the same derivative —at $-\pi$ and π.

Example 5C.5.

(a) $u(x) = \sin(x)$ and $v(x) = \cos(x)$ have periodic boundary conditions.

(b) For any $n \in \mathbb{N}$, the functions $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$ have periodic boundary conditions. (See Figure 6D.1 on page 113.)

(c) $\sin(3x) + 2\cos(4x)$ has periodic boundary conditions.

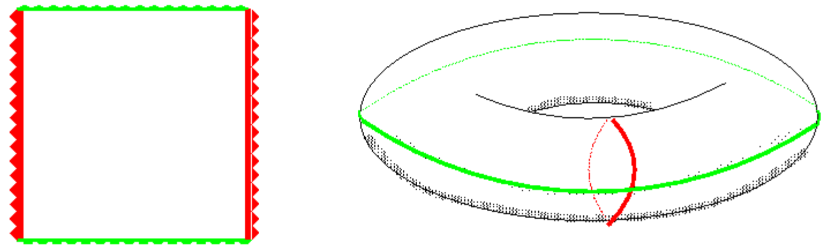


Figure 5C.5: If we 'glue' the opposite edges of a square together, we get a torus.

(d) If $u_1(x)$ and $u_2(x)$ have periodic boundary conditions, and c_1, c_2 are any constants, then $u(x) = c_1 u_1(x) + c_2 u_2(x)$ also has periodic boundary conditions.

Exercise 5C.4 Verify these examples. ◊ Ⓔ

On the **square** $[-\pi, \pi] \times [-\pi, \pi]$, periodic boundary conditions are imposed by requiring:

(P1) $u(x, -\pi) = u(x, \pi)$ and $\partial_y u(x, -\pi) = \partial_y u(x, \pi)$, for all $x \in [-\pi, \pi]$.

(P2) $u(-\pi, y) = u(\pi, y)$ and $\partial_x u(-\pi, y) = \partial_x u(\pi, y)$ for all $y \in [-\pi, \pi]$.

Interpretation #1: Pretend that u is actually a small piece of an infinitely extended, doubly periodic function $\widetilde{u} : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where, for every $(x, y) \in \mathbb{R}^2$, and every $n, m \in \mathbb{Z}$, we have:

$$\widetilde{u}(x + 2n\pi, y + 2m\pi) \quad = \quad u(x, y).$$

Exercise 5C.5. Explain how conditions **(P1)** and **(P1)** arise naturally from this interpretation. ♦ Ⓔ

Interpretation #2: Glue the top edge of the square to the bottom edge, and the right edge to the left edge. In other words, pretend that the square is really a **torus** (Figure 5C.5).

Example 5C.6.

(a) The functions $u(x, y) = \sin(x)\sin(y)$ and $v(x, y) = \cos(x)\cos(y)$ have periodic boundary conditions. So do the functions $w(x, y) = \sin(x)\cos(y)$ and $w(x, y) = \cos(x)\sin(y)$

(b) For any $(n, m) \in \mathbb{N}^2$, the functions $\mathbf{S}_{n,m}(x) = \sin(nx)\sin(my)$ and $\mathbf{C}_{n,m}(x) = \cos(nx)\cos(mx)$ have periodic boundary conditions. (See Figure 9A.2 on page 181.)

(c) $\sin(3x)\sin(2y) + 2\cos(4x)\cos(7y)$ has periodic boundary conditions.

(d) If $u_1(x, y)$ and $u_2(x, y)$ have periodic boundary conditions, and c_1, c_2 are any constants, then $u(x, y) = c_1 u_1(x, y) + c_2 u_2(x, y)$ also has periodic boundary conditions.

Ⓔ **Exercise 5C.6** Verify these examples. ♢

On the D-dimensional **cube** $[-\pi, \pi]^D$, we require, for $d = 1, 2, \ldots, D$ and all $x_1, \ldots, x_D \in [-\pi, \pi]$, that

$$\begin{array}{rrcl} & u(x_1, \ldots, x_{d-1}, -\pi, x_{d+1}, \ldots, x_D) & = & u(x_1, \ldots, x_{d-1}, \pi, x_{d+1}, \ldots, x_D) \\ \text{and} & \partial_d\, u(x_1, \ldots, x_{d-1}, -\pi, x_{d+1}, \ldots, x_D) & = & \partial_d\, u(x_1, \ldots, x_{d-1}, \pi, x_{d+1}, \ldots, x_D). \end{array}$$

Again, the idea is that we are identifying $[-\pi, \pi]^D$ with the D-dimensional **torus**. The space of all functions satisfying these conditions will be denoted $\mathcal{C}^{\infty}_{\text{per}}[-\pi, \pi]^D$. Thus, for example,

$$\begin{aligned} \mathcal{C}^{\infty}_{\text{per}}[-\pi, \pi] &= \Big\{ f : [-\pi, \pi] \longrightarrow \mathbb{R};\ \ f \text{ is differentiable,} \\ &\qquad\qquad f(-\pi) = f(\pi) \text{ and } f'(-\pi) = f'(\pi) \Big\} \\ \mathcal{C}^{\infty}_{\text{per}}[-\pi, \pi]^2 &= \Big\{ f : [-\pi, \pi] \times [-\pi, \pi] \longrightarrow \mathbb{R};\ \ f \text{ is differentiable,} \\ &\qquad\qquad \text{and satisfies } \textbf{(P1)} \text{ and } \textbf{(P2)} \text{ above} \Big\} \end{aligned}$$

5D Uniqueness of solutions

Prerequisites: §1B, §2B, §1C, §5B, §5C.

Prerequisites (for proofs): §1E, §0E(iii), §0G.

Differential equations are interesting primarily because they can be used to express the laws governing physical phenomena (e.g. heat flow, wave motion, electrostatics, etc.). By specifying particular initial conditions and boundary conditions, we try to mathematically encode the physical conditions, constraints and external influences which are present in a particular situation. A solution to the differential equation which satisfies these initial/boundary conditions thus constitutes a *prediction* about what will occur under these physical conditions.

However, this strategy can only succeed if there is a *unique* solution to the differential equation with particular initial/boundary conditions. If there are many mathematically correct solutions, then we cannot make a clear prediction about *which* of them will really occur. Sometimes we can reject some solutions as being 'unphysical' (e.g. they are nondifferentiable, or discontinuous, or contain unacceptable infinities, or predict negative values for a necessarily positive quantity like density). However, these notions of 'unphysicality' really just represent further mathematical constraints which we are implicitly imposing on the solution. If multiple solutions still exist, we should try to impose further constraints (i.e. construct a more detailed or well-specified model) until we get a unique solution. Thus, the question of *uniqueness of solutions* is extremely important in the general theory of differential equations (both ordinary and partial). In this section, we will establish sufficient conditions for the uniqueness of solutions to I/BVPs for the Laplace, Poisson, Heat, and wave equations.

Let $\mathcal{S} \subset \mathbb{R}^D$. We say that $\mathcal{S}$ is a **smooth graph** if there is an open subset $\mathbb{U} \subset \mathbb{R}^{D-1}$, a function $f : \mathbb{U} \longrightarrow \mathbb{R}$, and some $d \in [1...D]$, such that $\mathcal{S}$ 'looks like' the graph of the function f, plotted over the domain $\mathbb{U}$, with the value of f plotted in the dth coordinate. In other words:

$$\mathcal{S} = \{(u_1, \ldots, u_{d-1}, y, u_d, \ldots, u_{D-1}) \; ; \; (u_1, \ldots, u_{D-1}) \in \mathbb{U}, \; y = f(u_1, \ldots, u_{D-1})\}.$$

Intuitively, this means that $\mathcal{S}$ looks like a smooth surface (oriented 'roughly perpendicular' to the dth dimension). More generally, if $\mathcal{S} \subset \mathbb{R}^D$, we say that $\mathcal{S}$ is a **smooth hypersurface** if, for each $\mathbf{s} \in \mathcal{S}$, there exists some $\epsilon > 0$ such that $\mathbb{B}(\mathbf{s}, \epsilon) \cap \mathcal{S}$ is a smooth graph.

Example 5D.1.

(a) Let $\mathbb{P} \subset \mathbb{R}^D$ be any $(D-1)$-dimensional hyperplane; then $\mathbb{P}$ is a smooth hypersurface.

(b) Let $\mathbb{S}^1 := \left\{\mathbf{s} \in \mathbb{R}^2 \; ; \; |\mathbf{s}| = 1\right\}$ be the unit circle in $\mathbb{R}^2$. Then $\mathbb{S}^1$ is a smooth hypersurface in $\mathbb{R}^2$.

(c) Let $\mathbb{S}^2 := \left\{\mathbf{s} \in \mathbb{R}^3 \; ; \; |\mathbf{s}| = 1\right\}$ be the unit sphere in $\mathbb{R}^3$. Then $\mathbb{S}^2$ is a smooth hypersurface in $\mathbb{R}^3$.

(d) Let $\mathbb{S}^{D-1} := \left\{\mathbf{s} \in \mathbb{R}^D \; ; \; |\mathbf{s}| = 1\right\}$ be the unit hypersphere in $\mathbb{R}^D$. Then $\mathbb{S}^{D-1}$ is a smooth hypersurface in $\mathbb{R}^D$.

(e) Let $\mathcal{S} \subset \mathbb{R}^D$ be any smooth hypersurface, and let $\mathbb{U} \subset \mathbb{R}^D$ be an open set. Then $\mathcal{S} \cap \mathbb{U}$ is also a smooth hypersurface (if it is nonempty).

Exercise 5D.1 Verify these examples. $\diamondsuit$ Ⓔ

A domain $\mathbb{X} \subset \mathbb{R}^D$ has **piecewise smooth boundary** if $\partial\mathbb{X}$ is a finite union of smooth hypersurfaces. If $u : \mathbb{X} \longrightarrow \mathbb{R}$ is some differentiable function, then this implies that the normal derivative $\partial_\perp u(\mathbf{s})$ is well-defined for $\mathbf{s} \in \partial\mathbb{X}$, except for those $\mathbf{s}$ on the (negligible) regions where two or more of these smooth hypersurfaces intersect. This means that it is meaningful to impose Neumann boundary conditions on u. It also means that certain methods from vector calculus can be applied to u (see §0E(iii) on page 561).

Example 5D.2. Every domain in Example 5C.1 on page 71 has a piecewise smooth boundary. (**Exercise 5D.2** Verify this.) $\diamondsuit$ Ⓔ

Indeed, every domain we will consider in this book will have a piecewise smooth boundary, as does any domain which is likely to arise in any physically realistic model. Hence, it suffices to obtain uniqueness results for such domains.

5D(i) Uniqueness for the Laplace and Poisson equations

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain and let $u : \mathbb{X} \longrightarrow \mathbb{R}$. We say that u is **continuous and harmonic** on $\mathbb{X}$ if u is continuous on $\mathbb{X}$ and $\triangle u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathsf{int}(\mathbb{X})$.

Lemma 5D.3. (Solution uniqueness for Laplace equation; homogeneous BC) *Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, and suppose $u : \mathbb{X} \longrightarrow \mathbb{R}$ is continuous and harmonic on $\mathbb{X}$. Then various homogeneous boundary conditions constrain the solution as follows:*

(a) (Homogeneous Dirichlet BC) *If $u(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$, then u must be the constant 0 function: i.e. $u(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{X}$.*

(b) (Homogeneous Neumann BC) *Suppose $\mathbb{X}$ has a piecewise smooth boundary. If $\partial_\perp u(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$, then u must be a constant: i.e. $u(\mathbf{x}) = C$, for all $\mathbf{x} \in \mathbb{X}$.*

(c) (Homogeneous Robin BC) *Suppose $\mathbb{X}$ has a piecewise smooth boundary, and let $h, h_\perp : \partial\mathbb{X} \longrightarrow \mathbb{R}_{\not-}$ be two other continuous nonnegative functions such that $h(\mathbf{s}) + h_\perp(\mathbf{s}) > 0$ for all $\mathbf{s} \in \partial\mathbb{X}$. If $h(\mathbf{s})u(\mathbf{s}) + h_\perp(\mathbf{s})\partial_\perp u(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$, then u must be a constant function.*

Furthermore, if h is nonzero somewhere on $\partial\mathbb{X}$, then $u(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{X}$.

Proof. **(a)** If $u : \mathbb{X} \longrightarrow \mathbb{R}$ is harmonic, then the Maximum Principle (Corollary 1E.2 on page 17) says that any maximum/minimum of u occurs somewhere on $\partial\mathbb{X}$. But $u(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$; thus, $\max_{\mathbb{X}}(u) = 0 = \min_{\mathbb{X}}(u)$; thus, $u \equiv 0$.

(If $\mathbb{X}$ has a piecewise smooth boundary, then another proof of **(a)** arises by setting $h \equiv 1$ and $h_\perp \equiv 0$ in part **(c)**.)

To prove **(b)**, set $h \equiv 0$ and $h_\perp \equiv 1$ in part **(c)**.

To prove **(c)**, we will use Green's Formula. We begin with the following claim.

Claim 1: *For all $\mathbf{s} \in \partial\mathbb{X}$, we have $u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \leq 0$.*

Proof. The homogeneous Robin boundary conditions say $h(\mathbf{s})u(\mathbf{s}) + h_\perp(\mathbf{s})\partial_\perp u(\mathbf{s}) = 0$. Multiplying by $u(\mathbf{s})$, we get

$$h(\mathbf{s})u^2(\mathbf{s}) + u(\mathbf{s})h_\perp(\mathbf{s})\partial_\perp u(\mathbf{s}) \quad = \quad 0. \qquad (5D.1)$$

If $h_\perp(\mathbf{s}) = 0$, then $h(\mathbf{s})$ must be nonzero, and equation (5D.1) reduces to $h(\mathbf{s})u^2(\mathbf{s}) = 0$, which means $u(\mathbf{s}) = 0$, which means $u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \leq 0$, as desired.

If $h_\perp(\mathbf{s}) \neq 0$, then we can rearrange equation (5D.1) to get

$$u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \quad = \quad \frac{-h(\mathbf{s})u^2(\mathbf{s})}{h_\perp(\mathbf{s})} \quad \underset{(*)}{\leq} \quad 0,$$

where $(*)$ is because because $h(\mathbf{s}), h_\perp(\mathbf{s}) \geq 0$ by hypothesis, and of course $u^2(\mathbf{s}) \geq 0$. The claim follows. $\diamond_{\text{Claim 1}}$

Now, if u is harmonic, then u is infinitely differentiable, by Proposition 1E.4 on page 18. Thus, we can apply vector calculus techniques from Appendix 0E(iii). We have

$$0 \quad \underset{(*)}{\geq} \quad \int_{\partial\mathbb{X}} u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \, d\mathbf{s} \quad \underset{(\dagger)}{=} \quad \int_{\mathbb{X}} u(\mathbf{x}) \triangle u(\mathbf{x}) \; + \; |\nabla u(\mathbf{x})|^2 \, d\mathbf{x}$$
$$\underset{(\ddagger)}{=} \quad \int_{\mathbb{X}} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \quad \underset{(\diamond)}{\geq} \quad 0. \tag{5D.2}$$

Here, $(*)$ is by Claim 1, $(\dagger)$ is by Green's Formula (Theorem 0E.5(b) on page 564), $(\ddagger)$ is because $\triangle u \equiv 0$, and $(\diamond)$ is because $|\nabla u(\mathbf{x})|^2 \geq 0$ for all $\mathbf{x} \in \mathbb{X}$.

The inequalities (5D.2) imply that

$$\int_{\mathbb{X}} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \quad = \quad 0.$$

But this implies that $|\nabla u(\mathbf{x})| = 0$ for all $\mathbf{x} \in \mathbb{X}$, which means $\nabla u \equiv 0$, which means u is a constant on $\mathbb{X}$, as desired.

Now, if $\nabla u \equiv 0$, then clearly $\partial_\perp u(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$. Thus, the Robin boundary conditions reduce to $h(\mathbf{s})u(\mathbf{s}) = 0$. If $h(\mathbf{s}) \neq 0$ for some $\mathbf{s} \in \partial\mathbb{X}$, then we get $u(\mathbf{s}) = 0$. But since u is a constant, this means that $u \equiv 0$. $\Box$

One of the nice things about *linear* differential equations is that linearity enormously simplifies the problem of solution uniqueness. First we show that the only solution satisfying *homogeneous* boundary conditions (and, if applicable, *zero* initial conditions) is the constant zero function (as in Lemma 5D.3 above). Then it is easy to deduce uniqueness for arbitrary initial/boundary conditions.

Corollary 5D.4. (Solution uniqueness: Laplace equation, nonhomogeneous BC)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, and let $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ be continuous.

(a) *There exists at most one continuous, harmonic function $u : \mathbb{X} \longrightarrow \mathbb{R}$ which satisfies the* nonhomogeneous Dirichlet BC $u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$.

(b) *Suppose $\mathbb{X}$ has a piecewise smooth boundary.*

[i] *If $\int_{\partial\mathbb{X}} b(\mathbf{s}) \, d\mathbf{s} \neq 0$, then there is no continuous harmonic function u : $\mathbb{X} \longrightarrow \mathbb{R}$ which satisfies the* nonhomogeneous Neumann BC $\partial_\perp u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$.

[ii] *Suppose* $\int_{\partial\mathbb{X}} b(\mathbf{s})\ d\mathbf{s} = 0$. *If* $u_1, u_2 : \mathbb{X} \longrightarrow \mathbb{R}$ *are two continuous harmonic functions which both satisfy the* nonhomogeneous Neumann BC $\partial_\perp u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$, *then* $u_1 = u_2 + C$ *for some constant* C.

(c) *Suppose* $\mathbb{X}$ *has a piecewise smooth boundary, and let* $h, h_\perp : \partial\mathbb{X} \longrightarrow \mathbb{R}_{\not-}$ *be two other continuous nonnegative functions such that* $h(\mathbf{s}) + h_\perp(\mathbf{s}) > 0$ *for all* $\mathbf{s} \in \partial\mathbb{X}$. *If* $u_1, u_2 : \mathbb{X} \longrightarrow \mathbb{R}$ *are two continuous harmonic functions which both satisfy the* nonhomogeneous Robin BC $h(\mathbf{s})u(\mathbf{s}) + h_\perp(\mathbf{s})\partial_\perp u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$, *then* $u_1 = u_2 + C$ *for some constant* C. *Furthermore, if* h *is nonzero somewhere on* $\partial\mathbb{X}$, *then* $u_1 = u_2$.

Ⓔ *Proof.* **Exercise 5D.3** Hint: for (a), (c), and (b)[ii], suppose that $u_1, u_2 : \mathbb{X} \longrightarrow \mathbb{R}$ are two continuous harmonic functions with the desired nonhomogeneous boundary conditions. Then $(u_1 - u_2)$ is a continuous harmonic function satisfying *homogeneous* boundary conditions of the same kind; now apply the appropriate part of Lemma 5D.3 to conclude that $(u_1 - u_2)$ is zero or a constant.

For (b)[i], use Green's Formula (Theorem 0E.5(a) on page 564). □

Ⓔ **Exercise 5D.4.** Let $\mathbb{X} = \mathbb{D} = \{(r, \theta)\ ;\ \theta \in [-\pi, \pi),\ r \leq 1\}$ be the closed unit disk (in polar coordinates). Consider the function $h : \mathbb{D} \longrightarrow \mathbb{R}$ defined by $h(r, \theta) = \log(r)$. In Cartesian coordinates, h has the form $h(x, y) = \log(x^2 + y^2)$ (see Figure 1C.1(A) on page 10). In Example 1C.2 we observed that h is harmonic. But h satisfies homogeneous Dirichlet BC on $\partial\mathbb{D}$, so it seems to be a counterexample to Lemma 5D.3(a). Also, $\partial_\perp h(x) = 1$ for all $x \in \partial\mathbb{D}$, so h seems to be a counterexample to Corollary 5D.4(b)[i].

Why is this function *not* a counterexample to Lemma 5D.3 or Corollary 5D.4(b)[i].?

♦

Theorem 5D.5. (Solution uniqueness: Poisson equation, Nonhomogeneous BC) *Let* $\mathbb{X} \subset \mathbb{R}^D$ *be a bounded domain with a piecewise smooth boundary. Let* $q : \mathbb{X} \longrightarrow \mathbb{R}$ *be a continuous function* (e.g. describing an electric charge or heat source), *and let* $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ *be another continuous function* (a boundary condition). *Then there is at most one continuous function* $u : \mathbb{X} \longrightarrow \mathbb{R}$ *satisfying the* Poisson Equation $\triangle u = q$, *and satisfying either of the following nonhomogeneous boundary conditions:*

(a) (Nonhomogeneous Dirichlet BC) $u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$.

(b) (Nonhomogeneous Robin BC) $h(\mathbf{s}))u(\mathbf{s}) + h_\perp(\mathbf{s})\partial_\perp u(\mathbf{s}) = b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$, *where* $h, h_\perp : \partial\mathbb{X} \longrightarrow \mathbb{R}_{\not-}$ *are two other nonnegative functions, and* h *is nontrivial.*

Furthermore, if u_1 *and* u_2 *are two functions satisfying* $\triangle u = q$, *and also satisfying:*

(c) (Nonhomogeneous Neumann BC) $\partial_\perp u(\mathbf{s}) \ = \ b(\mathbf{s})$ *for all* $\mathbf{s} \in \partial\mathbb{X}$.

....then $u_1 = u_2 + C$, *where* C *is a constant.*

Proof. Suppose u_1 and u_2 were two continuous functions satisfying one of **(a)** or **(b)**, and such that $\triangle u_1 = q = \triangle u_2$. Let $u = u_1 - u_2$. Then u is continuous, harmonic, and satisfies one of **(a)** or **(c)** in Lemma 5D.3. Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$. Hence, there can be at most one solution. The proof for **(c)** is **Exercise 5D.5** . □ Ⓔ

5D(ii) Uniqueness for the heat equation

Throughout this section, if $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is a time-varying scalar field, and $t \in \mathbb{R}_{\not-}$, then define the function $u_t : \mathbb{X} \longrightarrow \mathbb{R}$ by $u_t(\mathbf{x}) := u(\mathbf{x}; t)$, for all $\mathbf{x} \in \mathbb{X}$. (*Note:* u_t does *not* denote the time-derivative).

If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is any integrable function, then the **L^2-norm** of f is defined

$$\|f\|_2 \quad := \quad \left(\int_{\mathbb{X}} |f(\mathbf{x})|^2 \ d\mathbf{x} \right)^{1/2}.$$

(See §6B for more information). We begin with a result which reinforces our intuition that the heat equation resembles 'melting' or 'erosion'.

Lemma 5D.6. (L^2-norm decay for heat equation)
Let $\mathbb{X} \subset \mathbb{R}^D$ *be a bounded domain with a piecewise smooth boundary. Suppose that* $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *satisfies the following three conditions:*

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $\partial_t u$ and $\partial_1^2 u, \ldots, \partial_D^2 u$ are continuous on* $\mathsf{int}(\mathbb{X}) \times \mathbb{R}_+$*;*

(b) (Heat equation) $\partial_t u \ = \ \triangle u$*;*

(c) (Homogeneous Dirichlet/Neumann BC) *For all* $\mathbf{s} \in \partial\mathbb{X}$ *and* $t \in \mathbb{R}_{\not-}$, *either* $u_t(\mathbf{s}) \ = \ 0$ *or* $\partial_\perp u_t(\mathbf{s}) \ = \ 0$.[6]

Define the function $E : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}_{\not-}$ *by*

$$E(t) \quad := \quad \|u_t\|_2^2 \quad = \quad \int_{\mathbb{X}} |u_t(\mathbf{x})|^2 \ d\mathbf{x}, \qquad \text{for all } t \in \mathbb{R}_{\not-}. \tag{5D.3}$$

Then E is differentiable and nonincreasing —that is, $E'(t) \leq 0$ for all $t \in \mathbb{R}_{\not-}$.

[6]Note that this allows different boundary points to satisfy different homogeneous boundary conditions at different times.

Proof. For any $\mathbf{x} \in \mathbb{X}$ and $t \in \mathbb{R}_{\not-}$, we have

$$\partial_t \, |u_t(\mathbf{x})|^2 \underset{(*)}{=} 2u_t(\mathbf{x}) \cdot \partial_t \, u_t(\mathbf{x}) \underset{(\dagger)}{=} 2u_t(\mathbf{x}) \cdot \triangle u_t(\mathbf{x}), \tag{5D.4}$$

where $(*)$ is the Leibniz rule, and $(\dagger)$ is because u satisfies the heat equation by hypothesis (b). Thus,

$$E'(t) \underset{(*)}{=} \int_{\mathbb{X}} \partial_t \, |u_t(\mathbf{x})|^2 \; d\mathbf{x} \underset{(\dagger)}{=} 2 \int_{\mathbb{X}} u_t(\mathbf{x}) \cdot \triangle u_t(\mathbf{x}) \; d\mathbf{x}, \tag{5D.5}$$

Here $(*)$ comes from differentiating the integral (5D.3) using Proposition 0G.1 on page 567. Meanwhile, $(\dagger)$ is by eqn.(5D.4).

Claim 1: *For all* $t \in \mathbb{R}_{\not-}$, $\displaystyle\int_{\mathbb{X}} u_t(\mathbf{x}) \cdot \triangle u_t(\mathbf{x}) \; d\mathbf{x} \; = \; -\int_{\mathbb{X}} \|\nabla u_t(\mathbf{x})\|^2 \; d\mathbf{x}$.

Proof. For all $\mathbf{s} \in \partial\mathbb{X}$, either $u_t(\mathbf{s}) = 0$ or $\partial_\perp u_t(\mathbf{s}) = 0$ by hypothesis (c). But $\partial_\perp u_t(\mathbf{s}) = \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})$ (where $\vec{\mathbf{N}}(\mathbf{s})$ is the unit normal vector at $\mathbf{s}$), so this implies that $u_t(\mathbf{s}) \cdot \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}) = 0$ for all $\mathbf{s} \in \partial\mathbb{X}$. Thus,

$$\begin{aligned} 0 \;&=\; \int_{\partial\mathbb{X}} u_t(\mathbf{s}) \cdot \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}) \; d\mathbf{s} \underset{(*)}{=} \int_{\mathbb{X}} \operatorname{div}\left(u_t \cdot \nabla u_t\right)(\mathbf{x}) \; d\mathbf{x} \\ &\underset{(\dagger)}{=} \int_{\mathbb{X}} \Big(u_t \cdot \operatorname{div} \nabla u_t \; + \; \nabla u_t \bullet \nabla u_t \Big)(\mathbf{x}) \; d\mathbf{x} \\ &\underset{(\ddagger)}{=} \int_{\mathbb{X}} u_t(\mathbf{x}) \cdot \triangle u_t(\mathbf{x}) \; d\mathbf{x} \; + \; \int_{\mathbb{X}} \|\nabla u_t(\mathbf{x})\|^2 \; d\mathbf{x}. \end{aligned}$$

Here, $(*)$ is the Divergence Theorem 0E.4 on page 563, $(\dagger)$ is by the Leibniz rule for divergences (Proposition 0E.2(b) on page 560) and $(\ddagger)$ is because $\operatorname{div} \nabla u = \triangle u$, while $\nabla u_t \bullet \nabla u_t = \|\nabla u_t(\mathbf{x})\|^2$. We thus have

$$\int_{\mathbb{X}} u_t \cdot \triangle u_t \; + \; \int_{\mathbb{X}} \|\nabla u_t\|^2 \quad = \quad 0.$$

Rearranging this equation yields the claim. $\diamond_{\text{Claim 1}}$

Applying Claim 1 to equation (5D.5), we get

$$E'(t) \quad = \quad -2 \int_{\mathbb{X}} \|\nabla u_t(\mathbf{x})\|^2 \; d\mathbf{x} \quad \leq \quad 0.$$

because $\|\nabla u_t(\mathbf{x})\|^2 \geq 0$ for all $\mathbf{x} \in \mathbb{X}$. $\square$

Lemma 5D.7. (Solution uniqueness for heat equation; homogeneous I/BC)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain with a piecewise smooth boundary. Suppose that $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ satisfies the following four conditions:

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $\partial_t u$ and $\partial_1^2 u, \ldots, \partial_D^2 u$ are continuous on $\mathsf{int}(\mathbb{X}) \times \mathbb{R}_+$;*

(b) (Heat equation) $\partial_t u = \triangle u$;

(c) (Zero initial condition) $u_0(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in \mathbb{X}$;

(d) (Homogeneous Dirichlet/Neumann BC) *For all $\mathbf{s} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$, either $u_t(\mathbf{s}) = 0$ or $\partial_\perp u_t(\mathbf{s}) = 0$.*[7]

Then u must be the constant 0 function: $u \equiv 0$.

Proof. Define $E : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}_{\not-}$ as in Lemma 5D.6. Then E is a nonincreasing function. But $E(0) = 0$, because $u_0 \equiv 0$ by hypothesis (c). Thus, $E(t) = 0$ for all $t \in \mathbb{R}_{\not-}$. Thus, we must have $u_t \equiv 0$ for all $t \in \mathbb{R}_{\not-}$. □

Theorem 5D.8. (Uniqueness: forced heat equation, nonhomogeneous I/BC)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain with a piecewise smooth boundary. Let $\mathcal{I} : \mathbb{X} \longrightarrow \mathbb{R}$ be a continuous function (describing an initial condition), *and let $b : \partial\mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$, and $h, h_\perp : \partial\mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be three other continuous functions* (describing time-varying boundary conditions). *Let $f : \mathsf{int}(\mathbb{X}) \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be another continuous function* (describing exogenous heat being 'forced' into or out of the system). *Then there is at most one solution function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ satisfying the following four conditions:*

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $\partial_t u$ and $\partial_1^2 u, \ldots, \partial_D^2 u$ are continuous on $\mathsf{int}(\mathbb{X}) \times \mathbb{R}_+$;*

(b) (Heat equation with forcing) $\partial_t u = \triangle u + f$;

(c) (Initial condition) $u(\mathbf{x}, 0) = \mathcal{I}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$;

(d) (Nonhomogeneous Mixed BC) $h(\mathbf{s}, t) \cdot u_t(\mathbf{s}) + h_\perp(\mathbf{s}, t) \cdot \partial_\perp u_t(\mathbf{s}) = b(x, t)$, *for all $\mathbf{s} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$.*[8]

[7] Note that this allows different boundary points to satisfy different homogeneous boundary conditions at different times.

[8] Note that this includes nonhomogeneous Dirichlet BC (set $h_\perp \equiv 0$) and nonhomogeneous Neumann BC (set $h \equiv 0$) as special cases. Also note that by varying h and $h_\perp$, we can allow different boundary points to satisfy different nonhomogeneous boundary conditions at different times.

Proof. Suppose u_1 and u_2 were two functions satisfying all of **(a)-(d)**. Let $u = u_1 - u_2$. Then u satisfies all of **(a)-(d)** in Lemma 5D.7. Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$. Hence, there can be at most one solution. □

5D(iii) Uniqueness for the wave equation

Throughout this section, if $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is a time-varying scalar field, and $t \in \mathbb{R}_{\not-}$, then define the function $u_t : \mathbb{X} \longrightarrow \mathbb{R}$ by $u_t(\mathbf{x}) := u(\mathbf{x}; t)$, for all $\mathbf{x} \in \mathbb{X}$. (*Note:* u_t does *not* denote the time-derivative). For all $t \geq 0$, the **energy** of u is defined:

$$E(t) \quad := \quad \frac{1}{2}\int_{\mathbb{X}} |\partial_t\, u_t(\mathbf{x})|^2 + \|\nabla u_t(\mathbf{x})\|^2 \; d\mathbf{x}. \tag{5D.6}$$

We begin with a result which has an appealing physical interpretation.

Lemma 5D.9. (Conservation of Energy for wave equation)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain with a piecewise smooth boundary. Suppose $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ satisfies the following three conditions:

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $u \in \mathcal{C}^2\left(\mathsf{int}\,(\mathbb{X}) \times \mathbb{R}_+\right)$;*

(b) (Wave equation) $\partial_t^2\, u \;=\; \triangle u$;

(c) (Homogeneous Dirichlet/Neumann BC) *For all $\mathbf{s} \in \partial\mathbb{X}$, either $u_t(\mathbf{s}) \;=\; 0$ for all $t \geq 0$, or $\partial_\perp u_t(\mathbf{s}) \;=\; 0$ for all $t \geq 0$.*[9]

Then E is constant in time —that is, $\partial_t\, E(t) = 0$ for all $t > 0$.

Proof. The Leibniz rule says that

$$\begin{aligned} \partial_t\, |\partial_t\, u|^2 &= (\partial_t^2\, u)\cdot(\partial_t\, u) + (\partial_t\, u)\cdot(\partial_t^2\, u) \\ &= 2\cdot(\partial_t\, u)\cdot(\partial_t^2\, u), && \text{(5D.7)} \\ \text{and}\quad \partial_t\, \|\nabla u\|^2 &= (\partial_t\,\nabla u)\bullet(\nabla u) \;+\; (\nabla u)\bullet(\partial_t\,\nabla u) \\ &= 2\cdot(\nabla u)\bullet(\partial_t\,\nabla u) \\ &= 2\cdot(\nabla u)\bullet(\nabla\partial_t\, u). && \text{(5D.8)} \\ \text{Thus,}\quad \partial_t\, E &\underset{(*)}{=} \frac{1}{2}\int_{\mathbb{X}}\left(\partial_t\, |\partial_t\, u|^2 + \partial_t\, \|\nabla u\|^2\right) \\ &\underset{(\dagger)}{=} \int_{\mathbb{X}}\left(\partial_t\, u\cdot\partial_t^2\, u \;+\; (\nabla u)\bullet(\nabla\partial_t\, u)\right). && \text{(5D.9)} \end{aligned}$$

[9] This allows different boundary points to satisfy different homogeneous boundary conditions; but each particular boundary point must satisfy the *same* homogeneous boundary condition at all times.

Here $(*)$ comes from differentiating the integral (5D.6) using Proposition 0G.1 on page 567). Meanwhile, $(\dagger)$ comes from substituting (5D.7) and (5D.8).

Claim 1: *Fix $\mathbf{s} \in \partial\mathbb{X}$ and let $\vec{\mathbf{N}}(\mathbf{s})$ be the outward unit normal vector to $\partial\mathbb{X}$ at $\mathbf{s}$. Then $\partial_t u_t(\mathbf{s}) \cdot \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}) = 0$, for all $t > 0$.*

Proof. By hypothesis (c), either $\partial_\perp u_t(\mathbf{s}) = 0$ for all $t > 0$, or $u_t(\mathbf{s}) = 0$ for all $t > 0$. Thus, either $\nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}) = 0$ for all $t > 0$, or $\partial_t u_t(\mathbf{s}) = 0$ for all $t > 0$. In either case, $\partial_t u_t(\mathbf{s}) \cdot \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}) = 0$ for all $t > 0$. $\diamond_{\text{Claim 1}}$

Claim 2: *For any $t \in \mathbb{R}_{\not -}$, $\displaystyle\int_{\mathbb{X}} \nabla u_t \bullet \nabla\partial_t u_t \;=\; -\int_{\mathbb{X}} \partial_t u_t \cdot \triangle u_t$.*

Proof. Integrating Claim 1 over $\partial\mathbb{X}$, we get

$$
\begin{aligned}
0 \;&=\; \int_{\partial\mathbb{X}} \partial_t u_t(\mathbf{s}) \cdot \nabla u_t(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})\; d\mathbf{s} \;\underset{(*)}{=}\; \int_{\mathbb{X}} \operatorname{div}\left(\partial_t u_t \cdot \nabla u_t\right)(\mathbf{x})\; d\mathbf{x} \\
&\underset{(\dagger)}{=}\; \int_{\mathbb{X}} \Big(\partial_t u_t \cdot \operatorname{div}\nabla u_t \;+\; \nabla\partial_t u_t \bullet \nabla u_t\Big)(\mathbf{x})\; d\mathbf{x} \\
&\underset{(\ddagger)}{=}\; \int_{\mathbb{X}} \left(\partial_t u_t \cdot \triangle u_t\right)(\mathbf{x})\; d\mathbf{x} \;+\; \int_{\mathbb{X}} \left(\nabla\partial_t u_t \bullet \nabla u_t\right)(\mathbf{x})\; d\mathbf{x}.
\end{aligned}
$$

Here, $(*)$ is the Divergence Theorem 0E.4 on page 563, $(\dagger)$ is by the Leibniz rule for divergences (Proposition 0E.2(b) on page 560) and $(\ddagger)$ is because $\operatorname{div}\nabla u_t = \triangle u_t$. We thus have

$$
\int_{\mathbb{X}} \nabla u_t \bullet \nabla\partial_t u_t \;+\; \int_{\mathbb{X}} \partial_t u_t \cdot \triangle u_t \quad=\quad 0.
$$

Rearranging this equation yields the claim. $\diamond_{\text{Claim 2}}$

Putting it all together, we get:

$$
\begin{aligned}
\partial_t E \;&\underset{(\dagger)}{=}\; \int_{\mathbb{X}} \partial_t u \cdot \partial_t^2 u \;+\; \int_{\mathbb{X}} (\nabla u) \bullet (\nabla\partial_t u) \\
&\underset{(\ddagger)}{=}\; \int_{\mathbb{X}} \partial_t u \cdot \partial_t^2 u \;-\; \int_{\mathbb{X}} \partial_t u \cdot \triangle u \quad=\quad \int_{\mathbb{X}} \partial_t u \cdot \left(\partial_t^2 u \;-\; \triangle u\right) \\
&\underset{(*)}{=}\; \int_{\mathbb{X}} \partial_t u \cdot 0 \quad=\quad 0,
\end{aligned}
$$

as desired. Here, $(\dagger)$ is by equation (5D.9), $(\ddagger)$ is by Claim 2, and $(*)$ is because $\partial_t^2 u \;-\; \triangle u \equiv 0$ because u satisfies the wave equation by hypothesis (b). $\square$

Physical interpretation. $E(t)$ can be interpreted as the *total energy* in the system at time t. The first term in the integrand of (5D.6) measures the *kinetic* energy of the wave motion, while the second term measures the *potential* energy stored in the deformation of the medium. With this physical interpretation, Lemma 5D.9 simply asserts the principle of *Conservation of Energy*: E must be constant in time, because no energy enters or leaves the system, by hypotheses (b) and (c).

Lemma 5D.10. (Solution uniqueness for wave equation; homogeneous I/BC)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain with a piecewise smooth boundary. Suppose $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ satisfies all five of the following conditions:

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $u \in \mathcal{C}^2(\mathsf{int}(\mathbb{X}) \times \mathbb{R}_+)$;*

(b) (Wave equation) $\partial_t^2 u = \triangle u$*;*

(c) (Zero initial position) $u_0(\mathbf{x}) = 0$*, for all $\mathbf{x} \in \mathbb{X}$;*

(d) (Zero initial velocity) $\partial_t u_0(\mathbf{x}) = 0$ *for all $\mathbf{x} \in \mathbb{X}$;*

(e) (Homogeneous Dirichlet/Neumann BC) *For all $\mathbf{s} \in \partial\mathbb{X}$, either $u_t(\mathbf{s}) = 0$ for all $t \geq 0$, or $\partial_\perp u_t(\mathbf{s}) = 0$ for all $t \geq 0$.*[10]

Then u must be the constant 0 function: $u \equiv 0$.

Proof. Let $E : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}_{\not-}$ be the energy function from Lemma 5D.9. Then E is a constant. But $E(0) = 0$ because $u_0 \equiv 0$ and $\partial_t u_0 \equiv 0$, by hypotheses (c) and (d). Thus, $E(t) = 0$ for all $t \geq 0$. But this implies that $|\partial_t u_t(\mathbf{x})|^2 = 0$, and hence $\partial_t u_t(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{X}$ and $t > 0$. Thus, u is constant in time. Since $u_0 \equiv 0$, we conclude that $u_t \equiv 0$ for all $t \geq 0$, as desired. □

Theorem 5D.11. (Uniqueness: forced wave equation, nonhomogeneous I/BC)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain with a piecewise smooth boundary. Let $\mathcal{I}_0, \mathcal{I}_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be continuous functions (describing initial position and velocity). *Let $b : \partial\mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be another continuous function* (describing a time-varying boundary condition). *Let $f : \mathsf{int}(\mathbb{X}) \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be another continuous function* (describing exogenous vibrations being 'forced' into the system). *Then there is at most one solution function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ satisfying all five of the following conditions:*

(a) (Regularity) *u is continuous on $\mathbb{X} \times \mathbb{R}_{\not-}$, and $u \in \mathcal{C}^2(\mathsf{int}(\mathbb{X}) \times \mathbb{R}_+)$;*

[10] This allows different boundary points to satisfy different homogeneous boundary conditions; but each particular boundary point must satisfy the *same* homogeneous boundary condition at all times.

(b) (Wave equation with forcing) $\partial_t^2 u \;=\; \triangle u + f$;

(c) (Initial position) $u(\mathbf{x}, 0) \;=\; \mathcal{I}_0(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$.

(d) (Initial velocity) $\partial_t u(\mathbf{x}, 0) \;=\; \mathcal{I}_1(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$.

(e) (Nonhomogeneous Dirichlet/Neumann BC) *For all* $\mathbf{s} \in \partial\mathbb{X}$, *either* $u(\mathbf{s}, t) \;=\; b(\mathbf{s}, t)$ *for all* $t \geq 0$, *or* $\partial_\perp u(\mathbf{s}, t) \;=\; b(\mathbf{s}, t)$ *for all* $t \geq 0$.

Proof. Suppose u_1 and u_2 were two functions satisfying all of **(a)**-**(e)**. Let $u = u_1 - u_2$. Then u satisfies all of **(a)**-**(e)**, in Lemma 5D.10. Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$. Hence, there can be at most one solution. □

Remark. (a) Earlier, we observed that the *initial position* problem for the (unforced) wave equation represents a 'plucked string' (e.g. in a guitar), while the *initial velocity* problem represents a 'struck string' (e.g. in a piano). Continuing the musical analogy, the *forced* wave equation represents a *rubbed* string (e.g. in a violin or cello), as well as any other musical instrument driven by an exogenous vibration (e.g. any wind instrument).

(b) Notice Theorems 5D.5, 5D.8, and 5D.11 apply under much more general conditions than any of the solution methods we will actually develop in this book (i.e. they work for almost any 'reasonable' domain, we allow for possible 'forcing', and we even allow the boundary conditions to vary in time). This is a recurring theme in differential equation theory; it is generally possible to prove 'qualitative' results (e.g. about existence, uniqueness, or general properties of solutions) in much more general settings than it is possible to get 'quantitative' results (i.e. explicit formulae for solutions). Indeed, for most *nonlinear* differential equations, qualitative results are pretty much all you can ever get.

5E* Classification of second order linear PDEs

Prerequisites: §5A. **Recommended:** §1B, §1C, §1F, §2B.

5E(i) ...in two dimensions, with constant coefficients

Recall that $\mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$ is the space of all differentiable scalar fields on the plane $\mathbb{R}^2$. In general, a second-order linear differential operator L on $\mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$ with constant coefficients looks like:

$$\mathsf{L}u \;=\; a \cdot \partial_x^2 u \;+\; b \cdot \partial_x \partial_y u \;+\; c \cdot \partial_y^2 u \;+\; d \cdot \partial_x u \;+\; e \cdot \partial_y u \;+\; f \cdot u \quad (5E.1)$$

where a, b, c, d, e, f are constants. Define:

$$\alpha = f, \qquad \boldsymbol{\beta} \;=\; \begin{bmatrix} d \\ e \end{bmatrix} \qquad \text{and} \qquad \Gamma \;=\; \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix} \;=\; \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.$$

Then we can rewrite (5E.1) as:

$$\mathsf{L}u \;=\; \sum_{c,d=1}^{2} \gamma_{c,d} \cdot \partial_c \partial_d u \;+\; \sum_{d=1}^{2} \beta_d \cdot \partial_d u \;+\; \alpha \cdot u,$$

Any 2×2 symmetric matrix Γ defines a **quadratic form** $G : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$G(x, y) \;=\; [x\ y] \cdot \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \gamma_{11} \cdot x^2 + \Big(\gamma_{12} + \gamma_{21}\Big) \cdot xy + \gamma_{22} \cdot y^2.$$

We say Γ is **positive definite** if, for all $x, y \in \mathbb{R}$, we have:

- $G(x, y) \geq 0$;
- $G(x, y) = 0$ if and only if $x = 0 = y$.

Geometrically, this means that the graph of G defines an *elliptic paraboloid* in $\mathbb{R}^2 \times \mathbb{R}$, which curves upwards in every direction. Equivalently, Γ is positive definite if there is a constant $K > 0$ such that

$$G(x, y) \geq K \cdot (x^2 + y^2)$$

for every $(x, y) \in \mathbb{R}^2$. We say Γ is **negative definite** if $-\Gamma$ is positive definite.

The differential operator L from equation (5E.1) is called **elliptic** if the matrix Γ is either positive definite or negative definite.

Example 5E.1. If $\mathsf{L} = \triangle$, then $\Gamma = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ is just the identity matrix. while $\boldsymbol{\beta} = 0$ and $\alpha = 0$. The identity matrix is clearly positive definite; thus, $\triangle$ is an elliptic differential operator. ♢

Suppose that L is an elliptic differential operator. Then:

- An **elliptic** PDE is one of the form: $\mathsf{L}u = 0$ (or $\mathsf{L}u = g$). For example, the **Laplace equation** is elliptic.
- A **parabolic** PDE is one of the form: $\partial_t = \mathsf{L}u$. For example, the two-dimensional **heat equation** is parabolic.
- A **hyperbolic** PDE is one of the form: $\partial_t^2 = \mathsf{L}u$. For example, the two-dimensional **wave equation** is hyperbolic.

(See Remark 16F.4 on page 371 for a partial justification of this terminology).

Ⓔ **Exercise 5E.1.** Show that Γ is positive definite if and only if $0 < \det(\Gamma) = ac - \frac{1}{4}b^2$. In other words, L is elliptic if and only if $4ac - b^2 > 0$. ♦

5E(ii) ...in general

Recall that $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ is the space of all differentiable scalar fields on D-dimensional space. The general second-order linear differential operator on $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ has the form

$$\mathsf{L}u \quad = \quad \sum_{c,d=1}^{D} \gamma_{c,d} \cdot \partial_c \partial_d u \quad + \quad \sum_{d=1}^{D} \beta_d \cdot \partial_d u \quad + \quad \alpha \cdot u, \tag{5E.2}$$

where $\alpha : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ is some time-varying scalar field, $(\beta_1, \ldots, \beta_D) = \boldsymbol{\beta} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}^D$ is a time-varying vector field, and $\gamma_{c,d} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ are functions such that, for any $\mathbf{x} \in \mathbb{R}^D$ and $t \in \mathbb{R}$, the matrix

$$\Gamma(\mathbf{x};t) = \begin{bmatrix} \gamma_{11}(\mathbf{x};t) & \ldots & \gamma_{1D}(\mathbf{x};t) \\ \vdots & \ddots & \vdots \\ \gamma_{D1}(\mathbf{x};t) & \ldots & \gamma_{DD}(\mathbf{x};t) \end{bmatrix}$$

is **symmetric** (i.e. $\gamma_{cd} = \gamma_{dc}$).

Example 5E.2.

(a) If $\mathsf{L} = \triangle$, then $\boldsymbol{\beta} \equiv 0$, $\alpha = 0$, and $\Gamma \quad \equiv \quad \mathbf{Id} \quad = \quad \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$.

(b) The Fokker-Plank Equation (see § 1F on page 18) has the form $\partial_t u = \mathsf{L}u$, where $\alpha = -\mathrm{div}\, \vec{\mathbf{V}}(\mathbf{x})$, $\boldsymbol{\beta}(\mathbf{x}) = -\nabla\vec{\mathbf{V}}(\mathbf{x})$, and $\Gamma \quad \equiv \quad \mathbf{Id}$. (**Exercise 5E.2**) Ⓔ
♢

If the functions $\gamma_{c,d}$, β_d and α are independent of $\mathbf{x}$, then we say L is **spatially homogeneous**. If they are also independent of t, we say that L has **constant coefficients**.

Any symmetric matrix Γ defines a **quadratic form** $G : \mathbb{R}^D \longrightarrow \mathbb{R}$ by

$$G(\mathbf{x}) \quad = \quad [x_1 ... x_D] \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1D} \\ \vdots & \ddots & \vdots \\ \gamma_{D1} & \cdots & \gamma_{DD} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_D \end{bmatrix} \quad = \quad \sum_{c,d=1}^{D} \gamma_{c,d} \cdot x_c \cdot x_d$$

Γ is called **positive definite** if, for all $\mathbf{x} \in \mathbb{R}^D$, we have:

- $G(\mathbf{x}) \geq 0$;

- $G(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

Equivalently, Γ is positive definite if there is a constant $K_\Gamma > 0$ such that $G(\mathbf{x}) \geq K_\Gamma \cdot \|\mathbf{x}\|^2$ for every $\mathbf{x} \in \mathbb{R}^D$. On the other hand, Γ is **negative definite** if $-\Gamma$ is positive definite.

The differential operator L from equation (5E.2) is **elliptic** if the matrix $\Gamma(\mathbf{x};t)$ is either positive definite or negative definite for every $(\mathbf{x};t) \in \mathbb{R}^D \times \mathbb{R}_{\not -}$, and furthermore, there is some $K > 0$ such that $K_{\Gamma(\mathbf{x};t)} \geq K$ for all $(\mathbf{x};t) \in \mathbb{R}^D \times \mathbb{R}_{\not -}$. For example, the Laplacian and the Fokker-Plank operator are both elliptic. (**Exercise 5E.3**) Ⓔ

Suppose that L is an elliptic differential operator. Then:

- An **elliptic** PDE is one of the form: $\mathsf{L}u = 0$ (or $\mathsf{L}u = g$).
- A **parabolic** PDE is one of the form: $\partial_t = \mathsf{L}u$.
- A **hyperbolic** PDE is one of the form: $\partial_t^2 = \mathsf{L}u$.

Example 5E.3.

(a) Laplace's Equation and Poisson's Equation are *elliptic* PDEs.

(b) The heat equation and the Fokker-Plank Equation are *parabolic*.

(c) The wave equation is *hyperbolic*. ◇

Parabolic equations are "generalized heat equations", describing *diffusion through an inhomogeneous*[11]*, anisotropic*[12] *medium with drift*. The terms in $\Gamma(\mathbf{x};t)$ describe the inhomogeneity and anisotropy of the diffusion[13], while the vector field $\boldsymbol{\beta}$ describes the drift.

Hyperbolic equations are "generalized wave equations", describing *wave propagation* through an inhomogeneous, anisotropic medium with drift —for example, sound waves propagating through an air mass with variable temperature and pressure and wind blowing.

5F Practice problems

Evolution equations and initial value problems. For each of the following equations: u is an unknown function; q is always some fixed, predetermined function; and λ is always a constant. In each case, determine the *order* of the equation, and decide: is this an *evolution equation*? Why or why not?

[11] **Homogeneous** means, "Looks the same everywhere in space", whereas **inhomogeneous** is the opposite.

[12] **Isotropic** means "looks the same in every direction"; **anisotropic** means the opposite.

[13] If the medium is homogeneous, then Γ is constant. If the medium is isotropic, then $\Gamma = \mathbf{Id}$.

1. heat equation: $\partial_t u(\mathbf{x}) \;=\; \triangle u(\mathbf{x})$.
2. Poisson Equation: $\triangle u(\mathbf{x}) \;=\; q(\mathbf{x})$.
3. Laplace Equation: $\triangle u(\mathbf{x}) \;=\; 0$.
4. Monge-Ampère Equation: $q(x,y) \;=\; \det \begin{bmatrix} \partial_x^2 u(x,y) & \partial_x \partial_y u(x,y) \\ \partial_x \partial_y u(x,y) & \partial_y^2 u(x,y) \end{bmatrix}$.
5. Reaction-Diffusion $\partial_t u(\mathbf{x};t) \;=\; \triangle u(\mathbf{x};t) \;+\; q\Big(u(\mathbf{x};t)\Big)$.
6. Scalar conservation Law $\partial_t u(x;t) \;=\; -\partial_x (q \circ u)(x;t)$.
7. Helmholtz Equation: $\triangle u(\mathbf{x}) \;=\; \lambda \cdot u(\mathbf{x})$.
8. Airy's Equation: $\partial_t u(x;t) \;=\; -\partial_x^3 u(x;t)$.
9. Beam Equation: $\partial_t u(x;t) \;=\; -\partial_x^4 u(x;t)$.
10. Schrödinger Equation: $\partial_t u(\mathbf{x};t) \;=\; \mathbf{i} \triangle u(\mathbf{x};t) \;+\; q(\mathbf{x};t) \cdot u(\mathbf{x};t)$.
11. Burger's Equation: $\partial_t u(x;t) \;=\; -u(x;t) \cdot \partial_x u(x;t)$.
12. Eikonal Equation: $|\partial_x u(x)| \;=\; 1$.

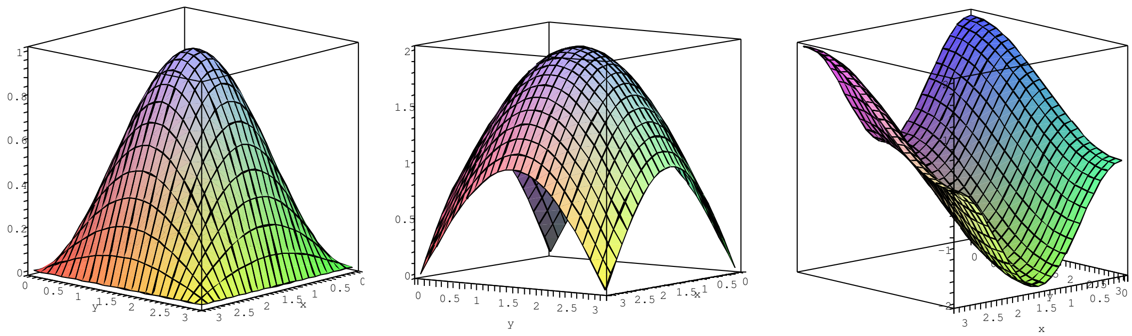
Boundary value problems.

1. Each of the following functions is defined on the interval $[0, \pi]$, in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[14] BC? Periodic BC? Justify your answers.

   (a) $u(x) \;=\; \sin(3x)$.
   (b) $u(x) \;=\; \sin(x) + 3\sin(2x) - 4\sin(7x)$.
   (c) $u(x) \;=\; \cos(x) + 3\sin(3x) - 2\cos(6x)$.
   (d) $u(x) \;=\; 3 + \cos(2x) - 4\cos(6x)$.
   (e) $u(x) \;=\; 5 + \cos(2x) - 4\cos(6x)$.

2. Each of the following functions is defined on the interval $[-\pi, \pi]$, in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[14] BC? Periodic BC? Justify your answers.

   (a) $u(x) \;=\; \sin(x) + 5\sin(2x) - 2\sin(3x)$.
   (b) $u(x) \;=\; 3\cos(x) - 3\sin(2x) - 4\cos(2x)$.



(A) $f(x,y) = \sin(x)\sin(y)$ **(B)** $g(x,y) = \sin(x)+\sin(y)$ **(C)** $h(x,y) = \cos(2x)+\cos(y)$.

Figure 5F.1: Problems #3a, #3b and #3c

(c) $u(x) = 6 + \cos(x) - 3\cos(2x)$.

3. Each of the following functions is defined on the box $[0,\pi]^2$. in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[14] BC? Periodic BC? Justify your answers.

   (a) $f(x,y) = \sin(x)\sin(y)$ (Figure 5F.1(A))
   (b) $g(x,y) = \sin(x)+\sin(y)$ (Figure 5F.1(B))
   (c) $h(x,y) = \cos(2x)+\cos(y)$ (Figure 5F.1(C))
   (d) $u(x,y) = \sin(5x)\sin(3y)$.
   (e) $u(x,y) = \cos(-2x)\cos(7y)$.

4. Each of the following functions is defined on the unit disk

$$\mathbb{D} = \{(r,\theta)\ ;\ 0 \leq r \leq 1, \quad \text{and} \quad \theta \in [0,2\pi)\}$$

   in polar coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[14] BC? Justify your answers.

   (a) $u(r,\theta) = (1-r^2)$.
   (b) $u(r,\theta) = 1-r^3$.
   (c) $u(r,\theta) = 3+(1-r^2)^2$.

[14] Here, 'Robin' B.C. means *nontrivial* Robin B.C. —i.e. *not* just homogenous Dirichlet or Neumann.

(d) $u(r,\theta) \;=\; \sin(\theta)(1-r^2)^2$.

(e) $u(r,\theta) \;=\; \cos(2\theta)(e-e^r)$.

5. Each of the following functions is defined on the 3-dimensional unit ball

$$\mathbb{B} \;=\; \left\{(r,\theta,\varphi)\;;\; 0\le r\le 1,\; \theta\in[0,2\pi), \quad \text{and} \quad \varphi\in\left[\frac{-\pi}{2},\,\frac{\pi}{2}\right]\right\}$$

in spherical coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[14] BC? Justify your answers.

(a) $u(r,\theta,\varphi) \;=\; (1-r)^2$.

(b) $u(r,\theta,\varphi) \;=\; (1-r)^3+5$.

6. Which Neumann BVP has solution(s) on the domain $\mathbb{X}=[0,1]$?

(a) $u''(x)=0$, $u'(0)=1$, $u'(1)=1$.

(b) $u''(x)=0$, $u'(0)=1$, $u'(1)=2$.

(c) $u''(x)=0$, $u'(0)=1$, $u'(1)=-1$.

(d) $u''(x)=0$, $u'(0)=1$, $u'(1)=-2$.

7. Which BVP of Laplace equation on the unit disk $\mathbb{D}$ has a solution? Which BVP has more than one solution?

(a) $\triangle u=0$, $u(1,\theta)=0$, for all $\theta\in[-\pi,\pi)$.

(b) $\triangle u=0$, $u(1,\theta)=\sin\theta$, for all $\theta\in[-\pi,\pi)$.

(c) $\triangle u=0$, $\partial_\perp u(1,\theta)=\sin(\theta)$, for all $\theta\in[-\pi,\pi)$.

(d) $\triangle u=0$, $\partial_\perp u(1,\theta)=1+\cos(\theta)$, for all $\theta\in[-\pi,\pi)$.

III Fourier series on bounded domains

Any complex sound is a combination of simple 'pure tones' of different frequencies. For example, a musical *chord* is a superposition of three (or more) musical notes, each with a different frequency. In fact, a musical note itself is not really a single frequency at all; a note consists of a 'fundamental' frequency, plus a cascade of higher frequency 'harmonics'. The energy distribution of these harmonics is part of what gives each musical instrument its distinctive sound. The decomposition of a sound into separate frequencies is sometimes called its *power spectrum*. A crude graphical representation of this power spectrum is visible on most modern stereo systems (the little jiggling red bars).

Fourier theory is based on the idea that a real-valued function is like a sound, which can be represented as a superposition of 'pure tones' (i.e. sine waves and/or cosine waves) of distinct frequencies. This provides a 'coordinate system' for expressing functions, and within this coordinate system, we can express the solutions for many partial differential equations in a simple and elegant way. Fourier theory is also an essential tool in probability theory and signal analysis (although we will not discuss these applications in this book).

The idea of Fourier theory is simple, but to make this idea rigorous enough to be useful, we must deploy some formidable mathematical machinery. So we will begin by developing the necessary background concerning inner products, orthogonality, and the convergence of functions.

Chapter 6

Some functional analysis

"Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts." —David Hilbert

6A Inner products

Prerequisites: §4A.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$, with $\mathbf{x} = (x_1, \ldots, x_D)$ and $\mathbf{y} = (y_1, \ldots, y_D)$. The **inner product**[1] of $\mathbf{x}, \mathbf{y}$ is defined:

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad := \quad x_1 y_1 + x_2 y_2 + \ldots + x_D y_D.$$

The inner product describes the geometric relationship between $\mathbf{x}$ and $\mathbf{y}$, via the formula:

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad := \quad \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \cos(\theta)$$

where $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are the lengths of vectors $\mathbf{x}$ and $\mathbf{y}$, and θ is the angle between them. (**Exercise 6A.1** Verify this). In particular, if $\mathbf{x}$ and $\mathbf{y}$ are *perpendicular*, then $\theta = \pm\frac{\pi}{2}$, and then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$; we then say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal**. Ⓔ
For example, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal in $\mathbb{R}^2$, while

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are all orthogonal to one another in $\mathbb{R}^4$. Indeed, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ also have unit norm; we call any such collection an **orthonormal set** of vectors. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal set. However, $\{\mathbf{x}, \mathbf{y}\}$ is orthogonal but *not* orthonormal (because $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2} \neq 1$).

[1] This is sometimes this is called the **dot product**, and denoted "$\mathbf{x} \bullet \mathbf{y}$".

The **norm** of a vector satisfies the equation:

$$\|\mathbf{x}\| \;=\; \left(x_1^2 + x_2^2 + \ldots + x_D^2\right)^{1/2} \;=\; \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}.$$

If $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are a collection of mutually orthogonal vectors, and $\mathbf{x} = \mathbf{x}_1 + \ldots + \mathbf{x}_N$, then we have the generalized **Pythagorean formula**:

$$\|\mathbf{x}\|^2 \;=\; \|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \ldots + \|\mathbf{x}_N\|^2.$$

Ⓔ (**Exercise 6A.2** Verify the Pythagorean formula.)

An **orthonormal basis** of $\mathbb{R}^D$ is any collection of mutually orthogonal vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_D\}$, all of norm 1, such that, for any $\mathbf{w} \in \mathbb{R}^D$, if we define $\omega_d = \langle \mathbf{w}, \mathbf{v}_d \rangle$ for all $d \in [1..D]$, then:

$$\mathbf{w} \quad = \quad \omega_1 \mathbf{v}_1 + \omega_2 \mathbf{v}_2 + \ldots + \omega_D \mathbf{v}_D.$$

In other words, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_D\}$ defines a *coordinate system* for $\mathbb{R}^D$, and in this coordinate system, the vector $\mathbf{w}$ has coordinates $(\omega_1, \omega_2, \ldots, \omega_D)$. If $\mathbf{x} \in \mathbb{R}^D$ is another vector, and $\xi_d = \langle \mathbf{x}, \mathbf{v}_d \rangle$ all $d \in [1..D]$, then we also have

$$\mathbf{x} \quad = \quad \xi_1 \mathbf{v}_1 + \xi_2 \mathbf{v}_2 + \ldots + \xi_D \mathbf{v}_D.$$

We can then compute $\langle \mathbf{w}, \mathbf{x} \rangle$ using **Parseval's Equality:**

$$\langle \mathbf{w}, \mathbf{x} \rangle \quad = \quad \omega_1 \xi_1 + \omega_2 \xi_2 + \cdots \omega_D \xi_D.$$

Ⓔ (**Exercise 6A.3** Prove Parseval's equality.) In particular, if $\mathbf{x} = \mathbf{w}$, we get the the following version of the generalized Pythagorean formula:

$$\|\mathbf{w}\|^2 \quad = \quad \omega_1^2 + \omega_2^2 + \ldots + \omega_D^2.$$

Example 6A.1.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^D$.

(b) If $\mathbf{v}_1 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis of $\mathbb{R}^2$.

If $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, then $\omega_1 = \sqrt{3} + 2$ and $\omega_2 = 2\sqrt{3} - 1$, so that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \;=\; \omega_1 \mathbf{v}_1 + \omega_2 \mathbf{v}_2 \;=\; \left(\sqrt{3} + 2\right) \cdot \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \;+\; \left(2\sqrt{3} - 1\right) \cdot \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}.$$

Thus, $\|\mathbf{w}\|_2^2 = 2^2 + 4^2 = 20$, and also, by Parseval's equality, $20 \;=\; \omega_1^2 +$
Ⓔ $\omega_2^2 \;=\; \left(\sqrt{3} + 2\right)^2 + \left(1 - 2\sqrt{3}\right)^2$. (**Exercise 6A.4** Verify these claims.) ◇

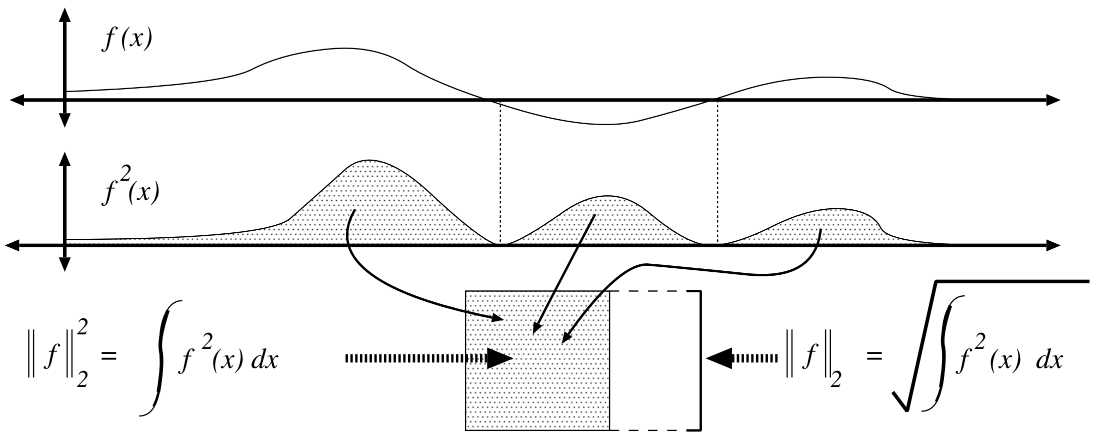


Figure 6A.1: The L^2 norm of f: $\|f\|_2 = \sqrt{\int_{\mathbb{X}} |f(x)|^2 \, dx}$

6B L^2 space

The ideas of section 6A generalize to spaces of functions. Suppose $\mathbb{X} \subset \mathbb{R}^D$ is some bounded domain, and let $M := \int_{\mathbb{X}} 1 \, d\mathbf{x}$ be the *volume*[2] of the domain $\mathbb{X}$. (The second column of Table 6.1 provides examples of M for various domains.)

Domain		M	Inner Product
Unit interval	$\mathbb{X} = [0,1] \quad \subset \mathbb{R}$	length $\quad M = 1$	$\langle f,g \rangle = \int_0^1 f(x) \cdot g(x) \, dx$
π interval	$\mathbb{X} = [0,\pi] \quad \subset \mathbb{R}$	length $\quad M = \pi$	$\langle f,g \rangle = \frac{1}{\pi} \int_0^\pi f(x) \cdot g(x) \, dx$
Unit square	$\mathbb{X} = [0,1] \times [0,1] \quad \subset \mathbb{R}^2$	area $\quad M = 1$	$\langle f,g \rangle = \int_0^1 \int_0^1 f(x,y) \cdot g(x,y) \, dx \, dy$
$\pi \times \pi$ square	$\mathbb{X} = [0,\pi] \times [0,\pi] \quad \subset \mathbb{R}^2$	area $\quad M = \pi^2$	$\langle f,g \rangle = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi f(x,y) \cdot g(x,y) \, dx \, dy$
Unit Disk (polar coords)	$\mathbb{X} = \{(r,\theta) \, ; \, r \leq 1\} \quad \subset \mathbb{R}^2$	area $\quad M = \pi$	$\langle f,g \rangle = \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi f(r,\theta) \cdot g(r,\theta) \, r \cdot d\theta \, dr$
Unit cube	$\mathbb{X} = [0,1] \times [0,1] \times [0,1] \quad \subset \mathbb{R}^3$	volume $\quad M = 1$	$\langle f,g \rangle = \int_0^1 \int_0^1 \int_0^1 f(x,y,z) \cdot g(x,y,z) \, dx \, dy \, dz$

Table 6.1: Inner products on various domains.

If $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then the **inner product** of f and g is defined:

$$\langle f, g \rangle \ := \ \frac{1}{M} \int_{\mathbb{X}} f(\mathbf{x}) \cdot g(\mathbf{x}) \, d\mathbf{x}. \tag{6B.1}$$

[2] Or *length*, if $D = 1$, or *area* if $D = 2$....

Example 6B.1.

(a) Suppose $\mathbb{X} = [0,3] = \{x \in \mathbb{R}\ ;\ 0 \le x \le 3\}$. Then $M = 3$. If $f(x) = x^2 + 1$ and $g(x) = x$ for all $x \in [0,3]$, then

$$\langle f, g\rangle \quad = \quad \frac{1}{3}\int_0^3 f(x)g(x)\ dx \quad = \quad \frac{1}{3}\int_0^3 (x^3 + x)\ dx \quad = \quad \frac{27}{4} + \frac{3}{2}.$$

(b) The third column of Table 6.1 provides examples of $\langle f, g\rangle$ for various other domains. ♢

The L^2**-norm** of an integrable function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined

$$\|f\|_2 \quad := \quad \langle f, f\rangle^{1/2} \quad = \quad \left(\frac{1}{M}\int_{\mathbb{X}} f^2(\mathbf{x})\ d\mathbf{x}\right)^{1/2}. \tag{6B.2}$$

(See Figure 6A.1. Of course, this integral may not converge.) The set of all integrable functions on $\mathbb{X}$ with finite L^2-norm is denoted $\mathbf{L}^2(\mathbb{X})$, and is called L^2**-space**. For example, any bounded, continuous function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is in $\mathbf{L}^2(\mathbb{X})$.

Example 6B.2. (a) Suppose $\mathbb{X} = [0,3]$, as in Example 6B.1, and let $f(x) = x+1$. Then $f \in \mathbf{L}^2[0,3]$, because

$$\begin{aligned}\|f\|_2^2 \quad &= \quad \langle f, f\rangle \quad = \quad \frac{1}{3}\int_0^3 (x+1)^2\, dx \\ &= \quad \frac{1}{3}\int_0^3 x^2 + 2x + 1\, dx \quad = \quad \frac{1}{3}\left(\frac{x^3}{3} + x^2 + x\right)_{x=0}^{x=3} \quad = \quad 7,\end{aligned}$$

hence $\|f\|_2 = \sqrt{7} < \infty$.

(b) Let $\mathbb{X} = (0,1]$, and suppose $f \in \mathcal{C}^\infty(0,1]$ is defined $f(x) := 1/x$. Then $\|f\|_2 = \infty$, so$f \notin \mathbf{L}^2(0,1]$. ♢

Remark. Some authors define the inner product as $\langle f, g\rangle := \int_{\mathbb{X}} f(\mathbf{x})\cdot g(\mathbf{x})\ d\mathbf{x}$, and define the L^2-norm as $\|f\|_2 := \left(\int_{\mathbb{X}} f^2(\mathbf{x})\ d\mathbf{x}\right)^{1/2}$. In other words, these authors do *not* divide by the volume M of the domain. This yields a mathematically equivalent theory. The advantage of our definition is greater computational convenience in some situations. (For example, if $\mathbb{1}_{\mathbb{X}}$ is the constant 1-valued function, then in our definition, $\|\mathbb{1}_{\mathbb{X}}\|_2 = 1$.) When comparing formulae from different books, you should always check their respective definitions of L^2 norm.

L^2 space on an infinite domain. Suppose $\mathbb{X} \subset \mathbb{R}^D$ is a region of *infinite* volume (or length, area, etc.). For example, maybe $\mathbb{X} = \mathbb{R}_{\not-}$ is the *positive half-line*, or perhaps $\mathbb{X} = \mathbb{R}^D$. In this case, $M = \infty$, so it doesn't make any sense to divide by M. If $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then the **inner product** of f and g is defined:

$$\langle f, g \rangle \quad := \quad \int_{\mathbb{X}} f(\mathbf{x}) \cdot g(\mathbf{x}) \; d\mathbf{x} \tag{6B.3}$$

Example 6B.3. Suppose $\mathbb{X} = \mathbb{R}$. If $f(x) = e^{-|x|}$ and $g(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 7 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$, then

$$\langle f, g \rangle \quad = \quad \int_{-\infty}^{\infty} f(x) g(x) \; dx \quad = \quad \int_0^7 e^{-x} \; dx \quad = \quad -(e^{-7} - e^0) \quad = \quad 1 - \frac{1}{e^7}.$$ ◊

The **L^2-norm** of an integrable function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined

$$\|f\|_2 \quad = \quad \langle f, f \rangle^{1/2} \quad = \quad \left(\int_{\mathbb{X}} f^2(\mathbf{x}) \; d\mathbf{x} \right)^{1/2}. \tag{6B.4}$$

Again, this integral may not converge. Indeed, even if f is bounded and continuous everywhere, this integral may still equal infinity. The set of all integrable functions on $\mathbb{X}$ with finite L^2-norm is denoted $\mathbf{L}^2(\mathbb{X})$, and called **L^2-space**. (You may recall that on page 40 of §3A, we discussed how L^2-space arises naturally in quantum mechanics as the space of 'physically meaningful' wavefunctions.)

Proposition 6B.4. Properties of the inner product

Whether it is defined using equation (6B.1) *or* (6B.3), *the inner product has the following properties.*

Bilinearity. *For any $f_1, f_2, g_1, g_2 \in \mathbf{L}^2(\mathbb{X})$, and any constants $r_1, r_2, s_1, s_2 \in \mathbb{R}$,*

$$\langle r_1 f_1 + r_2 f_2, \; s_1 g_1 + s_2 g_2 \rangle \; = \; r_1 s_1 \langle f_1, g_1 \rangle + r_1 s_2 \langle f_1, g_2 \rangle + r_2 s_1 \langle f_2, g_1 \rangle + r_2 s_2 \langle f_2, g_2 \rangle \, .$$

Symmetry. *For any $f, g \in \mathbf{L}^2(\mathbb{X})$, $\langle f, g \rangle = \langle g, f \rangle$.*

Positive-definite. *For any $f \in \mathbf{L}^2(\mathbb{X})$, $\langle f, f \rangle \geq 0$. Also, $\langle f, f \rangle = 0$ if and only if $f = 0$.*

Proof. <u>**Exercise 6B.1**</u> □ Ⓔ

If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^D$, recall that $\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos(\theta)$, where θ is the angle between $\mathbf{v}$ to $\mathbf{w}$. In particular, this implies that

$$\Big| \langle \mathbf{v}, \mathbf{w} \rangle \Big| \quad \leq \quad \|\mathbf{v}\| \cdot \|\mathbf{w}\|. \tag{6B.5}$$

If $f, g \in \mathbf{L}^2(\mathbb{X})$ are two functions, then it doesn't make sense to talk about the 'angle' between f and g [as 'vectors' in $\mathbf{L}^2(\mathbb{X})$]. But an inequality analogous to (6B.5) is still true.

Theorem 6B.5. (Cauchy-Bunyakowski-Schwarz Inequality)
Let $f, g \in \mathbf{L}^2(\mathbb{X})$. *Then* $\Big| \langle f, g \rangle \Big| \leq \|f\|_2 \cdot \|g\|_2$.

Proof. Let $A = \|g\|_2^2$, $B := \langle f, g \rangle$, and $C := \|f\|_2^2$; thus, we are trying to show that $B \leq \sqrt{A} \cdot \sqrt{C}$. Define $q : \mathbb{R} \longrightarrow \mathbb{R}$ by $q(t) \;:=\; \|f - t \cdot g\|_2^2$. Then

$$\begin{aligned} q(t) &= \langle f - t \cdot g, f - t \cdot g \rangle \underset{(\flat)}{=} \langle f, f \rangle - t \langle f, g \rangle - t \langle g, f \rangle + t^2 \langle g, g \rangle \\ &= \|f\|_2^2 + 2 \langle f, g \rangle t + \|g\|_2^2 t^2 \quad = \quad C + 2Bt + At^2, \end{aligned} \tag{6B.6}$$

a quadratic polynomial in t. (Here, step $(\flat)$ is by Proposition 6B.4(a)).

Now, $q(t) = \|f - t \cdot g\|_2^2 \geq 0$ for all $t \in \mathbb{R}$; thus, $q(t)$ has at most one root, so the discriminant of the quadratic polynomial (6B.6) is not positive. That is $4B^2 - 4AC \leq 0$. Thus, $B^2 \leq AC$, and thus, $B \leq \sqrt{A} \cdot \sqrt{C}$, as desired. □

Note. The CBS inequality involves three integrals: $\langle f, g \rangle$, $\|f\|_2$, and $\|g\|_2$. But the proof of Theorem 6B.5 *does not involve any integrals at all.* Instead, it just uses simple algebraic manipulations of the inner product operator. In particular, this means the same proof works whether we define the inner product using (6B.1) or using (6B.3). Indeed, the CBS inequality is not really about L^2 spaces, *per se* —it is actually a theorem about a much broader class of abstract geometric structures, called *inner product spaces.* An enormous amount of knowledge about $\mathbf{L}^2(\mathbb{X})$ can be obtained from this abstract geometric approach, usually through simple algebraic arguments like the proof of Theorem 6B.5 (i.e. without lots of messy integration technicalities). This is the beginning of a beautiful area of mathematics called *Hilbert space theory* (see [Con90] for an excellent introduction).

6C* More about L^2 space

Prerequisites: §6B, §0C.

This section contains some material which is not directly germane to the solution methods we present later in the book, but may be interesting to some students who want a broader perspective.

6C(i) Complex L^2 space

§6B introduced the inner product for real-valued functions. The inner product for complex-valued functions is slightly different. For any $z = x + y\mathbf{i} \in \mathbb{C}$, let $\overline{z} := x - y\mathbf{i}$ denote the *complex conjugate* of z. Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let $f, g : \mathbb{X} \longrightarrow \mathbb{C}$ be complex-valued functions. We define

$$\langle f, g \rangle \quad := \quad \int_{\mathbb{X}} f(\mathbf{x}) \cdot \overline{g(\mathbf{x})}\, d\mathbf{x}. \qquad (6C.1)$$

If g is real-valued, then $\overline{g} = g$, and then eqn.(6C.1) is equivalent to eqn.(6B.4).

For any $z \in \mathbb{C}$, recall that $z \cdot \overline{z} = |z|^2$. Thus, if f is a complex-valued function, then $f(x)\overline{f}(x) = |f(x)|^2$. It follows that we can define the L^2**-norm** of an integrable function $f : \mathbb{X} \longrightarrow \mathbb{C}$ just as before:

$$\|f\|_2 \quad = \quad \langle f, f \rangle^{1/2} \quad = \quad \left(\int_{\mathbb{X}} |f|^2(\mathbf{x})\, d\mathbf{x} \right)^{1/2},$$

and this quantity will always be a real number (when the integral converges). We define $\mathbf{L}^2(\mathbb{X}; \mathbb{C})$ to be the set of all integrable functions $f : \mathbb{X} \longrightarrow \mathbb{C}$ such that $\|f\|_2 < \infty$.[3]

Proposition 6C.1. Properties of the complex inner product
The inner product on $\mathbf{L}^2(\mathbb{X}; \mathbb{C})$ *has the following properties.*

Sesquilinearity. *For any* $f_1, f_2, g_1, g_2 \in \mathbf{L}^2(\mathbb{X}; \mathbb{C})$, *and any constants* $b_1, b_2, c_1, c_2 \in \mathbb{C}$,

$$\langle b_1 f_1 + b_2 f_2,\ c_1 g_1 + c_2 g_2 \rangle \ = \ b_1 \overline{c}_1 \langle f_1, g_1 \rangle + b_1 \overline{c}_2 \langle f_1, g_2 \rangle + b_2 \overline{c}_1 \langle f_2, g_1 \rangle + b_2 \overline{c}_2 \langle f_2, g_2 \rangle .$$

Hermitian. *For any* $f, g \in \mathbf{L}^2(\mathbb{X}; \mathbb{C})$, $\langle f, g \rangle = \overline{\langle g, f \rangle}$.

Positive-definite. *For any* $f \in \mathbf{L}^2(\mathbb{X}; \mathbb{C})$, $\langle f, f \rangle$ *is a real number and* $\langle f, f \rangle \geq 0$. *Also,* $\langle f, f \rangle = 0$ *if and only if* $f = 0$.

CBS Inequality. *For any* $f, g \in \mathbf{L}^2(\mathbb{X}; \mathbb{C})$, $\left| \langle f, g \rangle \right| \leq \|f\|_2 \cdot \|g\|_2$.

Proof. **Exercise 6C.1** Hint: Imitate the proofs of Proposition 6B.4 and Theorem 6B.5. In your proof of the CBS inequality, don't forget that $\langle f, g \rangle + \overline{\langle f, g \rangle} = 2\mathrm{Re}\left[\langle f, g \rangle\right]$. Ⓔ
□

[3] We are using $\mathbf{L}^2(\mathbb{X})$ to refer to only real-valued functions. In more advanced books, the notation $\mathbf{L}^2(\mathbb{X})$ denotes the set of *complex*-valued L^2 functions; if one wants to refer only to *real*-valued L^2 functions, one must use the notation $\mathbf{L}^2(\mathbb{X}; \mathbb{R})$.

6C(ii) Riemann vs. Lebesgue integrals

We have defined $\mathbf{L}^2(\mathbb{X})$ to be the set of all 'integrable' functions on $\mathbb{X}$ with finite L^2-norm, but we have been somewhat vague about what we mean by 'integrable'. The most familiar and elementary integral is the *Riemann integral*. For example, if $\mathbb{X} = [a, b]$, and $f : \mathbb{X} \longrightarrow \mathbb{R}$, then the Riemann integral of f is defined

$$\int_a^b f(x)\, dx \quad := \quad \lim_{N\to\infty} \frac{b-a}{N} \sum_{n=1}^{N} f\left(a + \frac{n(b-a)}{N}\right) \tag{6C.2}$$

A similar (but more complicated) definition can be given if $\mathbb{X}$ is an arbitrary domain in $\mathbb{R}^D$. We say f is *Riemann integrable* if the limit (6C.2) exists and is finite.

However, this is not what we mean here by 'integrable'. The problem is that the limit (6C.2) only exists if the function f is reasonably 'nice' (e.g. piecewise continuous). We need an integral which works even for extremely 'nasty' functions (e.g. functions which are discontinuous everywhere; functions which have a 'fractal' structure, etc.). This object is called the *Lebesgue integral*; its definition is similar to (6C.2) but much more complicated.

Loosely speaking, the 'Riemann sum' in (6C.2) chops the interval $[a, b]$ up into N equal subintervals. The corresponding sum in the Lebesgue integral allows us to chop $[a, b]$ into any number of 'Borel-measurable subsets'. A 'Borel-measurable subset' is any open set, any closed set, any (countably infinite) union or intersection of open or closed sets, any (countably infinite) union or intersection of *these* sets, etc. Clearly 'measurable subsets' can become very complex. The Lebesgue integral is obtained by taking a limit over all possible 'Riemann sums' obtained using such 'measurable partitions' of $[a, b]$. This is a very versatile and powerful construction, which can integrate incredibly bizarre and pathological functions. (See Remark 10D.3 on page 211 for further discussion of Riemann vs. Lebesgue integration).

You might ask, 'Why would I want to integrate bizarre and pathological functions?' Indeed, the sorts of functions which arise in applied mathematics are almost always piecewise continuous, and for them, the Riemann integral works just fine. To answer this, consider the difference between the following two equations:

$$\text{(a)}\ \ x^2 \ = \ \frac{16}{9}; \qquad\qquad \text{(b)}\ \ x^2 \ = \ 2.$$

Both equations have solutions, but they are different. The solutions to (a) are rational numbers, for which we have an *exact* expression $x = \pm 4/3$. The solutions to (b) are irrational numbers, for which we have only approximate expressions: $x = \pm\sqrt{2} \approx \pm 1.414213562....$

Irrational numbers are 'pathological': they do not admit nice, simple, exact expressions like $4/3$. We might be inclined to ignore such pathological objects in our mathematics —to pretend they don't exist. Indeed, this was precisely the attitude of the ancient Greeks, whose mathematics was based entirely on rational numbers. The problem is: in this 'ancient Greek' mathematical universe, equation (b) *has no solution.* This is not only inconvenient, it is profoundly counterintuitive; after all, $\sqrt{2}$ is simply the length of the hypotenuse of a right angle triangle whose other sides both have length 1. And surely the *sidelength* of a triangle should be a number.

Furthermore we can find rational numbers which seem to be arbitrarily good *approximations* to a solution of equation (b). For example,

$$\begin{aligned} \left(\frac{1,414}{100}\right)^2 &= 1.999396; \\ \left(\frac{1,414,213}{100,000}\right)^2 &= 1.999998409; \\ \left(\frac{141,421,356}{10,000,000}\right)^2 &= 1.999999993; \\ \vdots \quad &\vdots \quad \vdots \end{aligned}$$

It certainly seems like this sequence of rational numbers is converging to 'something'. Our name for that 'something' is $\sqrt{2}$. In fact, this is the only way we can *ever* specify $\sqrt{2}$. Since we cannot express $\sqrt{2}$ as a fraction or some simple decimal expansion, we can only say, '$\sqrt{2}$ is the number to which the above sequence of rational numbers seems to be converging.'

But how do we know that any such number exists? Couldn't there just be a 'hole' in the real number line where we think $\sqrt{2}$ is supposed to be? The answer is that the set $\mathbb{R}$ is *complete* —that is, any sequence in $\mathbb{R}$ which 'looks like it is converging'[4] does, in fact, converge to some limit point in $\mathbb{R}$. Because $\mathbb{R}$ is complete, we are confident that $\sqrt{2}$ exists, even though we can never precisely specify its value.

Now let's return to $\mathbf{L}^2(\mathbb{X})$. Like the real line $\mathbb{R}$, the space $\mathbf{L}^2(\mathbb{X})$ has a *geometry*: a notion of 'distance' defined by the L^2-norm $\|\bullet\|_2$. This geometry provides us with a notion of *convergence* in $\mathbf{L}^2(\mathbb{X})$ (see §6E(i) on page 117). Like $\mathbb{R}$, we would like $\mathbf{L}^2(\mathbb{X})$ to be *complete*, so that any sequence of functions which 'looks like it is converging' does, in fact, converge to some limit point in $\mathbf{L}^2(\mathbb{X})$.

Unfortunately, a sequence of perfectly 'nice' functions in $\mathbf{L}^2(\mathbb{X})$ can converge to a totally 'pathological' limit function, the same way that a sequence of 'nice' rational numbers can converge to an irrational number. If we exclude the pathological functions from $\mathbf{L}^2(\mathbb{X})$, we will be like the ancient Greeks, who excluded irrational numbers from their mathematics. We will encounter situations where a certain equation 'should' have a solution, but *doesn't*, just as the Greeks discovered that the equation $x^2 = 2$ had no solution in their mathematics.

[4] Technically, any *Cauchy sequence*.

Thus our definition of $\mathbf{L}^2(\mathbb{X})$ *must* include some pathological functions. But if these pathological functions are in $\mathbf{L}^2(\mathbb{X})$, and $\mathbf{L}^2(\mathbb{X})$ is defined as the set of elements with finite norm, and the norm $\|f\|_2$ is defined using an integral like (6B.2), then we must have a way of integrating these pathological functions. Hence the necessity of the Lebesgue integral.

Fortunately, all the functions we will encounter in this book are Riemann integrable. For the purposes of solving PDEs, you do not need to know how to compute the Lebesgue integral. But it is important to know that it exists, and that somewhere in the background, its presence is making all the mathematics work properly.

6D Orthogonality

Prerequisites: §6A.

Two functions $f, g \in \mathbf{L}^2(\mathbb{X})$ are **orthogonal** if $\langle f, g\rangle = 0$. Intuitively, this means that f and g are 'perpendicular' vectors in the infinite-dimensional vector space $\mathbf{L}^2(\mathbb{X})$.

Example 6D.1. Treat sin and cos as elements of $\mathbf{L}^2[-\pi,\pi]$. Then they are orthogonal:

Ⓔ

$$\langle \sin, \cos\rangle \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x)\cos(x)\; dx \;=\; 0. \qquad (\underline{\textbf{Exercise 6D.1}}). \qquad \diamondsuit$$

An **orthogonal set** of functions is a set $\{f_1, f_2, f_3, \ldots\}$ of elements in $\mathbf{L}^2(\mathbb{X})$ such that $\langle f_j, f_k\rangle = 0$ whenever $j \neq k$. If, in addition, $\|f_j\|_2 = 1$ for all j, then we say this is an **orthonormal set** of functions. Fourier analysis is based on the orthogonality of certain families of trigonometric functions. Example 6D.1 was an example of this, which generalizes as follows....

Proposition 6D.2. Trigonometric Orthogonality on $[-\pi,\pi]$

For every $n \in \mathbb{N}$, define the functions $\mathbf{S}_n, \mathbf{C}_n : [-\pi,\pi] \longrightarrow \mathbb{R}$ by $\mathbf{S}_n(x) := \sin(nx)$ and $\mathbf{C}_n(x) := \cos(nx)$, for all $x \in [-\pi,\pi]$. (See Figure 6D.1). *Then the set $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots; \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an orthogonal set of functions for $\mathbf{L}^2[-\pi,\pi]$. In other words:*

(a) $\langle \mathbf{S}_n, \mathbf{S}_m\rangle \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\; dx \;=\; 0$, *whenever $n \neq m$.*

(b) $\langle \mathbf{C}_n, \mathbf{C}_m\rangle \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\; dx \;=\; 0$, *whenever $n \neq m$.*

(c) $\langle \mathbf{S}_n, \mathbf{C}_m\rangle \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\; dx = 0$, *for any n and m.*

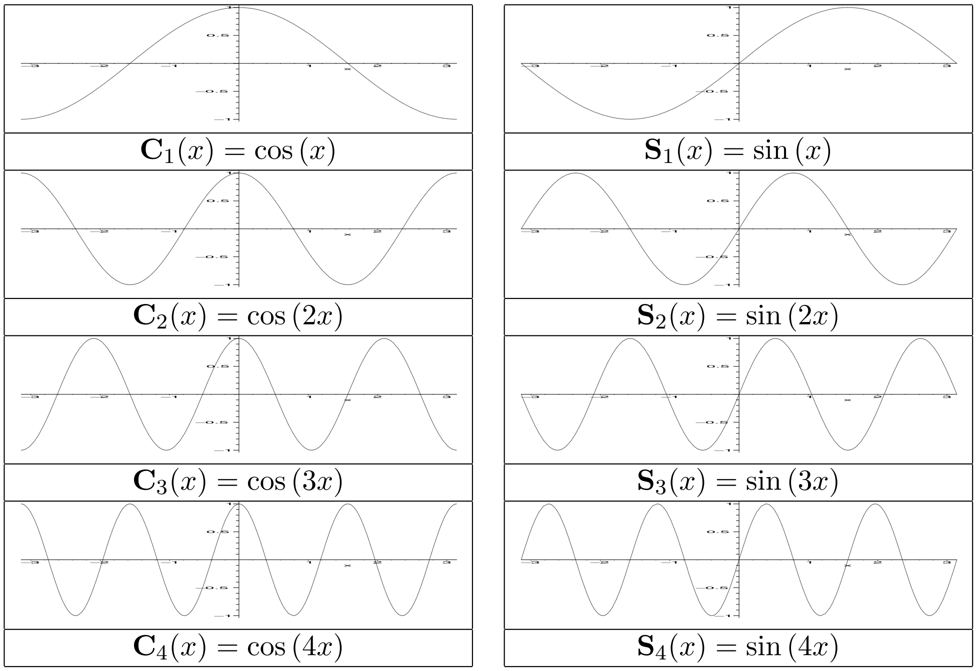


Figure 6D.1: $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, and $\mathbf{C}_4$; $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{S}_4$

(d) *However, these functions are not* orthonormal, *because they do* not *have unit norm. Instead, for any $n \neq 0$,*

$$\|\mathbf{C}_n\|_2 \;=\; \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)^2 \, dx} \;=\; \frac{1}{\sqrt{2}}, \quad \textit{and} \quad \|\mathbf{S}_n\|_2 \;=\; \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)^2 \, dx} \;=\; \frac{1}{\sqrt{2}}.$$

Proof. <u>**Exercise 6D.2**</u> Hint: Use the trigonometric identities: $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta)+\sin(\alpha-\beta)$, $2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)-\cos(\alpha+\beta)$, and $2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta)+\cos(\alpha-\beta)$. Ⓔ □

Remark. Notice that $\mathbf{C}_0(x) = 1$ is just the *constant* function.

It is important to remember that the statement, "f and g are orthogonal" depends upon the *domain* $\mathbb{X}$ which we are considering. For example, compare the following theorem to the preceeding one...

Proposition 6D.3. Trigonometric Orthogonality on $[0, L]$

Let $L > 0$, and, for every $n \in \mathbb{N}$, define the functions $\mathbf{S}_n, \mathbf{C}_n : [0, L] \longrightarrow \mathbb{R}$ by $\mathbf{S}_n(x) := \sin\left(\frac{n\pi x}{L}\right)$ and $\mathbf{C}_n(x) := \cos\left(\frac{n\pi x}{L}\right)$, for all $x \in [0, L]$.

(a) *The set* $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ *is an orthogonal set of functions for* $\mathbf{L}^2[0,L]$. *In other words:* $\langle \mathbf{C}_n, \mathbf{C}_m \rangle = \frac{1}{L}\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = 0$, *whenever* $n \neq m$.

However, these functions are **not** *orthonormal, because they do not have unit norm. Instead, for any* $n \neq 0$, $\|\mathbf{C}_n\|_2 = \sqrt{\frac{1}{L}\int_0^L \cos\left(\frac{n\pi}{L}x\right)^2\,dx} = \frac{1}{\sqrt{2}}$.

(b) *The set* $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ *is an orthogonal set of functions for* $\mathbf{L}^2[0,L]$. *In other words:* $\langle \mathbf{S}_n, \mathbf{S}_m \rangle = \frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx = 0$, *whenever* $n \neq m$.

However, these functions are **not** *orthonormal, because they do* **not** *have unit norm. Instead, for any* $n \neq 0$, $\|\mathbf{S}_n\|_2 = \sqrt{\frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)^2\,dx} = \frac{1}{\sqrt{2}}$.

(c) *The functions* $\mathbf{C}_n$ *and* $\mathbf{S}_m$ *are* **not** *orthogonal to one another on* $[0,L]$. *Instead:*

$$\langle \mathbf{S}_n, \mathbf{C}_m \rangle = \frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \begin{cases} 0 & \text{if } n+m \text{ is even} \\ \dfrac{2n}{\pi(n^2-m^2)} & \text{if } n+m \text{ is odd.} \end{cases}$$

Ⓔ *Proof.* **Exercise 6D.3** . □

Remark. The trigonometric functions are just one of several important orthogonal sets of functions. Different orthogonal sets are useful for different domains or different applications. For example, in some cases, it is convenient to use a collection of orthogonal *polynomial* functions. Several orthogonal polynomial families exist, including the *Legendre Polynomials* (see § 16D on page 359), the *Chebyshev polynomials* (see Exercise 14B.1(e) on page 278 of §14B(i)), the *Hermite polynomials* and the *Laguerre polynomials*. See [Bro89, Chap.3] for a good introduction.

In the study of partial differential equations, the following fact is particularly important:

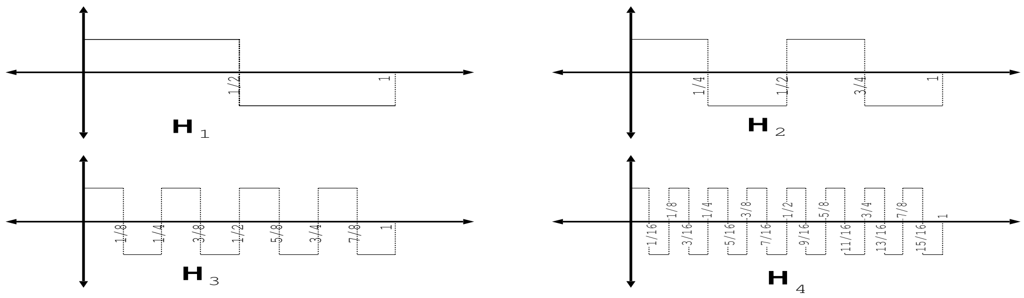


Figure 6D.2: Four Haar basis elements: $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4$

> *Let $\mathbb{X} \subset \mathbb{R}^D$ be any domain. If $f, g : \mathbb{X} \longrightarrow \mathbb{C}$ are two eigenfunctions of the Laplacian with different eigenvalues, then f and g are orthogonal in $\mathbf{L}^2(\mathbb{X})$.*

(See Proposition 15E.9 on page 345 for a precise statement of this.) Because of this, we can get orthogonal sets whose members are eigenfunctions of the Laplacian (see Theorem 15E.12 on page 347). These orthogonal sets are the 'building blocks' with which we can construct solutions to a PDE satisfying prescribed initial conditions or boundary conditions. This is the basic strategy behind the solution methods of Chapters 11-14.

Exercise 6D.4. Figure 6D.2 portrays the **The Haar Basis**. We define $\mathbf{H}_0 \equiv 1$, and for any natural number $N \in \mathbb{N}$, we define the Nth **Haar function** $\mathbf{H}_N : [0,1] \longrightarrow \mathbb{R}$ by: Ⓔ

$$\mathbf{H}_N(x) = \begin{cases} 1 & \text{if } \dfrac{2n}{2^N} \leq x < \dfrac{2n+1}{2^N}, \quad \text{for some } n \in \big[0...2^{N-1}\big); \\[2ex] -1 & \text{if } \dfrac{2n+1}{2^N} \leq x < \dfrac{2n+2}{2^N}, \quad \text{for some } n \in \big[0...2^{N-1}\big). \end{cases}$$

(a) Show that the set $\{\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \ldots\}$ is an orthonormal set in $\mathbf{L}^2[0,1]$.
(b) There is another way to define the Haar Basis. First recall that any number $x \in [0,1]$ has a unique **binary expansion** of the form

$$x = \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} + \frac{x_4}{16} + \cdots + \frac{x_n}{2^n} + \cdots$$

where $x_1, x_2, x_3, x_4, \ldots$ are all either 0 or 1. Show that, for any $n \geq 1$,

$$\mathbf{H}_n(x) = (-1)^{x_n} = \begin{cases} 1 & \text{if} \quad x_n = 0; \\ -1 & \text{if} \quad x_n = 1. \end{cases}$$

♦

Exercise 6D.5 Figure 6D.3 portrays a **Wavelet Basis.** We define $\mathbf{W}_0 \equiv 1$, Ⓔ

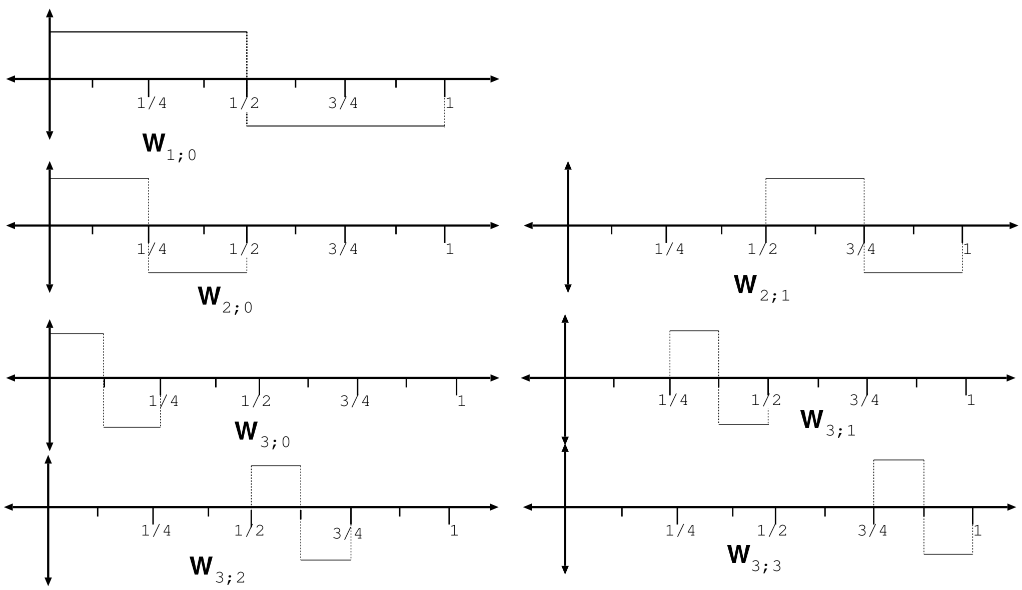


Figure 6D.3: Seven Wavelet basis elements: $\mathbf{W}_{1,0}$; $\mathbf{W}_{2,0}$, $\mathbf{W}_{2,1}$; $\mathbf{W}_{3,0}$, $\mathbf{W}_{3,1}$, $\mathbf{W}_{3,2}$, $\mathbf{W}_{3,3}$

and for any $N \in \mathbb{N}$ and $n \in \left[0...2^{N-1}\right)$, we define

$$\mathbf{W}_{n;N}(x) = \begin{cases} 1 & \text{if } \dfrac{2n}{2^N} \leq x < \dfrac{2n+1}{2^N}; \\[2ex] -1 & \text{if } \dfrac{2n+1}{2^N} \leq x < \dfrac{2n+2}{2^N}; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the the set

$$\{\mathbf{W}_0; \mathbf{W}_{1,0}; \mathbf{W}_{2,0}, \mathbf{W}_{2,1}; \mathbf{W}_{3,0}, \mathbf{W}_{3,1}, \mathbf{W}_{3,2}, \mathbf{W}_{3,3}; \mathbf{W}_{4,0}, \ldots, \mathbf{W}_{4,7}; \mathbf{W}_{5,0}, \ldots, \mathbf{W}_{5,15}; \ldots\}$$

is an *orthogonal* set in $\mathbf{L}^2[0,1]$, but is *not* orthonormal: for any N and n, we have $\|\mathbf{W}_{n;N}\|_2 = \dfrac{1}{2^{(N-1)/2}}$.

6E Convergence concepts

Prerequisites: §4A.

If $\{x_1, x_2, x_3, \ldots\}$ is a sequence of numbers, we know what it means to say "$\lim_{n\to\infty} x_n = x$". We can think of convergence as a kind of "approximation".

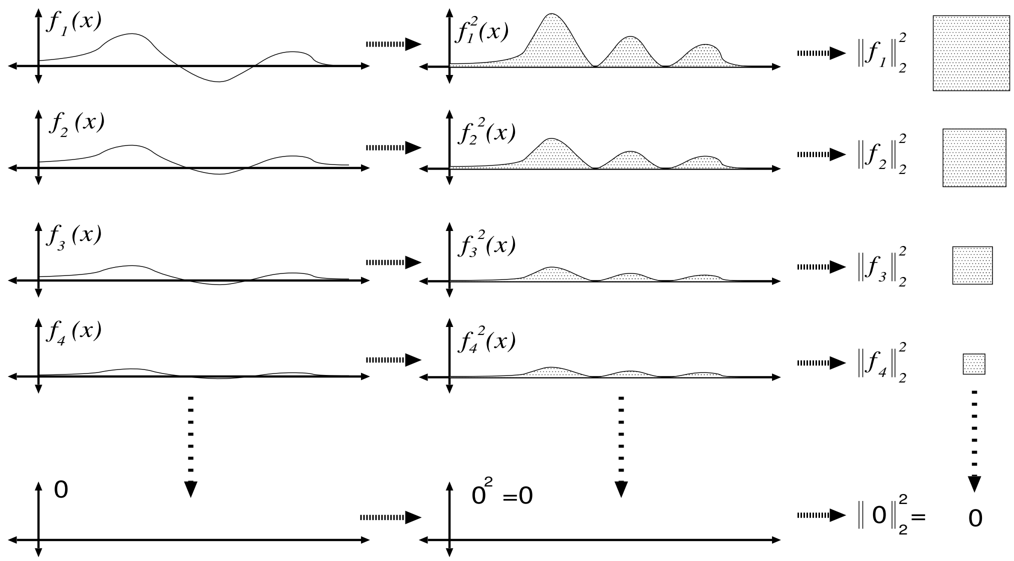


Figure 6E.1: The sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2(\mathbb{X})$.

Heuristically speaking, if the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, then, for very large n, the number x_n is a *good approximation* of x.

If $\{f_1, f_2, f_3, \ldots\}$ was a sequence of functions, and f was some other function, then we might want to say that "$\lim_{n\to\infty} f_n = f$". We again imagine convergence as a kind of "approximation". Heuristically speaking, if the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f, then, for very large n, the function f_n is a *good approximation* of f.

However, there are several ways we can interpret "good approximation", and these in turn lead to several different notions of "convergence". Thus, convergence of *functions* is a much more subtle concept that convergence of *numbers*. We will deal with *three* kinds of convergence here: L^2-convergence, **pointwise** convergence, and **uniform** convergence.

6E(i) L^2 convergence

Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and define

$$M \quad := \quad \begin{cases} \int_{\mathbb{X}} 1 \, d\mathbf{x} & \text{if } \mathbb{X} \text{ is a finite domain;} \\ 1 & \text{if } \mathbb{X} \text{ is an infinite domain.} \end{cases}$$

If $f, g \in \mathbf{L}^2(\mathbb{X})$, then the **$L^2$-distance** between f and g is just

$$\|f - g\|_2 \quad := \quad \left(\frac{1}{M} \int_{\mathbb{X}} |f(\mathbf{x}) - g(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2},$$

If we think of f as an "approximation" of g, then $\|f-g\|_2$ measures the *root-mean-squared error* of this approximation.

Lemma 6E.1. $\|\bullet\|_2$ *is a norm. That is:*

(a) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *and* $r \in \mathbb{R}$, $\|r \cdot f\|_2 = |r| \cdot \|f\|_2$.

(b) (Triangle Inequality) *For any* $f, g : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

(c) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f\|_2 = 0$ *if and only if* $f \equiv 0$.

Ⓔ *Proof.* **Exercise 6E.1** □

If $\{f_1, f_2, f_3, \ldots\}$ is a sequence of successive approximations of f, then we say the sequence **converges to** f **in** L^2 if $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ (sometimes this is called *convergence in mean square*). See Figure 6E.1. We then write $f = \mathbf{L}^2\text{–}\lim_{n\to\infty} f_n$.

Example 6E.2. In each of the following examples, let $\mathbb{X} = [0,1]$.

(a) Suppose $f_n(x) = \begin{cases} 1 & \text{if } 1/n < x < 2/n \\ 0 & \text{otherwise} \end{cases}$ (Figure 6E.2A). Then $\|f_n\|_2 = \frac{1}{\sqrt{n}}$ (**Exercise 6E.2**). Hence, $\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so the sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2[0,1]$. Ⓔ

(b) For all $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} n & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$ (Figure 6E.2B). Then $\|f_n\|_2 = \sqrt{n}$ (**Exercise 6E.3**). Hence, $\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \sqrt{n} = \infty$, so the sequence $\{f_1, f_2, f_3, \ldots\}$ does *not* converge to zero in $\mathbf{L}^2[0,1]$. Ⓔ

(c) For each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } \left|\frac{1}{2} - x\right| < \frac{1}{n}; \\ 0 & \text{otherwise} \end{cases}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges to 0 in L^2. (**Exercise 6E.4**) Ⓔ

(d) For all $n \in \mathbb{N}$, let $f_n(x) = \dfrac{1}{1 + n \cdot \left|x - \frac{1}{2}\right|}$. Figure 6E.3 portrays elements f_1, f_{10}, f_{100}, and f_{1000}; these picture strongly suggest that the sequence is converging to the constant 0 function in $\mathbf{L}^2[0,1]$. The proof of this is **Exercise 6E.5** . Ⓔ

(e) Recall the **Wavelet** functions from Example 6D.4(b). For any $N \in \mathbb{N}$ and $n \in \left[0..2^{N-1}\right)$, we had $\|\mathbf{W}_{N,n}\|_2 = \frac{1}{2^{(N-1)/2}}$. Thus, the sequence of wavelet basis elements converges to the constant 0 function in $\mathbf{L}^2[0,1]$. ◇

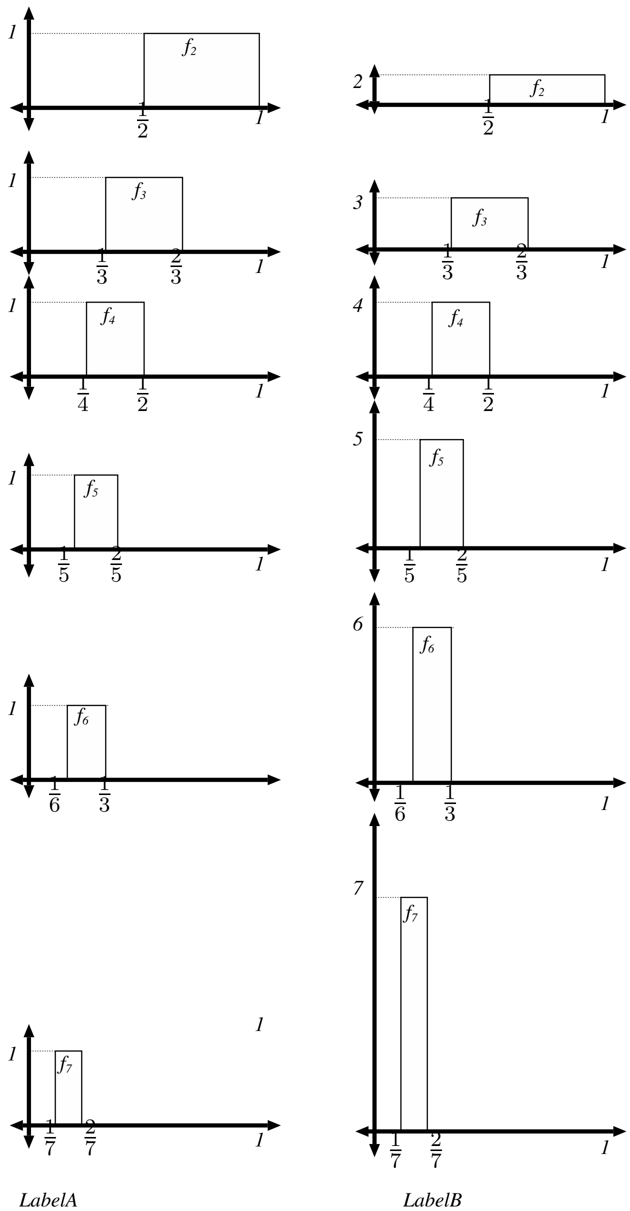
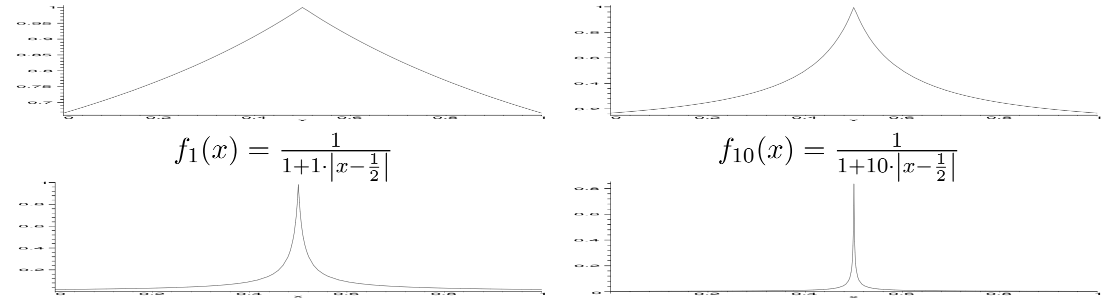


Figure 6E.2: **(A)** Examples 6E.2(a), 6E.5(a), and 6E.9(a); **(B)** Examples 6E.2(b) and 6E.5(b).



$f_1(x) = \frac{1}{1+1\cdot\left|x-\frac{1}{2}\right|}$ $f_{10}(x) = \frac{1}{1+10\cdot\left|x-\frac{1}{2}\right|}$

$f_{100}(x) = \frac{1}{1+100\cdot\left|x-\frac{1}{2}\right|}$ $f_{1000}(x) = \frac{1}{1+1000\cdot\left|x-\frac{1}{2}\right|}$

Figure 6E.3: Examples 6E.2(c) and 6E.5(c): If $f_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2}\right|}$, then the sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2[0,1]$.

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ in } L^2 \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^2 \right)$$

Hence, to understand L^2-convergence in general, it is sufficient to understand L^2-convergence to the constant 0 function.

Lemma 6E.3. *The inner product function $\langle \bullet, \bullet \rangle$ is continuous with respect to L^2 convergence. That is: if $\{f_1, f_2, f_3, \ldots\}$ and $\{g_1, g_2, g_3, \ldots\}$ are two sequences of functions in $\mathbf{L}^2(\mathbb{X})$, and $\mathbf{L}^2\text{-}\lim_{n\to\infty} f_n = f$ and $\mathbf{L}^2\text{-}\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \langle f_n, g_n \rangle = \langle f, g \rangle$.*

E *Proof.* **Exercise 6E.6** □

6E(ii) Pointwise convergence

Convergence in L^2 only means that the *average* approximation error gets small. It does *not* mean that $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{X}$. If *this* equation is true, then we say that the sequence $\{f_1, f_2, \ldots\}$ converges **pointwise** to f (see Figure 6E.4). We then write $f \equiv \lim_{n\to\infty} f_n$. Pointwise convergence is generally considered stronger than L^2 convergence because of the following result:

Theorem 6E.4. *Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, and let $\{f_1, f_2, \ldots\}$ be a sequence of functions in $\mathbf{L}^2(\mathbb{X})$. Suppose:*

(a) *All the functions are uniformly bounded —that is, there is some $M > 0$ such that $|f_n(x)| < M$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{X}$.*

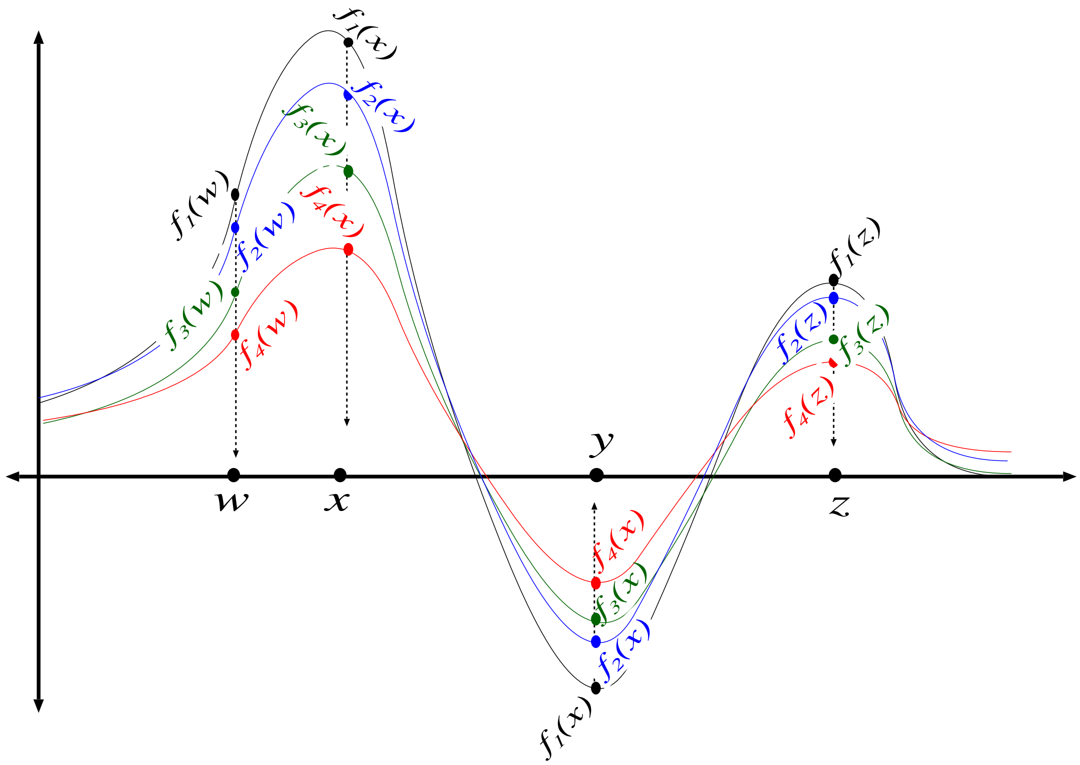


Figure 6E.4: The sequence $\{f_1, f_2, f_3, \ldots\}$ converges **pointwise** to the constant 0 function. Thus, if we pick some random points $w, x, y, z \in \mathbb{X}$, then we see that $\lim_{n\to\infty} f_n(w) = 0$, $\lim_{n\to\infty} f_n(x) = 0$, $\lim_{n\to\infty} f_n(y) = 0$, and $\lim_{n\to\infty} f_n(z) = 0$.

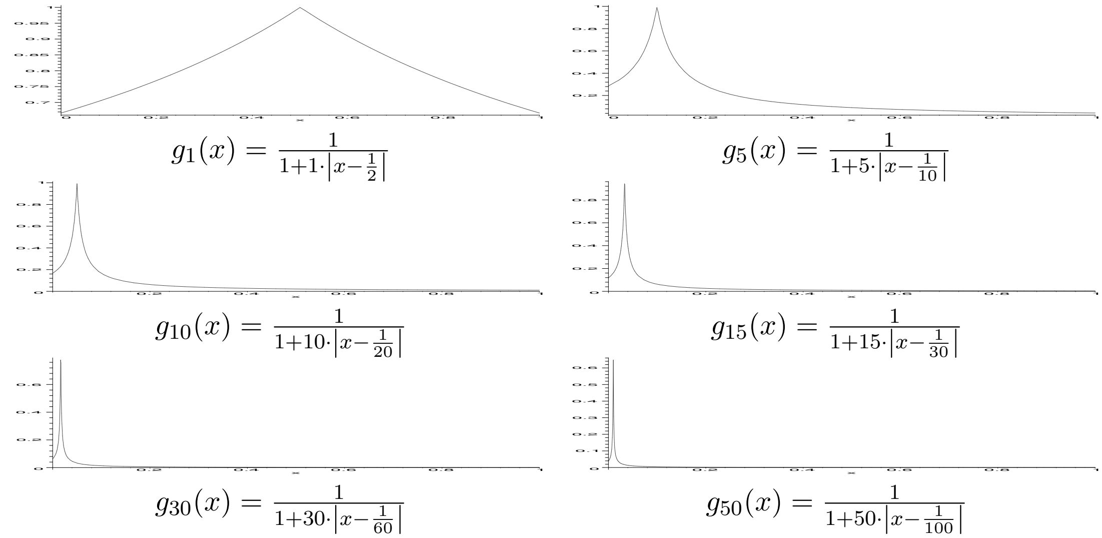
(b) *The sequence* $\{f_1, f_2, \ldots\}$ *converges pointwise to some function* $f \in \mathbf{L}^2(\mathbb{X})$.

Then the sequence $\{f_1, f_2, \ldots\}$ *also converges to* f *in* L^2. □

Proof. **Exercise 6E.7** Hint: You may use the following special case of Lebesgue's Dominated Convergence Theorem:[5] Ⓔ

> *Let* $\{g_1, g_2, \ldots\}$ *be a sequence of integrable functions on the domain* $\mathbb{X}$. *Let* $g : \mathbb{X} \longrightarrow \mathbb{R}$ *be another such function. Suppose that*
> (a) *There is some some* $L > 0$ *such that* $|g_n(x)| < L$ *for all* $n \in \mathbb{N}$ *and all* $x \in \mathbb{X}$.
> (b) *For all* $x \in \mathbb{X}$, $\lim_{n\to\infty} g_n(x) = g(x)$.
> *Then* $\lim_{n\to\infty} \int_{\mathbb{X}} g_n(x)\, dx = \int_{\mathbb{X}} g(x)\, dx$.

[5]See [Fol84, Thm.2.24, p.53] or [KF75, §30.1, p.303].



$g_1(x) = \frac{1}{1+1\cdot\left|x-\frac{1}{2}\right|}$ $\quad g_5(x) = \frac{1}{1+5\cdot\left|x-\frac{1}{10}\right|}$

$g_{10}(x) = \frac{1}{1+10\cdot\left|x-\frac{1}{20}\right|}$ $\quad g_{15}(x) = \frac{1}{1+15\cdot\left|x-\frac{1}{30}\right|}$

$g_{30}(x) = \frac{1}{1+30\cdot\left|x-\frac{1}{60}\right|}$ $\quad g_{50}(x) = \frac{1}{1+50\cdot\left|x-\frac{1}{100}\right|}$

Figure 6E.5: Examples 6E.5(d) and 6E.9(d): If $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$, then the sequence $\{g_1, g_2, g_3, \ldots\}$ converges pointwise to the constant 0 function on $[0,1]$.

Let $g_n := |f - f_n|^2$ for all $n \in \mathbb{N}$, and let $g = 0$. Apply the Dominated Convergence Theorem. □

Example 6E.5. In each of the following examples, let $\mathbb{X} = [0,1]$.

(a) As in Example 6E.2(a), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$. (Fig.6E.2A). The sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the constant 0 function on $[0,1]$. Also, as predicted by Theorem 6E.4, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to the constant 0 function in L^2 (see Example 6E.2(a)).

(b) As in Example 6E.2(b), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} n & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$ (Fig.6E.2B). Then this sequence converges *pointwise* to the constant 0 function, but does *not* converge to zero in $\mathbf{L}^2[0,1]$. This illustrates the importance of the *boundedness* hypothesis in Theorem 6E.4.

(c) As in Example 6E.2(c), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } \left|\frac{1}{2} - x\right| < \frac{1}{n}; \\ 0 & \text{otherwise} \end{cases}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ does *not* converges to 0 in pointwise, although it *does* converge in L^2.

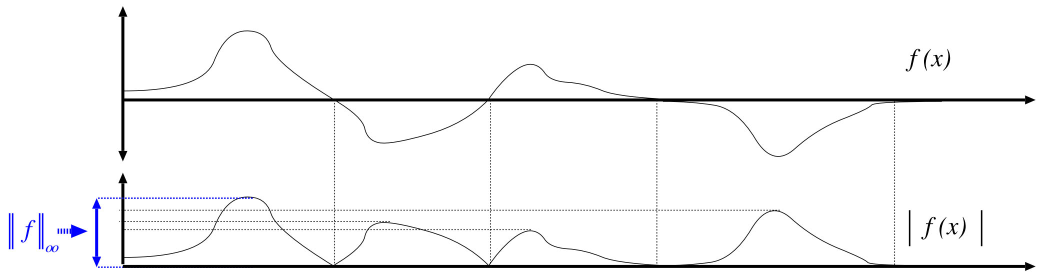


Figure 6E.6: The uniform norm of f is defined: $\|f\|_\infty := \sup_{x\in\mathbb{X}} |f(x)|$.

(d) Recall the functions $f_n(x) = \dfrac{1}{1 + n\cdot\left|x - \frac{1}{2}\right|}$ from Example 6E.2(d). This sequence of functions converges to zero in $\mathbf{L}^2[0,1]$, however, it does *not* converge to zero pointwise (**Exercise 6E.8**). Ⓔ

(e) For all $n \in \mathbb{N}$, let $g_n(x) = \dfrac{1}{1 + n\cdot\left|x - \frac{1}{2n}\right|}$. Figure 6E.5 on the facing page portrays elements g_1, g_5, g_{10}, g_{15}, g_{30}, and g_{50}; These picture strongly suggest that the sequence is converging pointwise to the constant 0 function on $[0,1]$. The proof of this is **Exercise 6E.9** . Ⓔ

(f) Recall from Example 6E.2(e) that the sequence of Wavelet basis elements $\{\mathbf{W}_{N;n}\}$ converges to zero in $\mathbf{L}^2[0,1]$. Note, however, that it does *not* converge to zero pointwise (**Exercise 6E.10**). ◇ Ⓔ

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ pointwise} \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ pointwise} \right)$$

Hence, to understand pointwise convergence in general, it is sufficient to understand pointwise convergence to the constant 0 function.

6E(iii) Uniform convergence

There is an even stronger form of convergence. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a function, then the **uniform norm** of f is defined:

$$\|f\|_\infty \quad := \quad \sup_{\mathbf{x}\in\mathbb{X}} \left| f(\mathbf{x}) \right|.$$

This measures the farthest deviation of the function f from zero (see Figure 6E.6).

Example 6E.6. Suppose $\mathbb{X} = [0,1]$, and $f(x) = \frac{1}{3}x^3 - \frac{1}{4}x$ (as in Figure 6E.8A). The minimal point of f is $x = \frac{1}{2}$, where $f'(\frac{1}{2}) = 0$ and $f(\frac{1}{2}) = \frac{-1}{12}$. The maximal point of f is $x = 1$, where $f(1) = \frac{1}{12}$. Thus, $|f(x)|$ takes a maximum value of $\frac{1}{12}$ at either point, so that $\|f\|_\infty = \sup_{0\le x\le 1} \left|\frac{1}{3}x^3 - \frac{1}{4}x\right| = \frac{1}{12}$. ♢

Lemma 6E.7. $\|\bullet\|_\infty$ *is a norm. That is:*

(a) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *and* $r \in \mathbb{R}$, $\|r \cdot f\|_\infty = |r| \cdot \|f\|_\infty$.

(b) (Triangle Inequality) *For any* $f, g : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

(c) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f\|_\infty = 0$ *if and only if* $f \equiv 0$.

Ⓔ *Proof.* **Exercise 6E.11** □

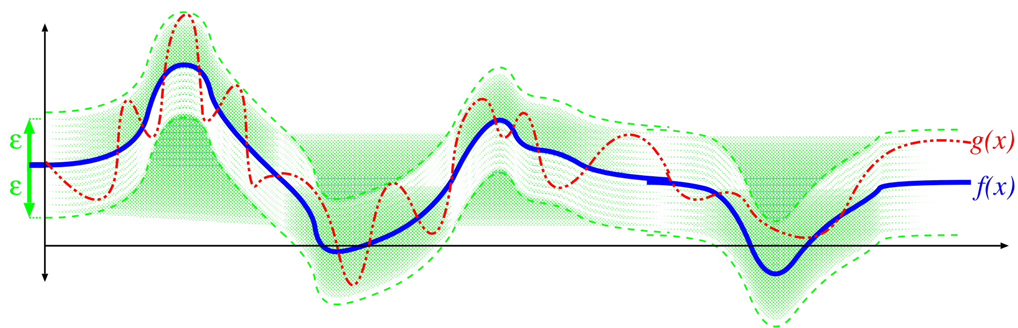


Figure 6E.7: If $\|f - g\|_\infty < \epsilon$, this means that $g(x)$ is confined within an ϵ-tube around f for all x.

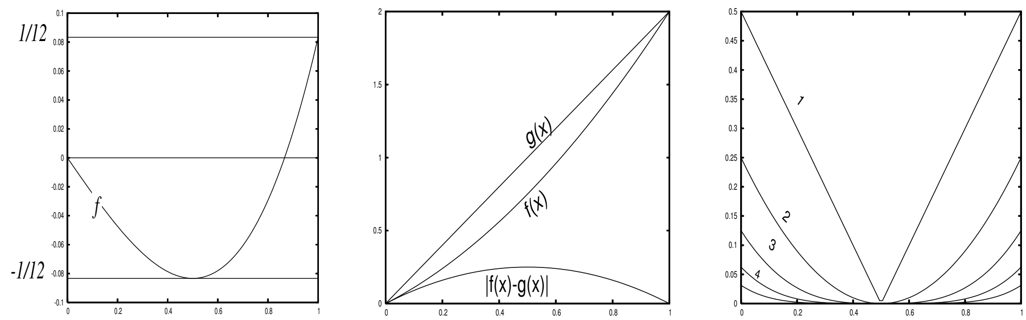
The **uniform distance** between two functions f and g is then given by:

$$\|f - g\|_\infty = \sup_{\mathbf{x}\in\mathbb{X}} \left|f(\mathbf{x}) - g(\mathbf{x})\right|.$$

One way to interpret this is portrayed in Figure 6E.7. Define a "tube" of width ϵ around the function f. If $\|f - g\|_\infty < \epsilon$, this means that $g(x)$ is confined within this tube for all x.

Example 6E.8. Let $\mathbb{X} = [0,1]$, and suppose $f(x) = x(x+1)$ and $g(x) = 2x$ (as in Figure 6E.8B). For any $x \in [0,1]$,

$$|f(x) - g(x)| = \left|x^2 + x - 2x\right| = \left|x^2 - x\right| = x - x^2.$$



(A) **(B)** **(C)**

Figure 6E.8: **(A)** The uniform norm of $f(x) = \frac{1}{3}x^3 - \frac{1}{4}x$ (Example 6E.6). **(B)** The uniform distance between $f(x) = x(x+1)$ and $g(x) = 2x$ (Example 6E.8). **(C)** $g_n(x) = \left|x - \frac{1}{2}\right|^n$, for $n = 1, 2, 3, 4, 5$ (Example (7b))

(because it is nonnegative). This expression takes its maximum at $x = \frac{1}{2}$ (to see this, solve for $f'(x) = 0$), and its value at $x = \frac{1}{2}$ is $\frac{1}{4}$. Thus, $\|f - g\|_\infty = \sup_{x \in \mathbb{X}} \left| x(x-1) \right| = \dfrac{1}{4}$. ♢

Let $\{g_1, g_2, g_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. The sequence $\{g_1, g_2, g_3, \ldots\}$ **converges uniformly** to f if $\lim_{n\to\infty} \|g_n - f\|_\infty = 0$. We then write $f = \text{unif}-\lim_{n\to\infty} g_n$. This means not only that $\lim_{n\to\infty} g_n(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{X}$, but furthermore, that the functions g_n converge to f everywhere at the same "speed". This is portrayed in Figure 6E.9. For any $\epsilon > 0$, we can define a "tube" of width ϵ around f, and, no matter how small we make this tube, the sequence $\{g_1, g_2, g_3, \ldots\}$ will eventually enter this tube and remain there. To be precise: there is some N such that, for all $n > N$, the function g_n is confined within the ϵ-tube around f —i.e. $\|f - g_n\|_\infty < \epsilon$.

Example 6E.9. In each of the following examples, let $\mathbb{X} = [0, 1]$.

(a) Suppose, as in Example 6E.5(a) on page 122, and Figure 6E.2B on page 119, that

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < x < \frac{2}{n}; \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $\{g_1, g_2, \ldots\}$ converges *pointwise* to the constant zero function, but does *not* converge to zero uniformly on $[0, 1]$. (**Exercise 6E.12** Verify these claims.). Ⓔ

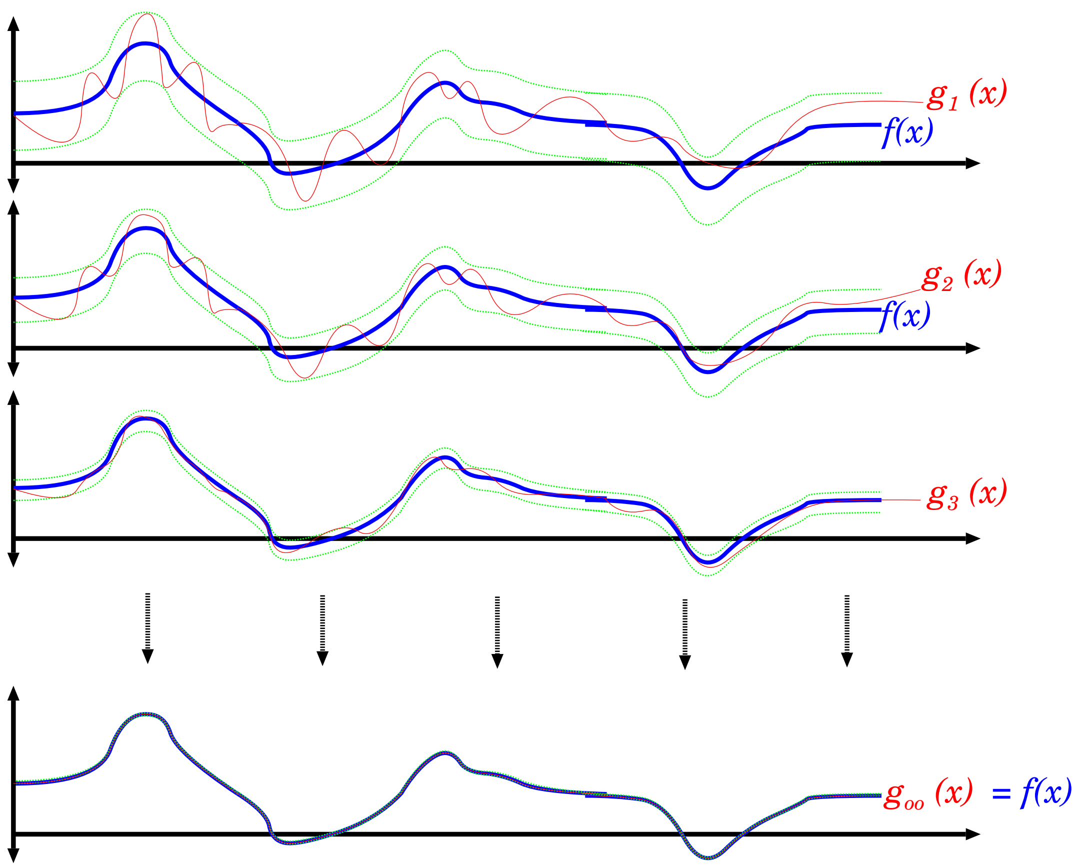


Figure 6E.9: The sequence $\{g_1, g_2, g_3, \ldots\}$ converges **uniformly** to f.

(b) If $g_n(x) = \left|x - \frac{1}{2}\right|^n$ (see Figure 6E.8C), then $\|g_n\|_\infty = \frac{1}{2^n}$ (**Exercise 6E.13**). Thus, the sequence $\{g_1, g_2, \ldots\}$ converges to zero uniformly on $[0, 1]$, because $\lim_{n\to\infty} \|g_n\|_\infty = \lim_{n\to\infty} \dfrac{1}{2^n} = 0$. Ⓔ

(c) If $g_n(x) = 1/n$ for all $x \in [0, 1]$, then the sequence $\{g_1, g_2, \ldots\}$ converges to zero uniformly on $[0, 1]$ (**Exercise 6E.14**). Ⓔ

(d) Recall the functions $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$ from Example 6E.5(e) (Figure 6E.5 on page 122). The sequence $\{g_1, g_2, \ldots\}$ converges *pointwise* to the constant zero function, but does *not* converge to zero uniformly on $[0, 1]$. (**Exercise 6E.15** Verify these claims.). Ⓔ ♢

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ uniformly} \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ uniformly} \right)$$

Hence, to understand uniform convergence in general, it is sufficient to understand uniform convergence to the constant 0 function.

Uniform convergence is the 'best' kind of convergence. It has the most useful consequences, but it is also the most difficult to achieve. (In many cases, we must settle for pointwise or L^2 convergence instead.) For example, the following consequences of uniform convergence are extremely useful.

Proposition 6E.10. *Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain. Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. Suppose $f_n \xrightarrow[n\to\infty]{} f$ uniformly.*

(a) *If $\{f_n\}_{n=1}^{\infty}$ are all continuous on $\mathbb{X}$, then f is also continuous on $\mathbb{X}$.*

(b) *If $\mathbb{X}$ is compact* (that is, closed and bounded), *then* $\lim_{n\to\infty} \int_{\mathbb{X}} f_n(x)\, dx = \int_{\mathbb{X}} f(x)\, dx$.

(c) *Suppose the functions $\{f_n\}_{n=1}^{\infty}$ are all differentiable on $\mathbb{X}$, and suppose $f_n' \xrightarrow[n\to\infty]{} F$ uniformly. Then f is also differentiable, and $f' = F$.*

Proof. (a) **Exercise 6E.16** (Slightly challenging; for students with some analysis background). Ⓔ

For (b,c) see e.g. [Asm05, Theorems 4 and 5, p.91-92 of §2.9]. □

Note that Proposition 6E.10(a,c) are *false* if we replace 'uniformly' with 'pointwise' or 'in L^2.' (Proposition 6E.10(b) is sometimes true under these conditions, but only if we also add additional hypotheses.) Indeed, the next result says that uniform convergence is logically stronger than either pointwise or L^2 convergence.

Corollary 6E.11. *Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function.*

(a) *If $f_n \xrightarrow[n\to\infty]{} f$ uniformly, then $f_n \xrightarrow[n\to\infty]{} f$ pointwise.*

(b) *Suppose $\mathbb{X}$ is compact* (that is, closed and bounded). *If $f_n \xrightarrow[n\to\infty]{} f$ uniformly, then:*

[i] $f_n \xrightarrow[n\to\infty]{} f$ *in* L^2.

[ii] *For any $g \in \mathbf{L}^2(\mathbb{X})$, we have* $\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle$.

Proof. **Exercise 6E.17** (a) is easy. For (b), use Proposition 6E.10(b). □ Ⓔ

Sometimes, uniform convergence is a little too much to ask for, and we must settle for a slightly weaker form of convergence. Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain. Let $\{g_1, g_2, g_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. The sequence $\{g_1, g_2, g_3, \ldots\}$ **converges semiuniformly** to f if:

(a) $\{g_1, g_2, g_3, \ldots\}$ converges *pointwise* to f on $\mathbb{X}$; i.e. $f(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in \mathbb{X}$.

(b) $\{g_1, g_2, g_3, \ldots\}$ converges *uniformly* to f on any *closed subset* of $\mathsf{int}(\mathbb{X})$. In other words, if $\mathbb{Y} \subset \mathsf{int}(\mathbb{X})$ is any closed set, then

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{Y}} |f(y) - g_n(y)| \right) \quad = \quad 0.$$

Heuristically speaking, this means that the sequence $\{g_n\}_{n=1}^{\infty}$ is 'trying' to converge to f uniformly on $\mathbb{X}$, but it is maybe getting 'stuck' at some of the boundary points of $\mathbb{X}$.

Example 6E.12. Let $\mathbb{X} := (0,1)$. Recall the functions $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$ from Figure 6E.5 on page 122. By Example 6E.9(d) on page 126, we know that this sequence *doesn't* converge uniformly to 0 on $(0,1)$. However, it does converge *semi*uniformly to 0. First, we know it converges pointwise on $(0,1)$, by Example 6E.5(e) on page 123. Second, if $0 < a < b < 1$, it is easy to check that $\{g_n\}_{n=1}^{\infty}$ converges to f uniformly on the closed interval $[a,b]$ (**Exercise 6E.18**). It follows that $\{g_n\}_{n=1}^{\infty}$ converges to f uniformly on any closed subset of $(0,1)$. ♢

Ⓔ

Summary. The various forms of convergence are logically related as follows:

$$\Big(\text{Uniform convergence}\Big) \Rightarrow \Big(\text{Semiuniform convergence}\Big) \Rightarrow \Big(\text{Pointwise convergence}\Big).$$

Also, if $\mathbb{X}$ is compact, then

$$\Big(\text{Uniform convergence}\Big) \quad \Longrightarrow \quad \Big(\text{Convergence in } L^2\Big).$$

Finally, if the sequence of functions is uniformly bounded and $\mathbb{X}$ is compact, then

$$\Big(\text{Pointwise convergence}\Big) \quad \Longrightarrow \quad \Big(\text{Convergence in } L^2\Big).$$

However, the opposite implications are *not* true. In general:

$$\Big(\text{Convergence in } L^2\Big) \not\Longrightarrow \Big(\text{Pointwise convergence}\Big) \not\Longrightarrow \Big(\text{Uniform convergence}\Big)$$

6E(iv) Convergence of function series

Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. The **function series** $\sum_{n=1}^{\infty} f_n$ is the formal infinite summation of these functions; we would like to think of this series as defining another function from $\mathbb{X}$ to $\mathbb{R}$. . Intuitively, the symbol "$\sum_{n=1}^{\infty} f_n$" should represent the function which arises as the limit $\lim_{N\to\infty} F_N$, where, for each $N \in \mathbb{N}$, $F_N(x) := \sum_{n=1}^{N} f_n(x) \;=\; f_1(x) + f_2(x) + \cdots + f_N(x)$ is the Nth *partial sum*. To make this precise, we must specify the sense in which the partial sums $\{F_1, F_2, \ldots\}$ converge. If $F : \mathbb{X} \longrightarrow \mathbb{R}$ is this putative limit function, then we say that the series $\sum_{n=1}^{\infty} f_n$....

- ...converges **in** L^2 to F if $F \;=\; \mathbf{L}^2\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} f_n$. We then write $F \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} f_n$.
- ...converges **pointwise** to F if, for each $x \in \mathbb{X}$, $\quad F(x) \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)$.

  We then write $F \;\equiv\; \sum_{n=1}^{\infty} f_n$.
- ...converges **uniformly** to F if $\;F \;=\; \text{unif}-\lim_{N\to\infty} \sum_{n=1}^{N} f_n$. We then write

  $F \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} f_n$.

The next result provides a useful condition for the uniform convergence of an infinite summation of functions; we will use this result often in our study of Fourier series and other eigenfunction expansions in Chapters 7 to 9:

Proposition 6E.13. Weierstrass M-test

Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. For every $n \in \mathbb{N}$, let $M_n := \|f_n\|_\infty$. If $\sum_{n=1}^{\infty} M_n \;<\; \infty$, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{X}$.

Proof. <u>**Exercise 6E.19**</u> (a) Show that the series converges *pointwise* to some limit function $f : \mathbb{X} \longrightarrow \mathbb{R}$. Ⓔ

(b) For any $N \in \mathbb{N}$, show that $\left\| F - \sum_{n=1}^{N} f_n \right\|_\infty \;\leq\; \sum_{n=N+1}^{\infty} M_n$.

(c) Show that $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} M_n = 0$. □

The next three sufficient conditions for convergence are also sometimes useful (but they are not used later in this book).

Proposition 6E.14. Dirichlet Test

(Optional) *Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers. Then the series $\sum_{n=1}^{\infty} c_n f_n$ converges uniformly on $\mathbb{X}$ if:*

- $\lim_{n\to\infty} c_k = 0$; *and*
- *There is some $M > 0$ such that, for all $N \in \mathbb{N}$, we have* $\left\|\sum_{n=1}^{N} f_n\right\|_{\infty} < M$.

Proof. See [Asm05, Appendix to §2.10, p.99] □

Proposition 6E.15. Cauchy's Criterion

(Optional) *Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. For every $N \in \mathbb{N}$, let* $C_N := \sup_{M>N} \left\|\sum_{n=N}^{M} f_n\right\|_{\infty}$.

$$\text{Then} \quad \left(\text{The series } \sum_{n=1}^{\infty} f_n \text{ converges uniformly on } \mathbb{X}\right) \iff \left(\lim_{N\to\infty} C_N = 0\right).$$

Proof. See [CB87, §88]. □

Proposition 6E.16. Abel's Test

(Optional) *Let $\mathbb{X} \subset \mathbb{R}^N$ and $\mathbb{Y} \subset \mathbb{R}^M$ be two domains. Let $\{f_1, f_2, f_3, \ldots\}$ be a sequence of functions from $\mathbb{X}$ to $\mathbb{R}$, such that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{X}$. Let $\{g_1, g_2, g_3, \ldots\}$ be another sequence of functions from $\mathbb{Y}$ to $\mathbb{R}$, and consider the sequence $\{h_1, h_2, \ldots\}$ of functions from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{R}$, defined by $h_n(x, y) := f_n(x) g_n(y)$. Suppose:*

(a) *The sequence $\{g_n\}_{n=1}^{\infty}$ is* uniformly bounded; *i.e. there is some $M > 0$ such that $|g_n(y)| < M$ for all $n \in \mathbb{N}$ and $y \in \mathbb{Y}$.*

(b) *The sequence* $\{g_n\}_{n=1}^{\infty}$ *is monotonic; i.e. either* $g_1(y) \leq g_2(y) \leq g_3(y) \leq \cdots$ *for all* $y \in \mathbb{Y}$*, or* $g_1(y) \geq g_2(y) \geq g_3(y) \geq \cdots$ *for all* $y \in \mathbb{Y}$*.*

Then the series $\sum_{n=1}^{\infty} h_n$ *converges uniformly on* $\mathbb{X} \times \mathbb{Y}$*.*

Proof. See [CB87, §88]. □

6F Orthogonal and orthonormal Bases

Prerequisites: §6A, §6E(i). **Recommended:** §6E(iv).

An **orthogonal set** in $\mathbf{L}^2(\mathbb{X})$ is a (finite or infinite) collection of functions $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ such that $\langle \mathbf{b}_k, \mathbf{b}_j \rangle = 0$ whenever $k \neq j$. Intuitively, the vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ are all 'perpendicular' to one another in the infinite-dimensional geometry of $\mathbf{L}^2(\mathbb{X})$. One consequence is an L^2-version of the **Pythagorean Formula:** For any $N \in \mathbb{N}$ and any real numbers $r_1, r_2, \ldots, r_N \in \mathbb{R}$, we have

$$\|r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_N\mathbf{b}_n\|_2^2 \quad = \quad r_1^2 \|\mathbf{b}_1\|_2^2 + r_2^2 \|\mathbf{b}_N\|_2^2 + \cdots + r_N^2 \|\mathbf{b}_N\|_2^2 . \tag{6F.1}$$

(**Exercise 6F.1** Verify the L^2 Pythagorean formula). Ⓔ

An **orthogonal basis** for $\mathbf{L}^2(\mathbb{X})$ is an infinite collection of functions $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ such that:

- $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ form an orthogonal set (i.e. $\langle \mathbf{b}_k, \mathbf{b}_j \rangle = 0$ whenever $k \neq j$.)
- For any $\mathbf{g} \in \mathbf{L}^2(\mathbb{X})$, if we define $\gamma_n = \dfrac{\langle \mathbf{g}, \mathbf{b}_n \rangle}{\|\mathbf{b}_n\|_2^2}$, for all $n \in \mathbb{N}$, then $\mathbf{g} \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} \gamma_n \mathbf{b}_n$.

Recall that this means that $\lim_{N\to\infty} \left\| \mathbf{g} - \sum_{n=1}^{N} \gamma_n \mathbf{b}_n \right\|_2 = 0$. In other words, we can approximate $\mathbf{g}$ as closely as we want in L^2 norm with a partial sum $\sum_{n=1}^{N} \gamma_n \mathbf{b}_n$, if we make N large enough.

An **orthonormal basis** for $\mathbf{L}^2(\mathbb{X})$ is an infinite collection of functions $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ such that:

- $\|\mathbf{b}_k\|_2 = 1$ for every k.

- $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$. In other words, $\langle \mathbf{b}_k, \mathbf{b}_j \rangle = 0$ whenever $k \neq j$, and, for any $\mathbf{g} \in \mathbf{L}^2(\mathbb{X})$, if we define $\gamma_n = \langle \mathbf{g}, \mathbf{b}_n \rangle$ for all $n \in \mathbb{N}$, then $\mathbf{g} \underset{\text{L2}}{\widetilde{\approx}} \sum_{n=1}^{\infty} \gamma_n \mathbf{b}_n$.

One consequence of this is

Theorem 6F.1. Parseval's Equality

Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots\}$ be an orthonormal basis for $\mathbf{L}^2(\mathbb{X})$, and let $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\mathbb{X})$. Let $\varphi_n := \langle \mathbf{f}, \mathbf{b}_n \rangle$ and $\gamma_n := \langle \mathbf{g}, \mathbf{b}_n \rangle$ for all $n \in \mathbb{N}$. Then

(a) $\langle \mathbf{f}, \mathbf{g} \rangle \;=\; \sum_{n=1}^{\infty} \varphi_n \gamma_n$.

(b) $\|\mathbf{g}\|_2^2 \;=\; \sum_{n=1}^{\infty} |\gamma_n|^2$.

Ⓔ *Proof.* **Exercise 6F.2** Hint: For all $N \in \mathbb{N}$, let $\mathbf{F}_N := \sum_{n=1}^{N} \varphi_n \mathbf{b}_n$ and $\mathbf{G}_N := \sum_{n=1}^{N} \gamma_n \mathbf{b}_n$.

(i) Show that $\langle \mathbf{F}_N, \mathbf{G}_N \rangle = \sum_{n=1}^{N} \varphi_n \gamma_n$ (Hint: the functions $\{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ are orthonormal).

(ii) To prove (a), show that $\langle \mathbf{f}, \mathbf{g} \rangle = \lim_{N\to\infty} \langle \mathbf{F}_N, \mathbf{G}_N \rangle$ (Hint: Use Lemma 6E.3).

(iii) To prove (b), set $\mathbf{f} = \mathbf{g}$ in (a). □

The idea of Fourier analysis is to find an *orthogonal basis* for an L^2-space, using familiar trigonometric functions. We will return to this in Chapter 7.

Further reading:

Most of the mathematically rigorous texts on partial differential equations (such as [CB87], [Asm05] or [Eva91, Appendix D]) contain detailed and thorough discussions of L^2 space, orthogonal basis, and the various convergence concepts discussed in this chapter. This is because almost all solutions to partial differential equations arise through some sort of infinite series or approximating sequence; hence it is essential to properly understand the various forms of function convergence and their relationships.

The convergence of sequences of functions is part of a subject called *real analysis*, and any advanced textbook on real analysis will contain extensive material on convergence. There are many other forms of function convergence we haven't even mentioned in this chapter, including $\mathbf{L}^p$ *convergence* (for any value of p between 1 and ∞), convergence *in measure*, convergence *almost everywhere*, and *weak** convergence. Different convergence modes are useful in different contexts,

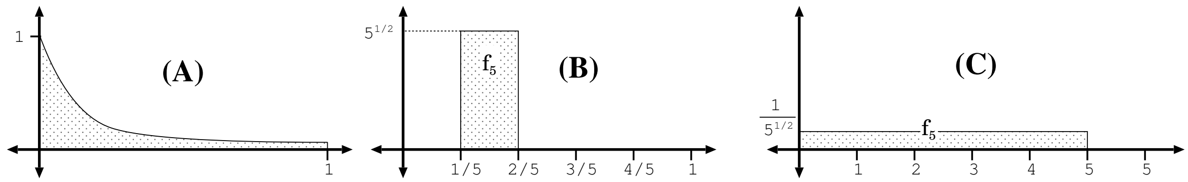


Figure 6G.1: Problems for Chapter 6

and the logical relationships between them are fairly subtle. See [Fol84, §2.4] for a good summary. Other standard references are [WZ77, Chap.8], [KF75, §28.4-§28.5; §37], [Rud87] or [Roy88].

The geometry of infinite-dimensional vector spaces is called *functional analysis*, and is logically distinct from the convergence theory for functions (although of course, most of the important infinite dimensional spaces are spaces of functions). Infinite-dimensional vector spaces fall into several broad classes, depending upon the richness of the geometric and topological structure, which include *Hilbert spaces* [such as $\mathbf{L}^2(\mathbb{X})$], *Banach Spaces* [such as $\mathcal{C}(\mathbb{X})$ or $\mathbf{L}^1(\mathbb{X})$] and *locally convex spaces*. An excellent introduction to functional analysis is [Con90]. Other standard references are [Fol84, Chap.5] and [KF75, Chap.4]. Hilbert spaces are the mathematical foundation of quantum mechanics; see [Pru81, BEH94].

6G Practice problems

1. Let $\mathbb{X} = (0, 1]$. For any $n \in \mathbb{N}$, define the function $f_n : (0, 1] \longrightarrow \mathbb{R}$ by $f_n(x) = \exp(-nx)$. (Fig. 6G.1A)

   (a) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (b) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(0, 1]$? Explain.

   (c) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.

   (d) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $(0, 1]$? Explain.

   (e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $(0, 1]$? Explain.

2. Let $\mathbb{X} = [0, 1]$. For any $n \in \mathbb{N}$, define $f_n : [0, 1] \longrightarrow \mathbb{R}$ by $f_n(x) = \begin{cases} \sqrt{n} & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$. (Fig. 6G.1B)

   (a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $[0, 1]$? Explain.

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

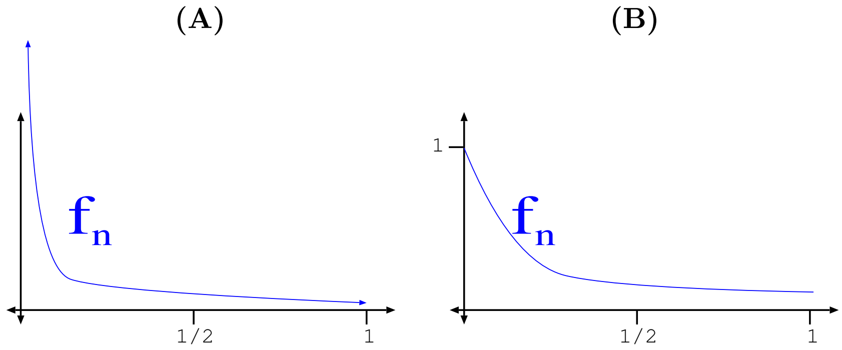


Figure 6G.2: Problems for Chapter 6

(c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2[0,1]$? Explain.

(d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.

(e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $[0,1]$? Explain.

3. Let $\mathbb{X} = \mathbb{R}$. For any $n \in \mathbb{N}$, define $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad 0 \leq x < n \\ 0 & \text{otherwise} \end{cases}$.

   (Fig. 6G.1C)

   (a) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.

   (b) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $\mathbb{R}$? Explain.

   (c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $\mathbb{R}$? Explain.

   (d) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(\mathbb{R})$? Explain.

4. Let $\mathbb{X} = (0,1]$. For all $n \in \mathbb{N}$, define $f_n : (0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \dfrac{1}{\sqrt[3]{nx}}$ (for all $x \in (0,1]$). (Figure 6G.2A)

   (a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function *pointwise* on $(0,1]$? Why or why not?

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(0,1]$? Why or why not?

(d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.

(e) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function uniformly on $(0,1]$? Explain.

5. Let $\mathbb{X} = [0,1]$. For all $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \dfrac{1}{(nx+1)^2}$ (for all $x \in [0,1]$). (Figure 6G.2B)

   (a) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function pointwise on $[0,1]$? Explain.

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (c) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function in $\mathbf{L}^2[0,1]$? Explain.

   (d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.
   Hint: *Look at the picture. Where is the value of $f_n(x)$ largest?*

   (e) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function uniformly on $[0,1]$? Explain.

6. In each of the following cases, you are given two functions $f, g : [0,\pi] \longrightarrow \mathbb{R}$. Compute the inner product $\langle f, g \rangle$.

   (a) $f(x) = \sin(3x), \quad g(x) = \sin(2x)$.

   (b) $f(x) = \sin(nx), \quad g(x) = \sin(mx)$, with $n \neq m$.

   (c) $f(x) = \sin(nx) = g(x)$ for some $n \in \mathbb{N}$. Question: What is $\|f\|_2$?

   (d) $f(x) = \cos(3x), \quad g(x) = \cos(2x)$.

   (e) $f(x) = \cos(nx), \quad g(x) = \cos(mx)$, with $n \neq m$.

   (f) $f(x) = \sin(3x), \quad g(x) = \cos(2x)$.

7. In each of the following cases, you are given two functions $f, g : [-\pi,\pi] \longrightarrow \mathbb{R}$. Compute the inner product $\langle f, g \rangle$.

   (a) $f(x) = \sin(nx), \quad g(x) = \sin(mx)$, with $n \neq m$.

   (b) $f(x) = \sin(nx) = g(x)$ for some $n \in \mathbb{N}$. Question: What is $\|f\|_2$?

   (c) $f(x) = \cos(nx), \quad g(x) = \cos(mx)$, with $n \neq m$.

   (d) $f(x) = \sin(3x), \quad g(x) = \cos(2x)$.

8. Determine if f_n converges to f pointwise, in $L^2(\mathbb{X})$, or uniformly.

   (a) $f_n(x) = e^{-nx^2}, \; f(x) = 0, \; \mathbb{X} = [-1,1]$.

   (b) $f_n(x) = n\sin(x/n), \; f(x) = x, \; \mathbb{X} = [-\pi,\pi]$.

Chapter 7

Fourier sine series and cosine series

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."
—David Hilbert

7A Fourier (co)sine series on $[0, \pi]$

Prerequisites: §6E(iv), §6F.

Throughout this section, for all $n \in \mathbb{N}$, we define the functions $\mathbf{S}_n : [0, \pi] \longrightarrow \mathbb{R}$ and $\mathbf{C}_n : [0, \pi] \longrightarrow \mathbb{R}$ by $\mathbf{S}_n(x) := \sin(nx)$ and $\mathbf{C}_n(x) := \cos(nx)$, for all $x \in [0, \pi]$ (see Figure 6D.1 on page 113).

7A(i) Sine series on $[0, \pi]$

Recommended: §5C(i).

Suppose $f \in \mathbf{L}^2[0, \pi]$ (i.e. $f : [0, \pi] \longrightarrow \mathbb{R}$ is a function with $\|f\|_2 < \infty$). We define the **Fourier sine coefficients** of f:

$$B_n \quad := \quad \frac{\langle f, \mathbf{S}_n \rangle}{\|\mathbf{S}_n\|_2^2} \quad = \quad \boxed{\frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx,} \quad \text{for all } n \geq 1. \tag{7A.1}$$

The **Fourier sine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=1}^{\infty} B_n \mathbf{S}_n(x).} \tag{7A.2}$$

A function $f : [0, \pi] \longrightarrow \mathbb{R}$ is **continuously differentiable** on $[0, \pi]$ if f is continuous on $[0, \pi]$ (hence, bounded), $f'(x)$ exists for all $x \in (0, \pi)$, and furthermore, the function $f' : (0, \pi) \longrightarrow \mathbb{R}$ is itself bounded and continuous on $(0, \pi)$. Let $\mathcal{C}^1[0, \pi]$ be the space of all continuously differentiable functions.

We say f is **piecewise continuously differentiable** (or **piecewise $\mathcal{C}^1$**, or **sectionally smooth**) if there exist points $0 = j_0 < j_1 < j_2 < \cdots < j_{M+1} = \pi$ (for some $M \in \mathbb{N}$) such that f is bounded and continuously differentiable on each of the open intervals (j_m, j_{m+1}); these are called **$\mathcal{C}^1$ intervals** for f. In particular, any continuously differentiable function on $[0,\pi]$ is piecewise continuously differentiable (in this case, $M = 0$ and the set $\{j_1, \ldots, j_M\}$ is empty, so all of $(0,\pi)$ is a $\mathcal{C}^1$ interval).

Ⓔ **Exercise 7A.1.** (a) Show that any continuously differentiable function has finite L^2-norm. In other words, $\mathcal{C}^1[0,\pi] \subset \mathbf{L}^2[0,\pi]$.

(b) Show that any piecewise $\mathcal{C}^1$ function on $[0,\pi]$ is in $\mathbf{L}^2[0,\pi]$. ♦

Theorem 7A.1. Fourier Sine Series Convergence on $[0,\pi]$

(a) *The set $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2[0,\pi]$. Thus, if $f \in \mathbf{L}^2[0,\pi]$, then the sine series (7A.2) converges to f in L^2-norm, i.e.* $f \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \mathbf{S}_n$.

Furthermore, the coefficient sequence $\{B_n\}_{n=1}^{\infty}$ is the unique sequence of coefficients with this property. In other words, if $\{B'_n\}_{n=1}^{\infty}$ is some other sequence of coefficients such that $f \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B'_n \mathbf{S}_n$, then we must have $B'_n = B_n$ for all $n \in \mathbb{N}$.

(b) *If $f \in \mathcal{C}^1[0,\pi]$, then the sine series (7A.2) converges pointwise on $(0,\pi)$.*

More generally, if f is piecewise $\mathcal{C}^1$, then the sine series (7A.2) converges to f pointwise on each $\mathcal{C}^1$ interval for f. In other words, if $\{j_1, \ldots, j_m\}$ is the set of discontinuity points of f and/or f', and $j_m < x < j_{m+1}$, then

$$f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} B_n \sin(nx).$$

(c) *If $\sum_{n=1}^{\infty} |B_n| < \infty$, then the sine series (7A.2) converges to f uniformly on $[0,\pi]$.*

(d) [i] *If f is continuous and piecewise differentiable on $[0,\pi]$, and $f' \in \mathbf{L}^2[0,\pi]$, and f satisfies homogeneous Dirichlet boundary conditions* (i.e. $f(0) = f(\pi) = 0$), *then the sine series (7A.2) converges to f uniformly on $[0,\pi]$.*

[ii] *Conversely, if the sine series (7A.2) converges to f uniformly on $[0,\pi]$, then f is continuous on $[0,\pi]$, and satisfies homogeneous Dirichlet boundary conditions.*

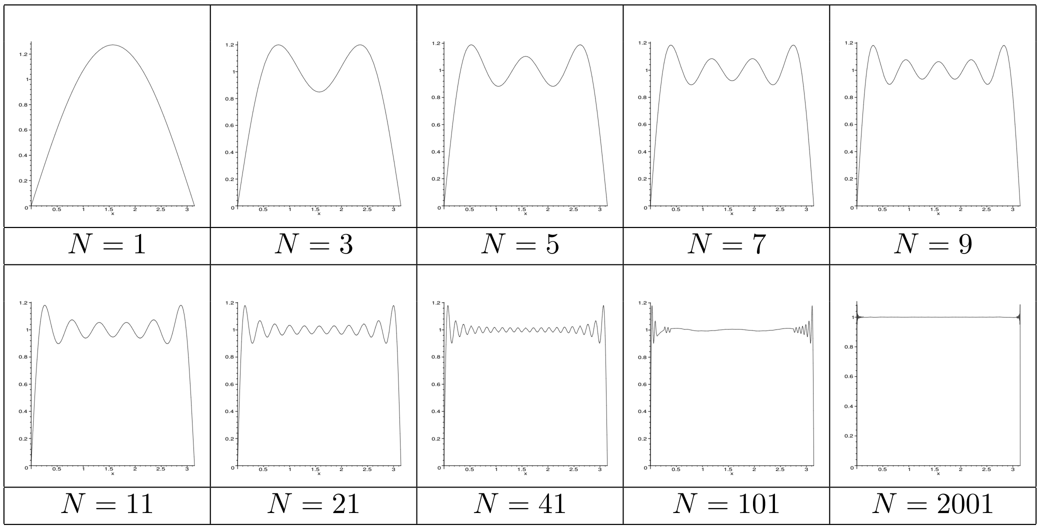


Figure 7A.1: $\frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{1}{n} \sin(nx)$, for $N = 1, 3, 5, 7, 9, 11, 21, 41$, and 2001. Notice the Gibbs phenomenon in the plots for large N.

(e) *If f is piecewise $\mathcal{C}^1$, and $\mathbb{K} \subset (j_m, j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the series* (7A.2) *converges* ***uniformly*** *to f on $\mathbb{K}$.*

(f) *Suppose $\{B_n\}_{n=1}^{\infty}$ is a nonnegative sequence decreasing to zero.* (That is, $B_1 \geq B_2 \geq \cdots \geq 0$ and $\lim_{n \to \infty} B_n = 0$). *If $0 < a < b < \pi$, then the series* (7A.2) *converges* ***uniformly*** *to f on $[a, b]$.*

Proof. **(c)** is <u>**Exercise 7A.2**</u> (Hint: Use the Weierstrass M-test, Proposition 6E.13 on page 129.) Ⓔ

(a,b,e) and **(d)**[i] are <u>**Exercise 7A.3**</u> (Hint: use Theorem 8A.1(a,b,d,e) on page 162, and Proposition 8C.5(a) and Lemma 8C.6(a) on page 171). Ⓔ

(d)[ii] is <u>**Exercise 7A.4**</u> . **(f)** is [Asm05, Thm.2, p.97 of §2.10]. □ Ⓔ

Example 7A.2.

(a) If $f(x) = \sin(5x) - 2\sin(3x)$, then the Fourier sine series of f is just "$\sin(5x) - 2\sin(3x)$". In other words, the Fourier coefficients B_n are all zero, except that $B_3 = -2$ and $B_5 = 1$.

(b) Suppose $f(x) \equiv 1$. For all $n \in \mathbb{N}$,

$$B_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\, dx = \frac{-2}{n\pi}\cos(nx)\Big|_{x=0}^{x=\pi} = \frac{2}{n\pi}\Big[1-(-1)^n\Big]$$
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is } \textbf{odd} \\ 0 & \text{if } n \text{ is } \textbf{even} \end{cases}.$$

Thus, the Fourier sine series is:

$$\frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin(nx) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right) \tag{7A.3}$$

Theorem 7A.1**(a)** says that $1 \underset{\text{L2}}{\widetilde{\approx}} \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin(nx)$. Figure 7A.1 displays some partial sums of the series (7A.3). The function $f \equiv 1$ is clearly continuously differentiable, so, by Theorem 7A.1**(b)**, the Fourier sine series converges pointwise to 1 on the interior of the interval $[0,\pi]$. However, the series does *not* converge to f at the points 0 or π. This is betrayed by the violent oscillations of the partial sums near these points; this is an example of the **Gibbs phenomenon**.

Since the Fourier sine series does not converge at the endpoints 0 and π, we know automatically that it does not converge to f uniformly on $[0,\pi]$. However, we could have also deduced this fact by noticing that f does *not* have homogeneous Dirichlet boundary conditions (because $f(0) = 1 = f(\pi)$), whereas every finite sum of $\sin(nx)$-type functions *does* have homogeneous Dirichlet BC. Thus, the series (7A.3) is 'trying' to converge to f, but it is 'stuck' at the endpoints 0 and π. (This is the idea behind Theorem 7A.1**(d)**).

(c) If $f(x) = \cos(mx)$, then the Fourier sine series of f is: $\frac{4}{\pi}\sum_{\substack{n=1 \\ n+m \text{ odd}}}^{\infty} \frac{n}{n^2-m^2}\sin(nx)$.

Ⓔ (**<u>Exercise 7A.5</u>** Hint: Use Theorem 6D.3 on page 113). ◊

Example 7A.3: $\sinh(\alpha x)$

If $\alpha > 0$, and $f(x) = \sinh(\alpha x)$, then its Fourier sine series is given by:

$$\sinh(\alpha x) \underset{\text{L2}}{\widetilde{\approx}} \frac{2\sinh(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{\alpha^2+n^2}\cdot\sin(nx)$$

To prove this, we must show that, for all $n > 0$,

$$B_n = \frac{2}{\pi}\int_0^\pi \sinh(\alpha x)\cdot\sin(nx)\, dx = \frac{2\sinh(\alpha\pi)}{\pi}\frac{n(-1)^{n+1}}{\alpha^2+n^2}.$$

To begin with, let $I := \int_0^\pi \sinh(\alpha x)\cdot\sin(nx)\ dx$. Then, applying integration by parts:

$$\begin{aligned}
I &= \frac{-1}{n}\left[\sinh(\alpha x)\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - \alpha\cdot\int_0^\pi \cosh(\alpha x)\cdot\cos(nx)\ dx\right] \\
&= \frac{-1}{n}\left[\sinh(\alpha\pi)\cdot(-1)^n - \frac{\alpha}{n}\cdot\left(\cosh(\alpha x)\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \alpha\int_0^\pi \sinh(\alpha x)\cdot\sin(nx)\ dx\right)\right] \\
&= \frac{-1}{n}\left[\sinh(\alpha\pi)\cdot(-1)^n - \frac{\alpha}{n}\cdot(0 - \alpha\cdot I)\right] \\
&= \frac{-\sinh(\alpha\pi)\cdot(-1)^n}{n} - \frac{\alpha^2}{n^2}I.
\end{aligned}$$

$$\begin{aligned}
\text{Hence:}\quad I &= \frac{-\sinh(\alpha\pi)\cdot(-1)^n}{n} - \frac{\alpha^2}{n^2}I; \\
\text{thus}\quad \left(1 + \frac{\alpha^2}{n^2}\right)I &= \frac{-\sinh(\alpha\pi)\cdot(-1)^n}{n}; \\
\text{i.e.}\quad \left(\frac{n^2+\alpha^2}{n^2}\right)I &= \frac{\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n}; \\
\text{so that}\quad I &= \frac{n\cdot\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n^2+\alpha^2}.
\end{aligned}$$

Thus, $\quad B_n = \frac{2}{\pi}I = \frac{2}{\pi}\frac{n\cdot\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n^2+\alpha^2}.$

The function sinh is clearly continuously differentiable, so Theorem 7A.1**(b)** implies that the Fourier sine series converges to $\sinh(\alpha x)$ pointwise on the open interval $(0,\pi)$. However, the series does *not* converge uniformly on $[0,\pi]$ (**Exercise 7A.6** Hint: What is $\sinh(\alpha\pi)$?). ♢ Ⓔ

7A(ii) Cosine series on $[0,\pi]$

Recommended: §5C(ii).

If $f \in \mathbf{L}^2[0,\pi]$, we define the **Fourier cosine coefficients** of f:

$$A_0 := \langle f, \mathbb{1}\rangle = \boxed{\frac{1}{\pi}\int_0^\pi f(x)\ dx,} \qquad \text{and}$$

$$A_n := \frac{\langle f, \mathbf{C}_n\rangle}{\|\mathbf{C}_n\|_2^2} = \boxed{\frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\ dx,} \quad \text{for all } n\in\mathbb{N}. \qquad (7A.4)$$

The **Fourier cosine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=0}^\infty A_n\mathbf{C}_n(x).} \qquad (7A.5)$$

Theorem 7A.4. Fourier Cosine Series Convergence on $[0,\pi]$

(a) *The set* $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ *is an* orthogonal basis *for* $\mathbf{L}^2[0,\pi]$. *Thus, if* $f \in \mathbf{L}^2[0,\pi]$, *then the cosine series* (7A.5) *converges to* f *in* L^2*-norm, i.e.* $f \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A_n \mathbf{C}_n$.

Furthermore, the coefficient sequence $\{A_n\}_{n=0}^{\infty}$ *is the* unique *sequence of coefficients with this property. In other words, if* $\{A'_n\}_{n=1}^{\infty}$ *is some other sequence of coefficients such that* $f \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A'_n \mathbf{C}_n$, *then we must have* $A'_n = A_n$ *for all* $n \in \mathbb{N}$.

(b) *If* $f \in \mathcal{C}^1[0,\pi]$, *then the cosine series* (7A.5) *converges* pointwise *on* $(0,\pi)$.

If f *is piecewise* $\mathcal{C}^1$ *on* $[0,\pi]$, *then the cosine series* (7A.5) *converges to* f pointwise *on each* $\mathcal{C}^1$ *interval for* f. *In other words, if* $\{j_1, \ldots, j_m\}$ *is the set of discontinuity points of* f *and/or* f', *and* $j_m < x < j_{m+1}$, *then*

$$f(x) \;=\; \lim_{N\to\infty} \sum_{n=0}^{N} A_n \cos(nx).$$

(c) *If* $\sum_{n=0}^{\infty} |A_n| < \infty$, *then the cosine series* (7A.5) *converges to* f uniformly *on* $[0,\pi]$.

(d) [i] *If* f *is continuous and piecewise differentiable on* $[0,\pi]$, *and* $f' \in \mathbf{L}^2[0,\pi]$, *then the cosine series* (7A.5) *converges to* f uniformly *on* $[0,\pi]$.

[ii] *Conversely, if* $\sum_{n=0}^{\infty} n\,|A_n| < \infty$, *then* $f \in \mathcal{C}^1[0,\pi]$ *and* f *satisfies homogeneous Neumann boundary conditions* (i.e. $f'(0) = f'(\pi) = 0$).

(e) *If* f *is piecewise* $\mathcal{C}^1$, *and* $\mathbb{K} \subset (j_m, j_{m+1})$ *is any closed subset of a* $\mathcal{C}^1$ *interval of* f, *then the series* (7A.5) *converges* uniformly *to* f *on* $\mathbb{K}$.

(f) *Suppose* $\{A_n\}_{n=0}^{\infty}$ *is a nonnegative sequence decreasing to zero.* (That is, $A_0 \geq A_1 \geq A_2 \geq \cdots \geq 0$ and $\lim_{n\to\infty} A_n = 0$). *If* $0 < a < b < \pi$, *then the series* (7A.5) *converges* uniformly *to* f *on* $[a,b]$.

Ⓔ *Proof.* **(c)** is **Exercise 7A.7** (Hint: Use the Weierstrass M-test, Proposition 6E.13 on page 129.)

Ⓔ **(a,b,e)** and **(d)**[i] are **Exercise 7A.8** (Hint: use Theorem 8A.1(a,b,d,e) on page 162, and Proposition 8C.5(b) and Lemma 8C.6(b) on page 171).

Ⓔ **(d)**[ii] is **Exercise 7A.9** (Hint: Use Theorem 7C.10(b) on page 158).

(f) is [Asm05, Thm.2, p.97 of §2.10]. $\Box$

Example 7A.5.

(a) If $f(x) = \cos(13x)$, then the Fourier cosine series of f is just "$\cos(13x)$". In other words, the Fourier coefficients A_n are all zero, except that $A_{13} = 1$.

(b) Suppose $f(x) \equiv 1$. Then $f = \mathbf{C}_0$, so the Fourier cosine coefficients are: $A_0 = 1$, while $A_1 = A_2 = A_3 = \ldots 0$.

(c) Let $f(x) = \sin(mx)$. If m is *even*, then the Fourier cosine series of f is:

$$\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{n}{n^2-m^2}\cos(nx).$$

If m is *odd*, then the Fourier cosine series of f is: $\displaystyle\frac{2}{\pi m} + \frac{4}{\pi}\sum_{\substack{n=2\\ n \text{ even}}}^{\infty} \frac{n}{n^2-m^2}\cos(nx)$.

(**<u>Exercise 7A.10</u>** Hint: Use Theorem 6D.3 on page 113). $\diamondsuit$ Ⓔ

Example 7A.6: $\cosh(x)$

Suppose $f(x) = \cosh(x)$. Then the Fourier cosine series of f is given by:

$$\cosh(x) \underset{\text{L2}}{\approx} \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\cos(nx)}{n^2+1}.$$

To see this, first note that $A_0 = \displaystyle\frac{1}{\pi}\int_0^{\pi}\cosh(x)\,dx = \frac{1}{\pi}\sinh(x)\Big|_{x=0}^{x=\pi} = \frac{\sinh(\pi)}{\pi}$ (because $\sinh(0) = 0$).

Next, let $I := \displaystyle\int_0^{\pi}\cosh(x)\cdot\cos(nx)\,dx$. Then applying integration by parts, we get:

$$\begin{aligned}
I &= \frac{1}{n}\left(\cosh(x)\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^{\pi}\sinh(x)\cdot\sin(nx)\,dx\right)\\
&= \frac{-1}{n}\int_0^{\pi}\sinh(x)\cdot\sin(nx)\,dx\\
&= \frac{1}{n^2}\left(\sinh(x)\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - \int_0^{\pi}\cosh(x)\cdot\cos(nx)\,dx\right)\\
&= \frac{1}{n^2}\left(\sinh(\pi)\cdot\cos(n\pi) - I\right) = \frac{1}{n^2}\left((-1)^n\sinh(\pi) - I\right).
\end{aligned}$$

Thus, $I = \frac{1}{n^2}\Big((-1)^n \cdot \sinh(\pi) - I\Big)$. Hence, $(n^2+1)I = (-1)^n \cdot \sinh(\pi)$. Hence, $I = \frac{(-1)^n \cdot \sinh(\pi)}{n^2+1}$. Thus, $A_n = \frac{2}{\pi} I = \frac{2}{\pi}\frac{(-1)^n \cdot \sinh(\pi)}{n^2+1}$.
$\diamondsuit$

Remark. (a) Almost any introduction to the theory of partial differential equations will contain a discussion of the Fourier convergence theorems. For example, see [Pow99, §1.3-1.7, pp.59-85], [dZ86, Thm.6.1, p.72] or [Hab87, §3.2, p.91].

(b) Please see Remark 8D.3 on page 174 for further technical remarks about the (non)convergence of Fourier (co)sine series, in situations where the hypotheses of Theorems 7A.1 and 7A.4 are not satisfied.

7B Fourier (co)sine series on $[0, L]$

Prerequisites: §6E, §6F. **Recommended:** §7A.

Throughout this section, let $L > 0$ be some positive real number. For all $n \in \mathbb{N}$, we define the functions $\mathbf{S}_n : [0, L] \longrightarrow \mathbb{R}$ and $\mathbf{C}_n : [0, L] \longrightarrow \mathbb{R}$ by $\mathbf{S}_n(x) := \sin\left(\frac{n\pi x}{L}\right)$ and $\mathbf{C}_n(x) := \cos\left(\frac{n\pi x}{L}\right)$, for all $x \in [0, L]$ (see Figure 6D.1 on page 113). Notice that, if $L = \pi$, then $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$, as in §7A. The results in this section exactly parallel those in §7A, except that we replace π with L to obtain slightly greater generality. In principle, every statement in this section is equivalent to the corresponding statement in §7A, through the change of variables $y = x/\pi$ (it is a useful exercise to reflect on this as you read this section).

7B(i) Sine series on $[0, L]$

Recommended: §5C(i), §7A(i).

Fix $L > 0$, and let $[0, L]$ be an interval of length L. If $f \in \mathbf{L}^2[0, L]$, we define the **Fourier sine coefficients** of f:

$$B_n \quad := \quad \frac{\langle f, \mathbf{S}_n\rangle}{\|\mathbf{S}_n\|_2^2} \quad = \quad \boxed{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,} \quad \text{for all } n \geq 1.$$

The **Fourier sine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=1}^{\infty} B_n \mathbf{S}_n(x).} \tag{7B.1}$$

A function $f : [0, L] \longrightarrow \mathbb{R}$ is **continuously differentiable** on $[0, L]$ if f is continuous on $[0, L]$ (hence, bounded), and $f'(x)$ exists for all $x \in (0, L)$, and furthermore, the function $f' : (0, L) \longrightarrow \mathbb{R}$ is itself bounded and continuous on $(0, L)$. Let $\mathcal{C}^1[0, L]$ be the space of all continuously differentiable functions.

We say $f : [0, L] \longrightarrow \mathbb{R}$ is **piecewise continuously differentiable** (or **piecewise $\mathcal{C}^1$**, or **sectionally smooth**) if there exist points $0 = j_0 < j_1 < j_2 < \cdots < j_{M+1} = L$ such that f is bounded and continuously differentiable on each of the open intervals (j_m, j_{m+1}); these are called $\mathcal{C}^1$ **intervals** for f. In particular, any continuously differentiable function on $[0, L]$ is piecewise continuously differentiable (in this case, all of $(0, L)$ is a $\mathcal{C}^1$ interval).

Theorem 7B.1. Fourier Sine Series Convergence on $[0, L]$
Parts **(a-f)** *of* Theorem 7A.1 on page 138 *are all still true if you replace "π" with "L" everywhere.*

Proof. **Exercise 7B.1** Hint: Use the change-of-variables $y = \frac{\pi}{L}x$ to pass from $y \in [0, L]$ to $x \in [0, \pi]$. Ⓔ □

Example 7B.2.

(a) If $f(x) = \sin\left(\frac{5\pi}{L}x\right)$, then the Fourier sine series of f is just "$\sin\left(\frac{5\pi}{L}x\right)$". In other words, the Fourier coefficients B_n are all zero, except that $B_5 = 1$.

(b) Suppose $f(x) \equiv 1$. For all $n \in \mathbb{N}$,

$$B_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{x=L} = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is } \textbf{odd} \\ 0 & \text{if } n \text{ is } \textbf{even} \end{cases}.$$

Thus, the Fourier sine series is given: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)$. Figure 7A.1 displays some partial sums of this series (in the case $L = \pi$). The **Gibbs phenomenon** is clearly evident just as in Example 7A.2(b) on page 139.

(c) If $f(x) = \cos\left(\frac{m\pi}{L}x\right)$, then the Fourier sine series of f is: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1 \\ n+m \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2}\sin\left(\frac{n\pi}{L}x\right)$.

(**Exercise 7B.2** Hint: Use Theorem 6D.3 on page 113). Ⓔ

(d) If $\alpha > 0$, and $f(x) = \sinh\left(\frac{\alpha\pi x}{L}\right)$, then its Fourier sine coefficients are computed:

$$B_n = \frac{2}{L}\int_0^L \sinh\left(\frac{\alpha\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2\sinh(\alpha\pi)}{\pi}\frac{n(-1)^{n+1}}{\alpha^2 + n^2}.$$

(**Exercise 7B.3**). ♢ Ⓔ

7B(ii) Cosine series on $[0, L]$

Recommended: §5C(ii), §7A(ii).

If $f \in \mathbf{L}^2[0, L]$, we define the **Fourier cosine coefficients** of f:

$$A_0 \quad := \quad \langle f, \mathbb{1} \rangle \quad = \quad \boxed{\frac{1}{L}\int_0^L f(x)\, dx,}$$

and $$A_n \quad := \quad \frac{\langle f, \mathbf{C}_n \rangle}{\|\mathbf{C}_n\|_2^2} \quad = \quad \boxed{\frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx,} \quad \text{for all } n > 0.$$

The **Fourier cosine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=0}^{\infty} A_n \mathbf{C}_n(x).} \tag{7B.2}$$

Theorem 7B.3. Fourier Cosine Series Convergence on $[0, L]$

Parts **(a-f)** *of* Theorem 7A.4 on page 142 *are all still true if you replace "π" with "L" everywhere.*

Ⓔ *Proof.* **Exercise 7B.4** Hint: Use the change-of-variables $y := \frac{\pi}{L}x$ to pass from $x \in [0, L]$ to $y \in [0, \pi]$. □

Example 7B.4.

(a) If $f(x) = \cos\left(\frac{13\pi}{L}x\right)$, then the Fourier cosine series of f is just "$\cos\left(\frac{13\pi}{L}x\right)$". In other words, the Fourier coefficients A_n are all zero, except that $A_{13} = 1$.

(b) Suppose $f(x) \equiv 1$. Then $f = \mathbf{C}_0$, so the Fourier cosine coefficients are: $A_0 = 1$, while $A_1 = A_2 = A_3 = \ldots 0$.

(c) Let $f(x) = \sin\left(\frac{m\pi}{L}x\right)$. If m is *even*, then the Fourier cosine series of f is:

$$\frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2} \cos\left(\frac{n\pi}{L}x\right).$$

If m is *odd*, then the Fourier cosine series of f is: $\displaystyle \frac{2}{\pi m} + \frac{4}{\pi} \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{n}{n^2 - m^2} \cos(nx).$

Ⓔ (**Exercise 7B.5** Hint: Use Theorem 6D.3 on page 113). ♢

7C Computing Fourier (co)sine coefficients

Prerequisites: §7B.

When computing the Fourier sine coefficient $B_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx$, it is simpler to first compute the integral $\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx$, and then multiply the result by $\frac{2}{L}$. Likewise, to compute a Fourier cosine coefficients, first compute the integral $\int_0^L f(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx$, and then multiply the result by $\frac{2}{L}$. In this section, we review some useful techniques to compute these integrals.

7C(i) Integration by parts

Computing Fourier coefficients almost always involves integration by parts. Generally, if you can't compute it with integration by parts, you can't compute it. When evaluating a Fourier integral by parts, one almost always ends up with boundary terms of the form "$\cos(n\pi)$" or "$\sin\left(\frac{n}{2}\pi\right)$", etc. The following formulae are useful in this regard:

$$\boxed{\sin(n\pi) = 0 \quad \text{for any } n \in \mathbb{Z}.} \tag{7C.3}$$

For example, $\sin(-\pi) = \sin(0) = \sin(\pi) = \sin(2\pi) = \sin(3\pi) = 0$.

$$\boxed{\cos(n\pi) = (-1)^n \quad \text{for any } n \in \mathbb{Z}.} \tag{7C.4}$$

For example, $\cos(-\pi) = -1$, $\cos(0) = 1$, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, etc.

$$\boxed{\sin\left(\frac{n}{2}\pi\right) = \begin{cases} 0 & \text{if } n \text{ is } \textit{even} \\ (-1)^k & \text{if } n \text{ is } \textit{odd}, \text{ and } n = 2k+1 \end{cases}} \tag{7C.5}$$

For example, $\sin(0) = 0$, $\sin\left(\frac{1}{2}\pi\right) = 1$, $\sin(\pi) = 0$, $\sin\left(\frac{3}{2}\pi\right) = -1$, etc.

$$\boxed{\cos\left(\frac{n}{2}\pi\right) = \begin{cases} 0 & \text{if } n \text{ is } \textit{odd} \\ (-1)^k & \text{if } n \text{ is } \textit{even}, \text{ and } n = 2k \end{cases}} \tag{7C.6}$$

For example, $\cos(0) = 1$, $\cos\left(\frac{1}{2}\pi\right) = 0$, $\cos(\pi) = -1$, $\cos\left(\frac{3}{2}\pi\right) = 0$, etc.

Exercise 7C.1. Verify equations (7C.3), (7C.4), (7C.5), and (7C.6). ♦ Ⓔ

7C(ii) Polynomials

Theorem 7C.1. *Let $n \in \mathbb{N}$. Then*

$$(\mathbf{a}) \qquad \int_0^L \sin\left(\frac{n\pi}{L}x\right)\, dx \quad = \quad \begin{cases} \dfrac{2L}{n\pi} & \textit{if } n \textit{ is odd;} \\[2ex] 0 & \textit{if } n \textit{ is even.} \end{cases} \tag{7C.7}$$

$$(\mathbf{b}) \qquad \int_0^L \cos\left(\frac{n\pi}{L}x\right)\, dx \quad = \quad \begin{cases} L & \textit{if } \ n = 0 \\ 0 & \textit{if } \ n > 0. \end{cases} \tag{7C.8}$$

For any $k \in \{1, 2, 3, \ldots\}$, we have the following recurrence relations:

$$(\mathbf{c}) \quad \int_0^L x^k \cdot \sin\left(\frac{n\pi}{L}x\right)\, dx \quad = \quad \frac{(-1)^{n+1}}{n} \cdot \frac{L^{k+1}}{\pi} \quad + \quad \frac{k}{n} \cdot \frac{L}{\pi} \int_0^L x^{k-1} \cdot \cos\left(\frac{n\pi}{L}x\right), \tag{7C.9}$$

$$(\mathbf{d}) \quad \int_0^L x^k \cdot \cos\left(\frac{n\pi}{L}x\right)\, dx \quad = \quad \frac{-k}{n} \cdot \frac{L}{\pi} \int_0^L x^{k-1} \cdot \sin\left(\frac{n\pi}{L}x\right). \tag{7C.10}$$

Ⓔ *Proof.* **Exercise 7C.2** Hint: for **(c)** and **(d)**, use integration by parts. □

Example 7C.2. In all of the following examples, let $L = \pi$.

(a) $\displaystyle \frac{2}{\pi}\int_0^\pi \sin(nx)\, dx \quad = \quad \frac{2}{\pi}\frac{1-(-1)^n}{n}.$

(b) $\displaystyle \frac{2}{\pi}\int_0^\pi x \cdot \sin(nx)\, dx \quad = \quad (-1)^{n+1}\frac{2}{n}.$

(c) $\displaystyle \frac{2}{\pi}\int_0^\pi x^2 \cdot \sin(nx)\, dx \quad = \quad (-1)^{n+1}\frac{2\pi}{n} \; + \; \frac{4}{\pi n^3}\Big((-1)^n - 1\Big).$

(d) $\displaystyle \frac{2}{\pi}\int_0^\pi x^3 \cdot \sin(nx)\, dx \quad = \quad (-1)^n \left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right).$

(e) $\displaystyle \frac{2}{\pi}\int_0^\pi \cos(nx)\, dx \quad = \quad \begin{cases} 2 & \text{if } \ n = 0 \\ 0 & \text{if } \ n > 0. \end{cases}$

(f) $\displaystyle \frac{2}{\pi}\int_0^\pi x \cdot \cos(nx)\, dx \quad = \quad \frac{2}{\pi n^2}\Big((-1)^n - 1\Big)$, if $n > 0$.

(g) $\displaystyle \frac{2}{\pi}\int_0^\pi x^2 \cdot \cos(nx)\, dx \quad = \quad (-1)^n\frac{4}{n^2}$, if $n > 0$.

(h) $\displaystyle \frac{2}{\pi}\int_0^\pi x^3 \cdot \cos(nx)\, dx \quad = \quad (-1)^n\frac{6\pi}{n^2} - \frac{12}{\pi n^4}\Big((-1)^n - 1\Big)$, if $n > 0$. ♢

Proof. **(b)**: We will show this in two ways. First, by direct computation:

$$\begin{aligned}\int_0^\pi x\cdot\sin(nx)\ dx &= \frac{-1}{n}\left(x\cdot\cos(nx)\ \Big|_{x=0}^{x=\pi} - \int_0^\pi \cos(nx)\ dx\right)\\ &= \frac{-1}{n}\left(\pi\cdot\cos(n\pi) - \frac{1}{n}\sin(nx)\ \Big|_{x=0}^{x=\pi}\right)\\ &= \frac{-1}{n}(-1)^n\pi \quad = \quad \frac{(-1)^{n+1}\pi}{n}.\end{aligned}$$

Thus, $\dfrac{2}{\pi}\displaystyle\int_0^\pi x\cdot\sin(nx)\ dx \ = \ \frac{2(-1)^{n+1}}{n}$, as desired.

Next, we verify **(b)** using Theorem 7C.1. Setting $L=\pi$ and $k=1$ in (7C.9), we have:

$$\begin{aligned}\int_0^\pi x\cdot\sin(nx)\ dx &= \frac{(-1)^{n+1}}{n}\cdot\frac{\pi^{1+1}}{\pi} \quad + \quad \frac{1}{n}\cdot\frac{\pi}{\pi}\int_0^\pi x^{k-1}\cdot\cos(nx)\ dx\\ &= \frac{(-1)^{n+1}}{n}\cdot\pi \quad + \quad \frac{1}{n}\int_0^\pi \cos(nx)\ dx \quad = \quad \frac{(-1)^{n+1}}{n}\cdot\pi,\end{aligned}$$

because $\displaystyle\int_0^\pi \cos(nx)\ dx \ = \ 0$ by (7C.8). Thus, $\dfrac{2}{\pi}\displaystyle\int_0^\pi x\cdot\sin(nx)\ dx \ = \ \frac{2(-1)^{n+1}}{n}$, as desired.

Proof of **(c)**:

$$\begin{aligned}\int_0^\pi x^2\cdot\sin(nx)\ dx &= \frac{-1}{n}\left(x^2\cdot\cos(nx)\ \Big|_{x=0}^{x=\pi} - 2\int_0^\pi x\cos(nx)\ dx\right)\\ &= \frac{-1}{n}\left[\pi^2\cdot\cos(n\pi) - \frac{2}{n}\left(x\cdot\sin(nx)\ \Big|_{x=0}^{x=\pi} - \int_0^\pi \sin(nx)\ dx\right)\right]\\ &= \frac{-1}{n}\left[\pi^2\cdot(-1)^n + \frac{2}{n}\left(\frac{-1}{n}\cos(nx)\ \Big|_{x=0}^{x=\pi}\right)\right]\\ &= \frac{-1}{n}\left[\pi^2\cdot(-1)^n - \frac{2}{n^2}\Big((-1)^n - 1\Big)\right]\\ &= \frac{2}{n^3}\Big((-1)^n - 1\Big) \quad + \quad \frac{(-1)^{n+1}\pi^2}{n}.\end{aligned}$$

The result follows.

Exercise 7C.3 Verify **(c)** using Theorem 7C.1. Ⓔ

(g) We will show this in two ways. First, by direct computation:

$$\begin{aligned}\int_0^\pi x^2\cdot\cos(nx)\ dx &= \frac{1}{n}\left[x^2\cdot\sin(nx)\ \Big|_{x=0}^{x=\pi} \quad - \quad 2\int_0^\pi x\cdot\sin(nx)\ dx\right]\\ &= \frac{-2}{n}\int_0^\pi x\cdot\sin(nx)\ dx \qquad (\text{because } \sin(nx)=\sin(0)=0)\end{aligned}$$

$$= \frac{2}{n^2}\left[x \cdot \cos(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \cos(nx)\ dx\right]$$
$$= \frac{2}{n^2}\left[\pi \cdot (-1)^n - \frac{1}{n}\sin(nx)\Big|_{x=0}^{x=\pi}\right]$$
$$= \frac{2\pi \cdot (-1)^n}{n^2}.$$

Thus, $\frac{2}{\pi}\int_0^\pi x^2 \cdot \cos(nx)\ dx = \frac{4 \cdot (-1)^n}{n^2}$, as desired.

Next, we verify **(g)** using Theorem 7C.1. Setting $L = \pi$ and $k = 2$ in (7C.10), we have:

$$\int_0^\pi x^2 \cdot \cos(nx)\ dx = \frac{-k}{n} \cdot \frac{L}{\pi} \int_0^p ix^{k-1} \cdot \sin(nx) = \frac{-2}{n} \cdot \int_0^\pi x \cdot \sin(nx). \qquad (7C.11)$$

Next, applying (7C.9) with $k = 1$, we get:

$$\int_0^\pi x \cdot \sin(nx) = \frac{(-1)^{n+1}}{n} \cdot \frac{\pi^2}{\pi} + \frac{1}{n} \cdot \frac{\pi}{\pi} \int_0^\pi \cos(nx) = \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\int_0^\pi \cos(nx).$$

Substituting this into (7C.11), we get

$$\int_0^\pi x^2 \cdot \cos(nx)\ dx = \frac{-2}{n} \cdot \left[\frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\int_0^\pi \cos(nx)\right]. \qquad (7C.12)$$

We're assuming $n > 0$. But then (7C.8) says $\int_0^\pi \cos(nx) = 0$. Thus, we can simplify (7C.12) to conclude:

$$\frac{2}{\pi}\int_0^\pi x^2 \cdot \cos(nx)\ dx = \frac{2}{\pi} \cdot \frac{-2}{n} \cdot \frac{(-1)^{n+1}\pi}{n} = \frac{4(-1)^n}{n^2},$$

as desired. □

Ⓔ **Exercise 7C.4.** Verify all of the other parts of Example 7C.2, both using Theorem 7C.1, and through direct integration. ♦

To compute the Fourier series of an arbitrary polynomial, we integrate one term at a time.

Example 7C.3. Let $L = \pi$ and let $f(x) = x^2 - \pi \cdot x$. Then the Fourier sine series of f is:

$$\frac{-8}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^3}\sin(nx) = \frac{-8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{27} + \frac{\sin(5x)}{125} + \frac{\sin(7x)}{343} + \cdots\cdots\right)$$

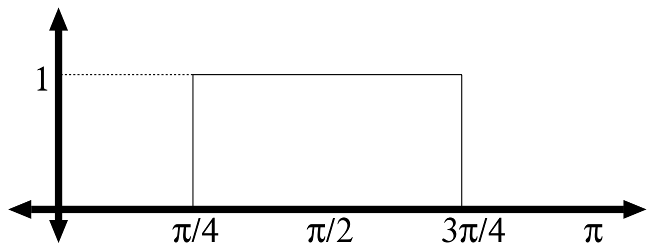


Figure 7C.1: Example 7C.4.

To see this, first, note that, by Example 7C.2(b)

$$\int_0^\pi x \cdot \sin(nx)\ dx \ = \ \frac{-1}{n}(-1)^n\pi \ = \ \frac{(-1)^{n+1}\pi}{n}.$$

Next, by Example 7C.2(c),

$$\int_0^\pi x^2 \cdot \sin(nx)\ dx \quad = \quad \frac{2}{n^3}\Big((-1)^n - 1\Big) \ + \ \frac{(-1)^{n+1}\pi^2}{n}.$$

Thus,

$$\begin{aligned}
\int_0^\pi \Big(x^2 - \pi x\Big) \cdot \sin(nx)\ dx &= \int_0^\pi x^2 \cdot \sin(nx)\ dx \ - \ \pi \cdot \int_0^\pi x \cdot \sin(nx)\ dx \\
&= \frac{2}{n^3}\Big((-1)^n - 1\Big) \ + \ \frac{(-1)^{n+1}\pi^2}{n} \ - \ \pi \cdot \frac{(-1)^{n+1}\pi}{n} \\
&= \frac{2}{n^3}\Big((-1)^n - 1\Big).
\end{aligned}$$

Thus,

$$B_n \ = \ \frac{2}{\pi}\int_0^\pi \Big(x^2 - \pi x\Big)\cdot\sin(nx)\ dx \ = \ \frac{4}{\pi n^3}\Big((-1)^n - 1\Big) \ = \ \begin{cases} -8/\pi n^3 & \text{if} \quad n \text{ is odd;} \\ 0 & \text{if} \quad n \text{ is even.} \end{cases}$$

♢

7C(iii) Step functions

Example 7C.4. Let $L = \pi$, and suppose $f(x) = \begin{cases} 1 & \text{if } \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$ (see Figure 7C.1). Then the Fourier sine coefficients of f are given:

$$B_n = \begin{cases} 0 & \text{if } n \text{ is } \textit{even}; \\ \frac{2\sqrt{2}(-1)^k}{n\pi} & \text{if } n \text{ is } \textit{odd}, \text{ and } n = 4k \pm 1 \text{ for some } k \in \mathbb{N}. \end{cases}$$

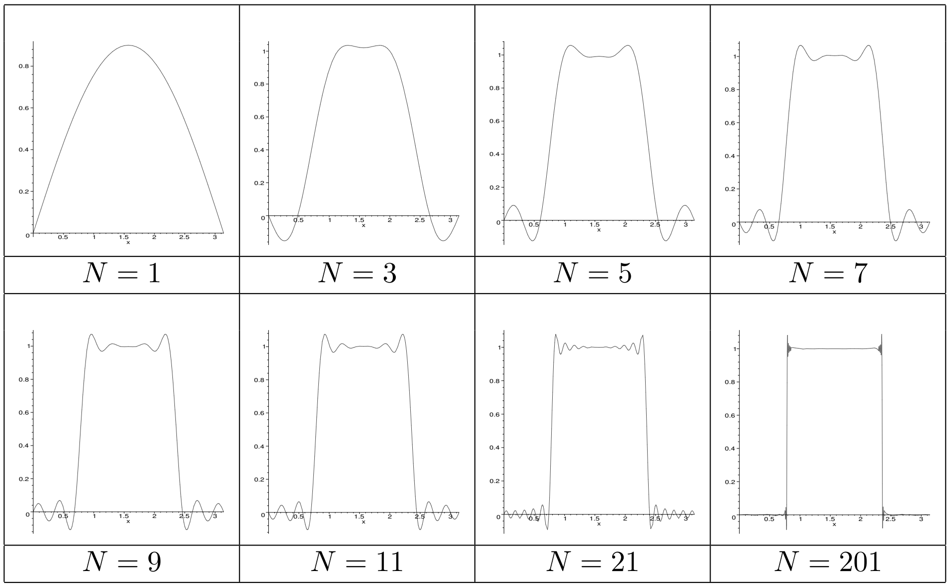


Figure 7C.2: Partial Fourier sine series for Example 7C.4, for $N = 0,1,2,3,4,5,10$ and 100. Notice the Gibbs phenomenon in the plots for large N.

To see this, observe that

$$\begin{aligned}
\int_0^{\pi} f(x)\sin(nx)\ dx &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(nx)\ dx \quad = \quad \frac{-1}{n}\cos(nx)\Big|_{x=\frac{\pi}{4}}^{x=\frac{3\pi}{4}} \\
&= \frac{-1}{n}\left(\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right)\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is } \textit{even}; \\ \frac{\sqrt{2}(-1)^{k+1}}{n} & \text{if } n \text{ is } \textit{odd}, \text{ and } n = 4k \pm 1 \text{ for some } k \in \mathbb{N}. \end{cases}
\end{aligned}$$

Ⓔ (**Exercise 7C.5**). Thus, the Fourier sine series for f is:

$$\frac{2\sqrt{2}}{\pi}\left(\sin(x) \;+\; \sum_{k=1}^{N}(-1)^k\left(\frac{\sin\Big((4k-1)x\Big)}{4k-1} + \frac{\sin\Big((4k+1)x\Big)}{4k+1}\right)\right)$$

Ⓔ (**Exercise 7C.6**).

Figure 7C.2 shows some of the partial sums of this series. The series converges *pointwise* to $f(x)$ in the interior of the intervals $\left[0, \frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, and $\left(\frac{3\pi}{4}, \pi\right]$. However, it does not converge to f at the discontinuity points $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. In the plots, this is betrayed by the violent oscillations of the partial sums near these discontinuity points –this is an example of the **Gibbs phenomenon**. ♢

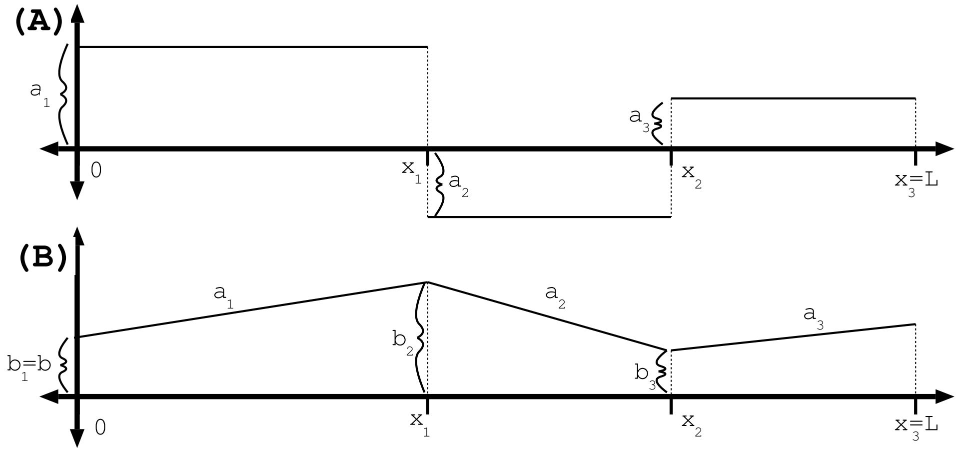


Figure 7C.3: (A) A step function. (B) A piecewise linear function.

Example 7C.4 is an example of a *step function.* A function $F : [0, L] \longrightarrow \mathbb{R}$ is a **step function** (see Figure 7C.3(A)) if there are numbers $0 = x_0 < x_1 < x_2 < x_3 < \ldots < x_{M-1} < x_M = L$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ such that

$$F(x) \quad = \quad \begin{cases} a_1 & \text{if } 0 \leq x \leq x_1; \\ a_2 & \text{if } x_1 < x \leq x_2; \\ \vdots & \vdots \\ a_m & \text{if } x_{m-1} < x \leq x_m; \\ \vdots & \vdots \\ a_M & \text{if } x_{M-1} < x \leq L. \end{cases} \tag{7C.13}$$

For instance, in Example 7C.4, $M = 3$; $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{3\pi}{4}$, and $x_3 = \pi$; $a_1 = 0 = a_3$, and $a_2 = 1$.

To compute the Fourier coefficients of a step function, we simply break the integral into 'pieces', as in Example 7C.4. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the idea.

Theorem 7C.5. *Suppose $F : [0, L] \longrightarrow \mathbb{R}$ is a step function like* (7C.13). *Then the Fourier coefficients of F are given:*

$$\frac{1}{L}\int_0^L F(x) \quad = \quad \frac{1}{L}\sum_{m=1}^{M} a_m \cdot (x_m - x_{m-1})\,;$$

$$\frac{2}{L}\int_0^L F(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx \quad = \quad \frac{-2}{\pi n}\sum_{m=1}^{M-1}\sin\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_{m+1} - a_m\Big)\,;$$

$$\frac{2}{L}\int_0^L F(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx \;=\; \frac{2}{\pi n}\left(a_1+(-1)^{n+1}a_M\right) + \frac{2}{\pi n}\sum_{m=1}^{M-1}\cos\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\left(a_{m+1}-a_m\right).$$

Ⓔ *Proof.* **Exercise 7C.7** Hint: Integrate the function piecewise. □

Remark. Note that the Fourier series of a step function f will converge uniformly to f on the *interior* of each "step", but will *not* converge to f at any of the step boundaries, because f is not continuous at these points.

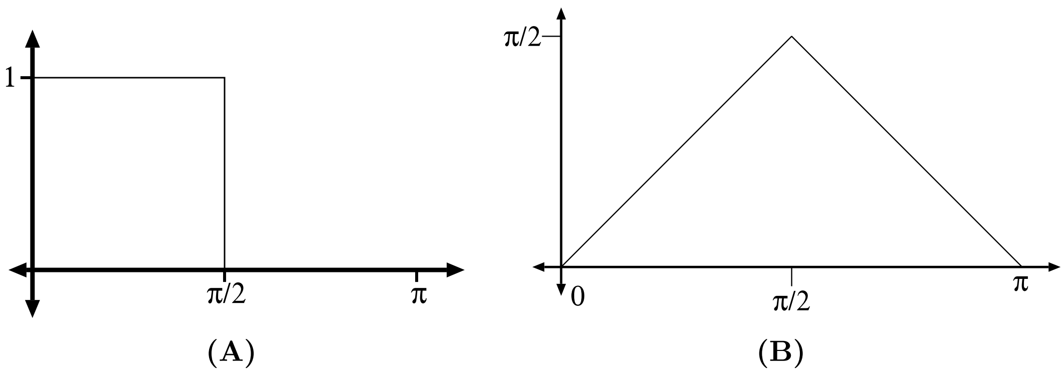


Figure 7C.4: **(A)** The step function $g(x)$ in Example 7C.6. **(B)** The tent function $f(x)$ in Example 7C.7.

Example 7C.6. Suppose $L=\pi$, and $g(x) = \begin{cases} 1 & \text{if } 0\le x<\frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2}\le x \end{cases}$ (see Figure 7C.4A). Then the Fourier cosine series of $g(x)$ is:

$$\frac{1}{2} \quad + \quad \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left((2k+1)x\right)$$

In other words, $A_0=\frac{1}{2}$ and, for all $n>0$, $A_n = \begin{cases} \frac{2}{\pi}\frac{(-1)^k}{2k+1} & \text{if } n \text{ is odd and } n=2k+1; \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Ⓔ **Exercise 7C.8** Show this in two ways: first by direct integration, and then by applying the formula from Theorem 7C.5. ♢

7C(iv) Piecewise linear functions

Example 7C.7: (The Tent Function)

Let $\mathbb{X} = [0,\pi]$ and let $f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2}; \\ \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$ (see Figure 7C.4B)

The Fourier sine series of f is: $\displaystyle \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd};\\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$

To prove this, we must show that, for all $n > 0$,

$$B_n \quad = \quad \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\, dx \quad = \quad \begin{cases} \dfrac{4}{n^2\pi}(-1)^k & \text{if } n \text{ is odd, } n = 2k+1; \\ \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

To verify this, we observe that

$$\int_0^\pi f(x)\sin(nx)\ dx \quad = \quad \int_0^{\pi/2} x\sin(nx)\ dx \ + \ \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\ dx.$$

<u>**Exercise 7C.9**</u> Complete the computation of B_n. ♢ Ⓔ

The tent function in Example 7C.7 is *piecewise linear*. A function $F : [0,L] \longrightarrow \mathbb{R}$ is **piecewise linear** (see Figure 7C.3(B) on page 153) if there are numbers $0 = x_0 < x_1 < x_1 < x_2 < \ldots < x_{M-1} < x_M = L$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$F(x) \quad = \quad \begin{cases} a_1(x - L) + b_1 & \text{if } 0 \le x \le x_1; \\ a_2(x - x_1) + b_2 & \text{if } x_1 < x \le x_2; \\ \vdots & \vdots \\ a_m(x - x_m) + b_{m+1} & \text{if } x_m < x \le x_{m+1}; \\ \vdots & \vdots \\ a_M(x - x_{M-1}) + b_M & \text{if } x_{M-1} < x \le L. \end{cases} \qquad (7C.14)$$

where $b_1 = b$, and, for all $m > 1$, $b_m = a_m(x_m - x_{m-1}) + b_{m-1}$.

For instance, in Example 7C.7, $M = 2$, $x_1 = \frac{\pi}{2}$ and $x_2 = \pi$; $a_1 = 1$ and $a_2 = -1$.

To compute the Fourier coefficients of a piecewise linear function, we can break the integral into 'pieces', as in Example 7C.7. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea*.

Theorem 7C.8. *Suppose $F : [0, L] \longrightarrow \mathbb{R}$ is a piecewise-linear function like (7C.14). Then the Fourier coefficients of F are given:*

$$\frac{1}{L}\int_0^L F(x) \quad = \quad \frac{1}{L}\sum_{m=1}^{M} \frac{a_m}{2}(x_m - x_{m-1})^2 \; + \; b_m \cdot (x_m - x_{m-1}).$$

$$\frac{2}{L}\int_0^L F(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx \quad = \quad \frac{2L}{(\pi n)^2}\sum_{m=1}^{M}\cos\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_m - a_{m+1}\Big)$$

$$\frac{2}{L}\int_0^L F(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx \quad = \quad \frac{2L}{(\pi n)^2}\sum_{m=1}^{M-1}\sin\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_m - a_{m+1}\Big)$$

(where we define $a_{M+1} := a_1$ for convenience).

Ⓔ *Proof.* **Exercise 7C.10** Hint: invoke Theorem 7C.5 and integration by parts. □

Note that the summands in this theorem read "$a_m - a_{m+1}$", not the other way around.

Example 7C.9: (Cosine series of the tent function)

Let Let $\mathbb{X} = [0, \pi]$ and let $f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2}; \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$ as in Example 7C.7. The Fourier cosine series of f is:

$$\frac{\pi}{4} - \frac{8}{\pi}\sum_{\substack{n=1 \\ n=4j+2, \\ \text{for some } j}}^{\infty} \frac{1}{n^2}\cos(nx).$$

In other words,

$$f(x) \quad = \quad \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{\cos(2x)}{4} + \frac{\cos(6x)}{36} + \frac{\cos(10x)}{100} + \frac{\cos(14x)}{196} + \frac{\cos(18x)}{324} + \ldots\right)$$

To see this, first observe that

$$\begin{aligned} A_0 \quad &= \quad \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad = \quad \frac{1}{\pi}\left(\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right) \\ &= \quad \frac{1}{\pi}\left(\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{\pi^2}{2} - \left[\frac{x^2}{2}\right]_{\pi/2}^{\pi}\right) \quad = \quad \frac{1}{\pi}\left(\frac{\pi^2}{8} + \frac{\pi^2}{2} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right) \\ &= \quad \frac{\pi^2}{4\pi} \quad = \quad \frac{\pi}{4}. \end{aligned}$$

Now let's compute A_n for $n > 0$.

$$\text{First,}\quad \int_0^{\pi/2} x\cos(nx)\,dx \;=\; \frac{1}{n}\left[x\sin(nx)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(nx)\,dx\right]$$
$$= \frac{1}{n}\left[\frac{\pi}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}\cos(nx)\Big|_0^{\pi/2}\right]$$
$$= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}.$$

$$\text{Next,}\quad \int_{\pi/2}^{\pi} x\cos(nx)\,dx \;=\; \frac{1}{n}\left[x\sin(nx)\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin(nx)\,dx\right]$$
$$= \frac{1}{n}\left[\frac{-\pi}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}\cos(nx)\Big|_{\pi/2}^{\pi}\right]$$
$$= \frac{-\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right).$$

$$\text{Finally,}\quad \int_{\pi/2}^{\pi} \pi\cos(nx)\,dx \;=\; \frac{\pi}{n}\sin(nx)\Big|_{\pi/2}^{\pi}$$
$$= \frac{-\pi}{n}\sin\left(\frac{n\pi}{2}\right).$$

Putting it all together, we have:

$$\int_0^{\pi} f(x)\cos(nx)\,dx \;=\; \int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}\pi\cos(nx)\,dx - \int_{\pi/2}^{\pi} x\cos(nx)\,dx$$
$$= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right)$$
$$+ \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)$$
$$= \frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1+(-1)^n}{n^2}.$$

Now,

$$\cos\left(\frac{n\pi}{2}\right) \;=\; \begin{cases} (-1)^k & \text{if } n \text{ is even and } n = 2k; \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$
$$\text{while}\quad 1+(-1)^n \;=\; \begin{cases} 2 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus,

$$2\cos\left(\frac{n\pi}{2}\right) - \left(1+(-1)^n\right) \;=\; \begin{cases} -4 & \text{if } n \text{ is even, } n = 2k \text{ and } k = 2j+1 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} -4 & \text{if } n = 4j+2 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}$$

(for example, $n = 2, 6, 10, 14, 18, \ldots$). Thus $A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$

$$= \begin{cases} \frac{-8}{n^2\pi} & \text{if } n = 4j+2 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases} \qquad \diamondsuit$$

7C(v) Differentiating Fourier (co)sine series

Prerequisites: §7B, §0F.

Suppose $f(x) = 3\sin(x) - 5\sin(2x) + 7\sin(3x)$. Then $f'(x) = 3\cos(x) - 10\cos(2x) + 21\cos(3x)$. Likewise, if $f(x) = 3 + 2\cos(x) - 6\cos(2x) + 11\cos(3x)$, then $f'(x) = -2\sin(x) + 12\sin(2x) - 33\sin(3x)$. This illustrates a general pattern.

Theorem 7C.10. *Suppose $f \in \mathcal{C}^\infty[0, L]$*

(a) *Suppose f has Fourier sine series $\sum_{n=1}^\infty B_n\mathbf{S}_n(x)$. If $\sum_{n=1}^\infty n|B_n| < \infty$, then f' has Fourier cosine series: $f'(x) = \frac{\pi}{L}\sum_{n=1}^\infty nB_n\mathbf{C}_n(x)$, and this series converges uniformly.*

(b) *Suppose f has Fourier cosine series $\sum_{n=0}^\infty A_n\mathbf{C}_n(x)$. If $\sum_{n=1}^\infty n|A_n| < \infty$, then f' has Fourier sine series: $f'(x) = \frac{-\pi}{L}\sum_{n=1}^\infty nA_n\mathbf{S}_n(x)$, and this series converges uniformly.*

Ⓔ *Proof.* **<u>Exercise 7C.11</u>** Hint: Apply Proposition 0F.1 on page 565. □

Consequence: If $f(x) = A\cos\left(\frac{n\pi x}{L}\right) + B\sin\left(\frac{n\pi x}{L}\right)$ for some $A, B \in \mathbb{R}$, then $f''(x) = -\left(\frac{n\pi}{L}\right)^2 \cdot f(x)$. In other words, f is an **eigenfunction**[1] for the differentation operator ∂_x^2, with eigenvalue $\lambda = -\left(\frac{n\pi}{L}\right)^2$. More generally, for any $k \in \mathbb{N}$, we have $\partial_x^{2k} f = \lambda^k \cdot f$.

7D Practice problems

In all of these problems, the domain is $\mathbb{X} = [0, \pi]$.

[1]See § 4B(iv) on page 63

1. Let $\alpha > 0$ be a constant. Compute the Fourier *sine* series of $f(x) = \exp(\alpha \cdot x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

2. Compute the Fourier *cosine* series of $f(x) = \sinh(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

3. Let $\alpha > 0$ be a constant. Compute the Fourier *sine* series of $f(x) = \cosh(\alpha x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

4. Compute the Fourier *cosine* series of $f(x) = x$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

5. Let $g(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \end{cases}$ (Fig. 7C.4A on p. 154)

   (a) Compute the Fourier *cosine* series of $g(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

   (b) Compute the Fourier *sine* series of $g(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

6. Compute the Fourier *cosine* series of $g(x) = \begin{cases} 3 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} \leq x \end{cases}$

   At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

7. Compute the Fourier *sine* series of $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases}$

   (Fig. 7C.4B on p.154) At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

   Hint: Note that $\displaystyle\int_0^{\pi} f(x)\sin(nx)\,dx = \int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx.$

8. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ be defined: $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$.

   Compute the Fourier **sine** series for $f(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

Chapter 8

Real Fourier series and complex Fourier series

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." —Bertrand Russell

8A Real Fourier series on $[-\pi, \pi]$

Prerequisites: §6E, §6F. **Recommended:** §7A, §5C(iv).

Throughout this section, for all $n \in \mathbb{N}$, we define the functions $\mathbf{S}_n : [-\pi, \pi] \longrightarrow \mathbb{R}$ and $\mathbf{C}_n : [-\pi, \pi] \longrightarrow \mathbb{R}$ by $\mathbf{S}_n(x) := \sin(nx)$ and $\mathbf{C}_n(x) := \cos(nx)$, for all $x \in [-\pi, \pi]$ (see Figure 6D.1 on page 113). If $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is any function with $\|f\|_2 < \infty$, we define the **(real) Fourier coefficients:**

$$\begin{aligned}
A_0 &:= \langle f, \ \mathbf{C}_0 \rangle \quad = \quad \langle f, \mathbb{1} \rangle \quad = \quad \boxed{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx,} \\
A_n &:= \frac{\langle f, \mathbf{C}_n \rangle}{\|\mathbf{C}_n\|_2^2} \quad = \quad \boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ \ dx,} \\
\text{and} \quad B_n &:= \frac{\langle f, \mathbf{S}_n \rangle}{\|\mathbf{S}_n\|_2^2} \quad = \quad \boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ \ dx,} \quad \text{for all } n \geq 1.
\end{aligned}$$

The **(real) Fourier series** of f is then the infinite summation of functions:

$$\boxed{A_0 \ + \ \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x) + \ + \ \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x).} \tag{8A.1}$$

We define *continuously differentiable* and *piecewise continuously differentiable* functions on $[-\pi, \pi]$ in a manner exactly analogous to the definitions on $[0, \pi]$

(page 138). Let $\mathcal{C}^1[-\pi,\pi]$ be the set of all continuously differentiable functions $f:[-\pi,\pi]\longrightarrow\mathbb{R}$.

Ⓔ **Exercise 8A.1.** (a) Show that any continuously differentiable function has finite L^2-norm. In other words, $\mathcal{C}^1[-\pi,\pi]\subset\mathbf{L}^2[\pi,\pi]$.

(b) Show that any piecewise $\mathcal{C}^1$ function on $[-\pi,\pi]$ is in $\mathbf{L}^2[-\pi,\pi]$. ♦

Theorem 8A.1. Fourier Convergence on $[-\pi,\pi]$

(a) *The set $\{\mathbb{1},\mathbf{S}_1,\mathbf{C}_1,\mathbf{S}_2,\mathbf{C}_2,\ldots\}$ is an* orthogonal basis *for $\mathbf{L}^2[-\pi,\pi]$. Thus, if $f\in\mathbf{L}^2[-\pi,\pi]$, then the Fourier series (8A.1) converges to f in L^2-norm.*

Furthermore, the coefficient sequences $\{A_n\}_{n=0}^\infty$ and $\{B_n\}_{n=1}^\infty$ are the unique *sequences of coefficients with this property. In other words, if $\{A'_n\}_{n=0}^\infty$ and $\{B'_n\}_{n=1}^\infty$ are two other sequences of coefficients such that $f \underset{\text{L2}}{\approx} \sum_{n=0}^\infty A'_n\mathbf{C}_n+\sum_{n=1}^\infty B'_n\mathbf{S}_n$, then we must have $A'_n=A_n$ and $B'_n=B_n$ for all $n\in\mathbb{N}$.*

(b) *If $f\in\mathcal{C}^1[-\pi,\pi]$ then the Fourier series (8A.1) converges* pointwise *on $(-\pi,\pi)$.*

More generally, if f is piecewise $\mathcal{C}^1$, then the real Fourier series (8A.1) converges to f pointwise *on each $\mathcal{C}^1$ interval for f. In other words, if $\{j_1,\ldots,j_m\}$ is the set of discontinuity points of f and/or f' in $[-\pi,\pi]$, and $j_m<x<j_{m+1}$, then $f(x) = A_0+\lim_{N\to\infty}\sum_{n=1}^N\Big(A_n\cos(nx)+B_n\sin(nx)\Big)$.*

(c) *If $\sum_{n=0}^\infty|A_n|+\sum_{n=1}^\infty|B_n| < \infty$, then the series (8A.1) converges to f* uniformly *on $[-\pi,\pi]$.*

(d) *Suppose $f:[-\pi,\pi]\longrightarrow\mathbb{R}$ is continuous and piecewise differentiable, $f'\in\mathbf{L}^2[-\pi,\pi]$, and $f(-\pi)=f(\pi)$. Then the series (8A.1) converges to f* uniformly *on $[-\pi,\pi]$.*

(e) *If f is piecewise $\mathcal{C}^1$, and $\mathbb{K}\subset(j_m,j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the series (8A.1) converges* uniformly *to f on $\mathbb{K}$.*

Proof. For a proof of **(a)** see § 10D on page 207. For a proof of **(b)**, see § 10B on page 197. (Alternately, **(b)** follows immediately from **(e)**.) For a proof of **(d)** see § 10C on page 204.

Ⓔ **(c)** is **Exercise 8A.2** (Hint: Use the Weierstrass M-test, Proposition 6E.13 on page 129.)

(e) is **Exercise 8A.3** (Hint: use Theorem 8D.1(e) and Proposition 8D.2 on page 173). Ⓔ □

There is nothing special about the interval $[-\pi, \pi]$. Real Fourier series can be defined for functions on an interval $[-L, L]$ for any $L > 0$. We chose $L = \pi$ because it makes the computations simpler. If $L \neq \pi$, then we can define a Fourier series analogous to (8A.1) using the functions $\mathbf{S}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ and $\mathbf{C}_n(x) = \cos\left(\frac{n\pi x}{L}\right)$.

Exercise 8A.4. Let $L > 0$, and let $f : [-L, L] \longrightarrow \mathbb{R}$. Generalize all parts of Theorem 8A.1 to characterize the convergence of the real Fourier series of f. Ⓔ ♦

Remark. Please see Remark 8D.3 on page 174 for further technical remarks about the (non)convergence of real Fourier series, in situations where the hypotheses of Theorem 8A.1 are not satisfied.

8B Computing real Fourier coefficients

Prerequisites: §8A. **Recommended:** §7C.

When computing the real Fourier coefficient $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\ dx$ (or $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\ dx$), it is simpler to first compute the integral $\int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\ dx$ (or $\int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\ dx$), and then multiply the result by $\frac{1}{\pi}$. In this section, we review some useful techniques to compute this integral.

8B(i) Polynomials

Recommended: §7C(ii).

Theorem 8B.1. $\int_{-\pi}^{\pi} \sin(nx)\ dx \ = \ 0 \ = \ \int_{-\pi}^{\pi} \cos(nx)\ dx.$

For any $k \in \{1, 2, 3, \ldots\}$, we have the following recurrence relations:

- *If k is even, then:*

$$\int_{-\pi}^{\pi} x^k \cdot \sin(nx)\, dx \ = \ 0 \quad \text{and} \quad \int_{-\pi}^{\pi} x^k \cdot \cos(nx)\, dx \ = \ \frac{-k}{n}\int_{-\pi}^{\pi} x^{k-1} \cdot \sin(nx)\, dx.$$

- *If $k > 0$ is **odd**, then:*

$$\int_{-\pi}^{\pi} x^k \cdot \sin(nx)\ dx \quad = \quad \frac{2(-1)^{n+1}\pi^k}{n} \quad + \quad \frac{k}{n}\int_{-\pi}^{\pi} x^{k-1}\cdot\cos(nx)\ dx$$

and

$$\int_{-\pi}^{\pi} x^k \cdot \cos(nx)\ dx \quad = \quad 0.$$

Ⓔ *Proof.* **Exercise 8B.1** Hint: use integration by parts. □

Example 8B.2.

(a) $p(x) = x$. Since $k = 1$ is *odd*, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\cos(nx)\ dx \quad = \quad 0,$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)\ dx \quad = \quad \frac{2(-1)^{n+1}\pi^0}{n} \quad + \quad \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)\ dx \quad \underset{(*)}{=} \quad \frac{2(-1)^{n+1}}{n}.$$

where equality $(*)$ follows from case $k = 0$ in Theorem 8B.1.

(b) $p(x) = x^2$. Since $k = 2$ is *even*, we have, for all n,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\ dx \quad = \quad 0,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\ dx \quad = \quad \frac{-2}{n\pi}\int_{-\pi}^{\pi} x^1\cdot\sin(nx)\ dx \quad \underset{(*)}{=} \quad \frac{-2}{n}\left(\frac{2(-1)^{n+1}}{n}\right) \quad = \quad \frac{4(-1)^n}{n^2}.$$

where equality $(*)$ follows from the previous example. ◇

8B(ii) Step functions

Recommended: §7C(iii).

A function $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a **step function** (see Figure 8B.1(A)) if there are numbers $-\pi = x_0 < x_1 < x_2 < x_3 < \cdots < x_{M-1} < x_M = \pi$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ such that

$$F(x) \quad = \quad \begin{cases} a_1 & \text{if } -\pi \leq x \leq x_1; \\ a_2 & \text{if } x_1 < x \leq x_2; \\ \vdots & \vdots \\ a_m & \text{if } x_{m-1} < x \leq x_m; \\ \vdots & \vdots \\ a_M & \text{if } x_{M-1} < x \leq \pi. \end{cases} \qquad (8B.1)$$

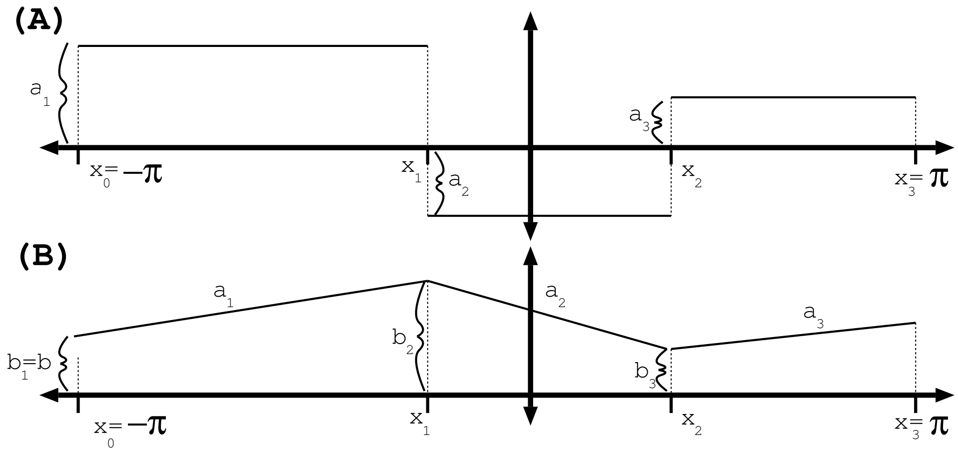


Figure 8B.1: (A) A step function. (B) A piecewise linear function.

To compute the Fourier coefficients of a step function, we break the integral into 'pieces'. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea*.

Theorem 8B.3. *Suppose $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a step function like* (8B.1). *Then the Fourier coefficients of F are given:*

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\ dx &= \frac{1}{2\pi}\sum_{m=1}^{M} a_m \cdot (x_m - x_{m-1});\\
\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\cos(nx)\ dx &= \frac{-1}{\pi n}\sum_{m=1}^{M-1} \sin(n\cdot x_m)\cdot\Big(a_{m+1} - a_m\Big);\\
\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\sin(nx)\ dx &= \frac{(-1)^n}{\pi n}\Big(a_1 - a_M\Big) + \frac{1}{\pi n}\sum_{m=1}^{M-1} \cos(n\cdot x_m)\cdot\Big(a_{m+1} - a_m\Big).
\end{aligned}$$

Proof. **Exercise 8B.2** Hint: Integrate the function piecewise. Use the fact that Ⓔ

$$\begin{aligned}
\int_{x_{m-1}}^{x_m} f(x)\sin(nx) &= \frac{a_m}{n}\Big(\cos(n\cdot x_{m-1}) - \cos(n\cdot x_m)\Big)\\
\text{and} \quad \int_{x_{m-1}}^{x_m} f(x)\cos(nx) &= \frac{a_m}{n}\Big(\cos(n\cdot x_m) - \cos(n\cdot x_{m-1})\Big).
\end{aligned}$$

□

Remark. Note that the Fourier series of a step function f will converge uniformly to f on the *interior* of each "step", but will *not* converge to f at any of the step boundaries, because f is not continuous at these points.

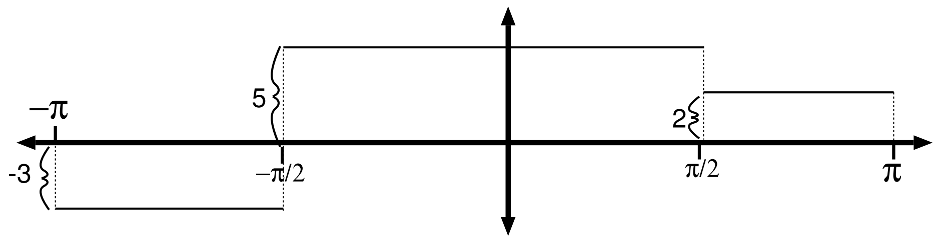


Figure 8B.2: The step function in Example 8B.4.

Example 8B.4. Suppose $f(x) = \begin{cases} -3 & \text{if} \quad -\pi \leq x < \frac{-\pi}{2}; \\ 5 & \text{if} \quad \frac{-\pi}{2} \leq x < \frac{\pi}{2}; \\ 2 & \text{if} \quad \frac{\pi}{2} \leq x \leq \pi. \end{cases}$ (see Figure 8B.2).

In the notation of Theorem 8B.3, we have $M = 3$, and

$$\begin{array}{rclrclrclrcl} x_0 & = & -\pi; & x_1 & = & \dfrac{-\pi}{2}; & x_2 & = & \dfrac{\pi}{2}; & x_3 & = & \pi; \\ & & & a_1 & = & -3; & a_2 & = & 5; & a_3 & = & 2. \end{array}$$

$$\begin{aligned} \text{Thus,} \qquad A_n &= \frac{-1}{\pi n}\left[8 \cdot \sin\left(n \cdot \frac{-\pi}{2}\right) - 3 \cdot \sin\left(n \cdot \frac{\pi}{2}\right)\right] \\ &= \begin{cases} 0 & \text{if} \quad n \text{ is even;} \\ (-1)^k \cdot \dfrac{11}{\pi n} & \text{if} \quad n = 2k+1 \text{ is odd.} \end{cases} \end{aligned}$$

$$\begin{aligned} \text{and} \qquad B_n &= \frac{1}{\pi n}\left[8 \cdot \cos\left(n \cdot \frac{-\pi}{2}\right) - 3 \cdot \cos\left(n \cdot \frac{\pi}{2}\right) - 5 \cdot \cos\left(n \cdot \pi\right)\right] \\ &= \begin{cases} \dfrac{5}{\pi n} & \text{if} \quad n \text{ is odd;} \\ \dfrac{5}{\pi n}\left((-1)^k - 1\right) & \text{if} \quad n = 2k \text{ is even.} \end{cases} \\ &= \begin{cases} \dfrac{5}{\pi n} & \text{if} \quad n \text{ is odd;} \\ 0 & \text{if} \quad n \text{ is divisible by } 4; \\ \dfrac{-10}{\pi n} & \text{if} \quad n \text{ is even but } \textit{not} \text{ divisible by } 4. \end{cases} \end{aligned} \qquad \diamondsuit$$

8B(iii) Piecewise linear functions

Recommended: §7C(iv).

A continuous function $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is **piecewise linear** (see Figure 8B.1(B)) if there are numbers $-\pi = x_0 < x_1 < x_1 < x_2 < \ldots < x_{M-1} < x_M = \pi$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$F(x) \quad = \quad \begin{cases} a_1(x-\pi) + b_1 & \text{if } -\pi < x < x_1; \\ a_2(x-x_1) + b_2 & \text{if } x_1 < x < x_2; \\ \vdots & \vdots \\ a_m(x-x_m) + b_{m+1} & \text{if } x_m < x < x_{m+1}; \\ \vdots & \vdots \\ a_M(x-x_{M-1}) + b_M & \text{if } x_{M-1} < x < \pi. \end{cases} \tag{8B.2}$$

where $b_1 = b$, and, for all $m > 1$, $b_m = a_m(x_m - x_{m-1}) + b_{m-1}$.

Example 8B.5. If $f(x) = |x|$, then f is piecewise linear, with: $x_0 = -\pi$, $x_1 = 0$, and $x_2 = \pi$; $a_1 = -1$ and $a_2 = 1$; $b_1 = \pi$, and $b_2 = 0$. ♢

To compute the Fourier coefficients of a piecewise linear function, we break the integral into 'pieces'. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea.*

Theorem 8B.6. *Suppose $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a piecewise-linear function like* (8B.2)*. Then the Fourier coefficients of F are given:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\ dx \quad = \quad \frac{1}{2\pi}\sum_{m=1}^{M} \frac{a_m}{2}\left(x_m - x_{m-1}\right)^2 \ + \ b_m \cdot \left(x_m - x_{m-1}\right);$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cdot \cos(nx)\ dx \quad = \quad \frac{1}{\pi n^2}\sum_{m=1}^{M} \cos(nx_m) \cdot \Big(a_m - a_{m+1}\Big);$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cdot \sin(nx)\ dx \quad = \quad \frac{1}{\pi n^2}\sum_{m=1}^{M-1} \sin(nx_m) \cdot \Big(a_m - a_{m+1}\Big).$$

(Here, we define $a_{M+1} := a_1$ for convenience.)

Proof. <u>**Exercise 8B.3**</u> Hint: invoke Theorem 8B.3 and integration by parts. □ Ⓔ

Note that the summands in this theorem read "$a_m - a_{m+1}$", not the other way around.

Example 8B.7. Recall $f(x) = |x|$, from Example 8B.5. Applying Theorem 8B.6, we have

$$A_0 \quad = \quad \frac{1}{2\pi}\left[\frac{-1}{2}(0+\pi)^2 + \pi\cdot(0+\pi) \ + \ \frac{1}{2}(\pi-0)^2 + 0\cdot(\pi-0)\right] \quad = \quad \frac{\pi}{2}.$$

$$A_n \quad = \quad \frac{\pi}{\pi n^2}\left[(-1-1)\cdot\cos\left(n0\right)(1+1)\cdot\cos\left(n\pi\right)\right]$$

$$= \quad \frac{1}{\pi n^2}\left[-2+2(-1)^n\right] \quad = \quad \frac{-2}{\pi n^2}\left[1-(-1)^n\right],$$

while $B_n = 0$ for all $n \in \mathbb{N}$, because f is an even function. ◊

8B(iv) Differentiating real Fourier series

Prerequisites: §8A, §0F.

Suppose $f(x) = 3 + 2\cos(x) - 6\cos(2x) + 11\cos(3x) + 3\sin(x) - 5\sin(2x) + 7\sin(3x)$. Then $f'(x) = -2\sin(x) + 12\sin(2x) - 33\sin(3x) + 3\cos(x) - 10\cos(2x) + 21\cos(3x)$. This illustrates a general pattern.

Theorem 8B.8. *Let $f \in \mathcal{C}^\infty[-\pi,\pi]$, and suppose f has Fourier series $\sum_{n=0}^{\infty} A_n \mathbf{C}_n + \sum_{n=1}^{\infty} B_n \mathbf{S}_n$. If $\sum_{n=1}^{\infty} n|A_n| < \infty$ and $\sum_{n=1}^{\infty} n|B_n| < \infty$, then f' has Fourier Series: $\sum_{n=1}^{\infty} n\left(B_n \mathbf{C}_n - A_n \mathbf{S}_n\right)$.*

Ⓔ *Proof.* **Exercise 8B.4** Hint: Apply Proposition 0F.1 on page 565. □

Consequence: If $f(x) = A\cos(nx) + B\sin(nx)$ for some $A, B \in \mathbb{R}$, then $f''(x) = -n^2 f(x)$. In other words, f is an **eigenfunction** for the differentation operator ∂_x^2, with eigenvalue $-n^2$. Hence, for any $k \in \mathbb{N}$, we have $\partial_x^{2k} f = (-n)^k \cdot f$.

8C Relation between (co)sine series and real series

Prerequisites: §7A, §8A.

We have seen in §8A how the collection $\{\mathbf{C}_n\}_{n=0}^{\infty} \cup \{\mathbf{S}_n\}_{n=1}^{\infty}$ forms an orthogonal basis for $\mathbf{L}^2[-\pi,\pi]$. However, if we confine our attention to *half* this interval —that is, to $\mathbf{L}^2[0,\pi]$ —then the results of §7A imply that we only need half as many basis elements; either the collection $\{\mathbf{C}_n\}_{n=0}^{\infty}$ or the collection $\{\mathbf{S}_n\}_{n=1}^{\infty}$ will suffice. Why is this? And what is the relationship between the Fourier (co)sine series of §7A and the Fourier series of §8A?

A function $f : [-L, L] \longrightarrow \mathbb{R}$ is **even** if $f(-x) = f(x)$ for all $x \in [0, L]$. For example, the following functions are even:

- $f(x) = 1$.
- $f(x) = |x|$.
- $f(x) = x^2$.
- $f(x) = x^k$ for any even $k \in \mathbb{N}$.
- $f(x) = \cos(x)$.

A function $f : [-L, L] \longrightarrow \mathbb{R}$ is **odd** if $f(-x) = -f(x)$ for all $x \in [0, L]$. For example, the following functions are odd:

- $f(x) = x$.
- $f(x) = x^3$.
- $f(x) = x^k$ for any odd $k \in \mathbb{N}$.
- $f(x) = \sin(x)$.

Every function can be 'split' into an 'even part' and an 'odd part'.

Proposition 8C.1. (a) *For any $f : [-L, L] \longrightarrow \mathbb{R}$, there is a unique even function $\check{f}$ and a unique odd function $\acute{f}$ such that $f = \check{f} + \acute{f}$. To be specific:*

$$\check{f}(x) \;=\; \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad \acute{f}(x) \;=\; \frac{f(x) - f(-x)}{2}$$

(b) *If f is even, then $f = \check{f}$, and $\acute{f} = 0$.*
(c) *If f is odd, then $\check{f} = 0$, and $f = \acute{f}$.*

Proof. **Exercise 8C.1** □ Ⓔ

The equation $f = \check{f} + \acute{f}$ is called the **even-odd decomposition** of f. Next, we define the vector spaces:

$$\begin{aligned} \mathbf{L}^2_{\mathsf{even}}[-\pi, \pi] &:= \{\text{all } \textit{even} \text{ elements in } \mathbf{L}^2[-\pi, \pi]\}. \\ \text{and} \quad \mathbf{L}^2_{\mathsf{odd}}[-\pi, \pi] &:= \{\text{all } \textit{odd} \text{ elements in } \mathbf{L}^2[-\pi, \pi]\}. \end{aligned}$$

Proposition 8C.1 implies that any $f \in \mathbf{L}^2[-\pi, \pi]$ can be written (in a unique way) as $f = \check{f} + \acute{f}$ for some $\check{f} \in \mathbf{L}^2_{\mathsf{even}}[-\pi, \pi]$ and $\acute{f} \in \mathbf{L}^2_{\mathsf{odd}}[-\pi, \pi]$. (This is sometimes indicated by writing: $\mathbf{L}^2[-\pi, \pi] = \mathbf{L}^2_{\mathsf{even}}[-\pi, \pi] \oplus \mathbf{L}^2_{\mathsf{odd}}[-\pi, \pi]$.)

Lemma 8C.2. *Let $n \in \mathbb{N}$.*

(a) *The function $\mathbf{C}_n(x) = \cos(nx)$ is even.*

(b) *The function $\mathbf{S}_n(x) = \sin(nx)$ is odd.*

Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be any function.

(c) *If $f(x) = \sum_{n=0}^{\infty} A_n \mathbf{C}_n(x)$, then f is even.*

(d) *If $f(x) = \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$, then f is odd.*

Proof. **Exercise 8C.2** □ Ⓔ

In other words, cosine series are even, and sine series are odd. The converse is also true. To be precise:

Proposition 8C.3. *Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be any function, and suppose f has real Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x) + \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$. Then:*

(a) *If f is odd, then $A_n = 0$ for every $n \in \mathbb{N}$.*

(b) *If f is even, then $B_n = 0$ for every $n \in \mathbb{N}$.*

Ⓔ *Proof.* **Exercise 8C.3** □

From this, it follows immediately:

Proposition 8C.4.

(a) *The set $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2_{\text{even}}[-\pi, \pi]$ (where $\mathbf{C}_0 = \mathbb{1}$).*

(b) *The set $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2_{\text{odd}}[-\pi, \pi]$.*

(c) *Suppose f has even-odd decomposition $f = \check{f} + \acute{f}$, and f has real Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x) + \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$. Then $\check{f}(x) = \sum_{n=0}^{\infty} A_n \mathbf{C}_n(x)$ and $\acute{f}(x) = \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$.*

Ⓔ *Proof.* **Exercise 8C.4** □

If $f : [0, \pi] \longrightarrow \mathbb{R}$, then we can "extend" f to a function on $[-\pi, \pi]$ in two ways:

- The **even** extension of f is defined: $f_{\text{even}}(x) = f(|x|)$ for all $x \in [-\pi, \pi]$.
- The **odd** extension of f is defined: $f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x < 0 \end{cases}$

Ⓔ **Exercise 8C.5.** (a) Show that f_{even} is even and f_{odd} is odd.
(b) For all $x \in [0, \pi]$, show that $f_{\text{even}}(x) = f(x) = f_{\text{odd}}(x)$. ♦

Proposition 8C.5. *Let $f : [0, \pi] \longrightarrow \mathbb{R}$ have even extension $f_{\mathsf{even}} : [-\pi, \pi] \longrightarrow \mathbb{R}$ and odd extension $f_{\mathsf{odd}} : [-\pi, \pi] \longrightarrow \mathbb{R}$.*

(a) *The Fourier* sine *series for f is the same as the* real *Fourier series for f_{odd}. In other words, the nth Fourier sine coefficient is given: $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{\mathsf{odd}}(x)\mathbf{S}_n(x)\, dx$.*

(b) *The Fourier* cosine *series for f is the same as the* real *Fourier series for f_{even}. In other words, the nth Fourier cosine coefficient is given: $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{\mathsf{even}}(x)\mathbf{C}_n(x)\, dx$.*

Proof. **Exercise 8C.6** □ Ⓔ

Let $f \in \mathcal{C}^1[0, \pi]$. Recall that Theorem 7A.1(d) (on page 138) says that the Fourier sine series of f converges to f *uniformly* on $[0, \pi]$ if and only if f satisfies homogeneous Dirichlet boundary conditions on $[0, \pi]$ (i.e. $f(0) = f(\pi) = 0$). On the other hand, Theorem 7A.4(d) (on page 142) says that the Fourier cosine series of f *always* converges to f uniformly on $[0, \pi]$ if $f \in \mathcal{C}^1[0, \pi]$; furthermore, if the formal derivative of this cosine series converges to f' uniformly on $[0, \pi]$, then f satisfies homogeneous *Neumann* boundary conditions on $[0, \pi]$ (i.e. $f'(0) = f'(\pi) = 0$). Meanwhile, if $F \in \mathcal{C}^1[-\pi, \pi]$, then Theorem 8A.1(d) (on page 162) says that the (real) Fourier series of F converges to F uniformly on $[-\pi, \pi]$ if F satisfies *periodic* boundary conditions on $[-\pi, \pi]$ (i.e. $F(-\pi) = F(\pi)$). The next result explains the logical relationship between these three statements.

Lemma 8C.6. *Let $f : [0, \pi] \longrightarrow \mathbb{R}$ have even extension $f_{\mathsf{even}} : [-\pi, \pi] \longrightarrow \mathbb{R}$ and odd extension $f_{\mathsf{odd}} : [-\pi, \pi] \longrightarrow \mathbb{R}$. Suppose f is right-continuous at 0 and left-continuous at π.*

(a) *f_{odd} is continuous at zero and satisfies periodic boundary conditions on $[-\pi, \pi]$, if and only if f satisfies homogeneous Dirichlet boundary conditions on $[0, \pi]$.*

(b) *f_{even} is* always *continuous at zero and* always *satisfies periodic boundary conditions on $[-\pi, \pi]$.*

However, the derivative f'_{even} is continuous at zero and satisfies periodic boundary conditions on $[-\pi, \pi]$ if and only if f satisfies homogeneous Neumann boundary conditions on $[0, \pi]$.

Proof. **Exercise 8C.7** □ Ⓔ

8D Complex Fourier series

Prerequisites: §6C(i), §6E, §6F, §0C. **Recommended:** §8A.

Let $f, g : \mathbb{X} \longrightarrow \mathbb{C}$ be complex-valued functions. Recall from §6C(i) that we define their **inner product**:

$$\langle f, g \rangle \quad := \quad \frac{1}{M} \int_{\mathbb{X}} f(\mathbf{x}) \cdot \overline{g(\mathbf{x})} \, d\mathbf{x},$$

where M is the length/area/volume of domain $\mathbb{X}$. Once again,

$$\|f\|_2 \quad := \quad \langle f, f \rangle^{1/2} \; = \; \left(\frac{1}{M} \int_{\mathbb{X}} f(\mathbf{x}) \overline{f(\mathbf{x})} \, d\mathbf{x} \right)^{1/2} \; = \; \left(\frac{1}{M} \int_{\mathbb{X}} |f(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.$$

The concepts of orthogonality, L^2 distance, and L^2 convergence are exactly the same as before. Let $\mathbf{L}^2([-L, L]; \mathbb{C})$ be the set of all complex-valued functions $f : [-L, L] \longrightarrow \mathbb{C}$ with $\|f\|_2 < \infty$. For all $n \in \mathbb{Z}$, let

$$\boxed{\mathbf{E}_n(x) \quad := \quad \exp\left(\frac{\pi \mathbf{i} n x}{L} \right).}$$

(thus, $\mathbf{E}_0 = \mathbb{1}$ is the constant unit function). For all $n > 0$, notice that Euler's Formula (see page 551) implies:

$$\begin{array}{rcl} \mathbf{E}_n(x) & = & \mathbf{C}_n(x) \; + \; \mathbf{i} \cdot \mathbf{S}_n(x) \\ \text{and} \;\; \mathbf{E}_{-n}(x) & = & \mathbf{C}_n(x) \; - \; \mathbf{i} \cdot \mathbf{S}_n(x) \end{array} \tag{8D.1}$$

Ⓔ Also, note that $\langle \mathbf{E}_n, \mathbf{E}_m \rangle = 0$ if $n \neq m$, and $\|\mathbf{E}_n\|_2 = 1$ (**Exercise 8D.1**), so these functions form an *orthonormal set*.

If $f : [-L, L] \longrightarrow \mathbb{C}$ is any function with $\|f\|_2 < \infty$, then we define the **(complex) Fourier coefficients** of f:

$$\widehat{f}_n \quad := \quad \langle f, \mathbf{E}_n \rangle \quad = \quad \boxed{\frac{1}{2L} \int_{-L}^{L} f(x) \cdot \exp\left(\frac{-\pi \mathbf{i} n x}{L} \right) \, dx.} \tag{8D.2}$$

The **(complex) Fourier Series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=-\infty}^{\infty} \widehat{f}_n \cdot \mathbf{E}_n.} \tag{8D.3}$$

(note that in this sum, n ranges from $-\infty$ to ∞).

Theorem 8D.1. Complex Fourier Convergence

(a) *The set $\{\ldots, \mathbf{E}_{-1}, \mathbf{E}_0, \mathbf{E}_1, \ldots\}$ is an orthonormal basis for $\mathbf{L}^2([-L, L];\ \mathbb{C})$. Thus, if $f \in \mathbf{L}^2([-L, L];\ \mathbb{C})$, then the complex Fourier series* (8D.3) *converges to f in L^2-norm.*

Furthermore, $\{\widehat{f}_n\}_{n=-\infty}^{\infty}$ is the unique sequence of coefficients with this property.

(b) *If f is continuously differentiable*[1] *on $[-\pi, \pi]$, then the Fourier series* (8D.3) *converges pointwise on $(-\pi, \pi)$.*

More generally, if f is piecewise $\mathcal{C}^1$, then the complex Fourier series (8D.3) *converges to f pointwise on each $\mathcal{C}^1$ interval for f. In other words, if $\{j_1, \ldots, j_m\}$ is the set of discontinuity points of f and/or f' in $[-L, L]$, and $j_m < x < j_{m+1}$, then $f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}_n \mathbf{E}_n(x)$.*

(c) *If $\sum_{n=-\infty}^{\infty} \left|\widehat{f}_n\right| < \infty$, then the series* (8D.3) *converges to f uniformly on $[-\pi, \pi]$.*

(d) *Suppose $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is continuous and piecewise differentiable, $f' \in \mathbf{L}^2[-\pi, \pi]$, and $f(-\pi) = f(\pi)$. Then the series* (8D.3) *converges to f uniformly on $[-\pi, \pi]$.*

(e) *If f is piecewise $\mathcal{C}^1$, and $\mathbb{K} \subset (j_m, j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the series* (8D.3) *converges uniformly to f on $\mathbb{K}$.*

Proof. For **(a)** is **Exercise 8D.2** (Hint: Use Theorem 8A.1(a) on page 162 and Proposition 8D.2 below). Ⓔ

For a direct proof of (a), see [Kat76, §I.5.5, p.29-30].

(b) is **Exercise 8D.3** (Hint: (i) use Theorem 8A.1(b) on page 162 and Proposition 8D.2 below. (ii) For a second proof, derive **(b)** from from **(e)**.) Ⓔ

(c) is **Exercise 8D.4** (Hint: Use the Weierstrass M-test, Proposition 6E.13 on page 129.) Ⓔ

(d) is **Exercise 8D.5** (Hint: use Theorem 8A.1(d) on page 162 and Proposition 8D.2 below). Ⓔ

For a direct proof of **(d)** see [WZ77, Theorem 12.20, p.219].

For **(e)** see [Fol84, Theorem 8.43, p.256] or [Kat76, Corollary on p.53 of §II.2.2]. □

[1] This means that $f(x) = f_r(x) + \mathbf{i} f_i(x)$, where $f_r : [-L, L] \longrightarrow \mathbb{R}$ and $f_i : [-L, L] \longrightarrow \mathbb{R}$ are both continuously differentiable, real-valued functions.

Proposition 8D.2. Relation between Real and Complex Fourier Series
Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be a real-valued function, and let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be its real Fourier coefficients, as defined on page 161. We can also regard f as a complex-valued function; let $\{\widehat{f}_n\}_{n=-\infty}^{\infty}$ be the complex Fourier coefficients of f, as defined by equation (8D.2) on page 172. Let $n \in \mathbb{N}_+$. Then

(a) $\widehat{f}_n = \frac{1}{2}(A_n - \mathbf{i}B_n)$, *and* $\widehat{f}_{-n} = \overline{\widehat{f}_n} = \frac{1}{2}(A_n + \mathbf{i}B_n)$.

(b) *Thus,* $A_n = \widehat{f}_n + \widehat{f}_{-n}$, *and* $B_n = \mathbf{i}(\widehat{f}_n - \widehat{f}_{-n})$.

(c) $\widehat{f}_0 = A_0$.

Ⓔ *Proof.* **Exercise 8D.6** Hint: use the equations (8D.1). □

Ⓔ **Exercise 8D.7.** Show that Theorem 8D.1(a) and Theorem 8A.1(a) are equivalent, using the Proposition 8D.2. ♦

Remark 8D.3: Further remarks on Fourier convergence

(a) In Theorems 7A.1(b), 7A.4(b), 8A.1(b) and 8D.1(b), if x is a discontinuity point of f, then the Fourier (co)sine series converges to the average of the 'left-hand' and 'right-hand' limits of f at x, namely:

$$\frac{f(x-) + f(x+)}{2}, \quad \text{where} \quad f(x-) := \lim_{y \nearrow x} f(y) \text{ and } f(x+) := \lim_{y \searrow x} f(y).$$

(b) If the hypothesis of Theorems 7A.1(c), 7A.4(c), 8A.1(c) or 8D.1(c) is satisfied, then we say that the Fourier series (real, complex, sine or cosine) converges *absolutely*. (In fact, Theorems 7A.1(d)[i], 7A.4(d)[i], 8A.1(d) or 8D.1(d) can be strengthened to yield absolute convergence). Absolute convergence is stronger than uniform convergence, and functions with absolutely convergent Fourier series form a special class; see [Kat76, §I.6, p.31-33] for more information.

(c) In Theorems 7A.1(e), 7A.4(e), 8A.1(e) and 8D.1(e), we don't quite need f to be *differentiable* to guarantee uniform convergence of the Fourier (co)sine series. Let $\alpha > 0$ be a constant; we say that f is *α-Hölder continuous* on $[-\pi, \pi]$ if there is some $M < \infty$ such that,

$$\text{For all } x, y \in [0, \pi], \quad \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \leq M.$$

Bernstein's Theorem says: If f is α-Hölder continuous for some $\alpha > \frac{1}{2}$, then the Fourier series (real, complex, sine or cosine) of f will converge uniformly (indeed, absolutely) to f; see [Fol84, Theorem 8.39] or [Kat76, Thm 6.3 on p.32]. (If f was differentiable, then f would be α-Hölder continuous with $\alpha = 1$, so Bernstein's Theorem immediately implies Theorems 7A.1(e) and 7A.4(e).)

(d) The *total variation* of f is defined

$$\mathrm{var}(f) \quad := \quad \sup_{N \in \mathbb{N}} \sup_{-\pi \leq x_0 < \cdots < x_N \leq \pi} \sum_{n=1}^{N} \left| f(x_n) - f(x_{n-1}) \right| \quad \underset{(*)}{=} \quad \int_{-\pi}^{\pi} \left| f'(x) \right| \, dx.$$

Here, the supremum is taken over all finite increasing sequences $\{-\pi \leq x_0 < x_1 < \cdots < x_N \leq \pi\}$ (for any $N \in \mathbb{N}$), and equality $(*)$ is true if and only if f is continuously differentiable. *Zygmund's Theorem* says: if $\mathrm{var}(f) < \infty$ (i.e. f has *bounded variation*) and f is α-Hölder continuous for some $\alpha > 0$, then the Fourier series of f will converge uniformly (indeed, absolutely) to f on $[-\pi, \pi]$; see [Kat76, Thm 6.4 on p.33].

(e) However, merely being *continuous* is *not* sufficient for uniform Fourier convergence, or even pointwise convergence. There exists a continuous function $f : [0, \pi] \longrightarrow \mathbb{R}$ whose Fourier series does *not* converge pointwise on $(0, \pi)$ —i.e. the series diverges at some points in $(0, \pi)$; see [WZ77, Theorem 12.35, p.227] or [Kat76, Theorem 2.1, p.51]. Thus, Theorems 7A.1(b), 7A.4(b), 8A.1(b) and 8D.1(b) are *false* if we replace 'differentiable' with 'continuous'.

(f) Fix $p \in [1, \infty)$. For any $f : [-\pi, \pi] \longrightarrow \mathbb{C}$, we define the L^p-*norm* of f:

$$\|f\|_p \quad = \quad \left(\int_{-\pi}^{\pi} |f(x)|^p \, dx \right)^{1/p}.$$

(Thus, if $p = 2$, we get the familiar L^2-norm $\|f\|_2$). Let $\mathbf{L}^p[-\pi, \pi]$ be the set of all integrable functions $f : [-\pi, \pi] \longrightarrow \mathbb{C}$ such that $\|f\|_p < \infty$. Theorem 8D.1(a) say that, if $f \in \mathbf{L}^2[-\pi, \pi]$, then the complex Fourier series of f converges to f in L^2-norm. The Fourier series of f also converges in L^p-norm for any other $p \in (1, \infty)$. That is, for any $p \in (1, \infty)$ and any $f \in \mathbf{L}^p[-\pi, \pi]$, we have

$$\lim_{N \to \infty} \left\| f \; - \sum_{n=-N}^{N} \widehat{f}_n \mathbf{E}_n \right\|_p \quad = \quad 0.$$

See [Kat76, Theorem 1.5, p.50]. If $f \in \mathbf{L}^p[-\pi, \pi]$ is purely real-valued, then the same statement holds for the real Fourier series:

$$\lim_{N \to \infty} \left\| f \; - \; \left(A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n + \sum_{n=1}^{N} B_n \mathbf{S}_n \right) \right\|_p \quad = \quad 0.$$

To understand the significance of L^p-convergence, we remark that if p is very large, then L^p convergence is 'almost' the same as uniform convergence. Also:

Ⓔ

- If $p > q$, then $\mathbf{L}^p[-\pi, \pi] \subset \mathbf{L}^q[-\pi, \pi]$. (**Exercise 8D.8**).

  For example, if $f \in \mathbf{L}^3[-\pi, \pi]$, then it follows that $f \in \mathbf{L}^2[-\pi, \pi]$ (but not vice versa). If $f \in \mathbf{L}^2[-\pi, \pi]$, then it follows that $f \in \mathbf{L}^{3/2}[-\pi, \pi]$ (but not vice versa).

- If $p > q$, and the Fourier series of f converges to f in L^p-norm, then it also converges to f in L^q-norm; see e.g. [Fol84, Proposition 6.12, p.178].

  For example, if $f \in \mathbf{L}^2[\pi, \pi]$, then Theorem 8D.1(a) implies that the Fourier series of f converges to f in L^q-norm for all $q \in [1, 2]$. (However, if $q < 2$, then there are functions in $\mathbf{L}^q[-\pi, \pi]$ to which Theorem 8D.1(a) does not apply).

Finally, similar L^p-convergence statements hold for the Fourier (co)sine series of real-valued functions in $\mathbf{L}^p[0, \pi]$.

(g) The pointwise convergence of a Fourier series is a somewhat subtle and complicated business, once you depart from the realm of $\mathcal{C}^1$ functions. In particular, the Fourier series of continuous (but non-differentiable) functions can be badly behaved. This is perplexing, because we know that Fourier series converge in L^2 norm for any function in $\mathbf{L}^2[-\pi, \pi]$ (which includes all sorts of strange functions which are not differentiable anywhere). To bridge the gap between L^2 and pointwise convergence, a variety of other 'summation schemes' have been introduced for Fourier coefficients. These include:

- The *Cesáro mean* $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} S_N(f)$, where $S_N(f) := \sum_{n=-N}^{N} \widehat{f}_n \mathbf{E}_n$ is the Nth partial sum of the complex Fourier series (8D.3).

- The *Abel mean* $\lim_{r \nearrow 1} \sum_{n=-\infty}^{\infty} r^{|n|} \widehat{f}_n \mathbf{E}_n$.

These sums have somewhat nicer convergence properties than the 'standard' Fourier series (8D.3). (See § 18F on page 461 for further discussion of the Abel mean.)

(h) There is a close relationship between the Fourier series of complex-valued functions on $[-\pi, \pi]$, and the Laurent series of complex-analytic functions defined near the unit circle; see § 18E on page 454.

(i) Remark (h) and the periodic boundary conditions required for Theorem 8D.1(d) both suggest that the Fourier series 'wants' us to identify the interval $(-\pi, \pi]$ with the unit circle $\mathbb{S}$ in the complex plane, via the bijection $\phi : (-\pi, \pi] \longrightarrow \mathbb{S}$ defined by $\phi(x) = e^{ix}$. Now, $\mathbb{S}$ is an *abelian group* under the complex multiplication operator. That is: if $s, t \in \mathbb{S}$, then their product $s \cdot t$ is also in $\mathbb{S}$, the multiplicative inverse s^{-1} is in $\mathbb{S}$, and the identity element 1 is an element of $\mathbb{S}$. Furthermore, $\mathbb{S}$ is a compact subset of $\mathbb{C}$, and the multiplication operation is continuous with respect to the topology of $\mathbb{S}$. In summary, $\mathbb{S}$ is a *compact abelian topological group.* The functions $\{\mathbf{E}_n\}_{n=-\infty}^{\infty}$ are then *continuous homomorphisms* from $\mathbb{S}$ into $\mathbb{S}$ (these are called the *characters* of the group).

The existence of the Fourier series (8D.3) and the convergence properties enumerated in Theorem 8D.1 are actually a *consequence* of these facts. In fact, if $\mathbb{G}$ is *any* compact abelian topological group, then one can develop a version of Fourier analysis on $\mathbb{G}$. The *characters* of $\mathbb{G}$ are the continuous homomorphisms from $\mathbb{G}$ into the unit circle group $\mathbb{S}$. The set of all characters of $\mathbb{G}$ forms an orthonormal basis for $\mathbf{L}^2(\mathbb{G})$, so that almost any 'reasonable' function $f : \mathbb{G} \longrightarrow \mathbb{C}$ can be expressed as a complex-linear combination of these characters.

The study of Fourier series, their summability, and their generalizations to other compact abelian groups is called *harmonic analysis*, and is a crucial tool in many areas of mathematics, including the ergodic theory of dynamical systems and the representation theory of Lie groups. See [Fol84, Ch.8], [WZ77, Ch.12] or the book [Kat76] to learn more about this vast and fascinating area of mathematics.

Chapter 9

Multidimensional Fourier series

"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living."
—Henri Poincaré

9A ...in two dimensions

Prerequisites: §6E, §6F. **Recommended:** §7B.

Let $X, Y > 0$, and let $\mathbb{X} := [0, X] \times [0, Y]$ be an $X \times Y$ rectangle in the plane. Suppose $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a real-valued function of two variables. For all $n, m \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$, we define the **two-dimensional Fourier sine coefficients:**

$$B_{n,m} \quad := \quad \boxed{\frac{4}{XY} \int_0^X \int_0^Y f(x, y) \sin\left(\frac{\pi n x}{X}\right) \sin\left(\frac{\pi m y}{Y}\right)\, dx\, dy.}$$

The **two-dimensional Fourier sine series** of f is the doubly infinite summation:

$$\boxed{\sum_{n,m=1}^{\infty} B_{n,m} \sin\left(\frac{\pi n x}{X}\right) \sin\left(\frac{\pi m y}{Y}\right).} \tag{9A.1}$$

Notice that we are now summing over *two* independent indices, n and m.

Example 9A.1. Let $X = \pi = Y$, so that $\mathbb{X} = [0, \pi] \times [0, \pi]$, and let $f(x, y) = x \cdot y$. Then f has two-dimensional Fourier sine series:

$$4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m}}{nm} \sin(nx) \sin(my).$$

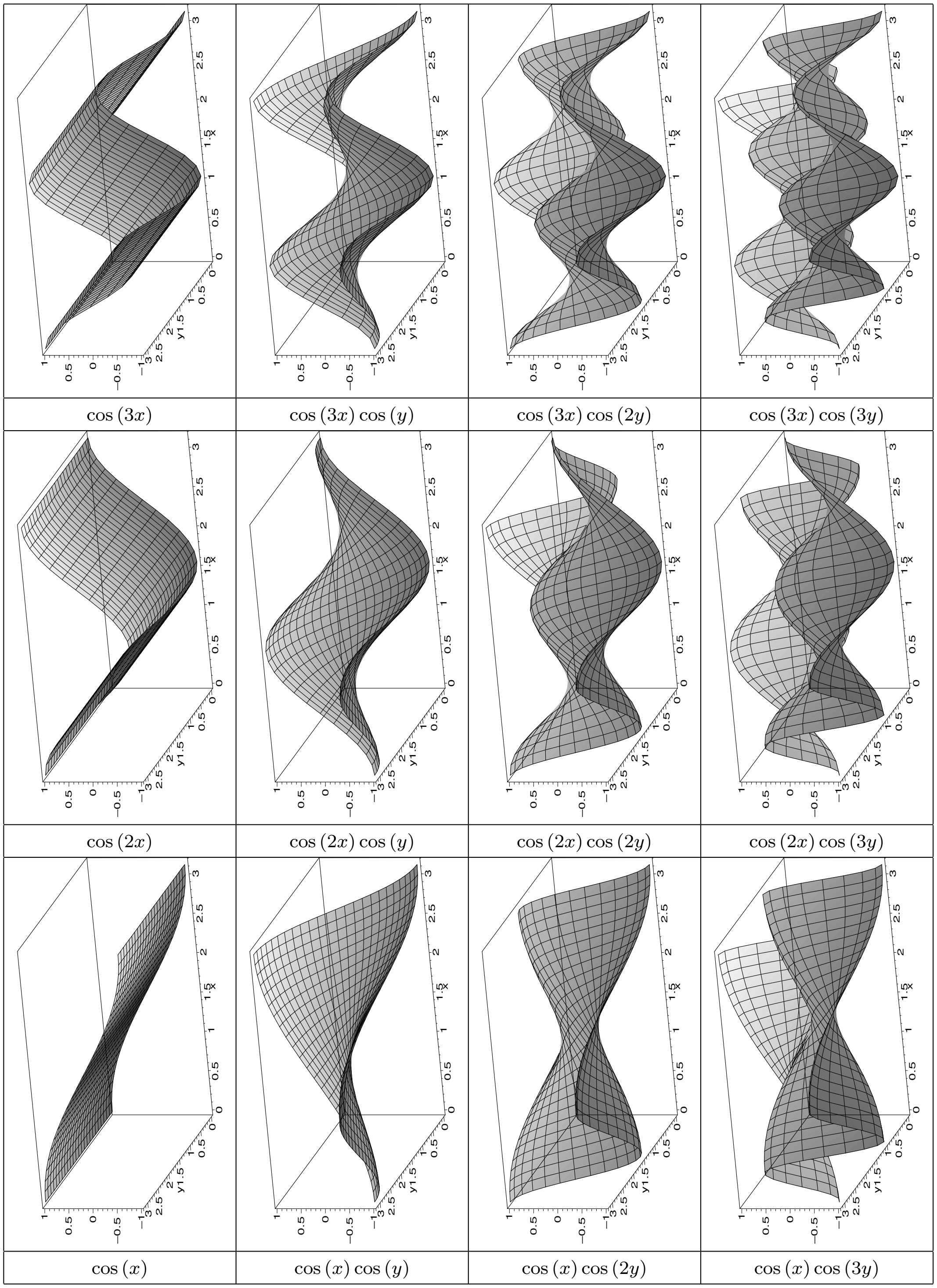


Figure 9A.1: $\mathbf{C}_{n,m}$ for $n = 1...3$ and $m = 0...3$ (rotate page).

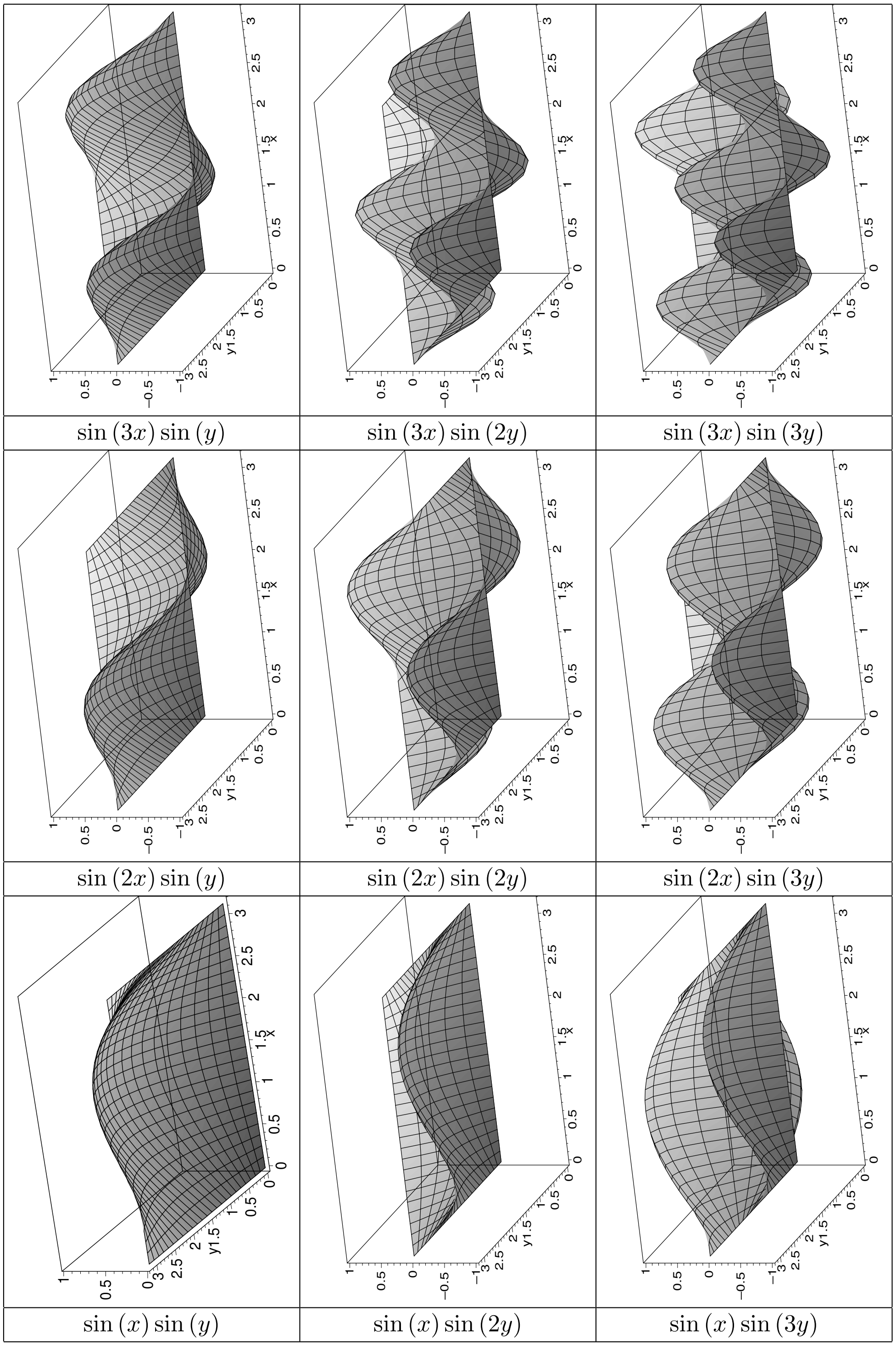


Figure 9A.2: $\mathbf{S}_{n,m}$ for $n = 1...3$ and $m = 1...3$ (rotate page).

To see this, recall from By Example 7C.2(c) on page 148, we know that

$$\frac{2}{\pi}\int_0^\pi x\sin(x)\ dx \ = \ \frac{2(-1)^{n+1}}{n}.$$

$$\begin{aligned}\text{Thus,}\quad B_{n,m} &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\cdot\sin(nx)\sin(my)\ dx\ dy \\ &= \left(\frac{2}{\pi}\int_0^\pi x\sin(nx)\ dx\right)\cdot\left(\frac{2}{\pi}\int_0^\pi y\sin(my)\ dy\right) \\ &= \left(\frac{2(-1)^{n+1}}{n}\right)\cdot\left(\frac{2(-1)^{m+1}}{m}\right) \quad = \quad \frac{4(-1)^{m+n}}{nm}\ . \qquad \diamondsuit\end{aligned}$$

Example 9A.2.

Let $X = \pi = Y$, so that $\mathbb{X} = [0,\pi]\times[0,\pi]$, and let $f(x,y) = 1$ be the constant 1 function. Then f has two-dimensional Fourier sine series:

$$\frac{4}{\pi^2}\sum_{n,m=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{[1-(-1)^m]}{m}\sin(nx)\sin(my) \quad = \quad \frac{16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty}\frac{1}{n\cdot m}\sin(nx)\sin(my)$$

Ⓔ **Exercise 9A.1** Verify this. ♢

For all $n,m\in\mathbb{N} := \{0,1,2,3,\ldots\}$, we define the **two-dimensional Fourier cosine coefficients** of f:

$$\begin{aligned} A_0 &:= \boxed{\frac{1}{XY}\int_0^X\int_0^Y f(x,y)\ dx\,dy,} \\ A_{n,0} &:= \boxed{\frac{2}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi nx}{X}\right)\ dx\,dy} \quad \text{for } n>0; \\ A_{0,m} &:= \boxed{\frac{2}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi my}{X}\right)\ dx\,dy} \quad \text{for } m>0;\ \text{ and} \\ A_{n,m} &:= \boxed{\frac{4}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi nx}{X}\right)\cos\left(\frac{\pi my}{Y}\right)\ dx\,dy} \quad \text{for } n,m>0. \end{aligned}$$

The **two-dimensional Fourier cosine series** of f is the doubly infinite summation:

$$\boxed{\sum_{n,m=0}^{\infty} A_{n,m}\cos\left(\frac{\pi nx}{X}\right)\cos\left(\frac{\pi my}{Y}\right).} \qquad (9A.2)$$

In what sense do these series converge to f? For any $n, m \in \mathbb{N}$, define the functions $\mathbf{C}_{n,m}, \mathbf{S}_{n,m} : [0,X] \times [0,Y] \longrightarrow \mathbb{R}$ by

$$\mathbf{C}_{n,m}(x,y) \quad := \quad \boxed{\cos\left(\frac{\pi n x}{X}\right) \cdot \cos\left(\frac{\pi m y}{Y}\right),}$$

$$\text{and} \quad \mathbf{S}_{n,m}(x,y) \quad := \quad \boxed{\sin\left(\frac{\pi n x}{X}\right) \cdot \sin\left(\frac{\pi m y}{Y}\right),}$$

for all $(x,y) \in [0,X] \times [0,Y]$ (see Figures 9A.1 and 9A.2).

Theorem 9A.3. Two-dimensional Co/Sine Series Convergence
Let $X, Y > 0$, and let $\mathbb{X} := [0,X] \times [0,Y]$.

(a) [i] *The set $\{\mathbf{S}_{n,m} \; ; \; n,m \in \mathbb{N}_+\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$.*

[ii] *The set $\{\mathbf{C}_{n,m} \; ; \; n,m \in \mathbb{N}\}$ is also an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$.*

[iii] *Thus, if $f \in \mathbf{L}^2(\mathbb{X})$, then the series (9A.1) and (9A.2) both converge to f in L^2-norm. Furthermore, the coefficient sequences $\{A_{n,m}\}_{n,m=0}^{\infty}$ and $\{B_{n,m}\}_{n,m=1}^{\infty}$ are the unique sequences of coefficients with this property.*

(b) *If $f \in \mathcal{C}^1(\mathbb{X})$ (i.e. f is continuously differentiable on $\mathbb{X}$), then the series (9A.1) and (9A.2) both converge to f pointwise on $(0,X) \times (0,Y)$.*

(c) [i] *If $\displaystyle\sum_{n,m=1}^{\infty} |B_{n,m}| < \infty$, then the two-dimensional Fourier sine series (9A.1) converges to f uniformly on $\mathbb{X}$.*

[ii] *If $\displaystyle\sum_{n,m=0}^{\infty} |A_{n,m}| < \infty$, then the two-dimensional Fourier cosine series (9A.2) converges to f uniformly on $\mathbb{X}$.*

(d) [i] *If $f \in \mathcal{C}^1(\mathbb{X})$, and the derivative functions $\partial_x f$ and $\partial_y f$ are both in $\mathbf{L}^2(\mathbb{X})$, and f satisfies homogeneous Dirichlet boundary conditions*[1] *on $\mathbb{X}$, then the two-dimensional Fourier sine series (9A.1) converges to f uniformly on $\mathbb{X}$.*

[ii] *Conversely, if the series (9A.1) converges to f uniformly on $\mathbb{X}$, then f is continuous and satisfies homogeneous Dirichlet boundary conditions.*

(e) [i] *If $f \in \mathcal{C}^1(\mathbb{X})$, the derivative functions $\partial_x f$ and $\partial_y f$ are both in $\mathbf{L}^2(\mathbb{X})$, then the two-dimensional Fourier cosine series (9A.2) converges to f uniformly on $\mathbb{X}$.*

[1] That is, $f(0,y) = 0 = f(X,y)$ for all $y \in [0,Y]$, and $f(x,0) = 0 = f(x,Y)$ for all $x \in [0,X]$.

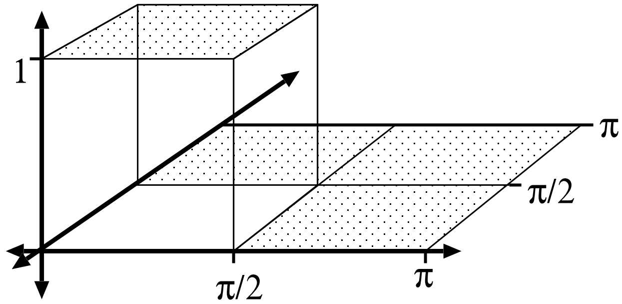


Figure 9A.3: The box function $f(x, y)$ in Example 9A.4.

[ii] *Conversely, if* $\sum_{n,m=1}^{\infty} n\,|A_{nm}| < \infty$ *and* $\sum_{n,m=1}^{\infty} m\,|A_{nm}| < \infty$, *then* $f \in \mathcal{C}^1(\mathbb{X})$, *and* f *satisfies homogeneous Neumann boundary conditions*[2] *on* $\mathbb{X}$.

Proof. This is just the case $D = 2$ of Theorem 9B.1 on page 187. □

Example 9A.4. Suppose $X = \pi = Y$, and $f(x, y) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \text{ and } 0 \le y < \frac{\pi}{2}; \\ 0 & \text{if } \frac{\pi}{2} \le x \text{ or } \frac{\pi}{2} \le y. \end{cases}$

(See Figure 9A.3). Then the two-dimensional Fourier cosine series of f is:

$$\frac{1}{4} \quad + \quad \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\Big((2k+1)x\Big) \quad + \quad \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \cos\Big((2j+1)y\Big)$$
$$+ \quad \frac{4}{\pi^2} \sum_{k,j=0}^{\infty} \frac{(-1)^{k+j}}{(2k+1)(2j+1)} \cos\Big((2k+1)x\Big) \cdot \cos\Big((2j+1)y\Big)$$

To see this, note that $f(x, y) = g(x)\cdot g(y)$, where $g(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \end{cases}$.
Recall from Example 7C.6 on page 154 that the (one-dimensional) Fourier cosine series of $g(x)$ is

$$g(x) \quad \underset{L2}{\approx} \quad \frac{1}{2} \quad + \quad \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\Big((2k+1)x\Big)$$

Thus, the cosine series for $f(x, y)$ is given:

$$f(x, y) \quad = \quad g(x) \cdot g(y)$$

[2]That is, $\partial_x f(0, y) = 0 = \partial_x f(X, y)$ for all $y \in [0, Y]$, and $\partial_y f(x, 0) = 0 = \partial_y f(x, Y)$ for all $x \in [0, X]$.

$$\underset{\text{L2}}{\approx} \left[\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)\right]\cdot\left[\frac{1}{2} + \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\cos\Big((2j+1)y\Big)\right].$$

$\diamondsuit$

Mixed Fourier series. (Optional)

We can also define the **mixed Fourier sine/cosine coefficients:**

$$\begin{aligned}
C^{[sc]}_{n,0} &:= \frac{2}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi n x}{X}\right)\,dx\,dy, \quad \text{for } n>0.\\
C^{[sc]}_{n,m} &:= \frac{4}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } n,m>0.\\
C^{[cs]}_{0,m} &:= \frac{2}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } m>0.\\
C^{[cs]}_{n,m} &:= \frac{4}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } n,m>0.
\end{aligned}$$

The **mixed Fourier sine/cosine series** of f are then:

$$\sum_{n=1,m=0}^{\infty} C^{[sc]}_{n,m}\sin\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right) \tag{9A.3}$$

$$\text{and} \quad \sum_{n=0,m=1}^{\infty} C^{[cs]}_{n,m}\cos\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right)$$

For any $n,m\in\mathbb{N}$, define the functions $\mathbf{M}^{[sc]}_{n,m}, \mathbf{M}^{[cs]}_{n,m} : [0,X]\times[0,Y] \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
\mathbf{M}^{[sc]}_{n,m}(x,y) &:= \sin\left(\frac{\pi n_1 x}{X}\right)\cos\left(\frac{\pi n_2 y}{Y}\right)\\
\text{and}\quad \mathbf{M}^{[cs]}_{n,m}(x,y) &:= \cos\left(\frac{\pi n_1 x}{X}\right)\sin\left(\frac{\pi n_2 y}{Y}\right).
\end{aligned}$$

for all $(x,y)\in[0,X]\times[0,Y]$.

Proposition 9A.5. Two-dimensional Mixed Co/Sine Series Convergence

Let $\mathbb{X} := [0,X]\times[0,Y]$. The sets of "mixed" functions, $\left\{\mathbf{M}^{[sc]}_{n,m}\ ;\ n\in\mathbb{N}_+,\ m\in\mathbb{N}\right\}$ and $\left\{\mathbf{M}^{[cs]}_{n,m}\ ;\ n\in\mathbb{N},\ m\in\mathbb{N}_+\right\}$ are both orthogonal basis for $\mathbf{L}^2(\mathbb{X})$. In other words, if $f\in\mathbf{L}^2(\mathbb{X})$, then the series (9A.3) both converge to f in L^2. $\square$

Exercise 9A.2. Formulate conditions for pointwise and uniform convergence of the mixed series. Ⓔ $\blacklozenge$

9B ...in many dimensions

Prerequisites: §6E, §6F. **Recommended:** §9A.

Let $X_1, \ldots, X_D > 0$, and let $\mathbb{X} := [0, X_1] \times \cdots \times [0, X_D]$ be an $X_1 \times \cdots \times X_D$ box in D-dimensional space. For any $\mathbf{n} \in \mathbb{N}^D$, define the functions $\mathbf{C_n} : \mathbb{X} \longrightarrow \mathbb{R}$ and $\mathbf{S_n} : \mathbb{X} \longrightarrow \mathbb{R}$ by

$$\mathbf{C_n}(x_1, \ldots, x_D) \quad := \quad \boxed{\cos\left(\frac{\pi n_1 x_1}{X_1}\right)\cos\left(\frac{\pi n_2 x_2}{X_2}\right)\cdots\cos\left(\frac{\pi n_D x_D}{X_D}\right),} \tag{9B.1}$$

$$\mathbf{S_n}(x_1, \ldots, x_D) \quad := \quad \boxed{\sin\left(\frac{\pi n_1 x_1}{X_1}\right)\sin\left(\frac{\pi n_2 x_2}{X_2}\right)\cdots\sin\left(\frac{\pi n_D x_D}{X_D}\right),} \tag{9B.2}$$

for any $\mathbf{x} = (x_1, x_2, \ldots, x_D) \in \mathbb{X}$. Also, for any sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_D)$ of D symbols "s" and "c", we can define the "mixed" functions, $\mathbf{M}_\mathbf{n}^{\boldsymbol{\omega}} : \mathbb{X} \longrightarrow \mathbb{R}$. For example, if $D = 3$, then define

$$\mathbf{M}_\mathbf{n}^{[scs]}(x, y, z) \; := \; \sin\left(\frac{\pi n_1 x}{X_x}\right)\cos\left(\frac{\pi n_2 y}{X_y}\right)\sin\left(\frac{\pi n_3 z}{X_z}\right).$$

If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is any function with $\|f\|_2 < \infty$, then, for all $\mathbf{n} \in \mathbb{N}_+^D$, we define the **multiple Fourier sine coefficients:**

$$B_\mathbf{n} \quad := \quad \frac{\langle f, \mathbf{S_n}\rangle}{\|\mathbf{S_n}\|_2^2} \quad = \quad \boxed{\frac{2^D}{X_1 \cdots X_D}\int_\mathbb{X} f(\mathbf{x}) \cdot \mathbf{S}_n(\mathbf{x}) \; d\mathbf{x}.}$$

The **multiple Fourier sine series** of f is then:

$$\boxed{\sum_{\mathbf{n}\in\mathbb{N}_+^D} B_\mathbf{n}\mathbf{S_n}}. \tag{9B.3}$$

For all $\mathbf{n} \in \mathbb{N}^D$, we define the **multiple Fourier cosine coefficients:**

$$A_0 \quad := \quad \langle f, \mathbb{1}\rangle \quad = \quad \boxed{\frac{1}{X_1 \cdots X_D}\int_\mathbb{X} f(\mathbf{x}) \; d\mathbf{x},}$$

$$\text{and} \quad A_\mathbf{n} \quad := \quad \frac{\langle f, \mathbf{C_n}\rangle}{\|\mathbf{C_n}\|_2^2} \quad = \quad \boxed{\frac{2^{d_\mathbf{n}}}{X_1 \cdots X_D}\int_\mathbb{X} f(\mathbf{x}) \cdot \mathbf{C}_n(\mathbf{x}) \; d\mathbf{x}.}$$

where, for each $\mathbf{n} \in \mathbb{N}^D$, the number $d_\mathbf{n}$ is the number of nonzero entries in $\mathbf{n} = (n_1, n_2, \ldots, n_D)$. The **multiple Fourier cosine series** of f is then:

$$\boxed{\sum_{\mathbf{n}\in\mathbb{N}^D} A_\mathbf{n}\mathbf{C_n}}, \qquad \text{where } \mathbb{N} := \{0, 1, 2, 3, \ldots\}. \tag{9B.4}$$

Finally, we define the **mixed Fourier Sine/Cosine coefficients:**

$$C_{\mathbf{n}}^{\boldsymbol{\omega}} \quad := \quad \frac{\langle f, \mathbf{M}_{\mathbf{n}}^{\boldsymbol{\omega}} \rangle}{\|\mathbf{M}_{\mathbf{n}}^{\omega}\|_2^2} \quad = \quad \frac{2^{d_{\mathbf{n}}}}{X_1 \cdots X_D} \int_{\mathbb{X}} f(\mathbf{x}) \cdot \mathbf{M}_n^{\boldsymbol{\omega}}(\mathbf{x}) \, d\mathbf{x},$$

where, for each $\mathbf{n} \in \mathbb{N}^D$, the number $d_{\mathbf{n}}$ is the number of nonzero entries n_i in $\mathbf{n} = (n_1, \ldots, n_D)$. The **mixed Fourier Sine/Cosine series** of f is then:

$$\boxed{\sum_{\mathbf{n} \in \mathbb{N}^D} C_{\mathbf{n}}^{\boldsymbol{\omega}} \mathbf{M}_{\mathbf{n}}^{\boldsymbol{\omega}}.} \tag{9B.5}$$

Theorem 9B.1. Multidimensional Co/Sine Series Convergence on $\mathbb{X}$

Let $\mathbb{X} := [0, X_1] \times \cdots \times [0, X_D]$ be a D-dimensional box.

(a) [i] *The set $\{\mathbf{S}_{\mathbf{n}} \; ; \; \mathbf{n} \in \mathbb{N}_+^D\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$.*

[ii] *The set $\{\mathbf{C}_{\mathbf{n}} \; ; \; \mathbf{n} \in \mathbb{N}^D\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$.*

[iii] *For any sequence $\boldsymbol{\omega}$ of D symbols "s" and "c", the set of "mixed" functions, $\{\mathbf{M}_{\mathbf{n}}^{\boldsymbol{\omega}} \; ; \; \mathbf{n} \in \mathbb{N}^D\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$.*

[iv] *In other words, if $f \in \mathbf{L}^2(\mathbb{X})$, then the series* (9B.3), (9B.4), *and* (9B.5) *all converge to f in L^2-norm. Furthermore, the coefficient sequences $\{A_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}^D}$, $\{B_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_+^D}$, and $\{C_{\mathbf{n}}^{\boldsymbol{\omega}}\}_{\mathbf{n} \in \mathbb{N}^D}$ are the unique sequences of coefficients with these properties.*

(b) *If $f \in \mathcal{C}^1(\mathbb{X})$* (i.e. f is continuously differentiable on $\mathbb{X}$), *then the series* (9B.3), (9B.4), *and* (9B.5) *converge pointwise on the interior of $\mathbb{X}$.*

(c) [i] *If $\sum_{\mathbf{n} \in \mathbb{N}_+^D} |B_{\mathbf{n}}| < \infty$, then the multidimensional Fourier sine series* (9B.3) *converges to f uniformly on $\mathbb{X}$.*

[ii] *If $\sum_{\mathbf{n} \in \mathbb{N}^D} |A_{\mathbf{n}}| < \infty$, then the multidimensional Fourier cosine series* (9B.4) *converges to f uniformly on $\mathbb{X}$.*

(d) [i] *If $f \in \mathcal{C}^1(\mathbb{X})$, and the derivative functions $\partial_k f$ are themselves in $\mathbf{L}^2(\mathbb{X})$ for all $k \in [1...D]$, and f satisfies homogeneous Dirichlet boundary conditions on $\mathbb{X}$, then the multidimensional Fourier sine series* (9B.3) *converges to f uniformly on $\mathbb{X}$.*

[ii] *Conversely, if the series* (9B.3) *converges to f uniformly on $\mathbb{X}$, then f is continuous and satisfies homogeneous Dirichlet boundary conditions.*

(e) [i] *If $f \in \mathcal{C}^1(\mathbb{X})$, and the derivative functions $\partial_k f$ are themselves in $\mathbf{L}^2(\mathbb{X})$ for all $k \in [1...D]$, then the multidimensional Fourier cosine series* (9B.4) *converges to f uniformly on $\mathbb{X}$.*

[ii] *Conversely, if* $\sum_{\mathbf{n}\in\mathbb{N}^D}(n_1+\cdots+n_D)\,|A_\mathbf{n}|<\infty$*, then* $f\in\mathcal{C}^1(\mathbb{X})$*, and* f *satisfies homogeneous Neumann boundary conditions.*

Ⓔ *Proof.* The proof of **(c)** is **Exercise 9B.1** (*Hint:* Use the Weierstrass M-test, Proposition 6E.13 on page 129.)

Ⓔ The proofs of **(d,e)**[ii] are **Exercise 9B.2** (*Hint:* Generalize the solutions to Exercises 7A.4 and 7A.9 on pages 139 and 142).

Ⓔ The proof of **(a)** is **Exercise 9B.3** (*Hint:* Prove this by induction on the dimension D. The base case ($D=1$) is Theorems 7A.1(a) and 7A.4(a) on pages 138 and 142. Use Lemma 15C.2(f) (on page 330) to handle the induction step.)

We will prove **(b)**, **(d)**[i] and **(e)**[i] by induction on the dimension D. The base cases ($D=1$) are Theorems 7A.1(b,d[i]) and 7A.4(b,d[i]) on pages 138 and 142.

For the induction step, suppose the theorem is true for D, and consider $D+1$. Let $\mathbb{X}:=[0,X_0]\times[0,X_1]\times\cdots\times[0,X_D]$ be a $(D+1)$-dimensional box. Note that $\mathbb{X}:=[0,X_0]\times\mathbb{X}^*$, where $\mathbb{X}^*:=[0,X_1]\times\cdots\times[0,X_D]$ is a D-dimensional box. If $f:\mathbb{X}\longrightarrow\mathbb{R}$, then for all $y\in[0,X_0]$, let $f^y:\mathbb{X}^*\longrightarrow\mathbb{R}$ be the function defined by $f^y(\mathbf{x}):=f(y,\mathbf{x})$ for all $\mathbf{x}\in\mathbb{X}^*$.

Claim 1: (a) *If* $f\in\mathcal{C}^1(\mathbb{X})$*, then* $f^y\in\mathcal{C}^1(\mathbb{X}^*)$ *for all* $y\in[0,X_0]$.

(b) *Furthermore, if* $\partial_k f\in\mathbf{L}^2(\mathbb{X})$ *for all* $k\in[1...D]$*, then* $\partial_k(f^y)\in\mathbf{L}^2(\mathbb{X}^*)$ *for all* $k\in[1...D]$ *and all* $y\in[0,X_0]$.

(c) *If* f *satisfies homogenous Dirichlet BC on* $\mathbb{X}$*, then* f^y *satisfies homogenous Dirichlet BC on* $\mathbb{X}^*$*, for all* $y\in[0,X_0]$.

Ⓔ *Proof.* **Exercise 9B.4** $\diamond_{\text{Claim 1}}$

For all $\mathbf{n}\in\mathbb{N}^D$, define $\mathbf{C}^*_\mathbf{n},\mathbf{S}^*_\mathbf{n}:\mathbb{X}^*\longrightarrow\mathbb{R}$ as in equations (9B.1) and (9B.2). For every $y\in[0,X_0]$, let

$$A^y_\mathbf{n} \;:=\; \frac{\langle f^y,\mathbf{C}^*_\mathbf{n}\rangle}{\|\mathbf{C}^*_\mathbf{n}\|_2^2} \quad\text{and}\quad B^y_\mathbf{n} \;:=\; \frac{\langle f^y,\mathbf{S}^*_\mathbf{n}\rangle}{\|\mathbf{S}^*_\mathbf{n}\|_2^2}$$

be the D-dimensional Fourier (co)sine coefficients for f^y, so that f^y has D-dimensional Fourier (co)sine series:

$$\sum_{\mathbf{n}\in\mathbb{N}^D_+} B^y_\mathbf{n}\,\mathbf{S}^*_\mathbf{n} \;\underset{\text{L2}}{\widetilde{\approx}}\; f^y \;\underset{\text{L2}}{\widetilde{\approx}}\; \sum_{\mathbf{n}\in\mathbb{N}^D} A^y_\mathbf{n}\,\mathbf{C}^*_\mathbf{n}. \tag{9B.6}$$

Claim 2: *For all* $y\in[0,X_0]$*, the two series in eqn.*(9B.6) *converge to* f^y *in the desired fashion* (i.e. pointwise or uniform) *on* $\mathbb{X}^*$.

Ⓔ *Proof.* **Exercise 9B.5** (Hint: Use the induction hypothesis and Claim 1). $\diamond_{\text{Claim 2}}$

Fix $\mathbf{n} \in \mathbb{N}^D$. Define $\alpha_\mathbf{n}, \beta_\mathbf{n} : [0, X_0] \longrightarrow \mathbb{R}$ by $\alpha_\mathbf{n}(y) := A^y_\mathbf{n}$ and $\beta_\mathbf{n}(y) := B^y_\mathbf{n}$ for all $y \in [0, X_0]$.

Claim 3: *For all* $\mathbf{n} \in \mathbb{N}^D$, $\alpha_\mathbf{n} \in \mathbf{L}^2[0, X_0]$ *and* $\beta_\mathbf{n} \in \mathbf{L}^2[0, X_0]$.

Proof. We have

$$\begin{aligned}
\|\alpha_\mathbf{n}\|_2^2 &= \frac{1}{X_0}\int_0^{X_0} |\alpha_\mathbf{n}(y)|^2 \ dy \quad = \quad \frac{1}{X_0}\int_0^{X_0} \left|\frac{\langle f^y, \mathbf{C}^*_\mathbf{n}\rangle}{\|\mathbf{C}^*_\mathbf{n}\|_2^2}\right|^2 \ dy \\
&= \frac{1}{X_0 \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^4}\int_0^{X_0} |\langle f^y, \mathbf{C}^*_\mathbf{n}\rangle|^2 \ dy \\
&\underset{(*)}{\leq} \frac{1}{X_0 \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^4}\int_0^{X_0} \|f^y\|_2^2 \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^2 \ dy \\
&= \frac{1}{X_0 \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^2}\int_0^{X_0} \|f^y\|_2^2 \ dy \\
&= \frac{1}{X_0 \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^2}\left(\int_0^{X_0} \frac{1}{X_1 \cdots X_D}\int_{\mathbb{X}^*} |f^y(\mathbf{x})|^2 \ d\mathbf{x}\right) \ dy \\
&= \frac{1}{X_0 \cdots X_D \cdot \|\mathbf{C}^*_\mathbf{n}\|_2^2}\int_{\mathbb{X}} |f(y, \mathbf{x})|^2 \ d(y; \mathbf{x}) \quad = \quad \frac{1}{\|\mathbf{C}^*_\mathbf{n}\|_2^2}\|f\|_2^2.
\end{aligned}$$

Here, $(*)$ is the Cauchy-Bunyakowski-Schwarz Inequality (Theorem 6B.5 on page 108).

Thus, $\|\alpha_\mathbf{n}\|_2^2 < \infty$ because $\|f\|_2^2 < \infty$ because $f \in \mathbf{L}^2(\mathbb{X})$ by hypothesis. Thus, $\alpha_\mathbf{n} \in \mathbf{L}^2[0, X_0]$. The proof that $\beta_\mathbf{n} \in \mathbf{L}^2[0, X_0]$ is similar. $\diamond_{\text{Claim 3}}$

For all $m \in \mathbb{N}$, define $\mathbf{S}_m, \mathbf{C}_m : [0, X_0] \longrightarrow \mathbb{R}$ by $\mathbf{S}_m(y) := \sin(\pi m y/X_0)$ and $\mathbf{C}_m(y) := \cos(\pi m y/X_0)$, for all $y \in [0, X_0]$. For all $m \in \mathbb{N}$, let

$$A^\mathbf{n}_m \quad := \quad \frac{\langle \alpha_\mathbf{n}, \mathbf{C}_m\rangle}{\|\mathbf{C}_m\|_2^2} \quad \text{and} \quad B^\mathbf{n}_m \quad := \quad \frac{\langle \beta_\mathbf{n}, \mathbf{S}_m\rangle}{\|\mathbf{S}_m\|_2^2}$$

be the one-dimensional Fourier (co)sine coefficients for the functions $\alpha_\mathbf{n}$ and $\beta_\mathbf{n}$, so that we get one-dimensional Fourier (co)sine series

$$\alpha_\mathbf{n} \underset{\text{L2}}{\approx} \sum_{m=0}^{\infty} A^\mathbf{n}_m \mathbf{C}_m \quad \text{and} \quad \beta_\mathbf{n} \underset{\text{L2}}{\approx} \sum_{m=1}^{\infty} B^\mathbf{n}_m \mathbf{S}_m. \tag{9B.7}$$

For all $\mathbf{n} \in \mathbb{N}^D$ and all $m \in \mathbb{N}$, define $\mathbf{S}_{m;\mathbf{n}}, \mathbf{C}_{m;\mathbf{n}} : \mathbb{X} \longrightarrow \mathbb{R}$ by $\mathbf{S}_{m;\mathbf{n}}(y; \mathbf{x}) := \mathbf{S}_m(y) \cdot \mathbf{S}_\mathbf{n}(\mathbf{x})$ and $\mathbf{C}_{m;\mathbf{n}}(y; \mathbf{x}) := \mathbf{C}_m(y) \cdot \mathbf{C}_\mathbf{n}(\mathbf{x})$, for all $y \in [0, X_0]$ and $\mathbf{x} \in \mathbb{X}^*$. Then let

$$A_{m;\mathbf{n}} \quad := \quad \frac{\langle f, \mathbf{C}_{m;\mathbf{n}}\rangle}{\|\mathbf{C}_{m;\mathbf{n}}\|_2^2} \quad \text{and} \quad B_{m;\mathbf{n}} \quad := \quad \frac{\langle f, \mathbf{S}_{m;\mathbf{n}}\rangle}{\|\mathbf{S}_{m;\mathbf{n}}\|_2^2}$$

be the $(D+1)$-dimensional Fourier (co)sine coefficients for the function f, so that we get $(D+1)$–dimensional Fourier (co)sine series

$$\sum_{m=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D} A_{m;\mathbf{n}} \mathbf{C}_{m;\mathbf{n}} \quad \widetilde{\underset{L2}{\approx}} \quad f \quad \widetilde{\underset{L2}{\approx}} \quad \sum_{m=1}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}_+^D} B_{m;\mathbf{n}} \mathbf{S}_{m;\mathbf{n}}. \tag{9B.8}$$

Claim 4: *For all $\mathbf{n} \in \mathbb{N}^D$ and all $m \in \mathbb{N}$, $A_m^{\mathbf{n}} = A_{m;\mathbf{n}}$ and $B_m^{\mathbf{n}} = B_{m;\mathbf{n}}$.*

Ⓔ *Proof.* **Exercise 9B.6** $\diamond_{\text{Claim 4}}$

Let $\partial_0 f$ be the derivative of f in the 0th (or 'y') coordinate, which we regard as a function $\partial_0 f : \mathbb{X} \longrightarrow \mathbb{R}$.

Claim 5: (a) *If $f \in \mathcal{C}^1(\mathbb{X})$, then for all $\mathbf{n} \in \mathbb{N}^D$, $\alpha_{\mathbf{n}} \in \mathcal{C}^1[0, X_0]$ and $\beta_{\mathbf{n}} \in \mathcal{C}^1[0, X_0]$.*

(b) *Furthermore, if $\partial_0 f \in \mathbf{L}^2(\mathbb{X})$, then for all $\mathbf{n} \in \mathbb{N}^D$, $\alpha'_{\mathbf{n}} \in \mathbf{L}^2[0, X_0]$ and $\beta'_{\mathbf{n}} \in \mathbf{L}^2[0, X_0]$.*

(c) *If f satisfies homogeneous Dirichlet BC on $\mathbb{X}$, then $\beta_{\mathbf{n}}$ satisfies homogeneous Dirichlet BC on $[0, X_0]$, for all $\mathbf{n} \in \mathbb{N}_+^D$.*

Proof. To prove (a), we proceed as follows.

Ⓔ **Exercise 9B.7** (a) Show that f is *uniformly* continuous on $\mathbb{X}$. (*Hint:* f is continuous on $\mathbb{X}$, and $\mathbb{X}$ is compact.)

(b) Show: for any $y_0 \in [0, X_0]$, the functions f^y converge *uniformly* to f^{y_0} as $y \to y_0$.

(c) For any fixed $\mathbf{n} \in \mathbb{N}$, deduce that $\lim_{y\to y_0} A_{\mathbf{n}}^y = A_{\mathbf{n}}^{y_0}$ and $\lim_{y\to y_0} B_{\mathbf{n}}^y = B_{\mathbf{n}}^{y_0}$. (*Hint:* Use Corollary 6E.11(b)[ii] on page 127.)

(d) Conclude that the functions $\alpha_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ are continuous at y_0.

The conclusion of Exercise 9B.7(d) holds for all $y_0 \in [0, X_0]$ and all $\mathbf{n} \in \mathbb{N}$. Thus, the functions $\alpha_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ are continuous on $[0, X_0]$, for all $\mathbf{n} \in \mathbb{N}$.

For all $y \in [0, X_0]$, let $(\partial_0 f)^y : \mathbb{X}^* \longrightarrow \mathbb{R}$ be the function defined by $(\partial_0 f)^y(\mathbf{x}) := \partial_0 f(y, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}^*$.

Ⓔ **Exercise 9B.8** Suppose $f \in \mathcal{C}^1(\mathbb{X})$. Use Proposition 0G.1 on page 567 to show, for all $\mathbf{n} \in \mathbb{N}^D$, that the functions $\alpha_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ are differentiable on $[0, X_0]$; furthermore, for all $y \in [0, X_0]$,

$$\alpha'_{\mathbf{n}}(y) \quad = \quad \frac{\langle (\partial_0 f)^y, \mathbf{C}_{\mathbf{n}} \rangle}{\|\mathbf{C}_{\mathbf{n}}\|_2^2} \quad \text{and} \quad \beta'_{\mathbf{n}}(y) \quad = \quad \frac{\langle (\partial_0 f)^y, \mathbf{S}_{\mathbf{n}} \rangle}{\|\mathbf{S}_{\mathbf{n}}\|_2^2}. \tag{9B.9}$$

Ⓔ **Exercise 9B.9** Using the same technique as Exercise 9B.7, use eqn.(9B.9) to prove that the functions $\alpha'_{\mathbf{n}}$ and $\beta'_{\mathbf{n}}$ are continuous on $[0, X_0]$.

Thus, $\alpha_{\mathbf{n}} \in \mathcal{C}^1[0, X_0]$ and $\beta_{\mathbf{n}} \in \mathcal{C}^1[0, X_0]$; this proves part (a) of the Claim.

Ⓔ The proof of (b) is **Exercise 9B.10** (*Hint.* Imitate the proof of Claim 3).

Ⓔ The proof of (c) is **Exercise 9B.11** . $\diamond_{\text{Claim 5}}$

Claim 6: *The one-dimensional Fourier cosine series in eqn.*(9B.7) *converges to* $\alpha_{\mathbf{n}}$, *and the one-dimensional Fourier sine series in eqn.*(9B.7) *converges to* $\beta_{\mathbf{n}}$ *in the desired fashion* (i.e. pointwise or uniform), *for all* $\mathbf{n} \in \mathbb{N}^D$.

Proof. **Exercise 9B.12** (Hint: Use Theorems 7A.1(b,d[i]) and 7A.4(b,d[i]) and Claim 5). Ⓔ

Now, Claim 4 implies that the $(D+1)$–dimensional Fourier (co)sine series in (9B.8) can be rewritten as

$$\sum_{m=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D} A_m^{\mathbf{n}}\, \mathbf{C}_m\, \mathbf{C}_{\mathbf{n}} \quad \underset{\text{L2}}{\approx} \quad f \quad \underset{\text{L2}}{\approx} \quad \sum_{m=1}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}_+^D} B_m^{\mathbf{n}}\, \mathbf{S}_m\, \mathbf{S}_{\mathbf{n}}. \tag{9B.10}$$

Exercise 9B.13 Suppose $f \in \mathcal{C}^1(\mathbb{X})$. Use the 'pointwise' versions of Claims 2 and 6 to show that the two series in eqn.(9B.10) converges to f pointwise on the interior of $\mathbb{X}$. Ⓔ

This proves part (b) of the Theorem.

Exercise 9B.14 (hard) Suppose $f \in \mathcal{C}^1(\mathbb{X})$, the derivative functions $\partial_0 f,\ \partial_1 f, \ldots, \partial_0 f$ are all in $\mathbf{L}^2(\mathbb{X})$, and (for the sine series) f satisfies homogeneous Dirichlet boundary conditions. Use the 'uniform' versions of Claims 2 and 6 to show that the two series in eqn.(9B.10) converges to f uniformly if $f \in \mathcal{C}^1(\mathbb{X})$. Ⓔ

This proves parts (d)[i] and (e)[i] of the Theorem. □

Remarks. (a) If f is a *piecewise* $\mathcal{C}^1$ function on the interval $[0,\pi]$, then Theorems 7A.1 and 7A.4 also yield pointwise convergence and 'local' uniform convergence of one-dimensional Fourier (co)sine to f inside the '$\mathcal{C}^1$ intervals' of f. Likewise, if f is a "piecewise $\mathcal{C}^1$ function" on the D-dimensional domain $\mathbb{X}$, then one can extend Theorem 9B.1 to get pointwise convergence and 'local' uniform convergence of D-dimensional Fourier (co)sine to f inside the '$\mathcal{C}^1$ regions' of f; however, it is too technically complicated to formally state this here.

(b) Remark 8D.3 on page 174 provided some technical remarks about the (non)convergence of one-dimensional Fourier (co)sine series, when the hypotheses of Theorems 7A.1 and 7A.4 are further weakened. Similar remarks apply to D-dimensional Fourier series.

(c) It is also possible to define D-dimensional complex Fourier series on the D-dimensional box $[-\pi,\pi]^D$, in a manner analogous to the results of Section 8D, and then state and prove a theorem analogous to Theorem 9B.1 for such D-dimensional complex Fourier series. (**Exercise 9B.15** (Challenging) Do this.) Ⓔ

In Chapters 11-14, we will often propose a multiple Fourier series (or similar object) as the solution to some PDE, perhaps with certain boundary conditions. To verify that the Fourier series really satisfies the PDE, we must be able to compute its Laplacian. If we also require the Fourier series solution to satisfy some Neumann boundary conditions, then we must be able to compute its normal derivatives on the boundary of the domain. For these purposes, the next result is crucial.

Proposition 9B.2. The Derivatives of a Multiple Fourier (co)sine series
Let $\mathbb{X} := [0, X_1] \times \cdots \times [0, X_D]$. Let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be have uniformly convergent Fourier series

$$f \underset{\text{unif}}{\equiv} A_0 + \sum_{\mathbf{n}\in\mathbb{N}^D} A_\mathbf{n}\mathbf{C}_\mathbf{n} + \sum_{\mathbf{n}\in\mathbb{N}^D_+} B_\mathbf{n}\mathbf{S}_\mathbf{n}.$$

(a) *Fix $i \in [1...D]$. Suppose that $\sum_{\mathbf{n}\in\mathbb{N}^D} n_i|A_\mathbf{n}| + \sum_{\mathbf{n}\in\mathbb{N}^D_+} n_i|B_\mathbf{n}| < \infty$. Then the function $\partial_i f$ exists, and*

$$\partial_i f \underset{\text{L2}}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^D} \left(\frac{\pi n_i}{X_i}\right) \cdot \Big(B_\mathbf{n}\mathbf{S}'_\mathbf{n} - A_\mathbf{n}\mathbf{C}'_\mathbf{n}\Big).$$

Here, for all $\mathbf{n} \in \mathbb{N}^D$, and all $\mathbf{x} \in \mathbb{X}$, we define

$$\mathbf{C}'_\mathbf{n}(\mathbf{x}) := \sin\left(\frac{\pi n_i x_i}{X_i}\right) \cdot \mathbf{C}_\mathbf{n}(\mathbf{x}) / \cos\left(\frac{\pi n_i x_i}{X_i}\right),$$
$$\text{and} \quad \mathbf{S}'_\mathbf{n}(\mathbf{x}) := \cos\left(\frac{\pi n_i x_i}{X_i}\right) \cdot \mathbf{S}_\mathbf{n}(\mathbf{x}) / \sin\left(\frac{\pi n_i x_i}{X_i}\right).$$

(b) *Fix $i \in [1...D]$. Suppose that $\sum_{\mathbf{n}\in\mathbb{N}^D} n_i^2|A_\mathbf{n}| + \sum_{\mathbf{n}\in\mathbb{N}^D_+} n_i^2|B_\mathbf{n}| < \infty$. Then the function $\partial_i^2 f$ exists, and*

$$\partial_i^2 f \underset{\text{L2}}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^D} -\left(\frac{\pi n_i}{X_i}\right)^2 \cdot \Big(A_\mathbf{n}\mathbf{C}_\mathbf{n} + B_\mathbf{n}\mathbf{S}_\mathbf{n}\Big).$$

(c) *Suppose that $\sum_{\mathbf{n}\in\mathbb{N}^D} |\mathbf{n}|^2|A_\mathbf{n}| + \sum_{\mathbf{n}\in\mathbb{N}^D} |\mathbf{n}|^2|B_\mathbf{n}| < \infty$* (where we define $|\mathbf{n}|^2 := n_1^2 + ... + n_D^2$). *Then f is twice-differentiable, and*

$$\triangle f \underset{\text{L2}}{\approx} -\pi^2 \sum_{\mathbf{n}\in\mathbb{N}^D} \left[\left(\frac{n_1}{X_1}\right)^2 + \cdots + \left(\frac{n_D}{X_D}\right)^2\right] \cdot \Big(A_\mathbf{n}\mathbf{C}_\mathbf{n} + B_\mathbf{n}\mathbf{S}_\mathbf{n}\Big).$$

Ⓔ *Proof.* **Exercise 9B.16** Hint: Apply Proposition 0F.1 on page 565 □

Example 9B.3. Fix $\mathbf{n} \in \mathbb{N}^D$. If $f = A \cdot \mathbf{C_n} + B \cdot \mathbf{S_n}$, then

$$\triangle f \quad = \quad -\pi^2 \left[\left(\frac{n_1}{X_1} \right)^2 + \cdots + \left(\frac{n_D}{X_D} \right)^2 \right] \cdot f.$$

In particular, if $X_1 = \cdots = X_D = \pi$, then this simplifies to: $\triangle f \quad = \quad -|\mathbf{n}|^2 \cdot f$. In other words, f is an *eigenfunction* of the Laplacian operator, with eigenvalue $\lambda = -|\mathbf{n}|^2$. ♢

9C Practice problems

Compute the two-dimensional Fourier sine transforms of the following functions. For each question, also determine: at which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

1. $f(x, y) \ = \ x^2 \cdot y$.
2. $g(x, y) \ = \ x + y$.
3. $f(x, y) \ = \ \cos(Nx) \cdot \cos(My)$, for some integers $M, N > 0$.
4. $f(x, y) \ = \ \sin(Nx) \cdot \sinh(Ny)$, for some integer $N > 0$.

Chapter 10

Proofs of the Fourier convergence theorems

"The profound study of nature is the most fertile source of mathematical discoveries."
—Jean Joseph Fourier

In this section, we will prove Theorem 8A.1(a,b,d) on page 162 (and thus, indirectly prove 7A.1(a,b,d) and 7A.4(a,b,d) on pages 138 and 142). Along the way, we will introduce some ideas which are of independent interest: Bessel's inequality, the Riemann-Lebesgue lemma, the Dirichlet kernel, convolutions and mollifiers, and the relationship between the smoothness of a function and the asymptotic decay of its Fourier coefficients. This chapter assumes no prior knowledge of analysis, beyond some background from Chapter 6. However, the presentation is slightly more abstract than most of the book, and is intended for more 'theoretically inclined' students.

10A Bessel, Riemann and Lebesgue

Prerequisites: §6D. **Recommended:** §7A, §8A.

We begin with a general result which is true for any orthonormal set in any L^2 space.

Theorem 10A.1. (Bessel's Inequality)
Let $\mathbb{X} \subset \mathbb{R}^D$ be any bounded domain. Let $\{\phi_n\}_{n=1}^{\infty}$ be any orthonormal set of functions in $\mathbf{L}^2(\mathbb{X})$. Let $f \in \mathbf{L}^2(\mathbb{X})$, and for all $n \in \mathbb{N}$, let $c_n := \langle f, \phi_n \rangle$. Then for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} |c_n|^2 \quad \leq \quad \|f\|_2^2 .$$

In particular, $\lim_{n\to\infty} c_n \;=\; 0.$

Proof. Without loss of generality, suppose $|\mathbb{X}| = 1$, so that $\langle f, g\rangle = \int_{\mathbb{X}} f(x)\, g(x)\, dx$ for any $f, g \in \mathbf{L}^2(\mathbb{X})$. First note that

$$\left[f(x) - \sum_{n=1}^{N} c_n\phi_n(x)\right]^2$$
$$= f(x)^2 - 2f(x)\sum_{n=1}^{N} c_n\phi_n(x) + \left(\sum_{n=1}^{N} c_n\phi_n(x)\right)\cdot\left(\sum_{m=1}^{N} c_m\phi_m(x)\right)$$
$$= f(x)^2 - 2\sum_{n=1}^{N} c_n f(x)\phi_n(x) + \sum_{n,m=1}^{N} c_n c_m \phi_n(x)\phi_m(x). \qquad (10A.1)$$

Thus,

$$\begin{aligned}
0 \;\le\; \left\| f - \sum_{n=1}^{N} c_n\phi_n\right\|_2^2 \;&=\; \int_{\mathbb{X}} \left[f(x) - \sum_{n=1}^{N} c_n\phi_n(x)\right]^2 dx \\
&\underset{(*)}{=} \int_{\mathbb{X}} \left(f(x)^2 - 2\sum_{n=1}^{N} c_n f(x)\phi_n(x) + \sum_{n,m=1}^{N} c_n c_m\phi_n(x)\phi_m(x)\right) dx \\
&= \int_{\mathbb{X}} f(x)^2\, dx \;-\; 2\sum_{n=1}^{N} c_n \int_{\mathbb{X}} f(x)\phi_n(x)\, dx \;+\; \sum_{n,m=1}^{N} c_n c_m \int_{\mathbb{X}} \phi_n(x)\phi_m(x)\, dx \\
&= \underbrace{\langle f, f\rangle}_{\|f\|_2^2} - 2\sum_{n=1}^{N} c_n \underbrace{\langle f, \phi_n\rangle}_{c_n} + \sum_{n,m=1}^{N} c_n c_m \underbrace{\langle \phi_n, \phi_m\rangle}_{\substack{=1 \text{ if } n=m \\ =0 \text{ if } n\neq m}} \\
&= \|f\|_2^2 \;-\; 2\sum_{n=1}^{N} c_n^2 \;+\; \sum_{n=1}^{N} c_n^2 \;=\; \|f\|_2^2 \;-\; \sum_{n=1}^{N} c_n^2.
\end{aligned}$$

Here, $(*)$ is by eqn.(10A.1). Thus, $0 \le \|f\|_2^2 - \sum_{n=1}^{N} c_n^2$. Thus $\sum_{n=1}^{N} c_n^2 \le \|f\|_2^2$, as desired. □

Example 10A.2. Suppose $f \in \mathbf{L}^2[-\pi, \pi]$ has real Fourier coefficients $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, as defined on page 161. Then for all $N \in \mathbb{N}$,

$$A_0^2 \;+\; \sum_{n=1}^{N} \frac{|A_n|^2}{2} \;+\; \sum_{n=1}^{N} \frac{|B_n|^2}{2} \;\le\; \|f\|_2^2.$$

Ⓔ <u>**Exercise 10A.1**</u> Prove this. (Hint: Let $\mathbb{X} = [-\pi, \pi]$ and let $\{\phi_k\}_{k=1}^{\infty} = \left\{\sqrt{2}\mathbf{C}_n\right\}_{n=0}^{\infty} \sqcup \left\{\sqrt{2}\mathbf{S}_n\right\}_{n=1}^{\infty}$. Show that $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal set of functions (Use Proposition 6D.2 on page 112). Now apply Bessel's Inequality). ♢

Corollary 10A.3. (Riemann-Lebesgue Lemma)

(a) *Suppose* $f \in \mathbf{L}^2[-\pi,\pi]$ *has real Fourier coefficients* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=1}^{\infty}$, *as defined on page 161. Then* $\lim_{n\to\infty} A_n = 0$ *and* $\lim_{n\to\infty} B_n = 0$.

(b) *Suppose* $f \in \mathbf{L}^2[0,\pi]$ *has Fourier cosine coefficients* $\{A_n\}_{n=0}^{\infty}$, *as defined by eqn.*(7A.4) *on page 141, and Fourier sine coefficients* $\{B_n\}_{n=1}^{\infty}$, *as defined by eqn.*(7A.1) *on page 137. Then* $\lim_{n\to\infty} A_n = 0$ *and* $\lim_{n\to\infty} B_n = 0$.

Proof. **Exercise 10A.2** Hint: Use Example 10A.2. □ Ⓔ

10B Pointwise convergence

Prerequisites: §8A, §10A. **Recommended:** §17B.

In this section we will prove Theorem 8A.1(b), through a common strategy in harmonic analysis: the use of a *summation kernel*. For all $N \in \mathbb{N}$, the Nth **Dirichlet kernel** is the function $\mathbf{D}_N : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ defined by

$$\mathbf{D}_N(x) \quad := \quad 1 + 2\sum_{n=1}^{N} \cos(nx) \qquad \text{(see Figure 10B.1).}$$

Note that $\mathbf{D}_N$ is 2π-periodic (i.e. $\mathbf{D}_N(x+2\pi) = \mathbf{D}_N(x)$ for all $x \in [-2\pi, 0]$). Thus, we could represent $\mathbf{D}_N$ as a function from $[-\pi,\pi]$ into $\mathbb{R}$. However, it is sometimes convenient to extend $\mathbf{D}_N$ to $[-2\pi, 2\pi]$. For example, for any function $f : [-\pi,\pi] \longrightarrow \mathbb{R}$, the **convolution** of $\mathbf{D}_N$ and f is the function $\mathbf{D}_N * f : [-\pi,\pi] \longrightarrow \mathbb{R}$ defined by

$$\mathbf{D}_N * f(x) \quad := \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,\mathbf{D}_N(x-y)\,dy, \qquad \text{for all } x \in [-\pi,\pi].$$

(Note that, to define $\mathbf{D}_N * f$, we must evaluate $\mathbf{D}_N(z)$ for all $z \in [-2\pi, 2\pi]$). The connection between Dirichlet kernels and Fourier series is given by the next lemma:

Lemma 10B.1. *Let* $f \in \mathbf{L}^2[-\pi,\pi]$, *and for all* $n \in \mathbb{N}$, *let*

$$A_n \ := \ \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(ny) f(y)\, dy \qquad \text{and} \qquad B_n \ := \ \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(ny) f(y)\, dy$$

be the real Fourier coefficients of f. *Then for any* $N \in \mathbb{N}$, *and every* $x \in [-\pi,\pi]$, *we have*

$$A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n(x) + \sum_{n=1}^{N} B_n \mathbf{S}_n(x) \quad = \quad \mathbf{D}_N * f(x).$$

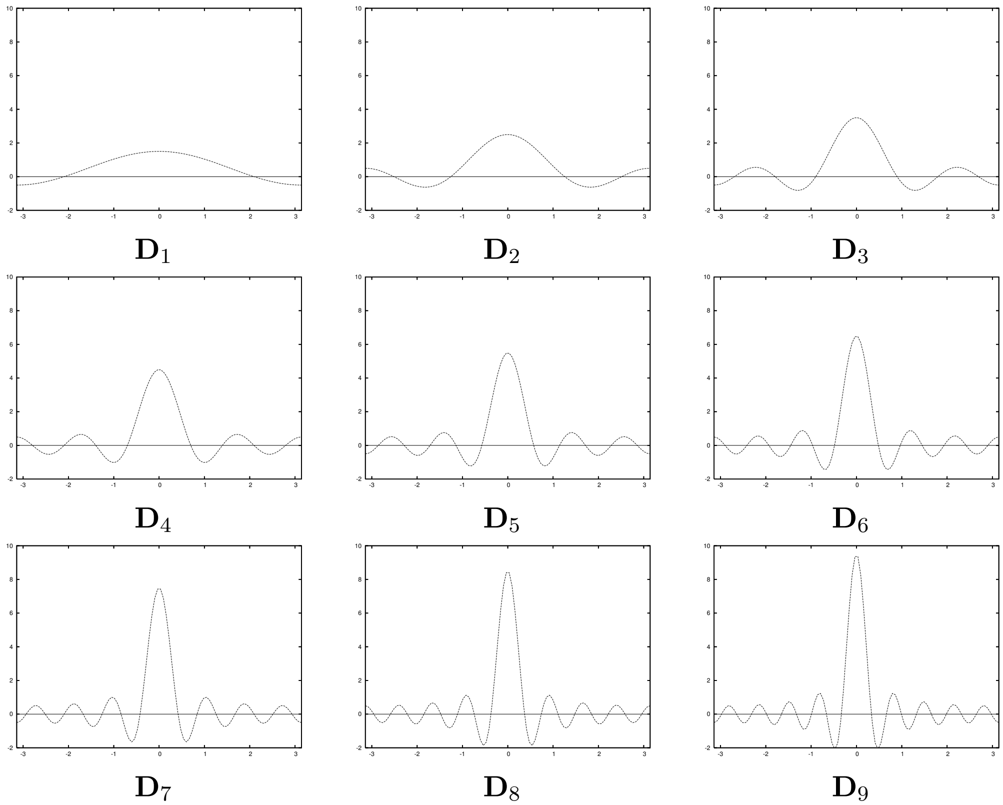


Figure 10B.1: The Dirichlet kernels $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_9$ plotted on interval $[-\pi, \pi]$. Note the increasing concentration of the function near $x = 0$. (In the terminology of Section 10D(ii) and 17B, the sequence $\{\mathbf{D}_1, \mathbf{D}_2, \ldots\}$ is like an *approximation of the identity*.)

Proof. For any $x \in [-\pi, \pi]$, we have

$$\begin{aligned}
&A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n(x) + \sum_{n=1}^{N} B_n \mathbf{S}_n(x) \\
&= A_0 + \sum_{n=1}^{N} \cos(nx) \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ny) f(y)\, dy \right) + \sum_{n=1}^{N} \sin(nx) \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(ny) f(y)\, dy \right) \\
&= A_0 + \sum_{n=1}^{N} \frac{1}{\pi} \left(\int_{-\pi}^{\pi} \cos(nx)\cos(ny) f(y)\, dy + \int_{-\pi}^{\pi} \sin(nx)\sin(ny) f(y)\, dy \right) \\
&= A_0 + \sum_{n=1}^{N} \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\cos(nx)\cos(ny) + \sin(nx)\sin(ny) \Big) f(y)\, dy
\end{aligned}$$

$$\underset{(*)}{=} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\ dy \ + \ \sum_{n=1}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\Big(n(x-y)\Big)\, f(y)\ dy$$
$$= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(y) \ + \ 2\sum_{n=1}^{N} \cos\Big(n(x-y)\Big)\, f(y)\right) dy$$
$$= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,\mathbf{D}_N(x-y)\ dy \quad = \quad \mathbf{D}_N * f(x).$$

Here, $(*)$ uses the fact that $A_0 := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\ dy$, and also the well-known trigonometric identity $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$ (with $u = nx$ and $v = ny$). □

Remark. See Exercise 18F.7 on page 464 for another proof of Lemma 10B.1 for complex Fourier series. ♢

Figure 10B.1 shows how the 'mass' of the Dirichlet kernel $\mathbf{D}_N$ becomes increasingly concentrated near $x = 0$ as $N \to \infty$. In the terminology of Sections 10D and 17B (pages 207 and 379), the sequence $\{\mathbf{D}_1, \mathbf{D}_2, \ldots\}$ is like an *approximation of the identity.* Thus, our strategy is to show that $\mathbf{D}_N * f(x) \to f(x)$ as $N \to \infty$, whenever f is continuous at x. Indeed, we will go further: when f is *discontinuous* at x, we will show that $\mathbf{D}_N * f(x)$ converges to the average of the *left-hand* and *right-hand limits* of f at x. First we need some technical results.

Lemma 10B.2.

(a) *For any* $N \in \mathbb{N}$, *we have* $\displaystyle\int_0^{\pi} \mathbf{D}_N(x)\ dx \ = \ \pi.$

(b) *For any* $N \in \mathbb{N}$ *and* $x \in (-\pi, 0) \sqcup (0, \pi)$, *we have* $\mathbf{D}_N(x) \ = \ \dfrac{\sin((2N+1)x/2)}{\sin(x/2)}$.

(c) *Let* $g : [0,\pi] \longrightarrow \mathbb{R}$ *be a piecewise continuous function. Then*

$$\lim_{N\to\infty} \int_0^{\pi} g(x)\ \sin\left(\frac{(2N+1)x}{2}\right)\ dx \quad = \quad 0.$$

Proof. The proof of (b) is **Exercise 10B.1** (Hint: Use trigonometric identities). Ⓔ

To prove (a), note that

$$\int_0^{\pi} \mathbf{D}_N(x)\ dx \quad = \quad \int_0^{\pi} 1 + 2\sum_{n=1}^{N}\cos(nx)\ dx \quad = \quad \int_0^{\pi} 1\ dx + 2\sum_{n=1}^{N}\int_0^{\pi}\cos(nx)\ dx$$
$$= \quad \pi + 2\sum_{n=1}^{N} 0 \quad = \quad \pi.$$

To prove (c), first observe that

$$\sin\left(\frac{(2N+1)x}{2}\right) = \sin\left(Nx + \frac{x}{2}\right) = \sin(Nx)\cos(x/2) + \cos(Nx)\sin(x/2), \quad (10B.1)$$

where the last step uses the well-known trigonometric identity $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$ (with $u := Nx$ and $v := x/2$). Thus,

$$\begin{aligned}
&\int_0^\pi g(x)\ \sin\left(\frac{(2N+1)x}{2}\right)\ dx \\
&\underset{(\dagger)}{=} \int_0^\pi g(x)\left(\sin(Nx)\cos(x/2) + \cos(Nx)\sin(x/2)\right)\ dx \\
&= \int_0^\pi \underbrace{g(x)\cos(x/2)}_{G_1(x)}\ \underbrace{\sin(Nx)}_{\mathbf{S}_N(x)}\ dx \ + \ \int_0^\pi \underbrace{g(x)\sin(x/2)}_{G_2(x)}\ \underbrace{\cos(Nx)}_{\mathbf{C}_N(x)}\ dx \\
&\underset{(*)}{=} \frac{2\pi}{2\pi}\int_0^\pi G_1(x)\,\mathbf{S}_N(x)\ dx \ + \ \frac{2\pi}{2\pi}\int_0^\pi G_2(x)\,\mathbf{C}_N(x)\ dx \\
&\underset{(\ddagger)}{=} \frac{\pi}{2}\langle G_1, \mathbf{S}_N\rangle + \frac{\pi}{2}\langle G_2, \mathbf{C}_N\rangle \\
&\xrightarrow[N\to\infty]{} 0 + 0, \quad \text{by Corollary 10A.3(b) (the Riemann-Lebesgue Lemma).}
\end{aligned}$$

Here $(\dagger)$ is by eqn.(10B.1) and $(\ddagger)$ is by definition of the inner product on $\mathbf{L}^2[0,\pi]$. In $(*)$, we define the functions $G_1(x) := g(x)\cos(x/2)$ $G_2(x) := g(x)\sin(x/2)$; these functions are piecewise continuous because g is piecewise continuous; thus they are in $\mathbf{L}^2[0,\pi]$, so the Riemann-Lebesgue Lemma is applicable. □

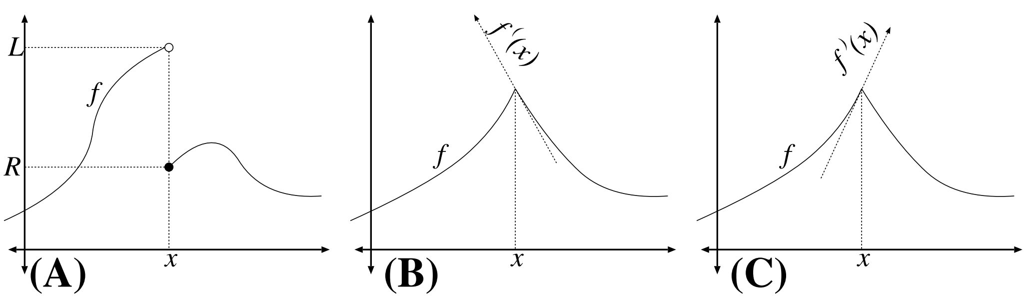


Figure 10B.2: **(A)** Left-hand and right-hand limits. Here, $L := \lim_{y\nearrow x} f(x)$ and $R := \lim_{y\searrow x} f(x)$. **(B)** The right-hand derivative $f^{\langle}(x)$. **(C)** The left-hand derivative $f^{\rangle}(x)$.

Let $f : [-\pi,\pi] \longrightarrow \mathbb{R}$ be a function. For any $x \in [-\pi,\pi)$, the **right-hand limit** of f at x is defined

$$\lim_{y\searrow x} f(y) \quad := \quad \lim_{\epsilon\to 0} f(x+|\epsilon|) \qquad \text{(if this limit exists).}$$

Likewise, for any $x \in (-\pi, \pi]$, the **left-hand limit** of f at x is defined

$$\lim_{y \nearrow x} f(y) \quad := \quad \lim_{\epsilon \to 0} f(x - |\epsilon|) \qquad \text{(if this limit exists).}$$

See Figure 10B.2(A). Clearly, if f is continuous at x, then the left-hand and right-hand limits both exist, and $\lim_{y \searrow x} f(y) = f(x) = \lim_{y \nearrow x} f(y)$. However, the left-hand and right-hand limits may exist even when f is not continuous.

For any $x \in [-\pi, \pi)$, let $f(x^+) := \lim_{y \searrow x} f(y)$. The **right-hand derivative** of f at x is defined

$$f^{\langle}(x) \quad := \quad \lim_{y \searrow x} \frac{f(y) - f(x^+)}{y - x} \quad = \quad \lim_{\epsilon \to 0} \frac{f(x + |\epsilon|) - f(x^+)}{|\epsilon|} \quad \text{(if this limit exists).}$$

See Figure 10B.2(B). Likewise, for any $x \in (-\pi, \pi]$, let $f(x^-) := \lim_{y \nearrow x} f(y)$. The **left-hand derivative** of f at x is defined

$$f^{\rangle}(x) \quad := \quad \lim_{y \nearrow x} \frac{f(y) - f(x^-)}{y - x} \quad = \quad \lim_{\epsilon \to 0} \frac{f(x - |\epsilon|) - f(x^-)}{-|\epsilon|} \quad \text{(if this limit exists).}$$

See Figure 10B.2(C). If $f^{\langle}(x)$ and $f^{\rangle}(x)$ both exist, then we say f is **semidifferentiable** at x. Clearly, f is differentiable at x if and only if f is continuous at x (so that $f(x^-) = f(x^+)$), and f semidifferentiable at x, *and* $f^{\langle}(x) = f^{\rangle}(x)$. In this case, $f'(x) = f^{\langle}(x) = f^{\rangle}(x)$. However, f can be semidifferentiable at x even when f is not differentiable (or even continuous) at x.

Lemma 10B.3. *Let $\widetilde{f} : [-\pi, \pi] \longrightarrow \mathbb{R}$ be a piecewise continuous function which is semidifferentiable at 0. Then*

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \widetilde{f}(x)\, \mathbf{D}_N(x)\, dx \quad = \quad \pi \cdot \left(\lim_{x \nearrow 0} \widetilde{f}(x) + \lim_{x \searrow 0} \widetilde{f}(x) \right).$$

Proof. It suffices to show that

$$\lim_{N \to \infty} \int_{-\pi}^{0} \widetilde{f}(x)\, \mathbf{D}_N(x)\, dx \quad = \quad \pi \cdot \lim_{x \nearrow 0} \widetilde{f}(x) \qquad (10B.2)$$

$$\text{and} \qquad \lim_{N \to \infty} \int_{0}^{\pi} \widetilde{f}(x)\, \mathbf{D}_N(x)\, dx \quad = \quad \pi \cdot \lim_{x \searrow 0} \widetilde{f}(x). \qquad (10B.3)$$

We will prove eqn.(10B.3). Let $\widetilde{f}(0^+) := \lim_{x \searrow 0} \widetilde{f}(x)$, and consider the function $g : [0, \pi] \longrightarrow \mathbb{R}$ defined by $g(x) := \dfrac{\widetilde{f}(x) - \widetilde{f}(0^+)}{\sin(x/2)}$ if $x > 0$, while $g(0) := 2\, \widetilde{f}^{\langle}(0)$.

Claim 1: *g is piecewise continuous on $[0, \pi]$.*

Proof. Clearly, g is piecewise continuous on $(0, \pi]$ because $\widetilde{f}$ is piecewise continuous, while $\sin(x/2)$ is nonzero on $(0, \pi]$. The only potential location of an unbounded discontinuity is at 0. But

$$\lim_{x \searrow 0} g(x) \;=\; \lim_{x \searrow 0} \frac{\widetilde{f}(x) - \widetilde{f}(0^+)}{\sin(x/2)} \;=\; \lim_{x \searrow 0} \left(\frac{\widetilde{f}(x) - \widetilde{f}(0^+)}{x} \right) \cdot \left(\frac{x}{\sin(x/2)} \right)$$

$$= \underbrace{\left(\lim_{x \searrow 0} \frac{\widetilde{f}(x) - \widetilde{f}(0^+)}{x - 0} \right)}_{\widetilde{f}^{\langle}(0)} \cdot 2 \cdot \underbrace{\left(\lim_{x \searrow 0} \frac{x/2}{\sin(x/2)} \right)}_{=1} \;=\; 2\, \widetilde{f}^{\langle}(0) \;=:\; g(0).$$

Thus, g is (right-)continuous at 0, as desired. $\diamond_{\text{Claim 1}}$

Now,

$$\begin{aligned}
&\lim_{N \to \infty} \int_0^{\pi} \widetilde{f}(x)\, \mathbf{D}_N(x)\, dx \\
&= \lim_{N \to \infty} \int_0^{\pi} \left(\widetilde{f}(0^+) + \widetilde{f}(x) - \widetilde{f}(0^+) \right) \mathbf{D}_N(x)\, dx \\
&= \lim_{N \to \infty} \int_0^{\pi} \widetilde{f}(0^+)\, \mathbf{D}_N(x)\, dx \;+\; \int_0^{\pi} \left(\widetilde{f}(x) - \widetilde{f}(0^+) \right) \mathbf{D}_N(x)\, dx \\
&\underset{\text{(a)}}{=} \pi\, \widetilde{f}(0^+) \;+\; \int_0^{\pi} \left(\widetilde{f}(x) - \widetilde{f}(0^+) \right) \mathbf{D}_N(x)\, dx \\
&\underset{\text{(b)}}{=} \pi\, \widetilde{f}(0^+) \;+\; \lim_{N \to \infty} \int_0^{\pi} \frac{\widetilde{f}(x) - \widetilde{f}(0^+)}{\sin(x/2)} \cdot \sin\left(\frac{(2N+1)x}{2} \right) dx \\
&= \pi\, \widetilde{f}(0^+) \;+\; \lim_{N \to \infty} \int_0^{\pi} g(x) \cdot \sin\left(\frac{(2N+1)x}{2} \right) dx \\
&\underset{\text{(c)}}{=} \pi\, \widetilde{f}(0^+) \;+\; 0 \;=\; \pi\, \widetilde{f}(0^+),
\end{aligned}$$

as desired. Here, (a) is by Lemma 10B.2(a), (b) is by Lemma 10B.2(b), and (c) is by Lemma 10B.2(c), which is applicable because g is piecewise continuous by Claim 1.

Ⓔ This proves eqn.(10B.3). The proof of eqn.(10B.2) is **Exercise 10B.2**. Adding together equations (10B.2) and (10B.3) proves the lemma. $\square$

Lemma 10B.4. *Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be piecewise continuous, and suppose that f is semidifferentiable at x* (i.e. $f^{\langle}(x)$ and $f^{\rangle}(x)$ exist). *Then*

$$\lim_{N \to \infty} \mathbf{D}_N * f(x) \;=\; \frac{1}{2} \left(\lim_{y \searrow x} f(y) \;+\; \lim_{y \nearrow x} f(y) \right). \tag{10B.4}$$

In particular, if f is continuous and semidifferentiable at x, then $\lim_{N \to \infty} \mathbf{D}_N * f(x) \;=\; f(x)$.

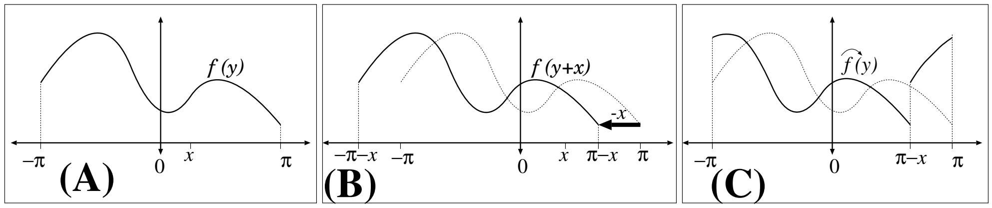


Figure 10B.3: The 2π-periodic phase-shift of a function. **(A)** A function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$. **(B)** The function $y \mapsto f(x+y)$. **(C)** The function $\overset{\curvearrowright}{f} \colon [-\pi, \pi] \longrightarrow \mathbb{R}$.

Proof. Suppose $x \in [0, \pi]$ (the case $x \in [-\pi, 0]$ is handled similarly). Define $\overset{\curvearrowright}{f} \colon [-\pi, \pi] \longrightarrow \mathbb{R}$ by

$$\overset{\curvearrowright}{f}(y) \quad := \quad \begin{cases} f(y+x) & \text{if} \quad y \in [-\pi, \pi - x]; \\ f(y+x-2\pi) & \text{if} \quad y \in [\pi - x, \pi]. \end{cases}$$

(Effectively, we are treating f as a 2π-periodic function, and 'phase-shifting' f by x; see Figure 10B.3). Then

$$\begin{aligned} 2\pi \cdot \mathbf{D}_N * f(x) &= \int_{-\pi}^{\pi} f(y)\mathbf{D}_N(x-y)\, dy \ \underset{(*)}{=} \ \int_{-\pi}^{\pi} f(y)\mathbf{D}_N(y-x)\, dy \\ &\underset{(c)}{=} \int_{-\pi-x}^{\pi-x} f(z+x)\mathbf{D}_N(z)\, dz \\ &= \int_{-\pi-x}^{-\pi} f(z+x)\mathbf{D}_N(z)\, dz \ + \ \int_{-\pi}^{\pi-x} f(z+x)\mathbf{D}_N(z)\, dz \\ &\underset{(@)}{=} \int_{\pi-x}^{\pi} f(w+x-2\pi)\mathbf{D}_N(w-2\pi)\, dw \ + \ \int_{-\pi}^{\pi-x} f(z+x)\mathbf{D}_N(z)\, dz \\ &\underset{(\dagger)}{=} \int_{\pi-x}^{\pi} \overset{\curvearrowright}{f}(w)\mathbf{D}_N(w)\, dw \ + \ \int_{-\pi}^{\pi-x} \overset{\curvearrowright}{f}(z)\mathbf{D}_N(z)\, dz \\ &= \int_{-\pi}^{\pi} \overset{\curvearrowright}{f}(z)\mathbf{D}_N(z)\, dz \ \underset{(\diamond)}{=} \ \pi \cdot \left(\lim_{z \searrow 0} \overset{\curvearrowright}{f}(z) \ + \ \lim_{z \nearrow 0} \overset{\curvearrowright}{f}(z) \right) \\ &\underset{(\ddagger)}{=} \pi \cdot \left(\lim_{z \searrow 0} f(z+x) \ + \ \lim_{z \nearrow 0} f(z+x) \right) \\ &\underset{(c)}{=} \pi \cdot \left(\lim_{y \searrow x} f(y) \ + \ \lim_{y \nearrow x} f(y) \right). \end{aligned}$$

Now divide both sides by 2π to get equation (10B.4).

Here, $(*)$ is because $\mathbf{D}_N$ is *even* (i.e. $\mathbf{D}_N(-r) = \mathbf{D}_N(r)$ for all $r \in \mathbb{R}$). Both equalities marked (c) are the change of variables $z := y - x$ (so that $y = z + x$). Likewise, equality (@) is the change of variables $w := z + 2\pi$ (so that $z = w - 2\pi$). Both $(\dagger)$ and $(\ddagger)$ use the definition of $\overset{\curvearrowright}{f}$, and $(\dagger)$ also uses the fact that $\mathbf{D}_N$ is 2π-periodic, so that $\mathbf{D}_N(w-2\pi) = \mathbf{D}_N(w)$ for all $w \in [\pi - x,\ \pi]$. Finally, $(\diamond)$ is by Lemma 10B.3 applied to $\overset{\frown}{f}$ (which is continuous and semidifferentiable at 0 because f is continuous and semidifferentiable at x). $\square$

Proof of Theorem 8A.1(b). Let $x \in [-\pi, \pi]$, and suppose f is continuous and differentiable at x. Then

$$\lim_{N\to\infty} A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n(x) + \sum_{n=1}^{N} B_n \mathbf{S}_n(x) \underset{(*)}{=} \lim_{N\to\infty} \mathbf{D}_N * f(x) \underset{(\dagger)}{=} f(x),$$

as desired. Here, $(*)$ is by Lemma 10B.1 and $(\dagger)$ is by Lemma 10B.4. $\square$

Remarks. (a) Note that we have actually proved a slightly stronger result than Theorem 8A.1(b). If f is discontinuous, but semidifferentiable at x, then Lemmas 10B.1 and 10B.4 together imply that

$$\lim_{N\to\infty} A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n(x) + \sum_{n=1}^{N} B_n \mathbf{S}_n(x) \quad = \quad \frac{1}{2}\left(\lim_{y \searrow x} f(y) \ + \ \lim_{y \nearrow x} f(y)\right).$$

This is how the 'Pointwise Fourier Convergence Theorem' is stated in some texts.

(b) For other good expositions of this material, see [CB87, §30-31, pp.87-92]. [Asm05, Thm. 1, p.30 of §2.2], [Pow99, §1.7, p.79], or [Bro89, Corollary 1.4.5, p.16].

10C Uniform convergence

Prerequisites: §8A, §10A.

In this section, we will prove Theorem 8A.1(d). First we state a 'discrete' version of the Cauchy-Bunyakowski-Schwarz Inequality (Theorem 6B.5 on page 108).

Lemma 10C.1. Cauchy-Bunyakowski-Schwarz Inequality in $\ell^2(\mathbb{N})$
Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two infinite sequences of real numbers. Then

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \quad \leq \quad \left(\sum_{n=1}^{\infty} a_n^2\right) \cdot \left(\sum_{n=1}^{\infty} b_n^2\right),$$

whenever these sums are finite.

Ⓔ *Proof.* **Exercise 10C.1** Hint: imitate the proof of Theorem 6B.5 on page 108 $\square$

Remark. For any infinite sequences of real numbers $\mathbf{a} := (a_n)_{n=1}^{\infty}$ and $\mathbf{b} := (b_n)_{n=1}^{\infty}$, we can define $\langle \mathbf{a}, \mathbf{b} \rangle := \sum_{n=1}^{\infty} a_n b_n$ and $\|\mathbf{a}\|_2 := \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{n=1}^{\infty} a_n^2}$. The set of all sequences $\mathbf{a}$ such that $\|\mathbf{a}\|_2 < \infty$ is denoted $\ell^2(\mathbb{N})$. Lemma 10C.1 can then be reformulated as the statement: "For all $\mathbf{a}, \mathbf{b} \in \ell^2(\mathbb{N})$, $|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2$". $\diamondsuit$

Next, we will prove a result which relates the 'smoothness' of the function f to the 'asymptotic decay rate' of its Fourier coefficients.

Lemma 10C.2. *Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be continuous, with $f(-\pi) = f(\pi)$. Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be the real Fourier coefficients of f, as defined on page 161. If f is piecewise differentiable on $[-\pi, \pi]$, and $f' \in \mathbf{L}^2[-\pi, \pi]$, then the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ converge to zero fast enough that $\sum_{n=1}^{\infty} |A_n| < \infty$ and $\sum_{n=1}^{\infty} |B_n| < \infty$.*

Proof. If $f' \in \mathbf{L}^2[-\pi, \pi]$, then we can compute its real Fourier coefficients $\{A'_n\}_{n=0}^{\infty}$ and $\{B'_n\}_{n=1}^{\infty}$.

Claim 1: *For all $n \in \mathbb{N}$, $A_n = -B'_n/n$ and $B_n = A'_n/n$.*

Proof. By definition,

$$\begin{aligned} A'_n &:= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x)\ \cos(nx)\ dx \\ &\underset{\text{(p)}}{=} \frac{1}{\pi} f(x) \cos(nx) \Big|_{x=-\pi}^{x=\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ n \sin(nx)\ dx \\ &\underset{\text{(c)}}{=} \frac{(-1)^n}{\pi} \Big(f(\pi) - f(-\pi) \Big) + n\, B_n \underset{(*)}{=} n\, B_n. \end{aligned}$$

Likewise,

$$\begin{aligned} B'_n &:= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x)\ \sin(nx)\ dx \\ &\underset{\text{(p)}}{=} \frac{1}{\pi} f(x) \sin(nx) \Big|_{x=-\pi}^{x=\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ n \cos(nx)\ dx \\ &\underset{\text{(s)}}{=} \Big(0 - 0 \Big) - n\, A_n = -n\, A_n. \end{aligned}$$

Here, (p) is integration by parts, (c) is because $\cos(-n\pi) = (-1)^n = \cos(n\pi)$, $(*)$ is because $f(-\pi) = f(\pi)$, and (s) is because $\sin(-n\pi) = 0 = \sin(n\pi)$.

Thus, $B'_n = -n\, A_n$ and $A'_n = n\, B_n$; hence $A_n = -B'_n/n$ and $B_n = A'_n/n$. $\diamond_{\text{Claim 1}}$

Let $K := \sum_{n=1}^{\infty} \frac{1}{n^2}$ (a finite value). Then

$$\left(\sum_{n=1}^{\infty}|A_n|\right)^2 \underset{(*)}{=} \left(\sum_{n=1}^{\infty}\frac{1}{n}|B_n'|\right)^2 \underset{\text{(CBS)}}{\leq} \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\cdot\left(\sum_{n=1}^{\infty}|B_n'|^2\right)$$
$$= K\cdot\sum_{n=1}^{\infty}|B_n'|^2 \underset{\text{(B)}}{\leq} K\cdot\left\|f'\right\|_2^2 \underset{(\dagger)}{<} \infty.$$

Here, $(*)$ is by Claim 1, (CBS) is by Lemma 10C.1, and (B) is by Bessel's inequality (Theorem 10A.1 on page 195). Finally, $(\dagger)$ is because $f' \in \mathbf{L}^2[-\pi,\pi]$ by hypothesis. It follows that $\sum_{n=1}^{\infty}|A_n| < \infty$. The proof that $\sum_{n=1}^{\infty}|B_n| < \infty$ is similar. □

Proof of Theorem 8A.1(d). If $f : [-\pi,\pi] \longrightarrow \mathbb{R}$ is continuous and piecewise differentiable, $f' \in \mathbf{L}^2[-\pi,\pi]$, and $f(-\pi) = f(\pi)$, then Lemma 10C.2 implies that $\sum_{n=1}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| < \infty$. But then Theorem 8A.1(c) says that the Fourier series of f converges uniformly. (Theorem 8A.1(c), in turn, is a direct consequence of the Weierstrass M test, Proposition 6E.13 on page 129.). □

Remarks. (a) For other treatments of the material in this section, see [CB87, §34-35, pp.105-109] or [Asm05, Thm. 3, p.90 of §2.9].

(b) The connection between smoothness of f and the asymptotic decay of its Fourier coefficients is a recurring theme in harmonic analysis. In general, the 'smoother' a function is, the 'faster' its Fourier coefficients decay to zero. The weakest statement of this kind is the Riemann-Lebesgue Lemma (Corollary 10A.3 on page 197), which says that if f is merely in L^2, then its Fourier coefficients must converge to zero —although perhaps very slowly. (In the context of Fourier *transforms* of functions on $\mathbb{R}$, the corresponding statement is Theorem 19B.1 on page 492). If f is 'slightly smoother' —specifically, if f is *absolutely continuous* or if f has *bounded variation* —then its Fourier coefficients decay to zero with speed comparable to the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$; see [Kat76, Thm.4.3 and 4.5, pp.24-25]. If f is differentiable, then Lemma 10C.2 says that its Fourier coefficients must decay fast enough that the sums $\sum_{n=1}^{\infty}|A_n|$ and $\sum_{n=1}^{\infty}|B_n|$ converge. (For Fourier *transforms* of functions on $\mathbb{R}$, the corresponding result is Theorem 19B.7 on page 496.) More generally, if f is k times differentiable on $[-\pi,\pi]$, then its Fourier coefficients must decay fast enough that the sums

$\sum_{n=1}^{\infty} n^{k-1}|A_n|$ and $\sum_{n=1}^{\infty} n^{k-1}|B_n|$ converge; see [Kat76, Thm.4.3, p.24]. Finally, if f is *analytic*[1], on $[-\pi,\pi]$, then its Fourier coefficients must decay *exponentially quickly* to zero; that is, for small enough $r > 0$, we have $\lim_{n\to\infty} r^n|A_n| = 0$ and $\lim_{n\to\infty} r^n|B_n| = 0$ (see Proposition 18E.3 on page 458).

At the other extreme, what about a sequence of Fourier coefficients which does *not* satisfy the Riemann-Lebesgue lemma —that is, which does not converge to zero? This corresponds to the Fourier series of an object which is more 'singular' than any function can be: a *Laurent distribution* or a *measure* on $[-\pi,\pi]$, which can have 'infinitely dense' concentrations of mass at some points. See [Kat76, §1.7, pp.34-46] or [Fol84, §8.5 and §8.8 on p.258 and p.281].

10D L^2 convergence

Prerequisites: §6B.

In this section, we will prove Theorem 8A.1(a) (concerning the L^2 convergence of Fourier series). For any $k \in \mathbb{N}$, let $\mathcal{C}^k_{\text{per}}[-\pi,\pi]$ be the set of functions f which are k times continuously differentiable on $[-\pi,\pi]$, and such that $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$, $f''(-\pi) = f''(\pi)$, ..., and $f^{(k)}(-\pi) = f^{(k)}(\pi)$. If $f \in \mathcal{C}^1_{\text{per}}[-\pi,\pi]$, then Theorem 8A.1(d) (which we just proved in §10C) says the Fourier series of f converges *uniformly*. Then Corollary 6E.11(b)[i] (on page 127) immediately implies that the Fourier series of f converges in L^2 norm. Unfortunately, this argument does not work for most functions in $\mathbf{L}^2[-\pi,\pi]$, which are *not* in $\mathcal{C}^1_{\text{per}}[-\pi,\pi]$. Our strategy will be to show that $\mathcal{C}^1_{\text{per}}[-\pi,\pi]$ is *dense* in $\mathbf{L}^2[-\pi,\pi]$; thus, the L^2 convergence of Fourier series in $\mathcal{C}^1_{\text{per}}[-\pi,\pi]$ can be 'leveraged' to obtain L^2 convergence for all functions in $\mathbf{L}^2[-\pi,\pi]$.

A subset $\mathcal{G} \subset \mathbf{L}^2[-\pi,\pi]$ is **dense** in $\mathbf{L}^2[-\pi,\pi]$ if, for any $f \in \mathbf{L}^2[-\pi,\pi]$, and any $\epsilon > 0$, we can find some $g \in \mathcal{G}$ such that $\|f - g\|_2 < \epsilon$. In other words, any element of $\mathbf{L}^2[-\pi,\pi]$ can be approximated arbitrarily closely[2] by elements of $\mathcal{G}$. Aside from Theorem 8A.1(a), the major goal of this section is to prove the following result:

Theorem 10D.1. *For all $k \in \mathbb{N}$, the subset $\mathcal{C}^k_{\text{per}}[-\pi,\pi]$ is dense in $\mathbf{L}^2[-\pi,\pi]$.*

To achieve this goal, we must first examine the structure of integrable functions, and develop some useful machinery involving 'convolutions' and 'mollifiers'. Then we will prove Theorem 10D.1. Once Theorem 10D.1 is established,

[1] See page 570 of Appendix 0H

[2] In the same way, the set $\mathbb{Q}$ of rational numbers is *dense* in the set $\mathbb{R}$ of real numbers: any real number can be approximated arbitrarily closely by rational numbers. Indeed, we exploit this fact every time we approximate a real number using a decimal expansion —e.g. $\pi \approx 3.141592653 = \frac{3141592653}{1000000000}$.

we will prove Theorem 8A.1(a) by using Theorem 8A.1(d) and the triangle inequality.

10D(i) Integrable functions and step functions in $\mathbf{L}^2[-\pi,\pi]$

Prerequisites: §6B, §6E(i).

We have defined $\mathbf{L}^2[-\pi,\pi]$ to be the set of 'integrable' functions $f:[-\pi,\pi]\longrightarrow\mathbb{R}$ such that $\int_{-\pi}^{\pi}|f(x)|^2\,dx<\infty$. But what exactly does *integrable* mean? To explain this, let $\mathsf{Step}[-\pi,\pi]$ be the set of all **step functions** on $[-\pi,\pi]$ (see § 8B(ii) on page 164 for the definition of step functions). If $f:[-\pi,\pi]\longrightarrow\mathbb{R}$ is any bounded function, then we can 'approximate' f using step functions in a natural way. First, let $\mathcal{Y}:=\{-\pi=y_0<y_1<y_2<y_3<\cdots<y_{M-1}<y_M=\pi\}$ be some finite 'mesh' of points in $[-\pi,\pi]$. For all $n\in\mathbb{N}$, let $\underline{a}_n:=\inf_{y_{n-1}\leq x\leq y_n} f(x)$ and $\overline{a}_n:=\sup_{y_{n-1}\leq x\leq y_n} f(x)$. Then define step functions $\underline{S}_{\mathcal{Y}}:[-\pi,\pi]\longrightarrow\mathbb{R}$ and $\overline{S}_{\mathcal{Y}}:[-\pi,\pi]\longrightarrow\mathbb{R}$ by

$$\underline{S}_{\mathcal{Y}}(x) \quad := \quad \begin{cases} \underline{a}_1 & \text{if} \quad -\pi\leq x\leq y_1; \\ \underline{a}_2 & \text{if} \quad y_1<x\leq y_2; \\ & \vdots \\ \underline{a}_m & \text{if} \quad y_{m-1}<x\leq y_m; \\ & \vdots \\ \underline{a}_M & \text{if} \quad y_{M-1}<x\leq \pi. \end{cases}$$

$$\text{and} \qquad \overline{S}_{\mathcal{Y}}(x) \quad := \quad \begin{cases} \overline{a}_1 & \text{if} \quad -\pi\leq x\leq y_1; \\ \overline{a}_2 & \text{if} \quad y_1<x\leq y_2; \\ & \vdots \\ \overline{a}_m & \text{if} \quad y_{m-1}<x\leq y_m; \\ & \vdots \\ \overline{a}_M & \text{if} \quad y_{M-1}<x\leq \pi. \end{cases}$$

It is easy to compute the integrals of $\underline{S}_{\mathcal{Y}}$ and $\overline{S}_{\mathcal{Y}}$:

$$\int_{-\pi}^{\pi}\underline{S}_{\mathcal{Y}}(x)\,dx \quad = \quad \sum_{n=1}^{N}\underline{a}_n\cdot|y_n-y_{n-1}| \qquad \text{and} \qquad \int_{-\pi}^{\pi}\overline{S}_{\mathcal{Y}}(x)\,dx \quad = \quad \sum_{n=1}^{N}\overline{a}_n\cdot|y_n-y_{n-1}|\,.$$

(You may recognize these as upper and lower *Riemann sums* of f). If the mesh $\{y_0,\,y_1,\,y_2\,,\ldots,\,y_M\}$ is 'dense' enough in $[-\pi,\pi]$, so that $\underline{S}_{\mathcal{Y}}$ and $\overline{S}_{\mathcal{Y}}$ are 'good approximations' of f, then we might expect $\int_{-\pi}^{\pi}\underline{S}_{\mathcal{Y}}(x)\,dx$ and $\int_{-\pi}^{\pi}\overline{S}_{\mathcal{Y}}(x)\,dx$ to be good approximations of $\int_{-\pi}^{\pi}f(x)\,dx$ (if the integral of f is well-defined). Furthermore, it is clear from their definitions that $\underline{S}_{\mathcal{Y}}(x)\leq f(x)\leq\overline{S}_{\mathcal{Y}}(x)$ for all

$x \in [-\pi, \pi]$; thus we would expect

$$\int_{-\pi}^{\pi} \underline{S}_{\mathcal{Y}}(x)\, dx \quad \leq \quad \int_{-\pi}^{\pi} f(x)\, dx \quad \leq \quad \int_{-\pi}^{\pi} \overline{S}_{\mathcal{Y}}(x)\, dx,$$

whenever the integral $\int_{-\pi}^{\pi} f(x)\, dx$ exists. Let $\mathfrak{Y}$ be the set of all finite 'meshes' of points in $[-\pi, \pi]$. Formally:

$$\mathfrak{Y} \;:=\; \left\{ \begin{array}{lll} \mathcal{Y} \subset [-\pi,\pi] & ; & \mathcal{Y} = \{y_0, y_1, y_2, \ldots, y_M\}, \;\; \text{for some} \;\; M \in \mathbb{N} \\ & & \text{and} \;\; -\pi = y_0 < y_1 < y_2 < \cdots < y_M = \pi \end{array} \right\}.$$

We define the **lower** and **upper semi-integrals** of f:

$$\begin{aligned} \underline{I}(f) \quad &:= \quad \sup_{\mathcal{Y} \in \mathfrak{Y}} \int_{-\pi}^{\pi} \underline{S}_{\mathcal{Y}}(x)\, dx && \text{(10D.1)} \\ &= \quad \sup \left\{ \sum_{n=1}^{N} \left(|y_n - y_{n-1}| \cdot \inf_{y_{n-1} \leq x \leq y_n} f(x) \right) \; ; \; M \in \mathbb{N}, \; -\pi = y_0 < y_1 < \cdots < y_M = \pi \right\}. \\ \overline{I}(f) \quad &:= \quad \inf_{\mathcal{Y} \in \mathfrak{Y}} \int_{-\pi}^{\pi} \overline{S}_{\mathcal{Y}}(x)\, dx && \text{(10D.2)} \\ &= \quad \inf \left\{ \sum_{n=1}^{N} \left(|y_n - y_{n-1}| \cdot \sup_{y_{n-1} \leq x \leq y_n} f(x) \right) \; ; \; M \in \mathbb{N}, \; -\pi = y_0 < y_1 < \cdots < y_M = \pi \right\}. \end{aligned}$$

It is easy to see that $\underline{I}(f) \leq \overline{I}(f)$ (**Exercise 10D.1** Check this.). Indeed, if f is a sufficiently 'pathological' function, then we may have $\underline{I}(f) < \overline{I}(f)$. If $\underline{I}(f) = \overline{I}(f)$, then we say that f is **(Riemann) integrable**, and we define the **(Riemann) integral** of f: Ⓔ

$$\int_{-\pi}^{\pi} f(x)\, dx \quad := \quad \underline{I}(f) \quad = \quad \overline{I}(f).$$

For example:

- Any bounded, piecewise continuous function on $[-\pi, \pi]$ is Riemann-integrable.
- Any continuous function on $[-\pi, \pi]$ is Riemann-integrable.
- Any step function on $[-\pi, \pi]$ is Riemann-integrable.

If $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is *not* bounded, then the definitions of $\underline{S}_{\mathcal{Y}}$ and/or $\overline{S}_{\mathcal{Y}}$ make no sense (because at least one of them is defined as "∞" or "$-\infty$" on some interval). Thus, at least one of the expressions (10D.1) and (10D.2) is not well-defined if f is unbounded. In this case, for any $N \in \mathbb{N}$, we define the 'truncated' functions $f_N^+ : [-\pi, \pi] \longrightarrow [0, N]$ and $f_N^- : [-\pi, \pi] \longrightarrow [-N, 0]$ as follows

$$f_N^+(x) \quad := \quad \left\{ \begin{array}{rll} 0 & \text{if} & f(x) \leq 0; \\ f(x) & \text{if} & 0 \leq f(x) \leq N; \\ N & \text{if} & N \leq f(x). \end{array} \right.$$

$$\text{and} \qquad f_N^-(x) \quad := \quad \begin{cases} -N & \text{if} \quad f(x) \le -N; \\ f(x) & \text{if} \quad -N \le f(x) \le 0; \\ 0 & \text{if} \quad 0 \le f(x). \end{cases}$$

The functions f_N^+ and f_N^- are clearly bounded, so their Riemann integrals are potentially well-defined. If f_N^+ and f_N^- are integrable for all $N \in \mathbb{N}$, then we say that f is **(Riemann) measurable**. We then define

$$\int_{-\pi}^{\pi} f(x)\ dx \quad := \quad \lim_{N\to\infty} \int_{-\pi}^{\pi} f_N^+(x)\ dx \ + \ \lim_{N\to\infty} \int_{-\pi}^{\pi} f_N^-(x)\ dx.$$

If both these limits are finite, then $\int_{-\pi}^{\pi} f(x)\ dx$ is well-defined, and we say that the unbounded function f is **(Riemann) integrable**. The set of all integrable functions (bounded or unbounded) is denoted $\mathbf{L}^1[-\pi,\pi]$, and for any $f \in \mathbf{L}^1[-\pi,\pi]$, we define

$$\|f\|_1 \quad = \quad \int_{-\pi}^{\pi} |f(x)|\ dx.$$

We can now define $\mathbf{L}^2[-\pi,\pi]$:

$$\mathbf{L}^2[-\pi,\pi] \quad := \quad \left\{ \begin{array}{c} \text{all measurable functions } f:[-\pi,\pi] \longrightarrow \mathbb{R} \text{ such that} \\ f^2:[-\pi,\pi] \longrightarrow \mathbb{R} \text{ is integrable —i.e. } \int_{-\pi}^{\pi} |f(x)|^2\ dx \ < \ \infty \end{array} \right\}.$$

Proposition 10D.2. *$\mathsf{Step}[-\pi,\pi]$ is dense in $\mathbf{L}^2[-\pi,\pi]$.*

Proof. Let $f \in \mathbf{L}^2[-\pi,\pi]$ and let $\epsilon > 0$. We want to find some $S \in \mathsf{Step}[-\pi,\pi]$ such that $\|f - S\|_2 < \epsilon$.

First suppose that f is *bounded*. Since f^2 is integrable, we know that $\underline{I}(f^2) = \int_{-\pi}^{\pi} f^2(x)\ dx$, where $\underline{I}(f^2)$ is defined by expression (10D.1). Thus, we can find some step function $S_0 \in \mathsf{Step}[-\pi,\pi]$ such that $0 \le S_0(x) \le f^2(x)$ for all $x \in [-\pi,\pi]$, and such that

$$0 \quad \le \quad \int_{-\pi}^{\pi} f(x)^2\ dx \ - \ \int_{-\pi}^{\pi} S_0(x)\ dx \quad < \quad \epsilon. \tag{10D.3}$$

Define the step function $S \in \mathsf{Step}[-\pi,\pi]$ by $S(x) := \text{sign}[f(x)] \cdot \sqrt{S_0(x)}$. Thus, $S^2(x) = S_0(x)$, and the sign of S agrees with that of f everywhere. Observe that

$$\begin{aligned} (f-S)^2 \quad &= \quad \frac{f-S}{f+S} \cdot (f-S)(f+S) \quad = \quad \frac{f-S}{f+S} \cdot (f^2 - S^2) \\ &\underset{(*)}{\le} \quad f^2 - S^2 \quad = \quad f^2 - S_0. \end{aligned} \tag{10D.4}$$

Here, $(*)$ is because $0 < \frac{f-S}{f+S} < 1$ (because for all x, either $f(x) \leq S(x) \leq 0$, or $0 \leq S(x) \leq f(x)$), while $f^2 - S^2 \geq 0$ (because $f^2 \geq S_0 = S^2$). Thus,

$$0 \quad \leq \quad \|f - S\|_2^2 \quad = \quad \int_{-\pi}^{\pi} (f(x) - S(x))^2 \, dx \quad \underset{(*)}{\leq} \quad \int_{-\pi}^{\pi} f(x)^2 - S_0(x) \, dx$$
$$= \quad \int_{-\pi}^{\pi} f(x)^2 \, dx \; - \; \int_{-\pi}^{\pi} S_0(x) \, dx \quad \underset{(\dagger)}{<} \quad \epsilon,$$

where $(*)$ is by eqn.(10D.4) and $(\dagger)$ is by eqn.(10D.3).

This works for any $\epsilon > 0$; thus the set $\mathsf{Step}[-\pi, \pi]$ is dense in the space of bounded elements of $\mathbf{L}^2[-\pi, \pi]$.

The case when f is unbounded is **Exercise 10D.2** (Hint: approximate f with bounded functions). □ Ⓔ

Remark 10D.3: To avoid developing a considerable amount of technical background, we have defined $\mathbf{L}^2[-\pi, \pi]$ using the *Riemann* integral. The 'true' definition of $\mathbf{L}^2[-\pi, \pi]$ involves the more powerful and versatile *Lebesgue* integral. (See § 6C(ii) on page 110 for an earlier discussion of Lebesgue integration). The definition of the Lebesgue integral is similar to the Riemann integral, but instead of approximating f using step functions, we use *simple* functions. A simple function is a piecewise-constant function, like a step function, but instead of open intervals, the 'pieces' of a simple function are *Borel-measurable subsets* of $[-\pi, \pi]$. A Borel measurable subset is a countable union of countable intersections of countable unions of countable intersections of .... of countable unions/intersections of open and/or closed subsets of $[-\pi, \pi]$. In particular, any interval is Borel measurable (so any step function is a simple function), but Borel measurable subsets can be very complicated indeed. Thus, 'simple' functions are capable of approximating even pathological, wildly discontinuous functions on $[-\pi, \pi]$, so that the Lebesgue integral can be evaluated even on such crazy functions. The set of Lebesgue-integrable functions is thus much larger than the set of Riemann-integrable functions. Every Riemann-integrable function is Lebesgue integrable (and its Lebesgue integral is the same as its Riemann integral), but not vice versa.

The analogy of Proposition 10D.2 is still true if we define $\mathbf{L}^2[-\pi, \pi]$ using *Lebesgue*-integrable functions, and if we replace $\mathsf{Step}[-\pi, \pi]$ with the set of all *simple* functions. The other results in this section can also be extended to the Lebesgue version of $\mathbf{L}^2[-\pi, \pi]$, but at the cost of considerable technical complexity. ♢

Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$. Let $\overset{\circlearrowleft}{f} : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ be the 2π-**periodic exten-**

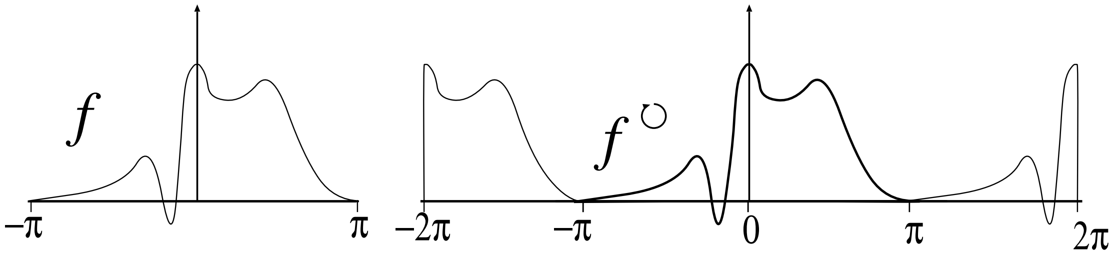


Figure 10D.1: $f^{\circlearrowleft} : [-2\pi, 2\pi]$ is the 2π-periodic extension of $f : [-\pi, \pi] \longrightarrow \mathbb{R}$.

sion of f, defined:

$$f^{\circlearrowleft}(x) \quad := \quad \begin{cases} f(x+2\pi) & \text{if} \quad -2\pi \le x < -\pi; \\ f(x) & \text{if} \quad -\pi \le x \le \pi; \\ f(x-2\pi) & \text{if} \quad \pi < x \le 2\pi. \end{cases} \qquad \text{(See Figure 10D.1)}$$

(Observe that $f^{\circlearrowleft}$ is continuous if and only if f is continous and $f(-\pi) = f(\pi)$). For any $t \in \mathbb{R}$, define the function $f^{\overset{\curvearrowleft}{t}} : [-\pi, \pi] \longrightarrow \mathbb{R}$ by $f^{\overset{\curvearrowleft}{t}}(x) = f^{\circlearrowleft}(x - t)$. (For example, the function $\overset{\curvearrowright}{f}$ defined on page 203 could be written: $\overset{\curvearrowright}{f} = f^{\overset{\curvearrowleft}{-x}}$; see Figure 10B.3 on page 203).

Lemma 10D.4. *Let $f \in \mathbf{L}^2[-\pi, \pi]$. Then* $f \;=\; \mathbf{L}^2\text{–}\lim_{t\to 0} f^{\overset{\curvearrowleft}{t}}$.

Proof. We will employ a classic strategy in real analysis: first prove the result for some 'nice' class of functions, and then prove it for all functions by approximating them with these nice functions. In this case, the nice functions are the step functions.

Claim 1: *Let $S \in \mathsf{Step}[-\pi, \pi]$. Then* $S \;=\; \mathbf{L}^2\text{–}\lim_{t\to 0} S^{\overset{\curvearrowleft}{t}}$.

Ⓔ *Proof.* **Exercise 10D.3** $\diamond_{\text{Claim 1}}$

Now, let $f \in \mathbf{L}^2[-\pi, \pi]$, and let $\epsilon > 0$. Proposition 10D.2 says there is some $S \in \mathsf{Step}[-\pi, \pi]$ such that

$$\|S - f\|_2 \quad < \quad \frac{\epsilon}{3}. \tag{10D.5}$$

Claim 2: *For all $t \in \mathbb{R}$,* $\left\| S^{\overset{\curvearrowleft}{t}} - f^{\overset{\curvearrowleft}{t}} \right\|_2 = \|S - f\|_2$.

Ⓔ *Proof.* **Exercise 10D.4** $\diamond_{\text{Claim 2}}$

Now, using Claim 1, find $\delta > 0$ such that, if $|t| < \delta$, then

$$\left\| S - S^{\overset{\curvearrowleft}{t}} \right\|_2 \quad < \quad \frac{\epsilon}{3}. \tag{10D.6}$$

Then

$$\left\| f - f^{\frown t} \right\|_2 \quad = \quad \left\| f - S \;+\; S - S^{\frown t} \;+\; S^{\frown t} - f^{\frown t} \right\|_2$$
$$\underset{(\triangle)}{\leq} \quad \|f-S\|_2 \;+\; \left\| S - S^{\frown t} \right\|_2 \;+\; \left\| S^{\frown t} - f^{\frown t} \right\|_2 \quad \underset{(*)}{\leq} \quad \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad = \quad \epsilon.$$

Here $(\triangle)$ is the triangle inequality, and $(*)$ is by equations (10D.5) and (10D.6) and Claim 2.

This works for all $\epsilon > 0$; thus, $f \;=\; \mathbf{L}^2\text{–}\lim_{t\to 0} f^{\frown t}$. □

There is one final technical result we will need about $\mathbf{L}^2[-\pi,\pi]$. If $f_1, f_2, \ldots, f_N \in \mathbf{L}^2[-\pi,\pi]$ and $r_1, r_2, \ldots, r_N \in \mathbb{R}$ are real numbers, then the triangle inequality implies that

$$\|r_1 f_1 + r_2 f_2 + \cdots + r_N f_N\|_2 \quad \leq \quad |r_1| \cdot \|f_1\|_2 + |r_2| \cdot \|f_2\|_2 + \cdots + |r_N| \cdot \|f_N\|_2 \,.$$

This is a special case of *Minkowski's inequality*. The next result says that the same inequality holds if we sum together a 'continuum' of functions.

Theorem 10D.5. (Minkowski's inequality for integrals)
Let $a < b$, and for all $t \in [a,b]$, let $f_t \in \mathbf{L}^2[-\pi,\pi]$. Define $F : [a,b]\times[-\pi,\pi] \longrightarrow \mathbb{R}$ by $F(t,x) = f_t(x)$ for all $(t,x) \in [a,b] \times [-\pi,\pi]$, and suppose that the family $\{f_t\}_{t\in[a,b]}$ is such that the function F is integrable on $[a,b] \times [-\pi,\pi]$. Let $R : [a,b] \longrightarrow \mathbb{R}$ be some other integrable function, and define $G : [-\pi,\pi] \longrightarrow \mathbb{R}$ by

$$G(x) \quad := \quad \int_a^b R(t) f_t(x)\; dt, \qquad \text{for all } x \in [-\pi,\pi].$$

Then $G \in \mathbf{L}^2[-\pi,\pi]$, and

$$\|G\|_2 \quad \leq \quad \int_a^b |R(t)| \cdot \|f_t\|_2 \; dt,$$

In particular, if $\|f_t\|_2 < M$ for all $t \in [a,b]$, then $\|G\|_2 \;\leq\; M \cdot \|R\|_1$, where $\|R\|_1 := \int_a^b |R(t)|\; dt$.

Proof. See [Fol84, Thm 6.18, p.186]. □

10D(ii) Convolutions and mollifiers

Prerequisites: §10D(i). **Recommended:** §17B.

Let $f, g : [-\pi, \pi] \longrightarrow \mathbb{R}$ be two integrable functions. Let $\overset{\circlearrowleft}{g} : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ be the 2π-**periodic extension** of g (see Figure 10D.1 on page 212). The (2π-periodic) **convolution** of f and g is the function $f * g : [-\pi, \pi] \longrightarrow \mathbb{R}$ defined by

$$f * g(x) \quad := \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \overset{\circlearrowleft}{g}(x-y)\ dy, \qquad \text{for all } x \in [-\pi, \pi].$$

Convolution is an important and versatile mathematical operation, which appears frequently in harmonic analysis, probability theory, and the study of partial differential equations. We will encounter it again in Chapter 17, in the context of 'impulse-response' solutions to boundary value problems. In this subsection, we will develop the theory of convolutions on $[-\pi, \pi]$. We will actually develop slightly more than we need in order to prove Theorems 10D.1 and 8A.1(a). Results which are not logically required for the proofs of Theorems 10D.1 and 8A.1(a) are marked with the margin symbol '(Optional)' and can be skipped on a first reading; however, we feel that these results are interesting enough in themselves to be worth including in the exposition.

Lemma 10D.6. (Properties of convolutions)
Let $f, g : [-\pi, \pi] \longrightarrow \mathbb{R}$ be integrable functions. The convolution of f and g has the following properties:

(a) (Commutativity) $f * g = g * f$.

(b) (Linearity) *If $h : [-\pi, \pi] \longrightarrow \mathbb{R}$ is another integrable function, then $f * (g + h) = f * g + f * h$ and $(f + g) * h = f * h + g * h$.*

(Optional) **(c)** *If $f, g \in \mathbf{L}^2[-\pi, \pi]$, then $f * g$ is bounded: for all $x \in [-\pi, \pi]$, we have $|f * g(x)| \ \le \ \|f\|_2 \cdot \|g\|_2$.* (In other words, $\|f * g\|_\infty \le \|f\|_2 \cdot \|g\|_2$.)

Ⓔ *Proof.* **(a)** is **Exercise 10D.5** . To prove **(b)**, let $x \in [-\pi, \pi]$. Then

$$\begin{aligned} f * (g+h)(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\overset{\circlearrowleft}{g}(x-y) + \overset{\circlearrowleft}{h}(x-y) \right) dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \overset{\circlearrowleft}{g}(x-y)\ dy \ + \ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \overset{\circlearrowleft}{h}(x-y)\ dy \\ &= f * g(x) + f * h(x). \end{aligned}$$

(Optional) **(c)** Let $x \in [-\pi, \pi]$. Define $h \in \mathbf{L}^2[-\pi, \pi]$ by $h(y) := \overset{\circlearrowleft}{g}(x-y)$. Then

$$f * g(x) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \overset{\circlearrowleft}{g}(x-y)\ dy \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) h(y)\ dy \quad = \quad \langle f, h \rangle.$$

Thus, $\quad |f * g(x)| \quad = \quad |\langle f, h \rangle| \quad \underset{\text{(CBS)}}{\le} \quad \|f\|_2 \cdot \|h\|_2, \qquad \text{(10D.7)}$

where (CBS) is the Cauchy-Bunyakowski-Schwarz inequality (Theorem 6B.5 on page 108). But

$$\begin{aligned}\|h\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(y)^2\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overset{\circlearrowleft}{g}(x-y)^2\,dy \underset{(*)}{=} \frac{-1}{2\pi}\int_{x+\pi}^{x-\pi} \overset{\circlearrowleft}{g}(z)^2\,dz \\ &= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} \overset{\circlearrowleft}{g}(z)^2\,dz \underset{(\dagger)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(z)^2\,dz = \|g\|_2^2. \qquad (10D.8)\end{aligned}$$

Here, $(*)$ is the change of variables $z = x - y$ (so that $dz = -dy$) and $(\dagger)$ is by definition of the periodic extension $\overset{\circlearrowleft}{g}$ of g.

Combining equations (10D.7) and (10D.8) we conclude that $|f * g(x)| < \|f\|_2 \cdot \|g\|_2$, as claimed. □

Remarks. (a) Proposition 17G.1 on page 409 provides an analog to Lemma 10D.6 for convolutions on $\mathbb{R}^D$.

(b) There is also an interesting relationship between convolution and complex Fourier coefficients; see Lemma 18F.3 on page 463. ♢

Elements of $\mathbf{L}^1[-\pi,\pi]$ and $\mathbf{L}^2[-\pi,\pi]$ need not be differentiable, or even continuous (indeed, some of these functions are discontinuous 'almost everywhere'). But the convolution of even two highly discontinuous elements of $\mathbf{L}^2[-\pi,\pi]$ will be a continuous function. Furthermore, convolution with a smooth function has a powerful 'smoothing' effect on even the nastiest elements of $\mathbf{L}^1[-\pi,\pi]$.

Lemma 10D.7. *Let $f, g \in \mathbf{L}^1[-\pi,\pi]$.*

(a) $f * g(-\pi) = f * g(\pi)$.

(b) *If $f \in \mathbf{L}^1[-\pi,\pi]$ and g is continuous with $g(-\pi) = g(\pi)$, then $f * g$ is continuous.*

(c) *If $f, g \in \mathbf{L}^2[-\pi,\pi]$, then $f * g$ is continuous.* (Optional)

(d) *If g is differentiable on $[-\pi,\pi]$, then $f * g$ is also differentiable on $[-\pi,\pi]$, and $(f * g)' = f * (g')$.*

(e) *If $g \in \mathcal{C}^1[-\pi,\pi]$, then $f * g \in \mathcal{C}^1_{\mathrm{per}}[-\pi,\pi]$.*

(f) *For any $k \in \mathbb{N}$, if $g \in \mathcal{C}^k[-\pi,\pi]$, then[3] $f * g \in \mathcal{C}^k_{\mathrm{per}}[-\pi,\pi]$. Furthermore, $(f * g)' = f * g'$, $(f * g)'' = f * g''$, ..., and $(f * g)^{(k)} = f * g^{(k)}$.* (Optional)

[3]See page 207 for the definition of $\mathcal{C}^k_{\mathrm{per}}[-\pi,\pi]$.

Proof. **(a)** $f * g(\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\overset{\circlearrowleft}{g}(\pi - y)\ dy \underset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\overset{\circlearrowleft}{g}(\pi - y - 2\pi)\ dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\overset{\circlearrowleft}{g}(-\pi - y)\ dy = f * g(-\pi)$. Here, $(*)$ is because $\overset{\circlearrowleft}{g}$ is 2π-periodic.

(b) Fix $x \in [-\pi, \pi]$ and let $\epsilon > 0$. We must find some $\delta > 0$ such that, for any $x_1 \in [-\pi, \pi]$, if $|x - x_1| < \delta$ then $|f * g(x) - f * g(x_1)| < \epsilon$. But

$$\begin{aligned} f * g(x) - f * g(x_1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, \overset{\circlearrowleft}{g}(x - y)\ dy \ - \ \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, \overset{\circlearrowleft}{g}(x_1 - y)\ dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, \overset{\circlearrowleft}{g}(x - y) - f(y)\overset{\circlearrowleft}{g}(x_1 - y)\ dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\ \left(\overset{\circlearrowleft}{g}(x - y) - \overset{\circlearrowleft}{g}(x_1 - y)\right)\ dy. \end{aligned} \tag{10D.9}$$

Since g is continuous on $[-\pi, \pi]$ and $g(-\pi) = g(\pi)$, it follows that $\overset{\circlearrowleft}{g}$ is continuous on $[-2\pi, 2\pi]$; since $[-2\pi, 2\pi]$ is a closed and bounded set, it then follows that $\overset{\circlearrowleft}{g}$ is *uniformly* continuous on $[-2\pi, 2\pi]$. That is, there is some $\delta > 0$ such that, for any $z, z_1 \in [-2\pi, 2\pi]$,

$$\text{if } |z - z_1| < \delta, \text{ then } \quad |\overset{\circlearrowleft}{g}(z) - \overset{\circlearrowleft}{g}(z_1)| \quad < \quad \frac{2\pi\epsilon}{\|f\|_1}. \tag{10D.10}$$

Now, suppose $|x - x_1| < \delta$. Then

$$\begin{aligned} |f * g(x) - f * g(x_1)| &\underset{(*)}{=} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\overset{\circlearrowleft}{g}(x - y) - \overset{\circlearrowleft}{g}(x_1 - y)\right)\ dy\right| \\ &\underset{(\triangle)}{\leq} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot \left|\overset{\circlearrowleft}{g}(x - y) - \overset{\circlearrowleft}{g}(x_1 - y)\right|\ dy \\ &\underset{(\dagger)}{\leq} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot \frac{2\pi\epsilon}{\|f\|_1}\ dy \quad = \quad \frac{\epsilon}{\|f\|_1} \cdot \int_{-\pi}^{\pi} |f(y)|\ dy \\ &= \frac{\epsilon}{\|f\|_1} \cdot \|f\|_1 \quad = \quad \epsilon. \end{aligned}$$

Here, $(*)$ is by eqn.(10D.9), $(\triangle)$ is the triangle inequality for integrals, and $(\dagger)$ is by eqn.(10D.10), because $|(x - y) - (x_1 - y)| < \delta$ for all $y \in [-\pi, \pi]$, because $|x - x_1| < \delta$.

Thus, if $|x - x_1| < \delta$ then $|f * g(x) - f * g(x_1)| < \epsilon$. This argument works for any $\epsilon > 0$ and $x \in [-\pi, \pi]$. Thus, $f * g$ is continuous, as desired.

(Optional) **(c)** Fix $x \in [-\pi, \pi]$ and let $\epsilon > 0$. We must find some $\delta > 0$ such that, for any $t \in [-\pi, \pi]$, if $|t| < \delta$ then $|f * g(x) - f * g(x - t)| < \epsilon$. But

$$f * g(x - t) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, \overset{\circlearrowleft}{g}(x - t - y)\ dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,(g^{\stackrel{\curvearrowleft}{t}})^{\circ}(x-y)\;dy \quad = \quad f * g^{\stackrel{\curvearrowleft}{t}}(x).$$

$$\text{Thus, } f*g(x) - f*g(x-t) \quad = \quad f*g(x) - f*g^{\stackrel{\curvearrowleft}{t}}(x) \underset{(*)}{=} f*(g - g^{\stackrel{\curvearrowleft}{t}})(x),$$

$$\text{so } \left|f*g(x) - f*g(x-t)\right| \quad = \quad \left|f*(g-g^{\stackrel{\curvearrowleft}{t}})(x)\right|$$

$$\underset{(\dagger)}{\leq} \quad \|f\|_2 \cdot \left\|g - g^{\stackrel{\curvearrowleft}{t}}\right\|_2. \tag{10D.11}$$

Here, $(*)$ is by Lemma 10D.6(b) and $(\dagger)$ is by Lemma 10D.6(c).

However, Lemma 10D.4 on page 212 says that $g = \mathbf{L}^2\text{–}\lim_{t\to 0} g^{\stackrel{\curvearrowleft}{t}}$. Thus, there exists some $\delta > 0$ such that, if $|t| < \delta$, then $\left\|g - g^{\stackrel{\curvearrowleft}{t}}\right\|_2 < \epsilon / \|f\|_2$. Thus, if $|t| < \delta$, then

$$\left|f*g(x) - f*g(x-t)\right| \quad \underset{(*)}{\leq} \quad \|f\|_2 \cdot \left\|g - g^{\stackrel{\curvearrowleft}{t}}\right\|_2 \quad \leq \quad \|f\|_2 \cdot \frac{\epsilon}{\|f\|_2} \quad = \quad \epsilon.$$

where $(*)$ is by eqn.(10D.11). This argument works for any $\epsilon > 0$ and $x \in [-\pi, \pi]$. Thus, $f*g$ is continuous, as desired.

(d) We have

$$\begin{aligned}
2\pi\,(f*g)'(x) \quad &= \quad 2\pi\,\partial_x\,(f*g)(x) \quad = \quad \partial_x \int_{-\pi}^{\pi} f(y)\cdot g(x-y)\;dy \\
&\underset{(*)}{=} \quad \int_{-\pi}^{\pi} f(y)\cdot \partial_x\, g(x-y)\;dy \quad = \quad \int_{-\pi}^{\pi} f(y)\cdot g'(x-y)\;dy \\
&= \quad 2\pi\, f*(g')(x).
\end{aligned}$$

Here, $(*)$ is by Proposition 0G.1 on page 567.

(e) Follows immediately from **(a)**, **(b)** and **(d)**.

(f) is <u>**Exercise 10D.6**</u> Hint: Use proof by induction, along with parts **(b)** and **(d)**. □ Ⓔ

Remarks. (a) Proposition 17G.2 on page 410 provides an analog to Lemma 10D.7 for convolutions on $\mathbb{R}^D$.

(b) (for algebraists) Let $\mathcal{C}_{\text{per}}[-\pi,\pi]$ be the set of all continuous functions $f : [-\pi,\pi] \longrightarrow \mathbb{R}$ such that $f(-\pi) = f(\pi)$. Then Lemmas 10D.6(a,b) and 10D.7(a,b) imply that $\mathcal{C}_{\text{per}}[-\pi,\pi]$ is a *commutative ring*, where functions are added pointwise, and where the convolution operator '$*$' plays the role of 'multiplication'. Furthermore, Lemma 10D.7(f) says that, for all $k \in \mathbb{N}$, the set $\mathcal{C}^k_{\text{per}}[-\pi,\pi]$ is an *ideal* of the ring $\mathcal{C}_{\text{per}}[-\pi,\pi]$. Note that this ring does *not* have

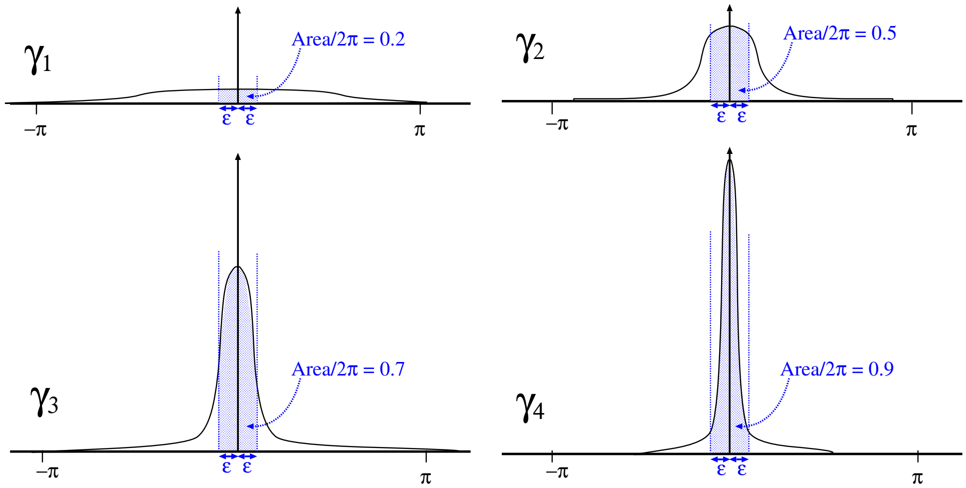


Figure 10D.2: An approximation of identity on $[-\pi, \pi]$. Here, $\epsilon > 0$ is fixed, and $\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \gamma_n(x)\, dx = 1$.

a multiplicative identity element. However, it does have 'approximations' of identity, as we shall now see. ♢

For all $n \in \mathbb{N}$, let $\gamma_n : [-\pi, \pi] \longrightarrow \mathbb{R}$ be a nonnegative function. The sequence $\{\gamma_n\}_{n=1}^{\infty}$ is called an **approximation of identity** if it has the following properties:

(AI1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_n(y)\, dy = 1$ for all $n \in \mathbb{N}$.

(AI2) For any $\epsilon > 0$, $\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \gamma_n(x)\, dx = 1$. (See Figure 10D.2).

Example 10D.8. Let $\Gamma : [-\pi, \pi] \longrightarrow \mathbb{R}$ be any nonnegative function with $\frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma(x)\, dx = 1$. For all $n \in \mathbb{N}$, define $\gamma_n : [-\pi, \pi] \longrightarrow \mathbb{R}$ by

$$\gamma_n(x) \quad := \quad \begin{cases} 0 & \text{if} \quad x < -\pi/n; \\ n\Gamma(nx) & \text{if} \quad -\pi/n \leq x \leq \pi/n; \\ 0 & \text{if} \quad \pi/n < x. \end{cases} \qquad \text{(see Figure 10D.3).}$$

Ⓔ Then $\{\gamma_n\}_{n=1}^{\infty}$ is a 2π-periodic approximation of identity (**Exercise 10D.7**). ♢

The term 'approximation of identity' is due to the following result:

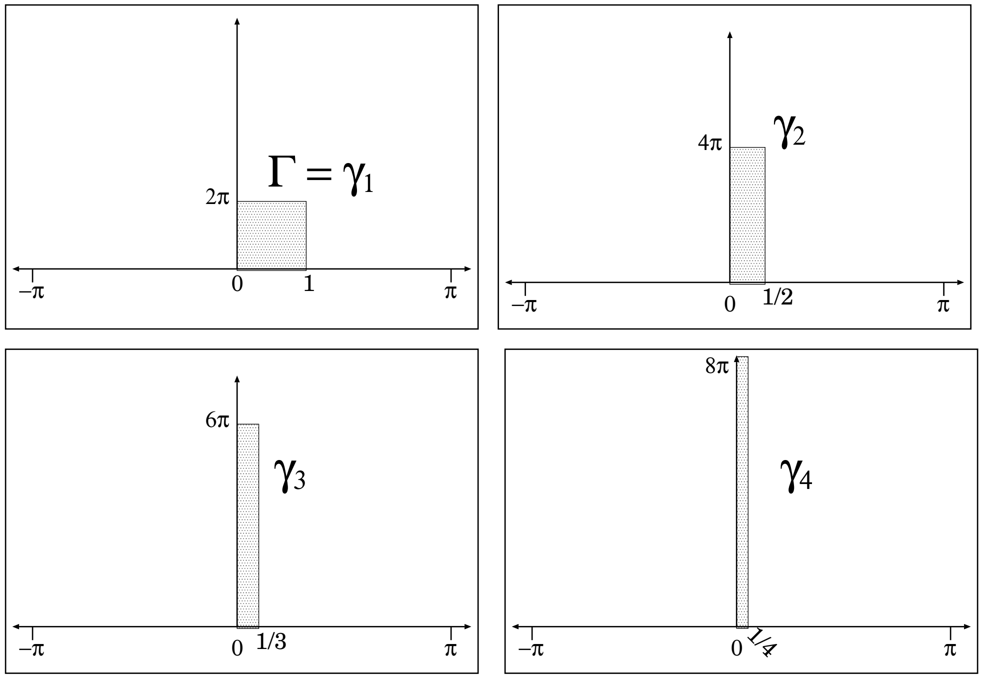


Figure 10D.3: Example 10D.8.

Proposition 10D.9. *Let $\{\gamma_n\}_{n=1}^{\infty}$ be an approximation of identity. Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be some integrable function.*

(a) *If $f \in \mathbf{L}^2[-\pi, \pi]$ then $f \;=\; \mathbf{L}^2\text{-}\lim_{n\to\infty} \gamma_n * f$.*

(b) *If $x \in (-\pi, \pi)$ and f is continuous at x, then $f(x) = \lim_{n\to\infty} \gamma_n * f(x)$.* (Optional)

Proof. **(a)** Fix $\epsilon > 0$. We must find $N \in \mathbb{N}$ such that, for all $n > N$, $\|f - \gamma_n * f\|_2 < \epsilon$. First, find some $\eta > 0$ which is small enough that

$$(2\|f\|_2 + 1) \cdot \eta \quad < \quad \epsilon. \tag{10D.12}$$

Now, Lemma 10D.4 on page 212 says that there is some $\delta > 0$ such that,

$$\text{For any } t \in (-\delta, \delta) \qquad \left\| f - f^{\overset{\curvearrowleft}{t}} \right\|_2 \quad < \quad \eta. \tag{10D.13}$$

Next, property **(AI2)** says there is some $N \in \mathbb{N}$ such that,

$$\text{For all } n > N, \qquad 1 - \eta \quad < \quad \frac{1}{2\pi}\int_{-\delta}^{\delta} \gamma_n(y)\, dy \quad \leq \quad 1. \tag{10D.14}$$

Now, for any $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$, observe that

$$f(x) \quad = \quad f(x)\cdot 1 \quad \underset{(*)}{=} \quad f(x)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma_n(y)\, dy \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\gamma_n(y)\, dy \tag{10D.15}$$

where $(*)$ is by property **(AI1)**. Thus, for all $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
f(x) - \gamma_n * f(x) &\underset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\gamma_n(t)\ dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \overset{\circlearrowleft}{f}(x-t)\,\gamma_n(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(f(x) - \overset{\circlearrowleft}{f}(x-t)\Big)\ \gamma_n(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(f(x) - \overset{\curvearrowleft}{f^{t}}(x)\Big)\ \gamma_n(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma_n(t)\cdot F_t(x)\ dt,
\end{aligned}
$$

where, $(*)$ is by eqn.(10D.15), and where, for all $t \in [-\pi, \pi]$ we define the function $F_t : [-\pi, \pi] \longrightarrow \mathbb{R}$ by $F_t(x) := f(x) - \overset{\curvearrowleft}{f^{t}}(x)$ for all $x \in [-\pi, \pi]$. Thus

$$
\begin{aligned}
\|f - \gamma_n * f\|_2 \quad &= \quad \left\| \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma_n(t)\cdot F_t\ dt \right\|_2 \quad \underset{(\mathrm{M})}{\leq} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\gamma_n(t)|\cdot \|F_t\|_2\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\delta} \gamma_n(t)\cdot\|F_t\|_2\ dt \ + \ \frac{1}{2\pi}\int_{-\delta}^{\delta} \gamma_n(t)\cdot\|F_t\|_2\ dt \ + \ \frac{1}{2\pi}\int_{\delta}^{\pi} \gamma_n(t)\cdot\|F_t\|_2\ dt \\
&\underset{(*)}{\leq} \frac{\|f\|_2}{\pi}\int_{-\pi}^{-\delta} \gamma_n(t)\ dt \ + \ \frac{1}{2\pi}\int_{-\delta}^{\delta} \gamma_n(t)\cdot\|F_t\|_2\ dt \ + \ \frac{\|f\|_2}{\pi}\int_{\delta}^{\pi} \gamma_n(t)\ dt \\
&\underset{(\dagger)}{\leq} \frac{\|f\|_2}{\pi}\left(\int_{-\pi}^{-\delta} \gamma_n(t)\ dt + \int_{\delta}^{\pi} \gamma_n(t)\ dt\right) \ + \ \frac{\eta}{2\pi}\int_{-\delta}^{\delta} \gamma_n(t)\ dt \\
&\underset{(\diamond)}{<} 2\|f\|_2\cdot\eta \ + \ \eta\cdot 1 \quad = \quad (2\|f\|_2 + 1)\cdot\eta \quad \underset{(\ddagger)}{\leq} \quad \epsilon.
\end{aligned}
$$

Here, (M) is Minkowski's inequality for integrals (Theorem 10D.5 on page 213). Next, $(*)$ is because

$$
\|F_t\|_2 \quad = \quad \left\| f - \overset{\curvearrowleft}{f^{t}} \right\|_2 \quad \underset{(\triangle)}{\leq} \quad \|f\|_2 + \left\| \overset{\curvearrowleft}{f^{t}} \right\|_2 \quad = \quad \|f\|_2 + \|f\|_2 \quad = \quad 2\|f\|_2 .
$$

Next, $(\dagger)$ is because $\|F_t\|_2 < \eta$ for all $t \in (-\delta, \delta)$ by equation (10D.13). Inequality $(\diamond)$ is because equation (10D.14) says $1 - \eta < \frac{1}{2\pi}\int_{-\delta}^{\delta} \gamma_n(t)\ dt \leq 1$; thus, we must have $\frac{1}{2\pi}\int_{-\pi}^{-\delta} \gamma_n(t)\ dt + \frac{1}{2\pi}\int_{\delta}^{\pi} \gamma_n(t)\ dt < \eta$. Finally, $(\ddagger)$ is by eqn.(10D.12).

This argument works for any $\epsilon > 0$. We conclude that $f \ = \ \mathbf{L}^2\text{-}\lim_{n\to\infty} \gamma_n * f$.

Ⓔ **(b)** is **<u>Exercise 10D.8</u>** . □

For any $k \in \mathbb{N}$, a $\mathcal{C}^k$-**mollifier** is an approximation of identity $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n \in \mathcal{C}^k_{\text{per}}[-\pi, \pi]$ for all $n \in \mathbb{N}$. Lemma 10D.7(f) says that you can 'mollify' some initially pathological function f into a nice smooth approximation by convolving it with γ_n. Our last task in this section is to show how to construct such a $\mathcal{C}^k$-mollifier.

Lemma 10D.10. *Let $\Gamma \in \mathcal{C}^k[-\pi,\pi]$ be such that $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \Gamma(x)\,dx = 1$, and such that*

$$\begin{array}{rcccccccl} \Gamma(-\pi) & = & \Gamma'(-\pi) & = & \Gamma''(-\pi) & = \cdots = & \Gamma^{(k)}(-\pi) & = & 0 \\ \text{and}\quad \Gamma(\pi) & = & \Gamma'(\pi) & = & \Gamma''(\pi) & = \cdots = & \Gamma^{(k)}(\pi) & = & 0. \end{array}$$

Define $\{\gamma_n\}_{n=1}^{\infty}$ as in Example 10D.8. Then $\{\gamma_n\}_{n=1}^{\infty}$ is a $\mathcal{C}^k$-mollifier.

Proof. **Exercise 10D.9** □ Ⓔ

Example 10D.11. Let $g(x) = (x+\pi)^{k+1}(x-\pi)^{k+1}$, let $G = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$, and then let $\Gamma(x) := g(x)/G$. Then Γ satisfies the hypotheses of Lemma 10D.10 (**Exercise 10D.10**). ♢ Ⓔ

Remark. For more information about convolutions and mollifiers, see [Fol84, §8.2, pp.230-237] or [WZ77, Chap.9, pp.145-160].

10D(iii) Proof of Theorems 8A.1(a) and 10D.1.

Prerequisites: §8A, §10A, §10D(ii).

Proof of Theorem 10D.1. Let $\{\gamma_n\}_{n=1}^{\infty} \subset \mathcal{C}^k_{\text{per}}[-\pi,\pi]$ be the $\mathcal{C}^k$-mollifier from Lemma 10D.10. Then Proposition 10D.9(a) says that $f = \mathbf{L}^2\text{-}\lim_{n\to\infty} \gamma_n * f$. Thus, for any $\epsilon > 0$, we can find some $n \in \mathbb{N}$ such that $\|f - \gamma_n * f\|_2 < \epsilon$. Furthermore, $\gamma_n \in \mathcal{C}^k[-\pi,\pi]$, so Lemma 10D.7(f) says that $\gamma_n * f \in \mathcal{C}^k_{\text{per}}[-\pi,\pi]$, for all $n \in \mathbb{N}$. □

The proof of Theorem 8A.1(a) now follows a standard strategy in analysis: approximate the function f with a 'nice' function $\widetilde{f}$, establish convergence for the Fourier series of $\widetilde{f}$ first, and then use the triangle inequality to 'leverage' this into convergence for the Fourier series of f.

Proof of Theorem 8A.1(a). Let $f \in \mathbf{L}^2[-\pi,\pi]$. Fix $\epsilon > 0$. Theorem 10D.1 says there exists some $\widetilde{f} \in \mathcal{C}^1_{\text{per}}[-\pi,\pi]$ such that

$$\left\|f - \widetilde{f}\right\|_2 \quad < \quad \frac{\epsilon}{3}. \tag{10D.16}$$

Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be the real Fourier coefficients for f, and let $\{\widetilde{A}_n\}_{n=0}^{\infty}$ and $\{\widetilde{B}_n\}_{n=1}^{\infty}$ be the real Fourier coefficients for $\widetilde{f}$. Let $\overline{f} := f - \widetilde{f}$, and let $\{\overline{A}_n\}_{n=0}^{\infty}$ and $\{\overline{B}_n\}_{n=1}^{\infty}$ be the real Fourier coefficients for $\overline{f}$. Then for all $n \in \mathbb{N}$, we have

$$A_n \quad = \quad \overline{A}_n + \widetilde{A}_n \qquad \text{and} \qquad B_n \quad = \quad \overline{B}_n + \widetilde{B}_n. \tag{10D.17}$$

Also, for any $N \in \mathbb{N}$, we have

$$\begin{aligned}\left\| \overline{A}_0 \; + \; \sum_{n=0}^{N} \overline{A}_n \mathbf{C}_n \; + \; \sum_{n=1}^{N} \overline{B}_n \mathbf{S}_n \right\|_2^2 \quad & \underset{(\triangle)}{\leq} \quad \overline{A}_0^2 \; + \; \sum_{n=0}^{\infty} |\overline{A}_n|^2 \cdot \|\mathbf{C}_n\|_2^2 \; + \; \sum_{n=1}^{\infty} |\overline{B}_n|^2 \cdot \|\mathbf{C}_n\|_2^2 \\ & \underset{(\dagger)}{=} \quad \overline{A}_0^2 \; + \; \sum_{n=0}^{\infty} \frac{|\overline{A}_n|^2}{2} \; + \; \sum_{n=1}^{\infty} \frac{|\overline{B}_n|^2}{2} \\ & \underset{(\mathrm{B})}{\leq} \quad \|\overline{f}\|_2^2 \quad = \quad \left\| f - \widetilde{f} \right\|_2^2 \quad \underset{(*)}{<} \quad \left(\frac{\epsilon}{3}\right)^2. \end{aligned} \tag{10D.18}$$

Here, $(\triangle)$ is by the triangle inequality[4], and $(\dagger)$ is because $\|\mathbf{C}_n\|_2^2 = \frac{1}{2} = \|\mathbf{S}_n\|_2^2$ for all $n \in \mathbb{N}$ (by Proposition 6D.2 on page 112). (B) is Bessel's Inequality (Example 10A.2 on page 196), and $(*)$ is by eqn.(10D.16).

Now, $\widetilde{f} \in \mathcal{C}^1_{\text{per}}[-\pi, \pi]$, so Theorem 8A.1(d) (which we proved in Section 10C) says that

$$\operatorname*{unif-lim}_{N\to\infty} \left(\widetilde{A}_0 \; + \; \sum_{n=0}^{N} \widetilde{A}_n \mathbf{C}_n \; + \; \sum_{n=1}^{N} \widetilde{B}_n \mathbf{S}_n \right) \quad = \quad \widetilde{f}.$$

Thus Corollary 6E.11(b)[i] on page 127 implies that

$$\mathbf{L}^2\!\!-\!\lim_{N\to\infty} \left(\widetilde{A}_0 \; + \; \sum_{n=0}^{N} \widetilde{A}_n \mathbf{C}_n \; + \; \sum_{n=1}^{N} \widetilde{B}_n \mathbf{S}_n \right) \quad = \quad \widetilde{f}.$$

Thus, there exists some $N \in \mathbb{N}$ such that

$$\left\| \widetilde{A}_0 + \sum_{n=0}^{N} \widetilde{A}_n \mathbf{C}_n + \sum_{n=1}^{N} \widetilde{B}_n \mathbf{S}_n \; - \; \widetilde{f} \right\|_2 \quad < \quad \frac{\epsilon}{3}. \tag{10D.19}$$

Thus,

$$\begin{aligned} & \left\| A_0 + \sum_{n=0}^{N} A_n \mathbf{C}_n + \sum_{n=1}^{N} B_n \mathbf{S}_n \; - \; f \right\|_2 \\ & \underset{(\dagger)}{=} \quad \left\| (\overline{A}_0 + \widetilde{A}_0) + \sum_{n=0}^{N} (\overline{A}_n + \widetilde{A}_n) \mathbf{C}_n + \sum_{n=1}^{N} (\overline{B}_n + \widetilde{B}_n) \mathbf{S}_n \; - \; \widetilde{f} + \widetilde{f} - f \right\|_2 \end{aligned}$$

[4]Actually, this is an equality, because of the L^2 Pythagorean formula (equation (6F.1) on page 131)

$$\begin{aligned} &= \left\| \overline{A}_0 + \sum_{n=0}^{N} \overline{A}_n \mathbf{C}_n + \sum_{n=1}^{N} \overline{B}_n \mathbf{S}_n \;+\; \widetilde{A}_0 + \sum_{n=0}^{N} \widetilde{A}_n \mathbf{C}_n + \sum_{n=1}^{N} \widetilde{B}_n \mathbf{S}_n - \widetilde{f} \;+\; \widetilde{f} - f \right\|_2 \\ &\underset{(\triangle)}{\leq} \left\| \overline{A}_0 + \sum_{n=0}^{N} \overline{A}_n \mathbf{C}_n + \sum_{n=1}^{N} \overline{B}_n \mathbf{S}_n \right\|_2 + \left\| \widetilde{A}_0 + \sum_{n=0}^{N} \widetilde{A}_n \mathbf{C}_n + \sum_{n=1}^{N} \widetilde{B}_n \mathbf{S}_n - \widetilde{f} \right\|_2 + \left\| \widetilde{f} - f \right\|_2 \\ &\underset{(*)}{\leq} \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad = \quad \epsilon. \end{aligned}$$

Here, (†) is by eqn.(10D.17), $(\triangle)$ is the Triangle inequality, and $(*)$ is by inequalities (10D.16), (10D.18) and (10D.19).

This argument works for any $\epsilon > 0$. We conclude that $A_0 + \sum_{n=0}^{\infty} A_n \mathbf{C}_n + \sum_{n=1}^{\infty} B_n \mathbf{S}_n \underset{\text{L2}}{\approx} f$. □

Recall that a function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is **analytic** if f is infinitely differentiable, and the Taylor expansion of f around any $x \in [-\pi, \pi]$ has a nonzero radius of convergence.[5] Let $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ be the set of all analytic functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$, and $f^{(k)}(-\pi) = f^{(k)}(\pi)$ for all $k \in \mathbb{N}$. For example, the functions sin and cos are in $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$. Elements of $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ are some of the 'nicest' possible functions on $[-\pi, \pi]$. On the other hand, arbitrary elements of $\mathbf{L}^2[-\pi, \pi]$ can by quite 'nasty' (i.e. nondifferentiable, discontinuous). Thus, the following result is quite striking.

Corollary 10D.12. *$\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ is dense in $\mathbf{L}^2[-\pi, \pi]$.*

Proof. Theorem 8A.1(a) says that any function in $\mathbf{L}^2[-\pi, \pi]$ can be approximated arbitrarily closely by a 'trigonometric polynomial' of the form $A_0 + \sum_{n=1}^{N} A_n \mathbf{C}_n + \sum_{n=1}^{N} B_n \mathbf{S}_n$. But all trigonometric polynomials are in $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ (because they are finite linear combinations of the functions $\mathbf{S}_n(x) := \sin(nx)$ and $\mathbf{C}_n(x) := \cos(nx)$, which are all in $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$). Thus, any function in $\mathbf{L}^2[-\pi, \pi]$ can be approximated arbitrarily closely by an element of $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ —in other words, $\mathcal{C}^{\omega}_{\text{per}}[-\pi, \pi]$ is dense in $\mathbf{L}^2[-\pi, \pi]$. □

Remarks. (a) Proposition 17G.3 on page 411 provides a 'pointwise' version of the Theorem 10D.1 for convolutional smoothing on $\mathbb{R}^D$.

(b) For another proof of the L^2-convergence of real Fourier series, see [Bro89, Theorems 1.5.4 (p.20) and 2.3.10 (p.35)]. For a proof of the L^2-convergence of complex Fourier series (which is very similar), see [Kat76, §I.5.5, p.29-30].

[5] See Appendix 0H(i) on page 568.

IV BVP solutions via eigenfunction expansions

A powerful and general method for solving linear PDEs is to represent the solutions using *eigenfunction expansions*. Rather than first deploying this idea in full abstract generality, we will start by illustrating it in a variety of special cases. We will gradually escalate the level of abstraction, so that the general theory is almost obvious when it is finally stated explicitly.

The orthogonal trigonometric functions $\mathbf{S_n}$ and $\mathbf{C_n}$ in a Fourier series are *eigenfunctions* of the Laplacian operator $\triangle$. Furthermore, the eigenfunctions $\mathbf{S_n}$ and $\mathbf{C_n}$ are particularly 'well-adapted' to domains like the interval $[0, \pi]$, the square $[0, \pi]^2$, or the cube $[0, \pi]^3$, for two reasons:

- The functions $\mathbf{S_n}$ and $\mathbf{C_n}$ and the domain $[0, \pi]^k$ are both easily expressed in a Cartesian coordinate system.
- The functions $\mathbf{S_n}$ and $\mathbf{C_n}$ satisfy desirable boundary conditions (e.g. homogeneous Dirichlet/Neumann) on the boundaries of the domain $[0, \pi]^k$.

Thus, we can use $\mathbf{S}_n$ and $\mathbf{C}_n$ as 'building blocks' to construct a solution to a given partial differential equation —a solution which also satisfies specified initial conditions and/or boundary conditions on $[0, \pi]^k$. In particular, we will use Fourier sine series to obtain homogeneous *Dirichlet* boundary conditions [by Theorems 7A.1(d), 9A.3(d) and 9B.1(d)] , and Fourier cosine series to obtain homogeneous *Neumann* boundary conditions [by Theorems 7A.4(d), 9A.3(e) and 9B.1(e)]. This basic strategy underlies all the solution methods developed in Chapters 11 to 13.

When we consider other domains (e.g. disks, annuli, balls, etc.), the functions $\mathbf{C}_n$ and $\mathbf{S}_n$ are no longer so 'well-adapted'. In Chapter 14, we discover that, in polar coordinates, the 'well-adapted' eigenfunctions are combinations of trigonometric functions ($\mathbf{C}_n$ and $\mathbf{S}_n$) with another class of transcendental functions called *Bessel functions*. This yields another orthogonal system of eigenfunctions. We can then represent most functions on the disks and annuli using *Fourier-Bessel expansions* (analogous to Fourier series), and we can then mimic the solution methods of Chapters 11 to 13.

Chapter 11

Boundary value problems on a line segment

"Mathematics is the music of reason." —James Joseph Sylvester

Prerequisites: §7A, §5C.

This chapter concerns boundary value problems on the line segment $[0, L]$, and provides solutions in the form of infinite series involving the functions $\mathbf{S}_n(x) := \sin\left(\frac{n\pi}{L}x\right)$ and $\mathbf{C}_n(x) := \cos\left(\frac{n\pi}{L}x\right)$. For simplicity, we will assume throughout the chapter that $L = \pi$. Thus $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$. We will also assume that (through an appropriate choice of time units) the physical constants in the various equations are all equal to one. Thus, the heat equation becomes "$\partial_t u = \triangle u$", the wave equation is "$\partial_t^2 u = \triangle u$", etc.

This does not limit the generality of our results. For example, faced with a general heat equation of the form "$\partial_t u(x,t) = \kappa \cdot \triangle u$" for $x \in [0, L]$, (with $\kappa \neq 1$ and $L \neq \pi$) you can simply replace the coordinate x with a new space coordinate $y = \frac{\pi}{L}x$ and replace t with a new time coordinate $s = \kappa t$, to reformulate the problem in a way compatible with the following methods.

11A The heat equation on a line segment

Prerequisites: §7B, §5B, §5C, §1B(i),§0F. **Recommended:** §7C(v).

Proposition 11A.1. (Heat equation; homogeneous Dirichlet boundary)
Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2[0, \pi]$ be some function describing an initial heat distribution. Suppose f has Fourier Sine Series $f(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx)$, *and*

define the function $u : \mathbb{X} \times \mathbb{R}_{\neq} \longrightarrow \mathbb{R}$ by

$$u(x;t) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx) \cdot \exp\Big(-n^2 \cdot t\Big), \qquad \textit{for all } x \in [0,\pi] \textit{ and } t \geq 0.$$

Then u is the unique solution to the one-dimensional heat equation "$\partial_t u = \partial_x^2 u$", with homogeneous Dirichlet boundary conditions

$$u(0;t) = u(\pi;t) \;=\; 0, \qquad \textit{for all } t > 0.$$

and initial conditions: $u(x;0) = f(x)$, for all $x \in [0,\pi]$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.

Ⓔ *Proof.* **Exercise 11A.1** Hint:

(a) Show that, when $t = 0$, the Fourier series of $u(x;0)$ agrees with that of $f(x)$; hence $u(x;0) = f(x)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n=1}^{\infty} \left| n^2 \cdot B_n \cdot e^{-n^2 t} \right| \;<\; \infty.$

(c) For any $T > 0$, apply Proposition 0F.1 on page 565 to conclude that

$$\partial_t u(x;t) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} -n^2 B_n \sin(nx) \cdot \exp\Big(-n^2 \cdot t\Big) \underset{\text{unif}}{\equiv} \triangle u(x;t) \qquad \text{on } [T,\infty).$$

(d) Observe that for any fixed $t > 0$, $\displaystyle\sum_{n=1}^{\infty} \left| B_n e^{-n^2 t} \right| \;<\; \infty.$

(e) Apply part (c) of Theorem 7A.1 on page 138 to show that the Fourier series of $u(x;t)$ converges uniformly for all $t > 0$.

(f) Apply part (d) of Theorem 7A.1 on page 138 to conclude that $u(0;t) = 0 = u(\pi,t)$ for all $t > 0$.

(g) Apply Theorem 5D.8 on page 91 to show that this solution is unique. □

Example 11A.2. Consider a metal rod of length π, with initial temperature distribution $f(x) = \tau \cdot \sinh(\alpha x)$ (where $\tau, \alpha > 0$ are constants), and homogeneous Dirichlet boundary condition. Proposition 11A.1 tells us to get the Fourier sine series for $f(x)$. In Example 7A.3 on page 140, we computed this to be $\displaystyle\frac{2\tau\sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\alpha^2 + n^2} \cdot \sin(nx)$. The evolving temperature distribution is therefore given:

$$u(x;t) \;=\; \frac{2\tau\sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\alpha^2 + n^2} \cdot \sin(nx) \cdot e^{-n^2 t}. \qquad \diamondsuit$$

Proposition 11A.3. (Heat equation; homogeneous Neumann boundary)
Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2[0, \pi]$ be some function describing an initial heat distribution. Suppose f has Fourier Cosine Series $f(x) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A_n \cos(nx)$, and define the function $u : \mathbb{X} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ by

$$u(x;t) \quad \underset{\text{L2}}{\approx} \quad \sum_{n=0}^{\infty} A_n \cos(nx) \cdot \exp\Big(-n^2 \cdot t\Big), \qquad \text{for all } x \in [0, \pi] \text{ and } t \geq 0.$$

Then u is the unique solution to the one-dimensional heat equation "$\partial_t u = \partial_x^2 u$", with homogeneous Neumann boundary conditions

$$\partial_x u(0;t) = \partial_x u(\pi;t) \;=\; 0, \qquad \text{for all } t > 0.$$

and initial conditions: $u(x;0) = f(x)$, for all $x \in [0, \pi]$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.

Proof. Setting $t = 0$, we get:

$$\begin{aligned} u(x;0) \quad &= \quad \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp\Big(-n^2 \cdot 0\Big) \quad = \quad \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp(0) \\ &= \quad \sum_{n=1}^{\infty} A_n \cos(nx) \cdot 1 \quad = \quad \sum_{n=1}^{\infty} A_n \cos(nx) \quad = \quad f(x), \end{aligned}$$

so we have the desired initial conditions.

Let $M := \max_{n \in \mathbb{N}} |A_n|$. Then $M < \infty$ (because $f \in \mathbf{L}^2$).

Claim 1: *For all $t > 0$, $\displaystyle\sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| \;<\; \infty$.*

Proof. Since $M = \max_{n \in \mathbb{N}} |A_n|$, we know that $|A_n| < M$ for all $n \in \mathbb{N}$. Thus,

$$\sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| \quad \leq \quad \sum_{n=0}^{\infty} \left|n^2\right| \cdot M \cdot \left|e^{-n^2 t}\right| \quad = \quad M \cdot \sum_{n=0}^{\infty} n^2 \cdot e^{-n^2 t}$$

Hence,it suffices to show that $\displaystyle\sum_{n=0}^{\infty} n^2 \cdot e^{-n^2 t} \;<\; \infty$. To see this, let $E = e^t$. Then $E > 1$ (because $t > 0$). Also, $n^2 \cdot e^{-n^2 t} \;=\; \dfrac{n^2}{E^{n^2}}$, for each $n \in \mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} n^2 e^{-n^2 t} \quad = \quad \sum_{n=1}^{\infty} \frac{n^2}{E^{n^2}} \quad \leq \quad \sum_{m=1}^{\infty} \frac{m}{E^m} \qquad (11A.1)$$

We must show that right-hand series in (11A.1) converges. We apply the Ratio Test:

$$\lim_{m\to\infty} \frac{\frac{m+1}{E^{m+1}}}{\frac{m}{E^m}} \quad = \quad \lim_{m\to\infty} \frac{m+1}{m}\frac{E^m}{E^{m+1}} \quad = \quad \lim_{m\to\infty} \frac{1}{E} \quad < \quad 1.$$

Hence the right-hand series in (11A.1) converges. $\diamond_{\text{Claim 1}}$

Claim 2: *For any $T > 0$, we have $\partial_x u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} nA_n \sin(nx) \cdot \exp\left(-n^2 \cdot t\right)$ on $\mathbb{X} \times [T,\infty)$, and also $\partial_x^2 u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} n^2 A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right)$ on $\mathbb{X} \times [T,\infty)$.*

Proof. This follows from Claim 1 and two applications of Proposition 0F.1 on page 565. $\diamond_{\text{Claim 2}}$

Claim 3: *For any $T > 0$, we have $\partial_t u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} n^2 A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right)$ on $[T,\infty)$.*

Proof.
$$\begin{aligned}
\partial_t u(x;t) &= \partial_t \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right) \\
&\underset{(*)}{=} \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \partial_t \exp\left(-n^2 \cdot t\right) \\
&= \sum_{n=1}^{\infty} A_n \cos(nx) \cdot (-n^2) \exp\left(-n^2 \cdot t\right),
\end{aligned}$$

where $(*)$ is by Claim 1 and Proposition 0F.1 on page 565. $\diamond_{\text{Claim 3}}$

Combining Claims 2 and 3, we conclude that $\partial_t u(x;t) = \triangle u(x;t)$.

Finally Claim 1 also implies that, for any $t > 0$,

$$\sum_{n=0}^{\infty} \left| n \cdot A_n \cdot e^{-n^2 t} \right| \quad < \quad \sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| \; < \; \infty.$$

Hence, Theorem 7A.4(d)[ii] on p.142 implies that $u(x;t)$ satisfies homogeneous Neumann boundary conditions for any $t > 0$. (This can also be seen directly via Claim 2).

Finally, Theorem 5D.8 on page 91 implies that this solution is unique. □

Example 11A.4. Consider a metal rod of length π, with initial temperature distribution $f(x) = \cosh(x)$ and homogeneous Neumann boundary condition. Proposition 11A.3 tells us to get the Fourier cosine series for $f(x)$. In Example 7A.6 on page 143, we computed this to be $\frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cdot \cos(nx)}{n^2+1}$. The evolving temperature distribution is therefore given:

$$u(x;t) \underset{\text{L2}}{\approx} \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cdot \cos(nx)}{n^2+1} \cdot e^{-n^2 t}. \qquad \diamondsuit$$

Exercise 11A.2. Let $L > 0$ and let $\mathbb{X} := [0, L]$. Let $\kappa > 0$ be a diffusion constant, and consider the general one-dimensional heat equation Ⓔ

$$\partial_t u = \kappa\, \partial_x^2 u. \qquad (11A.2)$$

(a) Generalize Proposition 11A.1 to find the solution to eqn.(11A.2) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 11A.3 to find the solution to eqn.(11A.2) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Neumann boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11A.2) (Hint: imitate the strategy suggested in Exercise 11A.1) ♦

Exercise 11A.3 Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2(\mathbb{X})$ be a function whose Fourier sine series satisfies $\sum_{n=1}^{\infty} n^2 |B_n| < \infty$. Imitate Proposition 11A.1, to find a 'Fourier series' solution to the initial value problem for the one-dimensional *free Schrödinger equation* Ⓔ

$$\mathbf{i}\partial_t\, \omega = \frac{-1}{2}\partial_x^2\, \omega, \qquad (11A.3)$$

on $\mathbb{X}$, with initial conditions $\omega_0 = f$, and satisfying homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11A.3). (Hint: imitate the strategy suggested in Exercise 11A.1).

11B The wave equation on a line (the vibrating string)

Prerequisites: §7B(i), §5B, §5C, §2B(i). **Recommended:** §17D(ii).

Imagine a piano string stretched tightly between two points. At equilibrium, the string is perfectly flat, but if we pluck or strike the string, it will vibrate, meaning there will be a vertical displacement from equilibrium. Let $\mathbb{X} = [0, \pi]$ represent the string, and for any point $x \in \mathbb{X}$ on the string and time $t > 0$, let $u(x; t)$ be the vertical displacement of the string. Then u will obey the one-dimensional wave equation:

$$\partial_t^2 u(x;t) \;=\; \triangle u(x;t). \qquad (11B.1)$$

However, since the string is fixed at its endpoints, the function u will also exhibit homogeneous **Dirichlet** boundary conditions

$$u(0;t) = u(\pi;t) \;=\; 0 \qquad \text{(for all } t > 0). \qquad (11B.2)$$

Proposition 11B.1. (Initial Position Problem for Vibrating String with fixed endpoints)

$f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial displacement of the string. Suppose f_0 has Fourier Sine Series $f_0(x) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx)$, and define the function $w : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by

$$w(x;t) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx) \cdot \cos(nt), \qquad \textit{for all } x \in [0,\pi] \textit{ and } t \geq 0. \qquad (11B.3)$$

Then w is the unique solution to the wave equation (11B.1), *satisfying the Dirichlet boundary conditions* (11B.2), *as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & w(x,0) & = & f_0(x), \\ \textbf{Initial Velocity:} & \partial_t w(x,0) & = & 0, \end{array}\right\} \quad \textit{for all } x \in [0,\pi].$$

Ⓔ *Proof.* <u>**Exercise 11B.1**</u> Hint:

(a) Prove the trigonometric identity $\sin(nx)\cos(nt) \;=\; \frac{1}{2}\Big(\sin(n(x-t)) + \sin(n(x+t))\Big)$.

(b) Use this identity to show that the Fourier sine series (11B.3) converges in L^2 to the d'Alembert solution from Theorem 17D.8(a) on page 401.

(c) Apply Theorem 5D.11 on page 94 to show that this solution is unique. □

Example 11B.2. Let $f_0(x) = \sin(5x)$. Thus, $B_5 = 1$ and $B_n = 0$ for all $n \neq 5$. Proposition 11B.1 tells us that the corresponding solution to the wave equation is $w(x,t) = \cos(5t)\sin(5x)$. To see that w satisfies the wave equation, note that, for any $x \in [0,\pi]$ and $t > 0$,

$$\begin{array}{lrcl} & \partial_t w(x,t) \;=\; -5\sin(5t)\sin(5x) & \text{and} & 5\cos(5t)\cos(5x) \;=\; \partial_x w(x,t); \\ \text{Thus} & \partial_t^2 w(x,t) \;=\; -25\cos(5t)\sin(5x) & = & -25\cos(5t)\cos(5x) \;=\; \partial_x^2 w(x,t). \end{array}$$

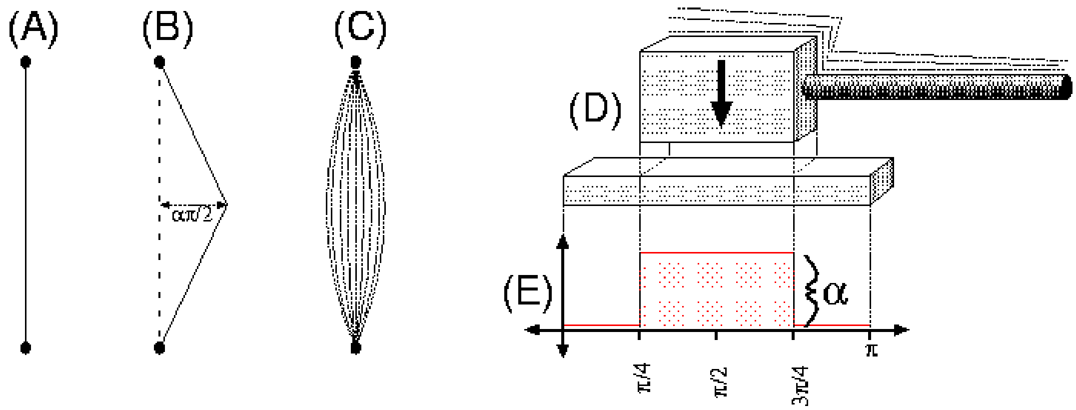


Figure 11B.1: **(A)** A harpstring at rest. **(B)** A harpstring being plucked. **(C)** The harpstring vibrating. **(D)** A big hammer striking a xylophone. **(E)** The initial velocity of the xylophone when struck.

Also w has the desired initial position because, for any $x \in [0, \pi]$, we have $w(x; 0) = \cos(0)\sin(5x) = \sin(5x) = f_0(x)$, because $\cos(0) = 1$.

Next, w has the desired initial velocity because for any $x \in [0, \pi]$, we have $\partial_t w(x; 0) = 5\sin(0)\sin(5x) = 0$, because $\sin(0) = 0$.

Finally w satisfies homogeneous Dirichlet BC because, for any $t > 0$, we have $w(0, t) = \cos(5t)\sin(0) = 0$ and $w(\pi, t) = \cos(5t)\sin(5\pi) = 0$, because $\sin(0) = 0 = \sin(5\pi)$. ♢

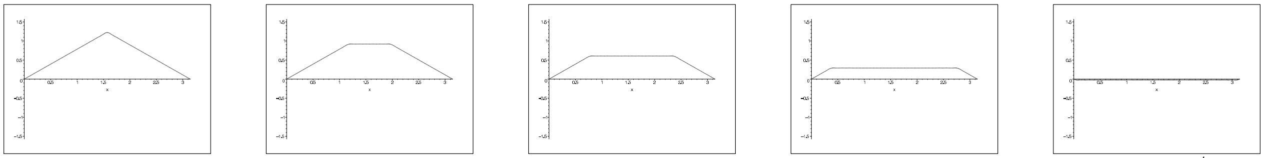
Example 11B.3: (The plucked harp string)

A harpist places her fingers at the midpoint of a harp string and plucks it. What is the formula describing the vibration of the string?

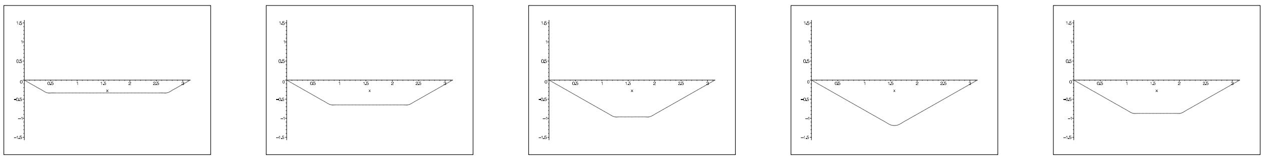
Solution: For simplicity, we imagine the string has length π. The tight string forms a straight line when at rest (Figure 11B.1A); the harpist plucks the string by pulling it away from this resting position and then releasing it. At the moment she releases it, the string's *initial velocity* is zero, and its *initial position* is described by a **tent function** like the one in Example 7C.7 on page 155

$$f_0(x) \quad = \quad \begin{cases} \alpha x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \alpha(\pi - x) & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases} \qquad \text{(Figure 11B.1B)}$$

where $\alpha > 0$ is a constant describing the force with which she plucks the string (and its resulting amplitude).



$t = 0$ $t = 0.4$ $t = 0.8$ $t = 1.2$ $t = 1.6 \approx \pi/2$



$t = 2.0$ $t = 2.4$ $t = 2.8$ $t = 3.2 \approx \pi$ $t = 3.6$

Figure 11B.2: The plucked harpstring of Example 11B.3. From $t = 0$ to $t = \pi/2$, the initially triangular shape is blunted; at $t = \pi/2$ it is totally flat. From $t = \pi/2$ to $t = \pi$, the process happens in reverse, only the triangle grows back upside down. At $t = \pi$, the original triangle reappears, upside down. Then the entire process happens in reverse, until the original triangle reappears at $t = 2\pi$.

The endpoints of the harp string are fixed, so it vibrates with *homogeneous Dirichlet* boundary conditions. Thus, Proposition 11B.1 tells us to find the Fourier sine series for f_0. In Example 7C.7, we computed this to be:

$$f_0 \quad \underset{\text{L2}}{\approx} \quad \frac{4 \cdot \alpha}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$$

Thus, the resulting solution is: $u(x;t) \underset{\text{L2}}{\approx} \dfrac{4 \cdot \alpha}{\pi} \displaystyle\sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx)\cos(nt);$

(See Figure 11B.2). This is not a very accurate model because we have not accounted for energy loss due to friction. In a real harpstring, these 'perfectly triangular' waveforms rapidly decay into gently curving waves depicted in Figure 11B.1(C); these slowly settle down to a stationary state. $\diamondsuit$

Proposition 11B.4. (Initial Velocity Problem for Vibrating String with fixed endpoints)

$f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial velocity of the string. Suppose f_1 has Fourier Sine Series $f_1(x) \underset{\text{L2}}{\approx} \displaystyle\sum_{n=1}^{\infty} B_n \sin(nx)$, *and define the*

function $v : \mathbb{X} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ *by*

$$v(x;t) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{B_n}{n} \sin(nx) \cdot \sin(nt), \qquad \textit{for all } x \in [0,\pi] \textit{ and } t \geq 0. \quad (11B.4)$$

Then v is the unique solution to the wave equation (11B.1)*, satisfying the Dirichlet boundary conditions* (11B.2)*, as well as*

$$\left.\begin{array}{rrcl} \textbf{Initial Position:} & v(x,0) & = & 0; \\ \textbf{Initial Velocity:} & \partial_t v(x,0) & = & f_1(x), \end{array}\right\} \quad \textit{for all } x \in [0,\pi].$$

Proof. **Exercise 11B.2** Hint: Ⓔ

(a) Prove the trigonometric identity $-\sin(nx)\sin(nt) \;=\; \frac{1}{2}\Big(\cos(n(x+t)) - \cos(n(x-t))\Big)$.

(b) Use this identity to show that the Fourier sine series (11B.4) converges in L^2 to the d'Alembert solution from Theorem 17D.8(b) on page 401.

(c) Apply Theorem 5D.11 on page 94 to show that this solution is unique. □

Example 11B.5. Let $f_1(x) = 3\sin(8x)$. Thus, $B_8 = 3$ and $B_n = 0$ for all $n \neq 8$. Proposition 11B.4 tells us that the corresponding solution to the wave equation is $w(x,t) = \frac{3}{8}\sin(8t)\sin(8x)$. To see that w satisfies the wave equation, note that, for any $x \in [0,\pi]$ and $t > 0$,

$$\begin{array}{lrclcrcl} & \partial_t w(x,t) & = & 3\sin(8t)\cos(8x) & \text{and} & 3\cos(8t)\sin(8x) & = & \partial_x w(x,t); \\ \text{Thus} & \partial_t^2 w(x,t) & = & -24\cos(8t)\cos(8x) & = & -24\cos(8t)\cos(8x) & = & \partial_x^2 w(x,t). \end{array}$$

Also w has the desired initial position because, for any $x \in [0,\pi]$, we have $w(x;0) \;=\; \frac{3}{8}\sin(0)\sin(8x) = 0$, because $\sin(0) = 0$.

Next, w has the desired initial velocity because for any $x \in [0,\pi]$, we have $\partial_t w(x;0) \;=\; \frac{3}{8} 8\cos(0)\sin(8x) = 3\sin(8x) = f_1(x)$, because $\cos(0) = 1$.

Finally w satisfies homogeneous Dirichlet BC because, for any $t > 0$, we have $w(0,t) \;=\; \frac{3}{8}\sin(8t)\sin(0) \;=\; 0$ and $w(\pi,t) \;=\; \frac{3}{8}\sin(8t)\sin(8\pi) \;=\; 0$, because $\sin(0) = 0 = \sin(8\pi)$. ♢

Example 11B.6: (The Xylophone)

A musician strikes the midpoint of a xylophone bar with a broad, flat hammer. What is the formula describing the vibration of the string?

Solution: For simplicity, we imagine the bar has length π and is fixed at its endpoints (actually most xylophones satisfy neither requirement). At the moment when the hammer strikes it, the string's *initial position* is zero, and its *initial velocity* is determined by the distribution of force imparted by the hammer head. For simplicity, we will assume the hammer head has width $\pi/2$, and hits the bar squarely at its midpoint (Figure 11B.1D). Thus, the initial velocity is given by the function:

$$f_1(x) \quad = \quad \begin{cases} \alpha & \text{if } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Figure 11B.1E)}$$

where $\alpha > 0$ is a constant describing the force of the impact. Proposition 11B.4 tells us to find the Fourier sine series for $f_1(x)$. From Example 7C.4 on page 151, we know this to be

$$f_1(x) \underset{\text{L2}}{\approx} \frac{2\alpha\sqrt{2}}{\pi}\left(\sin(x) + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k-1)x\Big)}{4k-1} + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k+1)x\Big)}{4k+1}\right).$$

The resulting vibrational motion is therefore described by:

$$v(x,t) \quad \underset{\text{L2}}{\approx} \quad \frac{2\alpha\sqrt{2}}{\pi}\left(\sin(x)\sin(t) + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k-1)x\Big)\sin\Big((4k-1)t\Big)}{(4k-1)^2}\right.$$
$$\left. + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k+1)x\Big)\sin\Big((4k+1)t\Big)}{(4k+1)^2}\right).$$

$\diamond$

Ⓔ **Exercise 11B.3** Let $L > 0$ and let $\mathbb{X} := [0, L]$. Let $\lambda > 0$ be a parameter describing wave velocity (determined by the string's tension, elasticity, density, etc.), and consider the general one-dimensional wave equation

$$\partial_t^2 u \quad = \quad \lambda^2 \, \partial_x^2 u. \qquad (11B.5)$$

(a) Generalize Proposition 11B.1 to find the solution to eqn.(11B.5) on $\mathbb{X}$ having zero initial velocity and a prescribed initial position, and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 11B.4 to find the solution to eqn.(11B.5) on $\mathbb{X}$ having zero initial position and a prescribed initial velocity, and homogeneous Dirichlet boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11B.5) (Hint: imitate the strategy suggested in Exercises 11B.1 and 11B.2.)

11C The Poisson problem on a line segment

Prerequisites: §7B, §5C, §1D. **Recommended:** §7C(v).

We can also use Fourier series to solve the one-dimensional Poisson problem on a line segment. This is not usually a practical solution method, because we already have a simple, complete solution to this problem using a double integral (see Example 1D.1 on page 13). However, we include this result anyways, as a simple illustration of Fourier techniques.

Proposition 11C.1. *Let $\mathbb{X} = [0, \pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function, with* semiuniformly convergent *Fourier sine series: $q(x) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} Q_n \sin(nx)$. Define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by*

$$u(x) \quad \underset{\text{unif}}{\equiv} \quad \sum_{n=1}^{\infty} \frac{-Q_n}{n^2} \sin(nx), \qquad \text{for all } x \in [0, \pi].$$

Then u is the unique solution to the Poisson equation *"$\triangle u(x) = q(x)$" satisfying homogeneous* Dirichlet *boundary conditions: $u(0) = u(\pi) = 0$.*

Proof. **Exercise 11C.1** Hint: (a) Apply Proposition 0F.1 on page 565 twice to show that $\triangle u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} Q_n \sin(nx) = q(x)$, for all $x \in \mathsf{int}(\mathbb{X})$. (Hint: The Fourier series of q is semiuniformly convergent). Ⓔ

(b) Observe that $\displaystyle\sum_{n=1}^{\infty} \left| \frac{Q_n}{n^2} \right| < \infty$.

(c) Apply Theorem 7A.1(c) (p.138) to show that the given Fourier sine series for $u(x)$ converges uniformly.

(d) Apply Theorem 7A.1(d)[ii] (p.138) to conclude that $u(0) = 0 = u(\pi)$.

(e) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique. □

Proposition 11C.2. *Let $\mathbb{X} = [0, \pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some some function, with* semiuniformly convergent *Fourier cosine series: $q(x) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} Q_n \cos(nx)$, and suppose that $Q_0 = 0$. Fix any constant $K \in \mathbb{R}$, and define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by*

$$u(x) \quad \underset{\text{unif}}{\equiv} \quad \sum_{n=1}^{\infty} \frac{-Q_n}{n^2} \cos(nx) \quad + \quad K, \qquad \text{for all } x \in [0, \pi]. \tag{11C.1}$$

*Then u is a solution to the **Poisson equation** "$\triangle u(x) = q(x)$", satisfying homogeneous **Neumann** boundary conditions $u'(0) = u'(\pi) = 0$.*

*Furthermore, **all** solutions to this Poisson equation with these boundary conditions have the form (11C.1) for some choice of K.*

If $Q_0 \neq 0$, however, the problem has no solution.

Ⓔ *Proof.* **Exercise 11C.2** Hint: (a) Apply Proposition 0F.1 on page 565 twice to show that $\triangle u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} Q_n \cos(nx) = q(x)$, for all $x \in \mathsf{int}(\mathbb{X})$. (Hint: The Fourier series of q is semiuniformly convergent).

(b) Observe that $\sum_{n=1}^{\infty} \left| \frac{Q_n}{n} \right| < \infty$.

(c) Apply Theorem 7A.4(d)[ii] (p.142) to conclude that $u'(0) = 0 = u'(\pi)$.

(d) Apply Theorem 5D.5(c) on page 88 to conclude that this solution is unique up to addition of a constant. □

Ⓔ **Exercise 11C.3.** Mathematically, it is clear that the solution of Proposition 11C.2 cannot be well-defined if $Q_0 \neq 0$. Provide a physical explanation for why this is to be expected. ◆

11D Practice problems

1. Let $g(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \end{cases}$. (see problem #5 of §7D)

   (a) Find the solution to the one-dimensional heat equation $\partial_t u(x,t) = \triangle u(x,t)$ on the interval $[0,\pi]$, with initial conditions $u(x,0) = g(x)$ and homogeneous **Dirichlet** Boundary conditions.

   (b) Find the solution to the one-dimensional heat equation $\partial_t u(x,t) = \triangle u(x,t)$ on the interval $[0,\pi]$, with initial conditions $u(x,0) = g(x)$ and homogeneous **Neumann** Boundary conditions.

   (c) Find the solution to the one-dimensional wave equation $\partial_t^2 w(x,t) = \triangle w(x,t)$ on the interval $[0,\pi]$, satisfying homogeneous **Dirichlet** Boundary conditions, with initial **position** $w(x,0) = 0$ and initial **velocity** $\partial_t w(x,0) = g(x)$.

2. Let $f(x) = \sin(3x)$, for $x \in [0,\pi]$.

   (a) Compute the Fourier **Sine** Series of $f(x)$ as an element of $\mathbf{L}^2[0,\pi]$.

   (b) Compute the Fourier **Cosine** Series of $f(x)$ as an element of $\mathbf{L}^2[0,\pi]$.

(c) Solve the one-dimensional **heat equation** ($\partial_t u = \triangle u$) on the domain $\mathbb{X} = [0, \pi]$, with **initial conditions** $u(x; 0) = f(x)$, and the following boundary conditions:

i. Homogeneous **Dirichlet** boundary conditions.

ii. Homogeneous **Neumann** boundary conditions.

(d) Solve the the one-dimensional **wave equation** ($\partial_t^2 v = \triangle v$) on the domain $\mathbb{X} = [0, \pi]$, with homogeneous **Dirichlet** boundary conditions, and with

Initial position: $v(x; 0) = 0$,
Initial velocity: $\partial_t v(x; 0) = f(x)$.

3. Let $f : [0, \pi] \longrightarrow \mathbb{R}$, and suppose f has

$$\text{Fourier cosine series:} \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(nx)$$

$$\text{Fourier sine series:} \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin(nx)$$

(a) Find the solution to the one-dimensional heat equation $\partial_t u = \triangle u$, with homogeneous **Neumann** boundary conditions, and initial conditions $u(x; 0) = f(x)$ for all $x \in [0, \pi]$.

(b) **Verify** your solution in part (a). Check the heat equation, the initial conditions, and boundary conditions. [Hint: Use Proposition 0F.1 on page 565]

(c) Find the solution to the one-dimensional **wave equation** $\partial_t^2 u(x; t) = \triangle u(x; t)$ with homogeneous **Dirichlet** boundary conditions, and

$$\begin{aligned} \text{Initial position} \quad u(x; 0) &= f(x), \quad \text{for all } x \in [0, \pi]. \\ \text{Initial velocity} \quad \partial_t u(x; 0) &= 0, \quad \text{for all } x \in [0, \pi]. \end{aligned}$$

4. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ be defined by $f(x) = x$.

(a) Compute the Fourier *sine* series for f.

(b) Does the Fourier sine series converge *pointwise* to f on $(0, \pi)$? Justify your answer.

(c) Does the Fourier sine series converge *uniformly* to f on $[0, \pi]$? Justify your answer in *two different* ways.

(d) Compute the Fourier *cosine* series for f.

(e) Solve the one-dimensional **heat equation** ($\partial_t u = \triangle u$) on the domain $\mathbb{X} := [0, \pi]$, with initial conditions $u(x, 0) := f(x)$, and with the following boundary conditions:

[i] Homogeneous **Dirichlet** boundary conditions.

[ii] Homogeneous **Neumann** boundary conditions.

(f) **Verify** your solution to question (e) part [i]. That is: check that your solution satisfies the heat equation, the desired initial conditions, and homogeneous Dirichlet BC. [You may assume that the relevent series converge uniformly, if necessary. You may differentiate Fourier series termwise, if necessary.]

(g) Find the solution to the one-dimensional **wave equation** on the domain $\mathbb{X} := [0, \pi]$, with homogeneous Dirichlet boundary conditions, and with

$$\begin{array}{rrcl} \text{Initial position} & u(x;0) & = & f(x), \quad \text{for all } x \in [0,\pi]. \\ \text{Initial velocity} & \partial_t\, u(x;0) & = & 0, \quad \text{for all } x \in [0,\pi]. \end{array}$$

Chapter 12

Boundary value problems on a square

"Each problem that I solved became a rule which served afterwards to solve other problems."

—René Descartes

Prerequisites: §9A, §5C. **Recommended:** §11.

Multiple Fourier series can be used to find solutions to boundary value problems on a box $[0, X] \times [0, Y]$. The key idea is that the functions $\mathbf{S}_{n,m}(x, y) := \sin\left(\frac{n\pi}{X}x\right)\sin\left(\frac{m\pi}{Y}y\right)$ and $\mathbf{C}_{n,m}(x, y) := \cos\left(\frac{n\pi}{X}x\right)\cos\left(\frac{m\pi}{Y}y\right)$ are *eigenfunctions* of the Laplacian operator. Furthermore, $\mathbf{S}_{n,m}$ satisfies *Dirichlet* boundary conditions, so any (uniformly convergent) Fourier sine series will also do so. Likewise, $\mathbf{C}_{n,m}$ satisfies *Neumann* boundary conditions, so any (sufficiently convergent) Fourier cosine series will also do so.

For simplicity, we will assume throughout that $X = Y = \pi$. Thus $\mathbf{S}_{n,m}(x) = \sin(nx)\sin(my)$ and $\mathbf{C}_{n,m}(x) = \cos(nx)\cos(my)$. We will also assume that (through an appropriate choice of time units) the physical constants in the various equations are all equal to one. Thus, the heat equation becomes "$\partial_t u = \triangle u$", the wave equation is "$\partial_t^2 u = \triangle u$", etc. This will allow us to develop the solution methods in the simplest possible scenario, without a lot of distracting technicalities.

The extension of these solution methods to equations with arbitrary physical constants on an arbitrary rectangular domain $[0, X] \times [0, Y]$ (for some $X, Y > 0$) are left as exercises. These exercises are quite straightforward, but are an effective test of your understanding of the solution techniques.

Remark on Notation: Throughout this chapter (and the following ones) we will often write a function $u(x, y; t)$ in the form $u_t(x, y)$. This emphasizes the distinguished role of the 'time' coordinate t, and makes it natural to think of fixing t at some value and applying the 2-dimensional Laplacian $\triangle = \partial_x^2 + \partial_y^2$ to the resulting 2-dimensional function u_t.

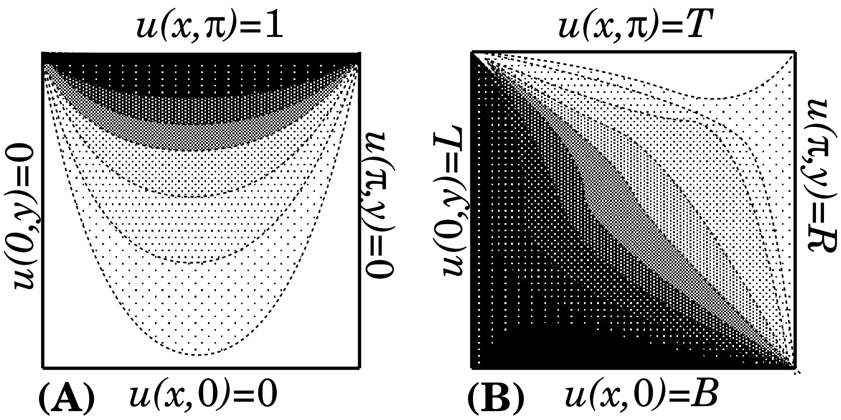


Figure 12A.1: The Dirichlet problem on a square. **(A)** Proposition 12A.1; **(B)** Propositions 12A.2 and 12A.4.

Some authors use the subscript notation "u_t" to denote the partial derivative $\partial_t u$. We *never* use this notation. In this book, partial derivatives are always denoted by "$\partial_t u$", etc.

12A The Dirichlet problem on a square

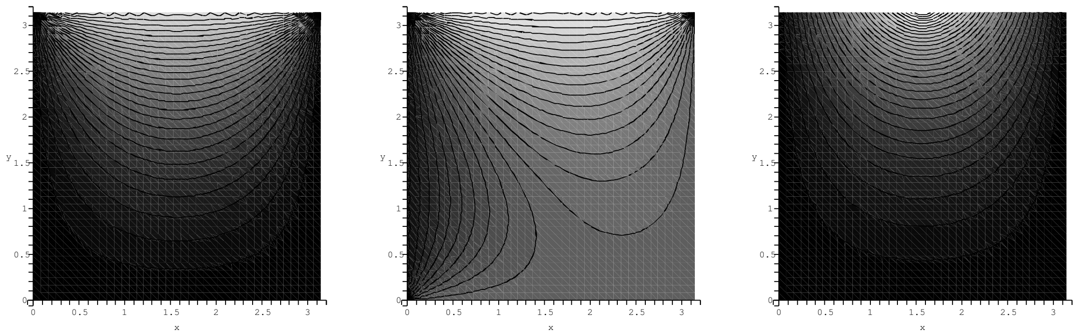
Prerequisites: §9A, §5C(i), §1C, §0F. **Recommended:** §7C(v).

In this section we will learn to solve the **Dirichlet problem** on a square domain $\mathbb{X}$: that is, to find a function which is harmonic on the interior of $\mathbb{X}$ and which satisfies specified Dirichlet boundary conditions on the boundary $\mathbb{X}$. Solutions to the Dirichlet problem have several physical interpretations.

Heat: Imagine that the boundaries of $\mathbb{X}$ are perfect heat conductors, which are in contact with external 'heat reservoirs' with fixed temperatures. For example, one boundary might be in contact with a heat source, and another, in contact with a coolant liquid. The solution to the Dirichlet problem is then the equilibrium temperature distribution on the interior of the box, given these constraints.

Electrostatic: Imagine that the boundaries of $\mathbb{X}$ are electrical conductors which are held at some fixed voltage by the application of an external electric potential (different boundaries, or different parts of the same boundary, may be held at different voltages). The solution to the Dirichlet problem is then the electric potential field on the interior of the box, given these constraints.

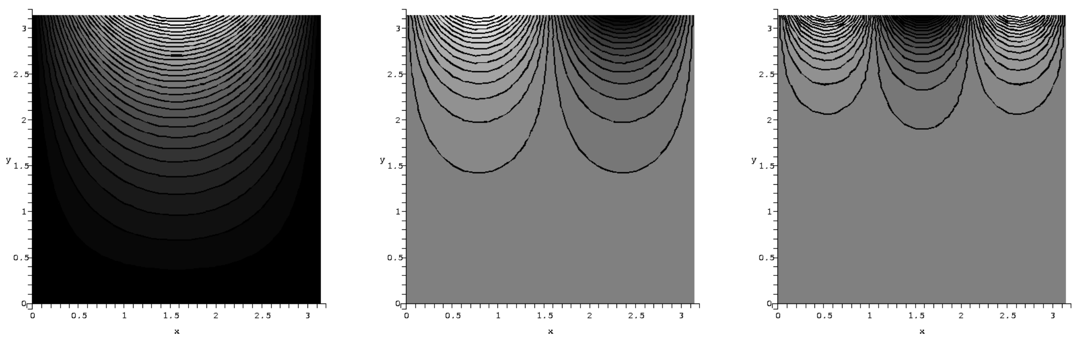
Minimal surface: Imagine a squarish frame of wire, which we have bent in the vertical direction to have some shape. If we dip this wire frame in a soap



Proposition 12A.1 — $T = 1$, $R = L = B = 0$

Example 12A.3 — $T = -3$, $L = 5$, $R = B = 0$

Example 12A.6 — $T =$ tent function , $R = L = B = 0$



$\sin(x)\sinh(y)$ — $T(x) = \sin(x)$, $R = L = B = 0$

$\sin(2x)\sinh(2y)$ — $T(x) = \sin(2x)$, $R = L = B = 0$

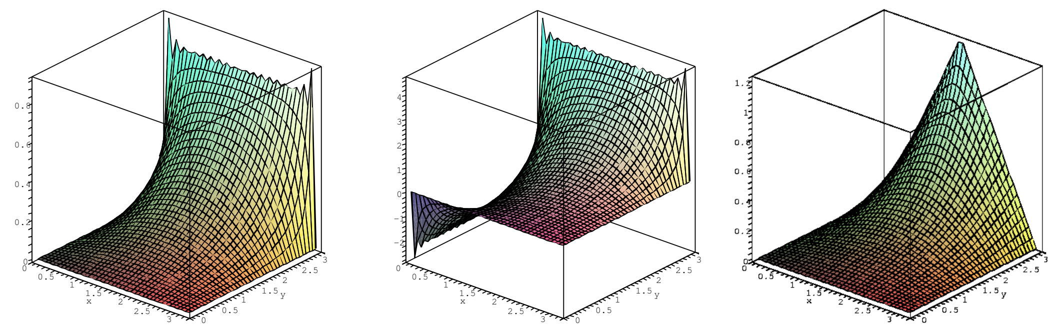
$\sin(3x)\sinh(3y)$ — $T(x) = \sin(3x)$, $R = L = B = 0$

Figure 12A.2: The Dirichlet problem on a box. The curves represent isothermal contours (of a temperature distribution) or equipotential lines (of an electric voltage field).

solution, we can form a soap bubble (i.e. minimal-energy surface) which must obey the 'boundary conditions' imposed by the shape of the wire. The differential equation describing a minimal surface is not *exactly* the same as the Laplace equation; however, when the surface is not too steeply slanted (i.e. when the wire frame is not too bent), the Laplace equation is a good approximation; hence the solution to the Dirichlet problem is a good approximation of the shape of the soap bubble.

We will begin with the simplest problem: a constant, nonzero Dirichlet boundary condition on one side of the box, and zero boundary conditions on the other three sides.

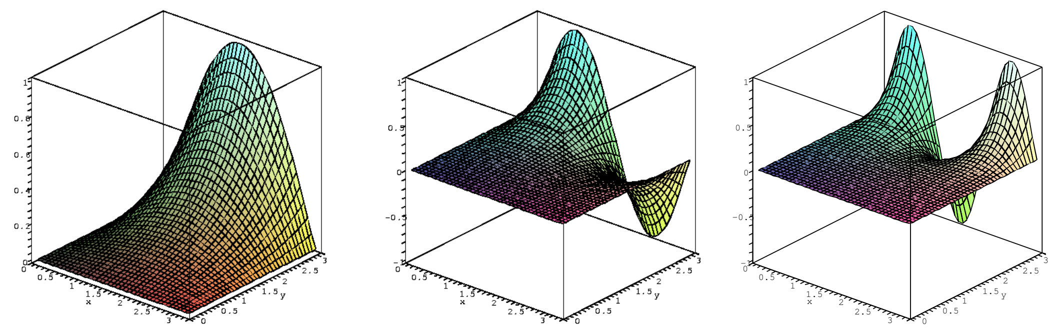
Proposition 12A.1. (Dirichlet problem; one constant nonhomog. BC)
Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, *and consider the Laplace equation* "$\triangle u = 0$", *with*



Proposition 12A.1 $T = 1$, $R = L = B = 0$

Example 12A.3 $T = -3$, $L = 5$, $R = B = 0$

Example 12A.6 $T =$ tent function , $R = L = B = 0$



$\sin(x)\sinh(y)$ $T(x) = \sin(x)$, $R = L = B = 0$

$\sin(2x)\sinh(2y)$ $T(x) = \sin(2x)$, $R = L = B = 0$

$\sin(3x)\sinh(3y)$ $T(x) = \sin(3x)$, $R = L = B = 0$

Figure 12A.3: The Dirichlet problem on a box: 3-dimensional plots. You can imagine these as soap films.

nonhomogeneous Dirichlet boundary conditions [see Figure 12A.1(A)]:

$$u(0, y) \;=\; u(\pi, y) \;=\; 0, \quad \text{for all } y \in [0, \pi). \qquad (12A.1)$$

$$u(x, 0) \;=\; 0 \quad \text{and} \quad u(x, \pi) \;=\; 1, \quad \text{for all } x \in [0, \pi]. \qquad (12A.2)$$

The unique solution to this problem is the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ *defined:*

$$u(x, y) \quad \underset{\text{L2}}{\approx} \quad \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(ny), \qquad \text{for all } (x, y) \in \mathbb{X}.$$

[See Figures 12A.2(a) and 12A.3(a).] *Furthermore, this series converges semiuniformly on* $\mathsf{int}(\mathbb{X})$.

Ⓔ *Proof.* **Exercise 12A.1**

(a) Check that, for all $n \in \mathbb{N}$, the function $u_n(x, y) = \sin(nx) \cdot \sinh(ny)$ satisfies the Laplace equation and the first boundary condition (12A.1). See Figures 12A.2(d,e,f) and 12A.3(d,e,f).

(b) Show that $\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} n^2 \left| \frac{\sinh(ny)}{n \sinh(n\pi)} \right| < \infty$, for any fixed $y < \pi$. (*Hint.* If $y < \pi$, then $\sinh(ny)/\sinh(n\pi)$ decays like $\exp(n(y-\pi))$ as $n \to \infty$.)

(c) Apply Proposition 0F.1 on page 565 to conclude that $\triangle u(x,y) = 0$.

(d) Observe that $\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left| \frac{\sinh(ny)}{n \sinh(n\pi)} \right| < \infty$, for any fixed $y < \pi$.

(e) Apply part (c) of Theorem 7A.1 on page 138 to show that the series given for $u(x,y)$ converges uniformly for any fixed $y < \pi$.

(f) Apply part (d) of Theorem 7A.1 on page 138 to conclude that $u(0,y) = 0 = u(\pi,y)$ for all $y < \pi$.

(g) Observe that $\sin(nx) \cdot \sinh(n \cdot 0) = 0$ for all $n \in \mathbb{N}$ and all $x \in [0,\pi]$. Conclude that $u(x,0) = 0$ for all $x \in [0,\pi]$.

(h) To check that the solution also satisfies the boundary condition (12A.2), subsititute $y = \pi$ to get:

$$u(x,\pi) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(n\pi) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin(nx) \underset{\text{L2}}{\approx} 1.$$

because $\frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin(nx)$ is the (one-dimensional) Fourier sine series for the function $b(x) = 1$ (see Example 7A.2(b) on page 139).

(i) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique.
□

Proposition 12A.2. (Dirichlet Problem; four constant nonhomog. BC)
Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions [see Figure 12A.1(B)]:

$$\begin{array}{rclcrcll} u(0,y) &=& L & \text{and} & u(\pi,y) &=& R, & \text{for all } y \in (0,\pi); \\ u(x,\pi) &=& T & \text{and} & u(x,0) &=& B, & \text{for all } x \in (0,\pi). \end{array}$$

where L, R, T, and B are four constants. The unique solution to this problem is the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ defined:

$$u(x,y) \quad := \quad l(x,y) + r(x,y) + t(x,y) + b(x,y), \quad \text{for all } (x,y) \in \mathbb{X}.$$

where, for all $(x,y) \in \mathbb{X}$,

$$l(x,y) \underset{\text{L2}}{\approx} L \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sinh\Big(n(\pi - x)\Big) \cdot \sin(ny), \qquad r(x,y) \underset{\text{L2}}{\approx} R \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sinh(nx) \cdot \sin(ny),$$

$$t(x,y) \underset{\text{L2}}{\approx} T \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sin(nx) \cdot \sinh(ny), \qquad b(x,y) \underset{\text{L2}}{\approx} B \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sin(nx) \cdot \sinh\Big(n(\pi - y)\Big).$$

where $c_n \;:=\; \dfrac{4}{n\pi\,\sinh(n\pi)}$, *for all* $n \in \mathbb{N}$.

Furthermore, these four series converge semiuniformly on $\mathsf{int}(\mathbb{X})$.

Ⓔ *Proof.* **Exercise 12A.2**

(a) Apply Proposition 12A.1 to show that each of the functions $l(x,y)$, $r(x,y)$, $t(x,y)$, $b(x,y)$ satisfies a Dirichlet problem where one side has nonzero temperature and the other three sides have zero temperature.

(b) Add these four together to get a solution to the original problem.

(c) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique. □

Ⓔ **Exercise 12A.3.** What happens to the solution at the four corners $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π)? ♦

Example 12A.3. Suppose $R = 0 = B$, $T = -3$, and $L = 5$. Then the solution is:

$$u(x,y) \;\underset{\text{L2}}{\approx}\; L \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} c_n \sinh\Big(n(\pi - x)\Big)\cdot\sin(ny) \;+\; T\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} c_n \sin(nx)\cdot\sinh(ny)$$

$$= \;\frac{20}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\sinh\Big(n(\pi-x)\Big)\cdot\sin(ny)}{n\sinh(n\pi)} \;-\; \frac{12}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{\sin(nx)\cdot\sinh(ny)}{n\sinh(n\pi)}.$$

See Figures 12A.2(b) and 12A.3(b). ♢

Proposition 12A.4. (Dirichlet Problem; arbitrary nonhomogeneous boundaries)

Let $\mathbb{X} = [0,\pi]\times[0,\pi]$, *and consider the Laplace equation "*$\triangle u = 0$*", with nonhomogeneous Dirichlet boundary conditions* [see Figure 12A.1(B)]:

$$\begin{array}{rclcrclc} u(0,y) &=& L(y) & \text{and} & u(\pi,y) &=& R(y), & \text{for all } y\in(0,\pi);\\ u(x,\pi) &=& T(x) & \text{and} & u(x,0) &=& B(x), & \text{for all } x\in(0,\pi). \end{array}$$

where $L, R, T, B : [0,\pi] \longrightarrow \mathbb{R}$ *are four arbitrary functions. Suppose these functions have (one-dimensional) Fourier sine series:*

$$\begin{array}{llll} L(y) \;\underset{\text{L2}}{\approx}\; \displaystyle\sum_{n=1}^{\infty} L_n \sin(ny), & & R(y) \;\underset{\text{L2}}{\approx}\; \displaystyle\sum_{n=1}^{\infty} R_n\sin(ny), & \text{for all } y\in[0,\pi];\\ T(x) \;\underset{\text{L2}}{\approx}\; \displaystyle\sum_{n=1}^{\infty} T_n\sin(nx), & \text{and} & B(x) \;\underset{\text{L2}}{\approx}\; \displaystyle\sum_{n=1}^{\infty} B_n\sin(nx), & \text{for all } x\in[0,\pi]. \end{array}$$

The unique solution to this problem is the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ *defined:*

$$u(x,y) \quad := \quad l(x,y) \; + \; r(x,y) \; + \; t(x,y) \; + \; b(x,y), \quad \textit{for all } (x,y) \in \mathbb{X}.$$

where, for all $(x,y) \in \mathbb{X}$,

$$\begin{aligned} l(x,y) \quad &\underset{\text{L2}}{\approx} \quad \sum_{n=1}^{\infty} \frac{L_n}{\sinh(n\pi)} \sinh\Big(n(\pi - x)\Big) \cdot \sin(ny), \\ r(x,y) \quad &\underset{\text{L2}}{\approx} \quad \sum_{n=1}^{\infty} \frac{R_n}{\sinh(n\pi)} \sinh(nx) \cdot \sin(ny), \\ t(x,y) \quad &\underset{\text{L2}}{\approx} \quad \sum_{n=1}^{\infty} \frac{T_n}{\sinh(n\pi)} \sin(nx) \cdot \sinh(ny), \\ \textit{and} \;\; b(x,y) \quad &\underset{\text{L2}}{\approx} \quad \sum_{n=1}^{\infty} \frac{B_n}{\sinh(n\pi)} \sin(nx) \cdot \sinh\Big(n(\pi - y)\Big). \end{aligned}$$

Furthermore, these four series converge semiuniformly on $\mathsf{int}(\mathbb{X})$.

Proof. **Exercise 12A.4** First we consider the function $t(x,y)$. Ⓔ

(a) Same as Exercise 12A.1(a)

(b) For any fixed $y < \pi$, show that $\sum_{n=1}^{\infty} n^2 T^n \left| \frac{\sinh(ny)}{\sinh(n\pi)} \right| < \infty$. (*Hint.* If $y < \pi$, then $\sinh(ny)/\sinh(n\pi)$ decays like $\exp(n(y-\pi))$ as $n \to \infty$.)

(c) Combine part (b) and Proposition 0F.1 on page 565 to conclude that $t(x,y)$ is harmonic —i.e. $\triangle t(x,y) = 0$.

Through symmetric reasoning, conclude that the functions $\ell(x,y)$, $r(x,y)$ and $b(x,y)$ are also harmonic.

(d) Same as Exercise 12A.1(d)

(e) Apply part (c) of Theorem 7A.1 on page 138 to show that the series given for $t(x,y)$ converges uniformly for any fixed $y < \pi$.

(f) Apply part (d) of Theorem 7A.1 on page 138 to conclude that $t(0,y) = 0 = t(\pi,y)$ for all $y < \pi$.

(g) Observe that $\sin(nx) \cdot \sinh(n \cdot 0) = 0$ for all $n \in \mathbb{N}$ and all $x \in [0,\pi]$. Conclude that $t(x,0) = 0$ for all $x \in [0,\pi]$.

(h) To check that the solution also satisfies the boundary condition (12A.2), subsititute $y = \pi$ to get:

$$t(x,\pi) \quad = \quad \sum_{n=1}^{\infty} \frac{T_n}{\sinh(n\pi)} \sin(nx) \cdot \sinh(n\pi) \quad = \quad \frac{4}{\pi} \sum_{n=1}^{\infty} T_n \sin(nx) \quad = \quad T(x).$$

(j) At this point, we know that $t(x,\pi) = T(x)$ for all $x \in [0,\pi]$, and $t \equiv 0$ on the other three sides of the square. Through symmetric reasoning, show that:

- $\ell(0, y) = L(y)$ for all $y \in [0, \pi]$, and $\ell \equiv 0$ on the other three sides of the square.
- $r(\pi, y) = R(y)$ for all $y \in [0, \pi]$, and $r \equiv 0$ on the other three sides of the square.
- $b(x, 0) = B(x)$ for all $x \in [0, \pi]$, and $b \equiv 0$ on the other three sides of the square.

(k) Conclude that $u = t + b + r + \ell$ is harmonic and satisfies the desired boundary conditions.

(l) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique. □

Example 12A.5. If $T(x) = \sin(3x)$, and $B \equiv L \equiv R \equiv 0$, then $u(x, y) = \dfrac{\sin(3x)\sinh(3y)}{\sinh(3\pi)}$. ♢

Example 12A.6. *Let* $\mathbb{X} = [0, \pi] \times [0, \pi]$. *Solve the 2-dimensional Laplace Equation on* $\mathbb{X}$, *with* inhomogeneous Dirichlet *boundary conditions:*

$$u(0, y) \;=\; 0; \qquad u(\pi, y) \;=\; 0; \qquad u(x, 0) \;=\; 0;$$

$$u(x, \pi) \;=\; T(x) \;=\; \left\{ \begin{array}{cl} x & \textit{if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \textit{if } \frac{\pi}{2} < x \leq \pi \end{array} \right. \qquad \text{(see Figure 7C.4(B) on page 154)}$$

Solution: Recall from Example 7C.7 on page 155 that $T(x)$ has Fourier series:

$$T(x) \;\underset{L2}{\approx}\; \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$$

Thus, the solution is $u(x, y) \;\underset{L2}{\approx}\; \dfrac{4}{\pi} \displaystyle\sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2 \sinh(n\pi)} \sin(nx)\sinh(ny).$

See Figures 12A.2(c) and 12A.3(c). ♢

Ⓔ **Exercise 12A.5.** Let $X, Y > 0$ and let $\mathbb{X} := [0, X] \times [0, Y]$. Generalize Proposition 12A.4 to find the solution to the Laplace equation on $\mathbb{X}$, satisfying arbitrary nonhomogeneous Dirichlet boundary conditions on the four sides of $\partial\mathbb{X}$. ♦

12B The heat equation on a square

12B(i) Homogeneous boundary conditions

Prerequisites: §9A, §5B, §5C, §1B(ii), §0F. **Recommended:** §11A, §7C(v).

Proposition 12B.1. (Heat equation; homogeneous Dirichlet boundary)
Consider the box $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier Sine Series

$$f(x,y) \underset{\mathrm{L2}}{\approx} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$$

and define the function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by

$$u_t(x,y) \underset{\mathrm{L2}}{\approx} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx) \cdot \sin(my) \cdot \exp\Big(-(n^2+m^2)\cdot t \Big),$$

for all $(x,y) \in \mathbb{X}$ and $t \geq 0$. Then u is the unique solution to the heat equation "$\partial_t u = \triangle u$", with homogeneous Dirichlet boundary conditions

$$u_t(x,0) = u_t(0,y) = u_t(\pi,y) = u_t(x,\pi) = 0, \qquad \text{for all } x,y \in [0,\pi] \text{ and } t > 0.$$

and initial conditions: $u_0(x,y) = f(x,y)$, for all $(x,y) \in \mathbb{X}$.
Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.

Proof. **<u>Exercise 12B.1</u>** Hint: Ⓔ

(a) Show that, when $t = 0$, the two-dimensional Fourier series of $u_0(x,y)$ agrees with that of $f(x,y)$; hence $u_0(x,y) = f(x,y)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n,m=1}^{\infty} \left|(n^2+m^2)\cdot B_{n,m}\cdot e^{-(n^2+m^2)t}\right| < \infty.$

(c) For any $T > 0$, apply Proposition 0F.1 on page 565 to conclude that

$$\partial_t\, u_t(x,y) \underset{\mathrm{unif}}{\equiv} \sum_{n,m=1}^{\infty} -(n^2+m^2) B_{n,m} \sin(nx)\cdot\sin(my)\cdot\exp\Big(-(n^2+m^2)\cdot t\Big) \underset{\mathrm{unif}}{\equiv} \triangle u_t(x,y),$$

for all $(x,y;t) \in \mathbb{X} \times [T,\infty)$.

(d) Observe that for all $t > 0$, $\displaystyle\sum_{n,m=1}^{\infty} \left|B_{n,m} e^{-(n^2+m^2)t}\right| < \infty.$

(e) Apply part (c)[i] of Theorem 9A.3 on page 183 to show that the two-dimensional Fourier series of u_t converges uniformly for any fixed $t > 0$.

(f) Apply part (d)[ii] of Theorem 9A.3 on page 183 to conclude that u_t satisfies homogeneous Dirichlet boundary conditions, for all $t > 0$.

(g) Apply Theorem 5D.8 on page 91 to show that this solution is unique. □

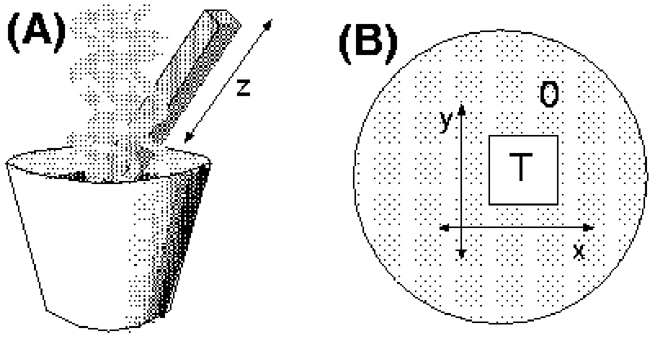


Figure 12B.1: **(A)** A hot metal rod quenched in a cold bucket. **(B)** A cross section of the rod in the bucket.

Example 12B.2: (The quenched rod)

On a cold January day, a blacksmith is tempering an iron rod. He pulls it out of the forge and plunges it, red-hot, into ice-cold water (Figure 12B.1A). *The rod is very long and narrow, with a square cross section. We want to compute how the rod cooled.*

Answer: The rod is immersed in freezing cold water, and is a good conductor, so we can assume that its outer surface takes the the surrounding water temperature of 0 degrees. Hence, we assume homogeneous Dirichlet boundary conditions.

Endow the rod with coordinate system (x, y, z), where z runs along the length of the rod. Since the rod is extremely long relative to its cross-section, we can neglect the z coordinate, and reduce to a 2-dimensional equation (Figure 12B.1B). Assume the rod was initially uniformly heated to a temperature of T. The initial temperature distribution is thus a constant function: $f(x, y) = T$. From Example 9A.2 on page 182, we know that the constant function 1 has two-dimensional Fourier sine series:

$$1 \underset{\text{L2}}{\approx} \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx) \sin(my).$$

Thus, $f(x, y) \underset{\text{L2}}{\approx} \dfrac{16T}{\pi^2} \displaystyle\sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx) \sin(my)$. Thus, the time-varying thermal profile of the rod is given:

$$u_t(x, y) \quad \underset{\text{L2}}{\approx} \quad \frac{16T}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx) \sin(my) \exp\Big(-(n^2 + m^2) \cdot t \Big). \qquad \diamondsuit$$

Proposition 12B.3. (Heat equation; homogeneous Neumann boundary)
Consider the box $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier Cosine Series

$$f(x,y) \underset{\text{L2}}{\approx} \sum_{n,m=0}^{\infty} A_{n,m} \cos(nx)\cos(my)$$

and define the function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by:

$$u_t(x,y) \underset{\text{L2}}{\approx} \sum_{n,m=0}^{\infty} A_{n,m} \cos(nx)\cdot\cos(my)\cdot \exp\Big(-(n^2+m^2)\cdot t\Big),$$

for all $(x,y) \in \mathbb{X}$ and $t \geq 0$. Then u is the unique solution to the heat equation "$\partial_t u = \triangle u$", with homogeneous Neumann boundary conditions

$$\partial_y u_t(x,0) = \partial_y u_t(x,\pi) = \partial_x u_t(0,y) = \partial_x u_t(\pi,y) = 0, \text{ for all } x,y \in [0,\pi] \text{ and } t > 0.$$

and initial conditions: $u_0(x,y) = f(x,y)$, for all $(x,y) \in \mathbb{X}$. □

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.

Proof. __**Exercise 12B.2**__ Hint: Ⓔ

(a) Show that, when $t = 0$, the two-dimensional Fourier cosine series of $u_0(x,y)$ agrees with that of $f(x,y)$; hence $u_0(x,y) = f(x,y)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n,m=0}^{\infty} \left|(n^2+m^2)\cdot A_{n,m}\cdot e^{-(n^2+m^2)t}\right| < \infty.$

(c) Apply Proposition 0F.1 on page 565 to conclude that

$$\partial_t u_t(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=0}^{\infty} -(n^2+m^2)A_{n,m}\cos(nx)\cdot\cos(my)\cdot\exp\Big(-(n^2+m^2)\cdot t\Big) \underset{\text{unif}}{\equiv} \triangle u_t(x,y),$$

for all $(x,y) \in \mathbb{X}$ and $t > 0$.

(d) Observe that for all $t > 0$, $\displaystyle\sum_{n,m=0}^{\infty} n \cdot \left|A_{n,m}e^{-(n^2+m^2)t}\right| < \infty$ and $\displaystyle\sum_{n,m=0}^{\infty} m \cdot \left|A_{n,m}e^{-(n^2+m^2)t}\right| < \infty.$

(e) Apply part (e)[ii] of Theorem 9A.3 on page 183 to conclude that u_t satisfies homogeneous Neumann boundary conditions, for any fixed $t > 0$.

(f) Apply Theorem 5D.8 on page 91 to show that this solution is unique. □

Example 12B.4. Suppose $\mathbb{X} = [0, \pi] \times [0, \pi]$

(a) Let $f(x, y) = \cos(3x)\cos(4y) + 2\cos(5x)\cos(6y)$. Then $A_{3,4} = 1$ and $A_{5,6} = 2$, and all other Fourier coefficients are zero. Thus, $u(x, y; t) = \cos(3x)\cos(4y) \cdot e^{-25t} + \cos(5x)\cos(6y) \cdot e^{-59t}$.

(b) Suppose $f(x, y) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \text{ and } 0 \le y < \frac{\pi}{2}; \\ 0 & \text{if } \frac{\pi}{2} \le x \text{ or } \frac{\pi}{2} \le y. \end{cases}$ We know from Example 9A.4 on page 184 that the two-dimensional Fourier cosine series of f is:

$$f(x, y) \underset{\text{L2}}{\approx} \frac{1}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\Big((2k+1)x\Big) + \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \cos\Big((2j+1)y\Big)$$
$$+ \frac{4}{\pi^2} \sum_{k,j=1}^{\infty} \frac{(-1)^{k+j}}{(2k+1)(2j+1)} \cos\Big((2k+1)x\Big) \cdot \cos\Big((2j+1)y\Big)$$

Thus, the solution to the heat equation, with initial conditions $u_0(x, y) = f(x, y)$ and homogeneous Neumann boundary conditions is given:

$$u_t(x, y) \underset{\text{L2}}{\approx} \frac{1}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\Big((2k+1)x\Big) \cdot e^{-(2k+1)^2 t}$$
$$+ \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \cos\Big((2j+1)y\Big) \cdot e^{-(2j+1)^2 t}$$
$$+ \frac{4}{\pi^2} \sum_{k,j=1}^{\infty} \frac{(-1)^{k+j}}{(2k+1)(2j+1)} \cos\Big((2k+1)x\Big) \cdot \cos\Big((2j+1)y\Big) \cdot e^{-\left[(2k+1)^2+(2j+1)^2\right] \cdot t}$$

◇

Ⓔ **Exercise 12B.3.** Let $X, Y > 0$ and let $\mathbb{X} := [0, X] \times [0, Y]$. Let $\kappa > 0$ be a diffusion constant, and consider the general two-dimensional heat equation

$$\partial_t u = \kappa \triangle u. \tag{12B.1}$$

(a) Generalize Proposition 12B.1 to find the solution to eqn.(12B.1) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 12B.3 to find the solution to eqn.(12B.1) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Neumann boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(12B.1) (Hint: imitate the strategy suggested in Exercises 12B.1 and 12B.2) ♦

Ⓔ **Exercise 12B.4.** Let $f : \mathbb{X} \longrightarrow \mathbb{R}$ and suppose the Fourier sine series of f satisfies the constraint $\sum_{n,m=1}^{\infty} (n^2 + m^2)|B_{nm}| < \infty$. Imitate Proposition 12B.1 to find a Fourier series solution to the initial value problem for the two-dimensional *free Schrödinger equation*

$$\mathbf{i}\partial_t \omega \quad = \quad \frac{-1}{2} \triangle \omega \qquad (12B.2)$$

on the box $\mathbb{X} = [0, \pi]^2$, with homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(12B.2). (Hint: imitate the strategy suggested in Exercise 12B.1, and also Exercise 12D.1 on page 260). ♦

12B(ii) Nonhomogeneous boundary conditions

Prerequisites: §12B(i), §12A. **Recommended:** §12C(ii).

Proposition 12B.5. (Heat equation on box; nonhomogeneous Dirichlet BC)
Let $\mathbb{X} = [0, \pi] \times [0, \pi]$. Let $f : \mathbb{X} \longrightarrow \mathbb{R}$ and let $L, R, T, B : [0, \pi] \longrightarrow \mathbb{R}$ be functions. Consider the Heat equation

$$\partial_t u(x, y; t) \ = \ \triangle u(x, y; t),$$

with initial conditions

$$u(x, y; 0) \quad = \quad f(x, y), \qquad \text{for all } (x, y) \in \mathbb{X}, \qquad (12B.3)$$

and nonhomogeneous Dirichlet boundary conditions:

$$\left.\begin{array}{rclcrclc} u(x, \pi; t) & = & T(x) & \text{and} & u(x, 0; t) & = & B(x), & \text{for all } x \in [0, \pi] \\ u(0, y; t) & = & L(y) & \text{and} & u(\pi, y; t) & = & R(y), & \text{for all } y \in [0, \pi] \end{array}\right\} \text{for all } t > 0. \qquad (12B.4)$$

This problem is solved as follows:

1. *Let $w(x, y)$ be the solution*[1] *to the Laplace Equation "$\triangle w(x, y) = 0$", with the nonhomogeneous Dirichlet BC* (12B.4).

2. *Define $g(x, y) \ := \ f(x, y) - w(x, y)$. Let $v(x, y; t)$ be the solution*[2] *to the heat equation "$\partial_t v(x, y; t) \ = \ \triangle v(x, y; t)$" with initial conditions $v(x, y; 0) = g(x, y)$, and homogeneous Dirichlet BC.*

3. *Define $u(x, y; t) \ := \ v(x, y; t) + w(x, y)$. Then $u(x, y; t)$ is a solution to the heat equation with initial conditions* (12B.3) *and nonhomogeneous Dirichlet BC* (12B.4).

Proof. **Exercise 12B.5** □ Ⓔ

Interpretation: In Proposition 12B.5, the function $w(x,y)$ represents the *long-term thermal equilibrium* that the system is 'trying' to attain. The function $g(x,y) = f(x,y) - w(x,y)$ thus measures the *deviation* between the current state and this equilibrium, and the function $v(x,y;t)$ thus represents how this 'transient' deviation decays to zero over time.

Example 12B.6. Suppose $T(x) = \sin(2x)$ and $R \equiv L \equiv 0$ and $B \equiv 0$. Then Proposition 12A.4 on page 244 says

$$w(x,y) \quad = \quad \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}.$$

Suppose $f(x,y) := \sin(2x)\sin(y)$. Then

$$\begin{aligned} g(x,y) \quad &= \quad f(x,y) - w(x,y) \quad = \quad \sin(2x)\sin(y) - \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)} \\ &\underset{(*)}{=} \quad \sin(2x)\sin(y) \quad - \quad \left(\frac{\sin(2x)}{\sinh(2\pi)}\right)\left(\frac{2\sinh(2\pi)}{\pi}\right)\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{2^2+m^2}\cdot\sin(my) \\ &= \quad \sin(2x)\sin(y) \quad - \quad \frac{2\sin(2x)}{\pi}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{4+m^2}\cdot\sin(my). \end{aligned}$$

Here $(*)$ is because Example 7A.3 on page 140 says $\sinh(2y) \;=\; \dfrac{2\sinh(2\pi)}{\pi}\displaystyle\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{2^2+m^2}\cdot \sin(my)$. Thus, Proposition 12B.1 on page 247 says that

$$v(x,y;t) \quad = \quad \sin(2x)\sin(y)e^{-5t} - \frac{2\sin(2x)}{\pi}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{4+m^2}\cdot\sin(mx)\exp(-(4+m^2)t).$$

Finally, Proposition 12B.5 says the solution is $u(x,y;t) \;:=\; v(x,y;t) \;+\; \dfrac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$. ♢

Example 12B.7. *A freshly baked baguette is removed from the oven and left on a wooden plank to cool near the window. The baguette is initially at a uniform temperature of* $90^o\,C$*; the air temperature is* $20^o\,C$*, and the temperature of the wooden plank (which was sitting in the sunlight) is* $30^o\,C$*.*

Mathematically model the cooling process near the center of the baguette. How long will it be before the baguette is cool enough to eat? (assuming 'cool enough' is below $40^o\,C$*.)*

[1] Obtained from Proposition 12A.4 on page 244, for example.
[2] Obtained from Proposition 12B.1 on page 247, for example.

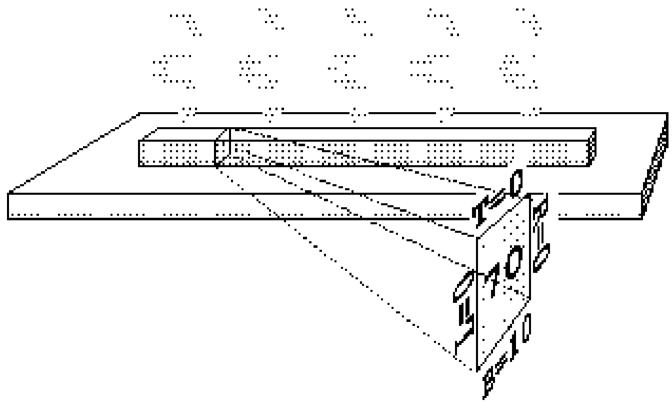


Figure 12B.2: The temperature distribution of a baguette

Answer: For simplicity, we will assume the baguette has a square cross-section (and dimensions $\pi \times \pi$, of course). If we confine our attention to the middle of the baguette, we are far from the endpoints, so that we can neglect the longitudinal dimension and treat this as a two-dimensional problem.

Suppose the temperature distribution along a cross section through the center of the baguette is given by the function $u(x, y; t)$. To simplify the problem, we will subtract $20^o\,C$ off all temperatures. Thus, in the notation of Proposition 12B.5 the boundary conditions are:

$$\begin{array}{rcl} L(y) \;=\; R(y) \;=\; T(x) & = & 0 \qquad \text{(the air)} \\ \text{and} \qquad B(x) & = & 10. \qquad \text{(the wooden plank)} \end{array}$$

and our initial temperature distribution is $f(x, y) = 70$ (see Figure 12B.2).

From Proposition 12A.1 on page 241, we know that the long-term equilibrium for these boundary conditions is given by:

$$w(x,y) \underset{L2}{\approx} \frac{40}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(n(\pi - y)).$$

We want to represent this as a two-dimensional Fourier sine series. To do this, we need the (one-dimensional) Fourier sine series for $\sinh(nx)$. We set $\alpha = n$ in Example 7A.3 on page 140, and get:

$$\sinh(nx) \quad \underset{L2}{\approx} \quad \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m(-1)^{m+1}}{n^2 + m^2} \cdot \sin(mx). \tag{12B.5}$$

Thus,

$$\sinh\Big(n(\pi - y)\Big) \quad \underset{L2}{\approx} \quad \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m(-1)^{m+1}}{n^2 + m^2} \cdot \sin(m\pi - my)$$

$$= \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m}{n^2+m^2} \cdot \sin(my),$$

because $\sin(m\pi - ny) = \sin(m\pi)\cos(ny) - \cos(m\pi)\sin(ny) = (-1)^{m+1}\sin(ny)$. Substituting this into (12B.5) yields:

$$\begin{aligned} w(x,y) &\underset{\mathrm{L2}}{\approx} \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sinh(n\pi)}{n \cdot \sinh(n\pi)(n^2+m^2)} \sin(nx) \cdot \sin(my) \\ &= \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sin(nx) \cdot \sin(my)}{n \cdot (n^2+m^2)} \end{aligned} \tag{12B.6}$$

Now, the initial temperature distribution is the constant function with value 70. Take the two-dimensional sine series from Example 9A.2 on page 182, and multiply it by 70, to obtain:

$$f(x,y) = 70 \underset{\mathrm{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus,

$$\begin{aligned} g(x,y) &= f(x,y) - w(x,y) \\ &\underset{\mathrm{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n \cdot m} - \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sin(nx) \cdot \sin(my)}{n \cdot (n^2+m^2)} \end{aligned}$$

Thus,

$$\begin{aligned} v(x,y;t) &\underset{\mathrm{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n \cdot m} \exp\Big(-(n^2+m^2)t\Big) \\ &\qquad - \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sin(nx) \cdot \sin(my)}{n \cdot (n^2+m^2)} \exp\Big(-(n^2+m^2)t\Big) \end{aligned}$$

If we combine the second term in this expression with (12B.6), we get the final answer:

$$\begin{aligned} u(x,y;t) &= v(x,y;t) + w(x,y) \\ &\underset{\mathrm{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n \cdot m} \exp\Big(-(n^2+m^2)t\Big) \\ &\qquad + \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sin(nx) \cdot \sin(my)}{n \cdot (n^2+m^2)} \left[1 - \exp\Big(-(n^2+m^2)t\Big)\right] \end{aligned}$$

◊

12C The Poisson problem on a square

12C(i) Homogeneous boundary conditions

Prerequisites: §9A, §5C, §1D. **Recommended:** §11C, §7C(v).

Proposition 12C.1. *Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function with* **semiuniformly convergent** *Fourier sine series:*

$$q(x,y) \underset{\text{L2}}{\approx} \sum_{n,m=1}^{\infty} Q_{n,m} \sin(nx)\sin(my).$$

Define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by $u(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} \frac{-Q_{n,m}}{n^2+m^2} \sin(nx)\sin(my)$,

for all $(x,y) \in \mathbb{X}$.

Then u is the unique solution to the **Poisson equation** *"$\triangle u(x,y) = q(x,y)$", satisfying homogeneous Dirichlet boundary conditions $u(x,0) = u(0,y) = u(x,\pi) = u(\pi,y) = 0$.*

Proof. <u>**Exercise 12C.1**</u> (a) Use Proposition 0F.1 on page 565 to show that u satisfies the Poisson equation on $\mathsf{int}(\mathbb{X})$. Ⓔ

(b) Use Proposition 9A.3(e) on page 183 to show that u satisfies homogeneous Dirichlet BC.

(c) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique. □

Example 12C.2. *A nuclear submarine beneath the Arctic Ocean has jettisoned a fuel rod from its reactor core (Figure 12C.1). The fuel rod is a very long, narrow, enriched uranium bar with square cross section. The radioactivity causes the fuel rod to be uniformly heated from within at a rate of Q, but the rod is immersed in freezing Arctic water. We want to compute its internal temperature distribution.*

Answer: The rod is immersed in freezing cold water, and is a good conductor, so we can assume that its outer surface takes the the surrounding water temperature of 0 degrees. Hence, we assume homogeneous Dirichlet boundary conditions.

Endow the rod with coordinate system (x,y,z), where z runs along the length of the rod. Since the rod is extremely long relative to its cross-section, we can neglect the z coordinate, and reduce to a 2-dimensional equation. The uniform heating is described by a constant function: $q(x,y) = Q$. From Example 9A.2

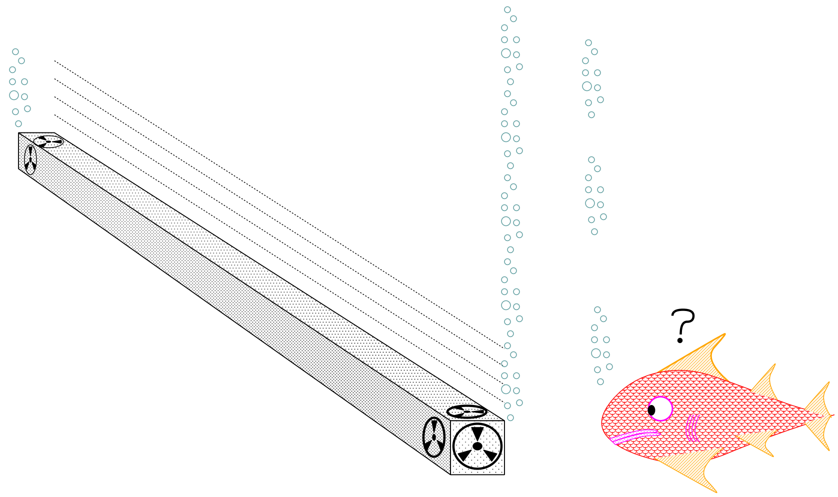


Figure 12C.1: A jettisoned fuel rod in the Arctic Ocean

on page 182, know that the constant function 1 has two-dimensional Fourier sine series:

$$1 \underset{L2}{\approx} \frac{16}{\pi^2} \sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus, $q(x,y) \underset{L2}{\approx} \frac{16Q}{\pi^2} \sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$. The temperature distribution must satisfy Poisson's equation. Thus, the temperature distribution is:

$$u(x,y) \underset{\text{unif}}{\equiv} \frac{-16Q}{\pi^2} \sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m \cdot (n^2+m^2)} \sin(nx)\sin(my). \qquad \diamond$$

Example 12C.3. Suppose $q(x,y) = x \cdot y$. Then the solution to the Poisson equation $\triangle u = q$ on the square, with homogeneous Dirichlet boundary conditions, is given by:

$$u(x,y) \underset{\text{unif}}{\equiv} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm \cdot (n^2+m^2)} \sin(nx)\sin(my)$$

To see this, recall from Example 9A.1 on page 179 that the two-dimensional Fourier sine series for $q(x,y)$ is:

$$xy \underset{L2}{\approx} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m}}{nm} \sin(nx)\sin(my).$$

Now apply Proposition 12C.1. $\diamond$

Proposition 12C.4. *Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function with semiuniformly convergent Fourier cosine series:*

$$q(x,y) \quad \underset{\text{L2}}{\approx} \quad \sum_{n,m=0}^{\infty} Q_{n,m} \cos(nx)\cos(my).$$

Suppose that $Q_{0,0} = 0$. Fix some constant $K \in \mathbb{R}$, and define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by

$$u(x,y) \quad \underset{\text{unif}}{\equiv} \quad \sum_{\substack{n,m=0 \\ \text{not both zero}}}^{\infty} \frac{-Q_{n,m}}{n^2+m^2} \cos(nx)\cos(my) \quad + \quad K, \tag{12C.1}$$

for all $(x,y) \in \mathbb{X}$. Then u is a solution to the Poisson equation "$\triangle u(x,y) = q(x,y)$", satisfying homogeneous Neumann boundary conditions $\partial_y u(x,0) = \partial_x u(0,y) = \partial_y u(x,\pi) = \partial_x u(\pi,y) = 0$.

Furthermore, all solutions to this Poisson equation with these boundary conditions have the form (12C.1).

If $Q_{0,0} \neq 0$, however, the problem has no solution.

Proof. **Exercise 12C.2** (a) Use Proposition 0F.1 on page 565 to show that u satisfies the Poisson equation on $\mathsf{int}(\mathbb{X})$. Ⓔ

(b) Use Proposition 9A.3 on page 183 to show that u satisfies homogeneous Neumann BC.

(c) Apply Theorem 5D.5(c) on page 88 to conclude that this solution is unique up to addition of a constant. □

Exercise 12C.3. Mathematically, it is clear that the solution of Proposition 12C.4 cannot be well-defined if $Q_{0,0} \neq 0$. Provide a physical explanation for why this is to be expected. ♦ Ⓔ

Example 12C.5. Suppose $q(x,y) = \cos(2x)\cdot\cos(3y)$. Then the solution to the Poisson equation $\triangle u = q$ on the square, with homogeneous Neumann boundary conditions, is given by:

$$u(x,y) \quad = \quad \frac{-\cos(2x)\cdot\cos(3y)}{13}.$$

To see this, note that the two-dimensional Fourier Cosine series of $q(x,y)$ is just $\cos(2x)\cdot\cos(3y)$. In other words, $A_{2,3} = 1$, and $A_{n,m} = 0$ for all other n and m. In particular, $A_{0,0} = 0$, so we can apply Proposition 12C.4 to conclude: $u(x,y) = \frac{-\cos(2x)\cdot\cos(3y)}{2^2+3^2} = \frac{-\cos(2x)\cdot\cos(3y)}{13}$. ♢

12C(ii) Nonhomogeneous boundary conditions

Prerequisites: §12C(i), §12A. **Recommended:** §12B(ii).

Proposition 12C.6. (Poisson equation on box; nonhomogeneous Dirichlet BC) *Let $\mathbb{X} = [0,\pi] \times [0,\pi]$. Let $q : \mathbb{X} \longrightarrow \mathbb{R}$ and $L, R, T, B : [0,\pi] \longrightarrow \mathbb{R}$ be functions. Consider the Poisson equation*

$$\triangle\, u(x,y) = q(x,y), \tag{12C.2}$$

with nonhomogeneous Dirichlet boundary conditions:

$$\begin{array}{rclcrcll} u(x,\pi) & = & T(x) & \text{and} & u(x,0) & = & B(x), & \text{for all } x \in [0,\pi] \\ u(0,y) & = & L(y) & \text{and} & u(\pi,y) & = & R(y), & \text{for all } y \in [0,\pi] \end{array} \tag{12C.3}$$

(see Figure 12A.1(B) on page 240). This problem is solved as follows:

1. *Let $v(x,y)$ be the solution*[3] *to the Poisson equation (12C.2) with homogeneous Dirichlet BC: $v(x,0) = v(0,y) = v(x,\pi) = v(\pi,y) = 0$.*
2. *Let $w(x,y)$ be the solution*[4] *to the Laplace Eqation "$\triangle w(x,y) = 0$", with the nonhomogeneous Dirichlet BC (12C.3).*
3. *Define $u(x,y) := v(x,y) + w(x,y)$; then $u(x,y)$ is a solution to the Poisson problem with the nonhomogeneous Dirichlet BC (12C.3).*

Ⓔ *Proof.* **Exercise 12C.4** □

Example 12C.7. *Suppose $q(x,y) = x \cdot y$. Find the solution to the Poisson equation $\triangle u = q$ on the square, with nonhomogeneous Dirichlet boundary conditions:*

$$u(0,y) \;=\; 0; \qquad u(\pi,y) \;=\; 0; \qquad u(x,0) \;=\; 0; \tag{12C.4}$$

$$u(x,\pi) \;=\; T(x) \;=\; \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{if } \frac{\pi}{2} < x \le \pi \end{cases} \quad \text{(see Figure 7C.4(B) on page 154)} \tag{12C.5}$$

Solution: In Example 12C.3, we found the solution to the Poisson equation $\triangle v = q$, with *homogeneous* Dirichlet boundary conditions; it was:

$$v(x,y) \;\underset{\text{unif}}{\equiv}\; 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm \cdot (n^2+m^2)} \sin(nx)\sin(my).$$

[3] Obtained from Proposition 12C.1 on page 255, for example.
[4] Obtained from Proposition 12A.4 on page 244, for example.

In Example 12A.6 on page 246, we found the solution to the Laplace equation $\triangle w = 0$, with nonhomogeneous Dirichlet boundary conditions (12C.4) and (12C.5); it was:

$$w(x,y) \underset{L2}{\approx} \frac{4}{\pi} \sum_{\substack{n=1\\ n \text{ odd};\\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2 \sinh(n\pi)} \sin(nx)\sinh(ny).$$

Thus, according to Proposition 12C.6 on the facing page, the solution to the nonhomogeneous Poisson problem is:

$$\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &\underset{L2}{\approx} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm\cdot(n^2+m^2)} \sin(nx)\sin(my) + \frac{4}{\pi} \sum_{\substack{n=1\\ n \text{ odd};\\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2\sinh(n\pi)} \sin(nx)\sinh(ny). \end{aligned}$$

$\diamondsuit$

12D The wave equation on a square (the square drum)

Prerequisites: §9A, §5B, §5C, §2B(ii), §0F. **Recommended:** §11B, §7C(v).

Imagine a drumskin stretched tightly over a square frame. At equilibrium, the drumskin is perfectly flat, but if we strike the skin, it will vibrate, meaning that the membrane will experience vertical displacements from equilibrium. Let $\mathbb{X} = [0,\pi]\times[0,\pi]$ represent the square skin, and for any point $(x,y)\in\mathbb{X}$ on the drumskin and time $t>0$, let $u(x,y;t)$ be the vertical displacement of the drum. Then u will obey the two-dimensional wave equation:

$$\partial_t^2 u(x,y;t) = \triangle u(x,y;t). \tag{12D.1}$$

However, since the skin is held down along the edges of the box, the function u will also exhibit homogeneous **Dirichlet** boundary conditions

$$\left.\begin{array}{rclcrcll} u(x,\pi;t) &=& 0 & \text{and} & u(x,0;t) &=& 0, & \text{for all } x\in[0,\pi] \\ u(0,y;t) &=& 0 & \text{and} & u(\pi,y;t) &=& 0, & \text{for all } y\in[0,\pi] \end{array}\right\} \text{ for all } t>0. \tag{12D.2}$$

Proposition 12D.1. (Initial Position for Square Drumskin)

Let $\mathbb{X} = [0,\pi]\times[0,\pi]$, and let $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial displacement of the drumskin. Suppose f_0 has Fourier Sine Series

$f_0(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$, *such that:*

$$\sum_{n,m=1}^{\infty} (n^2+m^2)|B_{n,m}| < \infty. \tag{12D.3}$$

Define the function $w : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *by*

$$w(x,y;t) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right), \tag{12D.4}$$

for all $(x,y) \in \mathbb{X}$ *and* $t \geq 0$. *Then series* (12D.4) *converges uniformly, and* $w(x,y;t)$ *is the unique solution to the wave equation* (12D.1), *satisfying the Dirichlet boundary conditions* (12D.2), *as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & w(x,y,0) & = & f_0(x,y), \\ \textbf{Initial Velocity:} & \partial_t\, w(x,y,0) & = & 0, \end{array}\right\} \quad \textit{for all } (x,y) \in \mathbb{X}.$$

Ⓔ *Proof.* **Exercise 12D.1** (a) Use the hypothesis (12D.3) and Proposition 0F.1 on page 565 to conclude that

$$\partial_t^2\, w(x,y;t) \underset{\text{unif}}{\equiv} -\sum_{n,m=1}^{\infty} (n^2+m^2)\cdot B_{n,m} \sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right) \underset{\text{unif}}{\equiv} \triangle w(x,y;t)$$

for all $(x,y) \in \mathbb{X}$ and $t > 0$.

(b) Check that the Fourier series (12D.4) converges uniformly.

(c) Use Theorem 9A.3(d)[ii] on page 183 to conclude that w satisfies Dirichlet boundary conditions.

(d) Set $t = 0$ to check the initial position.

(e) Set $t = 0$ and use Proposition 0F.1 on page 565 to check initial velocity.

(f) Apply Theorem 5D.11 on page 94 to show that this solution is unique. □

Example 12D.2. Suppose $f_0(x,y) = \sin(2x)\cdot\sin(3y)$. Then the solution to the wave equation on the square, with initial position f_0, and homogeneous Dirichlet boundary conditions, is given by:

$$w(x,y;t) \quad = \quad \sin(2x)\cdot\sin(3y)\cdot\cos(\sqrt{13}\ t).$$

To see this, note that the two-dimensional Fourier sine series of $f_0(x,y)$ is just $\sin(2x)\cdot\sin(3y)$. In other words, $B_{2,3} = 1$, and $B_{n,m} = 0$ for all other n and m. Apply Proposition 12D.1 to conclude: $w(x,y;t) = \sin(2x)\cdot\sin(3y)\cdot\cos\left(\sqrt{2^2+3^2}\ t\right) = \sin(2x)\cdot\sin(3y)\cdot\cos(\sqrt{13}\ t)$. ♢

Proposition 12D.3. (Initial Velocity for Square Drumskin)

Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial velocity of the drumskin. Suppose f_1 has Fourier Sine Series $f_1(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$, such that

$$\sum_{n,m=1}^{\infty} \sqrt{n^2+m^2} \cdot |B_{n,m}| < \infty. \tag{12D.5}$$

Define the function $v : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by:

$$v(x,y;t) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} \frac{B_{n,m}}{\sqrt{n^2+m^2}} \sin(nx)\cdot\sin(my)\cdot\sin\left(\sqrt{n^2+m^2}\ \cdot t\right), \tag{12D.6}$$

for all $(x,y) \in \mathbb{X}$ and $t \geq 0$. Then the series (12D.6) *converges uniformly, and $v(x,y;t)$ is the unique solution to the wave equation* (12D.1)*, satisfying the Dirichlet boundary conditions* (12D.2)*, as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & v(x,y,0) & = & 0; \\ \textbf{Initial Velocity:} & \partial_t\, v(x,y,0) & = & f_1(x,y). \end{array}\right\} \quad \textit{for all } (x,y) \in \mathbb{X}.$$

Proof. **<u>Exercise 12D.2</u>** (a) Use the hypothesis (12D.5) and Proposition 0F.1 on page 565 to conclude that Ⓔ

$$\begin{aligned} \partial_t^2\, v(x,y;t) &\underset{\text{unif}}{\equiv} - \sum_{n,m=1}^{\infty} \sqrt{n^2+m^2}\cdot B_{n,m} \sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right) \\ &\underset{\text{unif}}{\equiv} \triangle v(x,y;t), \end{aligned}$$

for all $(x,y) \in \mathbb{X}$ and $t > 0$.

(b) Check that the Fourier series (12D.6) converges uniformly.

(c) Use Theorem 9A.3(d)[ii] on page 183 to conclude that $v(x,y;t)$ satisfies Dirichlet boundary conditions.

(d) Set $t = 0$ to check the initial position.

(e) Set $t = 0$ and use Proposition 0F.1 on page 565 to check initial velocity.

(f) Apply Theorem 5D.11 on page 94 to show that this solution is unique. □

Remark. Note that it is important in these theorems not only for the Fourier series (12D.4) and (12D.6) to converge uniformly, but also for their formal *second derivative* series to converge uniformly. This is not guaranteed. This is the reason for imposing the hypotheses (12D.3) and (12D.5).

Example 12D.4. Suppose

$$f_1(x,y) \quad = \quad \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{99} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

(This is a partial sum of the two-dimensional Fourier sine series for the constant function $\widetilde{f_1}(x,y) \equiv 1$, from Example 9A.2 on page 182). Then the solution to the two-dimensional wave equation, with homogeneous Dirichlet boundary conditions and initial velocity f_1, is given:

$$w(x,y;\,t) \quad \underset{\mathrm{L2}}{\widetilde{\approx}} \quad \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{99} \frac{1}{n \cdot m \cdot \sqrt{n^2+m^2}} \sin(nx)\sin(my)\sin\left(\sqrt{n^2+m^2} \;\cdot t\right).$$

Question: Why can't we apply Theorem 12D.3 to the full Fourier series for the function $f_1 = 1$? (Hint: Is (12D.5) satisfied?) ♢

Question: For the solutions of the heat equation and Poisson equation, in Propositions 12B.1, 12B.3, and 12C.1, we did not need to impose explicit hypotheses guaranteeing the uniform convergence of the given series (and its derivatives). But we *do* need explicit hypotheses to get convergence for the wave equation. Why is this?

12E Practice problems

1. Let $f(y) \;=\; 4\sin(5y)$ for all $y \in [0,\pi]$.

   (a) Solve the **two-dimensional Laplace Equation** ($\triangle u \;=\; 0$) on the square domain $\mathbb{X} = [0,\pi]\times[0,\pi]$, with **nonhomogeneous Dirichlet boundary conditions:**

   $$\begin{array}{rclcrcll} u(x,0) &=& 0 & \text{and} & u(x,\pi) &=& 0, & \text{for all } x \in [0,\pi] \\ u(0,y) &=& 0 & \text{and} & u(\pi,y) &=& f(y), & \text{for all } y \in [0,\pi]. \end{array}$$

   (b) *Verify* your solution to part (a) (i.e. check boundary conditions, Laplacian, etc.).

2. Let $f_1(x,y) \;=\; \sin(3x)\sin(4y)$.

(a) Solve the **two-dimensional wave equation** ($\partial_t^2 u = \triangle u$) on the square domain $\mathbb{X} = [0,\pi] \times [0,\pi]$, with on the square domain $\mathbb{X} = [0,\pi] \times [0,\pi]$, with **homogeneous Dirichlet boundary conditions**, and initial conditions:

$$\begin{array}{rrcll} \text{Initial position:} & u(x,y,0) & = & 0 & \text{for all } (x,y) \in \mathbb{X} \\ \text{Initial velocity:} & \partial_t u(x,y,0) & = & f_1(x,y) & \text{for all } (x,y) \in \mathbb{X} \end{array}$$

(b) *Verify* your that solution in part (a) satisfies the required initial conditions (don't worry about boundary conditions or checking the wave equation).

3. Solve the two-dimensional **Laplace Equation** $\triangle h = 0$ on the square domain $\mathbb{X} = [0,\pi]^2$, with **inhomogeneous Dirichlet** boundary conditions:

   (a) $h(\pi,y) = \sin(2y)$ and $h(0,y) = 0$, for all $y \in [0,\pi]$;
   $h(x,0) = 0 = h(x,\pi)$ for all $x \in [0,\pi]$.

   (b) $h(\pi,y) = 0$ and $h(0,y) = \sin(4y)$, for all $y \in [0,\pi]$;
   $h(x,\pi) = \sin(3x)$; $h(x,0) = 0$, for all $x \in [0,\pi]$.

4. Let $\mathbb{X} = [0,\pi]^2$ and let $q(x,y) = \sin(x)\cdot\sin(3y) + 7\sin(4x)\cdot\sin(2y)$. Solve the Poisson Equation $\triangle u(x,y) = q(x,y)$. with **homogeneous Dirichlet** boundary conditions.

5. Let $\mathbb{X} = [0,\pi]^2$. Solve the **heat equation** $\partial_t u(x,y;t) = \triangle u(x,y;t)$ on $\mathbb{X}$, with initial conditions $u(x,y;0) = \cos(5x)\cdot\cos(y)$. and **homogeneous Neumann** boundary conditions.

6. Let $f(x,y) = \cos(2x)\cos(3y)$. Solve the following boundary value problems on the square domain $\mathbb{X} = [0,\pi]^2$ (**Hint:** see problem #3 of §9C).

   (a) Solve the two-dimensional **heat equation** $\partial_t u = \triangle u$, with homogeneous **Neumann** boundary conditions, and initial conditions $u(x,y;0) = f(x,y)$.

   (b) Solve the two-dimensional **wave equation** $\partial_t^2 u = \triangle u$, with homogeneous **Dirichlet** boundary conditions, initial **position** $w(x,y;0) = f(x,y)$ and initial **velocity** $\partial_t w(x,y;0) = 0$.

   (c) Solve the two-dimensional **Poisson Equation** $\triangle u = f$ with homogeneous **Neumann** boundary conditions.

   (d) Solve the two-dimensional **Poisson Equation** $\triangle u = f$ with homogeneous **Dirichlet** boundary conditions.

(e) Solve the two-dimensional **Poisson Equation** $\triangle v = f$ with **inhomogeneous Dirichlet** boundary conditions:

$$\begin{array}{rcll} v(\pi, y) &=& \sin(2y); \qquad v(0, y) \;=\; 0 & \text{for all } y \in [0, \pi]. \\ v(x, 0) &=& 0 \;=\; v(x, \pi) & \text{for all } x \in [0, \pi]. \end{array}$$

7. $\mathbb{X} = [0, \pi]^2$ be the **box** of sidelength π. Let $f(x, y) = \sin(3x) \cdot \sinh(3y)$. (**Hint:** see problem #4 of §9C).

(a) Solve the **heat equation** on $\mathbb{X}$, with **initial conditions** $u(x, y; 0) = f(x, y)$, and **homogeneous Dirichlet** boundary conditions.

(b) Let $T(x) = \sin(3x)$. Solve the **Laplace Equation** $\triangle u(x, y) = 0$ on the box, with **inhomogeneous Dirichlet** boundary conditions: $u(x, \pi) = T(x)$ and $u(x, 0) = 0$ for $x \in [0, \pi]$; $u(0, y) = 0 = u(\pi, y)$, for $y \in [0, \pi]$.

(c) Solve the **heat equation** on the box with initial conditions on the box $\mathbb{X}$, with **initial conditions** $u(x, y; 0) = 0$, and the same **inhomogeneous Dirichlet** boundary conditions as in part (b).

Chapter 13

Boundary value problems on a cube

"Mathematical Analysis is as extensive as nature herself." —Jean Joseph Fourier

The Fourier series technique used to solve BVPs on a square box extends readily to 3-dimensional cubes, and indeed, to rectilinear domains in any number of dimensions. As in Chapter 12, we will confine our exposition to the cube $[0,\pi]^3$, and assume that the physical constants in the various equations are all set to one. Thus, the heat equation becomes "$\partial_t u = \triangle u$", the wave equation is "$\partial_t^2 u = \triangle u$", etc. This allows us to develop the solution methods with minimum technicalities. The extension of each solution method to equations with arbitrary physical constants on an arbitrary box $[0,X]\times[0,Y]\times[0,Z]$ (for some $X,Y,Z>0$) is left as a straightforward (but important!) exercise.

We will use the following notation:

- The cube of dimensions $\pi\times\pi\times\pi$ is denoted $\mathbb{X} = [0,\pi]\times[0,\pi]\times[0,\pi] = [0,\pi]^3$.
- A point in the cube will be indicated by a vector $\mathbf{x} = (x_1, x_2, x_3)$, where $0 \le x_1, x_2, x_3 \le \pi$.
- If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a function on the cube, then
$$\triangle f(\mathbf{x}) \;=\; \partial_1^2 f(\mathbf{x}) + \partial_2^2 f(\mathbf{x}) + \partial_3^2 f(\mathbf{x}).$$
- A triple of natural numbers will be denoted by $\mathbf{n} = (n_1, n_2, n_3)$, where $n_1, n_2, n_3 \in \mathbb{N} := \{0,1,2,3,4,\ldots\}$. Let $\mathbb{N}^3$ be the set of all triples $\mathbf{n} = (n_1, n_2, n_3)$, where $n_1, n_2, n_3 \in \mathbb{N}$. Thus, an expression of the form
$$\sum_{\mathbf{n}\in\mathbb{N}^3} (\text{something about } \mathbf{n})$$

should be read as: "$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty}$ (something about (n_1, n_2, n_3))".

Let $\mathbb{N}_+ := \{1, 2, 3, 4, \ldots\}$ be the set of *nonzero* natural numbers, and let $\mathbb{N}_+^3$ be the set of all such triples. Thus, an expression of the form

$$\sum_{\mathbf{n} \in \mathbb{N}_+^3} (\text{something about } \mathbf{n})$$

should be read as: "$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty}$ (something about (n_1, n_2, n_3))".

- For any $\mathbf{n} \in \mathbb{N}_+^3$, $\mathbf{S_n}(\mathbf{x}) = \sin(n_1 x_1) \cdot \sin(n_2 x_2) \cdot \sin(n_3 x_3)$. The Fourier *sine* series of a function $f(\mathbf{x})$ thus has the form: $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_+^3} B_\mathbf{n} \mathbf{S_n}(\mathbf{x})$
- For any $\mathbf{n} \in \mathbb{N}^3$, $\mathbf{C_n}(\mathbf{x}) = \cos(n_1 x_1) \cdot \cos(n_2 x_2) \cdot \cos(n_3 x_3)$. The Fourier *cosine* series of a function $f(\mathbf{x})$ thus has the form: $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^3} A_\mathbf{n} \mathbf{C_n}(\mathbf{x})$
- For any $\mathbf{n} \in \mathbb{N}^3$, let $\|\mathbf{n}\| = \sqrt{n_1^2 + n_2^2 + n_3^2}$. In particular, note that:

$$\triangle \mathbf{S_n} = -\|\mathbf{n}\|^2 \cdot \mathbf{S_n}, \qquad \text{and} \quad \triangle\, \mathbf{C_n} = -\|\mathbf{n}\|^2 \cdot \mathbf{C_n}$$

Ⓔ (**Exercise 13.1**)

13A The heat equation on a cube

Prerequisites: §9B, §5B, §5C, §1B(ii). **Recommended:** §11A, §12B(i), §7C(v).

Proposition 13A.1. (Heat equation; homogeneous Dirichlet BC)

Consider the cube $\mathbb{X} = [0, \pi]^3$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier sine series $f(\mathbf{x}) \underset{L2}{\approx} \sum_{\mathbf{n} \in \mathbb{N}_+^3} B_\mathbf{n} \mathbf{S_n}(\mathbf{x})$. Define the function $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by:

$$u(\mathbf{x}; t) \underset{L2}{\approx} \sum_{\mathbf{n} \in \mathbb{N}_+^3} B_\mathbf{n} \mathbf{S_n}(\mathbf{x}) \cdot \exp\Big(-\|\mathbf{n}\|^2 \cdot t\Big).$$

Then u is the unique solution to the heat equation "$\partial_t u = \triangle u$", with homogeneous Dirichlet boundary conditions

$$\begin{aligned} u(x_1, x_2, 0; t) &= u(x_1, x_2, \pi; t) = u(x_1, 0, x_3; t) \qquad \text{(see Figure 13A.1A)} \\ &= u(x_1, \pi, x_3; t) = u(0, x_2, x_3; t) = u(\pi, x_2, x_3, ; t) = 0, \end{aligned}$$

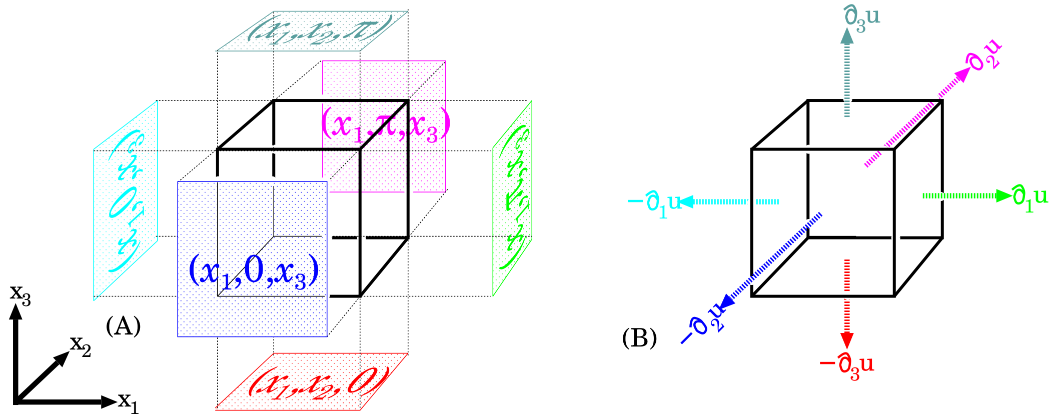


Figure 13A.1: Boundary conditions on a cube: **(A)** Dirichlet. **(B)** Neumann.

and initial conditions: $u(\mathbf{x};0) = f(\mathbf{x})$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.

Proof. **Exercise 13A.1** □ Ⓔ

Example: *An ice cube of dimensions $\pi \times \pi \times \pi$ is removed from a freezer (ambient temperature -10^o C) and dropped into a pitcher of freshly brewed tea (initial temperature $+90^o$ C). We want to compute how long it takes the ice cube to melt.*

Answer: We will assume that the cube has an initially uniform temperature of -10^o C and is completely immersed in the tea[1]. We will also assume that the pitcher is large enough that its temperature doesn't change during the experiment.

We assume the outer surface of the cube takes the temperature of the surrounding tea. By subtracting 90 from the temperature of the cube and the water, we can set the water to have temperature 0 and the cube, -100. Hence, we assume homogeneous Dirichlet boundary conditions; the initial temperature distribution is a constant function: $f(\mathbf{x}) = -100$. The constant function -100 has Fourier sine series:

$$-100 \underset{\mathrm{L2}}{\approx} \frac{-6400}{\pi^3} \sum_{\substack{\mathbf{n}\in\mathbb{N}^3_+ \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}(\mathbf{x}).$$

(**Exercise 13A.2** Verify this Fourier series). Let κ be the thermal conductivity Ⓔ

[1] Unrealistic, since actually the cube floats just at the surface.

of the ice. Thus, the time-varying thermal profile of the cube is given[2]

$$\boxed{u(\mathbf{x};t) \underset{\mathrm{L2}}{\approx} \frac{-6400}{\pi^3} \sum_{\substack{\mathbf{n}\in\mathbb{N}^3_+ \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}(\mathbf{x}) \exp\Big(-\|\mathbf{n}\|^2 \cdot \kappa \cdot t\Big)}.$$

Thus, to determine how long it takes the cube to melt, we must solve for the minimum value of t such that $u(\mathbf{x},t) > -90$ everywhere (recall than -90 corresponds to 0^o C.). The coldest point in the cube is always at its center (**Exercise 13A.3**), which has coordinates $\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)$, so we need to solve for t in the inequality $u\left(\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right);t\right) \geq -90$, which is equivalent to

$$\begin{aligned}
\frac{90\cdot\pi^3}{6400} &\geq \sum_{\substack{\mathbf{n}\in\mathbb{N}^3_+ \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) \exp\Big(-\|\mathbf{n}\|^2\cdot\kappa\cdot t\Big) \\
&= \sum_{\substack{\mathbf{n}\in\mathbb{N}^3_+ \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \sin\left(\frac{n_1\pi}{2}\right)\sin\left(\frac{n_2\pi}{2}\right)\sin\left(\frac{n_3\pi}{2}\right) \exp\Big(-\|\mathbf{n}\|^2\cdot\kappa\cdot t\Big) \\
&\underset{(7C.5)}{=\!=} \sum_{k_1,k_2,k_3\in\mathbb{N}_+} \frac{(-1)^{k_1+k_2+k_3}\exp\Big(-\kappa\cdot\left[(2k_1+1)^2+(2k_2+1)^2+(2k_3+1)^2\right]\cdot t\Big)}{(2k_1+1)\cdot(2k_2+1)\cdot(2k_3+1)}.
\end{aligned}$$

where **(7C.5)** is by eqn. (7C.5) on p. 147. The solution of this inequality is **Exercise 13A.4** .

Exercise 13A.5. Imitating Proposition 13A.1, find a Fourier series solution to the initial value problem for the *free Schrödinger equation*

$$\mathbf{i}\partial_t\,\omega = \frac{-1}{2}\triangle\,\omega,$$

on the cube $\mathbb{X} = [0,\pi]^3$, with homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies the Schrödinger equation. ♦

Proposition 13A.2. (Heat equation; homogeneous Neumann BC)

[2]Actually, this is physically unrealistic for two reasons. First, as the ice melts, additional thermal energy is absorbed in the phase transition from solid to liquid. Second, once part of the ice cube has melted, its thermal properties change; liquid water has a different thermal conductivity, and in addition, transports heat through convection.

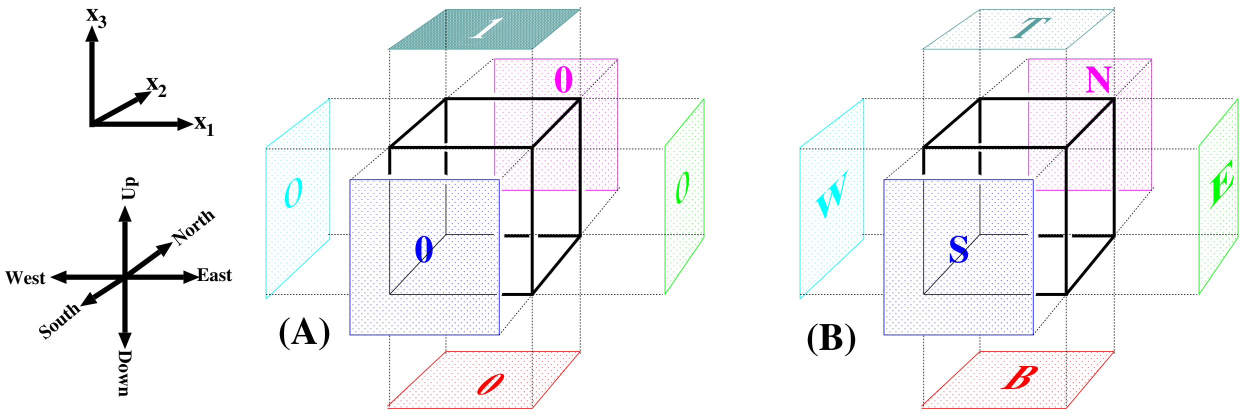


Figure 13B.1: Dirichlet boundary conditions on a cube **(A)** Constant; Nonhomogeneous on one side only. **(B)** Arbitrary nonhomogeneous on all sides.

Consider the cube $\mathbb{X} = [0,\pi]^3$, *and let* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *be some function describing an initial heat distribution. Suppose* f *has Fourier Cosine Series* $f(\mathbf{x}) \underset{L2}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^3} A_\mathbf{n}\mathbf{C}_\mathbf{n}(\mathbf{x})$. *Define the function* $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *by:*

$$u(\mathbf{x};t) \underset{L2}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^3} A_\mathbf{n}\mathbf{C}_\mathbf{n}(\mathbf{x}) \cdot \exp\Big(- \|\mathbf{n}\|^2 \cdot t\Big).$$

Then u *is the unique solution to the heat equation "*$\partial_t u = \triangle u$*", with homogeneous* Neumann *boundary conditions*

$$\begin{aligned} \partial_3 u(x_1,x_2,0;t) = \partial_3 u(x_1,x_2,\pi;t) = \partial_2 u(x_1,0,x_3;t) &= \\ \partial_2 u(x_1,\pi,x_3;t) = \partial_1 u(0,x_2,x_3;t) = \partial_1 u(\pi,x_2,x_3,;t) &= 0. \qquad \textit{(see Figure 13A.1B)} \end{aligned}$$

and initial conditions: $u(\mathbf{x};0) = f(\mathbf{x})$.

Furthermore, the series defining u *converges semiuniformly on* $\mathbb{X} \times \mathbb{R}_+$.

Proof. **Exercise 13A.6** □ Ⓔ

13B The Dirichlet problem on a cube

Prerequisites: §9B, §5C(i), §1C. **Recommended:** §7C(v), §12A.

Proposition 13B.1. (Laplace Equation; one constant nonhomog. Dirichlet BC)

Let $\mathbb{X} = [0, \pi]^3$, and consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions (see Figure 13B.1A):

$$\begin{aligned} u(x_1, 0, x_3) = u(x_1, \pi, x_3) = u(0, x_2, x_3) = u(\pi, x_2, x_3,) &= 0; \quad (13B.1) \\ u(x_1, x_2, 0) &= 0; \\ u(x_1, x_2, \pi) &= 1. \quad (13B.2) \end{aligned}$$

The unique solution to this problem is the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ defined

$$u(x_1, x_2, x_3) \underset{L2}{\approx} \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi \sinh(\pi\sqrt{n^2+m^2})} \sin(nx)\sin(my)\cdot\sinh(\sqrt{n^2+m^2}\cdot x_3).$$

for all $(x_1, x_2, x_3) \in \mathbb{X}$. Furthermore, this series converges semiuniformly on $\mathsf{int}(\mathbb{X})$.

Ⓔ *Proof.* **Exercise 13B.1** (a) Check that the series and its formal Laplacian both converge semiuniformly on $\mathsf{int}(\mathbb{X})$. (b) Check that each of the functions $u_{n,m}(\mathbf{x}) = \sin(nx)\sin(my)\cdot\sinh(\sqrt{n^2+m^2}x_3)$ satisfies the Laplace equation and the first boundary condition (13B.1). (c) To check that the solution also satisfies the boundary condition (13B.2), subsititute $x_2 = \pi$ to get:

$$\begin{aligned} u(x_1, x_2, \pi) &= \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi \sinh(\pi\sqrt{n^2+m^2})} \sin(nx)\sin(my)\cdot\sinh(\sqrt{n^2+m^2}\pi) \\ &= \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi} \sin(nx)\sin(my) \underset{L2}{\approx} 1, \end{aligned}$$

because this is the Fourier sine series for the function $b(x_1, x_2) = 1$, by Example 9A.2 on page 182.

(d) Apply Theorem 5D.5(a) on page 88 to conclude that this solution is unique. □

Proposition 13B.2. (Laplace Equation; arbitrary nonhomogeneous Dirichlet BC)

Let $\mathbb{X} = [0, \pi]^3$, and consider the Laplace equation "$\triangle h = 0$", with nonhomogeneous Dirichlet boundary conditions (see Figure 13B.1B):

$$\begin{aligned} h(x_1, x_2, 0) &= D(x_1, x_2) & h(x_1, x_2, \pi) &= U(x_1, x_2) \\ h(x_1, 0, x_3) &= S(x_1, x_3) & h(x_1, \pi, x_3) &= N(x_1, x_3) \\ h(0, x_2, x_3) &= W(x_2, x_3) & h(\pi, x_2, x_3,) &= E(x_2, x_3) \end{aligned}$$

where $D(x_1,x_2)$, $U(x_1,x_2)$, $S(x_1,x_3)$, $N(x_1,x_3)$, $W(x_2,x_3)$, and $E(x_2,x_3)$ are six functions. Suppose that these functions have two-dimensional Fourier sine series:

$$\begin{aligned}
D(x_1,x_2) &\underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} D_{n_1,n_2}\sin(n_1x_1)\sin(n_2x_2);\\
U(x_1,x_2) &\underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} U_{n_1,n_2}\sin(n_1x_1)\sin(n_2x_2);\\
S(x_1,x_3) &\underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} S_{n_1,n_3}\sin(n_1x_1)\sin(n_3x_3);\\
N(x_1,x_3) &\underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} N_{n_1,n_3}\sin(n_1x_1)\sin(n_3x_3);\\
W(x_2,x_3) &\underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} W_{n_2,n_3}\sin(n_2x_2)\sin(n_3x_3);\\
E(x_2,x_3) &\underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} E_{n_2,n_3}\sin(n_2x_2)\sin(n_3x_3).
\end{aligned}$$

Then the unique solution to this problem is the function:

$$h(\mathbf{x}) \;=\; d(\mathbf{x})+u(\mathbf{x})+s(\mathbf{x})+n(\mathbf{x})+w(\mathbf{x})+e(\mathbf{x})$$

$$\begin{aligned}
d(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} \frac{D_{n_1,n_2}}{\sinh\left(\pi\sqrt{n_1^2+n_2^2}\right)}\sin(n_1x_1)\sin(n_2x_2)\sinh\left(\sqrt{n_1^2+n_2^2}\;\cdot x_3\right);\\
u(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} \frac{U_{n_1,n_2}}{\sinh\left(\pi\sqrt{n_1^2+n_2^2}\right)}\sin(n_1x_1)\sin(n_2x_2)\sinh\left(\sqrt{n_1^2+n_2^2}\;\cdot(\pi-x_3)\right);\\
s(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} \frac{S_{n_1,n_3}}{\sinh\left(\pi\sqrt{n_1^2+n_3^2}\right)}\sin(n_1x_1)\sin(n_3x_3)\sinh\left(\sqrt{n_1^2+n_3^2}\;\cdot x_2\right);\\
n(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} \frac{N_{n_1,n_3}}{\sinh\left(\pi\sqrt{n_1^2+n_3^2}\right)}\sin(n_1x_1)\sin(n_3x_3)\sinh\left(\sqrt{n_1^2+n_3^2}\;\cdot(\pi-x_2)\right);\\
w(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} \frac{W_{n_2,n_3}}{\sinh\left(\pi\sqrt{n_2^2+n_3^2}\right)}\sin(n_2x_2)\sin(n_3x_3)\sinh\left(\sqrt{n_2^2+n_3^2}\;\cdot x_1\right);\\
e(x_1,x_2,x_3) &\underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} \frac{E_{n_2,n_3}}{\sinh\left(\pi\sqrt{n_2^2+n_3^2}\right)}\sin(n_2x_2)\sin(n_3x_3)\sinh\left(\sqrt{n_2^2+n_3^2}\;\cdot(\pi-x_1)\right).
\end{aligned}$$

Furthermore, these six series converge semiuniformly on $\mathsf{int}(\mathbb{X})$.

Ⓔ *Proof.* **Exercise 13B.2** □

13C The Poisson problem on a cube

Prerequisites: §9B, §5C, §1D. **Recommended:** §11C, §12C, §7C(v).

Proposition 13C.1. Poisson Problem on Cube; homogeneous Dirichlet BC

Let $\mathbb{X} = [0,\pi]^3$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function with semiuniformly convergent Fourier sine series: $q(\mathbf{x}) \underset{\text{L2}}{\approx} \sum_{\mathbf{n}\in\mathbb{N}_+^3} Q_{\mathbf{n}}\mathbf{S}_{\mathbf{n}}(\mathbf{x})$. Define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by

$$u(\mathbf{x}) \underset{\text{unif}}{\equiv} \sum_{\mathbf{n}\in\mathbb{N}_+^3} \frac{-Q_{\mathbf{n}}}{\|\mathbf{n}\|^2} \cdot \mathbf{S}_{\mathbf{n}}(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{X}.$$

Then u is the unique solution to the Poisson equation "$\triangle u(\mathbf{x}) = q(\mathbf{x})$", satisfying homogeneous Dirichlet boundary conditions $u(x_1, x_2, 0) = u(x_1, x_2, \pi) = u(x_1, 0, x_3) = u(x_1, \pi, x_3) = u(0, x_2, x_3) = u(\pi, x_2, x_3,) = 0$.

Ⓔ *Proof.* **Exercise 13C.1** □

Proposition 13C.2. Poisson Problem on Cube; homogeneous Neumann BC

Let $\mathbb{X} = [0,\pi]^3$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function with semiuniformly convergent Fourier cosine series: $q(\mathbf{x}) \underset{\text{L2}}{\approx} \sum_{\mathbf{n}\in\mathbb{N}_+^3} Q_{\mathbf{n}}\mathbf{C}_{\mathbf{n}}(\mathbf{x})$.

Suppose $Q_{0,0,0} = 0$. Fix some constant $K \in \mathbb{R}$, and define the function $u : \mathbb{X} \longrightarrow \mathbb{R}$ by

$$u(\mathbf{x}) \underset{\text{unif}}{\equiv} \sum_{\substack{\mathbf{n}\in\mathbb{N}^3 \\ n_1,n_2,n_3 \text{ not all zero}}} \frac{-Q_{\mathbf{n}}}{\|\mathbf{n}\|^2} \cdot \mathbf{C}_{\mathbf{n}}(\mathbf{x}) \quad + \quad K, \quad \text{for all } \mathbf{x} \in \mathbb{X}. \tag{13C.1}$$

Then u is a solution to the Poisson equation "$\triangle u(\mathbf{x}) = q(\mathbf{x})$", satisfying homogeneous Neumann boundary conditions $\partial_3 u(x_1, x_2, 0) = \partial_3 u(x_1, x_2, \pi) = \partial_2 u(x_1, 0, x_3) = \partial_2 u(x_1, \pi, x_3) = \partial_1 u(0, x_2, x_3) = \partial_1 u(\pi, x_2, x_3,) = 0$.

Furthermore, all solutions to this Poisson equation with these boundary conditions have the form (13C.1).

If $Q_{0,0,0} \neq 0$, however, the problem has no solution.

Ⓔ *Proof.* **Exercise 13C.2** □

Chapter 14

Boundary value problems in polar coordinates

"The source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case." —Paul Halmos

14A Introduction

Prerequisites: §0D(ii).

When solving a boundary value problem, the shape of the domain dictates the choice of coordinate system. Seek the coordinate system yielding the simplest description of the boundary. For rectangular domains, Cartesian coordinates are the most convenient. For disks and annuli in the plane, *polar* coordinates are a better choice. Recall that polar coordinates (r, θ) on $\mathbb{R}^2$ are defined by the transformation:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta). \qquad \text{(Figure 14A.1A)}$$

with reverse transformation:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Here, the coordinate r ranges over $\mathbb{R}_{\not-}$, while the variable θ ranges over $[-\pi, \pi)$. (Clearly, we could let θ range over *any* interval of length 2π; we just find $[-\pi, \pi)$ the most convenient).

The three domains we will examine are:

- $\mathbb{D} = \{(r, \theta) \; ; \; r \leq R\}$, the **disk** of radius R; see Figure 14A.1B. For simplicity we will usually assume $R = 1$.

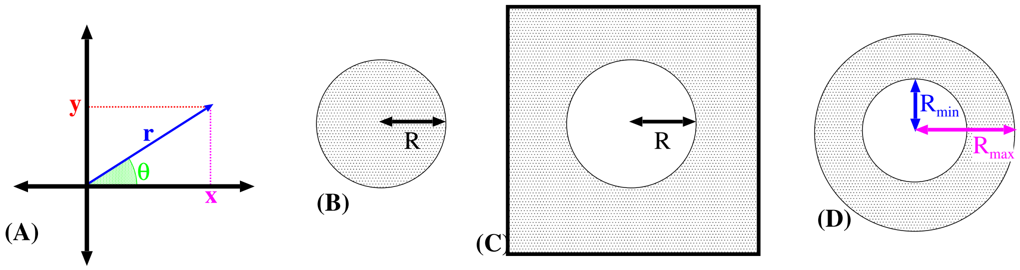


Figure 14A.1: **(A)** Polar coordinates; **(B)** The disk $\mathbb{D}$; **(C)** The codisk $\mathbb{D}^{\complement}$; **(D)** The annulus $\mathbb{A}$.

- $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ R \leq r\}$, the **codisk** or **punctured plane** of radius R; see Figure 14A.1C. For simplicity we will usually assume $R = 1$.

- $\mathbb{A} = \{(r,\theta)\ ;\ R_{\min} \leq r \leq R_{\max}\}$, the **annulus**, of inner radius $R_{\min}$ and outer radius $R_{\max}$; see Figure 14A.1D.

The boundaries of these domains are circles. For example, the boundary of the disk $\mathbb{D}$ of radius R is the **circle**:

$$\partial\mathbb{D} \quad = \quad \mathbb{S} \quad = \quad \{(r,\theta)\ ;\ r = R\}\,.$$

The circle can be parameterized by a single angular coordinate $\theta \in [-\pi,\pi)$. Thus, the boundary conditions will be specified by a function $b : [-\pi,\pi) \longrightarrow \mathbb{R}$. Note that, if $b(\theta)$ is to be *continuous* as a function on the circle, then it must be 2π*-periodic* as a function on $[-\pi,\pi)$.

In polar coordinates, the Laplacian is written:

$$\triangle\, u \quad = \quad \partial_r^2\, u \ + \ \frac{1}{r}\,\partial_r\, u \ + \ \frac{1}{r^2}\,\partial_\theta^2\, u. \tag{14A.1}$$

Ⓔ (**Exercise 14A.1**)

14B The Laplace equation in polar coordinates

14B(i) Polar harmonic functions

Prerequisites: §0D(ii), §1C.

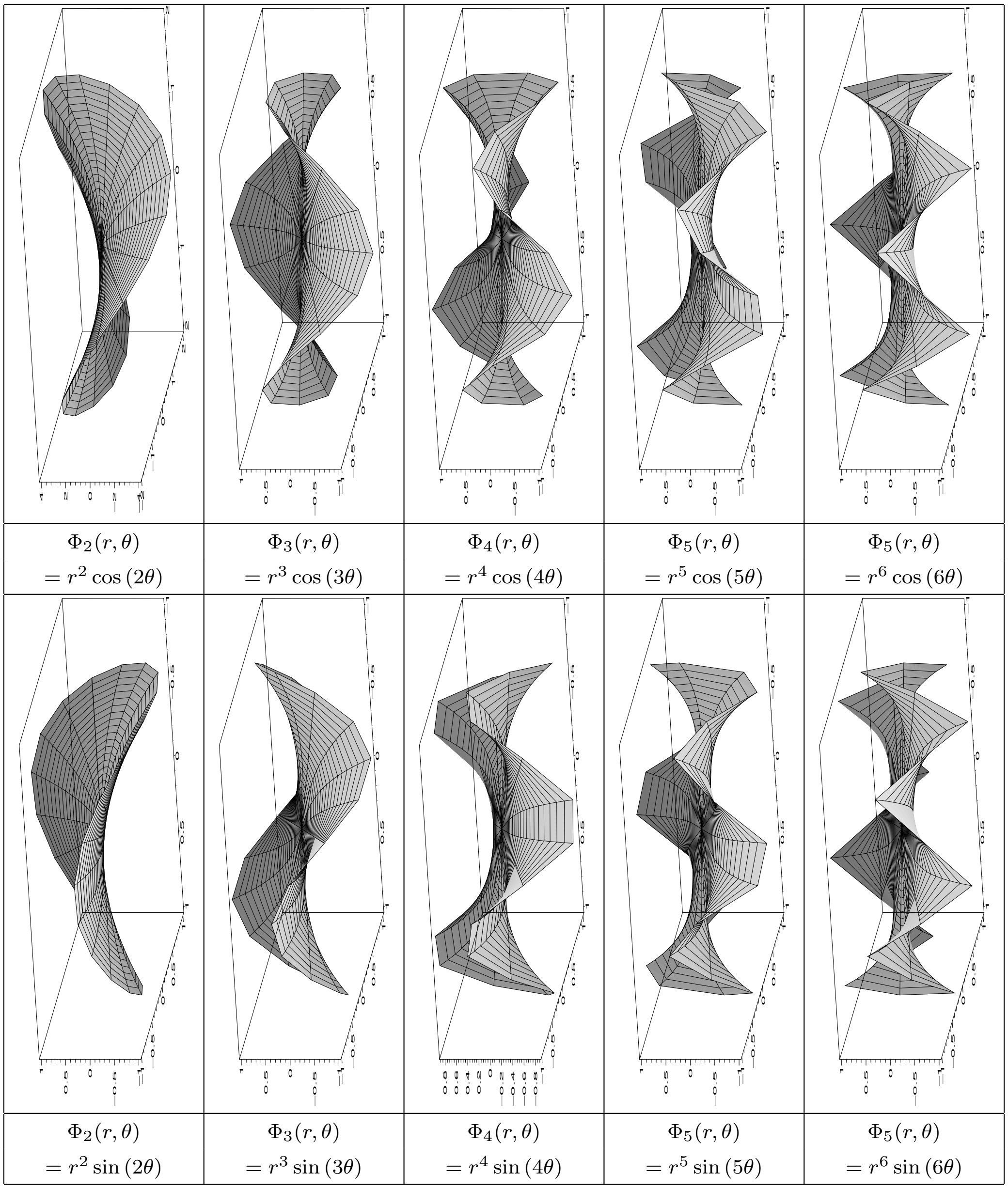


Figure 14B.1: Φ_n and Ψ_n for $n = 2...6$ (rotate page).

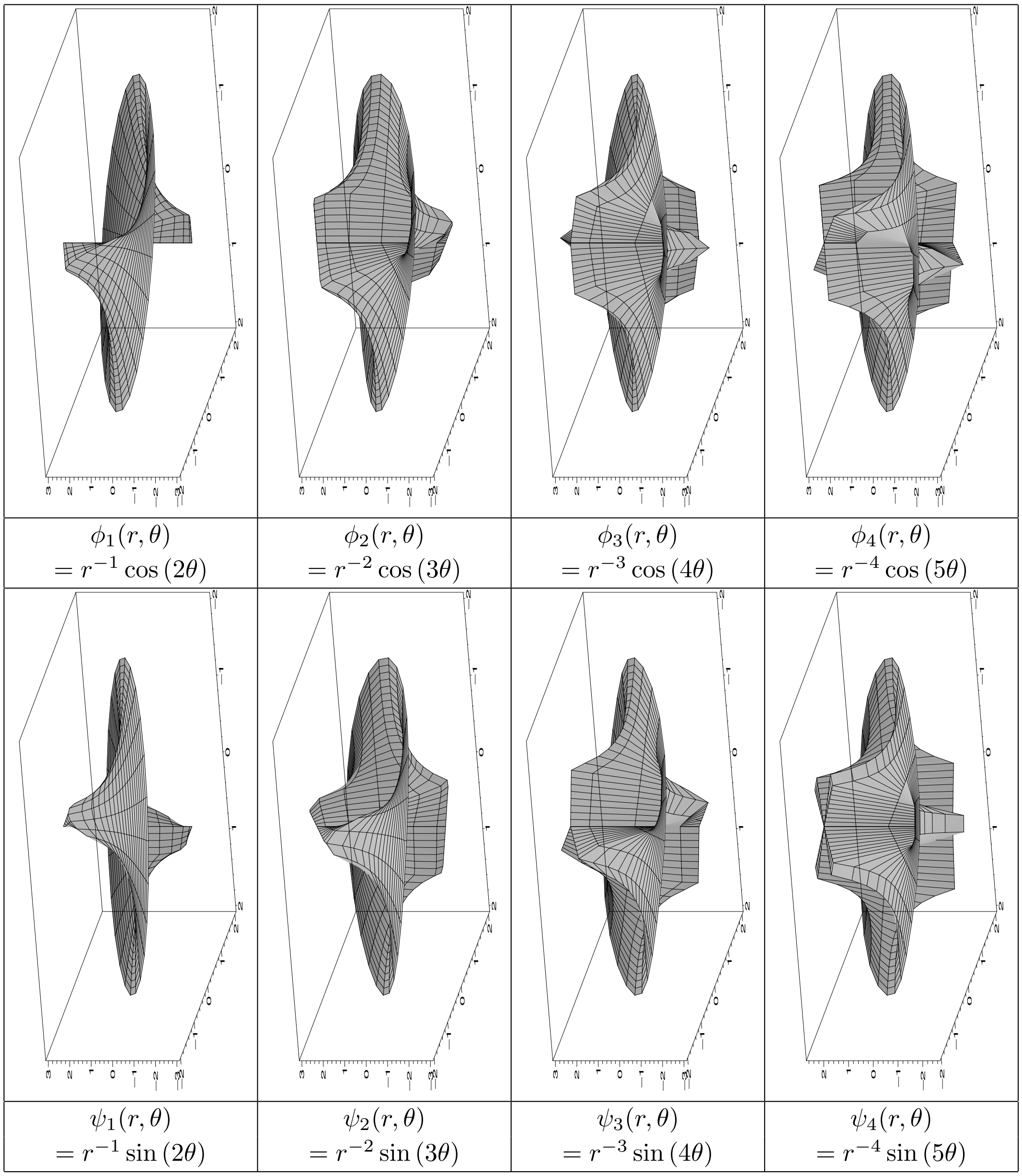


Figure 14B.2: ϕ_n and ψ_n for $n = 1...4$ (rotate page). Note that these plots have been 'truncated' to have vertical bounds ± 3, because these functions explode to $\pm\infty$ at zero.

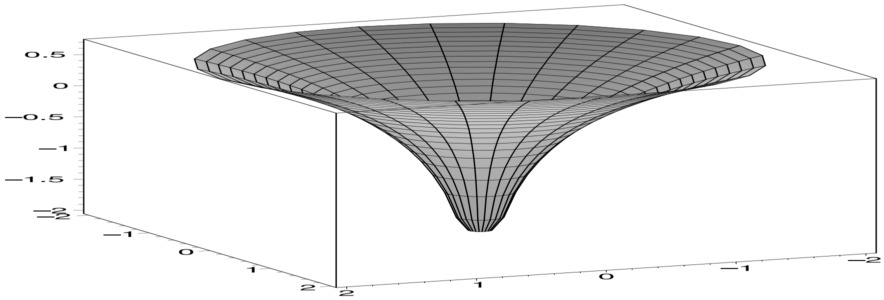
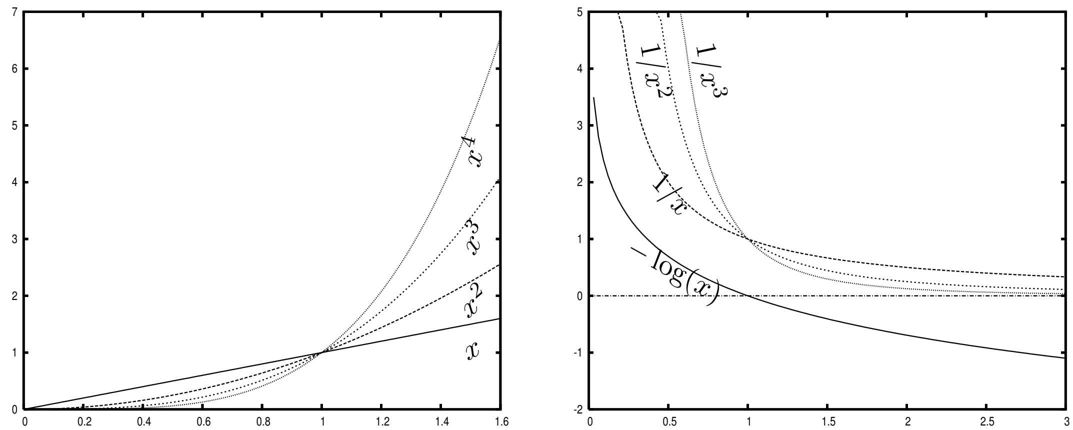


Figure 14B.3: $\phi_0(r,\theta) = \log|r|$ (vertically truncated near zero).



(A): x^n, for $n = 1, 2, 3, 4$;

(B): $-\log(x)$ and $\frac{1}{x^n}$, for $n = 1, 2, 3$ (these plots are vertically truncated).

Figure 14B.4: Radial growth/decay of polar-separated harmonic functions.

The following important harmonic functions *separate* in polar coordinates:

$$\Phi_n(r,\theta) = \cos(n\theta)\cdot r^n; \qquad \Psi_n(r,\theta) = \sin(n\theta)\cdot r^n; \quad \text{for } n \in \mathbb{N}_+ \qquad \text{(Fig.14B.1)}$$

$$\phi_n(r,\theta) = \frac{\cos(n\theta)}{r^n}; \qquad \psi_n(r,\theta) = \frac{\sin(n\theta)}{r^n}; \quad \text{for } n \in \mathbb{N}_+ \qquad \text{(Fig.14B.2)}$$

$$\Phi_0(r,\theta) = 1 \quad \text{and} \quad \phi_0(r,\theta) = \log(r) \qquad \text{(Fig.14B.3)}$$

Proposition 14B.1. *The functions Φ_n, Ψ_n, ϕ_n, and ψ_n are harmonic, for all $n \in \mathbb{N}$.*

Proof. See practice problems #1 to #5 in §14I. □

Exercise 14B.1. (a) Show that $\Phi_1(r,\theta) = x$ and $\Psi_1(r,\theta) = y$ in Cartesian coordinates. Ⓔ

(b) Show that $\Phi_2(r,\theta) = x^2 - y^2$ and $\Psi_2(r,\theta) = 2xy$ in Cartesian coordinates.

(c) Define $F_n : \mathbb{C} \longrightarrow \mathbb{C}$ by $F_n(z) := z^n$. Show that $\Phi_n(x,y) = \mathrm{Re}\left[F_n(x + y\mathbf{i})\right]$ and $\Psi_n(x,y) = \mathrm{Im}\left[F_n(x + y\mathbf{i})\right]$.

(d) (Hard) Show that Φ_n can be written as a homogeneous polynomial of degree n in x and y.

(e) Show that, if $(x,y) \in \partial\mathbb{D}$ (i.e. if $x^2 + y^2 = 1$), then $\Phi_N(x,y) = \zeta_N(x)$, where

$$\zeta_N(x) \quad := \quad 2^{(N-1)} x^N \; + \; \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n 2^{(N-1-2n)} \frac{N}{n} \binom{N-n-1}{n-1} x^{(N-2n)}.$$

is the Nth **Chebyshev polynomial**. (For more information, see [Bro89, §3.4]). ♦

We will solve the Laplace equation in polar coordinates by representing solutions as sums of these simple functions. Note that Φ_n and Ψ_n are *bounded* at zero, but *unbounded* at infinity (Figure 14B.4(A) shows the radial growth of Φ_n and Ψ_n). Conversely, ϕ_n and ψ_n are *unbounded* at zero, but *bounded* at infinity) (Figure 14B.4(B) shows the radial decay of ϕ_n and ψ_n). Finally, Φ_0 being constant, is bounded everywhere, while ϕ_0 is unbounded at both 0 and ∞ (see Figure 14B.4B). Hence, when solving BVPs in a neighbourhood around zero (e.g. the disk), it is preferable to use Φ_0, Φ_n and Ψ_n. When solving BVPs on an unbounded domain (i.e. one "containing infinity") it is preferable to use Φ_0, ϕ_n and ψ_n. When solving BVP's on a domain containing neither zero nor infinity (e.g. the annulus), we use all of Φ_n, Ψ_n, ϕ_n, ψ_n, Φ_0, and ϕ_0.

14B(ii) Boundary value problems on a disk

Prerequisites: §5C, §14A, §14B(i), §8A, §0F.

Proposition 14B.2. (Laplace Equation, Unit Disk, nonhomog. Dirichlet BC)

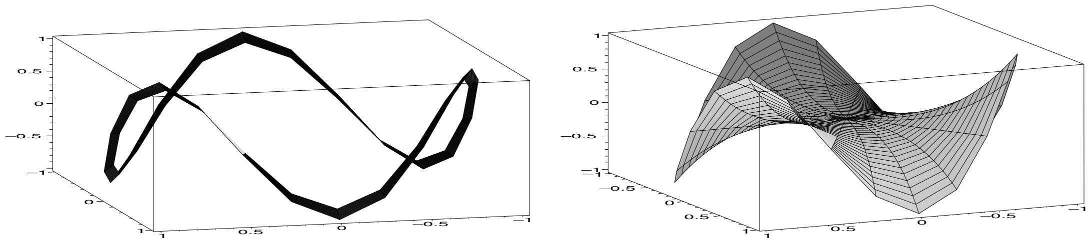
Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ be the unit disk, and let $b \in \mathbf{L}^2[-\pi,\pi)$ be some function. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions:

$$u(1,\theta) \;=\; b(\theta), \qquad \textit{for all } \theta \in [-\pi,\pi). \tag{14B.1}$$

Suppose b has real Fourier series: $b(\theta) \;\underset{L2}{\approx}\; A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta)$.

Then the unique solution to this problem is the function $u : \mathbb{D} \longrightarrow \mathbb{R}$ defined:

$$u(r,\theta) \quad \underset{L2}{\approx} \quad A_0 + \sum_{n=1}^{\infty} A_n \Phi_n(r,\theta) \; + \; \sum_{n=1}^{\infty} B_n \Psi_n(r,\theta)$$



(A): A bent circular wire frame: $b(\theta) = \sin(3\theta)$.

(B): A bubble in the frame: $u(r,\theta) \; = \; r^3 \sin(3\theta)$.

Figure 14B.5: A soap bubble in a bent wire frame.

$$= \quad A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \cdot r^n \; + \; \sum_{n=1}^{\infty} B_n \sin(n\theta) \cdot r^n \qquad (14B.2)$$

Furthermore, the series (14B.2) *converges semiuniformly to* u *on* $\mathsf{int}\,(\mathbb{D})$.

Proof. **Exercise 14B.2** (a) Fix $R < 1$ and let $\mathbb{D}(R) := \{(r,\theta)\ ;\ r < R\}$. Show that on the domain $\mathbb{D}(R)$, the conditions of Proposition 0F.1 on page 565 are satisfied; use this to show that Ⓔ

$$\triangle u(r,\theta) \quad \underset{\text{unif}}{\equiv} \quad \sum_{n=1}^{\infty} A_n \triangle \Phi_n(r,\theta) \; + \; \sum_{n=1}^{\infty} B_n \triangle \Psi_n(r,\theta)$$

for all $(r,\theta) \in \mathbb{D}(R)$. Now use Proposition 14B.1 on page 277 to deduce that $\triangle u(r,\theta) \; = \; 0$ for all $r \le R$. Since this works for any $R < 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}$.

(b) To check that u also satisfies the boundary condition (14B.1), substitute $r = 1$ into (14B.2) to get: $u(1,\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \; + \; \sum_{n=1}^{\infty} B_n \sin(n\theta) \; = \; b(\theta)$.

(c) Use Proposition 5D.5(a) on page 88 to conclude that this solution is unique. □

Example 14B.3. *Take a circular wire frame of radius* 1*, and warp it so that its vertical distortion is described by the function* $b(\theta) = \sin(3\theta)$*, shown in Figure 14B.5(A). Dip the frame into a soap solution to obtain a bubble with the bent wire as its boundary. What is the shape of the bubble?*

Solution: A soap bubble suspended from the wire is a *minimal surface*, and minimal surfaces of low curvature are well-approximated by harmonic functions. Let $u(r,\theta)$ be a function describing the bubble surface. As long as the distortion $b(\theta)$ is relatively small, $u(r,\theta)$ will be a solution to Laplace's equation, with boundary conditions $u(1,\theta) = b(\theta)$. Thus, as shown in Figure 14B.5(B), $u(r,\theta) \; = \; r^3 \sin(3\theta)$. ♢

Ⓔ **Exercise 14B.3.** Let $u(x,\theta)$ be a solution to the Dirichlet problem with boundary conditions $u(1,\theta)=b(\theta)$. Let $\mathbf{0}$ be the center of the disk (i.e. the point with radius 0). Use Proposition 14B.2 to prove that $u(\mathbf{0}) \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} b(\theta)\,d\theta$. ♦

Proposition 14B.4. (Laplace Equation, Unit Disk, nonhomog. Neumann BC)

Let $\mathbb{D}=\{(r,\theta)\;;\;r\le 1\}$ be the unit disk, and let $b\in\mathbf{L}^2[-\pi,\pi)$. Consider the Laplace equation "$\triangle u=0$", with nonhomogeneous Neumann boundary conditions:

$$\partial_r\, u(1,\theta) \;=\; b(\theta), \qquad \text{for all } \theta\in[-\pi,\pi). \tag{14B.3}$$

Suppose b has real Fourier series: $b(\theta) \;\underset{\text{L2}}{\approx}\; A_0+\sum_{n=1}^{\infty}A_n\cos(n\theta)+\sum_{n=1}^{\infty}B_n\sin(n\theta)$.

If $A_0=0$, then the solutions to this problem are all functions $u:\mathbb{D}\longrightarrow\mathbb{R}$ of the form

$$\begin{aligned} u(r,\theta) \;&\underset{\text{L2}}{\approx}\; C \;+\; \sum_{n=1}^{\infty}\frac{A_n}{n}\Phi_n(r,\theta) \;+\; \sum_{n=1}^{\infty}\frac{B_n}{n}\Psi_n(r,\theta) \\ &= \; C \;+\; \sum_{n=1}^{\infty}\frac{A_n}{n}\cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty}\frac{B_n}{n}\sin(n\theta)\cdot r^n \end{aligned} \tag{14B.4}$$

where C is any constant. Furthermore, the series (14B.4) converges semiuniformly to u on $\mathsf{int}(\mathbb{D})$.

However, if $A_0\neq 0$, then there is no solution.

Proof.

Claim 1: *For any $r<1$,* $\displaystyle\sum_{n=1}^{\infty}n^2\frac{|A_n|}{n}\cdot r^n \;+\; \sum_{n=1}^{\infty}n^2\frac{|B_n|}{n}\cdot r^n \;<\; \infty.$

Proof. Let $M=\max\left\{\max\{|A_n|\}_{n=1}^{\infty},\; \max\{|B_n|\}_{n=1}^{\infty}\right\}$. Then

$$\begin{aligned} \sum_{n=1}^{\infty}n^2\frac{|A_n|}{n}\cdot r^n \;+\; \sum_{n=1}^{\infty}n^2\frac{|B_n|}{n}\cdot r^n \;&\le\; \sum_{n=1}^{\infty}n^2\frac{M}{n}\cdot r^n \;+\; \sum_{n=1}^{\infty}n^2\frac{M}{n}\cdot r^n \\ &=\; 2M\sum_{n=1}^{\infty}nr^n. \end{aligned} \tag{14B.5}$$

Let $f(r)=\dfrac{1}{1-r}$. Then $f'(r)=\dfrac{1}{(1-r)^2}$. Recall that, for $|r|<1$, $f(r)=\displaystyle\sum_{n=0}^{\infty}r^n$. Thus, $f'(r)=\displaystyle\sum_{n=1}^{\infty}nr^{n-1}=\frac{1}{r}\sum_{n=1}^{\infty}nr^n$. Hence, the right hand side of eqn.(14B.5) is equal to

$$2M\sum_{n=1}^{\infty} nr^n \quad = \quad 2Mr\cdot f'(r) \quad = \quad 2Mr\cdot\frac{1}{(1-r)^2} \quad < \quad \infty,$$

for any $r<1$. $\diamond_{\text{Claim 1}}$

Let $R<1$ and let $\mathbb{D}(R)=\{(r,\theta)\ ;\ r\leq R\}$ be the disk of radius R. If $u(r,\theta) = C + \sum_{n=1}^{\infty}\frac{A_n}{n}\Phi_n(r,\theta) + \sum_{n=1}^{\infty}\frac{B_n}{n}\Psi_n(r,\theta)$, then for all $(r,\theta)\in\mathbb{D}(R)$,

$$\begin{aligned}\triangle u(r,\theta) \quad &\underset{\text{unif}}{\equiv} \quad \sum_{n=1}^{\infty}\frac{A_n}{n}\triangle\Phi_n(r,\theta) \quad + \quad \sum_{n=1}^{\infty}\frac{B_n}{n}\triangle\Psi_n(r,\theta)\\ &\underset{(*)}{=} \quad \sum_{n=1}^{\infty}\frac{A_n}{n}(0) \quad + \quad \sum_{n=1}^{\infty}\frac{B_n}{n}(0) \quad = \quad 0,\end{aligned}$$

on $\mathbb{D}(R)$. Here, "$\underset{\text{unif}}{\equiv}$" is by Proposition 0F.1 on page 565 and Claim 1, while $(*)$ is by Proposition 14B.1 on page 277.

To check boundary conditions, observe that, for all $R<1$ and all $(r,\theta)\in\mathbb{D}(R)$,

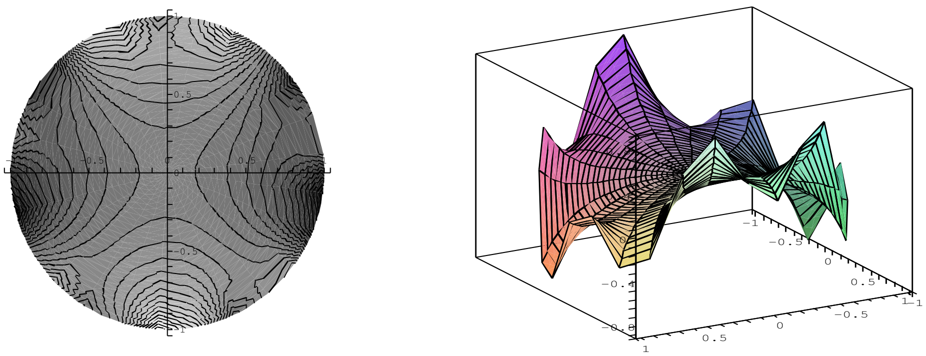
$$\begin{aligned}\partial_r u(r,\theta) \quad &\underset{\text{unif}}{\equiv} \quad \sum_{n=1}^{\infty}\frac{A_n}{n}\partial_r\Phi_n(r,\theta) \quad + \quad \sum_{n=1}^{\infty}\frac{B_n}{n}\partial_r\Psi_n(r,\theta)\\ &= \quad \sum_{n=1}^{\infty}\frac{A_n}{n}nr^{n-1}\cos(n\theta) \quad + \quad \sum_{n=1}^{\infty}\frac{B_n}{n}nr^{n-1}\sin(n\theta)\\ &= \quad \sum_{n=1}^{\infty}A_n r^{n-1}\cos(n\theta) \quad + \quad \sum_{n=1}^{\infty}B_n r^{n-1}\sin(n\theta).\end{aligned}$$

Here "$\underset{\text{unif}}{\equiv}$" is by Proposition 0F.1 on page 565. Hence, letting $R\to 1$, we get

$$\begin{aligned}\partial_\perp u(1,\theta) \quad = \quad \partial_r u(1,\theta) \quad &= \quad \sum_{n=1}^{\infty}A_n\cdot(1)^{n-1}\cos(n\theta) \quad + \quad \sum_{n=1}^{\infty}B_n\cdot(1)^{n-1}\sin(n\theta)\\ &= \quad \sum_{n=1}^{\infty}A_n\cos(n\theta) \quad + \quad \sum_{n=1}^{\infty}B_n\sin(n\theta) \quad \underset{\text{L2}}{\approx} \quad b(\theta),\end{aligned}$$

as desired. Here, "$\underset{\text{L2}}{\approx}$" is because this is the Fourier Series for $b(\theta)$, assuming $A_0=0$. (If $A_0\neq 0$, then this solution doesn't work.)

Finally, Proposition 5D.5(c) on page 88 implies that this solution is unique up to addition of a constant. $\Box$



(A) Contour plot **(B)** Surface plot

Figure 14B.6: The electric potential deduced from Scully's voltage measurements in Example 14B.5.

Remark. *Physically speaking, why must $A_0 = 0$?*

If $u(r,\theta)$ is an electric potential, then $\partial_r u$ is the *radial component* of the *electric field*. The requirement that $A_0 = 0$ is equivalent to requiring that the *net electric flux* entering the disk is zero, which is equivalent (via Gauss's law) to the assertion that the *net electric charge* contained in the disk is zero. If $A_0 \neq 0$, then the net electric charge within the disk must be nonzero. Thus, if $q : \mathbb{D} \longrightarrow \mathbb{R}$ is the charge density field, then we must have $q \not\equiv 0$. However, $q = \triangle u$ (see Example 1D.2 on page 14), so this means $\triangle u \neq 0$, which means u is not harmonic.

Example 14B.5. *While covertly investigating mysterious electrical phenomena on a top-secret military installation in the Nevada desert, Mulder and Scully are trapped in a cylindrical concrete silo by the Cancer Stick Man. Scully happens to have a voltimeter, and she notices an electric field in the silo. Walking around the (circular) perimeter of the silo, Scully estimates the radial component of the electric field to be the function $b(\theta) = 3\sin(7\theta) - \cos(2\theta)$. Estimate the electric potential field inside the silo.*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $\partial_r u(1,\theta) = 3\sin(7\theta) - \cos(2\theta)$. Thus,

$$u(r,\theta) \;=\; C \;+\; \frac{3}{7}\sin(7\theta)\cdot r^7 \;-\; \frac{1}{2}\cos(2\theta)\cdot r^2. \qquad \text{(see Figure 14B.6)}$$

Question: *Moments later, Mulder repeats Scully's experiment, and finds that the perimeter field has changed to $b(\theta) = 3\sin(7\theta) - \cos(2\theta) + 6$. He immediately suspects that an Alien Presence has entered the silo. Why?* $\diamondsuit$

14B(iii) Boundary value problems on a codisk

Prerequisites: §5C, §14A, §14B(i), §8A, §0F. **Recommended:** §14B(ii).

We will now solve the Dirichlet problem on an *unbounded* domain: the **codisk**

$$\mathbb{D}^{\complement} \quad := \quad \{(r,\theta)\ ;\ 1 \leq r,\ \theta \in [-\pi,\pi)\}.$$

Physical Interpretations:

Chemical Concentration: Suppose there is an unknown source of some chemical hidden inside the disk, and that this chemical diffuses into the surrounding medium. Then the solution function $u(r,\theta)$ represents the *equilibrium concentration* of the chemical. In this case, it is reasonable to expect $u(r,\theta)$ to be **bounded at infinity**, by which we mean:

$$\lim_{r\to\infty} |u(r,\theta)| \quad \neq \quad \infty, \quad \text{for all } \theta \in [-\pi,\pi). \tag{14B.6}$$

Otherwise the chemical concentration would become very large far away from the center, which is not realistic.

Electric Potential: Suppose there is an unknown charge distribution inside the disk. Then the solution function $u(r,\theta)$ represents the *electric potential field* generated by this charge. Even though we don't know the exact charge distribution, we can use the boundary conditions to extrapolate the shape of the potential field outside the disk.

If the net charge within the disk is zero, then the electric potental far away from the disk should be bounded (because from far away, the charge distribution inside the disk 'looks' neutral); hence, the solution $u(r,\theta)$ will again satisfy the Boundedness Condition (14B.6).

However, if there is a *nonzero* net charge within the the disk, then the electric potential will *not* be bounded (because, even from far away, the disk still 'looks' charged). Nevertheless, the electric *field* generated by this potential should still decay to zero (because the influence of the charge should be weak at large distances). This means that, while the potential is unbounded, the *gradient* of the potential must decay to zero near infinity. In other words, we must impose the **decaying gradient condition**:

$$\lim_{r\to\infty} \nabla u(r,\theta) \quad = \quad 0, \quad \text{for all } \theta \in [-\pi,\pi). \tag{14B.7}$$

Proposition 14B.6. (Laplace equation, Codisk, nonhomog. Dirichlet BC)

Let $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ 1 \leq r\}$ be the codisk, and let $b \in \mathbf{L}^2[-\pi,\pi)$. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions:

$$u(1,\theta) \;=\; b(\theta), \qquad \text{for all } \theta \in [-\pi,\pi). \tag{14B.8}$$

Suppose b has real Fourier series: $b(\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta)$.

Then the unique solution to this problem which is bounded at infinity as in (14B.6) is the function $u : \mathbb{D}^{\complement} \longrightarrow \mathbb{R}$ defined:

$$u(r,\theta) \underset{\text{L2}}{\approx} A_0 \;+\; \sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \tag{14B.9}$$

Furthermore, the series (14B.9) converges semiuniformly to u on $\mathsf{int}\left(\mathbb{D}^{\complement}\right)$.

Ⓔ *Proof.* **Exercise 14B.4** (a) To show that u is harmonic, apply eqn.(14A.1) on page 274 to get

$$\begin{aligned}\triangle u(r,\theta) \;=\; & \partial_r^2 \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right) + \frac{1}{r}\partial_r \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right) \\ & + \frac{1}{r^2}\partial_\theta^2 \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right).\end{aligned} \tag{14B.10}$$

Now let $R > 1$. Check that, on the domain $\mathbb{D}^{\complement}(R) = \{(r,\theta)\ ;\ r > R\}$, the conditions of Proposition 0F.1 on page 565 are satisfied; use this to simplify the expression (14B.10). Finally, apply Proposition 14B.1 on page 277 to deduce that $\triangle u(r,\theta) = 0$ for all $r \geq R$. Since this works for any $R > 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}^{\complement}$.

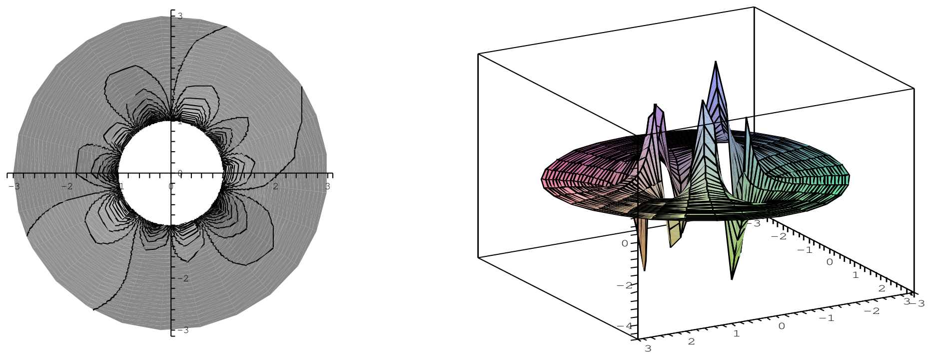
(b) To check that the solution also satisfies the boundary condition (14B.8), subsititute $r = 1$ into (14B.9) to get: $u(1,\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta) = b(\theta)$.

(c) Use Proposition 5D.5(a) on page 88 to conclude that this solution is unique. □

Example 14B.7. *An unknown distribution of electric charges lies inside the unit disk in the plane. Using a voltimeter, the electric potential is measured along the perimeter of the circle, and is approximated by the function $b(\theta) = \sin(2\theta) + 4\cos(5\theta)$. Far away from the origin, the potential is found to be close to zero. Estimate the electric potential field.*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $u(1,\theta) = \sin(2\theta) + 4\cos(5\theta)$. Far away, the potential apparently remains bounded. Thus, as shown in Figure 14B.7,

$$u(r,\theta) \;=\; \frac{\sin(2\theta)}{r^2} + \frac{4\cos(5\theta)}{r^5}. \qquad \diamondsuit$$



(A) Contour plot (unit disk occulted) **(B)** Surface plot (unit disk deleted)

Figure 14B.7: The electric potential deduced from voltage measurements in Example 14B.7.

Remark. Note that, for any constant $C \in \mathbb{R}$, another solution to the Dirichlet problem with boundary conditions (14B.8) is given by the function

$$u(r,\theta) \;=\; A_0 \;+\; C\log(r) \;+\; \sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n}.$$

(**Exercise 14B.5**). However, unless $C = 0$, this will *not* be bounded at infinity. Ⓔ

Proposition 14B.8. (Laplace equation, Codisk, nonhomog. Neumann BC)
Let $\mathbb{D}^{\complement} = \{(r,\theta)\,;\, 1 \leq r\}$ be the codisk, and let $b \in \mathbf{L}^2[-\pi,\pi)$. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Neumann boundary conditions:

$$-\,\partial_{\perp} u(1,\theta) \quad=\quad \partial_r u(1,\theta) \quad=\quad b(\theta), \quad \text{for all } \theta \in [-\pi,\pi). \tag{14B.11}$$

Suppose b has real Fourier series: $b(\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta).$

Fix a constant $C \in \mathbb{R}$, and define $u : \mathbb{D}^{\complement} \longrightarrow \mathbb{R}$ by:

$$u(r,\theta) \;\underset{\text{L2}}{\approx}\; C \;+\; A_0\log(r) \;+\; \sum_{n=1}^{\infty} \frac{-A_n}{n}\frac{\cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} \frac{-B_n}{n}\frac{\sin(n\theta)}{r^n} \tag{14B.12}$$

Then u is a solution to the Laplace equation, with nonhomogeneous Neumann boundary conditions (14B.11), and furthermore, obeys the Decaying Gradient Condition (14B.7) on p.283. Furthermore, all harmonic functions satisfying equations (14B.11) and (14B.7) must be of the form (14B.12). However, the solution (14B.12) is bounded at infinity as in (14B.6) if and only if $A_0 = 0$.

Finally, the series (14B.12) *converges semiuniformly to* u *on* $\mathsf{int}\left(\mathbb{D}^{\complement}\right)$.

Ⓔ *Proof.* **Exercise 14B.6** (a) To show that u is harmonic, apply eqn.(14A.1) on page 274 to get

$$\begin{aligned}\triangle u(r,\theta) \quad &= \quad \partial_r^2 \left(A_0 \log(r) \quad - \sum_{n=1}^{\infty} \frac{A_n}{n} \frac{\cos(n\theta)}{r^n} \quad - \sum_{n=1}^{\infty} \frac{B_n}{n} \frac{\sin(n\theta)}{r^n} \right) \\ &+ \frac{1}{r}\partial_r \left(A_0 \log(r) \quad - \sum_{n=1}^{\infty} \frac{A_n}{n} \frac{\cos(n\theta)}{r^n} \quad - \sum_{n=1}^{\infty} \frac{B_n}{n} \frac{\sin(n\theta)}{r^n} \right) \\ &+ \frac{1}{r^2}\partial_\theta^2 \left(A_0 \log(r) \quad - \sum_{n=1}^{\infty} \frac{A_n}{n} \frac{\cos(n\theta)}{r^n} \quad - \sum_{n=1}^{\infty} \frac{B_n}{n} \frac{\sin(n\theta)}{r^n} \right). \quad (14B.13)\end{aligned}$$

Now let $R > 1$. Check that, on the domain $\mathbb{D}^{\complement}(R) = \{(r,\theta)\ ;\ r > R\}$, the conditions of Proposition 0F.1 on page 565 are satisfied; use this to simplify the expression (14B.13). Finally, apply Proposition 14B.1 on page 277 to deduce that $\triangle u(r,\theta) = 0$ for all $r \geq R$. Since this works for any $R > 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}^{\complement}$.

(b) To check that the solution also satisfies the boundary condition (14B.11), substitute $r = 1$ into (14B.12) and compute the radial derivative (using Proposition 0F.1 on page 565) to get: $\partial_r u(1,\theta) \ = \ A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \ + \ \sum_{n=1}^{\infty} B_n \sin(n\theta) \ \underset{L2}{\approx} \ b(\theta)$.

(c) Use Proposition 5D.5(c) on page 88 to show that this solution is unique up to addition of a constant.

(d) What is the physical interpretation of $A_0 = 0$? □

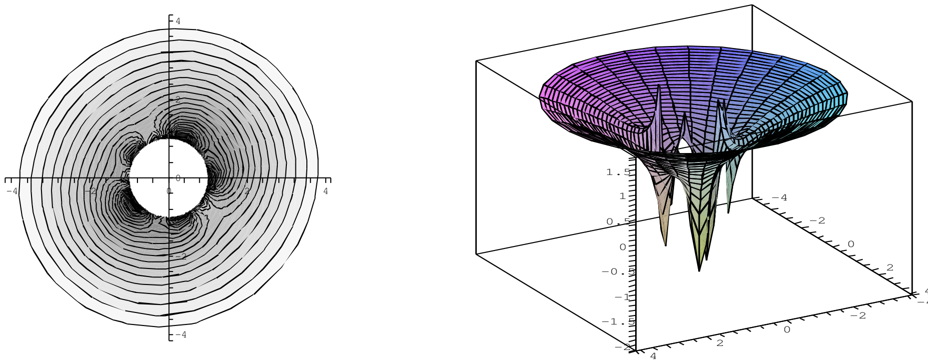
Example 14B.9. *An unknown distribution of electric charges lies inside the unit disk in the plane. The* radial *component of the electric field is measured along the perimeter of the circle, and is approximated by the function* $b(\theta) = 0.9 + \sin(2\theta) + 4\cos(5\theta)$. *Estimate the electric potential potential (up to a constant).*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $\partial_r\ u(1,\theta) = 0.9 + \sin(2\theta) + 4\cos(5\theta)$. Thus, as shown in Figure 14B.8,

$$u(r,\theta) \ = \ C \ + \ 0.9\log(r) + \frac{-\sin(2\theta)}{2\cdot r^2} + \frac{-4\cos(5\theta)}{5\cdot r^5}. \qquad \diamondsuit$$

14B(iv) Boundary value problems on an annulus

Prerequisites: §5C, §14A, §14B(i), §8A, §0F. **Recommended:** §14B(ii), §14B(iii).



(A) Contour plot (unit disk occulted) **(B)** Surface plot (unit disk deleted)

Figure 14B.8: The electric potential deduced from field measurements in Example 14B.9.

Proposition 14B.10. (Laplace Equation, Annulus, nonhomog. Dirichlet BC)

Let $\mathbb{A} = \{(r,\theta)\ ;\ R_{\min} \leq r \leq R_{\max}\}$ be an annulus, and let $b, B : [-\pi, \pi) \longrightarrow \mathbb{R}$ be two functions. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions:

$$u(R_{\min},\theta) \;=\; b(\theta) \quad \text{and} \quad u(R_{\max},\theta) \;=\; B(\theta), \qquad \text{for all } \theta \in [-\pi,\pi). \tag{14B.14}$$

Suppose b and B have real Fourier series:

$$\begin{aligned} b(\theta) \quad &\underset{\mathrm{L2}}{\approx} \quad a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} b_n \sin(n\theta) \\ \text{and} \quad B(\theta) \quad &\underset{\mathrm{L2}}{\approx} \quad A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta). \end{aligned}$$

Then the unique solution to this problem is the function $u : \mathbb{A} \longrightarrow \mathbb{R}$ defined

$$\begin{aligned} u(r,\theta) \;=\; \underset{\mathrm{L2}}{\approx} \quad U_0 \;+\; u_0 \log(r) \;+\; \sum_{n=1}^{\infty} \left(U_n r^n + \frac{u_n}{r^n} \right) \cos(n\theta) \\ +\; \sum_{n=1}^{\infty} \left(V_n r^n + \frac{v_n}{r^n} \right) \sin(n\theta). \end{aligned} \tag{14B.15}$$

where the coefficients $\{u_n, U_n, v_n, V_N\}_{n=1}^{\infty}$ are the unique solutions to the equa-

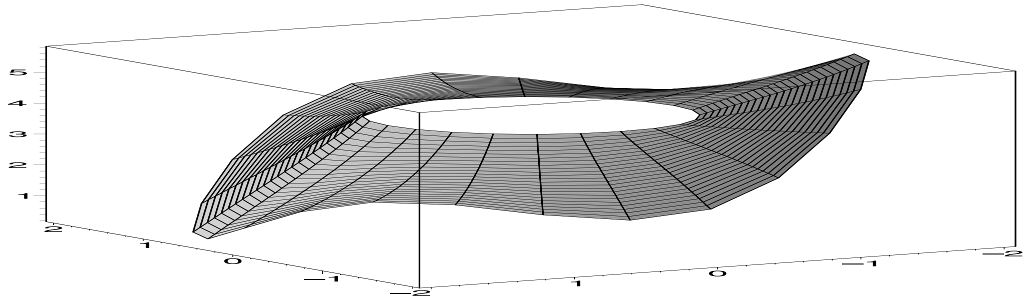


Figure 14B.9: A bubble between two concentric circular wires

tions:

$$\begin{array}{rclcrcl} U_0 + u_0 \log(R_{\text{min}}) &=& a_0; && U_0 + u_0 \log(R_{\text{max}}) &=& A_0; \\ U_n R^n_{\text{min}} + \dfrac{u_n}{R^n_{\text{min}}} &=& a_n; && U_n R^n_{\text{max}} + \dfrac{u_n}{R^n_{\text{max}}} &=& A_n; \\ V_n R^n_{\text{min}} + \dfrac{v_n}{R^n_{\text{min}}} &=& b_n; && V_n R^n_{\text{max}} + \dfrac{v_n}{R^n_{\text{max}}} &=& B_n. \end{array}$$

Furthermore, the series (14B.15) *converges semiuniformly to* u *on* $\text{int}(\mathbb{A})$.

Ⓔ *Proof.* **Exercise 14B.7** (a) To check that u is harmonic, generalize the strategies used to prove Proposition 14B.2 on page 278 and Proposition 14B.6 on page 284.

(b) To check that the solution also satisfies the boundary condition (14B.14), subsititute $r = R_{\text{min}}$ and $r = R_{\text{max}}$ into (14B.15) to get the Fourier series for b and B.

(c) Use Proposition 5D.5(a) on page 88 to show that this solution is unique. □

Example: *Consider an annular bubble spanning two concentric circular wire frames. The inner wire has radius* $R_{\text{min}} = 1$*, and is unwarped, but is elevated to a height of* 4cm*, while the outer wire has radius* $R_{\text{max}} = 2$*, and is twisted to have shape* $B(\theta) = \cos(3\theta) - 2\sin(\theta)$*. Estimate the shape of the bubble between the two wires.*

Solution: We have $b(\theta) = 4$, and $B(\theta) = \cos(3\theta) - 2\sin(\theta)$. Thus:

$$a_0 = 4; \qquad A_3 = 1; \qquad \text{and } B_1 = -2$$

and all other coefficients of the boundary conditions are zero. Thus, our solution will have the form:

$$u(r,\theta) = U_0 + u_0 \log(r) + \left(U_3 r^3 + \frac{u_3}{r^3}\right)\cdot\cos(3\theta) + \left(V_1 r + \frac{v_1}{r}\right)\cdot\sin(\theta),$$

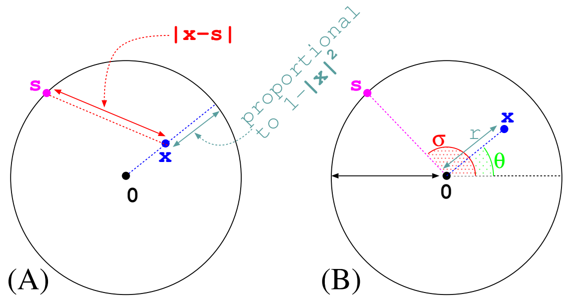


Figure 14B.10: The Poisson kernel (see also Figure 17F.1 on page 407)

where U_0, u_0, U_3, u_3, V_1, and v_1 are chosen to solve the equations:

$$\begin{aligned} U_0 + u_0 \log(1) &= 4; & U_0 + u_0 \log(2) &= 0; \\ U_3 + u_3 &= 0; & 8\,U_3 + \frac{u_3}{8} &= 1; \\ V_1 + v_1 &= 0; & 2V_1 + \frac{v_1}{2} &= -2. \end{aligned}$$

which is equivalent to:

$$\begin{aligned} U_0 &= 4; & u_0 &= \frac{-U_0}{\log(2)} = \frac{-4}{\log(2)}; \\ u_3 &= -U_3; & \left(8 - \frac{1}{8}\right) U_3 &= 1, \quad \text{and thus} \quad U_3 = \frac{8}{63}; \\ v_1 &= -V_1; & \left(2 - \frac{1}{2}\right) V_1 &= -2, \quad \text{and thus} \quad V_1 = \frac{-4}{3}. \end{aligned}$$

so that $u(r,\theta) \;=\; 4 - \dfrac{4\log(r)}{\log(2)} + \dfrac{8}{63}\left(r^3 - \dfrac{1}{r^3}\right)\cdot\cos(3\theta) - \dfrac{4}{3}\left(r - \dfrac{1}{r}\right)\cdot\sin(\theta).$

14B(v) Poisson's solution to Dirichlet problem on the disk

Prerequisites: §14B(ii). **Recommended:** §17F.[1]

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be the disk of radius R, and let $\partial\mathbb{D} = \mathbb{S} = \{(r,\theta)\ ;\ r = R\}$ be its boundary, the circle of radius R. Recall the *Dirichlet problem on the disk*

[1]See § 17F on page 406 for a different development of the material in this section, using impulse-response functions. For yet another approach, using complex analysis, see Corollary 18C.13 on page 445.

from §14B(ii). We will now construct an 'integral representation formula' for the solution to this problem. The **Poisson kernel** is the function $\mathcal{P} : \mathbb{D} \times \mathbb{S} \longrightarrow \mathbb{R}$ defined:

$$\mathcal{P}(\mathbf{x}, \mathbf{s}) \quad := \quad \frac{R^2 - \|\mathbf{x}\|^2}{\|\mathbf{x} - \mathbf{s}\|^2} \quad \text{for any } \mathbf{x} \in \mathbb{D} \text{ and } \mathbf{s} \in \mathbb{S}.$$

In polar coordinates (Figure 14B.10B), we can parameterize $\mathbf{s} \in \mathbb{S}$ with a single angular coordinate $\sigma \in [-\pi, \pi)$, and assign $\mathbf{x}$ the coordinates (r, θ). Poisson's kernel then takes the form:

$$\mathcal{P}(\mathbf{x}, \mathbf{s}) \;=\; \mathcal{P}(r, \theta;\ \sigma) \;=\; \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \sigma) + r^2}.$$

Ⓔ (**Exercise 14B.8**)

Proposition 14B.11. Poisson's Integral Formula

Let $\mathbb{D} = \{(r, \theta)\ ;\ r \leq R\}$ be the disk of radius R, and let $b \in \mathbf{L}^2[-\pi, \pi)$. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions $u(R, \theta) = b(\theta)$. The unique solution to this problem satisfies:

$$\text{For any } r \in [0, R) \text{ and } \theta \in [-\pi, \pi), \qquad u(r, \theta) \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}(r, \theta; \sigma) \cdot b(\sigma)\ d\sigma. \tag{14B.16}$$

$$\text{or, more abstractly,} \quad u(\mathbf{x}) \;=\; \frac{1}{2\pi} \int_{\mathbb{S}} \mathcal{P}(\mathbf{x}, \mathbf{s}) \cdot b(\mathbf{s})\ d\mathbf{s}, \quad \text{for any } \mathbf{x} \in \mathsf{int}(\mathbb{D}).$$

Proof. For simplicity, assume $R = 1$ (the general case can be obtained by rescaling). From Proposition 14B.2 on page 278, we know that

$$u(r, \theta) \;\underset{\text{L2}}{\approx}\; A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \cdot r^n \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta) \cdot r^n,$$

where A_n and B_n are the (real) Fourier coefficients for the function b. Substituting in the definition of these coefficients (see § 8A on page 161), we get:

$$\begin{aligned}
u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma)\ d\sigma \;+\; \sum_{n=1}^{\infty} \cos(n\theta) \cdot r^n \cdot \left(\frac{1}{\pi} \int_{-\pi}^{\pi} b(\sigma) \cos(n\sigma)\ d\sigma \right) \\
&\quad + \sum_{n=1}^{\infty} \sin(n\theta) \cdot r^n \cdot \left(\frac{1}{\pi} \int_{-\pi}^{\pi} b(\sigma) \sin(n\sigma)\ d\sigma \right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma) \left(1 \;+\; 2\sum_{n=1}^{\infty} r^n \cdot \cos(n\theta)\cos(n\sigma) + 2\sum_{n=1}^{\infty} r^n \cdot \sin(n\theta)\sin(n\sigma) \right) d\sigma \\
&\underset{(*)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma) \left(1 \;+\; 2\sum_{n=1}^{\infty} r^n \cdot \cos\Big(n(\theta - \sigma)\Big) \right)
\end{aligned} \tag{14B.17}$$

where $(*)$ is because $\cos(n\theta)\cos(n\sigma)+\sin(n\theta)\sin(n\sigma) \;=\; \cos\Big(n(\theta-\sigma)\Big)$.

It now suffices to prove:

Claim 1: $\quad 1 \;+\; 2\sum_{n=1}^{\infty} r^n \cdot \cos\Big(n(\theta-\sigma)\Big) \quad = \quad \mathcal{P}(r,\theta;\sigma).$

Proof. By Euler's Formula (see page 551), $2\cos\Big(n(\theta-\sigma)\Big) \;=\; e^{\mathbf{i}n(\theta-\sigma)} + e^{-\mathbf{i}n(\theta-\sigma)}$. Hence,

$$1 \;+\; 2\sum_{n=1}^{\infty} r^n \cdot \cos\Big(n(\theta-\sigma)\Big) \quad = \quad 1 \;+\; \sum_{n=1}^{\infty} r^n \cdot \left(e^{\mathbf{i}n(\theta-\sigma)} + e^{-\mathbf{i}n(\theta-\sigma)}\right). \tag{14B.18}$$

Now define complex number $z = r\cdot e^{\mathbf{i}(\theta-\sigma)}$; then observe that $r^n \cdot e^{\mathbf{i}n(\theta-\sigma)} = z^n$ and $r^n \cdot e^{-\mathbf{i}n(\theta-\sigma)} \;=\; \overline{z}^n$. Thus, we can rewrite the right hand side of (14B.18) as:

$$\begin{aligned}
&1 \;+\; \sum_{n=1}^{\infty} r^n \cdot e^{\mathbf{i}n(\theta-\sigma)} \;+\; \sum_{n=1}^{\infty} r^n \cdot e^{-\mathbf{i}n(\theta-\sigma)} \\
&= \; 1 \;+\; \sum_{n=1}^{\infty} z^n \;+\; \sum_{n=1}^{\infty} \overline{z}^n \quad \underset{\text{(a)}}{=} \quad 1 \;+\; \frac{z}{1-z} \;+\; \frac{\overline{z}}{1-\overline{z}} \\
&= \; 1 \;+\; \frac{z - z\overline{z} + \overline{z} - z\overline{z}}{1 - z - \overline{z} + z\overline{z}} \quad \underset{\text{(b)}}{=} \quad 1 \;+\; \frac{2\mathrm{Re}\,[z] - 2|z|^2}{1 - 2\mathrm{Re}\,[z] + |z|^2} \\
&= \; \frac{1 - 2\mathrm{Re}\,[z] + |z|^2}{1 - 2\mathrm{Re}\,[z] + |z|^2} \;+\; \frac{2\mathrm{Re}\,[z] - 2|z|^2}{1 - 2\mathrm{Re}\,[z] + |z|^2} \\
&= \; \frac{1 - |z|^2}{1 - 2\mathrm{Re}\,[z] + |z|^2} \quad \underset{\text{(c)}}{=} \quad \frac{1 - r^2}{1 \;-\; 2r\cos(\theta-\sigma) \;+\; r^2} \quad = \quad \mathcal{P}(r,\theta;\sigma).
\end{aligned}$$

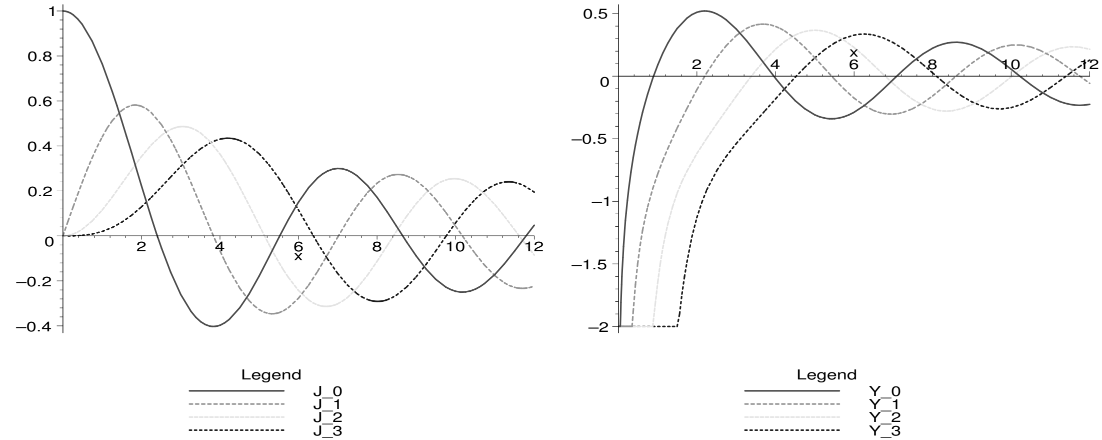
(a) is because $\sum_{n=1}^{\infty} x^n \;=\; \frac{x}{1-x}$ for any $x \in \mathbb{C}$ with $|x| < 1$. $\quad(b)$ is because $z + \overline{z} = 2\mathrm{Re}\,[z]$ and $z\overline{z} = |z|^2$ for any $z \in \mathbb{C}$. $\quad(c)$ is because $|z| = r$ and $\mathrm{Re}\,[z] = \cos(\theta-\sigma)$ by definition of z. $\diamond_{\text{Claim 1}}$

Now, use Claim 1 to substitute $\mathcal{P}(r,\theta;\sigma)$ into (14B.17); this yields the Poisson integral formula (14B.16). $\square$

14C Bessel functions

14C(i) Bessel's equation; Eigenfunctions of $\triangle$ in Polar Coordinates

Prerequisites: §4B, §14A. **Recommended:** §16C.



(A)
$\mathcal{J}_0(x)$, $\mathcal{J}_1(x)$, $\mathcal{J}_2(x)$, and $\mathcal{J}_3(x)$, for $x \in [0, 12]$;

(B)
$\mathcal{Y}_0(x)$, $\mathcal{Y}_1(x)$, $\mathcal{Y}_2(x)$, and $\mathcal{Y}_3(x)$ for $x \in [0, 12]$ (these four plots are vertically truncated).

Figure 14C.1: Bessel functions near zero.

Fix $n \in \mathbb{N}$. The (2-dimensional) **Bessel's Equation** (of **order** n) is the ordinary differential equation

$$x^2\mathcal{R}''(x) \;+\; x\mathcal{R}'(x) \;+\; (x^2 - n^2) \cdot \mathcal{R}(x) \;=\; 0, \qquad (14C.1)$$

where $\mathcal{R} : [0, \infty] \longrightarrow \mathbb{R}$ is an unknown function. In §16C, we will explain how this equation was first derived. In the present section, we will investigate its mathematical consequences.

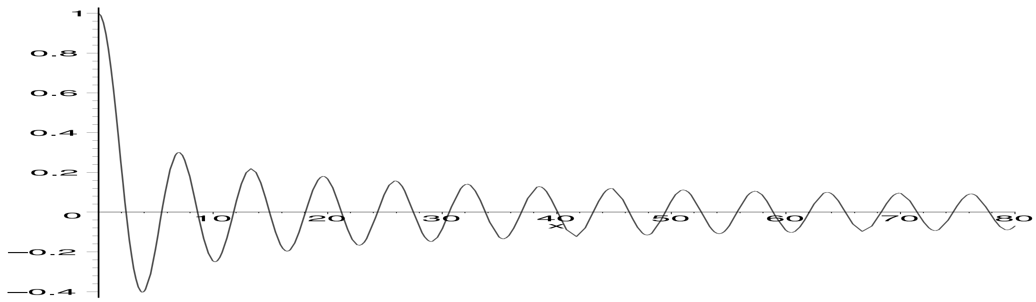
The Bessel equation has two solutions:

$\mathcal{R}(x) = \mathcal{J}_n(x)$ the nth order **Bessel function** of the **first kind**. [See Figures 14C.1(A) and 14C.2(A)]

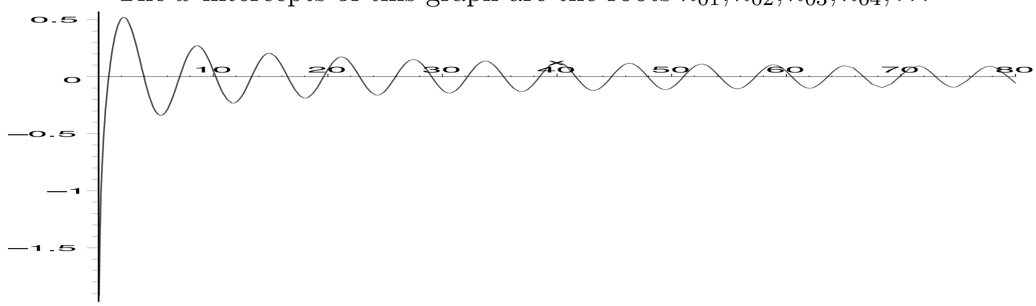
$\mathcal{R}(x) = \mathcal{Y}_n(x)$ the nth order **Bessel function** of the **second kind**, or **Neumann function**. [See Figures 14C.1(B) and 14C.2(B)]

Bessel functions are like trigonometric or logarithmic functions; the 'simplest' expression for them is in terms of a power series. Hence, you should treat the functions "$\mathcal{J}_n$" and "$\mathcal{Y}_n$" the same way you treat elementary functions like "sin", "tan" or "log". In §14G we will derive an explicit power series for Bessel's functions, and in §14H, we will derive some of their important properties. However, for now, we will simply take for granted that some solution functions $\mathcal{J}_n$ exists, and discuss how we can use these functions to build eigenfunctions for the Laplacian which *separate* in polar coordinates.

Proposition 14C.1.



(A): $\mathcal{J}_0(x)$, for $x \in [0, 100]$.
The x-intercepts of this graph are the roots $\kappa_{01}, \kappa_{02}, \kappa_{03}, \kappa_{04}, \ldots$



(B): $\mathcal{Y}_0(x)$, for $x \in [0, 80]$.

Figure 14C.2: Bessel functions are asymptotically periodic.

Fix $\lambda > 0$. For any $n \in \mathbb{N}$, define the functions $\Phi_{n,\lambda}, \Psi_{n,\lambda}, \phi_{n,\lambda}, \psi_{n,\lambda} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\begin{array}{rclcrcl} \Phi_{n,\lambda}(r,\theta) & = & \mathcal{J}_n(\lambda \cdot r) \cdot \cos(n\theta); & & \Psi_{n,\lambda}(r,\theta) & = & \mathcal{J}_n(\lambda \cdot r) \cdot \sin(n\theta); \\ \phi_{n,\lambda}(r,\theta) & = & \mathcal{Y}_n(\lambda \cdot r) \cdot \cos(n\theta); & \textit{and} & \psi_{n,\lambda}(r,\theta) & = & \mathcal{Y}_n(\lambda \cdot r) \cdot \sin(n\theta). \end{array}$$

(see Figures 14C.3 and 14C.4). *Then $\Phi_{n,\lambda}$, $\Psi_{n,\lambda}$, $\phi_{n,\lambda}$, and $\psi_{n,\lambda}$ are all eigenfunctions of the Laplacian with eigenvalue $-\lambda^2$:*

$$\begin{array}{rclcrcl} \triangle \Phi_{n,\lambda} & = & -\lambda^2 \Phi_{n,\lambda}; & & \triangle \Psi_{n,\lambda} & = & -\lambda^2 \Psi_{n,\lambda}; \\ \triangle \phi_{n,\lambda} & = & -\lambda^2 \phi_{n,\lambda}; & \textit{and} & \triangle \psi_{n,\lambda} & = & -\lambda^2 \psi_{n,\lambda}. \end{array}$$

Proof. See practice problems #12 to #15 of §14I. □

We can now use these eigenfunctions to solve PDEs in polar coordinates. Notice that $\mathcal{J}_n$ —and thus, eigenfunctions $\Phi_{n,\lambda}$ and $\Psi_{n,\lambda}$ —are *bounded* around zero (see Figure 14C.1A). On the other hand, $\mathcal{Y}_n$ —and thus, eigenfunctions $\phi_{n,\lambda}$ and $\psi_{n,\lambda}$ —are *unbounded* at zero (see Figure 14C.1B). Hence, when solving BVPs in a neighbourhood around zero (e.g. the disk), we should use $\mathcal{J}_n$, $\Phi_{n,\lambda}$ and $\Psi_{n,\lambda}$. When solving BVPs on a domain *away* from zero (e.g. the annulus), we can also use $\mathcal{Y}_n$, $\phi_{n,\lambda}$ and $\psi_{n,\lambda}$.

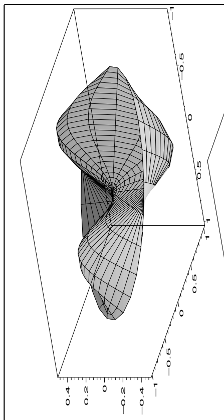
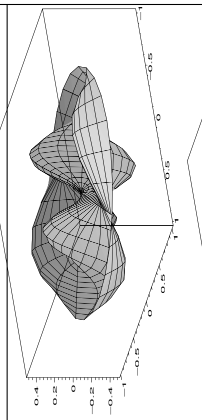
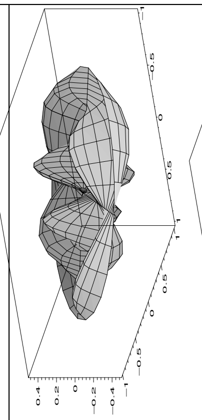
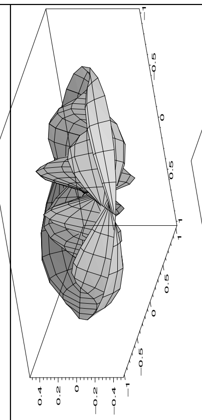
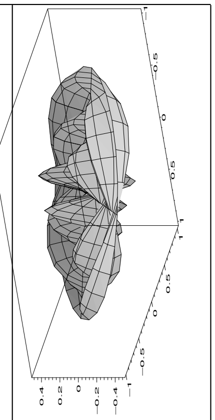
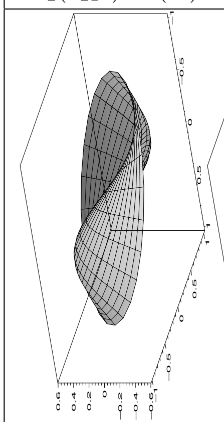
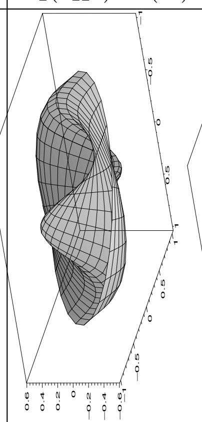
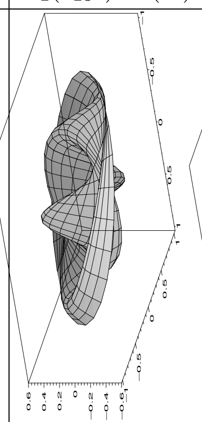
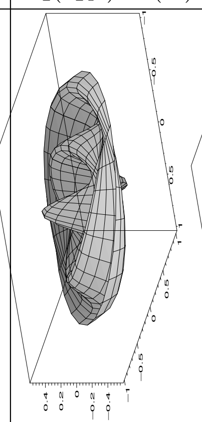
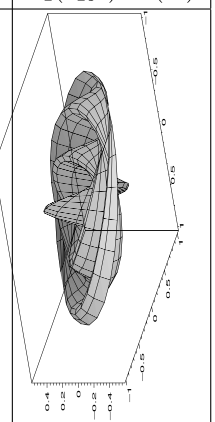
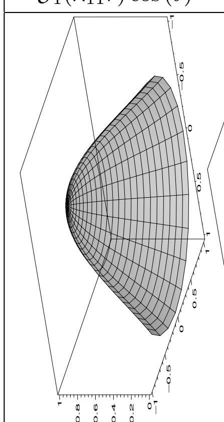
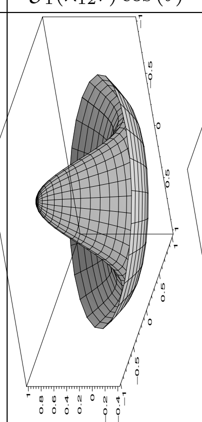
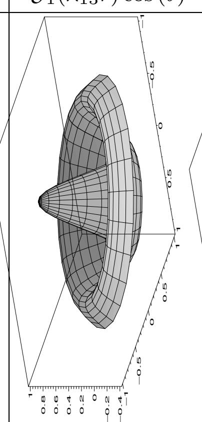
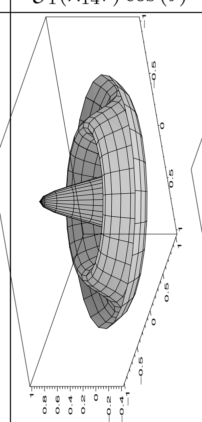
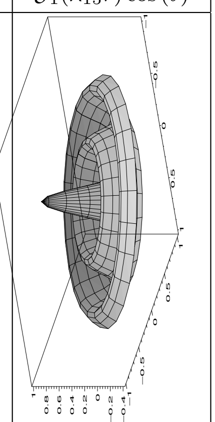
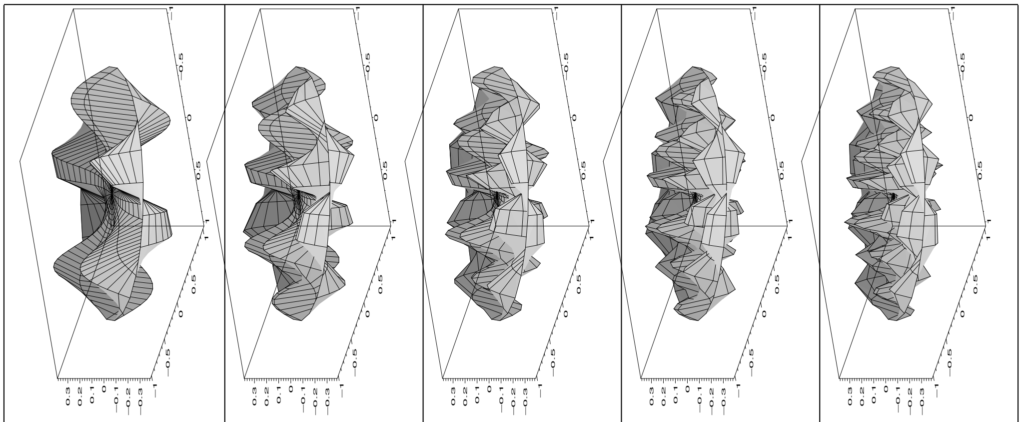
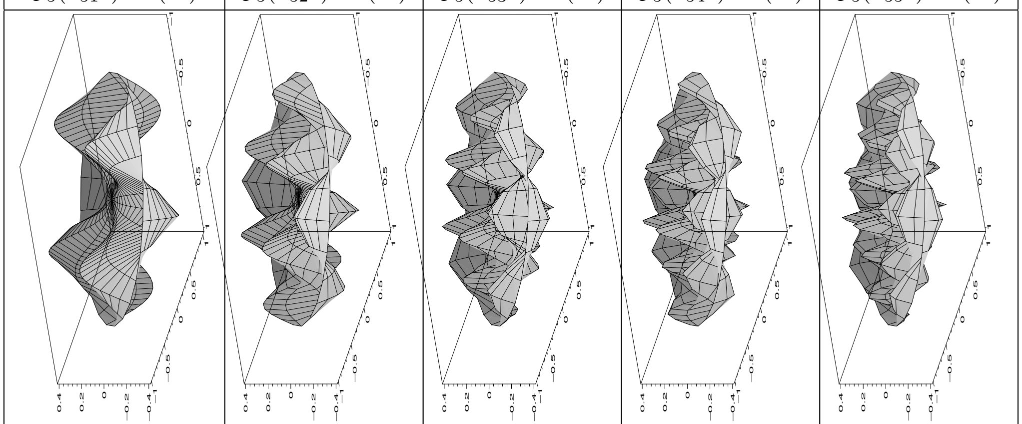
				
$\Phi_{21}(r,\theta) =$ $\mathcal{J}_2(\lambda_{21} r)\cos(2\theta)$	$\Phi_{22}(r,\theta) =$ $\mathcal{J}_2(\lambda_{22} r)\cos(2\theta)$	$\Phi_{23}(r,\theta) =$ $\mathcal{J}_2(\lambda_{23} r)\cos(2\theta)$	$\Phi_{24}(r,\theta) =$ $\mathcal{J}_2(\lambda_{24} r)\cos(2\theta)$	$\Phi_{25}(r,\theta) =$ $\mathcal{J}_2(\lambda_{25} r)\cos(2\theta)$
				
$\Phi_{11}(r,\theta) =$ $\mathcal{J}_1(\lambda_{11} r)\cos(\theta)$	$\Phi_{12}(r,\theta) =$ $\mathcal{J}_1(\lambda_{12} r)\cos(\theta)$	$\Phi_{13}(r,\theta) =$ $\mathcal{J}_1(\lambda_{13} r)\cos(\theta)$	$\Phi_{14}(r,\theta) =$ $\mathcal{J}_1(\lambda_{14} r)\cos(\theta)$	$\Phi_{15}(r,\theta) =$ $\mathcal{J}_1(\lambda_{15} r)\cos(\theta)$
				
$\Phi_{01}(r,\theta) =$ $\mathcal{J}_0(\lambda_{01} r)$	$\Phi_{02}(r,\theta) =$ $\mathcal{J}_0(\lambda_{02} r)$	$\Phi_{03}(r,\theta) =$ $\mathcal{J}_0(\lambda_{03} r)$	$\Phi_{04}(r,\theta) =$ $\mathcal{J}_0(\lambda_{04} r)$	$\Phi_{05}(r,\theta) =$ $\mathcal{J}_0(\lambda_{05} r)$

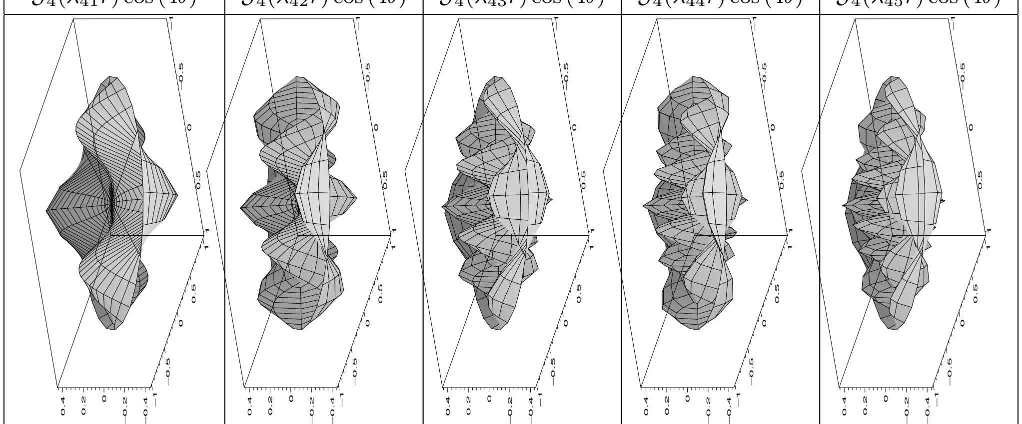
Figure 14C.3: $\Phi_{n,m}$ for $n = 0, 1, 2$ and for $m = 1, 2, 3, 4, 5$ (rotate page).



$\Phi_{51}(r,\theta) =$ $\mathcal{J}_5(\lambda_{51} r)\cos(5\theta)$	$\Phi_{52}(r,\theta) =$ $\mathcal{J}_5(\lambda_{52} r)\cos(5\theta)$	$\Phi_{53}(r,\theta) =$ $\mathcal{J}_5(\lambda_{53} r)\cos(5\theta)$	$\Phi_{54}(r,\theta) =$ $\mathcal{J}_5(\lambda_{54} r)\cos(5\theta)$	$\Phi_{55}(r,\theta) =$ $\mathcal{J}_5(\lambda_{55} r)\cos(5\theta)$



$\Phi_{41}(r,\theta) =$ $\mathcal{J}_4(\lambda_{41} r)\cos(4\theta)$	$\Phi_{42}(r,\theta) =$ $\mathcal{J}_4(\lambda_{42} r)\cos(4\theta)$	$\Phi_{43}(r,\theta) =$ $\mathcal{J}_4(\lambda_{43} r)\cos(4\theta)$	$\Phi_{44}(r,\theta) =$ $\mathcal{J}_4(\lambda_{44} r)\cos(4\theta)$	$\Phi_{45}(r,\theta) =$ $\mathcal{J}_4(\lambda_{45} r)\cos(4\theta)$



$\Phi_{31}(r,\theta) =$ $\mathcal{J}_3(\lambda_{31} r)\cos(3\theta)$	$\Phi_{32}(r,\theta) =$ $\mathcal{J}_3(\lambda_{32} r)\cos(3\theta)$	$\Phi_{33}(r,\theta) =$ $\mathcal{J}_3(\lambda_{33} r)\cos(3\theta)$	$\Phi_{34}(r,\theta) =$ $\mathcal{J}_3(\lambda_{34} r)\cos(3\theta)$	$\Phi_{35}(r,\theta) =$ $\mathcal{J}_3(\lambda_{35} r)\cos(3\theta)$

Figure 14C.4: $\Phi_{n,m}$ for $n = 3, 4, 5$ and for $m = 1, 2, 3, 4, 5$ (rotate page).

14C(ii) Boundary conditions; the roots of the Bessel function

Prerequisites: §5C, §14C(i).

To obtain *homogeneous Dirichlet boundary conditions* on a disk of radius R, we need an eigenfunction of the form $\Phi_{n,\lambda}$ (or $\Psi_{n,\lambda}$) such that $\Phi_{n,\lambda}(R,\theta) = 0$ for all $\theta \in [-\pi, \pi)$. Hence, we need:

$$\mathcal{J}_n(\lambda \cdot R) \;=\; 0. \tag{14C.2}$$

The **roots** of the Bessel function $\mathcal{J}_n$ are the values $\kappa \in \mathbb{R}_{\not -}$ such that $\mathcal{J}_n(\kappa) = 0$. These roots form an increasing sequence

$$0 \;\leq\; \kappa_{n1} \;<\; \kappa_{n2} \;<\; \kappa_{n3} \;<\; \kappa_{n4} \;<\; \ldots \tag{14C.3}$$

of irrational values[2]. Thus, to satisfy the homogeneous Dirichlet boundary condition (14C.2), we must set $\lambda := \kappa_{nm}/R$ for some $m \in \mathbb{N}$. This yields an increasing sequence of eigenvalues:

$$\lambda_{n1}^2 = \left(\frac{\kappa_{n1}}{R}\right)^2 \;<\; \lambda_{n2}^2 = \left(\frac{\kappa_{n2}}{R}\right)^2 \;<\; \lambda_{n3}^2 = \left(\frac{\kappa_{n3}}{R}\right)^2 \;<\; \lambda_{n4}^2 = \left(\frac{\kappa_{n4}}{R}\right)^2 \;<\; \ldots \tag{14C.4}$$

which are the eigenvalues which we can expect to see in this problem. The corresponding eigenfunctions will then have the form:

$$\Phi_{n,m}(r,\theta) \;=\; \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta) \qquad\qquad \Psi_{n,m}(r,\theta) \;=\; \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \sin(n\theta) \tag{14C.5}$$

(see Figures 14C.3 and 14C.4).

14C(iii) Initial conditions; Fourier-Bessel expansions

Prerequisites: §5B, §6F, §14C(ii).

To solve an initial value problem, while satisfying the desired boundary conditions, we express our initial conditions as a sum of the eigenfunctions from expression (14C.5). This is called a **Fourier-Bessel Expansion:**

$$\begin{aligned} f(r,\theta) \;=\; & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cdot \Phi_{nm}(r,\theta) \;+\; \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \cdot \Psi_{nm}(r,\theta) \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cdot \phi_{nm}(r,\theta) \;+\; \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \cdot \psi_{nm}(r,\theta), \end{aligned} \tag{14C.6}$$

where A_{nm}, B_{nm}, a_{nm}, and b_{nm} are all real-valued coefficients. Suppose we are considering boundary value problems on the unit disk $\mathbb{D}$. Then we want

[2] Computing these roots is difficult; tables of κ_{nm} can be found in most standard references on PDEs.

this expansion to be bounded at 0, so we don't want the second two types of eigenfunctions. Thus, expression (14C.6) simplifies to:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\Phi_{nm}(r,\theta) \quad + \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\Psi_{nm}(r,\theta). \tag{14C.7}$$

If we substitute the explicit expressions from (14C.5) for $\Phi_{nm}(r,\theta)$ and $\Psi_{nm}(r,\theta)$ into expression (14C.7), we get:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta). \tag{14C.8}$$

Now, if $f : \mathbb{D} \longrightarrow \mathbb{R}$ is some function describing initial conditions, is it always possible to express f using an expansion like (14C.8)? If so, how do we compute the coefficients A_{nm} and B_{nm} in expression (14C.8)? The answer to these questions lies in the following result:

Theorem 14C.2. *The collection* $\{\Phi_{n,m},\ \Psi_{\ell,m}\ ;\ n = 0...\infty,\ \ell\in\mathbb{N}, m\in\mathbb{N}\}$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{D})$. *Thus, suppose* $f\in\mathbf{L}^2(\mathbb{D})$, *and for all* $n,m\in\mathbb{N}$, *we define*

$$\begin{aligned}
A_{nm} &:= \frac{\langle f,\ \Phi_{nm}\rangle}{\|\Phi_n m\|_2^2} \\
&= \frac{2}{\pi R^2\cdot\mathcal{J}_{n+1}^2(\kappa_{nm})}\cdot\int_{-\pi}^{\pi}\int_0^R f(r,\theta)\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot r\ dr\ d\theta, \\
\text{and}\quad B_{nm} &:= \frac{\langle f,\ \Psi_{nm}\rangle}{\|\Psi_n m\|_2^2} \\
&= \frac{2}{\pi R^2\cdot\mathcal{J}_{n+1}^2(\kappa_{nm})}\cdot\int_{-\pi}^{\pi}\int_0^R f(r,\theta)\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot r\ dr\ d\theta.
\end{aligned}$$

Then the Fourier-Bessel series (14C.8) *converges to* f *in* L^2*-norm.*

Proof. (sketch) The fact that the collection $\{\Phi_{n,m},\ \Psi_{\ell,m}\ ;\ n = 0...\infty,\ \ell\in\mathbb{N}, m\in\mathbb{N}\}$ is an orthogonal set will be verified in Proposition 14H.4 on page 313 of §14H. The fact that this orthogonal set is actually a *basis* of $\mathbf{L}^2(\mathbb{D})$ is too complicated for us to prove here. Given that this is true, if we define $A_{nm} := \langle f,\ \Phi_{nm}\rangle/\|\Phi_n m\|_2^2$ and $B_{nm} := \langle f,\ \Psi_{nm}\rangle/\|\Psi_n m\|_2^2$, then the Fourier-Bessel series (14C.8) converges to f in L^2-norm, by definition of "orthogonal basis" (see § 6F on page 131).

It remains to verify the integral expressions given for the two inner products. To do this, recall that

$$\langle f,\ \Phi_{nm}\rangle \quad = \quad \frac{1}{\mathbf{Area}\,[\mathbb{D}]}\int_{\mathbb{D}} f(\mathbf{x})\cdot\Phi_{nm}(\mathbf{x})\ d\mathbf{x}$$

$$= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R f(r,\theta)\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot \cos(n\theta)\cdot r\ dr\ d\theta$$

$$\text{and}\qquad \|\Phi_{nm}\|_2^2 = \langle \Phi_{nm},\ \Phi_{nm}\rangle = \frac{1}{\pi R^2}\int_{-\pi}^{\pi}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot \cos^2(n\theta)\cdot r\ dr\ d\theta$$

$$= \left(\frac{1}{R^2}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot r\ dr\right)\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(n\theta)\ d\theta\right)$$

$$\underset{(\dagger)}{=} \frac{1}{R^2}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot r\ dr.$$

$$\underset{(\ddagger)}{=} \int_0^1 \mathcal{J}_n^2\left(\kappa_{nm}\cdot s\right)\cdot s\ ds.$$

$$\underset{(*)}{=} \frac{1}{2}\mathcal{J}_{n+1}^2\left(\kappa_{nm}\right)$$

here, (†) is by Proposition 6D.2 on page 112, (†) is the change of variables $s := \frac{r}{R}$, so that $dr = R\,ds$), and $(*)$ is by Lemma 14H.3(b) on page 310. □

To compute the integrals in Theorem 14C.2, one generally uses ‘integration by parts’ techniques similar to those used to compute trigonometic Fourier coefficients (see e.g. § 7C on page 147). However, instead of the convenient trigonometric facts that $\sin' = \cos$ and $\cos' = -\sin$, one must make use of slightly more complicated recurrence relations of Proposition 14H.1 on page 309 of §14H. See Remark 14H.2 on page 310.

We will do not have space in this book to properly develop integration techniques for computing Fourier-Bessel coefficients. Instead, in the remaining discussion, we will simply assume that f is given to us in the form (14C.8).

14D The Poisson equation in polar coordinates

Prerequisites: §1D, §14C(ii), §0F. **Recommended:** §11C, §12C, §13C, §14B .

Proposition 14D.1. (Poisson Equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be a disk, and let $q \in \mathbf{L}^2(\mathbb{D})$ be some function. Consider the Posson equation "$\triangle u(r,\theta) = q(r,\theta)$", with homogeneous Dirichlet *boundary conditions. Suppose q has* semiuniformly convergent *Fourier-Bessel series:*

$$q(r,\theta) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)$$

Then the unique solution to this problem is the function $u : \mathbb{D} \longrightarrow \mathbb{R}$ defined

$$u(r,\theta) \underset{\text{unif}}{\equiv\!\equiv} -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{R^2 \cdot A_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{R^2 \cdot B_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)$$

Proof. **Exercise 14D.1** □ Ⓔ

Remark. If $R = 1$, then the expression for q simplifies to:

$$q(r,\theta) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)$$

and the solution simplifies to

$$u(r,\theta) \underset{\text{unif}}{\equiv\!\equiv} -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_{nm}}{\kappa_{nm}^2}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B_{nm}}{\kappa_{nm}^2}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)$$

Example 14D.2. Suppose $R = 1$, and $q(r,\theta) = \mathcal{J}_0(\kappa_{0,3}\cdot r) + \mathcal{J}_5(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)$. Then

$$u(r,\theta) = \frac{-\mathcal{J}_0(\kappa_{0,3}\cdot r)}{\kappa_{0,3}^2} - \frac{\mathcal{J}_5(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)}{\kappa_{2,5}^2}.$$ ♢

Proposition 14D.3. (Poisson Equation on Disk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be a disk. Let $b \in \mathbf{L}^2[-\pi,\pi)$ and $q \in \mathbf{L}^2(\mathbb{D})$. Consider the Poisson equation "$\triangle u(r,\theta) = q(r,\theta)$", with nonhomogeneous *Dirichlet boundary conditions:*

$$u(R,\theta) = b(\theta), \qquad \text{for all } \theta \in [-\pi,\pi). \tag{14D.1}$$

1. *Let $w : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution[3] to the* Laplace Equation*; "$\triangle w = 0$", with the nonhomogeneous Dirichlet BC* (14D.1)*.*
2. *Let $v : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution[4] to the* Poisson Equation*; "$\triangle v = q$", with the* homogeneous *Dirichlet BC.*

3. *Define* $u(r,\theta) \;:=\; v(r,\theta;t) + w(r,\theta)$. *Then* $u(r,\theta)$ *is a solution to the Poisson Equation with inhomogeneous Dirichlet BC* (14D.1).

Proof. **Exercise 14D.2** □ Ⓔ

Example 14D.4. Suppose $R = 1$, and $q(r,\theta) \;=\; \mathcal{J}_0(\kappa_{0,3}\cdot r) + \mathcal{J}_2(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)$. Let $b(\theta) \;=\; \sin(3\theta)$.

From Example 14B.3 on page 279, we know that the (bounded) solution to the Laplace equation with Dirichlet BC $w(1,\theta) \;=\; b(\theta)$ is:

$$w(r,\theta) \quad=\quad r^3\sin(3\theta).$$

From Example 14D.2, we know that the solution to the Poisson equation "$\triangle v \;=\; q$", with homogeneous Dirichlet BC is:

$$v(r,\theta) \quad=\quad \frac{\mathcal{J}_0(\kappa_{0,3}\cdot r)}{\kappa_{0,3}^2} + \frac{\mathcal{J}_2(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)}{\kappa_{2,5}^2}.$$

Thus, by Proposition 14D.3, the the solution to the Poisson equation "$\triangle u \;=\; q$", with Dirichlet BC $w(1,\theta) \;=\; b(\theta)$, is given:

$$u(r,\theta) \quad=\quad v(r,\theta)+w(r,\theta) \quad=\quad \frac{\mathcal{J}_0(\kappa_{0,3}\cdot r)}{\kappa_{0,3}^2} + \frac{\mathcal{J}_2(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)}{\kappa_{2,5}^2} + r^3\sin(3\theta).$$

♢

14E The heat equation in polar coordinates

Prerequisites: §1B, §14C(iii), §0F. **Recommended:** §11A, §12B, §13A, §14B .

Proposition 14E.1. (Heat equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\,;\ r \le R\}$ *be a disk, and consider the heat equation* "$\partial_t u = \triangle u$", *with homogeneous* Dirichlet *boundary conditions, and initial conditions* $u(r,\theta;0) = f(r,\theta)$. *Suppose* f *has Fourier-Bessel series:*

$$f(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)$$

[3]Obtained from Proposition 14B.2 on page 278, for example.

[4]Obtained from Proposition 14D.1, for example.

Then the unique solution to this problem is the function $u : \mathbb{D} \times \mathbb{R}_{\neq} \longrightarrow \mathbb{R}$ *defined:*

$$u(r,\theta;\ t) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\exp\left(\frac{-\kappa_{nm}^2}{R^2}t\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\exp\left(\frac{-\kappa_{nm}^2}{R^2}t\right)$$

Furthermore, the series defining u *converges semiuniformly on* $\mathbb{D}\times\mathbb{R}_+$.

Proof. **Exercise 14E.1** □ Ⓔ

Remark. If $R = 1$, then the initial conditions simplify to:

$$f(r,\theta) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)$$

and the solution simplifies to:

$$u(r,\theta;\ t) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta)\cdot e^{-\kappa_{nm}^2 t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)\cdot e^{-\kappa_{nm}^2 t}.$$

Example 14E.2. Suppose $R = 1$, and $f(r,\theta) = \mathcal{J}_0(\kappa_{0,7}\cdot r) - 4\mathcal{J}_3(\kappa_{3,2}\cdot r)\cdot\cos(3\theta)$. Then

$$u(r,\theta;\ t) = \mathcal{J}_0(\kappa_{0,7}\cdot r)\cdot e^{-\kappa_{0,7}^2 t} - 4\mathcal{J}_3(\kappa_{3,2}\cdot r)\cdot\cos(3\theta)\cdot e^{-\kappa_{32}^2 t}. \qquad \diamondsuit$$

Proposition 14E.3. (Heat equation on Disk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \le R\}$ *be a disk, and let* $f : \mathbb{D} \longrightarrow \mathbb{R}$ *and* $b : [-\pi,\pi) \longrightarrow \mathbb{R}$ *be given functions. Consider the* Heat equation *"*$\partial_t u = \triangle u$*", with initial conditions* $u(r,\theta) = f(r,\theta)$, *and* nonhomogeneous *Dirichlet boundary conditions:*

$$u(R,\theta) = b(\theta), \qquad \text{for all } \theta \in [-\pi,\pi). \tag{14E.1}$$

1. *Let* $w : \mathbb{D} \longrightarrow \mathbb{R}$ *be the solution*[5] *to the* Laplace Equation*; "*$\triangle w = 0$*", with the nonhomogeneous Dirichlet BC* (14E.1).

[5] Obtained from Proposition 14B.2 on page 278, for example.

2. *Define* $g(r,\theta) := f(r,\theta) - w(r,\theta)$*. Let* $v : \mathbb{D} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *be the solution*[6] *to the heat equation "*$\partial_t v = \triangle v$*" with initial conditions* $v(r,\theta) = g(r,\theta)$*, and homogeneous Dirichlet BC.*

3. *Define* $u(r,\theta;t) := v(r,\theta;t) + w(r,\theta)$*. Then* $u(r,\theta;t)$ *is a solution to the heat equation with initial conditions* $u(r,\theta) = f(r,\theta)$*, and inhomogeneous Dirichlet BC* (14E.1).

Ⓔ *Proof.* **Exercise 14E.2** □

14F The wave equation in polar coordinates

Prerequisites: §2B, §14C(ii), §14C(iii), §0F. **Recommended:** §11B, §12D, §14E.

Imagine a drumskin stretched tightly over a circular frame. At equilibrium, the drumskin is perfectly flat, but if we strike the skin, it will vibrate, meaning that the membrane will experience vertical displacements from equilibrium. Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ represent the round skin, and for any point $(r,\theta) \in \mathbb{D}$ on the drumskin and time $t > 0$, let $u(r,\theta;t)$ be the vertical displacement of the drum. Then u will obey the two-dimensional wave equation:

$$\partial_t^2\, u(r,\theta;t) \;=\; \triangle u(r,\theta;t). \tag{14F.1}$$

However, since the skin is held down along the edges of the circle, the function u will also exhibit homogeneous *Dirichlet* boundary conditions:

$$u(R,\theta;t) \;=\; 0, \qquad \text{for all } \theta \in [-\pi,\pi) \text{ and } t \geq 0. \tag{14F.2}$$

Proposition 14F.1. (Wave equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ *be a disk, and consider the wave equation "*$\partial_t^2\, u = \triangle u$*", with homogeneous* **Dirichlet** *boundary conditions, and*

$$\begin{aligned} \textbf{Initial position:} \quad u(r,\theta;0) &= f_0(r,\theta); \\ \textbf{Initial velocity:} \quad \partial_t\, u(r,\theta;0) &= f_1(r,\theta) \end{aligned}$$

Suppose f_0 *and* f_1 *have Fourier-Bessel series:*

$$f_0(r,\theta) \;\underset{\text{L2}}{\approx}\; \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta);$$

[6] Obtained from Proposition 14E.1, for example.

$$\text{and } f_1(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A'_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta).$$

Assume that

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}^2|A_{nm}| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}^2|B_{nm}| < \infty,$$

$$\text{and } \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}|A'_{nm}| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}|B'_{nm}| < \infty.$$

Then the unique solution to this problem is the function $u : \mathbb{D}\times\mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *defined:*

$$\begin{aligned} u(r,\theta;\ t) \underset{L2}{\approx} & \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot\cos\left(\frac{\kappa_{nm}}{R}t\right) \\ & + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot\cos\left(\frac{\kappa_{nm}}{R}t\right) \\ & + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{R\cdot A'_{nm}}{\kappa_{nm}}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot\sin\left(\frac{\kappa_{nm}}{R}t\right) \\ & + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{R\cdot B'_{nm}}{\kappa_{nm}}\cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot\sin\left(\frac{\kappa_{nm}}{R}t\right). \end{aligned}$$

Proof. **Exercise 14F.1** □ Ⓔ

Remark. If $R = 1$, then the initial conditions would be:

$$\begin{aligned} f_0(r,\theta) \underset{L2}{\approx} & \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta), \\ \text{and } f_1(r,\theta) \underset{L2}{\approx} & \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A'_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta). \end{aligned}$$

and the solution simplifies to:

$$
\begin{aligned}
u(r,\theta;\ t) \quad \underset{\mathrm{L2}}{\approx} \quad & \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n\left(\kappa_{nm}\cdot r\right)\cdot\cos(n\theta)\cdot\cos\left(\kappa_{nm}t\right) \\
& + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n\left(\kappa_{nm}\cdot r\right)\cdot\sin(n\theta)\cdot\cos\left(\kappa_{nm}t\right) \\
& + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A'_{nm}}{\kappa_{nm}}\cdot \mathcal{J}_n\left(\kappa_{nm}\cdot r\right)\cdot\cos(n\theta)\cdot\sin\left(\kappa_{nm}t\right) \\
& + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B'_{nm}}{\kappa_{nm}}\cdot \mathcal{J}_n\left(\kappa_{nm}\cdot r\right)\cdot\sin(n\theta)\cdot\sin\left(\kappa_{nm}t\right).
\end{aligned}
$$

Acoustic Interpretation: The vibration of the drumskin is a superposition of distinct **modes** of the form

$$\Phi_{nm}(r,\theta) \;=\; \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) \qquad \text{and}\quad \Psi_{nm}(r,\theta) \;=\; \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta),$$

for all $m,n\in\mathbb{N}$. For fixed m and n, the modes Φ_{nm} and and Ψ_{nm} vibrate at (temporal) frequency $\lambda_{nm} = \dfrac{\kappa_{nm}}{R}$. In the case of the *vibrating string*, all the different modes vibrated at frequences that were *integer multiples* of the fundamental frequency; musically speaking, this means that they are 'in harmony'. In the case of a *drum*, however, the frequencies are all *irrational* multiples (because the roots κ_{nm} are all irrationally related). Acoustically speaking, this means we expect a drum to sound somewhat more 'discordant' than a string.

Notice also that, as the radius R gets *larger*, the frequency $\lambda_{nm} = \dfrac{\kappa_{nm}}{R}$ gets *smaller*. This means that larger drums vibrate at *lower* frequencies, which matches our experience.

Example 14F.2. *A circular membrane of radius $R=1$ is emitting a pure pitch at frequency κ_{35}. Roughly describe the space-time profile of the solution (as a pattern of distortions of the membrane).*

Answer: The spatial distortion of the membrane must be a combination of modes vibrating at this frequency. Thus, we expect it to be a function of the form:

$$
\begin{aligned}
u(r,\theta;\ t) \;=\; \mathcal{J}_3\left(\kappa_{35}\cdot r\right)\Bigg[&\Big(A\cdot\cos(3\theta) + \; B\cdot\sin(3\theta)\Big)\cdot\cos\left(\kappa_{35}t\right) \;+ \\
& \left(\frac{A'}{\kappa_{35}}\cdot\cos(3\theta) + \;\frac{B'}{\kappa_{35}}\cdot\sin(3\theta)\right)\cdot\sin\left(\kappa_{35}t\right)\Bigg]
\end{aligned}
$$

By introducing some constant angular phase-shifts ϕ and ϕ', as well as new
Ⓔ constants C and C', we can rewrite this (**Exercise 14F.2**) as:

$$u(r,\theta;\ t) \quad = \quad \mathcal{J}_3\left(\kappa_{35}\cdot r\right)\left(C\cdot\cos(3(\theta+\phi))\cdot\cos\left(\kappa_{35}t\right) \quad + \quad \frac{C'}{\kappa_{35}}\cdot\cos(3(\theta+\phi'))\cdot\sin\left(\kappa_{35}t\right)\right).$$

$\diamondsuit$

Example 14F.3. *An initially silent circular drum of radius $R=1$ is struck in its exact center with a drumstick having a spherical head. Describe the resulting pattern of vibrations.*

Solution: This is a problem with *nonzero* initial *velocity* and *zero* initial *position*. Since the initial velocity (the impact of the drumstick) is rotationally symmetric (dead centre, spherical head), we can write it as a Fourer-Bessel expansion with no angular dependence:

$$f_1(r,\theta) \quad = \quad f(r) \quad \underset{\text{L2}}{\approx} \quad \sum_{m=1}^{\infty} A'_m\cdot\mathcal{J}_0\left(\kappa_{0m}\cdot r\right) \qquad (A'_1, A'_2, A'_3, \ldots \text{ some constants})$$

(all the higher-order Bessel functions disappear, since $\mathcal{J}_n$ is always associated with terms of the form $\sin(n\theta)$ and $\cos(n\theta)$, which depend on θ.) Thus, the solution must have the form:

$$u(r,\theta;\ t) \quad = \quad u(r,t) \quad \underset{\text{L2}}{\approx} \quad \sum_{m=1}^{\infty}\frac{A'_m}{\kappa_{0m}}\cdot\mathcal{J}_0\left(\kappa_{0m}\cdot r\right)\cdot\sin\left(\kappa_{0m}t\right). \qquad \diamondsuit$$

14G The power series for a Bessel function

Prerequisites: §0H(iii). **Recommended:** §14C(i).

In §14C-§14F, we claimed that Bessel's equation had certain solutions called *Bessel functions*, and showed how to use these Bessel functions to solve differential equations in polar coordinates. Now we will derive an an explicit formula for these Bessel functions in terms of their power series.

Proposition 14G.1. *Set $\lambda := 1$. For any fixed $m\in\mathbb{N}$ there is a solution $\mathcal{J}_m:\mathbb{R}_{\not -}\longrightarrow\mathbb{R}$ to the Bessel Equation*

$$x^2\mathcal{J}''(x) + x\cdot\mathcal{J}'(x) + (x^2-m^2)\cdot\mathcal{J}(x) \quad = \quad 0, \quad \textit{for all } x>0. \qquad (14G.1)$$

with a power series expansion:

$$\mathcal{J}_m(x) \quad = \quad \left(\frac{x}{2}\right)^m\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\,k!\,(m+k)!}x^{2k} \qquad (14G.2)$$

($\mathcal{J}_m$ is called the mth order **Bessel function** of the **first kind.**)

Proof. The ODE (14G.1) satisfies the hypotheses of the Frobenius Theorem (see Example 0H.5 on page 574 of Appendix 0H(iii)). Thus, we can apply the *Method of Frobenius* to solve (14G.1). Suppose that $\mathcal{J}$ is a solution, with an (unknown) power series $\mathcal{J}(x) = x^M \sum_{k=0}^{\infty} a_k x^k$, where $a_0, a_1, \ldots$ are unknown coefficients, and $M \geq 0$. We assume that $a_0 \neq 0$. We substitute this power series into eqn.(14G.1) to get equations relating the coefficients. The details of this computation are shown in Table 14.1.

Claim 1: $M = m$.

Proof. If the Bessel equation is to be satisfied, the power series in the bottom row of Table 14.1 must be identically zero. In particular, this means that the coefficient labeled '(a)' must be zero; in other words $a_0(M^2 - m^2) = 0$. Since we know that $a_0 \neq 0$, this means $(M^2 - m^2) = 0$ —i.e. $M^2 = m^2$. But $M \geq 0$, so this means $M = m$. $\diamond_{\text{Claim 1}}$

Claim 2: $a_1 = 0$.

Proof. If the Bessel equation is to be satisfied, the power series in the bottom row of Table 14.1 must be identically zero. In particular, this means that the coefficient labeled '(b)' must be zero; in other words, $a_1 \left[(M+1)^2 - m^2\right] = 0$.

Claim 1 says that $M = m$; hence this is equivalent to $a_1 \left[(m+1)^2 - m^2\right] = 0$. Clearly, $\left[(m+1)^2 - m^2\right] \neq 0$; hence we conclude that $a_1 = 0$. $\diamond_{\text{Claim 2}}$

Claim 3: *For all $k \geq 2$, the coefficients $\{a_2, a_3, a_4, \ldots\}$ must satisfy the following recurrence relation:*

$$a_k = \frac{-1}{(m+k)^2 - m^2} a_{k-2}, \qquad \textit{for all even } k \in \mathbb{N} \textit{ with } k \geq 2. \tag{14G.3}$$

On the other hand, $a_k = 0$ for all odd $k \in \mathbb{N}$.

Proof. If the Bessel equation is to be satisfied, the power series in the bottom row of Table 14.1 must be identically zero. In particular, this means that all the coefficients b_k must be zero. In other words, $a_{k-2} + \left((M+k)^2 - m^2\right) a_k = 0$.

From Claim 1, we know that $M = m$; hence this is equivalent to $a_{k-2} + \left((m+k)^2 - m^2\right) a_k = 0$. In other words, $a_k = -a_{k-2}/\left((m+k)^2 - m^2\right) a_k$. From Claim 2, we know that $a_1 = 0$. It follows from this equation that $a_3 = 0$; hence $a_5 = 0$, etc. Inductively, $a_n = 0$ for all odd n. $\diamond_{\text{Claim 3}}$

Claim 4: *Assume we have fixed a value for a_0. Define*

$$a_{2j} := \frac{(-1)^j \cdot a_0}{2^{2j} j!(m+1)(m+2)\cdots(m+j)}, \qquad \textit{for all } j \in \mathbb{N}.$$

$$
\begin{array}{rcrcrcrcrc}
\mathcal{J}(x) & = & a_0 x^M & + & a_1 x^{M+1} & + & a_2 x^{M+2} & +\cdots+ & a_k x^{M+k} & +\cdots \\
\text{Then} & & & & & & & & & \\
-m^2\mathcal{J}(x) & = & -m^2 a_0 x^M & - & m^2 a_1 x^{M+1} & + & -m^2 a_2 x^{M+2} & +\cdots+ & -m^2 a_k x^{M+k} & +\cdots \\
x^2\mathcal{J}(x) & = & & & & & a_0 x^{M+2} & +\cdots+ & a_{k-2} x^{M+k} & +\cdots \\
x\mathcal{J}'(x) & = & M a_0 x^M & + & (M+1) a_1 x^{M+1} & + & (M+2) a_2 x^{M+2} & +\cdots+ & (M+k) a_k x^{M+k} & +\cdots \\
x^2\mathcal{J}''(x) & = & M(M-1) a_0 x^M & + & (M+1)M a_1 x^{M+1} & + & (M+2)(M+1) a_2 x^{M+2} & +\cdots+ & (M+k)(M+k-1) a_k x^{M+k} & +\cdots \\
\text{Thus} \quad 0 & = & \multicolumn{8}{l}{x^2\mathcal{J}''(x) + x\cdot\mathcal{J}'(x) + (x^2-m^2)\cdot\mathcal{J}(x)} \\
 & = & \underbrace{(M^2-m^2)}_{(a)} a_0 x^M & + & \underbrace{((M+1)^2-m^2)}_{(b)} a_1 x^{M+1} & + & \left(\begin{smallmatrix} a_0 + \\ ((M+2)^2-m^2)a_2 \end{smallmatrix}\right) x^{M+2} & +\cdots+ & b_k x^{M+k} & +\cdots
\end{array}
$$

where

$$
\begin{aligned}
b_k &:= a_{k-2} + (M+k)a_k + (M+k)(M+k-1)a_k - m^2 a_k \\
&= a_{k-2} + (M+k)(1+M+k-1)a_k - m^2 a_k \\
&= a_{k-2} + \left((M+k)^2 - m^2\right) a_k.
\end{aligned}
$$

Table 14.1: The method of Frobenius to solve Bessel's equation in the proof of 14G.1.

Then the sequence $\{a_0, a_2, a_4, \ldots\}$ satisfies the recurrence relation (14G.3).

Proof. Set $k = 2j$ in eqn.(14G.3). For any $j \geq 2$, we must show that $a_{2j} = \dfrac{-a_{2j-2}}{(m+2j)^2 - m^2}$. Now, by definition,

$$a_{2j-2} \quad = \quad a_{2(j-1)} \quad := \quad \frac{(-1)^{j-1} \cdot a_0}{2^{2j-2}(j-1)!(m+1)(m+2)\cdots(m+j-1)},$$

Also,

$$(m+2j)^2 - m^2 \quad = \quad m^2 + 4jm + 4j^2 - m^2 \quad = \quad 4jm + 4j^2 \quad = \quad 2^2 j(m+j).$$

Hence

$$\begin{aligned}
\frac{-a_{2j-2}}{(m+2j)^2 - m^2} &= \frac{-a_{2j-2}}{2^2 j(m+j)} \\
&= \frac{(-1)(-1)^{j-1} \cdot a_0}{2^2 j(m+j) \ \cdot \ 2^{2j-2}(j-1)!(m+1)(m+2)\cdots(m+j-1)} \\
&= \frac{(-1)^j \cdot a_0}{2^{2j-2+2} \cdot j(j-1)! \cdot (m+1)(m+2)\cdots(m+j-1)(m+j)} \\
&= \frac{(-1)^j \cdot a_0}{2^{2j} j!(m+1)(m+2)\cdots(m+j-1)(m+j)} \quad = \quad a_{2j},
\end{aligned}$$

as desired. $\diamond_{\text{Claim 4}}$

By convention we define $a_0 := \dfrac{1}{2^m}\dfrac{1}{m!}$. We claim that that the resulting coefficients yield the Bessel function $\mathcal{J}_m(x)$ defined by (14G.2) To see this, let b_{2k} be the $2k$th coefficient of the Bessel series. By definition,

$$\begin{aligned}
b_{2k} \quad := \quad \frac{1}{2^m} \cdot \frac{(-1)^k}{2^{2k}\, k!\,(m+k)!} \quad &= \quad \frac{1}{2^m} \cdot \frac{(-1)^k}{2^{2k}\, k!\, m!(m+1)(m+2)\cdots(m+k-1)(m+k)} \\
= \quad \frac{1}{2^m m!} \cdot \frac{(-1)^k}{2^{2k}\, k!\,(m+1)(m+2)\cdots(m+k-1)(m+k)} \\
= \quad a_0 \cdot \left(\frac{(-1)^{k+1}}{2^{2k} k!(m+1)(m+2)\cdots(m+k-1)(m+k)} \right) \quad &= \quad a_{2k},
\end{aligned}$$

as desired. □

Corollary 14G.2. *Fix $m \in \mathbb{N}$. For any $\lambda > 0$, the Bessel Equation* (16C.12) *has solution $\mathcal{R}(r) := \mathcal{J}_m(\lambda r)$.*

Ⓔ *Proof.* **Exercise 14G.1** . □

Remarks: (a) We can generalize the Bessel Equation be replacing m with an arbitrary real number $\mu \in \mathbb{R}$ with $\mu \geq 0$. The solution to this equation is the Bessel function

$$\mathcal{J}_\mu(x) \quad = \quad \left(\frac{x}{2}\right)^\mu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\, k!\, \Gamma(\mu+k+1)} x^{2k}$$

Here, Γ is the *Gamma function*; if $\mu = m \in \mathbb{N}$, then $\Gamma(m+k+1) = (m+k)!$, so this expression agrees with (14G.2).

(b) There is a second solution to (14G.1); a function $\mathcal{Y}_m(x)$ which is *unbounded* at zero. This is called a **Neumann function** (or a *Bessel function of the second kind* or a *Weber-Bessel function*). It's derivation is too complicated to discuss here. See [Bro89, §6.8, p.115] or [CB87, §68, p.233].

14H Properties of Bessel functions

Prerequisites: §14G. **Recommended:** §14C(i).

Let $\mathcal{J}_n(x)$ be the Bessel function defined by eqn.(14G.2) on page 305 of §14G. In this section, we will develop some computational tools to work with these functions. First, we will define Bessel functions with *negative* order as follows: for any $n \in \mathbb{N}$, we define

$$\mathcal{J}_{-n}(x) \quad := \quad (-1)^n \mathcal{J}_n(x). \tag{14H.4}$$

We can now state the following useful recurrence relations

Proposition 14H.1. *For any $m \in \mathbb{Z}$,*

(a) $\dfrac{2m}{x}\,\mathcal{J}_m(x) \;=\; \mathcal{J}_{m-1}(x) + \mathcal{J}_{m+1}(x)$.

(b) $2\mathcal{J}_m'(x) \;=\; \mathcal{J}_{m-1}(x) - \mathcal{J}_{m+1}(x)$.

(c) $\mathcal{J}_0'(x) \;=\; -\mathcal{J}_1(x)$.

(d) $\partial_x \left(x^m \cdot \mathcal{J}_m(x)\right) \;=\; x^m \cdot \mathcal{J}_{m-1}(x)$.

(e) $\partial_x \left(\dfrac{1}{x^m}\,\mathcal{J}_m(x)\right) \;=\; \dfrac{-1}{x^m} \cdot \mathcal{J}_{m+1}(x)$.

(f) $\mathcal{J}_m'(x) \;=\; \mathcal{J}_{m-1}(x) - \dfrac{m}{x}\mathcal{J}_m(x)$.

(g) $\mathcal{J}_m'(x) \;=\; -\mathcal{J}_{m+1}(x) + \dfrac{m}{x}\mathcal{J}_m(x)$.

Proof. **Exercise 14H.1** (i) Prove (d) for $m \geq 1$ by substituting in the power series (14G.2) and differentiating. Ⓔ

(ii) Prove (e) for $m \geq 0$ by substituting in the power series (14G.2) and differentiating.

(iii) Use the definition (14H.4) and (i) and (ii) to prove (d) for $m \leq 0$ and (e) for $m \leq -1$.

(iv) Set $m = 0$ in (e) to obtain (c).

(v) Deduce (f) and (g) from (d) and (e).

(vi) Compute the sum and difference of (f) and (g) to get (a) and (b). □

Remark 14H.2: (Integration with Bessel functions)

The recurrence relations of Proposition 14H.1 can be used to simplify integrals involving Bessel functions. For example, parts (d) and (e) immediately imply that

$$\int x^m \cdot \mathcal{J}_{m-1}(x)\, dx \quad = \quad x^m \cdot \mathcal{J}_m(x) + C$$

$$\text{and} \quad \int \frac{1}{x^m} \cdot \mathcal{J}_{m+1}(x)\, dx \quad = \quad \frac{-1}{x^m}\mathcal{J}_m(x) \; + C.$$

The other relations are sometimes useful in an 'integration by parts' strategy.♢

For any $n \in \mathbb{N}$, let $0 \leq \kappa_{n,1} < \kappa_{n,2} < \kappa_{n,3} < \cdots$ be the zeros of the nth Bessel function $\mathcal{J}_n$ (i.e. $\mathcal{J}_n(\kappa_{n,m}) = 0$ for all $m \in \mathbb{N}$). Proposition 14C.1 on page 292 of §14C(i) says we can use Bessel functions to define a sequence of polar-separated eigenfunctions of the Laplacian:

$$\Phi_{n,m}(r,\theta) \quad := \quad \mathcal{J}_n(\kappa_{n,m} \cdot r) \cdot \cos(n\theta); \qquad \Psi_{n,m}(r,\theta) \quad := \quad \mathcal{J}_n(\kappa_{n,m} \cdot r) \cdot \sin(n\theta).$$

In the proof of Theorem 14C.2 on page 297 of §14C(iii), we claimed that these eigenfunctions were *orthogonal* as elements of $\mathbf{L}^2(\mathbb{D})$. We will now verify this claim. First we must prove a technical lemma.

Lemma 14H.3. *Fix* $n \in \mathbb{N}$.

(a) *If* $m \neq M$, *then* $\displaystyle\int_0^1 \mathcal{J}_n(\kappa_{n,m} \cdot r) \cdot \mathcal{J}_n(\kappa_{n,M} \cdot r)\; r\; dr \quad = \quad 0.$

(b) $\displaystyle\int_0^1 \mathcal{J}_n(\kappa_{n,m} \cdot r)^2 \cdot \; r\; dr \quad = \quad \frac{1}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2.$

Proof. **(a)** Let $\alpha = \kappa_{n,m}$ and $\beta = \kappa_{n,M}$. Define $f(x) := \mathcal{J}_m(\alpha x)$ and $g(x) := \mathcal{J}_m(\beta x)$. Hence we want to show that

$$\int_0^1 f(x)g(x)x\; dx \quad = \quad 0.$$

Define $h(x) = x \cdot \Big(f(x)g'(x) - g(x)f'(x)\Big)$.

Claim 1: $h'(x) \;=\; (\alpha^2 - \beta^2) f(x) g(x) x.$

Proof. First observe that

$$\begin{aligned} h'(x) &= x \cdot \partial_x \Big(f(x)g'(x) - g(x)f'(x)\Big) \;+\; \Big(f(x)g'(x) - g(x)f'(x)\Big) \\ &= x \cdot \Big(f(x)g''(x) + f'(x)g'(x) - g'(x)f'(x) - g(x)f''(x)\Big) \\ &\qquad + \quad \Big(f(x)g'(x) - g(x)f'(x)\Big) \\ &= x \cdot \Big(f(x)g''(x) - g(x)f''(x)\Big) \;+\; \Big(f(x)g'(x) - g(x)f'(x)\Big). \end{aligned}$$

By setting $\mathcal{R} = f$ or $\mathcal{R} = g$ in Corollary 14G.2, we obtain:

$$\begin{aligned} x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x) &= 0, \\ \text{and} \;\; x^2 g''(x) + x g'(x) + (\beta^2 x^2 - n^2) g(x) &= 0. \end{aligned}$$

We multiply the first equation by $g(x)$ and the second by $f(x)$ to get

$$\begin{aligned} x^2 f''(x)g(x) + x f'(x)g(x) + \alpha^2 x^2 f(x)g(x) - n^2 f(x)g(x) &= 0, \\ \text{and} \;\; x^2 g''(x)f(x) + x g'(x)f(x) + \beta^2 x^2 g(x)f(x) - n^2 g(x)f(x) &= 0. \end{aligned}$$

We then subtract these two equations to get

$$x^2 \Big(f''(x)g(x) - g''(x)f(x)\Big) + x \Big(f'(x)g(x) - g'(x)f(x)\Big) + \Big(\alpha^2 - \beta^2\Big) f(x)g(x)x^2 \quad = \quad 0.$$

Divide by x to get

$$x \Big(f''(x)g(x) - g''(x)f(x)\Big) + \Big(f'(x)g(x) - g'(x)f(x)\Big) + \Big(\alpha^2 - \beta^2\Big) f(x)g(x)x \quad = \quad 0.$$

Hence we conclude

$$\begin{aligned} \Big(\alpha^2 - \beta^2\Big) f(x)g(x)x &= x \Big(g''(x)f(x) - f''(x)g(x)\Big) + \Big(g'(x)f(x) - f'(x)g(x)\Big) \\ &= h'(x), \end{aligned}$$

as desired $\diamond_{\text{Claim 1}}$

It follows from Claim 1 that

$$(\alpha^2 - \beta^2) \cdot \int_0^1 f(x)g(x)x\, dx \quad = \quad \int_0^1 h'(x)\, dx \quad = \quad h(1) - h(0) \;\underset{(*)}{=\!=}\; 0 - 0 \quad = \quad 0.$$

To see $(*)$, observe that $h(0) = 0 \cdot \Big(f(0)g'(0) - g(0)f'(0)\Big) \;=\; 0$. Also,

$$h(1) \;=\; (1) \cdot \Big(f(1)g'(1) - g(1)f'(1)\Big) \quad = \quad 0,$$

because $f(1) = \mathcal{J}_n(\kappa_{n,m}) \;=\; 0$ and $g(1) = \mathcal{J}_n(\kappa_{n,N}) \;=\; 0$.

(b) Let $\alpha = \kappa_{n,m}$ and $f(x) := \mathcal{J}_m(\alpha x)$. Hence we want to evaluate

$$\int_0^1 f(x)^2 x \; dx.$$

Define $h(x) := x^2(f'(x))^2 + (\alpha^2 x^2 - n^2) f^2(x)$.

Claim 2: $h'(x) \;=\; 2\alpha^2 f(x)^2 x.$

Proof. By setting $\mathcal{R} = f$ in Corollary 14G.2, we obtain:

$$0 \quad = \quad x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x).$$

We multiply by $f'(x)$ to get

$$\begin{aligned}
0 &= x^2 f'(x) f''(x) + x(f'(x))^2 + (\alpha^2 x^2 - n^2) f(x) f'(x) \\
&= x^2 f'(x) f''(x) + x(f'(x))^2 + (\alpha^2 x^2 - n^2) f(x) f'(x) + \alpha^2 x f^2(x) - \alpha^2 x f^2(x) \\
&= \frac{1}{2}\partial_x \left[x^2(f'(x))^2 + (\alpha^2 x^2 - n^2) f^2(x) \right] - \alpha^2 x f^2(x) \\
&= \frac{1}{2} h'(x) - \alpha^2 x f^2(x).
\end{aligned}$$

$\diamond_{\text{Claim 2}}$

It follows from Claim 2 that

$$\begin{aligned}
& 2\alpha^2 \int_0^1 f(x)^2 x \; dx \\
&= \int_0^1 h'(x)\; dx \quad = \quad h(1) - h(0) \\
&= 1^2(f'(1))^2 + (\alpha^2 1^2 - n^2) \underbrace{\cdot f^2(1)}_{\substack{\mathcal{J}_n^2(\kappa_{n,m}) \\ =0}} - \underbrace{0^2(f'(0))^2}_{0} + \underbrace{(\alpha^2 0^2 - n^2)}_{[0 \text{ if } n=0]} \underbrace{f^2(0)}_{[0 \text{ if } n \neq 0]} \\
&= f'(1)^2 \quad = \quad \left(\alpha \mathcal{J}_n'(\alpha)\right)^2 \quad = \quad \alpha^2 \mathcal{J}_n'(\alpha)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 f(x)^2 x \; dx &= \frac{1}{2}\mathcal{J}_n'(\alpha)^2 \underset{(*)}{=\!=} \frac{1}{2}\left(\frac{n}{\alpha}\mathcal{J}_n(\alpha) - \mathcal{J}_{n+1}(\alpha)\right)^2 \\
&\underset{(\dagger)}{=\!=} \frac{1}{2}\left(\frac{n}{\kappa_{n,m}} \underbrace{\mathcal{J}_n(\kappa_{n,m})}_{=0} - \mathcal{J}_{n+1}(\kappa_{n,m})\right)^2 \quad = \quad \frac{1}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2,
\end{aligned}$$

where $(*)$ is by Proposition 14H.1(g) and $(\dagger)$ is because $\alpha := \kappa_{n,m}$. $\square$

Proposition 14H.4. *Let $\mathbb{D} = \{(r,\theta) \,;\, r \le 1\}$ be the unit disk. Then the collection*

$$\{\Phi_{n,m},\ \Psi_{\ell,m} \,;\, n = 0...\infty,\ \ell \in \mathbb{N}, m \in \mathbb{N}\}$$

is an orthogonal set for $\mathbf{L}^2(\mathbb{D})$. In other words, for any $n, m, N, M \in \mathbb{N}$,

$$(\mathbf{a}) \qquad \langle \Phi_{n,m}, \Psi_{N,M} \rangle \quad = \quad \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Psi_{N,M}(r,\theta)\ d\theta\ r\ dr \quad = \quad 0.$$

Furthermore, if $(n,m) \neq (N,M)$, then

$$(\mathbf{b}) \qquad \langle \Phi_{n,m}, \Phi_{N,M} \rangle \quad = \quad \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Phi_{N,M}(r,\theta)\ d\theta\ r\ dr \quad = \quad 0.$$

$$(\mathbf{c}) \qquad \langle \Psi_{n,m}, \Psi_{N,M} \rangle \quad = \quad \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Psi_{n,m}(r,\theta) \cdot \Psi_{N,M}(r,\theta)\ d\theta\ r\ dr \quad = \quad 0.$$

Finally, for any (n,m),

$$(\mathbf{d}) \qquad \|\Phi_{n,m}\|_2 \quad = \quad \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)^2\ d\theta\ r\ dr \quad = \quad \frac{1}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2.$$

$$(\mathbf{e}) \qquad \|\Psi_{n,m}\|_2 \quad = \quad \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Psi_{n,m}(r,\theta)^2\ d\theta\ r\ dr \quad = \quad \frac{1}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2.$$

Proof. **(a)** $\Phi_{n,m}$ and $\Psi_{N,M}$ separate in the coordinates (r,θ), so the integral splits in two:

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Psi_{N,M}(r,\theta)\ d\theta\ r\ dr \\
&= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\kappa_{n,m}\cdot r)\cdot\cos(n\theta)\cdot\mathcal{J}_N(\kappa_{N,M}\cdot r)\cdot\sin(N\theta)\ d\theta\ r\ dr \\
&= \int_0^1 \mathcal{J}_n(\kappa_{n,m}\cdot r)\cdot\mathcal{J}_N(\kappa_{N,M}\cdot r)\ r\ dr \cdot \underbrace{\int_{-\pi}^{\pi}\cos(n\theta)\cdot\sin(N\theta)\ d\theta}_{=0 \text{ by Prop. 6D.2(c), p.112}} \quad = \quad 0.
\end{aligned}$$

(b) or **(c)** (Case $n \neq N$). Likewise, if $n \neq N$, then

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Phi_{N,M}(r,\theta)\ d\theta\ r\ dr \\
&= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\kappa_{n,m}\cdot r)\cdot\cos(n\theta)\cdot\mathcal{J}_N(\kappa_{N,M}\cdot r)\cdot\cos(N\theta)\ d\theta\ r\ dr \\
&= \int_0^1 \mathcal{J}_n(\kappa_{n,m}\cdot r)\cdot\mathcal{J}_N(\kappa_{N,M}\cdot r)\ r\ dr \cdot \underbrace{\int_{-\pi}^{\pi}\cos(n\theta)\cdot\cos(N\theta)\ d\theta}_{=0 \text{ by Prop. 6D.2(a), p.112}} \quad = \quad 0.
\end{aligned}$$

the case **(c)** is proved similarly.

(b) or **(c)** (Case $n = N$ but $m \neq M$). If $n = N$, then

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Phi_{n,M}(r,\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\kappa_{n,m} \cdot r) \cdot \cos(n\theta) \cdot \mathcal{J}_n(\kappa_{n,M} \cdot r) \cdot \cos(n\theta) \; d\theta \; r \; dr$$
$$= \underbrace{\int_0^1 \mathcal{J}_n(\kappa_{n,m} \cdot r) \cdot \mathcal{J}_n(\kappa_{n,M} \cdot r) \; r \; dr}_{= 0 \text{ by Lemma 14H.3(a)}} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta)^2 \; d\theta}_{\substack{= \pi \text{ by 6D.2(d),} \\ \text{on p.112.}}} \quad = \quad 0 \cdot \pi \quad = \quad 0.$$

(d): If $n = N$ and $m = M$ then

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)^2 \; d\theta \; r \; dr \quad = \quad \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\kappa_{n,m} \cdot r)^2 \cdot \cos(n\theta)^2 \; d\theta \; r \; dr$$
$$= \underbrace{\int_0^1 \mathcal{J}_n(\kappa_{n,m} \cdot r)^2 \; r \; dr}_{\substack{= \frac{1}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2 \\ \text{by Lemma 14H.3(b)}}} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta)^2 \; d\theta}_{\substack{= \pi \text{ by Prop 6D.2(d)} \\ \text{on p.112.}}} \quad = \quad \frac{\pi}{2}\mathcal{J}_{n+1}(\kappa_{n,m})^2.$$

Ⓔ The proof of **(e)** is **Exercise 14H.2** . □

Ⓔ **Exercise 14H.3.** (a) Use a 'separation of variables' argument (similar to Proposition 16C.2) to prove:

Proposition: *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a harmonic function —in other words suppose $\triangle f = 0$.*

*Suppose f **separates** in polar coordinates, meaning that there is a function $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$ (satisfying periodic boundary conditions) and a function $\mathcal{R} : \mathbb{R}_{\neq} \longrightarrow \mathbb{R}$ such that*

$$f(r,\theta) \quad = \quad \mathcal{R}(r) \cdot \Theta(\theta), \qquad \textit{for all } r \geq 0 \textit{ and } \theta \in [-\pi, \pi].$$

Then there is some $m \in \mathbb{N}$ such that

$$\Theta(\theta) \quad = \quad A\cos(m\theta) + B\sin(m\theta), \qquad \textit{(for constants } A, B \in \mathbb{R}.\textit{)}$$

and $\mathcal{R}$ is a solution to the **Cauchy-Euler Equation:**

$$r^2\mathcal{R}''(r) + r \cdot \mathcal{R}'(r) - m^2 \cdot \mathcal{R}(r) \quad = \quad 0, \quad \textit{for all } r > 0. \qquad (14H.5)$$

(b) Let $\mathcal{R}(r) = r^\alpha$ where $\alpha = \pm m$. Show that $\mathcal{R}(r)$ is a solution to the Cauchy-Euler equation (14H.5).

(c) Deduce that $\Psi_m(r,\theta) = r^m \cdot \sin(m\theta)$; $\Phi_m(r,\theta) = r^m \cdot \cos(m\theta)$; $\psi_m(r,\theta) = r^{-m} \cdot \sin(m\theta)$; and $\phi_m(r,\theta) = r^{-m} \cdot \cos(m\theta)$ are harmonic functions in $\mathbb{R}^2$. ♦

14I Practice problems

1. For all (r, θ), let $\Phi_n(r, \theta) = r^n \cos(n\theta)$. Show that Φ_n is harmonic.

2. For all (r, θ), let $\Psi_n(r, \theta) = r^n \sin(n\theta)$. Show that Ψ_n is harmonic.

3. For all (r, θ) with $r > 0$, let $\phi_n(r, \theta) = r^{-n} \cos(n\theta)$. Show that ϕ_n is harmonic.

4. For all (r, θ) with $r > 0$, let $\psi_n(r, \theta) = r^{-n} \sin(n\theta)$. Show that ψ_n is harmonic.

5. For all (r, θ) with $r > 0$, let $\phi_0(r, \theta) = \log |r|$. Show that ϕ_0 is harmonic.

6. Let $b(\theta) = \cos(3\theta) + 2\sin(5\theta)$ for $\theta \in [-\pi, \pi)$.

   (a) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Dirichlet** boundary conditions $u(1, \theta) = b(\theta)$. Is the solution unique?

   (b) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}^\complement$, with nonhomogeneous **Dirichlet** boundary conditions $u(1, \theta) = b(\theta)$. Is the solution unique?

   (c) Find the 'decaying gradient' solution(s) to the **Laplace equation** on $\mathbb{D}^\complement$, with nonhomogeneous **Neumann** boundary conditions $\partial_r u(1, \theta) = b(\theta)$. Is the solution unique?

7. Let $b(\theta) = 2\cos(\theta) - 6\sin(2\theta)$, for $\theta \in [-\pi, \pi)$.

   (a) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Dirichlet** boundary conditions: $u(1, \theta) = b(\theta)$ for all $\theta \in [-\pi, \pi)$. Is the solution unique?

   (b) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Neumann** boundary conditions: $\partial_r\, u(1, \theta) = b(\theta)$ for all $\theta \in [-\pi, \pi)$. Is the solution unique?

8. Let $b(\theta) = 4\cos(5\theta)$ for $\theta \in [-\pi, \pi)$.

   (a) Find the bounded solution(s) to the **Laplace equation** on the disk $\mathbb{D} = \{(r, \theta)\ ;\ r \leq 1\}$, with nonhomogeneous **Dirichlet** boundary conditions $u(1, \theta) = b(\theta)$. Is the solution unique?

   (b) *Verify* your answer in part (a) (i.e. check that the solution is harmonic and satisfies the prescribed boundary conditions.)
   (**Hint:** *Recall that* $\triangle = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$.)

9. Let $b(\theta) = 5 + 4\sin(3\theta)$ for $\theta \in [-\pi, \pi)$.

(a) Find the 'decaying gradient' solution(s) to the **Laplace equation** on the codisk $\mathbb{D}^{\complement} = \{(r, \theta) \; ; \; r \geq 1\}$, with nonhomogeneous **Neumann** boundary conditions $\partial_r \, u(1, \theta) = b(\theta)$. Is the solution unique?

(b) *Verify* that your answer in part (a) satisfies the prescribed boundary conditions. (Forget about the Laplacian).

10. Let $b(\theta) \; = \; 2\cos(5\theta) + \sin(3\theta)$, for $\theta \in [-\pi, \pi)$.

    (a) Find the solution(s) (if any) to the **Laplace equation** on the disk $\mathbb{D} \; = \; \{(r, \theta) \; ; \; r \leq 1\}$, with nonhomogeneous **Neumann** boundary conditions: $\partial_\perp \, u(1, \theta) \; = \; b(\theta)$, for all $\theta \in [-\pi, \pi)$.
    Is the solution unique? Why or why not?

    (b) Find the *bounded* solution(s) (if any) to the **Laplace equation** on the codisk $\mathbb{D}^{\complement} \; = \; \{(r, \theta) \; ; \; r \geq 1\}$, with nonhomogeneous **Dirichlet** boundary conditions: $u(1, \theta) \; = \; b(\theta)$, for all $\theta \in [-\pi, \pi)$.
    Is the solution unique? Why or why not?

11. Let $\mathbb{D}$ be the unit disk. Let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be some function, and let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution to the corresponding Dirichlet problem with boundary conditions $b(\sigma)$. Prove that

$$u(0, 0) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma) \; d\sigma.$$

Remark: This is a special case of the Mean Value Theorem for Harmonic Functions (Theorem 1E.1 on page 16), but do *not* simply 'quote' Theorem 1E.1 to solve this problem. Instead, apply Proposition 14B.11 on page 290.

12. Let $\Phi_{n,\lambda}(r, \theta) \; := \; \mathcal{J}_n(\lambda \cdot r) \cdot \cos(n\theta)$. Show that $\triangle \Phi_{n,\lambda} \; = \; -\lambda^2 \Phi_{n,\lambda}$.

13. Let $\Psi_{n,\lambda}(r, \theta) \; := \; \mathcal{J}_n(\lambda \cdot r) \cdot \sin(n\theta)$. Show that $\triangle \Psi_{n,\lambda} \; = \; -\lambda^2 \Psi_{n,\lambda}$.

14. Let $\phi_{n,\lambda}(r, \theta) \; := \; \mathcal{Y}_n(\lambda \cdot r) \cdot \cos(n\theta)$. Show that $\triangle \phi_{n,\lambda} \; = \; -\lambda^2 \phi_{n,\lambda}$.

15. $\psi_{n,\lambda}(r, \theta) \; := \; \mathcal{Y}_n(\lambda \cdot r) \cdot \sin(n\theta)$. Show that $\triangle \psi_{n,\lambda} \; = \; -\lambda^2 \psi_{n,\lambda}$.

Chapter 15

Eigenfunction methods on arbitrary domains

"Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house." —Henri Poincaré

The methods given in Chapters 11-14 are all special cases of a single, general technique: the solution of initial/boundary value problems using *eigenfunction expansions*. The time has come to explicate this technique in full generality. The exposition in this chapter is somewhat more abstract than in previous chapters, but that is because the concepts we introduce are of such broad applicability. Technically, this chapter can be read without having read Chapters 11-14; however, this chapter will be easier to understand if you have have already read Chapters 11-14.

15A General solution to Poisson, heat and wave equation BVPs

Prerequisites: §4B(iv), §5D, §6F, §0D. **Recommended:** Chapters 11, 12, 13, 14 .

Throughout this section:

- Let $\mathbb{X} \subset \mathbb{R}^D$ be any bounded domain (e.g. a line segment, box, disk, sphere, etc. —see §0D). When we refer to Neumann boundary conditions, we will also assume that $\mathbb{X}$ has a piecewise smooth boundary (so the normal derivative is well-defined).

- Let $\{\mathcal{S}_k\}_{k=1}^{\infty} \subset \mathbf{L}^2(\mathbb{X})$ be a **Dirichlet eigenbasis** —that is, $\{\mathcal{S}_k\}_{k=1}^{\infty}$ is an orthogonal basis of $\mathbf{L}^2(\mathbb{X})$, such that every $\mathcal{S}_k$ is an eigenfunction of the Laplacian, and satisfies homogeneous Dirichlet boundary conditions on $\mathbb{X}$ (i.e. $\mathcal{S}_k(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$). For every $k \in \mathbb{N}$, let $-\lambda_k < 0$ be the *eigenvalue* associated with $\mathcal{S}_k$ (i.e. $\triangle \mathcal{S}_k = -\lambda_k \mathcal{S}_k$). We can assume without loss of generality that $\lambda_k \neq 0$ for all $k \in \mathbb{N}$ (**<u>Exercise 15A.1</u>** Ⓔ

Why? Hint: Lemma 5D.3(a)).

- Let $\{\mathcal{C}_k\}_{k=0}^{\infty} \subset \mathbf{L}^2(\mathbb{X})$ be a **Neumann eigenbasis** —that is, $\{\mathcal{C}_k\}_{k=0}^{\infty}$ is an orthogonal basis $\mathbf{L}^2(\mathbb{X})$, such that every $\mathcal{C}_k$ is an eigenfunction of the Laplacian, and satisfies homogeneous Neumann boundary conditions on $\mathbb{X}$ (i.e. $\partial_\perp \mathcal{C}_k(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$). For every $k \in \mathbb{N}$, let $-\mu_k \leq 0$ be the *eigenvalue* associated with $\mathcal{C}_k$ (i.e. $\triangle \mathcal{C}_k = -\mu_k \mathcal{C}_k$). We can assume without loss of generality that $\mathcal{C}_0$ is a constant function (so that $\mu_0 = 0$), while $\mu_k \neq 0$ for all $k \geq 1$ (**Exercise 15A.2** Why? Hint: Lemma 5D.3(b)).

Ⓔ

Theorem 15E.12 (page 347 below) will guarantee that we will be able to find a Dirichlet eigenbasis for any domain $\mathbb{X} \subset \mathbb{R}^D$, and a Neumann eigenbasis for many domains. If $f \in \mathbf{L}^2(\mathbb{X})$ is some other function (describing, for example, an initial condition), then we can express f as a combination of these basis elements, as described in §6F:

$$f \underset{L2}{\approx} \sum_{k=0}^{\infty} A_k \mathcal{C}_k, \text{ where } A_k := \frac{\langle f, \mathcal{C}_k \rangle}{\|\mathcal{C}_k\|_2^2}, \text{ for all } k \in \mathbb{N}; \quad (15A.1)$$

$$\text{and } f \underset{L2}{\approx} \sum_{k=1}^{\infty} B_k \mathcal{S}_k, \text{ where } B_k := \frac{\langle f, \mathcal{S}_k \rangle}{\|\mathcal{S}_k\|_2^2}, \text{ for all } k \in \mathbb{N}. \quad (15A.2)$$

These expressions are called **eigenfunction expansions** for f.

Example 15A.1. (a) If $\mathbb{X} = [0, \pi] \subset \mathbb{R}$, then we could use the eigenbases $\{\mathcal{S}_k\}_{k=1}^{\infty} = \{\mathbf{S}_n\}_{n=1}^{\infty}$ and $\{\mathcal{C}_k\}_{k=0}^{\infty} = \{\mathbf{C}_n\}_{n=0}^{\infty}$, where $\mathbf{S}_n(x) := \sin(nx)$ and $\mathbf{C}_n(x) := \cos(nx)$ for all $n \in \mathbb{N}$. In this case, $\lambda_n = n^2 = \mu_n$ for all $n \in \mathbb{N}$. Also the eigenfunction expansions (15A.1) and (15A.2) are, respectively, the *Fourier Cosine Series* and *Fourier Sine Series* for f, from §7A.

(b) If $\mathbb{X} = [0, \pi]^2 \subset \mathbb{R}^2$, then we could use the eigenbases $\{\mathcal{S}_k\}_{k=1}^{\infty} = \{\mathbf{S}_{n,m}\}_{n,m=1}^{\infty}$ and $\{\mathcal{C}_k\}_{k=0}^{\infty} = \{\mathbf{C}_{n,m}\}_{n,m=0}^{\infty}$, where $\mathbf{S}_{n,n}(x, y) := \sin(nx)\sin(my)$ and $\mathbf{C}_{n,m}(x) := \cos(nx)\cos(my)$ for all $n, m \in \mathbb{N}$. In this case, $\lambda_{n,m} = n^2 + m^2 = \mu_{n,m}$ for all $(n, m) \in \mathbb{N}$. Also, the eigenfunction expansions (15A.1) and (15A.2) are, respectively, the two-dimensional Fourier Cosine Series and Fourier Sine Series for f, from §9A.

(c) If $\mathbb{X} = \mathbb{D} \subset \mathbb{R}^2$, then we could use the Dirichlet eigenbasis $\{\mathcal{S}_n\}_{k=1}^{\infty} = \{\Phi_{n,m}\}_{n=0,m=1}^{\infty} \sqcup \{\Psi_{n,m}\}_{n,m=1}^{\infty}$, where $\Phi_{n,m}$ and $\Psi_{n,m}$ are the type-1 Fourier-Bessel eigenfunctions defined by eqn.(14C.5) on page 296 of §14C(ii). In this case, we have eigenvalues $\lambda_{n,m} = \kappa_{n,m}^2$, as defined in equation (14C.4) on page 296. Then the eigenfunction expansion in (15A.2) is the *Fourier-Bessel expansion* for f, from §14C(iii). ♢

Theorem 15A.2. General Solution of the Poisson Equation

Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain. Let $f \in \mathbf{L}^2(\mathbb{X})$, and let $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ be some other function. Let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a solution to the Poisson equation "$\triangle\, u = f$".

(a) *Suppose $\{\mathcal{S}_k, \lambda_k\}_{k=1}^{\infty}$ is a Dirichlet eigenbasis, and $\{B_n\}_{n=1}^{\infty}$ are as in equation* (15A.2)*. Assume that $|\lambda_k| > 1$ for all but finitely many $k \in \mathbb{N}$. If u satisfies homogeneous Dirichlet BC* (i.e. $u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$)*, then $u \underset{\text{L2}}{\approx} -\sum_{n=1}^{\infty} \frac{B_n}{\lambda_n}\mathcal{S}_n$.*

(b) *Let $h : \mathbb{X} \longrightarrow \mathbb{R}$ be a solution to the Laplace equation "$\triangle\, h = 0$" satisfying the nonhomogeneous Dirichlet BC $h(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. If u is as in part* (a)*, then $w := u + h$ is a solution to the Poisson equation "$\triangle\, w = f$" and also satisfies Dirichlet BC $w(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$.*

(c) *Suppose $\{\mathcal{C}_k, \mu_k\}_{k=1}^{\infty}$ is a Neumann eigenbasis, and suppose that $|\mu_k| > 1$ for all but finitely many $k \in \mathbb{N}$. Let $\{A_n\}_{n=1}^{\infty}$ be as in equation* (15A.1)*, and suppose $A_0 = 0$. For any $j \in [1...D]$, let $\|\partial_j\, \mathcal{C}_k\|_\infty$ be the supremum of the j-derivative of $\mathcal{C}_k$ on $\mathbb{X}$, and suppose that*

$$\sum_{\substack{k=1 \\ \mu_k \neq 0}}^{\infty} \frac{|A_k|}{|\mu_k|} \|\partial_j\, \mathcal{C}_k\|_\infty \quad < \quad \infty. \tag{15A.3}$$

If u satisfies homogeneous Neumann BC (i.e. $\partial_\perp u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$)*, then $u \underset{\text{L2}}{\approx} C \ - \sum_{\substack{k=1 \\ \mu_k \neq 0}}^{\infty} \frac{A_k}{\mu_k}\mathcal{C}_k$, where $C \in \mathbb{R}$ is an arbitrary constant.*

However, if $A_0 \neq 0$, then there is no solution to this problem with homogeneous Neumann BC.

(d) *Let $h : \mathbb{X} \longrightarrow \mathbb{R}$ be a solution to the Laplace equation "$\triangle\, h = 0$" satisfying the nonhomogeneous Neumann BC $\partial_\perp h(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. If u is as in part* (c)*, then $w := u + h$ is a solution to the Poisson equation "$\triangle\, w = f$" and also satisfies Neumann BC $\partial_\perp w(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$.*

Proof. **Exercise 15A.3** Hint: To show solution uniqueness, use Theorem 5D.5. Ⓔ

For (a), imitate the proofs of Propositions 11C.1, 12C.1, 13C.1, and 14D.1.

For (b,d), imitate the proofs of Propositions 12C.6 and 14D.3.

For (c), imitate the proofs of Propositions 11C.2, 12C.4 and 13C.2. Note that you need hypothesis (15A.3) to apply Proposition 0F.1. □

Exercise 15A.4. Show how Propositions 11C.1, 11C.2, 12C.1, 12C.4,12C.6 13C.1, 13C.2, 14D.1, and 14D.3 are all special cases of Theorem 15A.2. For the results involving Neumann BC, don't forget to check that (15A.3) is satisfied. ♦ Ⓔ

Theorem 15A.3. General Solution of the heat equation

Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain. Let $f \in \mathbf{L}^2(\mathbb{X})$, and let $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ be some other function. Let $u : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be a solution to the heat equation "$\partial_t u = \triangle u$", with initial conditions $u(\mathbf{x}, 0) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$.

(a) *Suppose $\{\mathcal{S}_k, \lambda_k\}_{k=1}^{\infty}$ is a Dirichlet eigenbasis, and $\{B_n\}_{n=1}^{\infty}$ are as in equation* (15A.2). *If u satisfies homogeneous Dirichlet BC* (i.e. $u(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$), *then* $u \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \exp(-\lambda_n t)\, \mathcal{S}_n$.

(b) *Let $h : \mathbb{X} \longrightarrow \mathbb{R}$ be a solution to the Laplace equation "$\triangle h = 0$" satisfying the nonhomogeneous Dirichlet BC $h(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. If u is as in part* (a)*, then $w := u + h$ is a solution to the heat equation "$\partial_t w = \triangle w$", with initial conditions $w(\mathbf{x}, 0) = f(\mathbf{x}) + h(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$, and also satisfies Dirichlet BC $w(\mathbf{x}, t) = b(\mathbf{x})$ for all $(\mathbf{x}, t) \in \partial\mathbb{X} \times \mathbb{R}_+$.*

(c) *Suppose $\{\mathcal{C}_k, \mu_k\}_{k=0}^{\infty}$ is a Neumann eigenbasis, and $\{A_n\}_{n=0}^{\infty}$ are as in equation* (15A.1). *Suppose the sequence $\{\mu_k\}_{k=0}^{\infty}$ grows fast enough that*

$$\lim_{k\to\infty} \frac{\log(k)}{\mu_k} = 0, \quad \text{and, for all } j \in [1...D], \quad \lim_{k\to\infty} \frac{\log \|\partial_j \mathcal{C}_k\|_\infty}{\mu_k} = 0. \tag{15A.4}$$

If u satisfies homogeneous Neumann BC (i.e. $\partial_\perp u(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$), *then* $u \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A_n \exp(-\mu_n t)\, \mathcal{C}_n$.

(d) *Let $h : \mathbb{X} \longrightarrow \mathbb{R}$ be a solution to the Laplace equation "$\triangle h = 0$" satisfying the nonhomogeneous Neumann BC $\partial_\perp h(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. If u is as in part* (c)*, then $w := u + h$ is a solution to the heat equation "$\partial_t w = \triangle w$" with initial conditions $w(\mathbf{x}, 0) = f(\mathbf{x}) + h(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$, and also satisfies Neumann BC $\partial_\perp w(\mathbf{x}, t) = b(\mathbf{x})$ for all $(\mathbf{x}, t) \in \partial\mathbb{X} \times \mathbb{R}_+$.*

Furthermore, in parts (a) *and* (c)*, the series defining u converges semiuniformly on $\mathbb{X} \times \mathbb{R}_+$.*

Ⓔ *Proof.* **Exercise 15A.5** Hint: To show solution uniqueness, use Theorem 5D.8.

For part (a), imitate the proofs of Propositions 11A.1, 12B.1, 13A.1, and 14E.1.

For (b,d) imitate the proofs of Propositions 12B.5 and 14E.3.

For (c), imitate the proofs of Propositions 11A.3, 12B.3, and 13A.2. First use hypothesis (15A.4) to show that the sequence $\{e^{-\mu_n t}\, \|\partial_j \mathcal{C}_n\|_\infty\}_{n=0}^{\infty}$ is square-summable for any $t > 0$. Use Parseval's equality (Theorem 6F.1) to show that the sequence $\{|A_k|\}_{k=0}^{\infty}$ is also square-summable. Use the Cauchy-Bunyakowski-Schwarz inequality to conclude that the sequence $\{e^{-\mu_n t}\,|A_k|\,\|\partial_j\,\mathcal{C}_n\|_{\infty}\}_{n=0}^{\infty}$ is absolutely summable, which means the formal derivative $\partial_j u$ is absolutely convergent. Now apply Proposition 0F.1. □

Exercise 15A.6. Show how Propositions 11A.1, 11A.3, 12B.1, 12B.3, 12B.5, 13A.1, 13A.2, 14E.1, and 14E.3. are all special cases of Theorem 15A.3. For the results involving Neumann BC, don't forget to check that (15A.4) is satisfied. ♦ Ⓔ

Theorem 15A.4. General Solution of the wave equation

Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain and let $f \in \mathbf{L}^2(\mathbb{X})$. Suppose $u : \mathbb{X}\times\mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is a solution to the wave equation "$\partial_t^2\, u = \triangle\, u$", and has initial position $u(\mathbf{x};0) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$.

(a) *Suppose $\{\mathcal{S}_k, \lambda_k\}_{k=1}^{\infty}$ is a Dirichlet eigenbasis, and $\{B_n\}_{n=1}^{\infty}$ are as in equation (15A.2). Suppose $\sum_{n=1}^{\infty} |\lambda_n B_n| < \infty$. If u satisfies homogeneous Dirichlet BC* (i.e. $u(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$), *then* $u \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} B_n \cos(\sqrt{\lambda_n}\, t)\, \mathcal{S}_n$.

(b) *Suppose $\{\mathcal{C}_k, \mu_k\}_{k=0}^{\infty}$ is a Neumann eigenbasis, and $\{A_n\}_{n=0}^{\infty}$ are as in equation (15A.1). Suppose the sequence $\{A_n\}_{n=0}^{\infty}$ decays quickly enough that*

$$\sum_{n=0}^{\infty} |\mu_n A_n| \;<\; \infty, \qquad \text{and, for all } j \in [1...D], \qquad \sum_{n=0}^{\infty} |A_n| \cdot \|\partial_j\, \mathcal{C}_n\|_{\infty} \;<\; \infty. \tag{15A.5}$$

If u satisfies homogeneous Neumann BC (i.e. $\partial_\perp u(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \partial\mathbb{X}$ and $t \in \mathbb{R}_{\not-}$), *then* $u \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A_n \cos(\sqrt{\mu_n}\, t)\, \mathcal{C}_n$.

Now suppose $u : \mathbb{X}\times\mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ is a solution to the wave equation "$\partial_t^2\, u = \triangle\, u$", and has initial velocity $\partial_t\, u(\mathbf{x};0) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{X}$.

(c) *Suppose $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\,|B_n| < \infty$. If u satisfies homogeneous Dirichlet BC, then*

$$u \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} \frac{B_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}\, t)\, \mathcal{S}_n.$$

(d) *Suppose the sequence $\{A_n\}_{n=0}^{\infty}$ decays quickly enough that*

$$\sum_{n=1}^{\infty} \sqrt{\mu_n}\,|A_n| \;<\; \infty, \quad \text{and, for all } j \in [1...D], \quad \sum_{n=1}^{\infty} \frac{|A_n|}{\sqrt{\mu_n}} \|\partial_j\, \mathcal{C}_n\|_{\infty} \;<\; \infty. \tag{15A.6}$$

If u satisfies homogeneous Neumann BC, then there is some constant $C \in \mathbb{R}$ such that, for all $\mathbf{x} \in \mathbb{X}$, we have $u(\mathbf{x}; 0) = C$, and for all $t \in \mathbb{R}$, we have

$$u(\mathbf{x};t) \quad \underset{\text{L2}}{\approx} \quad A_0\, t \;+\; \sum_{n=1}^{\infty} \frac{A_n}{\sqrt{\mu_n}} \sin(\sqrt{\mu_n}\, t)\, \mathcal{C}_n(\mathbf{x}) \quad + \; C.$$

(e) *To obtain a solution with both a specified initial position and a specified initial velocity, add the solutions from* (a) *and* (c) *for homogeneous Dirichlet BC. Add the solutions from* (b) *and* (d) *for homogeneous Neumann BC* (setting $C = 0$ in part (d)).

Ⓔ *Proof.* **<u>Exercise 15A.7</u>** Hint: To show solution uniqueness, use Theorem 5D.11. For (a), imitate Propositions 12D.1 and 14F.1. For (c) imitate the proof of Propositions 12D.3 and 14F.1. For (b) and (d), use hypotheses (15A.5) and (15A.6) to apply Proposition 0F.1. □

Ⓔ **<u>Exercise 15A.8</u>.** Show how Propositions 11B.1, 11B.4 12D.1, 12D.3, and 14F.1 are all special cases of Theorem 15A.4(a,c). ♦

Ⓔ **<u>Exercise 15A.9</u>.** What is the physical meaning of a nonzero value of A_0 in Theorem 15A.4(d)? ♦

Theorems 15A.2, 15A.3, and 15A.4 allow us to solve I/BVPs on any domain, once we have a suitable eigenbasis. We illustrate with a simple example.

Proposition 15A.5. Eigenbases for a Triangle

Let $\mathbb{X} := \left\{(x, y) \in [0, \pi]^2 \;;\; y \le x\right\}$ be a **filled right-angle triangle** *(Figure 15A.1).*

(a) *For any two-element subset $\{n, m\} \subset \mathbb{N}$ (i.e. $n \neq m$), let $\mathcal{S}_{\{n,m\}} := \sin(nx)\sin(my) - \sin(mx)\sin(ny)$, and let $\lambda_{\{n,m\}} := n^2 + m^2$. Then:*

[i] *$\mathcal{S}_{\{n,m\}}$ is an eigenfunction of the Laplacian: $\triangle \mathcal{S}_{\{n,m\}} = -\lambda_{\{n,m\}} \mathcal{S}_{\{n,m\}}$.*

[ii] *$\{\mathcal{S}_{\{n,m\}}\}_{\{n,m\} \subset \mathbb{N}}$ is a Dirichlet eigenbasis for $\mathbf{L}^2(\mathbb{X})$.*

(b) *Let $\mathbf{C}_{0,0} = 1$, and for any two-element subset $\{n, m\} \subset \mathbb{N}$, let $\mathcal{C}_{\{n,m\}} := \cos(nx)\cos(my) + \cos(mx)\cos(ny)$, and let $\lambda_{\{n,m\}} := n^2 + m^2$. Then:*

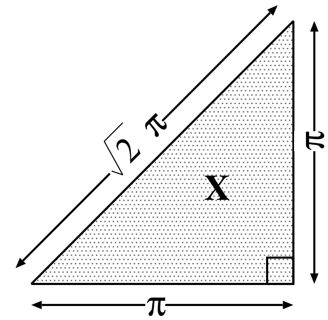


Figure 15A.1: Right-angle triangular domain of Proposition 15A.5

[i] $\mathcal{C}_{\{n,m\}}$ *is an eigenfunction of the Laplacian:* $\triangle\mathcal{C}_{\{n,m\}} = -\lambda_{\{n,m\}}\mathcal{C}_{\{n,m\}}$.

[ii] $\{\mathcal{C}_{\{n,m\}}\}_{\{n,m\}\subset\mathbb{N}}$ *is a Neumann eigenbasis for* $\mathbf{L}^2(\mathbb{X})$.

Proof. **Exercise 15A.10** Hint: Part [i] is a straightforward computation, as is the verification of the homogeneous boundary conditions (Hint: on the hypotenuse, $\partial_\perp = \partial_2 - \partial_1$). To verify that the specified sets are orthogonal bases, use Theorem 9A.3. □ Ⓔ

Exercise 15A.11. (a) Combine Proposition 15A.5 with Theorems 15A.2, 15A.3, and 15A.4 to provide a general solution method for solving the Poisson equation, heat equation, and wave equation on a right-angle triangle domain, with either Dirichlet or Neumann boundary conditions. Ⓔ

(b) Set up and solve some simple initial/boundary value problems using your method.

♦

Remark 15A.6. There is nothing special about the role of the Laplacian $\triangle$ in Theorems 15A.2, 15A.3, and 15A.4. If L is any *linear* differential operator, for which we have 'solution uniqueness' results analogous to the results of §5D, then Theorems 15A.2, 15A.3, and 15A.4 are still true if you replace "$\triangle$" with "L" everywhere (**Exercise 15A.12** Verify this). In particular, if L is an elliptic differential operator (see §5E), then: Ⓔ

- Theorem 15A.2 becomes the general solution to the boundary value problem for the *nonhomogeneous elliptic PDE* "$\mathsf{L}u = f$".
- Theorem 15A.3 becomes the general solution to the the initial/boundary value problem for the *homogeneous parabolic PDE* "$\partial_t u = \mathsf{L}u$".
- Theorem 15A.4 becomes the general solution to the initial value problem for the *homogeneous hyperbolic PDE* "$\partial_t^2 u = \mathsf{L}u$".

Theorem 15E.17 on page 349 (below) discusses the existence of Dirichlet eigenbases for other elliptic differential operators.

Ⓔ **Exercise 15A.13.** Let $\mathbb{X} \subset \mathbb{R}^3$ be a bounded domain, and consider a quantum particle confined to the domain $\mathbb{X}$ by an 'infinite potential well' $V : \mathbb{R}^3 \longrightarrow \mathbb{R} \cup \{\infty\}$, where $V(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{X}$, and $V(\mathbf{x}) = \infty$ for all $\mathbf{x} \notin \mathbb{X}$ (see Examples 3C.4 and 3C.5 on pages 49-50 for discussion of the physical meaning of this model). Modify Theorem 15A.3 to state and prove a theorem describing the general solution to the initial value problem for the Schrödinger equation with the potential V.

Hint. If $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ is a solution to the corresponding Schrödinger equation, then we can assume $\omega_t(\mathbf{x}) = 0$ for all $\mathbf{x} \notin \mathbb{X}$. If ω is also continuous, then we can model the particle using a function $\omega : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{C}$, which satisfies homogeneous Dirichlet boundary conditions on $\partial\mathbb{X}$. ◆

15B General solution to Laplace equation BVPs

Prerequisites: §4B(iv), §5B, §5C, §6F, §0D.

Recommended: §5D, §12A, §13B, §14B, §15A.

Theorems 15A.2(b,d) and 15A.3(b,d) both used the same strategy to solve a PDE with nonhomogeneous boundary conditions:

- Solve the original PDE with *homogeneous* boundary conditions.
- Solve the Laplace equation with the specified nonhomogeneous BC.
- Add these two solutions together to get a solution to the original problem.

However, we do not yet have a general method for solving the Laplace equation. That is the goal of this section. Throughout this section, we make the following assumptions.

- Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, whose boundary $\partial\mathbb{X}$ is piecewise smooth. This has two consequences: (1) The normal derivative on the boundary is well-defined (so we can meaningfully impose Neumann boundary conditions); and (2) We can meaningfully speak of integrating functions over $\partial\mathbb{X}$. For example, if $\mathbb{X} \subset \mathbb{R}^2$, then $\partial\mathbb{X}$ should be a finite union of smooth curves. If $\mathbb{X} \subset \mathbb{R}^3$, then $\partial\mathbb{X}$ should be a finite union of smooth surfaces, etc. If $b, c : \partial\mathbb{X} \longrightarrow \mathbb{R}$ are functions, then define

$$\langle b, c\rangle \quad := \quad \int_{\partial\mathbb{X}} b(\mathbf{x})\cdot c(\mathbf{x})\, dx \quad \text{and} \quad \|b\|_2 \quad := \quad \sqrt{\langle b, b\rangle} \;:=\; \left(\int_{\partial\mathbb{X}} |b(\mathbf{x})|^2 \; dx\right)^{1/2},$$

  where these are computed as contour integrals (or surface integrals, etc.) over $\partial\mathbb{X}$. As usual, let $\mathbf{L}^2(\partial\mathbb{X})$ be the set of all integrable functions $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ such that $\|b\|_2 < \infty$ (see §6B for further discussion).

- Let $\{\Xi_n\}_{n=1}^{\infty}$ be an orthogonal basis for $\mathbf{L}^2(\partial\mathbb{X})$. Thus, for any $b \in \mathbf{L}^2(\mathbb{X})$, we can write

$$b \underset{\mathrm{L2}}{\approx} \sum_{n=1}^{\infty} B_n \Xi_n \quad \text{where} \quad B_n \quad := \quad \frac{\langle b, \Xi_n\rangle}{\|\Xi_n\|_2^2}, \quad \text{for all } n \in \mathbb{N}. \tag{15B.1}$$

- For all $n \in \mathbb{N}$, let $\mathcal{H}_n : \mathbb{X} \longrightarrow \mathbb{R}$ be a harmonic function (i.e. $\triangle \mathcal{H}_n = 0$) satisfying the nonhomogeneous Dirichlet boundary condition $\mathcal{H}_n(\mathbf{x}) = \Xi_n(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. The system $\mathfrak{H} := \{\mathcal{H}_n\}_{n=1}^{\infty}$ is called a **Dirichlet harmonic basis** for $\mathbb{X}$.

- Suppose $\Xi_1 \equiv 1$ is the constant function. Then $\int_{\partial\mathbb{X}} \Xi_n(\mathbf{x})\, dx = \langle \Xi_n, 1\rangle = 0$, for all $n \geq 2$ (by orthogonality). For all $n \geq 2$, let $\mathcal{G}_n : \mathbb{X} \longrightarrow \mathbb{R}$ be a harmonic function (i.e. $\triangle \mathcal{G}_n = 0$) satisfying the nonhomogeneous Neumann boundary condition $\partial_\perp \mathcal{G}_n(\mathbf{x}) = \Xi_n(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$. The system $\mathfrak{G} := \{1\} \sqcup \{\mathcal{G}_n\}_{n=2}^{\infty}$ is called a **Neumann harmonic basis** for $\mathbb{X}$. (Note that $\partial_\perp 1 = 0$, *not* Ξ_1).

Note. Although they are called 'harmonic bases for $\mathbb{X}$', $\mathfrak{H}$ and $\{\partial_\perp \mathcal{G}_n\}_{n=2}^{\infty}$ are actually orthogonal bases for $\mathbf{L}^2(\partial\mathbb{X})$, *not* for $\mathbf{L}^2(\mathbb{X})$.

Exercise 15B.1. Show that there is no harmonic function $\mathcal{G}_1$ on $\mathbb{X}$ satisfying the Neumann boundary condition $\partial_\perp \mathcal{G}_1(\mathbf{x}) = 1$ for all $\mathbf{x} \in \partial\mathbb{X}$. *Hint:* Use Corollary 5D.4(b)[i] on page 87. ⓔ ♦

Example 15B.1. If $\mathbb{X} = [0,\pi]^2 \subset \mathbb{R}^2$, then $\partial\mathbb{X} = \mathbf{L} \cup \mathbf{R} \cup \mathbf{T} \cup \mathbf{B}$, where

$$\mathbf{L} := \{0\}\times[0,\pi], \quad \mathbf{R} := \{\pi\}\times[0,\pi], \quad \mathbf{B} := [0,\pi]\times\{0\}, \text{ and } \mathbf{T} := [0,\pi]\times\{\pi\}.$$

(See Figure 12A.1(B) on page 240).

(a) Let $\{\Xi_k\}_{k=1}^{\infty} := \{\mathcal{L}_n\}_{n=1}^{\infty} \sqcup \{\mathcal{R}_n\}_{n=1}^{\infty} \sqcup \{\mathcal{B}_n\}_{n=1}^{\infty} \sqcup \{\mathcal{T}_n\}_{n=1}^{\infty}$, where, for all $n \in \mathbb{N}$, the functions $\mathcal{L}_n, \mathcal{R}_n, \mathcal{B}_n, \mathcal{T}_n : \partial\mathbb{X} \longrightarrow \mathbb{R}$ are defined by

$$\mathcal{L}_n(x,y) := \begin{cases} \sin(ny) & \text{if } (x,y) \in \mathbf{L}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{R}_n(x,y) := \begin{cases} \sin(ny) & \text{if } (x,y) \in \mathbf{R}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{B}_n(x,y) := \begin{cases} \sin(nx) & \text{if } (x,y) \in \mathbf{B}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{and } \mathcal{T}_n(x,y) := \begin{cases} \sin(nx) & \text{if } (x,y) \in \mathbf{T}; \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\{\mathcal{L}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{L})$ (by Theorem 7A.1). Likewise, $\{\mathcal{R}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{R})$, $\{\mathcal{B}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{B})$, and $\{\mathcal{T}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{T})$. Thus, $\{\Xi_k\}_{k=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\partial\mathbb{X})$.

Let $\mathfrak{H} := \{\mathcal{H}_n^L\}_{n=1}^\infty \sqcup \{\mathcal{H}_n^R\}_{n=1}^\infty \sqcup \{\mathcal{H}_n^T\}_{n=1}^\infty \sqcup \{\mathcal{H}_n^B\}_{n=1}^\infty$, where for all $n \in \mathbb{N}$, and all $(x, y) \in [0, \pi]^2$, we define

$$\begin{aligned}
\mathcal{H}_n^L(x,y) &:= \frac{\sinh\left(n(\pi - x)\right)\sin(ny)}{\sinh(n\pi)};\\
\mathcal{H}_n^R(x,y) &:= \frac{\sinh(nx)\sin(ny)}{\sinh(n\pi)};\\
\mathcal{H}_n^B(x,y) &:= \frac{\sin(nx)\sinh\left(n(\pi - y)\right)}{\sinh(n\pi)};\\
\text{and}\quad \mathcal{H}_n^T(x,y) &:= \frac{\sin(nx)\sinh(ny)}{\sinh(n\pi)}.
\end{aligned}$$

(See Figures 12A.2 and 12A.3 on pages 241-242). Then $\mathfrak{H}$ is a Dirichlet harmonic basis for $\mathbb{X}$. This was the key fact employed by Proposition 12A.4 on page 244 to solve the Laplace Equation on $[0, \pi]^2$ with arbitrary nonhomogeneous Dirichlet boundary conditions.

(b) Let $\{\Xi_k\}_{k=1}^\infty := \{\Xi_1, \Xi_=, \Xi_{||}, \Xi_\diamond\} \sqcup \{\mathcal{L}_n\}_{n=1}^\infty \sqcup \{\mathcal{R}_n\}_{n=1}^\infty \sqcup \{\mathcal{B}_n\}_{n=1}^\infty \sqcup \{\mathcal{T}_n\}_{n=1}^\infty$. Here, for all $(x, y) \in \partial[0, \pi]^2$, we define

$$\begin{aligned}
\Xi_1(x,y) &:= 1;\\
\Xi_{||}(x,y) &:= \begin{cases} 1 & \text{if} \quad (x,y) \in \mathbf{R};\\ -1 & \text{if} \quad (x,y) \in \mathbf{L};\\ 0 & \text{if} \quad (x,y) \in \mathbf{B} \sqcup \mathbf{T}. \end{cases}\\
\Xi_=(x,y) &:= \begin{cases} 1 & \text{if} \quad (x,y) \in \mathbf{T};\\ -1 & \text{if} \quad (x,y) \in \mathbf{B};\\ 0 & \text{if} \quad (x,y) \in \mathbf{L} \sqcup \mathbf{R}. \end{cases}\\
\text{and} \qquad \Xi_\diamond(x,y) &:= \begin{cases} 1 & \text{if} \quad (x,y) \in \mathbf{L} \sqcup \mathbf{R};\\ -1 & \text{if} \quad (x,y) \in \mathbf{T} \sqcup \mathbf{B}. \end{cases}
\end{aligned}$$

Meanwhile, for all $n \in \mathbb{N}$, the functions $\mathcal{L}_n, \mathcal{R}_n, \mathcal{B}_n, \mathcal{T}_n : \partial\mathbb{X} \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned}
\mathcal{L}_n(x,y) &:= \begin{cases} \cos(ny) & \text{if } (x,y) \in \mathbf{L};\\ 0 & \text{otherwise.} \end{cases}\\
\mathcal{R}_n(x,y) &:= \begin{cases} \cos(ny) & \text{if } (x,y) \in \mathbf{R};\\ 0 & \text{otherwise.} \end{cases}\\
\mathcal{B}_n(x,y) &:= \begin{cases} \cos(nx) & \text{if } (x,y) \in \mathbf{B};\\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\text{and} \quad \mathcal{T}_n(x,y) \quad := \quad \begin{cases} \cos(nx) & \text{if } (x,y) \in \mathbf{T}; \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\{1\} \sqcup \{\mathcal{L}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{L})$ (by Theorem 7A.1). Likewise, $\{1\} \sqcup \{\mathcal{R}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{R})$, $\{1\} \sqcup \{\mathcal{B}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{B})$, and $\{1\} \sqcup \{\mathcal{T}_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbf{T})$. It follows that $\{\Xi_k\}_{k=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\partial\mathbb{X})$ (**Exercise 15B.2**). Ⓔ

Let $\mathfrak{G} := \{1, \mathcal{G}_=, \mathcal{G}_{||}, \mathcal{G}_\diamond\} \sqcup \{\mathcal{G}_n^L\}_{n=1}^\infty \sqcup \{\mathcal{G}_n^R\}_{n=1}^\infty \sqcup \{\mathcal{G}_n^B\}_{n=1}^\infty \sqcup \{\mathcal{G}_n^T\}_{n=1}^\infty$, where, for all $(x,y) \in [0,\pi]^2$,

$$\begin{aligned} \mathcal{G}_{||}(x,y) &:= x; \\ \mathcal{G}_=(x,y) &:= y; \\ \text{and} \quad \mathcal{G}_\diamond(x,y) &:= \frac{1}{\pi}\left(\left(x - \frac{\pi}{2}\right)^2 - \left(y - \frac{\pi}{2}\right)^2\right). \end{aligned}$$

The graphs of $\mathcal{G}_{||}(x,y)$ and $\mathcal{G}_=(x,y)$ are inclined planes at 45^o in the x and y directions respectively. The graph of $\mathcal{G}_\diamond$ is a 'saddle' shape very similar to Figure 1C.1(B) on page 10. Meanwhile, for all $n \geq 1$, and all $(x,y) \in [0,\pi]^2$, we define

$$\begin{aligned} \mathcal{G}_n^L(x,y) &:= \frac{\cosh(n(\pi - x))\cos(ny)}{n\sinh(n\pi)}; \\ \mathcal{G}_n^R(x,y) &:= \frac{\cosh(nx)\cos(ny)}{n\sinh(n\pi)}; \\ \mathcal{G}_n^B(x,y) &:= \frac{\cos(nx)\cosh(n(\pi - y))}{n\sinh(n\pi)}; \\ \text{and} \quad \mathcal{G}_n^T(x,y) &:= \frac{\cos(nx)\cosh(ny)}{n\sinh(n\pi)}. \end{aligned}$$

Then $\mathfrak{G}$ is a Neumann harmonic basis for $\mathbb{X}$ (**Exercise 15B.3**). $\diamond$ Ⓔ

Example 15B.2. (a) If $\mathbb{X} = \mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ (the unit disk in polar coordinates), then $\partial\mathbb{X} = \mathbb{S} = \{(r,\theta)\ ;\ r = 1\}$ (the unit circle). In this case, let $\{\Xi_k\}_{k=1}^\infty := \{\mathcal{C}_n\}_{n=0}^\infty \sqcup \{\mathcal{S}_n\}_{n=1}^\infty$, where, for all $n \in \mathbb{N}$ and $\theta \in [-\pi,\pi)$,

$$\mathcal{C}_n(\theta,1) \quad := \quad \cos(n\theta) \qquad \text{and} \qquad \mathcal{S}_n(\theta,1) \quad := \quad \sin(n\theta).$$

Then $\{\Xi_k\}_{k=1}^\infty$ is a basis of $\mathbf{L}^2(\mathbb{S})$, by Theorem 8A.1. Let $\mathfrak{H} := \{\Phi_n\}_{n=0}^\infty \sqcup \{\Psi_n\}_{n=1}^\infty$, where $\Phi_0 \equiv 1$, and where, for all $n \geq 1$ and $(r,\theta) \in \mathbb{D}$, we define

$$\Phi_n(r,\theta) \ := \ \cos(n\theta)\cdot r^n \qquad \text{and} \qquad \Psi_n(r,\theta) \ := \ \sin(n\theta)\cdot r^n.$$

(See Figure 14B.1 on page 275). Then $\mathfrak{H}$ is a Dirichlet harmonic basis for $\mathbb{D}$; this was the key fact employed by Proposition 14B.2 on page 278, to solve the

Laplace Equation on $\mathbb{D}$ with arbitrary nonhomogeneous Dirichlet boundary conditions.

Suppose $\Xi_1 = \mathcal{C}_0$ (i.e. $\Xi_1 \equiv 1$). Let $\mathfrak{G} := \{1\} \sqcup \{\Phi_n/n\}_{n=1}^{\infty} \sqcup \{\Psi_n/n\}_{n=1}^{\infty}$, where, for all $n \in \mathbb{N}$ and $(r,\theta) \in \mathbb{D}$, we have

$$\Phi_n(r,\theta)/n \;:=\; \frac{\cos(n\theta)\cdot r^n}{n} \qquad \text{and} \qquad \Psi_n(r,\theta)/n \;:=\; \frac{\sin(n\theta)\cdot r^n}{n}.$$

Then $\mathfrak{G}$ is a Neumann harmonic basis for $\mathbb{D}$; this was the key fact employed by Proposition 14B.4 on page 280, to solve the Laplace Equation on $\mathbb{D}$ with arbitrary nonhomogeneous Neumann boundary conditions.

(b) If $\mathbb{X} = \mathbb{D}^{\complement} = \{(r,\theta)\ ;\ r \geq 1\}$ (in polar coordinates)[1], then $\partial\mathbb{X} = \mathbb{S} = \{(r,\theta)\ ;\ r = 1\}$. In this case, let $\{\Xi_k\}_{k=1}^{\infty} := \{\mathcal{C}_n\}_{n=0}^{\infty} \sqcup \{\mathcal{S}_n\}_{n=1}^{\infty}$, just as in Example (a). However, this time, let $\mathfrak{H} := \{\Phi_0\} \sqcup \{\phi_n\}_{n=1}^{\infty} \sqcup \{\psi_n\}_{n=1}^{\infty}$, where $\Phi_0 \equiv 1$, and where, for all $n \geq 1$ and $(r,\theta) \in \mathbb{D}$, we define

$$\phi_n(r,\theta) \;:=\; \cos(n\theta)/r^n \qquad \text{and} \qquad \psi_n(r,\theta) \;:=\; \sin(n\theta)/r^n.$$

(See Figure 14B.2 on page 276). Then $\mathfrak{H}$ is a Dirichlet harmonic basis for $\mathbb{D}^{\complement}$; this was the key fact employed by Proposition 14B.6 on page 284, to solve the Laplace Equation on $\mathbb{D}^{\complement}$ with arbitrary nonhomogeneous Neumann boundary conditions.

Recall $\Xi_1 = \mathcal{C}_0 \equiv 1$. Let $\mathfrak{G} := \{1\} \sqcup \{-\phi_n/n\}_{n=1}^{\infty} \sqcup \{-\psi_n/n\}_{n=1}^{\infty}$, where ϕ_n and ψ_n are as defined above, for all $n \geq 1$. Then $\mathfrak{G}$ is a Neumann harmonic basis for $\mathbb{D}^{\complement}$; this was the key fact employed by Proposition 14B.8 on page 285, to solve the Laplace Equation on $\mathbb{D}^{\complement}$ with arbitrary nonhomogeneous Neumann boundary conditions.[2] $\diamondsuit$

Theorem 15B.3. General solution to Laplace equation
Let $b \in \mathbf{L}^2(\partial\mathbb{X})$ have orthogonal expansion (15B.1). *Let $\mathfrak{H} := \{\mathcal{H}_k\}_{k=1}^{\infty}$ be a Dirichlet harmonic basis for $\mathbb{X}$ and let $\mathfrak{G} := \{1\} \sqcup \{\mathcal{G}_k\}_{k=2}^{\infty}$ be a Neumann harmonic basis for $\mathbb{X}$.*

[1] Technically, we are here developing a theory for *bounded* domains, and $\mathbb{D}^{\complement}$ is obviously not bounded. But it is interesting to note that many our techniques still apply to $\mathbb{D}^{\complement}$. This is because $\mathbb{D}^{\complement}$ is *conformally isomorphic* to a bounded domain, once we regard $\mathbb{D}^{\complement}$ as a subset of the Riemann sphere by including the 'point at infinity'. See §18B on page 422 for an introduction to conformal isomorphism. See Remark 18G.4 on page 469 for a discussion of the Riemann sphere.

[2] Note that our Neumann harmonic basis does not include the element $\phi_0(r,\theta) \;:=\; \log(r)$. This is because $\partial_\perp \phi_0 = \Xi_1$. Of course, the domain $\mathbb{D}^{\complement}$ is not bounded, so Corollary 5D.4(b)[i] does not apply, and indeed ϕ_0 is a continuous harmonic function on $\mathbb{D}^{\complement}$. However, unlike the elements of $\mathfrak{G}$, the function ϕ_0 is not bounded, and thus does *not* extend to a continuous real-valued harmonic function when we embed $\mathbb{D}^{\complement}$ in the Riemann sphere by adding the 'point at infinity'.

(a) *Let* $u \underset{L2}{\approx} \sum_{k=1}^{\infty} B_k \mathcal{H}_k$. *If this series converges uniformly to* u *on the interior of* $\mathbb{X}$, *then* u *is the unique continuous harmonic function with nonhomogeneous Dirichlet BC* $u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$.

(b) *Suppose* $\Xi_1 \equiv 1$. *If* $B_1 \neq 0$, *then there is no continuous harmonic function on* $\mathbb{X}$ *with nonhomogeneous Neumann BC* $\partial_\perp u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$.

Suppose $B_1 = 0$. *Let* $u \underset{L2}{\approx} \sum_{k=2}^{\infty} B_k \mathcal{G}_k + C$, *where* $C \in \mathbb{R}$ *is any constant. If this series converges uniformly to* u *on the interior of* $\mathbb{X}$, *then it is a continuous harmonic function with nonhomogeneous Neumann BC* $\partial_\perp u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$. *Furthermore, all solutions to this BVP have this form, for some value of* $C \in \mathbb{R}$.

Proof. **Exercise 15B.4** Hint: The boundary conditions follow from expansion (15B.1). To verify that u is harmonic, use the Mean Value Theorem (Theorem 1E.1 on page 16). (Use Proposition 6E.10(b) on page 127 to guarantee that the integral of the sum is the sum of the integrals.) Finally, use Corollary 5D.4 on page 87 to show solution uniqueness. □ Ⓔ

Exercise 15B.5. Show how Propositions 12A.4, 13B.2, 14B.2, 14B.4 14B.6, 14B.8 and 14B.10 are all special cases of Theorem 15B.3. ♦ Ⓔ

Remark. There is nothing special about the role of the Laplacian $\triangle$ in Theorem 15B.3. If L is any *linear* differential operator, then something like Theorem 15B.3 is still true if you replace "$\triangle$" with "L" everywhere. In particular, if L is an *elliptic* differential operator (see §5E), then Theorem 15B.3 becomes the general solution to the boundary value problem for the *homogeneous elliptic PDE* "$\mathsf{L}\, u \equiv 0$".

However, if L is an arbitrary differential operator, then there is no guarantee that you will find a 'harmonic basis' $\{\mathcal{H}_k\}_{k=1}^{\infty}$ of functions such that $\mathsf{L}\, \mathcal{H}_k \equiv 0$ for all $k \in \mathbb{N}$, and such that the collection $\{\mathcal{H}_k\}_{k=1}^{\infty}$ (or $\{\partial_\perp \mathcal{H}_k\}_{k=1}^{\infty}$) provides an orthonormal basis for $\mathbf{L}^2(\partial\mathbb{X})$. (Even for the Laplacian, this is a nontrivial problem; see e.g. Corollary 15C.8 on page 333 below.)

Furthermore, once you define $u \underset{L2}{\approx} \sum_{k=1}^{\infty} B_k \mathcal{H}_k$ as in Theorem 15B.3, you might not be able to use something like the Mean Value Theorem to guarantee that $\mathsf{L}\, u = 0$. Instead you must 'formally differentiate' the series $\sum_{k=1}^{\infty} B_k \mathcal{H}_k$ and $\sum_{k=1}^{\infty} B_k \mathcal{G}_k$ using Proposition 0F.1 on page 565. For this to work, you need some convergence conditions on the 'formal derivatives' of these series. For example,

if L was an Nth order differential operator, it would be sufficient to require that $\sum_{k=1}^{\infty} |B_k| \cdot \left\|\partial_j^N \mathcal{H}_k\right\|_{\infty} < \infty$ and $\sum_{k=1}^{\infty} |B_k| \cdot \left\|\partial_j^N \mathcal{G}_k\right\|_{\infty} < \infty$ for all $j \in [1...D]$
Ⓔ (**Exercise 15B.6** Verify this).

Finally, for an arbitrary differential operator, there may not be a result like Corollary 5D.4 on page 87, which guarantees a unique solution to a Dirichlet/Neumann BVP. It may be necessary to impose further constraints to get a unique solution.

15C Eigenbases on Cartesian products

Prerequisites: §4B(iv), §5B, §5C, §6F, §0D.

If $\mathbb{X}_1 \subset \mathbb{R}^{D_1}$ and $\mathbb{X}_2 \subset \mathbb{R}^{D_2}$ are two domains, then their **Cartesian product** is the set

$$\mathbb{X}_1 \times \mathbb{X}_2 \quad := \quad \{(\mathbf{x}_1, \mathbf{x}_2)\ ;\ \mathbf{x}_1 \in \mathbb{X}_1 \ \text{ and } \ \mathbf{x}_2 \in \mathbb{X}_2\} \quad \subset \quad \mathbb{R}^{D_1+D_2}.$$

Example 15C.1. (a) if $\mathbb{X}_1 = [0,\pi] \subset \mathbb{R}$ and $\mathbb{X}_2 = [0,\pi]^2 \subset \mathbb{R}^2$ then $\mathbb{X}_1 \times \mathbb{X}_2 = [0,\pi]^3 \subset \mathbb{R}^3$.

(b) If $\mathbb{X}_1 = \mathbb{D} \subset \mathbb{R}^2$ and $\mathbb{X}_2 = [0,\pi] \subset \mathbb{R}$, then $\mathbb{X}_1 \times \mathbb{X}_2 = \{(r,\theta,z)\ ;\ (r,\theta) \in \mathbb{D} \ \text{ and } \ 0 \le z \le \pi\} \subset \mathbb{R}^3$ is the **cylinder** of height π. ◊

To apply the solution methods from Sections 15A and 15B, we must first construct eigenbases and/or harmonic bases on the domain $\mathbb{X}$; that is the goal of this section. We begin with some technical results which are useful and straightforward to prove.

Lemma 15C.2. *Let $\mathbb{X}_1 \subset \mathbb{R}^{D_1}$ and $\mathbb{X}_2 \subset \mathbb{R}^{D_2}$. Let $\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{D_1+D_2}$.*

(a) $\partial\mathbb{X} \ = \ [(\partial\mathbb{X}_1) \times \mathbb{X}_2] \cup [\mathbb{X}_1 \times (\partial\mathbb{X}_2)]$.

Let $\Phi_1 : \mathbb{X}_1 \longrightarrow \mathbb{R}$ and $\Phi_2 : \mathbb{X}_2 \longrightarrow \mathbb{R}$, and define $\Phi = \Phi_1 \cdot \Phi_2 : \mathbb{X} \longrightarrow \mathbb{R}$ by $\Phi(\mathbf{x}_1, \mathbf{x}_2) := \Phi_1(\mathbf{x}_1) \cdot \Phi_2(\mathbf{x}_2)$ for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{X}$.

(b) *If Φ_1 satisfies homogeneous Dirichlet BC on $\mathbb{X}_1$ and Φ_2 satisfies homogeneous Dirichlet BC on $\mathbb{X}_2$, then Φ satisfies homogeneous Dirichlet BC on $\mathbb{X}$.*

(c) *If Φ_1 satisfies homogeneous Neumann BC on $\mathbb{X}_1$ and Φ_2 satisfies homogeneous Neumann BC on $\mathbb{X}_2$, then Φ satisfies homogeneous Neumann BC on $\mathbb{X}$.*

(d) $\|\Phi\|_2 \;=\; \|\Phi_1\|_2 \cdot \|\Phi_2\|_2$. *Thus, if* $\Phi_1 \in \mathbf{L}^2(\mathbb{X}_1)$ *and* $\Phi_2 \in \mathbf{L}^2(\mathbb{X}_2)$ *then* $\Phi \in \mathbf{L}^2(\mathbb{X})$.

(e) *If* $\Psi_1 \in \mathbf{L}^2(\mathbb{X}_1)$ *and* $\Psi_2 \in \mathbf{L}^2(\mathbb{X}_2)$ *and* $\Psi = \Psi_1 \cdot \Psi_2$ *then* $\langle \Phi, \Psi \rangle \;=\; \langle \Phi_1, \Psi_1 \rangle \cdot \langle \Phi_2, \Psi_2 \rangle$.

(f) *Let* $\{\Phi_n^{(1)}\}_{n=1}^\infty$ *be an orthogonal basis for* $\mathbf{L}^2(\mathbb{X}_1)$ *and let* $\{\Phi_m^{(2)}\}_{m=1}^\infty$ *be an orthogonal basis for* $\mathbf{L}^2(\mathbb{X}_2)$. *For all* $(n,m) \in \mathbb{N}$, *let* $\Phi_{n,m} := \Phi_n^{(1)} \cdot \Phi_m^{(2)}$. *Then* $\{\Phi_{n,m}\}_{n,m=1}^\infty$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{X})$.

Let $\triangle_1$ *be the Laplacian operator on* $\mathbb{R}^{D_1}$, *let* $\triangle_2$ *be the Laplacian operator on* $\mathbb{R}^{D_2}$, *and let* $\triangle$ *be the Laplacian operator on* $\mathbb{R}^{D_1+D_2}$.

(g) $\triangle\Phi(\mathbf{x}_1, \mathbf{x}_2) \;=\; \Big(\triangle_1 \, \Phi_1(\mathbf{x}_1)\Big) \cdot \Phi_2(\mathbf{x}_2) \;+\; \Phi_1(\mathbf{x}_1) \cdot \Big(\triangle_2 \, \Phi_2(\mathbf{x}_2)\Big)$.

(h) *Thus, if* Φ_1 *is an eigenfunction of* $\triangle_1$ *with eigenvalue* λ_1, *and* Φ_2 *is an eigenfunction of* $\triangle_2$ *with eigenvalue* λ_2, *then* Φ *is an eigenfunction of* $\triangle$ *with eigenvalue* $(\lambda_1 + \lambda_2)$.

Proof. **Exercise 15C.1** (Remark: For part (f), just show that $\{\Phi_{n,m}\}_{n,m=1}^\infty$ is an orthogonal collection of functions. Showing that $\{\Phi_{n,m}\}_{n,m=1}^\infty$ is actually a *basis* for $\mathbf{L}^2(\mathbb{X})$ requires methods beyond the scope of this course.) □ Ⓔ

Corollary 15C.3. Eigenbases for Cartesian Products
Let $\mathbb{X}_1 \subset \mathbb{R}^{D_1}$ *and* $\mathbb{X}_2 \subset \mathbb{R}^{D_2}$. *Let* $\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{D_1+D_2}$. *Let* $\{\Phi_n^{(1)}\}_{n=1}^\infty$ *be a Dirichlet* (or Neumann) *eigenbasis for* $\mathbf{L}^2(\mathbb{X}_1)$, *and let* $\{\Phi_m^{(2)}\}_{m=1}^\infty$ *be a Dirichlet* (respectively Neumann) *eigenbasis for* $\mathbf{L}^2(\mathbb{X}_2)$. *For all* $(n,m) \in \mathbb{N}$, *define* $\Phi_{n,m} = \Phi_n^{(1)} \cdot \Phi_m^{(2)}$. *Then* $\{\Phi_{n,m}\}_{n,m=1}^\infty$ *Dirichlet* (respectively Neumann) *eigenbasis for* $\mathbf{L}^2(\mathbb{X})$.

Proof. **Exercise 15C.2** Just combine Lemma 15C.2(b,c,f,h). □ Ⓔ

Example 15C.4. Let $\mathbb{X}_1 = [0,\pi]$ and $\mathbb{X}_2 = [0,\pi]^2$, so $\mathbb{X}_1 \times \mathbb{X}_2 = [0,\pi]^3$. Note that $\partial([0,\pi]^3) \;=\; \Big(\{0,\pi\} \times [0,\pi]^2\Big) \cup \Big([0,\pi] \times \partial[0,\pi]^2\Big)$. For all $\ell \in \mathbb{N}$, define $\mathbf{C}_\ell$ and $\mathbf{S}_\ell \in \mathbf{L}^2[0,\pi]$ by $\mathbf{C}_\ell(x) := \cos(\ell x)$ and $\mathbf{S}_\ell(x) := \sin(\ell x)$. For all $m, n \in \mathbb{N}$, define $\mathbf{C}_{m,n}$ and $\mathbf{S}_{m,n} \in \mathbf{L}^2([0,\pi]^2)$ by $\mathbf{C}_{m,n}(y,z) := \cos(my)\cos(nz)$ and $\mathbf{S}_{m,n}(y,z) := \sin(my)\sin(nz)$.

For any $\ell, m, n \in \mathbb{N}$, define $\mathbf{C}_{\ell,m,n}$ and $\mathbf{S}_{\ell,m,n} \in \mathbf{L}^2(\mathbb{X})$ by $\mathbf{C}_{\ell,m,n}(x,y,z) := \mathbf{C}_\ell(x) \cdot \mathbf{C}_{m,n}(y,z) = \cos(\ell x)\cos(my)\cos(nz)$ and $\mathbf{S}_{\ell,m,n}(x,y,z) := \mathbf{S}_\ell(x) \cdot \mathbf{S}_{m,n}(y,z) = \sin(\ell x)\sin(my)\sin(nz)$.

Now, $\{\mathbf{S}_\ell\}_{\ell=1}^{\infty}$ is a Dirichlet eigenbasis for $[0,\pi]$ (by Theorem 7A.1), and $\{\mathbf{S}_{m,n}\}_{m,n=1}^{\infty}$ is a Dirichlet eigenbasis for $[0,\pi]^2$ (by Theorem 9A.3(a)); thus, Corollary 15C.3 says that $\{\mathbf{S}_{\ell,m,n}\}_{\ell,m,n=1}^{\infty}$ is a Dirichlet eigenbasis for $[0,\pi]^3$ (as earlier noted by Theorem 9B.1).

Likewise, $\{\mathbf{C}_\ell\}_{\ell=0}^{\infty}$ is a Neumann eigenbasis for $[0,\pi]$ (by Theorem 7A.4), and $\{\mathbf{C}_{m,n}\}_{m,n=0}^{\infty}$ is a Neumann eigenbasis for $[0,\pi]^2$; (by Theorem 9A.3(b)); thus, Corollary 15C.3 says that $\{\mathbf{C}_{\ell,m,n}\}_{\ell,m,n=0}^{\infty}$ is a Neumann eigenbasis for $[0,\pi]^3$ (as earlier noted by Theorem 9B.1). $\diamondsuit$

Example 15C.5. Let $\mathbb{X}_1 = \mathbb{D}$ and $\mathbb{X}_2 = [0,\pi]$, so that $\mathbb{X}_1 \times \mathbb{X}_2$ is the cylinder of height π and radius 1. Let $\mathbb{S} := \partial\mathbb{D}$ (the unit circle). Note that $\partial\mathbb{X} = \Big(\mathbb{S}\times[0,\pi]\Big) \cup \Big(\mathbb{D}\times\{0,\pi\}\Big)$. For all $n \in \mathbb{N}$, define $\mathbf{S}_n \in \mathbf{L}^2[0,\pi]$ as in Example 15C.4. For all $\ell, m \in \mathbb{N}$, let $\Phi_{\ell,m}$ and $\Psi_{\ell,m}$ be the type-1 Fourier-Bessel eigenfunctions defined by eqn.(14C.5) on page 296 of §14C(ii). For any $\ell, m, n \in \mathbb{N}$, define $\Phi_{\ell,m,n}$ and $\Psi_{\ell,m,n} \in \mathbf{L}^2(\mathbb{X})$ by $\Phi_{\ell,m,n}(r,\theta,z) := \Phi_{\ell,m}(r,\theta)\cdot\mathbf{S}_n(z)$ and $\Psi_{\ell,m,n}(r,\theta,z) := \Psi_{\ell,m}(r,\theta)\cdot\mathbf{S}_n(z)$.

Now $\{\Phi_{m,n}, \Psi_{m,n}\}_{m,n=1}^{\infty}$ is a Dirichlet eigenbasis for the disk $\mathbb{D}$ (by Theorem 14C.2) and $\{\mathbf{S}_n\}_{n=1}^{\infty}$ is a Dirichlet eigenbasis for the line $[0,\pi]$ (by Theorem 7A.1); thus, Corollary 15C.3 says that $\{\Phi_{\ell,m,n}, \Psi_{\ell,m,n}\}_{\ell,m,n=1}^{\infty}$ is a Dirichlet eigenbasis for the cylinder $\mathbb{X}$. $\diamondsuit$

Ⓔ **Exercise 15C.3.** (a) Combine Example 15C.5 with Theorems 15A.2, 15A.3, and 15A.4 to provide a general solution method for solving the Poisson equation, heat equation, and wave equation on a finite cylinder with Dirichlet boundary conditions.

(b) Set up and solve some simple initial/boundary value problems using your method.

♦

Ⓔ **Exercise 15C.4.** In cylindrical coordinates on $\mathbb{R}^3$, let $\mathbb{X} = \{(r,\theta,z);\ 1 \le r,\ 0 \le z \le \pi, \text{ and } -\pi \le \theta < \pi\}$ be the **punctured slab** of thickness π, having a cylindrical hole of radius 1.

(a) Express $\mathbb{X}$ as a Cartesian product of the punctured plane and a line segment.

(b) Use Corollary 15C.3 to obtain a Dirichlet eigenbasis for $\mathbb{X}$.

(c) Apply Theorems 15A.2, 15A.3, and 15A.4 to provide a general solution method for solving the Poisson equation, heat equation, and wave equation on the punctured slab with Dirichlet boundary conditions.

(d) Set up and solve some simple initial/boundary value problems using your method.

♦

Ⓔ **Exercise 15C.5.** Let $\mathbb{X}_1 = \Big\{(x,y) \in [0,\pi]^2\ ;\ y \le x\Big\}$ be the *right angle triangle* from Proposition 15A.5 on page 322, and let $\mathbb{X}_2 = [0,\pi] \subset \mathbb{R}$. Then $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ is a **right-angle triangular prism**.

(a) Use Proposition 15A.5 and Corollary 15C.3 to obtain Dirichlet and Neumann eigenbases for the prism $\mathbb{X}$.

(b) Apply Theorems 15A.2, 15A.3, and 15A.4 to provide a general solution method for solving the Poisson equation, heat equation, and wave equation on the prism with Dirichlet or Neumann boundary conditions.

(c) Set up and solve some simple initial/boundary value problems using your method.

♦

We now move on to the problem of constructing harmonic bases on a Cartesian product. We will need two technical lemmas.

Lemma 15C.6. Harmonic functions on Cartesian products

Let $\mathbb{X}_1 \subset \mathbb{R}^{D_1}$ *and* $\mathbb{X}_2 \subset \mathbb{R}^{D_2}$. *Let* $\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{D_1+D_2}$.

Let $\mathcal{E}_1 : \mathbb{X}_1 \longrightarrow \mathbb{R}$ *be an eigenfunction of* $\triangle_1$ *with eigenvalue* λ, *and let* $\mathcal{E}_2 : \mathbb{X}_2 \longrightarrow \mathbb{R}$ *be an eigenfunction of* $\triangle_2$ *with eigenvalue* $-\lambda$. *If we define* $\mathcal{H} := \mathcal{E}_1 \cdot \mathcal{E}_2 : \mathbb{X} \longrightarrow \mathbb{R}$, *as in Lemma 15C.2, then* $\mathcal{H}$ *is a harmonic function —that is,* $\triangle\mathcal{H} = 0$.

Proof. **<u>Exercise 15C.6</u>** Hint: Use Lemma 15C.2(h). □ Ⓔ

Lemma 15C.7. Orthogonal bases on almost-disjoint unions

Let $\mathbb{Y}_1, \mathbb{Y}_2 \subset \mathbb{R}^D$ *be two* $(D-1)$*-dimensional subsets* (e.g. two curves in $\mathbb{R}^2$, two surfaces in $\mathbb{R}^3$, etc.). *Suppose that* $\mathbb{Y}_1 \cap \mathbb{Y}_2$ *has dimension* $(D-2)$ (e.g. it is a discrete set of points in $\mathbb{R}^2$, or a curve in $\mathbb{R}^3$, etc.). *Let* $\{\Phi_n^{(1)}\}_{n=1}^{\infty}$ *be an orthogonal basis for* $\mathbf{L}^2(\mathbb{Y}_1)$, *such that* $\Phi_n^{(1)}(\mathbf{y}) = 0$ *for all* $\mathbf{y} \in \mathbb{Y}_2$ *and* $n \in \mathbb{N}$. *Likewise, let* $\{\Phi_n^{(2)}\}_{n=1}^{\infty}$ *be an orthogonal basis for* $\mathbf{L}^2(\mathbb{Y}_2)$, *such that* $\Phi_n^{(2)}(\mathbf{y}) = 0$ *for all* $\mathbf{y} \in \mathbb{Y}_1$ *and* $n \in \mathbb{N}$. *Then* $\{\Phi_n^{(1)}\}_{n=1}^{\infty} \sqcup \{\Phi_n^{(2)}\}_{n=1}^{\infty}$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{Y}_1 \cup \mathbb{Y}_2)$.

Proof. **<u>Exercise 15C.7</u>** Hint: the $(D-1)$-dimensional integral of any function on $\mathbb{Y}_1 \cap \mathbb{Y}_2$ must be zero. □ Ⓔ

For the rest of this section we adopt the following notational convention: if $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a function, then let $\widetilde{f}$ denote the restriction of f to a function $\widetilde{f} : \partial\mathbb{X} \longrightarrow \mathbb{R}$ (that is, $\widetilde{f} := f_{|\partial\mathbb{X}}$).

Corollary 15C.8. Harmonic bases on Cartesian products

Let $\mathbb{X}_1 \subset \mathbb{R}^{D_1}$ *and* $\mathbb{X}_2 \subset \mathbb{R}^{D_2}$. *Let* $\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{D_1+D_2}$.

Let $\{\Xi_m^2\}_{m\in\mathbb{M}_2}$ *be an orthogonal basis for* $\mathbf{L}^2(\partial\mathbb{X}_2)$ (here, $\mathbb{M}_2$ is some indexing set, either finite or infinite; e.g. $\mathbb{M}_2 = \mathbb{N}$). *Let* $\{\mathcal{E}_n^1\}_{n=1}^{\infty}$ *be a Dirichlet eigenbasis for* $\mathbb{X}_1$. *For all* $n \in \mathbb{N}$, *suppose* $\triangle_1\mathcal{E}_n^1 = -\lambda_n^{(1)}\mathcal{E}_n^1$, *and for all* $m \in \mathbb{M}_2$, *let*

$\mathcal{F}^2_{n,m} \in \mathbf{L}^2(\mathbb{X}_2)$ be an eigenfunction of $\triangle_2$ with eigenvalue $+\lambda^{(1)}_n$, such that $\widetilde{\mathcal{F}}^2_{n,m} = \Xi^2_m$. Let $\mathcal{H}^1_{n,m} := \mathcal{E}^1_n \cdot \mathcal{F}^2_{n,m} : \mathbb{X} \longrightarrow \mathbb{R}$, for all $n \in \mathbb{N}$ and $m \in \mathbb{M}_2$.

Likewise, let $\{\Xi^1_m\}_{m\in\mathbb{M}_1}$ be an orthogonal basis for $\mathbf{L}^2(\partial\mathbb{X}_1)$ (where $\mathbb{M}_1$ is some indexing set), *and let $\{\mathcal{E}^2_n\}_{n=1}^{\infty}$ be a Dirichlet eigenbasis for $\mathbb{X}_2$. For all $n \in \mathbb{N}$, suppose $\triangle_2\mathcal{E}^2_n = -\lambda^{(2)}_n \mathcal{E}^2_n$, and for all $m \in \mathbb{M}_1$, let $\mathcal{F}^1_{n,m} \in \mathbf{L}^2(\mathbb{X}_1)$ be an eigenfunction of $\triangle_1$ with eigenvalue $+\lambda^{(2)}_n$, such that $\widetilde{\mathcal{F}}^1_{n,m} = \Xi^1_m$. Define $\mathcal{H}^2_{n,m} := \mathcal{F}^1_{n,m} \cdot \mathcal{E}^2_n : \mathbb{X} \longrightarrow \mathbb{R}$, for all $n \in \mathbb{N}$ and $m \in \mathbb{M}_1$.*

Then $\mathfrak{H} := \{\mathcal{H}^1_{n,m} \ ; \ n \in \mathbb{N}, \ m \in \mathbb{M}_2\} \sqcup \{\mathcal{H}^2_{n,m}; \ n \in \mathbb{N}, \ m \in \mathbb{M}_1\}$ is a Dirichlet harmonic basis for $\mathbf{L}^2(\partial\mathbb{X})$.

Ⓔ *Proof.* **Exercise 15C.8** (a) Use Lemma 15C.6 to verify that all the functions $\mathcal{H}^1_{n,m}$ and $\mathcal{H}^2_{n,m}$ are harmonic on $\mathbb{X}$.

(b) Show that $\{\widetilde{\mathcal{H}}^1_{n,m}\}_{n\in\mathbb{N},m\in\mathbb{M}_2}$ is an orthogonal basis for $\mathbf{L}^2\Big(\mathbb{X}_1 \times (\partial\mathbb{X}_2)\Big)$, while $\{\widetilde{\mathcal{H}}^2_{n,m}\}_{n\in\mathbb{N},m\in\mathbb{M}_1}$ is an orthogonal basis for $\mathbf{L}^2\Big((\partial\mathbb{X}_1) \times \mathbb{X}_2\Big)$. Use Lemma 15C.2(f).

(c) Show that $\mathfrak{H}$ is an orthogonal basis for $\mathbf{L}^2(\partial\mathbb{X})$. Use Lemma 15C.2(a) and Lemma 15C.7. □

Example 15C.9. Let $\mathbb{X}_1 = [0,\pi] = \mathbb{X}_2$, so that $\mathbb{X} = [0,\pi]^2$. Observe that $\partial\Big([0,\pi]^2\Big) = \Big(\{0,\pi\} \times [0,\pi]\Big) \cup \Big([0,\pi] \times \{0,\pi\}\Big)$.

Observe that $\partial\mathbb{X}_1 = \{0,\pi\} = \partial\mathbb{X}_2$ (a two-element set), and $\mathbf{L}^2\{0,\pi\}$ is 2-dimensional vector space (isomorphic to $\mathbb{R}^2$). Let $\mathbb{M}_1 := \{1,2\} =: \mathbb{M}_2$. Let $\Xi^1_1 = \Xi^2_1 = \Xi_1$ and $\Xi^1_2 = \Xi^2_2 = \Xi_2$, where $\Xi_1, \Xi_2 : \{0,\pi\} \longrightarrow \mathbb{R}$ are defined:

$$\Xi_2(0) \ := \ 1 \ =: \ \Xi_1(\pi), \qquad \text{and} \qquad \Xi_2(\pi) \ := \ 0 \ =: \ \Xi_1(0).$$

Then $\{\Xi_1, \Xi_2\}$ is an orthogonal basis for $\mathbf{L}^2\{0,\pi\}$. For all $n \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{E}^1_n(x) = \mathcal{E}^2_n(x) = \mathcal{E}_n(x) &\ := \ \sin(nx). \\ \mathcal{F}^1_{n,1}(x) = \mathcal{F}^2_{n,1}(x) = \mathcal{F}_{n,1}(x) &\ := \ \sinh(nx)/\sinh(n\pi) \\ \mathcal{F}^1_{n,2}(x) = \mathcal{F}^2_{n,2}(x) = \mathcal{F}_{n,2}(x) &\ := \ \sinh\left(n(\pi - x)\right)/\sinh(n\pi). \end{aligned}$$

Then $\{\mathcal{E}_n\}_{n=1}^{\infty}$ is a Dirichlet eigenbasis for $[0,\pi]$ (by Theorem 7A.1), while $\widetilde{\mathcal{F}}_{n,1} = \Xi_1$ and $\widetilde{\mathcal{F}}_{n,2} = \Xi_2$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we have eigenvalue $\lambda_n := n^2$. That is $\triangle\mathcal{E}_n(x) = -n^2\mathcal{E}_n(x)$ while $\triangle\mathcal{F}_{n,m}(x) = n^2\mathcal{F}_{n,m}(x)$. Thus, the functions $\mathcal{H}_n(x,y) := \mathcal{E}_n(x)\mathcal{F}_{n,m}(y)$

are harmonic, by Lemma 15C.6. Thus, if we define

$$\begin{aligned}
\mathcal{H}^1_{n,1}(x,y) &:= \mathcal{E}^1_n(x)\cdot\mathcal{F}^2_{n,1}(y) = \frac{\sin(nx)\sinh(ny)}{\sinh(n\pi)},\\
\mathcal{H}^1_{n,2}(x,y) &:= \mathcal{E}^1_n(x)\cdot\mathcal{F}^2_{n,2}(y) = \frac{\sin(nx)\sinh\left(n(\pi-y)\right)}{\sinh(n\pi)},\\
\mathcal{H}^2_{n,1}(x,y) &:= \mathcal{F}^1_{n,1}(x)\cdot\mathcal{E}^2_n(y) = \frac{\sinh(nx)\sin(ny)}{\sinh(n\pi)}, \qquad \text{and}\\
\mathcal{H}^2_{n,2}(x,y) &:= \mathcal{F}^1_{n,2}(x)\cdot\mathcal{E}^2_n(y) = \frac{\sinh\left(n(\pi-x)\right)\sin(ny)}{\sinh(n\pi)},
\end{aligned}$$

then Corollary 15C.8 says that the collection $\{\mathcal{H}^1_{n,1}\}_{n\in\mathbb{N}}\sqcup\{\mathcal{H}^1_{n,2}\}_{n\in\mathbb{N}}\sqcup\{\mathcal{H}^2_{n,1}\}_{n\in\mathbb{N}}\sqcup\{\mathcal{H}^2_{n,2}\}_{n\in\mathbb{N}}$ is a Dirichlet harmonic basis for $[0,\pi]^2$ —a fact we already observed in Example 15B.1(a), and exploited earlier in Proposition 12A.4. $\diamondsuit$

Example 15C.10. Let $\mathbb{X}_1 = \mathbb{D} \subset \mathbb{R}^2$ and $\mathbb{X}_2 = [0,\pi]$, so that $\mathbb{X}_1\times\mathbb{X}_2 \subset \mathbb{R}^3$ is the cylinder of height π and radius 1. Note that $\partial\mathbb{X} = \Big(\mathbb{S}\times[0,\pi]\Big)\cup\Big(\mathbb{D}\times\{0,\pi\}\Big)$.

For all $n\in\mathbb{N}$ and $\ell\in\mathbb{N}$, let $\mathcal{E}^1_{\ell,n} := \Phi_{\ell,n}$, and $\mathcal{E}^1_{\ell,-n} := \Psi_{\ell,n}$, where $\Phi_{\ell,n}$ and $\Psi_{\ell,n}$ are the type-1 Fourier-Bessel eigenfunctions defined by eqn.(14C.5) on page 296 of §14C(ii). Then $\{\mathcal{E}^1_{\ell,n};\ \ell\in\mathbb{N}\ \text{ and }\ n\in\mathbb{Z}\}$ is a Dirichlet eigenbasis for $\mathbb{D}$, by Theorem 14C.2.

As in Example 15C.9, $\partial[0,\pi] = \{0,\pi\}$. Let $\mathbb{M}_2 := \{0,1\}$ and let $\Xi^2_1 : \{0,\pi\}\longrightarrow\mathbb{R}$ and $\Xi^2_2 : \{0,\pi\}\longrightarrow\mathbb{R}$ be as in Example 15C.9. Let $\{\kappa_{\ell,n}\}_{\ell,n=1}^{\infty}$ be the roots of the Bessel function $\mathcal{J}_n$, as described in equation (14C.3) on page 296. For every $(\ell,n)\in\mathbb{N}\times\mathbb{Z}$, define $\mathcal{F}^2_{\ell,n;1}$ and $\mathcal{F}^2_{\ell,n;2}\in\mathbf{L}^2[0,\pi]$ by

$$\mathcal{F}^2_{\ell,n;1}(z) := \frac{\sinh\left(\kappa_{\ell,|n|}\cdot z\right)}{\sinh(\kappa_{\ell,|n|}\,\pi)} \quad\text{and}\quad \mathcal{F}^2_{\ell,n;2}(z) := \frac{\sinh\left(\kappa_{\ell,|n|}\cdot(\pi-z)\right)}{\sinh(\kappa_{\ell,|n|}\,\pi)},$$

for all $z\in[0,\pi]$. Then clearly $\widetilde{\mathcal{F}}^2_{\ell,n;1} = \Xi^2_1$ and $\widetilde{\mathcal{F}}^2_{\ell,n;2} = \Xi^2_2$.

For each $(\ell,n)\in\mathbb{N}\times\mathbb{Z}$, we have eigenvalue $-\kappa^2_{\ell,|n|}$ by equation (14C.4) on page 296. That is $\triangle\Phi_{\ell,n}(r,\theta) = -\kappa^2_{\ell,n}\,\Phi_{\ell,n}(r,\theta)$ and $\triangle\Psi_{\ell,n}(r,\theta) = -\kappa^2_{\ell,n}\,\Psi_{\ell,n}(r,\theta)$; thus, $\triangle\mathcal{E}^1_{\ell,n}(z) = -\kappa^2_{\ell,|n|}\mathcal{E}^1_{\ell,n}(z)$ for all $(\ell,n)\in\mathbb{N}\times\mathbb{Z}$. Meanwhile, $\triangle\mathcal{F}^1_{\ell,n;m}(z) =$

$\kappa^2_{\ell,|n|}\mathcal{F}^1_{\ell,n;m}(z)$, for all $(\ell,n;m)\in\mathbb{N}\times\mathbb{Z}\times\{1,2\}$. Thus, the functions

$$\mathcal{H}^1_{\ell,n,1}(r,\theta,z) \;:=\; \mathcal{E}^1_{\ell,n}(r,\theta)\cdot\mathcal{F}^2_{\ell,n;1}(z) \;=\; \frac{\Phi_{\ell,n}(r,\theta)\,\sinh(\kappa_{\ell,n}\cdot z)}{\sinh(\kappa_{\ell,n}\,\pi)},$$

$$\mathcal{H}^1_{\ell,n,2}(r,\theta,z) \;:=\; \mathcal{E}^1_{\ell,n}(r,\theta)\cdot\mathcal{F}^2_{\ell,n;2}(z) \;=\; \frac{\Phi_{\ell,n}(r,\theta)\,\sinh\left(\kappa_{\ell,n}\cdot(\pi-z)\right)}{\sinh(\kappa_{\ell,n}\,\pi)},$$

$$\mathcal{H}^1_{\ell,-n,1}(r,\theta,z) \;:=\; \mathcal{E}^1_{\ell,-n}(r,\theta)\cdot\mathcal{F}^2_{\ell,-n;1}(z) \;=\; \frac{\Psi_{\ell,n}(r,\theta)\,\sinh(\kappa_{\ell,n}\cdot z)}{\sinh(\kappa_{\ell,n}\,\pi)}, \quad\text{and}$$

$$\mathcal{H}^1_{\ell,-n,2}(r,\theta,z) \;:=\; \mathcal{E}^1_{\ell,-n}(r,\theta)\cdot\mathcal{F}^2_{\ell,-n;2}(z) \;=\; \frac{\Psi_{\ell,n}(r,\theta)\,\sinh\left(\kappa_{\ell,n}\cdot(\pi-z)\right)}{\sinh(\kappa_{\ell,n}\,\pi)}$$

are all harmonic, by Lemma 15C.6.

Recall that $\partial\mathbb{D}=\mathbb{S}$. Let $\mathbb{M}_1:=\mathbb{Z}$, and for all $m\in\mathbb{Z}$, define $\Xi^1_m\in\mathbf{L}^2(\mathbb{S})$ by $\Xi^1_m(1,\theta):=\sin(m\theta)$ (if $m>0$) and $\Xi^1_m(1,\theta):=\cos(m\theta)$ (if $m\leq 0$), for all $\theta\in[-\pi,\pi]$; then $\{\Xi^1_m\}_{m\in\mathbb{Z}}$ is an orthogonal bass for $\mathbf{L}^2(\mathbb{S})$, by Theorem 8A.1. For all $n\in\mathbb{N}$ and $z\in[0,\pi]$, define $\mathcal{E}^2_n(z):=\sin(nz)$ as in Example 15C.9. Then $\{\mathcal{E}^2_n\}_{n=1}^{\infty}$ is a Dirichlet eigenbasis for $[0,\pi]$, by Theorem 7A.1. For all $n\in\mathbb{N}$, the eigenfunction $\mathcal{E}^2_n$ has eigenvalue $\lambda^{(2)}_n:=-n^2$. For all $m\in\mathbb{Z}$, let $\mathcal{F}^1_{n,m}:\mathbb{D}\longrightarrow\mathbb{R}$ be an eigenfunction of the Laplacian with eigenvalue n^2, and with boundary condition $\mathcal{F}^1_{n,m}(1,\theta)=\Xi^1_m(\theta)$ for all $\theta\in[-\pi,\pi]$ (see Exercise 15C.9(a) below). The function $\mathcal{H}^2_{n,m}(r,\theta,z):=\mathcal{F}^1_{n,m}(r,\theta)\cdot\mathcal{E}^2_n(z)$ is harmonic, by Lemma 15C.6. Thus, Corollary 15C.8 says that the collection

$$\left\{\mathcal{H}^1_{\ell,n,m}\ ;\ \ell\in\mathbb{N},\ n\in\mathbb{Z},\ m=1,2\right\}\quad\sqcup\quad\left\{\mathcal{H}^2_{n,m}\ ;\ n\in\mathbb{N},\ m\in\mathbb{Z}\right\}$$

is a Dirichlet harmonic basis for the cylinder $\mathbb{X}$. ♢

Ⓔ **Exercise 15C.9.** (a) Example 15C.10 posits the existence of eigenfunctions $\mathcal{F}^1_{n,m}:\mathbb{D}\longrightarrow\mathbb{R}$ of the Laplacian with eigenvalue n^2 and with boundary condition $\mathcal{F}^1_{n,m}(1,\theta)=\Xi^1_m(\theta)$ for all $\theta\in[-\pi,\pi]$. Assume $\mathcal{F}^1_{n,m}$ *separates* in polar coordinates —that is, $\mathcal{F}^1_{n,m}(r,\theta)=\mathcal{R}(r)\cdot\Xi_m(\theta)$, where $\mathcal{R}:[0,1]\longrightarrow\mathbb{R}$ is some unknown function with $\mathcal{R}(1)=1$. Show that $\mathcal{R}$ must satisfy the ordinary differential equation $r^2\mathcal{R}''(r)+r\mathcal{R}'(r)-(r^2+1)n^2\mathcal{R}(r)=0$. Use the *Method of Frobenius* (§0H(iii)) to solve this ODE and get an expression for $\mathcal{F}^1_{n,m}$.

(b) Combine Theorem 15B.3 with Example 15C.10 to obtain a general solution to the Laplace equation on a finite cylinder with nonhomogeneous Dirichlet boundary conditions.

(c) Set up and solve a few simple Dirichlet problems using your method. ♦

Ⓔ **Exercise 15C.10.** Let $\mathbb{X}=\{(r,\theta,z)\ ;\ 1\leq r\ \text{ and }\ 0\leq z\leq\pi\}$ be the **punctured slab** from Exercise 15C.4.

(a) Use Corollary 15C.8 to obtain a Dirichlet harmonic basis for $\mathbb{X}$.

(b) Apply Theorem 15B.3 to obtain a general solution to the Laplace equation on the punctured slab with nonhomogeneous Dirichlet boundary conditions.

(c) Set up and solve a few simple Dirichlet problems using your method. ♦

Exercise 15C.11. Let $\mathbb{X}$ be the *right-angle triangular prism* from Exercise 15C.5. Ⓔ

(a) Use Proposition 15A.5 and Corollary 15C.8 to obtain a Dirichlet harmonic basis for $\mathbb{X}$.

(b) Apply Theorem 15B.3 to obtain a general solution to the Laplace equation on the prism with nonhomogeneous Dirichlet boundary conditions.

(c) Set up and solve a few simple Dirichlet problems using your method. ♦

Exercise 15C.12. State and prove a theorem analogous to Corollary 15C.8 for *Neumann* harmonic bases. ♦ Ⓔ

15D General method for solving I/BVPs

Prerequisites: §15A, §15B. **Recommended:** §15C.

We now provide a general method for solving initial/boundary value problems. Throughout this section, let $\mathbb{X} \subset \mathbb{R}^D$ be a domain. Let L be a linear differential operator on $\mathbb{X}$ (e.g. $\mathsf{L} = \triangle$).

1. Pick a suitable coordinate system. Find the coordinate system where your problem can be expressed in simplest form. Generally, this is a coordinate system where the domain $\mathbb{X}$ can be described using a few simple inequalities. For example, if $\mathbb{X} = [0, L]^D$, then probably the Cartesian coordinate system is best. If $\mathbb{X} = \mathbb{D}$ or $\mathbb{D}^{\complement}$ or $\mathbb{A}$, then probably polar coordinates on $\mathbb{R}^2$ are the most suitable. If $\mathbb{X} = \mathbb{B}$ or $\mathbb{X} = \partial\mathbb{B}$, then probably spherical polar coordinates on $\mathbb{R}^3$ are best.

If the differential operator L has nonconstant coefficients, then you should also seek a coordinate system where these coefficients can be expressed using the simplest formulae. (If $\mathsf{L} = \triangle$, then it has constant coefficients, so this is not an issue).

Finally, if several coordinate systems are equally suitable for describing $\mathbb{X}$ and L, then find the coordinate system where the initial conditions and/or boundary conditions can be expressed most easily. For example, if $\mathbb{X} = \mathbb{R}^2$ and $\mathsf{L} = \triangle$, then either Cartesian or polar coordinates might be appropriate. However, if the initial conditions are rotationally symmetric around zero, then polar coordinates would be more appropriate. If the initial conditions are invariant under translation in some direction, then Cartesian coordinates would be more appropriate.

Note. Don't forget to find the correct expression for L in the new coordinate system. For example, in Cartesian coordinates on $\mathbb{R}^2$, we have $\triangle u(x,y) = \partial_y^2\, u(x,y) + \partial_y^2\, u(x,y)$. However, in polar coordinates, $\triangle u(r,\theta) = \partial_r^2\, u(r,\theta) + \frac{1}{r}\,\partial_r\, u(r,\theta) + \frac{1}{r^2}\,\partial_\theta^2\, u(r,\theta)$. If you apply the 'Cartesian' Laplacian to a function expressed in polar coordinates, the result will be nonsense.

2. Eliminate irrelevant coordinates. A coordinate x is "irrelevant" if:

(a) membership in the domain $\mathbb{X}$ does not depend on this coordinate; *and*

(b) the coefficients of L do not depend on this coordinate; *and*

(c) the initial and/or boundary conditions do not depend on this coordinate.

In this case, we can eliminate the x coordinate from all equations, by expressing the domain $\mathbb{X}$, the operator L and the initial/boundary conditions as functions of only the non-x coordinates. This reduces the dimension of the problem, thereby simplifying it.

To illustrate (a), suppose $\mathbb{X} = \mathbb{D}$ or $\mathbb{D}^{\complement}$ or $\mathbb{A}$, and we use the polar coordinate system (r, θ); then the angle coordinate θ is irrelevant to membership in $\mathbb{X}$. On the other hand, suppose $\mathbb{X} = \mathbb{R}^2 \times [0, L]$ is the 'slab' of thickness L in $\mathbb{R}^3$, and we use Cartesian coordinates (x, y, z). Then the coordinates x and y are irrelevant to membership in $\mathbb{X}$.

If $\mathsf{L} = \triangle$ or any other differential operator with constant coefficients, then (b) is automatically satisfied.

To illustrate (c), suppose $\mathbb{X} = \mathbb{D}$ and we use polar coordinates. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be some initial condition. If $f(r, \theta)$ is a function only of r, and doesn't depend on θ, then θ is a redundant coordinate and can be eliminated, thereby reducing the BVP to a one-dimensional problem, as in Example 14F.3 on page 305.

On the other hand, let $b : \mathbb{S} \longrightarrow \mathbb{R}$ be a boundary condition. Then θ is only irrelevant if b is a constant function (otherwise b has nontrivial dependence on θ).

Now, suppose $\mathbb{X} = \mathbb{R}^2 \times [0, L]$ is the 'slab' of thickness L in $\mathbb{R}^3$. If the boundary condition $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$ is constant on the top and bottom faces of the slab, then the x and y coordinates can be eliminated, thereby reducing the BVP to a one-dimensional problem: a BVP on the line segment $[0, L]$, which can be solved using the methods of Chapter 11.

In some cases, a certain coordinate can be eliminated if it is 'approximately' irrelevant. For example, if the domain $\mathbb{X}$ is particularly 'long' in the x dimension relative to its other dimensions, and the boundary conditions are roughly constant in the x dimension, then we can approximate 'long' with 'infinite' and 'roughly constant' with 'exactly constant', and eliminate the x dimension from the problem. This method was used in Example 12B.2 on page 248 (the 'quenched rod'), Example 12B.7 on page 252 (the 'baguette'), and Example 12C.2 on page 255 (the 'nuclear fuel rod').

3. Find an eigenbasis for $\mathbf{L}^2(\mathbb{X})$. If $\mathbb{X}$ is one of the 'standard' domains we have studied in this book, then use the eigenbases we have introduced in Chapters 7-9, Section 14C, or Section 15C. Otherwise, you must construct a suitable eigenbasis. Theorem 15E.12 (page 347) guarantees that such an eigenbasis exists, but it doesn't tell you how to construct it. The actual construction of eigenbases is usually done using *Separation of Variables*, discussed in Chapter 16. The separation of the "time" variable is really just a consequence of the fact that we have an eigenfunction. The separation of the "space" variables is not *necessary* to get an eigenfunction, but it is very *convenient*, for two reasons:

1. Separation of variables is a powerful strategy for finding the eigenfunctions; it reduces the problem to set of independent ODEs which can each be solved using classical ODE methods.

2. If an eigenfunction $\mathcal{E}_n$ appears in 'separated' form, then it is often easier to compute the inner product $\langle \mathcal{E}_n, f \rangle$, where f is some other function. This is important when the eigenfunctions form an orthogonal basis, and we want to compute the coefficients of f in this basis.

4. Find a harmonic basis for $\mathbf{L}^2(\partial\mathbb{X})$ (if there are nonhomogeneous boundary conditions). The same remarks apply as in Step 3.

5. Solve the problem Express any initial conditions in terms of the eigenbasis from step #3, as described in §15A Express any boundary conditions in terms of the harmonic basis from step #4, as described in §15B.

If $\mathsf{L} = \triangle$, then use Theorems 15A.2, 15A.3, 15A.4, and/or 15B.3. If L is some other linear differential operator, then use the appropriate analogues of these theorems (see Remark 15A.6).

6. Verify convergence. Note that Theorems 15A.2, 15A.3, 15A.4, and/or 15B.3 require the eigenvalue sequences $\{\lambda_n\}_{n=1}^\infty$ and/or $\{\mu_n\}_{n=1}^\infty$ to grow at a certain speed, or require the coefficient sequences $\{A_n\}_{n=0}^\infty$ and $\{B_n\}_{n=1}^\infty$ to decay at a certain speed, so as to guarantee that the solution series and its formal derivatives are absolutely convergent. These conditions are important, and must be checked. Typically, if $\mathsf{L} = \triangle$, the growth conditions on $\{\lambda_n\}_{n=1}^\infty$ and $\{\mu_n\}_{n=1}^\infty$ are easily satisfied. However, if you try to extend these theorems to some other linear differential operator, the conditions on $\{\lambda_n\}_{n=1}^\infty$ and $\{\mu_n\}_{n=1}^\infty$ must be checked.

7. Check the uniqueness of the solution. Section 5D describes conditions under which boundary value problems for the Poisson, Laplace, Heat, and wave equations will have a *unique* solution. Check that these conditions are satisfied. If $\mathsf{L} \neq \triangle$, then you will need to establish solution uniqueness using theorems analogous to those found in Section 5D. (General theorems for the existence/uniqueness of solutions to I/BVPs can be found in most advanced texts on PDE theory, such as [Eva91]).

If the solution is *not* unique, then it is important to *enumerate all solutions to the problem.* Remember that your ultimate goal here is to *predict* the behaviour of some physical system in response to some initial or boundary condition. If the solution to the I/BVP is not unique, then you cannot make a precise prediction; instead, your prediction must take the form of a precisely specified *range* of possible outcomes.

15E Eigenfunctions of self-adjoint operators

Prerequisites: §4B(iv), §5C, §6F. **Recommended:** §7A, §8A, §9B, §15A.

The solution methods of Section 15A are only relevant if we know that a suitable eigenbasis for the Laplacian exists on the domain of interest. If we want to develop similar methods for some other linear differential operator L (as described in Remark 15A.6 on page 323), then we must first know that suitable eigenbases exists for L. In this section, we will discuss the eigenfunctions and eigenvalues of an important class of linear operators: *self-adjoint* operators. This class includes the Laplacian and all other symmetric elliptic differential operators.

A linear operator $F : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is **self-adjoint** if, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$,

$$\langle F(\mathbf{x}),\ \mathbf{y}\rangle \;=\; \langle \mathbf{x},\ F(\mathbf{y})\rangle\,.$$

Example 15E.1. The matrix $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ defines a self-adjoint operator on $\mathbb{R}^2$, because for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, we have

$$\begin{aligned} \langle F(\mathbf{x}), \mathbf{y}\rangle &= \left\langle \begin{bmatrix} x_1-2x_2 \\ x_2-2x_1 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \;=\; y_1\Big(x_1 - 2x_2\Big) + y_2\Big(x_2 - 2x_1\Big) \\ &= x_1\Big(y_1 - 2y_2\Big) + x_2\Big(y_2 - 2y_1\Big) \;=\; \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1-2y_2 \\ y_2-2y_1 \end{bmatrix} \right\rangle \\ &= \langle \mathbf{x},\ F(\mathbf{y})\rangle\,. \end{aligned}$$ ♢

Theorem 15E.2. *Let $F : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ be a linear operator with matrix* $\mathbf{A}$. *Then F is self-adjoint if and only if $\mathbf{A}$ is symmetric* (i.e. $a_{ij} = a_{ji}$ for all j, i)

Ⓔ *Proof.* **Exercise 15E.1** . □

A linear operator $\mathsf{L} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is **self-adjoint** if, for any two functions $f, g \in \mathcal{C}^\infty$,

$$\langle \mathsf{L}[f],\ g\rangle \;=\; \langle f,\ \mathsf{L}[g]\rangle$$

whenever both sides are well-defined[3].

[3]This is an important point. Often, one of these inner products (say, the left one) will *not* be well-defined, because the integral $\displaystyle\int_{\mathbb{X}} \mathsf{L}[f]\cdot g\; dx$ does not converge, in which case "self-adjointness" is meaningless.

Example 15E.3: Multiplication Operators are Self-Adjoint.

Let $\mathbb{X} \subset \mathbb{R}^D$ be any bounded domain. Let $\mathcal{C}^\infty := \mathcal{C}^\infty(\mathbb{X};\mathbb{R})$. Fix $q \in \mathcal{C}^\infty(\mathbb{X})$, and define the operator $\mathsf{Q} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ by $\mathsf{Q}(f) := q \cdot f$ for any $f \in \mathcal{C}^\infty$. Then Q is self-adjoint. To see this, let $f, g \in \mathcal{C}^\infty$. Then

$$\langle q \cdot f,\ g\rangle \ = \ \int_{\mathbb{X}} (q \cdot f) \cdot g \ dx \ = \ \int_{\mathbb{X}} f \cdot (q \cdot g) \ dx \ = \ \langle f,\ q \cdot g\rangle \,.$$

(These integrals are all well-defined because q, f and g are all continuous and hence bounded on $\mathbb{X}$.) $\diamondsuit$

Let $L > 0$, and consider the interval $[0, L]$. Recall that $\mathcal{C}^\infty[0, L]$ is the set of all smooth functions from $[0, L]$ into $\mathbb{R}$, and that:

$\mathcal{C}_0^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying homogeneous *Dirichlet* boundary conditions: $f(0) = 0 = f(L)$ (see §5C(i)).

$\mathcal{C}_\perp^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying $f : [0, L] \longrightarrow \mathbb{R}$ satisfying homogeneous *Neumann* boundary conditions: $f'(0) = 0 = f'(L)$ (see §5C(ii)).

$\mathcal{C}_{\mathrm{per}}^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying $f : [0, L] \longrightarrow \mathbb{R}$ satisfying *periodic* boundary conditions: $f(0) = f(L)$ and $f'(0) = f'(L)$ (see §5C(iv)).

$\mathcal{C}_{h,h_\perp}^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying homogeneous *mixed* boundary conditions, for any fixed real numbers $h(0)$, $h_\perp(0)$, $h(L)$ and $h_\perp(L)$ (see §5C(iii)).

When restricted to these function spaces, the one-dimensional Laplacian operator ∂_x^2 is self-adjoint.

Proposition 15E.4. *Let $L > 0$, and consider the operator ∂_x^2 on $\mathcal{C}^\infty[0, L]$.*

(a) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}_0^\infty[0, L]$.*

(b) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}_\perp^\infty[0, L]$.*

(c) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}_{\mathrm{per}}^\infty[0, L]$.*

(d) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}_{h,h_\perp}^\infty[0, L]$, for any $h(0)$, $h_\perp(0)$, $h(L)$ and $h_\perp(L)$ in $\mathbb{R}$.*

Proof. Let $f, g : [0, L] \longrightarrow \mathbb{R}$ be smooth functions. We apply integration by parts to get:

$$\left\langle \partial_x^2 f,\ g\right\rangle \ = \ \int_0^L f''(x) \cdot g(x) \ dx \ = \ f'(x) \cdot g(x)\Big|_{x=0}^{x=L} \ - \ \int_0^L f'(x) \cdot g'(x) \ dx. \tag{15E.1}$$

But if we apply Dirichlet, Neumann, or Periodic boundary conditions, we get:

$$
\begin{aligned}
f'(x)\cdot g(x)\Big|_{x=0}^{x=L} &= f'(L)\cdot g(L) - f'(0)\cdot g(0) \\
&= \begin{cases} f'(L)\cdot 0 \; - \; f'(0)\cdot 0 & = 0 \quad \text{(if homog. Dirichlet BC)} \\ 0\cdot g(L) \; - \; 0\cdot g(0) & = 0 \quad \text{(if homog. Neumann BC)} \\ f'(0)\cdot g(0) \; - \; f'(0)\cdot g(0) & = 0 \quad \text{(if Periodic BC)} \end{cases} \\
&= 0 \qquad \text{in all cases.}
\end{aligned}
$$

Thus, the first term in (15E.1) is zero, so $\langle \partial_x^2 f,\ g\rangle \;=\; \int_0^L f'(x)\cdot g'(x)\ dx$.

But by the same reasoning, with f and g interchanged, $\int_0^L f'(x)\cdot g'(x)\ dx \;=\; \langle f,\ \partial_x^2 g\rangle$.

Thus, we've proved parts **(a)**, **(b)**, and **(c)**. To prove part **(d)**, first note that

$$
\begin{aligned}
f'(x)\cdot g(x)\Big|_{x=0}^{x=L} &= f'(L)\cdot g(L) \;-\; f'(0)\cdot g(0) \\
&= f(L)\cdot\frac{h(L)}{h_\perp(L)}\cdot g(L) \;+\; f(0)\cdot\frac{h(0)}{h_\perp(0)}\cdot g(0) \\
&= f(L)\cdot g'(L) \;-\; f(0)\cdot g'(0) \quad = \quad f(x)\cdot g'(x)\Big|_{x=0}^{x=L}.
\end{aligned}
$$

Hence, substituting $f(x)\cdot g'(x)\Big|_{x=0}^{x=L}$ for $f'(x)\cdot g(x)\Big|_{x=0}^{x=L}$ in (15E.1), we get: $\langle \partial_x^2 f,\ g\rangle \;=\; \int_0^L f''(x)\cdot g(x)\ dx \;=\; \int_0^L f(x)\cdot g''(x)\ dx \;=\; \langle f,\ \partial_x^2 g\rangle$. $\square$

Proposition 15E.4 generalizes to higher-dimensional Laplacians in the obvious way:

Theorem 15E.5. *Let $L > 0$.*

(a) *The Laplacian operator $\triangle$ is self-adjoint on any of the spaces: $\mathcal{C}_0^\infty[0,L]^D$, $\mathcal{C}_\perp^\infty[0,L]^D$, $\mathcal{C}_{h,h_\perp}^\infty[0,L]^D$ or $\mathcal{C}_{\mathrm{per}}^\infty[0,L]^D$.*

(b) *More generally, if $\mathbb{X}\subset\mathbb{R}^D$ is any bounded domain with a smooth boundary[4], then the Laplacian operator $\triangle$ is self-adjoint on any of the spaces: $\mathcal{C}_0^\infty(\mathbb{X})$, $\mathcal{C}_\perp^\infty(\mathbb{X})$, or $\mathcal{C}_{h,h_\perp}^\infty(\mathbb{X})$.*

In other words, the Laplacian is self-adjoint whenever we impose homogeneous Dirichlet, Neumann, or mixed boundary conditions, or (when meaningful) periodic boundary conditions.

Ⓔ *Proof.* **(a) Exercise 15E.2** Hint: The argument is similar to Proposition 15E.4. Apply integration by parts in each dimension, and cancel the "boundary" terms using the boundary conditions.

(b) Exercise 15E.3 Hint: Use Green's Formulae (Corollary 0E.5(c) on page 564) to set up an 'integration by parts' argument similar to Proposition 15E.4. □ Ⓔ

If L_1 and L_2 are two self-adjoint operators, then their sum $\mathsf{L}_1 + \mathsf{L}_2$ is also self-adjoint (**Exercise 15E.4**). Ⓔ

Example 15E.6. Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain (e.g. a cube), and let $V : \mathbb{X} \longrightarrow \mathbb{R}$ be a potential describing the force acting on a quantum particle (e.g. an electron) confined to the region $\mathbb{X}$ by an infinite potential barrier along $\partial\mathbb{X}$. Consider the *Hamiltonian* operator H defined in Section 3B on page 41:

$$\mathsf{H}\,\omega(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m} \triangle\,\omega(\mathbf{x}) \;+\; V(\mathbf{x})\cdot\omega(\mathbf{x}), \quad \text{for all } \mathbf{x}\in\mathbb{X}.$$

(Here, $\hbar$ is Plank's constant, m is the mass of the particle, and $\omega \in \mathcal{C}_0^\infty\mathbb{X}$ is its wavefunction.) The operator H is self-adjoint on $\mathcal{C}_0^\infty(\mathbb{X})$. To see this, note that $\mathsf{H}[\omega] \;=\; \frac{-\hbar^2}{2\,m}\triangle\,\omega + \mathsf{V}[\omega]$, where $\mathsf{V}[\omega] = V\cdot\omega$. Now, $\triangle$ is self-adjoint by Theorem 15E.5(b), and V is self-adjoint from Example 15E.3; thus, their sum H is also self-adjoint. The *stationary Schrödinger equation*[5] $\mathsf{H}\,\omega \;=\; \lambda\,\omega$ simply says that ω is an eigenfunction of H with eigenvalue λ. ♢

Example 15E.7. Let $s, q : [0, L] \longrightarrow \mathbb{R}$ be differentiable. The **Sturm-Liouville** operator

$$\mathsf{SL}_{s,q}[f] \quad := \quad s\cdot f'' + s'\cdot f' + q\cdot f$$

is self-adjoint on any of the spaces $\mathcal{C}_0^\infty[0,L]$, $\mathcal{C}_\perp^\infty[0,L]$, $\mathcal{C}_{h,h_\perp}^\infty[0,L]$ or $\mathcal{C}_{\mathrm{per}}^\infty[0,L]$. To see this, notice that

$$\mathsf{SL}_{s,q}[f] \;=\; (s\cdot f')' + (q\cdot f) \;=\; \mathsf{S}[f] + \mathsf{Q}[f], \tag{15E.2}$$

where $\mathsf{Q}[f] = q\cdot f$ is just a multiplication operator, and $\mathsf{S}[f] = (s\cdot f')'$. We know that Q is self-adjoint from Example 15E.3. We claim that S is also self-adjoint. To see this, note that:

$$\begin{aligned}\langle \mathsf{S}[f],\; g\rangle \quad &= \quad \int_0^L (s\cdot f')'(x)\cdot g(x)\;dx \\ &\underset{(*)}{=} \quad s(x)\cdot f'(x)\cdot g(x)\Big|_{x=0}^{x=L} \;-\; \int_0^L s(x)\cdot f'(x)\cdot g'(x)\;dx\end{aligned}$$

[4] See page 85 of §5D.
[5] See §3C.

$$\underset{(*)}{=} \quad s(x)\cdot f'(x)\cdot g(x)\Big|_{x=0}^{x=L} \;-\; s(x)\cdot f(x)\cdot g'(x)\Big|_{x=0}^{x=L} \;+\; \int_0^L f(x)\cdot(s\cdot g')'(x)\;dx$$

$$\underset{(\dagger)}{=} \quad \int_0^L f(x)\cdot(s\cdot g')'(x)\;dx \qquad = \qquad \langle f,\ \mathsf{S}[g]\rangle\,.$$

Here, each $(*)$ is integration by parts, and $(\dagger)$ follows from any of the cited
Ⓔ boundary conditions as in Proposition 15E.4 on page 341 (**Exercise 15E.5**).
Thus, S is self-adjoint, so $\mathsf{SL}_{s,q} = \mathsf{S}+\mathsf{Q}$ is self-adjoint. ♢

If $\mathsf{SL}_{s,q}$ is a Sturm-Liouville operator, then the corresponding **Sturm-Liouville equation** is the linear ordinary differential equation

$$\mathsf{SL}_{s,q}[f] \quad = \quad \lambda\, f. \tag{15E.3}$$

where $f:[0,L]\longrightarrow\mathbb{C}$ and $\lambda\in\mathbb{C}$ are unknown. Clearly, equation (15E.3) simply asserts that f is an eigenfunction of $\mathsf{SL}_{s,q}$, with eigenvalue λ. Sturm-Liouville equations appear frequently in the study of ordinary and partial differential equations.

Example 15E.8. (a) The one-dimensional *Helmholtz equation* $f''(x) = \lambda\, f(x)$ is a Sturm-Liouville equation, with $s\equiv 1$ (constant) and $q\equiv 0$.

(b) The one-dimensional *stationary Schrödinger equation*

$$\frac{-\hbar^2}{2\,m}\, f''(x) \;+\; V(x)\cdot f(x) \quad = \quad \lambda\, f(x), \qquad \text{for all } x\in[0,L].$$

is a Sturm-Liouville equation, with $s\equiv \frac{-\hbar^2}{2\,m}$ (constant) and $q(x) := V(x)$.

(c) The *Cauchy-Euler equation*[6] $x^2\, f''(x) + 2x\, f'(x) - \lambda\cdot f(x) \;=\; 0$ is a Sturm-Liouville equation: let $s(x) := x^2$ and $q\equiv 0$

(d) The *Legendre equation*[7] $(1-x^2)\, f''(x) - 2x\, f'(x) + \mu f(x) \;=\; 0$ is a Sturm-Liouville equation: let $s(x) := (1-x^2)$, $q\equiv 0$, and let $\lambda := -\mu$.

For more information about Sturm-Liouville problems, see [Bro89, §2.6, pp.39-44], [Pow99, §2.7, pp.146-150], [Con90, §II.6, pp.49-53], and especially [CB87, Chap.6, pp.159-203]. ♢

Examples 15E.6 and 15E.8, Theorem 15E.5, and the solution methods of §15A all illustrate the importance of the eigenfunctions of self-adjoint operators. One nice property of self-adjoint operators is that their eigenfunctions are orthogonal.

[6]See equation (14H.5) on page 314, and equation (16D.20) on page 361.
[7]See equation (16D.19) on page 361.

Proposition 15E.9. *Suppose* L *is a self-adjoint operator. If* f_1 *and* f_2 *are eigenfunctions of* L *with eigenvalues* $\lambda_1 \neq \lambda_2$, *then* f_1 *and* f_2 *are* ***orthogonal***.

Proof. By hypothesis, $\mathsf{L}[f_k] = \lambda_k \cdot f_k$, for $k = 1, 2$. Thus,

$$\lambda_1 \cdot \langle f_1, f_2 \rangle \;=\; \langle \lambda_1 \cdot f_1,\; f_2 \rangle \;=\; \langle \mathsf{L}[f_1],\; f_2 \rangle \;\underset{(*)}{=}\; \langle f_1,\; \mathsf{L}[f_2] \rangle \;=\; \langle f_1,\; \lambda_2 \cdot f_2 \rangle \;=\; \lambda_2 \cdot \langle f_1, f_2 \rangle\,,$$

where $(*)$ follows from self-adjointness. Since $\lambda_1 \neq \lambda_2$, this can only happen if $\langle f_1, f_2 \rangle = 0$. □

Example 15E.10. Eigenfunctions of ∂_x^2

(a) Let ∂_x^2 act on $\mathcal{C}^\infty[0, L]$. Then *all real numbers* $\lambda \in \mathbb{R}$ are eigenvalues of ∂_x^2. For any $\mu \in \mathbb{R}$,

- If $\lambda = \mu^2 > 0$, the eigenfunctions are of the form $\phi(x) = A\sinh(\mu \cdot x) + B\cosh(\mu \cdot x)$ for any constants $A, B \in \mathbb{R}$.
- If $\lambda = 0$, the eigenfunctions are of the form $\phi(x) = Ax + B$ for any constants $A, B \in \mathbb{R}$.
- If $\lambda = -\mu^2 < 0$, the eigenfunctions are of the form $\phi(x) = A\sin(\mu \cdot x) + B\cos(\mu \cdot x)$ for any constants $A, B \in \mathbb{R}$.

Note: Because we have not imposed any boundary conditions, Proposition 15E.4 does *not* apply; indeed ∂_x^2 is *not* a self-adjoint operator on $\mathcal{C}^\infty[0, L]$.

(b) Let ∂_x^2 act on $\mathcal{C}^\infty([0, L]; \mathbb{C})$. Then *all complex numbers* $\lambda \in \mathbb{C}$ are eigenvalues of ∂_x^2. For any $\mu \in \mathbb{C}$, with $\lambda = \mu^2$, the eigenvalue λ has eigenfunctions of the form $\phi(x) = A\exp(\mu \cdot x) + B\exp(-\mu \cdot x)$ for any constants $A, B \in \mathbb{C}$. (Note that the three cases of the previous example arise by taking $\lambda \in \mathbb{R}$.)

Again, Proposition 15E.4 does *not* apply in this case, because ∂_x^2 is *not* a self-adjoint operator on $\mathcal{C}^\infty([0, L]; \mathbb{C})$.

(c) Now let ∂_x^2 act on $\mathcal{C}_0^\infty[0, L]$. Then the eigenvalues of ∂_x^2 are $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$ for every $n \in \mathbb{N}$, each of multiplicity 1; the corresponding eigenfunctions are all scalar multiples of $\mathbf{S}_n(x) := \sin\left(\frac{n\pi x}{L}\right)$.

(d) If ∂_x^2 acts on $\mathcal{C}_\perp^\infty[0, L]$, then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$ for every $n \in \mathbb{N}$, each of multiplicity 1, but the corresponding eigenfunctions are now all scalar multiples of $\mathbf{C}_n(x) := \cos\left(\frac{n\pi x}{L}\right)$. Also, 0 is an eigenvalue, with eigenfunction $\mathbf{C}_0 = \mathbb{1}$.

(e) Let $h > 0$, and let ∂_x^2 act on $\mathcal{C} = \{f \in \mathcal{C}^\infty[0, L] \; ; \; f(0) = 0 \text{ and } h \cdot f(L) + f'(L) = 0\}$. Then the eigenfunctions of ∂_x^2 are all scalar multiples of

$$\Phi_n(x) \quad := \quad \sin(\mu_n \cdot x),$$

with eigenvalue $\lambda_n = -\mu_n^2$, where $\mu_n > 0$ is any real number such that

$$\tan(L \cdot \mu_n) \quad = \quad \frac{-\mu_n}{h}$$

This is a *transcendental equation* in the unknown μ_n. Thus, although there is an infinite sequence of solutions $\{\mu_0 < \mu_1 < \mu_2 < \ldots\}$, there is no closed-form algebraic expression for μ_n. At best, we can estimate μ_n through numerical methods.

(f) Let $h(0)$, $h_\perp(0)$, $h(L)$, and $h_\perp(L)$ be real numbers, and let ∂_x^2 act on $\mathcal{C}^\infty_{h,h_\perp}[0, L]$. Then the eigenfunctions of ∂_x^2 are all scalar multiples of

$$\Phi_n(x) \quad := \quad \sin\Big(\theta_n + \mu_n \cdot x\Big),$$

with eigenvalue $\lambda_n = -\mu_n^2$, where $\theta_n \in [0, 2\pi]$ and $\mu_n > 0$ are constants satisfying the transcendental equations:

$$\tan(\theta_n) \; = \; \mu_n \cdot \frac{h_\perp(0)}{h(0)} \qquad \text{and} \qquad \tan(\mu_n \cdot L + \theta_n) \; = \; -\mu_n \cdot \frac{h_\perp(L)}{h(L)}.$$

Ⓔ (**Exercise 15E.6**). In particular, if $h_\perp(0) = 0$, then we must have $\theta = 0$. If $h(L) = h$ and $h_\perp(L) = 1$, then we return to Example (e).

(g) Let ∂_x^2 act on $\mathcal{C}^\infty_{\text{per}}[-L, L]$. Then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, for every $n \in \mathbb{N}$, each having multiplicity 2. The corresponding eigenfunctions are of the form $A \cdot \mathbf{S}_n + B \cdot \mathbf{C}_n$ for any $A, B \in \mathbb{R}$. In particular, 0 is an eigenvalue, with eigenfunction $\mathbf{C}_0 = \mathbb{1}$.

(h) Let ∂_x^2 act on $\mathcal{C}^\infty_{\text{per}}([-L, L]; \; \mathbb{C})$. Then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, for every $n \in \mathbb{N}$, each having multiplicity 2. The corresponding eigenfunctions are of the form $A \cdot \mathbf{E}_n + B \cdot \mathbf{E}_{-n}$ for any $A, B \in \mathbb{R}$, where $\mathbf{E}_n(x) := \exp\left(\frac{\pi \mathbf{i} n x}{L}\right)$. In particular 0 is an eigenvalue, with eigenfunction $\mathbf{E}_0 = \mathbb{1}$. ♢

Example 15E.11. Eigenfunctions of $\triangle$

(a) Let $\triangle$ act on $\mathcal{C}_0^\infty[0,L]^D$. Then the eigenvalues of $\triangle$ are the numbers $\lambda_{\mathbf{m}} := -\left(\frac{\pi}{L}\right)^2 \cdot \|\mathbf{m}\|^2$ for all $\mathbf{m} \in \mathbb{N}_+^D$. (Here, if $\mathbf{m} = (m_1, \ldots, m_D)$, then $\|\mathbf{m}\|^2 := m_1^2 + \ldots + m_d^2$). The eigenspace of $\lambda_{\mathbf{m}}$ is spanned by all functions

$$\mathbf{S_n}(x_1, ..., x_D) \quad := \quad \sin\left(\frac{\pi n_1 x_1}{L}\right) \sin\left(\frac{\pi n_2 x_2}{L}\right) \cdots \sin\left(\frac{\pi n_D x_D}{L}\right),$$

for all $\mathbf{n} = (n_1, \ldots, n_D) \in \mathbb{N}_+^D$ such that $\|\mathbf{n}\| = \|\mathbf{m}\|$.

(b) Now let $\triangle$ act on $\mathcal{C}_\perp^\infty[0,L]^D$. Then the eigenvalues of $\triangle$ are $\lambda_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbb{N}^D$. The eigenspace of $\lambda_{\mathbf{m}}$ is spanned by all functions

$$\mathbf{C_n}(x_1, ..., x_D) \quad := \quad \cos\left(\frac{\pi n_1 x_1}{L}\right) \cos\left(\frac{\pi n_2 x_2}{L}\right) \cdots \cos\left(\frac{\pi n_D x_D}{L}\right),$$

for all $\mathbf{n} \in \mathbb{N}^D$ such that $\|\mathbf{n}\| = \|\mathbf{m}\|$. In particular, 0 is an eigenvalue whose eigenspace is the set of *constant* functions —i.e. multiples of $\mathbf{C}_0 = \mathbb{1}$.

(c) Let $\triangle$ act on $\mathcal{C}_{\text{per}}^\infty[-L,L]^D$. Then the eigenvalues of $\triangle$ are again $\lambda_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbb{N}^D$. The eigenspace of $\lambda_{\mathbf{m}}$ contains $\mathbf{C_n}$ and $\mathbf{S_n}$ for all $\mathbf{n} \in \mathbb{N}^D$ such that $\|\mathbf{n}\| = \|\mathbf{m}\|$.

(d) Let $\triangle$ act on $\mathcal{C}_{\text{per}}^\infty\left([-L,L]^D;\ \mathbb{C}\right)$. Then the eigenvalues of $\triangle$ are again $\lambda_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbb{N}^D$. The eigenspace of $\lambda_{\mathbf{m}}$ is spanned by all functions

$$\mathbf{E_n}(x_1, \ldots, x_D) \quad := \quad \exp\left(\frac{\pi \mathbf{i} n_1 x_1}{L}\right) \cdots \exp\left(\frac{\pi \mathbf{i} n_D x_D}{L}\right),$$

for all $\mathbf{n} \in \mathbb{Z}^D$ such that $\|\mathbf{n}\| = \|\mathbf{m}\|$. ♢

The alert reader will notice that, in each of the above scenarios (except Examples 15E.10(a) and 15E.10(b), where ∂_x^2 is not self-adjoint), the eigenfunctions are not only orthogonal, but actually form an *orthogonal basis* for the corresponding L^2-space. This is not a coincidence. If $\mathcal{C}$ is a subspace of $\mathbf{L}^2(\mathbb{X})$, and $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}$ is a linear operator, then a set $\{\Phi_n\}_{n=1}^\infty \subset \mathcal{C}$ is an **L-eigenbasis** for $\mathbf{L}^2(\mathbb{X})$ if $\{\Phi_n\}_{n=1}^\infty$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$, and for every $n \in \mathbb{N}$, Φ_n is an eigenfunction for L.

Theorem 15E.12. Eigenbases of the Laplacian

(a) *Let $L > 0$. Let $\mathcal{C}$ be any one of $\mathcal{C}_0^\infty[0,L]^D$, $\mathcal{C}_\perp^\infty[0,L]^D$, or $\mathcal{C}_{\text{per}}^\infty[0,L]^D$, and treat $\triangle$ as a linear operator on $\mathcal{C}$. Then there is a $\triangle$-eigenbasis for $\mathbf{L}^2[0,L]^D$ consisting of elements of $\mathcal{C}$. The corresponding eigenvalues of $\triangle$ are the values $\lambda_{\mathbf{m}}$ defined in Example* 15E.11(a)*, for all* $\mathbf{m} \in \mathbb{N}^D$.

(b) *More generally, if $\mathbb{X} \subset \mathbb{R}^D$ is any bounded open domain, then there is a $\triangle$-eigenbasis for $\mathbf{L}^2[\mathbb{X}]$ consisting of elements of $\mathcal{C}_0^\infty[\mathbb{X}]$. The corresponding eigenvalues of $\triangle$ on $\mathcal{C}$ form a decreasing sequence $0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ with $\lim_{n\to\infty} \lambda_n = -\infty$.*

In both (a) and (b), some of the eigenspaces may be many-dimensional.

Proof. **(a)** we have already established. The eigenfunctions of the Laplacian in these contexts are $\{\mathbf{C_n} \,;\, \mathbf{n} \in \mathbb{N}^D\}$ and/or $\{\mathbf{S_n} \,;\, \mathbf{n} \in \mathbb{N}_+^D\}$. Theorem 8A.1(a) on page 162 and Theorem 9B.1(a) on page 187 tell us that these form orthogonal bases for $\mathbf{L}^2[0,L]^D$.

(b) follows from Theorem 15E.17 on the next page. Alternately, see [War83], Chapter 6, p. 255; exercise 16(g), or [Cha93], Theorem 3.21, p. 156. □

Example 15E.13. (a) Let $\mathbb{B} = \{\mathbf{x} \in \mathbb{R}^D \,;\, \|\mathbf{x}\| < R\}$ be the ball of radius R. Then there is a $\triangle$-eigenbasis for $\mathbf{L}^2(\mathbb{B})$ consisting of functions which are zero on the spherical boundary of $\mathbb{B}$.

(b) Let $\mathbb{A} = \{(x,y) \in \mathbb{R}^2 \,;\, r^2 < x^2 + y^2 < R^2\}$ be the annulus of inner radius r and outer radius R in the plane. Then there is a $\triangle$-eigenbasis for $\mathbf{L}^2(\mathbb{A})$ consisting of functions which are zero on the inner and outer boundary circles of $\mathbb{A}$. ♢

Theorem 15E.14. Eigenbases for Sturm-Liouville operators

Let $L > 0$, let $s, q : [0,L] \longrightarrow \mathbb{R}$ be differentiable functions, and let $\mathsf{SL}_{s,q}$ be the Sturm-Liouville operator defined by s and q on $\mathcal{C}_0^\infty[0,L]$. Then there exists an $\mathsf{SL}_{s,q}$-eigenbasis for $\mathbf{L}^2[0,L]$ consisting of elements of $\mathcal{C}_0^\infty[0,L]$. The corresponding eigenvalues of $\mathsf{SL}_{s,q}$ form an infinite increasing sequence $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, with $\lim_{n\to\infty} \lambda_n = \infty$. Each eigenspace is one-dimensional.

Proof. See [Tit62, Theorem 1.9]. For a proof in the special case when $s \equiv 1$, see [Con90, Theorem 6.12, p.52]. □

Symmetric Elliptic Operators. The rest of this section concerns the eigenfunctions of symmetric elliptic operators. (Please see §5E for the definition of an elliptic operator.)

Lemma 15E.15. *Let $\mathbb{X} \subset \mathbb{R}^D$. If L is an elliptic differential operator on $\mathcal{C}^\infty(\mathbb{X})$, then there are functions $\omega_{cd} : \mathbb{X} \longrightarrow \mathbb{R}$ for all $c, d \in [1...D]$, and functions $\alpha, \xi_1, \ldots, \xi_D : \mathbb{X} \longrightarrow \mathbb{R}$ such that L can be written in* **divergence form:**

$$\begin{aligned} \mathsf{L}[u] &= \sum_{c,d=1}^{D} \partial_c(\omega_{cd} \cdot \partial_d u) \;+\; \sum_{d=1}^{D} \xi_d \cdot \partial_d u \;+\; \alpha \cdot u, \\ &= \operatorname{div}\left[\boldsymbol{\Omega} \cdot \nabla\phi\right] \;+\; \langle \Xi,\ \nabla\phi \rangle \;+\; \alpha \cdot u, \end{aligned}$$

where $\Xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_D \end{bmatrix}$, *and* $\boldsymbol{\Omega} = \begin{bmatrix} \omega_{11} \ldots \omega_{1D} \\ \vdots \ \ddots \ \vdots \\ \omega_{D1} \ldots \omega_{DD} \end{bmatrix}$ *is a symmetric, positive-definite matrix.*

Proof. **Exercise 15E.7** *Hint.* Use the same strategy as as equation (15E.2). □ Ⓔ

L is called **symmetric** if, in the divergence form, $\Xi \equiv 0$. For example, in the case when $\mathsf{L} = \triangle$, we have $\Omega = \mathbf{Id}$ and $\Xi = 0$, so $\triangle$ is symmetric.

Theorem 15E.16. *If $\mathbb{X} \subset \mathbb{R}^D$ is an open bounded domain, then any symmetric elliptic differential operator on $\mathcal{C}_0^\infty(\mathbb{X})$ is self-adjoint.*

Proof. This is a generalization of the integration-by-parts argument used to prove Proposition 15E.4 on page 341 and Theorem 15E.5 on page 342. See [Eva91, §6.5, p.334]. □

Theorem 15E.17. *Let $\mathbb{X} \subset \mathbb{R}^D$ be an open, bounded domain, and let L be any symmetric, elliptic differential operator on $\mathcal{C}_0^\infty(\mathbb{X})$. Then there exists an L-eigenbasis for $\mathbf{L}^2(\mathbb{X})$ consisting of elements of $\mathcal{C}_0^\infty(\mathbb{X})$. The corresponding eigenvalues of L form an infinite decreasing series $0 > \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots$, with $\lim_{n\to\infty} \lambda_n = -\infty$.*

Proof. See of [Eva91, Theorem 1, §6.5.1, p.335]. □

Remark. Theorems 15E.12, 15E.14, and 15E.17 are all manifestations of a far more general result, the *Spectral Theorem for Unbounded Self-Adjoint Operators*. Unfortunately, it would take us too far afield to even set up the necessary background to precisely *state* this theorem. See [Con90, §X.4 p. 319] for a good exposition.

Further reading

The study of eigenfunctions and eigenvalues is sometimes called *spectral theory*. For a good introduction to the spectral theory of linear operators on function spaces, see [Con90]. An analogy of the Laplacian can be defined on any Riemannian manifold; it is often called the *Laplace-Beltrami operator*, and its eigenfunctions reveal much about the geometry of the manifold; see [War83, Chap.6] or [Cha93, §3.9]. In particular, the eigenfunctions of the Laplacian on *spheres* have been extensively studied. These are called *spherical harmonics*, and a sort of "Fourier theory" can be developed on spheres, analogous to multivariate Fourier theory on the cube $[0, L]^D$, but with the spherical harmonics forming the orthonormal basis [Tak94, Mül66]. Much of this theory generalizes to a broader family of manifolds called *symmetric spaces* [Ter85, Hel81]. The eigenfunctions of the Laplacian on symmetric spaces are closely related to the theory of *Lie groups* and their representations [CW68, Sug75], a subject which is sometimes called *noncommutative harmonic analysis* [Tay86].

V Miscellaneous solution methods

In Chapters 11 to 15, we saw how initial/boundary value problems for linear partial differential equations could be solved by first identifying an orthogonal basis of eigenfunctions for the relevant differential operator (usually the Laplacian), and then representing the desired initial conditions or boundary conditions as an infinite summation of these eigenfunctions. For each bounded domain, each boundary condition, and each coordinate system we considered, we found a system of eigenfunctions that was 'adapted' to that domain, boundary conditions, and coordinate system.

This method is extremely powerful, but it raises several questions:

1. What if you are confronted with a new domain or coordinate system, where none of the known eigenfunction bases is applicable? Theorem 15E.12 on page 347 says that a suitable eigenfunction basis for this domain always exists, *in principle*. But how do you go about discovering such a basis *in practice*? For that matter, how were eigenfunctions bases like the Fourier-Bessel functions discovered in the first place? Where did Bessel's equation come from?

2. What if you are dealing with an *unbounded* domain, such as diffusion in $\mathbb{R}^3$? In this case, Theorem 15E.12 is not applicable, and in general, it may not be possible (or at least, not feasible) to represent initial/boundary conditions in terms of eigenfunctions. What alternative methods are available?

3. The eigenfunction method is difficult to connect to our physical intuitions. For example, intuitively, heat 'seaps' slowly through space, and temperature distributions gradually and irreversibly decay towards uniformity. It is thus impossible to send a long-distance 'signal' using heat. On the other hand, waves maintain their shape and propagate across great distances with a constant velocity; hence they can be used to send signals through space. These familiar intiutions are not explained or justified by the eigenfunction method. Is there an alternative solution method where these intuitions have a clear mathematical expression?

Part V provides answers to these questions. In Chapter 16, we introduce a powerful and versatile technique called *separation of variables*, to construct eigenfunctions adapted to any coordinate system. In Chapter 17, we develop the entirely different solution technology of *impulse-response functions*, which allows you to solve differential equations on unbounded domains, and which has an an appealing intuitive interpretation. Finally, in Chapter 18, we explore some surprising and beautiful applications of complex analysis to harmonic functions and Fourier theory.

Chapter 16

Separation of variables

"Before creation God did just pure mathematics. Then He thought it would be a pleasant change to do some applied." —J. E. Littlewood

16A Separation of variables in Cartesian coordinates on $\mathbb{R}^2$

Prerequisites: §1B, §1C.

A function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to **separate** if we can write $u(x, y) = X(x) \cdot Y(y)$ for some functions $X, Y : \mathbb{R} \longrightarrow \mathbb{R}$. If u is a solution to some partial differential equation, we say u is a **separated solution**.

Example 16A.1. *The heat equation on* $\mathbb{R}$

We wish to find $u : \mathbb{R} \times \mathbb{R}_{\neq} \longrightarrow \mathbb{R}$ such that $\partial_t u = \partial_x^2 u$. Suppose $u(x; t) = X(x) \cdot T(t)$, where

$$X(x) \;=\; \exp(\mathbf{i}\mu x) \quad \text{and} \quad T(t) \;=\; \exp(-\mu^2 t),$$

for some constant $\mu \in \mathbb{R}$. Then $u(x; t) = \exp(\mu \mathbf{i} x - \mu^2 t)$, so that $\partial_x^2 u = -\mu^2 \cdot u = \partial_t u$. Thus, u is a separated solution to the heat equation. ♢

Separation of variables is a strategy for for solving partial differential equations by specifically looking for separated solutions. At first, it seem like we are making our lives harder by insisting on a solution in separated form. However, often, we can use the hypothesis of separation to actually *simplify* the problem.

Suppose we are given some PDE for a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of two variables. *Separation of variables* is the following strategy:

1. Hypothesize that u can be written as a product of two functions, $X(x)$ and $Y(y)$, each depending on only one coordinate; in other words, assume that

$$u(x, y) \;=\; X(x) \cdot Y(y) \tag{16A.1}$$

2. When we evaluate the PDE on a function of type (16A.1), we may find that the PDE decomposes into two separate, *ordinary* differential equations for each of the two functions X and Y. Thus, we can solve these ODEs independently, and combine the resulting solutions to get a solution for u.

Example 16A.2. *Laplace's Equation in* $\mathbb{R}^2$

Suppose we want to find a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\triangle u \equiv 0$. If $u(x,y) = X(x) \cdot Y(y)$, then

$$\triangle u \;=\; \partial_x^2 (X \cdot Y) + \partial_y^2 (X \cdot Y) \;=\; \left(\partial_x^2 X\right)\cdot Y + X\cdot\left(\partial_y^2 Y\right) \;=\; X''\cdot Y + X\cdot Y'',$$

where we denote $X'' = \partial_x^2 X$ and $Y'' = \partial_y^2 Y$. Thus,

$$\begin{aligned}\triangle u(x,y) &= X''(x)\cdot Y(y) + X(x)\cdot Y''(y) \\ &= \Big(X''(x)\cdot Y(y) + X(x)\cdot Y''(y)\Big)\frac{X(x)Y(y)}{X(x)Y(y)} \\ &= \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right)\cdot u(x,y).\end{aligned}$$

Thus, dividing by $u(x,y)$, Laplace's equation is equivalent to:

$$0 \;=\; \frac{\triangle u(x,y)}{u(x,y)} \;=\; \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}.$$

This is a sum of two functions which depend on *different* variables. The only way the sum can be identically zero is if each of the component functions is constant:

$$\frac{X''}{X} \equiv \lambda, \qquad \frac{Y''}{Y} \equiv -\lambda$$

So, pick some **separation constant** $\lambda \in \mathbb{R}$, and then solve the two ordinary differential equations:

$$X''(x) = \lambda \cdot X(x) \qquad \text{and} \qquad Y''(y) = -\lambda \cdot Y(y) \tag{16A.2}$$

The (real-valued) solutions to (16A.2) depends on the sign of λ. Let $\mu = \sqrt{|\lambda|}$. Then the solutions of (16A.2) have the form:

$$X(x) \;=\; \begin{cases} A\sinh(\mu x) + B\cosh(\mu x) & \text{if } \lambda > 0; \\ Ax + B & \text{if } \lambda = 0; \\ A\sin(\mu x) + B\cos(\mu x) & \text{if } \lambda < 0; \end{cases}$$

where A and B are arbitrary constants. Assuming $\lambda < 0$, and $\mu = \sqrt{|\lambda|}$, we get:

$$X(x) \;=\; A\sin(\mu x) + B\cos(\mu x) \qquad \text{and} \qquad Y(y) \;=\; C\sinh(\mu x) + D\cosh(\mu x).$$

This yields the following separated solution to Laplace's equation:

$$u(x,y) \;=\; X(x)\cdot Y(y) \;=\; \Big(A\sin(\mu x) + B\cos(\mu x)\Big)\cdot\Big(C\sinh(\mu x) + D\cosh(\mu x)\Big) \tag{16A.3}$$

Alternately, we could consider the general *complex* solution to (16A.2), given by:

$$X(x) \;=\; \exp\left(\sqrt{\lambda}\cdot x\right),$$

where $\sqrt{\lambda} \in \mathbb{C}$ is some complex number. For example, if $\lambda < 0$ and $\mu = \sqrt{|\lambda|}$, then $\sqrt{\lambda} = \pm\mu\mathbf{i}$ are imaginary, and

$$\begin{array}{rrcccl} & X_1(x) & = & \exp(\mathbf{i}\mu x) & = & \cos(\mu x) + \mathbf{i}\sin(\mu x) \\ \text{and} & X_2(x) & = & \exp(-\mathbf{i}\mu x) & = & \cos(\mu x) - \mathbf{i}\sin(\mu x) \end{array}$$

are two linearly independent solutions to (16A.2). The general solution is then given by:

$$X(x) \quad = \; a\cdot X_1(x) + b\cdot X_2(x) \quad = \; (a+b)\cdot\cos(\mu x) \; + \; \mathbf{i}\cdot(a-b)\cdot\sin(\mu x).$$

Meanwhile, the general form for $Y(y)$ is

$$Y(y) \quad = \quad c\cdot\exp(\mu y) + d\cdot\exp(-\mu y) \;=\; (c+d)\cosh(\mu y) + (c-d)\sinh(\mu y).$$

The corresponding separated solution to Laplace's equation is:

$$u(x,y) \;=\; X(x)\cdot Y(y) \;=\; \Big(A\sin(\mu x) + B\mathbf{i}\cos(\mu x)\Big)\cdot\Big(C\sinh(\mu x) + D\cosh(\mu x)\Big), \tag{16A.4}$$

where $A = (a+b)$, $B = (a-b)$, $C = (c+d)$, and $D = (c-d)$. In this case, we just recover solution (16A.3). However, we could also construct separated solutions where $\lambda \in \mathbb{C}$ is an arbitrary complex number, and $\sqrt{\lambda}$ is one of its square roots. ♢

16B Separation of variables in Cartesian coordinates on $\mathbb{R}^D$

Recommended: §16A.

Given some PDE for a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$, we apply the strategy of *separation of variables* as follows:

1. Hypothesize that u can be written as a product of D functions, each depending on only one coordinate; in other words, assume that

$$u(x_1,\ldots,x_D) \;=\; u_1(x_1)\cdot u_2(x_2)\ldots u_D(x_D) \tag{16B.5}$$

2. When we evaluate the PDE on a function of type (16B.5), we may find that the PDE decomposes into D separate, *ordinary* differential equations for each of the D functions $u_1, \ldots, u_D$. Thus, we can solve these ODEs independently, and combine the resulting solutions to get a solution for u.

Example 16B.1. *Laplace's Equation in* $\mathbb{R}^D$*:*

Suppose we want to find a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ such that $\triangle u \equiv 0$. As in the two-dimensional case (Example 16A.2), we reason:

If $u(\mathbf{x}) = X_1(x_1) \cdot X_2(x_2) \ldots X_D(x_D)$, then $\triangle u = \left(\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \ldots + \frac{X_D''}{X_D} \right) \cdot u$.

Thus, Laplace's equation is equivalent to:

$$0 \quad = \quad \frac{\triangle u}{u}(\mathbf{x}) \quad = \quad \frac{X_1''}{X_1}(x_1) \; + \; \frac{X_2''}{X_2}(x_2) \; + \; \ldots \; + \; \frac{X_D''}{x_D}(x_D).$$

This is a sum of D distinct functions, each of which depends on a different variable. The only way the sum can be identically zero is if each of the component functions is constant:

$$\frac{X_1''}{X_1} \equiv \lambda_1, \qquad \frac{X_2''}{X_2} \equiv \lambda_2, \qquad \ldots, \qquad \frac{X_D''}{X_D} \equiv \lambda_D, \tag{16B.6}$$

such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_D = 0. \tag{16B.7}$$

So, pick some **separation constant** $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_D) \in \mathbb{R}^D$ satisfying (16B.7), and then solve the ODEs:

$$X_d'' = \lambda_d \cdot X_d \qquad \text{for d=1,2,}\ldots\text{,D} \tag{16B.8}$$

The (real-valued) solutions to (16B.8) depends on the sign of λ (and clearly, if (16B.7) is going to be true, either all λ_d are zero, or some are negative and some are positive). Let $\mu = \sqrt{|\lambda|}$. Then the solutions of (16B.8) have the form:

$$X(x) \;=\; \begin{cases} A\exp(\mu x) + B\exp(-\mu x) & \text{if } \lambda > 0; \\ Ax + B & \text{if } \lambda = 0; \\ A\sin(\mu x) + B\cos(\mu x) & \text{if } \lambda < 0; \end{cases}$$

where A and B are arbitrary constants. We then combine these as in Example 16A.2. $\diamondsuit$

16C Separation in polar coordinates: Bessel's equation

Prerequisites: §0D(ii), §1C. **Recommended:** §14C, §16A.

In §14C-§14F, we explained how to use solutions of Bessel's equation to solve the heat equation or wave equation in polar coordinates. In this section, we will see how Bessel derived his equation in the first place: it arises naturally when one uses 'separation of variables' to find eigenfunctions of the Laplacian in polar coordinates. First, a technical lemma from the theory of ordinary differential equations:

Lemma 16C.1. *Let $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$ be a function satisfying periodic boundary conditions [i.e. $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$]. Let $\mu > 0$ be some constant, and suppose Θ satisfies the linear ordinary differential equation:*

$$\Theta''(\theta) = -\mu \cdot \Theta(\theta), \qquad \text{for all } \theta \in [-\pi, \pi]. \tag{16C.9}$$

Then $\mu = m^2$ for some $m \in \mathbb{N}$, and Θ must be a function of the form:

$$\Theta(\theta) \quad = \quad A\cos(m\theta) + B\sin(m\theta), \qquad \text{(for constants } A, B \in \mathbb{C}.)$$

Proof. Eqn.(16C.9) is a second-order linear ODE, so the set of all solutions to eqn.(16C.9) is a two-dimensional vector space. This vector space is spanned by functions of the form $\Theta(\theta) = e^{r\theta}$, where r is any root of the characteristic polynomial $p(x) = x^2 + \mu$. The two roots of this polynomial are of course $r = \pm\sqrt{\mu}\mathbf{i}$. Let $m = \sqrt{\mu}$ (it will turn out that m is an integer, although we don't know this yet). Hence the general solution to (16C.9) is

$$\Theta(\theta) = C_1 e^{m\mathbf{i}\theta} + C_2 e^{-m\mathbf{i}\theta},$$

where C_1 and C_2 are any two constants. The periodic boundary conditions mean that

$$\Theta(-\pi) = \Theta(\pi) \qquad \text{and} \qquad \Theta'(-\pi) = \Theta'(\pi),$$

which means

$$C_1 e^{-m\mathbf{i}\pi} + C_2 e^{m\mathbf{i}\pi} = C_1 e^{m\mathbf{i}\pi} + C_2 e^{-m\mathbf{i}\pi}, \tag{16C.10}$$

$$\text{and} \quad m\mathbf{i}C_1 e^{-m\mathbf{i}\pi} - m\mathbf{i}C_2 e^{m\mathbf{i}\pi} = m\mathbf{i}C_1 e^{m\mathbf{i}\pi} - m\mathbf{i}C_2 e^{-m\mathbf{i}\pi}. \tag{16C.11}$$

If we divide both sides of the eqn.(16C.11) by $m\mathbf{i}$, we get

$$C_1 e^{-m\mathbf{i}\pi} - C_2 e^{m\mathbf{i}\pi} = C_1 e^{m\mathbf{i}\pi} - C_2 e^{-m\mathbf{i}\pi}.$$

If we add this to eqn.(16C.10), we get

$$2C_1 e^{-m\mathbf{i}\pi} \quad = \quad 2C_1 e^{m\mathbf{i}\pi},$$

which is equivalent to $e^{2m\mathbf{i}\pi} = 1$. Hence, m must be some integer, and $\mu = m^2$.

Now, let $A := C_1 + C_2$ and $B' := C_1 - C_2$. Then $C_1 = \frac{1}{2}(A + B')$ and $C_2 = \frac{1}{2}(A - B')$. Thus,

$$\begin{aligned}\Theta(\theta) &= C_1 e^{m\mathbf{i}\theta} + C_2 e^{-m\mathbf{i}\theta} \quad = \quad (A+B')e^{m\mathbf{i}\theta} + (A-B')e^{-m\mathbf{i}\theta} \\ &= \frac{A}{2}\left(e^{m\mathbf{i}\theta} + e^{-m\mathbf{i}\theta}\right) + \frac{B'\mathbf{i}}{2\mathbf{i}}\left(e^{m\mathbf{i}\theta} - e^{-m\mathbf{i}\theta}\right) \quad = \quad A\cos(m\theta) + B'\mathbf{i}\sin(m\theta)\end{aligned}$$

because of the Euler formulas: $\cos(x) = \frac{1}{2}(e^{\mathbf{i}x} + e^{-\mathbf{i}x})$ and $\sin(x) = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}x} - e^{-\mathbf{i}x})$.

Now let $B = B'\mathbf{i}$; then $\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$, as desired. □

Proposition 16C.2. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an eigenfunction of the Laplacian* [i.e. $\triangle f = -\lambda^2 \cdot f$ for some constant $\lambda \in \mathbb{R}$]. *Suppose f **separates** in polar coordinates, meaning that there is a function $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$* (satisfying periodic boundary conditions) *and a function $\mathcal{R} : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ such that*

$$f(r,\theta) \quad = \quad \mathcal{R}(r)\cdot\Theta(\theta), \qquad \textit{for all } r \geq 0 \textit{ and } \theta \in [-\pi,\pi].$$

Then there is some $m \in \mathbb{N}$ such that

$$\Theta(\theta) \quad = \quad A\cos(m\theta) + B\sin(m\theta), \qquad \textit{(for constants } A, B \in \mathbb{R}.)$$

and $\mathcal{R}$ is a solution to the (mth order) **Bessel Equation:**

$$r^2\mathcal{R}''(r) + r\cdot\mathcal{R}'(r) + (\lambda^2 r^2 - m^2)\cdot\mathcal{R}(r) \quad = \quad 0, \quad \textit{for all } r > 0. \tag{16C.12}$$

Proof. Recall that, in polar coordinates, $\triangle f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f$. Thus, if $f(r,\theta) = \mathcal{R}(r)\cdot\Theta(\theta)$, then the eigenvector equation $\triangle f = -\lambda^2 \cdot f$ becomes

$$\begin{aligned}-\lambda^2\cdot\mathcal{R}(r)\cdot\Theta(\theta) &= \triangle\mathcal{R}(r)\cdot\Theta(\theta) \\ &= \partial_r^2\,\mathcal{R}(r)\cdot\Theta(\theta) + \frac{1}{r}\partial_r\,\mathcal{R}(r)\cdot\Theta(\theta) + \frac{1}{r^2}\partial_\theta^2\,\mathcal{R}(r)\cdot\Theta(\theta) \\ &= \mathcal{R}''(r)\Theta(\theta) + \frac{1}{r}\mathcal{R}'(r)\Theta(\theta) + \frac{1}{r^2}\mathcal{R}(r)\Theta''(\theta),\end{aligned}$$

which is equivalent to

$$-\lambda^2 = \frac{\mathcal{R}''(r)\Theta(\theta) + \frac{1}{r}\mathcal{R}'(r)\Theta(\theta) + \frac{1}{r^2}\mathcal{R}(r)\Theta''(\theta)}{\mathcal{R}(r)\cdot\Theta(\theta)}$$

$$= \frac{\mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{\mathcal{R}'(r)}{r\mathcal{R}(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)}, \qquad (16C.13)$$

If we multiply both sides of (16C.13) by r^2 and isolate the Θ'' term, we get:

$$-\lambda^2 r^2 - \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{r\mathcal{R}'(r)}{\mathcal{R}(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)}. \qquad (16C.14)$$

Abstractly, equation (16C.14) has the form: $F(r) = G(\theta)$, where F is a function depending only on r and G is a function depending only on θ. The only way this can be true is if there is some constant $\mu \in \mathbb{R}$ such that $F(r) = -\mu$ for all $r > 0$ and $G(\theta) = -\mu$ for all $\theta \in [-\pi, \pi)$. In other words,

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu, \quad \text{for all } \theta \in [-\pi, \pi), \qquad (16C.15)$$

$$\text{and} \quad \lambda^2 r^2 + \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{r\mathcal{R}'(r)}{\mathcal{R}(r)} = \mu, \quad \text{for all } r \geq 0. \qquad (16C.16)$$

Multiply both sides of equation (16C.15) by $\Theta(\theta)$ to get:

$$\Theta''(\theta) = -\mu\cdot\Theta(\theta), \quad \text{for all } \theta \in [-\pi, \pi). \qquad (16C.17)$$

Multiply both sides of equation (16C.16) by $\mathcal{R}(r)$ to get:

$$r^2\mathcal{R}''(r) + r\cdot\mathcal{R}'(r) + \lambda^2 r^2\mathcal{R}(r) = \mu\mathcal{R}(r), \quad \text{for all } r > 0. \qquad (16C.18)$$

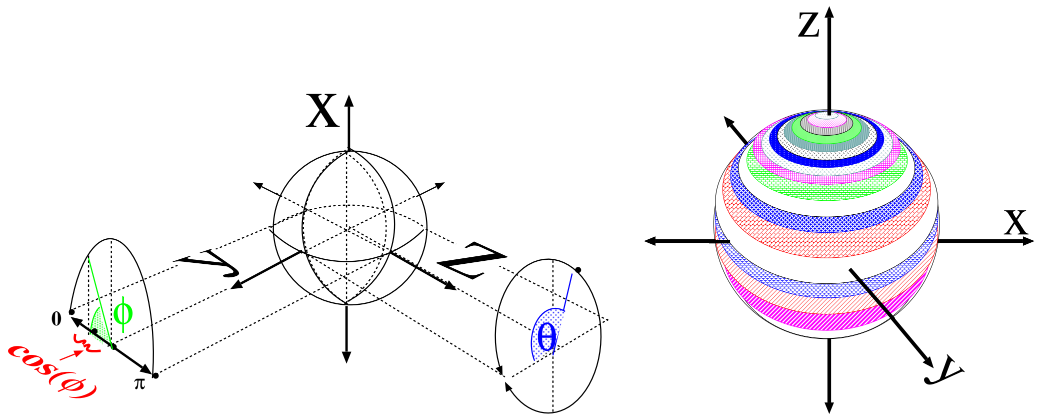
Apply Lemma 16C.1 to to eqn.(16C.17) to deduce that $\mu = m^2$ for some $m \in \mathbb{N}$, and that $\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$. Substitute $\mu = m^2$ into eqn.(16C.18) to get

$$r^2\mathcal{R}''(r) + r\cdot\mathcal{R}'(r) + \lambda^2 r^2\mathcal{R}(r) = m^2\mathcal{R}(r),$$

Now subtract $m^2\mathcal{R}(r)$ from both sides to get Bessel's equation (16C.12). □

16D Separation in spherical coordinates: Legendre's equation

Prerequisites: §0D(iv), §1C, §5C(i), §6F, §0H(iii). **Recommended:** §16C.



(A) **(B)**

Figure 16D.1: **(A)** Spherical coordinates. **(B)** Zonal functions.

Recall that *spherical coordinates* (r, θ, ϕ) on $\mathbb{R}^3$ are defined by the transformation:

$$x \quad = \quad r \cdot \sin(\phi) \cdot \cos(\theta), \qquad y \quad = \quad r \cdot \sin(\phi) \cdot \sin(\theta) \qquad \text{and} \qquad z \quad = \quad r \cdot \cos(\phi).$$

where $r \in \mathbb{R}_{\not -}$, $\theta \in [-\pi, \pi)$, and $\phi \in [0, \pi]$. The reverse transformation is defined:

$$r \quad = \quad \sqrt{x^2 + y^2 + z^2}, \qquad \theta \quad = \quad \arctan\left(\frac{y}{x}\right) \qquad \text{and } \phi \quad = \quad \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

[See Figure 16D.1(A)]. Geometrically, r is the radial distance from the origin. If we fix $r = 1$, then we get a sphere of radius 1. On the surface of this sphere, θ is *longitude* and ϕ is *latitude*. In terms of these coordinates, the Laplacian is written:

$$\triangle f(r, \theta, \phi) \quad = \quad \partial_r^2 f + \frac{2}{r} \partial_r f + \frac{1}{r^2 \sin(\phi)} \partial_\phi^2 f + \frac{\cot(\phi)}{r^2} \partial_\phi f + \frac{1}{r^2 \sin(\phi)^2} \partial_\theta^2 f.$$

Ⓔ (**Exercise 16D.1**)

A function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is called **zonal** if $f(r, \theta, \phi)$ depends only on on r and ϕ –in other words, $f(r, \theta, \phi) = F(r, \phi)$, where $F : \mathbb{R}_{\not -} \times [0, \pi] \longrightarrow \mathbb{R}$ is some other function. If we restrict f to the aforementioned sphere of radius 1, then f is invariant under rotations around the 'north-south axis' of the sphere. Thus, f is constant along lines of equal latitude around the sphere, so it divides the sphere into 'zones' from north to south [Figure 16D.1(B)].

Proposition 16D.1. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be zonal. Suppose f is a harmonic function (i.e. $\triangle f = 0$). Suppose f separates in spherical coordinates, meaning* *that there are (bounded) functions* $\Phi : [0, \pi] \longrightarrow \mathbb{R}$ *and* $\mathcal{R} : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ *such that*

$$f(r, \theta, \phi) \quad = \quad \mathcal{R}(r) \cdot \Phi(\phi), \qquad \textit{for all } r \geq 0, \;\; \phi \in [0, \pi], \textit{ and } \theta \in [-\pi, \pi].$$

Then there is some $\mu \in \mathbb{R}$ *such that* $\Phi(\phi) = \mathcal{L}[\cos(\phi)]$, *where* $\mathcal{L} : [-1, 1] \longrightarrow \mathbb{R}$ *is a (bounded) solution of the* **Legendre Equation:**

$$(1 - x^2)\mathcal{L}''(x) - 2x\mathcal{L}'(x) + \mu\mathcal{L}(x) \quad = \quad 0, \tag{16D.19}$$

and $\mathcal{R}$ *is a (bounded) solution to the* **Cauchy-Euler Equation:**

$$r^2\mathcal{R}''(r) + 2r \cdot \mathcal{R}'(r) - \mu \cdot \mathcal{R}(r) \quad = \quad 0, \quad \textit{for all } r > 0. \tag{16D.20}$$

Proof. By hypothesis

$$\begin{aligned} 0 \quad &= \quad \triangle f(r, \theta, \phi) \\ &= \quad \partial_r^2 f \; + \; \frac{2}{r}\partial_r f \; + \; \frac{1}{r^2 \sin(\phi)}\partial_\phi^2 f \; + \; \frac{\cot(\phi)}{r^2}\partial_\phi f \; + \; \frac{1}{r^2 \sin(\phi)^2}\partial_\theta^2 f \\ &\underset{(*)}{=} \quad \mathcal{R}''(r) \cdot \Phi(\phi) \; + \; \frac{2}{r}\mathcal{R}'(r) \cdot \Phi(\phi) \\ &\qquad\qquad + \; \frac{1}{r^2 \sin(\phi)}\mathcal{R}(r) \cdot \Phi''(\phi) \; + \; \frac{\cot(\phi)}{r^2}\mathcal{R}(r) \cdot \Phi'(\phi) \; + \; 0. \end{aligned}$$

[where $(*)$ is because $f(r, \theta, \phi) = \mathcal{R}(r) \cdot \Phi(\phi)$.] Hence, multiplying both sides by $\dfrac{r^2}{\mathcal{R}(r) \cdot \Phi(\phi)}$, we obtain

$$0 \quad = \quad \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} \; + \; \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} \; + \; \frac{1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} \; + \; \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)},$$

Or, equivalently,

$$\frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} \; + \; \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} \quad = \quad \frac{-1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} \; - \; \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)}. \tag{16D.21}$$

Now, the left hand side of (16D.21) depends only on the variable r, whereas the right hand side depends only on ϕ. The only way that these two expressions can be equal for *all* values of r and ϕ is if both expressions are constants. In other words, there is some constant $\mu \in \mathbb{R}$ (called a *separation constant*) such that

$$\begin{aligned} \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} \; + \; \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} \quad &= \quad \mu, \qquad \text{for all } r \geq 0, \\ \text{and} \quad \frac{1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} \; + \; \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)} \quad &= \quad -\mu, \qquad \text{for all } \phi \in [0, \pi]. \end{aligned}$$

Or, equivalently,

$$r^2\mathcal{R}''(r) \ + \ 2r\mathcal{R}'(r) \quad = \quad \mu\mathcal{R}(r), \quad \text{for all } r \geq 0, \tag{16D.22}$$

$$\text{and} \quad \frac{\Phi''(\phi)}{\sin(\phi)} \ + \ \cot(\phi)\Phi'(\phi) \quad = \quad -\mu\Phi(\phi), \quad \text{for all } \phi \in [0,\pi]. \tag{16D.23}$$

If we make the change of variables $x = \cos(\phi)$ (so that $\phi = \arccos(x)$, where $x \in [-1,1]$), then $\Phi(\phi) = \mathcal{L}(\cos(\phi)) = \mathcal{L}(x)$, where $\mathcal{L}$ is some other (unknown) function.

Claim 1: *The function Φ satisfies the ODE* (16D.23) *if and only if $\mathcal{L}$ satisfies the Legendre equation* (16D.19).

Ⓔ *Proof.* **<u>Exercise 16D.2</u>** (Hint: This is a straightforward application of the Chain Rule.) $\diamond_{\text{Claim 1}}$

Finally, observe that the ODE (16D.22) is equivalent to the Cauchy-Euler equation (16D.20). □

For all $n \in \mathbb{N}$, we define the nth **Legendre Polynomial** by

$$\mathcal{P}_n(x) \quad := \quad \frac{1}{n!\,2^n}\partial_x^n\left[(x^2-1)\right]^n. \tag{16D.24}$$

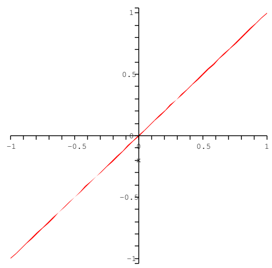
For example:

$$\begin{array}{lll lll l}
\mathcal{P}_0(x) & = & 1 & \mathcal{P}_3(x) & = & \frac{1}{2}(5x^3-3x) & \\
\mathcal{P}_1(x) & = & x & \mathcal{P}_4(x) & = & \frac{1}{8}(35x^4-30x^2+3) & \text{(see Figure 16D.2(A))} \\
\mathcal{P}_2(x) & = & \frac{1}{2}(3x^2-1) & \mathcal{P}_5(x) & = & \frac{1}{8}(63x^5-70x^3+15x). &
\end{array}$$

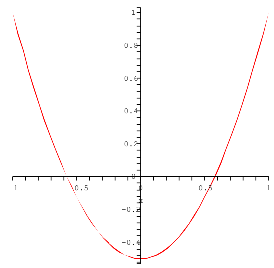
Lemma 16D.2. *Let $n \in \mathbb{N}$. Then the Legendre Polynomial $\mathcal{P}_n$ is a solution to the Legendre Equation* (16D.19) *with $\mu = n(n+1)$.*

Ⓔ *Proof.* **<u>Exercise 16D.3</u>** (Direct computation) □

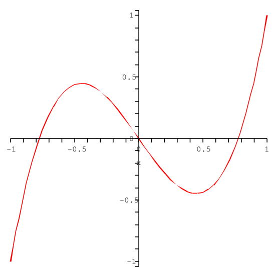
Is $\mathcal{P}_n$ the *only* solution to the Legendre Equation (16D.19)? No, because the Legendre Equation is an order-two linear ODE, so the set of solutions forms a two-dimensional vector space $\mathcal{V}$. The scalar multiples of $\mathcal{P}_n$ form a one-dimensional subspace of $\mathcal{V}$. However, to be physically meaningful, we need the solutions to be bounded at $x = \pm 1$. So instead we ask: is $\mathcal{P}_n$ the only *bounded* solution to the Legendre Equation (16D.19)? Also, what happens if $\mu \neq n(n+1)$ for any $n \in \mathbb{N}$?



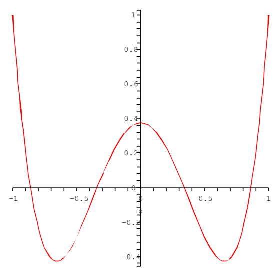
$\mathcal{P}_1(x)$



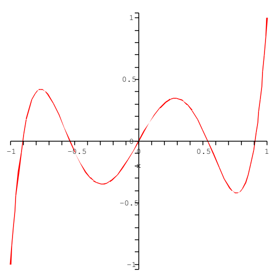
$\mathcal{P}_2(x)$



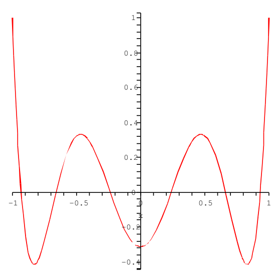
$\mathcal{P}_3(x)$



$\mathcal{P}_4(x)$



$\mathcal{P}_5(x)$



$\mathcal{P}_6(x)$

Above. The Legendre polynomials $\mathcal{P}_1(x)$ to $\mathcal{P}_6(x)$, plotted for $x \in [-1, 1]$.

Right. Substitution of the power series $\sum_{n=0}^{\infty} a_n x^n$ into the Legendre Equation (16D.19), in the proof of Claim 1 of Lemma 16D.3.

$$
\begin{array}{lrcccccccccc}
\text{If} & \mathcal{L}(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + \cdots + & a_k x^k & + \cdots \\
\text{then} & \mu\mathcal{L}(x) & = & \mu a_0 & + & \mu a_1 x & + & \mu a_2 x^2 & + \cdots + & \mu a_k x^k & + \cdots \\
\text{and} & -2x\mathcal{L}'(x) & = & & & -2a_1 x & + & -4a_2 x^2 & + \cdots + & -2k a_k x^k & + \cdots \\
\text{and} & \mathcal{L}''(x) & = & 2a_2 & + & 6a_3 x & + & 12a_4 x^2 & + \cdots + & (k+2)(k+1)a_{k+2} x^k & + \cdots \\
\text{and} & -x^2\mathcal{L}''(x) & = & & & & & -2a_2 x^2 & + \cdots + & -(k-1)k a_k x^k & + \cdots
\end{array}
$$

$$
\begin{array}{l}
\text{Thus,} \quad 0 = \\
(1-x^2)\mathcal{L}''(x) - 2x\mathcal{L}'(x) + \mu\mathcal{L}(x) = (\mu a_0 - 2a_2) + ((\mu-2)a_1 + 6a_3)\,x + ((\mu-6)a_2 + 12a_4)\,x^2 + \cdots + b_k x^k + \cdots
\end{array}
$$

where $b_k := (k+2)(k+1)a_{k+2} + \left[\mu - k(k+1)\right] a_k$ for all $k \in \mathbb{N}$.

Figure 16D.2:

Lemma 16D.3.

(a) *If* $\mu = n(n+1)$ *for some* $n \in \mathbb{N}$*, then* (up to multiplication by a scalar)*, the Legendre polynomial* $\mathcal{P}_n(x)$ *is the unique solution to the Legendre Equation* (16D.19) *which is bounded on* $[-1,1]$.

(b) *If* $\mu \neq n(n+1)$ *for any* $n \in \mathbb{N}$*, then all solutions to the Legendre Equation* (16D.19) *are infinite power series which diverge at* $x = \pm 1$ (and thus, are unsuitable for Proposition 16D.1).

Proof. We apply the *Power series method* (see § 0H(iii) on page 571). Suppose $\mathcal{L}(x) = \sum_{n=0}^{\infty} a_n x^n$ is some analytic function defined on $[-1,1]$ (where the coefficients $\{a_n\}_{n=1}^{\infty}$ are as yet unknown).

Claim 1: $\mathcal{L}(x)$ *satisfies the Legendre Equation* (16D.19) *if and only if the coefficients* $\{a_0, a_1, a_2, \ldots\}$ *satisfy the recurrence relation*

$$a_{k+2} \quad = \quad \frac{k(k+1)-\mu}{(k+2)(k+1)}\, a_k, \qquad \text{for all } k \in \mathbb{N}. \tag{16D.25}$$

In particular, $a_2 \;=\; \frac{-\mu}{2} a_0$ *and* $a_3 \;=\; \frac{2-\mu}{6} a_1$.

Proof. We will substitute the power series $\sum_{n=0}^{\infty} a_n x^n$ into the Legendre Equation (16D.19). The details of the computation are shown on the right side of Figure 16D.2. The computation yields the equation $0 = \sum_{k=0}^{\infty} b_k x_k$, where $b_k := (k+2)(k+1)a_{k+2} + \Big[\mu - k(k+1)\Big]\, a_k$ for all $k \in \mathbb{N}$. It follows that $b_k = 0$ for all $k \in \mathbb{N}$; in other words, that

$$(k+2)(k+1)a_{k+2} + \Big[\mu - k(k+1)\Big]\, a_k \quad = \quad 0, \quad \text{for all } k \in \mathbb{N}.$$

Rearranging this equation produces the desired recurrence relation (16D.25). $\diamond_{\text{Claim 1}}$

The space of all solutions to the Legendre Equation (16D.19) is a two-dimensional vector space, because the Legendre equation is a *linear* differential equation of order 2. We will now find a basis for this space. Recall that $\mathcal{L}$ is *even* if $\mathcal{L}(-x) = \mathcal{L}(x)$ for all $x \in [-1,1]$, and $\mathcal{L}$ is *odd* if $\mathcal{L}(-x) = -\mathcal{L}(x)$ for all $x \in [-1,1]$.

Claim 2: *There is a unique even analytic function* $\mathcal{E}(x)$ *and a unique odd analytic function* $\mathcal{O}(x)$ *which satisfy the Legendre Equation* (16D.19)*, such that* $\mathcal{E}(1) = 1 = \mathcal{O}(1)$*, and such that any other solution* $\mathcal{L}(x)$ *can be written as a linear combination* $\mathcal{L}(x) = a\,\mathcal{E}(x) + b\,\mathcal{O}(x)$*, for some constants* $a, b \in \mathbb{R}$.

Proof. Claim 1 implies that the power series $\mathcal{L}(x) = \sum_{n=0}^{\infty} a_n x^n$ is entirely determined by the coefficients a_0 and a_1. To be precise, $\mathcal{L}(x) = \mathcal{E}(x) + \mathcal{O}(x)$, where $\mathcal{E}(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$ and $\mathcal{O}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ both satisfy the recurrence relation (16D.25), and thus, are solutions to the Legendre Equation (16D.19). $\diamond_{\text{Claim 2}}$

Claim 3: *Suppose $\mu = n(n+1)$ for some $n \in \mathbb{N}$. Then the Legendre equation* (16D.19) *has a degree-n polynomial as one of its solutions. To be precise:*

(a) *If n is even, then $a_k = 0$ for all even $k > n$. Hence, $\mathcal{E}(x)$ is a degree-n polynomial.*

(b) *If n is odd, then $a_k = 0$ for all odd $k > n$. Hence, $\mathcal{O}(x)$ is a degree-n polynomial.*

Proof. **Exercise 16D.4** $\diamond_{\text{Claim 3}}$ Ⓔ

Thus, there is a one-dimensional space of *polynomial* solutions to the Legendre equation —namely all scalar multiples of $\mathcal{E}(x)$ (if n is even) or $\mathcal{O}(x)$ (if n is odd).

Claim 4: *If $\mu \neq n(n+1)$ for any $n \in \mathbb{N}$, the series $\mathcal{E}(x)$ and $\mathcal{O}(x)$ both diverge at $x = \pm 1$.*

Proof. **Exercise 16D.5** (a) First note that an infinite number of coefficients $\{a_n\}_{n=0}^{\infty}$ are nonzero. Ⓔ

(b) Show that $\lim_{n\to\infty} |a_n| = 1$.

(c) Conclude that the series $\mathcal{E}(x)$ and $\mathcal{O}(x)$ diverge when $x = \pm 1$. $\diamond_{\text{Claim 4}}$

So, there exist solutions to the Legendre equation (16D.19) that are bounded on $[-1, 1]$ if *and only if* $\mu = n(n+1)$ for some $n \in \mathbb{N}$, and in this case, the bounded solutions are all scalar multiples of a polynomial of degree n [either $\mathcal{E}(x)$ or $\mathcal{O}(x)$]. But Lemma 16D.2 says that the Legendre polynomial $\mathcal{P}_n(x)$ is a solution to the Legendre equation (16D.19). Thus, (up to multiplication by a constant), $\mathcal{P}_n(x)$ must be equal to $\mathcal{E}(x)$ (if n is even) or $\mathcal{O}(x)$ (if n is odd). □

Remark: Sometimes the Legendre polynomials are *defined* as the (unique) polynomial solutions to Legendre's equation; the definition we have given in eqn.(16D.24) is then *derived* from this definition, and is called *Rodrigues Formula.*

Lemma 16D.4. *Let $\mathcal{R} : \mathbb{R}_{\not=} \longrightarrow \mathbb{R}$ be a solution to the Cauchy-Euler equation*

$$r^2\mathcal{R}''(r) + 2r \cdot \mathcal{R}'(r) - n(n+1) \cdot \mathcal{R}(r) \quad = \quad 0, \quad \text{for all } r > 0. \qquad (16D.26)$$

Then $\mathcal{R}(r) = Ar^n + \frac{B}{r^{n+1}}$ for some constants A and B.

If $\mathcal{R}$ is bounded at zero, then $B = 0$, so $\mathcal{R}(r) = Ar^n$.

Proof. Check that $f(r) = r^n$ and $g(r) = r^{-n-1}$ are solutions to eqn.(16D.26). But (16D.26) is a second-order linear ODE, so the solutions form a 2-dimensional vector space. Since f and g are linearly independent, they span this vector space. □

Corollary 16D.5. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a zonal harmonic function that separates in spherical coordinates* (as in Proposition 16D.1). *Then there is some $m \in \mathbb{N}$ such that $f(r, \phi, \theta) = Cr^n \cdot \mathcal{P}_n[\cos(\phi)]$, where $\mathcal{P}_n$ is the nth Legendre Polynomial, and $C \in \mathbb{R}$ is some constant.* (See Figure 16D.3.)

Proof. Combine Proposition 16D.1 with Lemmas 16D.3 and 16D.4 □

Thus, the Legendre polynomials are important when solving the Laplace equation on spherical domains. We now describe some of their important properties

Proposition 16D.6. *Legendre polynomials satisfy the following recurrence relations:*

(a) $(2n+1)\mathcal{P}_n(x) \;=\; \mathcal{P}'_{n+1}(x) - \mathcal{P}'_{n-1}(x)$.

(b) $(2n+1)x\mathcal{P}_n(x) \;=\; (n+1)\mathcal{P}_{n+1}(x) + n\mathcal{P}'_{n-1}(x)$.

Ⓔ *Proof.* **Exercise 16D.6** □

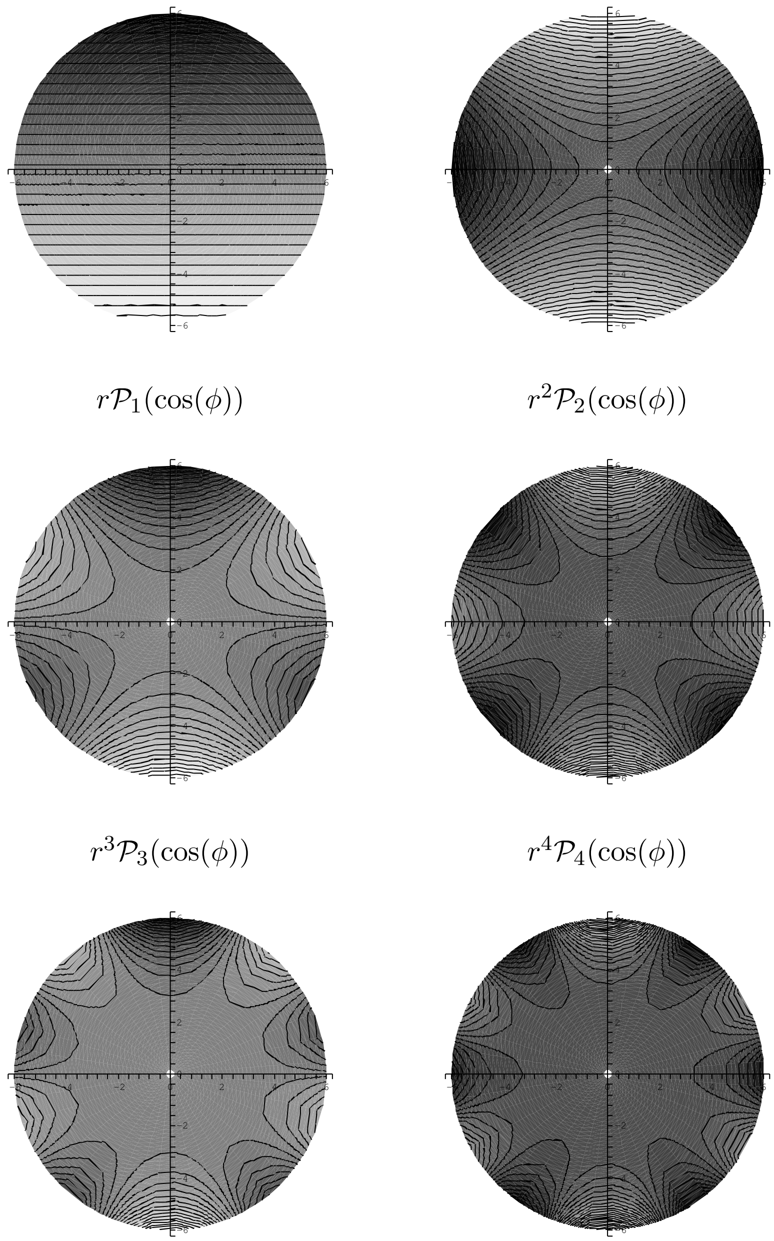
Proposition 16D.7. *The Legendre polynomials form an orthogonal set for $\mathbf{L}^2[-1,1]$. That is:*

(a) *For any $n \neq m$,* $\langle \mathcal{P}_n, \mathcal{P}_m \rangle \;=\; \dfrac{1}{2}\displaystyle\int_{-1}^{1} \mathcal{P}_n(x)\mathcal{P}_m(x)\, dx \;=\; 0.$

(b) *For any $n \in \mathbb{N}$,* $\|\mathcal{P}_n\|_2^2 \;=\; \dfrac{1}{2}\displaystyle\int_{-1}^{1} \mathcal{P}_n^2(x)dx \;=\; \dfrac{1}{2n+1}.$

Ⓔ *Proof.* **(a) Exercise 16D.7** (Hint: Start with the Rodrigues formula (16D.24). Apply integration by parts n times.)

Ⓔ **(b) Exercise 16D.8** (Hint: Use Proposition 16D.6(b).) □



$r\mathcal{P}_1(\cos(\phi))$ $\quad$ $r^2\mathcal{P}_2(\cos(\phi))$

$r^3\mathcal{P}_3(\cos(\phi))$ $\quad$ $r^4\mathcal{P}_4(\cos(\phi))$

$r^5\mathcal{P}_5(\cos(\phi))$ $\quad$ $r^6\mathcal{P}_6(\cos(\phi))$

Figure 16D.3: Planar cross-sections of the zonal harmonic functions $r\mathcal{P}_1(\cos(\phi))$ to $r^6\mathcal{P}_6(\cos(\phi))$, plotted for $r \in [0...6]$; see Corollary 16D.5 on the preceding page. Remember that these are functions in $\mathbb{R}^3$. To visualize these functions in three dimensions, take the above contour plots and mentally rotate them around the vertical axis.

Because of Proposition 16D.7, we can try to represent an arbitrary function $f \in \mathbf{L}^2[-1,1]$ in terms of Legendre polynomials, to obtain a **Legendre Series:**

$$f(x) \underset{L2}{\approx} \sum_{n=0}^{\infty} a_n \mathcal{P}_n(x), \tag{16D.27}$$

where $a_n := \dfrac{\langle f, \mathcal{P}_n \rangle}{\|\mathcal{P}_n\|_2^2} = \dfrac{2n+1}{2} \displaystyle\int_{-1}^{1} f(x)\mathcal{P}_n(x)\,dx$ is the nth **Legendre coefficient** of f.

Theorem 16D.8. *The Legendre polynomials form an orthogonal basis for $\mathbf{L}^2[-1,1]$. Thus, if $f \in \mathbf{L}^2[-1,1]$, then the Legendre series* (16D.27) *converges to f in L^2.*

Proof. See [Bro89, Thm 3.2.4, p.50]. □

Let $\mathbb{B} = \{(r,\theta,\phi)\ ;\ r \leq 1,\ \theta \in [-\pi,\pi],\ \phi \in [0,\pi]\}$ be the unit ball in spherical coordinates. Thus, $\partial\mathbb{B} = \{(1,\theta,\phi)\ ;\ \theta \in [-\pi,\pi],\ \phi \in [0,\pi]\}$ is the unit sphere. Recall that a *zonal* function on $\partial\mathbb{B}$ is a function which depends only on the 'latitude' coordinate ϕ, and not on the 'longitude' coordinate θ.

Theorem 16D.9. Dirichlet problem on a ball

Let $f : \partial\mathbb{B} \longrightarrow \mathbb{R}$ be some function describing a heat distribution on the surface of the ball. Suppose f is zonal –i.e. $f(1,\theta,\phi) = F(\cos(\phi))$, where $F \in \mathbf{L}^2[-1,1]$, and F has Legendre series

$$F(x) \underset{L2}{\approx} \sum_{n=0}^{\infty} a_n \mathcal{P}_n(x).$$

Define $u : \mathbb{B} \longrightarrow \mathbb{R}$ by $u(r,\phi,\theta) = \sum_{n=0}^{\infty} a_n r^n \mathcal{P}_n(\cos(\phi))$. Then u is the uniqe solution to the Laplace equation, satisfying the nonhomogeneous Dirichlet boundary conditions

$$u(1,\theta,\phi) \underset{L2}{\approx} f(\theta,\phi), \qquad \text{for all } (1,\theta,\phi) \in \partial\mathbb{B}.$$

Ⓔ *Proof.* **Exercise 16D.9** □

16E Separated vs. quasiseparated

Prerequisites: §16B.

If we use complex-valued functions like (16A.4) as the components of the separated solution (16B.5) on page 355, then we will still get mathematically valid solutions to Laplace's equation (as long as (16B.7) is true). However, these solutions are not physically meaningful —what does a *complex*-valued heat distribution feel like? This is not a problem, because we can extract *real*-valued solutions from the complex solution as follows.

Proposition 16E.1. *Suppose* L *is a linear differential operator with real-valued coefficients, and* $g : \mathbb{R}^D \longrightarrow \mathbb{R}$, *and consider the nonhomogeneous PDE "*$\mathsf{L}\, u \;=\; g$*".*

If $u : \mathbb{R}^D \longrightarrow \mathbb{C}$ *is a (complex-valued) solution to this PDE, and we define* $u_R(\mathbf{x}) = \mathrm{Re}\,[u(\mathbf{x})]$ *and* $u_I(\mathbf{x}) = \mathrm{Im}\,[u(\mathbf{x})]$, *then* $\mathsf{L}\, u_R \;=\; g$ *and* $\mathsf{L}\, u_I \;=\; 0$.

Proof. **Exercise 16E.1** □ Ⓔ

In this case, the solutions u_R and u_I are not themselves generally going to be in separated form. Since they arise as the real and imaginary components of a complex separated solution, we call u_R and u_I **quasiseparated** solutions.

Example 16E.2. Recall the separated solutions to the two-dimensional Laplace equation from Example 16A.2 on page 354. Here, $\mathsf{L} = \triangle$ and $g \equiv 0$, and, for any fixed $\mu \in \mathbb{R}$, the function

$$u(x,y) \;=\; X(x)\cdot Y(y) \;=\; \exp(\mu y)\cdot \exp(\mu \mathbf{i} y)$$

is a complex solution to Laplace's equation. Thus,

$$u_R(x,y) \;=\; \exp(\mu x)\cos(\mu y) \qquad \text{and} \qquad u_I(x,y) \;=\; \exp(\mu x)\sin(\mu y)$$

are real-valued solutions of the form obtained earlier. ♢

16F The polynomial formalism

Prerequisites: §16B, §4B.

Separation of variables seems like a bit of a miracle. Just how generally applicable is it? To answer this, it is convenient to adopt a **polynomial formalism** for differential operators. If L is a differential operator with *constant*[1]

[1] This is important.

coefficients, we will formally represent L as a "polynomial" in the "variables" $\partial_1, \partial_2, \ldots, \partial_D$. For example, we can write the Laplacian:

$$\triangle = \partial_1^2 + \partial_2^2 + \ldots + \partial_D^2 = \mathcal{P}(\partial_1, \partial_2, \ldots, \partial_D),$$

where $\mathcal{P}(x_1, x_2, \ldots, x_D) = x_1^2 + x_2^2 + \ldots + x_D^2$.

In another example, the general second-order linear PDE

$$A\partial_x^2 u \quad + \quad B\partial_x \partial_y u \quad + \quad C\partial_y^2 u \quad + \quad D\partial_x u \quad + \quad E\partial_y u \quad + \quad Fu \quad = \quad G$$

(where $A, B, C, \ldots, F$ are constants) can be rewritten:

$$\mathcal{P}(\partial_x, \partial_y)u \quad = \quad g$$

where $\mathcal{P}(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$.

The polynomial $\mathcal{P}$ is called the **polynomial symbol** of L, and provides a convenient method for generating separated solutions

Proposition 16F.1. *Suppose that L is a linear differential operator on $\mathbb{R}^D$ with polynomial symbol $\mathcal{P}$. Regard $\mathcal{P} : \mathbb{C}^D \longrightarrow \mathbb{C}$ as a function.*

Fix $\mathbf{z} = (z_1, \ldots, z_D) \in \mathbb{C}^D$, and define $u_{\mathbf{z}} : \mathbb{R}^D \longrightarrow \mathbb{R}$ by

$$u_{\mathbf{z}}(x_1, \ldots, x_D) \quad = \quad \exp(z_1 x_1) \cdot \exp(z_2 x_2) \ldots \exp(z_D x_D) \quad = \quad \exp(\mathbf{z} \bullet \mathbf{x}).$$

Then $\mathsf{L}u_{\mathbf{z}}(\mathbf{x}) \ = \ \mathcal{P}(\mathbf{z}) \cdot u_{\mathbf{z}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$.

In particular, if $\mathbf{z}$ is a root of $\mathcal{P}$ (that is, $\mathcal{P}(z_1, \ldots, z_D) = 0$), then $\mathsf{L}u = 0$.

Ⓔ *Proof.* **Exercise 16F.1** Hint: First, use formula (0C.1) on page 551 to show that $\partial_d u_{\mathbf{z}} = z_d \cdot u_{\mathbf{z}}$, and, more generally, $\partial_d^n u_{\mathbf{z}} = z_d^n \cdot u_{\mathbf{z}}$. □

Thus, many[2] separated solutions of the *differential* equation "$\mathsf{L}u = 0$" are defined by the the complex-valued solutions of the *algebraic* equation "$\mathcal{P}(\mathbf{z}) = 0$".

Example 16F.2. Consider again the two-dimensional Laplace equation

$$\partial_x^2 u + \partial_y^2 u = 0$$

The corresponding polynomial is $\mathcal{P}(x, y) = x^2 + y^2$. Thus, if $z_1, z_2 \in \mathbb{C}$ are any complex numbers such that $z_1^2 + z_2^2 = 0$, then

$$u(x, y) = \exp(z_1 x + z_2 y) = \exp(z_1 x) \cdot \exp(z_2 y)$$

is a solution to Laplace's equation. In particular, if $z_1 = 1$, then we must have $z_2 = \pm\mathbf{i}$. Say we pick $z_2 = \mathbf{i}$; then the solution becomes

$$u(x, y) \quad = \quad \exp(x) \cdot \exp(\mathbf{i}y) \quad = \quad e^x \cdot \Big(\cos(y) + \mathbf{i}\sin(y)\Big).$$

[2]But not all.

More generally, if we choose $z_1 = \mu \in \mathbb{R}$ to be a real number, then we must choose $z_2 = \pm\mu\mathbf{i}$ to be purely imaginary, and the solution becomes

$$u(x,y) \quad = \quad \exp(\mu x)\cdot\exp(\pm\mu\mathbf{i}y) \quad = \quad e^{\mu x}\cdot\Big(\cos(\pm\mu y)+\mathbf{i}\sin(\pm\mu y)\Big).$$

Compare this with the separated solutions obtained from Example 16A.2 on page 354. ♢

Example 16F.3. Consider the one-dimensional **telegraph equation:**

$$\partial_t^2 u + 2\partial_t u + u = \triangle u. \tag{16F.28}$$

We can rewrite this as

$$\partial_t^2 u + 2\partial_t u + u - \partial_x^2 u = 0,$$

which is equivalent to "$\mathsf{L}\,u = 0$", where L is the linear differential operator

$$\mathsf{L} \;=\; \partial_t^2 + 2\partial_t + u - \partial_x^2,$$

with polynomial symbol

$$\mathcal{P}(x,t) \;=\; t^2+2t+1-x^2 \;=\; (t+1+x)(t+1-x).$$

Thus, the equation "$\mathcal{P}(\alpha,\beta)=0$" has solutions:

$$\alpha = \pm(\beta+1)$$

So, if we define $u(x,t) = \exp(\alpha\cdot x)\exp(\beta\cdot t)$, then u is a separated solution to equation (16F.28). (**Exercise 16F.2** Check this.). In particular, suppose we choose $\alpha = -\beta-1$. Then the separated solution is $u(x,t) = \exp(\beta(t-x)-x)$. If $\beta = \beta_R + \beta_I\mathbf{i}$ is a complex number, then the quasiseparated solutions are: Ⓔ

$$\begin{aligned} u_R &= \exp\left(\beta_R(x+t)-x\right)\cdot\cos\left(\beta_I(x+t)\right) \\ u_I &= \exp\left(\beta_R(x+t)-x\right)\cdot\sin\left(\beta_I(x+t)\right)\ . \end{aligned}$$ ♢

Remark 16F.4: The polynomial formalism provides part of the motivation for the classification of PDEs as *elliptic*, *hyperbolic*[3], etc. Notice that, if L is an elliptic differential operator on $\mathbb{R}^2$, then the real-valued solutions to $\mathcal{P}(z_1,z_2)=0$ (if any) form an *ellipse* in $\mathbb{R}^2$. In $\mathbb{R}^D$, the solutions form an *ellipsoid*.

Similarly, if we consider the parabolic PDE "$\partial_t u = \mathsf{L}u$", the the corresponding differential operator $\mathsf{L}-\partial_t$ has polynomial symbol $\mathcal{Q}(\mathbf{x};t) = \mathcal{P}(\mathbf{x})-t$. The real-valued solutions to $\mathcal{Q}(\mathbf{x};t)=0$ form a *paraboloid* in $\mathbb{R}^D\times\mathbb{R}$. For example, the 1-dimensional heat equation "$\partial_x^2 u - \partial_t u = 0$" yields the classic equation "$t = x^2$" for a parabola in the (x,t)-plane. Similarly, with a hyperbolic PDE, the differential operator $\mathsf{L}-\partial_t^2$ has polynomial symbol $\mathcal{Q}(\mathbf{x};t) = \mathcal{P}(\mathbf{x})-t^2$, and the roots form a *hyperboloid*.

[3]See § 5E on page 95.

16G Constraints

Prerequisites: §16F.

Normally, we are not interested in just *any* solution to a PDE; we want a solution which satisfies certain constraints. The most common constraints are:

- **Boundary Conditions:** If the PDE is defined on some bounded domain $\mathbb{X} \subset \mathbb{R}^D$, then we may want the solution function u (or its derivatives) to have certain values on the boundary of this domain.
- **Boundedness:** If the domain $\mathbb{X}$ is unbounded (e.g. $\mathbb{X} = \mathbb{R}^D$), then we may want the solution u to be *bounded*; in other words, we want some finite $M > 0$ such that $|u(\mathbf{x})| < M$ for all values of some coordinate x_d.

16G(i) Boundedness

The solution obtained through Proposition 16F.1 is not generally going to be bounded, because the exponential function $f(x) = \exp(\lambda x)$ is not bounded as a function of x, unless λ is a purely imaginary number. More generally:

Proposition 16G.1. *Fix $\mathbf{z} = (z_1, \ldots, z_D) \in \mathbb{C}^D$, and suppose $u_\mathbf{z} : \mathbb{R}^D \longrightarrow \mathbb{R}$ is defined as in Proposition 16F.1:*

$$u_\mathbf{z}(x_1, \ldots, x_D) \;=\; \exp(z_1 x_1) \cdot \exp(z_2 x_2) \ldots \exp(z_D x_D) \;=\; \exp(\mathbf{z} \bullet \mathbf{x}).$$

Then:

1. *$u(\mathbf{x})$ is bounded for all values of the variable $x_d \in \mathbb{R}$ if and only if $z_d = \lambda\mathbf{i}$ for some $\lambda \in \mathbb{R}$.*
2. *$u(\mathbf{x})$ is bounded for all $x_d > 0$ if and only if $z_d = \rho + \lambda\mathbf{i}$ for some $\rho \leq 0$.*
3. *$u(\mathbf{x})$ is bounded for all $x_d < 0$ if and only if $z_d = \rho + \lambda\mathbf{i}$ for some $\rho \geq 0$.*

Ⓔ *Proof.* **Exercise 16G.1** □

Example 16G.2. Recall the one-dimensional telegraph equation of Example 16F.3:

$$\partial_t^2 u + 2\partial_t u + u = \triangle u$$

We constructed a separated solution of the form: $u(x, t) = \exp(\alpha x + \beta t)$, where $\alpha = \pm(\beta + 1)$. This solution will be bounded in time if and only if β is a purely imaginary number; i.e. $\beta = \beta_I \cdot \mathbf{i}$. Then $\alpha = \pm(\beta_I \cdot \mathbf{i} + 1)$, so that

$u(x,t) \;=\; \exp(\pm x)\cdot\exp\Big(\beta_I\cdot(t\pm x)\cdot\mathbf{i}\Big)$; thus, the quasiseparated solutions are:

$$u_R = \exp(\pm x)\cdot\cos\Big(\beta_I\cdot(t\pm x)\Big) \qquad \text{and} \qquad u_I = \exp(\pm x)\cdot\sin\Big(\beta_I\cdot(t\pm x)\Big).$$

Unfortunately, this solution is *unbounded* in *space*, which is probably not what we want. An alternative is to set $\beta = \beta_I\mathbf{i} - 1$, and then set $\alpha = \beta + 1 = \beta_I\mathbf{i}$. Then the solution becomes $u(x,t) = \exp(\beta_I\mathbf{i}(x+t) - t) = e^{-t}\exp(\beta_I\mathbf{i}(x+t))$, and the quasiseparated solutions are:

$$u_R = e^{-t}\cdot\cos\left(\beta_I(x+t)\right) \qquad \text{and} \qquad u_I = e^{-t}\cdot\sin\left(\beta_I(x+t)\right).$$

These solutions are exponentially decaying as $t\to\infty$, and thus, bounded in "forward time". For any fixed time t, they are also bounded (and actually periodic) functions of the space variable x. ♢

16G(ii) Boundary conditions

Prerequisites: §5C.

There is no cureall like Proposition 16G.1 for satisfying boundary conditions, since generally they are different in each problem. Generally, a single separated solution (say, from Proposition 16F.3) will *not* be able to satisfy the conditions; we must sum together several solutions, so that they "cancel out" in suitable ways along the boundaries. For these purposes, the following *Euler identities* are often useful:

$$\begin{aligned}
\sin(x) &= \frac{e^{x\mathbf{i}} - e^{-x\mathbf{i}}}{2\mathbf{i}}; & \cos(x) &= \frac{e^{x\mathbf{i}} + e^{-x\mathbf{i}}}{2\mathbf{i}}; \\
\sinh(x) &= \frac{e^{x} - e^{-x}}{2}; & \cosh(x) &= \frac{e^{x} + e^{-x}}{2}.
\end{aligned}$$

which we can utilize along with the following boundary information:

$$\begin{aligned}
-\cos'(n\pi) &= \sin(n\pi) = 0, && \text{for all } n\in\mathbb{Z}; \\
\sin'\left(\left(n+\frac{1}{2}\right)\pi\right) &= \cos\left(\left(n+\frac{1}{2}\right)\pi\right) = 0, && \text{for all } n\in\mathbb{Z}; \\
\cosh'(0) &= \sinh(0) = 0.
\end{aligned}$$

For "rectangular" domains, the boundaries are obtained by fixing a particular coordinate at a particular value; i.e. they are each of the form form $\left\{\mathbf{x}\in\mathbb{R}^D\;;\;x_d = K\right\}$ for some constant K and some dimension d. The convenient thing about a separated solution is that it is a product of D functions, and only *one* of them is involved in satisfying this boundary condition.

For example, recall Example 16F.2 on page 370, which gave the separated solution $u(x,y) \;=\; e^{\mu x}\cdot\Big(\cos(\pm\mu y) + \mathbf{i}\sin(\pm\mu y)\Big)$ for the two-dimensional Laplace equation, where $\mu \in \mathbb{R}$. Suppose we want the solution to satisfy *homogeneous Dirichlet boundary conditions*:

$$u(x,y) = 0 \quad \text{if} \quad x = 0, \quad \text{or} \quad y = 0, \quad \text{or} \quad y = \pi.$$

To satisfy these three conditions, we proceed as follows:

$$\begin{aligned} \text{First, let } u_1(x,y) &= e^{\mu x} \cdot \Big(\cos(\mu y) + \mathbf{i}\sin(\mu y)\Big), \\ \text{and } u_2(x,y) &= e^{\mu x} \cdot \Big(\cos(-\mu y) + \mathbf{i}\sin(-\mu y)\Big) \quad = \quad e^{\mu x} \cdot \Big(\cos(\mu y) - \mathbf{i}\sin(\mu y)\Big). \end{aligned}$$

If we define $v(x,y) = u_1(x,y) - u_2(x,y)$, then

$$v(x,y) = 2e^{\mu x} \cdot \mathbf{i}\sin(\mu y).$$

At this point, $v(x,y)$ already satisfies the boundary conditions for $\{y = 0\}$ and $\{y = \pi\}$. To satisfy the remaining condition:

$$\begin{aligned} \text{Let } v_1(x,y) &= 2e^{\mu x} \cdot \mathbf{i}\sin(\mu y), \\ \text{and } v_1(x,y) &= 2e^{-\mu x} \cdot \mathbf{i}\sin(\mu y). \end{aligned}$$

If we define $w(x,y) = v_1(x,y) - v_2(x,y)$, then

$$w(x,y) = 4\sinh(\mu x) \cdot \mathbf{i}\sin(\mu y)$$

also satisfies the boundary condition at $\{x = 0\}$.

Chapter 17

Impulse-response methods

"Nature laughs at the difficulties of integration." —Pierre-Simon Laplace

17A Introduction

A fundamental concept in science is *causality*: an initial event (an *impulse*) at some location $\mathbf{y}$ causes a later event (a *response*) at another location $\mathbf{x}$ (Figure 17A.1A). In an evolving, spatially distributed system (e.g. a temperature distribution, a rippling pond, etc.), the system state at each location results from a *combination* of the responses to the impulses from all other locations (as in Figure 17A.1B).

If the system is described by a linear PDE, then we expect some sort of 'superposition principle' to apply (Theorem 4C.3 on page 65). Hence, we can replace the word 'combination' with 'sum', and say:

The state of the system at $\mathbf{x}$ is a sum *of the responses to the impulses from all other locations.* (17A.1)

(See Figure 17A.1B). However, there are an infinite number —indeed, a continuum —of 'other locations', so we are 'summing' over a *continuum* of responses. But a 'sum' over a continuum is just an *integral*. Hence, statement (17A.1) becomes:

In a linear PDE, the solution at $\mathbf{x}$ is an integral *of the responses to the impulses from all other locations.* (17A.2)

The relation between impulse and response (i.e. between cause and effect) is described by *impulse-response function*, $\Gamma(\mathbf{y} \to \mathbf{x})$, which measures the degree of 'influence' which point $\mathbf{y}$ has on point $\mathbf{x}$. In other words, $\Gamma(\mathbf{y} \to \mathbf{x})$ measures the strength of the response at $\mathbf{x}$ to an impulse at $\mathbf{y}$. In a system which evolves in time, Γ may also depend on time (since it takes time for the effect from $\mathbf{y}$ to propagate to $\mathbf{x}$), so Γ also depends on time, and is written $\Gamma_t(\mathbf{y} \to \mathbf{x})$.

Intuitively, Γ should have four properties:

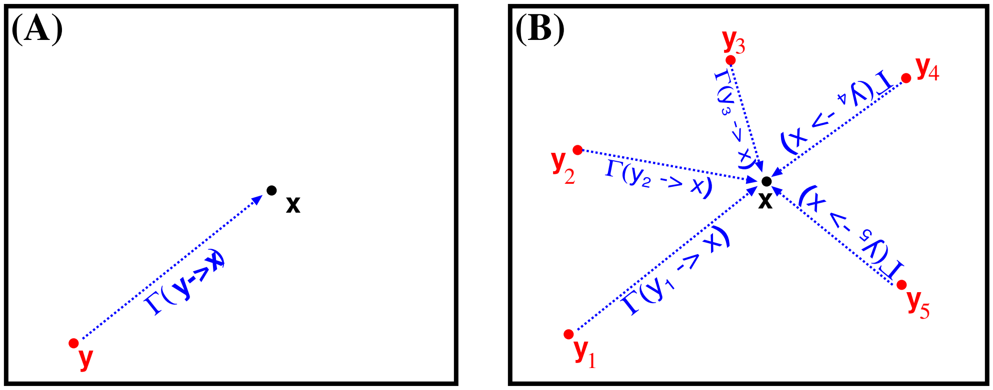


Figure 17A.1: **(A)** $\Gamma(\mathbf{y} \to \mathbf{x})$ describes the 'response' at $\mathbf{x}$ to an 'impulse' at $\mathbf{y}$. **(B)** The state at $\mathbf{x}$ is a sum of its responses to the impulses at $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_5$.

(i) Influence should *decay with distance*. In other words, if $\mathbf{y}$ and $\mathbf{x}$ are close together, then $\Gamma(\mathbf{y} \to \mathbf{x})$ should be large; if $\mathbf{y}$ and $\mathbf{x}$ are far apart, then $\Gamma(\mathbf{y} \to \mathbf{x})$ should be small (Figure 17A.2).

(ii) In a *spatially homogeneous* or *translation invariant* system (Figure 17A.3(A)), Γ should only depend on the *displacement* from $\mathbf{y}$ to $\mathbf{x}$, so that we can write $\Gamma(\mathbf{y} \to \mathbf{x}) = \gamma(\mathbf{x} - \mathbf{y})$, where γ is some other function.

(iii) In an *isotropic* or *rotation invariant* system system (Figure 17A.3(B)), Γ should only depend on the *distance* between $\mathbf{y}$ and $\mathbf{x}$, so that we can write $\Gamma(\mathbf{y} \to \mathbf{x}) = \psi\left(|\mathbf{x} - \mathbf{y}|\right)$, where ψ is a function of one real variable, and $\lim_{r\to\infty} \psi(r) = 0$.

(iv) In a *time-evolving* system, the value of $\Gamma_t(\mathbf{y} \to \mathbf{x})$ should first grow as t increases (as the effect 'propagates' from $\mathbf{y}$ to $\mathbf{x}$), reach a maximum value, and then decrease to zero as t grows large (as the effect 'dissipates' through space) (see Figure 17A.4).

Thus, if there is an 'impulse' of magnitude $\mathcal{I}$ at $\mathbf{y}$, and $\mathcal{R}(\mathbf{x})$ is the 'response' at $\mathbf{x}$, then

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I} \cdot \Gamma(\mathbf{y} \to \mathbf{x}) \qquad \text{(see Figure 17A.5A)}$$

What if there is an impulse $\mathcal{I}(\mathbf{y}_1)$ at $\mathbf{y}_1$, an impulse $\mathcal{I}(\mathbf{y}_2)$ at $\mathbf{y}_2$, and an impulse $\mathcal{I}(\mathbf{y}_3)$ at $\mathbf{y}_3$? Then statement (17A.1) implies:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I}(\mathbf{y}_1) \cdot \Gamma(\mathbf{y}_1 \to \mathbf{x}) \quad + \quad \mathcal{I}(\mathbf{y}_2) \cdot \Gamma(\mathbf{y}_2 \to \mathbf{x}) \quad + \quad \mathcal{I}(\mathbf{y}_3) \cdot \Gamma(\mathbf{y}_3 \to \mathbf{x}).$$

(see Figure 17A.5B). If $\mathbb{X}$ is the domain of the PDE, then suppose, for every $\mathbf{y}$

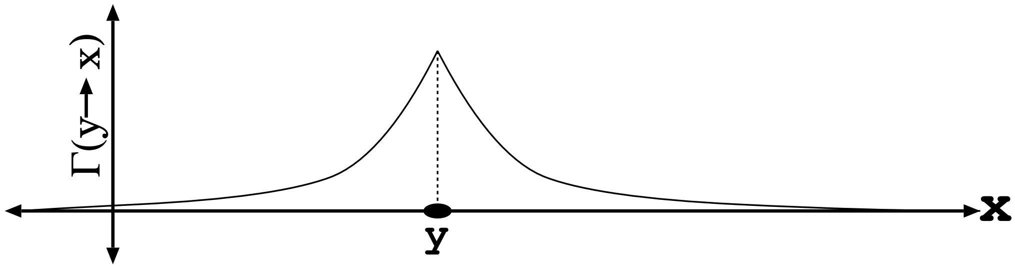


Figure 17A.2: The influence of $\mathbf{y}$ on $\mathbf{x}$ becomes small as the distance from $\mathbf{y}$ to $\mathbf{x}$ grows large.

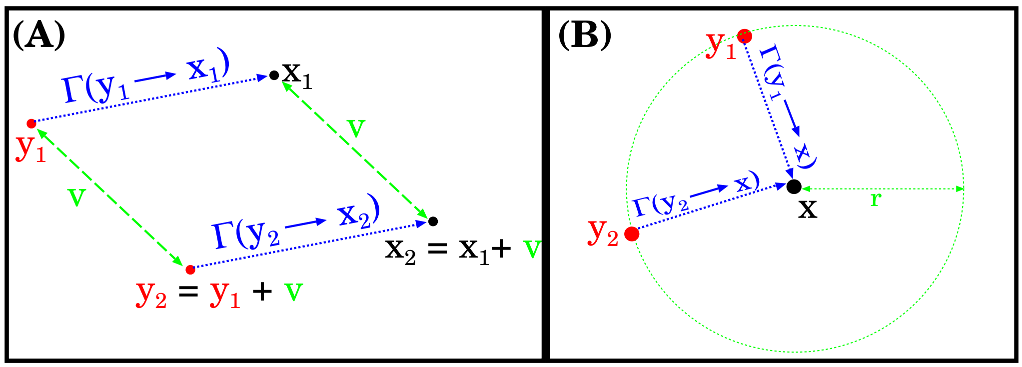


Figure 17A.3: **(A)** *Translation invariance*: If $\mathbf{y}_2 = \mathbf{y}_1 + \mathbf{v}$ and $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{v}$, then $\Gamma(\mathbf{y}_2 \to \mathbf{x}_2) = \Gamma(\mathbf{y}_1 \to \mathbf{x}_1)$. **(B)** *Rotation invariance*: If $\mathbf{y}_1$ and $\mathbf{y}_2$ are both the same distance from $\mathbf{x}$ (i.e. they lie on the circle of radius r around $\mathbf{x}$), then $\Gamma(\mathbf{y}_2 \to \mathbf{x}) = \Gamma(\mathbf{y}_1 \to \mathbf{x})$.

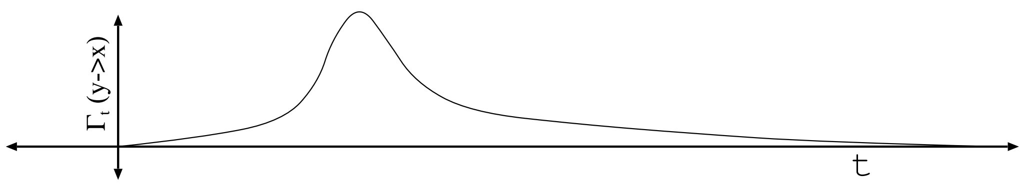


Figure 17A.4: The time-dependent impulse-response function first grows large, and then decays to zero.

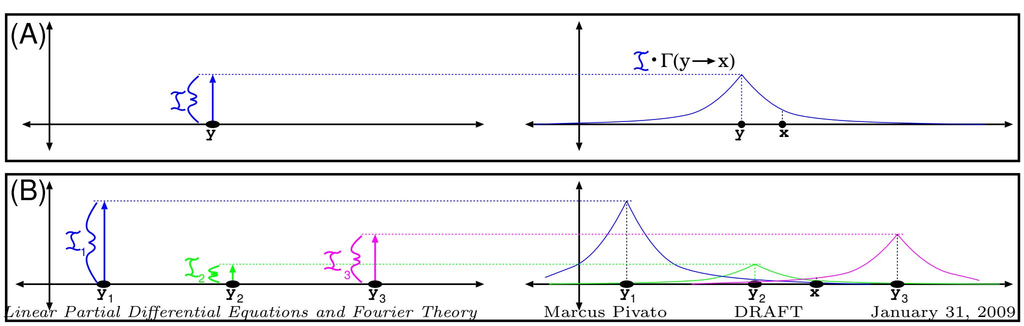


Figure 17A.5: **(A)** An 'impulse' of magnitude $\mathcal{I}$ at $\mathbf{y}$ triggers a 'response' of magnitude $\mathcal{I} \cdot \Gamma(\mathbf{y} \to \mathbf{x})$ at $\mathbf{x}$. **(B)** Multiple 'impulses' of magnitude $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ at $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$, respectively, triggers a 'response' at $\mathbf{x}$ of magnitude $\mathcal{I}_1 \cdot \Gamma(\mathbf{y}_1 \to \mathbf{x}) + \mathcal{I}_2 \cdot \Gamma(\mathbf{y}_2 \to \mathbf{x}) + \mathcal{I}_3 \cdot \Gamma(\mathbf{y}_3 \to \mathbf{x})$.

in $\mathbb{X}$, that $\mathcal{I}(\mathbf{y})$ is the impulse at $\mathbf{y}$. Then statement (17A.1) takes the form:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \sum_{\mathbf{y}\in\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma(\mathbf{y} \to \mathbf{x}). \tag{17A.3}$$

But now we are summing over all $\mathbf{y}$ in $\mathbb{X}$, and usually, $\mathbb{X} = \mathbb{R}^D$ or some subset, so the 'summation' in (17A.3) doesn't make mathematical sense. We must replace the sum with an *integral*, as in statement (17A.2), to obtain:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \int_{\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma(\mathbf{y} \to \mathbf{x})\ d\mathbf{y}. \tag{17A.4}$$

If the system is spatially homogeneous, then according to **(ii)**, this becomes

$$\mathcal{R}(\mathbf{x}) \quad = \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma(\mathbf{x} - \mathbf{y})\ d\mathbf{y}.$$

This integral is called a **convolution**, and is usually written as $\mathcal{I} * \gamma$. In other words,

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I} * \gamma(\mathbf{x}), \qquad \text{where} \qquad \mathcal{I} * \gamma(\mathbf{x}) \quad := \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma(\mathbf{x} - \mathbf{y})\ d\mathbf{y}. \tag{17A.5}$$

Note that $\mathcal{I} * \gamma$ is a function of $\mathbf{x}$. The variable $\mathbf{y}$ appears on the right hand side, but as only an *integration* variable.

In a time-dependent system, (17A.4) becomes:

$$\mathcal{R}(\mathbf{x};\ t) \quad = \quad \int_{\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma_t(\mathbf{y} \to \mathbf{x})\ d\mathbf{y}.$$

while (17A.5) becomes:

$$\mathcal{R}(\mathbf{x};\ t) \quad = \quad \mathcal{I} * \gamma_t(\mathbf{x}), \qquad \text{where} \qquad \mathcal{I} * \gamma_t(\mathbf{x}) \quad = \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma_t(\mathbf{x} - \mathbf{y})\ d\mathbf{y}. \tag{17A.6}$$

The following surprising property is often useful:

Proposition 17A.1. *If $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ are integrable functions, then $g * f = f * g$.*

Proof. (Case $D = 1$) Fix $x \in \mathbb{R}$. Then

$$\begin{aligned}(g * f)(x) \quad &= \quad \int_{-\infty}^{\infty} g(y) \cdot f(x-y)\ dy \quad \underset{\text{(s)}}{=} \quad \int_{\infty}^{-\infty} g(x-z) \cdot f(z) \cdot (-1)\ dz \\ &= \quad \int_{-\infty}^{\infty} f(z) \cdot g(x-z)\ dz \quad = \quad (f * g)(x).\end{aligned}$$

Here, step **(s)** was the substitution $z = x - y$, so that $y = x - z$ and $dy = -dz$.

Ⓔ <u>**Exercise 17A.1**</u> Generalize this proof to the case $D \geq 2$. □

Remarks: (a) Depending on the context, impulse-response functions are sometimes called *solution kernels*, or *Green's functions* or *impulse functions*.

(b) If f and g are analytic functions, then there is an efficient way to compute $f * g$ using complex analysis; see Corollary 18H.3 on page 474.

17B Approximations of identity

17B(i) ...in one dimension

Prerequisites: §17A.

Suppose $\gamma : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ was a one-dimensional *impulse response function*, as in equation (17A.6). Thus, if $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ is a function describing the initial 'impulse', then for any time $t > 0$, the 'response' is given by the function $\mathcal{R}_t$ defined:

$$\mathcal{R}_t(x) \quad := \quad \mathcal{I} * \gamma_t(x) \quad = \quad \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \gamma_t(x-y)\ dy. \tag{17B.1}$$

Intuitively, if t is close to zero, then the response $\mathcal{R}_t$ should be concentrated near the locations where the impulse $\mathcal{I}$ is concentrated (because the energy has not yet been able to propagate very far). By inspecting eqn.(17B.1), we see that this means that the mass of γ_t should be 'concentrated' near zero. Formally, we say that γ is an **approximation of the identity** if it has the following properties (Figure 17B.1):

(AI1) $\gamma_t(x) \geq 0$ everywhere, and $\displaystyle\int_{-\infty}^{\infty} \gamma_t(x)\ dx \;=\; 1$ for any fixed $t > 0$.

(AI2) For any $\epsilon > 0$, $\displaystyle\lim_{t \to 0} \int_{-\epsilon}^{\epsilon} \gamma_t(x)\ dx \;=\; 1.$

Property **(AI1)** says that γ_t is a probability density. **(AI2)** says that γ_t concentrates all of its "mass" at zero as $t \to 0$. (Heuristically speaking, the function γ_t is converging to the 'Dirac delta function' δ_0 as $t \to 0$.)

Example 17B.1.

(a) Let $\gamma_t(x) \;=\; \begin{cases} \frac{1}{t} & \text{if} \quad 0 \leq x \leq t; \\ 0 & \text{if} \quad x < 0 \text{ or } t < x. \end{cases}$ (Figure 17B.2)

Thus, for any $t > 0$, the graph of γ_t is a 'box' of width t and height $1/t$. Then γ is an approximation of identity. (See Practice Problem # 11 on page 413.)

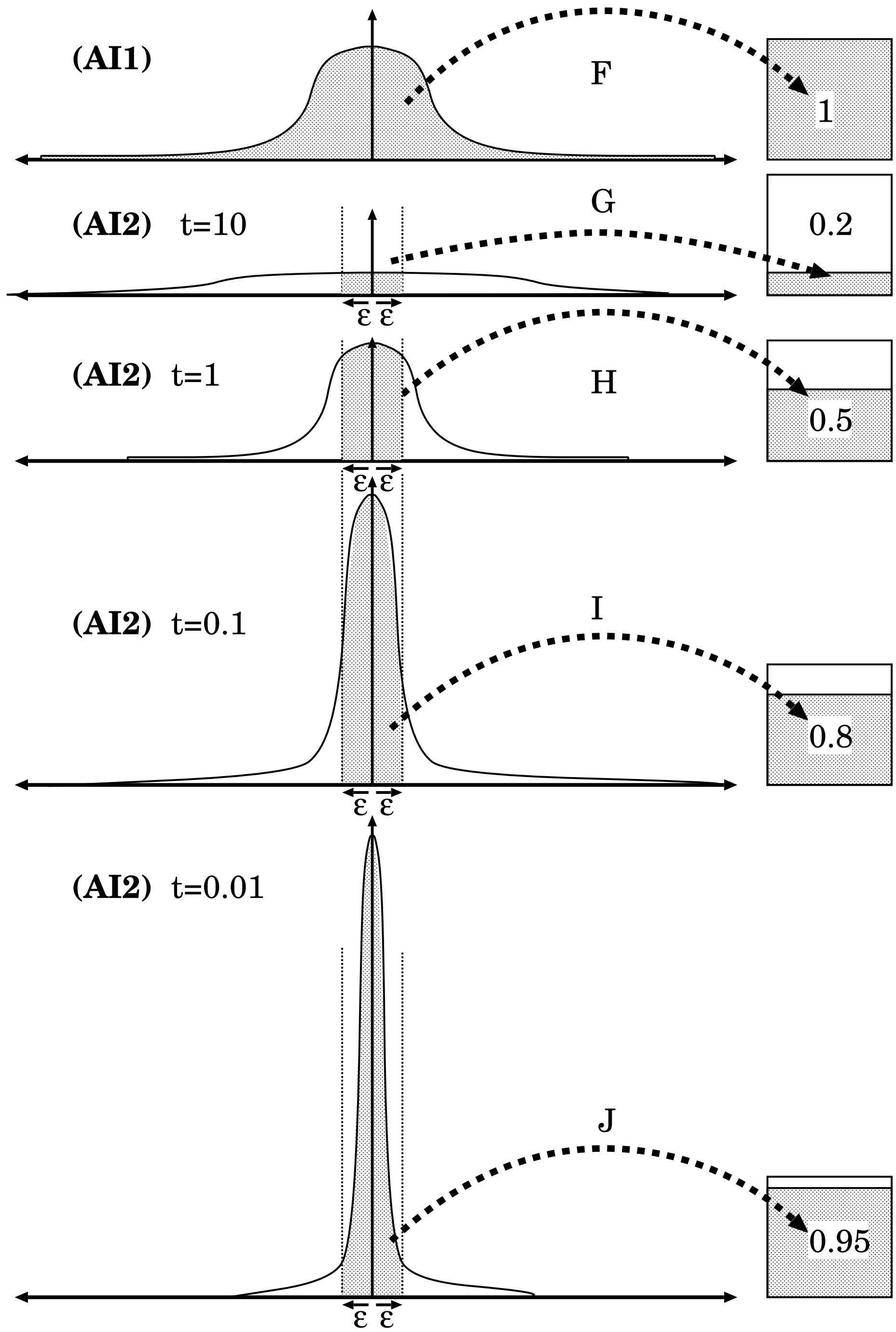


Figure 17B.1: γ is an approximation of the identity.

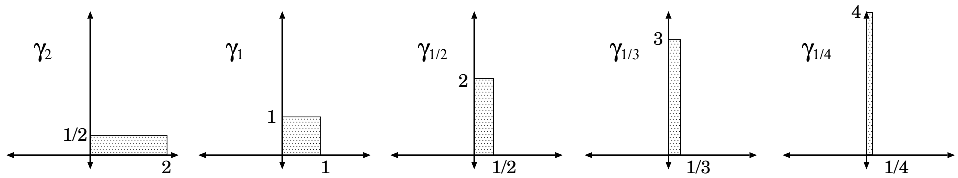


Figure 17B.2: Example 17B.1(a)

(b) Let $\gamma_t(x) \; = \; \begin{cases} \frac{1}{2t} & \text{if} \quad |x| \leq t \\ 0 & \text{if} \quad t < |x| \end{cases}$.

Thus, for any $t > 0$, the graph of γ_t is a 'box' of width $2t$ and height $1/2t$. Then γ is an approximation of identity. (See Practice Problem # 12 on page 413.) ♢

A function satisfying properties **(AI1)** and **(AI2)** is called an *approximation of the identity* because of the following theorem:

Proposition 17B.2. *Let $\gamma : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an approximation of identity.*

(a) *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function. Then for all $x \in \mathbb{R}$, $\lim_{t\to 0} \mathcal{I} * \gamma_t(x) \; = \mathcal{I}(x)$.*

(b) *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be any bounded integrable function. If $x \in \mathbb{R}$ is any continuity-point of $\mathcal{I}$, then $\lim_{t\to 0} \mathcal{I} * \gamma_t(x) \; = \mathcal{I}(x)$.*

Proof. **(a)** Fix $x \in \mathbb{R}$. Given any $\epsilon > 0$, find $\delta > 0$ such that,

$$\text{For all } y \in \mathbb{R}, \quad \Big(\, |y - x| < \delta \, \Big) \quad \Longrightarrow \quad \left(\left| \mathcal{I}(y) - \mathcal{I}(x) \right| \; < \; \tfrac{\epsilon}{3} \right).$$

(Such an ϵ exists because $\mathcal{I}$ is continuous). Thus,

$$\begin{aligned} & \left| \mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y) \; dy \quad - \quad \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y) \; dy \right| \\ = & \quad \left| \int_{x-\delta}^{x+\delta} \Big(\mathcal{I}(x) - \mathcal{I}(y) \Big) \cdot \gamma_t(x-y) \; dy \right| \quad \leq \quad \int_{x-\delta}^{x+\delta} \left| \mathcal{I}(x) - \mathcal{I}(y) \right| \cdot \gamma_t(x-y) \; dy \\ < & \quad \frac{\epsilon}{3} \int_{x-\delta}^{x+\delta} \gamma_t(x-y) \; dy \quad \underset{\text{(AI1)}}{<} \quad \frac{\epsilon}{3}. \end{aligned} \tag{17B.2}$$

(Here **(AI1)** is by property **(AI1)** of γ_t.)

Recall that $\mathcal{I}$ is *bounded*. Suppose $|\mathcal{I}(y)| < M$ for all $y \in \mathbb{R}$; using **(AI2)**, find some small $\tau > 0$ such that, if $t < \tau$, then $\int_{x-\delta}^{x+\delta} \gamma_t(y) \; dy \; > \; 1 - \frac{\epsilon}{3M}$; hence

$$\int_{-\infty}^{x-\delta} \gamma_t(y) \; dy \quad + \quad \int_{x+\delta}^{\infty} \gamma_t(y) \; dy \quad = \quad \int_{-\infty}^{\infty} \gamma_t(y) \; dy \quad - \quad \int_{x-\delta}^{x+\delta} \gamma_t(y) \; dy$$

$$\underset{\text{(AI1)}}{<} \quad 1-\left(1-\frac{\epsilon}{3M}\right) \;=\; \frac{\epsilon}{3M}. \qquad (17B.3)$$

(Here **(AI1)** is by property **(AI1)** of γ_t.) Thus,

$$\begin{aligned}
&\left|\mathcal{I}*\gamma_t(x) \;-\; \int_{x-\delta}^{x+\delta}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy\right| \\
&\le \left|\int_{-\infty}^{\infty}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy \;-\; \int_{x-\delta}^{x+\delta}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy\right| \\
&= \left|\int_{-\infty}^{x-\delta}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy \;+\; \int_{x+\delta}^{\infty}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy\right| \\
&\le \int_{-\infty}^{x-\delta}\left|\mathcal{I}(y)\cdot\gamma_t(x-y)\right|\,dy \;+\; \int_{x+\delta}^{\infty}\left|\mathcal{I}(y)\cdot\gamma_t(x-y)\right|\,dy \\
&\le \int_{-\infty}^{x-\delta}M\cdot\gamma_t(x-y)\,dy \;+\; \int_{x+\delta}^{\infty}M\cdot\gamma_t(x-y)\,dy \\
&\le M\cdot\left(\int_{-\infty}^{x-\delta}\gamma_t(x-y)\,dy \;+\; \int_{x+\delta}^{\infty}\gamma_t(x-y)\,dy\right) \\
&\underset{(*)}{\le} M\cdot\frac{\epsilon}{3M} \;=\; \frac{\epsilon}{3}. \qquad (17B.4)
\end{aligned}$$

(Here, $(*)$ is by eqn.(17B.3).) Combining equations (17B.2) and (17B.4) we have:

$$\begin{aligned}
&\left|\mathcal{I}(x)\cdot\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\,dy \;-\; \mathcal{I}*\gamma_t(x)\right| \\
&\le \left|\mathcal{I}(x)\cdot\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\,dy \;-\; \int_{x-\delta}^{x+\delta}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy\right| \\
&\qquad + \left|\int_{x-\delta}^{x+\delta}\mathcal{I}(y)\cdot\gamma_t(x-y)\,dy \;-\; \mathcal{I}*\gamma_t(x)\right| \\
&\le \frac{\epsilon}{3}+\frac{\epsilon}{3} \;=\; \frac{2\epsilon}{3}. \qquad (17B.5)
\end{aligned}$$

But if $t<\tau$,then $\left|1-\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\,dy\right| \;<\; \frac{\epsilon}{3M}$. Thus,

$$\begin{aligned}
\left|\mathcal{I}(x) \;-\; \mathcal{I}(x)\cdot\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\,dy\right| &\le |\mathcal{I}(x)|\cdot\left|1 \;-\; \int_{x-\delta}^{x+\delta}\gamma_t(x-y)\,dy\right| \\
&< |\mathcal{I}(x)|\cdot\frac{\epsilon}{3M} \;\le\; M\cdot\frac{\epsilon}{3M} \;=\; \frac{\epsilon}{3}. \qquad (17B.6)
\end{aligned}$$

Combining equations (17B.5) and (17B.6) we have:

$|\mathcal{I}(x) \;-\; \mathcal{I}*\gamma_t(x)|$

$$\leq \left|\mathcal{I}(x) \quad - \quad \mathcal{I}(x)\cdot\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\ dy\right| \quad + \quad \left|\mathcal{I}(x)\cdot\int_{x-\delta}^{x+\delta}\gamma_t(x-y)\ dy \quad - \quad \mathcal{I}*\gamma_t(x)\right|$$

$$\leq \frac{\epsilon}{3} + \frac{2\epsilon}{3}. \quad = \quad \epsilon.$$

Since ϵ can be made arbitrarily small, we're done.

(b) **Exercise 17B.1** (Hint: imitate part **(a)**). □ Ⓔ

In other words, as $t \to 0$, the convolution $\mathcal{I} * \gamma_t$ resembles $\mathcal{I}$ with arbitrarily high accuracy. Similar convergence results can be proved in other norms (e.g. L^2 convergence, uniform convergence).

Example 17B.3. Let $\gamma_t(x) \;=\; \begin{cases} \frac{1}{t} & \text{if} \quad 0 \leq x \leq t \\ 0 & \text{if} \quad x < 0 \text{ or } t < x \end{cases}$, as in Example 17B.1(a). Suppose $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Then for any $x \in \mathbb{R}$,

$$\mathcal{I}*\gamma_t(x) \quad = \quad \int_{-\infty}^{\infty}\mathcal{I}(y)\cdot\gamma_t(x-y)\ dy \quad = \quad \frac{1}{t}\int_{x-t}^{x}\mathcal{I}(y)\ dy \quad = \quad \frac{1}{t}\Big(\mathcal{J}(x) - \mathcal{J}(x-t)\Big),$$

where $\mathcal{J}$ is an antiderivative of $\mathcal{I}$. Thus, as implied by Proposition 17B.2,

$$\lim_{t\to 0}\mathcal{I}*\gamma_t(x) \quad = \quad \lim_{t\to 0}\frac{\mathcal{J}(x)-\mathcal{J}(x-t)}{t} \quad \underset{(*)}{=} \quad \mathcal{J}'(x) \quad \underset{(\dagger)}{=} \quad \mathcal{I}(x).$$

(Here $(*)$ is just the definition of differentiation, and $(\dagger)$ is because $\mathcal{J}$ is an antiderivative of $\mathcal{I}$.) ♢

17B(ii) ...in many dimensions

Prerequisites: §17B(i). **Recommended:** §17C(i).

A nonnegative function $\gamma : \mathbb{R}^D \times \mathbb{R}_+ \longrightarrow \mathbb{R}_{\not-}$ is called an **approximation of the identity** if it has the following two properties:

(AI1) $\displaystyle\int_{\mathbb{R}^D}\gamma_t(\mathbf{x})\ d\mathbf{x} \;=\; 1$ for all $t \in [0,\infty]$.

(AI2) For any $\epsilon > 0$, $\displaystyle\lim_{t\to 0}\int_{\mathbb{B}(0;\epsilon)}\gamma_t(\mathbf{x})\ d\mathbf{x} \;=\; 1.$

Property **(AI1)** says that γ_t is a probability density. **(AI2)** says that γ_t concentrates all of its "mass" at zero as $t \to 0$.

Example 17B.4. Define $\gamma : \mathbb{R}^2\times\mathbb{R}_+ \longrightarrow \mathbb{R}$ by $\gamma_t(x,y) = \begin{cases} \frac{1}{4t^2} & \text{if } |x| \leq t \text{ and } |y| \leq t; \\ 0 & \text{otherwise.}\end{cases}$.

Then γ is an approximation of the identity on $\mathbb{R}^2$. (**Exercise 17B.2**) ♢ Ⓔ

Proposition 17B.5. *Let $\gamma : \mathbb{R}^D \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an approximation of the identity.*

(a) *Let $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a bounded continuous function. Then for every $\mathbf{x} \in \mathbb{R}^D$, we have $\lim_{t\to 0} \mathcal{I} * \gamma_t(\mathbf{x}) = \mathcal{I}(\mathbf{x})$.*

(b) *Let $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ be any bounded integrable function. If $\mathbf{x} \in \mathbb{R}^D$ is any continuity-point of $\mathcal{I}$, then $\lim_{t\to 0} \mathcal{I} * \gamma_t(\mathbf{x}) = \mathcal{I}(\mathbf{x})$.*

Ⓔ *Proof.* **Exercise 17B.3** Hint: the argument is basically identical to that of Proposition 17B.2; just replace the interval $(-\epsilon, \epsilon)$ with a ball of radius ϵ. □

In other words, as $t \to 0$, the convolution $\mathcal{I} * \gamma_t$ resembles $\mathcal{I}$ with arbitrarily high accuracy. Similar convergence results can be proved in other norms (e.g. L^2 convergence, uniform convergence).

When solving partial differential equations, approximations of identity are invariably used in conjunction with the following result:

Proposition 17B.6. *Let L be a linear differential operator on $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$.*

(a) *If $\gamma : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a solution to the homogeneous equation "$\mathsf{L}\,\gamma = 0$", then for any function $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, the function $u = \mathcal{I} * \gamma$ satisfies: $\mathsf{L}\,u = 0$.*

(b) *If $\gamma : \mathbb{R}^D \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies the evolution equation "$\partial_t^n \gamma = \mathsf{L}\,\gamma$", and we define $\gamma_t(\mathbf{x}) := \gamma(\mathbf{x}; t)$, then for any function $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, the function $u_t = \mathcal{I} * \gamma_t$ satisfies: $\partial_t^n u = \mathsf{L}\,u$.*

Ⓔ *Proof.* **Exercise 17B.4** Hint: Generalize the proof of Proposition 17C.1 on the facing page, by replacing the one-dimensional convolution integral with a D-dimensional convolution integral, and by replacing the Laplacian with an arbitrary linear operator L. □

Corollary 17B.7. *Suppose γ is an approximation of the identity* and *satisfies the evolution equation "$\partial_t^n \gamma = \mathsf{L}\,\gamma$". For any $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, define $u : \mathbb{R}^D \times \mathbb{R}_{\not+} \longrightarrow \mathbb{R}$ by:*

- *$u(\mathbf{x}; 0) = \mathcal{I}(\mathbf{x})$.*
- *$u_t = \mathcal{I} * \gamma_t$, for all $t > 0$.*

Then u is a solution to the equation "$\partial_t^n u = \mathsf{L}\,u$", and u satisfies the initial conditions $u(\mathbf{x}, 0) = \mathcal{I}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$.

Proof. Combine Propositions 17B.5 and 17B.6. □

We say that γ is the **fundamental solution** (or **solution kernel**, or **Green's function** or **impulse function**) for the PDE. For example, the D-dimensional Gauss-Weierstrass kernel is a fundamental solution for the D-dimensional heat equation.

17C The Gaussian convolution solution (heat equation)

17C(i) ...in one dimension

Prerequisites: §1B(i), §17B(i), §0G. **Recommended:** §17A, §20A(ii) .

Given two functions $\mathcal{I}, \mathcal{G} : \mathbb{R} \longrightarrow \mathbb{R}$, recall (from §17A) that their **convolution** is the function $\mathcal{I} * \mathcal{G} : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$\mathcal{I} * \mathcal{G}(x) \;:=\; \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}(x-y)\; dy, \qquad \text{for all } x \in \mathbb{R}.$$

Recall the **Gauss-Weierstrass kernel** from Example 1B.1 on page 6:

$$\mathcal{G}_t(x) \quad := \quad \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right), \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

We will use $\mathcal{G}_t(x)$ as an *impulse-response function* to solve the one-dimensional heat equation.

Proposition 17C.1. *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded integrable function. Define $u : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $u(x;t) \;:=\; \mathcal{I} * \mathcal{G}_t(x)$ for all $x \in \mathbb{R}$ and $t > 0$. Then u is a solution to the one-dimensional heat equation.*

Proof. For any fixed $y \in \mathbb{R}$, define $u_y(x;t) \;=\; \mathcal{I}(y) \cdot \mathcal{G}_t(x-y)$.

Claim 1: *$u_y(x;t)$ is a solution of the one-dimensional heat equation.*

Proof. First note that $\partial_t\, \mathcal{G}_t(x-y) \;=\; \partial_x^2\, \mathcal{G}_t(x-y)$ (**Exercise 17C.1**). Ⓔ

Now, y is a constant, so we treat $\mathcal{I}(y)$ as a constant when differentiating by x or by t. Thus,

$$\begin{aligned} \partial_t\; u_y(x,t) &= \mathcal{I}(y) \cdot \partial_t\; \mathcal{G}_t(x-y) \quad = \quad \mathcal{I}(y) \cdot \partial_x^2\, \mathcal{G}_t(x-y) \\ &= \partial_x^2\, u_y(x,t) \quad = \quad \triangle u_y(x,t), \end{aligned}$$

as desired. $\diamond_{\text{Claim 1}}$

Now, $u(x,t) \;=\; \mathcal{I} * \mathcal{G}_t \;=\; \displaystyle\int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}_t(x-y)\; dy \;=\; \int_{-\infty}^{\infty} u_y(x;t)\; dy$. Thus,

$$\partial_t\; u(x,t) \;\underset{(*)}{=}\; \int_{-\infty}^{\infty} \partial_t\; u_y(x;t)\; dy \;\underset{(\dagger)}{=}\; \int_{-\infty}^{\infty} \triangle u_y(x;t)\; dy \;\underset{(*)}{=}\; \triangle\, u(x,t).$$

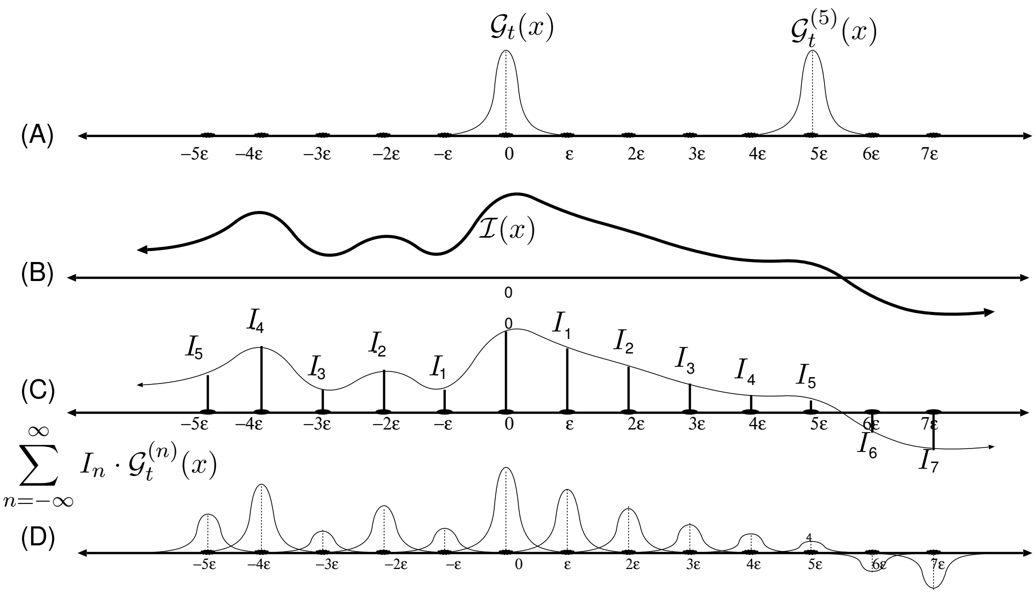


Figure 17C.1: Discrete convolution: a superposition of Gaussians

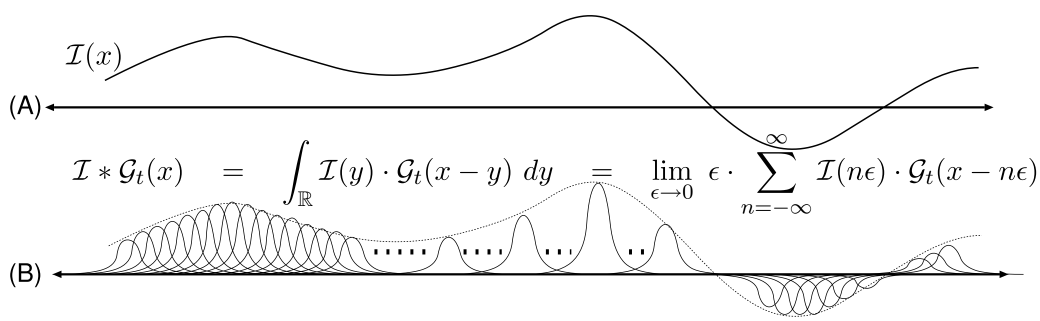


Figure 17C.2: Convolution as a limit of 'discrete' convolutions.

Here, (†) is by Claim 1, and ($*$) is by Proposition 0G.1 on page 567.

(**Exercise 17C.2** Verify that the conditions of Proposition 0G.1 are satisfied.) □ Ⓔ

Remark. One way to visualize the 'Gaussian convolution' $u(x;t) = \mathcal{I} * \mathcal{G}_t(x)$ is as follows. Consider a finely spaced "ϵ-mesh" of points on the real line,

$$\epsilon \cdot \mathbb{Z} \quad = \quad \{n\epsilon \; ; \; n \in \mathbb{Z}\}.$$

For every $n \in \mathbb{Z}$, define the function $\mathcal{G}_t^{(n)}(x) = \mathcal{G}_t(x - n\epsilon)$. For example, $\mathcal{G}_t^{(5)}(x) = \mathcal{G}_t(x - 5\epsilon)$ looks like a copy of the Gauss-Weierstrass kernel, but centered at 5ϵ (see Figure 17C.1A).

For each $n \in \mathbb{Z}$, let $I_n = \mathcal{I}(n \cdot \epsilon)$ (see Figure 17C.1C). Now consider the infinite linear combination of Gauss-Weierstrass kernels (see Figure 17C.1D):

$$u_\epsilon(x;t) \quad = \quad \epsilon \cdot \sum_{n=-\infty}^{\infty} I_n \cdot \mathcal{G}_t^{(n)}(x).$$

Now imagine that the ϵ-mesh become 'infinitely dense', by letting $\epsilon \to 0$. Define $u(x;t) = \lim_{\epsilon \to 0} u_\epsilon(x;t)$. I claim that $u(x;t) = \mathcal{I} * \mathcal{G}_t(x)$. To see this, note that

$$u(x;t) \quad = \quad \lim_{\epsilon \to 0} \epsilon \cdot \sum_{n=-\infty}^{\infty} I_n \cdot \mathcal{G}_t^{(n)}(x) \quad = \quad \lim_{\epsilon \to 0} \epsilon \cdot \sum_{n=-\infty}^{\infty} \mathcal{I}(n\epsilon) \cdot \mathcal{G}_t(x - n\epsilon)$$
$$\underset{(*)}{=} \quad \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}_t(x-y) \, dy \quad = \quad \mathcal{I} * \mathcal{G}_t(y),$$

as shown in Figure 17C.2.

Exercise 17C.3. Rigorously justify step ($*$) in the previous computation. (*Hint.* Use a Riemann sum.) ♦ Ⓔ

Proposition 17C.2. *The Gauss-Weierstrass kernel is an approximation of identity* (see §17B(i)), *meaning that it satisfies the following two properties:*

(AI1) $\mathcal{G}_t(x) \geq 0$ *everywhere, and* $\int_{-\infty}^{\infty} \mathcal{G}_t(x) \, dx = 1$ *for any fixed* $t > 0$.

(AI2) *For any* $\epsilon > 0$, $\lim_{t \to 0} \int_{-\epsilon}^{\epsilon} \mathcal{G}_t(x) \, dx = 1$.

Proof. **Exercise 17C.4** □ Ⓔ

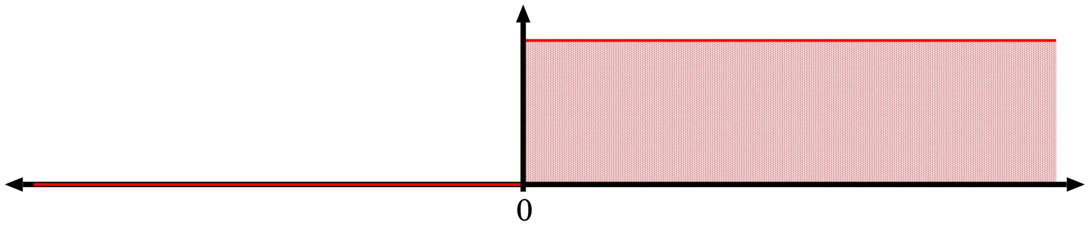


Figure 17C.3: The Heaviside step function $\mathcal{H}(x)$.

Corollary 17C.3. *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded integrable function. Define the function $u : \mathbb{R} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

- $u_0(x) := \mathcal{I}(x)$ *for all $x \in \mathbb{R}$ (initial conditions).*
- $u_t := \mathcal{I} * \mathcal{G}_t$, *for all $t > 0$.*

Then u is a solution to the one-dimensional heat equation. Furthermore:

(a) *If $\mathcal{I}$ is continuous on $\mathbb{R}$, then u is continuous on $\mathbb{R} \times \mathbb{R}_{\not-}$.*

(b) *Even if $\mathcal{I}$ is not continuous, the function u is still continuous on $\mathbb{R} \times \mathbb{R}_+$, and u is also continuous at $(x, 0)$ for any $x \in \mathbb{R}$ where $\mathcal{I}$ is continuous.*

Proof. Propositions 17C.1 says that u is a solution to the heat equation. Combine Proposition 17C.2 with Proposition 17B.2 on page 381 to verify the continuity assertions (a) and (b). □

The 'continuity' part of Corollary 17C.3 means that u is the solution to the *initial value problem* for the heat equation with *initial conditions* $\mathcal{I}$. Because of Corollary 17C.3, we say that $\mathcal{G}$ is the **fundamental solution** (or **solution kernel**, or **Green's function** or **impulse function**) for the heat equation.

Example 17C.4: The Heaviside Step function

Consider the Heaviside **step function** $\mathcal{H}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ (see Figure 17C.3). The solution to the one-dimensional heat equation with initial conditions $u(x, 0) = \mathcal{H}(x)$ is given by:

$$\begin{aligned} u(x,t) \underset{(*)}{=} \mathcal{H} * \mathcal{G}_t(x) \underset{(\dagger)}{=} \mathcal{G}_t * \mathcal{H}(x) &= \int_{-\infty}^{\infty} \mathcal{G}_t(y) \cdot \mathcal{H}(x-y)\, dy \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{4t}\right) \mathcal{H}(x-y)\, dy \quad \underset{(\ddagger)}{=} \quad \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{x} \exp\left(\frac{-y^2}{4t}\right) dy \\ &\underset{(\diamond)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2t}} \exp\left(\frac{-z^2}{2}\right) dz \quad = \quad \Phi\left(\frac{x}{\sqrt{2t}}\right). \end{aligned}$$

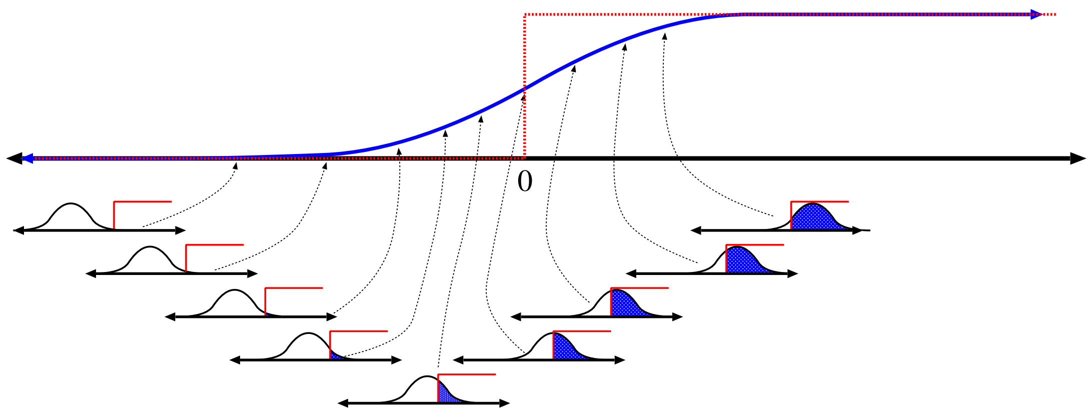


Figure 17C.4: $u_t(x) \;=\; (\mathcal{H} * \mathcal{G}_t)(x)$ evaluated at several $x \in \mathbb{R}$.

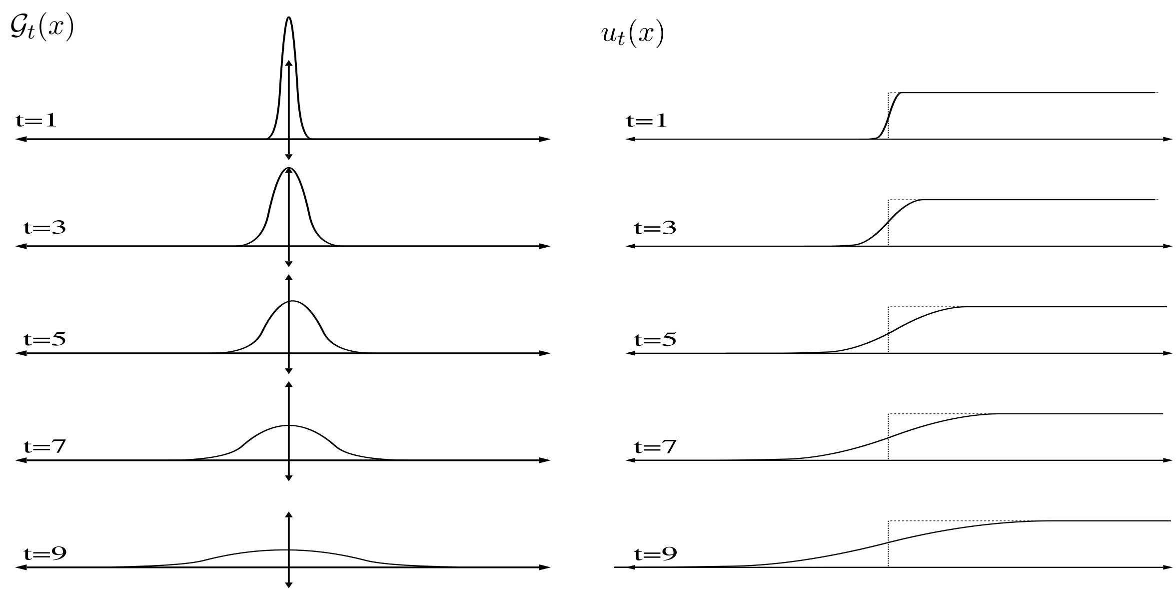


Figure 17C.5: $u_t(x) \;=\; (\mathcal{H} * \mathcal{G}_t)(x)$ for several $t > 0$.

Here, $(*)$ is by Prop. 17C.1 on page 385; $(\dagger)$ is by Prop. 17A.1 on page 378; $(\ddagger)$ is because $\mathcal{H}(x-y) = \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$, and $(\diamond)$ is where we make the substitution $z = \frac{y}{\sqrt{2t}}$; thus, $dy = \sqrt{2t}\ dz$.

Here, $\Phi(x)$ is the **cumulative distribution function** of the standard normal probability measure[1], defined:

$$\boxed{\Phi(x) \quad := \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-z^2}{2}\right)\ dz.}$$

(see Figure 17C.4). At time zero, $u(x,0) = \mathcal{H}(x)$ is a step function. For $t > 0$, $u(x,t)$ looks like a compressed version of $\Phi(x)$: a steep sigmoid function. As t increases, this sigmoid becomes broader and flatter. (see Figure 17C.5). $\diamondsuit$

When computing convolutions, you can often avoid a lot of messy integrals by exploiting the following properties:

Proposition 17C.5. *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Then:*

(a) *If $h : \mathbb{R} \longrightarrow \mathbb{R}$ is another integrable function, then $f * (g+h) \;=\; (f*g) + (f*h)$.*

(b) *If $r \in \mathbb{R}$ is a constant, then $f * (r \cdot g) \;=\; r \cdot (f * g)$.*

(c) *Suppose $d \in \mathbb{R}$ is some 'displacement', and we define $f_{\triangleright d}(x) \;=\; f(x-d)$. Then $(f_{\triangleright d} * g)(x) \;=\; (f*g)(x-d)$. (i.e. $(f_{\triangleright d}) * g \;=\; (f*g)_{\triangleright d}$.)*

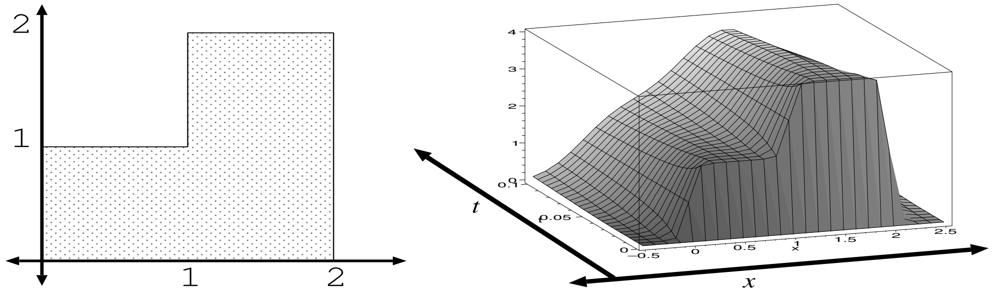
Proof. See Practice Problems #2 and # 3 on page 411 of §17H. $\square$

Example 17C.6: A staircase function

Suppose $\mathcal{I}(x) \;=\; \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x < 2 \\ 0 & \text{if } 2 \leq x \end{cases}$ (see Figure 17C.6A). Let $\Phi(x)$ be the sigmoid function from Example 17C.4. Then

$$u(x,t) \quad = \quad \Phi\left(\frac{x}{\sqrt{2t}}\right) + \Phi\left(\frac{x-1}{\sqrt{2t}}\right) - 2\cdot\Phi\left(\frac{x-2}{\sqrt{2t}}\right) \qquad \text{(see Figure 17C.6B)}$$

[1] This is sometimes called the **error function** or **sigmoid** function. Unfortunately, no simple formula exists for $\Phi(x)$. It can be computed with arbitrary accuracy using a Taylor series, and tables of values for $\Phi(x)$ can be found in most statistics texts.



(A) **(B)**

Figure 17C.6: **(A)** A staircase function. **(B)** The resulting solution to the heat equation.

To see this, observe that we can write:

$$\mathcal{I}(x) \;=\; \mathcal{H}(x) \;+\; \mathcal{H}(x-1) \;-\; 2\cdot\mathcal{H}(x-2) \tag{17C.1}$$

$$\;=\; \mathcal{H} \;+\; \mathcal{H}_{\triangleright 1}(x) \;-\; 2\mathcal{H}_{\triangleright 2}(x), \tag{17C.2}$$

where eqn. (17C.2) uses the notation of Proposition 17C.5(c). Thus,

$$\begin{aligned}
u(x;t) &\underset{(*)}{=} \mathcal{I} * \mathcal{G}_t(x) \quad \underset{(\dagger)}{=} \quad \Big(\mathcal{H} \;+\; \mathcal{H}_{\triangleright 1} \;-\; 2\mathcal{H}_{\triangleright 2}\Big) * \mathcal{G}_t(x) \\
&\underset{(\ddagger)}{=} \mathcal{H} * \mathcal{G}_t(x) \;+\; \mathcal{H}_{\triangleright 1} * \mathcal{G}_t(x) \;-\; 2\mathcal{H}_{\triangleright 2} * \mathcal{G}_t(x) \\
&\underset{(\diamond)}{=} \mathcal{H} * \mathcal{G}_t(x) \;+\; \mathcal{H} * \mathcal{G}_t(x-1) \;-\; 2\mathcal{H} * \mathcal{G}_t(x-2) \\
&\underset{(\P)}{=} \Phi\left(\frac{x}{\sqrt{2t}}\right) \;+\; \Phi\left(\frac{x-1}{\sqrt{2t}}\right) \;-\; 2\Phi\left(\frac{x-2}{\sqrt{2t}}\right).
\end{aligned} \tag{17C.3}$$

Here, $(*)$ is by Proposition 17C.1 on page 385; $(\dagger)$ is by eqn. (17C.2); $(\ddagger)$ is by Proposition 17C.5(a) and (b); $(\diamond)$ is by Proposition 17C.5(c); and $(\P)$ is by Example 17C.4.

Another approach. Begin with eqn. (17C.1), and, rather than using Proposition 17C.5, use instead the linearity of the heat equation, along with Theorem 4C.3 on page 65, to deduce that the solution must have the form:

$$u(x,t) = u_0(x,t) \;+\; u_1(x,t) \;-\; 2\cdot u_2(x,t), \tag{17C.4}$$

where

- $u_0(x,t)$ is the solution with initial conditions $u_0(x,0) = \mathcal{H}(x)$,
- $u_1(x,t)$ is the solution with initial conditions $u_1(x,0) = \mathcal{H}(x-1)$,
- $u_2(x,t)$ is the solution with initial conditions $u_2(x,0) = \mathcal{H}(x-2)$,

But then we know, from Example 17C.4 that

$$u_0(x,t) \;=\; \Phi\left(\frac{x}{\sqrt{2t}}\right); \quad u_1(x,t) \;=\; \Phi\left(\frac{x-1}{\sqrt{2t}}\right); \quad \text{and } u_2(x,t) \;=\; \Phi\left(\frac{x-2}{\sqrt{2t}}\right); \tag{17C.5}$$

Now combine (17C.4) with (17C.5) to again obtain the solution (17C.3). ♢

Remark. The Gaussian convolution solution to the heat equation is revisited in § 20A(ii) on page 530, using the methods of Fourier transforms.

17C(ii) ...in many dimensions

Prerequisites: §1B(ii), §17B(ii). **Recommended:** §17A, §17C(i).

Given two functions $\mathcal{I}, \mathcal{G} : \mathbb{R}^D \longrightarrow \mathbb{R}$, their **convolution** is the function $\mathcal{I} * \mathcal{G} : \mathbb{R}^D \longrightarrow \mathbb{R}$ defined:

$$\mathcal{I} * \mathcal{G}(\mathbf{x}) \quad := \quad \int_{\mathbb{R}^D} \mathcal{I}(\mathbf{y}) \cdot \mathcal{G}(\mathbf{x} - \mathbf{y}) \, d\mathbf{y}.$$

Note that $\mathcal{I} * \mathcal{G}$ is a function of $\mathbf{x}$. The variable $\mathbf{y}$ appears on the right hand side, but as an *integration* variable.

Consider the the D-dimensional Gauss-Weierstrass kernel:

$$\mathcal{G}_t(\mathbf{x}) \quad := \quad \frac{1}{(4\pi t)^{D/2}} \exp\left(\frac{-\|\mathbf{x}\|^2}{4t}\right), \quad \text{for all } \mathbf{x} \in \mathbb{R}^D \text{ and } t > 0.$$

(See Examples 1B.2(b,c) on page 8). We will use $\mathcal{G}_t(x)$ as an *impulse-response function* to solve the D-dimensional heat equation.

Theorem 17C.7.

Suppose $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ *is a bounded continuous function. Define the function* $u : \mathbb{R}^D \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ *by:*

- $u_0(\mathbf{x}) := \mathcal{I}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^D$ *(initial conditions).*
- $u_t := \mathcal{I} * \mathcal{G}_t$, *for all* $t > 0$.

Then u *is a continuous solution to the heat equation on* $\mathbb{R}^D$ *with initial conditions* $\mathcal{I}$.

Proof.

Claim 1: $u(\mathbf{x};t)$ *is a solution to the D-dimensional heat equation.*

Ⓔ *Proof.* **<u>Exercise 17C.5</u>** Hint: Combine Example 1B.2(c) on page 8 with Proposition 17B.6(b) on page 384. $\diamond_{\text{Claim 1}}$

Claim 2: *$\mathcal{G}$ is an approximation of the identity on $\mathbb{R}^D$.*

Ⓔ *Proof.* **Exercise 17C.6** $\diamond_{\text{Claim 2}}$

Now apply Corollary 17B.7 on page 384 □

Because of Theorem 17C.7, we say that $\mathcal{G}$ is the **fundamental solution** for the heat equation.

Exercise 17C.7. In Theorem 17C.7, suppose the initial condition $\mathcal{I}$ had some points of discontinuity in $\mathbb{R}^D$. What can you say about the continuity of the function u? In what sense is u still a solution to the initial value problem with initial conditions $u_0 = \mathcal{I}$? ♦ Ⓔ

17D d'Alembert's solution (one-dimensional wave equation)

"Algebra is generous; she often gives more than is asked of her." —Jean le Rond d'Alembert

d'Alembert's method provides a solution to the one-dimensional wave equation

$$\partial_t^2 u \quad = \quad \partial_x^2 u \tag{17D.1}$$

with any initial conditions, using combinations of *travelling waves* and *ripples*. First we'll discuss this in the infinite domain $\mathbb{X} = \mathbb{R}$, then we'll consider a finite domain like $\mathbb{X} = [a, b]$.

17D(i) Unbounded domain

Prerequisites: §2B(i). **Recommended:** §17A.

Lemma 17D.1. (Travelling Wave Solution)

Let $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be any twice-differentiable function. Define the functions $w_L, w_R : \mathbb{R} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by $w_L(x,t) := f_0(x+t)$ and $w_R(x,t) := f_0(x-t)$, for any $x \in \mathbb{R}$ and any $t \geq 0$ (see Figure 17D.1). *Then w_L and w_R are solutions to the wave equation* (17D.1), *with*

$$\begin{array}{rrcl} \textbf{Initial Position:} & w_L(x,0) & = & f_0(x) \;=\; w_R(x,0), \\ \textbf{Initial Velocities:} & \partial_t\, w_L(x,0) & = & f_0'(x); \qquad \partial_t\, w_R(x,0) \;=\; -f_0'(x). \end{array}$$

Define $w : \mathbb{R} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ *by* $w(x,t) := \frac{1}{2}\Big(w_L(x,t) + w_R(x,t)\Big)$, *for all* $x \in \mathbb{R}$ *and* $t \geq 0$. *Then* w *is a solution to the wave equation* (17D.1), *with* **Initial Position** $w(x,0) = f_0(x)$ *and* **Initial Velocity** $\partial_t w(x,0) = 0$.

Proof. See Practice Problem # 5 on page 412. □

Physically, w_L represents a leftwards-travelling wave: take a copy of the function f_0 and just rigidly translate it to the left. Similarly, w_R represents a rightwards-travelling wave. (Naïvely, it seems that $w_L(x,t) = f_0(x+t)$ should be a *rightwards* travelling wave, while w_R should be *leftwards* travelling wave. Yet the opposite is true. Think about this until you understand it. It may be helpful to do the following: Let $f_0(x) = x^2$. Plot $f_0(x)$, and then plot $w_L(x,5) = f(x+5) = (x+5)^2$. Observe the 'motion' of the parabola.)

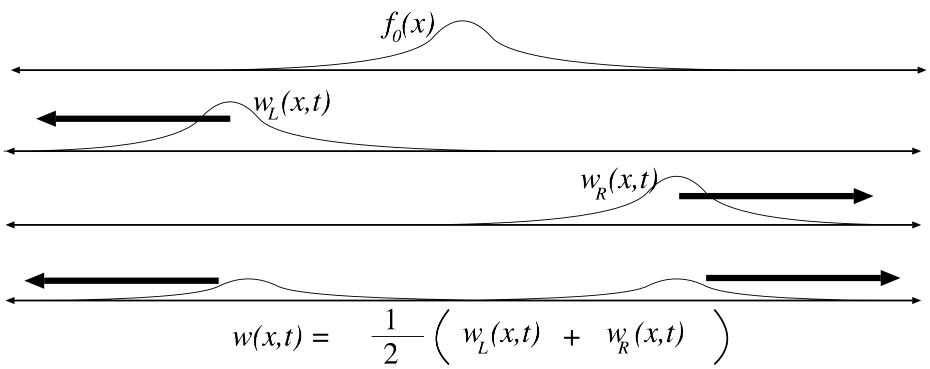


Figure 17D.1: The d'Alembert travelling wave solution; $f_0(x) = \frac{1}{x^2+1}$ from Example 17D.2.

Example 17D.2. (a) If $f_0(x) = \dfrac{1}{x^2+1}$, then $w(x) = \dfrac{1}{2}\left(\dfrac{1}{(x+t)^2+1} + \dfrac{1}{(x-t)^2+1}\right)$ (Figure 17D.1)

(b) If $f_0(x) = \sin(x)$, then

$$\begin{aligned} w(x;t) &= \frac{1}{2}\Big(\sin(x+t) \;+\; \sin(x-t)\Big) \\ &= \frac{1}{2}\Big(\sin(x)\cos(t) + \cos(x)\sin(t) \;+\; \sin(x)\cos(t) - \cos(x)\sin(t)\Big) \\ &= \frac{1}{2}\Big(2\sin(x)\cos(t)\Big) \quad = \quad \cos(t)\sin(x), \end{aligned}$$

In other words, two sinusoidal waves, traveling in opposite directions, when superposed, result in a sinusoidal *standing* wave.

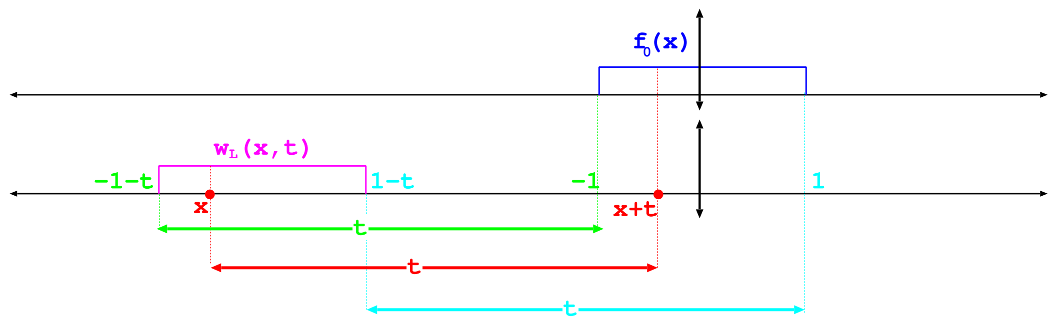


Figure 17D.2: The travelling box wave $w_L(x,t) = f_0(x+t)$ from Example 17D.2(c).

(c) (see Figure 17D.2) Suppose $f_0(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$. Then:

$$w_L(x,t) = f_0(x+t) = \begin{cases} 1 & \text{if } -1 < x+t < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } -1-t < x < 1-t; \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that the solutions w_L and w_R are continuous (or differentiable) only when f_0 is continuous (or differentiable). But the formulae of Lemma 17D.1 make sense even when the original wave equation itself ceases to make sense, as in Example (c). This is an example of a *generalized solution* of the wave equation.) $\diamondsuit$

Lemma 17D.3. (Ripple Solution)

Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Define the function $v : \mathbb{R} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by $v(x,t) := \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} f_1(y)\, dy$, for any $x \in \mathbb{R}$ and any $t \geq 0$. Then v is a solution to the wave equation (17D.1), *with*

Initial Position: $v(x,0) = 0$; **Initial Velocity:** $\partial_t v(x,0) = f_1(x)$.

Proof. See Practice Problem #6 in §17H. $\square$

Physically, v represents a "ripple". You can imagine that f_1 describes the energy profile of an "impulse" which is imparted into the vibrating medium at time zero; this energy propagates outwards, leaving a disturbance in its wake (see Figure 17D.5).

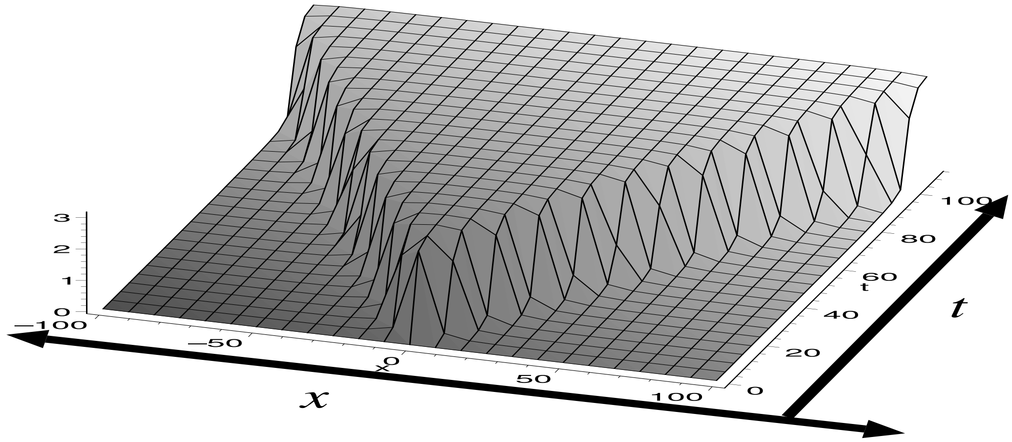


Figure 17D.3: The ripple solution with initial velocity $f_1(x) = \frac{1}{1+x^2}$ (see Example 17D.4(a)).

Example 17D.4. (a) If $f_1(x) = \dfrac{1}{1+x^2}$, then the d'Alembert solution to the initial velocity problem is

$$\begin{aligned} v(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\ dy \quad = \quad \frac{1}{2}\int_{x-t}^{x+t} \frac{1}{1+y^2}\ dy \\ &= \frac{1}{2}\arctan(y)\Big|_{y=x-t}^{y=x+t} \quad = \quad \frac{1}{2}\Big(\arctan(x+t) - \arctan(x-t)\Big). \end{aligned}$$

(see Figure 17D.3).

(b) If $f_1(x) = \cos(x)$, then

$$\begin{aligned} v(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} \cos(y)\ dy \quad = \quad \frac{1}{2}\Big(\sin(x+t) - \sin(x-t)\Big) \\ &= \frac{1}{2}\Big(\sin(x)\cos(t) + \cos(x)\sin(t) \ - \ \sin(x)\cos(t) + \cos(x)\sin(t)\Big) \\ &= \frac{1}{2}\Big(2\cos(x)\sin(t)\Big) \quad = \quad \sin(t)\cos(x). \end{aligned}$$

(c) Let $f_1(x) \ = \ \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ (Figures 17D.4 and 17D.5). If $t > 2$, then

$$v(x,t) \ = \ \begin{cases} 0 & \text{if} \qquad\qquad x+t \ < \ -1; \\ x+t+1 & \text{if } \ -1 \ \leq \ x+t \ < \ 1; \\ 2 & \text{if } \ x-t \ \leq \ -1 < 1 \ \leq \ x+t; \\ t+1-x & \text{if } \ -1 \ \leq \ x-t \ < \ 1; \\ 0 & \text{if } \ 1 \ \leq \ x-t. \end{cases}$$

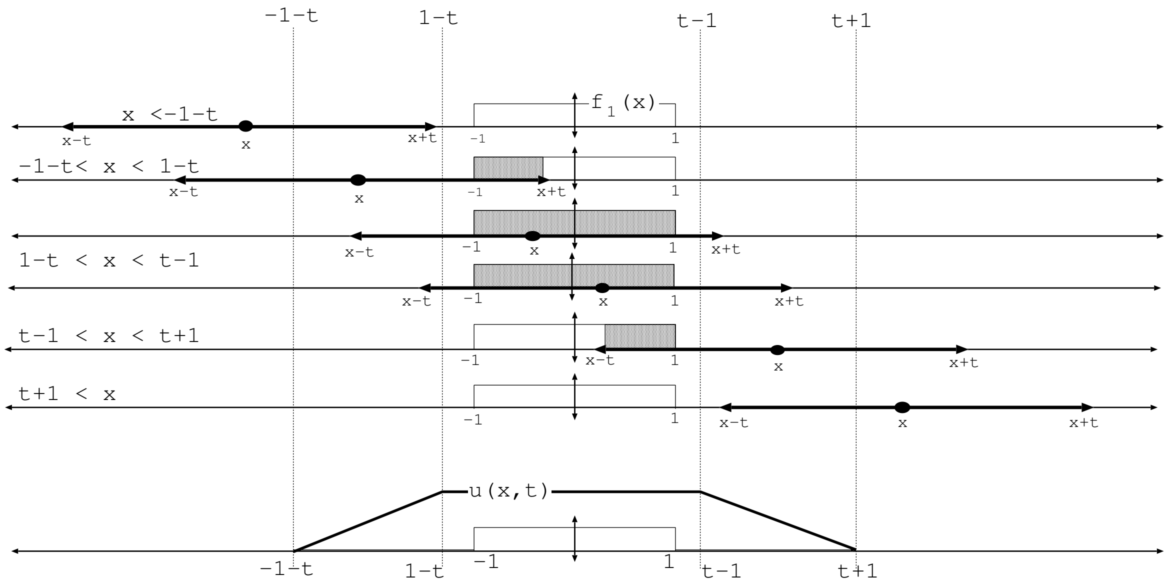


Figure 17D.4: The d'Alembert ripple solution from Example 17D.4(c), evaluated for various $x \in \mathbb{R}$, assuming $t > 2$.

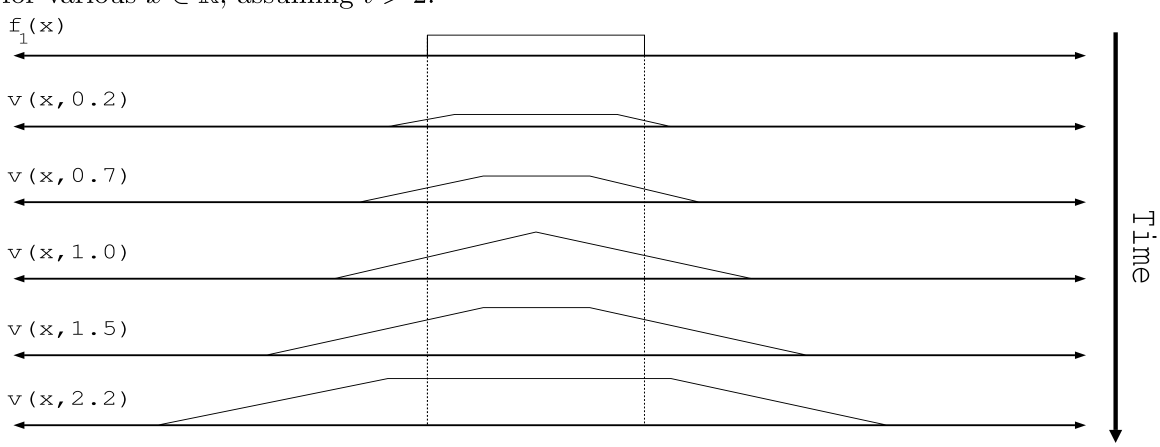


Figure 17D.5: The d'Alembert ripple solution from Example 17D.4(c), evolving in time.

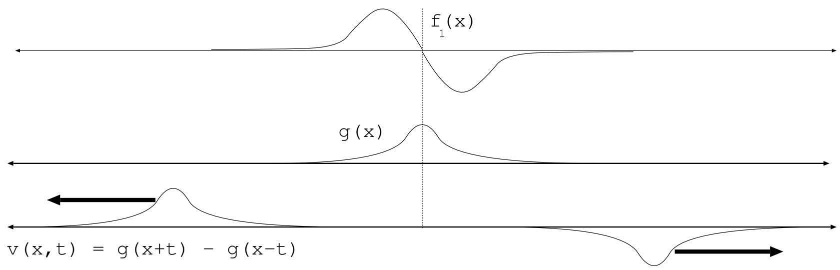


Figure 17D.6: The ripple solution with initial velocity: $f_1(x) = \frac{-2x}{(x^2+1)^2}$ (Example 17D.4(d)).

$$= \begin{cases} 0 & \text{if} \quad x < -1-t; \\ x+t+1 & \text{if} \quad -1-t \leq x < 1-t; \\ 2 & \text{if} \quad 1-t \leq x < t-1; \\ t+1-x & \text{if} \quad t-1 \leq x < t+1; \\ 0 & \text{if} \quad t+1 \leq x. \end{cases}$$

Ⓔ **Exercise 17D.1** Verify this formula. Find a similar formula for when $t < 2$.

Notice that, in this example, the wave of displacement propagates outwards through the medium, and the medium *remains displaced.* The model contains no "restoring force" which would cause the displacement to return to zero.

(d) If $f_1(x) = \dfrac{-2x}{(x^2+1)^2}$, then $g(x) = \dfrac{1}{x^2+1}$, and $v(x) = \dfrac{1}{2}\left(\dfrac{1}{(x+t)^2+1} - \dfrac{1}{(x-t)^2+1}\right)$ (see Figure 17D.6) ♢

Remark. If $g : \mathbb{R} \longrightarrow \mathbb{R}$ is an antiderivative of f_1 (i.e. $g'(x) = f_1(x)$, then $v(x,t) = g(x+t) - g(x-t)$. Thus, the d'Alembert "ripple" solution looks like the d'Alembert "travelling wave" solution, but with the rightward travelling wave being vertically *inverted.*

Ⓔ **Exercise 17D.2.** (a) Express the d'Alembert "ripple" solution as a *convolution*, as described in § 17A on page 375. *Hint:* Find an impulse-response function $\Gamma_t(x)$, such that $f_1 * \Gamma_t(x) \quad = \quad \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} f_1(y)\, dy.$

(b) Is Γ_t an approximation of identity? Why or why not? ♦

Proposition 17D.5. (d'Alembert Solution on an infinite wire)

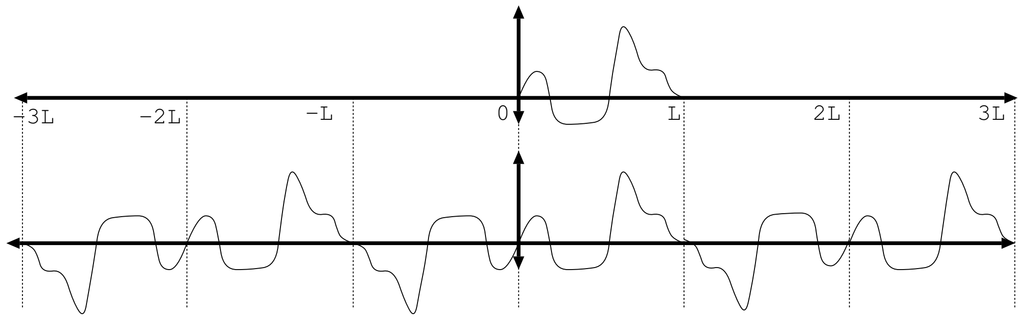


Figure 17D.7: The odd $2L$-periodic extension.

Let $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be twice-differentiable, and $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Define the function $u : \mathbb{R} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ by

$$u(x,t) \quad := \quad \frac{1}{2}\Big(w_{_L}(x,t) \ + \ w_{_R}(x,t)\Big) \ + \ v(x,t), \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0,$$

where $w_{_L}$, $w_{_R}$, and v are as in Lemmas 17D.1 and 17D.3. Then u satisfies the wave equation, with

Initial Position: $v(x,0) \ = \ f_0(x)$; **Initial Velocity:** $\partial_t \, v(x,0) \ = \ f_1(x)$.

Furthermore, all solutions to the wave equation with these initial conditions are of this type.

Proof. This follows from Lemmas 17D.1 and 17D.3. □

Remark. There is no nice extension of the d'Alembert solution in higher dimensions. The closest analogy is Poisson's **spherical mean** solution to the three-dimensional wave equation in free space, which is discussed in § 20B(ii) on page 534.

17D(ii) Bounded domain

Prerequisites: §17D(i), §5C(i).

The d'Alembert solution in §17D(i) works fine if $\mathbb{X} = \mathbb{R}$, but what if $\mathbb{X} = [0, L)$? We must "extend" the initial conditions in some way. If $f : [0, L) \longrightarrow \mathbb{R}$ is any function, then an **extension** of f is any function $\overline{f} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\overline{f}(x) = f(x)$ whenever $0 \leq x \leq L$. If f is continuous and differentiable, then we normally require its extension to also be continuous and differentiable.

The extension we want is the **odd, $2L$-periodic extension**, which is defined as the unique function $\overline{f} : \mathbb{R} \longrightarrow \mathbb{R}$ with the following three properties (see Figure 17D.7):

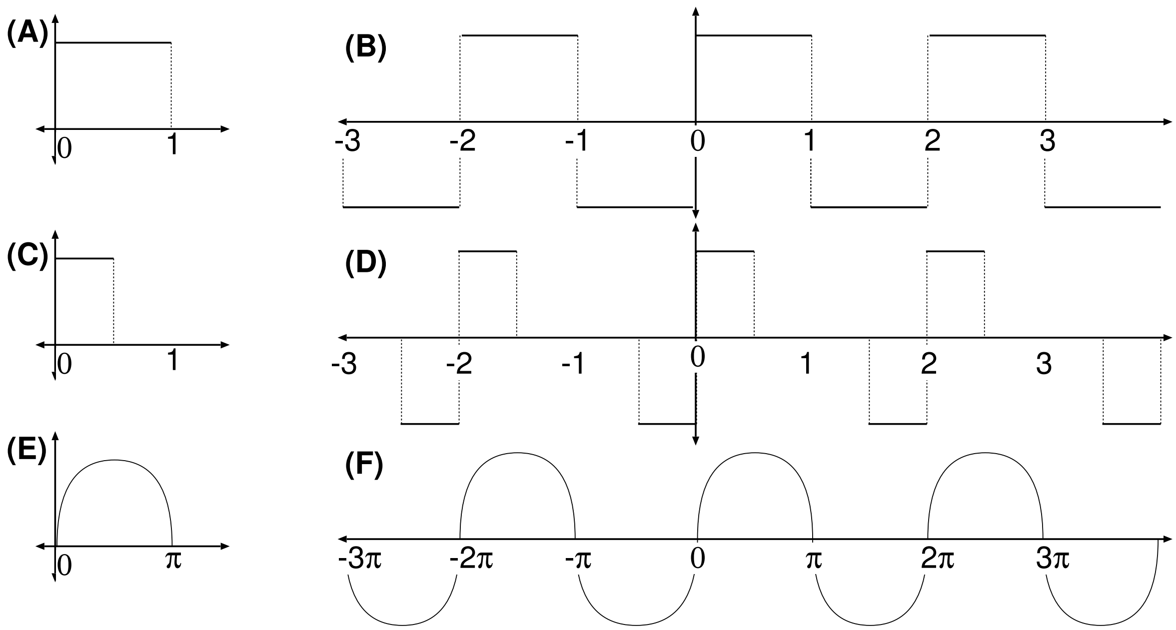


Figure 17D.8: The odd, $2L$-periodic extension.

1. $\overline{f}(x) = f(x)$ whenever $0 \leq x \leq L$.
2. $\overline{f}$ is an *odd* function,[2] meaning: $\overline{f}(-x) = -\overline{f}(x)$ for all $x \in \mathbb{R}$.
3. $\overline{f}$ is $2L$-*periodic*, meaning $\overline{f}(x + 2L) = \overline{f}(x)$ for all $x \in \mathbb{R}$.

Example 17D.6.

(a) Suppose $L = 1$, and $f(x) = 1$ for all $x \in [0, 1)$ (Figure 17D.8A). Then the odd, 2-periodic extension is defined:

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in \ldots \cup [-2, -1) \cup [0, 1) \cup [2, 3) \cup \ldots \\ -1 & \text{if } x \in \ldots \cup [-1, 0) \cup [1, 2) \cup [3, 4) \cup \ldots \end{cases} \qquad \text{(Figure 17D.8B)}$$

(b) Suppose $L = 1$, and $f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right) \end{cases}$ (Figure 17D.8C).
Then the odd, 2-periodic extension is defined:

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in \ldots \cup \left[-2, -1\frac{1}{2}\right) \cup \left[0, \frac{1}{2}\right) \cup \left[2, 2\frac{1}{2}\right) \cup \ldots \\ -1 & \text{if } x \in \ldots \cup \left[-\frac{1}{2}, 0\right) \cup \left[1\frac{1}{2}, 2\right) \cup \left[3\frac{1}{2}, 4\right) \cup \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Figure 17D.8D)}$$

[2]See § 8C on page 168 for more information about odd functions.

(c) Suppose $L = \pi$, and $f(x) = \sin(x)$ for all $x \in [0, \pi)$ (Figure 17D.8E) Then the odd, 2π-periodic extension is given by $\overline{f}(x) = \sin(x)$ for all $x \in \mathbb{R}$ (Figure 17D.8E).

Exercise 17D.3 Verify this. ♢ Ⓔ

We will now provide a general formula for the odd periodic extension, and characterize its continuity and/or differentiability. First some terminology. If $f : [0, L) \longrightarrow \mathbb{R}$ is a function, then we say that f is **right-differentiable** at 0 if the *right-hand derivative* $f^{\langle}(0)$ is well-defined (see page 201). We can usually extend f to a function $f : [0, L] \longrightarrow \mathbb{R}$ by defining $f(L^-) := \lim_{x \nearrow L} f(x)$, where this denotes the *left-hand limit* of f at L, if this limit exists (see page 201 for definition). We then say that f is **left-differentiable** at L if the *left-hand derivative* $f^{\rangle}(L)$ exists.

Proposition 17D.7. *Let $f : [0, L) \longrightarrow \mathbb{R}$ be any function*

(a) *The odd, $2L$-periodic extension of f is given:*

$$\overline{f}(x) = \begin{cases} f(x) & \text{if} \quad 0 \leq x < L \\ -f(-x) & \text{if} \quad -L \leq x < 0 \\ f(x - 2nL) & \text{if} \quad 2nL \leq x \leq (2n+1)L, \quad \text{for some } n \in \mathbb{Z} \\ -f(2nL - x) & \text{if} \quad (2n-1)L \leq x \leq 2nL, \quad \text{for some } n \in \mathbb{Z} \end{cases}$$

(b) *$\overline{f}$ is continuous at 0, L, $2L$ etc. if and only if $f(0) = f(L^-) = 0$.*

(c) *$\overline{f}$ is differentiable at 0, L, $2L$, etc. if and only if it is continuous, f is right-differentiable at 0, and f and left-differentiable at L.*

Proof. **Exercise 17D.4** □ Ⓔ

Proposition 17D.8. (d'Alembert solution on a finite string)

Let $f_0 : [0, L) \longrightarrow \mathbb{R}$ and $f_1 : [0, L) \longrightarrow \mathbb{R}$ be differentiable functions, and let their odd periodic extensions be $\overline{f}_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\overline{f}_1 : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) *Define $w : [0, L] \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

$$w(x, t) := \frac{1}{2}\left(\overline{f}_0(x - t) + \overline{f}_0(x + t)\right), \quad \text{for all } x \in [0, L] \text{ and } t \geq 0.$$

Then w is a solution to the wave equation (17D.1) with initial conditions:

$$w(x, 0) = f_0(x) \quad \text{and} \quad \partial_t w(x, 0) = 0, \quad \text{for all } x \in [0, L],$$

and homogeneous Dirichlet boundary conditions:

$$w(0,t) \;=\; 0 \;=\; w(L,t), \qquad \text{for all } t \geq 0.$$

The function w is continuous if and only if f_0 satisfies homogeneous Dirichlet boundary conditions (i.e. $f(0) = f(L^-) = 0$). *In addition, w is differentiable if and only if f_0 is also right-differentiable at 0 and left-differentiable at L.*

(b) *Define $v : [0, L] \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

$$v(x,t) \quad := \quad \frac{1}{2}\int_{x-t}^{x+t} \overline{f}_1(y)\; dy, \quad \text{for all } x \in [0, L] \text{ and } t \geq 0.$$

Then v is a solution to the wave equation (17D.1) *with initial conditions:*

$$v(x,0) \;=\; 0 \quad \text{and} \quad \partial_t\, v(x,0) \;=\; f_1(x), \qquad \text{for all } x \in [0, L],$$

and homogeneous Dirichlet boundary conditions:

$$v(0,t) = 0 = v(L,t), \qquad \text{for all } t \geq 0.$$

The function v is always continuous. However, v is differentiable if and only if f_1 satisfies homogeneous Dirichlet boundary conditions.

(c) *Define $u : [0, L] \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by $u(x,t) := w(x,t) + v(x,t)$, for all $x \in [0, L]$ and $t \geq 0$. Then $u(x,t)$ is a solution to the wave equation* (17D.1) *with initial conditions:*

$$u(x,0) \;=\; f_0(x) \quad \text{and} \quad \partial_t\, u(x,0) = f_1(x), \qquad \text{for all } x \in [0, L],$$

and homogeneous Dirichlet boundary conditions:

$$u(0,t) \;=\; 0 \;=\; u(L,t), \qquad \text{for all } t \geq 0.$$

Clearly, u is continuous (respectively, differentiable) whenever v and w are continuous (respectively, differentiable).

Proof. The fact that u, w, and v are solutions to their respective initial value problems follows from Proposition 17D.5 on page 398. The verification
Ⓔ of homogeneous Dirichlet conditions is **Exercise 17D.5** . The conditions for
Ⓔ continuity/differentiability are **Exercise 17D.6** . □

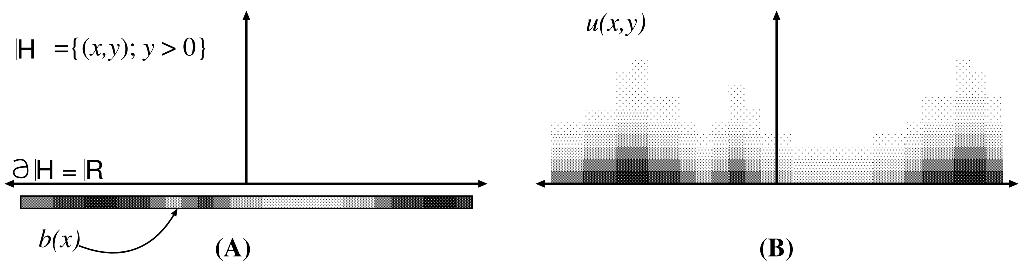


Figure 17E.1: The Dirichlet problem on a half-plane.

17E Poisson's solution (Dirichlet problem on half-plane)

Prerequisites: §1C, §5C, §0G, §17B(i). **Recommended:** §17A.

Consider the **half-plane** domain $\mathbb{H} := \left\{(x, y) \in \mathbb{R}^2 \;;\; y \geq 0\right\}$. The boundary of this domain is just the x axis: $\partial\mathbb{H} = \{(x, 0) \;;\; x \in \mathbb{R}\}$. Thus, we impose boundary conditions by choosing some function $b : \mathbb{R} \longrightarrow \mathbb{R}$. Figure 17E.1 illustrates the corresponding **Dirichlet problem**: find a continuous function $u : \mathbb{H} \longrightarrow \mathbb{R}$ such that

1. u is *harmonic* —i.e. u satisfies the Laplace equation: $\triangle u(x, y) = 0$ for all $x \in \mathbb{R}$ and $y > 0$.

2. u satisfies the *nonhomogeneous Dirichlet boundary condition*: $u(x, 0) = b(x)$, for all $x \in \mathbb{R}$.

Physical Interpretation: Imagine that $\mathbb{H}$ is an infinite 'ocean', so that $\partial\mathbb{H}$ is the beach. Imagine that $b(x)$ is the concentration of some chemical which has soaked into the sand of the beach. The harmonic function $u(x, y)$ on $\mathbb{H}$ describes the equilibrium concentration of this chemical, as it seeps from the sandy beach and diffuses into the water[3]. The boundary condition '$u(x, 0) = b(x)$' represents the chemical content of the sand. Note that $b(x)$ is constant in time; this represents the assumption that the chemical content of the sand is large compared to the amount seeping into the water; hence, we can assume the sand's chemical content remains effectively constant over time, as small amounts diffuse into the water.

[3]Of course this an unrealistic model: in a *real* ocean, currents, wave action, and weather transport chemicals far more quickly than mere diffusion alone.

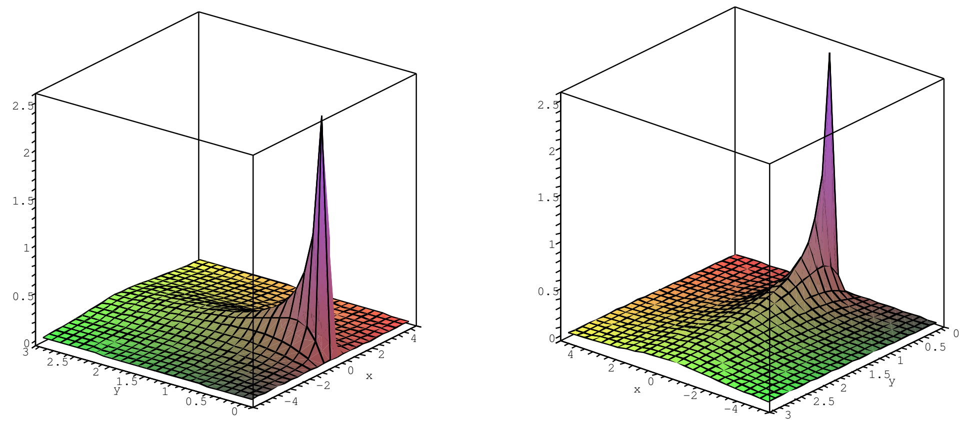


Figure 17E.2: Two views of the Poisson kernel $\mathcal{K}_y(x)$.

We will solve the half-plane Dirichlet problem using the impulse-response method. For any $y > 0$, define the **Poisson kernel** $\mathcal{K}_y : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$\mathcal{K}_y(x) \quad := \quad \frac{y}{\pi(x^2+y^2)}. \qquad \text{(Figure 17E.2)} \qquad (17E.1)$$

Observe that:

- $\mathcal{K}_y(x)$ is smooth for all $y > 0$ and $x \in \mathbb{R}$.
- $\mathcal{K}_y(x)$ has a singularity at $(0,0)$. That is: $\lim_{(x,y)\to(0,0)} \mathcal{K}_y(x) \;=\; \infty$,
- $\mathcal{K}_y(x)$ decays near infinity. That is, for any fixed $y > 0$, $\lim_{x\to\pm\infty} \mathcal{K}_y(x) \;=\; 0$, and also, for any fixed $x \in \mathbb{R}$, $\lim_{y\to\infty} \mathcal{K}_y(x) \;=\; 0$.

Thus, $\mathcal{K}_y(x)$ has the profile of an *impulse-response function* as described in § 17A on page 375. Heuristically speaking, you can think of $\mathcal{K}_y(x)$ as the solution to the Dirichlet problem on $\mathbb{H}$, with boundary condition $b(x) = \delta_0(x)$, where δ_0 is the infamous 'Dirac delta function'. In other words, $\mathcal{K}_y(x)$ is the equilibrium concentration of a chemical diffusing into the water from an 'infinite' concentration of chemical localized at a single point on the beach (say, a leaking barrel of toxic waste).

Proposition 17E.1. Poisson Kernel Solution to Half-Plane Dirichlet problem
Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded, continuous, integrable function. Define $u : \mathbb{H} \longrightarrow \mathbb{R}$ as follows:

$$u(x,y) \quad := \quad b * \mathcal{K}_y(x) \quad = \quad \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{b(z)}{(x-z)^2+y^2}\,dz,$$

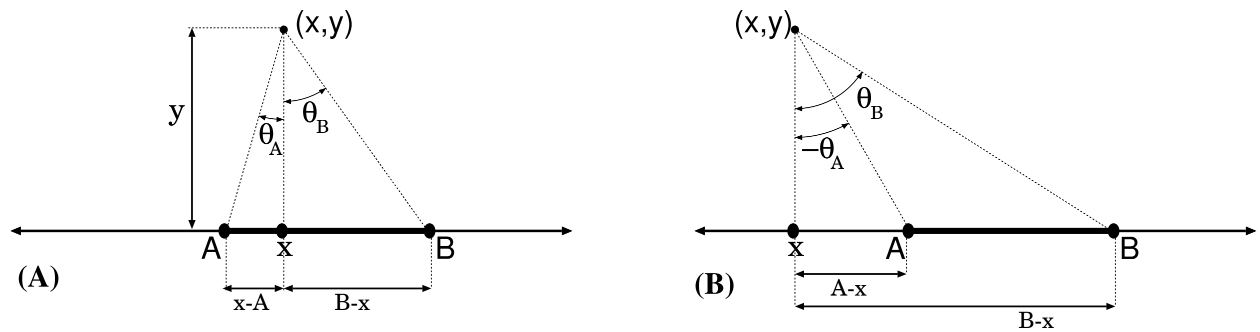


Figure 17E.3: Example 17E.2.

for all $x \in \mathbb{R}$ *and* $y > 0$, *while for all* $x \in \mathbb{R}$, *we define* $u(x, 0) := b(x)$. *Then* u *is the solution to the Laplace equation* ($\triangle u = 0$) *which is bounded at infinity and which satisfies the nonhomogeneous Dirichlet boundary condition* $u(x, 0) = b(x)$, *for all* $x \in \mathbb{R}$.

Proof. (sketch)

Claim 1: *Define* $\mathcal{K}(x, y) = \mathcal{K}_y(x)$ *for all* $(x, y) \in \mathbb{H}$, *except* $(0, 0)$. *Then the function* $\mathcal{K} : \mathbb{H} \longrightarrow \mathbb{R}$ *is harmonic on the interior of* $\mathbb{H}$.

Proof. See Practice Problem # 14 on page 414 of §17H. $\diamond_{\text{Claim 1}}$

Claim 2: *Thus, the function* $u : \mathbb{H} \longrightarrow \mathbb{R}$ *is harmonic on the interior of* $\mathbb{H}$.

Proof. **Exercise 17E.1** Hint: Combine Claim 1 with Proposition 0G.1 on page 567 $\diamond_{\text{Claim 2}}$ Ⓔ

Recall that we defined u on the boundary of $\mathbb{H}$ by $u(x, 0) = b(x)$. It remains to show that u is *continuous* when defined in this way.

Claim 3: *For any* $x \in \mathbb{R}$, $\lim_{y \to 0} u(x, y) = b(x)$.

Proof. **Exercise 17E.2** Show that the kernel $\mathcal{K}_y$ is an *approximation of the identity* as $y \to 0$. Then apply Proposition 17B.2 on page 381 to conclude that $\lim_{y \to 0} (b * \mathcal{K}_y)(x) = b(x)$ for all $x \in \mathbb{R}$. $\diamond_{\text{Claim 3}}$ Ⓔ

Finally, this solution is unique by Theorem 5D.5(a) on page 88. □

Example 17E.2. Let $A < B$ be real numbers. Let $b(x) := \begin{cases} 1 & \text{if } A < x < B; \\ 0 & \text{otherwise.} \end{cases}$.

Then Proposition 20C.3 yields solution:

$$U(x,y) \underset{(*)}{=} b * \mathcal{K}_y(x) \underset{(\dagger)}{=} \frac{y}{\pi}\int_A^B \frac{1}{(x-z)^2+y^2}\,dz \underset{(S)}{=} \frac{y^2}{\pi}\int_{\frac{A-x}{y}}^{\frac{B-x}{y}} \frac{1}{y^2w^2+y^2}\,dw$$
$$= \frac{1}{\pi}\int_{\frac{A-x}{y}}^{\frac{B-x}{y}} \frac{1}{w^2+1}\,dw = \frac{1}{\pi}\arctan(w)\Big|_{w=\frac{A-x}{y}}^{w=\frac{B-x}{y}}$$
$$= \frac{1}{\pi}\arctan\left(\frac{B-x}{y}\right) - \arctan\left(\frac{A-x}{y}\right) \underset{(T)}{=} \frac{1}{\pi}\Big(\theta_B - \theta_A\Big),$$

where θ_B and θ_A are as in Figure 17E.3. Here, $(*)$ is Proposition 20C.3; $(\dagger)$ is eqn.(17E.1); **(S)** is the substitution $w = \frac{z-x}{y}$, so that $dw = \frac{1}{y}\,dz$ and $dz = y\,dw$; and **(T)** follows from elementary trigonometry.

Note that, if $A < x$ (as in Fig. 17E.3A), then $A - x < 0$, so θ_A is negative, so that $U(x,y) = \frac{1}{\pi}\Big(\theta_B + |\theta_A|\Big)$. If $A > x$, then we have the situation in Fig. 17E.3B. In either case, the interpretation is the same:

$$U(x,y) = \frac{1}{\pi}\Big(\theta_B - \theta_A\Big) = \frac{1}{\pi}\left(\begin{array}{l}\textit{the angle subtended by interval } [A,B], \textit{ as}\\ \textit{seen by an observer at the point } (x,y)\end{array}\right).$$

This is reasonable, because if this observer moves far away from the interval $[A,B]$, or views it at an acute angle, then the subtended angle $\Big(\theta_B - \theta_A\Big)$ will become small —hence, the value of $U(x,y)$ will also become small. ♢

Remark. We will revisit the Poisson kernel solution to the half-plane Dirichlet problem in § 20C(ii) on page 539, where we will prove Proposition 17E.1 using Fourier transform methods.

17F Poisson's solution (Dirichlet problem on the disk)

Prerequisites: §1C, §0D(ii), §5C, §0G. **Recommended:** §17A, §14B(v).[4]

Let $\mathbb{D} := \left\{(x,y) \in \mathbb{R}^2 \; ; \; \sqrt{x^2+y^2} \leq R\right\}$ be the **disk** of radius R in $\mathbb{R}^2$. Thus, $\mathbb{D}$ has boundary $\partial\mathbb{D} = \mathbb{S} := \left\{(x,y) \in \mathbb{R}^2 \; ; \; \sqrt{x^2+y^2} = R\right\}$ (the circle of radius R). Suppose $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ is some function on the boundary. The **Dirichlet problem** on $\mathbb{D}$ asks for a continuous function $u : \mathbb{D} \longrightarrow \mathbb{R}$ such that:

- u is *harmonic*—i.e. u satisfies the Laplace equation $\triangle u \equiv 0$.

[4] See § 14B(v) on page 289 for a different development of the material in this section, using the methods of polar-separated harmonic functions. For yet another approach, using complex analysis, see Corollary 18C.13 on page 445.

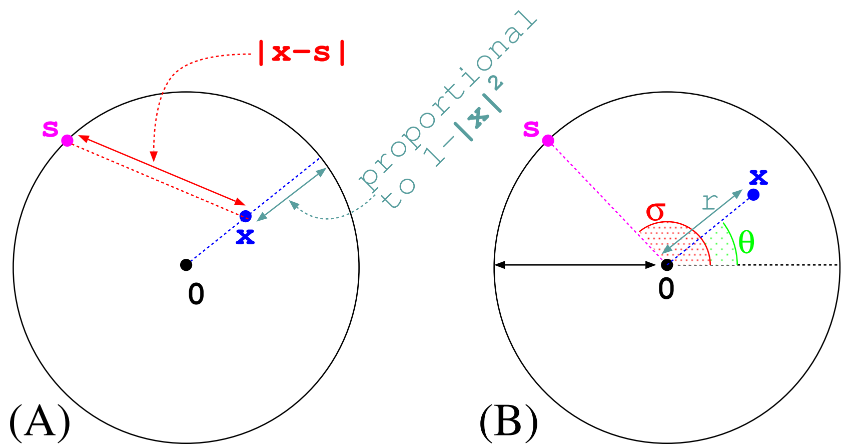


Figure 17F.1: The Poisson kernel

- u satisfies the *nonhomogeneous Dirichlet Boundary Condition* $u(x,y) = b(x,y)$ for all $(x,y) \in \partial\mathbb{D}$.

If u represents the concentration of some chemical diffusing into $\mathbb{D}$ from the boundary, then the value of $u(x,y)$ at any point (x,y) in the interior of the disk should represent some sort of 'average' of the chemical reaching (x,y) from all points on the boundary. This is the inspiration of *Poisson's Solution*. We define the **Poisson kernel** $\mathcal{P} : \mathbb{D} \times \mathbb{S} \longrightarrow \mathbb{R}$ as follows:

$$\mathcal{P}(\mathbf{x}, \mathbf{s}) \quad := \quad \frac{R^2 - \|\mathbf{x}\|^2}{\|\mathbf{x} - \mathbf{s}\|^2}, \qquad \text{for all } \mathbf{x} \in \mathbb{D} \text{ and } \mathbf{s} \in \mathbb{S}.$$

As shown in Figure 17F.1(A), the denominator, $\|\mathbf{x} - \mathbf{s}\|^2$, is just the squared-distance from $\mathbf{x}$ to $\mathbf{s}$. The numerator, $R^2 - \|\mathbf{x}\|^2$, roughly measures the distance from $\mathbf{x}$ to the boundary $\mathbb{S}$; if $\mathbf{x}$ is close to $\mathbb{S}$, then $R^2 - \|\mathbf{x}\|^2$ becomes very small. Intuitively speaking, $\mathcal{P}(\mathbf{x}, \mathbf{s})$ measures the 'influence' of the boundary condition at the point $\mathbf{s}$ on the value of u at $\mathbf{x}$; see Figure 17F.2.

In polar coordinates (Figure 17F.1B), we can parameterize $\mathbf{s} \in \mathbb{S}$ with a single angular coordinate $\sigma \in [-\pi, \pi)$, so that $\mathbf{s} = \Big(R\cos(\sigma),\ R\sin(\sigma)\Big)$. If $\mathbf{x}$ has coordinates (x,y), then Poisson's kernel takes the form:

$$\mathcal{P}(\mathbf{x}, \mathbf{s}) \;=\; \mathcal{P}_\sigma(x,y) \;=\; \frac{R^2 - x^2 - y^2}{(x - R\cos(\sigma))^2 \;+\; (y - R\sin(\sigma))^2}.$$

Proposition 17F.1. Poisson's Integral Formula

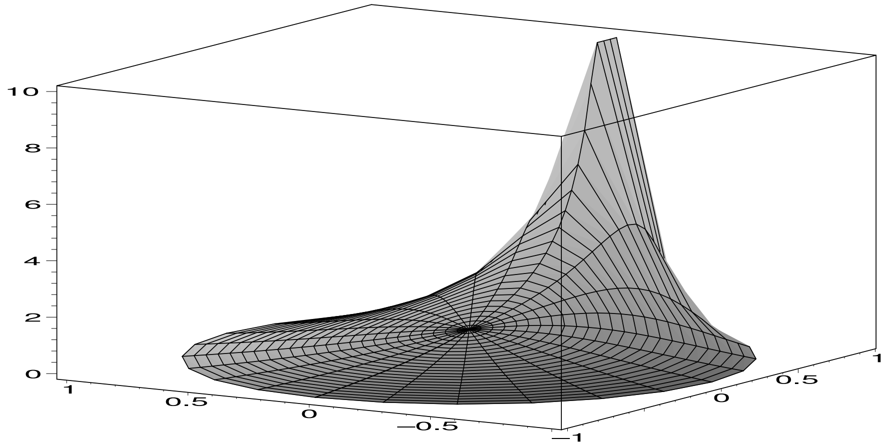


Figure 17F.2: The Poisson kernel $\mathcal{P}(\mathbf{x}, \mathbf{s})$ as a function of $\mathbf{x}$. (for some fixed value of $\mathbf{s}$). This surface illustrates the 'influence' of the boundary condition at the point $\mathbf{s}$ on the point $\mathbf{x}$. (The point $\mathbf{s}$ is located at the 'peak' of the surface.)

Let $\mathbb{D} = \{(x,y)\ ;\ x^2+y^2 \leq R^2\}$ be the disk of radius R, and let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be continuous. The unique solution to the corresponding Dirichlet problem is the function $u : \mathbb{D} \longrightarrow \mathbb{R}$ defined as follows:

$$\text{For any } (x,y) \text{ on the interior of } \mathbb{D} \qquad u(x,y) \ := \ \frac{1}{2\pi}\int_{-\pi}^{\pi} b(\sigma)\cdot \mathcal{P}_\sigma(x,y)\ d\sigma,$$

while, for $(x,y) \in \partial\mathbb{D}$, we define $u(x,y) := b(x,y)$.

$$\text{That is, for any } \mathbf{x} \in \mathbb{D},\ u(\mathbf{x}) \ := \ \begin{cases} \displaystyle\frac{1}{2\pi}\int_{\mathbb{S}} b(\mathbf{s})\cdot\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} & \text{if} \quad \|x\| < R; \\ \\ b(\mathbf{x}) & \text{if} \quad \|x\| = R. \end{cases}$$

Proof. (sketch) For simplicity, assume $R = 1$ (the proof for $R \neq 1$ is similar). Thus,

$$\mathcal{P}_\sigma(x,y) \ = \ \frac{1 - x^2 - y^2}{(x-\cos(\sigma))^2 + (y - \sin(\sigma))^2}.$$

Claim 1: *Fix $\sigma \in [-\pi, \pi)$. The function $\mathcal{P}_\sigma : \mathbb{D} \longrightarrow \mathbb{R}$ is harmonic on the interior of $\mathbb{D}$.*

Ⓔ *Proof.* **Exercise 17F.1** $\diamond_{\text{Claim 1}}$

Claim 2: *Thus, the function u is harmonic on the interior of $\mathbb{D}$.*

Ⓔ *Proof.* **Exercise 17F.2** Hint: Combine Claim 1 with Proposition 0G.1 on page 567. $\diamond_{\text{Claim 2}}$

Recall that we defined u on the boundary $\mathbb{S}$ of $\mathbb{D}$ by $u(\mathbf{s}) = b(\mathbf{s})$. It remains to show that u is *continuous* when defined in this way.

Claim 3: *For any* $\mathbf{s} \in \mathbb{S}$, $\displaystyle\lim_{(x,y)\to\mathbf{s}} u(x,y) \;=\; b(\mathbf{s})$.

Proof. **Exercise 17F.3** (Hard) Ⓔ

Hint: Write (x,y) in polar coordinates as (r,θ). Thus, our claim becomes $\displaystyle\lim_{\theta\to\sigma}\lim_{r\to 1} u(r,\theta) \;=\; b(\sigma)$.

(a) Show that $\mathcal{P}_\sigma(x,y) \;=\; \mathcal{P}_r(\theta-\sigma)$, where, for any $r \in [0,1)$, we define

$$\mathcal{P}_r(\phi) \quad = \quad \frac{1-r^2}{1-2r\cos(\phi)+r^2}, \quad \text{for all } \phi \in [-\pi,\pi).$$

(b) Thus, $u(r,\theta) \;=\; \displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} b(\sigma)\cdot\mathcal{P}_r(\theta-\sigma)\, d\sigma$ is a sort of 'convolution on a circle'. We can write this: $u(r,\theta) \;=\; (b \star \mathcal{P}_r)(\theta)$.

(c) Show that the function $\mathcal{P}_r$ is an 'approximation of the identity' as $r \to 1$, meaning that, for any continuous function $b : \mathbb{S} \longrightarrow \mathbb{R}$, $\displaystyle\lim_{r\to 1}(b\star\mathcal{P}_r)(\theta) \;=\; b(\theta)$. For your proof, borrow from the proof of Proposition 17B.2 on page 381 $\diamond_{\text{Claim 3}}$

Finally, this solution is unique by Theorem 5D.5(a) on page 88. □

17G* Properties of convolution

Prerequisites: §17A. **Recommended:** §17C.

We have introduced the convolution operator to solve the Heat Equation, but it is actually ubiquitous, not only in the theory of PDEs, but in other areas of mathematics, especially probability theory, harmonic analysis, and group representation theory. We can define an *algebra* of functions using the operations of convolution and addition; this algebra is as natural as the one you would form using 'normal' multiplication and addition.[5]

Proposition 17G.1. Algebraic Properties of Convolution

Let $f, g, h : \mathbb{R}^D \longrightarrow \mathbb{R}$ be integrable functions. Then the convolutions of f, g, and h have the following relations:

Commutativity: $f * g \;=\; g * f$.

Associativity: $f * (g * h) \;=\; (f * g) * h$.

Distribution: $f * (g + h) \;=\; (f * g) + (f * h)$.

[5] Indeed, in a sense, it is *the same* algebra, seen through the prism of the Fourier transform; see Theorem 19B.2 on page 494.

Linearity: $f * (r \cdot g) \;=\; r \cdot (f * g)$ *for any constant* $r \in \mathbb{R}$.

Proof. *Commutativity* is just Proposition 17A.1. In the case $D = 1$, the proofs of the other three properties are Practice Problems #1 and #2 in §17H. The proofs for $D \geq 2$ are **Exercise 17G.1** . □

Remark. Let $\mathbf{L}^1(\mathbb{R}^D)$ be the set of all integrable functions on $\mathbb{R}^D$. The properties of *Commutativity*, *Associativity*, and *Distribution* mean that the set $\mathbf{L}^1(\mathbb{R}^D)$, together with the operations '+' (pointwise addition) and '$*$' (convolution), is a *ring* (in the language of abstract algebra). This, together with *Linearity*, makes $\mathbf{L}^1(\mathbb{R}^D)$ an *algebra* over $\mathbb{R}$.

Example 17C.4 on page 388 exemplifies the convenient "smoothing" properties of convolution. If we convolve a "rough" function with a "smooth" function, then this "smooths out" the rough function.

Proposition 17G.2. Regularity Properties of Convolution
Let $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ *be integrable functions.*

(a) *If* f *is continuous, then so is* $f * g$ *(regardless of whether* g *is.)*

(b) *If* f *is differentiable, then so is* $f * g$*. Furthermore,* $\partial_d(f * g) \;=\; (\partial_d f) * g$*.*

(c) *If* f *is* N *times differentiable, then so is* $f * g$*, and*

$$\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_D^{n_D}(f * g) \quad = \quad \left(\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_D^{n_D} f\right) * g,$$

for any $n_1, n_2, \ldots, n_D$ *such that* $n_1 + \ldots + n_D \leq N$*.*

(d) *More generally, if* L *is any linear differential operator of degree* N *or less, with constant coefficients, then* $\mathsf{L}\,(f * g) \;=\; (\mathsf{L}\,f) * g$*.*

(e) *Thus, if* f *is a solution to the homogeneous linear equation "*$\mathsf{L}\,f = 0$*", then so is* $f * g$*.*

(f) *If* f *is infinitely differentiable, then so is* $f * g$*.*

Proof. **Exercise 17G.2** □

This has a convenient consequence: any function, no matter how "rough", can be approximated arbitrarily closely by smooth functions.

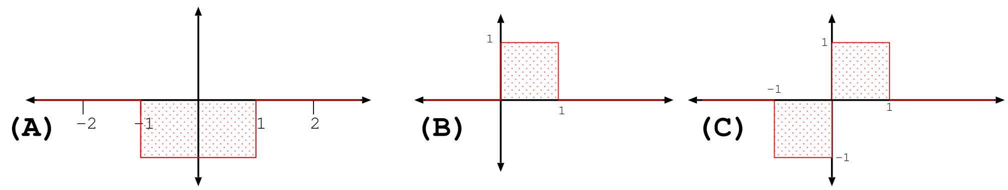


Figure 17H.1: Problems #1(a), #1(b), #1(c) and #2(a).

Proposition 17G.3. *Suppose $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ is integrable. Then there is a sequence $f_1, f_2, f_3, \ldots$ of infinitely differentiable functions which converges pointwise to f. In other words, for every $\mathbf{x} \in \mathbb{R}^D$, $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$.*

Ⓔ *Proof.* **Exercise 17G.3** Hint: Use the fact that the Gauss-Weierstrass kernel is infinitely differentiable, and is also an approximation of identity. Then use **Part 6** of the previous theorem. □

Remarks. (a) We have formulated Proposition 17G.3 in terms of *pointwise* convergence, but similar results hold for L^2 convergence, L^1 convergence, uniform convergence, etc. We're neglecting these to avoid technicalities.

(b) In § 10D(ii) on page 214, we discuss the convolution of periodic functions on the interval $[-\pi, \pi]$, and develop a theory quite similar to the theory developed here. In particular, Lemma 10D.6 on page 214 is analogous to Proposition 17G.1, Lemma 10D.7 on page 215 is analogous to Proposition 17G.2, and Theorem 10D.1 on page 207 is analogous to Proposition 17G.3, except that the convergence is in L^2 norm.

17H Practice problems

1. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Show that $f * (g * h) = (f * g) * h$.

2. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions, and let $r \in \mathbb{R}$ be a constant. Prove that $f * (r \cdot g + h) = r \cdot (f * g) + (f * h)$.

3. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Let $d \in \mathbb{R}$ be some 'displacement' and define $f_{\triangleright d}(x) = f(x - d)$. Prove that $(f_{\triangleright d}) * g = (f * g)_{\triangleright d}$.

4. In each of the following, use the method of Gaussian convolutions to find the solution to the one-dimensional **heat equation** $\partial_t u(x;t) = \partial_x^2 u(x;t)$ with **initial conditions** $u(x, 0) = \mathcal{I}(x)$.

   (a) $\mathcal{I}(x) = \begin{cases} -1 & \text{if} \quad -1 \leq x \leq 1 \\ 0 & \text{if} \quad x < -1 \text{ or } 1 < x \end{cases}$. (see Figure 17H.1A).

(In this case, sketch your solution evolving in time.)

(b) $\mathcal{I}(x) \;=\; \begin{cases} 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$ (see Figure 17H.1B).

(c) $\mathcal{I}(x) \;=\; \begin{cases} -1 & \text{if} \quad -1 \le x \le 0 \\ 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$ (see Figure 17H.1C).

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be some differentiable function. Define $v(x;t) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big)$.

   (a) Show that $v(x;t)$ satisfies the one-dimensional wave equation $\partial_t^2\, v(x;t) = \partial_x^2\, v(x;t)$

   (b) Compute the **initial position** $v(x;0)$.

   (c) Compute the **initial velocity** $\partial_t\, v(x;0)$.

6. Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. For any $x \in \mathbb{R}$ and any $t \ge 0$, define $v(x,t) = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} f_1(y)\; dy$.

   (a) Show that $v(x;t)$ satisfies the one-dimensional wave equation $\partial_t^2\, v(x;t) = \partial_x^2\, v(x;t)$

   (b) Compute the **initial position** $v(x;0)$.

   (c) Compute the **initial velocity** $\partial_t\, v(x;0)$.

7. In each of the following, use the d'Alembert method to find the solution to the one-dimensional **wave equation** $\partial_t^2\, u(x;t) \;=\; \partial_x^2\, u(x;t)$ with **initial position** $u(x,0) \;=\; f_0(x)$ and **initial velocity** $\partial_t\, u(x,0) \;=\; f_1(x)$.

   In each case, identify whether the solution satisfies homogeneous Dirichlet boundary conditions when treated as a function on the interval $[0,\pi]$. Justify your answer.

   (a) $f_0(x) \;=\; \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$; and $f_1(x) = 0$ (see Figure 17H.1B).

   (b) $f_0(x) = \sin(3x)$ and $f_1(x) = 0$.

   (c) $f_0(x) = 0$ and $f_1(x) = \sin(5x)$.

   (d) $f_0(x) = \cos(2x)$ and $f_1(x) = 0$.

   (e) $f_0(x) = 0$ and $f_1(x) = \cos(4x)$.

   (f) $f_0(x) = x^{1/3}$ and $f_1(x) = 0$.

   (g) $f_0(x) = 0$ and $f_1(x) = x^{1/3}$.

(h) $f_0(x) = 0$ and $f_1(x) = \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

8. Let $\mathcal{G}_t(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$ be the Gauss-Weierstrass Kernel. Fix $s, t > 0$; we claim that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$. (For example, if $s = 3$ and $t = 5$, this means that $\mathcal{G}_3 * \mathcal{G}_5 = \mathcal{G}_8$).

   (a) Prove that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$ by directly computing the convolution integral.

   (b) Use Corollary 17C.3 on page 388 to find a short and elegant proof that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$ *without* computing any convolution integrals.

   Remark. Because of this result, probabilists say that the set $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ forms a *stable family of probability distributions* on $\mathbb{R}$. Analysts say that $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ is a *one-parameter semigroup* under convolution.

9. Let $\mathcal{G}_t(x,y) = \dfrac{1}{4\pi t} \exp\left(\dfrac{-(x^2+y^2)}{4t}\right)$ be the 2-dimensional Gauss-Weierstrass Kernel. Suppose $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a harmonic function. Show that $h * \mathcal{G}_t = h$ for all $t > 0$.

10. Let $\mathbb{D}$ be the unit disk. Let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be some function, and let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution to the corresponding Dirichlet problem with boundary conditions $b(\sigma)$. Prove that

$$u(0,0) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma)\, d\sigma.$$

   Remark. This is a special case of the Mean Value Theorem for Harmonic Functions (Theorem 1E.1 on page 16), but do *not* simply 'quote' Theorem 1E.1 to solve this problem. Instead, apply Proposition 17F.1 on page 407.

11. Let $\gamma_t(x) = \begin{cases} \frac{1}{t} & \text{if} \quad 0 \le x \le t; \\ 0 & \text{if} \quad x < 0 \text{ or } t < x. \end{cases}$ (Figure 17B.2). Show that γ is an approximation of identity.

12. Let $\gamma_t(x) = \begin{cases} \frac{1}{2t} & \text{if} \quad |x| \le t \\ 0 & \text{if} \quad t < |x| \end{cases}$. Show that γ is an approximation of identity.

13. Let $\mathbb{D} = \left\{\mathbf{x} \in \mathbb{R}^2 \,;\, |\mathbf{x}| \le 1\right\}$ be the unit disk.

   (a) Let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the unique solution to the **Laplace equation** ($\triangle u = 0$) satisfying the **nonhomogeneous Dirichlet** boundary conditions $u(\mathbf{s}) = 1$, for all $\mathbf{s} \in \mathbb{S}$. Show that u must be constant: $u(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{D}$.

(b) Recall that the Poisson Kernel $\mathcal{P} : \mathbb{D}\times\mathbb{S} \longrightarrow \mathbb{R}$ is defined by $\mathcal{P}(\mathbf{x},\mathbf{s}) = \frac{1-\|\mathbf{x}\|^2}{\|\mathbf{x}-\mathbf{s}\|^2}$, for any $\mathbf{x} \in \mathbb{D}$ and $\mathbf{s} \in \mathbb{S}$. Show that, for any fixed $\mathbf{x} \in \mathbb{D}$, $\frac{1}{2\pi}\int_{\mathbb{S}} \mathcal{P}(\mathbf{x},\mathbf{s})\, d\mathbf{s} = 1.$

(c) Let $b : \mathbb{S} \longrightarrow \mathbb{R}$ be any function, and $\triangle u = 0)$ satisfying the nonhomogeneous Dirichlet boundary conditions $u(\mathbf{s}) = b(\mathbf{s})$, for all $\mathbf{s} \in \mathbb{S}$. Let $m := \min_{\mathbf{s}\in\mathbb{S}} b(\mathbf{s})$, and $M := \max_{\mathbf{s}\in\mathbb{S}} b(\mathbf{s})$. Show that:

$$\text{For all } \mathbf{x} \in \mathbb{D}, \qquad m \quad \le \quad u(\mathbf{x}) \quad \le \quad M.$$

[In other words, the harmonic function u must take its *maximal* and *minimal* values on the *boundary* of the domain $\mathbb{D}$. This is a special case of the *Maximum Principle* for harmonic functions; see Corollary 1E.2 on page 17]

14. Let $\mathbb{H} := \left\{(x,y) \in \mathbb{R}^2 \; ; \; y \ge 0\right\}$ be the *half-plane*. Recall that the *half-plane Poisson kernel* is the function $\mathcal{K} : \mathbb{H} \longrightarrow \mathbb{R}$ defined $\mathcal{K}(x,y) := \dfrac{y}{\pi(x^2+y^2)}$ for all $(x,y) \in \mathbb{H}$ except $(0,0)$ (where it is not defined). Show that $\mathcal{K}$ is harmonic on the interior of $\mathbb{H}$.

Chapter 18

Applications of complex analysis

"The shortest path between two truths in the real domain passes through the complex domain."
—Jacques Hadamard

Complex analysis is one of the most surprising and beautiful areas of mathematics. It also has some unexpected applications to PDEs and Fourier theory, which we will briefly survey in this chapter. Our survey is far from comprehensive —that would require another entire book. Instead, our goal in this chapter is to merely to sketch the possibilities. If you are interested in further exploring the interactions between complex analysis and PDEs, we suggest [Asm02] and [CB03], as well as [Asm05, Chapter 12], [Fis99, Chapters 4 and 5], [Lan85, Chapter VIII], or the innovative and lavishly illustrated [Nee97, Chapter 12].

This chapter assumes no prior knowledge of complex analysis. However, the presentation is slightly more abstract than most of the book, and is intended for more 'theoretically inclined' students. Nevertheless, someone who only wants the computational machinery of residue calculus can skip Sections 18B, 18E and 18F, and skim the proofs in Sections 18C, 18D, and 18G, proceeding rapidly to Section 18H.

18A Holomorphic functions

Prerequisites: §0C, §1C.

Let $\mathbb{U} \subset \mathbb{C}$ be a open set, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be a complex-valued function. If $u \in \mathbb{U}$, then the (complex) **derivative** of f at u is defined:

$$f'(u) \quad := \quad \lim_{\substack{c \to u \\ c \in \mathbb{C}}} \frac{f(c) - f(u)}{c - u}, \tag{18A.1}$$

where all terms in this formula are understood as complex numbers. We say that f is **complex-differentiable** at u if $f'(u)$ exists.

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the obvious way, then we might imagine f as a function from a domain $\mathbb{U} \subset \mathbb{R}^2$ into $\mathbb{R}^2$, and assume that the complex derivate f' was just another way of expressing the (real-valued) Jacobian matrix of f. But this is not the case. Not all (real-)differentiable functions on $\mathbb{R}^2$ can be regarded as complex-differentiable functions on $\mathbb{C}$. To see this, let $f_r := \mathrm{Re}[f] : \mathbb{U} \longrightarrow \mathbb{R}$ and $f_i := \mathrm{Im}[f] : \mathbb{U} \longrightarrow \mathbb{R}$ be the real and imaginary parts of f, so that we can write $f(u) = f_r(u) + f_i(u)\mathbf{i}$ for any $u \in \mathbb{U}$. For any $u \in \mathbb{U}$, let $u_r := \mathrm{Re}[u]$ and $u_i := \mathrm{Im}[u]$, so that $u = u_r + u_i\mathbf{i}$. Then the (real-valued) Jacobian matrix of f has the form

$$\begin{bmatrix} \partial_r f_r & \partial_r f_i \\ \partial_i f_r & \partial_i f_i \end{bmatrix}. \qquad (18A.2)$$

The relationship between the complex derivative (18A.1) and the Jacobian (18A.2) is the subject of the following fundamental result:

Theorem 18A.1. (Cauchy-Riemann)
Let $f : \mathbb{U} \longrightarrow \mathbb{C}$ and let $u \in \mathbb{U}$. Then f is complex-differentiable at u if and only if the partial derivatives $\partial_r f_r(u)$, $\partial_r f_i(u)$, $\partial_i f_r(u)$ and $\partial_i f_i(u)$ all exist, and furthermore, satisfy the **Cauchy-Riemann differential equations** *(CRDEs)*

$$\partial_r f_r(u) \;=\; \partial_i f_i(u) \qquad \text{and} \qquad \partial_i f_r(u) \;=\; -\partial_r f_i(u). \qquad (18A.3)$$

In this case, $f'(u) \;=\; \partial_r f_r(u) - \mathbf{i}\partial_i f_r(u) \;=\; \partial_i f_i(u) + \mathbf{i}\partial_r f_i(u)$.

Ⓔ *Proof.* **Exercise 18A.1** (a) Compute the limit (18A.1) along the 'real' axis —that is, let $c = u + \epsilon$ where $\epsilon \in \mathbb{R}$, and show that $\lim\limits_{\mathbb{R}\ni\epsilon\to 0} \dfrac{f(u+\epsilon) - f(u)}{\epsilon} = \partial_r f_r(u) + \mathbf{i}\partial_r f_i(u)$.

(b) Compute the limit (18A.1) along the 'imaginary' axis —that is, let $c = u + \epsilon\mathbf{i}$ where $\epsilon \in \mathbb{R}$, and show that $\lim\limits_{\mathbb{R}\ni\epsilon\to 0} \dfrac{f(u+\epsilon\mathbf{i}) - f(u)}{\epsilon\mathbf{i}} = \partial_i f_i(u) - \mathbf{i}\partial_i f_r(u)$.

(c) If the limit (18A.1) is well-defined, then it must be the same no matter the direction from which c approaches u. Conclude that the results of (a) and (b) must be equal. Derive equation (18A.3). □

Thus, the complex-differentiable functions are actually a very special subclass of the set of all (real-)differentiable functions on the plane. The function f is called **holomorphic** on $\mathbb{U}$ if f is complex-differentiable at all $u \in \mathbb{U}$. This is actually a *much* stronger requirement than merely requiring a real-valued function to be (real-)differentiable everywhere in some open subset of $\mathbb{R}^2$. For example, later we will show that every holomorphic function is *analytic* (Theorem 18D.1 on page 450). But one immediate indication of the special nature of holomorphic functions is their close relationship to two-dimensional harmonic functions.

Proposition 18A.2. *Let $\mathbb{U} \subset \mathbb{C}$ be an open set, and also regard $\mathbb{U}$ as a subset of $\mathbb{R}^2$ in the obvious way. If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is any holomorphic function, then $f_r : \mathbb{U} \longrightarrow \mathbb{R}$ and $f_i : \mathbb{U} \longrightarrow \mathbb{R}$ are both harmonic functions.*

Proof. **Exercise 18A.2** Hint: apply the Cauchy-Riemann differential equations (18A.3) twice to get Laplace's equation. □ Ⓔ

So, we can convert any holomorphic map into a pair of harmonic functions. Conversely, we can convert any harmonic function into a holomorphic map. To see this, suppose $h : \mathbb{U} \longrightarrow \mathbb{R}$ is a harmonic function. A **harmonic conjugate** for h is a function $g : \mathbb{U} \longrightarrow \mathbb{R}$ which satisfies the differential equation:

$$\partial_2 \, g(u) \;=\; \partial_1 \, h(u) \quad \text{and} \quad \partial_1 \, g(u) \;=\; -\partial_2 \, h(u), \quad \text{for all } u \in \mathbb{U}. \qquad (18A.4)$$

Proposition 18A.3. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a convex open set* (e.g. a disk or a rectangle)*. Let $h : \mathbb{U} \longrightarrow \mathbb{R}$ be any harmonic function.*

(a) *There exist harmonic conjugates for h on $\mathbb{U}$ —that is, the equations* (18A.4) *have solutions.*

(b) *Any two harmonic conjugates for h differ by a constant.*

(c) *If g is a harmonic conjugate to h, and we define $f : \mathbb{U} \longrightarrow \mathbb{C}$ by $f(u) = h(u) + g(u)\mathbf{i}$, then f is holomorphic.*

Proof. **Exercise 18A.3** Hint: (a) Define $g(0)$ arbitrarily, and then for any $u = (u_1, u_2) \in \mathbb{U}$, define $g(u) = -\int_0^{u_1} \partial_2 \, h(0, x) \; dx + \int_0^{u_2} \partial_1 \, h(u_1, y) \; dy$. Show that g is differentiable and satisfies eqn.(18A.4). Ⓔ

For (b), suppose g_1 and g_2 both satisfy eqn.(18A.4); show that $g_1 - g_2$ is a constant by showing that $\partial_1(g_1 - g_2) = 0 = \partial_2(g_1 - g_2)$.

For (c), derive the CRDEs (18A.3) from the harmonic conjugacy equation (18A.4). □

Remark. (a) If h satisfies a Dirichlet boundary condition on $\partial\mathbb{U}$, then its harmonic conjugate satisfies an associated Neumann boundary condition on $\partial\mathbb{U}$, and vice versa; see Exercise 18A.7 on page 421. Thus, harmonic conjugation can be used to convert a Dirichlet BVP into a Neumann BVP, and vice versa.

(b) The 'convexity' requirement in Proposition 18A.2 can be weakened to 'simply connected'. However, Proposition 18A.2 is *not* true if the domain $\mathbb{U}$ is not simply connected (i.e. has a 'hole'); see Exercise 18C.16(e) on page 448. ♢

Holomorphic functions have a rich and beautiful geometric structure, with many surprising properties. The study of such functions is called *complex analysis*. Propositions 18A.2 and 18A.3 imply that *every fact about harmonic functions in* $\mathbb{R}^2$ *is also a fact about complex analysis, and vice versa.*

Complex analysis also has important applications to fluid dynamics and electrostatics, because any holomorphic function can be interpreted as *sourceless, irrotational flow*, as we now explain. Let $\mathbb{U} \subset \mathbb{R}^2$ and let $\vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}^2$ be a two-dimensional vector field. Recall that the *divergence* of $\vec{\mathbf{V}}$ is the scalar field $\operatorname{div} \vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}$ defined by $\operatorname{div} \vec{\mathbf{V}}(\mathbf{u}) := \partial_1 V_1(\mathbf{u}) + \partial_2 V_2(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{U}$ (see § 0E(ii) on page 558). We say $\vec{\mathbf{V}}$ is **locally sourceless** if $\operatorname{div} \vec{\mathbf{V}} \equiv 0$. If $\vec{\mathbf{V}}$ represents the two-dimensional flow of an incompressible fluid (e.g. water) in $\mathbb{U}$, then $\operatorname{div} \vec{\mathbf{V}} \equiv 0$ means there are no sources or sinks in $\mathbb{U}$. If $\vec{\mathbf{V}}$ represents a two-dimensional electric (or gravitational) field, then $\operatorname{div} \vec{\mathbf{V}} \equiv 0$ means there are no charges (or masses) inside $\mathbb{U}$.

The *curl* of $\vec{\mathbf{V}}$ is the scalar field $\operatorname{curl} \vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}$ defined by $\operatorname{curl} \vec{\mathbf{V}}(\mathbf{u}) := \partial_1 V_2(\mathbf{u}) - \partial_2 V_1(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{U}$. We say $\vec{\mathbf{V}}$ is **locally irrotational** if $\operatorname{curl} \vec{\mathbf{V}} \equiv 0$. If $\vec{\mathbf{V}}$ represents a force field, then $\operatorname{curl} \vec{\mathbf{V}} \equiv 0$ means that the net energy absorbed by a particle moving around a closed path in $\vec{\mathbf{V}}$ is zero (i.e. the field is 'conservative'). If $\vec{\mathbf{V}}$ represents the flow of a fluid, then $\operatorname{curl} \vec{\mathbf{V}} \equiv 0$ means there are no 'vortices' in $\mathbb{U}$. (Note that this does *not* mean the fluid must move in straight lines without turning. It simply means that the fluid turns in a uniform manner, without turbulence).

Regard $\mathbb{U}$ as a subset of $\mathbb{C}$, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be some function, with real and imaginary parts $f_r : \mathbb{U} \longrightarrow \mathbb{R}$ and $f_i : \mathbb{U} \longrightarrow \mathbb{R}$. The **complex conjugate** of f is the function $\overline{f} : \mathbb{U} \longrightarrow \mathbb{C}$ defined by $\overline{f}(u) = f_r(u) - \mathbf{i} f_i(u)$. We can treat $\overline{f}$ as vector field $\vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}^2$, where $V_1 \equiv f_r$ and $V_2 \equiv -f_i$.

Proposition 18A.4. (Holomorphic $\iff$ sourceless irrotational flow)
The function f is holomorphic on $\mathbb{U}$ if and only if $\vec{\mathbf{V}}$ is locally sourceless and irrotational on $\mathbb{U}$.

Ⓔ *Proof.* <u>**Exercise 18A.4**</u> □

In §18B, we shall see that Proposition 18A.4 yields a powerful technique for studying fluids (or electric fields) confined to a subset of the plane (see Proposition 18B.6 on page 430). In §18C, we shall see that Proposition 18A.4 is also the key to understanding complex contour integration, through its role in the proof of Cauchy's Theorem 18C.5 on page 438.

To begin our study of complex analysis, we will verify that all the standard facts about the differentiation of real-valued functions carry over to complex differentiation, pretty much verbatim.

Proposition 18A.5. (Closure properties of holomorphic functions)
Let $\mathbb{U} \subset \mathbb{C}$ be an open set. Let $f, g : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic functions.

(a) *The function $h(u) := f(u) + g(u)$ is also holomorphic on $\mathbb{U}$, and $h'(u) = f'(u) + g'(u)$ for all $u \in \mathbb{U}$.*

(b) (Leibniz rule) *The function $h(u) := f(u) \cdot g(u)$ is also holomorphic on $\mathbb{U}$, and $h'(u) = f'(u)g(u) + g'(u)f(u)$ for all $u \in \mathbb{U}$.*

(c) (Quotient rule) *Let $\mathbb{U}^* := \{u \in \mathbb{U} \; ; \; g(u) \neq 0\}$. The function $h(u) := f(u)/g(u)$ is also holomorphic on $\mathbb{U}^*$, and $h'(u) = [g(u)f'(u) - f(u)g'(u)]/g(u)^2$ for all $u \in \mathbb{U}^*$.*

(d) *For any $n \in \mathbb{N}$, the function $h(u) := f^n(u)$ is holomorphic on $\mathbb{U}$, and $h'(u) = n\, f^{n-1}(u) \cdot f'(u)$ for all $u \in \mathbb{U}$.*

(e) *Thus, for any $c_0, c_1, \ldots, c_n \in \mathbb{C}$, the polynomial function $h(z) := c_n z^n + \cdots + c_1 z + c_0$ is holomorphic on $\mathbb{C}$.*

(f) *For any $n \in \mathbb{N}$, the function $h(u) := 1/g^n(u)$ is holomorphic on $\mathbb{U}^* := \{u \in \mathbb{U} \; ; \; g(u) \neq 0\}$ and $h'(u) = -ng'(u)/g^{n+1}(u)$ for all $u \in \mathbb{U}^*$.*

(g) *For all $n \in \mathbb{N}$, let $f_n : \mathbb{U} \longrightarrow \mathbb{C}$ be a holomorphic function. Let $f, F : \mathbb{U} \longrightarrow \mathbb{C}$ be two other functions. If $\operatorname{unif-}\lim_{n\to\infty} f_n = f$ and $\operatorname{unif-}\lim_{n\to\infty} f_n' = F$, then f is holomorphic on $\mathbb{U}$, and $f' = F$.*

(h) *Let $\{c_n\}_{n=0}^{\infty}$ be any sequence of complex numbers, and consider the power series*

$$\sum_{n=0}^{\infty} c_n z^n \quad = \quad c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots$$

Suppose this series converges on $\mathbb{U}$ to define a function $f : \mathbb{U} \longrightarrow \mathbb{C}$. Then f is holomorphic on $\mathbb{U}$. Furthermore, f' is given by the 'formal derivative' of the power series. That is:

$$f'(u) \quad = \quad \sum_{n=1}^{\infty} n c_n z^{n-1} \quad = \quad c_1 + 2c_2 z + 3c_3 z^2 + 4c_4 z^3 + \cdots$$

(i) *Let $\mathbb{X} \subset \mathbb{R}$ be open, let $f : \mathbb{X} \longrightarrow \mathbb{R}$, and suppose f is analytic at $x \in \mathbb{X}$, with a Taylor series*[1] *$T_x f$ which converges in the interval $(x - R, x + R)$ for some $R > 0$. Let $\mathbb{D} := \{c \in \mathbb{C} \; ; \; |c - x| < R\}$ be the open disk of radius R around x in the complex plane. Then the Taylor series $T_x f$ converges uniformly on $\mathbb{D}$, and defines a holomorphic function $F : \mathbb{D} \longrightarrow \mathbb{C}$ which extends f* (i.e. $F(r) = f(r)$ for all $r \in (x - R, x + R) \subset \mathbb{R}$).

[1]See § 0H(ii) on page 569.

(j) (Chain rule) *Let $\mathbb{U}, \mathbb{V} \subset \mathbb{C}$ be open sets. Let $g : \mathbb{U} \longrightarrow \mathbb{V}$ and $f : \mathbb{V} \longrightarrow \mathbb{C}$ be holomorphic functions. Then the function $h(u) = f \circ g(u) = f[g(u)]$ is holomorphic on $\mathbb{U}$, and $h'(u) = f'[g(u)] \cdot g'(u)$ for all $u \in \mathbb{U}$.*

(k) (Inverse function rule) *Let $\mathbb{U}, \mathbb{V} \subset \mathbb{C}$ be open sets. Let $g : \mathbb{U} \longrightarrow \mathbb{V}$ be a holomorphic function. Let $f : \mathbb{V} \longrightarrow \mathbb{U}$ be be an inverse for g —that is, $f[g(u)] = u$ for all $u \in \mathbb{U}$. Let $u \in \mathbb{U}$ and $v = g(u) \in \mathbb{V}$. If $g'(u) \neq 0$, then f is holomorphic in a neighbourhood of v, and $f'(v) = 1/g'(u)$.*

Ⓔ *Proof.* **Exercise 18A.5** Hint: For each part, the proof from single-variable (real) differential calculus generally translates verbatim to complex numbers. □

Theorem 18A.5(i) implies that all the standard real-analytic functions have natural extensions to the complex plane, obtained by evaluating their Taylor series on $\mathbb{C}$.

Example 18A.6. (a) We define $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ by $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$.

The function defined by this power series is the same as the exponential function defined by Euler's formula (0C) on page 551 in Appendix 0C. It satisfies the same properties as the real exponential function —that is, $\exp'(z) = \exp(z)$, $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$, etc. (See Exercise 18A.8 on the next page.)

(b) We define $\sin : \mathbb{C} \longrightarrow \mathbb{C}$ by $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for all $z \in \mathbb{C}$.

(c) We define $\cos : \mathbb{C} \longrightarrow \mathbb{C}$ by $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ for all $z \in \mathbb{C}$.

(d) We define $\sinh : \mathbb{C} \longrightarrow \mathbb{C}$ by $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ for all $z \in \mathbb{C}$.

(e) We define $\cosh : \mathbb{C} \longrightarrow \mathbb{C}$ by $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ for all $z \in \mathbb{C}$. ♢

The complex trigonometric functions satisfy the same algebraic relations and differentiation rules as the real trigonometric functions (see Exercise 18A.9 on page 422). We will later show that any analytic function on $\mathbb{R}$ has a *unique* extension to a holomorphic function on some open subset of $\mathbb{C}$ (see Corollary 18D.4 on page 453).

Ⓔ **Exercise 18A.6.** Proposition 18A.2 says that the real and imaginary parts of any holomorphic function will be harmonic functions.

(a) Let $r_0, r_1, \ldots, r_n \in \mathbb{R}$, and consider the real-valued polynomial $f(x) = r_n x^n + \cdots + r_1 x + r_0$. Proposition 18A.5(e) says that f extends to a holomorphic function $f : \mathbb{C} \longrightarrow \mathbb{C}$. Express the real and imaginary parts of f in terms of the polar harmonic functions $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ introduced in § 14B on page 274.

(b) Express the real and imaginary parts of each of the holomorphic functions sin, cos, sinh and cosh (from Example 18A.6) in terms of the harmonic functions introduced in § 12A on page 240. ♦

Exercise 18A.7. (Harmonic conjugation of boundary conditions) Ⓔ

Let $\mathbb{U} \subset \mathbb{R}^2$ be an open subset whose boundary $\partial\mathbb{U}$ is a smooth curve. Let $\gamma : [0, S] \longrightarrow \partial\mathbb{U}$ be a clockwise, arc-length parameterization of $\partial\mathbb{U}$. That is: γ is a differentiable bijection from $[0, S)$ into $\partial\mathbb{U}$ with $\gamma(0) = \gamma(S)$, and $|\dot{\gamma}(s)| = 1$ for all $s \in [0, S]$. Let $b : \partial\mathbb{U} \longrightarrow \mathbb{R}$ be a continuous function describing a Dirichlet boundary condition on $\mathbb{U}$, and define $B := b \circ \gamma : [0, S] \longrightarrow \mathbb{R}$. Suppose B is differentiable; let $B' : [0, S] \longrightarrow \mathbb{R}$ be its derivative, and then define the function $b' : \partial\mathbb{U} \longrightarrow \mathbb{R}$ by $b'(\gamma(s)) = B'(s)$ for all $s \in [0, S)$ (this defines b' on $\partial\mathbb{U}$ because γ is a bijection). Thus, we can regard b' as the derivative of b 'along' the boundary of $\mathbb{U}$.

Let $h : \mathbb{U} \longrightarrow \mathbb{R}$ be a harmonic function, and let $g : \mathbb{U} \longrightarrow \mathbb{R}$ be a harmonic conjugate for h. Show that h satisfies the Dirichlet boundary condition[2] $h(x) = b(x) + C$ for all $x \in \partial\mathbb{U}$ (where C is some constant) if and only if g satisfies the Neumann boundary condition $\partial_\perp g(x) = b'(x)$ for all $x \in \partial\mathbb{U}$.

Hint: For all $s \in [0, S]$, let $\vec{\mathbf{N}}(s)$ denote the outward unit normal vector of $\partial\mathbb{U}$ at $\gamma(s)$. Let $\mathbf{R} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ (thus, left-multiplying the matrix $\mathbf{R}$ rotates a vector clockwise by 90^o).

(a) Show that $(\nabla g) \cdot \mathbf{R} = \nabla h$. (Here we regard ∇h and ∇g as 2×1 'row matrices').

(b) Show that $\mathbf{R} \cdot \dot{\gamma}(s) = \vec{\mathbf{N}}(s)$ for all $s \in [0, S]$ (Here we regard $\dot{\gamma}$ and $\vec{\mathbf{N}}$ as a 1×2 'column matrices'. *Hint:* recall that γ is a *clockwise* parameterization).

(c) Show that $(h \circ \gamma)'(s) = \nabla h[\gamma(s)] \cdot \dot{\gamma}(s)$, for all $s \in [0, S]$. (To make sense of this, recall that ∇h is 2×1 matrix, while $\dot{\gamma}$ is a 1×2 matrix. *Hint:* use the chain rule).

(d) Show that $(\partial_\perp g)[\gamma(s)] = (h \circ \gamma)'(s)$ for all $s \in [0, S]$. (*Hint:* Recall that $(\partial_\perp g)[\gamma(s)] = (\nabla g)[\gamma(s)] \cdot \vec{\mathbf{N}}(s)$).

(e) Conclude that $\partial_\perp g[\gamma(s)] = b'[\gamma(s)]$ for all $s \in [0, S]$ if and only if $h[\gamma(s)] = b(s) + C$ for all $s \in [0, S]$ (where C is some constant). ♦

Exercise 18A.8. (a) Show that $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$. Ⓔ

(b) Fix $x \in \mathbb{R}$, and consider the smooth path $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$\gamma(t) \quad := \quad [\exp_r(x + \mathbf{i}t),\ \exp_i(x + \mathbf{i}t)],$$

where $\exp_r(z)$ and $\exp_i(z)$ denote the real and imaginary parts of $\exp(z)$. Let $R := e^x$; note that $\gamma(0) = (R, 0)$. Use (a) to show that γ satisfies the ordinary differential equation

$$\begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \end{bmatrix} \quad = \quad \begin{bmatrix} -\gamma_2(t) \\ \gamma_1(t) \end{bmatrix}$$

[2]See § 5C(i) on page 73 and § 5C(ii) on page 76.

Conclude that $\gamma(t) = [R\cos(t), R\sin(t)]$ for all $t \in \mathbb{R}$.

(c) For any $x, y \in \mathbb{R}$, use (b) to show that $\exp(x + \mathbf{i}y) = e^x(\cos(y) + \mathbf{i}\sin(y))$.

(d) Deduce that $\exp(c_1 + c_2) = \exp(c_1) \cdot \exp(c_2)$ for all $c_1, c_2 \in \mathbb{C}$. ♦

Ⓔ **Exercise 18A.9.** (a) Show that $\sin'(z) = \cos(z)$, $\cos'(z) = -\sin(z)$, $\sinh'(z) = \cos(z)$, and $\cosh'(z) = -\sinh(z)$, for all $z \in \mathbb{C}$.

(b) For all $z \in \mathbb{C}$, verify the Euler Identities:

$$\sin(z) = \frac{\exp(z\mathbf{i}) - \exp(-z\mathbf{i})}{2\mathbf{i}} \qquad \cos(z) = \frac{\exp(-z\mathbf{i}) + \exp(z\mathbf{i})}{2}$$

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2} \qquad \cosh(z) = \frac{\exp(z) + \exp(-z)}{2}$$

(c) Deduce that $\sinh(z) = \mathbf{i}\sin(\mathbf{i}z)$ and $\cosh(z) = \cos(\mathbf{i}z)$.

(d) For all $x, y \in \mathbb{R}$, prove the following identities:

$$\begin{aligned} \cos(x + y\mathbf{i}) &= \cos(x)\cosh(y) - \mathbf{i}\sin(x)\sinh(y); \\ \sin(x + y\mathbf{i}) &= \sin(x)\cosh(y) + \mathbf{i}\cos(x)\sinh(y). \end{aligned}$$

(e) For all $z \in \mathbb{C}$, verify the Pythagorean Identities:

$$\cos(z)^2 + \sin(z)^2 = 1 \quad \text{and} \quad \cosh(z)^2 - \sinh(z)^2 = 1.$$

(Later we will show that pretty much every 'trigonometric identity' which is true on $\mathbb{R}$ will also be true over all of $\mathbb{C}$; see Exercise 18D.4 on page 454.) ♦

18B Conformal maps

Prerequisites: §1B, §5C, §18A.

A linear map $f : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is called **conformal** if it preserves the angles between vectors. Thus, for example, *rotations*, *reflections*, and *dilations* are all conformal maps.

Let $\mathbb{U}, \mathbb{V} \subset \mathbb{R}^D$ be open subsets of $\mathbb{R}^D$. A differentiable map $f : \mathbb{U} \longrightarrow \mathbb{V}$ is called **conformal** if its derivative $\mathsf{D} f(\mathbf{x})$ is a conformal linear map, for every $\mathbf{x} \in \mathbb{U}$. One way to interpret this is depicted in Figure 18B.1. Suppose two smooth paths γ_1 and γ_2 cross at $\mathbf{x}$, and their velocity vectors $\dot{\gamma}_1$ and $\dot{\gamma}_2$ form an angle θ at $\mathbf{x}$. Let $\alpha_1 = f \circ \gamma_1$ and $\alpha_2 = f \circ \gamma_2$, and let $\mathbf{y} = f(\mathbf{x})$. Then α_1 and α_2 are smooth paths, and cross at $\mathbf{y}$, forming an angle ϕ. The map f is conformal if, for every $\mathbf{x}$, γ_1, and γ_2, the angles θ and ϕ are equal.

Complex analysis could be redefined as 'the study of two-dimensional conformal maps', because of the next result.

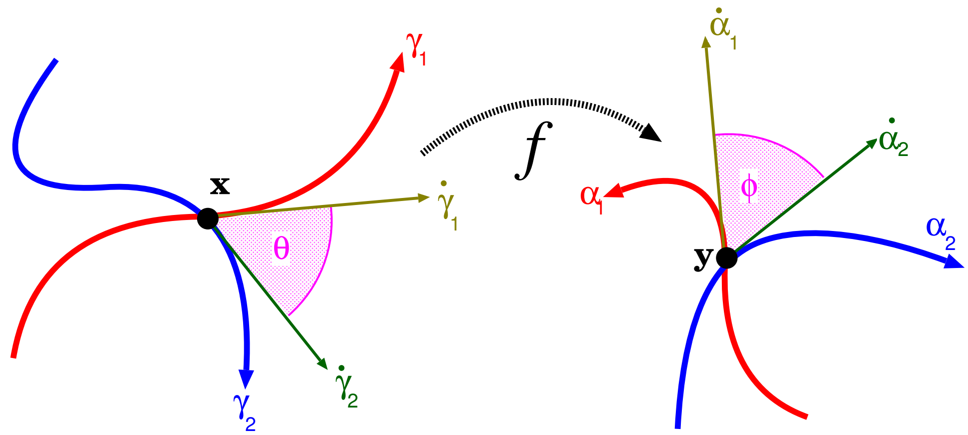


Figure 18B.1: A conformal map preserves the angle of intersection between two paths.

Proposition 18B.1. (Holomorphic $\iff$ conformal)
Let $\mathbb{U} \subset \mathbb{R}^2$ be an open subset, and let $f : \mathbb{U} \longrightarrow \mathbb{R}^2$ be a differentiable function, with $f(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}))$ for all $\mathbf{u} \in \mathbb{U}$. Identify $\mathbb{U}$ with a subset $\widetilde{\mathbb{U}}$ of the plane $\mathbb{C}$ in the obvious way, and define $\widetilde{f} : \widetilde{\mathbb{U}} \longrightarrow \mathbb{C}$ by $\widetilde{f}(x + y\mathbf{i}) = f_1(x, y) + f_2(x, y)\mathbf{i}$ —that is, $\widetilde{f}$ is just the representation of f as a complex-valued function on $\mathbb{C}$. Then $\Big(f \text{ is conformal}\Big) \iff \Big(\widetilde{f} \text{ is holomorphic}\Big)$.

Proof. **Exercise 18B.1** (*Hint:* The derivative $\mathsf{D}\, f$ is a linear map on $\mathbb{R}^2$. Show that $\mathsf{D}\, f$ is conformal if and only if $\widetilde{f}$ satisfies the Cauchy-Riemann differential equations (18A.3) on page 416.). □ Ⓔ

If $\mathbb{U} \subset \mathbb{C}$ is open, then Proposition 18B.1 means that every holomorphic map $f : \mathbb{U} \longrightarrow \mathbb{C}$ can be treated as a conformal transformation of $\mathbb{U}$. In particular we can often conformally identify $\mathbb{U}$ with some other domain in the complex plane via a suitable holomorphic map. A function $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a **conformal isomorphism** if f is conformal, invertible, and $f^{-1} : \mathbb{V} \longrightarrow \mathbb{U}$ is also conformal. Proposition 18B.1 says that this is equivalent to requiring f and f^{-1} to be holomorphic.

Example 18B.2. (a) In Figure 18B.2, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x \in \mathbb{R},\ 0 < y < \pi\}$ is a bi-infinite horizontal strip, and $\mathbb{C}_+ = \{x + y\mathbf{i}\ ;\ x \in \mathbb{R},\ y > 0\}$ is the open upper half-plane, and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{C}_+$ is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{C}_+$.

(b) In Figure 18B.3, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 0,\ y \in \mathbb{R}\}$ is the open right half of the complex plane, and $\mathbb{D}^{\complement} = \left\{x + y\mathbf{i}\ ;\ x^2 + y^2 > 1\right\}$ is the complement of the

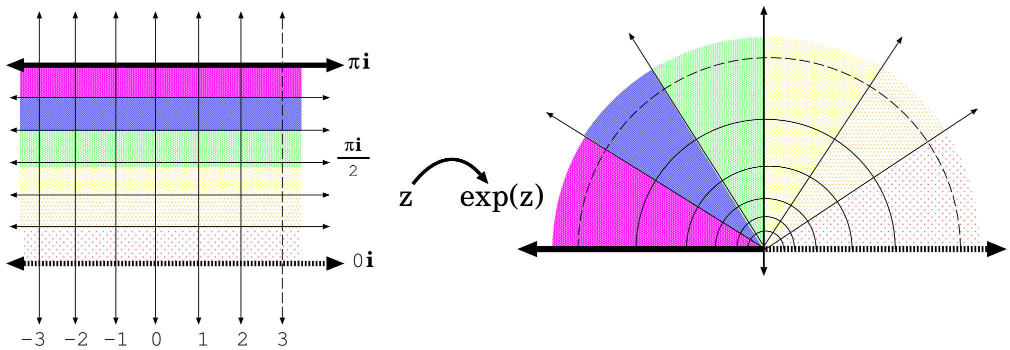


Figure 18B.2: Example 18B.2(a). The map $f(z) = \exp(z)$ conformally identifies a bi-infinite horizontal strip with the upper half-plane.

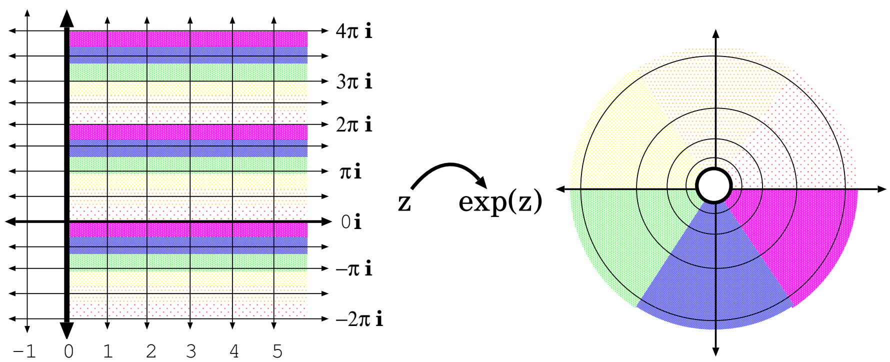


Figure 18B.3: Example 18B.2(b). The map $f(z) = \exp(z)$ conformally projects the right half-plane onto the complement of the unit disk.

closed unit disk, and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \complement\mathbb{D}$ is not a conformal isomorphism (because it is many-to-one). However, f *is* a conformal **covering map**. This means that f is *locally* one-to-one: for any point $u \in \mathbb{U}$, with $v = f(u) \in \complement\mathbb{D}$, there is a neighbourhood $\mathcal{V} \subset \complement\mathbb{D}$ of v and a neighbourhood $\mathcal{U} \subset \mathbb{U}$ of u such that $f_| : \mathcal{U} \longrightarrow \mathcal{V}$ *is* one-to-one. (Note that f is *not* globally one-to-one because it is periodic in the imaginary coordinate).

(c) In Figure 18B.4, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x < 0,\ 0 < y < \pi\}$ is a left half-infinite rectangle, and $\mathbb{V} = \left\{x + y\mathbf{i}\ ;\ y > 1,\ x^2 + y^2 < 1\right\}$ is the open half-disk, and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{V}$.

(d) In Figure 18B.5, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 0,\ 0 < y < \pi\}$ is a right half-infinite rectangle, and $\mathbb{V} = \left\{x + y\mathbf{i}\ ;\ y > 1,\ x^2 + y^2 > 1\right\}$ is the "amphitheatre", and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{V}$.

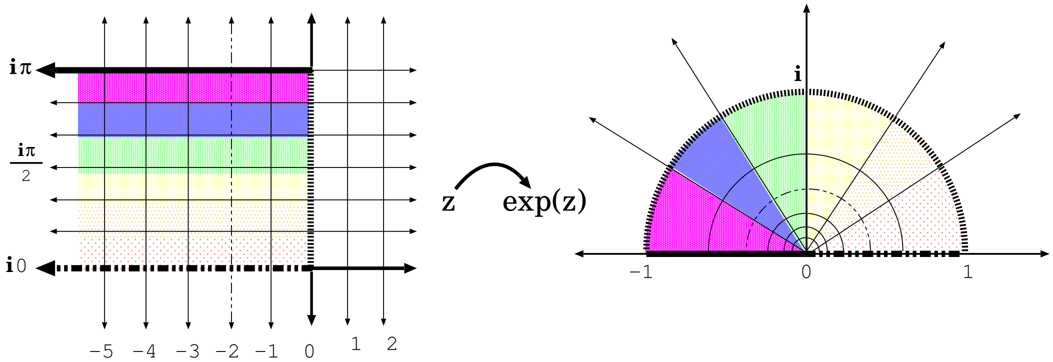


Figure 18B.4: Example 18B.2(c). The map $f(z) = \exp(z)$ conformally identifies a left half-infinite rectangle with the half-disk.

(e) In Figure 18B.6(A,B), $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x, y > 0\}$ is the open upper right quarter-plane, and $\mathbb{C}_+ = \{x + y\mathbf{i}\ ;\ y > 0\}$ is the open upper half-plane, and $f(z) = z^2$. Then f is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{C}_+$.

(f) Let $\mathbb{C}_+ := \{x + y\mathbf{i}\ ;\ y > 0\}$ be the upper half-plane, and $\mathbb{U} := \mathbb{C}_+ \setminus \{y\mathbf{i}\ ;\ 0 < y < 1\}$; that is, $\mathbb{U}$ is is the upper half-plane with a vertical line-segment of length 1 removed above the origin. Let $f(z) = (z^2 + 1)^{1/2}$; then f is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{C}_+$, as shown in Figure 18B.7(a).

(g) Let $\mathbb{U} := \{x+y\mathbf{i}\,;\text{either } y \neq 0 \text{ or } -1 < x < 1\}$, and let $\mathbb{V} := \left\{x + y\mathbf{i}\ ;\ \frac{-\pi}{2} < y < \frac{\pi}{2}\right\}$ be a bi-infinite horizontal strip of width π. Let $f(z) := \mathbf{i} \cdot \arcsin(z)$; then f is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{V}$, as shown in Figure 18B.7(b).

Exercise 18B.2 Verify each of examples (a)-(g). ◊ Ⓔ

Conformal maps are very useful for solving boundary value problems, because of the following result:

Proposition 18B.3. *Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ be open domains with closures $\overline{\mathbb{X}}$ and $\overline{\mathbb{Y}}$. Let $f : \overline{\mathbb{X}} \longrightarrow \overline{\mathbb{Y}}$ be a continuous surjection which conformally maps $\mathbb{X}$ into $\mathbb{Y}$. Let $h : \overline{\mathbb{Y}} \longrightarrow \mathbb{R}$ be some smooth function, and define $H = h \circ f : \overline{\mathbb{X}} \longrightarrow \mathbb{R}$.*

(a) *h is harmonic on $\mathbb{X}$ if and only if H is harmonic on $\mathbb{Y}$.*

(b) *Let $b : \partial\mathbb{Y} \longrightarrow \mathbb{R}$ be some function on the boundary of $\mathbb{Y}$. Then $B = b \circ f : \partial\mathbb{X} \longrightarrow \mathbb{R}$ is a function on the boundary of $\mathbb{X}$. The function h satisfies the nonhomogeneous Dirichlet boundary condition[3] "$h(\mathbf{y}) = b(\mathbf{y})$ for all*

[3] See § 5C(i) on page 73.

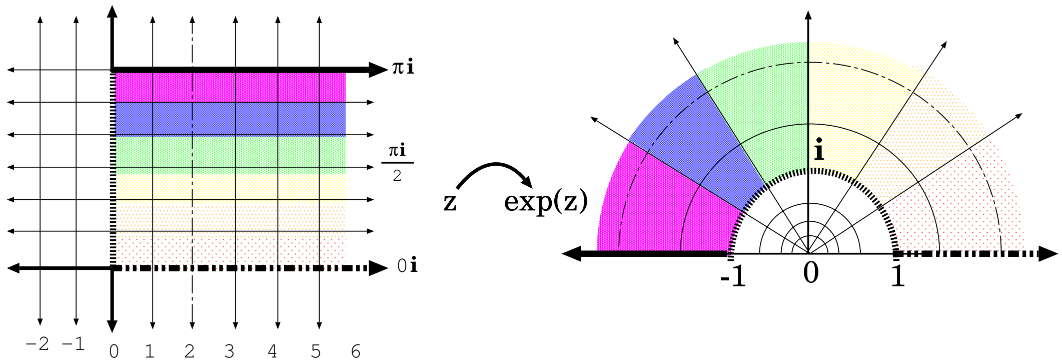


Figure 18B.5: Example 18B.2(d). The map $f(z) = \exp(z)$ conformally identifies a right half-infinite rectangle with the "amphitheatre"

$\mathbf{y} \in \partial\mathbb{Y}$" if and only if H satisfies the nonhomogeneous Dirichlet boundary condition "$H(\mathbf{x}) = B(\mathbf{x})$" for all $\mathbf{x} \in \partial\mathbb{X}$".

(c) *For all $\mathbf{x} \in \partial\mathbb{X}$, let $\vec{\mathbf{N}}_{\mathbb{X}}(\mathbf{x})$ be the outward unit normal vector to $\partial\mathbb{X}$ at $\mathbf{x}$, let $\vec{\mathbf{N}}_{\mathbb{Y}}(\mathbf{x})$ be the outward unit normal vector to $\partial\mathbb{Y}$ at $f(\mathbf{x})$, and let $\mathsf{D}f(\mathbf{x})$ be the derivative of f at $\mathbf{x}$* (a linear transformation of $\mathbb{R}^D$). *Then $\mathsf{D}f(\mathbf{x})[\vec{\mathbf{N}}_{\mathbb{X}}(\mathbf{x})] = \phi(\mathbf{x}) \cdot \vec{\mathbf{N}}_{\mathbb{Y}}(\mathbf{x})$ for some scalar $\phi(\mathbf{x}) > 0$.*

(d) *Let $b : \partial\mathbb{Y} \longrightarrow \mathbb{R}$ be some function on the boundary of $\mathbb{Y}$, and define $B : \partial\mathbb{X} \longrightarrow \mathbb{R}$ by $B(\mathbf{x}) := \phi(\mathbf{x}) \cdot b[f(\mathbf{x})]$ for all $\mathbf{x} \in \partial\mathbb{X}$. Then h satisfies the nonhomogeneous Neumann boundary condition[4] "$\partial_\perp h(\mathbf{y}) = b(\mathbf{y})$ for all $\mathbf{y} \in \partial\mathbb{Y}$" if and only if H satisfies the nonhomogeneous Neumann boundary condition "$\partial_\perp H(\mathbf{x}) = B(\mathbf{x})$" for all $\mathbf{x} \in \partial\mathbb{X}$".*

Ⓔ *Proof.* **Exercise 18B.3** Hint: (a) Combine Propositions 18A.2, 18A.3, and 18B.1. For (c), use the fact that f is a conformal map, so $\mathsf{D}f(\mathbf{x})$ is a conformal linear transformation; thus, if $\vec{\mathbf{N}}_{\mathbb{X}}(\mathbf{x})$ is normal to $\partial\mathbb{X}$, then $\mathsf{D}f(\mathbf{x})[\vec{\mathbf{N}}_{\mathbb{X}}(\mathbf{x})]$ must be normal to $\partial\mathbb{Y}$. To prove (d), use (c) and the chain rule. □

We can apply Proposition 18B.3 as follows: given a boundary value problem on some "nasty" domain $\mathbb{X}$, find a "nice" domain $\mathbb{Y}$ (e.g. a box, a disk, or a half-plane), and a conformal isomorphism $f : \mathbb{X} \longrightarrow \mathbb{Y}$. Solve the boundary value problem in $\mathbb{Y}$ (e.g. using the methods from Chapters 12-17), to get a solution function $h : \mathbb{Y} \longrightarrow \mathbb{R}$. Finally, "pull back" this solution to get a solution $H = h \circ f : \mathbb{X} \longrightarrow \mathbb{R}$ to the original BVP on $\mathbb{X}$. We can obtain a suitable

[4]See § 5C(ii) on page 76.

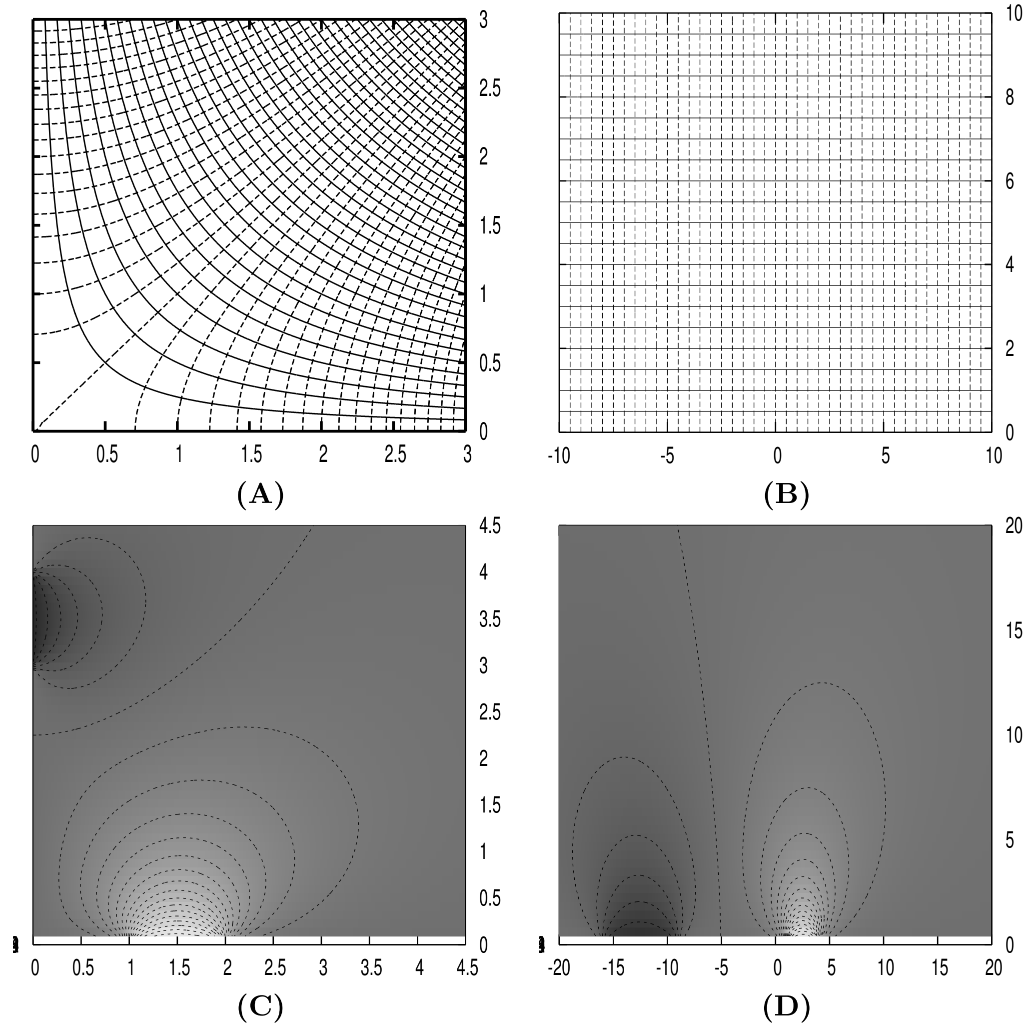


Figure 18B.6: **(A,B):** Example 18B.2(e). The map $f(z) = z^2$ conformally identifies the quarter-plane **(A)** and the half-plane **(B)**. The mesh of curves in **(A)** is the preimage of the Cartesian grid in **(B)**. Note that these curves always intersect at right angles; this is because f is a conformal map. The solid curves are the *streamlines*: the preimages of horizontal grid lines. The streamlines describe a sourceless, irrotational flow confined to the quarter-plane (see Proposition 18B.6 on page 430). The dashed curves are the *equipotential contours*: the preimages of vertical grid lines. The streamlines and equipotentials can be interpreted as the level curves of two harmonic functions (by Proposition 18A.2). They can also be interpreted as the voltage contours and field lines of an electric field in a quarter-plane bounded by perfect conductors on the x and y axes. **(C,D):** Example 18B.4 on the following page. The map $f(z) = z^2$ can be used to 'pull back' solutions to BVPs from the half-plane to the quarter-plane. Figure **(C)** shows a greyscale plot of the harmonic function H defined on the quarter-plane by eqn.(18B.2). Figure **(D)** shows a greyscale plot of the harmonic function h defined on the half-plane by eqn.(18B.1); the two functions are related by $H = h \circ f$.

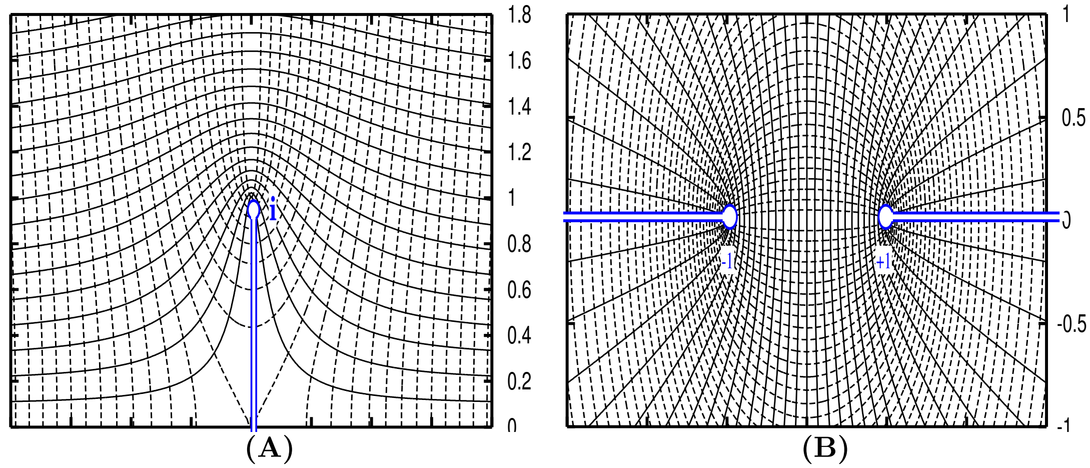


Figure 18B.7: **(A)** Example 18B.2(f). The map $f(z) = (z^2+1)^{1/2}$ is a conformal isomorphism from the set $\mathbb{C}_+ \setminus \{y\mathbf{i}\,;\, 0 < y < 1\}$ to the upper half-plane $\mathbb{C}_+$. **(B)** Example 18B.2(g). The map $f(z) := \mathbf{i} \cdot \arcsin(z)$ is a conformal isomorphism from the domain $\mathbb{U} := \{x + y\mathbf{i}\,;$ either $y \neq 0$ or $-1 < x < 1\}$ to a bi-infinite horizontal strip. In these figures, as in Figure 18B.6(A,B), the mesh is the preimage of a Cartesian grid on the image domain; the solid lines are streamlines, and the dashed lines are equipotential contours. In Figure **(A)**, we can interpret these streamlines as the flow of fluid over an obstacle; in Figure **(B)**; they represent the flow of fluid through a narrow aperature between two compartments. Alternately, we can interpret these curves as the voltage contours and field lines of an electric field, where the domain boundaries are perfect conductors.

conformal isomorphism from $\mathbb{X}$ to $\mathbb{Y}$ using holomorphic mappings, by Proposition 18B.1.

Example 18B.4. Let $\mathbb{X} = \left\{(x_1, x_2) \in \mathbb{R}^2\,;\, x_1, x_2 > 0\right\}$ be the open upper right quarter-plane. Suppose we want to find a harmonic function $H : \mathbb{X} \longrightarrow \mathbb{R}$ satisfying the nonhomogeneous Dirichlet boundary conditions $H(\mathbf{x}) = B(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{X}$, where $B : \partial\mathbb{X} \longrightarrow \mathbb{R}$ is defined:

$$B(x_1, 0) = \begin{cases} 3 & \text{if } 1 \leq x_1 \leq 2; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad B(0, x_2) = \begin{cases} -1 & \text{if } 3 \leq x_2 \leq 4; \\ 0 & \text{otherwise.} \end{cases}$$

Identify $\mathbb{X}$ with the complex right quarter-plane $\mathbb{U}$ from Example 18B.2(e). Let $f(z) := z^2$; then f is a conformal isomorphism from $\mathbb{U}$ to the upper half-plane $\mathbb{C}^+$. If we identify $\mathbb{C}^+$ with the real upper half-plane $\mathbb{Y} := \left\{(y_1, y_2) \in \mathbb{R}^2\,;\, y_2 > 0\right\}$, then we can treat f as a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$, given by the formula $f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$.

Since f is bijective, the inverse $f^{-1} : \mathbb{Y} \longrightarrow \mathbb{X}$ is well-defined. Thus, we can define a function $b := B \circ f^{-1} : \partial\mathbb{Y} \longrightarrow \mathbb{R}$. To be concrete:

$$b(y_1, 0) = \begin{cases} 3 & \text{if } 1 \leq y_1 \leq 4; \\ -1 & \text{if } -16 \leq y_1 \leq -9; \\ 0 & \text{otherwise.} \end{cases}$$

Now we must find a harmonic function $h : \mathbb{Y} \longrightarrow \mathbb{R}$ satisfying the Dirichlet boundary conditions $h(y_1, 0) = b(y_1, 0)$ for all $y_1 \in \mathbb{R}$. By adding together two copies of the solution from Example 17E.2 on page 405, we deduce that

$$\begin{aligned} h(y_1, y_2) \quad = \quad & \frac{3}{\pi}\left[\arcsin\left(\frac{4 - y_1}{y_2}\right) - \arcsin\left(\frac{1 - y_1}{y_2}\right)\right] \\ & - \frac{1}{\pi}\left[\arcsin\left(\frac{-9 - y_1}{y_2}\right) - \arcsin\left(\frac{-16 - y_1}{y_2}\right)\right], \end{aligned} \tag{18B.1}$$

for all $(y_1, y_2) \in \mathbb{Y}$; see Figure 18B.6(D). Finally, define $H := h \circ f : \mathbb{X} \longrightarrow \mathbb{R}$. That is,

$$\begin{aligned} H(x_1, x_2) \quad = \quad & \frac{3}{\pi}\left[\arcsin\left(\frac{4 - x_1^2 + x_2^2}{2x_1x_2}\right) - \arcsin\left(\frac{1 - x_1^2 + x_2^2}{2x_1x_2}\right)\right] \\ & - \frac{1}{\pi}\left[\arcsin\left(\frac{-9 - x_1^2 + x_2^2}{2x_1x_2}\right) - \arcsin\left(\frac{-16 - x_1^2 + x_2^2}{2x_1x_2}\right)\right], \end{aligned} \tag{18B.2}$$

for all $(x_1, x_2) \in \mathbb{X}$; see Figure 18B.6(C). Proposition 18B.3(a) says that H is harmonic on $\mathbb{X}$, because h is harmonic on $\mathbb{Y}$. Finally, h satisfies the Dirichlet boundary conditions specified by b, and $B = b \circ f$; thus Proposition 18B.3(b) says that H satisfies the Dirichlet boundary conditions specified by B, as desired. ♢

For Proposition 18B.3 to be useful, we must find a conformal map from our original domain $\mathbb{X}$ to some 'nice' domain $\mathbb{Y}$ where we are able to easily solve BVPs. For example, ideally, $\mathbb{Y}$ should be a disk or a half-plane, so that we can apply the Fourier techniques of Section 14B, or the Poisson kernel methods from Sections 14B(v), 17F and 17E. If $\mathbb{X}$ is a simply connected open subset of the plane, then a deep result in complex analysis says that it is always possible to find such a conformal map. An open subset $\mathbb{U} \subset \mathbb{C}$ is **simply connected** if any closed loop in $\mathbb{U}$ can be continuously shrunk down to a point without ever leaving $\mathbb{U}$. Heuristically speaking, this means that $\mathbb{U}$ has no 'holes'. (For example, the open disk is simply connected, and so is the upper half-plane. However, the open annulus is *not* simply connected.)

Theorem 18B.5. Riemann Mapping Theorem

Let $\mathbb{U}, \mathbb{V} \subset \mathbb{C}$ be two open, simply connected regions of the complex plane. Then there is always a holomorphic bijection $f : \mathbb{U} \longrightarrow \mathbb{V}$.

Proof. See [Lan85, Chapter XIV, pp.340-358]. □

In particular, this means that any simply connected open subset of $\mathbb{C}$ is conformally isomorphic to the disk, and also conformally isomorphic to the upper half-plane. Thus, in theory, a technique like Example 18B.4 can be applied to any such region.

Unfortunately, the Riemann Mapping Theorem does not tell you how to construct the conformal isomorphism —it merely tells you that such an isomorphism *exists*. This is not very useful when we want to solve a specific boundary value problem on a specific domain. If $\mathbb{V}$ is a region bounded by a polygon, and $\mathbb{U}$ is the upper half-plane, then it is possible to construct an explicit conformal isomorphism from $\mathbb{U}$ to $\mathbb{V}$ using *Schwarz-Christoffel transformations*; see [Fis99, §3.5, p.227] or [Asm05, §12.6]. For further information about conformal maps in general, see [Fis99, §3.4], [Lan85, Chapter VII], or the innovatively visual [Nee97, Chapter 12]. Older, but still highly respected references are [Neh75], [Bie53] and [Sch79].

Application to fluid dynamics. Let $\mathbb{U} \subset \mathbb{C}$ be an open connected set, and let $\vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}^2$ be a two-dimensional vector field (describing a flow). Define $f : \mathbb{U} \longrightarrow \mathbb{C}$ by $f(u) = V_1(u) - \mathbf{i}V_2(u)$. Recall that Proposition 18A.4 on page 418 says that $\vec{\mathbf{V}}$ is sourceless and irrotational (e.g. describing a nonturbulent, incompressible fluid) if and only if f is holomorphic. Suppose F is a *complex antiderivative*[5] of f on $\mathbb{U}$ —that is $F : \mathbb{U} \longrightarrow \mathbb{C}$ is a holomorphic map such that $F' \equiv f$. Then F is called a **complex potential** for $\vec{\mathbf{V}}$. The function $\phi(u) = \mathrm{Re}\,[F(u)]$ is called the (real) **potential** of the flow. An **equipotential contour** of F is a level curve of ϕ. That is, it is a set $\mathcal{E}_x = \{u \in \mathbb{U};\, \mathrm{Re}\,[F(u)] = x\}$ for some fixed $x \in \mathbb{R}$. For example, in Figures 18B.6(A) and 18B.7(A,B), the equipotential contours are the dashed curves. A **streamline** of F is a level curve of the imaginary part of F. That is, it is a set $\mathcal{S}_y = \{u \in \mathbb{U};\, \mathrm{Im}\,[F(u)] = y\}$ for some fixed $y \in \mathbb{R}$. For example, in Figures 18B.6(A) and 18B.7(A,B), the streamlines are the solid curves.

A **trajectory** of $\vec{\mathbf{V}}$ is the path followed by a particle carried in the flow —that is, it is a smooth path $\alpha : (-T, T) \longrightarrow \mathbb{U}$ (for some $T \in (0, \infty]$) such that $\dot{\alpha}(t) = \vec{\mathbf{V}}[\alpha(t)]$ for all $t \in (-T, T)$. The flow $\vec{\mathbf{V}}$ is **confined** to $\mathbb{U}$ if no trajectories of $\vec{\mathbf{V}}$ ever pass through the boundary $\partial\mathbb{U}$. (Physically, $\partial\mathbb{U}$ represents an 'impermeable barrier'). The equipotentials and streamlines of F are important for understanding the flow defined by $\vec{\mathbf{V}}$, because the following result:

Proposition 18B.6. *Let $\vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}^2$ be a sourceless, irrotational flow, and let $F : \mathbb{U} \longrightarrow \mathbb{C}$ be a complex potential for $\vec{\mathbf{V}}$.*

[5] We will discuss how to construct complex antiderivatives in Exercise 18C.15 on page 447; for now, just assume that F exists.

(a) *If $\phi = \mathrm{Re}[F]$, then $\nabla\phi = \vec{\mathbf{V}}$. Thus, particles in the flow can be thought of as descending the 'potential energy landscape' determined by ϕ. In particular, every trajectory of the flow cuts orthogonally through every equipotential contour of F.*

(b) *Every streamline of F also cuts orthogonally through every equipotential contour.*

(c) *Every trajectory of $\vec{\mathbf{V}}$ parameterizes a streamline of F, and every streamline can be parameterized by a trajectory. (Thus, by plotting the streamlines of F, we can visualize the flow $\vec{\mathbf{V}}$).*

(d) *$\vec{\mathbf{V}}$ is confined to $\mathbb{U}$ if and only if F conformally maps $\mathbb{U}$ to a bi-infinite horizontal strip $\mathbb{V} \subset \mathbb{C}$, and maps each connected component of $\partial\mathbb{U}$ to a horizontal line in $\mathbb{V}$.*

Proof. **Exercise 18B.4** Hint: (a) Follows from the definitions of F and $\vec{\mathbf{V}}$. To prove (b) use the fact that F is a conformal map. (c) follows by combining (a) and (b), and then (d) follows from (c). □ Ⓔ

Thus, the set of conformal mappings from $\mathbb{U}$ onto such horizontal strips describes all possible sourceless, irrotational flows confined to $\mathbb{U}$. If $\partial\mathbb{U}$ is simply connected, then we can assume F maps $\mathbb{U}$ to the upper half-plane and maps $\partial\mathbb{U}$ to $\mathbb{R}$ (as in Example 18B.2(e)). Or, if we are willing to allow one 'point source' (or sink) p in $\partial\mathbb{U}$, we can find a mapping from $\mathbb{U}$ to a bi-infinite horizontal strip, which maps the half of the boundary on one side of p to the top edge of strip, maps the other half of the boundary to the bottom edge, and maps p itself to ∞ (as in Example 18B.2(a); in this case, the 'point source' is at 0).

Application to electrostatics. Proposition 18B.6 has another important physical interpretation. The function $\phi = \mathrm{Im}[f]$ is harmonic (by Proposition 18A.2 on page 417). Thus, it can be interpreted as an electrostatic potential (see Example 1D.2 on page 14). In this case, we can regard the streamlines of F as the *voltage contours* of the resulting electric field; then the 'equipotentials' F of are the *field lines* (note the reversal of roles here). If $\partial\mathbb{U}$ is a perfect conductor (e.g. a metal), then the field lines must always intersect $\partial\mathbb{U}$ orthogonally, and the voltage contours (i.e. the 'streamlines') can never intersect $\partial\mathbb{U}$ —thus, in terms of our fluid dynamical model, the 'flow' is confined to $\mathbb{U}$. Thus, the streamlines and equipotentials in Figures 18B.6(A) and 18B.7(A,B) portray the (two-dimensional) electric field generated by charged metal plates.

For more about the applications of complex analysis to fluid dynamics and electrostatics, see [Fis99, §4.2, pp.261-278] or [Nee97, §12.V, pp.527-540].

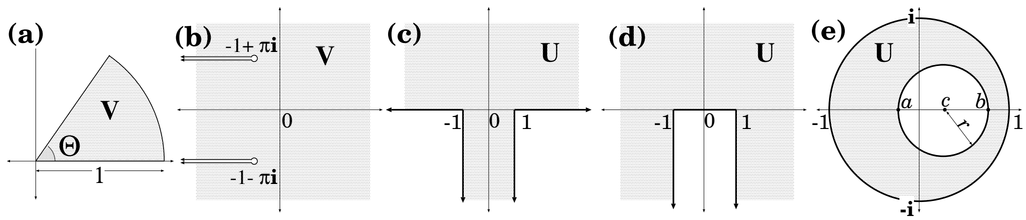


Figure 18B.8: Exercise 18B.5.

(E) **Exercise 18B.5.** (a) Let $\Theta \in (0, 2\pi]$, and consider the 'pie-wedge' domain $\mathbb{V} := \{r\,\mathtt{cis}\,\theta;\ 0 < r < 1,\ 0 < \theta < \Theta\}$ (in polar coordinates); see Figure 18B.8(a). Find a conformal isomorphism from $\mathbb{V}$ to the left half-infinite rectangle $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 0,\ 0 < y < \pi\}$.

(b) Let $\mathbb{U} := \{x + y\mathbf{i}\ ;\ -\pi < y < \pi\}$ be a bi-infinite horizontal strip of width 2π, and let $\mathbb{V} := \{x + y\mathbf{i}\,;$ either $y \neq \pm\mathbf{i}$ or $x > -1\}$, as shown in Figure 18B.8(b). Show that $f(z) := z + \exp(z)$ is a conformal isomorphism from $\mathbb{U}$ to $\mathbb{V}$.

(c) Let $\mathbb{C}_+ := \{x + y\mathbf{i}\ ;\ y > 0\}$ be the upper half-plane. Let $\mathbb{U} := \{x + y\mathbf{i};$ either $y > 0$ or $-1 < x < 1\}$. That is, $\mathbb{U}$ is the complex plane with two lower quarter-planes removed, leaving a narrow 'chasm' in between them, as shown in Figure 18B.8(c). Show that $f(z) = \frac{2}{\pi}\left(\sqrt{z^2 - 1} + \arcsin(1/z)\right)$ is a conformal isomorphism from $\mathbb{C}_+$ to $\mathbb{U}$.

(d) Let $\mathbb{C}_+ := \{x + y\mathbf{i}\ ;\ y > 0\}$ be the upper half-plane. Let $\mathbb{U} := \{x+y\mathbf{i};$ either $y > 0$ or $x < 1$ or $1 < x\}$. That is, $\mathbb{U}$ is the complex plane with a vertical half-infinite rectangle removed, as shown in Figure 18B.8(d). Show that $f(z) = \frac{2}{\pi}\left(z(1 - z^2)^{1/2} + \arcsin(z)\right)$ is a conformal isomorphism from $\mathbb{C}_+$ to $\mathbb{U}$.

(e) Let $c > 0$, let $0 < r < 1$, and let $\mathbb{U} := \{x+y\mathbf{i}\,;\ x^2+y^2 < 1$ and $(x-c)^2+y^2 < r^2\}$. That is, $\mathbb{U}$ is the 'off-centre annulus', obtained by removing from the unit disk a smaller smaller disk of radius r centered at $(c, 0)$, as shown in Figure 18B.8(e). Let $a := c - r$ and $b := c + r$, and define

$$\lambda \;:=\; \frac{1 + ab - \sqrt{(1 - a^2)(1 - b^2)}}{a + b} \qquad \text{and} \qquad R \;:=\; \frac{1 - ab - \sqrt{(1 - a^2)(1 - b^2)}}{b - a}.$$

Let $\mathbb{A} := \{x + y\mathbf{i}\,;\ R < x^2 + y^2 < 1\}$ be an annulus with inner radius R and outer radius 1, and let $f(z) := \dfrac{z - \lambda}{1 - \lambda z}$. Show that f is a conformal isomorphism from $\mathbb{U}$ into $\mathbb{A}$.

(f) Let $\mathbb{U}$ be the upper half-disk shown on the right side of Figure 18B.4, and let $\mathbb{D}$ be the unit disk. Show that the function $f(z) = -\mathbf{i}\dfrac{z^2 + 2\mathbf{i}z + 1}{z^2 - 2\mathbf{i}z + 1}$ is a conformal isomorphism from $\mathbb{U}$ into $\mathbb{D}$.

(g) Let $\mathbb{D} = \left\{x + y\mathbf{i}\ ;\ x^2 + y^2 < 1\right\}$ be the open unit disk and let $\mathbb{C}_+ = \{x + y\mathbf{i}\ ;\ y > 0\}$ be the open upper half-plane. Define $f : \mathbb{D} \longrightarrow \mathbb{C}_+$ by $f(z) = \mathbf{i}\frac{1+z}{1-z}$. Show that f is a conformal isomorphism from $\mathbb{D}$ into $\mathbb{C}_+$. ♦

(E) **Exercise 18B.6.** (a) Combine Example 18B.2(a) with Proposition 17E.1 on page 404 (or Proposition 20C.1 on page 538) to propose a general method for solving the Dirichlet problem on the bi-infinite strip $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x \in \mathbb{R},\ 0 < y < \pi\}$.

(b) Now combine Exercise (a) with Exercise 18B.5(b) to propose a general method for solving the Dirichlet problem on the domain portrayed in Figure 18B.8(b). (*Note:* despite the fact that the horizontal barriers are lines of zero thickness, your method allows you to assign different 'boundary conditions' to the two sides of these barriers.) Use your method to find the equilibrium heat distribution when the two barriers are each a different constant temperature. Reinterpret this solution as the electric field between two charged electrodes.

(c) Combine Exercise 18B.5(c) with Proposition 17E.1 on page 404 (or Proposition 20C.1 on page 538) to propose a general method for solving the Dirichlet problem on the 'chasm' domain portrayed in Figure 18B.8(c). Use your method to find the equilibrium heat distribution when the boundaries on either side of the chasm are two different constant temperatures. Reinterpret this solution as the electric field near the edge of a narrow gap between two large, oppositely charged parallel plates.

(d) Combine Exercise 18B.5(d) with Proposition 17E.1 on page 404 (or Proposition 20C.1 on page 538) to propose a general method for solving the Dirichlet problem on the domain portrayed in Figure 18B.8(d). Use your method to find the equilibrium heat distribution when the left side of the rectangle has temperature -1, the right side has temperature $+1$, and the top has temperature 0.

(e) Combine Exercise 18B.5(e) with Proposition 14B.10 on page 287 to propose a general method for solving the Dirichlet problem on the off-centre annulus portrayed in Figure 18B.8(e). Use your method to find the equilibrium heat distribution when the inner and outer circles are two different constant temperatures. Reinterpret this solution as an electric field between two concentric, oppositely charged cylinders.

(f) Combine Exercise 18B.5(f) with the methods of Sections 14B, 14B(v), and/or 17F to propose a general method for solving the Dirichlet and Neumann problems on the half-disk portrayed in Figure 18B.4. Use your method to find the equilibrium temperature distribution when the semicircular top of the half-disk is one constant temperature, and the base is another constant temperature.

(g) Combine Exercise 18B.5(g) with the Poisson Integral Formula on a disk (Proposition 14B.11 on page 290 or Proposition 17F.1 on page 407) to obtain another solution to the Dirichlet problem on a half-plane. Show that this is actually equivalent to the Poisson Integral Formula on a half-plane (Proposition 17E.1 on page 404).

(h) Combine Example 18B.2(f) with Proposition 17E.1 on page 404 (or Proposition 20C.1 on page 538) to propose a general method for solving the Dirichlet problem on the domain portrayed in Figure 18B.7(a). (*Note:* despite the fact that the vertical obstacle is a line of zero thickness, your method allows you to assign different 'boundary conditions' to the two sides of this line.) Use your method to find the equilibrium temperature distribution when the 'obstacle' has one constant temperature and the the real line has another constant temperature. Reinterpret this as the electric field generated by a charged electrode protruding but insulated from a horizontal, neutrally charged conducting barrier.

(i) Combine Exercise (a) with Example 18B.2(g) to propose a general method for solving the Dirichlet problem on the domain portrayed in Figure 18B.7(b). (*Note:* despite the fact that the horizontal barriers are lines of zero thickness, your method allows you to assign different 'boundary conditions' to the two sides of these barriers.) Use your method to find the equilibrium temperature distribution when the two horizontal barriers have different constant temperatures. Reinterpret this as the electric field between two charged electrodes. ♦

Ⓔ **Exercise 18B.7.** (a) Figure 18B.6(A) portrays the map $f(z) = z^2$ from Example 18B.2(e). Show that in this case, the equipotential contours are all curves of the form $\{x + \mathbf{i}y;\ y = \sqrt{x^2 - c}\}$ for some fixed $c > 0$. Show that the streamlines are all curves of the form $\{x + \mathbf{i}y;\ y = c/x\}$ for some fixed $c > 0$.

(b) Figure 18B.7(B) portrays the map $f(z) = \mathbf{i}\arcsin(z)$ from Example 18B.2(f). Show that in this case, the equipotential contours are all *ellipses* of the form

$$\left\{ x + \mathbf{i}y\ ;\ \frac{x^2}{\cosh(r)^2} + \frac{y^2}{\sinh(r)^2} = 1 \right\},$$

for some fixed $r \in \mathbb{R}$. Likewise, show that the streamlines are all *hyperbolas*

$$\left\{ x + \mathbf{i}y\ ;\ \frac{x^2}{\sin(r)^2} - \frac{y^2}{\cos(r)^2} = 1 \right\},$$

for some fixed $r \in \mathbb{R}$. *Hint:* Use Exercises 18A.9(d,e) on page 422.

(c) Find an equation describing all streamlines and equipotentials of the conformal map in Example 18B.2(a). Sketch the streamlines. (This describes a flow into a large body of water, from a point source on the boundary).

(d) Fix $\Theta \in (-\pi, \pi)$, and consider the infinite wedge-shaped region $\mathbb{U} = \{r \,\mathtt{cis}\, \theta;\ r \geq 0,\ 0 < \theta < 2\pi - \Theta\}$. Find a conformal isomorphism from $\mathbb{U}$ to the upper half-plane. Sketch the streamlines of this map. (This describes the flow near the bank of a wide river, at a corner where the river bends by angle of Θ).

(e) Suppose $\Theta = 2\pi/3$. Find an exact equation to describe the streamlines and equipotentials from question (d) (analogous to the equations "$y = \sqrt{x^2 - c}$" and "$y = c/x$" from question (a)).

(f) Sketch the streamlines and equipotentials defined by the conformal map in Exercise 18B.5(b). (This describes the flow out of a long pipe or channel into a large body of water).

(g) Sketch the streamlines and equipotentials defined by the inverse of the conformal map f in Exercise 18B.5(c). (In other words, sketch the f-images of vertical and horizontal lines in $\mathbb{C}_+$). This describes the flow over a deep 'chasm' in the streambed.

(h) Sketch the streamlines and equipotentials defined by the inverse of the conformal map f in Exercise 18B.5(d). (In other words, sketch the f-images of vertical and horizontal lines in $\mathbb{C}_+$). This describes the flow around a long rectangular peninsula in an ocean. ♦

18C Contour integrals and Cauchy's Theorem

Prerequisites: §18A.

A **contour** in $\mathbb{C}$ is a continuous function $\gamma : [0, S] \longrightarrow \mathbb{C}$ (for some $S > 0$) such that $\gamma(0) = \gamma(S)$, and such that γ does not 'self-intersect' —that is, $\gamma : [0, S) \longrightarrow \mathbb{C}$ is injective.[6]. Let $\gamma_r, \gamma_i : [0, S] \longrightarrow \mathbb{R}$ be the real and imaginary parts of γ (so $\gamma(s) = \gamma_r(s) + \gamma_i(s)\mathbf{i}$, for all $s \in \mathbb{R}$). For any $s \in (0, S)$, we

[6]What we are calling a contour is sometimes called a *simple, closed curve.*

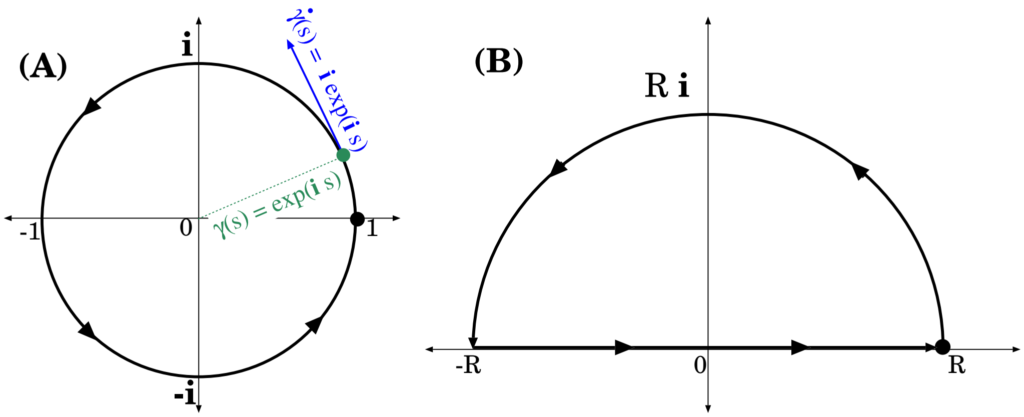


Figure 18C.1: **(A)** The counterclockwise unit circle contour from Example 18C.1. **(B)** The 'D' contour from Example 18C.3

define the (complex) **velocity vector** if γ at s by $\dot\gamma(s) := \gamma_r'(s) + \gamma_i'(s)\mathbf{i}$ (if these derivatives exist). We say that γ is **smooth** if $\dot\gamma(s)$ exists for all $s \in (0, S)$.

Example 18C.1. Define $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ by $\gamma(s) = \exp(\mathbf{i}s)$; then γ is a counterclockwise parameterization of the unit circle in the complex plane, as shown in Figure 18C.1(A). For any $s \in [0, 2\pi]$, we have $\gamma(s) = \cos(s) + \mathbf{i}\sin(s)$, so that $\dot\gamma(s) \ = \ \cos'(s) + \mathbf{i}\sin'(s) \ = \ -\sin(s) + \mathbf{i}\cos(s) \ = \ \mathbf{i}\gamma(s)$. ♢

Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset, let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be a complex function, and let $\gamma : [0, S] \longrightarrow \mathbb{U}$ be a smooth contour. The **contour integral** of f along γ is defined:

$$\oint_\gamma f \quad := \quad \int_0^S f[\gamma(s)] \cdot \dot\gamma(s)\ ds.$$

(Recall that $\dot\gamma(s)$ is a complex number, so $f[\gamma(s)] \cdot \dot\gamma(s)$ is a product of two complex numbers). Another notation we will sometimes use is $\displaystyle\oint_\gamma f(z)\ dz$.

Example 18C.2. Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ be the unit circle contour from Example 18C.1.

(a) Let $\mathbb{U} := \mathbb{C}$ and let $f(z) := 1$, a constant function. Then

$$\begin{aligned}\oint_\gamma f \quad &= \quad \int_0^{2\pi} 1 \cdot \dot\gamma(s)\ ds \quad = \quad \int_0^{2\pi} -\sin(s) + \mathbf{i}\cos(s) \ \ ds \\ &= \quad -\int_0^{2\pi} \sin(s)\ ds + \mathbf{i}\int_0^{2\pi} \cos(s) \ \ ds \quad = \quad -0 + \mathbf{i}0 \quad = \quad 0.\end{aligned}$$

(b) Let $\mathbb{U} := \mathbb{C}$ and let $f(z) := z^2$. Then

$$\begin{aligned}\oint_\gamma f &= \int_0^{2\pi} \gamma(s)^2 \cdot \dot{\gamma}(s)\, ds = \int_0^{2\pi} \exp(\mathbf{i}s)^2 \cdot \mathbf{i}\exp(\mathbf{i}s)\, ds \\ &= \mathbf{i}\int_0^{2\pi} \exp(\mathbf{i}s)^3\, ds = \mathbf{i}\int_0^{2\pi} \exp(3\mathbf{i}s)\, ds \\ &= \mathbf{i}\int_0^{2\pi} \cos(3s) + \mathbf{i}\sin(3s)\, ds = \mathbf{i}\int_0^{2\pi} \cos(3s)\, ds - \int_0^{2\pi} \sin(3s)\, ds \\ &= \mathbf{i}0 - 0 = 0.\end{aligned}$$

(c) More generally, for any $n \in \mathbb{Z}$ *except* $n = -1$, we have $\oint_\gamma z^n\, dz = 0$ (**Exercise 18C.1**). Ⓔ

(What happens if $n = -1$? See Example 18C.6 below).

(d) It follows that, if $c_n, \ldots, c_2, c_1, c_0 \in \mathbb{C}$, and $f(z) = c_n z^n + \cdots + c_2 z^2 + c_1 z + c_0$ is a complex polynomial function, then $\oint_\gamma f = 0$. ♢

A contour $\gamma : [0, S] \longrightarrow \mathbb{U} \subseteq \mathbb{C}$ is **piecewise smooth** if $\dot{\gamma}(s)$ exists for all $s \in [0, S]$, except for perhaps finitely many points $0 = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_N = S$. If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is a complex function, we define the **contour integral**

$$\oint_\gamma f := \sum_{n=1}^{N} \int_{s_{n-1}}^{s_n} f[\gamma(s)] \cdot \dot{\gamma}(s)\, ds.$$

Example 18C.3. Fix $R > 0$, and define $\gamma_R : [0, \pi + 2R] \longrightarrow \mathbb{C}$ as follows:

$$\gamma_R(s) := \begin{cases} R \cdot \exp(\mathbf{i}s) & \text{if} \quad 0 \leq s \leq \pi; \\ s - \pi - R & \text{if} \quad \pi \leq s \leq \pi + 2R. \end{cases} \tag{18C.1}$$

This contour looks like a 'D' turned on its side; see Figure 18C.1(B). The first half of the contour parameterizes the upper half of the circle from R to $-R$. The second half parameterizes a straight horizontal line segment from $-R$ back to R. It follows that

$$\dot{\gamma}_R(s) := \begin{cases} R\mathbf{i} \cdot \exp(\mathbf{i}s) & \text{if} \quad 0 \leq s \leq \pi; \\ 1 & \text{if} \quad \pi \leq s \leq \pi + 2R. \end{cases} \tag{18C.2}$$

(a) Let $\mathbb{U} := \mathbb{C}$ and let $f(z) := z$. Then

$$\begin{aligned}\oint_{\gamma_R} f &= \int_0^{\pi} \gamma(s) \cdot \dot{\gamma}(s)\, ds + \int_\pi^{\pi+2R} \gamma(s) \cdot \dot{\gamma}(s)\, ds \\ &\underset{(*)}{=} \int_0^{\pi} R\exp(\mathbf{i}s) \cdot R\mathbf{i}\exp(\mathbf{i}s)\, ds + \int_\pi^{\pi+2R} (s - \pi - R)\, ds\end{aligned}$$

$$\begin{aligned}
&= R^2\mathbf{i}\int_0^{\pi} \exp(\mathbf{i}s)^2 \, ds \;+\; \int_{-R}^{R} t \, dt \\
&= R^2\mathbf{i}\int_0^{\pi} \cos(2s) + \mathbf{i}\sin(2s) \, ds \;+\; \left.\frac{t^2}{2}\right|_{t=-R}^{t=R} \\
&= \frac{R^2\mathbf{i}}{2}\Big(\sin(2s) - \mathbf{i}\cos(2s)\Big)_{s=0}^{s=\pi} \;+\; \frac{1}{2}\Big(R^2 - (-R)^2\Big) \\
&= \frac{R^2\mathbf{i}}{2}\Big((0-0) - \mathbf{i}(1-1)\Big) \;+\; 0 \\
&= 0+0 \quad = \quad 0.
\end{aligned}$$

Here, $(*)$ is by equations (18C.1) and (18C.2).

(b) For generally, for any $n \in \mathbb{Z}$, if $n \neq -1$, then $\oint_{\gamma_R} z^n \, dz \;=\; 0$ (**Exercise 18C.2**). Ⓔ

Thus, if f is any complex polynomial, then $\oint_{\gamma_R} f \;=\; 0$. ♢

Any contour $\gamma : [0, S] \longrightarrow \mathbb{C}$ cuts the complex plane into exactly two pieces. Formally the set $\mathbb{C} \setminus \gamma[0, S]$ has exactly two connected components, and exactly one of these components (the one 'inside' γ) is bounded.[7] The bounded component is called the **purview** of γ; see Figure 18C.2(A). For example, the purview of the unit circle is the unit disk. If $\mathbb{G}$ is the purview of γ, then clearly $\partial\mathbb{G} = \gamma[0, S]$. We say that γ is **counterclockwise** if the outward normal vector of $\mathbb{G}$ is always on the righthand side of the vector $\dot{\gamma}$. We say γ is **clockwise** if the outward normal vector of $\mathbb{G}$ is always on the lefthand side of the vector $\dot{\gamma}$; see Figure 18C.2(C).

The contour γ is called **nullhomotopic** in $\mathbb{U}$ if the purview of γ is entirely contained in $\mathbb{U}$; see Figure 18C.2(B). Equivalently: it is possible to continuously 'shrink' γ down to a point without the any part of the contour leaving $\mathbb{U}$; this is called a **nullhomotopy** of γ, and is portrayed in see Figure 18C.2(D). Heuristically speaking, γ is nullhomotopic in $\mathbb{U}$ if and only if γ does not encircle any 'holes' in the domain $\mathbb{U}$.

Example 18C.4. (a) The unit circle from Examples 18C.1 and the 'D' contour from Example 18C.3 are both counterclockwise, and both are nullhomotopic in the domain $\mathbb{U} = \mathbb{C}$.

(b) The unit circle is *not* nullhomotopic on the domain $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The purview of γ (the unit disk) is *not* entirely contained in $\mathbb{C}^*$, because the point 0 is missing. Equivalently, it is *not* possible to shrink γ down to a point without passing the curve through 0 at some moment; at this moment the curve would not be contained in $\mathbb{U}$. ♢

The 'zero' outcomes of Examples 18C.2 and 18C.3 not accidents; they are consequences of one of the fundamental results of complex analysis.

[7]This seemingly innocent statement is actually the content of the *Jordan Curve Theorem,* which is a surprisingly difficult and deep result in planar topology.

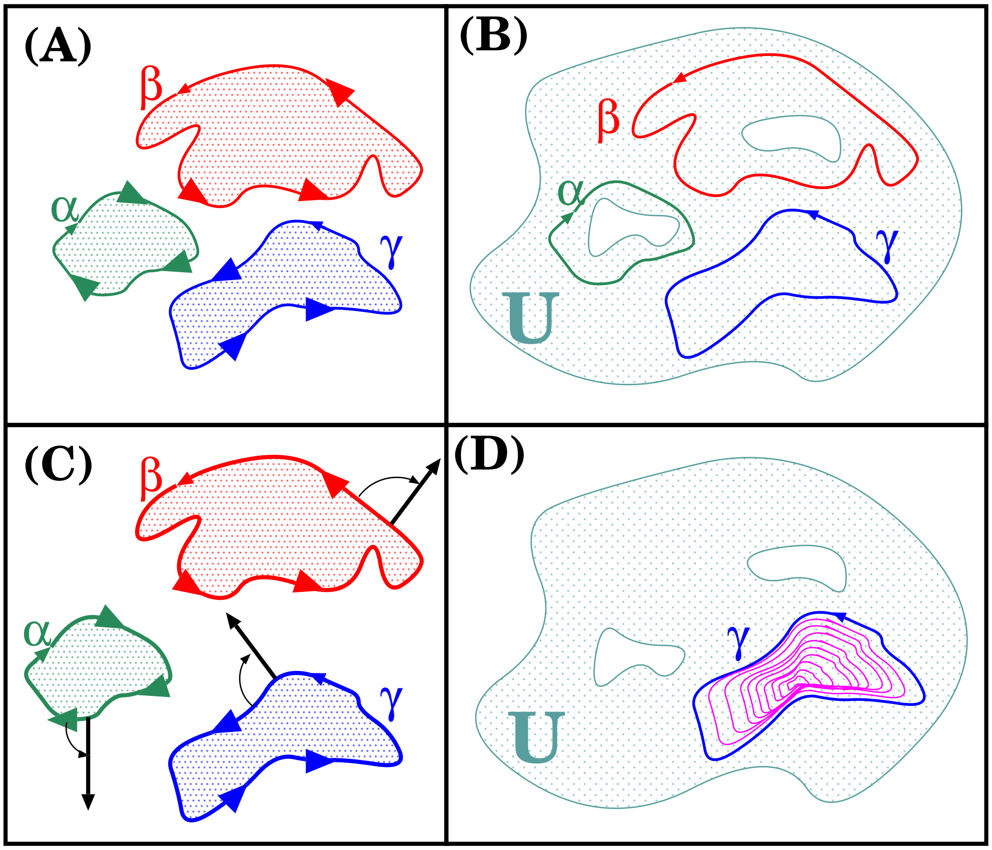


Figure 18C.2: **(A)** Three contours and their purviews. **(B)** Contour γ is nullhomotopic in $\mathbb{U}$, but contours α and β are not nullhomotopic in $\mathbb{U}$. **(C)** Contour α is clockwise; contours β and γ are counterclockwise. **(D)** A nullhomotopy of γ.

Theorem 18C.5. (Cauchy's Theorem)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic on $\mathbb{U}$. If $\gamma : [0, S] \longrightarrow \mathbb{U}$ is a contour which is nullhomotopic in $\mathbb{U}$, then $\oint_\gamma f = 0$.

Proof. Let $\mathbb{G}$ be the purview of γ. If γ is nullhomotopic in $\mathbb{U}$, then $\mathbb{G} \subseteq \mathbb{U}$ and γ parameterizes the boundary $\partial\mathbb{G}$. Treat $\mathbb{U}$ as a subset of $\mathbb{R}^2$. Let $f_r : \mathbb{U} \longrightarrow \mathbb{R}$ and $f_i : \mathbb{U} \longrightarrow \mathbb{R}$ be the real and imaginary parts of f. The function f can be expressed as a vector field $\vec{\mathbf{V}} : \mathbb{U} \longrightarrow \mathbb{R}^2$ defined by $V_1(u) := f_r(u)$ and $V_2(u) := -f_i(u)$. For any $\mathbf{b} \in \partial\mathbb{G}$, let $\vec{\mathbf{N}}[\mathbf{b}]$ denote the outward unit normal vector to $\partial\mathbb{G}$ at $\mathbf{b}$. We define

$$\mathrm{Flux}(\vec{\mathbf{V}}, \gamma) := \int_0^S \vec{\mathbf{V}}[\gamma(s)] \bullet \vec{\mathbf{N}}[\gamma(s)]\, ds, \quad \text{and} \quad \mathrm{Work}(\vec{\mathbf{V}}, \gamma) := \int_0^S \vec{\mathbf{V}}[\gamma(s)] \bullet \dot{\gamma}(s)\, ds.$$

The first integral is the *flux* of $\vec{\mathbf{V}}$ across the boundary of $\mathbb{G}$; this is just a reformulation of equation (0E.1) on page 562 (see Figure 0E.1(B) on page 561). The second integral is the *work* of $\vec{\mathbf{V}}$ along the contour γ.

Claim 1: *(a)* $\mathrm{Re}\left[\oint_\gamma f\right] = \mathrm{Work}(\vec{\mathbf{V}}, \gamma)$ *and* $\mathrm{Im}\left[\oint_\gamma f\right] = \mathrm{Flux}(\vec{\mathbf{V}}, \gamma)$.

(b) If $\mathrm{div}\,(\vec{\mathbf{V}}) \equiv 0$, *then* $\mathrm{Flux}(\vec{\mathbf{V}}, \gamma) = 0$.

(c) If $\mathrm{curl}\,(\vec{\mathbf{V}}) \equiv 0$, *then* $\mathrm{Work}(\vec{\mathbf{V}}, \gamma) = 0$.

Proof. (a) is **Exercise 18C.3**. (b) is Green's Theorem (Theorem 0E.3 on page 562). (c) is **Exercise 18C.4** (Hint: it's a variant of Green's Theorem). Ⓔ Ⓔ

$\diamond_{\text{Claim 1}}$

Now, if f is holomorphic on $\mathbb{U}$, then Proposition 18A.4 on page 418 says that $\mathrm{div}\,(\vec{\mathbf{V}}) \equiv 0$ and $\mathrm{curl}\,(\vec{\mathbf{V}}) \equiv 0$. Then Claim 1 implies $\oint_\gamma f = 0$.

(For other proofs, see [Fis99, Theorem 1, §2.3, p.107], [Lan85, §IV.3, p.137], or [Nee97, §8.X, p.410]). □

At this point you are wondering: what are complex contour integrals good for, if they are always equal to zero? The answer is that $\oint_\gamma f$ is only zero if the function f is holomorphic in the purview of γ. If f has a **singularity** inside this purview (i.e. a point where f is *not* complex-differentiable, or perhaps not even defined), then $\oint_\gamma f$ might be nonzero.

Example 18C.6. Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ be the unit circle contour from Example 18C.1. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and define $f : \mathbb{C}^* \longrightarrow \mathbb{C}$ by $f(z) := 1/z$. Then

$$\oint_\gamma f \quad = \quad \int_0^{2\pi} \frac{\dot\gamma(s)}{\gamma(s)}\, ds \quad = \quad \int_0^{2\pi} \frac{\mathbf{i}\exp(\mathbf{i}s)}{\exp(\mathbf{i}s)} ds \quad = \quad \int_0^{2\pi} \mathbf{i}\, ds \quad = \quad 2\pi\mathbf{i}.$$

Notice that γ is *not* nullhomotopic on $\mathbb{C}^*$. Of course, we could extend f to all of $\mathbb{C}$ by defining $f(0)$ in some arbitrary way. But no matter how we do this, f will never be complex-differentiable at zero —in other words, 0 is a *singularity* of f. ♢

If the purview of γ contains one or more singularities of f, then the value of $\oint_\gamma f$ reveals important information about these singularities. Indeed, the value of $\oint_\gamma f$ depends *only* on the singularities within the purview of γ, and *not* on the shape of γ itself. This is a consequence of the *homotopy-invariance* of contour integration.

Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset, and let $\gamma_0, \gamma_1 : [0, S] \longrightarrow \mathbb{U}$ be two contours. We say that γ_0 is **homotopic to** γ_1 **in** $\mathbb{U}$ if γ_0 can be 'continuously deformed' into γ_1 without ever moving outside of $\mathbb{U}$; see Figure 18C.3. (In particular, γ is

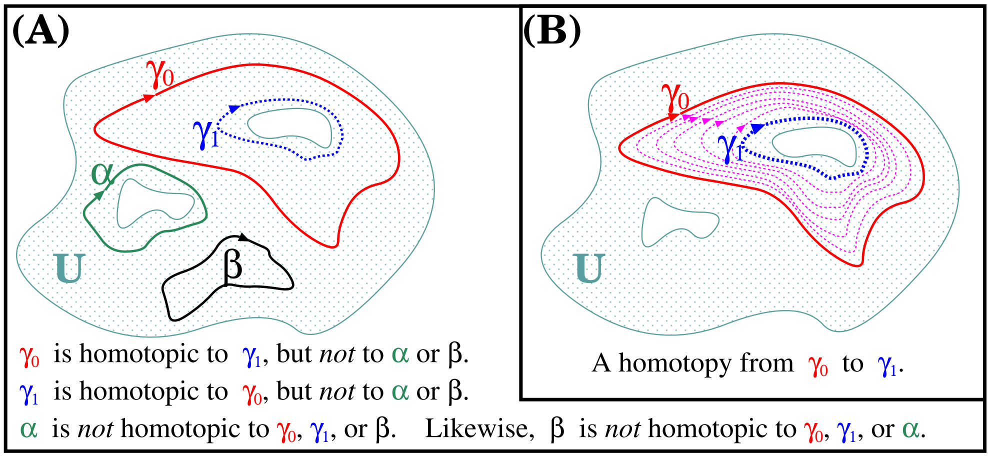


Figure 18C.3: Homotopy

nullhomotopic if γ is homotopic to a constant path in $\mathbb{U}$.) Formally, this means there is a continuous function $\Gamma : [0,1] \times [0,S] \longrightarrow \mathbb{U}$ such that:

- For all $s \in [0,S]$, $\Gamma(0,s) = \gamma_0(s)$.
- For all $s \in [0,S]$, $\Gamma(1,s) = \gamma_1(s)$.
- For all $t \in [0,1]$, if we fix t and define $\gamma_t : [0,S] \longrightarrow \mathbb{U}$ by $\gamma_t(s) := \Gamma(t,s)$ for all $s \in [0,S]$, then γ_t is a contour in $\mathbb{U}$.

The function Γ is called a **homotopy** of γ_0 into γ_1. See Figure 18C.4(A).

Proposition 18C.7. (Homotopy invariance of contour integration)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be a holomorphic function. Let $\gamma_0, \gamma_1 : [0,S] \longrightarrow \mathbb{U}$ be two contours. If γ_0 is homotopic to γ_1 in $\mathbb{U}$, then $\displaystyle\oint_{\gamma_0} f = \oint_{\gamma_1} f.$

Before proving this result, it will be useful to somewhat extend our definition of contour integration. A **chain** is a piecewise-continuous, piecewise differentiable function $\alpha : [0,S] \longrightarrow \mathbb{C}$ (for some $S > 0$). (Thus, a chain α is a *contour* if α is continuous, $\alpha(S) = \alpha(0)$, and α is not self-intersecting). If $\alpha : [0,S] \longrightarrow \mathbb{U} \subseteq \mathbb{C}$ is a chain, and $f : \mathbb{U} \longrightarrow \mathbb{C}$ is a complex-valued function, then the **integral** of f along α is defined

$$\oint_\alpha f \quad = \quad \int_0^S f[\alpha(s)] \cdot \dot{\alpha}(s)\, ds. \tag{18C.3}$$

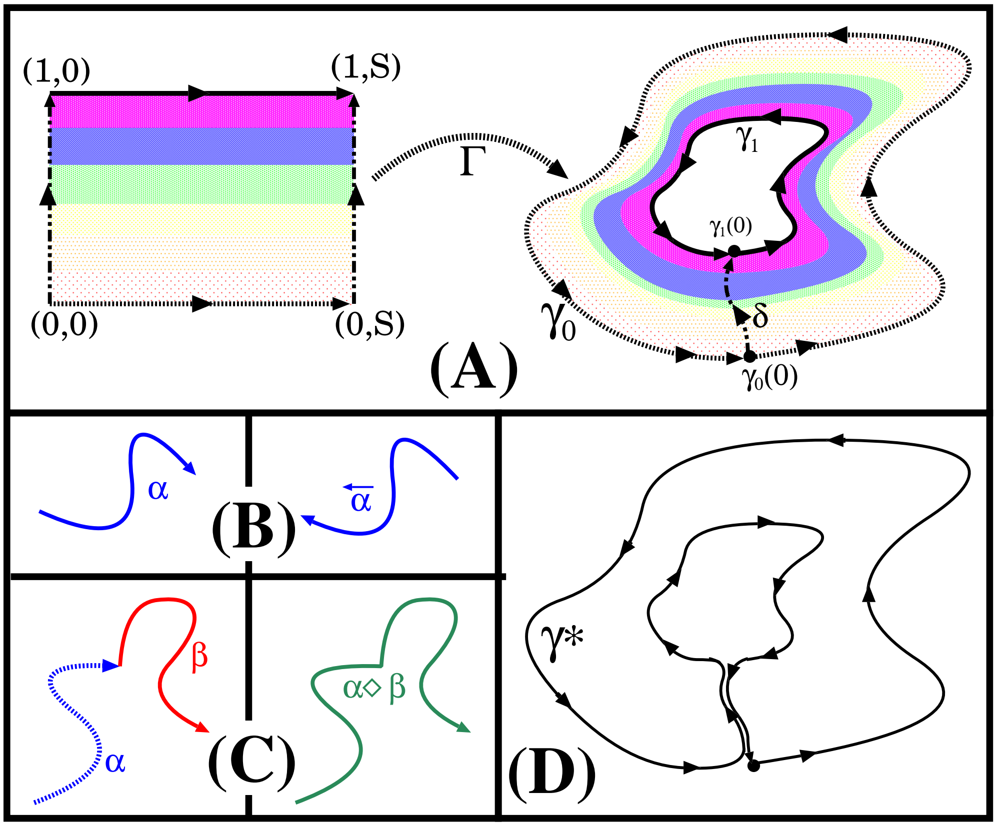


Figure 18C.4: **(A)** Γ is a homotopy from γ_0 to γ_1. **(B)** The reversal $\overleftarrow{\alpha}$ of α. **(C)** The linking $\alpha \diamond \beta$. **(D)** The contour γ^* defined by the 'boundary' of the homotopy map Γ from Figure (A).

Here we define $\dot{\alpha}(s) = 0$ whenever s is one of the (finitely many) points where α is nondifferentiable or discontinuous. (Thus, if α is a contour, then (18C.3) is just the contour integral $\oint_\alpha f$).

The **reversal** of chain α is the chain $\overleftarrow{\alpha}$: $[0, S] \longrightarrow \mathbb{C}$ defined by $\overleftarrow{\alpha}\ (s) := \alpha(S-s)$; see Figure 18C.4(B). If $\alpha : [0,S] \longrightarrow \mathbb{C}$ and $\beta : [0,T] \longrightarrow \mathbb{C}$ are two chains, then the **linking** of α and β is the chain $\alpha \diamond \beta : [0, S+T] \longrightarrow \mathbb{C}$ defined

$$\alpha \diamond \beta(s) \quad := \quad \begin{cases} \alpha(s) & \text{if} \quad 0 \leq s \leq S; \\ \beta(s-S) & \text{if} \quad S \leq s \leq S+T. \end{cases} \qquad \text{(Figure 18C.4(C))}$$

Lemma 18C.8. *Let $\mathbb{U} \subseteq \mathbb{C}$ be an open set and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be a complex-valued function. Let $\alpha : [0,S] \longrightarrow \mathbb{U}$ be a chain.*

(a) $\displaystyle\oint_{\overleftarrow{\alpha}} f \;=\; -\oint_{\alpha} f.$

(b) *If* $\beta : [0,T] \longrightarrow \mathbb{C}$ *is another chain, then* $\displaystyle\oint_{\alpha\diamond\beta} f \;=\oint_{\alpha} f + \oint_{\beta} f.$

(c) $\alpha \diamond \beta$ *is continuous if and only if* α *and* β *are both continuous, and* $\beta(0) = \alpha(S)$.

(d) *The linking operation is* associative*: that is, if* γ *is another chain, then* $(\alpha \diamond \beta) \diamond \gamma \;=\; \alpha \diamond (\beta \diamond \gamma)$.

(Thus, we normally drop the brackets and just write $\alpha \diamond \beta \diamond \gamma$).

Ⓔ *Proof.* **Exercise 18C.5** □

Proof of Proposition 18C.7. Define the continuous path $\delta : [0,1] \longrightarrow \mathbb{C}$ by

$$\delta(t) \quad := \quad \Gamma(0,t) \quad = \quad \Gamma(S,t), \qquad \text{for all } t \in [0,1].$$

Figure 18C.4(A) shows how δ traces the path defined by the homotopy Γ from $\gamma_0(S)$ $(= \gamma_0(0))$ to $\gamma_1(S)$ $(= \gamma_1(0))$. We assert (without proof) that the homotopy Γ can always be chosen such that δ is piecewise smooth; thus we regard δ as a chain. Figure 18C.4(D) portrays the contour $\gamma^* \;:=\; \gamma_0 \diamond \delta \diamond \overleftarrow{\gamma_1} \diamond \overleftarrow{\delta}$, which traces the Γ-image of the four sides of the rectangle $[0,1] \times [0, 2\pi]$.

Claim 1: γ^* *is nullhomotopic in* $\mathbb{U}$.

Proof. The purview of γ^* is simply the image of the open rectangle $(0,1) \times (0, 2\pi)$ under the function Γ. But by definition, Γ maps $(0,1) \times (0,2\pi)$ into $\mathbb{U}$; thus the purview of γ^* is contained in $\mathbb{U}$, so γ^* is nullhomotopic in $\mathbb{U}$. $\diamond_{\text{Claim 1}}$

$$\text{Thus,} \qquad 0 \;\underset{\text{(C)}}{=\!=}\; \oint_{\gamma^*} f \;\underset{(*)}{=\!=}\; \oint_{\gamma_0\diamond\delta\diamond\overleftarrow{\gamma_1}\diamond\overleftarrow{\delta}} f$$
$$\underset{(\dagger)}{=\!=}\; \oint_{\gamma_0} f + \oint_{\delta} f - \oint_{\gamma_1} f - \oint_{\delta} f \quad = \quad \oint_{\gamma_0} f - \oint_{\gamma_1} f.$$

Here (C) is by Cauchy's Theorem and Claim 1, $(*)$ is by definition of γ, and $(\dagger)$ is by Lemma 18C.8(a,b).

Thus, we have $\displaystyle\oint_{\gamma_0} f - \oint_{\gamma_1} f \;=\; 0$, which means $\displaystyle\oint_{\gamma_0} f \;=\; \oint_{\gamma_1} f$, as claimed. □

Example 18C.6 is a special case of the following important result:

Theorem 18C.9. (Cauchy's Integral Formula)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset, let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic, let $u \in \mathbb{U}$, and let $\gamma : [0, S] \longrightarrow \mathbb{U}$ be any counterclockwise contour whose purview contains u and is contained in $\mathbb{U}$. Then $f(u) = \dfrac{1}{2\pi\mathbf{i}} \oint_\gamma \dfrac{f(z)}{z-u}\, dz$.

In other words: if $\mathbb{U}^* := \mathbb{U} \setminus \{u\}$, and we define $F_u : \mathbb{U}^* \longrightarrow \mathbb{C}$ by $F_u(z) := \dfrac{f(z)}{z-u}$ for all $z \in \mathbb{U}^*$, then $f(u) = \dfrac{1}{2\pi\mathbf{i}} \oint_\gamma F_u$.

Proof. For simplicity, we will prove this in the case $u = 0$. We must show that $\dfrac{1}{2\pi\mathbf{i}} \oint_\gamma \dfrac{f(z)}{z}\, dz = f(0)$.

Let $\mathbb{G} \subset \mathbb{U}$ be the purview of γ. For any $r > 0$, let $\mathbb{D}_r$ be the disk of radius r around 0, and let β_r be a counterclockwise parameterization of $\partial\mathbb{D}_r$ (e.g. $\beta_r(s) := re^{\mathbf{i}s}$ for all $s \in [0, 2\pi]$). Let $\mathbb{U}^* := \mathbb{U} \setminus \{0\}$.

Claim 1: *If $r > 0$ is small enough, then $\mathbb{D}_r \subset \mathbb{G}$. In this case, γ is homotopic to β_r in $\mathbb{U}^*$.*

Proof. **Exercise 18C.6** $\diamond_{\text{Claim 1}}$ Ⓔ

Now, define $\phi : \mathbb{U} \longrightarrow \mathbb{C}$ as follows.

$$\phi(u) \quad := \quad \frac{f(z) - f(0)}{z} \qquad \text{for all } z \in \mathbb{U}^*, \text{ and} \quad \phi(0) \quad := \quad f'(0).$$

Then ϕ is holomorphic on $\mathbb{U}^*$. Observe that

$$1 \underset{(*)}{=} \frac{1}{2\pi\mathbf{i}} \oint_{\beta_r} \frac{1}{z}\, dz \underset{(\dagger)}{=} \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{1}{z}\, dz. \tag{18C.4}$$

where $(*)$ is by Example 18C.6 on page 439, and $(\dagger)$ is by Claim 1 and Proposition 18C.7. Thus,

$$\begin{aligned}
f(0) &\underset{(*)}{=} \frac{f(0)}{2\pi\mathbf{i}} \oint_\gamma \frac{1}{z}\, dz \quad = \quad \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(0)}{z}\, dz. \\
\text{Thus,} \quad \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z}\, dz - f(0) &= \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z}\, dz - \frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{f(0)}{z}\, dz \\
&= \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z} - \frac{f(0)}{z}\, dz \\
&= \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z) - f(0)}{z}\, dz \quad = \quad \frac{1}{2\pi\mathbf{i}} \oint_\gamma \phi \\
&\underset{(\dagger)}{=} \frac{1}{2\pi\mathbf{i}} \oint_{\beta_r} \phi, \quad \text{for any } r > 0.
\end{aligned}$$

Here, $(*)$ is by eqn.(18C.4), and $(\dagger)$ is again by Claim 1 and Proposition 18C.7. Thus, we have

$$\frac{1}{2\pi\mathbf{i}}\oint_\gamma \frac{f(z)}{z}\, dz \; - \; f(0) \quad = \quad \lim_{r\to 0} \frac{1}{2\pi\mathbf{i}}\oint_{\beta_r} \phi. \tag{18C.5}$$

Thus, it suffices to show that $\lim_{r\to 0} \oint_{\beta_r} \phi \;=\; 0$. To see this, first note that ϕ is continuous at 0 (because $\lim_{z\to 0} \phi(z) = f'(0)$ by definition of the derivative), and ϕ is also continuous on the rest of $\mathbb{U}$ (where ϕ is just another holomorphic function). Thus, ϕ is bounded on $\mathbb{G}$ (because $\mathbb{G}$ is a bounded set whose closure is inside $\mathbb{U}$). Thus, if $M := \sup_{z\in\mathbb{G}} |\phi(z)|$, then $M < \infty$. But then

$$\left|\oint_{\beta_r} \phi\right| \quad \underset{(*)}{\leq} \quad M\cdot \mathrm{length}(\beta_r) \quad = \quad M\cdot 2\pi r \quad \xrightarrow[r\to 0]{} \quad 0,$$

where $(*)$ is by Lemma 18C.10 (below).

Thus, $\lim_{r\to 0} \oint_{\beta_r} \phi \;=\; 0$, so eqn. (18C.5) implies that $\frac{1}{2\pi\mathbf{i}}\oint_\gamma \frac{f(z)}{z}\, dz \;=\; f(0)$, as desired. $\square$

If $\gamma : [0, S] \longrightarrow \mathbb{C}$ is a chain, then we define $\mathrm{length}(\gamma) \;:=\; \int_0^S |\dot{\gamma}(s)|\, ds$. The proof of Theorem 18C.9 invoked the following useful lemma.

Lemma 18C.10. *Let $f : \mathbb{U} \longrightarrow \mathbb{C}$ and let γ be a chain in $\mathbb{U}$. If $M := \sup_{u\in\mathbb{U}} |f(u)|$, then* $\left|\int_\gamma f\right| \;\leq\; M\cdot \mathrm{length}(\gamma)$.

Ⓔ *Proof.* **Exercise 18C.7** $\square$

Ⓔ **Exercise 18C.8.** Prove the general case of Theorem 18C.9, for an arbitrary $u \in \mathbb{C}$. ♦

Corollary 18C.11. (Mean Value Theorem for holomorphic functions)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open set and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic. Let $r > 0$ be small enough that the circle $\mathbb{S}(r)$ of radius r around u is contained in $\mathbb{U}$. Then

$$f(u) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{S}(r)} f(s)\, ds \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} f(u + r\, e^{\mathbf{i}\theta})\, d\theta.$$

Proof. Define $\gamma : [0, 2\pi] \longrightarrow \mathbb{U}$ by $\gamma(s) := u + re^{\mathbf{i}s}$ for all $s \in [0, 2\pi]$; thus, γ is a counterclockwise parameterization of $\mathbb{S}(r)$, and $\dot{\gamma}(s) = \mathbf{i}re^{\mathbf{i}s}$ for all $s \in [0, 2\pi]$. Then

$$f(u) \underset{(*)}{=} \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z-u}\, dz = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{f[\gamma(\theta)]}{\gamma(\theta)-u}\, \dot{\gamma}(\theta)\, d\theta$$
$$= \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{f(u + r\, e^{\mathbf{i}\theta})}{re^{\mathbf{i}\theta}}\, \mathbf{i}re^{\mathbf{i}\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(u + r\, e^{\mathbf{i}\theta})\, d\theta,$$

as desired. Here $(*)$ is by Cauchy's Integral Formula. □

Exercise 18C.9. Using Proposition 18C.11, derive another proof of the Mean Value Theorem for *harmonic* functions on $\mathbb{U}$ (Theorem 1E.1 on page 16). (Hint: Use Proposition 18A.3 on page 417). ♦ Ⓔ

Corollary 18C.12. (Maximum Modulus Principle)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open set and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic. Then the function $m(z) := |f(z)|$ has no local maxima inside $\mathbb{U}$.

Proof. **Exercise 18C.10** Hint: Use the Mean Value Theorem. □ Ⓔ

Exercise 18C.11. Let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic. Show that the functions $R(z) := \mathrm{Re}\,[f(z)]$ and $I(z) := \mathrm{Im}\,[f(z)]$ have no local maxima or minima inside $\mathbb{U}$. ♦ Ⓔ

Let $\mathbb{D} := \{z \in \mathbb{C}\ ;\ |z| < 1\}$ be the open unit disk in the complex plane, and let $\mathbb{S} := \partial\mathbb{D}$ be the unit circle. The **Poisson kernel** for $\mathbb{D}$ is the function $\mathcal{P} : \mathbb{S} \times \mathbb{D} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{P}(s, u) := \frac{1 - |u|^2}{|s-u|^2}, \qquad \text{for all } s \in \mathbb{S} \text{ and } u \in \mathbb{D}.$$

Corollary 18C.13. (Poisson Integral Formula for holomorphic functions)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open subset containing the unit disk $\mathbb{D}$, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic. Then for all $u \in \mathbb{D}$,

$$f(u) = \frac{1}{2\pi} \int_{\mathbb{S}} f(s)\, \mathcal{P}(s, u)\, ds = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}\theta})\, \mathcal{P}(e^{\mathbf{i}\theta}, u)\, d\theta.$$

Proof. If $u \in \mathbb{D}$, then $\overline{u}^{-1}$ is outside $\mathbb{D}$ (because $|\overline{u}^{-1}| = |\overline{u}|^{-1} = |u|^{-1} > 1$ if $|u| < 1$). Thus, the set $\mathbb{C}_u := \mathbb{C} \setminus \{\overline{u}^{-1}\}$ contains $\mathbb{D}$. Fix $u \in \mathbb{D}$ and define the function $g_u : \mathbb{C}_u \longrightarrow \mathbb{C}$ by

$$g_u(z) \quad := \quad \frac{f(z) \cdot \overline{u}}{1 - \overline{u}z}.$$

Claim 1: *g_u is holomorphic on $\mathbb{C}_u$.*

Ⓔ *Proof.* **Exercise 18C.12** $\diamond_{\text{Claim 1}}$

Now, define $F_u : \mathbb{U} \longrightarrow \mathbb{C}$ by $F_u(z) := \dfrac{f(z)}{z-u}$, and let $\gamma : [0, 2\pi] \longrightarrow \mathbb{S}$ be the unit circle contour from Example 18C.1 (i.e. $\gamma(s) = e^{\mathbf{i}s}$ for all $s \in [0, 2\pi]$). Then

$$\begin{aligned}
f(u) &= \frac{1}{2\pi\mathbf{i}} \oint_\gamma F_u \qquad \text{by Cauchy's Integral Formula (Theorem 18C.9),} \\
\text{and} \quad 0 &= \frac{1}{2\pi\mathbf{i}} \oint_\gamma g_u \qquad \text{by Cauchy's Theorem (Theorem 18C.5),} \\
\text{Thus,} \quad f(u) &= \frac{1}{2\pi\mathbf{i}} \oint_\gamma (F_u + g_u) \quad = \quad \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \Big(F_u[\gamma(\theta)] + g_u[\gamma(\theta)] \Big) \, \dot{\gamma}(\theta) \, d\theta \\
&= \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \left(\frac{f(e^{\mathbf{i}\theta})}{e^{\mathbf{i}\theta} - u} + \frac{f(e^{\mathbf{i}\theta}) \cdot \overline{u}}{1 - \overline{u}e^{\mathbf{i}\theta}} \right) \mathbf{i}e^{\mathbf{i}\theta} \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}\theta}) \cdot \left(\frac{e^{\mathbf{i}\theta}}{e^{\mathbf{i}\theta} - u} + \frac{e^{\mathbf{i}\theta}\overline{u}}{1 - \overline{u}e^{\mathbf{i}\theta}} \right) d\theta \\
&\underset{(*)}{=} \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}\theta}) \cdot \frac{1 - |u|^2}{|e^{\mathbf{i}\theta} - \overline{u}|^2} \, d\theta \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}\theta}) \mathcal{P}(e^{\mathbf{i}\theta}, u) \, d\theta,
\end{aligned}$$

as desired. Here, $(*)$ uses the fact that, for any $s \in \mathbb{S}$ and $u \in \mathbb{C}$,

$$\begin{aligned}
\frac{s}{s-u} + \frac{s\overline{u}}{1 - \overline{u}s} &= \frac{s}{s-u} + \frac{\overline{s}s\overline{u}}{\overline{s} - \overline{u}\overline{s}s} \quad = \quad \frac{s}{s-u} + \frac{|s|^2\overline{u}}{\overline{s} - \overline{u}|s|^2} \\
&\underset{(*)}{=} \frac{s}{s-u} + \frac{\overline{u}}{\overline{s} - \overline{u}} \quad = \quad \frac{s \cdot (\overline{s} - \overline{u}) + \overline{u} \cdot (s-u)}{(s-u) \cdot (\overline{s} - \overline{u})} \\
&= \frac{|s|^2 - s\overline{u} + \overline{u}s - |u|^2}{(s-u) \cdot \overline{(s-u)}} \quad = \quad \frac{|s|^2 - |u|^2}{|s-u|^2} \quad \underset{(*)}{=} \quad \frac{1 - |u|^2}{|s-u|^2}.
\end{aligned}$$

where both $(*)$ are because $|s| = 1$. $\square$

Ⓔ **Exercise 18C.13.** Using Corollary 18C.13, derive yet another proof of the Poisson Integral Formula for *harmonic* functions on $\mathbb{D}$. (See Proposition 14B.11 on page 290, and also Proposition 17F.1 on page 407.) Hint: Use Proposition 18A.3 on page 417. ♦

At this point, we have proved the Poisson Integral Formula three entirely different ways: using Fourier series (Proposition 14B.11), using impulse-response methods (Proposition 17F.1), and now, using complex analysis (Corollary 18C.13). In § 18F on page 461 below, we will encounter the Poisson Integral Formula yet again, while studying the Abel mean of a Fourier series.

An equation which expresses the solution to a boundary value problem in terms of an integral over the boundary of the domain is called an **integral representation formula**. For example, Poisson Integral Formula is such a formula, as is Poisson's solution to the Dirichlet problem on a half-space (Proposition 17E.1 on page 404). Cauchy's Integral Formula provides an integral representation formula for any holomorphic function on *any* domain in $\mathbb{C}$ which is bounded by a contour. Our proof of Corollary 18C.13 shows how this can be used to obtain integral representation formulae for harmonic functions on planar domains.

Exercise 18C.14. (*Liouville's Theorem*) Ⓔ

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic and *bounded* —i.e. there is some $M > 0$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$. Show that f must be a constant function.

Hint. Define $g(z) := \frac{f(z)-f(0)}{z}$.

(a) Show that g is holomorphic on $\mathbb{C}$.

(b) Show that $|g(z)| < 2M/|z|$ for all $z \in \mathbb{C}$.

(c) Let $z \in \mathbb{C}$. Let γ be a circle of radius $R > 0$ around 0, where R is large enough that z is in the purview of γ. Use Cauchy's Integral Formula and Lemma 18C.10 on page 444 (below) to show that $|g(z)| < \dfrac{1}{2\pi}\dfrac{2M}{R}\dfrac{2\pi R}{R-|z|}$. Now let $R \to \infty$. ♦

Exercise 18C.15. (*Complex antiderivatives*) Ⓔ

Let $\mathbb{U} \subset \mathbb{C}$ be an open connected set. We say that $\mathbb{U}$ is **simply connected** if every contour in $\mathbb{U}$ is nullhomotopic. Heuristically speaking, this means $\mathbb{U}$ doesn't have any 'holes'. For any $u_0, u_1 \in \mathbb{U}$, a **path in** $\mathbb{U}$ from u_0 to u_1 is a continuous function $\gamma : [0, S] \longrightarrow \mathbb{U}$ such that $\gamma(0) = u_0$ and $\gamma(S) = u_1$.

Let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic. Pick a 'basepoint' $b \in \mathbb{U}$, and define a function $F : \mathbb{U} \longrightarrow \mathbb{C}$ as follows.

$$\text{For all } u \in \mathbb{U},\ F(u) \quad := \quad \oint_\gamma f, \quad \text{where } \gamma \text{ is any path in } \mathbb{U} \text{ from } b \text{ to } u. \qquad (18C.6)$$

(a) Show that $F(u)$ is well-defined by expression (18C.6), independent of the path γ you use to get from b to u.

(*Hint.* If γ_1 and γ_2 are two paths from b to u, show that $\gamma_1 \diamond \overleftarrow{\gamma_2}$ is a contour. Then apply Cauchy's Theorem).

(b) For any $u_1, u_2 \in \mathbb{U}$, show that $F(u_2) - F(u_1) = \int_\gamma f$, where γ is any path in $\mathbb{U}$ from u_1 to u_2.

(c) Show that F is a holomorphic function, and $F'(u) = f(u)$ for all $u \in \mathbb{U}$.

(*Hint.* Write $F'(u)$ as the limit (18A.1) on page 415. For any c close to u let $\gamma : [0,1] \longrightarrow \mathbb{U}$ be the straight-line path linking u to c (i.e. $\gamma(s) = sc + (1-s)u$.). Deduce from part (b) that $\dfrac{F(c) - F(u)}{c - u} = \dfrac{1}{c-u} \oint_\gamma f$. Now take the limit as $c \to u$.)

The function F is called a **complex antiderivative** of f, based at b. Part (c) is the complex version of the Fundamental Theorem of Calculus.

(d) Let $\mathbb{U} = \mathbb{C}$, and let $b \in \mathbb{U}$, and let $f(u) = \exp(u)$. Let F be the complex antiderivative of f based at b. Show that $F(u) = \exp(u) - \exp(b)$ for all $u \in \mathbb{C}$.

(e) Let $\mathbb{U} = \mathbb{C}$, and let $b \in \mathbb{U}$, and let $f(u) = u^n$ for some $n \in \mathbb{N}$. Let F be the complex antiderivative of f based at b. Show that $F(u) = \frac{1}{n+1}(u^{n+1} - b^{n+1})$, for all $u \in \mathbb{C}$.

We already encountered one application of complex antiderivatives in Proposition 18B.6 on page 430. The next two exercises describe another important application. ♦

Ⓔ **Exercise 18C.16.** *Complex logarithms* (follows Exercise 18C.15).

(a) Let $\mathbb{U} \subset \mathbb{C}$ be an open, simply connected set which does not contain 0. Define a 'complex logarithm function' $\log : \mathbb{U} \longrightarrow \mathbb{C}$ as the complex antiderivative of $1/z$ based at 1. That is, $\log(u) := \oint_\gamma 1/z \, dz$, where γ is any path in $\mathbb{U}$ from 1 to z. Show that log is a *right-inverse* of the exponential function —that is, $\exp(\log(u)) = u$ for all $u \in \mathbb{U}$.

(b) What goes wrong with part (a) if $0 \in \mathbb{U}$? What goes wrong if $0 \notin \mathbb{U}$, but $\mathbb{U}$ contains an annulus which encircles 0? (*Hint.* Consider Example 18C.6)

Remark: This is the reason why we required $\mathbb{U}$ to be simply connected in Exercise 18C.15.

(c) Suppose our definition of 'complex logarithm' is 'any right-inverse of the complex exponential function' —that is, any holomorphic function $L : \mathbb{U} \longrightarrow \mathbb{C}$ such that $\exp(L(u)) = u$ for all $u \in \mathbb{U}$. Suppose $L_0 : \mathbb{U} \longrightarrow \mathbb{C}$ is one such 'logarithm' function (defined as in part (a), for example). Define $L_1 : \mathbb{U} \longrightarrow \mathbb{C}$ by $L_1(u) = L_0(u) + 2\pi\mathbf{i}$. Show that L_1 is also a 'logarithm'. Relate this to the problem you found in part (b).

(d) Indeed, for any $n \in \mathbb{Z}$, define $L_n : \mathbb{U} \longrightarrow \mathbb{C}$ by $L_n(u) = L_0(u) + 2n\pi\mathbf{i}$. Show that L_n is also a 'logarithm' in the sense of part (c). Make a sketch of the surface described by the functions $\mathrm{Im}[L_n] : \mathbb{C} \longrightarrow \mathbb{R}$, for all $n \in \mathbb{Z}$ at once.

(e) Proposition 18A.2 on page 417 asserted that any harmonic function on a convex domain $\mathbb{U} \subset \mathbb{R}^2$ can be represented as the real part of a holomorphic function on $\mathbb{U}$, treated as a subset of $\mathbb{C}$. The Remark following Proposition 18A.2 said that $\mathbb{U}$ actually doesn't need to be convex, but it *does* need to be simply connected. We will not prove that simple-connectedness is sufficient, but we can now show that it is necessary.

Consider the harmonic function $h(x,y) = \log(x^2+y^2)$ defined on $\mathbb{R}^2 \setminus \{0\}$. Show that, on any simply connected subset $\mathbb{U} \subset \mathbb{C}^*$, there is a holomorphic function $L : \mathbb{U} \longrightarrow \mathbb{C}$ with $h = \mathrm{Re}\,[L]$. However, show that there is no holomorphic function $L : \mathbb{C}^* \longrightarrow \mathbb{C}$ with $h = \mathrm{Re}\,[L]$.

The functions L_n (for $n \in \mathbb{Z}$) are called the **branches of the complex logarithm** . This exercise shows that the 'complex logarithm' is a much more complicated object than the real logarithm —indeed, the complex log is best understood as a holomorphic 'multifunction' which takes countably many distinct values at each point in $\mathbb{C}^*$. The surface in part (d) is an abstract representation of the 'graph' of this multifunction —it is called a **Riemann surface** . ♦

Exercise 18C.17. *Complex root functions* (follows Exercise 18C.16). Ⓔ

(a) Let $\mathbb{U} \subset \mathbb{C}$ be an open, simply connected set which does not contain 0, and let $\log : \mathbb{U} \longrightarrow \mathbb{C}$ be any complex logarithm function, as defined in Exercise 18C.16. Fix $n \in \mathbb{N}$. Show that $\exp(n \cdot \log(u)) = u^n$ for all $u \in \mathbb{N}$.

(b) Fix $n \in \mathbb{N}$ and now define $\sqrt[n]{\bullet} : \mathbb{U} \longrightarrow \mathbb{C}$ by $\sqrt[n]{u} = \exp(\log(u)/n)$ for all $u \in \mathbb{N}$. Show that $\sqrt[n]{\bullet}$ is a complex 'nth root' function. That is, $\left(\sqrt[n]{u}\right)^n = u$ for all $u \in \mathbb{U}$.

Different branches of logarithm define different 'branches' of the nth root function. However, while there are infinitely many distinct branches of logarithm, there are exactly n distinct branches of the nth root function.

(c) Fix $n \in \mathbb{N}$, and consider the equation $z^n = 1$. Show that the set of all solutions to this equation is $\mathcal{Z}_n := \{1, e^{2\pi\mathbf{i}/n}, e^{4\pi\mathbf{i}/n}, e^{6\pi\mathbf{i}/n}, \ldots e^{2(n-1)\pi\mathbf{i}/n}\}$. (These numbers are called the n**th roots of unity**). For example, $\mathcal{Z}_2 = \{\pm 1\}$ and $\mathcal{Z}_4 = \{\pm 1, \pm\mathbf{i}\}$.

(d) Suppose $r_1 : \mathbb{U} \longrightarrow \mathbb{C}$ and $r_2 : \mathbb{U} \longrightarrow \mathbb{C}$ are two branches of the square root function (defined by applying the definition in part (b) to different branches of the logarithm). Show that $r_1(u) = -r_2(u)$ for all $u \in \mathbb{U}$.

Sketch the Riemann surface for the complex square root function.

(e) More generally, let $n \geq 2$, and suppose $r_1 : \mathbb{U} \longrightarrow \mathbb{C}$ and $r_2 : \mathbb{U} \longrightarrow \mathbb{C}$ are two branches of the nth root function (defined by applying the definition in part (b) to different branches of the logarithm). Show that there is some $\zeta \in \mathcal{Z}_n$ (the set of nth roots of unity from part (c)) such that $r_1(u) = \zeta \cdot r_2(u)$ for all $u \in \mathbb{U}$.

Bonus: Sketch the Riemann surface for the complex nth root function. (Note that it is not possible to embed this surface in three dimensions without some self-intersection). ♦

18D Analyticity of holomorphic maps

Prerequisites: §18C, §0H(ii).

In §18A, we said that the holomorphic functions formed a very special subclass within the set of all (real)-differentiable functions on the plane. One indication of this was Proposition 18A.2 on page 417. Another indication is the following surprising and important result.

Theorem 18D.1. (Holomorphic ⇒ Analytic)
Let $\mathbb{U} \subset \mathbb{C}$ be an open subset. If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is holomorphic on $\mathbb{U}$, then f is infinitely (complex-)differentiable everywhere in $\mathbb{U}$. Thus, the functions $f', f'', f''', \ldots$ are also holomorphic on $\mathbb{U}$. Finally, for all $u \in \mathbb{U}$, the (complex) Taylor series of f at u converges uniformly to f in open disk around u.

Proof. Since any analytic function is $\mathcal{C}^\infty$, it suffices to prove the last sentence, and the rest of the theorem follows. Suppose $0 \in \mathbb{U}$; we will prove that f is analytic at $u = 0$ (the general case $u \neq 0$ is similar).

Let γ be a counterclockwise circular contour in $\mathbb{U}$ centered at 0 (e.g. define $\gamma : [0, 2\pi] \longrightarrow \mathbb{U}$ by $\gamma(s) = re^{\mathbf{i}s}$ for some $r > 0$). Let $\mathbb{W} \subset \mathbb{U}$ be the purview of γ (an open disk centered at 0). For all $n \in \mathbb{N}$, let

$$c_n \quad := \quad \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z^{n+1}}\, dz.$$

We will show that the power series $\sum_{n=0}^{\infty} c_n w^n$ converges to f for all $w \in \mathbb{W}$. For any $w \in \mathbb{W}$, we have

$$\begin{aligned} f(w) \;&\underset{(*)}{=}\; \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z-w}\, dz \;\underset{(\dagger)}{=}\; \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n\, dz \\ &= \frac{1}{2\pi\mathbf{i}} \oint_\gamma \sum_{n=0}^{\infty} \frac{f(z)}{z^{n+1}} \cdot w^n\, dz \;\underset{(\diamond)}{=}\; \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z^{n+1}}\, dz\right) w^n \;\underset{(\ddagger)}{=}\; \sum_{n=0}^{\infty} c_n w^n, \end{aligned}$$

as desired. Here, $(*)$ is Cauchy's Integral Formula (Theorem 18C.9 on page 443), and $(\ddagger)$ is by the definition of c_n. Step $(\dagger)$ is because

$$\frac{1}{z-w} \quad = \quad \left(\frac{1}{z}\right) \cdot \left(\frac{1}{1-\frac{w}{z}}\right) \quad = \quad \frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n. \tag{18D.1}$$

Here, the last step is the geometric series expansion $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$ (with $x := w/z$), which is valid because $|w/z| < 1$ because $|w| < |z|$ because w is inside the disk $\mathbb{W}$ and z is a point on the boundary of $\mathbb{W}$.

It remains to justify step $(\diamond)$. For any $N \in \mathbb{N}$, observe that

$$\frac{1}{2\pi\mathbf{i}} \oint_\gamma \sum_{n=0}^{\infty} \frac{f(z)}{z^{n+1}} \cdot w^n\, dz \quad = \quad \sum_{n=0}^{N} \left(\frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z^{n+1}}\, dz\right) w^n + \frac{1}{2\pi\mathbf{i}} \oint_\gamma \sum_{n=N+1}^{\infty} \frac{f(z)}{z^{n+1}} \cdot w^n\, dz \tag{18D.2}$$

Thus, to justify ($\diamond$), it suffices to show that the second term on the right hand side of (18D.2) tends to zero as $N \to \infty$. Let L be the length of γ (i.e. $L = 2\pi r$ if γ describes a circle of radius r). The function $z \mapsto f(z)/z$ is continuous on the boundary of $\mathbb{W}$, so it is bounded. Let $M := \sup_{z \in \partial\mathbb{W}} \left|\frac{f(z)}{z}\right|$. Fix $\epsilon > 0$ and find some $N \in \mathbb{N}$ such that $\left|\sum_{n=N}^{\infty} \left(\frac{w}{z}\right)^n\right| < \frac{\epsilon}{LM}$. (Such an N exists because the geometric series (18D.1) converges because $|w/z| < 1$.) It follows that:

$$\text{For all } z \in \partial\mathbb{W}, \quad \left|\frac{f(z)}{z} \cdot \sum_{n=N}^{\infty} \left(\frac{w}{z}\right)^n\right| \quad < \quad M \cdot \frac{\epsilon}{LM} \quad = \quad \frac{\epsilon}{L}. \qquad (18D.3)$$

Thus

$$\left|\oint_\gamma \sum_{n=N+1}^{\infty} \frac{f(z)}{z^{n+1}} \cdot w^n \, dz\right| \quad = \quad \left|\oint_\gamma \frac{f(z)}{z} \cdot \sum_{n=N+1}^{\infty} \left(\frac{w}{z}\right)^n \, dz\right| \quad \underset{(*)}{\leq} \quad \frac{\epsilon}{L} \cdot L \quad = \quad \epsilon.$$

Here, ($*$) is by equation (18D.3) above and Lemma 18C.10 on page 444. This works for any $\epsilon > 0$, so we conclude that the second term on the right side of (18D.2) tends to zero as $N \to \infty$. This justifies step ($\diamond$), which completes the proof. □

Corollary 18D.2. (Case $D = 2$ of Proposition 1E.5 on page 18)
Let $\mathbb{U} \subseteq \mathbb{R}^2$ be open. If $h : \mathbb{U} \longrightarrow \mathbb{R}$ is a harmonic function, then h is analytic on $\mathbb{U}$.

Proof. **Exercise 18D.1** *Hint.* Combine Theorem 18D.1 with Proposition 18A.3 on page 417. Note that this is not quite as trivial as it sounds: you must show how to translate the (complex) Taylor series of a holomorphic function on $\mathbb{C}$ into the (real) Taylor series of a real-valued function on $\mathbb{R}^2$. □ Ⓔ

Because of Theorem 18D.1 and Proposition 18A.5(h), holomorphic functions are also called **complex-analytic** functions (or even simply **analytic** functions) in some books. Analytic functions are extremely 'rigid': for any $u \in \mathbb{U}$, the behaviour of f in a tiny neighbourhood around u determines the structure of f everywhere on $\mathbb{U}$, as we now explain. Recall that a subset $\mathbb{U} \subset \mathbb{C}$ is **connected** if it is not possible to write $\mathbb{U}$ as a union of two nonempty disjoint open subsets. A subset $\mathbb{X} \subset \mathbb{C}$ is **perfect** if, for every $x \in \mathbb{X}$, every open neighbourhood around x contains other points in $\mathbb{X}$ besides x. (Equivalently: every point in $\mathbb{X}$ is a *cluster point* of $\mathbb{X}$). In particular, any open subset of $\mathbb{C}$ is perfect. Also, $\mathbb{R}$ and $\mathbb{Q}$ are a perfect subsets of $\mathbb{C}$. Any disk, annulus, line segment, or unbroken curve in $\mathbb{C}$ is both connected and perfect.

Theorem 18D.3. (Identity Theorem)
Let $\mathbb{U} \subset \mathbb{C}$ be a connected open set, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ and $g : \mathbb{U} \longrightarrow \mathbb{C}$ be two holomorphic functions.

(a) *Suppose there is some $a \in \mathbb{U}$ such that $f(a) = g(a)$, $f'(a) = g'(a)$, $f''(a) = g''(a)$, and in general, $f^{(n)}(a) = g^{(n)}(a)$ for all $n \in \mathbb{N}$. Then $f(u) = g(u)$ for all $u \in \mathbb{U}$.*

(b) *Suppose there is a perfect subset $\mathbb{X} \subset \mathbb{U}$ such that $f(x) = g(x)$ for all $x \in \mathbb{X}$. Then $f(u) = g(u)$ for all $u \in \mathbb{U}$.*

Proof. **(a)** Let $h := f - g$. It suffices to show that $h \equiv 0$. Let

$$\mathbb{W} \quad := \quad \left\{u \in \mathbb{U}\ ;\ h(u) = 0, \text{ and } h^{(n)}(u) = 0 \text{ for all } n \in \mathbb{N}\right\}.$$

The set $\mathbb{W}$ is nonempty, because $a \in \mathbb{W}$ by hypothesis. We will show that $\mathbb{W} = \mathbb{U}$; it follows that $h \equiv 0$.

Claim 1: $\mathbb{W}$ *is an open subset of* $\mathbb{U}$.

Proof. Let $w \in \mathbb{W}$; we must show that there is a nonempty open disk around w that is also in $\mathbb{W}$. Now, h is analytic at w because f and g are analytic at w. Thus, there is some nonempty open disk $\mathbb{D}$ centered at w such that the Taylor expansion of h converges to $h(z)$ for all $z \in \mathbb{D}$. The Taylor expansion of h at w is $c_0+c_1(z-u)+c_2(z-u)^2+c_3(z-u)^3+\cdots$, where $c_n := h^{(n)}(w)/n!$, for all $n \in \mathbb{N}$. But for all $n \in \mathbb{N}$, $c_n = 0$ because $h^{(n)}(w) = 0$ because $w \in \mathbb{W}$. Thus, the Taylor expansion is $0+0(z-w)+0(z-w)^2+\cdots$; hence it converges to zero. Thus, h is equal to the constant zero function on $\mathbb{D}$. Thus, $\mathbb{D} \subset \mathbb{W}$. This holds for any $w \in \mathbb{W}$; hence $\mathbb{W}$ is an open subset of $\mathbb{C}$. $\diamond_{\text{Claim 1}}$

Claim 2: $\mathbb{W}$ *is a closed subset of* $\mathbb{U}$.

Proof. For all $n \in \mathbb{N}$, the function $h^{(n)} : \mathbb{U} \longrightarrow \mathbb{C}$ is continuous (because $f^{(n)}$ and $g^{(n)}$ are continuous, since they are differentiable). Thus, the set $\mathbb{W}_n := \left\{u \in \mathbb{U}\ ;\ h^{(n)}(u) = 0\right\}$ is a closed subset of $\mathbb{U}$ (because $\{0\}$ is a closed subset of $\mathbb{C}$). But clearly $\mathbb{W} = \mathbb{W}_0 \cap \mathbb{W}_1 \cap \mathbb{W}_2 \cap \cdots$. Thus, $\mathbb{W}$ is also closed (because it is an intersection of closed sets). $\diamond_{\text{Claim 2}}$

Thus, $\mathbb{W}$ nonempty, and is both open and closed in $\mathbb{U}$. Thus, the set $\mathbb{V} := \mathbb{U} \setminus \mathbb{W}$ is also open and closed in $\mathbb{U}$, and $\mathbb{U} = \mathbb{V} \sqcup \mathbb{W}$. If $\mathbb{V} \neq \emptyset$, then we have expressed $\mathbb{U}$ as a union of two nonempty disjoint open sets, which contradicts the hypothesis that $\mathbb{U}$ is connected. Thus, $\mathbb{V} = \emptyset$, which means $\mathbb{W} = \mathbb{U}$. Thus, $h \equiv 0$. Thus, $f \equiv g$.

(b) Fix $x \in \mathbb{X}$, and let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{X}$ be a sequence converging to x (which exists because $\mathbb{X}$ is perfect).

Claim 3: $f'(x) = g'(x)$.

Ⓔ *Proof.* **Exercise 18D.2** Hint: Compute the limit (18A.1) on page 415 along the sequence $\{x_n\}_{n=1}^{\infty}$. $\diamond_{\text{Claim 3}}$

This argument works for any $x \in \mathbb{X}$; thus $f'(x) = g'(x)$ for all $x \in \mathbb{X}$. Repeating the same argument, we get $f''(x) = g''(x)$ for all $x \in \mathbb{X}$. By induction, $f^{(n)}(x) = g^{(n)}(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{X}$. But then part (a) implies that $f \equiv g$. □

Remark. The Identity Theorem is true (with pretty much the same proof) for any $\mathbb{R}^M$-valued, analytic function on any connected open subset $\mathbb{U} \subset \mathbb{R}^N$, for any $N, M \in \mathbb{N}$. (**Exercise 18D.3** Verify this.) In particular, the Identity Theorem holds for any harmonic functions defined on a connected open subset of $\mathbb{R}^N$ (for any $N \in \mathbb{N}$). This result nicely complements Corollary 5D.4 on page 87, which established the uniqueness of the harmonic function which satisfies specified boundary conditions. (Note that neither Corollary 5D.4 nor the Identity Theorem for harmonic functions is a special case of the other; they apply to distinct situations.) Ⓔ ♢

In Proposition 18A.5(i) and Example 18A.6 on pages 419 and 420, we showed how the 'standard' real-analytic functions on $\mathbb{R}$ can be extended to holomorphic functions on $\mathbb{C}$ in a natural way. We now show that these are the *only* holomorphic extensions of these functions. In other words, there is a one-to-one relationship between real-analytic functions and their holomorphic extensions.

Corollary 18D.4. (Analytic extension)
Let $\mathbb{X} \subseteq \mathbb{R}$ be an open subset, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be an analytic function. There exists some open subset $\mathbb{U} \subseteq \mathbb{C}$ containing $\mathbb{X}$, and a unique holomorphic function $F : \mathbb{U} \longrightarrow \mathbb{C}$ such that $F(x) = f(x)$ for all $x \in \mathbb{X}$.

Proof. For any $x \in \mathbb{X}$, Proposition 18A.5(i) says that the (real) Taylor series of f around x can be extended to define a holomorphic function $F_x : \mathbb{D}_x \longrightarrow \mathbb{C}$, where $\mathbb{D}_x \subset \mathbb{C}$ is an open disk centered at x. Let $\mathbb{U} := \bigcup_{x\in\mathbb{X}} \mathbb{D}_x$; then $\mathbb{U}$ is an open subset of $\mathbb{C}$ containing $\mathbb{X}$. We would like to define $F : \mathbb{U} \longrightarrow \mathbb{C}$ as follows:

$$F(u) \quad := \quad F_x(u), \qquad \text{for any } x \in \mathbb{X} \text{ and } u \in \mathbb{D}_x. \tag{18D.4}$$

But there's a problem: what if $u \in \mathbb{D}_x$ and also $u \in \mathbb{D}_y$ for some $x, y \in \mathbb{X}$. We must make sure that $F_x(u) = F_y(u)$ —otherwise F will not be well-defined by equation (18D.4).

So, let $x, y \in \mathbb{X}$, and suppose the disks $\mathbb{D}_x$ and $\mathbb{D}_y$ overlap. Then $\mathbb{P} := \mathbb{X} \cap \mathbb{D}_x \cap \mathbb{D}_y$ is a nonempty open subset of $\mathbb{R}$ (hence perfect). The functions F_x

and F_y both agree with f on $\mathbb{P}$; thus, they agree with each other on $\mathbb{P}$. Thus, the Identity Theorem says that F_x and F_y agree everywhere on $\mathbb{D}_x \cap \mathbb{D}_y$.

Thus, F is well-defined by equation (18D.4). By construction, F is a holomorphic function on $\mathbb{U}$ which extends f. Furthermore, F is the *only* holomorphic extension of f, by the Identity Theorem. □

Ⓔ **Exercise 18D.4.** Let $I : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be any real-analytic function, and suppose $I(\sin(r), \cos(r)) = 0$ for all $r \in \mathbb{R}$. Use the Identity Theorem to show that $I(\sin(c), \cos(c)) = 0$ for all $c \in \mathbb{C}$.

Conclude that any algebraic relation between sin and cos (i.e. any 'trigonometric identity') which is true on $\mathbb{R}$ will also be true over all of $\mathbb{C}$. ♦

18E Fourier series as Laurent series

Prerequisites: §18D, §8D. **Recommended:** §10D(ii).

For any $r \geq 0$, let $\mathbb{D}(r) := \{z \in \mathbb{C} \,;\, |z| < r\}$ be the open disk of radius r around 0, and let $\mathbb{D}^\complement(r) := \{z \in \mathbb{C} \,;\, |z| > r\}$ be the open *codisk* of 'coradius' r. Let $\mathbb{S}(r) := \{z \in \mathbb{C} \,;\, |z| = r\}$ be the circle of radius r; then $\partial\mathbb{D}(r) = \mathbb{S}(r) = \partial\mathbb{D}^\complement(r)$. Finally, for any $R > r \geq 0$, let $\mathbb{A}(r, R) := \{z \in \mathbb{C} \,;\, r < |z| < R\}$ be the open *annulus* of inner radius r and outer radius R.

Let $c_0, c_1, c_2, \ldots$ be complex numbers, and consider the complex-valued power series:

$$\sum_{n=0}^{\infty} c_n z^n \quad = \quad c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots \tag{18E.1}$$

For any coefficients $\{c_n\}_{n=0}^{\infty}$, there is some **radius of convergence** $R \geq 0$ (possibly zero) such that the power series (18E.1) converges uniformly on the open disk $\mathbb{D}(R)$ and diverges for all $z \in \mathbb{D}^\complement(R)$. (The series (18E.1) may or may not converge on the boundary circle $\mathbb{S}(R)$). The series (18E.1) then defines a holomorphic function on $\mathbb{D}(R)$. (**Exercise 18E.1** Prove the preceding three sentences.) Conversely, if $\mathbb{U} \subseteq \mathbb{C}$ is any open set containing 0, and $f : \mathbb{U} \longrightarrow \mathbb{C}$ is holomorphic, then Theorem 18D.1 on page 450 says that f has a power series expansion like (18E.1) which converges to f in a disk of nonzero radius around 0.

Next, let $c_0, c_{-1}, c_{-2}, c_{-3} \ldots$ be complex numbers, and consider the complex-valued *inverse power series*

$$\sum_{n=-\infty}^{0} c_n z^n \quad = \quad c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} + \frac{c_{-4}}{z^4} + \cdots \tag{18E.2}$$

For any coefficients $\{c_n\}_{n=-\infty}^{0}$, there is some **coradius of convergence** $r \geq 0$ (possibly infinity) such that the inverse power series (18E.2) converges uniformly on the open codisk ${}^{\circ}\mathbb{D}^{\complement}(r)$ and diverges for all $z \in {}^{\circ}\mathbb{D}(r)$. (The series (18E.2) may or may not converge on the boundary circle $\mathbb{S}(r)$). The series (18E.2) then defines a holomorphic function on ${}^{\circ}\mathbb{D}^{\complement}(r)$. Conversely, if $\mathbb{U} \subseteq \mathbb{C}$ is any open set, then we say that $\mathbb{U}$ is a **neighbourhood of infinity** if $\mathbb{U}$ contains ${}^{\circ}\mathbb{D}^{\complement}(r)$ for some $r < \infty$. If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is holomorphic, and $\lim_{z \to \infty} f(z)$ exists and is finite, then f has a inverse power series expansion like (18E.2) which converges to f in a codisk of finite coradius (i.e. a nonempty 'open disk around infinity').[8]

Exercise 18E.2. Prove all statements in the paragraph above. *Hint:* Consider the change of variables $w := 1/z$. Now use the results about power series from the paragraph between equations (18E.1) and (18E.2). ♦ Ⓔ

Finally, let $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ be complex numbers, and consider the complex-valued **Laurent series:**

$$\sum_{n=-\infty}^{\infty} c_n z^n \quad = \quad \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (18E.3)$$

For any coefficients $\{c_n\}_{n=-\infty}^{\infty}$, there exist $0 \leq r \leq R \leq \infty$ such that the Laurent series (18E.3) converges uniformly on the open annulus[9] ${}^{\circ}\mathbb{A}(r,R)$ and diverges for all $z \in {}^{\circ}\mathbb{D}(r)$ and all $z \in {}^{\circ}\mathbb{D}^{\complement}(R)$. (The series (18E.3) may or may not converge on the boundary circles $\mathbb{S}(r)$ and $\mathbb{S}(R)$.) The series (18E.3) then defines a holomorphic function on ${}^{\circ}\mathbb{A}(r,R)$.

Exercise 18E.3. Prove all statements in the paragraph above. *Hint:* Combine the results about power series and inverse power series from the from the paragraphs between equations (18E.1) and (18E.3). ♦ Ⓔ

Proposition 18E.1. *Let $0 \leq r < R \leq \infty$, and suppose the Laurent series* (18E.3) *converges on ${}^{\circ}\mathbb{A}(r,R)$ to define the function $f : {}^{\circ}\mathbb{A}(r,R) \longrightarrow \mathbb{C}$. Let γ be a counterclockwise contour in ${}^{\circ}\mathbb{A}(r,R)$ which encircles the origin* (for example, γ could be a counterclockwise circle of radius r_0, where $r < r_0 < R$). *Then for all $n \in \mathbb{Z}$,*

$$c_n \quad = \quad \frac{1}{2\pi\mathbf{i}} \oint_{\gamma} \frac{f(z)}{z^{n+1}}\, dz.$$

[8] This is not merely fanciful terminology; see Remark 18G.4 on page 469.

[9] Note that ${}^{\circ}\mathbb{A}(r,R) = \emptyset$ if $r = R$.

Proof. For all $z \in {}^\circ\!\mathbb{A}(r,R)$, we have $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$. Thus, for any $n \in \mathbb{Z}$,

$$\frac{f(z)}{z^{n+1}} = \frac{1}{z^{n+1}} \sum_{k=-\infty}^{\infty} c_k z^k = \sum_{k=-\infty}^{\infty} c_k z^{k-n-1} \underset{(*)}{=} \sum_{m=-\infty}^{\infty} c_{m+n+1} z^m,$$

where $(*)$ is the change of variables $m := k-n-1$, so that $k = m+n+1$. In other words,

$$\frac{f(z)}{z^{n+1}} = \cdots + \frac{c_{n-1}}{z^2} + \frac{c_n}{z} + c_{n+1} + c_{n+2}z + c_{n+3}z^2 + \cdots \tag{18E.4}$$

Thus,

$$\begin{aligned}
\oint_\gamma \frac{f(z)}{z^{n+1}} &\underset{(*)}{=} \cdots + \oint_\gamma \frac{c_{n-1}}{z^2} + \oint_\gamma \frac{c_n}{z} + \oint_\gamma c_{n+1} + \oint_\gamma c_{n+2}z + \oint_\gamma c_{n+3}z^2 + \cdots \\
&\underset{(\dagger)}{=} \cdots + 0 + 2\pi\mathbf{i}\, c_n + 0 + 0 + 0 + \cdots \\
&= 2\pi\mathbf{i}\, c_n, \quad \text{as desired.}
\end{aligned}$$

Here, $(*)$ is because the series (18E.4) converges uniformly on ${}^\circ\!\mathbb{A}(r,R)$ (because the Laurent series (18E.3) converges uniformly on ${}^\circ\!\mathbb{A}(r,R)$); thus, Proposition 6E.10(b) on page 127 implies we can compute the contour integral of series (18E.4) term-by-term. Next, $(\dagger)$ is by Examples 18C.2(c) and 18C.6 (pages 435 and 439). □

Laurent series are closely related to Fourier series.

Proposition 18E.2. *Suppose $0 \le r < 1 < R$ and suppose the Laurent series (18E.3) converges to the function $f : {}^\circ\!\mathbb{A}(r,R) \longrightarrow \mathbb{C}$. Define $g : [-\pi,\pi] \longrightarrow \mathbb{C}$ by $g(x) := f(e^{\mathbf{i}x})$ for all $x \in [-\pi,\pi]$. For all $n \in \mathbb{Z}$, let*

$$\widehat{g}_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\ \exp(-n\mathbf{i}x)\ dx \tag{18E.5}$$

be the nth complex Fourier coefficient of g (see § 8D on page 172). *Then:*

(a) *$\widehat{g}_n = c_n$ for all $n \in \mathbb{Z}$.*

(b) *For any $x \in [-\pi,\pi]$, if $z = e^{\mathbf{i}x} \in \mathbb{S}(1)$, then for all $N \in \mathbb{N}$, the Nth partial Fourier sum of $g(x)$ equals the Nth partial Laurent sum of $f(z)$; that is:*

$$\sum_{n=-N}^{N} \widehat{g}_n \exp(n\mathbf{i}x) = \sum_{n=-N}^{N} c_n z^n.$$

Thus, the Fourier Series for g converges on $[-\pi,\pi]$ in exactly the same ways (i.e. uniformly, in L^2, etc.), *and at exactly the same speed, as the Laurent series for f converges on $\mathbb{S}(1)$.*

Proof. **(a)** As in Example 18C.1 on page 435, define the 'unit circle' contour $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ by $\gamma(s) := \exp(\mathbf{i}s)$ for all $s \in [0, 2\pi]$. Then

$$\begin{aligned} c_n \underset{(*)}{=} \frac{1}{2\pi\mathbf{i}} \oint_\gamma \frac{f(z)}{z^{n+1}} &= \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{f[\gamma(s)])}{\gamma(s)^{n+1}} \cdot \dot{\gamma}(s)\ ds \\ = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{f(e^{\mathbf{i}s})}{e^{\mathbf{i}s(n+1)}} \cdot \mathbf{i}e^{\mathbf{i}s}\ ds &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}s}) \cdot e^{-n\mathbf{i}s}\ ds \\ \underset{(\dagger)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{\mathbf{i}x}) \cdot e^{-n\mathbf{i}x}\ dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \cdot e^{-n\mathbf{i}x}\ dx \underset{(\ddagger)}{=} \widehat{g}_n. \end{aligned}$$

Here, $(*)$ is by Proposition 18E.1, and $(\dagger)$ is because the function $s \mapsto e^{\mathbf{i}s}$ is 2π-periodic. Finally, $(\ddagger)$ is by equation (18E.5).

(b) follows immediately from (a), because if $z = e^{\mathbf{i}x}$, then for all $n \in \mathbb{Z}$ we have $z^n = \exp(n\mathbf{i}x)$. $\square$

We can also reverse this logic: given the Fourier series for a function $g : [-\pi, \pi] \longrightarrow \mathbb{C}$, we can interpret it as the Laurent series of some hypothetical function f defined on an open annulus in $\mathbb{C}$ (which may or may not contain $\mathbb{S}(1)$); then by studying f and its Laurent series, we can draw conclusions about g and its Fourier series.

Let $g : [-\pi, \pi] \longrightarrow \mathbb{C}$ be some function, let $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ be its Fourier coefficients, as defined by equation (18E.5), and consider the complex Fourier series[10] $\sum_{n=-\infty}^{\infty} \widehat{g}_n \mathbf{E}_n$. If $g \in \mathbf{L}^2[-\pi, \pi]$, then the Riemann-Lebesgue Lemma (Corollary 10A.3 on page 197) says that $\lim_{n\to\pm\infty} \widehat{g}_n = 0$; however, the sequence $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ might converge to zero very slowly, if g is nondifferentiable and/or discontinuous. We would like the sequence $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ to converges to zero as quickly as possible, for two reasons:

1. The faster the sequence $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ converges to zero, the easier it will be to approximate the function g using a 'partial sum' of the form $\sum_{n=-N}^{N} \widehat{g}_n \mathbf{E}_n$, for some $N \in \mathbb{N}$. (This is important for numerical computations.)

2. The faster the sequence $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ converges to zero, the more computations we can perform with g by 'formally manipulating' its Fourier series. For example, if $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ converges to zero faster than $\frac{1}{n^k}$, then we can compute the derivatives $g', g'', g''', \ldots, g^{(k-1)}$ by 'formally differentiating' the Fourier series for g (see § 8B(iv) on page 168). This is necessary to verify the 'Fourier series' solutions to I/BVPs which we constructed in Chapters 11- 14.

[10]For all $n \in \mathbb{Z}$, recall that $\mathbf{E}_n(x) := \exp(n\mathbf{i}x)$ for all $x \in [-\pi, \pi]$.

We say the sequence $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$ has **exponential decay** if there is some $a > 1$ such that

$$\lim_{n\to\infty} a^n |\widehat{g}_n| \;=\; 0 \qquad \text{and} \qquad \lim_{n\to\infty} a^n |\widehat{g}_{-n}| \;=\; 0.$$

This is an extremely rapid decay rate, which causes the partial sum $\sum_{n=-N}^{N} \widehat{g}_n \mathbf{E}_n$ to uniformly converge to g very quickly as $N \to \infty$. This means we can 'formally differentiate' the Fourier series of g as many times as we want. In particular, any 'formal solution to an I/BVP which we obtain through such formal differentiation will converge to the correct solution.

Suppose $g : [-\pi, \pi] \longrightarrow \mathbb{C}$ has real and imaginary parts $g_r, g_i : [-\pi, \pi] \longrightarrow \mathbb{R}$ (so that $g(x) = g_r(x) + g_i(x)\mathbf{i}$ for all $x \in [-\pi, \pi]$). We say that g is **analytic and periodic** if the functions g_r and g_i are (real)-analytic everywhere on $[-\pi, \pi]$, and if we have $g(-\pi) = g(\pi)$, $g'(-\pi) = g'(\pi)$, $g''(-\pi) = g''(\pi)$, etc. (where $g'(x) = g'_r(x) + g'_i(x)\mathbf{i}$, etc.).

Proposition 18E.3. *Let $g : [-\pi, \pi] \longrightarrow \mathbb{C}$ have complex Fourier coefficients $\{\widehat{g}_n\}_{n=-\infty}^{\infty}$. Then*

$$\Big(g \text{ is analytic and periodic} \Big) \iff \Big(\text{The sequence } \{\widehat{g}_n\}_{n=-\infty}^{\infty} \text{ decays exponentially} \Big).$$

Proof. "$\Longrightarrow$" Define the function $f : \mathbb{S}(1) \longrightarrow \mathbb{C}$ by $f(e^{\mathbf{i}x}) := g(x)$ for all $x \in [-\pi, \pi]$.

Claim 1: *f can be extended to a holomorphic function $F : \mathbb{A}(r, R) \longrightarrow \mathbb{C}$, for some $0 \le r < 1 < R \le \infty$.*

Proof. Let $\widetilde{g} : \mathbb{R} \longrightarrow \mathbb{C}$ be the 2π-periodic extension of g (i.e. $\widetilde{g}(x + 2n\pi) := g(x)$ for all $x \in [-\pi, \pi]$ and $n \in \mathbb{Z}$). Then $\widetilde{g}$ is analytic on $\mathbb{R}$, so Corollary 18D.4 on page 453 says that there is an open subset $\mathbb{U} \subset \mathbb{C}$ containing $\mathbb{R}$ and a holomorphic function $G : \mathbb{U} \longrightarrow \mathbb{C}$ which agrees with g on $\mathbb{R}$. Without loss of generality, we can assume that $\mathbb{U}$ is a horizontal strip of width $2W$ around $\mathbb{R}$, for some $W > 0$ —that is, $\mathbb{U} = \{x + y\mathbf{i} \,;\, x \in \mathbb{R},\ y \in (-W, W)\}$.

Claim 1.1: *G is horizontally 2π-periodic (i.e. $G(u + 2\pi) = E(u)$ for all $u \in \mathbb{U}$).*

Ⓔ *Proof.* **Exercise 18E.4** Hint: Use the Identity Theorem 18D.3 on page 452, and the fact that g is 2π-periodic. $\triangle_{\text{Claim 1.1}}$

Define $E : \mathbb{C} \longrightarrow \mathbb{C}$ by $E(z) := \exp(\mathbf{i}z)$; thus, E maps $\mathbb{R}$ to the unit circle $\mathbb{S}(1)$. Let $r := e^{-W}$ and $R := e^{W}$; then $r < 1 < R$. Then E maps the strip $\mathbb{U}$ to the open annulus $\mathbb{A}(r, R) \subseteq \mathbb{C}$. Note that E is horizontally 2π-periodic (i.e. $E(u + 2\pi) = E(u)$ for all $u \in \mathbb{U}$). Define $F : \mathbb{A} \longrightarrow \mathbb{C}$ by $F(E(u)) := G(u)$ for all $u \in \mathbb{U}$.

Claim 1.2: *F is well-defined on $\mathbb{A}(r, R)$.*

Proof. **Exercise 18E.5** Hint: use the fact that both G and E are 2π-periodic. Ⓔ

$\diamond_{\text{Claim 1}}$

Claim 1.3: *F is holomorphic on $\mathbb{A}(r,R)$.*

Proof. Let $a \in \mathbb{A}(r,R)$; we must show that F is differentiable at a. Suppose $a = G(u)$ for some $u \in \mathbb{U}$. There are open sets $\mathbb{V} \subset \mathbb{U}$ (containing u) and $\mathbb{B} \subset \mathbb{A}(r,R)$ (containing a) such that $E : \mathbb{V} \longrightarrow \mathbb{B}$ is bijective. Let $L : \mathbb{B} \longrightarrow \mathbb{V}$ be a local inverse of E. Then L is holomorphic on $\mathbb{V}$ by Proposition 18A.5(k) on page 419 (because $E'(v) \neq 0$ for all $v \in \mathbb{V}$). But by definition, $F(b) = G(L(b))$ for all $b \in \mathbb{B}$; Thus, F is holomorphic on $\mathbb{B}$ by Proposition 18A.5(j) (the chain rule).

This argument works for any $a \in \mathbb{A}(r,R)$; thus, F is holomorphic on $\mathbb{A}(r,R)$. $\triangle_{\text{Claim 1.3}}$

It remains to show that F is an extension of f. But by definition, $f(E(x)) = g(x)$ for all $x \in [-\pi,\pi]$. Since G is an extension of g, and $F \circ E = G$, it follows that F is an extension of f. $\diamond_{\text{Claim 1}}$

Let $\{c_n\}_{n=-\infty}^{\infty}$ be the Laurent coefficients of F. Then Proposition 18E.2 on page 456 says that $c_n = \widehat{g}_n$ for all $n \in \mathbb{Z}$. However, the Laurent series (18E.3) of F (on page 455) converges on $\mathbb{A}(r,R)$. Thus, if $|z| < R$, then the power series (18E.1) on page 454 converges absolutely at z. This means that, if $1 < a < R$, then $\sum_{n=0}^{\infty} a^n|\widehat{g}_n|$ is finite. Thus, $\lim_{n\to\infty} a^n|\widehat{g}_n| \ = \ 0$.

Likewise, if $r < |z|$, then the inverse power series (18E.2) on page 454 converges absolutely at z. This means that, if $1 < a < 1/r$, then $\sum_{n=0}^{\infty} a^n|\widehat{g}_{-n}|$ is finite. Thus, $\lim_{n\to\infty} a^n|\widehat{g}_{-n}| \ = \ 0$.

"$\Longleftarrow$" Define $c_n := \widehat{g}_n$ for all $n \in \mathbb{Z}$, and consider the resulting Laurent series (18E.3). Suppose there is some $a > 1$ such that

$$\lim_{n\to\infty} a^n|c_n| \ = \ 0 \qquad \text{and} \qquad \lim_{n\to\infty} a^n|c_{-n}| \ = \ 0. \tag{18E.6}$$

Claim 2: *Let $r := 1/a$ and $R := a$. For all $z \in \mathbb{A}(r,R)$, the Laurent series (18E.3) converges absolutely at z.*

Proof. Let $z_+ := z/a$; then $|z_+| < 1$ because $|z| < R := a$. Also, let $z_- := 1/az$; then $|z_-| < 1$, because $|z| > r := 1/a$. Thus,

$$\sum_{n=1}^{\infty} |z_-|^n \quad < \quad \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} |z_+|^n \quad < \quad \infty. \tag{18E.7}$$

Evaluating the Laurent series (18E.3) at z, we see that

$$\sum_{n=-\infty}^{\infty} c_n z^n \;=\; \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n} + \sum_{n=0}^{\infty} c_n\, z^n \;=\; \sum_{n=1}^{\infty} \frac{a^n c_{-n}}{(az)^n} + \sum_{n=0}^{\infty} a^n c_n\,(z/a)^n$$
$$= \sum_{n=1}^{\infty} (a^n c_{-n})\, z_-^n + \sum_{n=0}^{\infty} a^n c_n\, z_+^n.$$

Thus, $\displaystyle\sum_{n=-\infty}^{\infty} |c_n z^n| \;\le\; \sum_{n=1}^{\infty} a^n\,|c_{-n}|\,|z_-|^n + \sum_{n=0}^{\infty} a^n |c_n|\,|z_+|^n \underset{(*)}{<} \infty,$

where $(*)$ is by equations (18E.6) and (18E.7). $\diamond_{\text{Claim 2}}$

Claim 2 implies that the Laurent series (18E.3) converges to some holomorphic function $f : \mathbb{A}(r, R) \longrightarrow \mathbb{C}$. But $g(x) = f(e^{\mathbf{i}x})$ for all $x \in [-\pi, \pi]$; thus, g is (real)-analytic on $[-\pi, \pi]$, because f is (complex-)analytic on $\mathbb{A}(r, R)$, by Theorem 18D.1 on page 450. □

Ⓔ **Exercise 18E.6.** Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$, and consider the real Fourier series for f (see § 8A on page 161). Show that the real Fourier coefficients $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ have exponential decay if and only if f is analytic and periodic on $[-\pi, \pi]$. (*Hint:* Use Proposition 8D.2 on page 174.) ♦

Ⓔ **Exercise 18E.7.** Let $f : [0, \pi] \longrightarrow \mathbb{R}$, and consider the Fourier sine series and cosine series for f (see § 7A(i) on page 137 and § 7A(ii) on page 141).

(a) Show that the Fourier cosine coefficients $\{A_n\}_{n=0}^{\infty}$ have exponential decay if and only if f is analytic on $[0, \pi]$ and $f'(0) = 0 = f'(\pi)$, and $f^{(n)}(0) = 0 = f^{(n)}(\pi)$ for all odd $n \in \mathbb{N}$.

(b) Show that the Fourier sine coefficients $\{B_n\}_{n=1}^{\infty}$ have exponential decay if and only if f is analytic on $[0, \pi]$ and $f(0) = 0 = f(\pi)$, and $f^{(n)}(0) = 0 = f^{(n)}(\pi)$ for all even $n \in \mathbb{N}$.

(c) Conclude that if both the sine and cosine series have exponential decay, then $f \equiv 0$.

(*Hint.* Use the previous exercise and Proposition 8C.5 on page 171.) ♦

Ⓔ **Exercise 18E.8.** Let $\mathbb{X} = [0, L]$ be an interval, let $f \in \mathbf{L}^2(\mathbb{X})$ be some initial temperature distribution, and let $F : \mathbb{X} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ be the solution to the one-dimensional heat equation ($\partial_t F = \partial_x^2 F$) with initial conditions $F(x; 0) = f(x)$ for all $x \in \mathbb{X}$, and satisfying either homogeneous Dirichlet boundary conditions, or homogeneous Neumann boundary conditions, or periodic boundary conditions on $\mathbb{X}$, for all $t > 0$. Show that, for any fixed $t > 0$, the function $F_t(x) := F(x, t)$ is analytic on $\mathbb{X}$. (*Hint:* Apply Propositions 11A.1 and 11A.3 on pages 225 and 227.)

This shows how the action of the heat equation can rapidly ‘smooth’ even a highly irregular initial condition. ♦

Exercise 18E.9. Compute the complex Fourier series of $f : [-\pi, \pi] \longrightarrow \mathbb{C}$ when f is defined as follows: Ⓔ

(a) $f(x) = \sin(e^{\mathbf{i}x})$.
(b) $f(x) = \cos(e^{-3\mathbf{i}x})$.
(c) $f(x) = e^{2\mathbf{i}x} \cdot \cos(e^{-3\mathbf{i}x})$.
(d) $f(x) = (5 + e^{2\mathbf{i}x}) \cdot \cos(e^{-3\mathbf{i}x})$.
(e) $f(x) = \dfrac{1}{e^{2\mathbf{i}x} - 4}$.
(f) $f(x) = \dfrac{e^{\mathbf{i}x}}{e^{2\mathbf{i}x} - 4}$. ♦

Exercise 18E.10. (a) Show that the Laurent series (18E.3) can be written in the form $P_+(z) + P_-(1/z)$, where P_+ and P_- are both power series. Ⓔ

(b) Suppose P_+ has radius of convergence R_+, and P_- has radius of convergence R_-. Let $R := R_+$ and $r := 1/R_-$, and show that the Laurent series (18E.3) converges on $\mathbb{A}(r, R)$. ♦

18F* Abel means and Poisson kernels

Prerequisites: §18E. **Prerequisites** (for proofs): §10D(ii).

Theorem 18E.3 showed that, if $g : [-\pi, \pi] \longrightarrow \mathbb{C}$ is analytic, then its Fourier series $\sum_{n=-\infty}^{\infty} \widehat{g}_n \mathbf{E}_n$ will converge uniformly and extremely quickly to g. At the opposite extreme, if g is not even differentiable, then $\sum_{n=-\infty}^{\infty} \widehat{g}_n \mathbf{E}_n$ might not converge uniformly, or even pointwise, to g. To address this problem, we introduce the *Abel mean.* For any $r < 1$, the rth **Abel mean** of the Fourier series $\displaystyle\sum_{n=-\infty}^{\infty} \widehat{g}_n \mathbf{E}_n$ is defined:

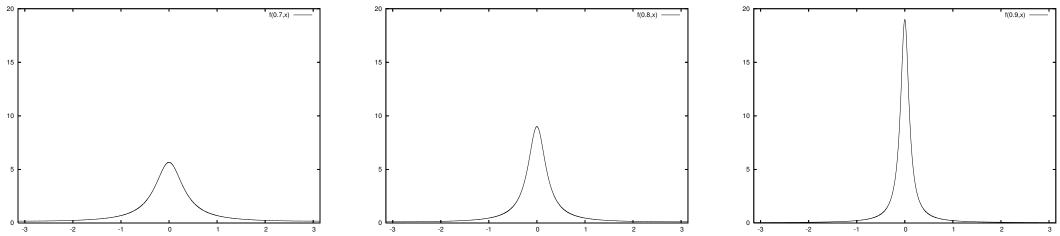
$$\mathbf{A}_r[g] \quad := \quad \sum_{n=-\infty}^{\infty} r^{|n|} \widehat{g}_n \mathbf{E}_n.$$

As $r \nearrow 1$, each summand $r^{|n|} \widehat{g}_n \mathbf{E}_n$ in the Abel mean converges to the corresponding summand $\widehat{g}_n \mathbf{E}_n$ in the Fourier series for g. Thus, we expect that $\mathbf{A}_r[g]$ should converge to g as $r \nearrow 1$. The goal of this section is to verify this intuition.

For any $r \in [0, 1)$, we define the **Poisson kernel** $\mathbf{P}_r : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ by

$$\mathbf{P}_r(x) \quad := \quad \frac{1 - r^2}{1 - 2r\cos(x) + r^2}, \qquad \text{for all } x \in [-2\pi, 2\pi].$$

(See Figure 18F.1). Note that $\mathbf{P}_r$ is 2π-periodic (i.e. $\mathbf{P}_r(x + 2\pi) = \mathbf{P}_r(x)$ for all $x \in [-2\pi, 0]$). For any function $g : [-\pi, \pi] \longrightarrow \mathbb{C}$, the **convolution** of $\mathbf{P}_r$ and g



$\mathbf{P}_{0.7}$ $\mathbf{P}_{0.8}$ $\mathbf{P}_{0.9}$

Figure 18F.1: The Poisson kernels $\mathbf{P}_{0.7}$, $\mathbf{P}_{0.8}$, and $\mathbf{P}_{0.9}$, plotted on interval $[-\pi, \pi]$. Note the increasing concentration of $\mathbf{P}_r$ near $x = 0$ as $r \nearrow 1$. (In the terminology of Section 10D(ii), the system $\{\mathbf{P}_r\}_{0<r<1}$ is like an *approximation of the identity*.)

is the function $\mathbf{P}_r * g : [-\pi, \pi] \longrightarrow \mathbb{C}$ defined by

$$\mathbf{P}_r * g(x) \quad := \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y)\, \mathbf{P}_r(x-y)\; dy, \qquad \text{for all } x \in [-\pi, \pi].$$

The next result tells us that $\lim_{r\nearrow 1} \mathbf{A}_r[g](x) \;=\; g(x)$, whenever the function g is continuous at x. Furthermore, for all $r < 1$, the functions $\mathbf{A}_r[g] : [-\pi, \pi] \longrightarrow \mathbb{C}$ are extremely smooth, and two-variable function $G(x, r) := \mathbf{A}_r[g](x)$ is also extremely smooth.

Proposition 18F.1. *Let $g \in \mathbf{L}^2[-\pi, \pi]$.*

(a) *For any $r \in [0, 1)$ and $x \in [-\pi, \pi]$, $\mathbf{P}_r * g(x) \;=\; \mathbf{A}_r[g](x)$.*

(b) *For any $x \in (-\pi, \pi)$, if g is continuous at x, then $\lim_{r\nearrow 1} \mathbf{A}_r[g](x) \;=\; g(x)$.*

(c) *Let $\mathbb{D}$ be the closed unit disk, and define $f : \mathbb{D} \longrightarrow \mathbb{C}$ by*

$$f(r\, e^{\mathbf{i}\theta}) \quad := \quad \begin{cases} \mathbf{P}_r * g(\theta) & \text{if} \quad r < 1, \\ g(\theta) & \text{if} \quad r = 1, \end{cases} \qquad \text{for all } \theta \in [-\pi, \pi] \text{ and } r \in [0, 1].$$

Then f is holomorphic on $\mathring{\mathbb{D}}$.

(d) *Thus, for any fixed $r < 1$, the function $\mathbf{A}_r[g] : [-\pi, \pi] \longrightarrow \mathbb{C}$ is analytic.*

(e) *Let $\theta \in (-\pi, \pi)$ and let $s = e^{\mathbf{i}\theta} \in \mathbb{S}$. If g is continuous in a neighbourhood of θ, then f is continuous at s —i.e. $\lim_{\substack{u \to s \\ u \in \mathbb{D}}} f(u) \;=\; f(s)$.*

(f) *If g is continuous on $[-\pi, \pi]$ and $g(-\pi) = g(\pi)$, then f is continuous on $\mathbb{D}$.*

Ⓔ *Proof.* **(a)** **Exercise 18F.1** (Hint: Use Lemmas 18F.2 and 18F.3 below).

Ⓔ **(b)** is **Exercise 18F.2** (Hint: Use Lemma 18F.4 below, and Proposition 10D.9(b) on page 219).

(e) and **(f)** follow immediately from **(b)**, while **(d)** follows from **(c)**.

(c) is **Exercise 18F.3** Hint: (i) Let $\mathcal{P} : \mathbb{S} \times \mathbb{D} \longrightarrow \mathbb{R}$ be the Poisson kernel defined on page 445. For any $s \in \mathbb{S}$ and $u \in \mathbb{D}$, if $s = e^{\mathbf{i}y}$ and $u = r \cdot e^{\mathbf{i}x}$, show that $\mathcal{P}(s,u) = \mathbf{P}_r(x-y)$. Ⓔ

(ii) Use this to show that $\mathbf{P}_r * g(x) = \frac{1}{2\pi}\int_0^{2\pi} g(y)\,\mathcal{P}(e^{\mathbf{i}y}, u)\, dy$.

(ii) Now apply the Poisson Integral Formula for holomorphic functions (Corollary 18C.13). □

To prove parts (a) and (b) of Proposition 18F.1, we require the following three lemmas.

Lemma 18F.2. *Fix $r \in [0,1)$. Then*

$$\mathbf{P}_r(x) \quad = \quad 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx) \quad = \quad \sum_{n=-\infty}^{\infty} r^{|n|}\exp(\mathbf{i}nx).$$

Thus, if $\{\widehat{\mathbf{P}}_r^n\}_{n=-\infty}^{\infty}$ are the complex Fourier coefficients of the Poisson kernel $\mathbf{P}_r$, then $\widehat{\mathbf{P}}_r^n = r^{|n|}$ for all $n \in \mathbb{Z}$.

Proof. **Exercise 18F.4** □ Ⓔ

For any $f, g : [-\pi, \pi] \longrightarrow \mathbb{C}$, recall the definition of the convolution $f * g$ from § 10D(ii) on page 214. The passage from a function to its Fourier coefficients converts the convolution operator into multiplication, as follows:[11]

Lemma 18F.3. *Let $f, g \in \mathbf{L}^2[-\pi,\pi]$ and suppose $h = f * g \in \mathbf{L}^2[-\pi,\pi]$ also. Then for all $n \in \mathbb{Z}$, we have $\widehat{h}_n = \widehat{f}_n \cdot \widehat{g}_n$.*

Proof. **Exercise 18F.5** □ Ⓔ

[11] For the corresponding result for Fourier transforms of functions on $\mathbb{R}$, see Theorem 19B.2(b) on page 494.

Lemma 18F.4. *The set of Poisson kernels* $\{\mathbf{P}_r\}_{0\leq r<1}$ *is an* **approximation of identity**, *in the following sense:*

(AI1) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_r(x)\ dx \ = \ 1$ *for all* $r\in[0,1)$.

(AI2) *For any* $\epsilon > 0$, $\displaystyle\lim_{r\nearrow 1}\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\mathbf{P}_r(x)\ dx \ = \ 1.$ *(See Figure 10D.2 on page 218).*

Ⓔ *Proof.* **Exercise 18F.6** □

Ⓔ **Exercise 18F.7.** For all $N\in\mathbb{N}$, the Nth *Dirichlet kernel* is the function $\mathbf{D}_N : [-\pi,\pi]\longrightarrow\mathbb{R}$ defined by

$$\mathbf{D}_N(x) \quad := \quad 1+2\sum_{n=1}^{N}\cos(nx) \qquad \text{(see Figure 10B.1 on page 198).}$$

(a) Show that $\displaystyle\mathbf{D}_N(x) = \sum_{n=-N}^{N}\exp(nx\mathbf{i})$.

(b) Let $g:[-\pi,\pi]\longrightarrow\mathbb{C}$ have complex Fourier series $\displaystyle\sum_{n=-\infty}^{\infty}\widehat{g}_n\mathbf{E}_n$. Use Lemma 18F.3 to show that $\displaystyle\mathbf{D}_N * g \ = \ \sum_{n=-N}^{N}\widehat{g}_n\mathbf{E}_n$. (Compare this with Lemma 10B.1). ♦

18G Poles and the residue theorem

Prerequisites: §18D.

Let $\mathbb{U}\subset\mathbb{C}$ be an open subset, let $p\in\mathbb{U}$, and let $\mathbb{U}^* := \mathbb{U}\setminus\{p\}$. Let $f:\mathbb{U}^*\longrightarrow\mathbb{C}$ be a holomorphic function. We say that p is an **isolated singularity** of f, because f is well-defined and holomorphic for all points *near* p, but not *at* p itself.

Now, it might be possible to 'extend' f to a holomorphic function $f:\mathbb{U}\longrightarrow\mathbb{C}$ by defining $f(p)$ in some suitable way. In this case, we say that p is a **removable singularity** of f; it is merely a point we 'forgot' when defining f on $\mathbb{U}^*$. However, sometimes there is no way to define $f(p)$ such that the resulting function $f:\mathbb{U}\longrightarrow\mathbb{C}$ is complex-differentiable (or even continuous) at p; in this case, we say that p is an **indelible singularity**. In this section, we will be concerned with a particularly 'nice' class of indelible singularities, called *poles*.

Define $F_1:\mathbb{U}^*\longrightarrow\mathbb{C}$ by $F_1(u) = (p-u)\cdot f(u)$. We say that p is a **simple pole** of f if p is a removable singularity of F_1 —i.e. if $F_1(p)$ can be defined such that F_1 *is* complex-differentiable at p. Now, F_1 is already holomorphic on $\mathbb{U}^*$ (because it is a product of two holomorphic functions f and $z \mapsto (z-u)$). Thus, if F_1 is differentiable at p, then F_1 is holomorphic on all of $\mathbb{U}$. Then Theorem 18D.1 on page 450 says that F_1 is *analytic* at p —i.e. F_1 has a Taylor expansion near p:

$$F_1(z) \quad = \quad a_0 \quad + \quad a_1(z-p) \quad + \quad a_2(z-p)^2 \quad + \quad a_3(z-p)^3 \quad + \quad a_4(z-p)^4 \quad + \cdots$$

Thus,

$$\begin{aligned} f(z) \quad &= \quad \frac{F_1(z)}{z-p} \\ &= \quad \frac{a_0}{z-p} \quad + \quad a_1 \quad + \quad a_2(z-p) \quad + \quad a_3(z-p)^2 \quad + \quad a_4(z-p)^3 \quad + \cdots \end{aligned}$$

This expression is called a **Laurent expansion** (of order 1) for f at the pole p. The coefficient a_0 is called the **residue** of f at the pole p, and denoted $\mathrm{res}_p(f)$.

But suppose p is not a simple pole (i.e. it is not a removable singularity for F_1). Let $n \in \mathbb{N}$, and define $F_n : \mathbb{U}^* \longrightarrow \mathbb{C}$ by $F_n(u) = (p-u)^n \cdot f(u)$. We say that p is a **pole** if there is some $n \in \mathbb{N}$ such that p is a removable singularity of F_n —i.e. if $F_n(p)$ can be defined such that F_n *is* complex-differentiable at p. The smallest value of n for which this is true is called the **order** of the pole p.

Now, F_n is already holomorphic on $\mathbb{U}^*$. Thus, if F_n is differentiable at p, then F_n is holomorphic on all of $\mathbb{U}$. Again, Theorem 18D.1 says that F_n is *analytic* at p, with Taylor expansion

$$\begin{aligned} F_n(z) \quad &= \\ a_0 \quad &+ \quad a_1(z-p) \quad + \cdots + \quad a_{n-1}(z-p)^{n-1} \quad + \quad a_n(z-p)^n \quad + \quad a_{n+1}(z-p)^{n+1} \quad + \cdots \end{aligned}$$

Thus,

$$\begin{aligned} f(z) \quad = \quad \frac{F_n(z)}{(z-p)^n} \quad &= \\ \frac{a_0}{(z-p)^n} \quad + \quad \frac{a_1}{(z-p)^{n-1}} \quad + \cdots + \quad \frac{a_{n-1}}{(z-p)} \quad &+ \quad a_n \quad + \quad a_{n+1}(z-p) \quad + \cdots \end{aligned}$$

This expression is called a **Laurent expansion** (of order n) for f at the pole p. The coefficient a_{n-1} is called the **residue** of f at the pole p, and denoted $\mathrm{res}_p(f)$.

Let $\widehat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$, where the symbol "$\infty$" represents a 'point at infinity'. If $f : \mathbb{U}^* \longrightarrow \mathbb{C}$ has a pole at p, then it is easy to check that $\lim_{z \to p} |f(z)| = \infty$ (**Exercise 18G.1**). Thus, it is natural and convenient to extend f to a function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ by defining $f(p) = \infty$. (Later, in Remark 18G.4 on page 469, we will explain why this is not merely a cute notational device, but is actually the 'correct' thing to do). The extended function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ is called a *meromorphic* function. Ⓔ

Formally, if $\mathbb{U} \subset \mathbb{C}$ is an open set, then a function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ is **meromorphic** if there is a discrete subset $\mathbb{P} \subset \mathbb{U}$ (possibly empty) such that, if $\mathbb{U}^* := \mathbb{U} \setminus \mathbb{P}$, then:

1. $f : \mathbb{U}^* \longrightarrow \mathbb{C}$ is holomorphic.
2. Every $p \in \mathbb{P}$ is a pole of f (hence $\lim_{z\to p} |f(z)| = \infty$).
3. $f(p) = \infty$ for all $p \in \mathbb{P}$.

Example 18G.1. (a) Any holomorphic function is meromorphic, since it has no poles.

(b) Let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic, let $p \in \mathbb{U}$, and define $F : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ by $F(z) = f(z)/(z-p)$. Then F is meromorphic on $\mathbb{U}$, with a single pole at p, and $\mathrm{res}_p(F) \ = \ f(p)$.

(c) Fix $y > 0$, and define $\mathcal{K}_y : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ by

$$\mathcal{K}_y(z) \quad = \quad \frac{y}{\pi(z^2+y^2)} \quad = \quad \frac{y}{\pi(z+y\mathbf{i})(z-y\mathbf{i})}, \quad \text{for all } z \in \mathbb{C}.$$

Then $\mathcal{K}_y$ is meromorphic on $\mathbb{C}$, with simple poles at $z = \pm y\mathbf{i}$. Observe that

$$\mathcal{K}_y(z) \quad = \quad \frac{f_+(z)}{(z-y\mathbf{i})}, \quad \text{where} \quad f_+(z) \ := \ \frac{y}{\pi(z+y\mathbf{i})}, \quad \text{for all } z \in \mathbb{C}.$$

Note that f_+ is holomorphic near $y\mathbf{i}$, so Example (b) says that $\mathrm{res}_{y\mathbf{i}}(\mathcal{K}_y) \ = \ f_+(y\mathbf{i}) \ = \ \frac{y}{\pi(2y\mathbf{i})} \ = \ \frac{1}{2\pi\mathbf{i}}$. Likewise,

$$\mathcal{K}_y(z) \quad = \quad \frac{f_-(z)}{(z+y\mathbf{i})}, \quad \text{where} \quad f_-(z) \ := \ \frac{y}{\pi(z-y\mathbf{i})}, \quad \text{for all } z \in \mathbb{C}.$$

Note that f_- is holomorphic near $-y\mathbf{i}$, so Example (b) says that $\mathrm{res}_{-y\mathbf{i}}(\mathcal{K}_y) \ = \ f_-(-y\mathbf{i}) \ = \ \frac{y}{\pi(-2y\mathbf{i})} \ = \ \frac{-1}{2\pi\mathbf{i}}$.

(d) More generally, let $P(z) = (z-p_1)^{n_1}(z-p_2)^{n_2}\cdots(z-p_J)^{n_J}$ be any complex polynomial with roots $p_1, \ldots, p_n \in \mathbb{C}$. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be any other holomorphic function (e.g. another polynomial), and define $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ by $F(z) = f(z)/P(z)$ for all $z \in \mathbb{C}$. Then F is a meromorphic function, whose poles are located at $\{p_1, p_2, \ldots, p_J\}$. For any $j \in [1...J]$, define $F_j(z) := f(z)/(z-p_1)^{n_1}\cdots(z-p_{j-1})^{n_{j-1}}(z-p_{j+1})^{n_{j+1}}\cdots(z-p_J)^{n_J}$. Then

$$\mathrm{res}_{p_j}(F) \quad = \quad \frac{F_j^{(n_j-1)}(p_j)}{(n_j-1)!}$$

(i.e. the $(n_j - 1)$th term in the Taylor expansion of F_j at p_j). In particular, if $n_j = 1$ (i.e. p_j is a simple pole), then $\mathrm{res}_{p_j}(F) = F_j(p_j)$.

(e) Let $g : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ be meromorphic and let $p \in \mathbb{U}$. Suppose g has a simple pole at p. If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is holomorphic, and $f(p) \neq 0$, then the function $f \cdot g$ is meromorphic, with a pole at p, and $\mathrm{res}_p(f \cdot g) \;=\; f(p) \cdot \mathrm{res}_p(g)$.

Exercise 18G.2 Verify Examples (d) and (e). ◊ Ⓔ

We now come to one of the most important results in complex analysis.

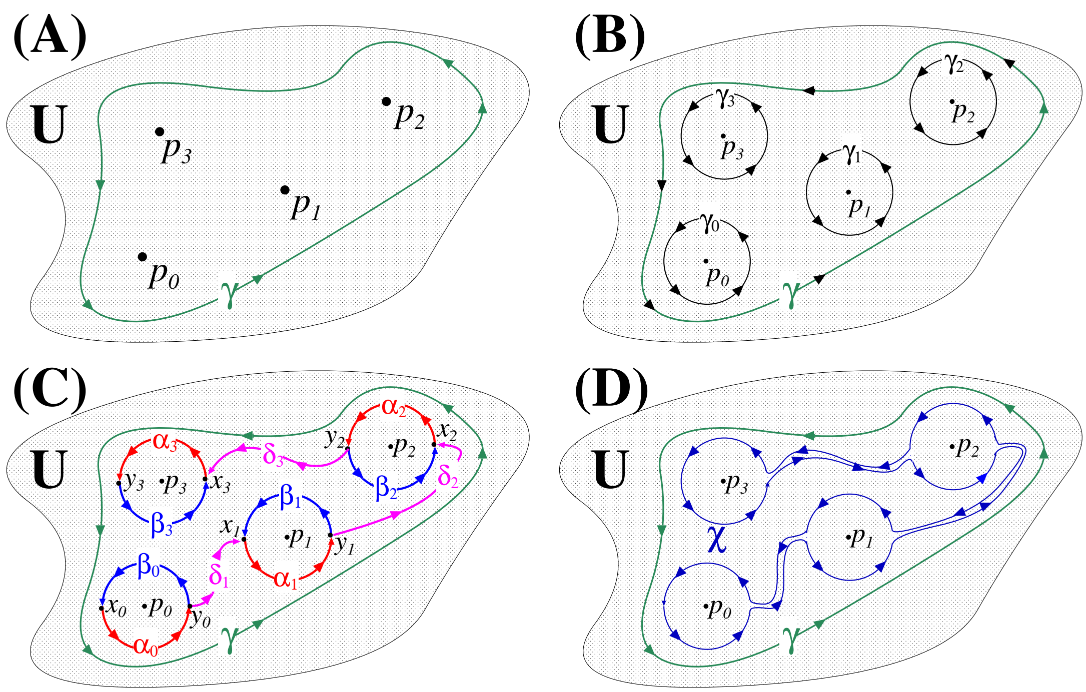


Figure 18G.1: **(A)** The hypotheses of the Residue Theorem. **(B)** For all $j \in [0...J]$, γ_j is a small, counterclockwise circular contour around the pole p_j. **(C)** The paths $\alpha_0, \ldots, \alpha_J$, $\beta_0, \ldots, \beta_J$, and $\delta_1, \ldots, \delta_J$. **(D)** The chain χ is a contour homotopic to γ

Theorem 18G.2. (Residue Theorem)
Let $\mathbb{U} \subseteq \mathbb{C}$ be an open, simply-connected subset of the plane. Let $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ be meromorphic on $\mathbb{U}$. Let $\gamma : [0, S] \longrightarrow \mathbb{U}$ be a counterclockwise contour which is nullhomotopic in $\mathbb{U}$, and suppose the purview of γ contains the poles $p_0, p_1, \ldots, p_J \in \mathbb{U}$, and no other poles, as shown in Figure 18G.1(A). *Then*

$$\oint_\gamma f \quad = \quad 2\pi\mathbf{i} \sum_{j=0}^{J} \mathrm{res}_{p_j}(f).$$

Proof. For all $j \in [0...J]$, let $\gamma_j : [0, 2\pi] \longrightarrow \mathbb{U}$ be a small, counterclockwise circular contour around the pole p_j, as shown in Figure 18G.1(B).

Claim 1: *For all* $j \in [0...J]$, $\displaystyle\oint_{\gamma_j} f \; = \; \mathrm{res}_{p_j}(f)$.

Proof. Suppose f has the following Laurent expansion around p_j:

$$f(z) = \frac{a_{-n}}{(z-p_j)^n} + \frac{a_{1-n}}{(z-p_j)^{n-1}} + \cdots + \frac{a_{-1}}{(z-p_j)} + a_0 + a_1(z-p_j) + \cdots$$

This series converges uniformly, so $\displaystyle\oint_{\gamma_j} f$ can integrated term-by-term to get:

$$\oint_{\gamma_j} \frac{a_{-n}}{(z-p_j)^n} + \oint_{\gamma_j} \frac{a_{1-n}}{(z-p_j)^{n-1}} + \cdots + \oint_{\gamma_j} \frac{a_{-1}}{(z-p_j)} + \oint_{\gamma_j} a_0 + \oint_{\gamma_j} a_1(z-p_j) + \cdots$$

$$\underset{(\dagger)}{=} \quad 0 + 0 + \cdots + a_{-1}\cdot 2\pi\mathbf{i} + 0 + 0 + \cdots$$

$$= \; 2\pi\mathbf{i}\, a_{-1} \underset{(\ddagger)}{=} 2\pi\mathbf{i}\cdot \mathrm{res}_{p_j}(f).$$

Here, (†) is by Examples 18C.2(c) and 18C.6 on pages 435 and 439. Meanwhile, (‡) is because $a_{-1} = \mathrm{res}_{p_j}(f)$ by definition. $\diamond_{\text{Claim 1}}$

Figure 18G.1(C) portrays the smooth paths $\alpha_j : [0, \pi] \longrightarrow \mathbb{U}$ and $\beta_j : [0, \pi] \longrightarrow \mathbb{U}$ defined by

$$\alpha_j(s) \;:=\; \gamma_j(s) \quad \text{and} \quad \beta_j(s) \;:=\; \gamma_j(s+\pi), \qquad \text{for all } s \in [0, \pi].$$

That is: α_j and β_j parameterize the 'first half' and the 'second half' of γ_j, respectively, so that

$$\gamma_j \;=\; \alpha_j \diamond \beta_j. \tag{18G.1}$$

For all $j \in [0...J]$, let $x_j := \alpha_j(0) = \beta_j(\pi)$. and let $y_j := \alpha_j(\pi) = \beta_j(0)$. For all $j \in [1...J]$, let $\delta_j : [0,1] \longrightarrow \mathbb{U}$ be a smooth path from y_{j-1} to x_j. For all $i \in [0...J]$, we can assume that δ_j is drawn so as not to intersect α_i or β_i, and for all $i \in [1...J]$, $i \neq j$, we can likewise assume that δ_j does not intersect δ_i. Figure 18G.1(D) portrays the chain

$$\chi \;:=\; \alpha_0 \diamond \delta_1 \diamond \alpha_1 \diamond \delta_2 \diamond \cdots \diamond \delta_J \diamond \gamma_J \diamond \overleftarrow{\delta}_J \diamond \beta_{J-1} \diamond \overleftarrow{\delta}_{J-1} \diamond \cdots \diamond \overleftarrow{\delta}_3 \diamond \beta_2 \diamond \overleftarrow{\delta}_2 \diamond \beta_1 \diamond \overleftarrow{\delta}_1 \diamond \beta_0. \tag{18G.2}$$

The chain χ is actually a *contour*, by Lemma 18C.8(c) on page 441.

Claim 2: *γ is homotopic to χ in $\mathbb{U}$.*

Ⓔ *Proof.* **<u>Exercise 18G.3</u>** (Not as easy as it looks) $\diamond_{\text{Claim 2}}$

Thus,

$$
\begin{aligned}
\oint_\gamma f \quad & \overline{\overline{\scriptstyle (*)}} \quad \oint_\chi f \\
& \overline{\overline{\scriptstyle (\dagger)}} \quad \int_{\alpha_0} f + \int_{\delta_1} f + \int_{\alpha_1} f + \int_{\delta_2} f \ + \cdots + \int_{\delta_J} f \ + \oint_{\gamma_J} f - \int_{\delta_J} f + \int_{\beta_{J-1}} f \\
& \qquad - \int_{\delta_{J-1}} f \ + \cdots - \int_{\delta_3} f + \int_{\beta_2} f - \int_{\delta_2} f + \int_{\beta_1} f - \int_{\delta_1} f + \int_{\beta_0} f. \\
& = \quad \int_{\alpha_0} f + \int_{\alpha_1} f \ + \cdots + \int_{\alpha_{J-1}} f \ + \oint_{\gamma_J} f + \int_{\beta_{J-1}} f \ + \cdots + \int_{\beta_1} f + \int_{\beta_0} f. \\
& \overline{\overline{\scriptstyle (@)}} \quad \int_{\alpha_0 \diamond \beta_0} f + \int_{\alpha_1 \diamond \beta_1} f \ + \cdots + \int_{\alpha_{J-1} \diamond \beta_{J-1}} f \ + \oint_{\gamma_J} f \\
& \overline{\overline{\scriptstyle (\ddagger)}} \quad \oint_{\gamma_0} f \ + \oint_{\gamma_1} f \ + \cdots + \oint_{\gamma_{J-1}} f \ + \oint_{\gamma_J} f \\
& \overline{\overline{\scriptstyle (\diamond)}} \quad 2\pi\mathbf{i} \cdot \mathrm{res}_{p_0}(f) \ + \ 2\pi\mathbf{i} \cdot \mathrm{res}_{p_1}(f) \ + \ \cdots \ + \ 2\pi\mathbf{i} \cdot \mathrm{res}_{p_{J-1}}(f) \ + \ 2\pi\mathbf{i} \cdot \mathrm{res}_{p_J}(f),
\end{aligned}
$$

as desired. Here, $(*)$ is by Claim 2 and Proposition 18C.7 on page 440. $(\dagger)$ is by eqn.(18G.2) and Lemma 18C.8(a,b) on page 441, and (@) is again by Lemma 18C.8(b). Finally, $(\ddagger)$ is by eqn.(18G.1), and $(\diamond)$ is by Claim 1. □

Example 18G.3. (a) Suppose f is holomorphic inside the purview of γ. Then it has no poles, so the residue-sum in the Residue Theorem is zero. Thus, we get $\oint_\gamma f \ = \ 0$, in agreement with Cauchy's Theorem (Theorem 18C.5 on page 438).

(b) Suppose $f(z) = 1/z$, and γ encircles 0. Then f has exactly one pole in the purview of γ (namely, at 0), and $\mathrm{res}_0(f) = 1$ (because the Laurent expansion of f is just $1/z$). Thus, we get $\oint_\gamma f \ = \ 2\pi\mathbf{i}$, in agreement with Example 18C.6 on page 439.

(c) Suppose f is holomorphic inside the purview of γ. Let p be in the purview of γ and define $F(z) := \frac{f(z)}{z-p}$. Then F has exactly one pole in the purview of γ (namely, at p), and $\mathrm{res}_p(F) = f(z)$, by Example 18G.1(b). Thus, we get $\oint_\gamma F \ = \ 2\pi\mathbf{i}\, f(z)$, in agreement with Cauchy's Integral Formula (Theorem 18C.9 on page 443). ♢

Remark 18G.4: (The Riemann Sphere) Earlier we introduced the notational convention of defining $f(p) = \infty$ whenever p was a pole of a holomorphic function

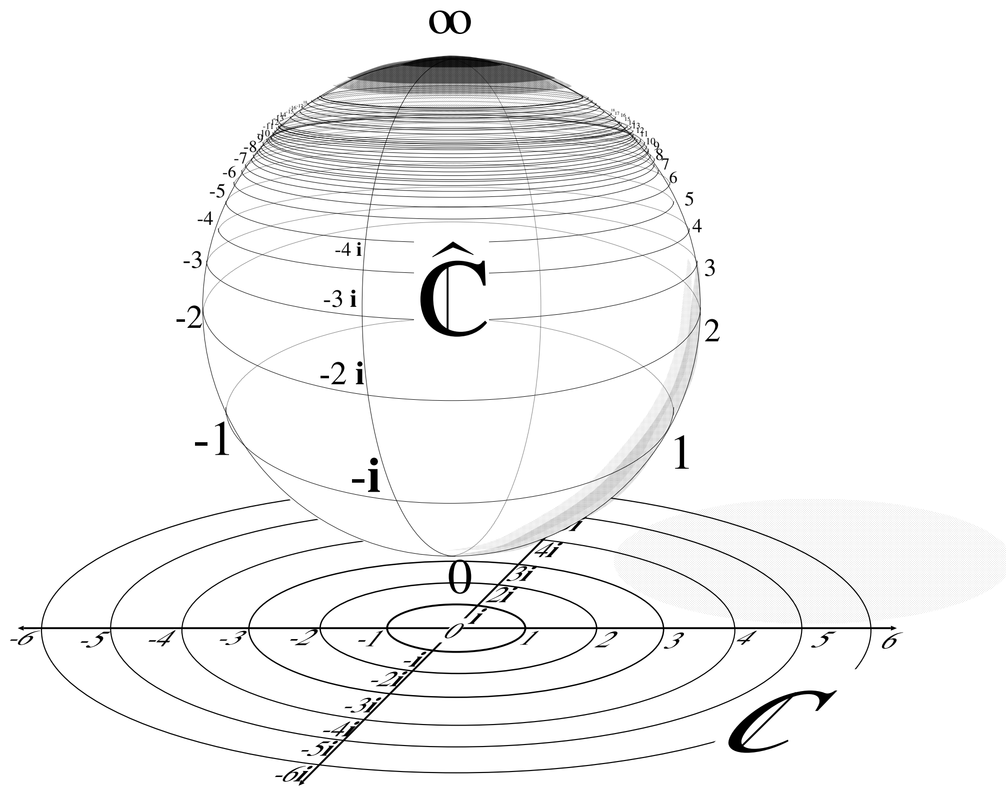


Figure 18G.2: The identification of the complex plane $\mathbb{C}$ with the Riemann sphere $\widehat{\mathbb{C}}$.

$f : \mathbb{U} \setminus \{p\} \longrightarrow \mathbb{C}$, thereby extending f to a 'meromorphic' function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$. We will now explain how this cute notation is actually quite sensible. The **Riemann sphere** is the topological space $\widehat{\mathbb{C}}$ constructed by taking the complex plane $\mathbb{C}$ and adding a 'point at infinity', denoted by "∞". An open set $\mathbb{U} \subset \mathbb{C}$ is considered a 'neighbourhood of ∞' if there is some $r > 0$ such that $\mathbb{U}$ contains the codisk $\mathbb{D}^{\complement}(r) := \{c \in \mathbb{C} \,;\, |c| > r\}$. See Figure 18G.2.

Now, let $\mathbb{U}^* := \mathbb{U} \setminus \{p\}$ and suppose $f : \mathbb{U}^* \longrightarrow \mathbb{C}$ is a continuous function with a singularity at p. Suppose we define $f(p) = \infty$, thereby extending f to a function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$. If $\lim_{z \to p} |f(p)| = \infty$ (e.g. if p is a pole of f), then this extended function will be *continuous* at p, with respect to the topology of the Riemann sphere. In particular, any meromorphic function $f : \mathbb{U} \longrightarrow \widehat{\mathbb{C}}$ is is a continuous mapping from $\mathbb{U}$ into $\widehat{\mathbb{C}}$.

If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is meromorphic, and $L := \lim_{c \to \infty} f(c)$ is well-defined, then we can extend f to a continuous function $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ by defining $f(\infty) := L$. We can then even define the complex *derivatives* of f at ∞; f effectively becomes a complex-differentiable transformation of the entire Riemann sphere.

Many of ideas in complex analysis are best understood by regarding meromorphic functions in this way. ♢

Remark. Not all indelible singularities are poles. Suppose p is a singularity of f, and the Laurent expansion of f at p has an infinite number of negative-power terms (i.e. it looks like the Laurent series (18E.3) on page 455). Then p is called a **essential singularity** of f. The *Casorati-Weierstrass Theorem* says that, if $\mathbb{B}$ is any open neighbourhood of p, however tiny, then the image $f[\mathbb{B}]$ is *dense* in $\mathbb{C}$. In other words, the value of $f(z)$ wildly oscillates all over the complex plane infinitely often as $z \to p$. This is a much more pathological behaviour than a pole, where we simply have $f(z) \to \infty$ as $z \to p$. ♢

Exercise 18G.4. Let $f(z) = \exp(1/z)$. Ⓔ

(a) Show that f has an essential singularity at 0.

(b) Verify the conclusion of the Casorati-Weierstrass Theorem for this function. In fact, show that, for any $\epsilon > 0$, if $\mathbb{D}(\epsilon)$ is the disk of radius ϵ around 0, then $f[\mathbb{D}(\epsilon)] = \mathbb{C} \setminus \{0\}$. ♦

Exercise 18G.5. For each of the following functions, find all poles and compute the residue at each pole. Then use the Residue Theorem to compute the contour integral along a counterclockwise circle of radius 1.8 around the origin. Ⓔ

(a) $f(z) = \dfrac{1}{z^4+1}$. (Hint: $z^4+1 = (z-e^{\pi\mathbf{i}/4})(z-e^{3\pi\mathbf{i}/4})(z-e^{5\pi\mathbf{i}/4})(z-e^{7\pi\mathbf{i}/4})$.)

(b) $f(z) = \dfrac{z^3-1}{z^4+5z^2+4}$. (Hint: $z^4+5z^2+4 = (z^2+1)(z^2+4)$.)

(c) $f(z) = \dfrac{z^4}{z^6+14z^4+49z^2+36}$. (Hint: $z^6+14z^4+49z^2+36 = (z^2+1)(z^2+4)(z^2+9)$.)

(d) $f(z) = \dfrac{z+\mathbf{i}}{z^4+5z^2+4}$. (Careful!)

(e) $f(z) = \tan(z) = \sin(z)/\cos(z)$.

(f) $f(z) = \tanh(z) = \sinh(z)/\cosh(z)$. ♦

Exercise 18G.6. (For algebraists) Ⓔ

(a) Let $\mathfrak{H}$ be the set of all holomorphic functions $f : \mathbb{C} \longrightarrow \mathbb{C}$. (These are sometimes called **entire** functions). Show that $\mathfrak{H}$ is an *integral domain* under the operations of pointwise addition and multiplication. That is: if $f, g \in \mathfrak{H}$, then the functions $(f+g)$ and $(f \cdot g)$ are in $\mathfrak{H}$. (Hint: Use Proposition 18A.5(a,b) on page 18A.5). Also, if $f \neq 0 \neq g$, then $f \cdot g \neq 0$. (Hint: Use the Identity Theorem 18D.3 on page 452).

(b) Let $\mathfrak{M}$ be the set of all meromorphic functions $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$. Show that $\mathfrak{M}$ is a *field* under the operations of pointwise addition and multiplication. That is: if $f, g \in \mathfrak{M}$, then the functions $(f+g)$ and $(f \cdot g)$ are in $\mathfrak{M}$, and if $g \not\equiv 0$, then the function (f/g) is also in $\mathfrak{M}$.

(c) Suppose $f \in \mathfrak{M}$ has only a finite number of poles. Show that f can be expressed in the form $f = g/h$, where $g, h \in \mathfrak{H}$. (Hint: you can make h a polynomial).

(d) (hard) Show that *any* function $f \in \mathfrak{M}$ can be expressed in the form $f = g/h$, where $g, h \in \mathfrak{H}$. (Thus, $\mathfrak{M}$ is related to $\mathfrak{H}$ the same way the field of rational functions is related to the ring of polynomials, and the same way that the field of rational numbers is related to the ring of integers. Technically, $\mathfrak{M}$ is the *field of fractions* of $\mathfrak{H}$.). ♦

18H Improper integrals and Fourier transforms

Prerequisites: §18G. **Recommended:** §17A, §19A.

The Residue Theorem is a powerful tool for evaluating contour integrals in the complex plane. We shall now see that it is also useful for computing improper integrals over the real line, such as convolutions and Fourier transforms. First some notation. Let $\mathbb{C}_+ := \{c \in \mathbb{C} \; ; \; \mathrm{Im}[c] > 0\}$ and $\mathbb{C}_- := \{c \in \mathbb{C} \; ; \; \mathrm{Im}[c] < 0\}$ be the upper and lower halves of the complex plane. If $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is some meromorphic function, then we say that F **uniformly decays at infinity** on $\mathbb{C}_+$ with **order** $o(1/z)$ if,[12] for any $\epsilon > 0$, there is some $r > 0$ such that:

$$\text{For all } z \in \mathbb{C}_+, \qquad \Big(\; |z| > r \; \Big) \Longrightarrow \Big(\; |z| \cdot |F(z)| < \epsilon \; \Big). \qquad (18H.1)$$

In other words, $\lim_{\mathbb{C}_+ \ni z \to \infty} |z| \cdot |F(z)| = 0$, and this convergence is 'uniform' as $z \to \infty$ in any direction in $\mathbb{C}_+$. We define **uniform decay** on $\mathbb{C}_-$ in the same fashion.

Example 18H.1. (a) The function $f(z) = 1/z^2$ uniformly decays at infinity on both $\mathbb{C}_+$ and $\mathbb{C}_-$ with order $o(1/z)$.

(b) However, the function $f(z) = 1/z$ does *not* uniformly decays at infinity with order $o(1/z)$ (it decays just a little bit too slowly).

(c) The function $f(z) = \exp(-\mathbf{i}z)/z^2$ uniformly decays at infinity with order $o(1/z)$ on $\mathbb{C}_+$, but does *not* decay on $\mathbb{C}_-$.

(d) If $P_1, P_2 : \mathbb{C} \longrightarrow \mathbb{C}$ are two complex polynomials of degree N_1 and N_2 respectively, and $N_2 \geq N_1 + 2$, then the rational function $f(z) = P_1(z)/P_2(z)$ uniformly decays with order $o(1/z)$ on both $\mathbb{C}_+$ and $\mathbb{C}_-$. ♢

Ⓔ **Exercise 18H.1.** Verify Examples 18H.1(a-d). ♦

[12] This is pronounced, 'small oh of $1/z$'.

Proposition 18H.2. (Improper integrals of analytic functions)
Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an analytic function, and let $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be an extension of f to a meromorphic function on $\mathbb{C}$.

(a) *Suppose that F uniformly decays with order $o(1/z)$ on $\mathbb{C}_+$. If $p_1, p_2, \ldots, p_J \in \mathbb{C}_+$ are all the poles of F in $\mathbb{C}_+$, then*

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad 2\pi\mathbf{i} \sum_{j=1}^{J} \mathrm{res}_{p_j}(F).$$

(b) *Suppose that F uniformly decays with order $o(1/z)$ on $\mathbb{C}_-$. If $p_1, p_2, \ldots, p_J \in \mathbb{C}_-$ are all the poles of F in $\mathbb{C}_-$, then*

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad -2\pi\mathbf{i} \sum_{j=1}^{J} \mathrm{res}_{p_j}(F).$$

Proof. (a) Note that F has no poles on the real line $\mathbb{R}$, because f is analytic on $\mathbb{R}$. For any $R > 0$, let γ_R be the 'D'-shaped contour of radius R from Example 18C.3 on page 436. If R is made large enough, then γ_R encircles all of $p_1, p_2, \ldots, p_J$. Thus, the Residue Theorem 18G.2 on page 467 says that

$$\oint_{\gamma_R} F \quad = \quad 2\pi\mathbf{i} \sum_{j=1}^{J} \mathrm{res}_{p_j}(F). \tag{18H.2}$$

But by definition,

$$\oint_{\gamma_R} F \quad = \quad \int_0^{\pi+R} F[\gamma_R(s)]\ \dot{\gamma}_R(s) \ \underset{(*)}{=} \ \int_0^{\pi} F(Re^{\mathbf{i}s}) \cdot R\mathbf{i}\, e^{\mathbf{i}s}\ ds \ + \ \int_{-R}^{R} f(x)\ dx,$$

where $(*)$ is by equations (18C.1) and (18C.2) on page 436. Thus,

$$\begin{aligned} \lim_{R\to\infty} \oint_{\gamma_R} F \quad &= \quad \lim_{R\to\infty} \int_0^{\pi} F(Re^{\mathbf{i}s}) \cdot R\mathbf{i}e^{\mathbf{i}s}\ ds \ + \ \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx \\ &\underset{(*)}{=} \quad \int_{-\infty}^{\infty} f(x)\ dx. \end{aligned} \tag{18H.3}$$

Now combine equations (18H.2) and (18H.3) to prove part **(a)**.

In equation (18H.3), step $(*)$ is because

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx \quad = \quad \int_{-\infty}^{\infty} f(x)\ dx, \tag{18H.4}$$

$$\text{while} \quad \lim_{R\to\infty} \left| \int_0^{\pi} F(Re^{\mathbf{i}s}) \cdot R\mathbf{i}e^{\mathbf{i}s}\ ds \right| \quad = \quad 0 \tag{18H.5}$$

Equation (18H.4) is just the definition of an improper integral. To see equation (18H.5), note that

$$\left|\int_0^\pi F(Re^{\mathbf{i}s})\, R\mathbf{i}e^{\mathbf{i}s}\ ds\right| \underset{(\triangle)}{\leq} \int_0^\pi \left|F(Re^{\mathbf{i}s})\, R\mathbf{i}e^{\mathbf{i}s}\right|\ ds \quad = \quad \int_0^\pi R\left|F(Re^{\mathbf{i}s})\right|\ ds, \tag{18H.6}$$

where $(\triangle)$ is just the triangle inequality for integrals. But for any $\epsilon > 0$, we can find some $r > 0$ satisfying equation (18H.1). Then for all $R > r$, and all $s \in [0, \pi]$, we have $R \cdot \left|F(Re^{\mathbf{i}s})\right| < \epsilon$, which means

$$\int_0^\pi R \cdot \left|F(Re^{\mathbf{i}s})\right|\ ds \quad \leq \quad \int_0^\pi \epsilon \quad = \quad \pi\epsilon. \tag{18H.7}$$

Since $\epsilon > 0$ can be made arbitrarily small, equations (18H.6) and (18H.7) imply (18H.5).

Ⓔ **Exercise 18H.2** Prove part **(b)** of the theorem. □

If $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ are integrable functions, recall that their **convolution** is the function $f * g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f * g(r) \ := \ \int_{-\infty}^{\infty} f(x)\ g(r-x)\ dx$, for any $r \in \mathbb{R}$. Chapter 17 showed how to solve I/BVPs by convolving with 'impulse-response' functions like the Poisson kernel.

Corollary 18H.3. (Convolutions of analytic functions)
Let $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ be analytic functions, with meromorphic extensions $F, G : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$. Suppose the function $z \mapsto F(z) \cdot G(-z)$ uniformly decays with order $o(1/z)$ on $\mathbb{C}_+$. Suppose F has simple poles $p_1, p_2, \ldots, p_J \in \mathbb{C}_+$, and no other poles in $\mathbb{C}_+$. Suppose G has simple poles $q_1, q_2, \ldots, q_K \in \mathbb{C}_-$, and no other poles in $\mathbb{C}_-$. Then for all $r \in \mathbb{R}$,

$$f * g(r) \quad = \quad 2\pi\mathbf{i}\sum_{j=1}^{J} G(r - p_j) \cdot \mathrm{res}_{p_j}(F) \ - \ 2\pi\mathbf{i}\sum_{k=1}^{K} F(r - q_k) \cdot \mathrm{res}_{q_j}(G).$$

Proof. Fix $r \in \mathbb{R}$, and consider the function $H(z) := F(z)G(r - z)$. For all $j \in [1...J]$, Example 18G.1(e) on page 466 says that H has a simple pole at $p_j \in \mathbb{C}_+$, with residue $G(r - p_j) \cdot \mathrm{res}_{p_j}(F)$. For all $k \in [1...K]$, the function $z \mapsto G(r - z)$ has a simple pole at $r - q_k$, with residue $-\mathrm{res}_{q_k}(G)$. Thus, Example 18G.1(e) says that H has a simple pole at $r - q_k$, with residue $-F(r - q_k) \cdot \mathrm{res}_{q_k}(G)$. Note that $(r - q_k) \in \mathbb{C}_+$, because $q_k \in \mathbb{C}_-$ and $r \in \mathbb{R}$. Now apply Proposition 18H.2. □

Example 18H.4. For any $y > 0$, recall the *half-plane Poisson kernel* $\mathcal{K}_y : \mathbb{R} \longrightarrow \mathbb{R}$ from §17E, defined by

$$\mathcal{K}_y(x) \quad := \quad \frac{y}{\pi(x^2+y^2)}, \qquad \text{for all } x \in \mathbb{R}.$$

Let $\mathbb{H} := \left\{(x,y) \in \mathbb{R}^2 \ ; \ y \geq 0\right\}$ (the upper half-plane). If $b : \mathbb{R} \longrightarrow \mathbb{R}$ is bounded and continuous, then Proposition 17E.1 on page 404 says that the function $h(x,y) := \mathcal{K}_y * b(x)$ is the unique continuous harmonic function on $\mathbb{H}$ which satisfies the Dirichlet boundary condition $h(x,0) = b(x)$ for all $x \in \mathbb{R}$. Suppose $b : \mathbb{R} \longrightarrow \mathbb{R}$ is analytic, with a meromorphic extension $B : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ which is **asymptotically bounded** near infinity in $\mathbb{C}_-$ —that is, there exist $K, R > 0$ such that $|B(z)| < K$ for all $z \in \mathbb{C}_-$ with $|z| > R$. Then the function $\mathcal{K}_y \cdot B$ asymptotically decays near infinity with order $o(1/z)$ on $\mathbb{C}_+$, so Corollary 18H.3 is applicable.

In Example 18G.1(c) on page 466, we saw that $\mathcal{K}_y$ has a simple pole at $y\mathbf{i}$, with $\operatorname{res}_{y\mathbf{i}}(\mathcal{K}_y) = 1/2\pi\mathbf{i}$, and no other poles in $\mathbb{C}_+$. Suppose B has simple poles $q_1, q_2, \ldots, q_K \in \mathbb{C}_-$, and no other poles in $\mathbb{C}_-$. Then setting $f := \mathcal{K}_y$, $g := b$, $J := 1$ and $p_1 := y\mathbf{i}$ in Corollary 18H.3, we get for any $(x,y) \in \mathbb{H}$,

$$h(x,y) \quad = \quad 2\pi\mathbf{i}\, B(x - y\mathbf{i}) \cdot \underbrace{\operatorname{res}_{y\mathbf{i}}(\mathcal{K}_y)}_{=1/2\pi\mathbf{i}} \; - \; 2\pi\mathbf{i} \sum_{k=1}^{K} \mathcal{K}_y(x - q_k) \cdot \operatorname{res}_{q_k}(B)$$

$$= \quad B(x - y\mathbf{i}) \; - \; 2y\mathbf{i} \sum_{k=1}^{k} \frac{\operatorname{res}_{q_k}(B)}{(x-q_k)^2 + y^2}. \qquad \diamondsuit$$

Exercise 18H.3. (a) Show that, in fact, $h(x,y) = \operatorname{Re}\left[B(x - y\mathbf{i})\right]$. (Thus, if we could compute B, then the BVP would already be solved, and we actually wouldn't need to apply Proposition 17E.1). Ⓔ

(b) Deduce that $\operatorname{Im}\left[B(x-y\mathbf{i})\right] \; = \; 2y \displaystyle\sum_{k=1}^{K} \frac{\operatorname{res}_{q_k}(B)}{(x-q_k)^2+y^2}$. ♦

Exercise 18H.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded analytic function, with meromorphic extension $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$. Let $u : \mathbb{R} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ be the unique solution to the one-dimensional heat equation ($\partial_t u \; = \; \partial_x^2 u$) with initial conditions $u(x;0) = f(x)$ for all $x \in \mathbb{R}$. Combine Proposition 18H.3 with Proposition 17C.1 on page 385 to find a formula for $u(x;t)$ in terms of the residues of F. ♦ Ⓔ

Exercise 18H.5. For any $t \geq 0$, let $\Gamma_t : \mathbb{R} \longrightarrow \mathbb{R}$ be the *d'Alembert kernel*: Ⓔ

$$\Gamma_t(x) \quad = \quad \begin{cases} \frac{1}{2} & \text{if} \quad -t < x < t; \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be an analytic function. Lemma 17D.3 on page 395 says that we can solve the initial velocity problem for the one-dimensional wave equation by defining $v(x;t) := \Gamma_t * f_1(x)$. Explain why Proposition 18H.3 is *not* suitable for computing $\Gamma_t * f_1$. ♦

If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is an integrable function, then its *Fourier transform* is the function $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\widehat{f}(\mu) \quad := \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\mu x \mathbf{i}) f(x)\, dx, \qquad \text{for all } \mu \in \mathbb{R}.$$

(See §19A for more information). Proposition 18H.2 can also be used to compute Fourier transforms, but it is not quite the strongest result for this purpose. If $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is some meromorphic function, then we say that F **uniformly decays at infinity with order** $\mathcal{O}(1/z)$ if[13] there exists some $M > 0$ and some $r > 0$ such that:

$$\text{For all } z \in \mathbb{C}, \qquad \Big(\ |z| > r \ \Big) \Longrightarrow \Big(\ |F(z)| < M/|z| \ \Big). \tag{18H.8}$$

In other words, the function $|z \cdot F(z)|$ is uniformly bounded (by M) as $z \to \infty$ in any direction in $\mathbb{C}$.

Example 18H.5. (a) The function $f(z) = 1/z$ uniformly decays at infinity with order $\mathcal{O}(1/z)$.

(b) If f uniformly decays at infinity on $\mathbb{C}_\pm$ with order $o(1/z)$, then it also uniformly decays at infinity with order $\mathcal{O}(1/z)$. (**Exercise 18H.6** Verify this).

Ⓔ

(c) In particular, if $P_1, P_2 : \mathbb{C} \longrightarrow \mathbb{C}$ are two complex polynomials of degree N_1 and N_2 respectively, and $N_2 \geq N_1 + 1$, then the rational function $f(z) = P_1(z)/P_2(z)$ uniformly decays at infinity with order $\mathcal{O}(1/z)$. ♢

Thus, decay with order $\mathcal{O}(1/z)$ is a slightly weaker requirement than decay with order $o(1/z)$.

Proposition 18H.6. (Fourier transforms of analytic functions)

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an analytic function. Let $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be an extension of f to a meromorphic function on $\mathbb{C}$ which uniformly decays with order $\mathcal{O}(1/z)$. Let $p_{-K}, \ldots, p_{-2}, p_{-1}, p_0, p_1, \ldots, p_J$ be all the poles of F in $\mathbb{C}$, where $p_{-K}, \ldots, p_{-2}, p_{-1} \in \mathbb{C}_-$ and $p_0, p_1, \ldots, p_J \in \mathbb{C}_+$. Then:

$$\widehat{f}(\mu) \quad = \quad \mathbf{i} \sum_{j=0}^{J} \mathrm{res}_{p_j}(\mathcal{E}_\mu \cdot F), \qquad \textit{for all } \mu < 0, \tag{18H.9}$$

$$\textit{and} \qquad \widehat{f}(\mu) \quad = \quad -\mathbf{i} \sum_{k=-1}^{-K} \mathrm{res}_{p_k}(\mathcal{E}_\mu \cdot F), \qquad \textit{for all } \mu > 0,$$

[13] This is pronounced, ‘big oh of $1/z$’.

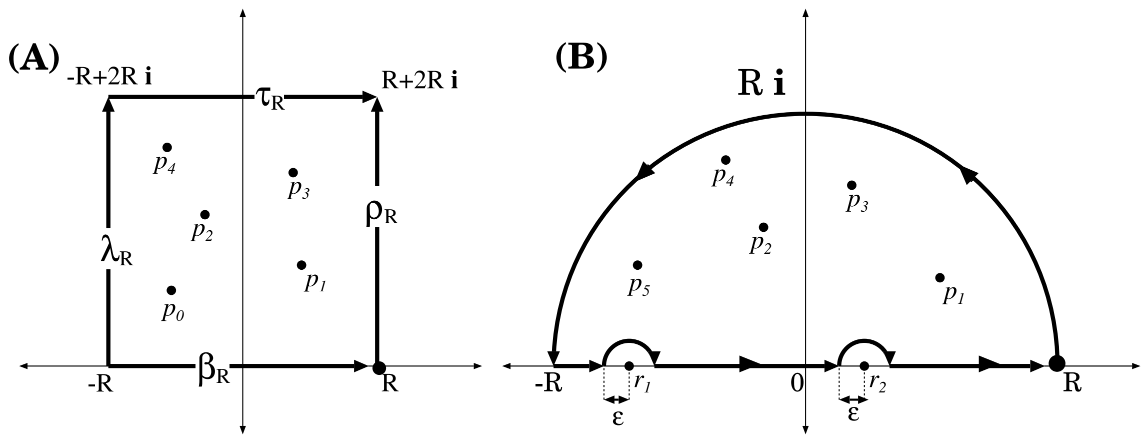


Figure 18H.1: **(A)** The square contour in the proof of Proposition 18H.6. **(B)** The contour in Exercise 18H.12 on page 480

where $\mathcal{E}_\mu : \mathbb{C} \longrightarrow \mathbb{C}$ *is the holomorphic function defined* $\mathcal{E}_\mu(z) := \exp(-\mu \cdot z \cdot \mathbf{i})$ *for all* $z \in \mathbb{C}$*. In particular, if all the poles of* F *are simple, then*

$$\widehat{f}(\mu) \quad = \quad \mathbf{i} \sum_{j=0}^{J} \exp(-\mu p_j \, \mathbf{i}) \cdot \operatorname{res}_{p_j}(F), \qquad \textit{for all } \mu < 0, \quad (18H.10)$$

$$\textit{and} \qquad \widehat{f}(\mu) \quad = \quad -\mathbf{i} \sum_{k=-1}^{-K} \exp(-\mu p_k \, \mathbf{i}) \cdot \operatorname{res}_{p_k}(F), \qquad \textit{for all } \mu > 0.$$

Proof. We will prove the theorem for $\mu < 0$. Fix $\mu < 0$ and define $G : \mathbb{C} \longrightarrow \mathbb{C}$ by $G(z) := \exp(-\mu z \, \mathbf{i}) \cdot F(z)$. For any $R > 0$, define the chains β_R, ρ_R, τ_r, and λ_R as shown in Figure 18H.1(A):

$$\begin{array}{rrclcrcl}
\text{For all } s \in [-R, R], & \beta_R(s) & := & s & \text{so that} & \dot{\beta}_R(s) & = & 1. \\
\text{For all } s \in [0, 2R], & \rho_R(s) & := & R + s\mathbf{i} & \text{so that} & \dot{\rho}_R(s) & = & \mathbf{i}. \\
\text{For all } s \in [-R, R], & \tau_R(s) & := & s + 2R\mathbf{i} & \text{so that} & \dot{\tau}_R(s) & = & 1. \\
\text{For all } s \in [0, 2R], & \lambda_R(s) & := & -R + s\mathbf{i} & \text{so that} & \dot{\lambda}_R(s) & = & \mathbf{i}.
\end{array} \quad (18H.11)$$

(Mnemonic: βottom, ρight, τop, λeft.) Thus, if $\gamma = \beta \diamond \rho \diamond \overleftarrow{\tau} \diamond \overleftarrow{\lambda}$, then γ_R traces a square in $\mathbb{C}_+$ of sidelength $2R$. If R is made large enough, then γ_R encloses all of $p_0, p_2, \ldots, p_J$. Thus, for any large enough $R > 0$,

$$2\pi\mathbf{i} \sum_{j=0}^{J} \operatorname{res}_{p_j}(G) \underset{(*)}{=} \oint_{\gamma_R} G \underset{(\dagger)}{=} \oint_{\beta_R} G + \oint_{\rho_R} G - \oint_{\tau_R} G - \oint_{\lambda_R} G \quad (18H.12)$$

where $(*)$ is by the Residue Theorem 18G.2 on page 467, and where $(\dagger)$ is by Lemma 18C.8(a,b) on page 441.

Claim 1: $\displaystyle\lim_{R\to\infty} \oint_{\beta_R} G \;=\; 2\pi\,\widehat{f}(\mu).$

Ⓔ *Proof.* **Exercise 18H.7** $\diamond_{\text{Claim 1}}$

Claim 2: **(a)** $\displaystyle\lim_{R\to\infty} \oint_{\rho_R} G \;=\; 0$ *and* **(b)** $\displaystyle\lim_{R\to\infty} \oint_{\lambda_R} G \;=\; 0.$

Proof. (a) By hypothesis, f decays with order $\mathcal{O}(1/z)$. Thus, we can find some $r > 0$ and $M > 0$ satisfying eqn.(18H.8). If $R > r$, then for all $s \in [0, 2R]$,

$$\begin{aligned} |G(R+s\mathbf{i})| &= |\exp[-\mu\mathbf{i}(R+s\mathbf{i})]| \cdot |F(R+s\mathbf{i})| \\ &\underset{(*)}{\leq} |\exp(\mu s - \mu R\mathbf{i})| \cdot \frac{M}{|R+s\mathbf{i}|} \;\leq\; e^{\mu s}\cdot\frac{M}{R}, \end{aligned} \tag{18H.13}$$

where $(*)$ is by equation (18H.8). Thus,

$$\begin{aligned} \left|\oint_{\rho_R} G\right| &\underset{(\diamond)}{=} \left|\int_0^{2R} G(R+s\mathbf{i})\,\mathbf{i}\;ds\right| \;\leq\; \int_0^{2R} |G(R+s\mathbf{i})|\;ds \;\underset{(*)}{\leq}\; \frac{M}{R}\int_0^{2R} e^{\mu s}\;ds \\ &= \frac{M}{\mu R} e^{\mu s}\Big|_{s=0}^{s=2R} \;=\; \frac{M}{-\mu R}(1-e^{2\mu R}) \;\underset{(\dagger)}{\leq}\; \frac{M}{-\mu R} \;\xrightarrow[R\to\infty]{}\; 0, \end{aligned}$$

as desired. Here, $(\diamond)$ is by eqn.(18H.11), $(*)$ is by eqn.(18H.13), and $(\dagger)$ is because $\mu < 0$. This proves **(a)**. The proof of **(b)** is similar. $\diamond_{\text{Claim 2}}$

Claim 3: $\displaystyle\lim_{R\to\infty} \oint_{\tau_R} G \;=\; 0.$

Proof. Again find $r > 0$ and $M > 0$ satisfying eqn.(18H.8). If $R > r$, then for all $s \in [-R, R]$,

$$\begin{aligned} |G(s+2R\mathbf{i})| &= |\exp[-\mu\mathbf{i}(s+2R\mathbf{i})]| \cdot |F(s+2R\mathbf{i})| \\ &\underset{(*)}{\leq} |\exp(2R\mu - s\mu\mathbf{i})| \cdot \frac{M}{|s+2R\mathbf{i}|} \\ &\leq e^{2R\mu}\cdot\frac{M}{2R}, \end{aligned} \tag{18H.14}$$

where $(*)$ is by equation (18H.8). Thus,

$$\left|\oint_{\tau_R} G\right| \;\underset{(*)}{\leq}\; e^{2R\mu}\cdot\frac{M}{2R}\cdot \text{length}(\tau_R) \;=\; e^{2R\mu}\cdot\frac{M}{2R}\cdot 2R \;=\; Me^{2R\mu} \;\xrightarrow[R\to\infty]{(\dagger)}\; 0,$$

as desired. Here, $(*)$ is by eqn.(18H.14) and Lemma 18C.10 on page 444, while $(\dagger)$ is because $\mu < 0$. $\diamond_{\text{Claim 3}}$

Now we put it all together:

$$2\pi\mathbf{i}\sum_{j=0}^{J}\mathrm{res}_{p_j}(G) \underset{(*)}{=} \lim_{R\to\infty}\left(\oint_{\beta_R} G + \oint_{\rho_R} G - \oint_{\tau_R} G - \oint_{\lambda_R} G\right)$$
$$\underset{(\dagger)}{=} 2\pi\,\widehat{f}(\mu) + 0 + 0 + 0 = 2\pi\,\widehat{f}(\mu).$$

Now divide both sides by 2π to get eqn.(18H.9). Here, $(*)$ is by eqn.(18H.12), and $(\dagger)$ is by Claims 1-3.

Finally, to see eqn.(18H.10), suppose all the poles $p_0, \ldots, p_J$ are simple. Then $\mathrm{res}_{p_j}(G) = \exp(-\mathbf{i}\mu p_j)\cdot \mathrm{res}_{p_j}(F)$ for all $j \in [0...J]$, by Example 18G.1(e) on page 466. Thus,

$$\mathbf{i}\sum_{j=0}^{J}\mathrm{res}_{p_j}(G) = \mathbf{i}\sum_{j=0}^{J}\exp(-\mu p_j\,\mathbf{i})\cdot\mathrm{res}_{p_j}(F),$$

so eqn.(18H.10) follows from (18H.9). □

Exercise 18H.8. Prove Proposition 18H.6 in the case $\mu > 0$. ♦ Ⓔ

Example 18H.7. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) := 1/(x^2+1)$ for all $x \in \mathbb{R}$. The meromorphic extension of f is simply the complex polynomial $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ defined

$$F(z) := \frac{1}{z^2+1} = \frac{1}{(z+\mathbf{i})(z-\mathbf{i})}, \qquad \text{for all } z \in \mathbb{C}.$$

Clearly F has simple poles at $\pm\mathbf{i}$, with $\mathrm{res}_{\mathbf{i}}(F) = \dfrac{1}{2\mathbf{i}}$ and $\mathrm{res}_{-\mathbf{i}}(F) = \dfrac{-1}{2\mathbf{i}}$. Thus, Proposition 18H.6 says

$$\text{If } \mu < 0, \text{ then } \quad \widehat{f}(\mu) = \mathbf{i}\exp(-\mu\mathbf{i}\cdot\mathbf{i})\frac{1}{2\mathbf{i}} = \frac{e^{\mu}}{2} = \frac{e^{-|\mu|}}{2}.$$
$$\text{If } \mu > 0, \text{ then } \quad \widehat{f}(\mu) = -\mathbf{i}\exp(-\mu\mathbf{i}\cdot(-\mathbf{i}))\frac{-1}{2\mathbf{i}} = \frac{e^{-\mu}}{2} = \frac{e^{-|\mu|}}{2}.$$

We conclude that $\widehat{g}(\mu) = \dfrac{e^{-|\mu|}}{2}$ for all $\mu \in \mathbb{R}$. ♢

Exercise 18H.9. Compute the Fourier transforms of the following rational functions Ⓔ

(a) $f(x) = \frac{1}{x^4+1}$. (Hint: $x^4+1 = (x-e^{\pi\mathbf{i}/4})(x-e^{3\pi\mathbf{i}/4})(x-e^{5\pi\mathbf{i}/4})(x-e^{7\pi\mathbf{i}/4})$.)

(b) $f(x) = \frac{x^3-1}{x^4+5x^2+4}$. (Hint: $x^4+5x^2+4 = (x^2+1)(x^2+4)$.)

(c) $f(x) = \frac{x^4}{x^6+14x^4+49x^2+36}$. (Hint: $x^6+14x^4+49x^2+36 = (x^2+1)(x^2+4)(x^2+9)$.)

(d) $f(x) = \frac{x+\mathbf{i}}{x^4+5x^2+4}$. (Careful!) ♦

Ⓔ **Exercise 18H.10.** Why is Proposition 18H.6 *not* suitable to compute the Fourier transforms of the following functions?

(a) $f(x) = \frac{1}{x^3+1}$.

(b) $f(x) = \frac{\sin(x)}{x^4+1}$.

(c) $f(x) = \frac{1}{|x|^3+1}$.

(d) $f(x) = \frac{\sqrt[3]{x}}{x^3+1}$. ♦

Ⓔ **Exercise 18H.11.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an analytic function whose meromorphic extension $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ decays with order $\mathcal{O}(1/z)$.

(a) State and prove a general formula for 'trigonometric integrals' of the form

$$\int_{-\infty}^{\infty} \cos(nx)\, f(x)\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \sin(nx)\, f(x)\, dx$$

(Hint: Use Proposition 18H.6 and the formula $\exp(\mu x) = \cos(\mu x) + \mathbf{i}\sin(\mu x)$).

Use your method to compute the following integrals:

(b) $\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1}\, dx$.

(c) $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+1}\, dx$.

(d) $\int_{-\infty}^{\infty} \frac{\sin(x)^2}{x^2+1}\, dx$. (Hint: $2\sin(x)^2 = 1-\cos(2x)$.) ♦

Ⓔ **Exercise 18H.12.** Proposition 18H.2 requires the function f to have no poles on the real line $\mathbb{R}$. However, this is not really necessary.

(a) Let $\mathcal{R} := \{r_1, \ldots, r_N\} \subset \mathbb{R}$. Let $f : \mathbb{R} \setminus \mathcal{R} \longrightarrow \mathbb{R}$ be an analytic function whose meromorphic extension $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ decays with order $o(1/z)$ on $\mathbb{C}_+$, and has poles $p_1, \ldots, p_J \in \mathbb{C}_+$, and also has simple poles at $r_1, \ldots, r_N$. Show that

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad 2\pi\mathbf{i} \sum_{j=1}^{J} \mathrm{res}_{p_j}(F) \; + \; \pi\mathbf{i} \sum_{n=1}^{N} \mathrm{res}_{r_n}(F).$$

Hint: For all $\epsilon > 0$ and $R > 0$, let $\gamma_{R,\epsilon}$ be the contour shown in Figure 18H.1(B) on page 477. This is like the 'D' contour in the proof of Propositions 18H.2, except that it makes a little semicircular 'detour' of radius ϵ around each of the poles $r_1, \ldots, r_N \in \mathbb{R}$. Show that the integral along each of these ϵ-detours tends to $-\pi\mathbf{i} \cdot \mathrm{res}_{r_n}(F)$ as $\epsilon \to 0$, while the integral over the remainder of the real line tends to $\int_{-\infty}^{\infty} f(x)\, dx$ as $\epsilon \to 0$ and $R \to \infty$.

(b) Use your method to compute $\displaystyle\int_{-\infty}^{\infty} \frac{\exp(\mathbf{i}\mu x)}{x^2}\, dx$. ♦

Exercise 18H.13. (a) Let $\mathcal{R} := \{r_1, \ldots, r_N\} \subset \mathbb{R}$. Let $f : \mathbb{R} \setminus \mathcal{R} \longrightarrow \mathbb{R}$ be an analytic function whose meromorphic extension $F : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ decays with order $\mathcal{O}(1/z)$ and has poles $p_1, \ldots, p_J \in \mathbb{C}_+$ and also has a simple poles at $r_1, \ldots, r_N$. Show that, for any $\mu < 0$, Ⓔ

$$\widehat{f}(\mu) \quad = \quad \mathbf{i} \sum_{j=1}^{J} \exp(-\mu p_j \mathbf{i}) \cdot \mathrm{res}_{p_j}(F) \; + \; \frac{\mathbf{i}}{2} \sum_{n=1}^{N} \exp(-\mu r_j \mathbf{i}) \cdot \mathrm{res}_{r_n}(F).$$

(If $\mu > 0$, it's a similar formula, only summing over the residues in $\mathbb{C}_-$ and multiplying by -1). *Hint.* Combine the method from Exercise 18H.12 with the proof technique from Proposition 18H.6.

(b) Use your method to compute $\widehat{f}(\mu)$ when $f(x) = \dfrac{1}{x(x^2+1)}$. ♦

Exercise 18H.14. The *Laplace inversion integral* is defined by equation (19H.3) on page 518. State and prove a formula similar to Theorem 18H.6 for the computation of Laplace inversion integrals. ♦ Ⓔ

18I* Homological extension of Cauchy's theorem

Prerequisites: §18C.

We have defined 'contours' to be non-self-intersecting curves only so as to simplify the exposition in Section 18C.[14] All of the results of Section 18C are true for any piecewise smooth closed curve in $\mathbb{C}$. Indeed, the results of Section 18C can be even extended to integrals on *chains,* as we now discuss.

Let $\mathbb{U} \subseteq \mathbb{C}$ be a connected open set, and let $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_N$ be the connected components of the boundary $\partial\mathbb{U}$. Suppose each $\mathbb{G}_n$ can be parameterized by a piecewise smooth contour γ_n, such that the outward normal vector field of $\mathbb{G}_n$ is always on the right-hand side of $\dot{\gamma}_n$. The chain $\gamma := \gamma_1 \diamond \gamma_2 \diamond \cdots \diamond \gamma_n$ is called the **positive boundary** of $\mathbb{U}$. Its reversal $\overleftarrow{\gamma}$ is called the **negative boundary** of $\mathbb{U}$. Both the negative and positive boundaries of a set are called

[14] To be precise, it made it simpler for us to define the 'purview' of the contour, by invoking the Jordan Curve Theorem. It also made it simpler to define 'clockwise' versus 'counterclockwise' contours.

oriented boundaries. For example, any contour γ is the oriented boundary of the purview of γ. Theorem 18C.5 on page 438 now extends to the following theorem.

Theorem 18I.1. Cauchy's Theorem on oriented boundaries
Let $\mathbb{U} \subseteq \mathbb{C}$ be any open set, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic on $\mathbb{U}$. If α is an oriented boundary of $\mathbb{U}$, then $\oint_\alpha f \;=\; 0.$ □

Let $\alpha_1, \alpha_2, \ldots, \alpha_N : [0,1] \longrightarrow \mathbb{C}$ be continuous, piecewise smooth curves in $\mathbb{C}$ (not necessarily closed), and consider the chain $\alpha = \alpha_1 \diamond \alpha_2 \diamond \cdots \alpha_N$ (note that any chain can be expressed in this way). Let's refer to the paths $\alpha_1, \ldots, \alpha_N$ as the the 'links' of the chain α. We say that α is a **cycle** if the endpoint of each link is the starting point of exactly one other link, and the starting point of each link is the endpoint of exactly one other link. In other words, for all $m \in [1..N]$, there exists a unique $\ell, n \in [1...C]$ such that $\alpha_\ell(1) = \alpha_m(0)$ and $\alpha_m(1) = \alpha_n(0)$.

Example 18I.2. (a) Any contour is a cycle.

(b) If α and β are two cycles, then $\alpha \diamond \beta$ is also a cycle.

(c) Thus, if $\gamma_1, \ldots, \gamma_N$ are contours, then $\gamma_1 \diamond \cdots \diamond \gamma_N$ is a cycle.

(d) In particular, the oriented boundary of an open set is a cycle.

(e) If α is a cycle, then $\overleftarrow{\alpha}$ is a cycle. ♢

Not all cycles are oriented boundaries. For example, let γ_1 and γ_2 be two concentric counterclockwise circles around the origin; then $\gamma_1 \diamond \gamma_2$ is *not* an oriented boundary. (Although $\gamma_1 \diamond \overleftarrow{\gamma_2}$ is.)

Let $\mathbb{U} \subseteq \mathbb{C}$ be an open set. Let α and β be two cycles in $\mathbb{U}$. We say that α is **homologous** to β in $\mathbb{U}$ if the cycle $\alpha \diamond \overleftarrow{\beta}$ is the oriented boundary of some open subset $\mathbb{V} \subseteq \mathbb{U}$. We then write "$\alpha \underset{\mathbb{U}}{\sim} \beta$"

Example 18I.3. (a) Let α be a clockwise circle of radius 1 around the origin, and let β be a clockwise circle of radius 2 around the origin. Then α is homologous to β in $\mathbb{C}^*$, because $\alpha \diamond \overleftarrow{\beta}$ is the positive boundary of the annulus $\mathbb{A} := \{c \in \mathbb{C} \,;\; 1 < |c| < 2\} \subseteq \mathbb{C}^*$.

(b) If γ_0 and γ_1 are contours, then they are cycles. If γ_0 is homotopic to γ_1 in $\mathbb{U}$, then γ_0 is also *homologous* to γ_1 in $\mathbb{U}$. To see this, let $\Gamma : [0,1] \times [0,S] \longrightarrow \mathbb{C}$ be a homotopy from γ_0 to γ_1, and let $\mathbb{V} := \Gamma((0,1) \times [0,S])$. Then $\mathbb{V}$ is an open subset of $\mathbb{U}$, and $\gamma_1 \diamond \overleftarrow{\gamma_2}$ is an oriented boundary of $\mathbb{V}$. ♢

Thus, homology can be seen as a generalization of homotopy. Proposition 18C.7 on page 440 can be extended as follows:

Proposition 18I.4. (Homology invariance of chain integrals)
Let $\mathbb{U} \subseteq \mathbb{C}$ be any open set, and let $f : \mathbb{U} \longrightarrow \mathbb{C}$ be holomorphic on $\mathbb{U}$. If α and β are two cycles which are homologous in $\mathbb{U}$, then $\oint_\alpha f = \oint_\beta f$. □

Proof. **Exercise 18I.1** Hint: Use Theorem 18I.1. □ Ⓔ

The relation "$\underset{\mathbb{U}}{\approx}$" is an *equivalence relation*. That is, for all cycles α, β, and γ,

- $\alpha \underset{\mathbb{U}}{\approx} \alpha$;.
- If $\alpha \underset{\mathbb{U}}{\approx} \beta$, then $\beta \underset{\mathbb{U}}{\approx} \alpha$;
- If $\alpha \underset{\mathbb{U}}{\approx} \beta$, and $\beta \underset{\mathbb{U}}{\approx} \gamma$, then $\alpha \underset{\mathbb{U}}{\approx} \gamma$.

(**Exercise 18I.2** Verify these three properties.) Ⓔ

For any cycle α, let $[\alpha]_\mathbb{U}$ denote its equivalence class under "$\underset{\mathbb{U}}{\approx}$" (this is called a **homology class**). Let $\mathcal{H}^1(\mathbb{U})$ denote the set of all homology classes of cycles. In particular, let $[\emptyset]_\mathbb{U}$ denote the homology class of the empty cycle —then $[\emptyset]_\mathbb{U}$ contains all cycles which are oriented boundaries of subsets of $\mathbb{U}$.

Corollary 18I.5. *Let $\mathbb{U} \subseteq \mathbb{C}$ be any open set.*

(a) *If $\alpha_1 \underset{\mathbb{U}}{\approx} \alpha_2$ and $\beta_1 \underset{\mathbb{U}}{\approx} \beta_2$, then $(\alpha_1 \diamond \beta_1) \underset{\mathbb{U}}{\approx} (\alpha_2 \diamond \beta_2)$. Thus, we can define an operation $\oplus$ on $\mathcal{H}^1(\mathbb{U})$ by $[\alpha]_\mathbb{U} \oplus [\beta]_\mathbb{U} := [\alpha \diamond \beta]_\mathbb{U}$.*

(b) *$\mathcal{H}^1(\mathbb{U})$ is an abelian group under the operation $\oplus$.*

(c) *If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is holomorphic, then the function $[\alpha]_\mathbb{U} \mapsto \oint_\alpha f$ is a group homomorphism from $(\mathcal{H}^1(\mathbb{U}), \oplus)$ to the group $(\mathbb{C}, +)$ of complex numbers under addition.*

Proof. **(a)** is **Exercise 18I.3** . Verify the following: Ⓔ

(i) The operation $\oplus$ is *commutative*. That is, for any cycles α and β, we have $\alpha \diamond \beta \underset{\mathbb{U}}{\approx} \beta \diamond \alpha$; thus, $[\alpha]_\mathbb{U} \oplus [\beta]_\mathbb{U} = [\beta]_\mathbb{U} \oplus [\alpha]_\mathbb{U}$.

(ii) The operation $\oplus$ is *associative*. That is, for any cycles α, β, and γ, we have $\alpha \diamond (\beta \diamond \gamma) \underset{\mathbb{U}}{\approx} (\alpha \diamond \beta) \diamond \gamma$; thus, $[\alpha]_\mathbb{U} \oplus ([\beta]_\mathbb{U} \oplus [\gamma]_\mathbb{U}) = ([\alpha]_\mathbb{U} \oplus [\beta]_\mathbb{U}) \oplus [\gamma]_\mathbb{U}$.

(iii) The cycle $[\emptyset]_\mathbb{U}$ is an *identity element*. For any cycle α, we have $[\alpha]_\mathbb{U} \oplus [\emptyset]_\mathbb{U} = [\alpha]_\mathbb{U}$.

(iv) For any cycle α, the class $[\overleftarrow{\alpha}]_\mathbb{U}$ is an *additive inverse* for $[\alpha]_\mathbb{U}$. That is: $[\alpha]_\mathbb{U} \oplus [\overleftarrow{\alpha}]_\mathbb{U} = [\emptyset]_\mathbb{U}$.

(b) follows immediately from **(a)**. **(c)** follows from Proposition 18I.4 and Lemma 18C.8(a,b). □

The group $\mathcal{H}^1(\mathbb{U})$ is called the **first homology group** of $\mathbb{U}$. In general, $\mathcal{H}^1(\mathbb{U})$ is a free abelian group of rank R, where R is the number of 'holes' in $\mathbb{U}$. One can similarly define homology groups for any subset $\mathbb{U} \subseteq \mathbb{R}^N$ for any $N \in \mathbb{N}$ (e.g. a surface or a manifold), or even for more abstract spaces. The algebraic properties of the homology groups of $\mathbb{U}$ encode the 'large-scale' topological properties of $\mathbb{U}$ (e.g. the presence of 'holes' or 'twists'). The study of homology groups is one aspect of a vast and beautiful area in mathematics called *algebraic topology*. Surprisingly, the algebraic topology of a differentiable manifold indirectly influences the behaviour of partial differential equations defined on this manifold; this is content of deep results such as the *Atiyah-Singer Index Theorem*. For an elementary introduction to algebraic topology, see [Hen94]. For a comprehensive text, see the beautiful book [Hat02].

VI Fourier transforms on unbounded domains

In Part III, we saw that trigonometric functions like *sin* and *cos* formed orthogonal bases of $\mathbf{L}^2(\mathbb{X})$, where $\mathbb{X}$ was one of several bounded subsets of $\mathbb{R}^D$. Thus, any function in $\mathbf{L}^2(\mathbb{X})$ could be expressed using a *Fourier series*. In Part IV we used these Fourier series to solve initial/boundary value problems on $\mathbb{X}$.

A *Fourier transform* is similar to a Fourier series, except that now $\mathbb{X}$ is an *unbounded* set (e.g. $\mathbb{X} = \mathbb{R}$ or $\mathbb{R}^D$). This introduces considerable technical complications. Nevertheless, the underlying philosophy is the same; we will construct something analogous to an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$, and use this to solve partial differential equations on $\mathbb{X}$.

It is technically convenient (although not strictly necessary) to replace *sin* and *cos* with the complex exponential functions like $\exp(x\mathbf{i}) = \cos(x) + \mathbf{i}\sin(x)$. The material on Fourier series in Part III could have also been developed using these complex exponentials, but in that context, this would have been a needless complication. In the context of Fourier transforms, however, it is actually a simplification.

Chapter 19

Fourier transforms

"There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world." —Nicolai Lobachevsky

19A One-dimensional Fourier transforms

Prerequisites: §0C. **Recommended:** §6C(i), §8D.

Fourier *series* help us to represent functions on a *bounded* domain, like $\mathbb{X} = [0, 1]$ or $\mathbb{X} = [0, 1] \times [0, 1]$. But what if the domain is *unbounded*, like $\mathbb{X} = \mathbb{R}$? Now, instead of using a *discrete* collection of Fourier coefficients like $\{A_0, A_1, B_1, A_2, B_2, \ldots\}$ or $\{\widehat{f}_{-1}, \widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \ldots\}$, we must use a continuously parameterized family.

For every $\mu \in \mathbb{R}$, we define the function $\mathcal{E}_\mu : \mathbb{R} \longrightarrow \mathbb{C}$ by $\mathcal{E}_\mu(x) := \exp(\mu \mathbf{i} x)$. You can visualize this function as a helix which spirals with frequency μ around the unit circle in the complex plane (see Figure 19A.1). Indeed, using Euler's Formula (see page 551), it is not hard to check that $\mathcal{E}_\mu(x) = \cos(\mu x) + \mathbf{i}\sin(\mu x)$ (**Exercise 19A.1**). In other words, the real and imaginary parts of $\mathcal{E}_\mu(x)$ act like a cosine wave and a sine wave, respectively, both of frequency μ. Ⓔ

Heuristically speaking, the (continuously parameterized) family of functions $\{\mathcal{E}_\mu\}_{\mu \in \mathbb{R}}$ acts as a kind of 'orthogonal basis' for a certain space of functions

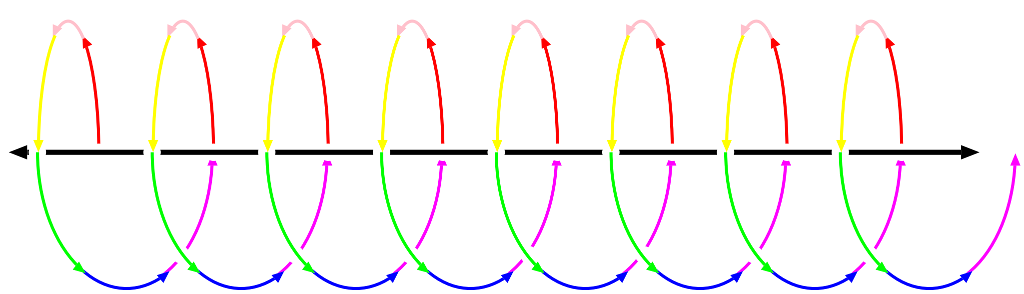


Figure 19A.1: $\mathcal{E}_\mu(x) := \exp(-\mu \cdot x \cdot \mathbf{i})$ as a function of x.

from $\mathbb{R}$ into $\mathbb{C}$ (although making this rigorous is very complicated). This is the motivating idea behind the *Fourier transform.*

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be some function. The **Fourier transform** of f is the function $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\widehat{f}(\mu) \quad := \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\overline{\mathcal{E}_\mu(x)}\ dx \quad = \quad \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cdot \exp(-\mu \cdot x \cdot \mathbf{i})\ dx,}$$

for any $\mu \in \mathbb{R}$. (In other words, $\widehat{f}(\mu) := \frac{1}{2\pi} \langle f, \mathcal{E}_\mu \rangle$, in the notation of § 6C(i) on page 109). Notice that this integral may not converge, in general. We need $f(x)$ to "decay fast enough" as x goes to $\pm\infty$. To be precise, we need f to be an **absolutely integrable** function, meaning that

$$\int_{-\infty}^{\infty} |f(x)|\ dx \ < \ \infty.$$

We indicate this by writing: "$f \in \mathbf{L}^1(\mathbb{R})$".

The Fourier transform $\widehat{f}(\mu)$ plays the same role that the complex Fourier coefficients $\{\ldots \widehat{f}_{-1},\ \widehat{f}_0,\ \widehat{f}_1,\ \widehat{f}_2,\ \ldots\}$ play for a function on an interval (see § 8D on page 172). In particular, we can express $f(x)$ as a sort of generalized "Fourier series". We would like to write something like:

$$\text{“}\ f(x) \quad = \quad \sum_{\mu \in \mathbb{R}} \widehat{f}(\mu)\mathcal{E}_\mu(x).\ \text{”}$$

However, this expression makes no mathematical sense, because you can't *sum* over all real numbers (there are too many). Instead of *summing* over all Fourier coefficients, we must *integrate*. For this to work, we need a technical condition. We say that f is **piecewise smooth** if there is a finite set of points $r_1 < r_2 < \cdots < r_N$ in $\mathbb{R}$ such that f is continuously differentiable on the open intervals $(-\infty, r_1)$, (r_1, r_2), $(r_1, r_2), \ldots, (r_{N-1}, r_N)$, and (r_N, ∞), and furthermore, the left-hand and right-hand limits[1] of f and f' exist at each of the points $r_1, \ldots, r_N$.

Theorem 19A.1. Fourier Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R})$ is piecewise smooth. For any $x \in \mathbb{R}$, if f is continuous at x, then

$$f(x) \quad = \quad \lim_{M\to\infty} \int_{-M}^{M} \widehat{f}(\mu) \cdot \mathcal{E}_\mu(x)\ d\mu \quad = \quad \lim_{M\to\infty} \int_{-M}^{M} \widehat{f}(\mu) \cdot \exp(\mu \cdot x \cdot \mathbf{i})\ d\mu. \tag{19A.1}$$

If f is discontinuous at x, then we have

$$\lim_{M\to\infty} \int_{-M}^{M} \widehat{f}(\mu) \cdot \mathcal{E}_\mu(x)\ d\mu \quad = \quad \frac{1}{2}\left(\lim_{y \searrow x} f(y) \ + \ \lim_{y \nearrow x} f(y) \right).$$

[1] See page 201 for definition.

Proof. See [Wal88, Theorem 5.17, p.244], [Kör88, Theorem 61.1, p.300], or [Fis99, §5.2, p.335-342]. $\square$

It follows that, under mild conditions, a function can be uniquely identified from its Fourier transform:

Proposition 19A.2. *Suppose $f, g \in \mathcal{C}(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ are continuous and integrable. Then* $\left(\widehat{f} = \widehat{g} \right) \iff \left(f = g \right)$.

Proof. "$\Longleftarrow$" is obvious. The proof of "$\Longrightarrow$" is **Exercise 19A.2** (*Hint.* (a) If f and g are piecewise smooth, then show that this follows immediately from Theorem 19A.1.

(b) In the general case (where f and g might not be piecewise smooth), proceed as follows. Let $h \in \mathcal{C}(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$. Suppose $\widehat{h} \equiv 0$; show that we must have $h \equiv 0$. Now let $h := f - g$; then $\widehat{h} = \widehat{f} - \widehat{g} \equiv 0$ (because $\widehat{f} = \widehat{g}$). Thus $h = 0$; thus, $f = g$.) $\square$

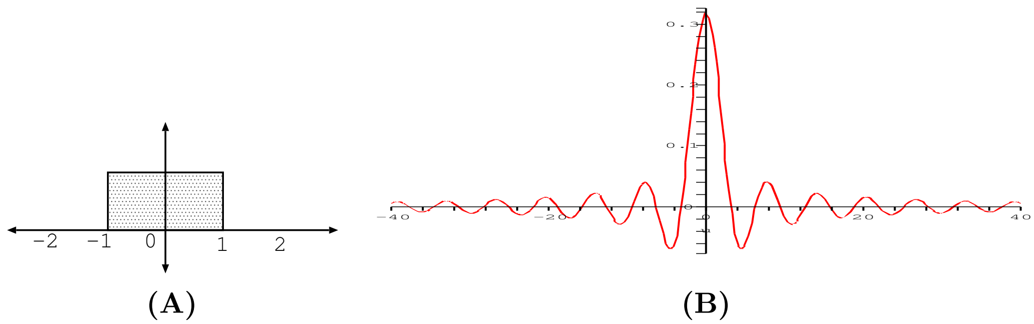


Figure 19A.2: **(A)** Example 19A.3. **(B)** The Fourier transform $\widehat{f}(x) = \frac{\sin(\mu)}{\pi\mu}$ from Example 19A.3.

Example 19A.3. Suppose $f(x) = \begin{cases} 1 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise} \end{cases}$ [see Figure 19A.2(A)]. Then

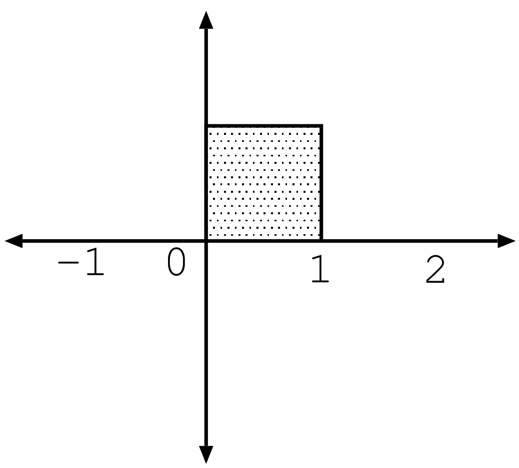
$$\text{For all } \mu \in \mathbb{R}, \quad \widehat{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-\mu\cdot x\cdot \mathbf{i})\, dx = \frac{1}{2\pi}\int_{-1}^{1} \exp(-\mu\cdot x\cdot\mathbf{i})\, dx$$
$$= \frac{1}{-2\pi\mu\mathbf{i}} \exp\Big(-\mu\cdot x\cdot\mathbf{i}\Big)_{x=-1}^{x=1} = \frac{1}{-2\pi\mu\mathbf{i}}\left(e^{-\mu\mathbf{i}} - e^{\mu\mathbf{i}}\right)$$
$$= \frac{1}{\pi\mu}\left(\frac{e^{\mu\mathbf{i}} - e^{-\mu\mathbf{i}}}{2\mathbf{i}}\right) \underset{\text{(Eu)}}{=\!=} \frac{1}{\pi\mu}\sin(\mu) \quad \text{[see Fig.19A.2(B)]}$$

where **(Eu)** is Euler's Formula (see page 551).

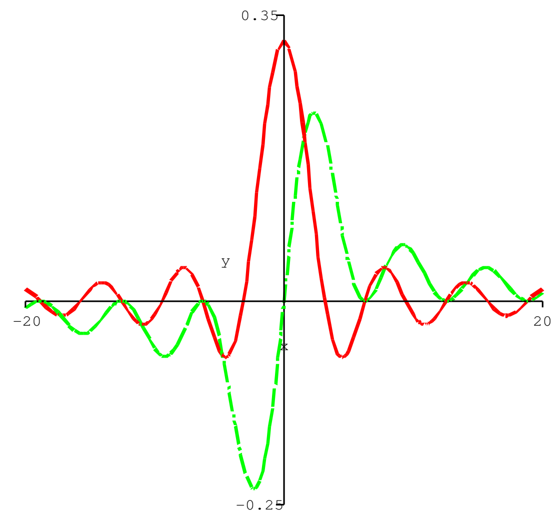
Thus, the Fourier Inversion Formula says, that, if $-1 < x < 1$, then

$$\lim_{M\to\infty} \int_{-M}^{M} \frac{\sin(\mu)}{\pi\mu} \exp(\mu \cdot x \cdot \mathbf{i}) \, d\mu \;=\; 1,$$

while, if $x < -1$ or $x > 1$, then $\lim_{M\to\infty} \int_{-M}^{M} \frac{\sin(\mu)}{\pi\mu} \exp(\mu \cdot x \cdot \mathbf{i}) \, d\mu \;=\; 0$. If $x = \pm 1$, then the Fourier inversion integral will converge to $\frac{1}{2}$. ♢



(A)



(B)

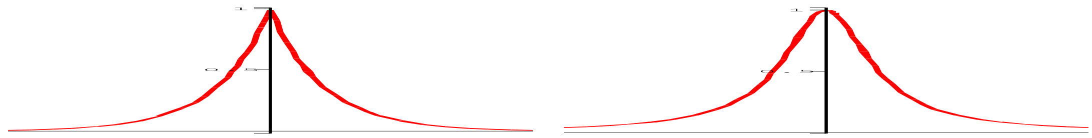
Figure 19A.3: **(A)** Example 19A.4. **(B)** The real and imaginary parts of the Fourier transform $\widehat{f}(x) = \frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$ from Example 19A.4.

Example 19A.4. Suppose $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise} \end{cases}$ [see Figure 19A.3(A)].

Then $\widehat{f}(\mu) = \frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$ [see Figure 19A.3(B)]; the verification of this is practice problem # 1 on page 523 of §19I. Thus, the Fourier inversion formula says, that, if $0 < x < 1$, then

$$\lim_{M\to\infty} \int_{-M}^{M} \frac{1 - e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}} \exp(\mu \cdot x \cdot \mathbf{i}) \, d\mu \;=\; 1,$$

while, if $x < 0$ or $x > 1$, then $\lim_{M\to\infty} \int_{-M}^{M} \frac{1 - e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}} \exp(\mu \cdot x \cdot \mathbf{i}) \, d\mu \;=\; 0$. If $x = 0$ or $x = 1$, then the Fourier inversion integral will converge to $\frac{1}{2}$. ♢



(A) **(B)**

Figure 19A.4: **(A)** The symmetric exponential tail function $f(x) = e^{-\alpha\cdot|x|}$ from Example 19A.7. **(B)** The Fourier transform $\widehat{f}(x) = \frac{a}{\pi(x^2+a^2)}$ of the symmetric exponential tail function from Example 19A.7.

In the Fourier Inversion Formula, it is important that the positive and negative bounds of the integral go to infinity at the same rate in the limit (19A.1). In particular, it is *not* the case that $f(x) = \lim_{N,M\to\infty} \int_{-N}^{M} \widehat{f}(\mu)\exp(\mu\cdot x\cdot \mathbf{i})\ d\mu$; in general, *this* integral may not converge. The reason is this: even if f is absolutely integrable, its Fourier transform $\widehat{f}$ may *not* be. If we assume that $\widehat{f}$ is *also* absolutely integrable, then things become easier.

Theorem 19A.5. Strong Fourier Inversion Formula
Suppose that $f \in \mathbf{L}^1(\mathbb{R})$, and that $\widehat{f}$ is also in $\mathbf{L}^1(\mathbb{R})$. If $x \in \mathbb{R}$, and f is continuous at x, then $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\mu)\cdot\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu$.

Proof. See [Kör88, Theorem 60.1, p.296], [Wal88, Theorem 4.11, p.236], [Fol84, Theorem 8.26, p. 243], or [Kat76, §VI.1.12, p.126]. □

Corollary 19A.6. *Suppose $f \in \mathbf{L}^1(\mathbb{R})$, and there exists some $g \in \mathbf{L}^1(\mathbb{R})$ such that $f = \widehat{g}$. Then $\widehat{f}(\mu) = \frac{1}{2\pi}g(-\mu)$ for all $\mu \in \mathbb{R}$.*

Proof. <u>**Exercise 19A.3**</u> □ Ⓔ

Example 19A.7. Let $\alpha > 0$ be a constant, and suppose $f(x) = e^{-\alpha\cdot|x|}$. [see Figure 19A.4(A)]. Then

$$\begin{aligned}
2\pi\widehat{f}(\mu) &= \int_{-\infty}^{\infty} e^{-\alpha\cdot|x|}\exp(-\mu x\mathbf{i})\ dx \\
&= \int_{0}^{\infty} e^{-\alpha\cdot x}\exp(-\mu x\mathbf{i})\ dx\ +\ \int_{-\infty}^{0} e^{\alpha\cdot x}\exp(-\mu x\mathbf{i})\ dx \\
&= \int_{0}^{\infty} \exp(-\alpha x-\mu x\mathbf{i})\ dx\ +\ \int_{-\infty}^{0} \exp(\alpha x-\mu x\mathbf{i})\ dx
\end{aligned}$$

$$= \frac{1}{-(\alpha+\mu\mathbf{i})} \exp\Big(-(\alpha+\mu\mathbf{i})\cdot x\Big)_{x=0}^{x=\infty} \quad + \quad \frac{1}{\alpha-\mu\mathbf{i}} \exp\Big((\alpha-\mu\mathbf{i})\cdot x\Big)_{x=-\infty}^{x=0}$$

$$\underset{(*)}{=} \frac{-1}{\alpha+\mu\mathbf{i}}(0-1) + \frac{1}{\alpha-\mu\mathbf{i}}(1-0) \quad = \quad \frac{1}{\alpha+\mu\mathbf{i}} + \frac{1}{\alpha-\mu\mathbf{i}} \quad = \quad \frac{\alpha-\mu\mathbf{i}+\alpha+\mu\mathbf{i}}{(\alpha+\mu\mathbf{i})(\alpha-\mu\mathbf{i})}$$

$$= \frac{2\alpha}{\alpha^2+\mu^2}.$$

Thus, we conclude: $\widehat{f}(\mu) \;=\; \dfrac{\alpha}{\pi(\alpha^2+\mu^2)}$. [see Figure 19A.4(B)].

To see equality $(*)$, recall that $\left|\exp\Big(-(\alpha+\mu\mathbf{i})\cdot x\Big)\right| = e^{-\alpha\cdot x}$. Thus,

$$\lim_{\mu\to\infty}\left|\exp\Big(-(\alpha+\mu\mathbf{i})\cdot x\Big)\right| \quad = \quad \lim_{\mu\to\infty} e^{-\alpha\cdot x} \quad = \quad 0.$$

Likewise, $\displaystyle\lim_{\mu\to-\infty}\left|\exp\Big((\alpha-\mu\mathbf{i})\cdot x\Big)\right| \;=\; \lim_{\mu\to-\infty} e^{\alpha\cdot x} \;=\; 0.$ ♢

Example 19A.8. Conversely, suppose $\alpha > 0$, and $g(x) \;=\; \dfrac{1}{(\alpha^2+x^2)}$. Then $\widehat{g}(\mu) \;=\; \dfrac{1}{2\alpha}e^{-\alpha\cdot|\mu|}$; the verification of this is practice problem # 6 on page 523 of §19I. ♢

Remark. Proposition 18H.6 on page 476 provides a powerful technique for computing the Fourier transform of any analytic function $f : \mathbb{R} \longrightarrow \mathbb{C}$, using residue calculus.

19B Properties of the (one-dimensional) Fourier transform

Prerequisites: §19A, §0G.

Theorem 19B.1. Riemann-Lebesgue Lemma
Let $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *The function $\widehat{f}$ is* continuous *and* bounded. *To be precise: If $B := \displaystyle\int_{-\infty}^{\infty} |f(x)|\,dx$, then, for all $\mu \in \mathbb{R}$, we have $\left|\widehat{f}(\mu)\right| < B$.*

(b) *$\widehat{f}$* asymptotically decays *near infinity. That is, $\displaystyle\lim_{\mu\to\pm\infty}\left|\widehat{f}(\mu)\right| = 0$.*

Ⓔ *Proof.* **(a) Exercise 19B.1** *Hint:* Boundedness follows from applying the triangle inequality to the integral defining $\widehat{f}(\mu)$. For continuity, fix $\mu_1, \mu_2 \in \mathbb{R}$, and define $E : \mathbb{R} \longrightarrow \mathbb{R}$ by $E(x) := \exp(-\mu_1 x\mathbf{i}) - \exp(-\mu_2 x\mathbf{i})$. For any $X > 0$, we can write

$$\widehat{f}(\mu_1) - \widehat{f}(\mu_2) \;=\; \frac{1}{2\pi}\Big(\underbrace{\int_{-\infty}^{-X} f(x)\cdot E(x)\ dx}_{(A)} + \underbrace{\int_{-X}^{X} f(x)\cdot E(x)\ dx}_{(B)} + \underbrace{\int_{X}^{\infty} f(x)\cdot E(x)\ dx}_{(C)} \Big).$$

(i) Show that, if X is large enough, then the integrals (A) and (C) can be made arbitrarily small, independent of the values of μ_1 and μ_2. (*Hint.* Recall that $f \in \mathbf{L}^1(\mathbb{R})$. Observe that $|E(x)| \leq 2$ for all $x \in \mathbb{R}$.)

(ii) Fix $X > 0$. Show that, if μ_1 and μ_2 are close enough, then integral (B) can also be made arbitrarily small (*Hint:* if μ_1 and μ_2 are 'close', then $|E(x)|$ is 'small' for all $x \in \mathbb{R}$.)

(iii) Show that, if μ_1 and μ_2 are close enough, then $\left|\widehat{f}(\mu_1) - \widehat{f}(\mu_2)\right|$ can be made arbitrarily small. (*Hint:* Combine (i) and (ii), using the triangle inequality). Hence, $\widehat{f}$ is continuous.

(b) (if f is continuous) **Exercise 19B.2** *Hint.* For any $X > 0$, we can write Ⓔ

$$\widehat{f}(\mu) \;=\; \frac{1}{2\pi}\Big(\underbrace{\int_{-\infty}^{-X} f(x)\cdot \mathcal{E}_\mu(x)\ dx}_{(A)} + \underbrace{\int_{-X}^{X} f(x)\cdot \mathcal{E}_\mu(x)\ dx}_{(B)} + \underbrace{\int_{X}^{\infty} f(x)\cdot \mathcal{E}_\mu(x)\ dx}_{(C)} \Big).$$

(i) Show that, if X is large enough, then the integrals (A) and (C) can be made arbitrarily small, independent of the value of μ. (*Hint.* Recall that $f \in \mathbf{L}^1(\mathbb{R})$. Observe that $|\mathcal{E}_\mu(x)| = 1$ for all $x \in \mathbb{R}$.)

(ii) Fix $X > 0$. Show that, if μ is large enough, then integral (B) can also be made arbitrarily small (*Hint:* f is uniformly continuous on the interval $[-X, X]$ (why?). Thus, for any $\epsilon > 0$, there is some M such that, for all $\mu > M$, and all $x \in [-X, X]$, we have $|f(x) - f(x + \pi/\mu)| < \epsilon$. But $\mathcal{E}_\mu(x + \pi/\mu) = -\mathcal{E}_\mu(x)$).

(iii) Show that, if μ is large enough, then $\left|\widehat{f}(\mu)\right|$ can be made arbitrarily small. (*Hint:* Combine (i) and (ii), using the triangle inequality).

For the proof of **(b)** when f is an arbitrary (discontinuous) element of $\mathbf{L}^1(\mathbb{R})$, see [Fol84, Theorem 8.22(f), p.241] or [Fis99, Exercise 15, §5.2, p.343] or [Kat76, Theorem 1.7, p.123]. □

Recall that, if $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions, then their **convolution** is the function $(f * g) : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$(f * g)(x) \quad := \quad \int_{-\infty}^{\infty} f(y)\cdot g(x-y)\ dy.$$

(see § 17A on page 375 for a discussion of convolutions). Similarly, if f has Fourier transform $\widehat{f}$ and g has Fourier transform $\widehat{g}$, we can convolve $\widehat{f}$ and $\widehat{g}$ to get a function $(\widehat{f} * \widehat{g}) : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$(\widehat{f} * \widehat{g})(\mu) \quad := \quad \int_{-\infty}^{\infty} \widehat{f}(\nu) \cdot \widehat{g}(\mu - \nu) \; d\nu.$$

Theorem 19B.2. Algebraic Properties of the Fourier Transform
Suppose $f, g \in \mathbf{L}^1(\mathbb{R})$ are two functions.

(a) *If $h := f + g$, then for all $\mu \in \mathbb{R}$, we have $\widehat{h}(\mu) = \widehat{f}(\mu) + \widehat{g}(\mu)$.*

(b) *If $h := f * g$, then for all $\mu \in \mathbb{R}$, we have $\widehat{h}(\mu) = 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{g}(\mu)$.*

(c) *Conversely, suppose $h := f \cdot g$. If $\widehat{f}, \widehat{g}$ and $\widehat{h}$ are in $\mathbf{L}^1(\mathbb{R})$, then for all $\mu \in \mathbb{R}$, we have $\widehat{h}(\mu) \; = \; (\widehat{f} * \widehat{g})(\mu)$.*

Proof. See practice problems #11 to # 13 on page 524. □

This theorem allows us to compute the Fourier transform of a complicated function by breaking it into a sum/product of simpler pieces.

Theorem 19B.3. Translation and Phase Shift

Suppose $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *If $\tau \in \mathbb{R}$ is fixed, and $g \in \mathbf{L}^1(\mathbb{R})$ is defined by: $g(x) := f(x + \tau)$, then for all $\mu \in \mathbb{R}$, we have $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}} \cdot \widehat{f}(\mu)$.*

(b) *Conversely, if $\nu \in \mathbb{R}$ is fixed, and $g \in \mathbf{L}^1(\mathbb{R})$ is defined: $g(x) := e^{\nu x \mathbf{i}} f(x)$, then for all $\mu \in \mathbb{R}$, we have $\widehat{g}(\mu) = \widehat{f}(\mu - \nu)$.*

Proof. See practice problems #14 and # 15 on page 524. □

Thus, *translating* a function by τ in physical space corresponds to *phase-shifting* its Fourier transform by $e^{\tau\mu\mathbf{i}}$, and vice versa. This means that, via a suitable translation, we can put the "center" of our coordinate system wherever it is most convenient to do so.

Example 19B.4. Suppose $g(x) \; = \; \begin{cases} 1 & \text{if } \; -1-\tau < x < 1-\tau; \\ 0 & \text{otherwise} \end{cases}$. Thus, $g(x) = f(x + \tau)$, where $f(x)$ is as in Example 19A.3 on page 489. We know that $\widehat{f}(\mu) \; = \; \dfrac{\sin(\mu)}{\pi\mu}$; thus, it follows from Theorem 19B.3 that $\widehat{g}(\mu) \; = \; e^{\tau\mu\mathbf{i}} \cdot \dfrac{\sin(\mu)}{\pi\mu}$. ♢

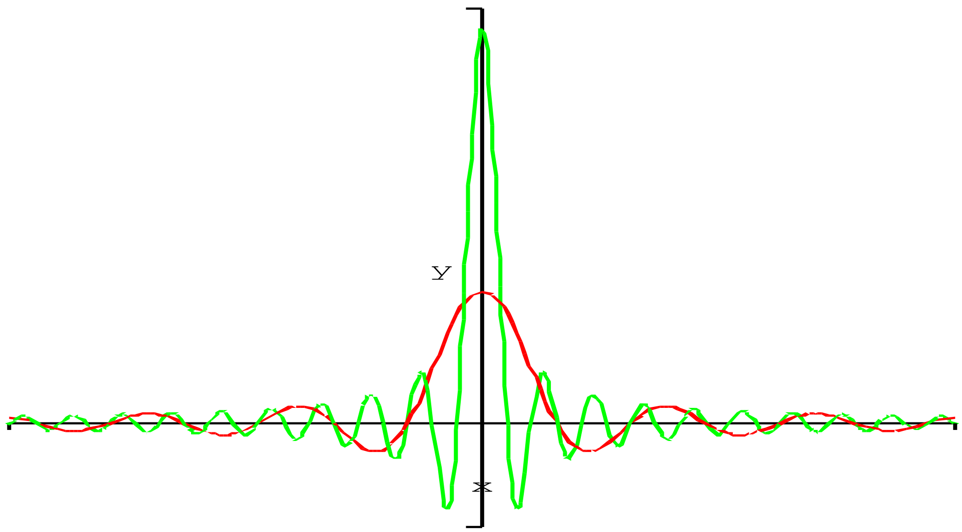


Figure 19B.1: Plot of $\widehat{f}$ (black) and $\widehat{g}$ (grey) in Example 19B.6, where $g(x) = f(x/3)$.

Theorem 19B.5. Rescaling Relation
Suppose $f \in \mathbf{L}^1(\mathbb{R})$. If $\alpha > 0$ is fixed, and g is defined by: $g(x) = f\left(\frac{x}{\alpha}\right)$, then for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \alpha \cdot \widehat{f}(\alpha \cdot \mu)$.

Proof. See practice problem # 16 on page 525. □

In Theorem 19B.5, the function g is the same as function f, but expressed in a coordinate system "rescaled" by a factor of α.

Example 19B.6. Suppose $g(x) = \begin{cases} 1 & \text{if } -3 < x < 3; \\ 0 & \text{otherwise} \end{cases}$. Thus, $g(x) = f(x/3)$, where $f(x)$ is as in Example 19A.3 on page 489. We know that $\widehat{f}(\mu) = \dfrac{\sin(\mu)}{\mu\pi}$; thus, it follows from Theorem 19B.5 that $\widehat{g}(\mu) = 3 \cdot \dfrac{\sin(3\mu)}{3\mu\pi} = \dfrac{\sin(3\mu)}{\mu\pi}$. See Figure 19B.1. ♢

A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is **continuously differentiable** if $f'(x)$ exists for all $x \in \mathbb{R}$, and the function $f' : \mathbb{R} \longrightarrow \mathbb{C}$ is itself continuous. Let $\mathcal{C}^1(\mathbb{R})$ be the set of all continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$. For any $n \in \mathbb{N}$ let $f^{(n)}(x) := \frac{d^n}{dx^n} f(x)$. The function f is n **times continuously differentiable** if $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$, and the function $f^{(n)} : \mathbb{R} \longrightarrow \mathbb{C}$ is itself continuous. Let $\mathcal{C}^n(\mathbb{R})$ be the set of all n-times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$.

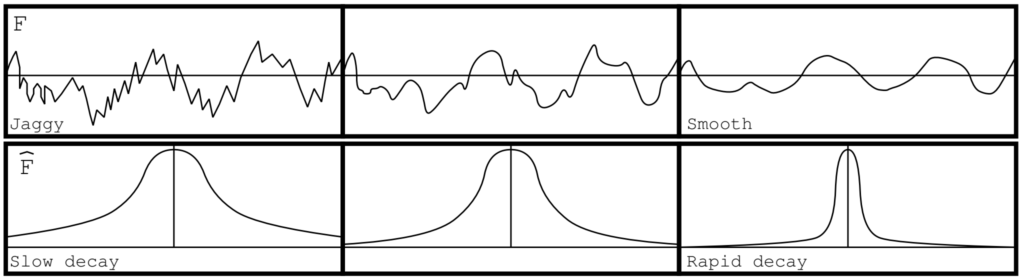


Figure 19B.2: Smoothness vs. asymptotic decay in the Fourier Transform.

Theorem 19B.7. Differentiation and Multiplication
Suppose $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *Suppose $f \in \mathcal{C}^1(\mathbb{R})$ and $\lim_{x\to\pm\infty} |f(x)| = 0$. Let $g(x) := f'(x)$. If $g \in \mathbf{L}^1(\mathbb{R})$, then for all $\mu \in \mathbb{R}$, we have $\widehat{g}(\mu) = \mathbf{i}\mu \cdot \widehat{f}(\mu)$.*

(b) *More generally, suppose $f \in \mathcal{C}^n(\mathbb{R})$ and $\lim_{x\to\pm\infty} |f^{(n-1)}(x)| = 0$. Let $g(x) := f^{(n)}(x)$. If $g \in \mathbf{L}^1(\mathbb{R})$, then for all $\mu \in \mathbb{R}$, we have $\widehat{g}(\mu) \;=\; (\mathbf{i}\mu)^n \cdot \widehat{f}(\mu)$.*

Thus, $\widehat{f}(\mu)$ asymptotically decays faster than $\dfrac{1}{\mu^n}$ as $\mu \to \pm\infty$. That is, $\lim_{\mu\to\pm\infty} \mu^n \widehat{f}(\mu) \;=\; 0$.

(c) *Conversely, let $g(x) := x^n \cdot f(x)$, and suppose that f decays "quickly enough" that g is also in $\mathbf{L}^1(\mathbb{R})$* [for example, this happens if $\lim_{x\to\pm\infty} x^{n+1} f(x) = 0$]. *Then the function $\widehat{f}$ is n times differentiable, and, for all $\mu \in \mathbb{R}$,*

$$\widehat{g}(\mu) \quad = \quad \mathbf{i}^n \cdot \frac{d^n}{d\mu^n} \widehat{f}(\mu).$$

Proof. **(a)** is practice problem # 17 on page 525 of §19I.

(b) is just the result of iterating **(a)** n times.

Ⓔ **(c)** is **Exercise 19B.3** (Hint: *either* 'reverse' the result of **(a)** using the Fourier Inversion Formula (Theorem 19A.1 on page 488), *or* use Proposition 0G.1 on page 567 to directly differentiate the integral defining $\widehat{f}(\mu)$.) □

This theorem says that the Fourier transform converts *differentiation-by-x* into *multiplication-by-μ***i**. This implies that the *smoothness* of a function f is closely related to the *asymptotic decay rate* of its Fourier transform. The "smoother" f is (i.e. the more times we can differentiate it), the more *rapidly* $\widehat{f}(\mu)$ decays as $\mu \to \infty$ (see Figure 19B.2).

Physically, we can interpret this as follows. If we think of f as a "signal", then $\widehat{f}(\mu)$ is the amount of "energy" at the "frequency" μ in the spectral decomposition of this signal. Thus, the magnitude of $\widehat{f}(\mu)$ for extremely large μ is the amount of "very high frequency" energy in f, which corresponds to very finely featured, "jaggy" structure in the shape of f. If f is "smooth", then we expect there will be very little of this "jagginess"; hence the high frequency part of the energy spectrum will be very small.

Conversely, the asymptotic decay rate of f determines the smoothness of its Fourier transform. This makes sense, because the Fourier inversion formula can be (loosely) intepreted as saying that f is itself a sort of "backwards" Fourier transform of $\widehat{f}$.

One very important Fourier transform is the following:

Theorem 19B.8. Fourier Transform of a Gaussian

(a) *If* $f(x) = \exp\left(-x^2\right)$, *then* $\widehat{f}(\mu) \;=\; \frac{1}{2\sqrt{\pi}} \cdot f\left(\frac{\mu}{2}\right) \;=\; \frac{1}{2\sqrt{\pi}} \cdot \exp\left(\frac{-\mu^2}{4}\right)$.

(b) *Fix* $\sigma > 0$. *If* $f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$ *is a Gaussian probability distribution with mean* 0 *and variance* σ^2, *then*

$$\widehat{f}(\mu) \quad = \quad \frac{1}{2\pi} \exp\left(\frac{-\sigma^2\mu^2}{2}\right).$$

(c) *Fix* $\sigma > 0$ *and* $\tau \in \mathbb{R}$. *If* $f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-|x-\tau|^2}{2\sigma^2}\right)$ *is a Gaussian probability distribution with mean* τ *and variance* σ^2, *then*

$$\widehat{f}(\mu) \quad = \quad \frac{e^{-\mathbf{i}\tau\mu}}{2\pi} \exp\left(\frac{-\sigma^2\mu^2}{2}\right).$$

Proof. We'll start with part **(a)**. Let $g(x) = f'(x)$. Then by Theorem 19B.7**(a)**,

$$\widehat{g}(\mu) \quad = \quad \mathbf{i}\mu \cdot \widehat{f}(\mu). \tag{19B.1}$$

However direct computation says $g(x) = -2x \cdot f(x)$, so $\frac{-1}{2} g(x) = x \cdot f(x)$, so Theorem 19B.7**(c)** implies

$$\frac{\mathbf{i}}{2}\widehat{g}(\mu) \quad = \quad (\widehat{f})'(\mu). \tag{19B.2}$$

Combining (19B.2) with (19B.1), we conclude:

$$(\widehat{f})'(\mu) \;\underset{\text{(19B.2)}}{=\!=}\; \frac{\mathbf{i}}{2}\widehat{g}(\mu) \;\underset{\text{(19B.1)}}{=\!=}\; \frac{\mathbf{i}}{2}\cdot\mathbf{i}\mu\cdot\widehat{f}(\mu) \;=\; \frac{-\mu}{2}\widehat{f}(\mu). \qquad (19B.3)$$

Define $h(\mu) = \widehat{f}(\mu)\cdot\exp\left(\frac{\mu^2}{4}\right)$. If we differentiate $h(\mu)$, we get:

$$h'(\mu) \;\underset{\text{(dL)}}{=\!=}\; \widehat{f}(\mu)\cdot\frac{\mu}{2}\exp\left(\frac{\mu^2}{4}\right) \;\underbrace{-\;\frac{\mu}{2}\widehat{f}(\mu)}_{(*)}\cdot\exp\left(\frac{\mu^2}{4}\right) \;=\; 0.$$

Here, **(dL)** is differentiating using the Leibniz rule, and $(*)$ is by eqn.(19B.3). In other words, $h(\mu) = H$ is a constant. Thus,

$$\widehat{f}(\mu) \;=\; \frac{h(\mu)}{\exp(\mu^2/4)} \;=\; H\cdot\exp\left(\frac{-\mu^2}{4}\right) \;=\; H\cdot f\left(\frac{\mu}{2}\right).$$

To evaluate H, set $\mu = 0$, to get

$$H \;=\; H\cdot\exp\left(\frac{-0^2}{4}\right) \;=\; \widehat{f}(0) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,dx \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-x^2\right) \;=\; \frac{1}{2\sqrt{\pi}}.$$

Ⓔ (where the last step is **Exercise 19B.4**). Thus, we conclude: $\widehat{f}(\mu) = \frac{1}{2\sqrt{\pi}}\cdot f\left(\frac{\mu}{2}\right)$.

Part **(b)** follows by setting $\alpha := \sqrt{2}\,\sigma$ in Theorem 19B.5 on page 495.

Ⓔ Part **(c)** is **Exercise 19B.5** (*Hint:* Apply Theorem 19B.3 on page 494). □

Loosely speaking, Theorem 19B.8 says, "The Fourier transform of a Gaussian is another Gaussian"[2]. However, notice that, in Part **(b)** of the theorem, as the **variance** of the Gaussian (that is, σ^2) gets bigger, the "variance" of it's Fourier transform (which is effectively $\frac{1}{\sigma^2}$) gets *smaller* (see Figure 19B.3). If we think of the Gaussian as the probability distribution of some unknown piece of information, then the variance measures the degree of "uncertainty". Hence, we conclude: the greater the uncertainty embodied in the Gaussian f, the *less* the uncertainty embodied in $\widehat{f}$, and vice versa. This is a manifestation of the so-called *Heisenberg Uncertainty Principle* (see Theorem 19G.2 on 513).

[2]This is only *loosely* speaking, however, because a proper Gaussian contains the multiplier "$\frac{1}{\sigma\sqrt{2\pi}}$" to make it a probability distribution, whereas the Fourier transform does not.

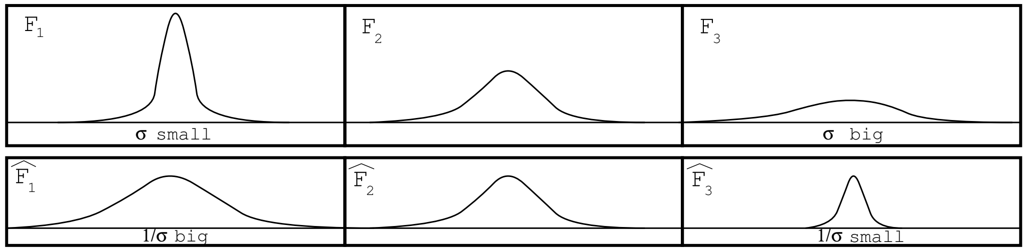


Figure 19B.3: The Uncertainty Principle.

Proposition 19B.9. Inversion and Conjugation
For any $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of z. Let $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *For all $\mu \in \mathbb{R}$, we have $\widehat{\overline{f}}(\mu) = \overline{\widehat{f}(-\mu)}$. In particular,*
$$\Big(f \text{ purely real-valued} \Big) \iff \Big(\text{for all } \mu \in \mathbb{R}, \text{ we have } \widehat{f}(-\mu) = \overline{\widehat{f}(\mu)} \Big).$$

(b) *Suppose $g(x) = f(-x)$ for all $x \in \mathbb{R}$. Then for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \widehat{f}(-\mu)$.*
In particular, if f purely real-valued, then $\widehat{g}(\mu) = \overline{\widehat{f}(\mu)}$. for all $\mu \in \mathbb{R}$.

(c) *If f is real-valued and **even*** (i.e. $f(-x) = f(x)$), *then $\widehat{f}$ is purely real-valued.*

(d) *If f is real-valued and **odd*** (i.e. $f(-x) = -f(x)$), *then $\widehat{f}$ is purely imaginary-valued.*

Proof. **Exercise 19B.6** □ Ⓔ

Example 19B.10: Autocorrelation and Power Spectrum

If $f : \mathbb{R} \longrightarrow \mathbb{R}$, then the **autocorrelation function** of f is defined by
$$\mathbf{A}f(x) \quad := \quad \int_{-\infty}^{\infty} f(y) \cdot f(x+y)\, dy.$$

Heuristically, if we think of $f(x)$ as a "random signal", then $\mathbf{A}f(x)$ measures the degree of correlation in the signal across time intervals of length x —i.e. it provides a crude measure of how well you can predict the value of $f(y+x)$ given information about $f(x)$. In particular, if f has some sort of "T-periodic" component, then we expect $\mathbf{A}f(x)$ to be large when $x = nT$ for any $n \in \mathbb{Z}$.

If we define $g(x) = f(-x)$, then we can see that $\mathbf{A}f(x) = (f * g)(-x)$ (**Exercise 19B.7**). Thus, Ⓔ

$$\widehat{\mathbf{A}f}(\mu) \underset{(*)}{=} \overline{\widehat{f * g}\ (\mu)} \underset{(\dagger)}{=} \overline{\widehat{f}(\mu) \cdot \widehat{g}(\mu)}$$
$$\underset{(*)}{=} \overline{\widehat{f}(\mu) \cdot \overline{\widehat{f}(\mu)}} \quad = \quad \overline{\widehat{f}(\mu)} \cdot \widehat{f}(\mu) \quad = \quad \left|\widehat{f}(\mu)\right|^2.$$

Here, both $(*)$ are by Proposition 19B.9(b), while $(\dagger)$ is by Theorem 19B.2**(b)**. The function $\left|\widehat{f}(\mu)\right|^2$ measures the absolute magnitude of the Fourier transform of $\widehat{f}$, and is sometimes called the **power spectrum** of $\widehat{f}$. ♢

Evil twins of the Fourier transform. Unfortunately, the mathematics literature contains at least four different definitions of the Fourier transform. In this book, the Fourier transform and its inversion are defined with the integrals

$$\widehat{f}(\mu) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-\mathbf{i}\,x\mu)\ d\mu \quad \text{and} \quad f(x) = \int_{-\infty}^{\infty} \widehat{f}(x)\exp(\mathbf{i}\,x\mu)\ d\mu.$$

Some books (e.g. [Kat76, Kör88, Fis99, Hab87]) instead use what we will call the *opposite Fourier transform*:

$$\check{f}(\mu) \quad := \quad \int_{-\infty}^{\infty} f(x)\exp(-\mathbf{i}\,x\mu)\ dx,$$

with inverse transform

$$f(x) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \check{f}(\mu)\exp(\mathbf{i}\,x\mu)\ d\mu.$$

Other books (e.g. [Asm05]) instead use what we will call the *symmetric Fourier transform*:

$$\widehat{f}(\mu) \quad := \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp(-\mathbf{i}\,x\mu)\ dx,$$

with inverse transform

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\mu)\exp(\mathbf{i}\,x\mu)\ d\mu.$$

Finally, some books (e.g. [Fol84, Wal88]) use what we will call the *canonical Fourier transform*:

$$\widetilde{f}(\mu) \quad := \quad \int_{-\infty}^{\infty} f(x)\exp(-2\pi\mathbf{i}\,x\mu)\ dx,$$

with inverse transform

$$f(x) \quad = \quad \int_{-\infty}^{\infty} \widetilde{f}(\mu)\exp(2\pi\mathbf{i}\,x\mu)\ d\mu.$$

All books use the symbol "$\widehat{f}$" to denote the Fourier transform of f —we are using four different 'accents' simply to avoid confusing the four definitions. It is easy to translate the Fourier transform in this book into its evil twins. For any $f \in \mathbf{L}^1(\mathbb{R})$ and any $\mu \in \mathbb{R}$, we have:

$$\begin{aligned}
\check{f}(\mu) &= 2\pi\,\widehat{f}(\mu) &\text{and}\quad \widehat{f}(\mu) &= \frac{1}{2\pi}\,\check{f}(\mu);\\
\widehat{f}(\mu) &= \sqrt{2\pi}\,\widehat{f}(\mu) &\text{and}\quad \widehat{f}(\mu) &= \frac{1}{\sqrt{2\pi}}\,\widehat{f}(\mu);\\
\widetilde{f}(\mu) &= 2\pi\,\widehat{f}(2\pi\,\mu) &\text{and}\quad \widehat{f}(\mu) &= \frac{1}{2\pi}\,\widetilde{f}\left(\frac{\mu}{2\pi}\right).
\end{aligned} \tag{19B.4}$$

(**Exercise 19B.8** Check this.) All of the formulae and theorems we have derived in this section are still true under these alternate definitions, except that one must multiply or divide by 2π or $\sqrt{2\pi}$ at certain key points, and replace $e^{\mathbf{i}}$ with $e^{2\pi\mathbf{i}}$ (or vice versa) at others. Ⓔ

Exercise 19B.9. (Annoying) Use the identities (19B.4) to reformulate all the formulae and theorems in this chapter in terms of (a) The opposite Fourier transform $\check{f}$; or (b) The symmetric Fourier transform $\widehat{f}$; or (c) The canonical Fourier transform $\widetilde{f}$. ♦ Ⓔ

Each of the four definitions has advantages and disadvantages; some formulae become simpler, others become more complex. Clearly, both the 'symmetric' and 'canonical' versions of the Fourier transform have some appeal because the Fourier transform and its inverse have 'symmetrical' formulae using these definitions. Furthermore, in both of these versions, the '2π' factor disappears from Parseval's and Plancherel's Theorems (see §19C below) —in other words, the Fourier transform becomes an *isometry* of $\mathbf{L}^2(\mathbb{R})$. The symmetric Fourier transform has the added advantage that it maps a Gaussian distribution into another Gaussian (no scalar multiplication required). The canonical Fourier transform has the added advantage that $\widetilde{f * g} = \widetilde{f} \cdot \widetilde{g}$ (without the 2π factor required in Theorem 19B.2(a)), while simultaneously, $\widetilde{f \cdot g} = \widetilde{f} * \widetilde{g}$ (unlike the symmetric Fourier transform).

The definition used in this book (and also in [Pin98, CB87, Pow99, Bro89, McW72], among others) has none of these advantages. Its major advantage is that it will yield simpler expressions for the abstract solutions to partial differential equations in Chapter 20. If one uses the 'symmetric' Fourier transform, then every one of the solution formulae in Chapter 20 must be multiplied by some power of $\frac{1}{\sqrt{2\pi}}$. If one uses the 'canonical' Fourier transform, then every spacetime variable (i.e. x, y, z, t) in every formula must by multiplied by 2π or sometimes by $4\pi^2$, which makes all the formulae look much more complicated.[3]

[3] Of course, when you actually apply these formulae to solve specific problem, you will end

We end with a warning. When comparing or combining formulae from two or more books, make sure to first *compare their definitions of the Fourier transform*, and make the appropriate conversions using formulae (19B.4), if necessary.

Further reading. Almost any book on PDEs contains a discussion of Fourier transforms, but for greater depth (and rigour) it is better to seek a text dedicated to Fourier analysis. Good introductions to Fourier transforms and their applications can be found in [Wal88, Chapter 6-7] and [Kör88, Part IV]. (In addition to a lot of serious mathematical content, Körner's book contains interesting and wide-ranging discussions about the history of Fourier theory and its many scientific applications, and is written in a delightfully informal style).

19C* Parseval and Plancherel

Prerequisites: §19A. **Recommended:** §6C(i), §6F.

Let $\mathbf{L}^2(\mathbb{R})$ be the set of all *square-integrable* complex-valued functions on $\mathbb{R}$ —that is, all integrable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\|f\|_2 < \infty$, where we define

$$\|f\|_2 \quad := \quad \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx\right)^{1/2} \qquad \text{(see §6C(i) for more information).}$$

Note that $\mathbf{L}^2(\mathbb{R})$ is neither a subset nor a superset of $\mathbf{L}^1(\mathbb{R})$; however, the two spaces do overlap. If $f, g \in \mathbf{L}^2(\mathbb{R})$, then we define

$$\langle f, g \rangle \quad := \quad \int_{-\infty}^{\infty} f(x)\, \overline{g}(x) \, dx.$$

The following identity is useful in many applications of Fourier theory, especially quantum mechanics. It can be seen as the 'continuum' analog of Parseval's equality for an orthonormal basis (Theorem 6F.1 on page 132).

Theorem 19C.1. Parseval's Equality for Fourier Transforms
If $f, g \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$, *then* $\langle f, g \rangle = 2\pi \left\langle \widehat{f}, \widehat{g} \right\rangle$.

Proof. Define $h : \mathbb{R} \longrightarrow \mathbb{R}$ by $h(x) := f(x)\overline{g}(x)$. Then $h \in \mathbf{L}^1(\mathbb{R})$ because $f, g \in \mathbf{L}^2(\mathbb{R})$. We have

$$\begin{aligned}\widehat{h}(0) \quad &= \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cdot \overline{g}(x) \cdot \exp(-\mathbf{i}0x) \, dx \\ &\underset{(*)}{=} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cdot \overline{g}(x) \;\; dx \quad = \quad \frac{\langle f, g \rangle}{2\pi},\end{aligned} \tag{19C.1}$$

up with exactly the same solution no matter which version of the Fourier transform you use —why?

where $(*)$ is because $\exp(-\mathbf{i}0x) = \exp(0) = 1$ for all $x \in \mathbb{R}$. But we also have

$$\widehat{h}(0) \underset{(*)}{=} \widehat{f} * \widehat{\overline{g}}(0) = \int_{-\infty}^{\infty} \widehat{f}(\nu) \cdot \widehat{\overline{g}}(-\nu)\, d\nu \underset{(\dagger)}{=} \int_{-\infty}^{\infty} \widehat{f}(\nu) \cdot \overline{\widehat{g}(\nu)}\, d\nu = \left\langle \widehat{f}, \widehat{g} \right\rangle. \quad (19C.2)$$

Here, $(*)$ is by Theorem 19B.2(c), because $h = f \cdot \overline{g}$ so $\widehat{h} = \widehat{f} * \widehat{\overline{g}}$. Meanwhile, $(\dagger)$ is by Proposition 19B.9(a).

Combining (19C.1) and (19C.2) yields $\left\langle \widehat{f}, \widehat{g} \right\rangle = \widehat{h}(0) = \langle f, g\rangle / 2\pi$. The result follows. □

Corollary 19C.2. Plancherel's Theorem

Suppose $f \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$. *Then* $\widehat{f} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ *also, and* $\|f\|_2 = \sqrt{2\pi}\left\|\widehat{f}\right\|_2$.

Proof. Set $f = g$ in the Parseval equality. Recall that $\|f\|_2 = \sqrt{\langle f, f\rangle}$. □

In fact, the Plancherel Theorem says much more than this. Define the linear operator $\mathsf{F}_1 : \mathbf{L}^1(\mathbb{R}) \longrightarrow \mathbf{L}^1(\mathbb{R})$ by $\mathsf{F}_1(f) := \sqrt{2\pi}\,\widehat{f}$ for all $f \in \mathbf{L}^1(\mathbb{R})$; then the full Plancherel Theorem says that F_1 extends uniquely to a *unitary isomorphism* $\mathsf{F}_2 : \mathbf{L}^2(\mathbb{R}) \longrightarrow \mathbf{L}^2(\mathbb{R})$ —that is, a bijective linear transformation from $\mathbf{L}^2(\mathbb{R})$ to itself such that $\|\mathsf{F}_2(f)\|_2 = \|f\|_2$ for all $f \in \mathbf{L}^2(\mathbb{R})$. For any $p \in [1, \infty)$, let $\mathbf{L}^p(\mathbb{R})$ be the set of all integrable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}.$$

For any $p \in [1, 2]$, let $\widehat{p} \in [2, \infty]$ be the unique number such that $\frac{1}{p} + \frac{1}{\widehat{p}} = 1$ (for example, if $p = 3/2$, then $\widehat{p} = 3$). Then, through a process called *Riesz-Thorin interpolation*, it is possible to extend the Fourier transform even further, to get a linear transformation $\mathsf{F}_p : \mathbf{L}^p(\mathbb{R}) \longrightarrow \mathbf{L}^{\widehat{p}}(\mathbb{R})$. For example, one can define a Fourier transform $\mathsf{F}_{3/2} : \mathbf{L}^{3/2}(\mathbb{R}) \longrightarrow \mathbf{L}^3(\mathbb{R})$. All these transformations agree on the overlaps of their domains, and satisfy the *Hausdorff-Young inequality:*

$$\|\mathsf{F}_p(f)\|_{\widehat{p}} \leq \|f\|_p, \qquad \text{for any } p \in [1, 2] \text{ and } f \in \mathbf{L}^p(\mathbb{R}).$$

However, the details are well beyond the scope of this text. For more information, see [Fol84, Chapter 8] or [Kat76, Chapter VI].

19D Two-dimensional Fourier transforms

Prerequisites: §19A. **Recommended:** §9A.

Let $\mathbf{L}^1(\mathbb{R}^2)$ be the set of all functions $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ which are **absolutely integrable** on $\mathbb{R}^2$, meaning that

$$\int_{\mathbb{R}^2} |f(x,y)|\ dx\ dy \ <\ \infty.$$

If $f \in \mathbf{L}^1(\mathbb{R}^2)$, then the **Fourier transform** of f is the function $\widehat{f} : \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined:

$$\widehat{f}(\mu,\nu) \quad := \quad \boxed{\frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(x,y)\cdot \exp\Big(-(\mu x+\nu y)\cdot \mathbf{i}\Big)\ dx\ dy,}$$

for all $(\mu,\nu) \in \mathbb{R}^2$.

Theorem 19D.1. Strong Fourier Inversion Formula
Suppose that $f \in \mathbf{L}^1(\mathbb{R}^2)$, and that $\widehat{f}$ is also in $\mathbf{L}^1(\mathbb{R}^2)$. For any $(x,y) \in \mathbb{R}^2$, if f is continuous at (x,y), then

$$f(x,y) \quad = \quad \int_{\mathbb{R}^2} \widehat{f}(\mu,\nu)\cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu.$$

Proof. [Fol84, Theorem 8.26, p. 243] or [Kat76, §VI.1.12, p.126]. □

Unfortunately, not all the functions one encounters have the property that their Fourier transform is in $\mathbf{L}^1(\mathbb{R}^2)$. In particular, $\widehat{f} \in \mathbf{L}^1(\mathbb{R}^2)$ only if f agrees 'almost everywhere' with a continuous function (thus, Theorem 19D.1 is inapplicable to step functions, for example). We want a result analogous to the 'weak' Fourier Inversion Theorem 19A.1 on page 488. It is surprisingly difficult to find clean, simple 'inversion theorems' of this nature for multidimensional Fourier transforms. The result given here is far from the most general one in this category, but it has the advantage of being easy to state and prove. First, we must define an appropriate class of functions. Let $\widehat{\mathbf{L}}^1(\mathbb{R}^2) := \Big\{f \in \mathbf{L}^1(\mathbb{R}^2)\ ;\ \widehat{f} \in \mathbf{L}^1(\mathbb{R}^2)\Big\}$; this is the class considered by Theorem 19D.1. Let $\widetilde{\mathbf{L}}^1(\mathbb{R})$ be the set of all *piecewise smooth* functions in $\mathbf{L}^1(\mathbb{R})$ (the class considered by Theorem 19A.1). Let $\mathcal{F}(\mathbb{R}^2)$ be the set of all functions $f \in \mathbf{L}^1(\mathbb{R}^2)$ such that there exist $f_1, f_2 \in \widetilde{\mathbf{L}}^1(\mathbb{R})$ with $f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$ for all $(x_1,x_2) \in \mathbb{R}^2$. Let $\mathcal{H}(\mathbb{R}^2)$ denote the set of all functions in $\mathbf{L}^1(\mathbb{R}^2)$ which can be written as a *finite sum* of elements in $\mathcal{F}(\mathbb{R}^2)$. Finally, we define

$$\widetilde{\mathbf{L}}^1(\mathbb{R}^2) \quad := \quad \Big\{f \in \mathbf{L}^1(\mathbb{R}^2)\ ;\ f \ =\ g+h \text{ for some } g \in \widehat{\mathbf{L}}^1(\mathbb{R}^2) \text{ and } h \in \mathcal{H}(\mathbb{R}^2)\Big\}.$$

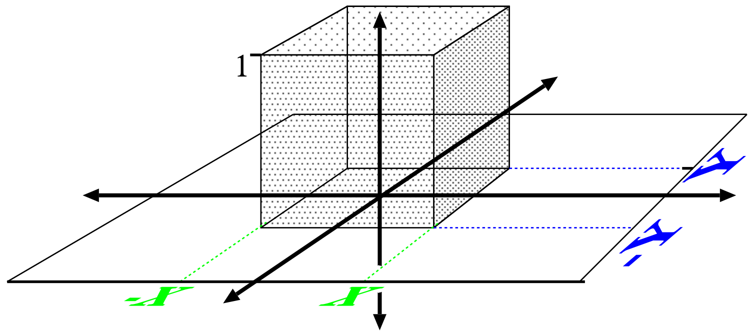


Figure 19D.1: Example 19D.4

Theorem 19D.2. 2-dimensional Fourier Inversion Formula
Suppose $f \in \widetilde{\mathbf{L}}^1(\mathbb{R}^2)$. If $(x,y) \in \mathbb{R}^2$ and f is is continuous at (x,y), then

$$f(x,y) \quad = \quad \lim_{M\to\infty} \int_{-M}^{M}\int_{-M}^{M} \widehat{f}(\mu,\nu)\cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu. \qquad (19D.1)$$

Proof. **Exercise 19D.1** (a) First show that eqn.(19D.1) holds for any element of $\mathcal{F}(\mathbb{R}^2)$. (*Hint.* If $f_1, f_2 \in \widetilde{\mathbf{L}}^1(\mathbb{R})$, and $f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$ for all $(x_1,x_2)\in\mathbb{R}^2$, then show that $\widehat{f}(\mu_1,\mu_2) = \widehat{f_1}(\mu_1)\cdot\widehat{f_2}(\mu_2)$. Substitute this expression into the right hand side of eqn.(19D.1); factor the integral into two one-dimensional Fourier inversion integrals, and then apply Theorem 19A.1 on page 488.) Ⓔ

(b) Deduce that eqn.(19D.1) holds for any element of $\mathcal{H}(\mathbb{R}^2)$. (*Hint.* The Fourier transform is linear.)

(c) Now combine (b) with Theorem 19D.1 to conclude that eqn.(19D.1) holds for any element of $\widetilde{\mathbf{L}}^1(\mathbb{R}^2)$. □

Proposition 19D.3. *If $f, g \in \mathcal{C}(\mathbb{R}^2)\cap\mathbf{L}^1(\mathbb{R}^2)$ are continuous, integrable functions, then* $\Big(\widehat{f} = \widehat{g}\Big) \iff \Big(f = g\Big)$. □

Example 19D.4. Let $X, Y > 0$, and let $f(x,y) = \begin{cases} 1 & \text{if} \quad -X \le x \le X \\ & \quad\ \text{and } -Y \le y \le Y; \\ 0 & \quad\ \text{otherwise.}\end{cases}$

(Figure 19D.1) Then:

$$\begin{aligned}\widehat{f}(\mu,\nu) \quad &= \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\exp\Big(-(\mu x+\nu y)\cdot\mathbf{i}\Big)\ dx\ dy \\ &= \quad \frac{1}{4\pi^2}\int_{-X}^{X}\int_{-Y}^{Y}\exp(-\mu x\mathbf{i})\cdot\exp(-\nu y\mathbf{i})\ dx\ dy\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2}\left(\int_{-X}^{X}\exp(-\mu x\mathbf{i})\,dx\right)\cdot\left(\int_{-Y}^{Y}\exp(-\nu y\mathbf{i})dy\right)\\
&= \frac{1}{4\pi^2}\cdot\left(\frac{-1}{\mu\mathbf{i}}\exp\Big(-\mu x\mathbf{i}\Big)_{x=-X}^{x=X}\right)\cdot\left(\frac{1}{\nu\mathbf{i}}\exp\Big(-\nu y\mathbf{i}\Big)_{y=-Y}^{y=Y}\right)\\
&= \frac{1}{4\pi^2}\left(\frac{e^{\mu X\mathbf{i}}-e^{-\mu X\mathbf{i}}}{\mu\mathbf{i}}\right)\left(\frac{e^{\nu Y\mathbf{i}}-e^{-\nu Y\mathbf{i}}}{\nu\mathbf{i}}\right)\\
&= \frac{1}{\pi^2\mu\nu}\left(\frac{e^{\mu X\mathbf{i}}-e^{-\mu X\mathbf{i}}}{2\mathbf{i}}\right)\left(\frac{e^{\nu Y\mathbf{i}}-e^{-\nu Y\mathbf{i}}}{2\mathbf{i}}\right)\\
&\overline{\overline{\scriptstyle(\mathrm{Eu})}}\ \frac{1}{\pi^2\mu\nu}\sin(\mu X)\cdot\sin(\nu Y),
\end{aligned}
$$

where **(Eu)** is by double application of Euler's formula (see page 551). Note that f is in $\mathcal{F}(\mathbb{R}^2)$ (why?), and thus, in $\widetilde{\mathbf{L}}^1(\mathbb{R}^2)$. Thus, Theorem 19D.2 says, that, if $-X < x < X$ and $-Y < y < Y$, then

$$\lim_{M\to\infty}\int_{-M}^{M}\int_{-M}^{M}\frac{\sin(\mu X)\cdot\sin(\nu Y)}{\pi^2\cdot\mu\cdot\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu \ = \ 1,$$

while, if $(x,y)\notin[-X,X]\times[-Y,Y]$, then

$$\lim_{M\to\infty}\int_{-M}^{M}\int_{-M}^{M}\frac{\sin(\mu X)\cdot\sin(\nu Y)}{\pi^2\cdot\mu\cdot\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu \ = \ 0.$$

At points on the *boundary* of the box $[0,X]\times[0,Y]$, however, the Fourier inversion integral will converge to neither of these values. ♢

Example 19D.5. If $f(x,y) = \dfrac{1}{2\sigma^2\pi}\exp\left(\dfrac{-x^2-y^2}{2\sigma^2}\right)$ is a two-dimensional Gaussian distribution, then $\widehat{f}(\mu,\nu) = \dfrac{1}{4\pi^2}\exp\left(\dfrac{-\sigma^2}{2}\left(\mu^2+\nu^2\right)\right)$.

Ⓔ (Exercise 19D.2) ♢

Ⓔ **Exercise 19D.3.** State and prove 2-dimensional versions of all results in §19B. ♦

19E Three-dimensional Fourier transforms

Prerequisites: §19A. **Recommended:** §9B, §19D.

In three or more dimensions, it is cumbersome to write vectors as an explicit list of coordinates. We will adopt a more compact notation. **Bold-face** letters will indicate vectors, and italic letters, their components. For example:

$$\mathbf{x} = (x_1,x_2,x_3), \qquad \mathbf{y} = (y_1,y_2,y_3), \qquad \boldsymbol{\mu} = (\mu_1,\mu_2,\mu_3), \quad \text{and} \quad \boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3)$$

We define the *inner product* $\mathbf{x} \bullet \mathbf{y} := x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$. Let $\mathbf{L}^1(\mathbb{R}^3)$ be the set of all functions $f : \mathbb{R}^3 \longrightarrow \mathbb{C}$ which are **absolutely integrable** on $\mathbb{R}^3$, meaning that

$$\int_{\mathbb{R}^3} |f(\mathbf{x})| \; d\mathbf{x} \quad < \quad \infty.$$

If $f \in \mathbf{L}^1(\mathbb{R}^3)$, then we can define

$$\int_{\mathbb{R}^3} f(\mathbf{x}) \; d\mathbf{x} \quad := \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3) \; dx_1 \; dx_2 \; dx_3,$$

where this integral is understood to be *absolutely convergent*. In particular if $f \in \mathbf{L}^1(\mathbb{R}^3)$, then the **Fourier transform** of f is the function $\widehat{f} : \mathbb{R}^3 \longrightarrow \mathbb{C}$ defined:

$$\widehat{f}(\boldsymbol{\mu}) \quad := \quad \boxed{\frac{1}{8\pi^3}\int_{\mathbb{R}^3} f(\mathbf{x}) \cdot \exp\left(-\mathbf{x} \bullet \boldsymbol{\mu} \cdot \mathbf{i}\right) \; d\mathbf{x},}$$

for all $\boldsymbol{\mu} \in \mathbb{R}^3$. Define $\widetilde{\mathbf{L}}^1(\mathbb{R}^3)$ in a manner analogous to the definition of $\widetilde{\mathbf{L}}^1(\mathbb{R}^2)$ on page 505.

Theorem 19E.1. 3-dimensional Fourier Inversion Formula

(a) *Suppose $f \in \widetilde{\mathbf{L}}^1(\mathbb{R}^3)$. For any $\mathbf{x} \in \mathbb{R}^3$, if f is continuous at $\mathbf{x}$, then*
$$f(\mathbf{x}) \;=\; \lim_{M\to\infty} \int_{-M}^{M}\int_{-M}^{M}\int_{-M}^{M} \widehat{f}(\boldsymbol{\mu}) \; \cdot \exp\left(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}\right) \; d\boldsymbol{\mu}.$$

(b) *Suppose $f \in \mathbf{L}^1(\mathbb{R}^3)$, and $\widehat{f}$ is also in $\mathbf{L}^1(\mathbb{R}^3)$. For any $\mathbf{x} \in \mathbb{R}^3$, if f is continuous at $\mathbf{x}$, then $f(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \widehat{f}(\boldsymbol{\mu}) \cdot \exp(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}) \; d\mu$.*

Proof. **(a)** **Exercise 19E.1** (*Hint:* Generalize the proof of Theorem 19D.2 on page 505. You may assume **(b)** is true.) Ⓔ

(b) See [Fol84, Thm 8.26, p. 243] or [Kat76, §VI.1.12, p.126]. □

Proposition 19E.2. *If $f, g \in \mathcal{C}(\mathbb{R}^3) \cap \mathbf{L}^1(\mathbb{R}^3)$ are continuous, integrable functions, then* $\left(\widehat{f} = \widehat{g}\right) \iff \left(f = g\right)$. □

Example 19E.3: A Ball

For any $\mathbf{x} \in \mathbb{R}^3$, let $f(\mathbf{x}) \;=\; \begin{cases} 1 & \text{if } \|\mathbf{x}\| \leq R; \\ 0 & \text{otherwise.} \end{cases}$. Thus, $f(\mathbf{x})$ is nonzero on a ball of radius R around zero. Then

$$\widehat{f}(\boldsymbol{\mu}) \;=\; \frac{1}{2\pi^2}\left(\frac{\sin(\mu R)}{\mu^3} \;-\; \frac{R\cos(\mu R)}{\mu^2}\right),$$

where $\mu := \|\boldsymbol{\mu}\|$. ♢

Ⓔ **Exercise 19E.2.** Verify Example 19E.3. *Hint:* Argue that, by spherical symmetry, we can rotate $\boldsymbol{\mu}$ without changing the integral, so we can assume that $\boldsymbol{\mu} = (\mu, 0, 0)$. Switch to the spherical coordinate system $(x_1, x_2, x_3) = (r \cdot \cos(\phi),\ r \cdot \sin(\phi)\sin(\theta),\ r \cdot \sin(\phi)\cos(\theta))$, to express the Fourier integral as

$$\frac{1}{8\pi^3}\int_0^R \int_0^\pi \int_{-\pi}^\pi \exp\left(\mu \cdot r \cdot \cos(\phi) \cdot \mathbf{i}\right) \cdot r\sin(\phi)\ d\theta\ d\phi\ dr.$$

Use *Claim 1* from Theorem 20B.6 on page 534 to simplify this to $\dfrac{1}{2\pi^2\mu}\displaystyle\int_0^R r{\cdot}\sin\left(\mu \cdot r\right)\ dr$. Now apply integration by parts. ♦

Ⓔ **Exercise 19E.3** The Fourier transform of Example 19E.3 contains the terms $\dfrac{\sin(\mu R)}{\mu^3}$ and $\dfrac{\cos(\mu R)}{\mu^2}$, both of which go to infinity as $\mu \to 0$. However, these two infinities "cancel out". Use l'Hôpital's rule to show that $\displaystyle\lim_{\boldsymbol{\mu}\to 0} \widehat{f}(\boldsymbol{\mu}) = \frac{1}{24\pi^3}$.

Example 19E.4: A spherically symmetric function

Suppose $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ was a spherically symmetric function; in other words, $f(\mathbf{x}) = \phi\left(\|\mathbf{x}\|\right)$ for some function $\phi : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$. Then for any $\boldsymbol{\mu} \in \mathbb{R}^3$,

$$\widehat{f}(\boldsymbol{\mu}) = \frac{1}{2\pi^2}\int_0^\infty \phi(r) \cdot r \cdot \sin\left(\|\boldsymbol{\mu}\| \cdot r\right)\ dr.$$

Ⓔ (**Exercise 19E.4**) ♢

D-dimensional Fourier transforms. Fourier transforms can be defined in an analogous way in higher dimensions. Let $\mathbf{L}^1(\mathbb{R}^D)$ be the set of all functions $f : \mathbb{R}^D \longrightarrow \mathbb{C}$ such that $\int_{\mathbb{R}^D} |f(\mathbf{x})|\ d\mathbf{x} < \infty$. If $f \in \mathbf{L}^1(\mathbb{R}^D)$, then the **Fourier transform** of f is the function $\widehat{f} : \mathbb{R}^D \longrightarrow \mathbb{C}$ defined:

$$\widehat{f}(\boldsymbol{\mu}) \quad := \quad \boxed{\frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} f(\mathbf{x}) \cdot \exp\left(-\mathbf{x} \bullet \boldsymbol{\mu} \cdot \mathbf{i}\right)\ d\mathbf{x},}$$

for all $\boldsymbol{\mu} \in \mathbb{R}^D$. Define $\widetilde{\mathbf{L}}^1(\mathbb{R}^D)$ in a manner analogous to the definition of $\widetilde{\mathbf{L}}^1(\mathbb{R}^2)$ on page 505.

Theorem 19E.5. D-dimensional Fourier Inversion Formula

(a) *Suppose $f \in \widetilde{\mathbf{L}}^1(\mathbb{R}^D)$. For any $\mathbf{x} \in \mathbb{R}^D$, if f is continuous at $\mathbf{x}$, then*

$$f(\mathbf{x}) = \lim_{M\to\infty}\int_{-M}^M \int_{-M}^M \cdots \int_{-M}^M \widehat{f}(\boldsymbol{\mu}) \cdot \exp\left(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}\right)\ d\boldsymbol{\mu}.$$

(b) *Suppose $f \in \mathbf{L}^1(\mathbb{R}^D)$, and $\widehat{f}$ is also in $\mathbf{L}^1(\mathbb{R}^D)$. For any $\mathbf{x} \in \mathbb{R}^D$, if f is continuous at $\mathbf{x}$, then $f(\mathbf{x}) = \int_{\mathbb{R}^D} \widehat{f}(\boldsymbol{\mu}) \cdot \exp(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i})\ d\mu.$*

Proof. **(a)** **Exercise 19E.5** Ⓔ

(b) See [Fol84, Thm 8.26, p. 243] or [Kat76, §VI.1.12, p.126]. □

Exercise 19E.6. State and prove D-dimensional versions of all results in §19B. ♦ Ⓔ

Evil twins of multidimensional Fourier transform. Just as with the one-dimensional Fourier transform, the mathematics literature contains at least four different definitions of multidimensional Fourier transform. Instead of the transform we have defined here, some books use what we will call the *opposite Fourier transform*:

$$\check{f}(\boldsymbol{\mu}) \quad := \quad \int_{\mathbb{R}^D} f(\mathbf{x}) \exp(-\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu})\ d\mathbf{x},$$

with inverse transform

$$f(\mathbf{x}) \quad = \quad \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \check{f}(\boldsymbol{\mu}) \exp(\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu})\ d\boldsymbol{\mu}.$$

Other books instead use the *symmetric Fourier transform*:

$$\widehat{\widehat{f}}(\boldsymbol{\mu}) \quad := \quad \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}^D} f(\mathbf{x}) \exp\left(-\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu}\right)\ d\mathbf{x},$$

with inverse transform

$$f(\mathbf{x}) \quad = \quad \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}^D} \widehat{\widehat{f}}(\boldsymbol{\mu}) \exp\left(\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu}\right)\ d\boldsymbol{\mu},$$

Finally, some books use the *canonical Fourier transform*:

$$\widetilde{f}(\boldsymbol{\mu}) \quad := \quad \int_{\mathbb{R}^D} f(\mathbf{x}) \exp\left(-2\pi\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu}\right)\ d\mathbf{x},$$

with inverse transform

$$f(\mathbf{x}) \quad = \quad \int_{\mathbb{R}^D} \widetilde{f}(\boldsymbol{\mu}) \exp\left(2\pi\mathbf{i}\,\mathbf{x} \bullet \boldsymbol{\mu}\right)\ d\boldsymbol{\mu},$$

For any $f \in \mathbf{L}^1(\mathbb{R}^D)$ and any $\boldsymbol{\mu} \in \mathbb{R}^D$, we have:

$$\begin{aligned}
\check{f}(\boldsymbol{\mu}) &= (2\pi)^D\,\widehat{f}(\boldsymbol{\mu}) \quad &\text{and}\quad \widehat{f}(\boldsymbol{\mu}) &= \frac{1}{(2\pi)^D}\,\check{f}(\boldsymbol{\mu});\\
\widehat{\widehat{f}}(\boldsymbol{\mu}) &= (2\pi)^{D/2}\,\widehat{f}(\boldsymbol{\mu}) \quad &\text{and}\quad \widehat{f}(\boldsymbol{\mu}) &= \frac{1}{(2\pi)^{D/2}}\,\widehat{\widehat{f}}(\boldsymbol{\mu}); \qquad (19\text{E}.1)\\
\widetilde{f}(\boldsymbol{\mu}) &= (2\pi)^D\,\widehat{f}(2\pi\,\boldsymbol{\mu}) \quad &\text{and}\quad \widehat{f}(\boldsymbol{\mu}) &= \frac{1}{(2\pi)^D}\,\widetilde{f}\left(\frac{\boldsymbol{\mu}}{2\pi}\right).
\end{aligned}$$

Ⓔ (**Exercise 19E.7** Check this.) When comparing or combining formulae from two or more books, *always compare their definitions of the Fourier transform*, and make the appropriate conversions using formulae (19E.1), if necessary.

19F Fourier (co)sine Transforms on the half-line

Prerequisites: §19A. **Recommended:** §7A, §8A.

In §8A, to represent a function on the *symmetric* interval $[-\pi, \pi]$, we used a real Fourier series (with both "sine" and "cosine" terms). However, to represent a function on the interval $[0, \pi]$, we found in §7A that it was only necessary to employ half as many terms, using either the Fourier sine series or the Fourier cosine series. A similar phenomenon occurs when we go from functions on the *whole* real line to functions on the positive *half*-line.

Let $\mathbb{R}_{\not-} := \{x \in \mathbb{R} \,;\, x \geq 0\}$ be the **half-line**: the set of all nonnegative real numbers. Let

$$\mathbf{L}^1(\mathbb{R}_{\not-}) \quad := \quad \left\{ f : \mathbb{R}_{\not-} \longrightarrow \mathbb{R} \,;\, \int_0^\infty |f(x)| \, dx \; < \infty \right\}$$

be the set of **absolutely integrable** functions on the half-line.

The "boundary" of the half-line is just the point 0. Thus, we will say that a function f satisfies homogeneous **Dirichlet** boundary conditions if $f(0) = 0$. Likewise, f satisfies homogeneous **Neumann** boundary conditions if $f'(0) = 0$.

If $f \in \mathbf{L}^1(\mathbb{R}_{\not-})$, then the **Fourier Cosine Transform** of f is the function $\widehat{f}_{\cos} : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ defined:

$$\widehat{f}_{\cos}(\mu) \quad := \quad \boxed{\frac{2}{\pi} \int_0^\infty f(x) \cdot \cos(\mu x) \, dx,} \quad \text{for all } \mu \in \mathbb{R}_{\not-}.$$

The **Fourier Sine Transform** of f is the function $\widehat{f}_{\sin} : \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ defined:

$$\widehat{f}_{\sin}(\mu) \quad := \quad \boxed{\frac{2}{\pi} \int_0^\infty f(x) \cdot \sin(\mu x) \, dx,} \quad \text{for all } \mu \in \mathbb{R}_{\not-}.$$

In both cases, for the transform to be well-defined, we require $f \in \mathbf{L}^1(\mathbb{R}_{\not-})$.

Theorem 19F.1. Fourier (co)sine Inversion Formula
Suppose that $f \in \mathbf{L}^1(\mathbb{R}_{\not-})$ be peicewise smooth. Then for any $x \in \mathbb{R}_+$ such that f is continuous at x,

$$\begin{aligned} f(x) \quad &= \quad \lim_{M \to \infty} \int_0^M \widehat{f}_{\cos}(\mu) \cdot \cos(\mu \cdot x) \, d\mu, \\ \textit{and} \quad f(x) \quad &= \quad \lim_{M \to \infty} \int_0^M \widehat{f}_{\sin}(\mu) \cdot \sin(\mu \cdot x) \, d\mu, \end{aligned}$$

The Fourier cosine series also converges at 0. If $f(0) = 0$, then the Fourier sine series converges at 0.

Proof. **Exercise 19F.1** Hint: Imitate the methods of §8C. □ Ⓔ

19G* Momentum representation & Heisenberg uncertainty

"Anyone who is not shocked by quantum theory has not understood it." —Niels Bohr

Prerequisites: §3B, §6B, §19C.

Let $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ be the wavefunction of a quantum particle (e.g. an electron). Fix $t \in \mathbb{R}$, and define the 'instantaneous wavefunction' $\omega_t : \mathbb{R}^3 \longrightarrow \mathbb{C}$ by $\omega_t(\mathbf{x}) = \omega(\mathbf{x}; t)$ for all $\mathbf{x} \in \mathbb{R}^3$. Recall from §3B that ω_t encodes the probability distribution for the classical *position* of the particle at time t. However, ω_t seems to say nothing about the classical *momentum* of the particle. In Example 3B.2 on page 42, we stated (without proof) the wavefunction of a particle with a particular known velocity. Now we make a more general assertion:

> *Suppose a particle has instantaneous wavefunction $\omega_t : \mathbb{R}^3 \longrightarrow \mathbb{C}$. Let $\widehat{\omega}_t : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be the* (3*-dimensional) Fourier transform of ω_t, and define $\widetilde{\omega}_t := \widehat{\omega}_t\left(\frac{\mathbf{p}}{\hbar}\right)$ for all $\mathbf{p} \in \mathbb{R}^3$. Then $\widetilde{\omega}_t$ is the wavefunction for the particle's classical momentum at time t. That is: if we define $\widetilde{\rho}_t(\mathbf{p}) := |\widetilde{\omega}_t|^2(\mathbf{p}) / \|\widetilde{\omega}_t\|_2^2$ for all $\mathbf{p} \in \mathbb{R}^3$, then $\widetilde{\rho}_t$ is the probability distribution for the particle's classical momentum at time t.*

Recall that we can reconstruct ω_t from $\widehat{\omega}_t$ via the inverse Fourier transform. Hence, the (positional) wavefunction ω_t implicitly encodes the (momentum) wavefunction $\widetilde{\omega}_t$, and conversely the (momentum) wavefunction $\widetilde{\omega}_t$ implicitly encodes the (positional) wavefunction ω_t. This answers the question we posed on page 38 of §3A. The same applies to multi-particle quantum systems:

> *Suppose an N-particle quantum system has instantaneous* (positional) *wavefunction $\omega_t : \mathbb{R}^{3N} \longrightarrow \mathbb{C}$. Let $\widehat{\omega}_t : \mathbb{R}^{3N} \longrightarrow \mathbb{C}$ be the* ($3N$-dimensional) *Fourier transform of ω_t, and define $\widetilde{\omega}_t := \widehat{\omega}_t\left(\frac{\mathbf{p}}{\hbar}\right)$ for all $\mathbf{p} \in \mathbb{R}^{3N}$. Then $\widetilde{\omega}_t$ is the joint wavefunction for the classical momenta of all the particles at time t. That is: if we define $\widetilde{\rho}_t(\mathbf{p}) := |\widetilde{\omega}_t|^2(\mathbf{p}) / \|\widetilde{\omega}_t\|_2^2$ for all $\mathbf{p} \in \mathbb{R}^{3N}$, then $\widetilde{\rho}_t$ is the joint probability distribution for the classical momenta of all the particles at time t.*

Because the momentum wavefunction contains exactly the same information as the positional wavefunction, we can reformulate the Schrödinger equation in momentum terms. For simplicity, we will only do this in the case of a single particle. Suppose the particle is subjected to a potential energy function $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Let $\widehat{V}$ be the Fourier transform of V, and define $\widetilde{V} := \frac{1}{\hbar^3}\widehat{V}\left(\frac{\mathbf{p}}{\hbar}\right)$ for all $\mathbf{p} \in \mathbb{R}^3$. Then the momentum wavefunction $\widetilde{\omega}$ evolves according to the **momentum Schrödinger Equation:**

$$\mathbf{i}\partial_t\, \widetilde{\omega}(\mathbf{p};t) \quad = \quad \frac{\hbar^2}{2m}|\mathbf{p}|^2 \cdot \widetilde{\omega}_t(\mathbf{p}) \ + \ (\widetilde{V} * \widetilde{\omega}_t)(\mathbf{p}). \tag{19G.1}$$

(here, if $\mathbf{p} = (p_1, p_2, p_3)$, then $|\mathbf{p}|^2 := p_1^2 + p_2^2 + p_3^2$). In particular, if the potential field is trivial, we get the *free* momentum Schrödinger equation:

$$\mathbf{i}\partial_t\, \widetilde{\omega}(p_1, p_2, p_3; t) \quad = \quad \frac{\hbar^2}{2m}(p_1^2 + p_2^2 + p_3^2) \cdot \widetilde{\omega}(p_1, p_2, p_3; t).$$

Ⓔ **Exercise 19G.1.** Verify eqn.(19G.1) by applying the Fourier transform to the (positional) Schrödinger equation eqn.(3B.3) on page 41. Hint: Use Theorem 19B.7 on page 496 to show that $\widehat{\triangle\omega_t}(\mathbf{p}) \ = \ -|\mathbf{p}|^2 \cdot \widehat{\omega}_t(\mathbf{p})$. Use Theorem 19B.2(c) to show that $(\widehat{V \cdot \omega_t})(\mathbf{p}/\hbar) \ = \widetilde{V} * \widetilde{\omega}_t(\mathbf{p})$. ♦

Ⓔ **Exercise 19G.2.** Formulate the momentum Schrödinger equation for an single particle confined to a 1-dimensional or 2-dimensional environment. Be careful how you define $\widetilde{V}$. ♦

Ⓔ **Exercise 19G.3.** Formulate the momentum Schrödinger equation for an N-particle quantum system. Be careful how you define $\widetilde{V}$. ♦

Recall that Theorem 19B.8 said: if f is a Gaussian distribution, then $\widehat{f}$ is also a 'Gaussian' (after multiplying by a scalar), but the variance of $\widehat{f}$ is inversely proportional to the variance of f. This is an example of a general phenomenon, called *Heisenberg's Inequality.* To state this formally, we need some notation. Recall from §6B that $\mathbf{L}^2(\mathbb{R})$ is the set of all *square-integrable* complex-valued functions on $\mathbb{R}$ —that is, all integrable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\|f\|_2 < \infty$, where

$$\|f\|_2 \quad := \quad \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx\right)^{1/2}.$$

If $f \in \mathbf{L}^2(\mathbb{R})$, and $x \in \mathbb{R}$, then define the **uncertainty** of f around x to be

$$\blacktriangle_x\ (f) \quad := \quad \frac{1}{\|f\|_2^2}\int_{-\infty}^{\infty} |f(y)|^2 \cdot |y - x|^2 \, dy.$$

(In most physics texts, the uncertainty of f is denoted by $\triangle_x f$; however, we will not use this symbol because it looks too much like the Laplacian operator.)

Example 19G.1. (a) If $f \in \mathbf{L}^2(\mathbb{R})$, then $\rho(x) := f(x)^2/\|f\|_2^2$ is a probability density function on $\mathbb{R}$ (why?). If $\overline{x}$ is the mean of the distribution ρ (i.e. $\overline{x} = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$), then

$$\blacktriangle_{\overline{x}}(f) \quad = \quad \int_{-\infty}^{\infty} \rho(y)\cdot|y-\overline{x}|^2\,dy$$

is the *variance* of the distribution. Thus, if ρ describes the probability density of a random variable $X \in \mathbb{R}$, then $\overline{x}$ is the expected value of X, and $\blacktriangle_{\overline{x}}(\omega)$ measures the degee of 'uncertainty' we have about the value of X. If $\blacktriangle_{\overline{x}}(\omega)$ is small, then the distribution is tightly concentrated around $\overline{x}$, so we can be fairly confident that X is close to $\overline{x}$. If $\blacktriangle_{\overline{x}}(\omega)$ is large, then the distribution is broadly dispersed around $\overline{x}$, so we really have only a vague idea where X might be.

(b) In particular, suppose $f(x) = \exp\left(\frac{-x^2}{4\sigma^2}\right)$. Then $f^2/\|f\|_2^2 = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\sigma^2}\right)$ is a *Gaussian distribution* with mean 0 and variance σ^2. It follows that $\blacktriangle_0(f) = \sigma^2$.

(c) Suppose $\omega : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ is a one-dimensional wavefunction, and fix $t \in \mathbb{R}$; thus, the function $\rho_t(x) = |\omega_t|^2(x)/\|\omega_t\|_2$ is the probability density for the classical position of the particle at time t in a one-dimensional environment (e.g. an electron in a thin wire). If $\overline{x}$ is the mean of this distribution, then $\blacktriangle_{\overline{x}}(\omega_t)$ is the variance of the distribution; this reflects our degree of uncertainty about the particle's classical position at time t. ◇

Why the subscript x in $\blacktriangle_x(f)$? Why not just measure the uncertainty around the *mean* of the distribution as in Example 19G.1? Three reasons. First, because the distribution might not *have* a well-defined mean (i.e. the integral $\int_{-\infty}^{\infty} x\,\rho(x)\,dx$ might not converge). Second, because it is sometimes useful to measure the uncertainty around other points in $\mathbb{R}$ besides the mean value. Third, because we do not need to use the mean value to state the next result.

Theorem 19G.2. Heisenberg's Inequality
Let $f \in \mathbf{L}^2(\mathbb{R})$ be a nonzero function, and let $\widehat{f}$ be its Fourier transform. Then for any $x, \mu \in \mathbb{R}$, we have $\blacktriangle_x(f)\cdot\blacktriangle_\mu(\widehat{f}) \;\geq\; \frac{1}{4}$.

Example 19G.3. (a) If $f(x) = \exp\left(\dfrac{-x^2}{4\sigma^2}\right)$, then $\widehat{f}(p) = \dfrac{\sigma}{\sqrt{\pi}}\exp\left(-\sigma^2p^2\right)$ (**Exercise 19G.4**).[4] Thus, $\widehat{f}(p)^2/\left\|\widehat{f}\right\|_2^2 = \dfrac{2\sigma}{\sqrt{2\pi}}\exp(-2\sigma^2p^2)$ is a Gaussian Ⓔ

distribution with mean 0 and variance $1/4\sigma^2$. Thus, Example 19G.1(b) says that $\blacktriangle_0(f) = \sigma^2$ and $\blacktriangle_0(\widehat{f}) = 1/4\sigma^2$. Thus,

$$\blacktriangle_0(f) \cdot \blacktriangle_0(\widehat{f}) \quad = \quad \frac{\sigma^2}{4\sigma^2} \quad = \quad \frac{1}{4}.$$

(b) Suppose $\omega_t \in \mathbf{L}^2(\mathbb{R})$ is the instantaneous wavefunction for the position of a particle at time t, so that $\widetilde{\omega}_t(\mathbf{p}) = \widehat{\omega}_t(\mathbf{p}/\hbar)$ is the instantaneous wavefunction for the momentum of the particle at time t. Then Heisenberg's Inequality becomes **Heisenberg's Uncertainty Principle:** For any $x, p \in \mathbb{R}$,

$$\blacktriangle_x(\omega_t) \quad \geq \quad \frac{\hbar^2}{4 \cdot \blacktriangle_p(\widetilde{\omega}_t)} \qquad \text{and} \qquad \blacktriangle_p(\widetilde{\omega}_t) \quad \geq \quad \frac{\hbar^2}{4 \cdot \blacktriangle_x(\omega_t)}$$

Ⓔ (**Exercise 19G.5**). In other words: if our uncertainty $\blacktriangle_\mu(\widetilde{\omega}_t)$ about the particle's momentum is small, then our uncertainty $\blacktriangle_x(\omega_t)$ about its position must be big. Conversely, if our uncertainty $\blacktriangle_x(\omega_t)$ about the particle's position is small, then our uncertainty $\blacktriangle_\mu(\widetilde{\omega}_t)$ about its momentum must be big.

In physics popularizations, the Uncertainty Principle is usually explained as a practical problem of measurement precision: any attempt to measure an electron's position (e.g. by deflecting photons off of it) will impart some unpredictable momentum into the particle. Conversely, any attempt to measure its momentum disturbs its position. However, as you can see, Heisenberg's Uncertainty Principle is actually an abstract mathematical *theorem* about Fourier transforms —it has nothing to do with the limitations of experimental equipment or the unpredictable consequences of photon bombardment. $\diamondsuit$

Proof of Heisenberg's Inequality. For simplicity, assume $\lim_{x\to\pm\infty} x\,|f(x)|^2 = 0$.

Case $x = \mu = 0$. Define $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ by $\xi(x) := x$ for all $x \in \mathbb{R}$. Thus, $\xi'(x) := 1$ for all $x \in \mathbb{R}$. Observe that

$$\begin{aligned} \|f \cdot \xi\|_2^2 &= \int_{-\infty}^{\infty} |f \cdot \xi|^2(x)\ dx \quad = \quad \frac{\|f\|_2^2}{\|f\|_2^2} \int_{-\infty}^{\infty} |f(x)|^2 |x|^2\ dx \\ &= \|f\|_2^2\ \blacktriangle_0(f). \end{aligned} \tag{19G.2}$$

Also, Theorem 19B.7 implies that

$$\widehat{(f')} \quad = \quad \mathbf{i} \cdot \xi \cdot \widehat{f}. \tag{19G.3}$$

Now,

$$\|f\|_2^2 \quad := \quad \int_{-\infty}^{\infty} |f|^2(x)\ dx \quad \underset{(\P)}{=} \quad \xi(x) \cdot |f(x)|^2 \Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \xi(x) \cdot (|f|^2)'(x)\ dx$$

[4]Hint: set $\alpha := 2\sigma$ in Theorem 19B.5 on page 495, and then apply it to Theorem 19B.8(a) on page 497.

$$
\underset{(\dagger)}{=} \; -\int_{-\infty}^{\infty} x \cdot (|f|^2)'(x)\ dx \;\underset{(*)}{=}\; -\int_{-\infty}^{\infty} x \cdot 2\,\mathrm{Re}\left[f'(x)\overline{f}(x)\right]\ dx
$$
$$
= \; -2\,\mathrm{Re}\left[\int_{-\infty}^{\infty} x\,\overline{f}(x) f'(x)\ dx\right] \qquad (19G.4)
$$

Here, (¶) is integration by parts, because $\xi'(x) = 1$. Next, (†) is because $\lim_{x\to\pm\infty} x\,|f(x)|^2 = 0$. Meanwhile, $(*)$ is because $|f|^2(x) = f(x)\overline{f}(x)$, so that $(|f|^2)'(x) = f'(x)\overline{f}(x) + f(x)\overline{f}'(x) \;=\; 2\,\mathrm{Re}\left[f'(x)\overline{f}(x)\right]$, where the last step uses the identity $z + \overline{z} = 2\,\mathrm{Re}\,[z]$, with $z = f'(x)\overline{f}(x)$. Thus,

$$
\begin{aligned}
\frac{1}{4}\,\|f\|_2^4 \;&\underset{(\ddagger)}{=}\; \frac{2^2}{4}\mathrm{Re}\left[\int_{-\infty}^{\infty} x\,\overline{f}(x) f'(x)\ dx\right]^2 \;\leq\; \left|\int_{-\infty}^{\infty} x\,\overline{f}(x) f'(x)\ dx\right|^2 \\
&= \; \left|\left\langle \xi\,\overline{f},\ f'\right\rangle\right|^2 \;\underset{(\mathrm{CBS})}{\leq}\; \left\|\xi\,\overline{f}\right\|_2^2 \cdot \left\|f'\right\|_2^2 \;=\; \|\xi\, f\|_2^2 \cdot \left\|f'\right\|_2^2 \\
&\underset{(*)}{=}\; \blacktriangle_0\,(f) \cdot \|f\|_2^2 \cdot \left\|f'\right\|_2^2 \;\underset{(\mathrm{Pl})}{=}\; \blacktriangle_0(f) \cdot \|f\|_2^2 \cdot (2\pi)\left\|\widehat{(f')}\right\|_2^2 \\
&\underset{(\dagger)}{=}\; 2\pi\ \blacktriangle_0(f) \cdot \|f\|_2^2 \cdot \left\|\mathbf{i}\xi\widehat{f}\right\|_2^2 \;=\; 2\pi\ \blacktriangle_0(f) \cdot \|f\|_2^2 \cdot \left\|\xi\widehat{f}\right\|_2^2 \\
&\underset{(*)}{=}\; 2\pi\ \blacktriangle_0(f) \cdot \|f\|_2^2 \cdot \blacktriangle_0\,(\widehat{f}) \cdot \left\|\widehat{f}\right\|_2^2 \;\underset{(\mathrm{Pl})}{=}\; \blacktriangle_0(f) \cdot \|f\|_2^2 \cdot \blacktriangle_0\,(\widehat{f}) \cdot \|f\|_2^2 \\
&= \; \blacktriangle_0\,(f) \cdot \blacktriangle_0\,(\widehat{f}) \cdot \|f\|_2^4 .
\end{aligned}
$$

Cancelling $\|f\|_2^4$ from both sides of this equation, we get $\frac{1}{4} \;\leq\; \blacktriangle_0\,(f) \cdot \blacktriangle_0\,(\widehat{f})$, as desired.

Here, (‡) is by eqn.(19G.4), while (CBS) is the Cauchy-Bunyakowski-Schwarz inequality (Theorem 6B.5 on page 108). Both $(*)$ are by eqn.(19G.2). Both (Pl) are by Plancharel's theorem (Corollary 19C.2 on page 503) . Finally, (†) is by eqn.(19G.3).

Case $x \neq 0$ and/or $\mu \neq 0$. **Exercise 19G.6** (Hint: Combine the case $x = \mu = 0$ with Theorem 19B.3). Ⓔ □

Exercise 19G.7. State and prove a form of Heisenberg's Inequality for a function $f \in \mathbf{L}^1(\mathbb{R}^D)$ for $D \geq 2$. *Hint:* You must compute the 'uncertainty' in one coordinate at a time. Integrate out all the other dimensions to reduce the D-dimensional problem to a one-dimensional problem, and then apply Theorem 19G.2. Ⓔ ♦

19H* Laplace transforms

Recommended: §19A, §19B.

The Fourier transform $\widehat{f}$ is only well-defined if $f \in \mathbf{L}^1(\mathbb{R})$, which implies that $\lim_{t\to\pm\infty} |f(t)| = 0$ relatively 'quickly'.[5] This is often inconvenient in physical models where the function $f(t)$ is bounded away from zero, or even grows without bound as $t \to \pm\infty$. In some cases, we can handle this problem using a *Laplace* transform, which can be thought of as a Fourier transform 'rotated by 90^o in the complex plane'. The price we pay is that we must work on the half-infinite line $\mathbb{R}_{\not<} := [0,\infty)$, instead of the entire real line.

Let $f : \mathbb{R}_{\not<} \longrightarrow \mathbb{C}$. We say that f has **exponential growth** if there are constants $\alpha \in \mathbb{R}$ and $K > 0$ such that

$$|f(t)| \quad \leq \quad K\, e^{\alpha t}, \qquad \text{for all } t \in \mathbb{R}_{\not<}. \tag{19H.1}$$

If $\alpha > 0$, then inequality (19H.1) even allows $\lim_{t\to\infty} f(t) = \infty$, as long as $f(t)$ doesn't grow 'too quickly'. (However, if $\alpha < 0$, then inequality (19H.1) requires $\lim_{t\to\infty} f(t) = 0$ exponentially fast). The **exponential order** of f is the infimum of all α satisfying inequality (19H.1). Thus, if f has exponential order α_0, then (19H.1) is true for all $\alpha > \alpha_0$ (but may or may not be true for $\alpha = \alpha_0$).

Example 19H.1. (a) Fix $r \geq 0$. If $f(t) = t^r$ for all $t \in \mathbb{R}_{\not<}$, then f has exponential order 0.

(b) Fix $r < 0$ and $t_0 > 0$. If $f(t) = (t + t_0)^r$ for all $t \in \mathbb{R}_{\not<}$, then f has exponential order 0. (However, if $t_0 \leq 0$, then $f(t)$ does *not* have exponential growth, because in this case $\lim_{t\to -t_0} f(t) = \infty$, so inequality (19H.1) is always false near $-t_0$).

(c) Fix $\alpha \in \mathbb{R}$. If $f(t) = e^{\alpha t}$ for all $t \in \mathbb{R}_{\not<}$, then f has exponential order α.

(d) Fix $\mu \in \mathbb{R}$. If $f(t) = \sin(\mu t)$ or $f(t) = \cos(\mu t)$, then f has exponential order 0.

(e) If $f : \mathbb{R}_{\not<} \longrightarrow \mathbb{C}$ has exponential order α, and $r \in \mathbb{R}$ is any constant, then $r\,f$ also has exponential order α. If $g : \mathbb{R}_{\not<} \longrightarrow \mathbb{C}$ has exponential order β, then $f + g$ has exponential order at most $\max\{\alpha, \beta\}$, and $f \cdot g$ has exponential order $\alpha + \beta$.

(f) Combining (a) and (e): any polynomial $f(t) = c_n t^n + \cdots c_2 t^2 + c_1 t + c_0$ has exponential order 0. Likewise, combining (d) and (e): any trigonometric polynomial has exponential order 0.

Ⓔ **Exercise 19H.1** Verify examples (a-f). ♢

If $c = x + y\mathbf{i}$ is a complex number, recall that $\mathrm{Re}\,[c] := x$. Let $\mathbb{H}_\alpha := \{c \in \mathbb{C}\ ;\ \mathrm{Re}\,[c] > \alpha\}$ be the half of the complex plane to the right of the vertical

[5] Actually, we can define $\widehat{f}$ if $f \in \mathbf{L}^p(\mathbb{R})$ for any $p \in [1,2]$, as discussed in §19C. However, this isn't really that much of an improvement; we still need $\lim_{t\to\pm\infty} |f(t)| = 0$ 'quickly'.

line $\{c \in \mathbb{C} \,;\, \mathrm{Re}\,[c] = \alpha\}$. In particular, $\mathbb{H}_0 := \{c \in \mathbb{C} \,;\, \mathrm{Re}\,[c] > 0\}$. If f has exponential order α, then the **Laplace transform** of f is the function $\mathcal{L}[f] : \mathbb{H}_\alpha \longrightarrow \mathbb{C}$ defined as follows:

$$\text{For all } s \in \mathbb{H}_\alpha, \qquad \mathcal{L}[f](s) \quad := \quad \boxed{\int_0^\infty f(t)\, e^{-ts}\, dt.} \tag{19H.2}$$

Lemma 19H.2. *If f has exponential order α, then the integral* (19H.2) *converges for all $s \in \mathbb{H}_\alpha$. Thus, $\mathcal{L}[f]$ is well-defined on $\mathbb{H}_\alpha$.*

Proof. **Exercise 19H.2** □ Ⓔ

Example 19H.3. (a) If $f(t) = 1$, then f has exponential order 0. For all $s \in \mathbb{H}_0$,

$$\mathcal{L}[f](s) \quad = \quad \int_0^\infty e^{-ts}\, dt \quad = \quad \frac{-e^{-ts}}{s}\Big|_{t=0}^{t=\infty} \quad \underset{(*)}{=} \quad \frac{-(0-1)}{s} \quad = \quad \frac{1}{s}.$$

Here $(*)$ is because $\mathrm{Re}\,[s] > 0$.

(b) If $\alpha \in \mathbb{R}$ and $f(t) = e^{\alpha t}$, then f has exponential order α. For all $s \in \mathbb{H}_\alpha$,

$$\begin{aligned}\mathcal{L}[f](s) \quad &= \quad \int_0^\infty e^{\alpha t}\, e^{-ts}\, dt \quad = \quad \int_0^\infty e^{t(\alpha - s)}\, dt \\ &= \quad = \quad \frac{e^{t(\alpha-s)}}{\alpha - s}\Big|_{t=0}^{t=\infty} \quad \underset{(*)}{=} \quad \frac{(0-1)}{\alpha - s} \quad = \quad \frac{1}{s-\alpha}.\end{aligned}$$

Here $(*)$ is because $\mathrm{Re}\,[\alpha - s] < 0$ because $\mathrm{Re}\,[s] > \alpha$ because $s \in \mathbb{H}_\alpha$.

(c) If $f(t) = t$, then f has exponential order 0. For all $s \in \mathbb{H}_0$,

$$\begin{aligned}\mathcal{L}[f](s) \quad &= \quad \int_0^\infty t\, e^{-ts}\, dt \quad \underset{(p)}{=} \quad \frac{-t\, e^{-ts}}{s}\Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{-e^{-ts}}{s}\, dt \\ &= \quad \frac{(0-0)}{s} - \frac{e^{-ts}}{s^2}\Big|_{t=0}^{t=\infty} \quad = \quad \frac{-(0-1)}{s^2} \quad = \quad \frac{1}{s^2},\end{aligned}$$

where (p) is integration by parts.

(d) Suppose $f : \mathbb{R}_{\not -} \longrightarrow \mathbb{C}$ has exponential order $\alpha < 0$. Extend f to a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ by defining $f(t) = 0$ for all $t < 0$. Then the Fourier transform $\widehat{f}$ of f is well-defined, and for all $\mu \in \mathbb{R}$, we have $2\pi\, \widehat{f}(\mu) = \mathcal{L}[f](\mu\mathbf{i})$ (**Exercise 19H.3**). Ⓔ

(e) Fix $\mu \in \mathbb{R}$. If $f(t) = \cos(\mu t)$, then $\mathcal{L}[f] \;=\; \dfrac{s}{s^2 + \mu^2}$. If $f(t) = \sin(\mu t)$, then $\mathcal{L}[f] \;=\; \dfrac{\mu}{s^2 + \mu^2}$.

Exercise 19H.4 Verify (e). Hint: recall that $\exp(\mathbf{i}\mu t) = \cos(\mu t) + \mathbf{i}\sin(\mu t)$. ♢ Ⓔ

Example 19H.3(d) suggests that most properties of the Fourier transform should translate into properties of the Laplace transform, and vice versa. First, like Fourier, the Laplace transform is invertible.

Theorem 19H.4. Laplace Inversion Formula
Suppose $f : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ has exponential order α, and let $F := \mathcal{L}[f]$. Then for any fixed $s_r > \alpha$, and any $t \in \mathbb{R}_{\not-}$, we have

$$f(t) \quad = \quad \frac{1}{2\pi\mathbf{i}} \int_{-\infty}^{\infty} F(s_r + s_i\mathbf{i}) \; \exp(ts_r + ts_i\mathbf{i}) \; ds_i. \tag{19H.3}$$

In particular, if $g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$, and $\mathcal{L}[g] = \mathcal{L}[f]$ on any infinite vertical strip in $\mathbb{C}$, then we must have $g = f$.

Ⓔ *Proof.* **Exercise 19H.5** (Hint: Use an argument similar to Example 19H.3(d) to represent F as a Fourier transform. Then apply the Fourier Inversion Formula.) □

The integral (19H.3) is called the **Laplace inversion integral**, and is denoted by $\mathcal{L}^{-1}[F]$. The integral (19H.3) is sometimes written

$$f(t) \quad = \quad \frac{1}{2\pi\mathbf{i}} \int_{s_r - \infty\mathbf{i}}^{s_r + \infty\mathbf{i}} F(s) \; \exp(ts) \; ds.$$

The integral (19H.3) is can be treated as a *contour integral* along the vertical line $\{c \in \mathbb{C} \; ; \; \mathrm{Re}\,[c] = s_r\}$ in the complex plane, and evaluated using residue calculus[6]. However, in many situations, it is neither easy nor particularly necessary to explicitly compute (19H.3); instead, we can determine the inverse Laplace transform 'by inspection', by simply writing F as a sum of Laplace transforms of functions we recognize. Most books on ordinary differential equations contain an extensive table of Laplace transforms and their inverses, which is useful for this purpose.

Example 19H.5. Suppose $F(s) \; = \; \frac{3}{s} + \frac{5}{s-2} + \frac{7}{s^2}$. Then by inspecting Example 19H.3(a,b,c), we deduce that $f(t) = \mathcal{L}^{-1}[F](t) \; = \; 3 + 5e^{2t} + 7t$. ♢

Most of the results about Fourier transforms from Section 19B have equivalent formulations for Laplace transforms.

Theorem 19H.6. Properties of the Laplace transform
Let $f : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ have exponential order α.

(a) (Linearity) *Let $g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ have exponential order β, and let $\gamma = \max\{\alpha, \beta\}$. Let $b, c \in \mathbb{C}$. Then $bf + cg$ has exponential order at most γ, and for all $s \in \mathbb{H}_\gamma$, $\mathcal{L}[bf + cg](s) \; = \; b\mathcal{L}[f](s) + c\mathcal{L}[g](s)$.*

[6]See § 18H on page 472 for a discussion of residue calculus and its application to improper integrals. See [Fis99, §5.3] for some examples of computing Laplace inversion integrals using this method.

(b) (Transform of a derivative)

(i) *Suppose $f \in \mathcal{C}^1(\mathbb{R}_{\not-})$ (i.e. f is continuously differentiable on $\mathbb{R}_{\not-}$) and f' has exponential order β. Let $\gamma = \max\{\alpha, \beta\}$. Then for all $s \in \mathbb{H}_\gamma$,*

$$\mathcal{L}[f'](s) \quad = \quad s\mathcal{L}[f](s) - f(0).$$

(ii) *Suppose $f \in \mathcal{C}^2(\mathbb{R}_{\not-})$, f' has exponential order α_1 and f'' has exponential order α_2. Let $\gamma = \max\{\alpha, \alpha_1, \alpha_2\}$. Then for all $s \in \mathbb{H}_\gamma$,*

$$\mathcal{L}[f''](s) \quad = \quad s^2\mathcal{L}[f](s) - f(0)s - f'(0).$$

(iii) *Let $N \in \mathbb{N}$, and suppose $f \in \mathcal{C}^N(\mathbb{R}_{\not-})$. Suppose $f^{(n)}$ has exponential order α_n for all $n \in [1...N]$. Let $\gamma = \max\{\alpha, \alpha_1, \ldots, \alpha_N\}$. Then for all $s \in \mathbb{H}_\gamma$,*

$$\mathcal{L}[f^{(N)}](s) \;=\; s^N\mathcal{L}[f](s) - f(0)s^{N-1} - f'(0)s^{N-2} - f''(0)s^{N-3} - \cdots - f^{(N-2)}(0)\,s - f^{(N-1)}(0).$$

(c) (Derivative of a transform) *For all $n \in \mathbb{N}$, the function $g_n(t) = t^n f(t)$ also has exponential order α.*

If f is piecewise continuous, then the function $F = \mathcal{L}[f] : \mathbb{H}_\alpha \longrightarrow \mathbb{C}$ is (complex)*-differentiable,*[7] *and for all $s \in \mathbb{H}_\alpha$, $F'(s) = -\mathcal{L}[g_1](s)$, $F''(s) = \mathcal{L}[g_2](s)$, and in general $F^{(n)}(s) = (-1)^n\mathcal{L}[g_n](s)$.*

(d) (Translation) *Fix $T \in \mathbb{R}_{\not-}$, and define $g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ by $g(t) = f(t-T)$ for $t \geq T$ and $g(t) = 0$ for $t \in [0, T)$. Then g also has exponential order α. For all $s \in \mathbb{H}_\alpha$, $\mathcal{L}[g](s) = e^{-Ts}\,\mathcal{L}[f](s)$.*

(e) (Dual translation) *For all $\beta \in \mathbb{R}$, the function $g(t) = e^{\beta t} f(t)$ has exponential order $\alpha + \beta$. For all $s \in \mathbb{H}_{\alpha+\beta}$, $\mathcal{L}[g](s) = \mathcal{L}[f](s - \beta)$.*

Proof. **Exercise 19H.6** (Hint: Imitate the proofs of Theorems 19B.2(a), 19B.7 and 19B.3.) □ Ⓔ

Exercise 19H.7. Show by a counterexample that Theorem 19H.6(d) is *false* if $T < 0$. ♦ Ⓔ

Corollary 19H.7. *Fix $n \in \mathbb{N}$. If $f(t) = t^n$ for all $t \in \mathbb{R}_{\not-}$, then $\mathcal{L}[f](s) = \dfrac{n!}{s^{n+1}}$ for all $s \in \mathbb{H}_0$.*

Proof. **Exercise 19H.8** (Hint: Combine Theorem 19H.6(c) with Example 19H.3(a).) □ Ⓔ

Ⓔ **Exercise 19H.9.** Combine Corollary 19H.7 with Theorem 19H.6(b) to get a formula for the Laplace transform of any polynomial. ♦

Ⓔ **Exercise 19H.10.** Combine Theorem 19H.6(c,d) with Example 19H.3(a) to get a formula for the Laplace transform of $f(t) = \dfrac{1}{(1+t)^n}$ for all $n \in \mathbb{N}$. ♦

Corollary 19H.7 does not help us to compute the Laplace transform of $f(t) = t^r$ when r is not an integer. To do this, we must introduce the **gamma function** $\Gamma : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$, which is defined

$$\Gamma(r) \quad := \quad \boxed{\int_0^\infty t^{r-1} e^{-t}\, dt,} \qquad \text{for all } r \in \mathbb{R}_{\not-}.$$

This is regarded as a 'generalized factorial' because of the following properties.

Lemma 19H.8.

(a) $\Gamma(1) = 1$.

(b) *For any* $r \in \mathbb{R}_{\not-}$, $\Gamma(r+1) \;=\; r \cdot \Gamma(r)$.

(c) *Thus, for any* $n \in \mathbb{N}$, $\Gamma(n+1) \;=\; n!$. (For example, $\Gamma(5) = 4! = 24$.)

Ⓔ *Proof.* **Exercise 19H.11** □

Ⓔ **Exercise 19H.12.** (a) Show that $\Gamma(1/2) = \sqrt{\pi}$.
(b) Deduce that $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. ♦

The gamma function is useful when computing Laplace transforms because of the next result.

Proposition 19H.9. Laplace transform of $f(t) = t^r$

Fix $r > -1$, *and let* $f(t) := t^r$ *for all* $t \in \mathbb{R}_{\not-}$. *Then* $\mathcal{L}[f](s) \;=\; \dfrac{\Gamma(r+1)}{s^{r+1}}$ *for all* $s \in \mathbb{H}_0$.

Ⓔ *Proof.* **Exercise 19H.13** □

Remark. If $r < 0$, then technically, $f(t) = t^r$ does *not* have exponential growth, as noted in Example 19H.1(b). Hence Lemma 19H.2 does not apply. However, the Laplace transform integral (19H.2) converges in this case anyways,

[7] See §18A on page 415 for more about complex differentiation.

because although $\lim_{t \searrow 0} t^r = \infty$, it goes to infinity 'slowly', so that the integral $\int_0^1 t^r \, dt$ is still finite.

Example 19H.10. (a) If $r \in \mathbb{N}$ and $f(t) = t^r$, then Proposition 19H.9 and Lemma 19H.8(c) together imply $\mathcal{L}[f](s) = \frac{r!}{s^{r+1}}$, in agreement with Corollary 19H.7.

(b) Let $r = 1/2$. Then $f(t) = \sqrt{t}$, and Proposition 19H.9 says $\mathcal{L}[f](s) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$, where the last step is by Exercise 19H.12(b).

(c) Let $r = -1/2$. Then $f(t) = \frac{1}{\sqrt{t}}$, and Proposition 19H.9 says $\mathcal{L}[f](s) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$, where the last step is by Exercise 19H.12(a). ♢

Theorems 19B.2(c) showed how the Fourier transform converts function convolution into multiplication. A similar property holds for the Laplace transform. Let $f, g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ be two functions. The **convolution** of f, g is the function $f * g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ defined

$$f * g(T) \quad := \quad \int_0^T f(T-t) g(t) \, dt, \qquad \text{for all } T \in \mathbb{R}_{\not-}.$$

Note that $f * g(T)$ is an integral over a *finite* interval $[0, T]$; thus it is well-defined no matter how fast $f(t)$ and $g(t)$ grow as $t \to \infty$.

Theorem 19H.11. *Let $f, g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ have exponential order. Then $\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$ wherever all these functions are defined.*

Proof. **Exercise 19H.14** □ Ⓔ

Theorem 19H.6(b) makes the Laplace transform a powerful tool for solving linear ordinary differential equations.

Proposition 19H.12. Laplace solution to linear ODE
Let $f, g : \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ have exponential order α and β respectively, and let $\gamma = \max\{\alpha, \beta\}$. Let $F := \mathcal{L}[f]$ and $G := \mathcal{L}[g]$. Let $c_0, c_1, \ldots, c_n \in \mathbb{C}$ be constants. Then f and g satisfies the linear ODE

$$g \quad = \quad c_0 f + c_1 f' + c_2 f'' + \cdots + c_n f^{(n)} \tag{19H.4}$$

if and only if F and G satisfy the algebraic equation

$$\begin{aligned} G(s) \quad = \quad & \left(c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots + c_n s^n\right) \; F(s) \\ & - \left(c_1 + c_2 s^1 + c_3 s^2 + \cdots + c_n s^{n-1}\right) \; f(0) \\ & - \left(c_2 + c_3 s + \cdots + c_n s^{n-2}\right) \; f'(0) \\ & \ddots \qquad \vdots \qquad \vdots \quad \vdots \\ & - (c_{n-1} + c_n s) \; f^{(n-2)}(0) \\ & - c_n \; f^{(n-1)}(0). \end{aligned}$$

for all $s \in \mathbb{H}_\gamma$. In particular, f satisfies ODE (19H.4) *and homogeneous boundary conditions $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$ if and only if*

$$F(s) \quad = \quad \frac{G(s)}{c_0 + c_1 s + c_2 s^2 + +c_3 s^3 + \cdots + c_n s^n}$$

for all $s \in \mathbb{H}_\gamma$

Ⓔ *Proof.* **Exercise 19H.15** (Hint: Apply Theorem 19H.6(b), then reorder terms.) □

Laplace transforms can also be used to solve *partial* differential equations. Let $\mathbb{X} \subseteq \mathbb{R}$ be some one-dimensional domain, let $f : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$, and write $f(x; t)$ as $f_x(t)$ for all $x \in \mathbb{X}$. Fix $\alpha \in \mathbb{R}$. For all $x \in \mathbb{X}$ suppose that f_x has exponential order α, so that $\mathcal{L}[f_x]$ is a function $\mathbb{H}_\alpha \longrightarrow \mathbb{C}$. Then we define $\mathcal{L}[f] : \mathbb{X} \times \mathbb{H}_\alpha \longrightarrow \mathbb{C}$ by $\mathcal{L}[f](x; s) = \mathcal{L}[f_x](s)$ for all $x \in \mathbb{X}$ and $s \in \mathbb{H}_\alpha$.

Proposition 19H.13. *Suppose $\partial_x f(x,t)$ is defined for all $(x,t) \in \mathsf{int}(\mathbb{X}) \times \mathbb{R}_+$. Then $\partial_x \mathcal{L}[f](x,s)$ is defined for all $(x,s) \in \mathsf{int}(\mathbb{X}) \times \mathbb{H}_\alpha$, and $\partial_x \mathcal{L}[f](x,s) = \mathcal{L}[\partial_x f](x,s)$.*

Ⓔ *Proof.* **Exercise 19H.16** (Hint: Apply Theorem 0G.1 (on page 567) to the Laplace transform integral (19H.2).) □

By iterating Proposition 19H.13, we have $\partial_x^n \mathcal{L}[f](x,s) = \mathcal{L}[\partial_x^n f](x,s)$ for any $n \in \mathbb{N}$. Through Proposition 19H.13 and Theorem 19H.6(b), we can convert a PDE about f into an ODE involving only the x-derivatives of $\mathcal{L}[f]$.

Example 19H.14. Let $f : \mathbb{X} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{C}$ and let $F := \mathcal{L}[f] : \mathbb{X} \times \mathbb{H}_\alpha \longrightarrow \mathbb{C}$.

(a) (*heat equation*) Define $f_0(x) := f(x,0)$ for all $x \in \mathbb{X}$ (the 'initial temperature distribution'). Then f satisfies the one-dimensional heat equation $\partial_t f(x,t) = \partial_x^2 f(x,t)$ if and only if, for every $s \in \mathbb{H}_\alpha$, the function $F_s(x) = F(x,s)$ satisfies the second-order linear ODE

$$\partial_x^2 F_s(x) \quad = \quad s\, F_s(x) - f_0(x), \qquad \text{for all } x \in \mathbb{X}.$$

(b) (*wave equation*) Define $f_0(x) := f(x,0)$ and $f_1(x) := \partial_t f(x,0)$ for all $x \in \mathbb{X}$ (the 'initial position' and 'initial velocity', respectively). Then f satisfies the one-dimensional wave equation $\partial_t^2 f(x,t) = \partial_x^2 f(x,t)$ if and only if, for every $s \in \mathbb{H}_\alpha$, the function $F_s(x) = F(x,s)$ satisfies the second-order linear ODE

$$\partial_x^2 F_s(x) \quad = \quad s^2 F_s(x) - s\, f_0(x) - f_1(x), \qquad \text{for all } x \in \mathbb{X}.$$

(**Exercise 19H.17** Verify examples (a) and (b).) ♢ Ⓔ

We can then use solution methods for ordinary differential equations to solve for F_s for all $s \in \mathbb{H}_\alpha$, obtain an expression for the function F, and then apply the Laplace Inversion Theorem 19H.4 to obtain an expression for f. We will not pursue this approach further here; however, we will develop a very similar approach in the Chapter 20 using Fourier transforms. For more information, see [Asm05, Chapt.8], [Fis99, §5.3-5.4], [Hab87, Chapt.13], or [Bro89, Chapt.5]

19I Practice problems

1. Suppose $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1; \\ 0 & \text{otherwise} \end{cases}$, as in Example 19A.4 on page 490.
   Check that $\widehat{f}(\mu) = \dfrac{1 - e^{-\mu \mathbf{i}}}{2\pi\mu\mathbf{i}}$

2. Compute the one-dimensional Fourier transforms of $g(x)$, when:

   (a) $g(x) = \begin{cases} 1 & \text{if} \quad -\tau < x < 1-\tau; \\ 0 & \text{otherwise} \end{cases}$,

   (b) $g(x) = \begin{cases} 1 & \text{if } 0 < x < \sigma; \\ 0 & \text{otherwise} \end{cases}$.

3. Let $X, Y > 0$, and let $f(x,y) = \begin{cases} 1 & \text{if} \quad 0 \le x \le X \text{ and } 0 \le y \le Y; \\ 0 & \text{otherwise.} \end{cases}$.
   Compute the two-dimensional Fourier transform of $f(x,y)$. What does the Fourier Inversion formula tell us?

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined: $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$
   (Fig.19I.1) Compute the **Fourier Transform** of f.

5. Let $f(x) = x \cdot \exp\left(\dfrac{-x^2}{2}\right)$. Compute the Fourier transform of f.

6. Let $\alpha > 0$, and let $g(x) = \dfrac{1}{\alpha^2 + x^2}$. Example 19A.8 claims that $\widehat{g}(\mu) = \frac{1}{2\alpha} e^{-\alpha|\mu|}$. Verify this. **Hint:** *Use the Fourier Inversion Theorem.*

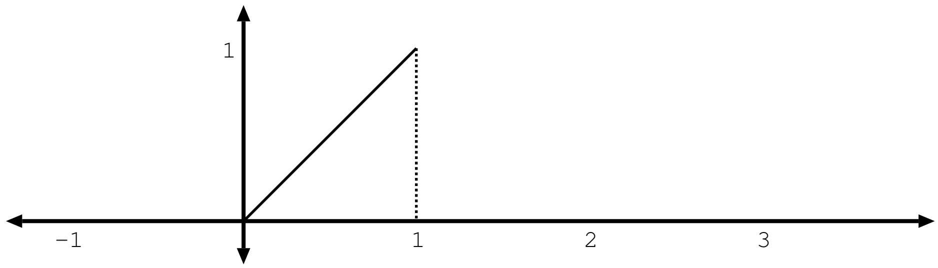


Figure 19I.1: Problem #4

7. Fix $y > 0$, and let $\mathcal{K}_y(x) = \dfrac{y}{\pi(x^2+y^2)}$ (this is the *half-space Poisson Kernel* from §17E and §20C(ii)). Compute the one-dimensional Fourier transform $\widehat{\mathcal{K}}_y(\mu) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathcal{K}_y(x) \exp\Big(-\mu \mathbf{i} x\Big)\ d\mu$.

8. Let $f(x) = \dfrac{2x}{(1+x^2)^2}$. Compute the Fourier transform of f.

9. Let $f(x) = \begin{cases} 1 & \text{if} \quad -4 < x < 5; \\ 0 & \quad\text{otherwise.} \end{cases}$ Compute the Fourier transform $\widehat{f}(\mu)$.

10. Let $f(x) = \dfrac{x\cos(x) - \sin(x)}{x^2}$. Compute the Fourier transform $\widehat{f}(\mu)$.

11. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h(x) = f(x) + g(x)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = \widehat{f}(\mu) + \widehat{g}(\mu)$.

12. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h = f * g$. Show that for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{g}(\mu)$.

    Hint: $\exp(-\mathbf{i}\mu x) = \exp(-\mathbf{i}\mu y) \cdot \exp\Big(-\mathbf{i}\mu(x-y)\Big)$.

13. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h(x) = f(x) \cdot g(x)$. Suppose $\widehat{h}$ is also in $\mathbf{L}^1(\mathbb{R})$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = (\widehat{f} * \widehat{g})(\mu)$.

    Hint: *Combine problem #12 with the Strong Fourier Inversion Formula (Theorem 19A.5 on page 491).*

14. Let $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\tau \in \mathbb{R}$, and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by: $g(x) = f(x + \tau)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}} \cdot \widehat{f}(\mu)$.

15. Let $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\nu \in \mathbb{R}$ and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by $g(x) = e^{\nu x\mathbf{i}} f(x)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \widehat{f}(\mu - \nu)$.

16. Suppose $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\sigma > 0$, and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by: $g(x) = f\left(\frac{x}{\sigma}\right)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \sigma \cdot \widehat{f}(\sigma \cdot \mu)$.

17. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable, and that $f \in \mathbf{L}^1(\mathbb{R})$ and $g := f' \in \mathbf{L}^1(\mathbb{R})$. Assume that $\lim_{x\to\pm\infty} f(x) = 0$. Show that $\widehat{g}(\mu) = \mathbf{i}\mu \cdot \widehat{f}(\mu)$.

18. Let $\mathcal{G}_t(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$ be the Gauss-Weierstrass kernel. Recall that $\widehat{\mathcal{G}_t}(\mu) = \frac{1}{2\pi} e^{-\mu^2 t}$. Use this to construct a simple proof that, for any $s, t > 0$, $\mathcal{G}_t * \mathcal{G}_s = \mathcal{G}_{t+s}$.

    (**Hint:** *Use problem #12. Do* **not** *compute any convolution integrals, and do* **not** *use the 'solution to the heat equation' argument from Problem # 8 on page 413.*)

    Remark. Because of this result, probabilists say that the set $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ forms a *stable family of probability distributions* on $\mathbb{R}$. Analysts say that $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ is a *one-parameter semigroup* under convolution.

Chapter 20

Fourier transform solutions to PDEs

"Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them." —Jean Joseph Fourier

We will now see that the 'Fourier series' solutions to the PDEs on a bounded domain (Chapters 11-14) generalize to 'Fourier transform' solutions on the unbounded domain in a natural way.

20A The heat equation

20A(i) Fourier transform solution

Prerequisites: §1B, §19A, §5B, §0G. **Recommended:** §11A, §12B, §13A, §19D, §19E.

Proposition 20A.1. Heat equation on an infinite rod
Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded function (of $\mu \in \mathbb{R}$).

(a) *Define $u : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by*

$$u(x;t) \quad := \quad \int_{-\infty}^{\infty} F(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \, d\mu, \tag{20A.1}$$

for all $t > 0$ and all $x \in \mathbb{R}$. Then u is a smooth function and satisfies the heat equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R})$, and $\widehat{f}(\mu) = F(\mu)$. If we define $u(x;0) := f(x)$ for all $x \in \mathbb{R}$, and define $u(x;t)$ by eqn.(20A.1), when $t > 0$, then u is continuous on $\mathbb{R} \times \mathbb{R}_{\not -}$, and is a solution to the heat equation with initial conditions $u(x;0) = f(x)$.*

Proof. **Exercise 20A.1** (Hint: Use Proposition 0G.1 on page 567.) □ Ⓔ

Example 20A.2. Suppose $f(x) = \begin{cases} 1 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$ We know from Example 19A.3 on page 489 that $\widehat{f}(\mu) = \dfrac{\sin(\mu)}{\pi\mu}$. Thus,

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\mu)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2 t}\, d\mu = \int_{-\infty}^{\infty} \frac{\sin(\mu)}{\pi\mu}\exp(\mu x\mathbf{i})\cdot e^{-\mu^2 t}\, d\mu.$$

Ⓔ (**Exercise 20A.2** Verify that u satisfies the one-dimensional heat equation and the specified initial conditions.) ♢

Example 20A.3: The Gauss-Weierstrass kernel

For all $x \in \mathbb{R}$ and $t > 0$, define the **Gauss-Weierstrass Kernel**: $\mathcal{G}_t(x) := \dfrac{1}{2\sqrt{\pi t}}\exp\left(\dfrac{-x^2}{4t}\right)$ (see Example 1B.1(c) on page 6). Fix $t > 0$; then setting $\sigma = \sqrt{2t}$ in Theorem 19B.8**(b)**, we get

$$\widehat{\mathcal{G}}_t(\mu) = \frac{1}{2\pi}\exp\left(\frac{-(\sqrt{2t})^2\mu^2}{2}\right) = \frac{1}{2\pi}\exp\left(\frac{-2t\mu^2}{2}\right) = \frac{1}{2\pi}e^{-\mu^2 t}.$$

Thus, applying the Fourier Inversion formula (Theorem 19A.1 on page 488), we have:

$$\mathcal{G}(x,t) = \int_{-\infty}^{\infty} \widehat{\mathcal{G}}_t(\mu)\exp(\mu x\mathbf{i})\, d\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mu^2 t}\exp(\mu x\mathbf{i})\, d\mu,$$

which, according to Proposition 20A.1, is a smooth solution of the heat equation, where we take $F(\mu)$ to be the *constant* function: $F(\mu) = 1/2\pi$. Thus, F is *not* the Fourier transform of any function f. Hence, the Gauss-Weierstrass kernel solves the heat equation, but the "initial conditions" $\mathcal{G}_0$ do not correspond to a function, but instead a define more singular object, rather like an infinitely dense concentration of mass at a single point. Sometimes $\mathcal{G}_0$ is called the **Dirac delta function**, but this is a misnomer, since it isn't really a function. Instead, $\mathcal{G}_0$ is an example of a more general class of objects called *distributions.* ♢

Proposition 20A.4. Heat equation on an infinite plane

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be some bounded function (of $(\mu,\nu) \in \mathbb{R}^2$).

(a) *Define $u : \mathbb{R}^2 \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by*

$$u(x,y;\ t) := \int_{\mathbb{R}^2} F(\mu,\nu)\cdot\exp\Big((\mu x + \nu y)\cdot\mathbf{i}\Big)\cdot e^{-(\mu^2+\nu^2)t}\, d\mu\, d\nu, \qquad (20A.2)$$

for all $t > 0$ and all $(x,y) \in \mathbb{R}^2$. Then u is continuous on $\mathbb{R}^3 \times \mathbb{R}_+$ and satisfies the two-dimensional heat equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R}^2)$, and $\widehat{f}(\mu,\nu) = F(\mu,\nu)$. If we define $u(x,y,0) := f(x,y)$ for all $(x,y) \in \mathbb{R}^2$, and define $u(x,y,t)$ by eqn.(20A.2) when $t > 0$, then u is continuous on $\mathbb{R}^2 \times \mathbb{R}_{\not-}$, and is a solution to the heat equation with initial conditions $u(x,y,0) = f(x,y)$.*

Proof. **Exercise 20A.3** (Hint: Use Proposition 0G.1 on page 567.) □ Ⓔ

Example 20A.5. Let $X, Y > 0$ be constants, and consider the initial conditions:

$$f(x,y) \quad = \quad \begin{cases} 1 & \text{if } -X \leq x \leq X \text{ and } -Y \leq y \leq Y; \\ 0 & \text{otherwise.} \end{cases}$$

From Example 19D.4 on page 505, the Fourier transform of $f(x,y)$ is given:

$$\widehat{f}(\mu,\nu) \quad = \quad \frac{\sin(\mu X) \cdot \sin(\nu Y)}{\pi^2 \cdot \mu \cdot \nu}.$$

Thus, the corresponding solution to the two-dimensional heat equation is:

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^2} \widehat{f}(\mu,\nu) \cdot \exp\Big((\mu x + \nu y) \cdot \mathbf{i}\Big) \cdot e^{-(\mu^2+\nu^2)t} \, d\mu \, d\nu \\ &= \int_{\mathbb{R}^2} \frac{\sin(\mu X) \cdot \sin(\nu Y)}{\pi^2 \cdot \mu \cdot \nu} \cdot \exp\Big((\mu x + \nu y) \cdot \mathbf{i}\Big) \cdot e^{-(\mu^2+\nu^2)t} \, d\mu \, d\nu \ . \quad \diamondsuit \end{aligned}$$

Proposition 20A.6. Heat equation in infinite space

Let $F : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be some bounded function (of $\boldsymbol{\mu} \in \mathbb{R}^3$).

(a) *Define $u : \mathbb{R}^3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by*

$$u(\mathbf{x};\ t) \ := \int_{\mathbb{R}^3} F(\boldsymbol{\mu}) \cdot \exp\Big(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}\Big) \cdot e^{-\|\boldsymbol{\mu}\|^2 t} \, d\boldsymbol{\mu}, \qquad (20A.3)$$

for all $t > 0$ and all $\mathbf{x} \in \mathbb{R}^3$, (where $\|\boldsymbol{\mu}\|^2 := \mu_1^2 + \mu_2^2 + \mu_3^2$). Then u is continuous on $\mathbb{R}^3 \times \mathbb{R}_+$ and satisfies the three-dimensional heat equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R}^3)$, and $\widehat{f}(\boldsymbol{\mu}) = F(\boldsymbol{\mu})$. If we define $u(\mathbf{x},0) := f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$, and define $u(\mathbf{x},t)$ by eqn.(20A.3) when $t > 0$, then u is continuous on $\mathbb{R}^3 \times \mathbb{R}_{\not-}$, and is a solution to the heat equation with initial conditions $u(\mathbf{x},0) = f(\mathbf{x})$.*

Proof. **Exercise 20A.4** (Hint: Use Proposition 0G.1 on page 567.) □ Ⓔ

Example 20A.7: A ball of heat

Suppose the initial conditions are: $f(\mathbf{x}) \;=\; \begin{cases} 1 & \text{if } \|\mathbf{x}\| \leq 1; \\ 0 & \text{otherwise.} \end{cases}$

Setting $R = 1$ in Example 19E.3 (p.507) yields the three-dimensional Fourier transform of f:

$$\widehat{f}(\boldsymbol{\mu}) \quad = \quad \frac{1}{2\pi^2}\left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right).$$

The resulting solution to the heat equation is:

$$u(\mathbf{x};\ t) \quad = \quad \int_{\mathbb{R}^3} \widehat{f}(\boldsymbol{\mu}) \cdot \exp\Big(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}\Big) \cdot e^{-\|\mu\|^2 t}\ d\boldsymbol{\mu}$$

$$= \quad \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right) \cdot \exp\Big(\boldsymbol{\mu} \bullet \mathbf{x} \cdot \mathbf{i}\Big) \cdot e^{-\|\mu\|^2 t}\ d\boldsymbol{\mu}. \quad \diamondsuit$$

20A(ii) The Gaussian convolution formula, revisited

Prerequisites: §17C(i), §19B, §20A(i).

Recall from § 17C(i) on page 385 that the Gaussian Convolution formula solved the initial value problem for the heat equation by "locally averaging" the initial conditions. Fourier methods provide another proof that this is a solution to the heat equation.

Theorem 20A.8. Gaussian convolutions and the heat equation

Let $f \in \mathbf{L}^1(\mathbb{R})$, and let $\mathcal{G}_t(x)$ be the Gauss-Weierstrass kernel from Example 20A.3. For all $t > 0$, define $U_t := f * \mathcal{G}_t$; in other words, for all $x \in \mathbb{R}$,

$$U_t(x) \quad := \quad \int_{-\infty}^{\infty} f(y) \cdot \mathcal{G}_t(x - y)\ dy.$$

Also, for all $x \in \mathbb{R}$, define $U_0(x) := f(x)$. Then U is continuous on $\mathbb{R} \times \mathbb{R}_{\not-}$, and is a solution to the Heat Equation with initial conditions $U(x, 0) = f(x)$.

Proof. $U(x, 0) = f(x)$ by definition. To show that U satisfies the heat equation, we will show that it is in fact *equal* to the Fourier solution u described in Theorem 20A.1 on page 527. Fix $t > 0$, and let $u_t(x) = u(x, t)$; recall that, by definition

$$u_t(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t}\ d\mu \;=\; \int_{-\infty}^{\infty} \widehat{f}(\mu) e^{-\mu^2 t} \cdot \exp(\mu x \mathbf{i})\ d\mu$$

Thus, Proposition 19A.2 on page 489 says that

$$\widehat{u}_t(\mu) \quad = \quad \widehat{f}(\mu) \cdot e^{-t\mu^2} \quad \underset{(*)}{=} \quad 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{\mathcal{G}}_t(\mu). \tag{20A.4}$$

Here, $(*)$ is because Example 20A.3 on page 528 says that $e^{-t\mu^2} = 2\pi \cdot \widehat{\mathcal{G}}_t(\mu)$.

But remember that $U_t = f * \mathcal{G}_t$, so, Theorem 19B.2(b) on page 494 says

$$\widehat{U}_t(\mu) \quad = \quad 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{\mathcal{G}}_t(\mu). \tag{20A.5}$$

Thus (20A.4) and (20A.5) mean that $\widehat{U}_t = \widehat{u}_t$. But then Proposition 19A.2 on page 489 implies that $u_t(x) = U_t(x)$. □

For more discussion and examples of the Gaussian convolution approach to the heat equation, see § 17C(i) on page 385.

Exercise 20A.5. State and prove a generalization of Theorem 20A.8 to solving the D-dimensional heat equation, for $D \geq 2$. ♦ Ⓔ

20B The wave equation

20B(i) Fourier transform solution

Prerequisites: §2B, §19A, §5B, §0G. **Recommended:** §11B, §12D, §19D, §19E, §20A(i).

Proposition 20B.1. Wave equation on an infinite wire

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R})$ be twice-differentiable, and suppose f_0 and f_1 have Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$, respectively. Define $u : \mathbb{R} \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ by

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \left(\widehat{f_0}(\mu)\cos(\mu t) + \frac{\widehat{f_1}(\mu)}{\mu}\sin(\mu t) \right) \cdot \exp(\mu x \mathbf{i})\; d\mu.$$

Then u is a solution to the one-dimensional wave equation with initial position *$u(x,0) \;=\; f_0(x)$, and* initial velocity *$\partial_t\, u(x,0) \;=\; f_1(x)$, for all $x \in \mathbb{R}$.*

Proof. **Exercise 20B.1** (Hint: Show that this solution is equivalant to the d'Alembert solution of Proposition 17D.5 on page 398.) □ Ⓔ

Example 20B.2. Fix $\alpha > 0$, and suppose we have initial position $f_0(x) := e^{-\alpha|x|}$ for all $x \in \mathbb{R}$, and initial velocity $f_1 \equiv 0$. From Example 19A.7 on page 491, we know that $\widehat{f_0}(\mu) = \dfrac{2\alpha}{(\alpha^2+\mu^2)}$. Thus, Proposition 20B.1 says:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \widehat{f_0}(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot \cos(\mu t)\ d\mu \\ &= \int_{-\infty}^{\infty} \frac{2\alpha}{(\alpha^2+\mu^2)} \cdot \exp(\mu x \mathbf{i}) \cdot \cos(\mu t)\ d\mu. \end{aligned}$$

Ⓔ (**Exercise 20B.2** Verify that u satisfies the one-dimensional wave equation and the specified initial conditions.) ♢

Proposition 20B.3. Wave equation on an infinite plane

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^2)$ be twice differentiable functions, whose Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$ decay fast enough that

$$\begin{aligned} &\int_{\mathbb{R}^2} (\mu^2+\nu^2) \cdot \left|\widehat{f_0}(\mu,\nu)\right|\ d\mu\ d\nu \quad < \quad \infty \\ \text{and} \quad &\int_{\mathbb{R}^2} \sqrt{\mu^2+\nu^2} \cdot \left|\widehat{f_1}(\mu,\nu)\right|\ d\mu\ d\nu \quad < \quad \infty. \end{aligned} \tag{20B.6}$$

Define $u : \mathbb{R}^2 \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} u(x,y,t) \quad = \quad &\int_{\mathbb{R}^2} \widehat{f_0}(\mu,\nu) \cos\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right) \cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu. \\ &+ \int_{\mathbb{R}^2} \frac{\widehat{f_1}(\mu,\nu)}{\sqrt{\mu^2+\nu^2}} \sin\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right) \cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu. \end{aligned}$$

Then u is a solution to the two-dimensional wave equation with initial position $u(x,y,0) = f_0(x,y)$, and initial velocity $\partial_t u(x,y,0) = f_1(x,y)$, for all $(x,y) \in \mathbb{R}^2$.

Ⓔ *Proof.* **Exercise 20B.3** (*Hint:* Equation (20B.6) makes the integral absolutely convergent, and also enables you to apply Proposition 0G.1 on page 567 to compute the relevant derivatives of u.) □

Example 20B.4. Let $\alpha, \beta > 0$ be constants, and suppose we have initial position $f_0 \equiv 0$, and initial velocity $f_1(x,y) = \dfrac{1}{(\alpha^2+x^2)(\beta^2+y^2)}$ for all $(x,y) \in \mathbb{R}^2$. By adapting Example 19A.8 on page 492, one can check that

$$\widehat{f_1}(\mu,\nu) \quad = \quad \frac{1}{4\alpha\beta} \exp\Big(-\alpha\cdot|\mu| - \beta\cdot|\nu|\Big).$$

Thus, Proposition 20B.3 says

$$u(x,y,t) \;=\; \int_{\mathbb{R}^2} \frac{\widehat{f_1}(\mu,\nu)}{\sqrt{\mu^2+\nu^2}} \sin\left(\sqrt{\mu^2+\nu^2}\;\cdot t\right)\cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\; d\mu\; d\nu$$

$$=\; \int_{\mathbb{R}^2} \frac{\sin\left(\sqrt{\mu^2+\nu^2}\;\cdot t\right)\cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i} - \alpha\cdot|\mu| - \beta\cdot|\nu|\Big)}{4\alpha\beta\sqrt{\mu^2+\nu^2}}\; d\mu\; d\nu.$$

(**Exercise 20B.4** Verify that u satisfies the two-dimensional wave equation and the specified initial conditions.) $\diamondsuit$ Ⓔ

Proposition 20B.5. Wave equation in infinite space
Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$ be twice differentiable functions whose Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$ decay fast enough that

$$\int_{\mathbb{R}^3} \|\boldsymbol{\mu}\|^2 \cdot \left|\widehat{f_0}(\boldsymbol{\mu})\right|\; d\boldsymbol{\mu} \;<\; \infty$$
$$\text{and} \qquad \int_{\mathbb{R}^3} \|\boldsymbol{\mu}\| \cdot \left|\widehat{f_1}(\boldsymbol{\mu})\right|\; d\boldsymbol{\mu} \;<\; \infty. \tag{20B.7}$$

Define $u : \mathbb{R}^3 \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ by

$$u(\mathbf{x},t) \;:=\; \int_{\mathbb{R}^3} \left(\widehat{f_0}(\boldsymbol{\mu})\cos\left(\|\boldsymbol{\mu}\|\cdot t\right) \;+\; \frac{\widehat{f_1}(\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|}\sin\left(\|\boldsymbol{\mu}\|\cdot t\right)\right)\cdot \exp\left(\boldsymbol{\mu} \bullet \mathbf{x}\mathbf{i}\right).\; d\boldsymbol{\mu}$$

Then u is a solution to the three-dimensional wave equation with initial position *$u(\mathbf{x},0) = f_0(\mathbf{x})$ and* initial velocity *$\partial_t\, u(\mathbf{x},0) = f_1(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^3$.*

Proof. **Exercise 20B.5** (*Hint:* Equation (20B.7) makes the integral absolutely convergent, and also enables you to apply Proposition 0G.1 on page 567 to compute the relevant derivatives of u.) $\square$ Ⓔ

Exercise 20B.6. Show that the decay conditions (20B.6) or (20B.7) are satisfied if f_0 and f_1 are *asymptotically flat* in the sense that $\lim_{|\mathbf{x}|\to\infty} |f(\mathbf{x})| = 0$ and $\lim_{|\mathbf{x}|\to\infty} |\nabla f(\mathbf{x})| = 0$, while $(\partial_i\,\partial_j\, f) \in \mathbf{L}^1(\mathbb{R}^2)$ for all $i,j \in \{1,...,D\}$ (where $D = 2$ or 3). Ⓔ

Hint. Apply Theorem 19B.7 on page 496 to compute the Fourier transforms of the derivative functions $\partial_i\,\partial_j\, f$; conclude that the function $\widehat{f}$ must itself decay at a certain speed. ♦

20B(ii) Poisson's spherical mean solution; Huygen's principle

Prerequisites: §17A, §19E, §20B(i). **Recommended:** §17D, §20A(ii).

The Gaussian Convolution formula of §20A(ii) solves the initial value problem for the heat equation in terms of a kind of "local averaging" of the initial conditions. Similarly, d'Alembert's formula (§17D) solves the initial value problem for the one-dimensional wave equation in terms of a local average.

There is an analogous result for higher-dimensional wave equations. To explain it, we must introduce the concept of *spherical averages*. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be some integrable function. If $\mathbf{x} \in \mathbb{R}^3$ is a point in space, and $R > 0$, then the **spherical average** of f at $\mathbf{x}$, of radius R, is defined:

$$\mathbf{M}_R\, f(\mathbf{x}) \quad := \quad \frac{1}{4\pi R^2} \int_{\mathbb{S}(R)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s}.$$

Here, $\mathbb{S}(R) := \left\{\mathbf{s} \in \mathbb{R}^3\ ;\ \|\mathbf{s}\| = R\right\}$ is the sphere around 0 of radius R. The total surface area of the sphere is $4\pi R^2$; we divide by this quantity to obtain an average. We adopt the notational convention that $\mathbf{M}_0\, f(\mathbf{x}) := f(\mathbf{x})$. This is justified by the next exercise.

Ⓔ **Exercise 20B.7.** Suppose f is continuous at $\mathbf{x}$. Show that $\lim_{R\to 0} \mathbf{M}_R\, f(\mathbf{x}) = f(\mathbf{x})$. ♦

Theorem 20B.6. Poisson's Spherical Mean Solution to wave equation

(a) *Let $f_1 \in \mathbf{L}^1(\mathbb{R}^3)$ be twice-differentiable, and such that $\widehat{f_1}$ satisfies eqn.(20B.7) on page 533. Define $v : \mathbb{R}^3 \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

$$v(\mathbf{x};t) \quad := \quad t \cdot \mathbf{M}_t\, f_1(\mathbf{x}), \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^3 \textit{ and } t \geq 0.$$

Then v is a solution to the wave equation with

Initial Position: $v(\mathbf{x},0) \ = \ 0$; **Initial Velocity:** $\partial_t\, v(\mathbf{x},0) \ = \ f_1(\mathbf{x})$.

(b) *Let $f_0 \in \mathbf{L}^1(\mathbb{R}^3)$ be twice-differentiable and such that $\widehat{f_0}$ satisfies eqn.(20B.7) on page 533. For all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, define $W(\mathbf{x};t) \ := \ t \cdot \mathbf{M}_t\, f_0(\mathbf{x})$, and then define $w : \mathbb{R}^3 \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

$$w(\mathbf{x};t) \quad := \quad \partial_t\, W(\mathbf{x};t), \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^3 \textit{ and } t \geq 0.$$

Then w is a solution to the wave equation with

Initial Position: $w(\mathbf{x},0) \ = \ f_0(\mathbf{x})$; **Initial Velocity:** $\partial_t\, w(\mathbf{x},0) \ = \ 0$.

(c) *Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$ be as in (a) and (b), and define $u : \mathbb{R}^3 \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by*

$$u(\mathbf{x};t) \quad := \quad w(\mathbf{x};t) \ + \ v(\mathbf{x};t), \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^3 \textit{ and } t \geq 0,$$

where w is as in Part **(b)** *and v is as in Part* **(a)**. *Then u is the unique solution to the wave equation with*

Initial Position: $u(\mathbf{x},0) \ = \ f_0(\mathbf{x})$; **Initial Velocity:** $\partial_t\, u(\mathbf{x},0) \ = \ f_1(\mathbf{x})$.

Proof. We will prove Part **(a)**. First we will need a certain calculation.

Claim 1: *For any $R > 0$, and any $\boldsymbol{\mu} \in \mathbb{R}^3$,*

$$\int_{\mathbb{S}(R)} \exp\Big(\boldsymbol{\mu} \bullet \mathbf{s}\mathbf{i}\Big) \ d\mathbf{s} \quad = \quad \frac{4\pi R \cdot \sin\left(\|\boldsymbol{\mu}\| \cdot R\right)}{\|\boldsymbol{\mu}\|}.$$

Proof. By spherical symmetry, we can rotate the vector $\boldsymbol{\mu}$ without affecting the value of the integral, so rotate $\boldsymbol{\mu}$ until it becomes $\boldsymbol{\mu} = (\mu, 0, 0)$, with $\mu > 0$. Thus, $\|\boldsymbol{\mu}\| = \mu$, and, if a point $\mathbf{s} \in \mathbb{S}(R)$ has coordinates (s_1, s_2, s_3) in $\mathbb{R}^3$, then $\boldsymbol{\mu} \bullet \mathbf{s} = \mu \cdot s_1$. Thus, the integral simplifies to:

$$\int_{\mathbb{S}(R)} \exp\left(\boldsymbol{\mu} \bullet \mathbf{s}\mathbf{i}\right) \ d\mathbf{s} \ = \ \int_{\mathbb{S}(R)} \exp\left(\mu \cdot s_1 \cdot \mathbf{i}\right) \ d\mathbf{s}$$

We will integrate using a spherical coordinate system (ϕ, θ) on the sphere, where $0 < \phi < \pi$ and $-\pi < \theta < \pi$, and where

$$(s_1, s_2, s_3) \ = \ R \cdot \left(\cos(\phi), \quad \sin(\phi)\sin(\theta), \quad \sin(\phi)\cos(\theta)\right).$$

On the sphere of radius R, the surface area element is $d\mathbf{s} = R^2 \sin(\phi) \ d\theta \ d\phi$. Thus,

$$\begin{aligned}
\int_{\mathbb{S}(R)} \exp\left(\mu \cdot s_1 \cdot \mathbf{i}\right) \ d\mathbf{s} \quad &= \quad \int_0^\pi \int_{-\pi}^{\pi} \exp\left(\mu \cdot R \cdot \cos(\phi) \cdot \mathbf{i}\right) \cdot R^2 \sin(\phi) \ d\theta \ d\phi \\
&\underset{(*)}{=} \quad 2\pi \int_0^\pi \exp\left(\mu \cdot R \cdot \cos(\phi) \cdot \mathbf{i}\right) \cdot R^2 \sin(\phi) \ d\phi \\
&\underset{(\diamond)}{=} \quad 2\pi \int_{-R}^{R} \exp\left(\mu \cdot s_1 \cdot \mathbf{i}\right) \cdot R \ ds_1 \\
&= \quad \frac{2\pi R}{\mu \mathbf{i}} \exp\Big(\mu \cdot s_1 \cdot \mathbf{i}\Big)_{s_1=-R}^{s_1=R} \\
&= \quad 2\frac{2\pi R}{\mu} \cdot \left(\frac{e^{\mu R \mathbf{i}} - e^{-\mu R \mathbf{i}}}{2\mathbf{i}}\right) \quad \underset{(\dagger)}{=} \quad \frac{4\pi R}{\mu} \sin\left(\mu R\right).
\end{aligned}$$

$(*)$ The integrand is constant in the θ coordinate. $(\diamond)$ Making substitution $s_1 = R\cos(\phi)$, so $ds_1 = -R\sin(\phi) \ d\phi$. $(\dagger)$ By Euler's Formula (see page 551).

$\diamond_{\text{Claim 1}}$

Now, by Proposition 20B.5 on page 533, a solution to the three-dimensional wave equation with zero initial position and initial velocity f_1 is given by:

$$u(\mathbf{x},t) \;=\; \int_{\mathbb{R}^3} \widehat{f_1}(\boldsymbol{\mu}) \frac{\sin\left(\|\boldsymbol{\mu}\|\cdot t\right)}{\|\boldsymbol{\mu}\|} \exp\left(\boldsymbol{\mu} \bullet \mathbf{x}\mathbf{i}\right)\; d\boldsymbol{\mu}. \tag{20B.8}$$

However, if we set $R = t$ in **Claim 1**, we have:

$$\frac{\sin\left(\|\boldsymbol{\mu}\|\cdot t\right)}{\|\boldsymbol{\mu}\|} \;=\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\left(\boldsymbol{\mu}\bullet \mathbf{s}\mathbf{i}\right)\; d\mathbf{s}.$$

Thus,

$$\begin{aligned}
\frac{\sin\left(\|\boldsymbol{\mu}\|\cdot t\right)}{\|\boldsymbol{\mu}\|}\cdot \exp\left(\boldsymbol{\mu}\bullet \mathbf{x}\mathbf{i}\right) \;&=\; \exp\left(\boldsymbol{\mu}\bullet \mathbf{x}\mathbf{i}\right)\cdot \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\left(\boldsymbol{\mu}\bullet \mathbf{s}\mathbf{i}\right)\; d\mathbf{s}\\
&=\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\boldsymbol{\mu}\bullet \mathbf{x}\mathbf{i} + \boldsymbol{\mu}\bullet \mathbf{s}\mathbf{i}\Big)\; d\mathbf{s}\\
&=\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\boldsymbol{\mu}\bullet (\mathbf{x}+\mathbf{s})\,\mathbf{i}\Big)\; d\mathbf{s}.
\end{aligned}$$

Substituting this into (20B.8), we get:

$$\begin{aligned}
u(\mathbf{x},t) \;&=\; \int_{\mathbb{R}^3} \frac{\widehat{f_1}(\boldsymbol{\mu})}{4\pi t}\cdot \left(\int_{\mathbb{S}(t)} \exp\Big(\boldsymbol{\mu}\bullet (\mathbf{x}+\mathbf{s})\,\mathbf{i}\Big)\; d\mathbf{s}\right)\; d\boldsymbol{\mu}\\
&\underset{(*)}{=}\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \int_{\mathbb{R}^3} \widehat{f_1}(\boldsymbol{\mu})\cdot \exp\Big(\boldsymbol{\mu}\bullet (\mathbf{x}+\mathbf{s})\,\mathbf{i}\Big)\; d\boldsymbol{\mu}\; d\mathbf{s}\\
&\underset{(\diamond)}{=}\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} f_1(\mathbf{x}+\mathbf{s})\; d\mathbf{s} \quad=\quad t\cdot \frac{1}{4\pi t^2}\int_{\mathbb{S}(t)} f_1(\mathbf{x}+\mathbf{s})\; d\mathbf{s}\\
&=\; t\cdot \mathbf{M}_t\, f_1(\mathbf{x}).
\end{aligned}$$

$(*)$ We simply interchange the two integrals[1]. $(\diamond)$ This is just the Fourier Inversion Theorem 19E.1 on page 507.

Ⓔ Part **(b)** is **<u>Exercise 20B.8</u>** . Part **(c)** follows by combining Part **(a)** and Part **(b)**. □

Corollary 20B.7. Huygen's principle

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$, and suppose there is some bounded region $\mathbb{K} \subset \mathbb{R}^3$ such that f_0 and f_1 are zero outside of $\mathbb{K}$ —that is: $f_0(\mathbf{y}) = 0$ and $f_1(\mathbf{y}) = 0$ for all $\mathbf{y} \notin \mathbb{K}$ (see Figure 20B.1A). Let $u : \mathbb{R}^3 \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ be the solution to the wave equation with initial position f_0 and initial velocity f_1, and let $\mathbf{x} \in \mathbb{R}^3$.

[1]This actually involves some subtlety, which we will gloss over.

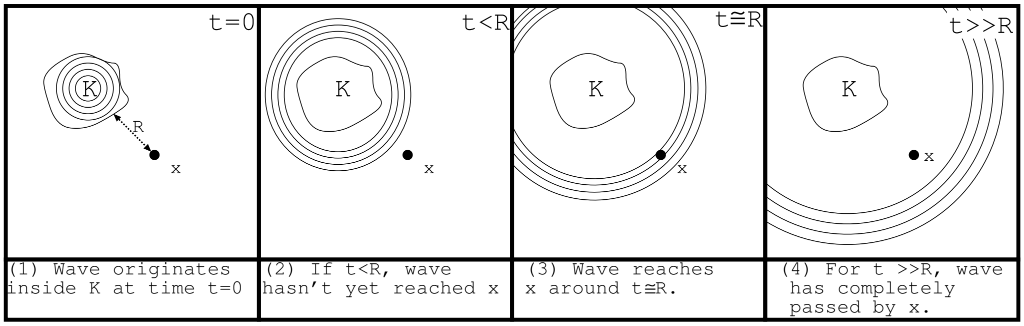


Figure 20B.1: Huygen's principle.

(a) *Let R be the distance from $\mathbb{K}$ to $\mathbf{x}$. If $t < R$ then $u(\mathbf{x}; t) = 0$ (Figure 20B.1B).*

(b) *If t is large enough that $\mathbb{K}$ is* entirely contained *in a ball of radius t around $\mathbf{x}$, then $u(\mathbf{x}; t) = 0$ (Figure 20B.1D).*

Proof. **Exercise 20B.9** □ Ⓔ

Part **(a)** of Huygen's Principle says that, if a sound wave originates in the region $\mathbb{K}$ at time 0, and $\mathbf{x}$ is of distance R then it does not reach the point $\mathbf{x}$ before time R. This is not surprising; it takes time for sound to travel through space. Part **(b)** says that the soundwave propagates through the point $\mathbf{x}$ in a *finite* amount of time, and leaves no wake behind it. This is somewhat more surprising, but corresponds to our experience; sounds travelling through open spaces do not "reverberate" (except due to echo effects). It turns out, however, that Part **(b)** of the theorem is *not* true for waves travelling in *two* dimensions (e.g. ripples on the surface of a pond).

20C The Dirichlet problem on a half-plane

Prerequisites: §1C, §19A, §5C, §0G. **Recommended:** §12A, §13B, §19D, §19E.

In §12A and §13B, we saw how to solve Laplace's equation on a bounded domain such as a rectangle or a cube, in the context of Dirichlet boundary conditions. Now consider the **half-plane** domain $\mathbb{H} := \left\{(x, y) \in \mathbb{R}^2 \; ; \; y \geq 0\right\}$. The boundary of this domain is just the x axis: $\partial\mathbb{H} = \{(x, 0) \; ; \; x \in \mathbb{R}\}$; thus, boundary conditions are imposed by choosing some function $b : \mathbb{R} \longrightarrow \mathbb{R}$. Figure 17E.1 on page 403 illustrates the corresponding **Dirichlet problem**: find a continuous function $u : \mathbb{H} \longrightarrow \mathbb{R}$ such that

1. u satisfies the Laplace equation: $\triangle u(x,y) = 0$ for all $x \in \mathbb{R}$ and $y > 0$.

2. u satisfies the nonhomogeneous Dirichlet boundary condition: $u(x,0) = b(x)$ for all $x \in \mathbb{R}$.

20C(i) Fourier solution

Heuristically speaking, we will solve the problem by defining $u(x,y)$ as a continuous sequence of horizontal "fibres", parallel to the x axis, and ranging over all values of $y > 0$. Each fibre is a function only of x, and thus, has a one-dimensional Fourier transform. The problem then becomes determining these Fourier transforms from the Fourier transform of the boundary function b.

Proposition 20C.1. Fourier Solution to Half-Plane Dirichlet problem
Let $b \in \mathbf{L}^1(\mathbb{R})$. Suppose that b has Fourier transform $\widehat{b}$, and define $u : \mathbb{H} \longrightarrow \mathbb{R}$ by

$$u(x,y) \quad := \quad \int_{-\infty}^{\infty} \widehat{b}(\mu) \cdot e^{-|\mu|\cdot y} \cdot \exp\Big(\mu \mathbf{i} x\Big) \, d\mu, \qquad \text{for all } x \in \mathbb{R} \text{ and } y \geq 0.$$

Then u is the solution to the Laplace equation ($\triangle u = 0$) which is bounded at infinity and which satisfies the nonhomogeneous Dirichlet boundary condition $u(x,0) = b(x)$, for all $x \in \mathbb{R}$.

Proof. For any fixed $\mu \in \mathbb{R}$, the function $f_\mu(x,y) = \exp\Big(-|\mu| \cdot y\Big) \exp\Big(-\mu \mathbf{i} x\Big)$ is harmonic (see practice problem # 10 on page 543). Thus, Proposition 0G.1 on page 567 implies that the function $u(x,y)$ is also harmonic. Finally, notice that, when $y = 0$, the expression for $u(x,0)$ is just the Fourier inversion integral for $b(x)$. □

Example 20C.2. Suppose $b(x) \;=\; \begin{cases} 1 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$ We already know from Example 19A.3 on page 489 that $\widehat{b}(\mu) \;=\; \dfrac{\sin(\mu)}{\pi\mu}$.

Thus, $u(x,y) \;=\; \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sin(\mu)}{\mu} \cdot e^{-|\mu|\cdot y} \cdot \exp\Big(\mu \mathbf{i} x\Big) \, d\mu.$ ◊

Ⓔ **Exercise 20C.1.** Note the 'boundedness' condition in Proposition 20C.1. Find another solution to the Dirichlet problem on $\mathbb{H}$ which is *unbounded* at infinity. ♦

20C(ii) Impulse-response solution

Prerequisites: §20C(i). **Recommended:** §17E.

For any $y > 0$, define the **Poisson kernel** $\mathcal{K}_y : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$\mathcal{K}_y(x) \quad := \quad \frac{y}{\pi(x^2+y^2)}. \qquad \text{(see Figure 17E.2 on page 404)} \tag{20C.9}$$

In §17E, we used the Poisson kernel to solve the half-plane Dirichlet problem using *impulse-response* methods (Proposition 17E.1 on page 404). We can now use the 'Fourier' solution to provide another proof Proposition 17E.1.

Proposition 20C.3. Poisson Kernel Solution to Half-Plane Dirichlet problem
Let $b \in \mathbf{L}^1(\mathbb{R})$. Define $u : \mathbb{H} \longrightarrow \mathbb{R}$ by

$$U(x,y) \quad = \quad b * \mathcal{K}_y(x) \quad = \quad \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{b(z)}{(x-z)^2+y^2}\, dz, \tag{20C.10}$$

for all $y > 0$ and $x \in \mathbb{R}$. Then U is the solution to the Laplace equation ($\triangle U = 0$) which is bounded at infinity and which can be continuously extended to satisfy the nonhomogeneous Dirichlet boundary condition $U(x,0) = b(x)$ for all $x \in \mathbb{R}$.

Proof. We'll show that the solution U in eqn. (20C.10) is actually equal to the 'Fourier' solution u from Proposition 20C.1.

Fix $y > 0$, and define $U_y(x) = U(x,y)$ for all $x \in \mathbb{R}$. Equation (20C.10) says $U_y \;=\; b * \mathcal{K}_y$; hence Theorem 19B.2(b) (p.494) says:

$$\widehat{U}_y \quad = \quad 2\pi \cdot \widehat{b} \cdot \widehat{\mathcal{K}}_y. \tag{20C.11}$$

Now, by practice problem # 7 on page 524 of §19I, we have:

$$\widehat{\mathcal{K}}_y(\mu) \quad = \quad \frac{e^{-y|\mu|}}{2\pi}, \tag{20C.12}$$

Combine (20C.11) and (20C.12) to get:

$$\widehat{U}_y(\mu) \quad = \quad e^{-y|\mu|} \cdot \widehat{b}(\mu). \tag{20C.13}$$

Now apply the Fourier inversion formula (Theorem 19A.1 on page 488) to eqn (20C.13) to obtain:

$$\begin{aligned} U_y(x) \quad &= \quad \int_{-\infty}^{\infty} \widehat{U}(\mu) \cdot \exp(\mu \cdot x \cdot \mathbf{i})\, d\mu \quad = \quad \int_{-\infty}^{\infty} e^{-y|\mu|} \cdot \widehat{b}(\mu)\; exp(\mu \cdot x \cdot \mathbf{i})\, d\mu \\ &= \quad u(x,y), \end{aligned}$$

where $u(x,y)$ is the solution from Proposition 20C.1. $\square$

20D PDEs on the half-line

Prerequisites: §1B(i), §19F, §5C, §0G.

Using the Fourier (co)sine transform, we can solve PDEs on the half-line.

Theorem 20D.1. The heat equation; Dirichlet boundary conditions
Let $f \in \mathbf{L}^1(\mathbb{R}_{\not-})$ have Fourier sine transform $\widehat{f}_{\sin}$, and define $u : \mathbb{R}_{\not-} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by:

$$u(x,t) \quad := \quad \int_0^\infty \widehat{f}_{\sin}(\mu) \cdot \sin(\mu \cdot x) \cdot e^{-\mu^2 t} \, d\mu$$

Then $u(x,t)$ is a solution to the heat equation, with initial conditions $u(x,0) = f(x)$, and satisfies the homogeneous Dirichlet boundary condition: $u(0,t) = 0$.

Ⓔ *Proof.* **<u>Exercise 20D.1</u>** (Hint: Use Proposition 0G.1 on page 567.) □

Theorem 20D.2. The heat equation; Neumann boundary conditions
Let $f \in \mathbf{L}^1(\mathbb{R}_{\not-})$ have Fourier cosine transform $\widehat{f}_{\cos}$, and define $u : \mathbb{R}_{\not-} \times \mathbb{R}_{\not-} \longrightarrow \mathbb{R}$ by:

$$u(x,t) \quad := \quad \int_0^\infty \widehat{f}_{\cos}(\mu) \cdot \cos(\mu \cdot x) \cdot e^{-\mu^2 t} \, d\mu$$

Then $u(x,t)$ is a solution to the heat equation, with initial conditions $u(x,0) = f(x)$, and the homogeneous Neumann boundary condition: $\partial_x u(0,t) = 0$.

Ⓔ *Proof.* **<u>Exercise 20D.2</u>** (Hint: Use Proposition 0G.1 on page 567.) □

20E General solution to PDEs using Fourier transforms

Prerequisites: §16F, §18A, §19A, §19D, §19E.

Recommended: §20A(i), §20B(i), §20C, §20D.

Most of the results of this chapter can be subsumed into a single abstraction, which makes use of the *polynomial formalism* developed in § 16F on page 369.

Theorem 20E.1. *Fix $D \in \mathbb{N}$, and let L be a linear differential operator on $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ with constant coefficients. Suppose L has polynomial symbol $\mathcal{P}$.*

(a) *If $f \in \mathbf{L}^1(\mathbb{R}^D)$ has Fourier Transform $\widehat{f} : \mathbb{R}^D \longrightarrow \mathbb{R}$, and $g = \mathsf{L} f$, then g has Fourier transform: $\widehat{g}(\boldsymbol{\mu}) \;=\; \mathcal{P}(\mathbf{i}\,\boldsymbol{\mu}) \cdot \widehat{f}(\boldsymbol{\mu})$, for all $\boldsymbol{\mu} \in \mathbb{R}^D$.*

(b) *If $q \in \mathbf{L}^1(\mathbb{R}^D)$ has Fourier transform $\widehat{q} \in \mathbf{L}^1(\mathbb{R}^D)$, and $f \in \mathbf{L}^1(\mathbb{R}^D)$ has Fourier transform*

$$\widehat{f}(\boldsymbol{\mu}) \quad = \quad \frac{\widehat{q}(\boldsymbol{\mu})}{\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})}, \quad \text{for all } \boldsymbol{\mu} \in \mathbb{R}^D,$$

then f is a solution to the Poisson-type nonhomogeneous equation "$\mathsf{L}\, f = q$."

Let $u : \mathbb{R}^D \times \mathbb{R}_{\not -} \longrightarrow \mathbb{R}$ be another function, and, for all $t \geq 0$, define $u_t : \mathbb{R}^D \longrightarrow \mathbb{R}$ by $u_t(\mathbf{x}) := u(\mathbf{x}, t)$ for all $\mathbf{x} \in \mathbb{R}^D$. Suppose $u_t \in \mathbf{L}^1(\mathbb{R}^D)$, and let u_t have Fourier transform $\widehat{u}_t$.

(c) *Let $f \in \mathbf{L}^1(\mathbb{R}^D)$, and suppose $\widehat{u}_t(\boldsymbol{\mu}) = \exp\left(\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu}) \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})$, for all $\boldsymbol{\mu} \in \mathbb{R}^D$ and $t \geq 0$. Then u is a solution to the first-order evolution equation*

$$\partial_t\, u(\mathbf{x}, t) \quad = \quad \mathsf{L}\, u(\mathbf{x}, t), \qquad \text{for all } \mathbf{x} \in \mathbb{R}^D \text{ and } t > 0,$$

with initial conditions $u(\mathbf{x}, 0) = f(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^D$.

(d) *Suppose $f \in \mathbf{L}^1(\mathbb{R}^D)$ has Fourier transform $\widehat{f}$ which decays fast enough that $\displaystyle\int_{\mathbb{R}^D} \left|\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu}) \cdot \cos\left(\sqrt{-\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})} \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})\right| \, d\boldsymbol{\mu} \; < \; \infty$, for all $t \geq 0$.[2]*

Suppose $\widehat{u}_t(\boldsymbol{\mu}) = \cos\left(\sqrt{-\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})} \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})$, for all $\boldsymbol{\mu} \in \mathbb{R}^D$ and $t \geq 0$. Then u is a solution to the second-order evolution equation

$$\partial_t^2\, u(\mathbf{x}, t) \quad = \quad \mathsf{L}\, u(\mathbf{x}, t), \qquad \text{for all } \mathbf{x} \in \mathbb{R}^D \text{ and } t > 0,$$

with initial position *$u(\mathbf{x}, 0) = f(\mathbf{x})$ and* initial velocity *$\partial_t\, u(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in \mathbb{R}^D$.*

(e) *Suppose $f \in \mathbf{L}^1(\mathbb{R}^D)$ has Fourier transform $\widehat{f}$ which decays fast enough that $\displaystyle\int_{\mathbb{R}^D} \left|\sqrt{\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})} \cdot \sin\left(\sqrt{-\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})} \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})\right| \, d\boldsymbol{\mu} \; < \; \infty$, for all $t \geq 0$.*

Suppose $\widehat{u}_t(\boldsymbol{\mu}) = \dfrac{\sin\left(\sqrt{-\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})} \cdot t\right)}{\sqrt{-\mathcal{P}(\mathbf{i}\,\boldsymbol{\mu})}} \cdot \widehat{f}(\boldsymbol{\mu})$, for all $\boldsymbol{\mu} \in \mathbb{R}^D$ and $t \geq 0$. Then the function $u(\mathbf{x}, t)$ is a solution to the second-order evolution equation

$$\partial_t^2\, u(\mathbf{x}, t) \quad = \quad \mathsf{L}\, u(\mathbf{x}, t), \qquad \text{for all } \mathbf{x} \in \mathbb{R}^D \text{ and } t > 0,$$

with initial position *$u(\mathbf{x}, 0) = 0$ and* initial velocity *$\partial_t\, u(\mathbf{x}, 0) = f(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^D$.*

Proof. **Exercise 20E.1** (Hint: Use Proposition 0G.1 on page 567. In each case, be sure to verify that convergence conditions of Proposition 0G.1 are satisfied.) □ Ⓔ

[2]See Example 18A.6(b,c) on page 420 for the definitions of complex sine and cosine functions. See Exercise 18C.17 on page 449 for a discussion of complex square roots.

Ⓔ **Exercise 20E.2.** Go back through this chapter and see how all of the different solution theorems for the heat equation (§20A(i)) wave equation (§20B(i)), and Poisson equation (§20C) are special cases of this result. What about the solution for the Dirichlet problem on a half-space in §20D? How does it fit into this formalism? ◆

Ⓔ **Exercise 20E.3.** State and prove a theorem analogous to Theorem 20E.1 for solving a D-dimensional Schrödinger equation using Fourier transforms. ◆

20F Practice problems

1. Let $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$, as in Example 19A.4 on page 490
   (a) Use the Fourier method to solve the Dirichlet problem on a half-space, with boundary condition $u(x,0) = f(x)$.
   (b) Use the Fourier method to solve the heat equation on a line, with initial condition $u_0(x) = f(x)$.

2. Solve the two-dimensional heat equation, with initial conditions
$$f(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq X \text{ and } 0 \leq y \leq Y; \\ 0 & \text{otherwise.} \end{cases}$$
   where $X, Y > 0$ are constants. (**Hint:** See problem # 3 on page 523 of §19I)

3. Solve the two-dimensional wave equation, with
$$\textbf{Initial Position:} \quad u(x,y,0) = 0,$$
$$\textbf{Initial Velocity:} \quad \partial_t u(x,y,0) = f_1(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases}.$$
   (**Hint:** See problem # 3 on page 523 of §19I)

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined: $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ (see Figure 19I.1 on page 524). Solve the **heat equation** on the real line, with initial conditions $u(x;0) = f(x)$. (Use the Fourier method; see problem # 4 on page 523 of §19I)

5. Let $f(x) = x \cdot \exp\left(\frac{-x^2}{2}\right)$. (See problem # 5 on page 523 of §19I.)
   (a) Solve the **heat equation** on the real line, with initial conditions $u(x;0) = f(x)$. (Use the Fourier method.)

(b) Solve the **wave equation** on the real line, with initial position $u(x;0) = f(x)$ and initial velocity $\partial_t u(x,0) = 0$. (Use the Fourier method.)

6. Let $f(x) = \dfrac{2x}{(1+x^2)^2}$. (See problem # 8 on page 524 of §19I.)

   (a) Solve the **heat equation** on the real line, with initial conditions $u(x;0) = f(x)$. (Use the Fourier method.)

   (b) Solve the **wave equation** on the real line, with initial position $u(x,0) = 0$ and initial velocity $\partial_t u(x,0) = f(x)$. (Use the Fourier method.)

7. Let $f(x) = \begin{cases} 1 & \text{if} \quad -4 < x < 5; \\ 0 & \quad\text{otherwise.} \end{cases}$ (See problem # 9 on page 524 of §19I.) Use the 'Fourier Method' to solve the one-dimensional heat equation ($\partial_t\, u(x;t) = \triangle u(x;t)$) on the domain $\mathbb{X} = \mathbb{R}$, with initial conditions $u(x;0) = f(x)$.

8. Let $f(x) = \dfrac{x\cos(x) - \sin(x)}{x^2}$. (See problem # 10 on page 524 of §19I.) Use the 'Fourier Method' to solve the one-dimensional heat equation ($\partial_t\, u(x;t) = \triangle u(x;t)$) on the domain $\mathbb{X} = \mathbb{R}$, with initial conditions $u(x;0) = f(x)$.

9. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ had Fourier transform $\widehat{f}(\mu) = \dfrac{\mu}{\mu^4 + 1}$.

   (a) Find the solution to the one-dimensional heat equation $\partial_t\, u = \triangle u$, with initial conditions $u(x;0) = f(x)$ for all $x \in \mathbb{R}$.

   (b) Find the solution to the one-dimensional wave equation $\partial_t^2\, u = \triangle u$, with

$$\begin{array}{rrcll} \text{Initial position} & u(x;0) & = & 0, & \text{for all } x \in \mathbb{R}. \\ \text{Initial velocity} & \partial_t\, u(x;0) & = & f(x), & \text{for all } x \in \mathbb{R}. \end{array}$$

   (c) Find the solution to the two-dimensional Laplace Equation $\triangle u(x,y) = 0$ on the half-space $\mathbb{H} = \{(x,y)\ ;\ x \in \mathbb{R},\ y \geq 0\}$, with boundary condition: $u(x,0) = f(x)$ for all $x \in \mathbb{R}$.

   (d) **Verify** your solution to question (c). That is: check that your solution satisfies the Laplace equation and the desired boundary conditions.

10. Fix $\mu \in \mathbb{R}$, and define $f_\mu : \mathbb{R}^2 \longrightarrow \mathbb{C}$ by $f_\mu(x,y) := \exp\Big(-|\mu| \cdot y\Big) \exp\Big(-\mu\mathbf{i}x\Big)$. Show that f is harmonic on $\mathbb{R}^2$.

    (This function appears in the Fourier solution to the half-plane Dirichlet problem; see Proposition 20C.1 on page 538.)

Chapter 0

Appendices

0A Sets and functions

Sets: A **set** is a collection of objects. If $\mathcal{S}$ is a set, then the objects in $\mathcal{S}$ are called **elements** of $\mathcal{S}$; if s is an element of $\mathcal{S}$, we write "$s \in \mathcal{S}$. A **subset** of $\mathcal{S}$ is a smaller set $\mathcal{R}$ such that every element of $\mathcal{R}$ is also an element of $\mathcal{S}$. We indicate this by writing "$\mathcal{R} \subset \mathcal{S}$".

Sometimes we can explicitly list the elements in a set; we write "$\mathcal{S} = \{s_1, s_2, s_3, \ldots\}$".

Example 0A.1.

(a) In Figure 0A.1(A), $\mathcal{S}$ is the set of all cities in the world, so $\underline{\mathsf{Toronto}} \in \mathcal{S}$. We might write $\mathcal{S} = \{\underline{\mathsf{Toronto}}, \underline{\mathsf{Beijing}}, \underline{\mathsf{London}}, \underline{\mathsf{Kuala\ Lampur}}, \underline{\mathsf{Nairobi}}, \underline{\mathsf{Santiago}}, \underline{\mathsf{Pisa}}, \underline{\mathsf{Sidney}}, \ldots\}$. If $\mathcal{R}$ is the set of all cities in Canada, then $\mathcal{R} \subset \mathcal{S}$.

(b) In Figure 0A.1(B), the set of **natural numbers** is $\mathbb{N} := \{0, 1, 2, 3, 4, \ldots\}$. The set of **positive natural numbers** is $\mathbb{N}_+ := \{1, 2, 3, 4, \ldots\}$.

Thus, $5 \in \mathbb{N}$, but $\pi \not\in \mathbb{N}$ and $-2 \not\in \mathbb{N}$.

(c) In Figure 0A.1(B), the set of **integers** is $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. Thus, $5 \in \mathbb{Z}$ and $-2 \in \mathbb{Z}$, but $\pi \not\in \mathbb{Z}$ and $\frac{1}{2} \not\in \mathbb{Z}$. Observe that $\mathbb{N}_+ \subset \mathbb{N} \subset \mathbb{Z}$.

(d) In Figure 0A.1(B), the set of **real numbers** is denoted by $\mathbb{R}$. It is best visualised as an infinite line. Thus, $5 \in \mathbb{R}$, $-2 \in \mathbb{R}$, $\pi \in \mathbb{R}$ and $\frac{1}{2} \in \mathbb{R}$. Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.

(e) In Figure 0A.1(B), the set of **nonnegative real numbers** is denoted by $[0, \infty)$ or $\mathbb{R}_{\not-}$. It is best visualised as a half-infinite line, including zero. Observe that $[0, \infty) \subset \mathbb{R}$.

(f) In Figure 0A.1(B), the set of **positive real numbers** is denoted by $(0, \infty)$ or $\mathbb{R}_+$. It is best visualised as a half-infinite line, excluding zero. Observe that $\mathbb{R}_+ \subset \mathbb{R}_{\not-} \subset \mathbb{R}$.

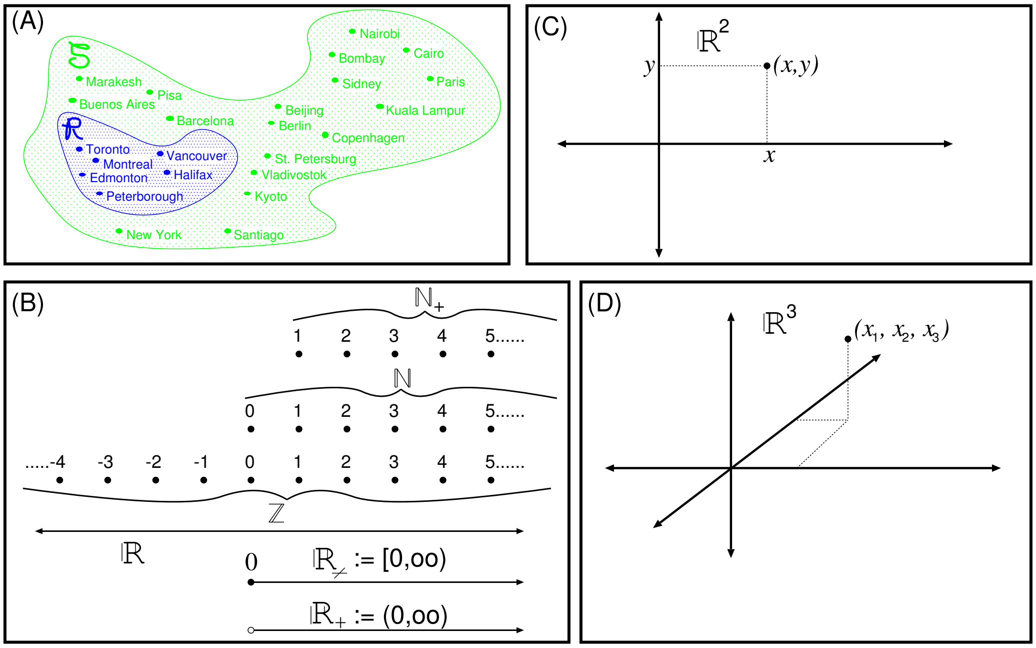


Figure 0A.1: **(A)** $\mathcal{R}$ is a subset of $\mathcal{S}$ **(B)** Important Sets: $\mathbb{N}_+, \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}_{\not-} := [0, \infty)$ and $\mathbb{R}_+ := (0, \infty)$. **(C)** $\mathbb{R}^2$ is two-dimensional space. **(D)** $\mathbb{R}^3$ is three-dimensional space.

(g) Figure 0A.1(C) depicts **two-dimensional space:** the set of all coordinate pairs (x, y), where x and y are real numbers. This set is denoted by $\mathbb{R}^2$, and is best visualised as an infinite plane.

(h) Figure 0A.1(D) depicts **three-dimensional space:** the set of all coordinate triples (x_1, x_2, x_3), where x_1, x_2, and x_3 are real numbers. This set is denoted by $\mathbb{R}^3$, and is best visualised as an infinite void.

(i) If D is any natural number, then D**-dimensional space** is the set of all coordinate triples $(x_1, x_2, \ldots, x_D)$, where $x_1, \ldots, x_D$ are all real numbers. This set is denoted by $\mathbb{R}^D$. It is hard to visualize when D is bigger than 3. ♢

Cartesian Products: If $\mathcal{S}$ and $\mathcal{T}$ are two sets, then their **Cartesian product** is the set of all pairs (s, t), where s is an element of $\mathcal{S}$, and t is an element of $\mathcal{T}$. We denote this set by $\mathcal{S} \times \mathcal{T}$.

Example 0A.2.

(a) $\mathbb{R} \times \mathbb{R}$ is the set of all pairs (x, y), where x and y are real numbers. In other words, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

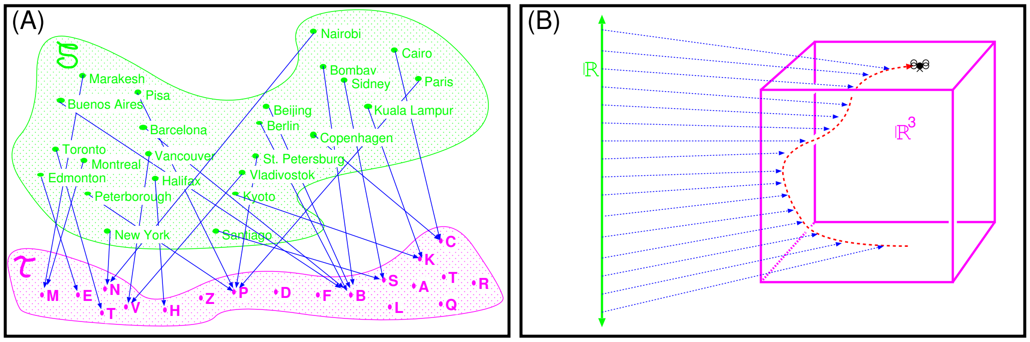


Figure 0A.2: **(A)** $f(C)$ is the first letter of city C. **(B)** $\mathbf{p}(t)$ is the position of the fly at time t.

(b) $\mathbb{R}^2 \times \mathbb{R}$ is the set of all pairs $(\mathbf{w}, z)$, where $\mathbf{w} \in \mathbb{R}^2$ and $z \in \mathbb{R}$. But if $\mathbf{w}$ is an element of $\mathbb{R}^2$, then $\mathbf{w} = (x, y)$ for some $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Thus, any element of $\mathbb{R}^2 \times \mathbb{R}$ is an object $\big((x, y),\ z\big)$. By suppressing the inner pair of brackets, we can write this as (x, y, z). In this way, we see that $\mathbb{R}^2 \times \mathbb{R}$ is the same as $\mathbb{R}^3$.

(c) In the same way, $\mathbb{R}^3 \times \mathbb{R}$ is the same as $\mathbb{R}^4$, once we write $\big((x, y, z),\ t\big)$ as (x, y, z, t). More generally, $\mathbb{R}^D \times \mathbb{R}$ is mathematically the same as $\mathbb{R}^{D+1}$.

Often, we use the final coordinate to store a 'time' variable, so it is useful to distinguish it, by writing $\big((x, y, z),\ t\big)$ as $(x, y, z;\ t)$. $\diamondsuit$

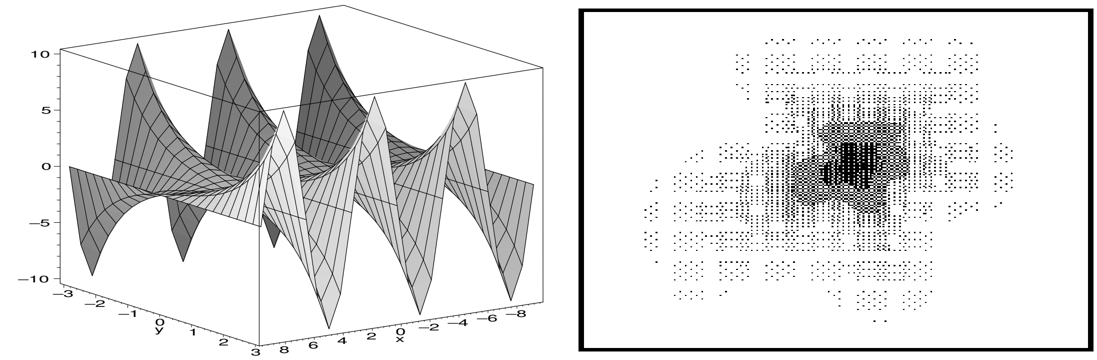
Functions: If $\mathcal{S}$ and $\mathcal{T}$ are sets, then a **function** from $\mathcal{S}$ to $\mathcal{T}$ is a rule which assigns a specific element of $\mathcal{T}$ to every element of $\mathcal{S}$. We indicate this by writing "$f : \mathcal{S} \longrightarrow \mathcal{T}$".

Example 0A.3.

(a) In Figure 0A.2(A), $\mathcal{S}$ is the cities in the world, and $\mathcal{T} = \{A, B, C, \ldots, Z\}$ is the letters of the alphabet, and f is the function which is the first letter in the name of each city. Thus $f(\underline{\text{Peterborough}}) = P$, $f(\underline{\text{Santiago}}) = S$, etc.

(b) if $\mathbb{R}$ is the set of real numbers, then $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ is a function: $\sin(0) = 0$, $\sin(\pi/2) = 1$, etc. $\diamondsuit$

Two important classes of functions are **paths** and **fields**.



(A) **(B)**

Figure 0A.3: **(A)** A height function describes a landscape. **(B)** A density distribution in $\mathbb{R}^2$.

Paths: Imagine a fly buzzing around a room. Suppose you try to represent its trajectory as a curve through space. This defines a a function $\mathbf{p}$ from $\mathbb{R}$ into $\mathbb{R}^3$, where $\mathbb{R}$ represents *time*, and $\mathbb{R}^3$ represents the (three-dimensional) room, as shown in Figure 0A.2(B). If $t \in \mathbb{R}$ is some moment in time, then $\mathbf{p}(t)$ is the position of the fly at time t. Since this $\mathbf{p}$ describes the path of the fly, we call $\mathbf{p}$ a **path**.

More generally, a **path** (or **trajectory** or **curve**) is a function $\mathbf{p} : \mathbb{R} \longrightarrow \mathbb{R}^D$, where D is any natural number. It describes the motion of an object through D-dimensional space. Thus, if $t \in \mathbb{R}$, then $\mathbf{p}(t)$ is the position of the object at time t.

Scalar Fields: Imagine a three-dimensional topographic map of Antarctica. The rugged surface of the map is obtained by assigning an altitude to every location on the continent. In other words, the map implicitly defines a function $\mathbf{h}$ from $\mathbb{R}^2$ (the Antarctic continent) to $\mathbb{R}$ (the set of altitudes, in metres above sea level). If $(x, y) \in \mathbb{R}^2$ is a location in Antarctica, then $\mathbf{h}(x, y)$ is the altitude at this location (and $\mathbf{h}(x, y) = 0$ means (x, y) is at sea level).

This is an example of a **scalar field**. A scalar field is a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$; it assigns a numerical quantity to every point in D-dimensional space.

Example 0A.4.

(a) In Figure 0A.3(A), a landscape is represented by a **height function h** : $\mathbb{R}^2 \longrightarrow \mathbb{R}$.

(b) Figure 0A.3(B) depicts a **concentration function** on a two-dimensional plane (e.g. the concentration of bacteria on a petri dish). This is a function $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}_{\not<}$ (where $\rho(x, y) = 0$ indicates zero bacteria at (x, y)).

(c) The **mass density** of a three-dimensional object is a function $\rho : \mathbb{R}^3 \longrightarrow \mathbb{R}_{\not-}$ (where $\rho(x_1, x_2, x_3) = 0$ indicates vacuum).

(d) The **charge density** is a function $\mathbf{q} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (where $\mathbf{q}(x_1, x_2, x_3) = 0$ indicates electric neutrality)

(e) The **electric potential** (or **voltage**) is a function $\mathbf{V} : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

(f) The **temperature distribution** in space is a function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (so $u(x_1, x_2, x_3)$ is the "temperature at location (x_1, x_2, x_3)") ♢

A **time-varying scalar field** is a function $u : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$, assigning a quantity to every point in space at each moment in time. Thus, for example, $u(\mathbf{x}; t)$ is the "temperature at location $\mathbf{x}$, at time t"

Vector Fields: A **vector field** is a function $\vec{\mathbf{V}} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$; it assigns a vector (i.e. an "arrow") at every point in space.

Example 0A.5.

(a) The **electric field** generated by a charge distribution (denoted by $\vec{\mathbf{E}}$).

(b) The **flux** of some material flowing through space (often denoted by $\vec{\mathbf{F}}$). ♢

Thus, for example, $\vec{\mathbf{F}}(\mathbf{x})$ is the "flux" of material at location $\mathbf{x}$.

0B Derivatives —notation

If $f : \mathbb{R} \longrightarrow \mathbb{R}$, then f' is the first derivative of f; f'' is the second derivative,... $f^{(n)}$ the nth derivative, etc. If $\mathbf{x} : \mathbb{R} \longrightarrow \mathbb{R}^D$ is a path, then the **velocity** of $\mathbf{x}$ at time t is the vector

$$\dot{\mathbf{x}}(t) = \left[x_1'(t),\ x_2'(t),\ \ldots, x_D'(t)\right]$$

If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then the following notations will be used interchangeably:

$$\text{for all } j \in [1...D], \quad \partial_j u \ := \ \frac{\partial u}{\partial x_j}$$

If $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (i.e. $u(x, y)$ is a function of two variables), then we will also write

$$\partial_x u \ := \ \frac{\partial u}{\partial x} \quad \text{and} \quad \partial_y u \ := \ \frac{\partial u}{\partial y}.$$

Multiple derivatives will be indicated by iterating this procedure. For example,

$$\partial_x^3 \partial_y^2 u \quad := \quad \frac{\partial^3}{\partial x^3} \frac{\partial^2 u}{\partial y^2}$$

A useful notational convention (which we rarely use) is **multiexponents**. If $\gamma_1, \ldots, \gamma_D$ are positive integers, and $\gamma = (\gamma_1, \ldots, \gamma_D)$, then

$$\mathbf{x}^\gamma \quad := \quad x_1^{\gamma_1} x_2^{\gamma_2} \ldots, x_D^{\gamma_D}$$

For example, if $\gamma = (3, 4)$, and $\mathbf{z} = (x, y)$ then $\mathbf{z}^\gamma = x^3 y^4$.

This generalizes to **multi-index** notation for derivatives. If $\gamma = (\gamma_1, \ldots, \gamma_D)$, then

$$\partial^\gamma u \quad := \quad \partial_1^{\gamma_1} \partial_2^{\gamma_2} \ldots \partial_D^{\gamma_D} u$$

For example, if $\gamma = (1, 2)$, then $\partial^\gamma u = \dfrac{\partial}{\partial x} \dfrac{\partial^2 u}{\partial y^2}$.

Remark. Many authors use subscripts to indicate partial derivatives. For example, they would write

$$u_x \; := \; \partial_x u, \qquad u_{xx} \; := \; \partial_x^2 u, \qquad u_{xy} \; := \; \partial_x \partial_y u, \text{ etc.}$$

This notation is very compact and intuitive, but it has two major disadvantages:

1. When dealing with an N-dimensional function $u(x_1, x_2, \ldots, x_N)$ (where N is either large or indeterminate), you have only two options. You can either either use awkward 'nested subscript' expressions like

   $$u_{x_3} \; := \; \partial_3 u, \qquad u_{x_5 x_5} \; := \; \partial_5^2 u, \qquad u_{x_2 x_3} \; := \; \partial_2 \partial_3 u, \text{ etc.,}$$

   or you must adopt the 'numerical subscript' convention that

   $$u_3 \; := \; \partial_3 u, \qquad u_{55} \; := \; \partial_5^2 u, \qquad u_{23} \; := \; \partial_2 \partial_3 u, \text{ etc.}$$

   But once 'numerical' subscripts are reserved to indicate derivatives in this fashion, they can no longer be used for other purposes (e.g. indexing a sequence of functions, or indexing the coordinates of a vector-valued function). This can create further awkwardness.

2. We will often be considering functions of the form $u(x, y; t)$, where (x, y) are 'space' coordinates and t is a 'time' coordinate. In this situation, it is often convenient to fix a value of t and consider the two-dimensional scalar field $u_t(x, y) := u(x, y; t)$. Normally, when we use t as a subscript, it will be indicate a 'time-frozen' scalar field of this kind.

Thus, in this book, *we will never use subscripts to indicate partial derivatives.* Partial derivatives will always be indicated by the notation "$\partial_x u$" or "$\frac{\partial u}{\partial x}$" (almost always the first one). However, when consulting other texts, you should be aware of the 'subscript' notation for derivatives, because it is used quite frequently.

0C Complex numbers

Complex numbers have the form $z = x + y\mathbf{i}$, where $\mathbf{i}^2 = -1$. We say that x is the **real part** of z, and y is the **imaginary part**; we write: $x = \mathrm{Re}[z]$ and $y = \mathrm{Im}[z]$.

If we imagine (x, y) as two real coordinates, then the complex numbers form a two-dimensional plane. Thus, we can also write a complex number in *polar coordinates* (see Figure 0C.1) If $r > 0$ and $0 \le \theta < 2\pi$, then we define

$$\boxed{r \,\mathtt{cis}\, \theta \ = \ r \cdot [\cos(\theta) + \mathbf{i}\sin(\theta)]}$$

Addition: If $z_1 = x_1 + y_1\mathbf{i}$, $z_2 = x_2 + y_2\mathbf{i}$, are two complex numbers, then $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\mathbf{i}$. (see Figure 0C.2)

Multiplication: If $z_1 = x_1 + y_1\mathbf{i}$, $z_2 = x_2 + y_2\mathbf{i}$, are two complex numbers, then $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\,\mathbf{i}$.

Multiplication has a nice formulation in polar coordinates; If $z_1 = r_1 \,\mathtt{cis}\, \theta_1$ and $z_2 = r_2 \,\mathtt{cis}\, \theta_2$, then $z_1 \cdot z_2 = (r_1 \cdot r_2)\,\mathtt{cis}\,(\theta_1 + \theta_2)$. In other words, multiplication by the complex number $z = r \,\mathtt{cis}\, \theta$ is equivalent to *dilating* the complex plane by a factor of r, and *rotating* the plane by an angle of θ. (see Figure 0C.3)

Exponential: If $z = x + y\mathbf{i}$, then $\exp(z) = e^x \,\mathtt{cis}\, y = e^x \cdot [\cos(y) + \mathbf{i}\sin(y)]$. (see Figure 0C.4) In particular, if $x \in \mathbb{R}$, then

- $\exp(x) = e^x$ is the standard real-valued exponential function.
- $\exp(y\mathbf{i}) = \cos(y) + \sin(y)\mathbf{i}$ is a periodic function; as y moves along the real line, $\exp(y\mathbf{i})$ moves around the unit circle. (This is *Euler's formula*.)

The complex exponential function shares two properties with the real exponential function:

- If $z_1, z_2 \in \mathbb{C}$, then $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$.
- If $w \in \mathbb{C}$, and we define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = \exp(w \cdot z)$, then $f'(z) = w \cdot f(z)$.

Consequence: If $w_1, w_2, \ldots, w_D \in \mathbb{C}$, and we define $f : \mathbb{C}^D \longrightarrow \mathbb{C}$ by

$$f(z_1, \ldots, z_D) \ = \ \exp(w_1z_1 + w_2z_2 + \ldots w_Dz_D),$$

then $\partial_d f(\mathbf{z}) = w_d \cdot f(\mathbf{z})$. More generally,

$$\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_D^{n_D}\, f(\mathbf{z}) \ = \ w_1^{n_1} \cdot w_2^{n_2} \cdot \ldots w_D^{n_D} \cdot f(\mathbf{z}). \qquad (0C.1)$$

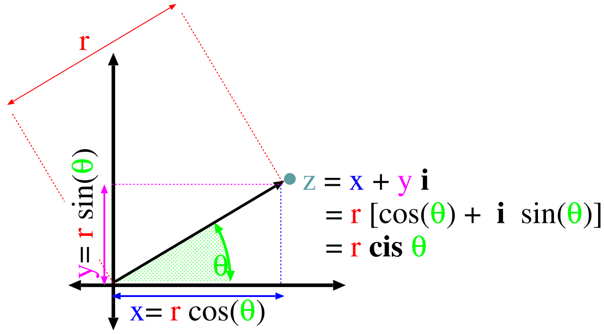


Figure 0C.1: $z = x + y\mathbf{i}$; $r = \sqrt{x^2 + y^2}$, $\theta = \tan(y/x)$.

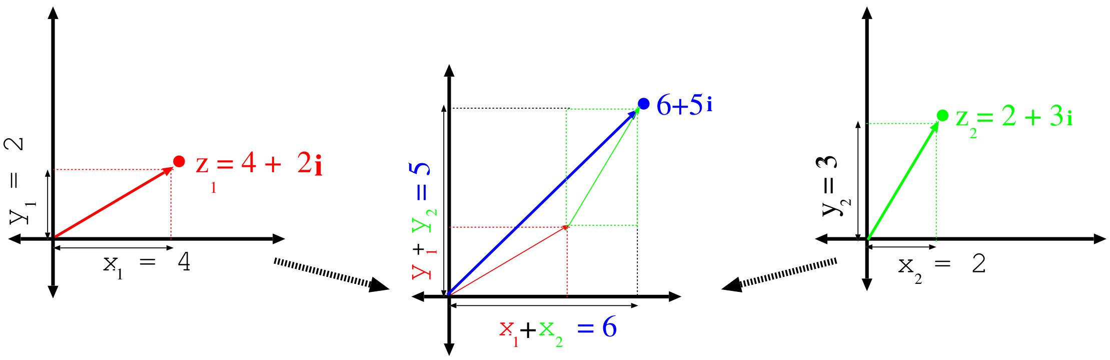


Figure 0C.2: The addition of complex numbers $z_1 = x_1 + y_1\mathbf{i}$ and $z_2 = x_2 + y_2\mathbf{i}$.

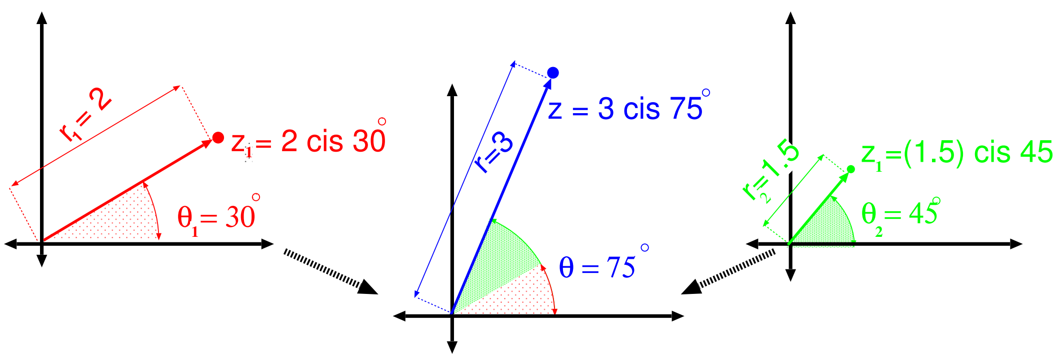


Figure 0C.3: The multiplication of complex numbers $z_1 = r_1\,\mathtt{cis}\,\theta_1$ and $z_2 = r_2\,\mathtt{cis}\,\theta_2$.

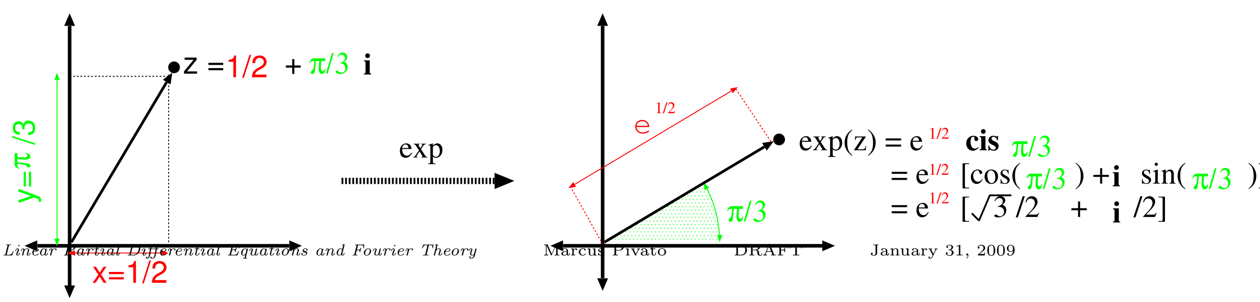


Figure 0C.4: The exponential of complex number $z = x + y\mathbf{i}$.

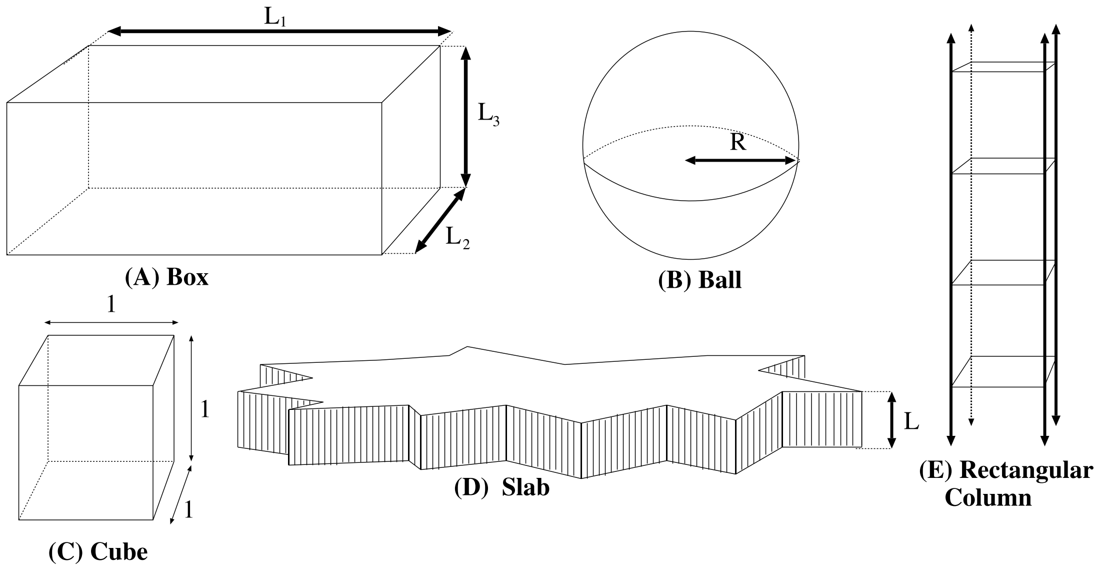


Figure 0D.1: Some domains in $\mathbb{R}^3$.

For example, if $f(x,y) = \exp(3x + 5\mathbf{i}y)$, then

$$f_{xxy}(x,y) \quad = \quad \partial_x^2 \, \partial_y \, f(x,y) \quad = \quad 45\,\mathbf{i} \cdot \exp(3x + 5\mathbf{i}y).$$

If $\mathbf{w} = (w_1, \ldots, w_D)$ and $\mathbf{z} = (z_1, \ldots, z_D)$, then we will sometimes write:

$$\exp(w_1 z_1 + w_2 z_2 + \ldots w_D z_D) \quad = \quad \exp \langle \mathbf{w}, \mathbf{z} \rangle \, .$$

Conjugation and Norm: If $z = x + y\mathbf{i}$, then the **complex conjugate** of z is $\overline{z} = x - y\mathbf{i}$. In polar coordinates, if $z = r \,\mathtt{cis}\, \theta$, then $\overline{z} = r \,\mathtt{cis}\, (-\theta)$.

The **norm** of z is $|z| = \sqrt{x^2 + y^2}$. We have the formula:

$$|z|^2 \;=\; z \cdot \overline{z}.$$

0D Coordinate systems and domains

Prerequisites: §0A.

Boundary Value Problems are usually posed on some "domain" —some region of space. To solve the problem, it helps to have a convenient way of mathematically representing these domains, which can sometimes be simplified by adopting a suitable coordinate system. We will first give a variety of examples of 'domains' in different coordinate systems in §0D(a,b,c,d). Then in §0D(e) we will give a formal definition of the word 'domain'.

0D(i) Rectangular coordinates

Rectangular coordinates in $\mathbb{R}^3$ are normally denoted (x, y, z). Three common domains in rectangular coordinates:

- The **slab** $\mathbb{X} = \left\{(x, y, z) \in \mathbb{R}^3 \; ; \; 0 \leq z \leq L\right\}$, where L is the thickness of the slab (see Figure 0D.1D).
- The **unit cube:** $\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3; \; 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad \text{and } 0 \leq z \leq 1\}$ (see Figure 0D.1C).
- The **box:** $\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3; \; 0 \leq x \leq L_1, \quad 0 \leq y \leq L_2, \quad \text{and } 0 \leq z \leq L_3\}$, where L_1, L_2, and L_3 are the sidelengths (see Figure 0D.1A).
- The **rectangular column:** $\mathbb{X} = \left\{(x, y, z) \in \mathbb{R}^3 \; ; \; 0 \leq x \leq L_1 \text{ and } 0 \leq y \leq L_2\right\}$ (see Figure 0D.1E).

0D(ii) Polar coordinates on $\mathbb{R}^2$

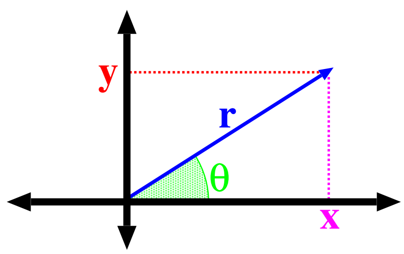


Figure 0D.2: Polar coordinates

Polar coordinates (r, θ) on $\mathbb{R}^2$ are defined by the transformation:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta).$$

with reverse transformation:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{Arctan}(y, x).$$

Here, the coordinate r ranges over $\mathbb{R}_{\not-}$, while the variable θ ranges over $[-\pi, \pi)$. Finally we define

$$\operatorname{Arctan}(y, x) \quad := \quad \begin{cases} \arctan(y/x) & \text{if} \quad x > 0; \\ \arctan(y/x) + \pi & \text{if} \quad x < 0 \;\text{ and }\; y > 0; \\ \arctan(y/x) - \pi & \text{if} \quad x < 0 \;\text{ and }\; y < 0. \end{cases}$$

Three common domains in polar coordinates are:

- $\mathbb{D} = \{(r, \theta) \; ; \; r \leq R\}$ is the **disk** of radius R (see Figure 0D.3A).

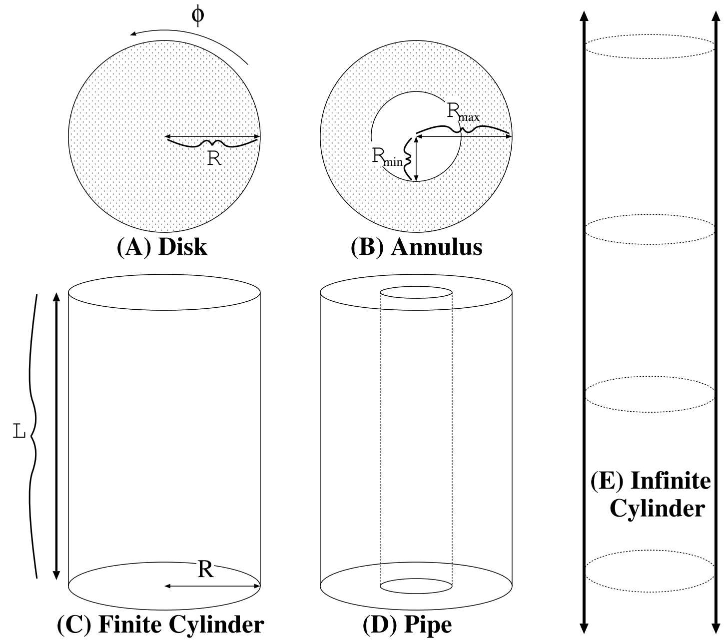


Figure 0D.3: Some domains in polar and cylindrical coordinates.

- $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ R \leq r\}$ is the **codisk** of inner radius R.
- $\mathbb{A} = \{(r,\theta)\ ;\ R_{\min} \leq r \leq R_{\max}\}$ is the **annulus**, of inner radius $R_{\min}$ and outer radius $R_{\max}$ (see Figure 0D.3B).

0D(iii) Cylindrical coordinates on $\mathbb{R}^3$

Cylindrical coordinates (r,θ,z) on $\mathbb{R}^3$, are defined by the transformation:

$$x = r\cdot\cos(\theta), \quad y = r\cdot\sin(\theta) \quad \text{and} \quad z = z$$

with reverse transformation:

$$r = \sqrt{x^2+y^2}, \quad \theta = \operatorname{Arctan}(y,x) \quad \text{and} \quad z = z.$$

Five common domains in cylindrical coordinates are:

- $\mathbb{X} = \{(r,\theta,z)\ ;\ r \leq R\}$ is the **(infinite) cylinder** of radius R (see Figure 0D.3E).
- $\mathbb{X} = \{(r,\theta,z)\ ;\ R_{\min} \leq r \leq R_{\max}\}$ is the **(infinite) pipe** of inner radius $R_{\min}$ and outer radius $R_{\max}$ (see Figure 0D.3D).
- $\mathbb{X} = \{(r,\theta,z)\ ;\ r > R\}$ is the **wellshaft** of radius R.

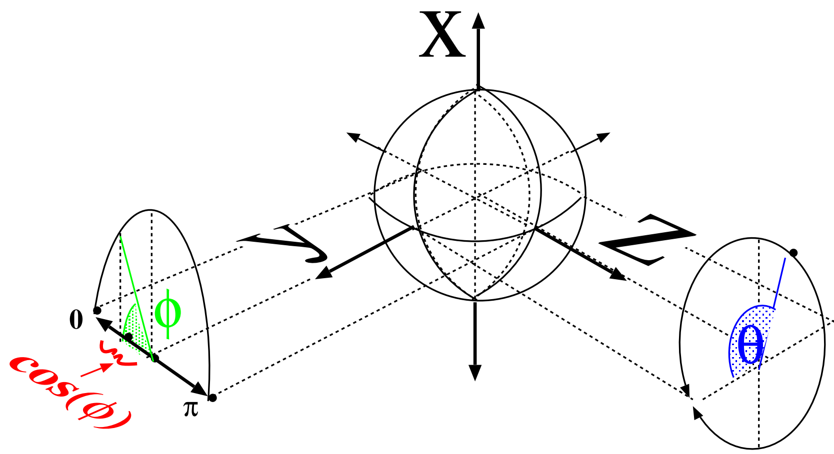


Figure 0D.4: Spherical coordinates

- $\mathbb{X} = \{(r, \theta, z) \ ; \ r \leq R \text{ and } 0 \leq z \leq L\}$ is the **finite cylinder** of radius R and length L (see Figure 0D.3C).
- In cylindrical coordinates on $\mathbb{R}^3$, we can write the **slab** as $\{(r, \theta, z) \ ; \ 0 \leq z \leq L\}$.

0D(iv) Spherical coordinates on $\mathbb{R}^3$

Spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ are defined by the transformation:

$$\begin{aligned} x &= r \cdot \sin(\phi) \cdot \cos(\theta), \qquad y = r \cdot \sin(\phi) \cdot \sin(\theta) \\ \text{and} \quad z &= r \cdot \cos(\phi). \end{aligned}$$

with reverse transformation:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \operatorname{Arctan}(y, x) \\ \text{and} \quad \phi &= \operatorname{Arctan}\left(\sqrt{x^2 + y^2}, \ z\right). \end{aligned}$$

In spherical coordinates, the set $\mathbb{B} = \{(r, \theta, \phi) \ ; \ r \leq R\}$ is the **ball** of radius R (see Figure 0D.1B).

0D(v) What is a 'domain'?

Formally, a **domain** is a subset $\mathbb{X} \subseteq \mathbb{R}^D$ which satisfies three conditions:

(a) $\mathbb{X}$ is *closed*. That is, $\mathbb{X}$ must contain all its boundary points.

(b) $\mathbb{X}$ has a *dense interior*. That is, every point in $\mathbb{X}$ is a limit point of a sequence $\{x_n\}_{n=1}^{\infty}$ of interior points of $\mathbb{X}$. (A point $x \in \mathbb{X}$ is an *interior point* if $\mathbb{B}(x, \epsilon) \subset \mathbb{X}$ for some $\epsilon > 0$).

(c) $\mathbb{X}$ is *connected*. That is, we cannot find two disjoint closed subsets $\mathbb{X}_1$ and $\mathbb{X}_2$ such that $\mathbb{X} = \mathbb{X}_1 \sqcup \mathbb{X}_2$.

Observe that all of the examples in §0D(a,b,c,d) satisfy these three conditions.

Why conditions (a), (b), and (c)? We are normally interested in finding a function $f : \mathbb{X} \longrightarrow \mathbb{R}$ which satisfies a certain partial differential equation on $\mathbb{X}$. However, such a PDE only makes sense on the interior of $\mathbb{X}$ (because the derivatives of f at x are only well-defined if x is an interior point of $\mathbb{X}$). Thus, first $\mathbb{X}$ must *have* a nonempty interior, and second, this interior must fill 'most' of $\mathbb{X}$. This is the reason for condition (b). We often represent certain physical constraints by requiring f to satisfy certain *boundary conditions* on the boundary of $\mathbb{X}$. (That's what a 'boundary value problem' means). But this cannot make sense unless $\mathbb{X}$ satisfies condition (a). Finally, we don't actually *need* condition (c). But if $\mathbb{X}$ is disconnected, then we could split $\mathbb{X}$ into two or more disconnected pieces and solve the equations separately on each piece. Thus, we can always assume without loss of generality that $\mathbb{X}$ is connected.

0E Vector calculus

Prerequisites: §0A, §0B.

0E(i) Gradient

....in two dimensions:

Suppose $\mathbb{X} \subset \mathbb{R}^2$ was a two-dimensional region. To define the topography of a "landscape" on this region, it suffices[1] to specify the *height* of the land at each point. Let $u(x, y)$ be the height of the land at the point $(x, y) \in \mathbb{X}$. (Technically, we say: "$u : \mathbb{X} \longrightarrow \mathbb{R}$ is a *two-dimensional scalar field*.")

The **gradient** of the landscape measures the *slope* at each point in space. To be precise, we want the gradient to be an arrow pointing in the direction of *most rapid ascent*. The *length* of this arrow should then measure the *rate* of ascent. Mathematically, we define the **two-dimensional gradient** of u by:

$$\boxed{\nabla u(x, y) \quad = \quad \left[\frac{\partial u}{\partial x}(x, y), \quad \frac{\partial u}{\partial y}(x, y)\right]}$$

The gradient arrow points in the direction where u is increasing the most rapidly. If $u(x, y)$ was the height of a mountain at location (x, y), and you were trying to climb the mountain, then your (naive) strategy would be to always walk in the direction $\nabla u(x, y)$. Notice that, for any $(x, y) \in \mathbb{X}$, the gradient $\nabla u(x, y)$ is a two-dimensional vector —that is, $\nabla u(x, y) \in \mathbb{R}^2$. (Technically, we say "$\nabla u : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional vector field*".)

[1] Assuming no overhangs!

....in many dimensions:

This idea generalizes to any dimension. If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then the **gradient** of u is the associated vector field $\nabla u : \mathbb{R}^D \longrightarrow \mathbb{R}^D$, where, for any $\mathbf{x} \in \mathbb{R}^D$,

$$\boxed{\nabla u(\mathbf{x}) \quad = \quad \Big[\partial_1 u,\ \partial_2 u, \ldots,\ \partial_D u\Big]\,(\mathbf{x})}$$

Proposition 0E.1. Algebra of gradients
Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain. Let $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ be differentiable scalar fields, and let $(f + g) : \mathbb{X} \longrightarrow \mathbb{R}$ and $(f \cdot g) : \mathbb{X} \longrightarrow \mathbb{R}$ denote the sum and product of f and g.

(a) (Linearity) *For all $\mathbf{x} \in \mathbb{X}$, and any $r \in \mathbb{R}$,*

$$\nabla(rf + g)(\mathbf{x}) \quad = \quad r\,\nabla f(\mathbf{x}) \ + \ \nabla g(\mathbf{x}).$$

(b) (Leibniz rule) *For all $\mathbf{x} \in \mathbb{X}$,*

$$\nabla(f \cdot g)(\mathbf{x}) \quad = \quad f(\mathbf{x}) \cdot \Big(\nabla g(\mathbf{x})\Big) \ + \ g(\mathbf{x}) \cdot \Big(\nabla f(\mathbf{x})\Big).$$

Ⓔ *Proof.* **Exercise 0E.1** □

0E(ii) Divergence

....in one dimension:

Imagine a current of 'fluid' (e.g. air, water, electricity) flowing along the real line $\mathbb{R}$. For each point $x \in \mathbb{R}$, let $V(x)$ describe the rate at which fluid is flowing past this point. Now, in places where the fluid *slows down*, we expect the derivative $V'(x)$ to be negative. We also expect the fluid to *accumulate* (i.e. become 'compressed') at such locations (because fluid is entering the region more quickly than it leaves). In places where the fluid *speeds up*, we expect the derivative $V'(x)$ to be positive, and we expect the fluid to be *depleted* (i.e. to decompress) at such locations (because fluid is leaving the region more quickly than it arrives).

If the fluid is incompressible (e.g. water), then we can assume that the quantity of fluid at each point is constant. In this case, the fluid cannot 'accumulate' or 'be depleted'. In this case, a negative value of $V'(x)$ means that fluid is somehow being 'absorbed' (e.g. being destroyed or leaking out of the system) at x. Likewise, a positive value of $V'(x)$ means that fluid is somehow being 'generated' (e.g. being created, or leaking into the system) at x.

In general, positive $V'(x)$ may represent some combination of fluid depletion, decompression, or generation at x, while negative $V'(x)$ may represent some combination of local accumulation, compression or absorption at x. Thus, if we define the **divergence** of the flow to be the *rate at which fluid is being depleted/decompressed/generated* (if positive) *or being accumulated/compressed/absorbed* (if positive), then mathematically speaking,

$$\boxed{\operatorname{div} V(x) \quad = \quad V'(x).}$$

This physical model yields an interesting interpretation of the Fundamental Theorem of Calculus. Suppose $a < b \in \mathbb{R}$, and consider the interval $[a, b]$. If $V : \mathbb{R} \longrightarrow \mathbb{R}$ describes the flow of fluid, then $V(a)$ is the amount of fluid flowing into the left end of the interval $[a, b]$ (or flowing *out*, if $V(a) < 0$). Likewise, $V(b)$ is the amount of fluid flowing out of the right end of the interval $[a, b]$ (or flowing *in*, if $V(b) < 0$). Thus, $V(b) - V(a)$ is the net amount of fluid flowing *out* through the endpoints of $[a, b]$ (or flowing *in*, if this quantity is negative). But the *Fundamental Theorem of Calculus* asserts that

$$V(b) - V(a) \quad = \quad \int_a^b V'(x)\, dx \quad = \quad \int_a^b \operatorname{div} V(x)\, dx.$$

In other words, the net amount of fluid instantaneously leaving/entering through the endpoints of $[a, b]$ is equal to the integral of the divergence over the interior. But if $\operatorname{div} V(x)$ is the amount of fluid being instantaneously 'generated' at x (or 'absorbed' if negative) this integral can be interpreted as the saying:

> *The net amount of fluid instantaneously leaving the endpoints of* $[a, b]$ *is equal to the net quantity of fluid being instantaneously generated throughout the interior of* $[a, b]$.

From a physical point of view, this makes perfect sense; it is simply 'conservation of mass'. This is the one-dimensional form of the *Divergence Theorem* (Theorem 0E.4 on page 563 below).

....in two dimensions:

Let $\mathbb{X} \subset \mathbb{R}^2$ be some planar region, and consider a fluid flowing through $\mathbb{X}$. For each point $(x, y) \in \mathbb{X}$, let $\vec{\mathbf{V}}(x, y)$ be a two-dimensional vector describing the current at that point[2].

Think of this two-dimensional current as a superposition of a *horizontal current* V_1 and a *vertical current* V_2. For each of the two currents, we can reason as in the one-dimensional case. If $\partial_x V_1(x, y) > 0$, then the horizontal current is accelerating at (x, y), so we expect it to deplete the fluid at (x, y) (or, if the fluid

[2]Technically, we say "$\vec{\mathbf{V}} : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional vector field*".

is incompressible, we interpret this to mean that additional fluid is being generated at (x, y)). If $\partial_x V_1(x, y) < 0$, then the horizontal current is decelarating, we expect it to deposit fluid at (x, y) (or, if the fluid is incompressible, we interpret this to mean that fluid is being absorbed or destroyed at (x, y)).

The same reasoning applies to $\partial_y V_2(x, y)$. The **divergence** of the two-dimensional current is thus just the sum of the divergences of the horizontal and vertical currents

$$\boxed{\operatorname{div} \vec{\mathbf{V}}(x, y) \quad = \quad \partial_x V_1(x, y) \quad + \quad \partial_y V_2(x, y).}$$

Notice that, although $\vec{\mathbf{V}}(x, y)$ was a *vector*, the divergence $\operatorname{div} \vec{\mathbf{V}}(x, y)$ is a *scalar*[3]. Just as in the one-dimensional case, we interpret $\operatorname{div} \vec{\mathbf{V}}(x, y)$ to be the the *instantaneous rate at which fluid is being depleted/decompressed/generated at* (x, y) (if positive) *or being accumulated/compressed/absorbed at* (x, y) (if negative).

For example, suppose $\mathbb{R}^2$ represents the ocean, and $\vec{\mathbf{V}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field representing ocean currents. If $\operatorname{div} \vec{\mathbf{V}}(x, y) > 0$, this means that there is a net injection of water into the ocean at the point (x, y) —e.g. due to rainfall or a river outflow. If $\operatorname{div} \vec{\mathbf{V}}(x, y) < 0$, this means that there is a net removal of water from the ocean at the point (x, y) —e.g. due to evaporation or hole in the bottom of the sea.

....in many dimensions:

We can generalize this idea to any number of dimensions. If $\vec{\mathbf{V}} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is a vector field, then the **divergence** of $\vec{\mathbf{V}}$ is the associated scalar field $\operatorname{div} \vec{\mathbf{V}} : \mathbb{R}^D \longrightarrow \mathbb{R}$, where, for any $\mathbf{x} \in \mathbb{R}^D$,

$$\boxed{\operatorname{div} \vec{\mathbf{V}}(\mathbf{x}) \;=\; \partial_1 V_1(\mathbf{x}) + \partial_2 V_2(\mathbf{x}) + \ldots + \partial_D V_D(\mathbf{x}).}$$

If $\vec{\mathbf{V}}$ represents the flow of some fluid through $\mathbb{R}^D$, then $\operatorname{div} \vec{\mathbf{V}}(\mathbf{x})$ represents the *instantaneous rate at which fluid is being depleted/decompressed/generated at* $\mathbf{x}$ (if positive) *or being accumulated/compressed/absorbed at* $\mathbf{x}$ (if negative). For example, if $\vec{\mathbf{E}}$ is the electric field, then $\operatorname{div} \vec{\mathbf{E}}(\mathbf{x})$ is the amount of electric field being "generated" at $\mathbf{x}$ —that is, $\operatorname{div} \vec{\mathbf{E}}(\mathbf{x}) = \mathbf{q}(\mathbf{x})$ is the **charge density** at $\mathbf{x}$.

Proposition 0E.2. Algebra of Divergences
Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain. Let $\vec{\mathbf{V}}, \vec{\mathbf{W}} : \mathbb{X} \longrightarrow \mathbb{R}^D$ be differentiable vector fields, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be a differentiable scalar field, and let $(f \cdot \vec{\mathbf{V}}) : \mathbb{X} \longrightarrow \mathbb{R}^D$ represent the product of f and $\vec{\mathbf{V}}$.

(a) (Linearity) *For all $\mathbf{x} \in \mathbb{X}$, and any $r \in \mathbb{R}$,*

$$\operatorname{div} (r\vec{\mathbf{V}} + \vec{\mathbf{W}})(\mathbf{x}) \quad = \quad r \operatorname{div} \vec{\mathbf{V}}(\mathbf{x}) \;+\; \operatorname{div} \vec{\mathbf{W}} g(\mathbf{x}).$$

[3] Technically, we say "$\operatorname{div} \vec{\mathbf{V}} : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional scalar field*".

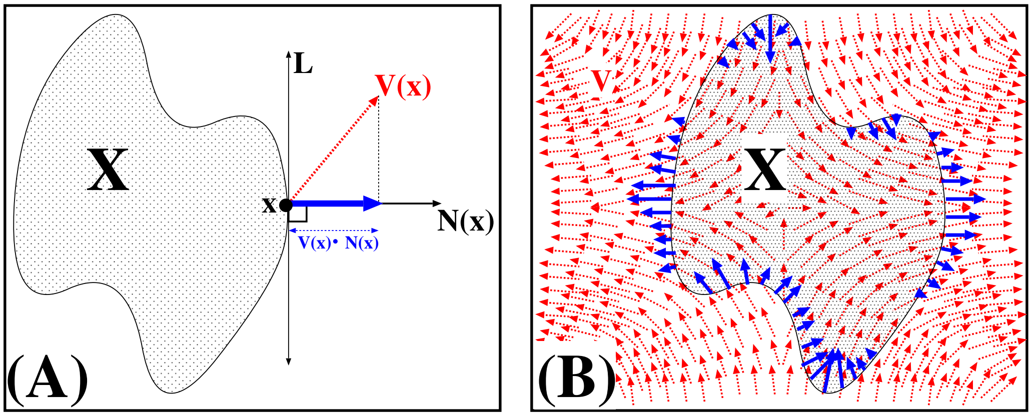


Figure 0E.1: **(A)** Line segment $\mathbb{L}$ is tangent to $\partial\mathbb{X}$ at $\mathbf{x}$. Vector $\vec{\mathbf{N}}(\mathbf{x})$ is normal to $\partial\mathbf{X}$ at $\mathbf{x}$. If $\vec{\mathbf{V}}(\mathbf{x})$ is another vector based at $\mathbf{x}$, then the dot product $\vec{\mathbf{V}}(\mathbf{x}) \bullet \vec{\mathbf{N}}(\mathbf{x})$ measures the orthogonal projection of $\vec{\mathbf{V}}(\mathbf{x})$ onto $\vec{\mathbf{N}}(\mathbf{x})$ —that is, the 'part of $\vec{\mathbf{V}}(\mathbf{x})$ which is normal to $\partial\mathbb{X}$'. **(B)** Here $\vec{\mathbf{V}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a differentiable vector field, and we portray the scalar field $\vec{\mathbf{V}} \bullet \vec{\mathbf{N}}$ along the curve $\partial\mathbb{X}$ (although we have visualized it as a 'vector field', to help your intuitions). The **flux** of $\vec{\mathbf{V}}$ across the boundary of $\mathbb{X}$ is obtained by integrating $\vec{\mathbf{V}} \bullet \vec{\mathbf{N}}$ along $\partial\mathbb{X}$.

(b) (Leibniz rule) *For all* $\mathbf{x} \in \mathbb{X}$,

$$\operatorname{div}(f \cdot \vec{\mathbf{V}})(\mathbf{x}) \quad = \quad f(\mathbf{x}) \cdot \left(\operatorname{div} \vec{\mathbf{V}}(\mathbf{x})\right) \; + \; \left(\nabla f(\mathbf{x})\right) \bullet \vec{\mathbf{V}}(\mathbf{x}).$$

Proof. **Exercise 0E.2** □ Ⓔ

Exercise 0E.3. Let $\vec{\mathbf{V}}, \vec{\mathbf{W}} : \mathbb{X} \longrightarrow \mathbb{R}^D$ be differentiable vector fields, and consider their dot product $(\vec{\mathbf{V}} \bullet \vec{\mathbf{W}}) : \mathbb{X} \longrightarrow \mathbb{R}^D$ (a differentiable scalar field). State and prove a Leibniz-like rule for $\nabla(\vec{\mathbf{V}} \bullet \vec{\mathbf{W}})$.... (a) In the case $D = 3$; ....(b) In the case $D \geq 4$. ♦ Ⓔ

0E(iii) The Divergence Theorem.

...in two dimensions

Let $\mathbb{X} \subset \mathbb{R}^2$ be some domain in the plane, and let $\partial\mathbb{X}$ be the **boundary** of $\mathbb{X}$. (For example, if $\mathbb{X}$ is the unit disk, then $\partial\mathbb{X}$ is the unit circle. If $\mathbb{X}$ is a square domain, then $\partial\mathbb{X}$ is the four sides of the square, etc.). Let $\mathbf{x} \in \partial\mathbb{X}$. A line segment $\mathbb{L}$ through $\mathbf{x}$ is **tangent** to $\partial\mathbb{X}$ if $\mathbb{L}$ touches $\partial\mathbb{X}$ only at $\mathbf{x}$; that is, $\mathbb{L} \cap \partial\mathbb{X} = \{\mathbf{x}\}$ (see Figure 0E.1(A)). A unit vector $\vec{\mathbf{N}}$ is **normal** to $\partial\mathbb{X}$ at $\mathbf{x}$ if there is a line segment through $\mathbf{x}$ which is orthogonal to $\vec{\mathbf{N}}$ and which is tangent to $\partial\mathbb{X}$. We say $\partial\mathbb{X}$ is **piecewise smooth** if there is a unique unit normal vector

$\vec{\mathbf{N}}(\mathbf{x})$ at every $\mathbf{x} \in \partial\mathbb{X}$, except perhaps at finitely many points (the 'corners' of the boundary). For example, the disk, the square, and any other polygonal domain have piecewise smooth boundaries. The function $\vec{\mathbf{N}} : \partial\mathbb{X} \longrightarrow \mathbb{R}^2$ is then called the **normal vector field** for $\partial\mathbb{X}$.

If $\vec{\mathbf{V}} = (V_1, V_2)$ and $\vec{\mathbf{N}} = (N_1, N_2)$ are two vectors, then define $\vec{\mathbf{V}} \bullet \vec{\mathbf{N}} := V_1N_1 + V_2N_2$. If $\vec{\mathbf{V}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field, and $\mathbb{X} \subset \mathbb{R}^2$ is a domain with a smooth boundary $\partial\mathbb{X}$, then we can define the **flux** of $\vec{\mathbf{V}}$ across $\partial\mathbb{X}$ as the integral:

$$\int_{\partial\mathbb{X}} \vec{\mathbf{V}}(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})\ d\mathbf{s}. \qquad \text{(see Figure 0E.1(B)).} \qquad (0E.1)$$

Here, by 'integrating over $\partial\mathbb{X}$', we are assuming that $\partial\mathbb{X}$ can be parameterized as a smooth curve or a union of smooth curves; this integral can then be computed (via this parameterization) as one or more one-dimensional integrals over intervals in $\mathbb{R}$. The value of integral (0E.1) is independent of the choice of parameterization you use. If $\vec{\mathbf{V}}$ describes the flow of some fluid, then the flux (0E.1) represents the net quantity of fluid flowing across the boundary of $\partial\mathbb{X}$.

On the other hand, if $\operatorname{div} \vec{\mathbf{V}}(\mathbf{x})$ represents the instantaneous rate at which fluid is being generated/destroyed at the point $\mathbf{x}$, then the two-dimensional integral

$$\int_{\mathbb{X}} \operatorname{div} \vec{\mathbf{V}}(\mathbf{x})\ d\mathbf{x}$$

is the net rate at which fluid is being generated/destroyed throughout the interior of the region $\mathbb{X}$. The next result then simply says that the total 'mass' of the fluid must be conserved when we combine these two processes:

Theorem 0E.3. (Green's Theorem)
If $\mathbb{X} \subset \mathbb{R}^2$ is an bounded domain with a piecewise smooth boundary, and $\vec{\mathbf{V}} : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a continuously differentiable vector field, then $\displaystyle\int_{\partial\mathbb{X}} \vec{\mathbf{V}}(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})\ d\mathbf{s} = \int_{\mathbb{X}} \operatorname{div} \vec{\mathbf{V}}(\mathbf{x})\ d\mathbf{x}$.

Proof. See any introduction to vector calculus; see e.g. [Ste08, §16.5, p.1067] □

...in many dimensions

Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let $\partial\mathbb{X}$ be the **boundary** of $\mathbb{X}$. (For example, if $\mathbb{X}$ is the unit ball, then $\partial\mathbb{X}$ is the unit sphere). If $D = 2$, then $\partial\mathbb{X}$ will be a 1-dimensional *curve*. If $D = 3$, then $\partial\mathbb{X}$ will be a 2-dimensional *surface*. In general, if $D \geq 4$, then $\partial\mathbb{X}$ will be a $(D-1)$-dimensional *hypersurface*.

Let $\mathbf{x} \in \partial\mathbb{X}$. A (hyper)plane segment $\mathbb{P}$ through $\mathbf{x}$ is **tangent** to $\partial\mathbb{X}$ if $\mathbb{P}$ touches $\partial\mathbb{X}$ only at $\mathbf{x}$; that is, $\mathbb{P} \cap \partial\mathbb{X} = \{\mathbf{x}\}$. A unit vector $\vec{\mathbf{N}}$ is **normal** to $\partial\mathbb{X}$

at $\mathbf{x}$ if there is a (hyper)plane segment through $\mathbf{x}$ which is orthogonal to $\vec{\mathbf{N}}$ and which is tangent to $\partial\mathbb{X}$. We say $\partial\mathbb{X}$ is **smooth** if there is a unique unit normal vector $\vec{\mathbf{N}}(\mathbf{x})$ at each $\mathbf{x} \in \partial\mathbb{X}$.[4] The function $\vec{\mathbf{N}} : \partial\mathbb{X} \longrightarrow \mathbb{R}^D$ is then called the **normal vector field** for $\partial\mathbb{X}$.

If $\vec{\mathbf{V}} = (V_1, \ldots, V_D)$ and $\vec{\mathbf{N}} = (N_1, \ldots, N_D)$ are two vectors, then define $\vec{\mathbf{V}} \bullet \vec{\mathbf{N}} := V_1 N_1 + \cdots + V_D N_D$. If $\vec{\mathbf{V}} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is a vector field, and $\mathbb{X} \subset \mathbb{R}^D$ is a domain with a smooth boundary $\partial\mathbb{X}$, then we can define the **flux** of $\vec{\mathbf{V}}$ across $\partial\mathbb{X}$ as the integral:

$$\int_{\partial\mathbb{X}} \vec{\mathbf{V}}(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})\ d\mathbf{s}. \tag{0E.2}$$

Here, by 'integrating over $\partial\mathbb{X}$', we are assuming that $\partial\mathbb{X}$ can be parameterized as a smooth (hyper)surface or a union of smooth (hyper)surfaces; this integral can then be computed (via this parameterization) as one or more $(D-1)$-dimensional integrals over open subsets of $\mathbb{R}^{D-1}$. The value of integral (0E.2) is independent of the choice of parameterization you use. If $\vec{\mathbf{V}}$ describes the flow of some fluid, then the flux (0E.2) represents the net quantity of fluid flowing across the boundary of $\partial\mathbb{X}$.

On the other hand, if $\operatorname{div} \vec{\mathbf{V}}(\mathbf{x})$ represents the instantaneous rate at which fluid is being generated/destroyed at the point $\mathbf{x}$, then the D-dimensional integral

$$\int_{\mathbb{X}} \operatorname{div} \vec{\mathbf{V}}(\mathbf{x})\ d\mathbf{x}$$

is the net rate at which fluid is being generated/destroyed throughout the interior of the region $\mathbb{X}$. The next result then simply says that the total 'mass' of the fluid must be conserved when we combine these two processes:

Theorem 0E.4. (Divergence Theorem)

If $\mathbb{X} \subset \mathbb{R}^D$ is a bounded domain with a piecewise smooth boundary, and $\vec{\mathbf{V}} : \mathbb{X} \longrightarrow \mathbb{R}^D$ is a continuously differentiable vector field, then $\displaystyle\int_{\partial\mathbb{X}} \vec{\mathbf{V}}(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s})\ d\mathbf{s} = \int_{\mathbb{X}} \operatorname{div} \vec{\mathbf{V}}(\mathbf{x})\ d\mathbf{x}$.

Proof. If $D = 1$, this just a restates the Fundamental Theorem of Calculus.

If $D = 2$, this just a restates of Green's Theorem (Theorem 0E.3).

For the case $D = 3$, this result can be found in any introduction to vector calculus; see e.g. [Ste08, §16.9, p.1099]. This theorem is often called *Gauss's*

[4]More generally, $\partial\mathbb{X}$ is **piecewise smooth** if there is a unique unit normal vector $\vec{\mathbf{N}}(\mathbf{x})$ at 'almost every' $\mathbf{x} \in \partial\mathbb{X}$, except perhaps for a subset of dimension $(D-2)$. For example, a surface in $\mathbb{R}^3$ is piecewise smooth if it has a normal vector field everywhere except at some union of curves, which represent the 'edges' between the smooth 'faces' the surface. In particular, a cube, a cylinder, or any other polyhedron is has a piecewise smooth boundary.

Theorem (after C.F. Gauss) or *Ostrogradsky's Theorem* (after Mikhail Ostrogradsky).

For the case $D \geq 4$, this is a special case of the *Generalized Stokes Theorem*, one of the fundamental results of modern differential geometry, which unifies the classic (2-dimensional) Stokes theorem, Green's theorem, Gauss' theorem, and the Fundamental Theorem of Calculus. A statement and proof can be found in any introduction to differential geometry or tensor calculus. See e.g. [BG80, Theorem 4.9.2, p.196].

Some texts on partial differential equations also review the Divergence Theorem, usually in an appendix. See for example [Eva91, Appendix C.2, p. 627]. □

Green's formulae. Let $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a scalar field. If $\mathbb{X}$ is a domain, and $\mathbf{s} \in \partial\mathbb{X}$, then the **outward normal derivative** of u at $\mathbf{s}$ is defined

$$\partial_\perp u(\mathbf{s}) \quad := \quad \nabla u(\mathbf{s}) \bullet \vec{\mathbf{N}}(\mathbf{s}).$$

(see §5C(ii) for more information). Meanwhile, the **Laplacian** of u is defined by

$$\triangle u \quad = \quad \operatorname{div}(\nabla(u)).$$

(see §1B(ii) on page 7 for more information). The Divergence Theorem then has the following useful consequences.

Corollary 0E.5. (Green's Formulae)
Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, and let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a scalar field which is $\mathcal{C}^2$ (i.e. twice continuously differentiable). *Then*

(a) $\displaystyle\int_{\partial\mathbb{X}} \partial_\perp u(\mathbf{s})\, d\mathbf{s} \;=\; \int_{\mathbb{X}} \triangle u(\mathbf{x})\, d\mathbf{x}.$

(b) $\displaystyle\int_{\partial\mathbb{X}} u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s})\, d\mathbf{s} \;=\; \int_{\mathbb{X}} u(\mathbf{x}) \triangle u(\mathbf{x}) \;+\; |\nabla u(\mathbf{x})|^2\, d\mathbf{x}.$

(c) *For any other $\mathcal{C}^2$ function $w : \mathbb{X} \longrightarrow \mathbb{R}$,*

$$\int_{\mathbb{X}} \Big(u(\mathbf{x}) \triangle w(\mathbf{x}) - w(\mathbf{x}) \triangle u(\mathbf{x}) \Big)\, d\mathbf{x} \quad = \quad \int_{\partial\mathbb{X}} \Big(u(\mathbf{s}) \cdot \partial_\perp w(\mathbf{s}) - w(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s}) \Big)\, d\mathbf{s}.$$

Ⓔ *Proof.* **(a)** is **Exercise 0E.4**. To prove **(b)**, note that

$$2\int_{\partial\mathbb{X}} u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s})\, d\mathbf{s} \;\underset{(\dagger)}{=}\; \int_{\partial\mathbb{X}} \partial_\perp (u^2)(\mathbf{s})\, d\mathbf{s} \;\underset{(*)}{=}\; \int_{\mathbb{X}} \triangle(u^2)(\mathbf{x})\, d\mathbf{x}.$$
$$\underset{(\diamond)}{=}\; 2\int_{\mathbb{X}} u(\mathbf{x}) \triangle u(\mathbf{x}) \;+\; |\nabla u(\mathbf{x})|^2\, d\mathbf{x}.$$

Here, (†) is because $\partial_\perp(u^2)(\mathbf{s}) = 2u(\mathbf{s}) \cdot \partial_\perp u(\mathbf{s})$, by the Leibniz rule for normal derivatives (Exercise 0E.6 below), while (⋄) is because $\triangle(u^2)(\mathbf{x}) = 2|\nabla u(\mathbf{x})|^2 + 2u(\mathbf{x}) \cdot \triangle u(\mathbf{x})$ by the Leibniz rule for Laplacians (Exercise 1B.4 on page 9). Finally, (∗) is by part (a). The result follows.

(c) is **Exercise 0E.5** . □ Ⓔ

Exercise 0E.6. Prove the *Leibniz rule* for normal derivatives: if $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are two scalar fields, and $(f \cdot g) : \mathbb{X} \longrightarrow \mathbb{R}$ is their product, then for all $\mathbf{s} \in \partial\mathbb{X}$, Ⓔ

$$\partial_\perp(f \cdot g)(\mathbf{s}) \quad = \quad \Big(\partial_\perp f(\mathbf{s})\Big) \cdot g(\mathbf{s}) \ + \ f(\mathbf{s}) \cdot \Big(\partial_\perp g(\mathbf{s})\Big).$$

Hint: Use the Leibniz rules for gradients (Propositions 0E.1(b) on page 558) and the linearity of the dot product. ♦

0F Differentiation of function series

Recommended: §6E(iii), §6E(iv).

Many of our methods for solving partial differential equations will involve expressing the solution function as an infinite *series* of functions (e.g. Taylor series, Fourier series, etc.). To make sense of such solutions, we must be able to differentiate them.

Proposition 0F.1. Differentiation of Series

Let $-\infty \leq a < b \leq \infty$. For all $n \in \mathbb{N}$, let $f_n : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function, and define $F : (a, b) \longrightarrow \mathbb{R}$ by

$$F(x) \quad = \quad \sum_{n=0}^{\infty} f_n(x), \qquad \text{for all } x \in (a, b).$$

(a) *Suppose that $\sum_{n=0}^{\infty} f_n$ converges uniformly*[5] *to F on (a, b), and that $\sum_{n=0}^{\infty} f_n'$ also converges uniformly on (a, b). Then F is differentiable, and $F'(x) = \sum_{n=0}^{\infty} f_n'(x)$ for all $x \in (a, b)$.*

(b) *Suppose there is a sequence $\{B_n\}_{n=1}^{\infty}$ of positive real numbers such that*

- $\sum_{n=1}^{\infty} B_n < \infty$.

[5] See §6E(iii) and §6E(iv) for the definition of 'uniform convergence' of a function series.

- *For all $x \in (a,b)$, and all $n \in \mathbb{N}$,* $|f_n(x)| \leq B_n$ *and* $|f_n'(x)| \leq B_n$.

Then F is differentiable, and $F'(x) = \sum_{n=0}^{\infty} f_n'(x)$ *for all* $x \in (a,b)$.

Proof. (a) follows immediately from Proposition 6E.10(c) on page 127.

(b) follows from (a) and the Weierstras M-test (Proposition 6E.13 on page 129).

For a direct proof, see [Asm05, Theorems 1 and 5, p.87 and p.92 of §2.9] or [Fol84, Theorem 2.27(b), p.54]. □

Example 0F.2. Let $a = 0$ and $b = 1$. For all $n \in \mathbb{N}$, let $f_n(x) = \frac{x^n}{n!}$. Thus,

$$F(x) \quad = \quad \sum_{n=0}^{\infty} f_n(x) \quad = \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad = \quad \exp(x),$$

(because this is the Taylor series for the exponential function). Now let $B_0 = 1$ and let $B_n = \frac{1}{(n-1)!}$ for $n \geq 1$. Then for all $x \in (0,1)$, and all $n \in \mathbb{N}$, $|f_n(x)| = \frac{1}{n!}x^n < \frac{1}{n!} < \frac{1}{(n-1)!} = B_n$ and $|f_n'(x)| = \frac{n}{n!}x^{n-1} = \frac{1}{(n-1)!}x^{n-1} < \frac{1}{(n-1)!} = B_n$. Also,

$$\sum_{n=1}^{\infty} B_n \; = \; \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \; < \; \infty.$$

Hence the conditions of Proposition 0F.1(b) are satisfied, so we conclude that

$$F'(x) \quad = \quad \sum_{n=0}^{\infty} f_n'(x) \quad = \quad \sum_{n=0}^{\infty} \frac{n}{n!} x^{n-1} \quad = \quad \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \quad \underset{\text{(c)}}{=} \quad \sum_{m=0}^{\infty} \frac{x^m}{m!} \quad = \quad \exp(x),$$

where **(c)** is the change of variables $m = n - 1$. In this case, the conclusion is a well-known fact. But the same technique can be applied to more mysterious functions. ♢

Remarks: (a) The series $\sum_{n=0}^{\infty} f_n'(x)$ is sometimes called the *formal derivative* of the series $\sum_{n=0}^{\infty} f_n(x)$. It is 'formal' because it is obtained through a purely symbolic operation; it is not true in general that the 'formal' derivative is *really* the derivative of the series, or indeed, if the formal derivative series even converges. Proposition 0F.1 essentially says that, under certain conditions, the 'formal' derivative equals the *true* derivative of the series.

(b) Proposition 0F.1 is also true if the functions f_n involve more than one variable and/or more than one index. For example, if $f_{n,m}(x,y,z)$ is a function of three variables and two indices, and

$$F(x,y,z) \quad = \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}(x,y,z), \qquad \text{for all } (x,y,z) \in (a,b)^3.$$

then under similar hypothesis, we can conclude that $\partial_y F(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \partial_y f_{n,m}(x,y,z)$, for all $(x,y,z) \in (a,b)^3$.

0G Differentiation of integrals

Recommended: §0F.

Many of our methods for solving partial differential equations will involve expressing the solution function $F(x)$ as an *integral* of functions; i.e. $F(x) = \int_{-\infty}^{\infty} f_y(x)\ dy$, where, for each $y \in \mathbb{R}$, $f_y(x)$ is a differentiable function of the variable x. This is a natural generalization of the 'solution series' mentioned in §0F. Instead of beginning with a *discretely* paramaterized family of functions $\{f_n\}_{n=1}^{\infty}$, we begin with a *continuously* paramaterized family, $\{f_y\}_{y\in\mathbb{R}}$. Instead of combining these functions through a *summation* to get $F(x) = \sum_{n=1}^{\infty} f_n(x)$, we combine them through *integration*, to get $F(x) = \int_{-\infty}^{\infty} f_y(x)\ dy$. However, to make sense of such integrals as the solutions of differential equations, we must be able to differentiate them.

Proposition 0G.1. Differentiation of Integrals
Let $-\infty \leq a < b \leq \infty$. For all $y \in \mathbb{R}$, let $f_y : (a,b) \longrightarrow \mathbb{R}$ be a differentiable function, and define $F : (a,b) \longrightarrow \mathbb{R}$ by

$$F(x) \quad = \quad \int_{-\infty}^{\infty} f_y(x)\ dy, \qquad \text{for all } x \in (a,b).$$

Suppose there is a function $\beta : \mathbb{R} \longrightarrow \mathbb{R}_{\not -}$ such that

(a) $\int_{-\infty}^{\infty} \beta(y)\ dy \ < \ \infty.$

(b) *For all $y \in \mathbb{R}$ and for all $x \in (a,b)$,* $|f_y(x)| \ \leq \ \beta(y)$ *and* $\left|f_y'(x)\right| \ \leq \ \beta(y)$.

Then F is differentiable, and, for all $x \in (a,b)$, $F'(x) \;=\; \int_{-\infty}^{\infty} f'_y(x)\, dy$.

Proof. See [Fol84, Theorem 2.27(b), p.54]. □

Example 0G.2. Let $a = 0$ and $b = 1$. For all $y \in \mathbb{R}$ and $x \in (0,1)$, let $f_y(x) \;=\; \dfrac{x^{|y|+1}}{1+y^4}$. Thus,

$$F(x) \quad = \quad \int_{-\infty}^{\infty} f_y(x)\, dy \quad = \quad \int_{-\infty}^{\infty} \frac{x^{|y|+1}}{1+y^4}\, dy.$$

Now, let $\beta(y) \;=\; \dfrac{1+|y|}{1+y^4}$. Then

(a) $\displaystyle\int_{-\infty}^{\infty} \beta(y)\, dy \quad = \quad \int_{-\infty}^{\infty} \frac{1+|y|}{1+y^4}\, dy \quad < \quad \infty$ (check this).

(b) For all $y \in \mathbb{R}$ and all $x \in (0,1)$, $|f_y(x)| \;=\; \dfrac{x^{|y|+1}}{1+y^4} \;<\; \dfrac{1}{1+y^4} \;<\; \dfrac{1+|y|}{1+y^4} \;=\; \beta(y)$, and $\left|f'_n(x)\right| \;=\; \dfrac{(|y|+1)\cdot x^{|y|}}{1+y^4} \;<\; \dfrac{1+|y|}{1+y^4} \;=\; \beta(y)$.

Hence the conditions of Proposition 0G.1 are satisfied, so we conclude that

$$F'(x) \quad = \quad \int_{-\infty}^{\infty} f'_n(x)\, dy \quad = \quad \int_{-\infty}^{\infty} \frac{(|y|+1)\cdot x^{|y|}}{1+y^4}\, dy. \qquad \diamondsuit$$

Remarks: Proposition 0G.1 is also true if the functions f_y involve more than one variable. For example, if $f_{v,w}(x,y,z)$ is a function of five variables, and

$$F(x,y,z) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{u,v}(x,y,z)\, du\, dv \qquad \text{for all } (x,y,z) \in (a,b)^3.$$

then under similar hypothesis, we can conclude that $\partial_y^2 F(x,y,z) \;=\; \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \partial_y^2 f_{u,v}(x,y,z)\, du\, dv$, for all $(x,y,z) \in (a,b)^3$.

0H Taylor polynomials

0H(i) Taylor polynomials in one dimension

Let $\mathbb{X} \subset \mathbb{R}$ be an open set and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be an N-times differentiable function. Fix $a \in \mathbb{X}$. The **Taylor polynomial of order** N for f around a is the function

$$T_a^N f(x) \;:=\; f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N. \qquad \text{(0H.1)}$$

Here, $f^{(n)}(a)$ denotes the nth derivative of f at a [e.g. $f^{(3)}(a) = f'''(a)$], and $n!$ (pronounced 'n factorial') is the product $n\cdot(n-1)\cdots 4\cdot 3\cdot 2\cdot 1$. For example,

$$\begin{array}{rcll} T_a^0 f(x) & = & f(a) & \text{(a constant);} \\ T_a^1 f(x) & = & f(a) + f'(a)\cdot(x-a) & \text{(a linear function);} \\ T^2_{,}a\, f(x) & = & f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 & \text{(a quadratic function);} \end{array}$$

Note that $T_a^1 f(x)$ parameterizes the *tangent line* to the graph of $f(x)$ at the point $(a, f(a))$ —that is, the best *linear approximation* of f in a neighbourhood of a. Likewise, $T_a^2 f(x)$ is the best *quadratic* approximation of f in a neighbourhood of a. In general $T_a^N f(x)$ is the polynomial of degree N which provides the best approximation of $f(x)$ if x is reasonably close to N. The formal statement of this is *Taylor's theorem*, which states that

$$f(x) \quad = \quad T_a^N f(x) + \mathcal{O}(|x-a|^{N+1}).$$

Here "$\mathcal{O}(|x-a|^{N+1})$" means some function which is smaller than a constant multiple of $|x-a|^{N+1}$. In other words, there is a constant $K > 0$ such that

$$\left|f(x) - T_a^N f(x)\right| \quad \leq \quad K\cdot|x-a|^{N+1}.$$

If $|x-a|$ is large, then $|x-a|^{N+1}$ is huge, so this inequality isn't particularly useful. However, as $|x-a|$ becomes small, $|x-a|^{N+1}$ becomes really, really small. For example, if $|x-a| < 0.1$, then $|x-a|^{N+1} < 10^{-N-1}$. In this sense, $T_a^N f(x)$ is a very good approximation of $f(x)$ if x is close enough to a.

Further reading. More information about Taylor polynomials can be found in any introduction to single-variable calculus; see e.g. [Ste08, p.253-254].

0H(ii) Taylor series and analytic functions

Prerequisites: §0H(i), §0F.

Let $\mathbb{X} \subset \mathbb{R}$ be an open set, let $f : \mathbb{X} \longrightarrow \mathbb{R}$, let $a \in \mathbb{X}$, and suppose f is infinitely differentiable at a. By letting $N \to \infty$ in equation (0H.1), we obtain the **Taylor series** (or **power series**) for f at a:

$$T_a^\infty f(x) \quad := \quad f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad \text{(0H.2)}$$

Taylor's Theorem suggests that $T_a^\infty f(x) = f(x)$ if x is close enough to a. Unfortunately, this is not true for all infinitely differentiable functions; indeed, the series (0H.2) might not even converge for any $x \neq a$. However, we have the following result:

Proposition 0H.1. *Suppose the series* (0H.2) *converges for some* $x \neq a$. *In that case, there is some* $R > 0$ *such that the series* (0H.2) *converges uniformly to* $f(x)$ *on the interval* $(a - R, a + R)$. *Thus,* $T_a^\infty f(x) = f(x)$ *for all* $x \in (a - R, a + R)$. *On the other hand,* (0H.2) *diverges for all* $x \in (-\infty, a - R)$ *and all* $x \in (a + R, \infty)$. □

The $R > 0$ in Proposition 0H.1 is called the **radius of convergence** of the power series (0H.2), and the interval $(a - R, a + R)$ is the **interval of convergence.** (Note that Proposition 0H.1 says nothing about the convergence of (0H.2) at $a \pm R$; this varies from case to case). When the conclusion of Proposition 0H.1 is true, we say that f is **analytic** at a.

Example 0H.2. (a) All the 'basic' functions of calculus are analytic everywhere on their domain: all polynomials, all rational functions, all trigonometric functions, the exponential function, the logarithm, and any sum, product, or quotient of these functions.

(b) More generally, if f and g are analytic at a, then $(f + g)$ and $(f \cdot g)$ are analytic at a. If $g(a) \neq 0$, then f/g is analytic at a.

(c) If g is analytic at a, and $g(a) = b$, and f is analytic at b, then $f \circ g$ is analytic at a. ♢

If f is infinitely differentiable at $a = 0$, then we can compute the Taylor series

$$T_0^\infty f(x) \quad := \quad c_0 \; + \; c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad = \quad \sum_{n=0}^{\infty} c_n x^n. \qquad (0H.3)$$

where $c_n := \dfrac{f^{(n)}(0)}{n!}$, for all $n \in \mathbb{N}$. This special case of the Taylor series (with $a = 0$) is sometimes called a **Maclaurin series**.

Differentiating a Maclaurin series. If f is analytic at $a = 0$, then there is some $R > 0$ such that $f(x) = T_0^\infty f(x)$ for all $x \in (-R, R)$. It follows that $f'(x) = (T_0^\infty f)'(x)$, and $f''(x) = (T_0^\infty f)''(x)$, and so on, for all $x \in (-R, R)$. Proposition 0F.1 says that we can compute $(T_0^\infty f)'(x)$, $(T_0^\infty f)''(x)$ etc. by 'formally differentiating' the Maclaurin series (0H.3). Thus, we get:

$$\begin{array}{rcl}
f(x) &=& c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots = \sum_{n=0}^{\infty} c_n x^n; \\
f'(x) &=& c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}; \\
f''(x) &=& 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2}; \\
f'''(x) &=& 6c_3 + 24 c_4 x + \cdots = \sum_{n=1}^{\infty} n(n-1)(n-2) c_n x^{n-3};
\end{array} \tag{0H.4}$$

etc.

Further reading. More information about Taylor series can be found in any introduction to single-variable calculus; see e.g. [Ste08, §11.10, p.734].

0H(iii) Taylor series to solve ordinary differential equations

Prerequisites: §0H(ii).

Suppose f is an unknown analytic function (so the coefficients $\{c_0, c_1, c_2, \ldots\}$ are unknown). An ordinary differential equation in f can be reformulated in terms of the Maclaurin series in (0H.4); this yields a set of equations involving the coefficients $\{c_0, c_1, c_2, \ldots\}$. For example, let $A, B, C \in \mathbb{R}$ be constants. The second-order linear ODE

$$A f(x) + B f'(x) + C f''(x) = 0 \tag{0H.5}$$

can be reformulated as a power-series equation

$$\begin{array}{rcl}
0 &=& Ac_0 + Ac_1 x + Ac_2 x^2 + Ac_3 x^3 + Ac_4 x^4 + \cdots \\
&& \quad + Bc_1 + 2Bc_2 x + 3Bc_3 x^2 + 4Bc_4 x^3 + \cdots \\
&& \quad\quad + 2Cc_2 + 6Cc_3 x + 12 Cc_4 x^2 + \cdots
\end{array} \tag{0H.6}$$

When we collect like terms in the x variable, this becomes:

$$0 = (Ac_0 + Bc_1 + 2Cc_2) + (Ac_1 + 2Bc_2 + 6Cc_3)x + (Ac_2 + 3Bc_3 + 12Cc_4)x^2 + \cdots \tag{0H.7}$$

This yields an (infinite) system of linear equations

$$\begin{array}{rcl}
0 &=& Ac_0 + Bc_1 + 2Cc_2; \\
0 &=& Ac_1 + 2Bc_2 + 6Cc_3; \\
0 &=& Ac_2 + 3Bc_3 + 12Cc_4; \\
\vdots & \vdots & \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array} \tag{0H.8}$$

If we define $\widetilde{c}_n := n!\, c_n$ for all $n \in \mathbb{N}$, then the system (0H.8) reduces to the simple linear recurrence relation:

$$A\widetilde{c}_n \ + \ B\widetilde{c}_{n+1} \ + \ C\widetilde{c}_{n+2} \quad = \quad 0, \quad \text{for all } n \in \mathbb{N}. \tag{0H.9}$$

[Note the relationship between (0H.9) and (0H.5); this is because, if f is analytic and has Maclaurin series (0H.3), then $\widetilde{c}_n = f^{(n)}(0)$ for all $n \in \mathbb{N}$.]

We can then solve the linear recurrence relation (0H.9) using standard methods (e.g. characteristic polynomials), and obtain the coefficients $\{c_0, c_1, c_2, \ldots\}$. If the resulting power series converges, then it is a solution of the ODE (0H.5) which is analytic in a neighbourhood of zero.

This technique for solving an ordinary differential equation is called the *Power Series Method*. It is not necessary to work in a neighbourhood of zero to apply this method; we assumed $a = 0$ only to simplify the exposition. The Power Series Method can be applied to a Taylor expansion around any point in $\mathbb{R}$.

We used the *constant-coefficient* linear ODE (0H.5) just to provide a simple example. In fact, there are much easier ways to solve these sorts of ODEs (e.g. characteristic polynomials, matrix exponentials). However, the Power Series Method is also applicable to linear ODEs with *nonconstant* coefficients. For example, if the coefficients A, B, and C in equation (0H.5) were themselves analytic functions in x, then we would simply substitute the Taylor series expansions of $A(x)$, $B(x)$ and $C(x)$ into the power series equation (0H.6). This would make the simplification into equation (0H.7) much more complicated, but we would still end up with a system of linear equations in $\{c_n\}_{n=0}^{\infty}$, like (0H.8). In general, this will not simplify into a neat linear recurrence relation like (0H.9). But it can still be solved one term at a time.

Indeed, the Power Series Method is also applicable to *nonlinear* ODEs. In this case, we may end up with a system of *nonlinear* equations in $\{c_n\}_{n=0}^{\infty}$ instead of the linear system (0H.8). For example, if the ODE (0H.5) contained a term like $f(x) \cdot f''(x)$, then the system of equations (0H.8) would contain quadratic terms like c_0c_2, c_1c_3, c_2c_4, etc.

Our analysis is actually incomplete, because we didn't check that the power series (0H.3) had a nonzero radius of convergence when we obtained the sequence $\{\widetilde{c}_n\}_{n=0}^{\infty}$ as solutions to (0H.9). If (0H.9) is a linear recurrence relation (as in the example here), then the sequence $\{c_n\}_{n=0}^{\infty}$ will grow subexponentially, and it is easy to show that the radius of convergence for (0H.3) will always be nonzero. However, in the case of *non*constant coefficients or a *non*linear ODE, the power series (0H.3) may not converge; this needs to be checked. For most of the second-order linear ODEs we will encounter in this book, convergence is assured by the following result.

Theorem 0H.3. (Fuchs)

Let $a \in \mathbb{R}$, let $R > 0$, and let $\mathbb{I} := (a - R, a + R)$. Let $p, q, r : \mathbb{I} \longrightarrow \mathbb{R}$ be analytic functions whose Taylor series at a all converges everywhere in $\mathbb{I}$. Then every solution of the ODE

$$f''(x) + p(x)f'(x) + q(x)f(x) \quad = \quad r(x) \tag{0H.10}$$

is an analytic function, whose Taylor series at a converges on $\mathbb{I}$. The coefficients of this Taylor series can be found using the Power Series Method.

Proof. See [RB69, Chapter 3]. □

If the conditions for Fuchs' theorem are satisfied (i.e. if p, q, r are all analytic at a), then a is called an **ordinary point** for the ODE (0H.10). Otherwise, if one of p, q, r is *not* analytic at a, then a is called a **singular point** for ODE (0H.10). In this case, we can sometimes use a modification of the Power Series Method: the **Method of Frobenius**. For simplicity, we will discuss this method in the case $a = 0$. Consider the homogeneous linear ODE

$$f''(x) + p(x)f'(x) + q(x)f(x) \quad = \quad 0. \tag{0H.11}$$

Suppose that $a = 0$ is a singular point —i.e. either p or q is not analytic at 0. Indeed, perhaps p and/or q are not even defined at zero (e.g. $p(x) = 1/x$). We say that 0 is a **regular singular point** if there are functions $P(x)$ and $Q(x)$ which *are* analytic at 0, such that $p(x) = P(x)/x$ and $q(x) = Q(x)/x^2$ for all $x \neq 0$. Let $p_0 := P(0)$ and $q_0 := Q(0)$ (the zeroth terms in the Maclaurin series of P and Q), and consider the **indicial polynomial**

$$x(x-1) + p_0 x + q_0.$$

The roots $r_1 \geq r_2$ of the indicial polynomial are called the **indicial roots** of the ODE (0H.11).

Theorem 0H.4. (Frobenius)
Suppose $x = 0$ is a regular singular point of the ODE (0H.11), *and let $\mathbb{I}$ be the largest open interval of 0 where the Taylor series of both $P(x)$ and $Q(x)$ converge. Let $\mathbb{I}^* := \mathbb{I} \setminus \{0\}$. Then there are two linearly independent functions $f_1, f_2 : \mathbb{I}^* \longrightarrow \mathbb{R}$ which satisfy the ODE* (0H.11), *and which depend on the indicial roots $r_1 \geq r_2$ as follows:*

(a) *If $r_1 - r_2$ is not an integer, then $f_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} b_n x^n$ and $f_2(x) = |x|^{r_2} \sum_{n=0}^{\infty} c_n x^n$.*

(b) *If* $r_1 = r_2 = r$, *then* $f_1(x) = |x|^r \sum_{n=0}^{\infty} b_n x^n$ *and* $f_2(x) = f_1(x)\ln|x| + |x|^r \sum_{n=0}^{\infty} c_n x^n$.

(c) *If* $r_1 - r_2 \in \mathbb{N}$, *then* $f_1(x) = |x|^r \sum_{n=0}^{\infty} b_n x^n$ *and* $f_2(x) = k \cdot f_1(x)\ln|x| + |x|^{r_2} \sum_{n=0}^{\infty} c_n x^n$, *for some* $k \in \mathbb{R}$.

In all three cases, to obtain explicit solutions, substitute the expansions for f_1 *and* f_2 *into the ODE* (0H.11), *along with the power series for* $P(x)$ *and* $Q(x)$, *to obtain recurrence relations characterizing the coefficients* $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$.

Proof. See [Asm05, Appendex A.6]. □

Example 0H.5: (Bessel's equation)

For any $n \in \mathbb{N}$, the (2-dimensional) **Bessel equation of order** n is the ordinary differential equation

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) \quad = \quad 0. \tag{0H.12}$$

To put this in the form of ODE (0H.11), we divide by x^2, to get

$$f''(x) + \frac{1}{x} f'(x) + \left(1 - \frac{n^2}{x^2}\right) f(x) \quad = \quad 0.$$

Thus, we have $p(x) = \frac{1}{x}$ and $q(x) = \left(1 - \frac{n^2}{x^2}\right)$; hence 0 is a singular point of ODE (0H.12), because p and q are not defined (and hence not analytic) at zero. However, clearly $p(x) = P(x)/x$ and $q(x) = Q(x)/x^2$, where $P(x) = 1$ and $Q(x) = (x^2 - n^2)$ *are* analytic at zero; thus 0 is a *regular* singular point of ODE (0H.12). We have $p_0 = 1$ and $q_0 = -n^2$, so the indicial polynomial is $x(x-1) + 1x - n^2 = x^2 - n^2$, which has roots $r_1 = n$ and $r_2 = -n$. Since $r_1 - r_2 = 2n \in \mathbb{N}$, we apply Case (c) of Frobenius' Theorem, and look for solutions of the form

$$f_1(x) = |x|^n \sum_{n=0}^{\infty} b_n x^n \qquad \text{and} \qquad f_2(x) = k \cdot f_1(x)\ln|x| + |x|^{-n} \sum_{n=0}^{\infty} c_n x^n, \tag{0H.13}$$

To identify the coefficients $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, we substitute the power series (0H.13) into ODE (0H.12) and simplify. The resulting solutions are called the **Bessel functions** of types 1 and 2, respectively. The details can be found in the proof of Proposition 14G.1 on page 305 of §14G. ◇

Finally, we remark that a multivariate version of the Power Series Method can be applied to a multivariate Taylor series, to obtain solutions to *partial* differential equations. (However, this book provides many other, much nicer methods for solving linear PDEs with constant coefficients).

Further reading. More information about the power series method and the method of Frobenius can be found in any introduction to ordinary differential equations. See e.g. [Cod89, §3.9,p.138 and §4.6,p.162]. Some books on partial differential equations also contain this information (usually in an appendix); see e.g. [Asm05, Appendix A.5-A.6].

0H(iv) Taylor polynomials in two dimensions

Prerequisites: §0B. **Recommended:** §0H(i).

Let $\mathbb{X} \subset \mathbb{R}^2$ be an open set and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be an N-times differentiable function. Fix $\mathbf{a} = (a_1, a_2) \in \mathbb{X}$. The **Taylor polynomial of order** N for f around $\mathbf{a}$ is the function

$$\begin{aligned}
T_{\mathbf{a}}^N f(x_1, x_2) \quad := \quad & f(\mathbf{a}) \ + \ \partial_1 f(\mathbf{a}) \cdot (x_1 - a_1) \ + \ \partial_2 f(\mathbf{a}) \cdot (x_2 - a_2) \\
& + \frac{1}{2}\Big(\partial_1^2 f(\mathbf{a}) \cdot (x_1 - a_1)^2 \ + \ 2\,\partial_1\partial_2 f(\mathbf{a}) \cdot (x_1 - a_1)(x_2 - a_2) \ + \ \partial_2^2 f(\mathbf{a}) \cdot (x_2 - a_2)^2\Big) \\
& + \frac{1}{6}\Big(\partial_1^3 f(\mathbf{a}) \cdot (x_1 - a_1)^3 \ + \ 3\,\partial_1^2\partial_2 f(\mathbf{a}) \cdot (x_1 - a_1)^2(x_2 - a_2) \\
& \qquad + \ 3\,\partial_1\partial_2^2 f(\mathbf{a}) \cdot (x_1 - a_1)(x_2 - a_2)^2 \ + \ \partial_2^3 f(\mathbf{a}) \cdot (x_1 - a_1)^3\Big) \\
& + \frac{1}{4!}\Big(\partial_1^4 f(\mathbf{a}) \cdot (x_1 - a_1)^4 + 4\,\partial_1^3\partial_2 f(\mathbf{a}) \cdot (x_1 - a_1)^3(x_2 - a_2) \\
& \qquad + \ 6\,\partial_1^2\partial_2^2 f(\mathbf{a}) \cdot (x_1 - a_1)^2(x_2 - a_2)^2 \\
& \qquad + \ 4\,\partial_1\partial_2^3 f(\mathbf{a}) \cdot (x_1 - a_1)(x_2 - a_2)^2 \ + \ \partial_2^4 f(\mathbf{a}) \cdot (x_1 - a_1)^4\Big) \ + \ \cdots\cdots \\
& \cdots\cdots \ + \ \frac{1}{N!}\sum_{n=0}^{N} \binom{N}{n} \partial_1^{(N-n)}\partial_2^n f(\mathbf{a}) \cdot (x_1 - a_1)^{N-n}(x_2 - a_2)^n.
\end{aligned}$$

For example, $T_{\mathbf{a}}^0 f(x_1, x_2) = f(\mathbf{a})$ is just a constant, while

$$T_{\mathbf{a}}^1 f(x_1, x_2) \quad = \quad f(\mathbf{a}) \ + \ \partial_1 f(\mathbf{a}) \cdot (x_1 - a_1) \ + \ \partial_2 f(\mathbf{a}) \cdot (x_2 - a_2)$$

is an affine function which parameterizes the *tangent plane* to the surface graph of $f(x)$ at the point $(\mathbf{a}, f(\mathbf{a}))$ —that is, the best *linear approximation* of f in a neighbourhood of $\mathbf{a}$. In general, $T_a^N f(\mathbf{x})$ is the 2-variable polynomial of degree N which provides the best approximation of $f(\mathbf{x})$ if $\mathbf{x}$ is reasonably close to N. The formal statement of this is *multivariate Taylor's theorem*, which states that

$$f(\mathbf{x}) \quad = \quad T_{\mathbf{a}}^N f(\mathbf{x}) \ + \ \mathcal{O}(|\mathbf{x} - \mathbf{a}|^{N+1}).$$

For example, if we set $N = 1$, we get:

$$f(x_1, x_2) \quad = \quad f(\mathbf{a}) \; + \; \partial_1 f(\mathbf{a}) \cdot (x_1 - a_1) \; + \; \partial_2 f(\mathbf{a}) \cdot (x_2 - a_2) \; + \; \mathcal{O}(|\mathbf{x} - \mathbf{a}|^2).$$

0H(v) Taylor polynomials in many dimensions

Prerequisites: §0E(i). **Recommended:** §0H(iv).

Let $\mathbb{X} \subset \mathbb{R}^D$ be an open set and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be an N-times differentiable function. Fix $\mathbf{a} = (a_1, \ldots, a_D) \in \mathbb{X}$. The **Taylor polynomial of order** N **for** f around $\mathbf{a}$ is the function

$$T_{\mathbf{a}}^N f(\mathbf{x}) \quad := \quad f(\mathbf{a}) + \nabla f(\mathbf{a})^\dagger \cdot (\mathbf{x} - \mathbf{a}) \; + \; \frac{1}{2}(\mathbf{x} - \mathbf{a})^\dagger \cdot \mathsf{D}^2 f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) \; + \; \cdots$$
$$\cdots + \; \frac{1}{N!} \sum_{n_1 + \cdots + n_D = N} \binom{N}{n_1 \ldots n_D} \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_D^{n_D} f(\mathbf{a}) \cdot (x_1 - a_1)^{n_1} (x_2 - a_2)^{n_2} \cdots (x_D - a_D)^{n_D}.$$

Here, we regard $\mathbf{x}$ and $\mathbf{a}$ as *column* vectors, and the transposes $\mathbf{x}^\dagger$, $\mathbf{a}^\dagger$ etc. as *row* vectors. $\nabla f(\mathbf{a})^\dagger := [\partial_1 f(\mathbf{a}),\; \partial_2 f(\mathbf{a}), \ldots,\; \partial_D f(\mathbf{a})]$ is the (transposed) *gradient vector* of f at $\mathbf{a}$, and

$$\mathsf{D}^2 f \quad := \quad \begin{bmatrix} \partial_1^2 f & \partial_1 \partial_2 f & \ldots & \partial_1 \partial_D f \\ \partial_2 \partial_1 f & \partial_2^2 f & \ldots & \partial_2 \partial_D f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_D \partial_1 f & \partial_D \partial_2 f & \ldots & \partial_D^2 f \end{bmatrix}$$

is the **Hessian derivative matrix** of f. For example, $T_{\mathbf{a}}^0 f(\mathbf{x}) = f(\mathbf{a})$ is just a constant, while

$$T_{\mathbf{a}}^1 f(x) \quad = \quad f(\mathbf{a}) + \nabla f(\mathbf{a})^\dagger \cdot (\mathbf{x} - \mathbf{a})$$

is an affine function which paramaterizes the *tangent hyperplane* to the hypersurface graph of $f(x)$ at the point $(\mathbf{a}, f(\mathbf{a}))$ —that is, the best *linear approximation* of f in a neighbourhood of $\mathbf{a}$. In general $T_a^N f(\mathbf{x})$ is the multivariate polynomial of degree N which provides the best approximation of $f(\mathbf{x})$ if $\mathbf{x}$ is reasonably close to N. The formal statement of this is *multivariate Taylor's theorem*, which states that

$$f(\mathbf{x}) \quad = \quad T_{\mathbf{a}}^N f(\mathbf{x}) \; + \; \mathcal{O}(|\mathbf{x} - \mathbf{a}|^{N+1}).$$

For example, if we set $N = 2$, we get

$$f(\mathbf{x}) \quad = \quad f(\mathbf{a}) + \nabla f(\mathbf{a})^\dagger \cdot (\mathbf{x} - \mathbf{a}) \; + \; \frac{1}{2}(\mathbf{x} - \mathbf{a})^\dagger \cdot \mathsf{D}^2 f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) \; + \; \mathcal{O}\left(|\mathbf{x} - \mathbf{a}|^3\right).$$

Bibliography

[Asm02] Nakhlé H. Asmar. *Applied Complex Analysis with Partial Differential Equations.* Prentice-Hall, Upper Saddle River, NJ, 1st edition, 2002.

[Asm05] Nakhlé H. Asmar. *Partial differential equations, with Fourier series and boundary value problems.* Prentice-Hall, Upper Saddle River, NJ, 2nd edition, 2005.

[BEH94] Jiri Blank, Pavel Exner, and Miloslav Havlicek. *Hilbert Space Operators in Quantum Physics.* AIP series in computational and applied mathematical science. American Institute of Physics, New York, 1994.

[BG80] Richard L. Bishop and Samuel I. Goldberg. *Tensor analysis on manifolds.* Dover, Mineola, NY, 1980.

[Bie53] L. Bieberbach. *Conformal mapping.* Chelsea Publishing Co., New York, 1953. Translated by F. Steinhardt.

[Boh79] David Bohm. *Quantum Theory.* Dover, Mineola, NY, 1979.

[Bro89] Arne Broman. *Introduction to partial differential equations.* Dover Books on Advanced Mathematics. Dover Publications Inc., New York, second edition, 1989. From Fourier series to boundary value problems.

[CB87] Ruel V. Churchill and James Ward Brown. *Fourier series and boundary value problems.* McGraw-Hill Book Co., New York, fourth edition, 1987.

[CB03] Ruel V. Churchill and James Ward Brown. *Complex Variables and Applications.* McGraw-Hill Book Co., New York, 7th edition, 2003.

[Cha93] Isaac Chavel. *Riemannian Geometry: A modern introduction.* Cambridge UP, Cambridge, MA, 1993.

[Cod89] Earl A. Coddington. *An introduction to ordinary differential equations.* Dover, Mineola, NY, 1989.

[Con90] John B. Conway. *A Course in Functional Analysis.* Springer-Verlag, New York, second edition, 1990.

[CW68] R. R. Coifman and G. Wiess. Representations of compact groups and spherical harmonics. *L'enseignment Math.*, 14:123–173, 1968.

[dZ86] Paul duChateau and David W. Zachmann. *Partial Differential Equations.* Schaum's Outlines. McGraw-Hill, New York, 1986.

[Eva91] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, Rhode Island, 1991.

[Fis99] Stephen D. Fisher. *Complex variables.* Dover Publications Inc., Mineola, NY, 1999. Corrected reprint of the second (1990) edition.

[Fol84] Gerald B. Folland. *Real Analysis.* John Wiley and Sons, New York, 1984.

[Hab87] Richard Haberman. *Elementary applied partial differential equations.* Prentice Hall Inc., Englewood Cliffs, NJ, second edition, 1987. With Fourier series and boundary value problems.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Hel81] Sigurdur Helgason. *Topics in Harmonic Analysis on Homogeneous Spaces.* Birkhäuser, Boston, Massachusetts, 1981.

[Hen94] Michael Henle. *A combinatorial introduction to topology.* Dover, Mineola, NY, 1994.

[Kat76] Yitzhak Katznelson. *An Introduction to Harmonic Analysis.* Dover, New York, second edition, 1976.

[KF75] A. N. Kolmogorov and S. V. Fomīn. *Introductory real analysis.* Dover Publications Inc., New York, 1975. Translated from the second Russian edition and edited by Richard A. Silverman, Corrected reprinting.

[Kör88] T. W. Körner. *Fourier Analysis.* Cambridge University Press, Cambridge, UK, 1988.

[Lan85] Serge Lang. *Complex Analysis.* Springer-Verlag, New York, second edition, 1985.

[McW72] Roy McWeeny. *Quantum Mechanics: Principles and Formalism.* Dover, Mineola, NY, 1972.

[Mül66] C. Müller. *Spherical Harmonics.* Number 17 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1966.

[Mur93] James D. Murray. *Mathematical Biology*, volume 19 of *Biomathematics.* Springer-Verlag, New York, second edition, 1993.

[Nee97] Tristan Needham. *Visual complex analysis.* The Clarendon Press Oxford University Press, New York, 1997.

[Neh75] Zeev Nehari. *Conformal mapping.* Dover Publications Inc., New York, 1975. Reprinting of the 1952 edition.

[Pin98] Mark A. Pinsky. *Partial Differential Equations and Boundary-Value Problems with Applications.* International Series in Pure and Applied Mathematics. McGraw-Hill, Boston, third edition, 1998.

[Pow99] David L. Powers. *Boundary value problems.* Harcourt/Academic Press, San Diego, CA, fourth edition, 1999.

[Pru81] Eduard Prugovecki. *Quantum Mechanicss in Hilbert Space.* Academic Press, New York, second edition, 1981.

[RB69] Gian-Carlo Rota and Garret Birkhoff. *Ordinary Differential Equations.* Wiley, 2nd edition, 1969.

[Roy88] H. L. Royden. *Real analysis.* Macmillan Publishing Company, New York, third edition, 1988.

[Rud87] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[Sch79] Hans Schwerdtfeger. *Geometry of complex numbers.* Dover Publications Inc., New York, 1979. Circle geometry, Moebius transformation, non-Euclidean geometry, A corrected reprinting of the 1962 edition, Dover Books on Advanced Mathematics.

[Ste95] Charles F. Stevens. *The six core theories of modern physics.* A Bradford Book. MIT Press, Cambridge, MA, 1995.

[Ste08] James Stewart. *Calculus, Early Transcendentals.* Thomson Brooks/Cole, Belmont, CA, 6e edition, 2008.

[Str93] Daniel W. Strook. *Probability Theory: An analytic view.* Cambridge University Press, Cambridge, UK, revised edition, 1993.

[Sug75] M. Sugiura. *Unitary Representations and Harmonic Analysis.* Wiley, New York, 1975.

[Tak94] Masaru Takeuchi. *Modern Spherical Functions*, volume 135 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1994.

[Tay86] Michael Eugene Taylor. *Noncommutative Harmonic Analysis*. American Mathematical Society, Providence, Rhode Island, 1986.

[Ter85] Audrey Terras. *Harmonic Analysis on Symmetric Spaces and Applications*, volume I. Springer-Verlag, New York, 1985.

[Tit62] E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations, Part I*. Oxford University Press, London, 2nd edition, 1962.

[Wal88] James S. Walker. *Fourier Analysis*. Oxford UP, Oxford, UK, 1988.

[War83] Frank M. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer-Verlag, New York, 1983.

[WZ77] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*. Marcel Dekker Inc., New York, 1977. An introduction to real analysis, Pure and Applied Mathematics, Vol. 43.

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Notation

Sets and domains:

$\mathbb{A}(r,R)$**:** The 2-dimensional **closed annulus** of inner radius r and outer radius R: the set of all $(x,y) \in \mathbb{R}^2$ such that $r \leq x^2+y^2 \leq R$.

${}^{\circ}\mathbb{A}(r,R)$**:** The 2-dimensional **open annulus** of inner radius r and outer radius R: the set of all $(x,y) \in \mathbb{R}^2$ such that $r < x^2+y^2 < R$.

$\mathbb{B}$**:** A D-dimensional **closed ball** (often the unit ball centred at the origin).

$\mathbb{B}(\mathbf{x},\epsilon)$**:** The D-dimensional **closed ball**; of radius ϵ around the point $\mathbf{x}$; the set of all $\mathbf{y} \in \mathbb{R}^D$ such that $\|\mathbf{x}-\mathbf{y}\| < \epsilon$.

$\mathbb{C}$**:** The set of **complex numbers** of the form $x+y\mathbf{i}$, where $x,y \in \mathbb{R}$, and $\mathbf{i}$ is the square root of -1.

$\mathbb{C}_+$**:** The set of complex numbers $x+y\mathbf{i}$ with $y>0$.

$\mathbb{C}_-$**:** The set of complex numbers $x+y\mathbf{i}$ with $y<0$.

$\widehat{\mathbb{C}}$ $= \mathbb{C} \sqcup \{\infty\}$, the **Riemann Sphere** (the range of a meromorphic function).

$\mathbb{D}$**:** A 2-dimensional **closed disk** (usually the unit disk centred at the origin).

$\mathbb{D}(R)$**:** A 2-dimensional **closed disk of radius** R, centred at the origin: the set of all $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 \leq R$.

${}^{\circ}\mathbb{D}(R)$**:** A 2-dimensional **open disk of radius** R, centred at the origin: the set of all $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 < R$.

$\mathbb{D}^{\complement}(R)$**:** A 2-dimensional **closed codisk of coradius** R, centred at the origin: the set of all $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 \geq R$.

${}^{\circ}\mathbb{D}^{\complement}(R)$**:** A 2-dimensional **open codisk of coradius** R, centred at the origin: the set of all $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 > R$.

$\mathbb{H}$**:** A **half-plane**. Usually $\mathbb{H} = \left\{(x,y) \in \mathbb{R}^2 \,;\, y \geq 0\right\}$ (the upper half-plane).

$\mathbb{N}$**:** $= \{0,1,2,3,\ldots\}$, the set of **natural numbers**.

$\mathbb{N}_+$**:** $= \{1,2,3,\ldots\}$, the set of **positive natural numbers**.

$\mathbb{N}^D$**:** The set of all $\mathbf{n} = (n_1,n_2,\ldots,n_D)$, where $n_1,\ldots,n_D$ are natural numbers.

$\emptyset$**:** The empty set, also denoted $\{\}$.

$\mathbb{Q}$**:** The **rational numbers**: the set of all fractions n/m, where $n,m \in \mathbb{Z}$, and $m \neq 0$.

$\mathbb{R}$**:** The set of **real numbers** (e.g. 2, -3, $\sqrt{7}+\pi$, etc.)

$\mathbb{R}_+ \;:= (0,\infty) = \{r \in \mathbb{R}\,;\, r \geq 0\}$.

$\mathbb{R}_{\not<} \;:= [0,\infty) = \{r \in \mathbb{R}\,;\, r \geq 0\}$.

$\mathbb{R}^2$**:** The 2-dimensional infinite plane —the set of all ordered pairs (x,y), where $x,y \in \mathbb{R}$.

$\mathbb{R}^D$**:** D-dimensional space —the set of all D**-tuples** $(x_1,x_2,\ldots,x_D)$, where $x_1,x_2,\ldots,x_D \in \mathbb{R}$. Sometimes we will treat these D-tuples as **points** (representing locations in physical space); normally points will be indicated in **bold face**, eg: $\mathbf{x} = (x_1,\ldots,x_D)$. Sometimes we will treat the D-tuples as **vectors** (pointing in a particular direction); then they will be indicated with arrows, eg: $\vec{\mathbf{V}} = (V_1,V_2,\ldots,V_D)$.

$\mathbb{R}^D \times \mathbb{R}$**:** The set of all pairs $(\mathbf{x};t)$, where $\mathbf{x} \in \mathbb{R}^D$ is a vector, and $t \in \mathbb{R}$ is a number. (Of course, mathematically, this is the same as $\mathbb{R}^{D+1}$, but sometimes it is useful to regard the last dimension as "time".)

$\mathbb{R} \times \mathbb{R}_{\not<}$**:** The **half-space** of all points $(x,y) \in \mathbb{R}^2$, where $y \geq 0$.

$\mathbb{S}$**:** The 2-dimensional **unit circle**; the set of all $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 = 1$.

$\mathbb{S}^{D-1}(\mathbf{x};R)$**:** The D-dimensional **sphere**; of radius R around the point $\mathbf{x}$; the set of all $\mathbf{y}\in\mathbb{R}^D$ such that $\|\mathbf{x}-\mathbf{y}\|=R$

$\mathbb{U}$**,** $\mathbb{V}$**,** $\mathbb{W}$ usually denote open subsets of $\mathbb{R}^D$ or $\mathbb{C}$.

$\mathbb{X}$**,** $\mathbb{Y}$**:** usually denote **domains** —closed connected subsets of $\mathbb{R}^D$ with dense interiors.

$\mathbb{Z}$**:** The **integers** $\{\ldots,-2,-1,0,\,1,\,2,\,3,\ldots\}$.

$\mathbb{Z}^D$**:** The set of all $\mathbf{n}=(n_1,n_2,\ldots,n_D)$, where $n_1,\ldots,n_D$ are integers.

$[1...D]$ $=\{1,2,3,\ldots,D\}$.

$[0,\pi]$**:** The **closed interval** of length π; the set of all real numbers x where $0\le x\le\pi$.

$(0,\pi)$**:** The **open interval** of length π; the set of all real numbers x where $0<x<\pi$.

$[0,\pi]^2$**:** The (closed) $\pi\times\pi$ **square**; the set of all points $(x,y)\in\mathbb{R}^2$ where $0\le x,y\le\pi$.

$[0,\pi]^D$**:** The D-dimensional **unit cube**; the set of all points $(x_1,\ldots,x_D)\in\mathbb{R}^D$ where $0\le x_d\le 1$ for all $d\in[1...D]$.

$[-L,L]$**:** The **interval** of all real numbers x with $-L\le X\le L$.

$[-L,L]^D$**:** The D-dimensional **cube** of all points $(x_1,\ldots,x_D)\in\mathbb{R}^D$ where $-L\le x_d\le L$ for all $d\in[1...D]$.

Set operations:

$\mathsf{int}(\mathbb{X})$ The **interior** of the set $\mathbb{X}$ (i.e. all points in $\mathbb{X}$ *not* on the boundary of $\mathbb{X}$).

$\bigcap$ **Intersection**. If $\mathbb{X}$ and $\mathbb{Y}$ are sets, then $\mathbb{X}\cap\mathbb{Y}:=\{z\,;\ z\in\mathbb{X}\text{ and }z\in\mathbb{Y}\}$. If $\mathbb{X}_1,\ldots,\mathbb{X}_N$ are sets, then $\displaystyle\bigcap_{n=1}^N\mathbb{X}_n\ :=\ \mathbb{X}_1\cap\mathbb{X}_2\cap\cdots\cap\mathbb{X}_N$.

$\bigcup$ **Union**. If $\mathbb{X}$ and $\mathbb{Y}$ are sets, then $\mathbb{X}\cup\mathbb{Y}:=\{z\,;\ z\in\mathbb{X}\text{ or }z\in\mathbb{Y}\}$. If $\mathbb{X}_1,\ldots,\mathbb{X}_N$ are sets, then $\displaystyle\bigcup_{n=1}^N\mathbb{X}_n\ :=\ \mathbb{X}_1\cup\mathbb{X}_2\cup\cdots\cup\mathbb{X}_N$.

$\bigsqcup$ **Disjoint union**. If $\mathbb{X}$ and $\mathbb{Y}$ are sets, then $\mathbb{X}\sqcup\mathbb{Y}$ means the same as $\mathbb{X}\cup\mathbb{Y}$, but conveys the added information that $\mathbb{X}$ and $\mathbb{Y}$ are *disjoint* —i.e. $\mathbb{X}\cap\mathbb{Y}=\emptyset$. Likewise, $\displaystyle\bigsqcup_{n=1}^N\mathbb{X}_n\ :=$ $\mathbb{X}_1\sqcup\mathbb{X}_2\sqcup\cdots\sqcup\mathbb{X}_N$.

$\setminus$ **Difference.** If $\mathbb{X}$ and $\mathbb{Y}$ are sets, then $\mathbb{X}\setminus\mathbb{Y}=\{x\in\mathbb{X}\,;\ x\not\in\mathbb{Y}\}$.

Spaces of Functions:

$\mathcal{C}^\infty$**:** A vector space of (infinitely) differentiable functions. Some examples:

- $\mathcal{C}^\infty[\mathbb{R}^2;\ \mathbb{R}]$: The space of differentiable scalar fields on the plane.
- $\mathcal{C}^\infty[\mathbb{R}^D;\ \mathbb{R}]$: The space of differentiable scalar fields on D-dimensional space.
- $\mathcal{C}^\infty[\mathbb{R}^2;\ \mathbb{R}^2]$: The space of differentiable vector fields on the plane.

$\mathcal{C}_0^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **Dirichlet boundary conditions**: $f(\mathbf{x})=0$ for all $\mathbf{x}\in\partial[0,1]^D$.

$\mathcal{C}_\perp^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **Neumann boundary conditions**: $\partial_\perp f(\mathbf{x})=0$ for all $\mathbf{x}\in\partial[0,1]^D$.

$\mathcal{C}_h^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **mixed boundary conditions**: $\frac{\partial_\perp f}{f}(\mathbf{x})=h(\mathbf{x})$ for all $\mathbf{x}\in\partial[0,1]^D$.

$\mathcal{C}_{\text{per}}^\infty[-\pi,\pi]$**:** The space of differentiable scalar fields on the interval $[-\pi,\pi]$ satisfying **periodic boundary conditions**.

$\mathbf{L}^1(\mathbb{R})$: The set of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x)|\ dx\ <\ \infty$.

$\mathbf{L}^1(\mathbb{R}^2)$: The set of all functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|\ dx\ dy\ <\ \infty$.

$\mathbf{L}^1(\mathbb{R}^3)$: The set of all functions $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^3} |f(\mathbf{x})|\ d\mathbf{x}\ <\ \infty$.

$\mathbf{L}^2(\mathbb{X})$: The set of all functions $f : \mathbb{X} \longrightarrow \mathbb{R}$ such that $\|f\|_2 = \left(\int_{\mathbb{X}} |f(\mathbf{x})|^2\ d\mathbf{x}\right)^{1/2} < \infty$.

$\mathbf{L}^2(\mathbb{X}; \mathbb{C})$: The set of all functions $f : \mathbb{X} \longrightarrow \mathbb{C}$ such that $\|f\|_2 = \left(\int_{\mathbb{X}} |f(\mathbf{x})|^2\ d\mathbf{x}\right)^{1/2} < \infty$.

Derivatives and Boundaries:

$\partial_k f\ = \frac{df}{dx_k}$.

$\nabla f\ = (\partial_1 f,\ \partial_2 f, \ldots, \partial_D f)$, the **gradient** of scalar field f.

$\operatorname{div} f\ = \partial_1 f_1\ +\ \partial_2 f_2\ +\ \ldots\ +\ \partial_D f_D$, the **divergence** of vector field f.

$\partial_\perp f$ is the derivative of f *normal* to the boundary of some region. Sometimes this is written as $\frac{\partial f}{\partial \mathbf{n}}$ or $\frac{\partial f}{\partial \nu}$, or as $\nabla f \cdot \mathbf{n}$.

$\triangle f\ = \partial_1^2 f + \partial_2^2 f + \ldots + \partial_D^2 f$. Sometimes this is written as $\nabla^2 f$.

$\mathsf{L}\, f$ sometimes means a general **linear differential operator** L being applied to the function f.

$\mathsf{SL}_{s,q}(\phi)\ = s \cdot \partial^2\phi + s' \cdot \partial\phi + q \cdot \phi$. Here, $s, q : [0, L] \longrightarrow \mathbb{R}$ are predetermined functions, and $\phi : [0, L] \longrightarrow \mathbb{R}$ is the function we are operating on by the **Sturm-Liouville operator** $\mathsf{SL}_{s,q}$.

$\dot{\gamma}\ = (\gamma_1', \ldots, \gamma_D')$ is the **velocity vector** of the path $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^D$.

$\partial\,\mathbb{X}$: If $\mathbb{X} \subset \mathbb{R}^D$ is some region in space, then $\partial\mathbb{X}$ is the **boundary** of that region. For example:

- $\partial\,[0, 1] =\ \{0,\ 1\}$.
- $\partial\,\mathbb{B}^2\,(0; 1) =\ \mathbb{S}^2(0; 1)$.
- $\partial\,\mathbb{B}^D\,(\mathbf{x}; R) =\ \mathbb{S}^D(\mathbf{x}; R)$.
- $\partial\ (\mathbb{R} \times \mathbb{R}_{\not-}) =\ \mathbb{R} \times \{0\}$.

Norms and Inner products:

$\|\mathbf{x}\|$: If $\mathbf{x} \in \mathbb{R}^D$ is a vector, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_D^2}$ is the **norm** (or **length**) of $\mathbf{x}$.

$\|f\|_2$: Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, with volume $M = \int_{\mathbb{X}} 1 d\mathbf{x}$. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is an integrable function, then $\|f\|_2 = \dfrac{1}{M}\left(\displaystyle\int_{\mathbb{X}} |f(\mathbf{x})|\ \ d\mathbf{x}\right)^{1/2}$ is the L^2**-norm** of f.

$\langle f, g\rangle$: If $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then their **inner product** is given by: $\langle f, g\rangle = \dfrac{1}{M}\displaystyle\int_{\mathbb{X}} f(\mathbf{x}) \cdot g(\mathbf{x})\ d\mathbf{x}$.

$\|f\|_1$: Let $\mathbb{X} \subseteq \mathbb{R}^D$ be any domain. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is an integrable function, then $\|f\|_\infty = \displaystyle\int_{\mathbb{X}} |f(\mathbf{x})|\ \ d\mathbf{x}$ is the L^1**-norm** of f.

$\|f\|_\infty$: Let $\mathbb{X} \subseteq \mathbb{R}^D$ be any domain. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a bounded function, then $\|f\|_\infty = \sup_{\mathbf{x}\in\mathbb{X}} |f(\mathbf{x})|$ is the L^∞**-norm** of f.

Other Operations on Functions:

A_n**:** normally denotes the nth **Fourier cosine coefficient** of a function f on $[0,\pi]$ or $[-\pi,\pi]$. That is, $A_n := \frac{2}{\pi}\int_0^\pi f(x)\ \cos(nx)\ dx$ or $A_n := \frac{1}{\pi}\int_{-\pi}^\pi f(x)\ \cos(nx)\ dx$.

$A_{n,m}$**:** normally denotes a 2-dimensional Fourier cosine coefficient, while $A_{\mathbf{n}}$ normally denotes a D-dimensional Fourier cosine coefficient.

B_n**:** normally denotes the nth **Fourier cosine coefficient** of a function f on $[0,\pi]$ or $[-\pi,\pi]$. That is, $A_n := \frac{2}{\pi}\int_0^\pi f(x)\ \sin(nx)\ dx$ or $A_n := \frac{1}{\pi}\int_{-\pi}^\pi f(x)\ \sin(nx)\ dx$.

$B_{n,m}$**:** normally denotes a 2-dimensional Fourier sine coefficient, while $B_{\mathbf{n}}$ normally denotes a D-dimensional Fourier sine coefficient.

$f * g$**:** If $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$, then their **convolution** is the function $f * g : \mathbb{R}^D \longrightarrow \mathbb{R}$ defined by $(f * g)(\mathbf{x}) \;=\; \displaystyle\int_{\mathbb{R}^D} f(\mathbf{y}) \cdot g(\mathbf{x}-\mathbf{y})\ d\mathbf{y}$.

$\widehat{f}(\boldsymbol{\mu}) \;=\; \dfrac{1}{(2\pi)^D}\displaystyle\int_{\mathbb{R}^D} f(\mathbf{x})\exp(\mathbf{i}\,\boldsymbol{\mu}\bullet\mathbf{x})\ d\mathbf{x}$ is the **Fourier transform** of the function $f : \mathbb{R}^D \longrightarrow \mathbb{C}$. It is defined for all $\boldsymbol{\mu} \in \mathbb{R}^D$.

$\widehat{f}_n \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\exp(\mathbf{i}\,nx)\ dx$ is the nth **complex Fourier coefficient** of a function $f : [-\pi,\pi] \longrightarrow \mathbb{C}$ (here $n \in \mathbb{Z}$).

$\displaystyle\int_\gamma f :\ = \int_0^S f[\gamma(s)] \cdot \dot{\gamma}(s)\ ds$ is a **chain integral**. Here, $f : \mathbb{U} \longrightarrow \mathbb{C}$ is some complex-valued function and $\gamma : [0,S] \longrightarrow \mathbb{U}$ is a **chain** (a piecewise-continuous, piecewise differentiable path).

$\displaystyle\oint_\gamma f$: A **contour integral**. The same definition as the chain integral $\displaystyle\int_\gamma f$, but γ is a contour.

$\alpha \diamond \beta$: If α and β are two chains, then $\alpha \diamond \beta$ represents the **linking** of the two chains.

$\mathcal{L}[f] \;=\; \displaystyle\int_0^\infty f(t)\, e^{-ts}\ dt$ is the **Laplace transform** of the function $f : \mathbb{R}_{\not -} \longrightarrow \mathbb{C}$; it is defined for all $s \in \mathbb{C}$ with $\mathrm{Re}\,[s] > \alpha$, where α is the exponential order of f.

$\mathbf{M}_R\, u(\mathbf{x}) \;=\; \dfrac{1}{4\pi R^2}\displaystyle\int_{\mathbb{S}(R)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s}$ is the **spherical average** of f at $\mathbf{x}$, of radius R. Here, $\mathbf{x} \in \mathbb{R}^3$ is a point in space, $R > 0$, and $\mathbb{S}(R) = \left\{\mathbf{s} \in \mathbb{R}^3\ ;\ \|\mathbf{s}\| = R\right\}$.

Special functions.:

$\mathbf{C}_n(x) \;= \cos(nx)$ for all $n \in \mathbb{N}$ and $x \in [-\pi,\pi]$.

$\mathbf{C}_{n,m}(\mathbf{x}) \;= \cos(nx)\cdot\cos(my)$ for all $n, m \in \mathbb{N}$ and $(x,y) \in [\pi,\pi]^2$.

$\mathbf{C}_{\mathbf{n}}(\mathbf{x}) \;= \cos(n_1x_1)\cdots\cos(n_Dx_D)$ for all $\mathbf{n} \in \mathbb{N}^D$ and $\mathbf{x} \in [\pi,\pi]^D$.

$\mathbf{D}_N(x) \;= 1 + 2\sum_{n=1}^N \cos(nx)$ is the nth **Dirichlet kernel**, for all $n \in \mathbb{N}$ and $x \in [-2\pi, 2\pi]$.

$\mathbf{E}_n(x) \;= \exp(\mathbf{i}\,nx)$ for all $n \in \mathbb{Z}$ and $x \in [-\pi,\pi]$.

$\mathcal{E}_\mu(x) \;= \exp(\mathbf{i}\,\mu x)$ for all $\mu \in \mathbb{R}$ and $x \in [-\pi,\pi]$.

$\mathcal{G}(x;t) \;=\; \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-x^2}{4t}\right)$ is the (one-dimensional) **Gauss-Weierstrass kernel**.

$\mathcal{G}(x,y;\ t) \;=\; \frac{1}{4\pi t}\exp\left(\frac{-x^2-y^2}{4t}\right)$ is the (two-dimensional) **Gauss-Weierstrass kernel**.

$\mathcal{G}(\mathbf{x};t) \;=\; \frac{1}{(4\pi t)^{D/2}}\exp\left(\frac{-\|\mathbf{x}\|^2}{4t}\right)$ is the (D-dimensional) **Gauss-Weierstrass kernel**.

$\mathcal{J}_n$ is the nth **Bessel function of the first kind**.

$\mathcal{K}_y(x) \;=\; \frac{y}{\pi(x^2+y^2)}$ is the **half-plane Poisson kernel**, for all $x \in \mathbb{R}$ and $y > 0$.

$\vec{\mathbf{N}}(\mathbf{x})$ is the **outward unit normal vector** to a domain $\mathbb{X}$ at a point $\mathbf{x} \in \partial\mathbb{X}$.

$\mathcal{P}(\mathbf{x}, \mathbf{s}) = \dfrac{R^2 - \|\mathbf{x}\|^2}{\|\mathbf{x} - \mathbf{s}\|^2}$ is the **Poisson kernel** on the disk, for all $\mathbf{x} \in \mathbb{D}$ and $\mathbf{s} \in \mathbb{S}$.

$\mathbf{P}_r(x) = \dfrac{1 - r^2}{1 - 2r\cos(x) + r^2}$ is the **Poisson kernel** in polar coordinates, for all $x \in [-2\pi, 2\pi]$ and $r < 1$.

Φ_n**,** ϕ_n**,** Ψ_n **and** ψ_n refer to the harmonic functions on the unit disk which separate in polar coordinates. $\Phi_0(r, \theta) = 1$ and $\phi_0(r, \theta) = \log(r)$, while for all $n \geq 1$, we have $\Phi_n(r, \theta) = \cos(n\theta) \cdot r^n$, $\Psi_n(r, \theta) = \sin(n\theta) \cdot r^n$, $\phi_n(r, \theta) = \dfrac{\cos(n\theta)}{r^n}$, and $\psi_n(r, \theta) = \dfrac{\sin(n\theta)}{r^n}$.

$\Phi_{n,\lambda}$**,** $\Psi_{n,\lambda}$**,** $\phi_{n,\lambda}$**, and** $\psi_{n,\lambda}$ refer to eigenfunctions of the Laplacian on the unit disk which separate in polar coordinates. For all $n \in \mathbb{N}$ and $\lambda > 0$, $\Phi_{n,\lambda}(r, \theta) = \mathcal{J}_n(\lambda \cdot r) \cdot \cos(n\theta)$, $\Psi_{n,\lambda}(r, \theta) = \mathcal{J}_n(\lambda \cdot r) \cdot \sin(n\theta)$, $\phi_{n,\lambda}(r, \theta) = \mathcal{Y}_n(\lambda \cdot r) \cdot \cos(n\theta)$, and $\psi_{n,\lambda}(r, \theta) = \mathcal{Y}_n(\lambda \cdot r) \cdot \sin(n\theta)$

$\mathbf{S}_n(x) = \sin(nx)$ for all $n \in \mathbb{N}$ and $x \in [\pi, \pi]$.

$\mathbf{S}_{n,m}(\mathbf{x}) = \sin(nx) \cdot \sin(my)$ for all $n, m \in \mathbb{N}$ and $(x, y) \in [\pi, \pi]^2$.

$\mathbf{S}_{\mathbf{n}}(\mathbf{x}) = \sin(n_1 x_1) \cdots \sin(n_D x_D)$ for all $\mathbf{n} \in \mathbb{N}^D$ and $\mathbf{x} \in [\pi, \pi]^D$.

$\mathcal{Y}_n$ is the nth **Bessel function of the second kind**.