UNIVERSITY OF TORONTO
3 1761 00184414 1

Digitized by the Internet Archive
in 2007 with funding from
Microsoft Corporation

http://www.archive.org/details/highermathematic00melluoft

HIGHER MATHEMATICS

FOR

STUDENTS OF CHEMISTRY AND PHYSICS

11

Mat
1527h

HIGHER MATHEMATICS

FOR

STUDENTS OF CHEMISTRY AND PHYSICS

WITH SPECIAL REFERENCE TO PRACTICAL WORK

BY

J. W. MELLOR, D.Sc.

LATE SENIOR SCHOLAR AND 1851 EXHIBITION SCHOLAR,
NEW ZEALAND UNIVERSITY; RESEARCH FELLOW,
THE OWENS COLLEGE, MANCHESTER

60056
2/9/03

LONGMANS, GREEN, AND CO.
39 PATERNOSTER ROW, LONDON
NEW YORK AND BOMBAY
1902

All rights reserved

QA
37
M45
Cop. 3

Dedicated

to

My Parents

PREFACE.

It is almost impossible to follow the later developments of physical or general chemistry without a working knowledge of higher mathematics. I have found that the regular textbooks of mathematics rather perplex than assist the chemical student who seeks a short road to this knowledge, for it is not easy to discover the relation which the pure abstractions of formal mathematics bear to the problems which every day confront the student of Nature's laws, and realize the complementary character of mathematical and physical processes.

During the last five years I have taken note of the chief difficulties met with in the application of the mathematician's x and y to physical chemistry, and, as these notes have grown, I have sought to make clear how experimental results lend themselves to mathematical treatment. I have found by trial that it is possible to interest chemical students and to give them a working knowledge of mathematics by manipulating the results of physical or chemical observations.

I should have hesitated to proceed beyond this experimental stage if I had not found at The Owens College a set of students eagerly pursuing work in different branches of physical chemistry, and most of them looking for help in the discussion of their results. When I told my plan to the Professor of Chemistry he encouraged me to write this book. It has been my aim to carry out his suggestion, so I quote his letter as giving the spirit of the book, which I only wish I could have carried out to the letter.

"The Owens College,
"Manchester.

"My Dear Mellor,

"If you will convert your ideas into words and write a book explaining the inwardness of mathematical operations as applied to chemical results, I believe you will confer a benefit on many students of chemistry. We chemists, as a tribe, fight shy of any symbols but our own. I know very well you have the power of winning new results in chemistry and discussing them mathematically. Can you lead us up the high hill by gentle slopes? Talk to us chemically to beguile the way? Dose us, if need be, 'with learning put lightly, like powder in jam'? If you feel you have it in you to lead the way we will try to follow, and perhaps some of the youngest of us may succeed. Wouldn't this be a triumph worth working for? Try.

"Yours very truly,
"H. B. Dixon."

The Owens College,
Manchester, *May*, 1901.

CONTENTS.

PART I.—ELEMENTARY.

CHAPTER I.—THE DIFFERENTIAL CALCULUS.

CHAPTER II.—COORDINATE OR ANALYTICAL GEOMETRY.

CHAPTER III.—FUNCTIONS WITH SINGULAR PROPERTIES.

CHAPTER IV.—THE INTEGRAL CALCULUS.

CHAPTER V.—INFINITE SERIES AND THEIR USES.

PART II.—ADVANCED.

CHAPTER VI.—HYPERBOLIC FUNCTIONS.

CHAPTER VII.—HOW TO SOLVE DIFFERENTIAL EQUATIONS.

CHAPTER VIII.—FOURIER'S THEOREM.

PART III.—USEFUL RESULTS FROM ALGEBRA AND TRIGNOMETRY.

CHAPTER IX.—HOW TO SOLVE NUMERICAL EQUATIONS.

CHAPTER X.—DETERMINANTS.

CHAPTER XI.—PROBABILITY AND THE THEORY OF ERRORS.

CHAPTER XII.—COLLECTION OF FORMULAE FOR REFERENCE.

CHAPTER XIII.—REFERENCE TABLES.

b

"The first thing to be attended to in reading any algebraic treatise is the gaining a perfect understanding of the different processes there exhibited, and of their connection with one another. This cannot be attained by a mere reading of the book, however great the attention which may be given. It is impossible in a mathematical work to fill up every process in the manner in which it must be filled up in the mind of the student before he can be said to have completely mastered it. Many results must be given of which the details are suppressed, such are the additions, multiplications, extractions of square root, etc., with which the investigations abound. These must not be taken on trust by the student, but must be worked by his own pen, which must never be out of his hand while engaged in any algebraical process."—De Morgan, *On the Study and Difficulties of Mathematics*, 1831.

PROLOGUE.

WHEN Sir Isaac Newton communicated the manuscript of his "Methodus fluxionem" to his friends in 1669 he furnished science with its most powerful and subtle instrument of research. The states and conditions of matter, as they occur in Nature, are in a state of perpetual flux, and these qualities may be effectively studied by the Newtonian method whenever they can be referred to number or subjected to measurement (real or imaginary). By the aid of Newton's calculus the mode of action of natural changes from moment to moment can be portrayed as faithfully as these words represent the thoughts at present in my mind. From this, the law which controls the whole process can be determined with unmistakable certainty by pure calculation—the so-called Higher Mathematics.

This work starts from the thesis that so far as the investigator is concerned,

Higher Mathematics is the art of reasoning about the numerical relations between natural phenomena; and the several sections of Higher Mathematics are different modes of viewing these relations.*

For instance, I have assumed that the purpose of the Differential Calculus is to inquire how natural phenomena

---

* In the new German *Annalen der Naturphilosophie*, **1**, 50, 1902, Ostwald maintains that mathematics is only a language in which the results of experiments may be conveniently expressed; and from this standpoint criticises Kant's *Metaphysical Foundations of Science*. *Cf.* footnote, page 1.

change from moment to moment. This change may be uniform and simple (Chapter I.); or it may be associated with certain so-called "singularities" (Chapter III.). The Integral Calculus (Chapters IV. and VII.) attempts to deduce the fundamental principle governing the whole course of any natural process from the law regulating the momentary states. Coordinate Geometry (Chapter II.) is concerned with the study of natural processes by means of "pictures" or geometrical figures. Infinite Series (Chapters V. and VIII.) furnish approximate ideas about natural processes when other attempts fail. From this, then, we proceed to study the various methods ("mathematical tools") to be employed in Higher Mathematics.

This limitation of the scope of Higher Mathematics enables us to dispense with many of the formal proofs of rules and principles. Much of Sidgwick's * trenchant indictment of the educational value of formal logic might be urged against the subtle formalities which prevail in "school" mathematics. While none but logical reasoning could be for a moment tolerated, yet too often "its most frequent work is to build a *pons asinorum* over chasms that shrewd people can bestride without such a structure".†

So far as the tyro is concerned theoretical demonstrations are by no means so convincing as is sometimes supposed. It is as necessary to learn to "think in letters" and to handle numbers and quantities by their symbols as it is to learn to swim or to ride a bicycle. The inutility of "general proofs" is an everyday experience to the teacher. The beginner only acquires confidence by reasoning about something which allows him to test whether his results are true or false; he is really convinced only after the principle has been verified by actual measurement—as in § 88, say—or by arithmetical illustration—as in § 188, say. "The best of all proofs," said Oliver Heaviside in a recent number of the *Electrician*, "is to set out the fact descriptively so that it can be seen to be a fact". Remembering also that the majority of students are only interested in mathematics so

---

* A. Sidgwick, *The Use of Words in Reasoning*. (A. & C. Black, London.)

† O. W. Holmes, *The Autocrat of the Breakfast Table*. (W. Scott, London.)

far as it is brought to bear directly on problems connected with their own work, I have, especially in Part I., explained any troublesome principle or rule in terms of some well-known natural process. For example, the meaning of the differential coefficient and of a limiting ratio is first explained in terms of the velocity of a chemical reaction; the differentiation of exponential functions leads us to compound interest and hence to the "Compound Interest Law" in Nature; the general equations of the straight line are deduced from solubility curves; discontinuous functions lead us to discuss Mendeléeff's work on the existence of hydrates in solutions; Wilhelmy's law of mass action prepares us for a detailed study of processes of integration; Harcourt and Esson's work introduces the study of simultaneous differential equations; Fourier's series is applied to diffusion phenomena, etc., etc. Unfortunately, this plan has caused the work to assume more formidable dimensions than if the precise and rigorous language of the mathematicians had been retained throughout.

I have sometimes found it convenient to evade a tedious demonstration by reference to the "regular textbooks". In such cases, if the student wants to "dig deeper," one of the following works, according to subject, will be found sufficient: Williamson's *Differential Calculus*, also the same author's *Integral Calculus* (Longmans, Green, & Co., London); Forsyth's *Differential Equations* (Macmillan & Co., London); Johnson's *Differential Equations* (Wiley & Son, New York).

Of course, it is not always advisable to evade proofs in this summary way. The fundamental assumptions—the so-called premises—employed in deducing some formulae must be carefully checked and clearly understood. However correct the reasoning may have been, any limitations introduced as premises must, of necessity, reappear in the conclusions. The resulting formulae can, in consequence, only be applied to data which satisfy the limiting conditions. The results deduced in Chapter XI. exemplify, in a forcible manner, the perils which attend the indiscriminate application of mathematical formulae to experimental data. Some formulae are particularly liable to mislead. The "probable error" is one of the greatest sinners in this respect.

The teaching of mathematics by means of abstract problems is a good old practice easily abused. The abuse has given rise to a widespread conviction that "mathematics is the art of problem solving," or, perhaps, the prejudice dates from certain painful reminiscences associated with the arithmetic of our school-days.

Under the heading "Examples" I have collected laboratory measurements, well-known formulae, practical problems and exercises to illustrate the text immediately preceding. A few of the problems are abstract exercises in pure mathematics, old friends which have run through dozens of textbooks. A great number, however, are based upon measurements, etc., recorded in papers in the current science journals (Continental, American or British), and are reproduced in this connection for the first time.

It can serve no useful purpose to disguise the fact that a certain amount of drilling, nay, even of drudgery, is necessary in some stages, if mathematics is to be of *real* use as a working tool, and not employed simply for quoting the results of others. The proper thing, obviously, is to make the beginner feel that he is gaining strength and power during the drilling. In order to guide the student along the right path, hints and explanations have been appended to those exercises which have been found to present any difficulty. The subject-matter contains no difficulty which has not been mastered by beginners of average ability without the help of a teacher.

The student of this work is supposed to possess a working knowledge of elementary algebra so far as to be able to solve a set of simple simultaneous equations, and to know the meaning of a few trignometrical formulae. If any difficulty should arise on this head, it is very possible that §§ 155, 156, or 188 to 194 will say what is required on the subject. I have, indeed, every reason to suppose that beginners in the study of Higher Mathematics most frequently find their ideas on the questions discussed in

§§ 188 to 194 have grown so rusty with neglect as to require refurbishing.

I have also assumed that the reader is acquainted with the elementary principles of chemistry and physics. Should any illustration involve some phenomenon with which he is not acquainted, there are two remedies—to skip it, or to look up some textbook. There is no special reason why the student should waste time with illustrations in which he has no interest.

It will be found necessary to procure a set of mathematical tables containing the common logarithms of numbers and numerical values of the natural and logarithmic trignometrical ratios. Such sets can be purchased for about eighteen pence. The other numerical tables required for reference in Higher Mathematics are reproduced in the last chapter.

Where I am consciously indebted to any particular authority for ideas, either in the design of a diagram or in the writing of the text, I have stated the original source so that the student may have the opportunity of consulting the original for a fuller and perhaps a more lucid discussion.

I have great pleasure in thanking my friends for assistance in reading over the proofs, more especially Mr. W. R. Anderson, B.Sc., who has verified a great number of the examples from the printed slips, and Mr. L. Bradshaw, B.Sc., who has carefully checked all the numerical tables. I am also pleased to acknowledge the general excellency of the printer's share of the work.

It is, perhaps, too much to hope that all errors have been eliminated from the text, and the writer will be grateful when apprised of any which may have escaped his notice.

J. W. M.

XXII

xxiii

ADDENDA AND CORRIGENDA.

P. 70, last sentence in footnote to read: "Gay Lussac says that Charles had worked on this subject some years before himself, hence, etc." "See also (27) p. 526."

P. 82, fig. 23, insert "R" as described in the text.

P. 85, fig. 25, the upper "T" should be "T'".

P. 112, fig. 51, *for* "$\frac{3}{4}$" *read* "$\frac{3}{2}$".

P. 113, fig. 52, *for* "ϵ" *read* "$r \sin \epsilon$".

P. 189, equation 3, the vinculum should not extend over "$d\phi$".

P. 203, line 10, *for* "5·40" *read* "5·76".

P. 269, line 25, add "see (30) p. 526".

P. 289, line 16, *for* "third" *read* "first".

P. 378, at end of line 1, insert "322".

PART I.

ELEMENTARY.

CHAPTER I.

THE DIFFERENTIAL CALCULUS.

§ 1. On the Nature of Mathematical Reasoning.

"The philosopher may be delighted with the extent of his views, the artificer with the readiness of his hands, but let the one remember that without mechanical performance, refined speculation is an empty dream, and the other that without theoretical reasoning, dexterity is little more than brute instinct."—DR. JOHNSON.

HERBERT SPENCER has defined a law of Nature as a proposition stating that a certain uniformity has been observed in the relations between certain phenomena. In this sense a law of Nature expresses a mathematical relation between the phenomena under consideration. Every physical law, therefore, can be represented in the form of a mathematical equation. One of the chief objects of scientific investigation is to find out how one thing depends on another, and to express this relationship in the form of a mathematical equation (symbolic or otherwise) is the experimenter's ideal goal.*

There is in some minds an erroneous notion that the methods of higher mathematics are prohibitively difficult. Any difficulty

* Thus Berthelot, in the preface to his celebrated *Essai de Mécanique Chimique fondée sur la thermochemie* of 1879, described his work as an attempt to base chemistry wholly on those mechanical principles which prevail in various branches of physical science. Kant, in the preface to his *Metaphysischen Anfangsgruenden der Naturwissenschaft*, has said that in every department of physical science there is only so much science, properly so called, as there is mathematics. As a consequence, he denied to chemistry the name "science". But there were no "journals of physical chemistry" in his time (1786).

that might arise is rather due to the complicated nature of the phenomena alone. Comte has said in his *Philosophie Positive*, "our feeble minds can no longer trace the logical consequences of the laws of natural phenomena whenever we attempt to simultaneously include more than two or three essential factors". * In consequence it is generally found expedient to introduce "simplifying assumptions" into the mathematical analysis. For example, in the theory of solutions we pretend that the dissolved substance behaves as if it were an indifferent gas. The kinetic theory of gases, thermodynamics, and other branches of applied mathematics are full of such assumptions.

By no process of sound reasoning can a conclusion drawn from limited data have more than a limited application. Even when the comparison between the observed and calculated results is considered satisfactory, the errors of observation may quite obscure the imperfections of the formula based on incomplete or simplified premises. Given a sufficient number of "if's," there is no end to the weaving of "cobwebs of learning admirable for the fineness of thread and work, but of no substance or profit" (Bacon). The only safeguard is to compare the deductions of mathematics with observation and experiment "for the very simple reason that they are only deductions, and the premises from which they are made may be inaccurate or incomplete. We must remember that we cannot get more out of the mathematical mill than we put into it, though we may get it in a form infinitely more useful for our purpose" (John Hopkinson's *James Forrest Lecture*, 1894).

The first clause of this last sentence is often quoted in a parrot-like way as an objection to mathematics. Nothing but real ignorance as to the nature of mathematical reasoning could give rise to such a thought. No process of sound reasoning can establish a result not contained in the premises.† It is

---

* I believe that this is the key to the interpretation of Comte's strange remarks: "Every attempt to employ mathematical methods in the study of chemical questions must be considered profoundly irrational and contrary to the spirit of chemistry. . . . If mathematical analysis should ever hold a prominent place in chemistry—an aberration which is happily almost impossible—it would occasion a rapid and a widespread degeneration of that science." (Freely translated from the fourth book of Auguste Comte's *Philosophie Positive*, 1830.)

† Inductive reasoning is, of course, good guessing, not *sound* reasoning, but the finest results in science have been obtained in this way. Calling the guess a "working hypothesis," its consequences are tested by experiment in every conceivable way. For

admitted on all sides that any demonstration is vicious if it contains in the conclusion anything *more* than was assumed in the premises. Why then is mathematics singled out and condemned for possessing the essential attribute of all sound reasoning?

It has been said that no science is established on a firm basis unless its generalisations can be expressed in terms of number, and it is the special province of mathematics to *assist* the investigator in finding numerical relations between phenomena. After experiment, then mathematics. While a science is in the experimental or observational stage, there is little scope for discerning numerical relations. It is only *after* the different workers have "collected data" that the mathematician is able to deduce the required generalisation. Thus a Maxwell followed Faraday and a Newton completed Kepler.

It must not be supposed, however, that these remarks are intended to imply that a law of Nature has ever been represented by a mathematical expression with perfect exactness. In the best of generalisations, hypothetical conditions invariably replace the complex state of things which actually obtains in Nature.

There is a prevailing impression that once a mathematical formula has been theoretically deduced, the law, embodied in the formula, has been sufficiently demonstrated, provided the differences between the "calculated" and the "observed" results fall within the limits of experimental error. The important point, already emphasized, is quite overlooked, namely, that any discrepancy between theory and fact is masked by errors of observation. With improved instruments, and better methods of measurement, more accurate data are from time to time available. The errors of observation being thus reduced, the approximate nature of the formulae becomes more and more apparent. Ultimately, the discrepancy between theory and fact becomes too great to be ignored. It is then necessary to "go over the fundamentals". New formulae must be obtained embodying less of hypothesis, more of fact. Thus, from the first bold guess of an original mind, succeeding genera-

---

example, the brilliant work of Fresnel was the sequel of Young's undulatory theory of light, and Hertz's recent work was suggested by Maxwell's electro-magnetic theories. J. Thomson's remarkable prediction of the influence of pressure on the melting point of ice was experimentally verified by Lord Kelvin. See also pages 42, 156, etc.

tions progress step by step towards a comprehensive and a complete formulation of the several laws of Nature.*

I shall proceed at once to explain the nature of the more important "tools" used in the application of mathematical processes to natural phenomena.

§ 2. The Differential Coefficient.

Higher mathematics, in general, deals with magnitudes which vary in a continuous manner. In order to render such a process susceptible of mathematical treatment the magnitude is supposed to change during a series of very short intervals of time. The shorter the interval the more uniform the process. This conception is of fundamental importance. To illustrate, let us consider the chemical reaction denoted by the equation:

$$\underset{(hydrogen)}{H_2} + \underset{(chlorine)}{Cl_2} = \underset{(hydrogen\ chloride)}{2HCl},$$

and suppose that the product of the action—hydrogen chloride—is removed from the sphere of the reaction the moment it is formed.†

If thirty cubic centimetres of hydrogen chloride are formed in *one minute* the reaction proceeds with a velocity of 30 c.c. per minute. This statement is not meant to imply that 0·5 c.c. of hydrogen chloride is formed during every second of the time of observation, for 0·2 c.c. may have been formed in the first second and 0·8 c.c. during some other second of time. The fact observed is that the mean rate of formation of hydrogen chloride is thirty cubic centimetres per minute.

---

* Most, if not all, the formulae of physics and chemistry are in the earlier stages of such a process of evolution. For example, some exact experiments by Forbes and by Tait indicate that Fourier's formula (page 375) for the conduction of heat gives somewhat discordant results on account of the inexact simplifying assumption: "the quantity of heat passing along a given line is proportional to the rate of change of temperature"; Weber has pointed out that Fick's equation (page 376) for the diffusion of salts in solution must be modified to allow for the decreasing diffusivity of the salt with increasing concentration; and finally, van der Waals, Clausius, Rankine, Sarrau, etc., have attempted to correct the simple gas equation: $pv = R\theta$, by making certain assumptions as to the internal structure of the gas.

† According to Bunsen and Roscoe these conditions are approximately realised when a mixture of hydrogen and chlorine gases is confined over water saturated with the two gases, and exposed to a constant source of light. The water absorbs the HCl as fast as it is formed.

If it were now possible to measure the amount formed in *thirty seconds* of time, and 18 c.c. of hydrogen chloride were obtained during that interval, this would be equivalent to a velocity of reaction of 36 c.c. per minute. In this case the calculated velocity would more accurately represent the actual velocity during the time of observation, because there would be less time for the rate of formation of this compound to vary during thirty than during sixty seconds.

Suppose further that 0·25 c.c. of hydrogen chloride were produced during an interval of *one second*, the observer would be perfectly certain that he had determined the rate of formation of this acid with a greater accuracy than before, because there would be less time for any variation to take place. Following out the consequences of this reasoning we are quite sure that if an observation could be made of the amount of hydrogen chloride formed during *one-millionth of a second*, the rate of formation of the compound at this moment would be still more accurate.

Using the symbol δx to denote the amount of hydrogen chloride formed during the very small interval of time δt, the quotient $\delta x/\delta t$ represents the velocity of the chemical reaction during this interval of time. If we could measure the amount of substance formed during an *infinitely short interval of time* the true velocity (v) of the chemical reaction would be denoted by the equation:

$$\frac{dx}{dt} = v, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where dx is a symbol used by mathematicians to represent an *infinitely small* amount of something (in this case of HCl), and dt a correspondingly short interval of time. Hence it follows that neither of these symbols *per se* is of any practical value, but their quotient stands for a perfectly definite conception, namely, the rate of formation of hydrogen chloride during an interval of time so small that all possibility of error due to variation of speed is eliminated.

The quotient dx/dt is known as the **differential coefficient** of x with respect to t. The value of x obviously depends on what value is assigned to t; for this reason x is called the **dependent variable,** t the **independent variable.** *The differential coefficient of x with respect to t thus measures a velocity.*

Just as the idea of the velocity of a chemical reaction represents the amount of substance formed in a given time, so the velocity of any motion can be expressed in terms of the differential coefficient of a distance with respect to time, be the motion that of a train, tramcar, bullet, sound-wave, water in a pipe or an electric current. Again, we may represent the differential coefficient of the volume of a gas, the length of a rod, or the electro-motive force of a galvanic element with respect to temperature to obtain the so-called *temperature coefficient* or *coefficient of expansion* as the case might be. The differential coefficient of the volume of a gas with respect to pressure furnishes the so-called *coefficient of compression*, which measures the compressibility of a gas.

From these and similar illustrations which will occur to the reader, it will be evident that the conception called by mathematicians the differential coefficient is not new. Every one consciously or unconsciously uses it whenever a "rate," "speed," or a "velocity" is in question.

NOTE ON VELOCITY.—In elementary dynamics, velocity (v) is defined as *rate of motion*, and is measured in terms of the distance (s) traversed in the time (t). That is to say,

$$\frac{\textit{distance traversed}}{\textit{time}} = \textit{velocity}; \ \therefore v = \frac{s}{t}.$$

It is specially important for us to start with a clear idea of what is meant by the terms "velocity," "rate of motion," etc.

A train is observed to travel a distance of 60 miles in one hour. We cannot therefore say that it has travelled 30 miles during the last half hour, nor yet that it will travel 30 miles during the next half hour. On the other hand, if the train, at any part of its journey, is going at the rate of a mile a minute, we can say that the velocity *at* that particular moment is 60 miles an hour.

Strictly speaking, it is a physical impossibility to actually measure the "**velocity at any instant,**" we must therefore understand by this term, the *mean* or **average velocity** during a very small interval of time, *with the proviso that we can get as near as we please to the actual "velocity at any instant," by taking the interval of time sufficiently small.*

We shall soon see that "methods of differentiation" will actually enable us to find the velocity or rate of change during an interval of time so small that the rate of motion has not time to change. The differential coefficient is the only true measure of the velocity at any instant of time.

It is important to distinguish between the average velocity *during* any given interval of time, and the actual velocity *at* any instant.

The term "velocity" not only includes the *rate* of motion, but also the *direction* of the motion. If we agree to represent the velocity of a train travelling southwards to London, positive, a train going northwards to

Aberdeen would be travelling with a negative velocity. Again, if we conventionally agree to consider the rate of formation of hydrogen chloride from hydrogen and chlorine as a positive velocity, the rate of decomposition of hydrogen chloride into chlorine and hydrogen will be a negative velocity.

It is not necessary, for our present purpose, to enter into refined distinctions between rate, speed, and velocity. I shall use these terms synonymously.

The concept velocity need not be associated with bodies. Every one is familiar with the terms "the velocity of light," "the velocity of sound," "the rate of propagation of an explosion wave," etc. The chemical student will soon adapt the idea to such phrases as, "the velocity of chemical action," "the speed of catalysis," "the rate of dissociation," "the velocity of diffusion," "the rate of evaporation," etc.

It requires no great mental effort to extend the notion still further. If a quantity of heat is added to a substance at a uniform rate, the quantity of heat (Q) added per degree rise of temperature (θ) corresponds exactly with the idea of a distance traversed per second of time. Specific heat, therefore, may be represented by the differential coefficient $dQ/d\theta$. Similarly, the increase in volume (V) (or length) per degree rise of temperature is represented by the differential coefficient $dV/d\theta$; the decrease in volume (V) per gram of pressure (p), is represented by the ratio $-dV/dp$, where the negative sign signifies that the volume decreases with increase of pressure.

In the above examples, it has been assumed that unit mass or unit volume of substance is operated upon, and therefore the differential coefficients respectively represent specific heat, coefficient of expansion, coefficient of compressibility. If we start with unit mass of substance, the coefficient of velocity of a chemical reaction would obviously be dx/dt. (What does this measure? The rate of transformation of unit mass of substance.)

But velocity is generally changing. The velocity of a falling stone gradually increases during its descent, while, if a stone is projected upwards, its velocity gradually decreases during its ascent. Instead of using the awkward term "the velocity of a velocity," the word "**acceleration**" is employed. If the velocity is increasing at a uniform rate, the acceleration, or rate of change of velocity, or rate of change of motion, is evidently

$$\frac{\textit{increase of velocity}}{\textit{time}} = \textit{acceleration}\ ;\ f = \frac{v_1 - v_0}{t},$$

where v_0 and v_1 respectively denote the velocities at the beginning and end of the interval of time under consideration.

Mathematicians have agreed to represent an increasing velocity with a positive sign, a decreasing velocity with a negative sign. If a clock gains one second an hour, the acceleration is positive, if it loses one second an hour, the acceleration is negative. This discussion is resumed in § 7.

§ 3. Differentials.

It is sometimes convenient to regard dx and dt, or more generally dx and dy, as very small quantities which determine the course of any particular process under investigation. These small magnitudes are called *differentials* or *infinitesimals*.* Differentials may be treated like ordinary algebraic magnitudes. The quantity of hydrogen chloride formed in the time dt is represented by the differential dx. Hence from (1), if $dx/dt = v$, we may write in the language of differentials

$$dx = v.dt.$$

§ 4. Orders of Magnitude.

If a small number n be divided into a million parts, each part $(n/10^6)$ is so very small that it may for all practical purposes be neglected in comparison with n. If we agree to call n a *magnitude of the first order*, the quantity $n/10^6$ is a *magnitude of the second order*. If one of these parts be again subdivided into a million parts, each part $(n/10^{12})$ is extremely small when compared with n, and the quantity $n/10^{12}$ is a *magnitude of the third order*. We thus obtain a series of magnitudes of the first, second, and higher orders,

$$n, n \times 10^{-6}, n \times 10^{-12}, \ldots,$$

each one of which is negligibly small in comparison with those which precede it, and very large relative to those which follow.†

This idea is of great practical use in the reduction of intricate expressions to a simpler form more easily manipulated. It is usual to reject magnitudes of a higher order than those under investigation when the resulting error is so small that it is outside the limits of the "errors of observation" peculiar to that method of investigation. (See §§ 96 and 189.)

In order to prevent any misconception it might be pointed out that "great" and "small" in mathematics, like "hot" and "cold" in physics, are purely relative terms. The astronomer in calculating interstellar distances comprising millions of miles takes no notice of a few thousand miles; while the physicist dare not neglect distances of the order of the ten thousandth of an inch in his measurements of the wave length of light.

A term, therefore, is not to be rejected simply because it *seems*

* Some one has defined differentials as small quantities "verging on nothing".

† Note 10^8 = unity followed by eight cyphers, or 100,000,000. 10^{-8} is a decimal point followed by seven cyphers and unity, or $10^{-8} = 1/10^8 = 0{\cdot}00000001$. This notation is in general use.

small in an absolute sense, but only when it appears small in comparison with a much larger magnitude, and when an exact determination of this small quantity has no appreciable effect on the magnitude of the larger. In making up a litre of normal oxalic acid solution, the weighing of the 63 grams of acid required need not be more accurate than to the tenth of a gram. In many forms of analytical work, however, the thousandth of a gram is of fundamental importance; an error of a tenth of a gram would stultify the result.

§ 5. Zero and Infinity.

The words "infinitely small" were used in the second paragraph. It is, of course, impossible to conceive of an infinitely small or of an infinitely great magnitude, for if it were possible to retain either of these quantities before the mind for a moment, it would be just as easy to think of a smaller or a greater as the case might be. In mathematical thought the word **"infinity"** (written ∞) signifies the *properties* possessed by a magnitude greater than any finite magnitude that can be named. For instance, the greater we make the radius of a circle, the more approximately does the circumference approach a straight line, until, when the radius is made infinitely great, the circumference may, without committing any sensible error, be taken to represent a straight line. The consequences of the above definition of infinity have led to some of the most important results of higher mathematics. To summarize, infinity represents neither the magnitude nor the value of any particular quantity. *The term "infinity" is simply an abbreviation for the property of growing large without limit. E.g.*, "$\tan 90° = \infty$" means that as an angle approaches 90°, its tangent grows indefinitely large. Now for the opposite of greatness—smallness.

In mathematics two meanings are given to the word "zero". The ordinary meaning of the word implies the total absence of magnitude (called **absolute zero**). Nothing remains when the thing spoken of or thought about is taken away. If four units be taken from four units absolutely nothing remains. There is, however, another meaning to be attached to the word different from the destruction of the thing itself. If a small number be divided by a billion we get a small fraction. If this fraction be raised to the billionth power we get a number still more nearly equal to absolute zero. By continuing this process as long as we please we continually approach, but never actually reach, the absolute zero. In this relative sense zero—**relative zero**—is defined as "an infinitely small" or "a vanishingly small number," or "a number smaller than any assignable fraction of unity". For example, we might consider a point as an infinitely small circle or an infinitely short line. To put these ideas tersely, *absolute zero implies that the thing and all its properties are absent; relative zero implies that however small the thing may be its property of growing small without limit is alone retained in the mind.* This will be more rigorously demonstrated in the next paragraph.

EXAMPLES.—After the reader has verified the following results he will understand the special meaning to be attached to the zero and infinity of mathematical reasoning. Let n be any finite number, and let "?" denote an indeterminate magnitude, that is, one whose exact value cannot be determined:—

(1) $\infty + \infty = \infty$; $\infty - \infty = ?$; $n \times 0 = 0$; $0 \times 0 = ?$; $n \times \infty = \infty$; $0/0 = ?$; $n/0 = \infty$; $0/n = 0$; $\infty/0 = \infty$; $0/\infty = 0$; $n/\infty = 0$; $\infty/n = \infty$; $0^n = 0$; $1/0^n = \infty$; $0^0 = ?$; $1/0^0 = ?$; $\infty^n = \infty$; $1/\infty^n = 0$; $\infty^0 = ?$; $1/\infty^0 = ?$; $n^\infty = \infty$ when $n > 1$*, and $n^\infty = 0$ when $n < 1$; $1/n^\infty = 0$ when $n > 1$, and $1/n^\infty = \infty$ when $n < 1$; $1^\infty = ?$; $1/1^\infty = ?$; $n^0 = 1$; $1/n^0 = 1$. The last two results are proved in "the theory of indices" of any algebraic textbook.

(2) Let $y = 1/(1 - x)$ and put $x = 1$, then $y = \infty$; if $x < 1$, y is positive, and y is negative when $x > 1$. Thus a variable magnitude may change its sign when it becomes infinite.

(3) $\log 1 = 0$; $\log 0 = - \infty$; $\log \infty = \infty$.

§ 6. Limiting Values.

(i) *The sum of an infinite number of terms may have a finite value.* Converting $\frac{1}{9}$ into a decimal fraction we obtain

$$\tfrac{1}{9} = 0{\cdot}1111\dot{1} \ . \ . \ . \text{ continued to infinity,}$$

or

$$\tfrac{1}{9} = \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \ . \ . \ . \text{ to infinity,}$$

that is to say, the sum of an infinite number of terms is equal to $\frac{1}{9}$ —a finite term! If we were to attempt to perform this summation we should find that as long as the number of terms is finite we could never actually obtain the result $\frac{1}{9}$.

* The *signs of inequality* are as follows: "$\neq$" denotes "is not equal to"; "$>$," "is greater than"; "$\not>$," "is not greater than"; "$<$," "is less than"; and "$\not<$," "is not less than". See p. 454.

For "$\equiv$" read "is equivalent to" or "is identical with".

If we omit all terms after the first, the result is $\frac{1}{90}$ less than $\frac{1}{9}$; if we omit all terms after the third, the result is $\frac{1}{9,000}$ too little; and if we omit all terms after the sixth, the result is $\frac{1}{9,000,000}$ less than $\frac{1}{9}$, that is to say, the sum of these terms continually approaches but is never actually equal to $\frac{1}{9}$, as long as the number of terms is finite. $\frac{1}{9}$ is then said to be the **limiting value** of the sum of this series of terms.

Again, the perimeter of a polygon inscribed in a circle is less than the sum of the arcs of the circle, *i.e.*, less than the circumference of the circle.

In figure 1, let the arcs AaB, BbC . . . be bisected at a, b . . . Join Aa, aA, Bb, . . . Although the perimeter of the second polygon is greater than the first, it is still less than the circumference of the circle. In a similar way, if the arcs of this second polygon are bisected, we get a third polygon whose perimeter approaches yet nearer to the circumference of the circle. By continuing this process, a polygon may be obtained as nearly equal to the circumference of a circle as we please. The circumference of the circle is thus the limiting value of the perimeter of an inscribed polygon, when the number of its sides is increased indefinitely.

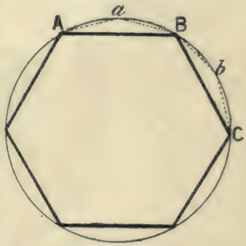


FIG. 1.

In general, *when a variable magnitude x continually approaches nearer and nearer to a constant value n so that x can be made to differ from n by a quantity less than any assignable magnitude, n is said to be the limiting value of x.*

From page 5, it follows that dx/dt is the limiting value of $\delta x/\delta t$, when t is made less than any finite quantity, however small. This is written for brevity

$$\frac{dx}{dt} = Lt_{\delta t = 0} \frac{\delta x}{\delta t};$$

in words "dx/dt* is the limiting value of $\delta x/\delta t$ when δt becomes zero" (relative zero, *i.e.*, small without limit). This notation is frequently employed.

---

* Although differential quotients are, in this work, written in the form "dx/dt," d^2x/dt^2 . . . , the student in working through the examples and demonstrations, should write $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ . . . The former method is used to economise space.

(ii) *The value of a limiting ratio depends on the relation between the two variables.* Strictly speaking, the limiting value of the ratio $\delta x/\delta t$ has the form $\frac{0}{0}$, and as such is indeterminate.* But for all practical purposes the differential coefficient dx/dt is to be regarded as a fraction or quotient (hence the German "Differentialquotient"). The quotient dx/dt may also be called a "rate-measurer," because it determines the velocity or rate with which one quantity varies when an extremely small variation is given to the other. *The actual value of the ratio dx/dt depends on the relation existing between x and t.*

Consider the following three examples (De Morgan):—

(1) If the point P move on the circumference of the circle towards a fixed point Q (Fig. 2), the arc x will diminish at the same time as the chord y. By bringing the point P sufficiently near to Q we obtain an arc and its chord

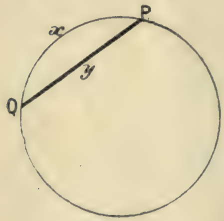


FIG. 2.

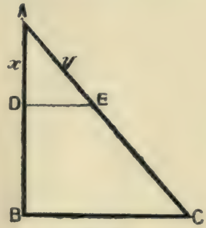


FIG. 3.

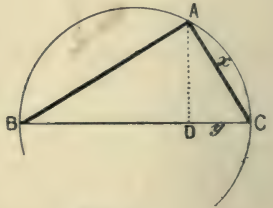


FIG. 4.

each less than any given line, that is, the arc and the chord continually approach a ratio of equality. Or, the limiting value of the ratio $\delta y/\delta x$ is unity.

$$\therefore Lt_{x=0}\frac{\delta y}{\delta x} = \frac{dy}{dx} = 1.$$

(2) If ABC (Fig. 3) be any right-angled triangle such that $AB = BC$. By Pythagoras' theorem or Euclid, i., 47, and vi., 4,

$$AB : AC = x : y = 1 : \sqrt{2}.$$

DE

If a line ~~BC~~, moving towards A, remains parallel to BC, this proportion will remain constant even though each side of the triangle ADE is made less than any assignable magnitude, however small. That is

$$Lt_{y=0}\frac{\delta x}{\delta y} = \frac{dx}{dy} = \frac{1}{\sqrt{2}}.$$

* Indeterminate, because $\frac{0}{0}$ may have any numerical value we please. It is not difficult to see this, *e.g.*,

$$\tfrac{0}{0} = 0, \text{ because } 0 \times 0 = 0;\ \tfrac{0}{0} = 1, \text{ because } 0 \times 1 = 0;$$
$$\tfrac{0}{0} = 2, \text{ because } 0 \times 2 = 0;\ \tfrac{0}{0} = 15, \text{ because } 0 \times 15 = 0;$$
$$\tfrac{0}{0} = 999{,}999, \text{ because } 0 \times 999{,}999 = 0, \text{ etc.}$$

(3) Let ABC be a triangle inscribed in a circle (Fig. 4). Draw AB perpendicular to BC. Then by Euclid, vi., 8

$$BC : AC = AC : DC = x : y.$$

If A approaches C until the chord AC becomes indefinitely small, DC will also become indefinitely small. The above proportion, however, remains. When the ratio $BC : AC$ becomes infinitely great, the ratio of AC to DC will also become infinitely great, or

$$Lt_{y=0}\frac{\delta x}{\delta y} = \frac{dx}{dy} = \infty.$$

It therefore follows at once that *although two quantities may become infinitely small their limiting ratio may have any finite or infinite value whatever.*

§ 7. The Differential Coefficient of a Differential Coefficient.

It will be evident from § 2, that the differential coefficient does not necessarily measure the absolute rate of increase during the whole process of formation of hydrogen chloride, but rather the rate of formation of that compound which would occur if the velocity remained during the whole interval the same as it was during the extremely short interval of time dt.

In the same reaction, if the hydrogen chloride had been allowed to remain mixed with the other reacting gases, the velocity of the chemical reaction would gradually decrease as the amount of hydrogen chloride present increased. In other words, the velocity of the reaction would be retarded.

If we consider the number of cubic centimetres of hydrogen chloride formed per second, the rate of change of the velocity of the reaction is evidently the limit of the ratio $\delta v/\delta t$. A retardation * is equivalent to a negative acceleration. If f denotes the acceleration, then a retardation must be denoted by f with a negative sign, or,

$$f = - Lt_{t=0}\frac{\delta v}{\delta t} = - \frac{dv}{dt}.$$

But from (1) § 2, v is equal to dx/dt, and hence

$$f = - d\left(\frac{dx}{dt}\right)\Big/dt,$$

* The meaning of the term "acceleration" is explained in elementary dynamics. If a body moves with an increasing velocity its motion is said to be accelerated. Acceleration means the rate at which the velocity of a body changes.

which is more conveniently written

$$f = -\frac{d^2x}{dt^2}, \left(\text{NOT } f = -\frac{dx^2}{dt^2}\right), \quad . \quad . \quad . \quad (2)$$

an expression denoting the momentary rate of increase in the velocity of the action due to the presence of increasing amounts of hydrogen chloride.

The ratio d^2x/dt^2 is called the **second differential coefficient** of x with respect to t.

Just as the first differential coefficient of x with respect to t signifies a "velocity," *the second differential coefficient of x with respect to t denotes an "acceleration"*.

In order to fix these ideas we shall consider a familiar experiment, namely, that of a stone falling from a vertical height. Observation shows that the velocity of the descending stone is changing from moment to moment. The above reasoning still holds good. We first find the distance (ds) traversed during any infinitely short interval of time (dt), that is

$$ds/dt = v.$$

We next consider the rate at which the velocity changes from one moment to another and obtain

$$dv/dt = f.$$

Substituting for v, we obtain the second differential coefficient

$$\frac{d^2s}{dt^2} = f,$$

which represents the rate of change of velocity or the acceleration at any instant of time. In this particular example the acceleration is due to the earth's gravitational force, and g is usually written instead of f.

In a similar way it could be shown that the *third differential coefficient* would represent the rate of change of acceleration from moment to moment, and so on for the higher differential coefficients d^nx/dt^n, which are seldom, if ever, used in practice. A few words on notation.

§ 8. Notation.

Strictly speaking the symbols δx, δt . . . should be reserved for small finite quantities; dx, dt . . . have no meaning *per se*. As a matter of fact the symbols dx, dt . . . are constantly used in place of δx, δt. . . . It is perhaps needless to remark that δ, d, d^2 . . . do not denote algebraic magnitudes.

In the ratio $\frac{dx}{dt}$, $\frac{d}{dt}$ is a **symbol of an operation** performed on x, as much as the symbols " ÷ " or " / " denote the operation of division. In the present case *the operation has been to find the limiting value of the ratio* $\frac{\delta x}{\delta t}$ *when* δt *is made smaller and smaller without limit;* but we constantly find that dx/dt is used when $\delta x/\delta t$ is intended. The notation we are using is due to Leibnitz. Newton, the discoverer of this calculus, superscribed a small dot over the dependent variable for the first differential coefficient, two dots for the second, thus $\dot{x}$, $\ddot{x}$ . . . In special cases, besides dy/dx and $\dot{y}$, we may have $\frac{d}{dx}(y)$, dy^x, x_y, x_1, x' . . . for the first differential coefficient; $\frac{d^2y}{dx^2}$, $\ddot{y}$, $\left(\frac{d}{dx}\right)^2 y$, x_2, x'' . . . for the second differential coefficient; and so on for the higher coefficients or derivatives as they are sometimes called. The operation of finding the value of the first differential coefficient of any expression is called **differentiation.** The differential calculus is that branch of mathematics which deals with these operations.

§ 9. Functions.

If the pressure to which a gas is subject be altered, it is known that the volume of the gas changes in a proportional way. The two magnitudes, pressure p and volume v, are interdependent. Any variation of the one is followed by a corresponding variation of the other. In mathematical language this idea is included in the word "*function*"; v is said to be a function of p. The two related magnitudes are called **variables.** Any magnitude which remains invariable during a given operation is called a **constant.**

In expressing Boyle's law for perfect gases we write this idea thus:

$$(\textit{dependent variable}) = f\,(\textit{independent variable}),$$

or

$$v = f(p),$$

meaning that "v is some function of p". There is, however, no particular reason why p was chosen as the independent variable. The choice of the dependent variable depends on the conditions of the experiment alone. We could here have written

$$p = f(v)$$

just as correctly as $v = f(p)$. In actions involving time it is customary, though not essential, to regard the latter as the independent variable, since time changes in a most uniform and independent way. Time is the natural independent variable.

In the same way the area of a circle is a function of the radius, so is the volume of a sphere; the pressure of a gas is a function of the density; the volume of a gas is a function of the temperature; the amount of substance formed in a chemical reaction is a function of the time; the velocity of an explosion wave is a function of the density of the medium; the boiling point of a liquid is a function of the atmospheric pressure; the resistance of a wire to the passage of an electric current is a function of the thickness of he wire; the solubility of a salt is a function of the temperature, etc.

The independent variable may be denoted by x when writing in general terms, and the dependent variable by y. The relation between these variables is variously denoted by the symbols:

$$y = f(x);\ y = \phi(x);\ y = F(x);\ y = \psi(x);\ y = f_1(x)\ \ldots\text{*}$$

Any one of these expressions means nothing more than that "*y is some function of x*".

If x_1, y_1; x_2, y_2; x_3, y_3, . . . are corresponding values of x and y, we may have

$$y = f(x);\ y_1 = f(x_1);\ y_2 = f(x_2)\ \ldots$$

"Let $y = f(x)$" means "take any equation which will enable you to calculate y when the value of x is known."

The word "function" in mathematical language thus implies that for every value of x there is a determinate value of y. If v_0 and p_0 are the corresponding values of the pressure and volume of a gas in any given state, v and p their respective values in some other state, Boyle's law states that

$$pv = p_0v_0.$$

Hence, $$p = p_0v_0/v;\ \text{or},\ v = p_0v_0/p.$$

The value of p or of v can therefore be determined for any assigned value of v or p as the case might be.

A similar rule applies for all physical changes in which two magnitudes simultaneously change their values according to some fixed law. It is quite immaterial, from our present point of view, whether or not any mathematical expression for the function $f(x)$ is known. For instance, although the pressure of the aqueous vapour in any vessel containing water and steam is a function of the temperature, the actual *form* of the expression or function

* For ". . ." read "etc." or "and so on".

showing this relation *is not known;* but the laws connecting the volume of a gas with its temperature and pressure are *known* expressions—Boyle and Gay Lussac's laws. The concept thus remains even though it is impossible to assign any rule for calculating the value of a function. In such cases the corresponding values of each variable can only be determined by actual observation and measurement. In other words, $f(x)$ *is a convenient symbol to denote any mathematical expression containing* x.

From pages 5 and 13, since

$$y = f(x),$$

the differential coefficient dy/dx is another function of x, say $f'(x)$,

$$dy/dx = f'(x), \text{ or } df(x)/dx = f'(x).$$

Similarly the second derivative, d^2y/dx^2, is another function of x, say $f''(x)$,

$$df'(x)/dx = f''(x);\ d^2y/dx^2 = f''(x);\ d^2f(x)/dx^2 = f''(x);$$

and so on for the higher differential functions.

The above investigation may be extended to functions of three or more variables. Thus the volume of a gas is a function of the pressure and temperature. We have tacitly assumed that the temperature was constant in our preceding illustration. If the pressure and temperature vary simultaneously,

$$v = f(p, \theta).$$

These ideas will be developed later on.

It might be pointed out that the methods of the calculus are usually applied to changes in which the independent variable varies continuously, or is a continuous function of the dependent variable; discontinuous functions when they do arise only occur for special values of x. See "Continuity and Discontinuity," page 118.

§ 10. Differentiation.

Before a knowledge of the instantaneous rate of change, dy/dx, can be of any *practical* use, it is necessary to know the actual relation, "law," or "form" of the function connecting the varying quantities one with another. (§ 69 may now be read.)

The differential calculus is not directly concerned with the establishment of any relation between the quantities themselves, but rather with the inquiry into the momentary state of the body.

This momentary state is symbolised by the differential coefficient, which thus conveys to the mind a perfectly clear and definite conception altogether apart from any numerical or practical application.

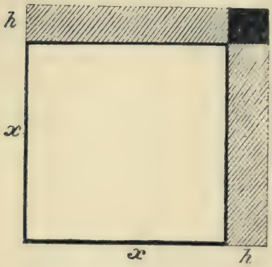


FIG. 5.

The mechanical operations of finding the differential coefficient of one variable with respect to another in any expression are no more difficult than ordinary algebraic processes. Before describing the practical methods of differentiation it will be instructive to study a geometrical illustration of the process.

Let x (Fig. 5) be the side of a square, and let there be an increment in the area of the square due to an increase of h in the variable x.

The original area of the square $= x^2$
The new area $= (x+h)^2$
The increment in the area $= (x + h)^2 - x^2 = 2xh + h^2$. (3)

This equation is true, whatever value be given to h. The smaller the increment h the less does the value of h^2 become. If this increment h ultimately become indefinitely small, then h^2, being of a very high order of magnitude, may be neglected. For example, if when $x = 1$,

$h = 1$, increment in area $= 2 + 1$;
$h = \frac{1}{10}$, ,, ,, $= \cdot 2 + \frac{1}{100}$;
$h = \frac{1}{1000}$, ,, ,, $= \cdot 002 + \frac{1}{1,000,000}$, etc.

If, therefore, dy denotes the infinitely small increment in the area (y) of the square corresponding to an infinitely small increment dx in two adjoining sides (x), then, in the language of differentials,

$$\textit{increment } y = 2xh, \text{ becomes, } dy = 2x.dx.$$

(See the historical note, page 20.)

The same result can be deduced by means of limiting ratios. For instance, consider the ratio of any increment in the area (y) to any increment in the length of a side of the square (x).

$$\frac{\textit{increment } y}{h} = \frac{(\textit{incr. } y)}{(\textit{incr. } x)} = \frac{\delta y}{\delta x} = \frac{2xh + h^2}{h},$$

and when the value of h is zero

$$\frac{dy}{dx} = Lt_{h=0}\frac{\delta y}{\delta x} = 2x.$$

Examples.—(1) Show, by similar reasoning to the above, that if the three adjoining sides (x) of a cube receive an increment h, then $Lt_{h=0}\frac{\delta y}{\delta x} = 3x^2$.

(2) Prove that if the radius (r) of a circle be increased by an amount h, the increment in the area of the circle will be $(2rh + h^2)\,\pi$. Show that the limiting ratio (dy/dx) in this case is $2\pi r$.

The former method of differentiation is known as "Leibnitz's method of differentials," the latter, "Newton's method of limits". It cannot be denied that while Newton's method is rigorous, exact, and satisfying, Leibnitz's at once raises the question :

§ 11. Is Differentiation a Method of Approximation only?

The method of differentiation might at first sight be regarded as a method of approximation, for these small quantities appear to be rejected only because this *may* be done without committing any sensible error. For this reason, in its early days, the calculus was subject to much opposition on metaphysical grounds. Bishop Berkeley called these limiting ratios "the ghosts of departed quantities". A little consideration, however, will show that these small quantities *must* be rejected in order that no error may be committed in the calculation. The process of elimination is essential to the operation.

Assuming that the quantities under investigation are continuous, and noting that the smaller the differentials the closer the approximation to absolute accuracy, our reason is satisfied to reject the differentials, when they become so small as to be no longer perceptible to our senses. The psychological process that gives rise to this train of thought leads to the inevitable conclusion that this mode of representing the process is the true one. Moreover, the validity of the reasoning is justified by its results.

The following remarks on this question are freely translated from Carnot's *Réflexions sur la Métaphysique du Calcul Infinitésimal*. "The essential merit, the sublimity, one may say, of the infinitesimal (or differential) method lies in the fact that it is as easily performed as a simple method of approximation, and as accurate as the results of an ordinary calculation. This immense advantage would be lost, or at any rate greatly diminished, if, under the pretence of obtaining a greater degree of accuracy throughout the whole process, we were to substitute for the simple method given by Leibnitz * one less convenient and less in accord with the probable course of the natural event. If this method is accurate in its results, as no one doubts at this day; if we always have recourse to it in difficult questions, what need is there to supplant it by complicated and indirect means? Why content ourselves with founding it on inductions and analogies with the results furnished by other means when it can be demonstrated directly and generally, more easily, perhaps, than any of these very methods?

"The objections which have been raised against it are based on the false supposition that the errors made by neglecting infinitesimally small quantities during the actual calculation are still to be found in the result of the calculation, however small they may be. Now this is not the case. The error is of necessity removed from the result by elimination. It is indeed a strange thing that every one did not from the very first realise the true character of infinitesimal quantities, and see that a conclusive answer to all objections lies in this indispensable process of elimination." (Paris, p. 215, 1813.)

Historical Note.—The beginner will have noticed that, unlike algebra and arithmetic, higher mathematics postulates that number is capable of gradual growth. The differential calculus is concerned with the rate at which quantities increase or diminish. There are three modes of viewing this growth:—

1. *Leibnitz's* "*method of infinitesimals or differentials*". According to this, a quantity is supposed to pass from one degree of magnitude to another by the continual addition of infinitely small parts, called *infinitesimals* or *differentials*. Infinitesimals may have different orders of magnitude. Thus, the product $dx \,.\, dy$ is an infinitesimal of the second order, infinitely small in comparison with the product $y \,.\, dx$, or $x \,.\, dy$.

In the preceding section (§ 10, see also Fig. 6, § 12, and Fig. 8, § 21) it was shown that when each of two sides of a square receives a small increment h, the corresponding increment in the area is $2xh + h^2$. When h is made indefinitely small and equal to say dx, then $(dx)^2$ is vanishingly small in comparison with $x \,.\, dx$. Hence,

$$dy = 2x \,.\, dx.$$

---

* Isaac Newton discovered the fundamental process of the "differential" calculus in 1665-69. Leibnitz improved the notation in 1677. Leibnitz is also said to have made the discovery independently of Newton.

In calculations involving quantities which are ultimately made to approach the limit zero, the higher orders of infinitesimals may be rejected at any stage of the process. Only the lowest orders of infinitesimals are, as a rule, retained. See (5), page 523.

2. *Newton's "method of rates or fluxions"*. Here, the velocity or rate with which the quantity is generated is employed. The measure of this velocity is called a *fluxion*. A fluxion, written $\dot{x}$, $\dot{y}$, . . . is equivalent to our dx/dt, dy/dt, . . .

These two methods are modifications of one idea. It is all a question of notation or definition. While Leibnitz referred the rate of change of a dependent variable y, to an independent variable x, Newton referred each variable to "uniformly flowing" time. Leibnitz assumed that when x receives an increment dx, y is increased by an amount dy. Newton conceived these changes to occupy a certain time dt, so that y increases with a velocity $\dot{y}$, as x increases with a velocity $\dot{x}$. This relation may be written symbolically,

$$dx = \dot{x}dt, \; dy = \dot{y}dt, \text{ and therefore, } \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}.$$

The method of fluxions is not in general use, perhaps because of its more abstruse character. It is occasionally employed in mechanics.

3. *Newton's "method of limits"*. This has been set forth in §§ 2, 6, *et seq.** The ultimate limiting ratio is considered as a fixed quantity to which the ratio of the two variables can be made to approximate as closely as we please.

The methods of limits and of infinitesimals are employed indiscriminately in this work, according as the one or the other appeared the more instructive or convenient. As a rule, it is easier to represent a process mathematically by the method of infinitesimals. The determination of the limiting ratio frequently involves more complicated operations than is required by Leibnitz's method. (Compare § 85, and § 86.)

"The limiting ratio," says Carnot (*l. c.*, p. 210), "is neither more nor less difficult to define than an infinitely small quantity. . . . To proceed rigorously by the method of limits it is necessary to lay down the definition of a limiting ratio. But this is the definition, or rather, this ought to be the definition, of an infinitely small quantity." It follows, therefore, that the psychological process of reducing quantities down to their limiting ratios is equivalent to the rejection of terms involving the higher orders of infinitesimals. These operations have been indicated side by side in § 10.

The earlier part of Professor Williamson's article on the "Infinitesimal Calculus," in the *Encyclopædia Britannica* (9th edit.), contains some interesting details on the evolution of the calculus.

We may now take up the routine processes of differentiation.

---

* The method of limits is sometimes said to have been suggested by d'Alembert. But this *savant* has stated positively in the *Encyclopédie Mathématique* (1784-1789), l'art. "Différentiel," that he has but interpreted the later views of Newton set forth in *The Principia*.

§ 12. The Differentiation of Algebraic Functions.

An *algebraic function* of x is an expression containing terms which involve only the operations of addition, subtraction, multiplication, division, evolution (root extraction) or involution. For instance, $x^2y + \sqrt[3]{x} + y^{\frac{1}{2}} - ax = 1$ is an algebraic function. Functions that cannot be so expressed are termed *transcendental functions*. Thus, $\sin x = y$, $\log x = y$, $e^x = y$ are transcendental functions.

On page 18 a method was described for finding the differential coefficient of $y = x^2$, by the following series of operations:—

(1) Give an arbitrary increment h to x in the original function;

(2) subtract the original function x^2 from the new value of $(x + h)^2$ found in (1);

(3) divide the result of (2) by h the increment of x;

(4) find the limiting value of this ratio when $h = 0$.

This procedure must be carefully noted; it lies at the basis of all processes of differentiation. In this way it can be shown that

$$\text{if } y = x^2,\ dy/dx = 2x,$$
$$\text{if } y = x^3,\ dy/dx = 3x^2,$$
$$\text{if } y = x^4,\ dy/dx = 4x^3, \text{ etc.}$$

(1) *To find the differential coefficient of any power of a variable.*

By actual multiplication we shall find that

$$(x + h)^2 = (x + h)(x + h) = x^2 + 2hx + h^2;$$
$$(x + h)^3 = (x + h)^2(x + h) = x^3 + 3hx^2 + 3h^2x + h^3;$$
$$\ldots \qquad \ldots \qquad \ldots$$

Continuing this process as far as we please, we shall find that

$$(x + h)^n = x^n + \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 + \ldots + \frac{n}{1}xh^{n-1} + h^n. \quad (1)$$

This result, known as the **binomial theorem**, enables us to raise any expression of the type $x + h$ to any power of n (where n is positive integer, *i.e.*, a positive whole number, not a fraction) without going through the actual process of successive multiplication. Exactly the same thing holds for $(x - h)^n$.

To find the differential coefficient of

$$y = x^n.$$

Let each side of this expression receive a small increment so that

$$(y + h') - y = (incr.\ y) = (x + h)^n - x^n.$$

From the binomial theorem, (1) above

$$(incr.\ y) = nx^{n-1}h + \tfrac{1}{2}n(n - 1)x^{n-2}h^2 + \ldots$$

Divide by increment x, namely h.

$$\frac{(incr.\ y)}{h} = \frac{(incr.\ y)}{(incr.\ x)} = nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}h + \ldots$$

Hence when h is made zero

$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = Limit_{h=0}\frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

That is to say

$$\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}. \quad . \quad . \quad . \qquad (2)$$

Hence the rule: *to find the differential coefficient of any power of x, diminish the index by unity and multiply the power of x so obtained by the original exponent* (*or index*).

EXAMPLES.—(1) If $y = x^6$, show that $dy/dx = 6x^5$.

(2) If $y = x^{20}$, show that $dy/dx = 20x^{19}$.

(3) If $y = a(x^5)$, show that $dy/dx = a(5x^4) = 5ax^4$.

(4) If the diameter of a spherical soap bubble increases uniformly at the rate of 0·1 centimetre per second, show that the capacity is increasing at the rate of $0{\cdot}2\pi$ centimetre per second when the diameter becomes 2 centimetres. Note: $y = \frac{1}{6}\pi D^3$, (23), page 492;

$$\therefore dy/dD = \tfrac{1}{2}\pi D^2, \therefore dy = \tfrac{1}{2} \times \pi \times 2^2 \times 0{\cdot}1 = 0{\cdot}2\pi.$$

(2) *To find the differential coefficient of the sum or difference of any number of functions.* Let $u, v, w \ldots$ be functions of x; y their sum. Let $u_1, v_1, w_1, \ldots, y_1$, be the respective values of these functions when x is changed to $x + h$, then

$$y = u + v + w + \ldots; \ y_1 = u_1 + v_1 + w_1 + \ldots$$

Hence $\quad y_1 - y = (u_1 - u) + (v_1 - v) + (w_1 - w) + \ldots,$

that is $\quad (incr.\ y) = (incr.\ u) + (incr.\ v) + (incr.\ w) + \ldots,$

dividing by h

$$\frac{(incr.\ y)}{h} = \frac{(incr.\ u)}{h} + \frac{(incr.\ v)}{h} + \frac{(incr.\ w)}{h} + \ldots,$$

or

$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots \quad . \qquad (3)$$

If some of the symbols had had a minus instead of a plus sign, a corresponding result would have been obtained. For instance, if

$$y = u - v - w - \ldots,$$

then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} - \frac{dw}{dx} - \ldots \quad . \quad . \qquad (4)$$

The differential coefficient of the sum or difference of any number of functions is, therefore, equal to the sum or difference of the differential coefficients of the several functions.

(3) *To differentiate a polynomial* * *raised to any power.* Let

$$y = (ax + x^2)^n.$$

Regarding the expression in brackets as one variable raised to the power of n, we get

$$dy = n(ax + x^2)^{n-1} d(ax + x^2).$$

Differentiating the last term,

$$\frac{dy}{dx} = n(ax + x^2)^{n-1}(a + 2x). \quad . \quad . \quad (5)$$

Thus, to find the differential coefficient of a polynomial raised to any power, diminish the exponent of the power by unity and multiply the expression so obtained by the differential coefficient of the polynomial, and the original exponent.

(4) *The differential coefficient of any constant term is zero.* Since a constant term is essentially a quantity that does not vary, if $y = c$, dy must be absolute zero. Let

$$y = x^n + c$$

then $$(incr.\ y) = (x + h)^n + c - (x^n + c)$$

$$= \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots \dagger$$

or $$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = \frac{dy}{dx} = nx^{n-1}, \quad . \quad . \quad (6)$$

where the constant term has disappeared.

(5) *To find the differential coefficient of the product of a variable and a constant quantity.* Let

$$y = ax^n;\ (incr.\ y) = a(x + h)^n - ax^n;$$

$$= anx^{n-1}h + \frac{an(n-1)}{2!}x^{n-2}h^2 + \ldots$$

Therefore

$$Lt_{h=0}\frac{(incr.\ y)}{h} = \frac{dy}{dx} = anx^{n-1} \quad . . \quad . \quad (7)$$

---

* A *polynomial* is an expression containing two or more terms connected by plus or minus signs. Thus, $a + bx$; $ax + by + z$, etc. A *binomial* contains two such terms.

† Note $1! = 1$; $2! = 1 \times 2$; $3! = 1 \times 2 \times 3$; $n! = 1 \times 2 \times 3 \times \ldots \times (n-2) \times (n-1) \times n$. Strictly speaking, 0! has no meaning; mathematicians, however, find it convenient to make $0! = 1$. This notation is due to Kramp. "$n!$" is read "factorial n".

The differential coefficient of the product of a variable quantity and a constant is thus equal to the constant multiplied by the differential coefficient of the variable.

EXAMPLES.—Some illustrations of this process have been given in preceding examples.

(1) If $y = (1 - x^2)^3$, show that $dy/dx = - 6x(1 - x^2)^2$.

(2) If $y = x - 2x^2$, show that $dy/dx = 1 - 4x$.

(3) *Young's formula* for the relation between the vapour pressure p and the temperature θ of isopentane at constant volume is, $p = b\theta - a$, where a and b are empirical constants. Hence show that the ratio of the change of pressure with temperature is constant and equal to b.

(4) *Mendeléeff's formula* for the superficial tension s of a perfect liquid at any temperature θ is, $s = a - b\theta$, where a and b are constants. Hence show that rate of change of s with θ is constant. Ansr. $- b$.

(5) *Callendar's formula* for the variation of the electrical resistance R of a platinum wire with temperature θ is, $R = R_0(1 + \alpha\theta + \beta\theta^2)$, where α and β are constant. Find the increase in the resistance of the wire for a small rise of temperature. Ansr. $dR = R_0(\alpha + 2\beta\theta)\,d\theta$.

(6) If the volume of a gramme of water varies as $1 + (\theta - 4)^2/144{,}000$, where θ denotes the temperature (°C), show that the coefficient of cubical expansion of water at any temperature θ is equal to $\cdot000013889\,\kappa(\theta - 4)$, where κ is the constant of proportion (2), page 487.

(7) A piston slides freely in a circular cylinder (diameter 6 in.). At what rate is the piston moving when steam is admitted into the cylinder at the rate of 11 cubic feet per second?

Let v denote the volume, x the height of the piston at any moment. From (25), page 492,

$$v = \pi(\tfrac{1}{4})^2x\,;\ \therefore\ dv = \pi(\tfrac{1}{4})^2dx.$$

But we require the value of dx/dt. Divide the last expression through with dt, let $\pi = \frac{22}{7}$,

$$\therefore \frac{dx}{dt} = \frac{dv}{dt} \times 16 \times \frac{7}{22} = 56 \text{ ft. per sec.}$$

(8) If the quantity of heat (Q) necessary to raise the temperature of a gram of solid from $0_0°$ to $\theta°$ is represented by

$$Q = a\theta + b\theta^2 + c\theta^3$$

(where a, b, c, are constants), what is the specific heat of the substance at $\theta°$? Hint. Compare the meaning of $dQ/d\theta$ with your definition of specific heat. Ansr. $a + 2b\theta + 3c\theta^2$.

(6) *To find the differential coefficient of the product of any number of functions.* Let

$$y = uv$$

where u and v are functions of x. When x becomes $x + h$, u, v and y become u_1, v_1, y_1, or $u_1 = u + h$, $v_1 = v + h$ . . . Then

$$y_1 = u_1v_1\,;\ y_1 - y = u_1v_1 - uv,$$

add and subtract uv_1 from the second member of this last equation, and transpose the terms so that

$$y_1 - y = u(v_1 - v) + v_1(u_1 - u),$$

or $$(incr.\ y) = u(incr.\ v) + (v + h)\ (incr.\ u).$$

Divide by h and find the limit when $h = 0$

$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = u\frac{dv}{dx} + v\frac{du}{dx},$$

or $$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad . \quad . \quad . \quad (8)$$

In the language of differentials

$$dy = d(uv) = udv + vdu. \quad . \quad . \quad . \quad (9)$$

Similarly, by taking any number of functions, say

$$y = uvw.$$

Put $vw = z$ then $$y = uz.$$

From (8) $$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}.$$

Divide through by y or its equivalent uz and

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{z}\frac{dz}{dx}.$$

Substituting vw for z we get

$$\frac{1}{z}\frac{dz}{dx} = \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx},$$

$$\frac{dy}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}, \quad . \quad . \quad (10)$$

and so on for the products of a greater number of terms.

To find the differential coefficient of any number of terms, multiply the differential coefficient of each separate function by the product of all the remaining functions and add up all the results.

This may be illustrated by a geometrical figure similar to that of page 18. In the rectangle (Fig. 6) let the unequal sides be denoted by x and y. Let x and y be increased by their differentials dx and dy. Then the increment of the area will be represented by the shaded parts, which are in turn represented by the areas of the parallelograms $xdy + ydx + dxdy$, but at the limit $dxdy$ vanishes, as previously shown.

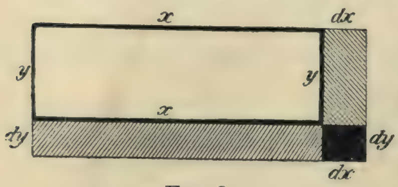


FIG. 6.

EXAMPLES.—(1) Show geometrically that the differential of a small increment in the capacity of a rectangular solid figure whose unequal sides are x, y, z is denoted by the expression $xydz + yzdx + zxdy$. Hence, show that if an ingot of gold expands uniformly in its linear dimensions at the rate of 0·001 units per second, its volume (V) is increasing at the rate of $dV/dt = 0·110$ units per second, when the dimensions of the ingot are 4 by 5 by 10 units.

(2) If $y = (x - 1)(x - 2)(x - 3)$, $dy/dx = 3x^2 - 12x + 11$.

(3) If $y = x^2(1 + ax^2)(1 - ax^2)$, $dy/dx = 2x - 6a^2x^5$.

(7) *To find the differential coefficient of a fraction or quotient.*

Let
$$y = \frac{u}{v},$$
where u and v are functions of x. When x becomes $x + h$, u v and y become respectively u_1 v_1 and y_1, such that $u_1 = u + h$, etc. Then $y = u_1/v_1$ and
$$y_1 - y = \frac{u_1}{v_1} - \frac{u}{v} = \frac{u_1v - v_1u}{v_1v}$$
add and subtract u/v_1. Then divide by h and
$$\frac{(incr.\ y)}{h} = \left(v\frac{u_1 - u}{h} - u\frac{v_1 - v}{h}\right)\Big/v_1v,$$
$$\frac{(incr.\ y)}{(incr.\ x)} = \left(v\frac{(incr.\ u)}{(incr.\ x)} - u\frac{(incr.\ v)}{(incr.\ x)}\right)\Big/(v + h)v,$$
$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Big/v^2;$$
or
$$\frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad . \quad . \quad . \quad (11)$$
In words, to find the differential coefficient of a fraction or of a quotient, subtract the product of the numerator into the differential coefficient of the denominator, from the product of the denominator into the differential coefficient of the numerator, and divide by the square of the denominator.

In the language of differentials the last result may be written in the more useful form:
$$dy = d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} \quad . \quad . \quad . \quad (12)$$

SPECIAL CASE.—If the numerator of the fraction be constant, say c, then
$$y = c/x.$$
$$dy = (xdc - cdx)/x^2 = - cdx/x^2;$$
or
$$\frac{dy}{dx} = -\frac{c}{x^2} \quad . \quad . \quad . \quad . \quad (13)$$

Examples.—(1) If $y = \dfrac{x}{1 - x}$, show that $dy/dx = 1/(1 - x)^2$.

(2) If $y = (1 + x^2)/(1 - x^2)$, show that $dy/dx = 4x/(1 - x^2)^2$.

(3) If $y = a/x^n$, show that $dy/dx = - na/x^{n+1}$.

(4) If $y = x^3/(x^2 - 1) - x^2/(x - 1)$, show that $dy/dx = 2x/(x^2 \quad 1)^2$.

(5) The refractive index (μ) of a ray of light of wave-length λ is, according to *Christoffel's dispersion formula*,

$$\mu = \mu_0 \sqrt{2}/\{\sqrt{1 + \lambda_0/\lambda} + \sqrt{1 - \lambda_0/\lambda}\},$$

where μ_0 and λ_0 are constants. Find the change in the refractive index corresponding to a small change in the wave-length of the light. Ansr. $d\mu/d\lambda = - \mu^3\lambda_0{}^2/\{2\lambda^3\mu_0{}^2\sqrt{(1 - \lambda_0{}^2/\lambda^2)}\}$. It is not often so difficult a differentiation occurs in practice. The most troublesome part of the work is to reduce

$$\frac{d\mu}{d\lambda} = - \frac{\sqrt{2}\mu_0\lambda_0\{\sqrt{(1 + \lambda_0/\lambda)} - \sqrt{(1 - \lambda_0/\lambda)}\}/\lambda^2}{\sqrt{(1 - \lambda_0{}^2/\lambda^2)}\{\sqrt{(1 + \lambda_0/\lambda)} + \sqrt{(1 - \lambda_0/\lambda)}\}^2},$$

to the answer given, by multiplying the numerator and denominator of the right member with the proper factors to get μ^3. Of course the student is *not* using this abbreviated symbol of division. See footnote, page 11.

(8) *To find the differential coefficient of a function affected with any exponent.* Since the binomial theorem is true for any exponent positive or negative, fractional or integral, formula (2) may be regarded as quite general. To illustrate this consider the three cases.

Case I. *When n is a positive integer.* It follows directly

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$

Case II. *When n is a positive fraction.* Let $n = p/q$, where p and q are any integers, then

$$y = x^{\frac{p}{q}}.$$

Raising each term to the qth power

$$y^q = x^p.$$

By differentiation, using the notation of differentials

$$qy^{q-1}dy = px^{p-1}dx,$$

$$\therefore \quad \frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{-(q-1)}.$$

But since $y = x^{\frac{p}{q}}$,

$$y^{q-1} = x^{\frac{pq-p}{q}}.$$

By substituting this value of y^{q-1} in the preceding result, we obtain

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}x^{p/q}}{x^p},$$

or simplified :
$$\frac{dy}{dx} = \frac{p}{q}x^{\frac{p}{q}-1}, \quad . \quad . \quad . \quad . \quad (14)$$
which has exactly the same form as if n were a positive integer.

CASE III. *When n is a negative integer or a negative fraction.* Let
$$y = x^{-n}, \text{ then } y = 1/x^n.$$
Differentiating this as if it were a fraction, (13) above
$$dy/dx = -nx^{n-1}/x^{2n},$$
or, on reduction,
$$\frac{dy}{dx} = \frac{d(x^{-n})}{dx} = -nx^{-n-1}. \quad . \quad . \quad (15)$$
Thus the method of page 23 is quite general.

EXAMPLES.—(1) *Matthiessen's formula* for the variation of the electrical resistance R of a platinum wire with temperature θ, between 0° and 100° is $R = R_0(1 - a\theta + b\theta^2)^{-1}$. Find the increase in the resistance of the wire for a small change of temperature. Ansr. $dR/d\theta = R^2(a - 2b\theta)/R_0$. Note a and b are constants.

(2) *Siemens' formula* for the relation between the electrical resistance of a metallic wire and temperature is, $R = R_0(1 + a\theta + b\sqrt{\theta})$. Hence find the rate of change of resistance with temperature. Ansr. $R_0(a + \frac{1}{2}b\theta^{-\frac{1}{2}})$.

(9) *To find the differential coefficient of a function of a function.* Let
$$y = f(x) \text{ and } u = \phi(y).$$
It is required to find the differential coefficient of u with respect to x. Let u and y receive small increments so that $y_1 = y + h$ and $u_1 = u + h$ and $x_1 = x + h$. Then
$$\frac{u_1 - u}{x_1 - x} = \frac{u_1 - u}{y_1 - y} \cdot \frac{y_1 - y}{x_1 - x}, \text{ or } \frac{(incr.\ u)}{(incr.\ x)} = \frac{(incr.\ u)}{(incr.\ y)} \cdot \frac{(incr.\ y)}{(incr.\ x)},$$
which is true, however small the increment may be. At the limit, therefore,
$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \quad . \quad . \quad . \quad . \quad (16)$$

In a similar way if
$$u = \phi(w)\,; \; w = \psi(y)\,; \text{ and } y = f(x),$$
and the differential coefficient of u with respect to x is required since
$$\frac{du}{dx} = \frac{du}{dw} \cdot \frac{dw}{dx}, \text{ and } \frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx}.$$
$$\therefore \frac{dy}{dx} = \frac{du}{dw} \cdot \frac{dw}{dy} \cdot \frac{dy}{dx} \quad . \quad . \quad . \quad (17)$$

(10) *To prove that* $\frac{dx}{dy} = 1\Big/\frac{dy}{dx}$. Since it has just been shown that

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

is true for all values of x, we may assume that when $u = x$

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = 1 \text{ ; or } \frac{dx}{dy} = 1\Big/\frac{dy}{dx}. \quad . \quad . \quad (18)$$

EXAMPLES.—(1) If $y = x^n/(1 + x)^n$, show that $dy/dx = nx^{n-1}/(1 + x)^{n+1}$.

(2) If $y = 1/\sqrt{(1 - x^2)}$, show that $dy/dx = x/\sqrt{(1 - x^2)^3}$.

(3) If $y = a + 1/\sqrt{x}$, show that $dy/dx = -\frac{1}{2}\sqrt{x^{-3}}$.

(4) If $y = a + bx/c$, show that $dy/dx = b/c$.

The use of formula (16) often simplifies the actual process of differentiation; for instance, it is required to differentiate the expression

(5) $u = \sqrt{(a^2 - x^2)}$. Assume $y = a^2 - x^2$. Then $u = \sqrt{y}$, $y = a^2 - x^2$, and $dy/dx = -x(a^2 - x^2)^{-\frac{1}{2}}$.

This is an easy example which could be done at sight; it is given here to illustrate the method.

Every type of algebraic expression has now been investigated, and by the application of these principles any algebraic function may be differentiated. Before proceeding to transcendental functions (that is to say, functions which contain trignometrical, logarithmic or other terms not algebraic), it seems a convenient opportunity to apply our knowledge to the well-known equations of Boyle and van der Waals. These equations will also be discussed from other points of view later on.

§ 13. The Gas Equations of Boyle and van der Waals.

In van der Waals' equation, at a constant temperature,

$$(p + a/v^2)(v - b) = \text{constant}, \quad . \quad . \quad (1)$$

where b is a constant depending on the volume of the molecule, a is a constant depending on intermolecular attraction. Differentiating with respect to p and v, we obtain, as on pages 24 and 25,

$$(v - b)d(p + a/v^2) + (p + a/v^2)d(v - b) = 0,$$

and therefore

$$\frac{dv}{dp} = -(v - b)\Big/\left(p - \frac{a}{v^2} + \frac{2ab}{v^3}\right). \quad . \quad . \quad (2)$$

The differential coefficient dv/dp measures the compressibility of the gas (page 7).

If the gas strictly obeyed Boyle's law, $a = b = 0$, and we should have

$$dv/dp = -v/p. \quad . \quad . \quad . \quad (3)$$

The negative sign in these equations means that the volume of the gas decreases with increase of pressure. Any gas, therefore, will be more or less sensitive to changes of pressure than Boyle's law indicates, according as the differential coefficient of (2) is greater or less than that of (3), that is according as

$$(v - b)/(p - a/v^2 + 2ab/v^3) \gtrless v/p,$$

or as
$$pv - pb \gtrless pv - a/v + 2ab/v^2,$$

or as
$$pb \gtrless a/v - 2ab/v^2,$$

or as
$$pv \gtrless \frac{a}{b} - \frac{2a}{v} \qquad (4)$$

If Boyle's law were strictly obeyed

$$pv = \text{constant}, \qquad (5)$$

but if the gas be less sensitive to pressure than Boyle's law indicates, so that, in order to produce a small contraction, the pressure has to be increased a little more than Boyle's law demands,

pv increases with increase of pressure;

while if the gas be more sensitive to pressure than Boyle's law provides for,

pv decreases with increase of pressure.

Some valuable deductions as to intermolecular action have been drawn by comparing the behaviour of gases under compression in the light of equations similar to (4) and (5). For this the reader is referred to the proper textbooks.

But this is not all. From (5), if c = constant, $v = c/p$, which gives on differentiation

$$dv/dp = - c/p^2,$$

or the ratio of the decrease in volume to the increase of pressure, is inversely as the square of the pressure. By simple substitution of $p = 2, 3, 4, \ldots$ in the last equation we obtain

$$dv/dp = \tfrac{1}{4}, \tfrac{1}{9}, \tfrac{1}{16} \ldots$$

when c = unity. In other words, the greater the pressure to which a gas is subjected the less the corresponding diminution in volume for any subsequent increase of pressure.

§ 14. The Differentiation of Trignometrical Functions.

A *trignometrical function* is any expression containing trignometrical ratios, sines, cosines, etc. The elementary definitions of trignometry are discussed on page 493. We may therefore pass at once *in medias res.* There is no new principle to be learned.

(1) *The differential coefficient of* $\sin x$ *is* $\cos x$. Let

$$y = \sin x, \text{ and } y_1 = \sin(x + h);$$
$$\therefore y_1 - y = \sin(x + h) - \sin x.$$

By the formula (36), page 500,

$$y_1 - y = 2\sin\frac{h}{2}\cos\left(x + \frac{h}{2}\right).$$

Divide by h and

$$\frac{y_1 - y}{h} = \frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cos\left(x + \frac{h}{2}\right).$$

But the limit of $\sin\theta/\theta$ is unity (page 499),

$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = \cos x;$$
$$\therefore \frac{dy}{dx} = \frac{d(\sin x)}{dx} = \cos x \quad . \quad . \quad . \quad (1)$$

(2) *The differential coefficient of* $\cos x$ *is* $-\sin x$. Let

$$y = \cos x, \text{ and } y_1 = \cos(x + h);$$
$$y_1 - y = \cos(x + h) - \cos x.$$

From page 499 $\quad y_1 - y = -2\sin\frac{h}{2}\sin\left(x + \frac{h}{2}\right),$

or
$$\frac{y_1 - y}{h} = -\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\sin\left(x + \frac{h}{2}\right);$$

and at the limit when $h = 0$,

$$Lt_{h=0}\frac{(incr.\ y)}{(incr.\ x)} = -\sin x;$$
$$\therefore \frac{dy}{dx} = \frac{d(\cos x)}{dx} = -\sin x \quad . \quad . \quad . \quad (2)$$

The meaning of the negative sign can readily be deduced from the definition of the differential coefficient. $d(\cos x)/dx$ represents the rate at which $\cos x$ increases when x is slightly increased. The negative sign shows that this rate of increase is negative, in other words, $\cos x$ diminishes as x increases.

(3) *The differential coefficient of* $\tan x$ *is* $\sec^2 x$. Using the results already deduced for $\sin x$ and $\cos x$,

$$\frac{d(\tan x)}{dx} = d\left(\frac{\sin x}{\cos x}\right)\Big/dx,$$
$$= \left\{\cos x\frac{d(\sin x)}{dx} - \sin x\frac{d(\cos x)}{dx}\right\}\Big/\cos^2 x,$$
$$= (\cos^2 x + \sin^2 x)/\cos^2 x.$$

But the numerator is equal to unity (formula (17), page 499). Hence

$$\frac{d(\tan x)}{dx} = \frac{1}{\cos^2 x} = \sec^2 x. \quad . \quad . \quad . \quad (3)$$

In the same way $\quad d(\cot x)/dx = -\operatorname{cosec}^2 x. \quad . \quad . \quad . \quad$ (4)

The remaining trignometrical functions may be left for the reader to work out himself. The results are given on page 158.

EXAMPLES.—(1) If $y = \cos^n x$, $dy/dx = -n\cos^{n-1}x \,.\, \sin x$.

(2) If $y = \sin^n x$, $dy/dx = n\sin^{n-1}x \,.\, \cos x$.

(3) If a particle vibrates according to the equation $y = a\sin(qt - \epsilon)$, what is its velocity at any instant when a, q and ϵ are constant?

Ansr. $v = dy/dt = aq\cos(qt - \epsilon)$.

(4) If $y = \sin^2(nx - a)$, $dy/dx = 2n\sin(nx - a)\cos(nx - a)$.

§ 15. The Differentiation of Inverse Trignometrical Functions. The Differentiation of Angles.

The equation, $\sin x = y$, means that x is an angle whose sine is y. It is sometimes convenient to write this another way, *viz.*,

$$\sin^{-1}y = x,$$

meaning that $\sin^{-1}y$ is an *angle* whose sine is y. Thus if $\sin 30° = \frac{1}{2}$, we say that $30°$ or $\sin^{-1}\frac{1}{2}$ is an angle whose sine is $\frac{1}{2}$. Trignometrical ratios written in this reversed way are called **inverse trignometrical functions.** The superscript " -1 " has no other signification when attached to the trignometrical ratios. Note, if $\tan 45° = 1$, then $\tan^{-1}1 = 45°$; $\therefore \tan(\tan^{-1}1) = \tan 45°$.

Their differentiation may be illustrated by proving that the differential coefficient of $\sin^{-1}x$ is $1/\sqrt{(1 - x^2)}$. If $y = \sin^{-1}x$, then $\sin y = x$, and

$$dx/dy = \cos y; \text{ or } dy/dx = 1/\cos y.$$

But we know that

$$\cos^2 y + \sin^2 y = 1, \text{ or } \cos y = \pm\sqrt{(1 - \sin^2 y)} = \pm\sqrt{(1 - x^2)},$$

for by hypothesis $\sin y = x$. Hence

$$\frac{d(\sin^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\cos y} = \pm\frac{1}{\sqrt{1 - x^2}}.$$

The ambiguity of the sign means that if any assigned value of x satisfies the equation $y = \sin^{-1}x$, so does $\pi - y$, $2\pi - y$ and in general $n\pi \pm y$. When y has its least value the angle whose sine is x is acute. The differential coefficient is then positive, that is to say,

$$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{(1 - x^2)}}. \quad . \quad . \quad . \quad (1)$$

Similarly $d(\cos^{-1}x)/dx = -1/\sqrt{1-x^2}$. . . . (2)

The differential coefficient of $\tan^{-1}x$ is an important function, since it appears very frequently in practical formulæ.

If $y = \tan^{-1}x$, $x = \tan y$, $dx/dy = 1/\cos^2 y$. But (page 499) $\cos^2 y = 1/(1 + \tan^2 y) = 1/(1 + x^2)$. Hence

$$\frac{d(\tan^{-1}x)}{dx} = \cos^2 y = \frac{1}{1+x^2} \quad . \quad . \quad . \quad (3)$$

$$d(\cot^{-1}x)/dx = -1/(1+x^2) \quad . \quad . \quad . \quad (4)$$

The remaining inverse trignometrical functions may be left to the reader. Their values will be found on page 158.

Examples.—(1) Differentiate $y = \sin^{-1}x/\sqrt{(1+x^2)}$. Put $\sin y = x/\sqrt{1+x^2}$ hence $\cos y dy = dx/\sqrt{(1+x^2)^3}$. But $\cos y = \sqrt{(1-\sin^2 y)} = \sqrt{[1 - x^2/(1+x^2)]}$. Substituting this value of $\cos y$ in the former result we get, on reduction, $dy/dx = 1/(1+x^2)$ – the answer required.

(2) If $y = \sin^{-1}x^2$, $\dfrac{dy}{dx} = \dfrac{2x}{\sqrt{1-x^4}}$.

(3) If $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$. See formula (19), page 499.

(4) If $y = \tan^{-1}x + \tan^{-1}\dfrac{1}{x}$, $\dfrac{dy}{dx} = 0$.

(5) If $y = \sin^{-1}(\cos x)$, $dy/dx = -1$.

§ 16. Logarithms and their Differentiation.

It is proved in elementary algebra that all numbers may be represented as different powers of one fundamental number. *E.g.*,

$$1 = 10^0,\ 2 = 10^{\cdot 301},\ 3 = 10^{\cdot 477},\ 4 = 10^{\cdot 602},\ 5 = 10^{\cdot 699},\ \ldots$$

The power, index or exponent is called a **logarithm**, the fundamental number is called the *base* of the system of logarithms. Thus if

$$a^x = b,$$

x is the logarithm of the number b to the base a, and is written

$$x = \log_a b.$$

For convenience in numerical calculations tables are used in which all numbers are represented as different powers of 10. The logarithm of any number taken from the table thus indicates what power of 10 the selected number represents. Thus if

$$10^3 = 1000,\ 10^{1\cdot 0413927} = 11;$$

then $3 = \log_{10}1000,\ 1\cdot 0413927 = \log_{10}11.$

Exactly the same thing is true if the base 10 be replaced by any other base. Read § 188.

Before finding the relation between the logarithms of a number to different bases, we shall proceed to deduce the differential coefficient of a logarithm. A *logarithmic function* is any expression containing logarithmic terms. *E.g.*, $y = \log x + x^3$.

(1) *To determine the differential coefficient of $\log x$.* Let

$$y = \log x, \text{ and } y_1 = \log(x + h).$$

Then
$$\frac{y_1 - y}{h} = \frac{\log(x + h) - \log x}{h};$$

but it is known (page 37) that $\log a - \log b = \log \frac{a}{b}$, therefore

$$\frac{(incr.\ y)}{(incr.\ x)} = \frac{1}{h}\log\left(\frac{x + h}{x}\right),$$
$$= \frac{1}{h}\log\left(1 + \frac{h}{x}\right),$$

and
$$\frac{dy}{dx} = Lt_{h=0}\frac{1}{h}\log\left(1 + \frac{h}{x}\right). \quad . \quad . \quad (1)$$

The limiting value of this expression cannot be determined in its present form by the processes hitherto used, owing to the nature of the terms $1/h$ and h/x. The calculation must therefore be made by an indirect process. See § 103.

Let $\frac{h}{x} = \frac{1}{u}$ then

$$\frac{1}{h}\log\left(1 + \frac{h}{x}\right) = \frac{1}{x} \cdot u\log\left(1 + \frac{1}{u}\right),$$
$$= \frac{1}{x} \cdot \log\left(1 + \frac{1}{u}\right)^u.$$

As h decreases u increases, and the limiting value of u when h becomes vanishingly small, is infinity. The problem now is to find what is the limiting value of $\log\left(1 + \frac{1}{u}\right)^u$ when u is infinitely great. That is to say,

$$\frac{dy}{dx} = Lt_{u=\infty}\frac{1}{x} \cdot \log\left(1 + \frac{1}{u}\right)^u \quad . \quad . \quad (2)$$

According to the binomial theorem (page 22)

$$\left(1 + \frac{1}{u}\right)^u = 1 + \frac{u}{1} \cdot \frac{1}{u} + \frac{u(u - 1)}{2!} \cdot \frac{1}{u^2} + \ldots,$$

dividing out the u's in each term we get

$$\left(1 + \frac{1}{u}\right)^u = 2 + \frac{\left(1 - \frac{1}{u}\right)}{2!} + \frac{\left(1 - \frac{1}{u}\right)\left(1 - \frac{2}{u}\right)}{3!} + \ldots$$

The limiting value of this expression when u is infinitely great is evidently equal to the sum of the infinite series of terms

$$1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \text{ to infinity} \quad . \quad (3)$$

(see page 230). Let the sum of this series of terms be denoted by the symbol e. By taking a sufficient number of these terms we can approximate as close as ever we please to the value of e. If taken to the ninth decimal place

$$e = 2{\cdot}718281828 \ldots$$

This number, like $\pi = 3{\cdot}14159265 \ldots$, plays an important *rôle* in mathematics. Both magnitudes are incommensurable and can only be evaluated in an approximate way (see page 454).

Returning now to (2), it is obvious that

$$\frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}\log e \quad . \quad . \quad . \quad (4)$$

This formula is true whatever base we adopt for our system of logarithms. If we use 10

$$\log_{10}e = 0{\cdot}43429 \ldots = \text{(say) } M,$$

and

$$\frac{dy}{dx} = \frac{d(\log_{10}x)}{dx} = \frac{M}{x}. \quad . \quad . \quad . \quad . \quad (5)$$

Since $\log_a a = 1$ (page 480) we can put expression (4) in a much simpler form by using a system of logarithms to the base e, then

$$\frac{dy}{dx} = \frac{d(\log_e x)}{dx} = \frac{1}{x} \quad . \quad . \quad . \quad (6)$$

Logarithms to the base e are called **natural or Napierian logarithms.** Logarithms to the base 10 are called **Briggsian or common logarithms.**

(2) *To find the relation between the logarithms of a number to different bases.* Let n be a number such that

$$\alpha^a = n, \text{ or } a = \log_\alpha n,$$

and

$$\beta^b = n, \text{ or } b = \log_\beta n.$$

Hence

$$\alpha^a = \beta^b.$$

"Taking logs" to the base α

$$a = b \log_\alpha \beta$$

Substitute for a and b and

$$\log_\alpha n = \log_\beta n \,.\, \log_\alpha \beta, \text{ or } \log_\beta n = \log_\alpha n/\log_\alpha \beta \quad . \quad (7)$$

In words, the logarithm of a number to the base β may be obtained from the logarithm of that number to the base α by multiplying it by $\frac{1}{\log_\alpha \beta}$. For example, suppose $\alpha = 10$ and $\beta = e$,

$$\log n = \log_{10}n/\log_{10}e,$$

where the subscript in $\log_e n$ is omitted. This is the usual practice. Hence :—

To pass from natural to common logarithms

$$\left.\begin{aligned}\text{common log} &= \text{natural log} \times 0{\cdot}4343 \\ \log_{10}a &= \log_e a \times 0{\cdot}4343\end{aligned}\right\} \quad . \quad . \quad \textbf{(8)}$$

To pass from common to natural logarithms

$$\left.\begin{aligned}\text{natural log} &= \text{common log} \times 2{\cdot}3026 \\ \log_e a &= \log_{10}a \times 2{\cdot}3026\end{aligned}\right\} \quad . \quad . \quad \textbf{(9)}$$

The number 0·4343 is called the **modulus** of the Briggsian or common system of logarithms. When required it is written M or μ.*

EXAMPLES.—(1) If $y = \log ax^4$, show that $dy/dx = 4/x$.

(2) If $y = x^n \log x$, show that $dy/dx = x^{n-1}(1 + n \log x)$.

(3) What is meant by the expression, $2{\cdot}71828^{n \times 2{\cdot}3026} = 10^n$? Ansr. If n is a common logarithm, then $n \times 2{\cdot}3026$ is a natural logarithm. Note, $e = 2{\cdot}71828$.

In seeking the differential coefficient of a complex function containing products and powers of polynomials, the work is often facilitated by taking the logarithm of each member separately before differentiation. The compound process is called **logarithmic differentiation.**

EXAMPLES.—(1) Differentiate $y = x^n/(1 + x)^n$.
Here $\log y = n \log x - n \log (1 + x)$, or $dy/y = ndx/x(1 + x)$. Hence $dy/dx = yn/x(1 + x) = nx^{n-1}/(1 + x)^{n+1}$.

(2) Differentiate $x^4(1 + x)^n/(x^3 - 1)$.

Ansr. $\{(n + 1)x^4 - x^3 - (n + 4)x - 4\}x^3.(1 + x)^{n-1}/(x^3 - 1)^2$.

(3) Establish (12), § 12, by log differentiation. In the same way, show that

$$d(xyz) = yzdx + zxdy + xydz.$$

The differential coefficient of complex transcendental functions can often be easily obtained in this way.

EXAMPLES.—The following standard results can now be verified :—

(1) If $y = \log \sin x$, $dy/dx = d(\sin x)/\sin x = \cot x$ . . . . (10)

(2) If $y = \log \tan x$, $dy/dx = 2/\sin 2x$ . . . . (11)

---

* Note : the *logarithm of the product ab* is $\log a + \log b$.

The *logarithm of the fraction a/b* is $\log a - \log b$.

The *logarithm of a power*, say a^n, is $n \log a$, and so on (see page 479). The use of logarithms is explained in the introductory pages of the table books. *Chambers's Mathematical Tables* is a convenient set to have at hand. Less cumbersome and cheaper tables are, however, quite as useful for most scientific calculations. See pages 484 and 520.

(3) If $y = \log \cos x$, $dy/dx = - \tan x$ . . . . (12)
(4) If $y = \log \cot x$, $dy/dx = - 2/\sin 2x$ . . . . (13)
(5) If $y = \log \sin^{-1}x$, $dy/dx = - x/(1 - x^2)$ . . . . . . (14)
(6) If $y = \log \cos^{-1}x$, $dy/dx = x/(1 - x^2)$ . . . . . . (15)
(7) If $y = \log \tan^{-1}x$, $dy/dx = - 2x/(1 + x^2)$ . . . . . . (16)
(8) If $y = \log \cot^{-1}x$, $dy/dx = 2x/(1 + x^2)$ . . . . . . (17)
(9) If $y = x(a^2 + x^2)\sqrt{a^2 - x^2}$, $dy/dx = (a^4 + a^2x^2 - 4x^4)(a^2 - x^2)^{-\frac{1}{2}}$.

§ 17. The Differential Coefficient of Exponential Functions.

Exponential functions are those in which the variable quantity occurs in the index. Thus, a^x, e^x and $(a + x)^x$ are exponential functions. A few words on the transformation of logarithmic into exponential functions may be needed.

It is required to transform $\log y = ax$ into an exponential function. Remembering that $\log a$ to the base a is unity, it makes no difference to any magnitude if we multiply it by such expressions as $\log_a a$, $\log_{10} 10$, $\log_e e$, and so on. Thus since $\log_e(e^{ax}) = ax \log_e e$, if $\log_e y = ax$,

$$\log_e y = ax \log_e e = \log_e e^{ax}; \ \therefore \ y = e^{ax},$$

when the logarithms are removed. In future "$\log_e$" will generally be written "log".

EXAMPLES.—(1) Show that if $\log y_0 - \log y = kct$, $y = y_0 e^{-kct}$.
(2) If $\log I = - an$, $I = e^{-an}$.
(3) If $\theta = be^{-at}$, $\log b - \log \theta = at$.
(4) If $\log_e \dfrac{1}{1 - x} = a\theta$, $\log_{10} \dfrac{1}{1 - x} = \cdot 4343 a\theta$.

The differentiation of exponential functions may be conveniently studied in three sections :

(i) Let $$y = e^x.$$
Taking logarithms and then differentiating

$$\log y = x \log e; \ \frac{dy}{y} = dx, \text{ or } \frac{dy}{dx} = e^x;$$

$$\therefore \ \frac{d(e^x)}{dx} = e^x \quad . \quad . \quad . \quad . \quad (1)$$

(ii) Let $$y = a^x.$$

As before $$\log y = x \log a; \ \frac{dy}{dx} = y \log a;$$

$$\therefore \ \frac{d(a^x)}{dx} = a^x \log_e a . \quad . \quad . \quad . \quad (2)$$

(iii) Let $$y = x^z,$$
where x and z are both variable. Taking logarithms and differentiating

$$\log y = z \log x; \ \frac{dy}{y} = \log x dz + \frac{zdx}{x};$$

$$\therefore \ dy = x^z \log x dz + zx^{z-1}dx \qquad . \qquad . \qquad . \qquad (3)$$

If x and z are functions of t

$$\frac{d(x^z)}{dt} = \frac{dy}{dt} = x^z \log x \frac{dz}{dt} + zx^{z-1}\frac{dx}{dt} \qquad . \qquad . \qquad (4)$$

EXAMPLES.—(1) If $y = a^{nx}$, $dy/dx = na^{nx} \log a$.

(2) If $y = (a^x + x)^2$, $dy/dx = 2(a^x + x)(a^x \log a + 1)$.

(3) If $y = x^{\frac{1}{x}}$, $dy/dx = x^{\frac{1}{x}}(1 - \log x)/x^2$.

(4) If $y = e^{e^x}$, $dy/dx = e^x . e^{e^x}$.

(5) If $y = x^{x^x}$, $dy/dx = x^{x^x} . x^x\{(\log x)^2 + \log x + 1/x\}$.

(6) *Magnus' empirical formula* for the relation between the pressure of aqueous vapour and temperature is

$$p = ab^{\frac{\theta}{\gamma + \theta}}$$

where a, b, γ are constants. Show that $dp/d\theta = \frac{a\gamma \log ab}{(\gamma + \theta)^2} . b^{\frac{\theta}{\gamma + \theta}}$. This formula represents the increase of pressure corresponding to a small rise of temperature from (say) $\theta°$ to $(\theta + 1)°$.

(7) *Biot's empirical formula* for the relation between the pressure of aqueous vapour (p) and the temperature (θ) is

$$\log p = a + b\alpha^\theta - c\beta^\theta; \text{ hence } \frac{dp}{d\theta} = pb\alpha^\theta \log \alpha - pc\beta^\theta \log \beta.$$

(8) Required the velocity of a point which moves according to the equation $y = ae^{-\lambda t} \cos 2\pi\left(\frac{t}{T} + \epsilon\right)$. Since velocity $= dy/dt$

$$\frac{dy}{dt} = - ae^{-\lambda t}\left\{\lambda \cos 2\pi\left(\frac{t}{T} + \epsilon\right) + \frac{2\pi}{T} \sin 2\pi\left(\frac{t}{T} + \epsilon\right)\right\}.$$

§ 18. The "Compound Interest Law" in Nature.

I cannot pass by the function e^x without indicating its great significance in physical processes. From the above equations it follows that if

$$y = Ce^{ax}; \text{ then } dy/dx = be^{ax} \qquad . \qquad . \qquad . \qquad (1)$$

where a, b and C are constants, b, by the way, being equal to $aC \log_e e$. C is the value of y when $x = 0$. (Why?) It will be proved later on that this may be reversed, and if

$$\frac{dy}{dx} = be^{ax}, \text{ then } y = Ce^{ax}, \qquad . \qquad . \qquad . \qquad (2)$$

where a, b and C are again constant.

All these results indicate that the rate of increase of the exponential function e^x is e^x itself. *If, therefore, in any physical investigation we find some function, say* ϕ, *varying at a rate proportional to itself* (*with or without some constant term*) *we guess at once that we are dealing with an exponential function.* Thus if

$$\frac{d\phi}{dx} = \pm a\phi; \text{ we may write } \phi = Ce^{ax}, \text{ or } Ce^{-ax},$$

according as the function is increasing or decreasing in magnitude.

Money lent at compound interest increases in this way, and hence the above property has been happily styled by Lord Kelvin * "the compound interest law". A great many natural phenomena possess this property. The following will repay study:—

Illustration 1.—*Compound interest.* If £100 is lent out at 5 °/ₒ per annum, at the end of the first year £105 remains. If this be the principal for a second year, the interest during that time will be charged not only on the original £100, but also on the additional £5. To put this in more general terms, let £p_0 be lent at r °/ₒ per annum, at the end of the first year the interest amounts to $p_0\frac{r}{100}$, and if p_1 be the principal for the second year, we have at the end of the first year

$$p_1 = p_0(1 + r/100);$$

and at the end of the second year,

$$p_2 = p_1(1 + r/100) = p_0(1 + r/100)^2.$$

If this be continued year after year, the interest charged on the increasing capital becomes greater and greater until at the end of t years

$$p = p_0\left(1 + \frac{r}{100}\right)^t. \qquad (3)$$

Instead of adding the interest to the capital every twelve months, we could do this monthly, weekly, daily, hourly, and so on. If we are to compare this process with natural phenomena, we must imagine the interest is added to the principal continuously from moment to moment. *Natura non facit saltus.* In this way we should approximate closely to what actually occurs in Nature.

As a first approximation, suppose the interest to be added to the principal every month. It can be shown in the same way that the principal at the end of twelve months, is

$$p = p_0(1 + r/12 \,.\, 100)^{12} \qquad (4)$$

* Quoted from Perry's *Calculus for Engineers* (E. Arnold, London).

If we consider now the interest is added to the principal every moment, say t, we may replace 12 by t, in (4), and

$$p = p_0\left(1 + \frac{r}{100 \,.\, t}\right)^t. \quad . \quad . \quad . \quad (5)$$

For convenience in subsequent calculation, let us put $\frac{r}{100t} = \frac{1}{u}$, so that $t = ur/100$. From (5) and formula (16), page 483,

$$p = p_0\left\{\left(1 + \frac{1}{u}\right)^u\right\}^{r/100}.$$

But $(1 + 1/u)^u$ has been shown in (3), page 36, to be equivalent to e when u is infinitely great; hence, writing $r/100 = x$,

$$p = p_0e^x; \quad . \quad . \quad . \quad . \quad (6)$$

which shows that the exponential function represents the rate of increase of the principal with time, when the principal is reckoned from moment to moment.

We could deduce this result in a simpler, but perhaps less instructive way. Note that $\log(1 + r/100)$, and also $\log p_0$, are constant. Put the former equal to a. From (3)

$$\frac{dp}{dt} = ap.$$

We guess at once that we are dealing with an exponential function. Hence we may put, as on page 40,

$$p = Ce^{at}.$$

To find the value of C, note that when $t = 0$, $p = p_0$, and therefore

$$p = p_0e^{at},$$

which is identical with (6), when we put $x = at$.

Illustration 2.—*Newton's law of cooling.* Let a body have a uniform temperature θ_1, higher than its surroundings, it is required to find the rate at which the body cools. Let θ_0 denote the temperature of the medium surrounding the body.

In consequence of the exchange of heat, the temperature of the body gradually falls from θ_1 to θ_0. Let t denote the time required by the body to fall from θ_1 to θ. The temperature of the body is then $\theta - \theta_0$ above that of its surroundings. The most probable supposition that we can now make is that the rate at which the body loses heat ($-dQ$) is proportional to the difference between its temperature and that of its surroundings. Hence

$$-\frac{dQ}{dt} = k(\theta - \theta_0),$$

where k is a coefficient depending on the nature of the substance.

From the definition of specific heat, if s denotes the specific heat of unit mass of substance

$$Q = s(\theta - \theta_0),$$

or

$$dQ = sd\theta.$$

Substitute this in the former expression. Since k/s = constant = a (say) and $\theta_0 = 0° C.$, we obtain,

$$-\frac{d\theta}{dt} = a\theta, \quad . \quad . \quad . \quad . \quad (7)$$

or, in words, the velocity of cooling of a body is proportional to the difference between its temperature and that of its surroundings. This is Newton's well-known law of cooling (Preston's *Theory of Heat*, p. 444).

Since the rate of diminution of θ is proportional to θ itself, we guess at once that we are dealing with the compound interest law, and from a comparison with (1) and (2) above, we get

$$\theta = be^{-at}, \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\log b - \log \theta = at. \quad . \quad . \quad . \quad (9)$$

If θ_1 represents the temperature at the time t_1, and θ_2 the temperature at the time t_2, we have

$$\log b - \log \theta_1 = at_1, \text{ and } \log b - \log \theta_2 = at_2.$$

By subtraction

$$a = \frac{1}{t_2 - t_1} \cdot \log \frac{\theta_2}{\theta_1}, \quad . \quad . \quad . \quad (10)$$

a being constant.

The validity of the original "simplifying assumption" as to the rate at which heat is lost by the body must be tested by comparing the result expressed in equation (10) *with the results of experiment.* If the logical consequence of the assumption agrees with facts, there is every reason to suppose that the working hypothesis is true. For the purpose of comparison we may use Winkelmann's data, published in Wiedemann's *Annalen* for 1891, for the rate of cooling of a body from a temperature of 19·9° C. to 0° C.*

Hence if θ denote the temperature of the body at any time $t_2 - t_1$, and $\theta_2 = 19·9$, $\theta_1 = \theta$, remembering that in practical work Briggsian logarithms are used, we obtain, from (10), the expression

$$\frac{1}{t_2 - t_1} \cdot \log_{10} \frac{\theta_2}{\theta} \doteq \text{constant, say } k.$$

---

* I was led to select this happy illustration of Newton's law from Winkelmann's papers (*Wied. Ann.*, **44**, 177, 429, 1891) after reading Nernst and Schönflies' *Introduction to the Mathematical Treatment of Science.*

The data are to be arranged as shown in the following table (after Winkelmann):—

θ.	$t_2 - t_1$.	k.
18·9	3·45	0·006490
16·9	10·85	0·006540
14·9	19·30	0·006509
12·9	28·80	0·006537
10·9	40·10	0·006519
8·9	53·75	0·006502
6·9	70·95	0·006483

k is therefore constant within the limits of certain small irregular variations due to experimental error. Thus the truth of Newton's supposition is established.

This is a typical example of the way in which the logical deductions of an hypothesis are tested.

This can be done another way. Dulong and Petit (*Ann. de Chim. et de Phys.*, [2], **7**, 225, 337, 1817) have made the series of exact measurements shown in columns 1 and 2 of the following table:—

θ, excess of temp. of body above that of medium.	v, velocity of cooling $= d\theta/dt$.			
	Observed.	Calculated by the formula of:		
		Newton.	Dulong and Petit	Stefan.
220°	8·81	6·88	8·89	8·92
200°	7·40	6·20	7·34	7·42
180°	6·10	5·58	6·03	6·09
160°	4·89	4·96	4·87	4·93
140°	3·88	4·34	3·89	3·92
120°	3·02	3·72	3·05	3·05
100°	2·30	3·10	2·33	2·30

If we knew the numerical value of the constant a in formula (7), this formula could be employed to calculate the value of $d\theta/dt$ for any given value of θ. To evaluate a, substitute the observed values of v and θ in (7) and take the mean of the different results so obtained (see § 106). Thus, $a = 0·31$. Column 3 shows the velocities of cooling calculated on the assumption that Newton's law is true. The agreement between the experimental and theoretical results is very poor. Hence it is necessary to seek a second approximation to the true law.

With this object, Dulong and Petit have proposed

$$v = b(c^\theta - 1),$$

as a second approximation. Here $b = 2{\cdot}037$, $c = 1{\cdot}0077$. Column 4 shows the velocity of cooling calculated from Dulong and Petit's law. The agreement between theory and fact is now very close. This formula, however, has no theoretical basis. It is the result of a guess.

Stefan's guess is that

$$v = a\{(273 + \theta)^4 - (273)^4\},$$

where $a = 10^{-9} \times 16{\cdot}72$. The calculated results (column 5) are quite as good as those attending the use of Dulong and Petit's formula. Galitzine has pointed out that Stefan's formula can be established on theoretical grounds.

It is a very common thing to find different formulae agree, so far as we can test them, equally well with facts. *The reader must, therefore, guard against implicit faith in this criterion—the agreement between observed and calculated results—as an infallible experimentum crucis.*

A little consideration will show that it is quite legitimate to deduce the numerical values of the above constants from the experiments themselves. For example, we might have taken the mean of the values of k in Winkelmann's table above, and applied the test by comparing the calculated with the observed values of either $t_2 - t_1$, or of θ.

EXAMPLE.—To again quote from Winkelmann's paper, if, when the temperature of the surrounding medium is 99·74°, the body cools so that when

$$\theta = 119{\cdot}97^\circ,\ 117{\cdot}97^\circ,\ 115{\cdot}97^\circ,\ 113{\cdot}97^\circ,\ 111{\cdot}97^\circ,\ 109{\cdot}97^\circ;$$
$$t = 0,\quad 12{\cdot}6,\quad 26{\cdot}7,\quad 42{\cdot}9,\quad 61{\cdot}2,\quad 83{\cdot}1.$$

Do you think that Newton's law is confirmed by these measurements? Hint. Instead of assuming that $\theta_0 = 0$, it will be found necessary to retain θ_0 in the above discussion. Do this and show that the above results must be tested by means of the formula

$$\frac{1}{t_2 - t_1} \cdot \log_{10} \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = \text{constant.}$$

To return to the compound interest law.

ILLUSTRATION 3.—*The variation of atmospheric pressure with altitude* above sea level can be shown to follow the compound interest law. Let p_0 be the pressure in centimetres of mercury at the so-called datum line, or sea level, p the pressure at a height h above this level. Let ρ_0 be the density of air at sea level ($Hg = 1$).

Now the pressure at the sea level is produced by the weight of the superincumbent air, that is, by the weight of a column of air of a height h and constant density ρ_0. This weight is equal to $h\rho_0$. If the downward pressure of the air were constant, the barometric pressure would be lowered ρ_0 centimetres for every centimetre rise above sea level. But by Boyle's law the decrease in the density of air is proportional to the pressure, and if ρ denote the density of air at a height dh above sea level, the pressure dp is given by the expression

$$dp = - \rho dh.$$

If we consider the air arranged in very thin strata, we may regard the density of the air in each strata as constant. By Boyle's law

$$\rho p_0 = \rho_0 p, \text{ or } \rho = \rho_0 p/p_0.$$

Substituting this value of ρ in the above formula, we get

$$\frac{dp}{dh} = - \frac{\rho_0 p}{p_0} \quad . \quad . \quad . \quad . \quad (11)$$

The negative sign indicates that the pressure decreases vertically upwards. This equation is the compound interest law in another guise. The variation in the pressure, as we ascend or descend, is proportional to the pressure itself. Since p_0/ρ_0 is constant, we have on applying the compound interest law to (11),

$$p = \text{constant} \times e^{-\frac{\rho_0}{p_0}h}.$$

We can readily find the value of the constant by noting that at sea level $h = 0$, and $p = p_0$. Substituting these values in the last equation, and remembering that $e^0 = 1$, constant $= p_0$,

$$p = p_0 e^{-\frac{\rho_0}{p_0}h}, \quad . \quad . \quad . \quad . \quad (12)$$

a relation known as *Halley's law*. Continued p. 213.

ILLUSTRATION 4.—*The absorption of actinic energy from light passing through an absorbing medium.* A beam of light of intensity I is changed by an amount dI after it has passed through a layer of absorbing medium dn thick. That is to say

$$dI = - aIdn,$$

where a is a constant depending on the nature of the absorbing medium and on the wave length of light. The rate of variation in the intensity of the light is therefore proportional to the intensity of the light itself, in other words, the compound interest law again appears. Hence

$$\frac{dI}{dn} = - aI; \text{ or } I = \text{constant} \times e^{an}.$$

If I_0 denote the intensity of the incident light, then when

$$n = 0,\ I = I_0 = \text{constant}.$$

Hence the intensity of the light after it has passed through a medium of thickness n, is

$$I = I_0e^{-an} \quad . \quad . \quad . \quad . \quad (13)$$

The student might profitably read Bunsen and Roscoe's work on the absorption of light by different media, in the *Philosophical Transactions of the Royal Society* for 1857.

ILLUSTRATION 5.—*Wilhelmy's law for the velocity of chemical reactions.* Wilhelmy as early as 1850 published the law of mass action in a form which will be recognised as still another example of the ubiquitous law of compound interest. The statement of the law of mass action put forward by Harcourt and Esson is probably the simplest possible. It is "the amount of chemical change in a given time is directly proportional to the quantity of reacting substance present in the system". (Wilhelmy, *Annalen der Physik und Chemie*, **81**, 413, 499, 1850. See page, 197.)

If x denote the quantity of changing substance, and dx the amount of substance which disappears in the time dt, the velocity of the chemical reaction is

$$\frac{dx}{dt} = - kx,$$

where k is a constant depending on the nature of the reacting substance. It has been called the *coefficient of the velocity of the reaction.* This equation is probably the simplest we have yet studied. It follows directly, since the rate of increase of x is proportional to x, that

$$x = be^{-kt},$$

where b is some constant to be determined.* The negative sign indicates that the velocity of the action diminishes as time goes on.

Harcourt and Esson's papers in the *Philosophical Transactions* for 1866, 1867 and 1895 might be read with advantage and profit.

EXAMPLES.—(1) If a volume v of mercury be heated to any temperature θ, the change of volume dv corresponding to a small increment of temperature $d\theta$, is found to be proportional to v, hence

$$dv = avd\theta.$$

* How? See page 162.

Prove *Bosscha's formula*, $v = e^{a\theta}$, for the volume of mercury at any temperature θ. Ansr. $v = be^{a\theta}$, where a, b, a are constants. If $b = 1$ we have the required result.

(2) According to *Nordenskjöld's solubility law*, in the absence of supersaturation, for a small change in the temperature ($d\theta$), there is a change in the solubility of a salt (ds) proportional to the amount of salt s contained in the solution at the temperature θ, or

$$ds = asd\theta$$

where a is a constant. Show that the equation connecting the amount of salt dissolved by the solution with the temperature is, $s = be^{a\theta}$, where b is a constant.

(3) If any dielectric (condenser) be subject to a difference of potential, the density ρ of the charge constantly diminishes according to the relation

$$\rho = be^{-at},$$

where b is an empirical constant and a is a constant equal to the product 4π into the coefficient of conductivity (c) of the dielectric and the time (t), divided by the specific inductive capacity (μ), *i.e.*, $b = 4\pi ct/\mu$. Hence show that the gradual discharge of a condenser follows the compound interest law. Ansr. Show $d\rho/dt = \rho \times$ a negative constant.

(4) One form of *Dalton's empirical law* for the pressure of saturated vapour (p) between certain limits of temperature (θ) is,

$$p = ae^{\theta}.$$

Show that this is an example of the compound interest law.

(5) The relation between the velocity v of a chemical reaction and temperature, $\theta°$, is

$$\log v = a + b\theta,$$

where a and b are constants. Show that we are dealing with the Compound Interest Law. What is the logical consequence of this law with reference to reactions which (like hydrogen and oxygen) take place at high temperatures (say 500°), but, so far as we can tell, not at ordinary temperatures? Look up "False Equilibrium" in your *Textbook of Physical Chemistry*.

§ 19. Successive Differentiation.

The differential coefficient derived from any function of a variable may be either another function of the variable, or a constant (page 17). The new function may be differentiated again in order to obtain the second differential coefficient. In the same way we may obtain the third and higher derivatives.

EXAMPLE.—Let $y = x^3$;

the first derivative is, $\dfrac{dy}{dx} = 3x^2$;

the second derivative is, $\dfrac{d^2y}{dx^2} = 6x$;

the third derivative is, $\dfrac{d^3y}{dx^3} = 6$;

the fourth derivative is, $\dfrac{d^4y}{dx^4} = 0$.

It will be observed that each differentiation reduces the index of the power by unity. If n is a positive integer the number of derivatives is finite.

In the symbols $\frac{d^2}{dx^2}(y)$, $\frac{d^3}{dx^3}(y)$ . . ., the superscripts simply denote that the differentiation has been repeated 2, 3 . . . times.

In differential notation we may write these results

$$d^2y = 6x \,.\, dx^2 *; \ d^3y = 6dx^3 *; \ \ldots$$

The successive differential coefficients of

$$y = \sin x$$

are

$$y^{\mathrm{I}} = \cos x; \ y^{\mathrm{II}} = -\sin x; \ y^{\mathrm{III}} = -\cos x; \ y^{\mathrm{IV}} = \sin x; \ \ldots$$

The fourth derivative is thus a repetition of the original function, the process of differentiation may thus be continued without end, every fourth derivative resembling the original function.

The simplest case of such a repetition is

$$y = e^x,$$

where $$y^{\mathrm{I}} = e^x; \ y^{\mathrm{II}} = e^x; \ y^{\mathrm{III}} = e^x; \ \ldots$$

The differential coefficients are all equal to each other and to the original function.

EXAMPLES.—(1) Show that every fourth derivative in the successive differentiation of $y = \cos x$ repeats itself.

(2) If $y = \log x$, $d^4y/dx^4 = -6/x^4$.

(3) If $y = x^n$, $d^4y/dx^4 = n(n-1)(n-2)(n-3)x^{n-4}$.

(4) If $y = x^{-2}$, $d^3y/dx^3 = -24x^{-5}$.

(5) If $y = \log(x+1)$, $d^2y/dx^2 = -(x+1)^{-2}$.

Just as the first derivative of x with respect to t measures a velocity, the second differential coefficient of x with respect to t measures an acceleration. See page 13.

EXAMPLES.—(1) If a material point (P) † move in a straight line AB (Fig. 7) so that its distance (s) from a fixed point O is given by the equation

$$s = a \sin t,$$

where a is constant, show that the acceleration due to the force acting on the particle is proportional to its distance from the fixed point.

FIG. 7.

---

* Do not confuse these with $dx^2 = 2x \,.\, dx$; $dx^3 = 3x^2 \,.\, dx$; . . .

† A *material point* is a fiction much used in applied mathematics for purposes of calculation, just as the atom is in chemistry. An atom may contain an infinite number of "material points" or particles.

The velocity is evidently

$$v = ds/dt = a \cos t;$$

and the acceleration

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -a \sin t = -s,$$

the negative sign showing that the force (f) is attractive, tending to lessen the distance of the moving point from O.

To get some idea of this motion find a set of corresponding values of f, s and v as shown in the following table:—

If t =	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π . . .
then v =	a	0	$-a$	0	a . . .
and s =	0	a	0	a	0 . . .
and f =	0	$-a$	0	a	0 . . .
and P is at	O	B	O	A	O etc.

A careful study of these facts will convince the reader that the point is oscillating regularly in a straight line, alternately right and left of the point O.

(2) Another illustration of the second derivative. If a body falls from a vertical height according to the law

$$s = \tfrac{1}{2}gt^2,$$

where g represents the acceleration due to the earth's gravity, show that g is equal to the second differential coefficient of s with respect to t.

(3) If the distance traversed by a moving point in the time t be denoted by the equation

$$s = at^2 + bt + c$$

(where a, b and c are arbitrary constants), show that the acceleration is constant.

The geometrical signification of the second differential coefficient is discussed on pages 133 to 135.

§ 20. Leibnitz' Theorem.

To find the nth differential coefficient of the product of two functions of x in terms of the differential coefficients of each function.

On page 26 the differential coefficient of the product of two variables was shown to be

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx},$$

where u and v are functions of x. By successive differentiation and analogy with the binomial theorem it may be shown that

$$\frac{d^n(uv)}{dx^n} = v\frac{d^nu}{dx^n} + n\frac{dv}{dx} \cdot \frac{d^{n-1}u}{dx^{n-1}} + \ldots + u\frac{d^nv}{dx^n} \quad . \quad . \quad (1)$$

This formula, due to Leibnitz, will be found very convenient in Chapter VII.—"How to Solve Differential Equations". The reader must himself prove the formula, as an exercise on § 19, by comparing the values of $d^2(uv)/dx^2$, $d^3(uv)/dx^3$, . . ., with the developments of $(x+h)^2$, $(x+h)^3$, . . ., of page 22.

EXAMPLES.—(1) If $y = x^4 . e^{ax}$, find the value of d^3y/dx^3. Substitute x^4 and e^{ax} respectively for v and u in (1). Thus,

$$v = x^4; \therefore dv/dx = 4x^3, d^2v/dx^2 = 12x^2, d^3v/dx^3 = 24x;$$
$$u = e^{ax}; \therefore du/dx = ae^{ax}, d^2u/dx^2 = a^2e^{ax}, d^3u/dx^3 = a^3e^{ax}.$$

From (1)

$$\frac{d^3y}{dx^3} = v\frac{d^3u}{dx^3} + n\frac{dv}{dx}\cdot\frac{d^2u}{dx^2} + \frac{n(n-1)}{2!}\cdot\frac{d^2v}{dx^2}\cdot\frac{du}{dx} + u\frac{n(n-1)(n-2)}{3!}\cdot\frac{d^3v}{dx^3};$$
$$= e^{ax}\left(a^3v + 3a^2\frac{dv}{dx} + 3a\frac{d^2v}{dx^2} + \frac{d^3v}{dx^3}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$= e^{ax}(a^3x^4 + 12a^2x^3 + 36ax^2 + 24x).$$

If we pretend, for the time being, that the symbols of operation—§ 8—$\frac{d}{dx}$, $\left(\frac{d}{dx}\right)^2$, $\left(\frac{d}{dx}\right)^3$, in (2), represent the magnitudes of an operation, in an algebraic sense, we can write

$$\frac{d^3(e^{ax}v)}{dx^3} = e^{ax}\left(a + \frac{d}{dx}\right)^3 v, \quad . \quad . \quad . \quad . \quad (3)$$

instead of (2). The expression $\left(a + \frac{d}{dx}\right)^3$ is supposed to be developed by the binomial theorem, page 22, and dv/dx, d^2v/dx^2, . . ., substituted in place of $\left(\frac{d}{dx}\right)v$, $\left(\frac{d}{dx}\right)^2 v$, . . ., in the result. Equation (3) would also hold good if 3 were replaced by any integer, say n. This result is known as the *symbolic form of Leibnitz' theorem.*

(2) If $y = \log x$, show that $d^6y/dx^6 = -5!/x^6$.

§ 21. Partial Differentiation.

Up to the present time we have been principally occupied with functions of one independent variable x, such that

$$u = f(x);$$

but functions of two, three or more variables may occur, say

$$u = f(x, y, z, \ldots),$$

where the variables $x, y, z, \ldots$ are independent of each other. Such functions are common. As illustrations, it might be pointed out that the area of a triangle depends on its base and altitude, the volume of a rectangular box depends on its three dimensions, and the volume of a gas depends on the temperature and pressure.

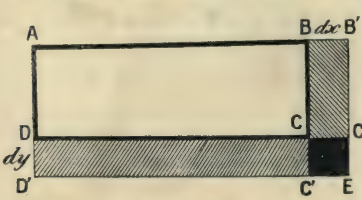


FIG. 8.

(1) *To find the differential of a function of two independent variables.* This can be best done in the following manner, partly

graphic and partly analytical. In figure 8, the area u of the rectangle $ABCD$, with the sides x, y, is given by the function

$$u = xy.$$

Since x and y are independent of each other, the one may be supposed to vary, while the other remains unchanged. The function, therefore, ought to furnish two differential coefficients, the one resulting from a variation in x, and the other from a variation in y.

First, let the side x vary while y remains unchanged. The area is then a function of x alone. y remains constant.

$$\therefore (du)_y = ydx, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $(du)_y$ represents the area of the rectangle $BB'CC''$. The subscript denoting that y is constant.

Second, in the same way, suppose the length of the side y changes, while x remains constant, then

$$(du)_x = xdy, \quad . \quad . \quad . \quad . \quad (2)$$

where $(du)_x$ represents the area of the rectangle $DD'CC'$.

Instead of using the differential form of these variables, we may write the differential coefficients

$$\left(\frac{du}{dx}\right)_y = y, \text{ and } \left(\frac{du}{dy}\right)_x = x;$$

or

$$\frac{\partial u}{\partial x} = y, \text{ and } \frac{\partial u}{\partial y} = x,$$

where "$\dfrac{\partial}{\partial x}$" is the symbol of differentiation when all the variables, other than x, are constant. Substituting these values of x and y in (1) and (2), we obtain

$$(du)_y = \frac{\partial u}{\partial x}dx\,;\ (du)_x = \frac{\partial u}{\partial y}dy.$$

Lastly, let us allow x and y to vary simultaneously, the total increment in the area of the rectangle is evidently represented by the figure $D'EB'BCD$.

$$(incr.\ u) = BB'CC'' + DD'CC' + CC'C''E$$
$$= ydx + xdy + dx\,.\,dy.$$

Neglecting infinitely small magnitudes of the second order, we get

$$du = ydx + xdy\,; \quad . \quad . \quad . \quad . \quad (3)$$

or

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad . \quad . \quad . \quad (4)$$

which is also written in the form

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy$$

the former for preference.

In equations (3) and (4) du is called the **total differential** of the function; $\frac{\partial u}{\partial x}dx$ the **partial differential** of u with respect to x when y is constant; and $\frac{\partial u}{\partial y}dy$ the partial differential of u with respect to y when x is constant. Hence the rule:

The total differential of two (or more) independent variables is equal to the sum of their partial differentials.

The physical meaning of this rule is that the total force acting on a body at any instant is the sum of every separate action. This is nothing more than the so-called *principle of superposition of small impulses.**

"According to this principle, the total force acting on a particle at any moment is the sum of all the infinitely small individual actions to which the particle is subjected. This, in reality, means nothing more than that the total differential represents the total change experienced by the mathematical function. For instance, if a gas is exposed to variable conditions of temperature and pressure, the total change in volume is the sum of the changes which occur at a constant temperature and varying pressure, and at a constant pressure and varying temperature. The total differential, therefore, is equal to the sum of the partial differentials corresponding respectively to a changing pressure and to a changing temperature. The mathematical process thus corresponds with the actual physical change." (Freely translated from Nernst and Schönflies' *Einführung in die mathematische Behandlung der Naturwissenschaften*, p. 180, 1898.)

In other words, the total change in u when x and y vary is made up of two parts: (1) the change which would occur in u if x alone varied, and (2) the change which would occur in u if y alone varied.

Total variation = variation due to x alone + variation due to y alone.

Equation (4) may be written in a more general manner if we put $u = f(x,y)$, thus

$$du = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy; \qquad (5)$$

or

$$du = f^1(x)dx + f^1(y)dy,$$

where the meaning of $f^1(x)$ and $f^1(y)$ is obvious.

* Ostwald calls this the "principle of the mutual independence of different processes," or the "principle of the coexistence of different reactions," meaning that if a number of forces act upon a material particle, each force produces its own motion independently of all the others. (Ostwald, *Grundriss der allgemeinen Chemie*, Walker's translation, p. 297.)

Examples.—(1) If $u = x^3 + x^2y + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy\,;\ \frac{\partial u}{\partial y} = x^2 + 3y^2$$

$$\therefore du = (3x^2 + 2xy)dx + (x^2 + 3y^2)dy.$$

(2) If $u = x\log y$; $du = \log y dx + x\,.\,dy/y$.

(3) If $u = \cos x\,.\,\sin y + \sin x\,.\,\cos y$,

$$du = (dx + dy)(\cos x\,.\,\cos y - \sin x\sin y)$$
$$= (dx + dy)\{\cos(x + y)\}.$$

(4) If $u = x^y$, $du = yx^{y-1}dx + x^y\log x dy$.

(5) The differentiation of a function of three independent variables may be left as an exercise to the reader. Neglecting quantities of a higher order, if u be the volume of a rectangular parallelopiped having the three dimensions x, y, z, independently variable, then

$$u = xyz,$$

and $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\,; \quad . \quad . \quad . \quad . \quad (6)$$

or an infinitely small increment in the volume of the solid is the sum of the infinitely small increments resulting when each variable changes independently of the others. In differential notation show also that

$$du = yzdx + xzdy + xydz. \quad . \quad . \quad . \quad . \quad (7)$$

(6) If the relation between the pressure p, and volume v, and temperature θ of a gas is given by the formula $pv = R(1 + a\theta)$, show that the total change in volume for a simultaneous change of pressure and temperature is $dv = aR\,.\,d\theta/p - R(1 + a\theta)\,.\,dp/p^2$.

(7) *Clairaut's formula* for the attraction of gravitation (g) at different latitudes (L) on the earth's surface, and at different altitudes (H) above mean tide level, is

$$g = 980{\cdot}6056 - 2{\cdot}5028\cos 2L - 0{\cdot}000003H,\ \text{dynes}.$$

Discuss the changes in the force of gravitation and in the weight of a substance with change of locality. Note, "weight" is nothing more than a measure of the force of gravitation.

(2) *To find the differential coefficient of a function of two interdependent variables.* If the meaning of the different terms in

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

is carefully noted, it will be found that *the equation is really expressed in differential notation*, not differential coefficients. In the partial derivative $\frac{\partial u}{\partial x}dx$, ∂u represents the infinitely small change that takes place in u when x is increased by an amount dx, y being constant; similarly ∂u in $\frac{\partial u}{\partial y}dy$ stands for the infinitely small change which occurs when y is increased by an amount dy, x being constant. If du is the total increment in a function of the variables when each variable is individually increased by an amount $(du)_y$ and $(du)_x$, then

$$du = (du)_y + (du)_x.$$

If the variables x and y are both functions of say t, we have

$$u = f(x, y)\,;\ x = \phi(t)\,;\ y = \psi(t),$$

and
$$du = \frac{\partial f(x, y)}{\partial x}dx + \frac{\partial f(x, y)}{\partial y}dy.$$

We may pass directly from differentials to differential coefficients by dividing through with dt, thus

$$\frac{du}{dt} = \frac{\partial f(x, y)}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y}\cdot\frac{dy}{dt},$$

which is more frequently written

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}\,;\ \text{or}\ \frac{du}{dt} = \left(\frac{du}{dx}\right)\frac{dx}{dt} + \left(\frac{du}{dy}\right)\frac{dy}{dt}. \qquad (8)$$

When there is likely to be any doubt as to what variables have been assumed constant, a subscript is appended to the lower corner on the right of the bracket. The brackets in the second of equations (8) may be omitted, when there is no chance of confusing $\partial u/\partial x$ . . . with differential coefficients.

The most general form of (8) for any number of variables is obtained as follows: If

$$u = f(x_1, x_2, \ . \ . \ . \ x_n),$$

where x_1, x_2, . . . are functions of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dx} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dx} + \ . \ . \ . + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dx} \qquad (9)$$

If, at the same time, x_1, x_2, . . . are functions of y,

$$\frac{du}{dy} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dy} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dy} + \ . \ . \ . + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dy} \qquad (10)$$

Equation (8) leads to some interesting results.

If $y = uv$, where u and v are functions of x, then $\partial y/\partial v = u$ and $\partial y/\partial u = v$; substituting these values in (8), and making the necessary changes in the letters,* we get our old formula, page 26,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (11)$$

If u is a function of x, such that $u = x$,

$$\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{dv}{dx}, \qquad (12)$$

since dx/dx is unity. Note the distinction between $\dfrac{\partial y}{\partial x}$ and $\dfrac{dy}{dx}$.

* Note that u, x, y, t of (8) are now to be replaced by y, u, v, and x respectively.

If u is constant,

$$\frac{dy}{dt} = \frac{\partial y}{\partial v} \cdot \frac{dv}{dx} \quad . \quad . \quad . \quad . \quad (13)$$

A formula previously obtained in a different way.

Many illustrations of functions with properties similar to those required in order to satisfy the conditions of equation (8) may occur to the reader. The following is typical:—

When rhombic crystals are heated they may have different coefficients of expansion in different directions. A cubical portion of one of these crystals at one temperature is not necessarily cubical at another. Suppose a rectangular parallelopiped is cut from such a crystal, with faces parallel to the three axes of dilation (see Preston's *Theory of Heat*, p. 199). The volume of the crystal is

$$V = xyz,$$

where x, y, z are the lengths of the different sides. Hence

$$\partial V/\partial x = yz; \ \partial V/\partial y = xz; \ \partial V/\partial z = xy.$$

Substitute in (6) and divide by $d\theta$, where $d\theta$ represents a slight rise of temperature, then

$$\frac{dV}{d\theta} = yz\frac{dx}{d\theta} + xz\frac{dy}{d\theta} + xy\frac{dz}{d\theta},$$

where $dx/d\theta$, $dy/d\theta$, $dz/d\theta$ represent the coefficients of expansion (page 7) along the three directions.

The *coefficient of cubical expansion* is obtained by putting $x = y = z = 1$, when

$$\alpha = \frac{dx}{d\theta} + \frac{dy}{d\theta} + \frac{dz}{d\theta},$$

where $dx/d\theta$ or $\partial x/\partial\theta$, etc., represent the linear expansions (λ) in each direction. For isotropic bodies

$$dx/d\theta = dy/d\theta = dz/d\theta, \text{ and hence } \alpha = 3\lambda.$$

Examples.—(1) *Loschmidt and Obermeyer's formula for the coefficient of diffusion of a gas* at $\theta°$ (absolute) is

$$k = k_0\left(\frac{\theta}{\theta_0}\right)^n \frac{p}{760},$$

where k_0 is the coefficient of diffusion at 0° C. and p is the pressure of the gas. Required the variation in the coefficient of diffusion of the gas corresponding to small changes of temperature and pressure. Note k_0 and θ_0 are constant. Put $a = k_0/760\theta_0{}^n$; $\frac{\partial k}{\partial\theta}d\theta = apn\theta^{n-1} . d\theta$; $\frac{\partial k}{\partial p}dp = a\theta^n dp$. But

$$dk = \frac{\partial k}{\partial\theta}d\theta + \frac{\partial k}{\partial p}dp. \ \therefore dk = a\theta^{n-1}(npd\theta + \theta dp).$$

(2) *Biot and Arago's formula for the index of refraction* (μ) *of a gas or vapour* at $\theta°$ and pressure p is

$$\mu - 1 = \frac{\mu_0 - 1}{1 + \alpha\theta} \cdot \frac{p}{760},$$

where μ_0 is the index of refraction at 0°, α the coefficient of expansion of the gas with temperature. What is the effect of small variations of temperature and pressure on the index of refraction? Ansr. To cause it to vary by an amount $d\mu = \frac{\mu_0 - 1}{760}\left(\frac{dp}{1 + \alpha\theta} - \frac{p\alpha d\theta}{(1 + \alpha\theta)^2}\right)$.

§ 22. Euler's Theorem on Homogeneous Functions.

The following discussion is convenient for reference:

To show that if u is an homogeneous function of the nth degree, say* $u = \Sigma a x^\alpha y^\beta$,† *where* $\alpha + \beta = n$, *then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

By differentiation of the homogeneous function,

$$u = a x^\alpha y^\beta + b x^{\alpha^1} y^{\beta^1} + \ . \ . \ . = \Sigma a x^\alpha y^\beta,$$

where $\alpha + \beta = \alpha^1 + \beta^1 = \ . \ . \ . = n$, we obtain

$$\frac{\partial u}{\partial x} = \Sigma a \alpha x^{\alpha - 1} y^\beta; \text{ and } \frac{\partial u}{\partial y} = \Sigma a \beta x^\alpha y^{\beta - 1}.$$

Hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \Sigma a(\alpha + \beta)x^\alpha y^\beta = n\Sigma a x^\alpha y^\beta = nu.$$

The theorem may be extended to include any number of variables (see footnote, page 340).

Examples.—(1) If $u = x^2y + xy^2 + 3xyz$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$. Prove this result by actual differentiation. It of course follows directly from Euler's theorem, since the equation is homogeneous and of the third degree.

(2) If $u = \frac{x^3 + x^2y + y^3}{x^2 + xy + y^2}$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$, since the equation is of the first degree and homogeneous.

(3) Put Euler's theorem into words. Ansr. *In any homogeneous function, the sum of the products of each variable with the partial differential coefficients of the original function with respect to that variable is equal to the product of the original function with its degree.*

---

* An **homogeneous function** is one in which all the terms containing the variables have the same degree. Examples: $x^2 + bxy + z^2$; $x^4 + xyz^2 + x^3y + x^2z^2$ are homogeneous functions of the second and fourth degrees respectively.

† The sign "Σ" is to be read "the sum of all terms of the same type as . . .," or here "the sum of all terms containing x, y and a constant". The symbol "Π" is sometimes used in the same way for "the product of all terms of the type".

§ 23. Successive Partial Differentiation.

We can get the higher partial derivatives by successive differentiation, using processes analogous to those used on page 47. Thus when

$$u = x^2 + y^2 + x^2y^3,$$

$$\frac{\partial u}{\partial x} = 2x + 2y^3x ; \ \frac{\partial u}{\partial y} = 2y + 3x^2y^2 ; \quad . \quad . \quad (1)$$

repeating the differentiation,

$$\frac{\partial^2 u}{\partial x^2} = 2(1 + y^3) ; \ \frac{\partial^2 u}{\partial y^2} = 2(1 + 3x^2y), \quad . \quad . \quad (2)$$

If we had differentiated $\partial u/\partial x$ with respect to y, and $\partial u/\partial y$ with respect to x, we should have obtained two identical results, *viz.* :—

$$\frac{\partial^2 u}{\partial y \partial x} = 6y^2x, \text{ and } \frac{\partial^2 u}{\partial x \partial y} = 6y^2x. \quad . \quad . \quad (3)$$

This rule is general.

The higher partial derivatives are independent of the order of differentiation. By differentiation of $\partial u/\partial x$ with respect to y, assuming x to be constant, we get $\partial\left(\frac{\partial u}{\partial x}\right)\Big/\partial y$, which is written $\frac{\partial^2 u}{\partial y \partial x}$; on the other hand, by the differentiation of $\partial u/\partial y$ with respect to x, assuming y to be constant, we obtain $\frac{\partial^2 u}{\partial x \partial y}$. That is to say

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}. \quad . \quad . \quad . \quad . \quad (4)$$

This was only proved in (3) for a special case. As soon as the reader has got familiar with the idea of differentiation, he will no doubt be able to deduce the general proof for himself, although it is given in the regular text books. The result stated in (4) is of great importance.

§ 24. Exact Differentials.

To find the condition that u may be a function of x and y in the equation

$$du = Mdx + Ndy, \quad . \quad . \quad . \quad . \quad (5)$$

where M and N are functions of x and y.

We have just seen that if u is a function of x and y

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad . \quad . \quad . \quad (6)$$

that is to say, by comparing (5) and (6)

$$M = \frac{\partial u}{\partial x} ; \ N = \frac{\partial u}{\partial y}.$$

Differentiating the first with respect to y, and the second with respect to x, we have, from (4)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (7)$$

In the chapter on differential equations this condition is shown to be necessary and sufficient in order that certain equations may be solved, or "integrated" as it is called. Equation (7) is therefore called the **criterion of integrability.** An equation that satisfies this condition is said to be a **complete** *or* **an exact differential.** For examples, see page 290.

§ 25. Integrating Factors.

The equation

$$Mdx + Ndy = 0 \qquad (8)$$

can always be made exact by multiplying through with some function of x, called an integrating factor. (M and N are functions of x and y.)

Since M and N are functions of x and y, (8) may be written

$$\frac{dy}{dx} = -\frac{M}{N}, \qquad (9)$$

or the variation of y with respect to x is as $-M$ is to N; that is to say, x is some function of y, say

$$f(x, y) = a,$$

then from (5), page 52,

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0. \qquad (10)$$

By a transformation of (10), and a comparison of the result with (9), we find that

$$\frac{dy}{dx} = -\frac{\partial f(x,y)}{\partial x} \Big/ \frac{\partial f(x,y)}{\partial y} = -\frac{M}{N} \qquad (11)$$

Hence $$\frac{\partial f(x, y)}{\partial x} = \mu M; \text{ and } \frac{\partial f(x, y)}{\partial y} = \mu N, \qquad (12)$$

where μ is either a function of x and y, or else a constant. Multiplying the original equation by the integrating factor μ, and substituting the values of μM, μN so obtained in (12), we obtain

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0,$$

which fulfils the condition of exactness.

Example.—Show that the equation $ydx - xdy = 0$ becomes exact when multiplied by $1/y^2$.

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}; \frac{\partial N}{\partial x} = -\frac{1}{y^2}.$$

Hence $\partial M/\partial y = \partial N/\partial x$, the condition required by (7). In the same way show that $1/xy$ and $1/x^2$ are also integrating factors.

§ 26. Illustrations from Thermodynamics.

It is proved in the theory of differential equations that the number of integrating factors for any equation, $Mdx + Ndy = 0$ is infinite. Integrating factors are very much used in solving certain forms of differential equations (*q.v.*), and in certain important equations which arise in thermodynamics. Several illustrations of partial derivatives will be found in subsequent parts of this work.

The change of state of every homogeneous liquid, or gaseous substance is completely defined by some law connecting the pressure (p), volume (v) and temperature (θ). This law, called the *characteristic equation*, or the *equation of state* of the substance, has the form

$$f(p, v, \theta) = 0.$$

Any change, therefore, is completely determined when any two of these three variables are known. Thus, we may have

$$p = f_1(v, \theta)\,;\ v = f_2(p, \theta)\,;\ \text{or } \theta = f_3(p, v). \quad . \quad (1)$$

Confining our attention to the first, we obtain, by partial differentiation,

$$dp = \frac{\partial p}{\partial v}dv + \frac{\partial p}{\partial \theta}d\theta, \quad . \quad . \quad . \quad (2)$$

where the partial derivative $\partial p/\partial v$ represents the coefficient of elasticity of the gas, $\partial p/\partial \theta$ is nothing but the so-called coefficient of increase of pressure with temperature at constant volume. If the change takes place at constant pressure, $dp = 0$, and (2) may be written

$$\frac{\partial p}{\partial \theta} = -\frac{\partial p}{\partial v}\cdot\frac{dv}{d\theta}\,;\ \text{or } \left(\frac{\partial v}{\partial \theta}\right)_p = -\left(\frac{\partial p}{\partial \theta}\right)_v \Big/ \left(\frac{\partial p}{\partial v}\right)_\theta \quad . \quad (3)$$

The subscript is added to show which factor has been supposed constant during the differentiation. Note the change of $dv/d\theta$ to $\partial v/\partial \theta$ at constant pressure. Equations (3) state that the coefficient of thermal expansion is equal to the ratio of the coefficient of the increase of pressure with temperature at constant volume, and the coefficient of elasticity of the gas.

EXAMPLES.—(1) Show that a pressure of 120 atmospheres is required to keep unit volume of mercury at constant volume when heated 2° C. (Coefficient of expansion of $Hg = 0{\cdot}00018$, of compressibility $0{\cdot}000003$.) (Planck.)

(2) *J. Thomsen's formula* for the amount of heat Q disengaged when one molecule of sulphuric acid (H_2SO_4) is mixed with n molecules of water (H_2O) is

$$Q = 17860\, n/(1{\cdot}798 + n) \text{ cals.}$$

Put $a = 17860$ and $b = 1{\cdot}798$, for the sake of brevity. If x of H_2SO_4 be mixed with y of H_2O, the quantity of heat disengaged by the mixture is x times as great as when one molecule of H_2SO_4 unites with y/x molecules of water. Substituting $y/x = n$ in Thomsen's formula

$$Q = ay/(bx + y) \text{ cals.}$$

If dx of acid is now mixed with x of H_2SO_4 and y of H_2O, show that the amount of heat liberated is

$$\frac{\partial Q}{\partial x}dx = \frac{ay^2}{(bx + y)^2} \cdot dx\,; \text{ or } \frac{an^2}{(b + n)^2} \cdot dx \text{ cals.}$$

In the same way the amount of heat liberated when dy of water is added to a similar mixture is

$$\frac{\partial Q}{\partial y}dy = \frac{ab}{(b + n)^2}dy \text{ cals.}$$

The student of thermodynamics is not likely to meet with anything more difficult than the seven following examples:—

(3) Apply equation (3) to the ordinary gas equation

$$pv = R\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where R is a constant, p, v, and θ have their usual meaning. Ansr.

$$R/p = (- R/v)/(- p/v).$$

What does this mean?

(4) Verify the following deductions: Let Q, θ, p, v, represent any four variable magnitudes whatever. By partial differentiation,

$$dQ = \frac{\partial Q}{\partial \theta}d\theta + \frac{\partial Q}{\partial v}dv = \frac{\partial Q}{\partial \theta}d\theta + \frac{\partial Q}{\partial p}dp = \frac{\partial Q}{\partial p}dp + \frac{\partial Q}{\partial v}dv. \quad . \quad (5)$$

Equate together the second and last members of (5), and substitute the value of dp from (2), in the result. Thus,

$$\left(\frac{\partial Q}{\partial \theta}\right)_v d\theta + \left(\frac{\partial Q}{\partial v}\right)_\theta dv = \left(\frac{\partial Q}{\partial p}\right)_v\left(\frac{\partial p}{\partial \theta}\right)_v d\theta + \left(\frac{\partial Q}{\partial p}\right)_v\left(\frac{\partial p}{\partial v}\right)_\theta dv + \left(\frac{\partial Q}{\partial v}\right)_p dv. \quad (6)$$

Put $dv = 0$, and divide by $d\theta$,

$$\left(\frac{dQ}{d\theta}\right)_v = \left(\frac{\partial Q}{\partial p}\right)_v\left(\frac{\partial p}{\partial \theta}\right)_v. \quad . \quad . \quad . \quad . \quad (7)$$

Again, by partial differentiation

$$d\theta = \left(\frac{\partial \theta}{\partial p}\right)_v dp + \left(\frac{\partial \theta}{\partial v}\right)_p dv. \quad . \quad . \quad . \quad . \quad (8)$$

Substitute this value of $d\theta$ in the last two members of (5),

$$\left(\frac{\partial Q}{\partial \theta}\right)_p\left(\frac{\partial \theta}{\partial p}\right)_v dp + \left(\frac{\partial Q}{\partial \theta}\right)_p\left(\frac{\partial \theta}{\partial v}\right)_p dv + \left(\frac{\partial Q}{\partial p}\right)_\theta dp = \left(\frac{\partial Q}{\partial p}\right)_v dp + \left(\frac{\partial Q}{\partial v}\right)_p dv. \quad (9)$$

Put $dp = 0$, and write the result

$$\left(\frac{\partial Q}{\partial \theta}\right)_p\left(\frac{\partial \theta}{\partial v}\right)_p = \left(\frac{\partial Q}{\partial v}\right)_p. \quad . \quad . \quad . \quad . \quad (10)$$

By proceeding in this way, the reader can deduce a great number of relations between Q, θ, p, v, quite apart from any physical meaning the letters might possess.

If Q denotes the quantity of heat added to a substance during any small changes of state, and p, v, θ, the pressure, volume and absolute temperature of the substance, the above formulae are then identical with corresponding formulae in thermodynamics. Here, however, the relations have been deduced without any reference to the theory of heat.

Under these circumstances, $(dQ/\partial\theta)_v d\theta$ represents the quantity of heat required for a small rise of temperature at constant volume; $(\partial Q/\partial\theta)_v$ is nothing but the *specific heat of the substance at constant volume*, usually written C_v; similarly, $(\partial Q/\partial\theta)_p$ is the *specific heat at constant pressure*, written C_p; and $(\partial Q/\partial v)_\theta$ and $(\partial Q/\partial p)_\theta$ refer to the two *latent heats*.

These results may be applied to any substance for which the relation (4) holds good. In this case

$$\left(\frac{\partial p}{\partial\theta}\right)_v = \frac{R}{v}, \ldots; \text{ and, therefore, } v\left(\frac{\partial Q}{\partial\theta}\right)_v = R\left(\frac{\partial\theta}{\partial p}\right)_v, \ldots$$

(5) A little ingenuity, and the reader should be able to deduce the so-called *Reech's Theorem:*

$$\gamma = \frac{C_p}{C_v} = \left(\frac{\partial p}{\partial v}\right)_Q \Big/ \left(\frac{\partial p}{\partial v}\right)_\theta, \quad \ldots \quad (11)$$

employed by Clément and Desormes for evaluating γ. See any text-book on physics for experimental detail.

(6) By the definition of adiabatic and isothermal elasticities (page 92),

$$E_\phi = -v(\partial p/\partial v)_\phi, \text{ and } E_\theta = -v(\partial p/\partial v)_\theta, \text{ respectively.}$$

The subscripts ϕ and θ indicating, in the former case, that there has been neither gain nor loss of heat, in other words that Q has remained constant, and in the latter case, that the temperature remained constant during the process $\partial p/\partial v$. Verify the following reasoning:—

From the first and last members of (5), when Q is constant,

$$v\left(\frac{\partial p}{\partial v}\right)_\phi = -v\left(\frac{\partial Q}{\partial v}\right)_p \Big/ \left(\frac{\partial Q}{\partial p}\right)_v.$$

From (7), (10) and (3),

$$\frac{E_\phi}{E_\theta} = -\left(\frac{\partial Q}{\partial v}\right)_p \Big/ \left(\frac{\partial Q}{\partial p}\right)_v \left(\frac{\partial Q}{\partial v}\right)_\theta = \left(\frac{\partial Q}{\partial\theta}\right)_p \left(\frac{\partial Q}{\partial v}\right)_p \left(\frac{\partial p}{\partial\theta}\right)_v \Big/ \left(\frac{\partial Q}{\partial\theta}\right)_v \left(\frac{\partial p}{\partial v}\right)_\theta,$$

$$= \left(\frac{\partial Q}{\partial\theta}\right)_p \Big/ \left(\frac{\partial Q}{\partial\theta}\right)_v = \frac{C_p}{C_v} = \gamma. \quad \ldots \quad (12)$$

An important result.

(7) According to the *second law of thermodynamics*, "the expression dQ/θ is a perfect differential". It is usually written $d\phi$, where ϕ is called the entropy of the substance. From the first two members of (5), therefore,

$$\frac{dQ}{\theta} = d\phi = \frac{1}{\theta}\cdot\left(\frac{\partial Q}{\partial\theta}\right)_v d\theta + \frac{1}{\theta}\cdot\left(\frac{\partial Q}{\partial v}\right)_\theta dv, \quad \ldots \quad (13)$$

is a perfect differential. From (7), page 58, therefore,

$$\frac{d}{dv}\left(\frac{1}{\theta}\cdot\frac{\partial Q}{\partial\theta}\right)_\theta = \frac{d}{d\theta}\left(\frac{1}{\theta}\cdot\frac{\partial Q}{\partial v}\right)_v; \text{ or } \left(\frac{\partial C_v}{\partial v}\right)_\theta = \left(\frac{\partial L}{\partial\theta}\right)_v - \frac{L}{\theta}, \quad . \quad (14)$$

where C_v has been written for $(\partial Q/\partial\theta)_v$, L for $(\partial Q/\partial v)_\theta$.

According to the *first law of thermodynamics*, when a quantity of heat dQ is added to a substance, part of the heat energy dU is spent in the doing of internal work among the molecules of the substance, and part is expended in the mechanical work of expansion ($p.dv$) against atmospheric pressure (see page 182). To put this symbolically,

$$dQ = dU + pdv; \text{ or } dU = dQ - pdv. \quad \ldots \quad (15)$$

Now dU is a perfect differential. This means that however much energy U, the substance absorbs, all will be given back again when the substance returns to its original state. In other words, U is a function of the state of the substance (see page 295). This state is determined, (2) above, when any two of the three variables p, v, θ, are known.

From the first two members of (5), and the last equation of (14), therefore,

$$dU = C_v . d\theta + L . dv - pdv = C_v . d\theta + (L - p)dv, \quad . \quad (16)$$

is a complete differential. In consequence, as before,

$$\left(\frac{\partial C_v}{\partial v}\right)_\theta = \left(\frac{\partial L}{\partial \theta}\right)_v - \left(\frac{\partial p}{\partial \theta}\right)_v. \quad . \quad . \quad . \quad . \quad (17)$$

From (14) and (17),

$$\left(\frac{\partial p}{\partial \theta}\right)_v = \frac{1}{\theta}\left(\frac{\partial Q}{\partial v}\right)_\theta. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

a "law" which has formed the starting point of some of the finest deductions in physical chemistry (see page 216).

(8) Establish *Mayer's formula*,

$$C_p - C_v = R, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

for a perfect gas.

Hints: (i.) Since $pv = R\theta$, $(\partial p/\partial \theta)_v = R/v$; $\therefore (\partial Q/\partial \theta)_v = R\theta/v = p$. (ii.) Evaluate dv as in (2), and substitute the result in the second and third members of (5). (iii.) Equate dv to zero. Find $\partial v/\partial \theta$ from the gas equation, etc. Thus,

$$\left(\frac{\partial Q}{\partial \theta}\right)_v + \left(\frac{\partial Q}{\partial v}\right)_\theta \left(\frac{\partial v}{\partial \theta}\right)_p = \left(\frac{\partial Q}{\partial \theta}\right)_p; \ \left(\frac{\partial Q}{\partial \theta}\right)_v + \left(\frac{\partial Q}{\partial v}\right)_\theta . \frac{R}{p} = \left(\frac{\partial Q}{\partial \theta}\right)_p; \text{ etc.}$$

(9) Assuming *Newton's formula* that the square of the velocity of propagation (V) of a compression wave (*e.g.*, of sound) in a gas varies directly as the adiabatic elasticity of the gas (E_ϕ) and inversely as the density (ρ), or

$$V^2 \propto E_\phi/\rho; \text{ show that } V^2 \propto \gamma R\theta.$$

Hints: Since the compression wave travels so rapidly, the changes of pressure and volume take place without gain or loss of heat. Therefore, instead of using Boyle's law, $pv =$ constant, we must employ $pv^\gamma =$ constant (page 212). Hence deduce $\gamma p = v . dp/dv = E_\phi$. Note that the volume varies inversely as the density of the gas. Hence, if

$$V^2 \propto E_\phi/\rho \propto E_\phi v \propto \gamma pv \propto \gamma R\theta. \quad . \quad . \quad . \quad . \quad (20)$$

Equations (19) and (20) can be employed to determine the two specific heats of any gas in which the velocity of sound is known. Let a be a constant to be evaluated from the known values of R, θ, V^2,

$$\therefore C_v = R/(1 - a), \text{ and } C_p = aC_v.$$

Boynton has employed van der Waals' equation in place of Boyle's. Perhaps the reader can do this for himself. It will simplify matters to neglect terms containing magnitudes of a high order (see Boynton, *Physical Review*, **12**, 353, 1901).

(10) If $y = e^{ax + \beta t + \gamma}$ is to satisfy the equation

$$\frac{\partial^2 y}{\partial x^2} = A\frac{\partial^2 y}{\partial t^2} + B\frac{\partial y}{\partial t},$$

show that $a^2 = A\beta^2 + B\beta$, where α, β, γ, are constants.

CHAPTER II.

COORDINATE OR ANALYTICAL GEOMETRY.

"Order and regularity are more readily and clearly recognised when exhibited to the eye in a picture than they are when presented to the mind in any other manner."—DR. WHEWELL.

§ 27. Cartesian Coordinates.

THE physical properties of a substance may, in general, be concisely represented by a geometrical figure. Such a figure furnishes an elegant method for studying certain natural changes, because the whole history of the process is thus brought vividly before the mind. At the same time the numerical relations between a series of tabulated numbers can be exhibited in the form of a picture and their true meaning seen at a glance.

Let xOx' and yOy' (Fig. 9) be two straight lines at right angles to each other, and intersecting at the point O, so as to divide the plane of this paper into four quadrants I, II, III and IV. Let P_1 be any * point in the first quadrant yOx; draw P_1M_1 parallel to Oy and P_1N parallel to Ox. Then, if the lengths OM_1 and P_1M_1 are known, the position of the point P with respect to these lines follows directly from the properties of the rectangle NP_1M_1O (Euclid, i., 34). For example, if OM_1 denotes three units, P_1M_1 four units, the position of the point P_1 is found by marking off three units along Ox to the right and four units along Oy vertically upwards. Then by drawing NP_1 parallel to Ox, and P_1M_1 parallel to Oy, the position of the given point is at P_1, since,

$$P_1M_1 = ON = 4 \text{ units}; \ NP_1 = OM_1 = 3 \text{ units}.$$

$x'Ox$, yOy' are called **coordinate axes.** If the angle yOx is a right angle the axes are said to be **rectangular.** Conditions may arise when it is more convenient to make yOx an oblique angle, the axes are then said to be **oblique.** xOx' is called the **abscissa**

* It is perhaps needless to remark that *what is true of any point is true of all.*

or x-axis, yOy' the **ordinate or y-axis.** The point O is called the **origin**; OM_1 the **abscissa** of the point P, and P_1M_1 the **ordinate** of the same point. In referring the position of a point to a pair of coordinate axes, the abscissa is always mentioned first, P_1 is spoken of as the point whose coordinates are 3 and 4; it is written "the point $P(3, 4)$".

In memory of its inventor, René Descartes, this system of notation is sometimes styled the system of **Cartesian coordinates.**

The usual conventions of trignometry are made with respect to the algebraic sign of a point in any of the four quadrants. Any

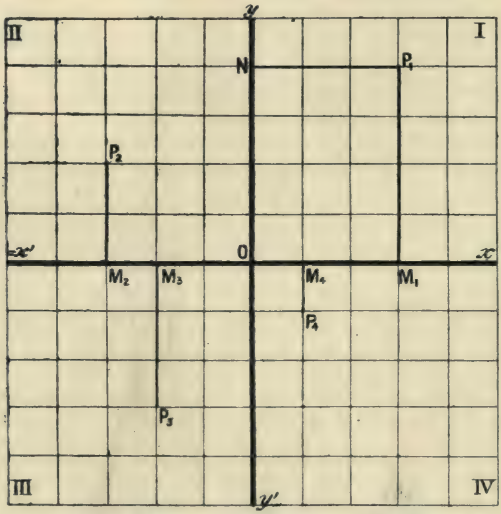


FIG. 9.—Rectangular Cartesian Coordinates.

abscissa measured from the origin to the right is positive, to the left, negative; ordinates measured vertically upward are positive, and in the opposite direction, negative. For example, if a and b be any assigned number of units corresponding respectively to the abscissa and ordinate of some given point, then the Cartesian coordinates of the point P_1 are represented as $P_1(a, b)$, of P_2 as $P_2(-a, b)$, of P_3 as $P_3(-a, -b)$ and of P_4 as $P_4(a, -b)$. Points falling in quadrants other than the first are not often met with in practical work.

Thus, any point in a plane represents two things, (1) its horizontal distance along some standard line of reference (x-axis), and

(2) its vertical distance along some other standard line of reference (y-axis).

When the position of a point is determined by two variable magnitudes (the coordinates), the point is said to be *two dimensional.*

We are always making use of coordinate geometry in a rough way. Thus, a book in a library is located by its shelf and number; the position of a town in a map is fixed by its latitude and longitude; etc.

§ 28. Graphical Representation.

Consider any straight or curved line OP situate, with reference to a pair of rectangular coordinate axes, as shown in figure 10. Take any abscissae OM_1, OM_2, OM_3, . . . OM, and through M_1, M_2 . . . M draw the ordinates P_1M_1, P_2M_2, . . . PM parallel to the y-axis. The ordinates all have a definite value dependent on the slope of the line * and on the value of the abscissae. If x be any abscissa and y any ordinate, x and y are connected by some definite law called **the equation of the curve.**

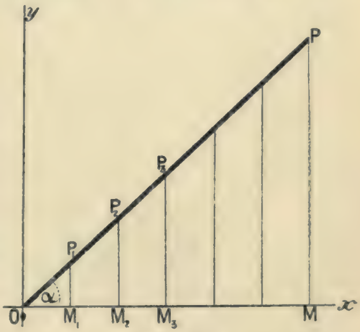


FIG. 10.

It is required to find the equation to the curve OP. In the triangle OPM

$$PM = OM \tan POM,$$

or $$y = x \tan \alpha, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α denotes the angle POM. But if $OM = PM$,

$$\therefore \tan POM = \frac{PM}{OM} = 1 = \tan 45^\circ.$$

The equation of the line OP is, therefore,

$$y = x; \quad . \quad . \quad . \quad . \quad (2)$$

and the line is inclined at an angle of 45° to the x-axis.

It follows directly that both the abscissa and ordinate of a point situate at the origin are zero. A point on the x-axis has a zero

* Any straight or curved line when referred to its coordinate axes, is called a "*curve*".

ordinate; a point on the y-axis has a zero abscissa. Any line parallel to the x-axis has an equation

$$y = b; \quad . \quad . \quad . \quad . \quad (3)$$

any line parallel to the y-axis has an equation

$$x = a, \quad . \quad . \quad . \quad . \quad (4)$$

where a and b denote the distances between the two lines and their respective axes.

§ 29. Practical Illustrations of Graphical Representation.

Suppose, in an investigation on the relation between the pressure (p) and the weight (w) of a gas dissolved by unit volume of a solution, we obtained the following successive pairs of observations,

$$p = \tfrac{1}{4}, 2, 4, 8 \ldots = x.$$
$$w = \tfrac{1}{8}, 1, 2, 4 \ldots = y.$$

By setting off on millimetre, coordinate or squared paper (Fig. 11) points $P_1(\frac{1}{4}, \frac{1}{8})$, $P_2(2, 1)$ . . . , and drawing a line to pass through all these points, we are said to **plot the curve.** This has been done in figure 11. The only difference between the lines OP of figures 10 and 11 is in their slope towards the two axes.

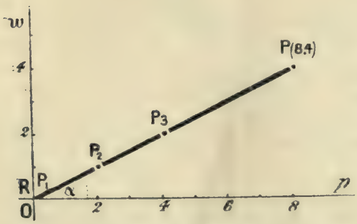


FIG. 11.—Solution of Gases in liquids.

From equation (1) we can put

$$w = p \tan \alpha, \text{ or } \tan \alpha = \tfrac{1}{2},$$

that is to say, an angle whose tangent is $\frac{1}{2}$. This can be found by reference to a table of natural tangents. It is 26° 33′ (approximately).

Putting $\tan \alpha = m$, we may write

$$w = mp, \quad . \quad . \quad . \quad . \quad (5)$$

where m is a constant depending on the nature of the gas and liquid used in the experiment.

Equation (5) is the mathematical expression for the solubility of a gas obeying *Henry's law*, *viz.*: "At constant temperature, the weight of a gas dissolved by unit volume of a liquid is proportional to the pressure". The curve OP is a graphical representation of Henry's law.

To take one more illustration. The solubility of potassium chloride (λ) in 100 parts of water at temperatures (θ) between 0° and 100° is approximately as follows:

$$\theta = 0°, \quad 20°, \quad 40°, \quad 60°, \quad 80°, \quad 100° = x,$$
$$\lambda = 28{\cdot}5, \quad 39{\cdot}7, \quad 49{\cdot}8, \quad 59{\cdot}2, \quad 69{\cdot}5, \quad 79{\cdot}5 = y.$$

By plotting these numbers, as in the preceding example, we obtain a curve PQ (Fig. 12) which, instead of passing through the origin at O, cuts the y-axis at the point Q such that

$$OQ = 28{\cdot}5 \text{ units} = b \text{ (say)}.$$

If OP' be drawn from the point O parallel to PQ, then the equation for this line is obviously, from (5),

$$\lambda = m\theta,$$

but since the line under consideration cuts the y-axis at Q,

$$\lambda = m\theta + b, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $b = OQ$. In these equations, b, λ and θ are known, the value of m is therefore obtained by a simple transposition of (6),

$$m = (\lambda - b)/\theta.$$

Substituting the values of b and m in (6), we can find the approximate solubility of potassium chloride at any temperature (θ) between 0° and 100° by the relation

$$\lambda = 0{\cdot}51280\theta + 28{\cdot}5.$$

The curve QP in figure 12 is a graphical representation of the variation in the solubility of KCl in water at different temperatures.

Knowing the equation to the curve, or even the form of the curve alone, the probable solubility of KCl for any unobserved temperature can be deduced, for if the solubility had been determined every 10° (say) instead of every 20°, the corresponding ordinates could still be connected in an unbroken line. The same relation holds however short the temperature interval. From this point of view the solubility curve may be regarded as the path of a point moving according to some fixed law. This law is defined by the equation to the curve, since the coordinates of every point on the curve satisfy the equation. The path described by such a point is called the **picture, locus** or **graph of the equation.**

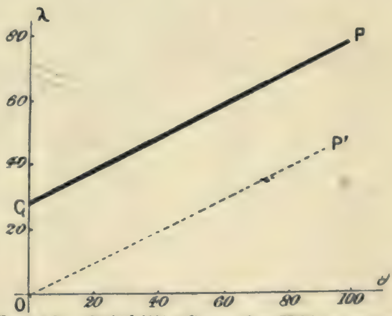


FIG. 12.—Solubility Curve for KCl in water.

EXAMPLES.—(1) Let the reader procure some "squared" paper and plot: $y = \frac{1}{2}x - 2$; $2y + 3x = 12$ . . .

(2) The following experimental results have been obtained:—

When $x =$	0,	1,	10,	20,	30, . . .
$y =$	− 3,	1·56,	11·40,	25·80,	40·20, . . .

(*a*) Plot the curve. (*b*) Show (i.) that the slope of the curve to the x-axis is $1·44 = \tan \alpha = \tan 60°$ (nearly), (ii.) that the equation to the curve is $y = 1·44x - 3$. (*c*) Measure off 5 and 15 units along the x-axis, and show that the distance of these points from the curve, measured vertically above the x-axis, represents the corresponding ordinates. (*d*) Compare the values of y so obtained with those deduced by substituting $x = 5$ and $x = 15$ in the above equation.

Note the laborious and roundabout nature of process (*c*) when contrasted with (*d*). The graphic process, called *graphic interpolation* (*q.v.*), is seldom resorted to when the equation connecting the two variables is available, but of this anon.

(3) Get some solubility determinations from any chemical text-book and plot the values of the composition of the solution (C, ordinate) at different temperatures ($\theta°$, abscissa), *e.g.*, Loewel's numbers for sodium sulphate are

$C = 5·0$,	19·4,	55·0,	46·7,	44·4,	43·1,	42·2;
$\theta° = 0°$,	20°,	34°,	50°,	70°,	90°,	103·5°.

What does the peculiar bend at 34° mean?

In this and analogous cases, a question of this nature has to be decided: *What is the best way to represent the composition of a solution?* Several methods are available. The right choice depends entirely on the judgment, or rather on the *finesse*, of the investigator. Most chemists (like Loewel above) follow Gay Lussac, and represent the composition of the solution as "parts of substance which would dissolve in 100 parts of the solvent". Etard found it more convenient to express his results as "parts of substance dissolved in 100 parts of saturated solution".

The right choice, at this day, seems to be to express the results in molecular proportions. This allows the solubility constant to be easily compared with the other physical constants. In this way, Gay Lussac's method becomes "the ratio of the number of molecules of dissolved substance to the number, say 100, molecules of solvent"; Etard's "the ratio of the number of molecules of dissolved substance to any number, say 100, molecules of solution".

§ 30. General Equations of the Straight Line.

If equations (1) and (6) be expressed in general terms, using x and y for the variables, m and b for the constants, we can deduce the following properties for straight lines referred to a pair of coordinate axes.

(1) *A straight line passing through the origin of a pair of rectangular coordinate axes, is represented by the equation*

$$y = mx, \quad . \quad . \quad . \quad . \quad (7)$$

where $m = \tan \alpha = y/x$, a constant representing the slope of the curve. The equation is obtained from (5) above.

(2) *A straight line which cuts one of the rectangular coordinate axes at a distance b from the origin, is represented by the equation*

$$y = mx + b \quad . \quad . \quad . \quad . \quad (8)$$

where m and b are any constants whatever. For every value of m there is an angle such that $\tan \alpha = m$. The position of the line is therefore determined by a point and a direction. Equation (8) follows immediately from (6).

(3) *A straight line is always represented by an equation of the first degree,*

$$Ax + By + C = 0; \quad . \quad . \quad . \quad (9)$$

*and conversely, any equation of the first degree between two variables represents a straight line.**

This conclusion is drawn from the fact that any equation containing only the first powers of x and y, represents a straight line. By substituting $m = -A/B$ and $b = -C/B$ in (8), and reducing the equation to its simplest form, we get the general equation of the first degree between two variables: $Ax + By + C = 0$. This represents a straight line inclined to the positive direction of the x-axis at an angle whose tangent is $-A/B$, and cutting the y-axis at a point $-C/B$ above the origin.

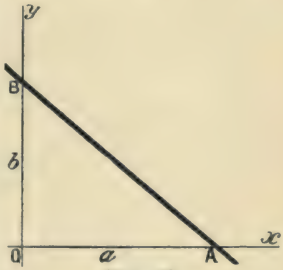


FIG. 13.

(4) *A straight line which cuts each coordinate axis at the respective distances a and b from the origin, is represented by the equation*

$$\frac{x}{a} + \frac{y}{b} = 1 \quad . \quad . \quad . \quad . \quad (10)$$

Consider the straight line AB (Fig. 13) which intercepts the x- and y-axes at the points A and B respectively. Let $OA = a$, $OB = b$. From equation (9) if

$$y = 0, x = a; \ Aa + C = 0, a = -C/A.$$

Similarly if $\quad x = 0, y = b; \ Bb + C = 0, b = -C/B.$

* If the reader has not previously met with the idea conveyed by a "general equation," he must pay careful attention to it now. By assigning suitable values to the constants A, B, C, he will be able to deduce every possible equation of the first degree between the two variables x and y. See page 481.

Substituting these values of a and b in (9), *i.e.*, in

$$-\frac{A}{C}x - \frac{B}{C}y = 1\text{; and we get }\frac{x}{a} + \frac{y}{b} = 1.$$

There are several proofs of this useful equation. Formula (10) is called the *intercept form of the equation of the straight line*, equations (7) and (8) *the tangent forms*.

Many equations can be readily transformed into the intercept form and their geometrical interpretation seen at a glance. For instance, the equation

$$x + y = 2 \text{ becomes } \tfrac{1}{2}x + \tfrac{1}{2}y = 1,$$

which represents a straight line cutting each axis at a distance of 2 units from the origin.

One way of stating *Gay Lussac's law* is that "the pressure of a given mass of gas at constant volume varies directly as the temperature". If, under these conditions, the temperature be raised $\theta°$, the pressure increases the $\frac{1}{273}\theta$rd part of what it was at the original temperature.* Let the original pressure, at 0° C., be unity; the final pressure p_1, then at $\theta°$

$$p_1 = 1 + \tfrac{1}{273}\theta.$$

This equation resembles the intercept form of the equation of a straight line (10) where $a = 273$ and $b = 1$. The intercepts a and b may be found by putting x and y, or rather their equivalents, θ and p, successively equal to zero. If $\theta = 0$, $p = 1$; if $p = 0$, $\theta = -273$, the well-known absolute zero.

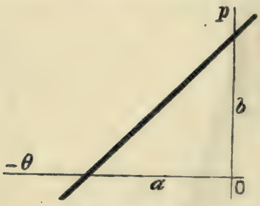


FIG. 14
(b much exaggerated).

If possible let θ fall below $-273°$, then we have a negative value of p in the above equation, which is physically impossible. The physical signification of this is that temperatures below $-273°$ are impossible, *if* the gas obeys Gay Lussac's law at temperatures approaching the absolute zero.

* Many students, and even some of the textbooks, appear to have hazy notions on this question. According to *Gay Lussac's law*, the increase in the volume of a gas at any temperature for a rise of temperature of 1°, is *a constant fraction of its initial volume at* 0°*C.*; *Dalton's law*, on the other hand, supposes the increase in the volume of a gas at any temperature for a rise of 1°, is *a constant fraction of its volume at that temperature* (the "Compound Interest Law," in fact). The former appears to approximate closer to the truth than the latter. Gay Lussac says that he got the idea from Charles, hence this property of gases is sometimes called *Charles' law*, or the law of Charles and Gay Lussac.

EXAMPLES.—(1) *To find the angle between the point of intersection of two straight lines whose equations are given.* Let the equations be

$$y = mx + b; \; y' = m'x' + b'.$$

Let ϕ be the angle required (see Fig. 15), $m = \tan \alpha$, $m' = \tan \alpha'$. From Euclid, i., 32, $\alpha' - \alpha = \phi$, $\therefore \tan (\alpha' - \alpha) = \tan \phi$. By formula, page 500,

$$\tan \phi = \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha \cdot \tan \alpha'} = \frac{m' - m}{1 + mm'} \quad . \quad . \quad . \quad (11)$$

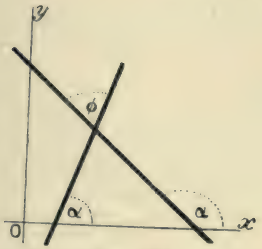


FIG. 15.

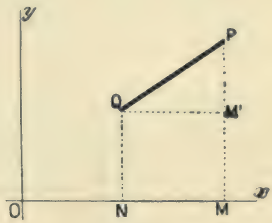


FIG. 16.

(2) *To find the distance between two points in terms of their coordinates.* In Fig. 16, let $P(x_1y_1)$ and $Q(x_2y_2)$ be the given points. Draw QM^1 parallel to NM. $OM = x_1$, $PM = y_1$; $ON = x_2$, $QN = y_2$;

$$M^1P = MP - MM^1 = MP - NQ = y_1 - y_2;$$

$$QM^1 = MN = OM - ON = x_1 - x_2.$$

Since QPM^1 is a right-angled triangle

$$(QP)^2 = (QM^1)^2 + (M^1P)^2.$$

$$\therefore \quad QP = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad . \quad . \quad . \quad (12)$$

§ 31. Differential Coefficient of a Point moving on a Straight Line.

If the amount of gas (v_1) consumed in a burner is proportional to the time (t_1), equal amounts of gas are consumed in equal times. Suppose that the amount of gas burnt in one second be denoted by V, then for time t_0, v_1 has a value v_0, and the gas consumed in $t_1 - t_0$ seconds amounts to $V(t_1 - t_0)$. Hence

$$v_1 = v_0 + V(t_1 - t_0),$$

whatever be the values of v and t. This equation can be written

$$(v_1 - v_0) = V(t_1 - t_0),$$

which resembles the equation to a straight line (7), when the ratio of the increments of x and y possesses a constant value.

Expressing the last equation in general symbols, we can put

$$\frac{x_1 - x_0}{y_1 - y_0} = \frac{\textit{increment } x}{\textit{increment } y} = \text{constant},$$

or, at the limit, the velocity of gas consumption may be represented by

$$V = \frac{dx}{dy} = \tan \alpha ; \quad . \quad . \quad . \quad (13)$$

that is to say, by a straight line with a slope, or inclination to the x-axis equal to $\tan \alpha$.

EXAMPLE.—Malard and Le Chatelier represent the relation between the molecular specific heat (s) of carbon dioxide and temperature (θ) by the expression

$$s = 6{\cdot}3 + 0{\cdot}00564\theta - 0{\cdot}000001{,}08\theta^2.$$

Plot the $\theta, ds/d\theta$-curve from $\theta = 0°$ to $\theta = 2{,}000°$ (abscissae). Possibly a few trials will have to be made before the "scale" of each coordinate will be properly proportioned to give the most satisfactory graph. The student must learn to do this sort of thing for himself. What is the difference in meaning between this curve and the s,θ – curve?

§ 32. Straight Lines Satisfying Conditions.

The reader should work through the following examples so as to familiarise himself with the conceptions of coordinate geometry. Many of the properties here developed for the straight line can easily be extended to curved lines.

(1) *The condition that a straight line may pass through a given point.* This evidently requires that the coordinates of the point should satisfy the equation of the line. Let the equation be in the tangent form

$$y = mx + b.$$

If the line is to pass through the point (x_1, y_1),

$$y_1 = mx_1 + b,$$

and by subtraction $(y - y_1) = m(x - x_1)$ . . . (14)
which is an equation of a straight line satisfying the required conditions.

(2) *The condition that a straight line may pass through two given points.* Continuing the preceding discussion, if the line is to pass through (x_2, y_2), substitute x_2, y_2, in (14)

$$(y_2 - y_1) = m(x_2 - x_1),$$

$$\therefore m = (y_2 - y_1)/(x_2 - x_1).$$

Substituting this value of m in (14), we get the equation,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}, \quad . \quad . \quad . \quad (15)$$

for a straight line passing through two given points (x_1, y_1) and (x_2, y_2).

(3) *To find the coordinates of the point of intersection of two given straight lines.* Let the given equations be

$$y = mx + b \text{ and } y = m'x + b'.$$

Now each equation is satisfied by an infinite number of pairs of values of x and y. These pairs of values are generally different in the two equations, but there can be one, and only one pair of values of x and y that satisfy the two equations, that is, the coordinates of the point of intersection. The coordinates at this point must satisfy the two equations, and this is true of no other point.

The roots of these two equations, obtained by a simple algebraic operation, are the coordinates of the point required. The point whose coordinates are $(b' - b)/(m - m')$, $(b'm - bm')/(m - m')$ satisfies the two equations.

(4) *To find the condition that three given lines may meet at a point.* The roots of the equations of two of the lines are the coordinates of their point of intersection, and in order that this point may be on a third line the roots of the equations of two of the lines must satisfy the equation of the third.

EXAMPLE.—If three lines are represented by the equations $5x + 3y = 7$, $3x - 4y = 10$, and $x + 2y = 0$, show that they will all intersect at a point whose coordinates are $x = 2$ and $y = -1$. Solving the last two equations, we get $x = 2$ and $y = -1$, but these values of x and y satisfy the first equation, hence these three lines meet at the point $(2, -1)$.

(5) *To find the condition that two lines may be parallel to one another.* Since the lines are to be parallel they must make equal angles with the x-axis,

$$\therefore \text{angle } \alpha' = \text{angle } \alpha, \text{ or } \tan\alpha' = \tan\alpha,$$

or

$$m = m', \quad . \quad . \quad . \quad . \quad (16)$$

that is to say, the coefficient of x in the two equations must be equal.

(6) *To find the condition that two lines may be perpendicular to one another.* If the angle between the lines is

$$\phi = 90^\circ \text{ [see (11)] } \alpha' - \alpha = 90^\circ,$$

$$\therefore \tan\alpha' = \tan(90 - \alpha) = -\cot\alpha = -1/\tan\alpha,$$

$$\therefore \quad m' = -\frac{1}{m'}, \quad . \quad . \quad . \quad . \quad (17)$$

or, the slope of the one line to the x-axis must be equal and opposite in sign to the reciprocal of the slope of the other.

§ 33. Changing the Coordinate Axes.

In plotting the graph of any function, the axes of reference should be so chosen that the resulting curve is represented in the most convenient position. It is frequently necessary to pass from one system of coordinate axes to another. In order to do this the equation of the given line referred to the new axes must be deduced from the corresponding equation referred to the old set of axes.

(1) *To transform from any system of coordinate axes to another set parallel to the former but having a different origin.* Let Ox, Oy (Fig. 17) be original axes, and KO_1x_1, HO_1y_1 the new axes parallel to Ox and Oy. Let MM_1P be the ordinate of any point P parallel to the axes Oy and O_1y_1. Let h, k be the ordinates of the

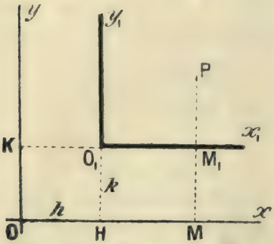


FIG. 17.—Transformation of Axes.

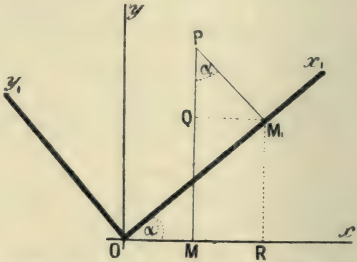


FIG. 18.—Transformation of Axes.

new origin O_1 referred to the old axes. Let (x, y) be the coordinates of P referred to the old axes Ox, Oy, and (x_1y_1) its coordinates referred to the new axes. Then $OH = h$, $O_1H = k$,

$$x = OM = OH + HM = OH + O_1M_1 = h + x_1;$$
$$y = MP = MM_1 + M_1P = O_1H + M_1P = k + y_1.$$

That is to say we must substitute

$$x = h + x_1 \text{ and } y = k + y_1, \quad . \quad . \quad (18)$$

in order to refer a curve to a new set of rectangular axes. The new coordinates of the point P being

$$x_1 = x - h \text{ and } y_1 = y - k. \quad . \quad . \quad (19)$$

(2) *To transform from one set of axes to another having the same origin but different directions.* Let the two straight lines x_1O and y_1O, passing through O (Fig. 18), be taken as the new system of coordinates. Let the coordinates of the point P (x, y) when referred to the new axes be x_1, y_1. Draw PM perpendicular to

the old x-axis, and PM_1 perpendicular to the new axes, so that the angle $MPM_1 = ROM_1 = \alpha$,

$$OM = x,\ OM_1 = x_1,\ PM = y,\ PM_1 = y_1.$$

Draw M_1R perpendicular and QM_1 parallel to the x-axis. Then

$$x = OM = OR - MR = OR - QM_1,$$
$$= OM_1 \cos\alpha - M_1P \sin\alpha;$$
$$\therefore x = x_1 \cos\alpha - y_1 \sin\alpha \quad . \quad . \quad . \quad . \quad \textbf{(20)}$$

Similarly $$y = MP = MQ + QP = RM_1 + QP,$$
$$= OM_1 \sin\alpha + M_1P \cos\alpha,$$
$$\therefore y = x_1 \sin\alpha + y_1 \cos\alpha \quad . \quad . \quad . \quad . \quad \textbf{(21)}$$

Equations (20) and (21) enable us to refer the coordinates of a point P from one set of axes to another. Solving equations (20) and (21) simultaneously,

$$\left.\begin{aligned} x_1 &= x\cos\alpha + y\sin\alpha \\ y_1 &= y\cos\alpha - x\sin\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

(3) *To transform from one set of axes to another set having a different origin and different directions.* Obviously this can be done by making the two preceding transformations one after another.

§ 34. The Circle and its Equation.

To find the equation of a circle referred to its centre as origin. Let r be radius of the circle (Fig. 19) whose centre is the origin of the rectangular coordinate axes xOx' and yOy'. Take any point $P(x, y)$ on the circle. Let PM be the ordinate of P. From the definition of a circle OP is constant and equal to r. Then by Euclid, i., 47.

$$(OM)^2 + (MP)^2 = (OP)^2,$$

or $$x^2 + y^2 = r^2, \quad . \quad \textbf{(1)}$$

which is the equation required.

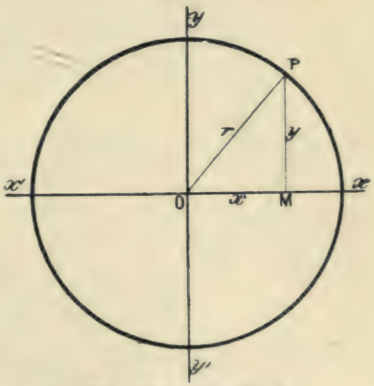


FIG. 19.—The Circle.

In connection with this equation it must be remembered that the abscissae and ordinates of some points have negative values, but, since the square of a negative quantity is always positive, the rule still holds good. Equation (1) therefore expresses the geometrical fact that all points on the circumference are at an equal distance from the centre.

EXAMPLES.—(1) Required the locus of a point moving in a path according to the equations $y = a \cos t$, $x = a \sin t$, where t denotes any given interval of time. Square each equation and add,

$$y^2 + x^2 = a^2(\cos^2 t + \sin^2 t).$$

The expression in brackets is unity (formula (17) page 499), and hence for all values of t

$$y^2 + x^2 = a^2,$$

i.e., the point moves on the perimeter of a circle of radius a.

(2) To find the equation of a circle whose centre, referred to a pair of rectangular axes, has the coordinates h and k. From (19), previous paragraph,

$$(x - h)^2 + (y - k)^2 = r^2, \quad . \quad . \quad . \quad . \quad (2)$$

where $P(x, y)$ is any point on the circumference. Note the product xy is absent. The coefficients of x^2 and y^2 are equal in magnitude and sign. These conditions are fulfilled by every equation to a circle. Such is

$$3x^2 + 3y^2 + 7x - 12 = 0.$$

The *general equation of a circle* is

$$x^2 + y^2 + ax + by + c = 0^*. \quad . \quad . \quad (3)$$

§ 35. The Parabola and its Equation.

There is a set of important curves whose shape can be obtained by cutting a cone at different angles. Hence the name *conic sections*. They include the parabola, hyperbola and ellipse, of which the circle is a special case. I shall very briefly describe their chief properties.

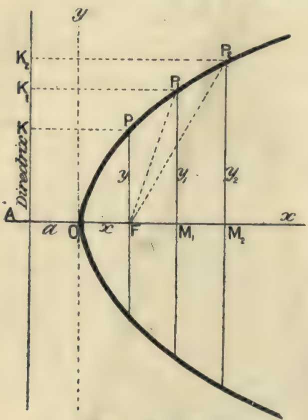


FIG. 20.—The Parabola.

A parabola is a curve such that any point on the curve is equi-distant from a given point and a given straight line.

The given point is called the *focus*, the straight line the *directrix*, the distance of any point on the curve from the focus is called the *focal radius*. O, Fig. 20, is called *vertex* of the parabola. AK is the directrix; OF, PF, P_1F . . . are focal radii;

$$K_3P_3 = P_3F,\ K_2P_2 = P_2F,\ KP = PF,\ AO = OF.$$

* The reader should verify all these equations by plotting on his "squared" paper.

(1) *To find the equation of the parabola.* Take vertex as origin of the coordinate axes. Let $OA = OF = a$. Take *any* point $P(x, y)$

$$FP = PK = AM = AO + OM = x + a;$$
$$FM = OM - OF \qquad = x - a;$$
$$PM = y.$$

In right-angled triangle FPM

$$(x - a)^2 + y^2 = (x + a)^2;$$
$$\therefore \quad y^2 = 4ax, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the standard equation of the parabola. The abscissae are proportional to the squares of the ordinates.

(2) *To find the shape of the parabola.* From (1)

$$y = \pm 2\sqrt{ax}.$$

1st. Every positive value of x gives two equal and opposite values of y, that is to say, there are two points at equal distances perpendicular to the x-axis. This being true for all values of x, the part of the curve lying on one side of the x-axis is the mirror image of that on the opposite side *; in this case the x-axis is said to be symmetrical with respect to the parabola. Hence any line perpendicular to the x-axis cuts the curve at two points equidistant from the x-axis.

2nd. When $x = 0$, the y-axis is tangent † to the curve.

3rd. a being positive when x is negative, there is no real value of y, for no real number is known whose square is negative; in consequence, the parabola lies wholly on the right side of the y-axis.

4th. As x increases without limit, y approaches infinity, that is to say, the parabola recedes indefinitely from the x or symmetrical-axis on both sides.

EXAMPLES.—(1) By a transformation of coordinates show that the parabola represented by equation (1), may be written in the form

$$x = a + by + cy^2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a, b, c are constants. Let x become $x + h$; $y = y + k$; $a = j$ where h, k and j are constants. Substitute the new values of x in (1) and multiply out. Collect the constants together and equate to a, b and c as the case might be.

(2) In the general equation of the second degree

$$ax^2 + bxy + cy^2 + fx + gy + h = 0, \quad . \quad . \quad . \quad (3)$$

if $b^2 - 4ac = 0$, the equation represents a parabola, one or two straight lines or an impossible curve. Trace the curve $x^2 - 2xy + y^2 - 8x + 16 = 0$ and show that the curve is a parabola, $b = 2$, $a = 1$, $c = 1$. What relations must exist between the coefficients in order that (3) may represent a circle?

---

* The student of stereo-chemistry would say the two sides were "enantiomorphic".

† A "tangent" is a straight line which touches but does not cut the curve (see pages 82 and 494).

§ 36. The Ellipse and its Equation.

An ellipse is a curve such that the sum of the distances of any point on the curve from two given points is always the same.

In Fig. 21 let P be the given point which moves on the curve PP' so that its distance from the two fixed points F_1, F_2, called the *foci*, has a constant value say $2a$. The distance of P from either focus is called the *focal radius* (or *radius vector*). $PF_2 + PF_1 = 2a$.

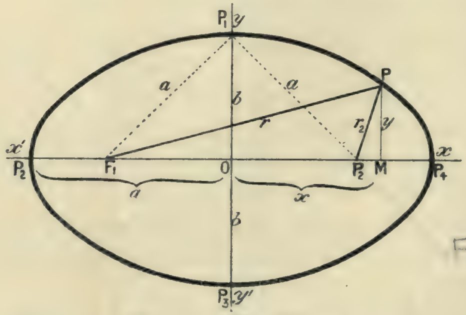


FIG. 21.—The Ellipse.

(1) *To find the equation of the ellipse.* This is rather a tedious deduction. If desired, the final equation may be taken without proof. In the same figure, let xOx' and yOy' be a pair of coordinate axes such that the centre O bisects the line F_2F_1 on the x-axis. Take any point $P(x, y)$ on the ellipse. Complete Fig. 21. Let $OF_2 = OF_1 = c$ so that $a > c$, otherwise F_2 and F_1 fall outside the ellipse. Let $F_2P = r_2$, $F_1P = r_1$. In the right-angled triangles PF_2M and F_1PM

$$(F_2P)^2 = (PM)^2 + (F_2M)^2, \text{ and } (F_1P)^2 = (PM)^2 + (MF_1)^2;$$

or
$$r_2^2 = y^2 + (x - c)^2, \text{ and } r_1^2 = y^2 + (x + c)^2. \quad . \quad . \quad (1)$$

Add and also subtract equations (1), when

$$r_1^2 + r_2^2 = 2(y^2 + x^2 + c^2); \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$r_1^2 - r_2^2 = 4cx; \text{ or } (r_1 + r_2)(r_1 - r_2) = 4cx. \quad . \quad . \quad (3)$$

By definition of the ellipse $r_1 + r_2 = 2a$ and substituting this value of $r_1 + r_2$ in (3), we get

$$r_1 - r_2 = 2cx/a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Adding and subtracting $r_1 + r_2 = 2a$ from this equation we obtain

$$r_1 = a + cx/a; \ r_2 = cx/a - a. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Squaring equations (5), and substituting in (2), we get

$$a^4 + c^2x^2 = a^2(y^2 + x^2 + c^2);$$

or
$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2) \quad . \quad . \quad . \quad . \quad (6)$$

If we take a point P_1 on the y-axis such that $P_1F_1 = P_1F_2 = a$ and let $OP_1 = b$;

in the right-angled triangles P_1OF_1 and P_1OF_2, $b^2 + c^2 = a^2$, or $a^2 - c^2 = b^2$. Substituting b^2 for $a^2 - c^2$, in (6), and dividing by a^2b^2, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is the required equation of the ellipse.

Obviously, if $a = b$, this equation passes into that of a circle (page 75). The circle is thus a special case of the ellipse.

The line P_2P_4, in Fig. 21, is called the *major axis*, P_1P_2 the *minor axis*, their respective lengths being $2a$ and $2b$; the magnitudes a and b are the *semi-axes*; each of the points P_1, P_2, P_3, P_4, is a *vertex*.

(2) To *find the shape of the ellipse*. From equation (7) it follows that

$$y = \pm b\sqrt{1 - x^2/a^2}, \text{ and } x = \pm a\sqrt{1 - y^2/b^2}. \quad . \quad (8)$$

1st. Since y^2 must be positive, $x^2/a^2 \not> 1$, that is to say, x cannot be numerically greater than a. Similarly it can be shown that y cannot be numerically greater than b.

2nd. Every positive value of x gives two equal and opposite values of y, that is to say, there are two points at equal distances perpendicularly above and below the x-axis. The ellipse is therefore symmetrical with respect to the x-axis. In the same way, it can be shown that the ellipse is symmetrical with respect to the y-axis.

3rd. If the value of x increases from zero until $x = \pm a$, then $y = 0$, and these two values of x furnish two points on the x-axis. If x now increases until $x > a$, there is no real corresponding value of y^2. Hence the ellipse lies in a strip bounded by the limits $x = \pm a$; similarly it can be shown that the ellipse is bounded by the limits $y = \pm b$.

The ellipse is not a very important curve. Its chief application will be discussed later on.

EXAMPLES.—(1) Let the point $P(x, y)$ move on a curve so that the position of the point, at any moment, is given by the equations, $x = a\cos t$ and $y = b\sin t$; required the path described by the moving point.

Square and add, since $\cos^2 t + \sin^2 t$ is unity (page 499),

$$x^2/a^2 + y^2/b^2 = 1.$$

The point therefore moves on an ellipse.

(2) The general equation of the second degree,

$$ax^2 + bxy + cy^2 + fx + gy + h = 0,$$

represents an ellipse when $b^2 - 4ac$ is negative, or else it represents a circle, point, or an imaginary curve. For instance, $x^2 - 2xy + 2y^2 - x + y + 2 = 0$ Here $b^2 - 4ac = -4$. Plot the curve to this equation.

(3) Find the relation between the constants a, b, m, c in the equations $x^2/a^2 + y^2/b^2 = 1$ and $y = mx + c$, in order that the line may cut the ellipse in two, one, or no point. For the first $a^2m^2 + b^2 - c^2$ must be greater than zero, for the second, equal to zero, for the third, less than zero.

§ 37. The Hyperbola and its Equation.

The hyperbola is a curve such that the difference of the distance of any point on the curve from two fixed points is always the same.

Let the point $P(x, y)$ (Fig. 22) move so that the difference of its distances from two fixed points F, F' (called the *foci*) is equal to $2a$. Then $PF' - PF = 2a$.

(1) *To find the equation of the hyperbola.* Let xOx', yOy' be rectangular axes intersecting at a point midway between F' and F so that $OF = OF' = c$, and let $FP = r$, $F'P = r'$. In the right-angled triangles FPM and $F'PM$,

$$(FP)^2 = (PM)^2 + (MF)^2, \text{ and } (F'P)^2 = (PM)^2 + (F_1M)^2;$$

or

$$r^2 = y^2 + (x - c)^2, \text{ and } r'^2 = y^2 + (x + c)^2. \quad . \quad . \quad . \quad (1)$$

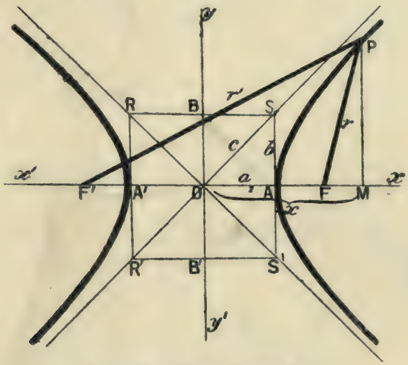


FIG. 22.—The Hyperbola.

Adding and subtracting equations (1) we get

$$r^2 + r'^2 = 2(y^2 + x^2 + c^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$r'^2 - r^2 = 4cx, \text{ or } (r' - r)(r + r') = 4cx. \quad . \quad . \quad . \quad (3)$$

By definition of the hyperbola, $r' - r = 2a$. Substituting this result in (3) we get

$$r + r' = 2cx/a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By addition and subtraction of $r' - r = 2a$, from (4),

$$r' = a + cx/a; \ r = -a + cx/a. \quad . \quad . \quad . \quad (5)$$

Squaring equations (5), and substituting in (2), we get

$$a^4 + c^2x^2 = a^2(y^2 + x^2 + c^2);$$

or

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2). \quad . \quad . \quad . \quad (6)$$

By Euclid, i., 20 (Cor.), the difference between any two sides of a triangle is smaller than the third side, and therefore

$$2a < 2c, \text{ or } a < c.$$

Let * $$c^2 = a^2 + b^2, \text{ or } a^2 - c^2 = - b^2.$$
Substituting this value of b^2 in (6) and dividing out, we obtain the equation to the hyperbola in the simple form
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The xOx'-axis is called the *transverse* or *real axes* of the hyperbola; yOy' the *conjugate* or *imaginary axes;* the points A, A' are the *vertices* of the hyperbolas, a is the *real semi-axis*, b the *imaginary semi-axis*.

(2) *To find the shape of the hyperbola.* From (7)
$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}, \text{ and } x = \pm \frac{a}{b}\sqrt{y^2 + b^2}. \quad . \quad (8)$$

1st. Since y^2 must be positive, $x^2 \not< a^2$, or x cannot be numerically less than a. No limit with respect to y can be inferred from equation (8).

2nd. For every positive value of x, there are two values of y differing only in sign. Hence these two points are perpendicular above and below the x-axis, that is to say, the hyperbola is symmetrical with respect to the x-axis. There are also two equal and opposite values of x for all values of y. The hyperbola is thus symmetrical with respect to the y-axis.

3rd. If the value of x changes from zero until $x = \pm a$, then $y = 0$, and these two values of x furnish two points on the x-axis. If $x > a$, there are two equal and opposite values of y. Similarly for every value of y there are two equal and opposite values of x. The curve is thus symmetrical with respect to both axes, and lies beyond the limits $x = \pm a$.

EXAMPLES.—(1) In the general equation of the second degree,
$$ax^2 + bxy + cy^2 + fx + gy + h = 0,$$
if $b^2 - 4ac$ is positive, the equation either represents an hyperbola or two intersecting straight lines. *E.g.*, $x^2 - 6xy + y^2 + 2x + 2y + 2 = 0$. Plot this curve.

(2) The equation to the hyperbola whose origin is at its vertex is
$$a^2y^2 = 2ab^2x + b^2x^2.$$
Substitute $x + a$ for x in the regular equation. Note that y does not change.

Before describing the properties of this interesting curve I shall discuss some fundamental properties of curves in general.

---

* With A or A' as centre, and radius equal to $OF = c$, describe a circle cutting the y-axis at the points B, B'. Complete Fig. 22. Hence $c^2 = a^2 + b^2$. *Note.*—For greater clearness in the drawing, F and F' have been removed a little further from the curve than their real position.

§ 38. A Study of Curves.

(1) The **tangent** to a curve (footnote, page 77). Let OPQ be a curve situated, with respect to a pair of coordinate axes, as shown in Fig. 23. Let P and Q be two points on the curve, PM and QN their perpendiculars on Ox. Let PR be drawn parallel to MN. Join PQ and produce QP to cut Ox produced at T. If Q is supposed to travel along the curve until it approaches infinitely near to the point P, the chord PQ becomes, at the limit, the tangent to the given curve at P. Hence the limit of the ratio RQ/PR is a tangent to the given curve. Or

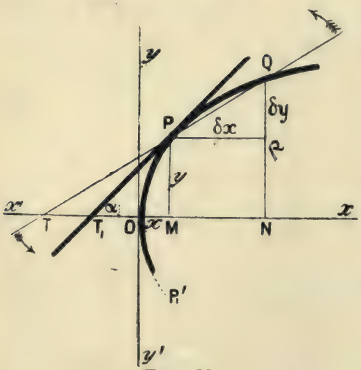


FIG. 23.

$$Lt\frac{RQ}{PR} = Lt \tan RPQ = Lt \tan NTP \quad . \quad . \quad (1)$$

Take any point $P(x, y)$ on the curve POP' represented by the equation

$$y = f(x). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the coordinates of P be increased by any arbitrary increments δx and δy, so that the particle occupies a new position,

$$Q(x + \delta x, y + \delta y).$$

$$OM = x;\ PR = MN = \delta x;\ ON = x + \delta x$$

$$MP = y;\ QR = \delta y;\ QN = QR + RN = QR + PM = y + \delta y.$$

Since the point Q also lies on the curve,

$$y + \delta y = f(x + \delta x), \quad . \quad . \quad . \quad . \quad (3)$$

and

$$RQ = \delta y = f(x + \delta x) - f(x).$$

$$\therefore\ Lt\frac{RQ}{PR} = Lt_{\delta x = 0}\frac{f(x + \delta x) - f(x)}{\delta x} = Lt\frac{(incr.\ y)}{(incr.\ x)}.$$

$$= Lt_{\delta x = 0}\frac{\delta y}{\delta x};$$

or

$$dy/dx = \tan \alpha \quad . \quad . \quad . \quad . \quad (4)$$

This is a most important result. *In words, the tangent of the angle made by the slope of any part of the curve towards the x-axis is the first differential coefficient of the ordinate of the curve with respect to the abscissa.* This rule applies to any curve.

EXAMPLES.—(1) Find the tangent of the angle (α) made by any point $P(x, y)$ on the parabolic curve. In other words, it is required to find a straight line which has the same slope as the curve has which passes through the point $P(x, y)$. Since

$$y^2 = 4ax\,;\ dy/dx = 2a/y = \tan\alpha.$$

If the tangent of the angle were to have any particular value, this value would have to be substituted in place of dy/dx. For instance, let the tangent to the point $P(x, y)$ make an angle of 45°. Since tan 45 = unity,

$$2a/y = \tan\alpha = 1, \therefore y = 2a.$$

Substituting in the original equation $y^2 = 4ax$, we get

$$x = a,$$

that is to say, the required tangent passes through the extremity of the ordinate perpendicular on the focus. If the tangent had to be parallel to the x-axis, tan 0 being zero, dy/dx is equated to zero; while if the tangent had to be perpendicular to the x-axis, since $\tan 90° = \infty$, $dy/dx = \infty$.

(2) Required the direction of motion at any moment of a point moving according to the equation, $y = a\cos 2\pi\left(\frac{t}{T} + \epsilon\right)$. The tangent at any time t has the slope, $-\frac{2\pi a}{T}\sin 2\pi\left(\frac{t}{T} + \epsilon\right)$.

(2) *Equation of the tangent line.* Let TP (Fig. 24) be a tangent to the curve at the point $P(x_1, y_1)$. Let $OM = x_1$, $PM = y_1$. Let $y = mx$, be the equation of the tangent line, and $y_1 = f(x_1)$ the equation of the curve. The condition that a straight line may pass through the point $P(x_1, y_1)$, is (equation (14), page 72) that

$$y - y_1 = m(x - x_1) \quad . \quad . \quad . \quad (5)$$

where m is the tangent of the angle which the line $y = mx$ makes with the x-axis. But we have just seen that this angle is equal to the first differential coefficient of the ordinate of the curve; hence by substitution

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1), \quad . \quad . \quad . \quad (6)$$

which is the required equation of the tangent to a curve at a point whose coordinates are x_1, y_1.

EXAMPLE.—Required the equation of the tangent to a parabola. Since

$$y_1^2 = 4ax_1,\ dy_1/dx_1 = 2a/y_1.$$

Substituting in (5) and rearranging terms,

$$yy_1 - y_1^2 = 2a(x - x_1).$$

Substituting for y_1^2, we get

$$yy_1 = 2a(x + x_1)$$

as the equation for the tangent line of a parabola. If $x = 0$, $\tan\alpha = \infty$, and the tangent is perpendicular to the x-axis and touches the y-axis. To get the point of intersection of the tangent with the x-axis put $y = 0$, then $x = -x_1$. The vertex of the parabola therefore bisects the x-axis between the point of intersection of the tangent and of the ordinate of the point of tangency.

(3) *Equation of the normal line.* A **normal** line is a perpendicular to the tangent at a given point on the curve, drawn to the x-axis.

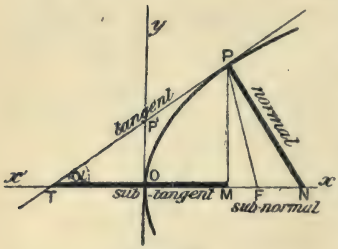


FIG. 24.

Let NP be normal to the curve (Fig. 24) at the point $P(x_1, y_1)$. Let $y \doteq mx$, be the equation to the normal line, $y_1 = f(x_1)$, the equation to the curve. The condition that any line may be perpendicular to the tangent line TP, is that $m' = -1/m$ (equation (17), page 73). From (5)

$$y - y_1 = -\frac{1}{m}(x - x_1),$$

or

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1) \quad . \quad . \quad . \quad (7)$$

(4) *Equation of the subnormal.* The **subnormal** of any curve is that part of the x-axis lying between the point of intersection of the normal and the ordinate drawn from the same point on the curve.

Let MN be the subnormal of the curve shown in figure 24, then

$$MN = x - x_1.$$

The corresponding value for the length of the subnormal is, from (7),

$$MN = x - x_1 = y_1\frac{dy_1}{dx_1} \quad . \quad . \quad . \quad (8)$$

the normal being drawn from the point $P(x_1, y_1)$.

(5) *Equation of the subtangent.* The **subtangent** of any curve is that part of the x-axis lying between the points of intersection of the tangent and the ordinate drawn from the given point.

Let MT (Fig. 24) be the subtangent, then

$$x_1 - x = MT.$$

Putting $y = 0$ in equation (6), the corresponding value for the length of the subtangent is

$$MT = x_1 - x \doteq y_1 dx_1/dy_1. \quad . \quad . \quad . \quad (9)$$

(6) *The length of the tangent and of the normal.* The length of the tangent can be readily found by substituting the values PM and TM in the equation for the hypotenuse of a right-angled

triangle TPM (Euclid, i., 47); and in the same way the length of the normal is obtained from the known values of MN and PM already deduced.

EXAMPLES.—(1) Find the length of the subtangent and subnormal lines in the parabola, $y^2 = 4ax$. Since

$$y\frac{dy}{dx} = 2a,$$

the subtangent is $2x$, the subnormal $2a$.

(2) Show that the subtangent of the curve pv = constant, is equal to $-p$.

§ 39. The Parabola (*resumed*).

Returning now to the special curves, let $P(x, y)$ be a point on the parabolic curve (Fig. 25) referred to the coordinate axes Ox, Oy; PT a tangent at the point P. Let F be the focus of the parabola $y^2 = 4ax$. Join PF. Draw KP parallel to Ox. Join KT. Then $KPFT$ is a rhombus (Euclid, i., 34), for it has been shown that the vertex of the parabola A bisects the subtangent (Example (1) above). Hence,

$$TA = AM, \text{ and, by definition, } OA = AF,$$
$$\therefore TO = FM, \text{ and } KP = TF,$$

$\therefore$ the sides KT and PF are parallel, and by definition of the parabola, $KP = PF$; $\therefore$ the two triangles KPT and PTF are equal in all respects, and (Euclid, i., 5) the angle KPT = angle TPF, that is to say, *the tangent to the parabola at any given point bisects the angle made by the focal radius and the perpendicular dropped on to the directrix from the given point.*

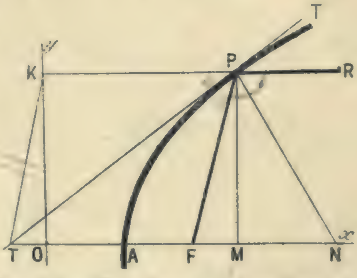


FIG. 25.—The Focus of the Parabola (after Nernst and Schönflies).

In Fig. 25, the angle TPF = angle TPK = opposite angle RPT' (Euclid, i., 15). But, by construction, the angles TPN and NPT' are right angles; take away the equal angles TPF and RPT' and the angle FPN is equal to the angle NPR, that is to say, *the normal at any point on the parabola, bisects the angle enclosed by the focal radius and a line drawn through the given point, parallel to the x-axis.*

This property is of great importance in physics. All light rays falling parallel to the principal (or x-) axis on to a parabolic mirror are reflected at the focus F, and conversely all light rays proceeding from the focus are reflected parallel to the x-axis. Hence the employment of parabolic mirrors for illumination and other purposes. In some of Marconi's recent experiments on wireless telegraphy, electrical radiations were directed by means of parabolic reflectors. Hertz, in his classical researches on the identity of light and electro-magnetic waves, employed large parabolic mirrors, in the focus of which a "generator," or "receiver" of the electrical oscillations was placed. See the translation of Hertz's *Electric Waves*, by Jones (1893), page 172.

§ 40. The Ellipse (*resumed*).

The deduction of an *equation for the tangent* at any point on the ellipse is a simple exercise on equation (6), page 83,

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating the equation of the ellipse, $x_1^2/a^2 + y_1^2/b^2 = 1$, we obtain

$$\frac{dy_1}{dx_1} = -\frac{b^2}{a^2}\frac{x_1}{y_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

substituting this value of dy_1/dx_1 in (1)

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

Multiply by y_1 and divide through by b^2, rearrange terms and combine the result with the equation to the ellipse. The result is the tangent to any point on the ellipse,

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where x_1, y_1 are coordinates of any point on the curve and x, y the coordinates of the tangent.

Now the tangent cuts the x-axis at a point where $y = 0$. Hence

$$xx_1 = a^2, \text{ or, } x = a^2/x_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In Fig. 26 let PT be a tangent to the ellipse, PN the normal. From (4)

$$F_1T = x + c = a^2/x_1 + c, \ FT = x - c = a^2/x_1 - c,$$

and

$$\frac{F_1T}{FT} = \frac{a^2 + cx_1}{a^2 - cx_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From equations (5), page 78,

$$\frac{PF_1}{PF} = \frac{r_1}{r} = \frac{a^2 + cx_1}{a^2 - cx_1} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (5) and (6), therefore,

$$F_1T : FT = F_1P : FP.$$

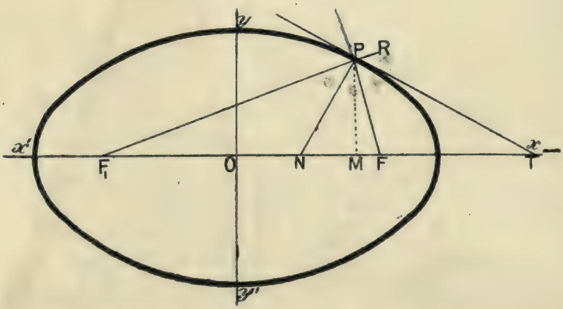


FIG. 26.—The Foci of the Ellipse (after Nernst and Schönflies).

By Euclid, vi., A: "If, in any triangle, the segments of the base produced have to one another the same ratio as the remaining sides of the triangle, the straight line drawn from the vertex to the point of section bisects the external angle". Hence in the triangle FPF_1, the tangent bisects the external angle FPR, and the normal bisects the angle FPF_1. That is to say, *the normal at any point on the ellipse bisects the angle enclosed by the focal radii; and the tangent at any point on the ellipse bisects the exterior angle formed by the focal radii.*

This property accounts for the fact that if F_1P be a ray of light emitted by some source F_1, the tangent at P represents the reflecting surface at that point, and the normal to the tangent is therefore normal to the surface of incidence. From a well-known optical law, "the angles of incidence and reflection are equal," and since F_1PN is equal to NPF when PF is the reflected ray, all rays emitted from one focus of the ellipse are reflected and concentrated at the other focus. This well-known physical phenomena applies to light, heat, sound and electro-magnetic waves.

The questions raised in §§ 39 and 40 are treated in any textbook on physical or geometrical optics.

§ 41. The Hyperbola (*resumed*).

(1) The *equation of the tangent* at any point $P(x_1, y_1)$ on the hyperbolic curve, is obtained, as before, by substituting the first differential coefficient of the tangent to the curve in the equation

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By differentiation of the equation $x^2/a^2 - y^2/b^2 = 1$, we get,

$$\frac{dy_1}{dx_1} = \frac{b^2}{a^2}\frac{x_1}{y_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore y - y_1 = \frac{b^2}{a^2}\frac{x}{y}(x - x_1).$$

Multiply this equation by y, divide by b^2, rearrange the terms and combine the result with (2). We thus find that the tangent to any point on the hyperbola has the equation

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At the point of intersection of the tangent with the x-axis, $y = 0$ and the corresponding value of x is

$$x = a^2/x_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the same as for the ellipse.

The limiting position of the tangent to the point on the hyperbola at an infinite distance away is interesting. Such a tangent is called an **asymptote.**

From (4) if x_1 is infinitely great, $x = 0$, and the tangent then passes through the origin.

(2) *To find the angle which the asymptote makes with the x-axis* we must determine the relative value of

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1.$$

Multiply both sides by b^2/x^2, and

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}.$$

If x be made infinitely great the desired ratio is

$$Lt_{x=\infty}\frac{y^2}{x^2}=\frac{b^2}{a^2}, \therefore Lt_{x=\infty}\frac{y}{x}=\frac{b}{a}.$$

Substituting this in equation (2) above we get, by writing x for x_1, y for y_1,

$$\frac{dy}{dx}=\tan\alpha \text{ (say)} = \frac{a}{b}\cdot\frac{b^2}{a^2}=\frac{b}{a} \qquad (5)$$

If we now construct the rectangle $RSS'R'$ (Fig. 22, page 80) with sides parallel to the axis and cut off $OA = OA' = a$, $OB = OB' = b$, the diagonal in the first quadrant and the asymptote, having the same relation to the two axes, are identical. Since the x- and y-axes are symmetrical, it follows that these conditions hold for every quadrant. See page 137 for a further discussion on the properties of asymptotes.

§ 42. The Rectangular or Equilateral Hyperbola.

If we put $a = b$ in the standard equation to the hyperbola, the result is a special case of the hyperbola for which

$$x^2 - y^2 = a^2, \qquad (6)$$

and from equation (2), page 496,

$$\tan\alpha = 1 = \tan 45°,$$

that is to say, each asymptote makes an angle of 45° with the x-, or y-axes. In other words, the asymptotes bisect the coordinate axes. This special form of the hyperbola is called an *equilateral* or *rectangular hyperbola.* It follows directly that the asymptotes are at right angles to each other. The asymptotes may, therefore, serve as a pair of rectangular coordinate axes. This is a very important property of the rectangular hyperbola.

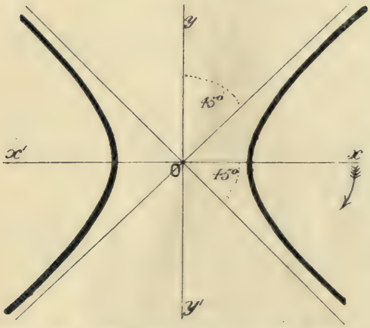


FIG. 27.—The Rectangular Hyperbola.

To find the equation of a rectangular hyperbola referred to its asymptotes as coordinate axes. This problem is most simply treated as one of transformation of coordinates from one system (page 74) to another inclined at an angle of 45° to the old set, but having the same origin.

On page 75 it was shown that if the coordinates of a point $P(x, y)$ referred to one set of axes, become x_1 and y_1 when referred to a new set, the equations of transformation are

$$x = x_1 \cos\alpha - y_1 \sin\alpha; \ y = x_1 \sin\alpha + y_1 \cos\alpha \qquad (7)$$

As shown in Fig. 27, the old axes yOx have to be rotated through an angle of $-45°$.*

But $\sin(-45°) = -1/\sqrt{2}$; $\cos(-45°) = 1/\sqrt{2}$ (page 497). Hence from (7) above

$$x = x_1/\sqrt{2} + y_1/\sqrt{2}\,;\ y = -x_1/\sqrt{2} + y_1/\sqrt{2} \quad . \qquad (8)$$

By addition and subtraction

$$x - y = 2x_1/\sqrt{2}\,;\ x + y = 2y_1/\sqrt{2} \quad . \quad . \qquad (9)$$

If $P(x, y)$ be any point on the rectangular hyperbola

$$x^2 - y^2 = a^2, \text{ or } (x - y)(x + y) = a^2.$$

Substituting these values of $(x - y)$ and $(x + y)$ in (9), we get

$$2x_1y_1 = a^2,$$

or, writing the constant term $a^2/2 = \kappa$, x for x_1, y for y_1,

$$xy = \text{constant} = \kappa \quad . \quad . \quad . \qquad \mathbf{(10)}$$

What is true of any point on the hyperbola is true for all points, that is to say, equation (10) is the equation for a rectangular hyperbola referred to its asymptotes as coordinate axes.

From (10) $y = \kappa/x$, and it follows that as y becomes smaller, x increases in magnitude. When $y = 0$, $x = \infty$, that is to say, the x-axis touches the hyperbola an infinite distance away. The same thing may be said of the y-axis.

§ 43. Illustrations of Hyperbolic Curves.

(1) *The graphical representation of the gas equation,*

$$pv = R\theta,$$

furnishes a rectangular hyperbola when θ is fixed or constant. The law as set forth in the above equation shows that the volume of a gas (v) varies inversely as the pressure (p) and directly as the temperature (θ). For any assigned value of θ, we can obtain a series of values of p and v. For the sake of simplicity, let the constant $R = 1$. Then if

$$\theta = 1 \begin{cases} p = 0{\cdot}1, & 0{\cdot}5, \quad 1{\cdot}0, \quad 5{\cdot}0, \quad 10{\cdot}0, \ldots; \\ v = 10{\cdot}0, & 2{\cdot}0, \quad 1{\cdot}0, \quad 0{\cdot}2, \quad 0{\cdot}1, \ldots \end{cases}$$

$$\theta = {\cdot}5 \begin{cases} p = 0{\cdot}1, & 0{\cdot}5, \quad 1{\cdot}0, \quad 5{\cdot}0, \quad 10{\cdot}0, \ldots; \\ v = 5{\cdot}0, & 1{\cdot}0, \quad 0{\cdot}5, \quad 0{\cdot}1, \quad 0{\cdot}05, \ldots \text{ etc.} \end{cases}$$

The "curves" of constant temperature obtained by plotting

* The trignometrical convention with regard to sign is that if a point rotates in the opposite direction to the hands of a watch, it is positive, if in the same direction, negative.

these numbers are called **isothermals.** Each isothermal (*i.e.*, curve at constant temperature) is a rectangular hyperbola obtained from the equation

$$pv = R\theta = \text{constant}, \quad . \quad . \quad . \quad (11)$$

similar to (10) above.

A series of isothermal curves, obtained by putting θ successively equal to θ_1, θ_2, θ_3 . . . and plotting the corresponding values of p and v, is shown in Fig. 28.

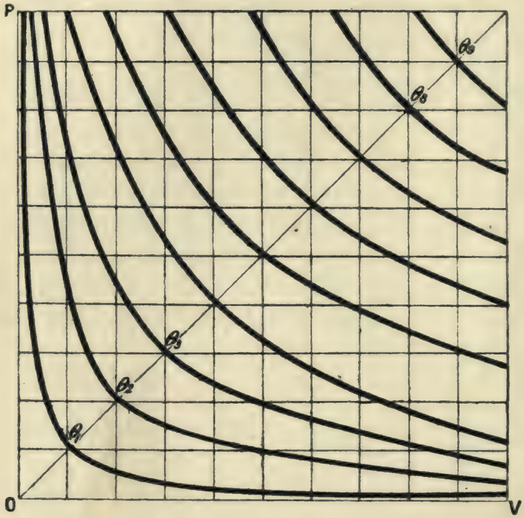


FIG. 28.—Isothermal pv-curves.

We could have obtained a series of curves from the variables p and θ, or v and θ, according as we assume v or p to be constant. If v be constant, the resulting curves are called **isometric lines,** or **isochores**; if p be constant the curves are **isopiestic lines,** or **isobars.** For van der Waals' equation, see page 398.

(2) *Exposure formula for a thermometer stem.* When a thermometer stem is not exposed to the same temperature as the bulb, the mercury in the exposed stem is cooled, and a small correction must be made for the consequent contraction of the mercury exposed in the stem. If x denotes the difference between the temperature registered by the thermometer and the temperature of the exposed stem, y the number of thermometer divisions exposed to the cooler atmosphere, then the correct temperature θ

can be obtained by the so-called *exposure formula of a thermometer*, namely,

$$\theta = 0{\cdot}00016xy, \quad . \quad . \quad . \quad . \quad (12)$$

which has the same form as equation (10). By assuming a series of suitable values for θ (say 0·1 . . .) and plotting the result for pairs of values of x and y, curves are obtained for use in the laboratory. These curves allow the required correction to be seen at a glance (see Ramsay, *Chemical Theory*, 1893, 11).

(3) *Dissociation isotherm.* Gaseous molecules under certain conditions dissociate into simpler parts. Nitrogen peroxide, for instance, dissociates into simpler molecules, thus:

$$N_2O_4 = 2NO_2.$$

Iodine at a high temperature does the same thing, I_2 becoming $2I$. In solution a similar series of phenomena occur, KCl becoming $K + Cl$, and so on. Let x denote the number of molecules of an

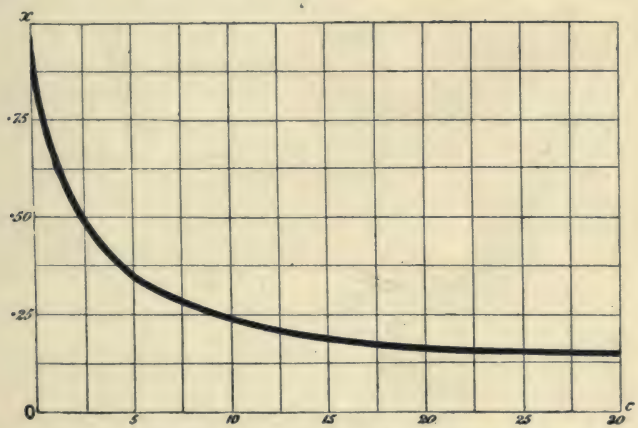


FIG. 29.—Dissociation Isotherm (after Nernst and Schönflies).

acid or salt which dissociates into two parts called *ions*, $(1 - x)$ the number of molecules of the acid, or salt resisting dissociation, c the quantity of substance contained in unit volume, that is the concentration of the solution. Nernst has shown that at constant temperature

$$K = c\frac{x^2}{1 - x} \quad . \quad . \quad . \quad . \quad (13)$$

where K is the so-called dissociation constant whose meaning is obtained by putting $x = 0{\cdot}5$. In this case $K = \frac{1}{2}c$, that is to say,

K is equal to half the quantity of acid or salt in solution when half of the acid or salt is dissociated.

Putting $K = 1$ we can obtain a series of corresponding values of c and x. For example, if

$$x = \cdot 16, \quad 0\cdot 25, \quad 0\cdot 5, \quad 0\cdot 75, \quad 0\cdot 94 \; \ldots;$$

then $$c = 32, \quad 12, \quad 2, \quad 0\cdot 44, \quad 0\cdot 07 \; \ldots$$

It thus appears that when the concentration is very great, the amount of dissociation is very small, and *vice versâ*, when the concentration is small the amount of dissociation is very great. Complete dissociation can perhaps never be obtained. The graphic curve (Fig. 29), called the **dissociation isotherm** (Nernst), is asymptotic towards the two axes, but when drawn on a small scale the curve appears to cut the ordinate axis.

(4) The *volume elasticity* of a substance is defined as the ratio of any small increase of pressure to the diminution of volume per unit volume of substance. If the temperature is kept constant during the change, we have **isothermal elasticity,** while if the change takes place without gain or loss of heat, **adiabatic elasticity.** If unit volume of gas (v) changes by an amount dv for an increase of pressure dp, the elasticity (E) is

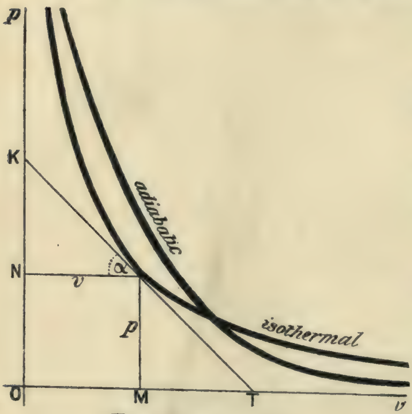


FIG. 30.—*pv*-curves.

$$E = - dp \Big/ \frac{dv}{v} = - v\frac{dp}{dv} \quad . \quad . \quad . \quad (14)$$

A similar equation is obtained by differentiating Boyle's law for an isothermal change of state,

$$pv = \text{constant}, \quad . \quad . \quad . \quad . \quad (15)$$

or

$$p = - v\frac{dp}{dv}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

an equation identical with that deduced for the definition of volume elasticity. Equation (16) is that of a rectangular hyperbola referred to its asymptotes as axes.

Let $P(p, v)$ (Fig. 30) be a point on the curve pv = constant. From the construction of figure 30, the triangles KNP and PMT

are equal and similar (Euclid, i., 26). See example (2) page 85, and note that KN is the vertical subtangent equivalent to $-p$.

$$KN = -NP \tan \alpha = -v \tan KPN,$$

$$= -v\frac{dp}{dv},$$

that is to say, *the isothermal elasticity of a gas in any assigned condition, is numerically equal to the vertical subtangent of the curve* corresponding to the substance in the given state.

But since in the rectangular hyperbola $KN = PM$, *the isothermal elasticity of a gas* is equal to the pressure (16). The adiabatic elasticity of a gas may be obtained by a similar method to that used for equation (14). If the gas be subject to an adiabatic change of pressure and volume it is known that

$$pv^\gamma = \text{constant} = C \text{ (say)}, \quad . \quad . \quad . \quad (17)$$

or

$$\log p + \gamma \log v = \log C.$$

Differentiating and arranging terms

$$E_Q{}^* = -v\frac{dp}{dv} = \gamma p, \quad . \quad . \quad . \quad (18)$$

in other words the adiabatic elasticity of a gas is γ times the pressure. A similar construction for the adiabatic curve furnishes

$$KN : MP = KP : PT$$

$$= \gamma : 1,$$

that is to say, the tangent to an adiabatic curve is divided at the point of contact in the ratio $\gamma : 1$.

§ 44. Polar Coordinates.

Instead of representing the position of a point in a plane in terms of its horizontal and vertical distances along two standard lines of reference, it is sometimes more convenient to define the position of the point by a length and a direction. For example, in Fig. 31 let the point O be fixed, and Ox a straight line through O. Then, the position of any other point P will be completely defined *if* (1) the length OP and (2) the angle OP makes with Ox, are known. These are called the polar coordinates of P, the first is called the **radius vector**, the latter the **vectorial angle**. The radius vector is

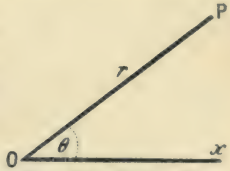


FIG. 31.—Polar Coordinates.

* From other considerations, E_Q is usually written E_ϕ.

generally represented by the symbol r, the vectorial angle by θ, and P is called the point $P(r, \theta)$, O is called the *pole* and Ox the *initial line.* As in trignometry, the vectorial angle is measured by supposing the angle θ has been swept out by a revolving line moving from a position coincident with Ox to OP. It is positive if the direction of revolution is contrawise to the motion of the hands of a clock.

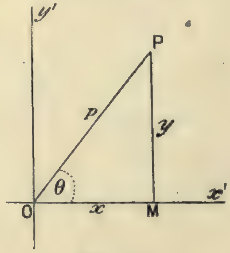


FIG. 32.

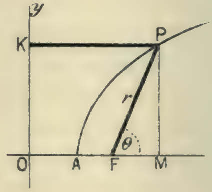


FIG. 33.

To change from polar to rectangular coordinates and vice versâ. In Fig. 32, let (r, θ) be the polar coordinates of the point $P(x, y)$. Let the angle $x'OP = \theta$.

First, to pass from polar to Cartesian coordinates.

$$\sin\theta = \frac{MP}{OP} = \frac{y}{r}; \ \cos\theta = \frac{OM}{OP} = \frac{x}{r};$$

$$\therefore \quad y = r\sin\theta \text{ and } x = r\cos\theta, \quad . \quad . \quad . \quad (1)$$

which determines x and y, when r and θ are known.

Second, to pass from Cartesian to polar coordinates. In the same figure

$$\tan\theta = \frac{MP}{OM} = \frac{y}{x};$$

$$r^2 = (OP)^2 = (OM)^2 + (MP)^2 = x^2 + y^2;$$

$$\therefore \theta = \tan^{-1}\frac{y}{x}; \ r = \pm\sqrt{x^2 + y^2} \quad . \quad . \quad . \quad (2)$$

which determines θ and r, when x and y are known. The sign of r is ambiguous, but, by taking any particular solution for θ, the preceding remarks will show which sign is to be taken.

Just as in Cartesian coordinates an equation between r and θ may represent one or more curves. The graph may be obtained by assigning convenient values to θ (say 0°, 30°, 45°, 60°, 90° . . .) and determining the corresponding value of r from the equation.

EXAMPLE.—Show that the polar equations of the hyperbola and ellipse are respectively $\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}$ and $\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$.

NOTE.—The parabola, ellipse and hyperbola are sometimes defined as curves such that the ratio of the distance of any point on the curve from a fixed line and from a fixed point, is constant. The ratio is called the *eccentricity*, and is denoted by the letter e, the fixed point is called the *focus*, the fixed line, the *directrix*.

In Fig. 33 let OK be directrix, F the focus, AP any curve, PK is a perpendicular from P on to the directrix, PM is perpendicular from P on to OF produced. A is vertex of curve. Then if

$$e = \frac{FP}{PK} = \text{constant} = 1, \text{ the curve is parabolic,}$$

$$e = \frac{FP}{PK} = \text{constant} < 1, \text{ the curve is elliptical,}$$

$$e = \frac{FP}{PK} = \text{constant} > 1, \text{ the curve is hyperbolic.}$$

These definitions ultimately furnish equations for the hyperbola, ellipse and parabola similar to those adopted above. Let $FP = r$, $OF = p$, then from these definitions

$$PK = OF + FM = p + r\cos\theta,$$

$$e = \frac{r}{p + r\cos\theta}; \text{ or } r = \frac{pe}{1 - e\cos\theta}, \quad . \quad . \quad . \quad (3)$$

which is true whether curve be hyperbolic, elliptical or parabolic.

§ 45. Logarithmic or Equiangular Spiral.

Equations to the spiral curves are considerably simplified by the use of polar coordinates. For instance, the curve for the logarithmic spiral, though somewhat complex in Cartesian coordinates, is represented in polar coordinates by the simple equation

$$r = a^\theta,$$

where a may have any constant value. Hence

$$\log r = \theta \log a.$$

Let C_1, C, c' . . . (Fig. 34) be a series of points on the spiral corresponding to the angles $\theta_1, \theta_2, \ldots$, then $r_1, r_2, \ldots$ will represent the corresponding radii vectores, so that

$$\log r_1 = \theta_1 \log a;\ \log r_2 = \theta_2 \log a \ \ldots$$

Since $\log a$ is constant, say equal to k,

$$\log\frac{r_1}{r_2} = (\theta_1 - \theta_2)k,$$

that is, the logarithm of the ratio of the distance of any two points on the curve from the pole is proportional to the angle between them. If r_1 and r_2 lie on the same straight line, then

$$\theta - \theta_2 = 2\pi = 360°,$$

π being the symbol used in trignometry to denote 180°,

$$\therefore \log\frac{r_1}{r_2} = 2k\pi.$$

Similarly, it can be shown that if r_3, r_4 . . . lie on the same straight line, the logarithm of the ratio of r_1 to r_3, r_4 . . . is given by $4k\pi$, $6k\pi$. . . . This is true for any straight line passing through O, that is to say, the spiral is made up of an infinite number of turns which extend inwards and outwards without limit.

If the radii vectores OC_1, OD, OE . . . OC, Od . . . be taken to represent the number of vibrations of a sounding body in a given time, the angles C_1OD, DOE . . . may be taken as a measure of the interval between the

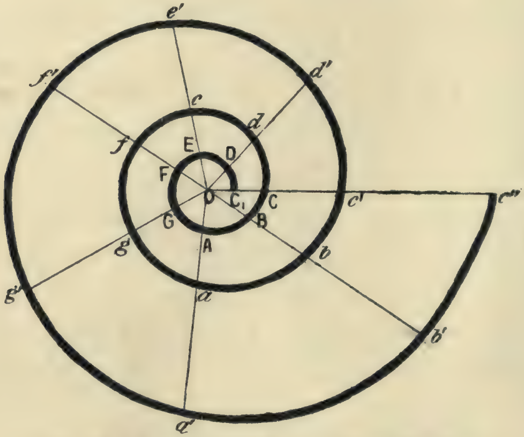


FIG. 34.—Logarithmic Spiral (after Donkin).

tones produced by these vibrations. A point travelling along the curve will then represent a tone continuously rising in pitch, and the curve, passing successively through the same line produced, represents the passage of the tone through successive octaves. The geometrical periodicity of the curve is a graphical representation of the periodicity perceived by the ear when a tone continuously rises in pitch.

In the above diagram the angles C_1OD, DOE . . . represent the intervals in the diatonic scale. The intervals

C_1 to D, F to G, A to B are major seconds, each 61° 10′ 22″;
D to E, G to A are minor seconds, each 54° 43′ 16″;
E to F, B to C are diatonic semitones, each 33° 31′ 11″
(Donkin's *Acoustics*, page 26).

This diagram may also be used to illustrate the Newlands-Mendeleeff law of octaves, by arranging the elements along the curve in the order of their atomic weights.

EXAMPLES.—(1) Plot Archimedes' spiral, $r = a\theta$. . . . . (4)
(2) Plot the hyperbolic spiral, $r\theta = a$, . . . (5)

§ 46. Trilinear Coordinates and Triangular Diagrams.

Another method of representing the position of a point in a plane is to refer it to its *perpendicular* distance from the sides of a triangle called the *triangle of reference*. The perpendicular distances of the point from the sides are called **trilinear coordinates.** In the equilateral triangle ABC (Fig. 35), let the perpendicular distance of the vertex A from the base BC be denoted by 100 units, and let P be any point within the triangle whose trilinear coordinates are Pa, Pb, Pc, then

$$Pa + Pb + Pc = 100.$$

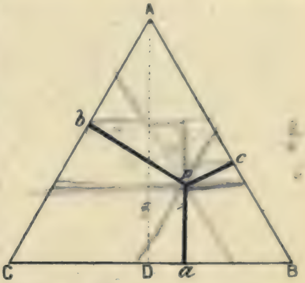


FIG. 35.—Trilinear Coordinates.

This property has been extensively used in the graphic representation of the composition of certain ternary alloys, and mixtures of salts. Each vertex is supposed to represent one constituent of the mixture. Any point within the triangle corresponds to that mixture whose percentage composition is represented by the trilinear coordinates of that point. Any point on a side of the triangle represents a binary mixture. Fig. 36 shows the melting points of ternary mixtures of isomorphous carbonates of barium, strontium and calcium. Such a diagram is sometimes called a *surface of fusibility*. A mixture melting at 670° may have the composition represented by any point on the isothermal curve marked 670°, and so on for the other isothermal curves.

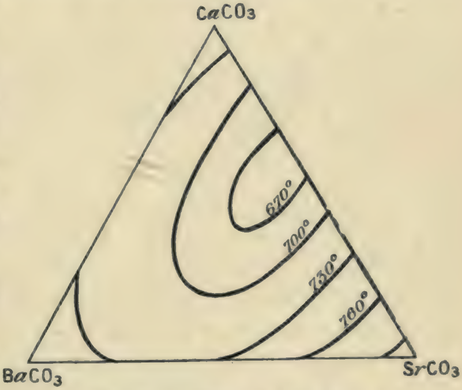


FIG. 36.—Surface of Fusibility.

In a similar way the composition of quaternary mixtures has been graphically represented by the perpendicular distance of a point from the four sides of a square.

Roozeboom, Bancroft and others have used triangular diagrams with lines ruled parallel to each side as shown in Fig. 37. Suppose we have a mixture of three salts, A, B, C, such that the three vertices of the triangle ABC represent phases * containing 100 °/₀ of each component. The composition of any binary mixture is given by a point on the boundary lines of the triangle, while the composition of any ternary mixture is represented by some point inside the triangle.

The position of any point inside the triangle is read directly from the coordinates *parallel* to the sides of the triangle. For instance, the composition of a mixture represented by the point O is given

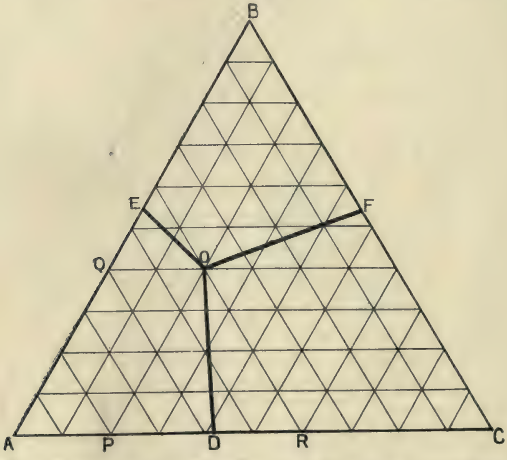


FIG. 37.—Concentration-Temperature diagram (after Bancroft).

by drawing lines from O, *parallel* to the three sides of the triangle OP, OR, OQ. Then start from one corner as origin and measure along the two sides, AP fixes the amount of C, AQ the amount of B, and, by difference, CR determines the amount of A. For the point chosen, therefore $A = 40$, $B = 40$, $C = 20$.

(1) Suppose the substance A melted at 320°, B at 300°, and C at 305°, and that the point D represents an eutectic alloy † melting

* A **phase** is a mass of uniform concentration. The number of phases in a system is the number of masses of different concentration present. For example, at the temperature of melting ice three phases may be present in the H_2O-system, *viz.*, solid ice, liquid water and steam ; if a salt is dissolved in water there is a solution and a vapour phase, if solid salt separates out, another phase appears in the system.

† An *eutectic alloy* is a mixture of two substances in such proportions that the alloy melts at a lower temperature than a mixture of the same two substances in any other proportions. The numbers chosen are based on Guthrie's work (*Philosophical Magazine* [5], **17**, 462, 1884) on the nitrates of potassium (A), lead (B), sodium (C).

at 215°; at E, A and B form an eutectic alloy melting at 207°; at F, B and C form an eutectic alloy melting at 268°.

(2) Along the line DO, the system A and C has a solid phase; along EO, A and B have a solid phase; and along FO, B and C have a solid phase.

(3) At the triple point O, the system A, B and C exists in the three solid, solution and vapour phases at a temperature of 186° (say).

(4) Any point in the area $ADOE$ represents a system comprising solid, solution and vapour of A,—in the solution, the two components B and C are dissolved in A. Any point in the area $CDOF$ represents a system comprising solid, solution and vapour of C,—in the solution, A and B are dissolved in C. Any point in the area $BEOF$ represents a system comprising solid solution and vapour of B,—in the solution, A and C are dissolved in B.

Each apex of the triangle not only represents 100 °/$_o$ of a substance, but also the temperature at which the respective substances A, B, or C melt; D, E, F also represent temperatures at which the respective eutectic alloys melt. It follows, therefore, that the temperature at D is lower than at either A or C. Similarly the temperature at E is lower than at A or B, and at F lower than at either B or C. The temperature, therefore, rises as we pass from one of the points D, E, F to an apex on either side.

For full details the reader is referred to Bancroft's *The Phase Rule*, 1897, p. 147.

§ 47. Orders of Curves.

The order of a curve corresponds with the degree of its equation. The degree of any term may be regarded as the sum of the exponents of the variables it contains; the degree of an equation is that of the highest term in it. For example, the equation $xy + x + b^3y = 0$, is of the second degree if b is constant; the equation $x^3 + xy = 0$, is of the third degree; $x^2yz^3 + ax = 0$, is of the sixth degree, and so on.

1st. **A line of the first order** is represented by the general equation of the first degree

$$ax + by + c = 0 \quad . \quad . \quad . \quad . \quad (1)$$

This equation is that of a *straight line* only.

2nd. **A line of the second order** is represented by the general equation of the second degree between two variables, namely,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad (2)$$

This equation includes, as particular cases, every possible form of equation in which no term contains x and y as factors more than twice. The term $2hxy$ can be made to disappear by changing the direction of the rectangular axes, and the terms containing $2gx$ and $2fy$ can be made to disappear by changing the origin of the coordinate axes. Every equation of the second degree can be made to assume one of the forms

$$ax^2 + by^2 = q, \qquad (3)$$

or

$$y^2 = px \qquad (4)$$

The first can be made to represent a *circle*,* *ellipse*, or *hyperbola*; the second a *parabola*. Hence *every equation of the second degree between two variables includes four species of curves—circle, ellipse, parabola and hyperbola.*

It must be here pointed out that if two equations of the first degree with all their terms collected on one side be multiplied together we obtain an equation of the second degree which is satisfied by a quantity which satisfies either of the two original equations. *An equation of the second degree may thus represent two straight lines*, as well as one of the above species of curves. The same thing applies for equations of the third and higher degrees.

The condition that the general equation of the second degree may represent two straight lines. Rearranging the terms of equation (2) in descending powers of y, we obtain

$$by^2 + 2(hx + f)y + ax^2 + 2gx + c = 0.$$

This may be solved like a quadratic in y with the result that

$$y = \pm \sqrt{\{(h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)\}}/b - (hx + f)/b \qquad (5)$$

an expression analogous to the tangent form of the equation to a straight line,

$$y = mx + b.$$

The two solutions in equation (5) can only represent straight lines if the quantity under the root sign can be extracted as a simple rational expression in x, that is to say, if

$$(h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)$$

is a perfect square (see page 388). This condition is satisfied when

$$(hf - bg)^2 = (h^2 - ab)(f^2 - bc).$$

Multiplying out and dividing by b (when b is not zero)

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \qquad (6)$$

If $b = 0$ and a is not zero, we obtain the same result by solving for x.

If $a = b = 0$, resolve the original expression into factors and

$$(x + f/h)(y + g/h) = 0,$$

provided $c/2h = fg/h^2$, or $2fgh - ch^2 = 0$. This agrees with equation (6). Under

---

* The circle may be regarded as an ellipse with major and minor axes equal.

these circumstances equation (2) represents two straight lines respectively parallel to the two axes.

The general equation of the second degree may represent a parabola, ellipse, or hyperbola, according as $h^2 - ab$, *is zero, negative, or positive.*

EXAMPLE.—Show that the equation

$$3x^2 - 3xy - 3y^2 + 5x + 4y + 4 = 0,$$

represents two straight lines as well as an ellipse.

3rd. **A line of the third order** is represented by the general equation of the third degree between two variables

$$ay^3 + by^2x + cyx^2 + fx^3 + \ldots + n = 0 \quad . \quad \textbf{(7)}$$

Sir Isaac Newton has shown that some eighty species of lines are included in this equation; these may be reduced to one of the following four classes:

$$ax^3 + bx^2 + cx + f = xy^2 + gy \quad . \quad . \quad . \quad (8)$$

$$ax^3 + bx^2 + cx + f = xy \quad . \quad . \quad . \quad . \quad (9)$$

$$ax^3 + bx^2 + cx + f = y^2 \quad . \quad . \quad . \quad . \quad (10)$$

$$ax^3 + bx^2 + cx + f = y \quad . \quad . \quad . \quad . \quad (11)$$

The last (equation 11) includes the *cubical parabola* $y^3 = ax$.

EXAMPLES.—The student will gain more information by plotting all these curves on squared paper, than by reading pages of descriptive matter. Use table of cubes, page 517.

4th. **A line of the fourth order** is represented by the general equation of the fourth degree between two variables, *viz.*,

$$ay^4 + by^3x + cy^2x^2 + fyx^3 + \ldots + t = 0 \quad . \quad \textbf{(12)}$$

Euler divided these into some 200 species which reduce to 146 classes. At the present time the number of species is said to exceed 5,000.

A **family of curves** is an assemblage of curves defined by one equation of an indeterminate degree. For example, the number of parabolas whose abscissae are proportional to any power of the ordinates is infinite. Their equation is

$$y^n = ax.$$

For the common or quadratic parabola $n = 2$, for the cubic parabola $n = 3$, and for the biquadratic parabola $n = 4$.

The study of curves of higher orders than the third is perhaps more interesting than useful, at least so far as practical work is concerned.

§ 48. Coordinate Geometry in Three Dimensions.—Geometry in Space.

(1) *The graphic representation of functions of three variables.* Methods have been described for representing changes in the state

of a system involving two variable magnitudes, by the locus of a point moving in a plane according to a fixed law defined by the equation of the curve. Such was the pv-diagram described on page 90. There, a series of isothermal curves were obtained, when θ was made constant during a set of corresponding changes of p and v in the equation

$$pv = R\theta,$$

where R is constant.

When any three magnitudes, x, y, z, are made to vary together, we can, by assigning arbitrary values to two of the variables, find corresponding values for the third, and refer the results so obtained to three fixed and intersecting planes called the **coordinate planes.** The lines of inter-section of these planes are the **coordinate axes.** Of the resulting eight quadrants, four of which are shown in Fig. 38, only the first is utilised to any great extent in mathematical physics. This mode of graphic representation is called *geometry in space*, or geometry in three dimensions.

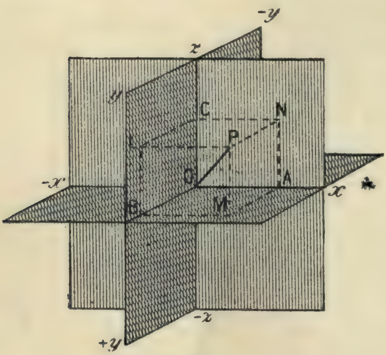


FIG. 38.—Cartesian Coordinates in Three Dimensions.

If we get a series of sets of corresponding values of x, y, z in the equation

$$x + y = z,$$

and refer them to coordinate axes in three dimensions, as described below, the result is a *plane* or *surface*. If one of the variables remains constant, the resulting figure is a *line*. A surface may, therefore, be considered to be the locus of a line moving in space.

The position of the point P with reference to the three coordinate planes xOy, xOz, yOz (Fig. 38) is obtained by dropping perpendiculars PL, PM, PN from the given point on to the three planes. Complete the parallelopiped, as shown in Fig. 38. Let OP be a diagonal. Then $LP = OA$, $NP = OB$, $MP = OC$.

To find the point whose coordinates OA, OB, OC are given. Draw three planes through A, B, C parallel respectively to the coordinate planes; the point of intersection of the three planes, namely P, will be the required point.

If the coordinates of P, parallel to Ox, Oy, Oz, are respectively x, y and z, then P is said to be the point x, y, z. A similar convention with regard to the sign is used as in analytical geometry of two dimensions, with the additional convention that y is positive when in front of O; negative, when behind. It is necessary that the reader shall have a clear idea of spatial geometry in working many physical problems.

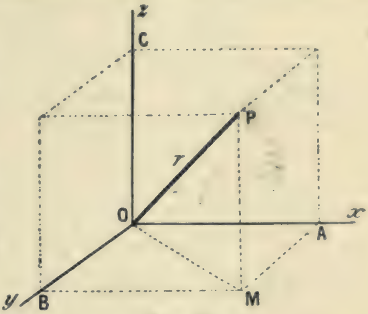


FIG. 39.

(2) *To find the distance of a point from the origin in terms of the rectangular coordinates of that point.* In Fig. 39, let Ox, Oy, Oz be three rectangular axes, $P(x, y, z)$ the given point such that $PM = z$, $MA = y$, $OA = x$. It is required to find the distance $OP = r$, say. From the construction (rectangular coordinates)

$$OP^2 = OM^2 + PM^2, \text{ or } r^2 = OM^2 + z^2,$$

but $$OM^2 = MA^2 + OA^2 = x^2 + y^2.$$

$$\therefore \quad r^2 = x^2 + y^2 + z^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the angle $POx = \alpha$; $POy = \beta$; $POz = \gamma$, then

$$x = r\cos\alpha;\ y = r\cos\beta;\ z = r\cos\gamma \quad . \quad . \quad (2)$$

These equations are true wherever the point P may lie, and therefore the signs of x, y, z are always the same as those of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively. Substituting these values in (1), and dividing through by r^2, we get the following relation between the three angles:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad (3)$$

These cosines are called the **direction cosines**, and are usually symbolised by the letters l, m, n. Thus (3) becomes

$$l^2 + m^2 + n^2 = 1 \quad . \quad (4)$$

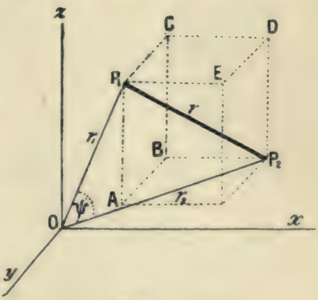


FIG. 40.

(3) *To find the distance between two points in terms of their rectangular coordinates.* Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be the given points, it is required to find the distance P_1P_2 in terms of the coordinates of the points P_1 and P_2. Draw planes through P_1

and P_2 parallel to the coordinate planes so as to form the parallelopiped $ABCDE$. By the construction (Fig. 40), the angle P_1EP_2 is a right angle. Hence

$$P_1P_2^2 = P_1E^2 + P_2E^2 = P_1E^2 + DE^2 + P_2D^2.$$

But P_1E is evidently the difference of the distance of the foot of the perpendiculars from P_1 and P_2 on the x-axis, or $P_1E = x_2 - x_1$. Similarly $DE = y_2 - y_1$, $P_2D = z_2 - z_1$. Hence

$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 . \qquad (5)$$

(4) *To find the angle between two straight lines whose direction cosines are given.* In the preceding diagram (Fig. 40) join OP_1 and OP_2. Let ψ be the angle between these two lines. In the triangle P_2OP_1 (formula 47, page 500) if $OP_1 = r_1$, $OP_2 = r_2$, $P_1P_2 = r$,

$$r^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \psi.$$

Rearranging terms and substituting

$$r_1^2 = x_1^2 + y_1^2 + z_1^2 ; \ r_2^2 = x_2^2 + y_2^2 + z_2^2,$$

we get

$$\cos \psi = (x_1x_2 + y_1y_2 + z_1z_2)/r_1r_2.$$

Substituting, as in (2),

$$x_1 = r_1 \cos \alpha_1 ; \ x_2 = r_2 \cos \alpha_2 ; \ y_2 = r_2 \cos \beta_2 \ . \ . \ .$$

$$\cos \psi = \cos \alpha_1 . \cos \alpha_2 + \cos \beta_1 . \cos \beta_2 + \cos \gamma_1 . \cos \gamma_2 \qquad (6)$$

or, from (4),

$$\cos \psi = l_1l_2 + m_1m_2 + n_1n_2, \qquad (7)$$

which represents the angle between two straight lines whose direction cosines are known.

If the lines are perpendicular, $\cos \psi = \cos 90° = 0$. Hence

$$\cos \alpha_1 . \cos \alpha_2 + \cos \beta_1 . \cos \beta_2 + \cos \gamma_1 . \cos \gamma_2 = 0 \qquad (8)$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad (9)$$

If the vectors r_1, r_2 (page 93) are known, multiply (6) by r_1r_2, and, remembering that

$$r_1 \cos \alpha_1 = x_1 ; \ r_2 \cos \alpha_2 = x_2 ; \ r_2 \cos \gamma_2 = z_2, \text{ etc.},$$

we may write a preceding result:

$$r_1r_2 \cos \psi = x_1x_2 + y_1y_2 + z_1z_2, \qquad (10)$$

and when the lines are perpendicular,

$$x_1x_2 + y_1y_2 + z_1z_2 = 0 \qquad (11)$$

(5) *Projection.* If a perpendicular be dropped from a given point upon a given plane the point where the perpendicular touches the plane is the *projection of the point P* upon that plane. For instance, in Fig. 38, the projection of the point P on the plane xOy is M, on the plane xOz is N, and on the plane yOz is L. Similarly, the projection of the point P upon the lines Ox, Oy, Oz is at A, B and C respectively.

In the same way the *projection of any curve* on a given plane is obtained by projecting every point in the curve on to the plane. The plane, which contains all the perpendiculars drawn from the

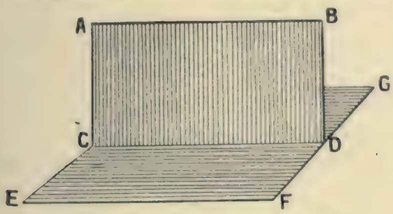


FIG. 41.—Projecting Plane.

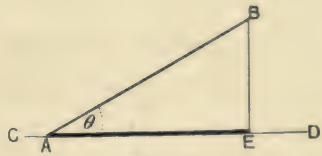


FIG. 42.

different points of the given curve, is called the *projecting plane*. In Fig. 41, CD is the projection of AB on the plane EFG; $ABCD$ is the projecting plane.

EXAMPLES.—(1) The projection of any given line on an intersecting line is equal to the product of the length of the given line into the cosine of the angle of intersection. In Fig. 42, the projection of AB on CD is AE, but $AE = AB \cos\theta$.

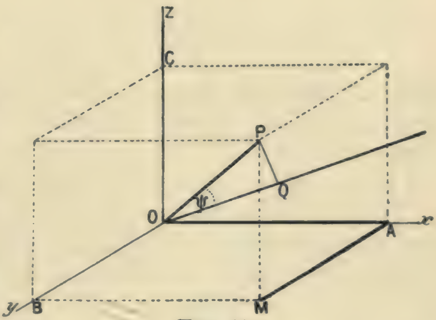


FIG. 43.

(2) In Fig. 43, show that the projection of OP on OQ is the algebraic sum of the projections of OA, AM, MP, taken in this order, on OQ. Hence, if $OA = x$, $OB = AM = y$, $OC = PM = z$ and $OP = r$, from (6)

$$r \cos\psi = x \cos\alpha + y \cos\beta + z \cos\gamma \quad . \quad . \quad . \quad (12)$$

(6) *To find the equation of a plane surface in rectangular coordinates.* Let ABC (Fig. 44) be the given plane whose equation is to be determined. Let the given plane cut the coordinate axes at points A, B, C, such that $OA = a$, $OB = b$, $OC = c$. From any point $P(x, y, z)$ drop the perpendicular

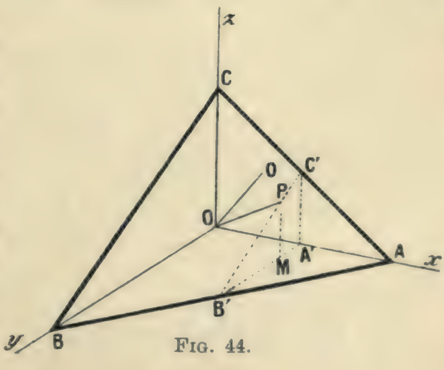


FIG. 44.

PM on to the yOx plane. Then $OA' = x$, $MA' = y$ and $PM = z$. It is required to find an equation connecting the coordinates with the intercepts a, b, c. From the similar triangles AOB, $AA'B'$,

$$OA : OB = A'A : A'B'; \text{ or } a : b = a - x : A'B',$$

$$\therefore A'B' = b - \frac{bx}{a}; \text{ similarly, } MB' = b - y - \frac{bx}{a}.$$

Again, from the similar triangles COB, $C'A'B'$, PMB',

$$CO : OB = PM : MB'; \text{ or } c : b = z : b - y - \frac{bx}{a},$$

$$\therefore bz = cb - cy - \frac{bcx}{a}.$$

Divide through by bc and rearrange for the required:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad . \quad . \quad . \quad (13)$$

an equation very similar in form to that developed on page 69.

We may write this equation in its most general form,

$$Ax + By + Cz + D = 0, \quad . \quad . \quad . \quad (14)$$

which is the most general equation of the first degree between three variables. Equation (14) is the **general equation of a plane surface.** It is easily converted into (13) by substituting $Aa + D = 0$, $Bb + D = 0$, $Cc + D = 0$.

If $OQ = r$ (Fig. 44) be a perpendicular on the plane ABC, the projection of OP on OQ is equal to the sum of the projections of OM, PM, MA' on OQ (see example (2), page 105). Hence

$$x \cos \alpha + y \cos \beta + z \cos \gamma = r \quad . \quad . \quad (15)$$

from (14)

$$\cos^2\alpha : \cos^2\beta : \cos^2\gamma = A^2 : B^2 : C^2;$$

componendo,*

$$(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) : \cos^2\alpha = A^2 + B^2 + C^2 : A^2.$$

But by (3), the term in brackets is unity,

$$\therefore \cos \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}}; \ \cos \beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}};$$

$$\cos \gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \quad . \quad . \quad . \quad (16)$$

---

* If a, b, c and d are proportionals,

$a : b = c : d$

$b : a = d : c$ . . . . . . (*invertendo*)

$a : c = b : d$ . . . . . . (*alternando*)

$a + b : b = c + d : d$ . . . . . (*componendo*)

$a - b : b = c - d : d$ . . . . . (*dividendo*)

$a : a - b = c : c - d$ . . . . . (*convertendo*)

$a \pm b : a \mp b = c \pm d : c \mp d$ . . . . . (*componendo et dividendo*).

(See any elementary text-book on algebra.)

Dividing equation (14) through with $\sqrt{A^2 + B^2 + C^2}$, we get, from (16),

$$x \cos \alpha + y \cos \beta + z \cos \gamma = - \frac{D}{\sqrt{A^2 + B^2 + C^2}}, \qquad (17)$$

where $-D / \sqrt{(A^2 + B^2 + C^2)}$ represents the distance of the plane from the origin.

If ABC (Fig. 44) represents the face, or plane of a crystal, the intercepts a, b, c on the x-, y- and z-axes are called the *parameters* of that plane. The parameters in crystallography are usually expressed in terms of certain axial lengths assumed unity. If $OA = a$, $OB = b$, $OC = c$, any other plane, whose intercepts on the x-, y- and z-axes are respectively p, q and r, is defined by the ratios

$$\frac{a}{p} : \frac{b}{q} : \frac{c}{r}.$$

These quotients are called the parameters of the new plane. The reciprocals of the parameters are the *indices* of a crystal face. The several systems of crystallographic notation which determine the position of the faces of a crystal with reference to the axes of the crystal are based on the use of parameters and indices.

(7) *To find the equation of a straight line in rectangular coordinates.* A line in space is represented in mathematics by two equations. If we consider a straight line in space to be formed by the intersection of two projecting planes, formed, in turn, by the projection of the given line on two coordinate planes, the equation to the straight line evidently consists of two parts. Let ab, $a'b'$ be the projection of the given line AB on the xOz and the yOz planes, then (Fig. 45)

$$x = mz + c\,; \quad y = m'z + c'. \qquad (18)$$

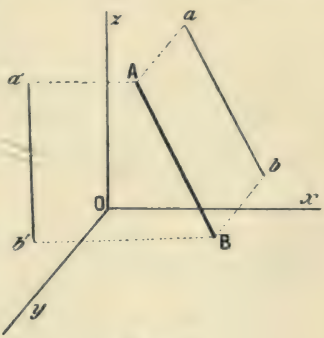


FIG. 45.

Here m represents the tangent of the angle which the projection of the given line on the xOz plane makes with the x-axis; m' the tangent of the angle made by the line projected on the yOz plane with the y-axis; c is the distance intercepted by the projection of the given line on the x-axis; c', a similar intersection on the y-axis.

If we eliminate z from equation (18),

$$y - c' = \frac{m'}{m}(x - c) \quad . \quad . \quad . \qquad (19)$$

represents the projection of the given line on the xOy plane.

(8) *Surfaces of revolution.* A surface is assumed to have been generated by the motion of a line in space. If the line rotates round a fixed axis, the rotating surface is called a *surface of revolution.* Thus, a *sphere* may be formed by the rotation of a circle about a diameter; a *cylinder* may be formed by the rotation of a rectangle about one of its sides as axis; a *cone* may be generated by the revolution of a triangle about its axis; an *ellipsoid* of revolution, by the rotation of an ellipse about its major or minor axes; a *paraboloid*, by the rotation of a parabola about its axis. If an hyperbola rotates about its transverse axis, two *hyperboloids* will be formed by the revolution of both branches of the hyperbola. On the other hand, only one hyperboloid is formed by rotating the hyperbolas about their conjugate axes. In the former case, the *hyperboloid* is said to be *of two sheets*, in the latter, *of one sheet.*

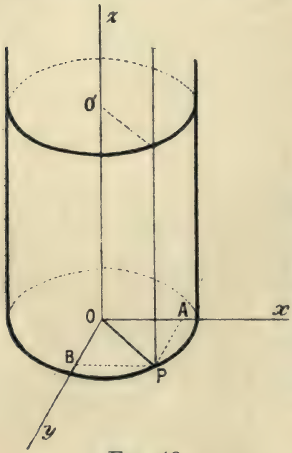


FIG. 46.

(9) *To find the equation of the surface of a right cylinder.* Let one side of a rectangle rotate about Oz as axis. Any point on the outer edge will describe the circumference of a circle. If $P(x, y, z)$ (Fig. 46) be any point on the surface, r the radius of the cylinder, then the required equation is

$$r^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \qquad (20)$$

The equation to a right cylinder is thus independent of z. This means that z may have any value whatever assigned to it.

EXAMPLES.—(1) Show that the equation of a right cone is $x^2 + y^2 - z^2\tan^2\phi = 0$, where ϕ represents half the angle at the apex of the cone. (Origin of axes is at apex of cone.)

(2) The equation of a sphere is $x^2 + y^2 + z^2 = r^2$. Prove this.

(10) *To find the equation of the tangent plane* at any point of a surface.* Let $P(x_1, y_1, z_1)$ be any point on the curved surface,

$$z = f(x, y). \quad . \quad . \quad . \quad . \qquad (21)$$

The tangent plane through the point $P(x_1, y_1, z_1)$ will be determined

* The *tangent plane* of a surface is a plane which touches the surface at a point, or a line.

when two *linear* tangents through this point have been determined. For the sake of simplicity, consider the linear tangents parallel to the xz- and the zy-planes. As an exercise after (6), page 83, the reader will be able to show that these two tangent lines have equations,

$$z - z_1 = \frac{dz_1}{dx_1}(x - x_1)\,;\ y = y_1\,; \qquad . \quad . \qquad (22)$$

$$z - z_1 = \frac{dz_1}{dy_1}(y - y_1)\,;\ x = x_1, \qquad . \quad . \qquad (23)$$

where the partial derivatives, dz_1/dx_1, dz_1/dy_1, obviously represent the trignometrical tangents of the lines of intersection of the tangent plane with the coordinate planes xz and zy respectively. Hence, the equation to this plane is,

$$z - z_1 = \frac{dz_1}{dx_1}(x - x_1) + \frac{dz_1}{dy_1}(y - y_1). \qquad . \qquad (24)$$

Example.—Prove that the tangent plane to the surface

$$u = f(x, y, z) = 0,$$

is

$$\frac{du}{dx}(x - x_1) + \frac{du}{dy}(y - y_1) + \frac{du}{dz}(z - z_1) = 0. \qquad . \qquad (25)$$

(11) *Polar coordinates.* Instead of referring the point to its Cartesian coordinates in three dimensions, we may use polar coordinates. In Fig. 47, let P be the given point whose rectangular coordinates are x, y, z; and whose polar coordinates are r, θ, ϕ, as shown in the figure.

(i) *To pass from rectangular to polar coordinates.*

$$\left.\begin{aligned} x &= OA = OM\cos\phi = r\sin\theta\,.\cos\phi \\ y &= AM = OM\sin\phi = r\sin\theta\,.\sin\phi \\ z &= PM = r\cos\theta. \end{aligned}\right\} \quad (26)$$

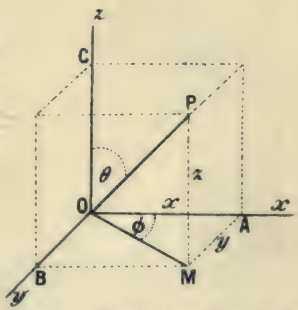


Fig. 47.—Polar Coordinates in Three Dimensions.

(ii) *To pass from polar to rectangular coordinates.*

$$\left.\begin{aligned} r &= \sqrt{(x^2 + y^2 + z^2)} \\ \theta &= \tan^{-1}\frac{\sqrt{(x^2 + y^2)}}{z} \\ \phi &= \tan^{-1}\frac{y}{x}. \end{aligned}\right\} \quad . \quad . \quad . \qquad (27)$$

§ 49. Orders of Surfaces.

Just as an equation of the first degree between two variables represents a straight line of the first order, so does an equation of the first degree between three variables represent a **surface of the first order.** Such an equation in its most general form is

$$Ax + By + Cz + D = 0,$$

the equation to a plane.

An equation of the second degree between three variables represents a **surface of the second order.** The most general equation of the second degree between three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + \ldots + N = 0.$$

All plane sections of surfaces of the second order are either circular, parabolic, hyperbolic, or elliptical, and comprised under the generic word *conicoids*, of which *spheroids*, *paraboloids*, *hyperboloids* and *ellipsoids* are special cases.

A surface of the second degree may be formed by plotting from the gas equation

$$f(p, v, \theta) = 0; \text{ or } pv = R\theta,$$

by causing p, v and θ to vary simultaneously. The surface *pabv* (Fig. 48) was developed in this way.

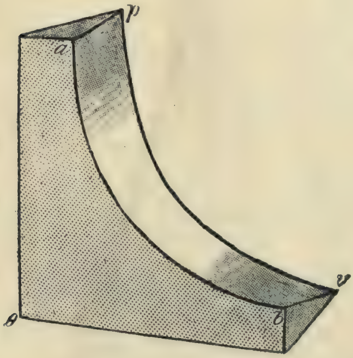


FIG. 48.—*pvθ*-surface.

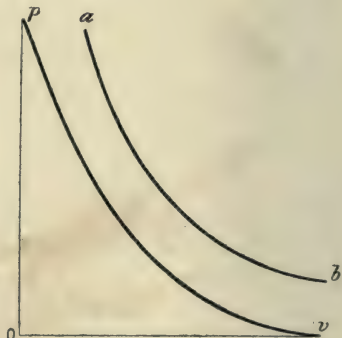


FIG. 49.

Since any section cut perpendicular to the θ-axis is a rectangular hyperbola, the surface is an hyperboloid. The isothermals θ, θ_2, θ_3, . . . (Fig. 28, page 90) may be looked upon as plane sections cut perpendicular to the θ-axis at points corresponding to θ_1, θ_2, . . . , and then projected upon the pv-plane. In Fig. 49, the curves corresponding to pv and ab have been so projected.

If a sufficient number of such projections were available, the characteristic equation, $f(p, v, \theta) = 0$, would be solved completely.

As a general rule, the surface generated by three variables is not so simple as the one represented by a gas obeying the simple laws of Boyle and Gay Lussac.

van der Waals "ψ" surfaces are developed by using the variables ψ, x, v, where ψ denotes the thermodynamic potential at constant volume $(U - \theta\phi)$; x the composition of the substance; v the volume of the system under investigation. The "ψ" surface is analogous to, but not identical with, *pabv* in the above figure. Full particulars are given in van der Waals' classic, *Die Continuität des gasförmigen und flüssigen Zustandes*, Theil II.

The so-called **thermodynamic surfaces of Gibbs** are obtained in the same way from the variables v, U, ϕ (or volume, internal energy and entropy) of a given system. They are described with some detail in Preston's *Theory of Heat*, page 685, or better still, Le Chatelier's *Équilibre des systèmes chimiques par J. Willard Gibbs*, p. 98 (see also page 343).

§ 50. Periodic or Harmonic Motion.

Let P (Fig. 50) be a point which starts to move from a position of rest with a uniform velocity on the perimeter of a circle. Let xOx', yOy' be coordinate axes about the centre O. Let $P_1, P_2 \ldots$ be positions occupied by the point after the elapse of intervals of time $t_1, t_2 \ldots$ From P_1 drop the perpendicular M_1P_1 on to the x-axis. Remembering that if the direction of M_1P_1, $M_2P_2 \ldots$ be positive, that of M_3P_3, M_4P_4 is negative, and the motion of OP as P revolves about the centre O in the opposite direction to the hands of a clock is conventionally reckoned positive, then

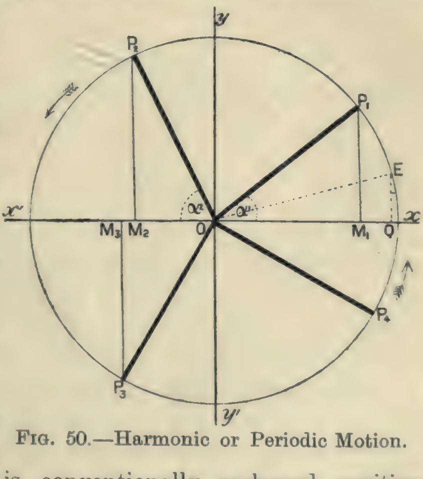


FIG. 50.—Harmonic or Periodic Motion.

$$\sin \alpha_1 = \frac{+ M_1P_1}{+ OP_1}; \ \sin \alpha_2 = \frac{+ M_2P_2}{+ OP_2}; \ \sin \alpha_3 = \frac{- M_3P_3}{+ OP_3}; \ \sin \alpha_4 = \frac{- M_4P_4}{+ OP_4}.$$

Or, if the circle have unit radius $r = 1$,

$$\sin \alpha_1 = + M_1P_1; \ \sin \alpha_2 = + M_2P_2; \ \sin \alpha_3 = - M_3P_3; \ \sin \alpha_4 = - M_4P_4.$$

If the point continues in motion after the first revolution, this series of changes is repeated again and again.

During the first revolution, if we put $\pi = 180°$, and let θ_1, θ_2, . . . represent certain angles described in the respective quadrants,

$$\theta_1 = \alpha_1; \ \theta_2 = \pi - \alpha_2; \ \theta_3 = \pi + \alpha_3; \ \theta_4 = 2\pi - \alpha_4.$$

During the second revolution,

$$\theta_1 = 2\pi + \alpha_1; \ \theta_2 = 2\pi + (\pi - \alpha_2); \ \theta_3 = 2\pi + (\pi + \alpha_3), \text{ etc.}$$

We may now plot the curve

$$y = \sin \alpha \quad . \quad . \quad . \quad . \quad (1)$$

by giving a series of values $0, \frac{1}{2}\pi, \frac{3}{4}\pi$ . . . to α and finding the corresponding values of y. Thus if

$x = \alpha =$	$0,$	$\frac{1}{2}\pi,$	$\pi,$	$\frac{3}{2}\pi,$	$2\pi,$	$\frac{5}{2}\pi, \ . \ . \ .;$
$y =$	$\sin 0,$	$\sin \frac{1}{2}\pi,$	$\sin \pi,$	$\sin \frac{3}{2}\pi,$	$\sin 2\pi,$	$\sin \frac{5}{2}\pi, \ . \ . \ .;$
$=$	$\sin 0°,$	$\sin 90°,$	$\sin 180°,$	$\sin 270°,$	$\sin 360°,$	$\sin 90°, \ . \ . \ .;$
$=$	$1,$	$0,$	$-1,$	$0,$	$1,$	$0, \ . \ . \ .$

Intermediate values are $\sin \frac{1}{4}\pi = \sin 45° = \cdot 707$, $\sin \frac{3}{4}\pi = - \cdot 707$ . . . The curve so obtained has the wavy or undulatory appearance shown in Fig. 51. It is called the **curve of sines** or **the harmonic curve.**

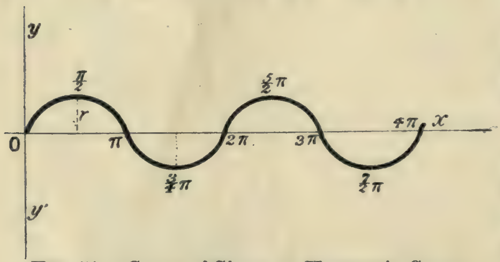


FIG. 51.—Curve of Sines, or Harmonic Curve.

A function whose value recurs at fixed intervals when the variable uniformly increases in magnitude is said to be a **periodic function.** Its mathematical expression is

$$f(t) = f(t + nt) \quad . \quad . \quad . \quad . \quad (2)$$

where n may be any positive or negative integer. In the present case $n = 2\pi$. The motion of the point P is said to be a *simple harmonic motion.* Equation (1) thus represents a simple harmonic motion.

If we are given a particular value of a periodic function of, say, t, we can find an unlimited number of different values of t which satisfy the original function. Thus $2t$, $3t$, $4t$, . . ., all satisfy equation (2).

EXAMPLES.—(1) Show that the graph of $y = \cos\alpha$ has the same form as the sine curve and would be identical with it if the y-axis of the sine curve were shifted a distance of $\frac{1}{2}\pi$ to the right. [Proof: $\sin(\frac{1}{2}\pi + x) = \cos x$, etc.] The physical meaning of this is that a point moving round the perimeter of the circle according to the equation $y = \cos\alpha$ is just $\frac{1}{2}\pi$, or 90° in advance of one moving according to $y = \sin\alpha$.

(2) Illustrate graphically the periodicity of the function $y = \tan\alpha$. (Note the passage through $\pm\infty$.)

Instead of taking a circle of unit radius, let r denote the magnitude of the radius, then

$$y = r\sin\alpha.$$

Since $\sin\alpha$ can never exceed the limits ± 1, the greatest and least values y can assume are $-r$ and $+r$; r is called the *amplitude* of the curve. The velocity of the motion of P determines the rate at which the angle α is described by OP (called the *angular velocity*). Let t denote the time, ω the angular velocity,

$$\frac{d\alpha}{dt} = \omega;\ \text{or}\ \alpha = \omega t,$$

and the time required for a complete revolution is

$$t = 2\pi/\omega,$$

which is called the *periodic value* of α, or *period of oscillation*, or *periodic time*; 2π is the *wave length*. If E (Fig. 50) denotes some arbitrary fixed point such that the periodic time is counted from the instant P passes through E, the angle $xOE = \epsilon$, is called the *epoch*, and the angle described by OP in the time $t = \omega t + \epsilon = \alpha$, or

$$y = r\sin(\omega t + \epsilon)\quad . \qquad (3)$$

Electrical engineers call ϵ the lead or, if negative, the lag of the electric current.

EXAMPLE. — Show that the graph of equation (3) may be represented by a curve of the form shown in Fig. 52.

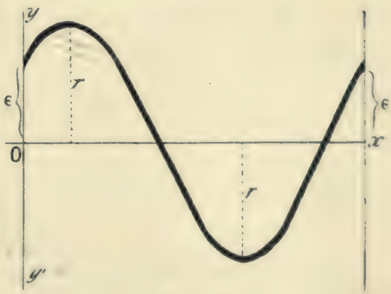


FIG. 52.

The motion of M (Fig. 50), that is to say, the projection of the moving point on the diameter of the circle xOx' is a good illustration of periodic motion, already discussed, page 48. The motion of an oscillating pendulum, of a galvanometer needle, of a tuning fork, the up and down motion of a water wave, the alternating electric current, sound, light, and electromagnetic waves are all periodic motions. Many of the properties of the chemical elements are periodic functions of their atomic weights (*Newlands-Mendeléeff law*). Some interesting phenomena have recently come to light which indicate that chemical action may assume a periodic character.* The evolution of hydrogen gas, when hydrochloric acid acts on one of the allotropic forms of chromium, has recently been studied by W. Ostwald.† He found that if the volume of gas evolved during the action be plotted as ordinate against the time as abscissa, a curve is obtained which shows regularly alternating periods of slow and rapid evolution of hydrogen. The particular form of these "waves" varies with the conditions of the experiment. One of Ostwald's curves is shown in Fig. 53.

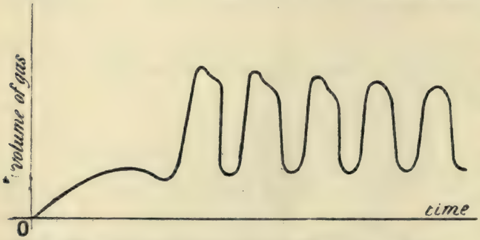


FIG. 53.—Ostwald's Curve of Chemical Action.

Composition of harmonic motions. It is important to remember that two or more simple harmonic motions may be compounded into one. Thus it can be shown that

$$a \sin (qt + \epsilon) + b \cos (qt + \epsilon) = A \sin (qt + \epsilon_1) \quad . \quad (4)$$

where q has the same meaning as ω above. Expand the left-hand side of (4) according to formulae (21) and (22), page 499; rearrange terms to obtain

$$= \sin qt(a \cos \epsilon - b \sin \epsilon) + \cos qt(b \cos \epsilon + a \sin \epsilon);$$
$$= A \sin (qt + \epsilon_1),$$

provided

$$A \cos \epsilon_1 = a \cos \epsilon - b \sin \epsilon; \; A \sin \epsilon_1 = b \cos \epsilon + a \sin \epsilon. \quad (5)$$

Square equations (5) and add

$$A^2 = a^2 + b^2. \quad . \quad . \quad . \quad . \quad (6)$$

* Abney has noticed that if a photographic film be "exposed" for a much longer period than is required it will after a certain interval return to a sensitive condition. Troost and Hautefeuille state that silicon hexachloride (Si_2Cl_6) is stable above 1,000° and below 350°; hydrazine hydrate ($N_2H_4.H_2O$); ozone (O_3), hydrogen selenide (H_2Se); cyanogen (C_2N_2); acetylene (C_2H_2); and nitrogen peroxide (N_2O_4) are said to exhibit similar phenomena; the action of chlorine on platinum, of oxygen on copper and on phosphorus is also said to be similar. Many of these statements, no doubt, arise from a faulty interpretation of experimental work. But the subject certainly merits a closer investigation.

† W. Ostwald, *Zeitschrift für physikalische Chemie*, **35**, 33, 204, 1900; Brauner, *ib.*, **38**, 441, 1901.

Divide equations (5), rearrange terms and show that

$$\tan (\epsilon - \epsilon_1) = -b/a, \quad . \quad . \quad . \quad . \quad (7)$$

from formulae (21) and (22), page 499. When $\epsilon = 0°$,

$$\tan \epsilon_1 = b/a. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equations (6) and (7) are the necessary conditions that (4) may hold good. Give a geometrical interpretation to (4), (6) and (8), by means of figure 54.

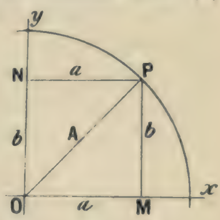


FIG. 54.

EXAMPLES.—(1) Draw the graphs of the two curves,

$$y = a \sin (qt + \epsilon) \text{ and } y_1 = a_1 \sin (qt + \epsilon_1).$$

Compare the result with the graph of

$$y_2 = a \sin (qt + \epsilon) + a_1 \sin (qt + \epsilon_1).$$

(2) Draw the graphs of

$$y_1 = \sin x,\ y_2 = \tfrac{1}{3} \sin x,\ y_3 = \tfrac{1}{5} \sin 5x,\ y = \sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x$$

(see page 363 for this and other examples).

§ 51. Generalised Forces and Coordinates.

When a mass of any substance is subject to some physical change, certain properties (mass, chemical composition) remain fixed and invariable, while other properties (temperature, pressure, volume) vary. When the value these variables assume in any given condition of the substance is known, we are said to have a complete knowledge of the state of the system. These variable properties are not necessarily independent of one another. We have just seen, for instance, that if two of the three variables defining the state of a perfect gas are known, the third variable can be determined from the equation

$$pv = R\theta,$$

where R is a constant. In such a case as this, the third variable is said to be a *dependent variable*, the other two, *independent variables. When the state of any material system can be defined in terms of n independent variables, the system is said to possess n* **degrees of freedom,** *and the n independent variables are called* **generalised coordinates.** For the system just considered $n = 2$, and the system possesses two degrees of freedom.

Again, in order that we may possess a knowledge of some systems, say gaseous nitrogen peroxide, not only must the variables given by the gas equation

$$\phi(p, v, \theta) = 0$$

be known, but also the mass of the N_2O_4 and of the NO_2 present. If these masses be respectively m_1 and m_2, there are five variables to be considered, namely,

$$\phi_1(p, v, \theta, m_1, m_2) = 0,$$

but these are not all independent. The pressure, for instance, may be fixed by assigning values to v, θ, m_1, m_2; p is thus a dependent variable, v, θ, m_1, m_2 are independent variables. Thus

$$p = f(v, \theta, m_1, m_2).$$

We know that the dissociation of N_2O_4 into $2NO_2$ depends on the volume, temperature and amount of NO_2 present in the system under consideration. At ordinary temperatures

$$m_1 = f_1(v, \theta, m_2),$$

and the number of independent variables is reduced to three. In this case the system is said to possess three degrees of freedom.* At temperatures over 135°—138° the system contains NO_2 alone, and behaves as a perfect gas with two degrees of freedom.

In general, if a system contains m dependent and n independent variables, say

$$x_1, x_2, x_3, \ldots x_{n+m}$$

variables, the state of the system can be determined by $m + n$ equations. As in the familiar condition for the solution of simultaneous equations in algebra, n independent equations are required for finding the value of n unknown quantities. But the state of the system is *defined* by the m dependent variables; the remaining n independent variables can therefore be determined from n independent equations.

Let a given system with n degrees of freedom be subject to external forces

$$X_1, X_2, X_3, \ldots X_n,$$

so that no energy enters or leaves the system except in the form of heat or work, and such that the n independent variables are displaced by amounts

$$dx, dx_1, dx_2, \ldots dx_n.$$

Since the amount of work done on or by a system is measured by the product of the force and the displacement (page 182), these

* If a system contains more than three degrees of freedom its state cannot be represented on a single diagram.

external forces X_1X_2 . . . perform a quantity of work dW which depends on the nature of the transformation. Hence

$$dW = X_1dx_1 + X_2dx_2 + \ldots X_ndx_n$$

where the coefficients X_1, X_2, X_3 . . . are called the **generalised forces** acting on the system. Duhem in his great work, *Traité Élémentaire de Mécanique Chimique fondée sur la Thermodynamique*, 4 vols., 1897-99, makes considerable use of generalised forces and generalised coordinates.

CHAPTER III.

FUNCTIONS WITH SINGULAR PROPERTIES.

§ 52. Continuous and Discontinuous Functions.

> "Although a physical law may never admit of a perfectly abrupt change, there is no limit to the approach which it may make to abruptness."—W. STANLEY JEVONS.

THE law of continuity affirms that no change can take place abruptly. The conception involved will have been familiar to the reader from the second section of this work. It was there shown that the amount of substance (x) formed in a given time becomes smaller as the interval of time (t) during which the change occurs is diminished, until finally, when the interval of time approaches zero, the amount of substance formed also approaches zero. In such a case x is not only a function of t, but it is a **continuous function** of t.

The course of such a reaction may be represented by the motion of a point along the curve

$$x = f(t).$$

According to the principle of continuity, in order that the moving point may pass from one end (a) of the curve to the other (b), it must successively assume all values intermediate between a and b, and never move off the curve. This is a characteristic property of continuous functions. Several examples have been considered in preceding chapters. Most natural processes can be represented by continuous functions. Hence the old empiricism: *Natura non agit per saltum.*

The law of continuity, though tacitly implied up to the present, is by no means always true. Even in some of the simplest phenomena exceptions may arise. In a general way, we can divide discontinuous functions into two classes: first, those in which the graph of the function suddenly stops to reappear in some other part of the plane, in other words a "**break**" occurs; second, those in which the graph suddenly changes its direction without exhibiting a break.* Other kinds of discontinuity may occur, but do not commonly arise in physical work. For example, a function is said to be discontinuous when the value of the function $y = f(x)$ becomes infinite for some particular value of x. Such a discontinuity occurs when $x = 0$ in the expression $y = 1/x$. The differential coefficient of this expression,

$$\frac{dy}{dx} = -\frac{1}{x^2},$$

is also discontinuous for $x = 0$. Other examples which may be verified by the reader are $\log x$, when $x = 0$, $\tan x$, when $x = \frac{1}{2}\pi$, . . . The graph for Boyle's equation, pv = constant, is also discontinuous at an infinite distance along either axis.

§ 53. Discontinuity accompanied by "Breaks".

The specific heat—that is to say, the amount of heat required to raise the temperature of one gram of a solid substance one degree—may be a known function of the temperature of the solid. As soon as the substance begins to melt, it absorbs a great amount of heat (latent heat), unaccompanied by any rise of temperature. When the substance has assumed the fluid state of aggregation the specific heat is again a function of the temperature until, at the boiling point, similar phenomena recur. Heat is absorbed unaccompanied by any rise of temperature (latent heat of vaporisation) until the liquid is completely vaporised.

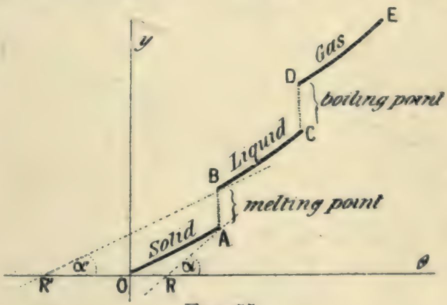


FIG. 55.

If the quantity of heat (Q) supplied be regarded as a function of the temperature (θ), the curve (Fig. 55), represented by the equation

$$y = \phi(x); \text{ or, } Q = \phi(\theta),$$

* Sometimes the word "break" is used indiscriminately for both kinds of discontinuity. It is, indeed, questionable if ever the "break" is real.

is said to be discontinuous between the values $Q = AB$ and CD, and breaks are said to occur in these positions. $f(\theta)$ is therefore a **discontinuous function,** for, if a small quantity of heat be now added to the substance, the temperature does not change in a corresponding way.

The geometrical signification of these phenomena is as follows: For the points A and B, corresponding to one abscissa, there are two, generally different, tangents to the curve, namely, tan α and tan α'. In other words (see page 82),

$$\frac{dQ}{d\theta} = \phi'(\theta) = \tan\alpha = \tan \text{ angle } \theta RA \text{ ;}$$

and
$$\frac{dQ}{d\theta} = \phi'(\theta) = \tan\alpha' = \tan \text{ angle } \theta R'A,$$

that is to say, *a function is discontinuous when the differential coefficient has two distinct values determined by the slope of the tangent to each curve at the point where the discontinuity occurs.*

The physical meaning of the discontinuity in this example, is that the substance may have two values for its specific heat at the melting point, the one corresponding to the solid and the other to the liquid state of aggregation. The tangent of the angle represented by the ratio $dQ/d\theta$ obviously represents the specific heat of the substance. An analogous set of changes occurs at the boiling point.

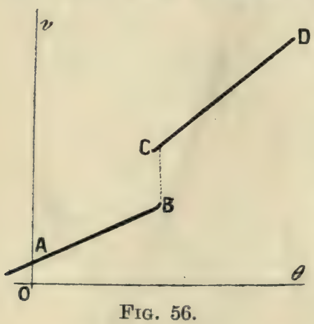


FIG. 56.

Fig. 56 shows the result of plotting the variations in the volume of phosphorus with temperatures in the neighbourhood of its melting point. AB represents the expansion curve of the solid, CD that of the liquid. A break occurs between B and C. Phosphorus at its melting point may thus have two distinct coefficients of expansion, the one corresponding to the solid and the other to the liquid state of aggregation.

§ 54. The Existence of Hydrates in Solution.

The fact (page 100) that an equation of the second (or nth) degree may include not only a single curve of the second (or nth) order, but also two (or n) straight lines, has been used in an ingenious way to indicate the probable existence of certain chemical compounds in solution. The following data are quoted at some length in order to explain an important application of mathematical methods for bringing these obscured lines into prominence:

If p denotes the percentage compositions of various aqueous solutions of ethyl alcohol and s the corresponding specific gravities *in vacuo* at 15° (sp. gr. H_2O at 15° = 9991·6), we have the following table compiled by Mendeléeff:—

p	s	p	s	p	s	p	s
5	9904·1	30	9570·2	55	9067·4	80	8479·8
10	9831·2	35	9484·5	60	8953·8	85	8354·8
15	9768·4	40	9389·6	65	8838·6	90	8225·0
20	9707·9	45	9287·8	70	8714·5	95	8086·9
25	9644·3	50	9179·0	75	8601·4	100	7936·6

It is found empirically that the experimental results are fairly well represented by the equation

$$s = a + bp + cp^2, \quad . \quad . \quad . \quad (1)$$

which is the general expression for a parabolic curve, a, b and c being constants (page 77). By plotting the experimental data the curve shown in Fig. 57 is obtained.

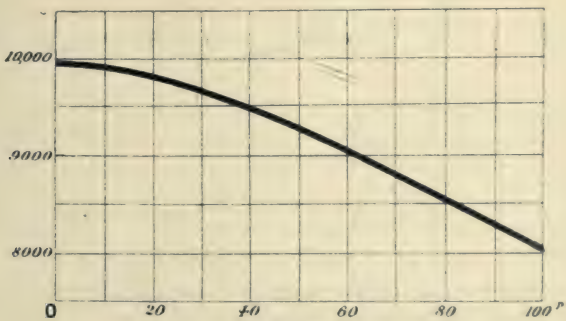


FIG. 57.

It is urged that just as compounds may be formed and decomposed at temperatures higher than that at which their dissociation commences, and that for any given temperature a definite relation exists between the relative amounts of the original compound and of the products of its dissociation, so may solutions contain definite but unstable hydrates at a temperature above their dissociation temperature. If the dissolved substance really enters into combination with the solvent to form different compounds according to the nature of the solution, many of the physical properties of the solution (density, thermal conductivity and such like) will naturally depend on the amount and nature of these compounds, because chemical combination is usually accompanied by volume, density, thermal and other changes. Assuming that the *amount* of such a definite compound is proportional to the concentration of the solution, the rate of change of, say, the density with change of concentration will be a linear function of p, that is to say, from the differentiation of (1)

$$\frac{ds}{dp} = b + 2cp, \quad . \quad . \quad . \quad . \quad (2)$$

where ds is the difference in the density of two experimental values corresponding to the difference in the percentage composition of two solutions of the same substance.*

The second member of (2) corresponds with the equation of a straight line (page 69). On treating the experimental data by this method, Mendeléeff † found that ds/dp was discontinuous, and that breaks were obtained by plotting ds/dp as ordinates against abscissa p for concentrations corresponding to 17·56, 46·00 and 88·46 per cent. of ethyl alcohol. These concentrations coincide with chemical compounds having the composition $C_2H_5OH \, . \, 12H_2O$, $C_2H_5OH \, . \, 3H_2O$ and $3C_2H_5OH \, . \, H_2O$ as shown in Fig. 58.

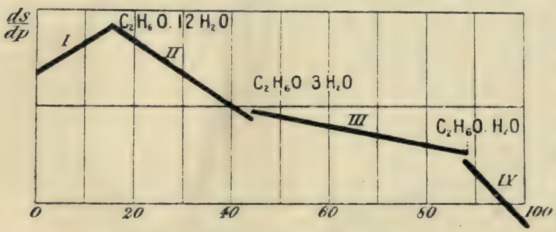


FIG. 58 (after Mendeléeff).

This procedure has been extensively used by Pickering ‡ in the treatment of an elaborate and painstaking series of determinations of the physical properties of solutions.

Crompton found that if the electrical conductivity of a solution

* See page 247 for the method of finding dy/dx from a set of tabulated measurements.

† Mendeléeff, *Journal of the London Chemical Society*, **51**, 778, 1887.

‡ Pickering, *Journal London Chemical Society* and *Philosophical Magazine*, about 1890. Crompton, *Journal Chemical Society*, **53**, 116, 1888.

is regarded as a function of its percentage composition, such that

$$K = a + bp + cp^2 + fp^3, \quad . \quad . \quad . \quad (3)$$

the first differential coefficient gives a parabolic curve of the type of (1) above, while the second differential coefficient, instead of being a continuous function of p,

$$\frac{d^2K}{dp^2} = A + Bp, \quad . \quad . \quad . \quad . \quad (4)$$

was found to consist of a series of straight lines, the position of the breaks being identical with those obtained by Mendeléeff for the first differential coefficient ds/dp. The values of the constants A and B are obvious.

The mathematical argument is that the differential coefficient of a continuous curve will differentiate into a straight line or another continuous curve; while if a curve is really discontinuous, or made up of a number of different curves, it will yield a series of straight lines. Each line represents the rate of change of the particular physical property under investigation with the amount of *hypothetical* unstable compound existing in solution at that concentration. An abrupt change in the direction of the curve leads to a breaking up of the first differential coefficient of that curve into two curves which do not meet. For the p, s-curve, ds/dp is discontinuous; for the $ds/dp, p$-curve, d^2s/dp^2 is discontinuous.

It must be pointed out that the differentiation of experimental results very often furnishes quantities of the same order of magnitude as the experimental errors themselves.* This is a very serious objection. Pickering has proposed to eliminate the experimental errors to some extent by differentiating the results obtained by "smoothing" the curve obtained by plotting the experimental results.† On the face of it this "smoothing"‡ of

* This will appear after reading Chapter V., § 104.

† See Horstmann (Liebig's *Annalen*, Suppl., 8, 125, 1872) for finding $dp/d\theta$ by drawing tangents to the graph of the experimental data; and *Berichte*, 2, 137, 1869, for finding $dp/d\theta$ by the differentiation of an empirical equation. See § 104.

‡ The results of the observation of a series of corresponding changes in two variables are plotted as abscissae and ordinates by light dots on a sheet of squared paper, and a curve is drawn to pass as nearly as possible through all these points. The resulting curve is assumed to be a graphic representation of the general formula (known or unknown) connecting the results of experiment. Points deviating from the curve are assumed to be due to errors of observation. As a general rule the curve with the least curvature is chosen to pass through or within a short distance of the

experimental results is a dangerous operation even in the hands of the most experienced workers. Indeed, it is supposed that that prince of experimenters, Regnault, overlooked an important phenomenon in applying this very smoothing process to his observations on the vapour pressure of saturated steam. Regnault supposed that the curve showed no singular point when water passed from the liquid to the solid state at 0°. It was reserved for J. Thomson to prove that the ice-steam curve is really different from the water-steam curve (see page 127).

§ 55. Discontinuity accompanied by Change of Direction.

The vapour pressure of a solid increases continuously with rising temperature, until at its melting point the vapour pressure suddenly changes. This is shown graphically in Fig. 59. The point P marks the melting point of the substance. The curve does not exhibit a break because the vapour pressure is the same at this point whether the substance be solid or liquid.

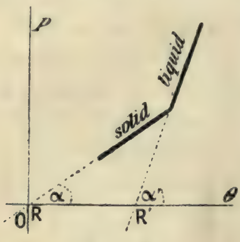


Fig. 59.

It is, however, quite clear that the tangents of the two curves differ from each other at the transition point P, because

$$\tan \alpha = f(\theta) = \frac{dp}{d\theta} \text{ and, } \tan \alpha' = f(\theta) = \frac{dp}{d\theta}.$$

If the equations to the two curves were $ax + by = 1$ and $bx + ay = 1$, the roots of the equations $x = 1/(a + b)$ and $y = 1/(a + b)$ would represent the coordinates of the point of intersection (see page 73).

To illustrate this kind of discontinuity we shall examine the following phenomena :—

(1) Critical temperature. Cailletet and Collardeau have an ingenious method for finding the **critical temperature** of a substance without seeing the liquid.* By plotting temperatures

---

greatest number of dots, so that an equal number of these dots (representing experimental observations) lies on each side of the curve. Such a curve is said to be a **smoothed curve.** The choice of the proper curve is more or less arbitrary. Pickering used a bent spring or steel lath held near its ends. Such a lath is shown in statical works to give a line of constant curvature. *E.g.*, Minchin's *A Treatise on Statics*, **2**, § 306, 1886.

* Cailletet and Collardeau, *Ann. de Chim. et de Phys.* [6], **25**, 522, 1891. Note that the *critical temperature* is the temperature above which a substance cannot exist other than in the gaseous state.

as abscissae against the vapour pressures of different weights of the same substance heated at constant volume, a series of curves are obtained which are coincident as long as part of the substance is liquid, for "the pressure exerted by a saturated vapour depends on temperature only and is independent of the quantity of liquid with which it is in contact". Above the critical temperature the different masses of the substance occupying the same volume give different pressures. From this point upwards the pressure-temperature curves are no longer superposable. A series of curves are thus obtained which coincide at a certain point P (Fig. 60), the abscissa of which denotes the critical temperature. As before, the tangent of each curve Pa, Pb . . . is different from that of OP.

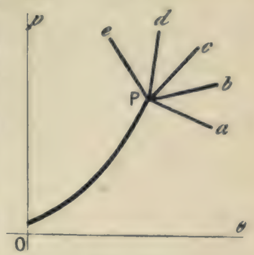


FIG. 60.

(2) Coexistence of the different states of aggregation. Another example which is also a good illustration of the beauty and comprehensive nature of the graphic method of representing natural processes may be given here.

(*a*) When water, partly liquid, partly vapour, is enclosed in a vessel, the relation between the pressure and the temperature can be represented by a curve PQ (Fig. 61), which gives the pressure corresponding to any given temperature when the liquid and vapour are in contact and in equilibrium. This curve is called the **steam line.**

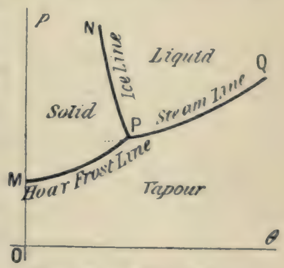


FIG. 61.—Triple Point.

(*b*) In the same way if the enclosure were filled with solid (ice) and liquid water the pressure of the mixture would be completely determined by the temperature. The relation between pressure and temperature is represented by the curve NP, called the **ice line.**

(*c*) Ice may be in stable equilibrium with its vapour, and we can plot the variation of the vapour pressure of ice with its temperature. The curve PM so obtained represents the variation of the vapour pressure of ice with temperature. It is called the **hoar frost line.**

The plane of the paper is thus divided into three parts bounded by the three curves PM, PN, PQ. If a point falls within one of these three parts of the plane, it represents some state in which the water may exist in the form of ice, liquid or steam as the case might be.* If the point falls on a boundary line it corresponds to the coexistence of two states of aggregation. Finally, at the point P, and only at this point, the three states of aggregation, ice, water and steam may coexist together. This point is called the **triple point.** For water the coordinates of the triple point are

$$p = 4{\cdot}57 \text{ mm.}, \ \theta = 0{\cdot}00747^\circ \text{ C.}$$

The two formulae,

$$dQ = \theta d\phi; \ (\partial Q/\partial v)_\theta = \theta(\partial p/\partial \theta)_v,$$

were discussed in one of the examples appended to § 26. Divide the former by dv and substitute the result in the latter. We thus obtain,

$$\left(\frac{\partial \phi}{\partial v}\right)_\theta = \left(\frac{\partial p}{\partial \theta}\right)_v, \quad . \quad . \quad . \quad . \quad (1)$$

which states that the change of entropy (ϕ) per unit change of volume (v), at constant temperature (θ° absolute), is equal to the change of pressure per unit change of temperature at constant volume.

If a small amount of heat (dQ) be added to a substance existing partly in one state, 1, and partly in another state, 2, a proportional quantity (dm) of the mass changes its state, such that

$$dQ = L_{12}dm,$$

where L_{12} is a constant representing the latent heat of the change from state 1 to state 2. By definition of entropy (ϕ),

$$dQ = \theta d\phi; \text{ hence } d\phi = \frac{L_{12}}{\theta}dm \quad . \quad . \quad (2)$$

If v_1, v_2 be the specific volumes of the substance in the first and second states respectively

$$dv = v_2dm - v_1dm = (v_2 - v_1)dm.$$

From (2) and (1)

$$\therefore \left(\frac{\partial \phi}{\partial v}\right)_\theta = \frac{L_{12}}{\theta(v_2 - v_1)}; \ \left(\frac{\partial p}{\partial \theta}\right)_v = \frac{L_{12}}{\theta(v_2 - v_1)}. \quad . \quad (3)$$

This last equation tells us at once how a change of pressure will change the temperature at which two states of a substance can coexist provided that we know v_1, v_2, θ and L_{12}.

* Certain unstable conditions (*metastable states*) are known in which a liquid may be found in the solid region. A supercooled liquid, for instance, may continue the QP curve along to S instead of changing its direction along PM.

EXAMPLES.—(1) If the specific volume of ice is 1·087, and that of water unity, find the lowering of the freezing point of water when the pressure increases one atmosphere (latent heat of ice = 80 cal.). Here $v_2 - v_1 = 0·87$, $\theta = 273$, $dp = 76$ cm. mercury. The specific gravity of mercury is 13·5, and the weight of a column of mercury of one square cm. cross section is $76 \times 13·5 = 1,033$ grams. Hence $dp = 1,033$ grams, $L_{12} = 80$ cal. $= 80 \times 47,600$ *C.G.S.* or dynamical units. From (3), $d\theta = 0·0072°$ C. per atmosphere.

(2) For naphthalene $\theta = 352·2$, $v_2 - v_1 = 0·146$; $L_{12} = 35·46$ cal. Find the change of melting point per atmosphere increase of pressure. $d\theta = 0·035$.

Let L_{12}, L_{23}, L_{31} be the latent heats of conversion of a substance from states 1 to 2, 2 to 3, 3 to 1 respectively; v_1, v_2, v_3 the respective volumes of the substance in states 1, 2, 3 respectively; let θ denote the absolute temperature at the triple point. Then $dp/d\theta$ is the slope of the tangent to these curves at the triple point, and

$$\left(\frac{\partial p}{\partial \theta}\right)_{12} = \frac{L_{12}}{\theta(v_2 - v_1)}; \left(\frac{\partial p}{\partial \theta}\right)_{23} = \frac{L_{23}}{\theta(v_3 - v_2)}; \left(\frac{\partial p}{\partial \theta}\right)_{31} = \frac{L_{31}}{\theta(v_1 - v_3)} \quad (4)$$

The specific volumes and the latent heats are generally quite different for the three changes of state, and therefore *the slopes of the three curves at the triple point are also different.*

The difference in the slopes of the tangents of the solid-vapour (hoar frost line) and liquid-vapour (steam line) curves of water (Fig. 59) is

$$\left(\frac{\partial p}{\partial \theta}\right)_{13} - \left(\frac{\partial p}{\partial \theta}\right)_{23} = \frac{1}{\theta}\left(\frac{L_{13}}{v_3 - v_1} - \frac{L_{23}}{v_3 - v_2}\right) \quad . \quad (5)$$

At the triple point $L_{13} = L_{12} + L_{23}$, and $(v_3 - v_1) = (v_2 - v_1) + (v_3 - v_2)$.

EXAMPLE.—As a general rule, the change of volume on melting, $(v_2 - v_1)$, is very small compared with the change in volume on evaporation, $(v_3 - v_2)$, or sublimation, $(v_3 - v_1)$; hence $v_2 - v_1$ may be neglected in comparison with the other volume changes. Then,

$$\left(\frac{\partial p}{\partial \theta}\right)_{13} - \left(\frac{\partial p}{\partial \theta}\right)_{23} = \frac{L_{12}}{\theta(v_3 - v_2)}.$$

Hence calculate the difference in the slope of the hoar frost and steam lines for water at the triple point. Latent heat of water = 80; $L_{12} = 80 \times 42,700$; $\theta = 273$, $v_3 - v_2 = 209,400$ c.c. Substitute these values on the right-hand side of the last equation. Ansr. 0·059.

The above deductions have been tested experimentally in the case of water, sulphur and phosphorus; the results are in close agreement with theory. A full discussion of the properties of sulphur, water and phosphorus, etc., in relation to the triple point, are given by Duhem in his *Traité Elémentaire de Mécanique*

Chimique fondée sur la Thermodynamique, **2**, 93; an outline sketch will also be found in Preston's *Theory of Heat* (1894), pp. 677-8.

(3) Cooling curves. If the temperature of cooling of pure liquid bismuth be plotted against time, the resulting curve will be continuous (*ab*, Fig. 62), but the moment a part of the metal solidifies, the curve will take another direction *bc*, and continue so until all the metal is solidified, when the direction of the curve again changes, and then continues quite regular along *cd*. For bismuth the point *b* is at 268°.

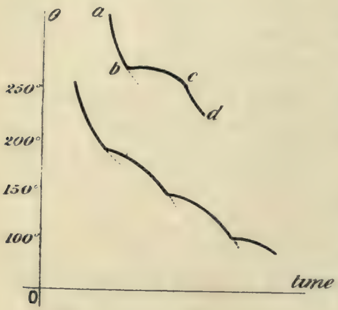


FIG. 62.—Cooling Curves.

If the cooling curve of an alloy of bismuth, lead and tin (*Bi*, 21; *Pb*, 5·5; *Sn*, 75·5) is similarly plotted, the first change of direction is observed at 175°, when solid bismuth is deposited; at 125° the curve again changes its direction, with a simultaneous deposition of solid bismuth and tin; and finally at 96° another change occurs corresponding to the solidification of the eutectic alloy of these three metals.

These cooling curves are of great importance in investigations on the constitution of metals and alloys. The cooling curve of iron from a white heat is particularly interesting, and has given rise to much discussion. The curve shows changes of direction at about 1,130°, at about 850° (called Ar_3 critical point), at about 770° (called Ar_2 critical point), at about 500° (called the Ar_1 critical point), at about 450°—500° C., and at about 400° C. (below redness). The magnitude of these changes varies according to the purity of the iron. Some are very marked even with the purest iron. This sudden evolution of heat (recalescence) at different points of the cooling curve has led many to believe that iron exists in some allotropic state in the neighbourhood of

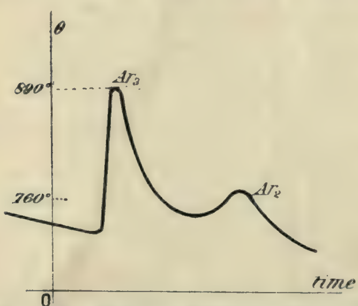
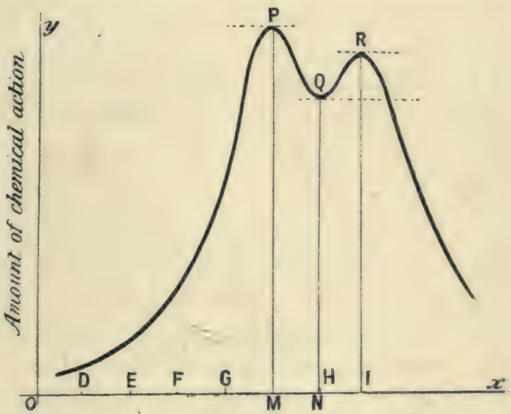


FIG. 63.—Portion of Cooling Curve of Iron.

these temperatures.* Fig. 63 shows part of a cooling curve of iron in the most interesting region, namely, the Ar_3 and Ar_2 critical points.

§ 56. Maximum and Minimum Values of a Function.

If a mixture of hydrogen and chlorine gases is exposed to a ray of light, the amount of chemical action which takes place in a given time depends on the wave length of the light, that is to say, if y denotes the amount of hydrogen chloride formed in unit time, and x the wave length of light, $y = f(x)$. Experiment shows that as x changes from one value to another, y changes in such a way that it is sometimes increasing and sometimes decreasing. In consequence, there must be certain values of the function for which y, which had previously been increasing, begins to decrease, that is to say, y is greater for this particular value of x than for any adjacent value; in this case y is said to have **a maximum value.** Conversely, there must be certain values of $f(x)$ for which y, having been decreasing, begins to increase. When the value of y, for some particular value of x, is less than for any adjacent value of x, y is said to be **a minimum value.**



Wave length of actinic rays referred to Fraunhofer's lines.

FIG. 64 (Diagrammatic).

Fig. 64 is a geometrical illustration of the action of light rays of different wave length on a mixture of hydrogen and chlorine. Imagine the variable ordinate of the curve to move perpendicularly along Ox, gradually increasing until it arrives at the position PM,

---

* Roberts-Austen's papers in the *Proceedings of the Society of Mechanical Engineers* for 1891, 543; 1893, 102; 1895, 238; 1897, 31; 1899, 35, may be consulted for fuller details.

and afterwards gradually decreasing. The ordinate at PM is said to have a maximum value. The decreasing ordinate, continuing its motion, arrives at the position QN, and after that gradually increases. In this case the ordinate at QN is said to have a minimum value.

The terms "maximum" and "minimum" do not necessarily denote the greatest and least possible values which the function can assume, for the same function may have several maximum and several minimum values, any particular one of which may be greater or less than another value of the same function.

Example.—If the y-axis represents the amount of hydrogen chloride formed in unit time; the x-axis, the wave length of the ray of light impinging on a mixture of hydrogen and chlorine gases, interpret the curve shown in Fig. 64.

The mathematical form of the function employed in the above illustration is unknown, the curve is an approximate representation of corresponding values of the two variables determined by actual measurements. (Bunsen and Roscoe, *Phil. Trans.*, **148,** 879, 1859.)

Example.—Plot the curve represented by the equation

$$y = \sin x.$$

Give x a series of values $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π, and so on.

Maximum values of y occur for $x = \frac{1}{2}\pi, \frac{5}{2}\pi, \frac{9}{2}\pi, \ldots$

Minimum values of y occur for $x = -\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{7}{2}\pi, \ldots$

The resulting curve is an harmonic or sine curve (see Fig. 51, page 112).

One of the most useful applications of the differential calculus is the determination of maximum and minimum values of a function. Many of the following examples can be solved by special algebraic or geometric devices. The calculus, however, offers a sure and easy method for the solution of these problems.

§ 57. How to find Maximum and Minimum Values of a Function.

Let us trace the different values which the tangent to the curve shown in Fig. 65 may assume. Firstly, when x is increasing, y is approaching a maximum value and the tangent to the curve makes an *acute* angle with the x-axis. In this case, Table XIII.,

$$\tan \alpha \text{ and } \therefore \frac{dy}{dx} \text{ is positive};$$

at P the tangent is parallel to the x-axis, that is to say,

$$\tan \alpha \text{ and also } \frac{dy}{dx} \text{ are zero} \quad . \quad . \quad . \qquad (1)$$

Secondly, immediately after passing P, the tangent to the curve makes an *obtuse* angle with the x-axis, that is to say,

$$\tan \alpha' \text{ and } \frac{dy}{dx} \text{ are negative} \quad . \quad . \quad . \qquad (2)$$

Finally, as the tangent to the curve approaches the minimum value QN, dy/dx remains negative; at Q the tangent is again parallel to x-axis, and

$$\tan \alpha', \text{ as well as } \frac{dy}{dx}, \text{ is zero.} \quad . \quad . \quad . \qquad (3)$$

After passing Q, dy/dx again becomes positive.

There are some curves which have maximum and minimum values very much resembling P' and Q' (Fig. 66). These curves are said to have cusps at P' and Q'.

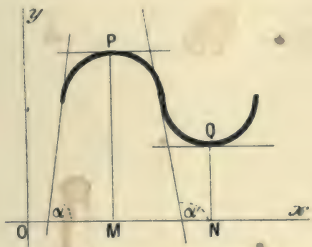


FIG. 65.—Maximum and Minimum.

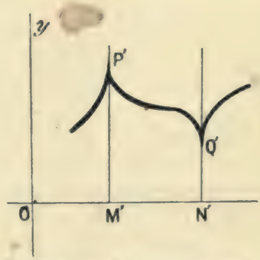


FIG. 66.—Maximum and Minimum Cusps.

It will be here observed that x increases and y approaches a maximum value while the tangent $P'M'$ makes an acute angle with the x-axis, that is to say, dy/dx is positive. At P' the tangent becomes perpendicular to the x-axis, and in consequence the ratio dy/dx becomes infinite. After passing P', dy/dx is negative. In the same way it can be shown that as the tangent approaches $Q'N'$, dy/dx is negative, at Q', dy/dx becomes infinite, and after passing Q', dy/dx is positive.

We thus deduce the following rules:

(1) *When the first differential coefficient changes its sign from a positive to a negative value the function has a maximum value, and when the first differential coefficient changes its sign from a negative to a positive value the function has a minimum value.*

(2) *Since a function can only change its sign by becoming zero or infinity, it is necessary for the first differential coefficient of the function to assume either of these values in order that it may have a maximum or a minimum value.*

(3) *In order to find all the values of x for which y possesses a maximum or a minimum value, the first differential coefficient must be equated to zero or infinity and the value of x which satisfies these conditions determined.*

Examples.—(1) Consider the equation $y = x^2 - 8x$,

$$\therefore \frac{dy}{dx} = 2x - 8.$$

Equating the first differential coefficient to zero, we have

$$2x - 8 = 0; \text{ or } x = 4.$$

Add ± 1 to this root and substitute for x in the original equation,

$$\begin{aligned} \text{when } x = 3, y &= 9 - 24 = -15; \\ x = 4, y &= 16 - 32 = -16; \\ x = 5, y &= 25 - 40 = -15. \end{aligned}$$

y is therefore a minimum when $x = 4$, since a slightly greater or a slightly less value of x makes y assume a greater value. If the values of y had been less for $x = 3$ and $x = 5$, than for $x = 4$, then, $x = 4$ would have made y a maximum. If one had been greater, and the other less than for $x = 4$, this root would have been neither a maximum nor a minimum.

The addition of ± 1 to the root gives only a first approximation, as will be shown later on (page 392). The minimum value of the function might, for all we can tell to the contrary, lie between 3 and 4 or 4 and 5. The approximation may be carried as close as we please by using less and less numerical values in the above substitution. Suppose we substitute in place of ± 1, $\pm \delta x$, then

$$\begin{aligned} \text{when } x = 4 - \delta x, y &= \delta x^2 - 16; \\ x = 4 , y &= - 16; \\ x = 4 + \delta x, y &= \delta x^2 - 16. \end{aligned}$$

Therefore, however small δx may be, the corresponding value of y is greater than -16. That is to say, $x = 4$ makes the function a minimum.

(2) Show that $y = 1 + 8x - 2x^2$, has a maximum value for $x = 2$.

§ 58. Points of Inflection.

Continuing the discussion in the preceding paragraph, to equate

$$\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = \infty,$$

is not a *sufficient* condition to establish the existence of maximum and minimum values of a function, although it is a rough practical test. Some of the values thus obtained do not necessarily make the function a maximum or a minimum, since a variable may become zero or infinite without changing its sign. This is obvious from a simple inspection of Fig. 67, where

$$\frac{dy}{dx} = 0, \text{ or } \infty, \text{ resp.},$$

for the points R and S. Yet neither maximum nor minimum values of the function exist. A further test is therefore required in order to decide whether individual values of x correspond to maximum or minimum values of the function. This is all the more essential in practical work where the function, not the curve, is to be operated upon.

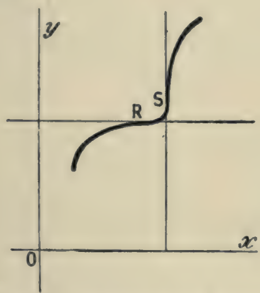


FIG. 67.—Points of Inflection.

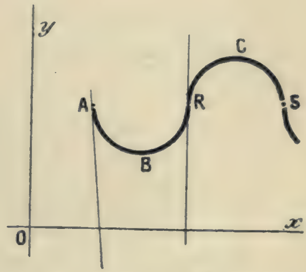


FIG. 68.—Concavity and Convexity.

By reference to Figs. 67 and 68 it will be noticed that the tangent crosses the curve at the points R and S. Such a point is called a **point of inflection.** The point of inflection (or inflexion) marks the spot where the curve passes from a convex to a concave, or from a concave to a convex configuration with regard to one of the coordinate axes. The terms concave and convex have here their ordinary meaning.

§ 59. How to find whether a Curve is Concave or Convex with respect to the x-Axis.

Referring to Fig. 68, along the convex part from A to B, the numerical value of $\tan \alpha$, regularly decreases to zero. At B the lowest point of the curve $\tan \alpha = 0$; from this point to R the tangent to the angle continually increases, for $\tan \alpha$ has now an increasing positive value.

The differential coefficient of $\tan \alpha$ with respect to x for the convex curve ABR is

$$\frac{d(\tan \alpha)}{dx} = \frac{d^2y}{dx^2} > 0; \quad . \quad . \quad . \quad (1)$$

because, *if a function, $y = f(x)$, increases with increasing values of x, dy/dx is positive; while if the function, $y = f(x)$, decreases with increasing values of x, dy/dx is negative.* Along the concave part of the curve RCS, $\tan \alpha$ regularly decreases in value; from R to C, $\tan \alpha$ has a decreasing positive value. At the point C, $\tan \alpha = 0$, and from C to S, $\tan \alpha$ has a continually increasing negative value.

The differential coefficient of $\tan \alpha$ with respect to the concave curve RCS is

$$\frac{d(\tan \alpha)}{dx} = \frac{d^2y}{dx^2} < 0. \quad . \quad . \quad . \quad (2)$$

Hence *a curve is concave or convex with respect to the upper side of the x-axis, according as the second differential coefficient is positive or negative.*

I have assumed that the curve is on the positive side of the x-axis; when the curve lies on the negative side, assume the x-axis to be displaced parallel with itself until the above condition is attained. A more general rule, which evades the above limitation, is proved in the regular text-books. The proof is of little importance for our purpose. The rule is to the effect that "a curve is concave or convex with respect to the x-axis according as the product of the ordinate of the curve and the second differential coefficient, *i.e.*, according as $y\frac{d^2y}{dx^2}$ is positive or negative".

EXAMPLES.—(1) Show that the curves $y = \log x$ and $y = x \log x$ are respectively concave and convex towards the x-axis.

(2) Show that the parabola is concave upwards below the x-axis (where y is negative) and convex upwards above the x-axis.

§ 60. How to find Points of Inflection.

From the above principles of curvature and points of inflection, it is clearly necessary, in order to locate a point of inflection, to find a value of x, for which $\tan \alpha$ assumes a maximum or a minimum value. But

$$\tan \alpha = \frac{dy}{dx},$$

$$\therefore \frac{d(\tan \alpha)}{dx} = \frac{d^2y}{dx^2} = 0 \quad . \quad . \quad . \quad (3)$$

Hence the rule: *In order to find a point of inflection at which the second differential coefficient changes sign, we must equate the second differential coefficient of the function to zero and find the value of x satisfying these conditions.*

EXAMPLES.—(1) Show that the curve

$$y = a + (x - b)^3$$

has a point of inflection at the point $y = a$, $x = b$. Differentiating twice we get $d^2y/dx^2 = 6(x - b)$. Equating this to zero we get $x = b$, and hence substituting in the original equation $y = a$. When $x < b$ the second differential coefficient is negative, when $x > b$ the second differential coefficient is positive. Hence there is an inflection at the point (b, a).

(2) For the special case of the harmonic curve

$$y = \sin x, \frac{d^2y}{dx^2} = - \sin x = - y,$$

that is to say, at the point of inflection the ordinate y changes sign. This occurs when the curve crosses the x-axis, and there are an infinite number of points of inflection for which $y = 0$.

(3) Show that the probability curve, $y = ke^{-h^2x^2}$, has a point of inflection for $x = \pm 1/h\sqrt{2}$.

§ 61. Multiple Points.

A multiple point is one through which two or more branches of a curve meet or intersect. There are two species:

(1) Two or more branches of the curve intersect.

(2) Two or more branches of the curve meet but do not intersect (point of osculation).

An algebraic equation of the *nth* degree has n roots corresponding to the different values of one of the variables. When two or more branches of a curve touch each other, the different values of y, corresponding to x, become equal to each other, while for slightly less values of x, the corresponding values of y are not equal.

First species of multiple point. If the first differential coefficient has two or more real values, the curve has more than one tangent, that is to say, the curves intersect. The number of intersecting branches is denoted by the number of real roots of the first differential coefficient.

EXAMPLE.—In the lemniscate curve, familiar to students of crystallography,

$$y^2 = a^2x^2 - x^4;\ y = \pm x\sqrt{a^2 - x^2}.$$

Here y has two values of opposite sign for every value of x between $\pm a$; the curve is therefore symmetrical with respect to the x-axis. When $x = \pm a$, these two values of y become zero; but these are not multiple points since the curve does not extend beyond these limits, and therefore cannot satisfy the above conditions. When $x = 0$ the two values of y become zero, and since there are two values of y one on each side of the point $x = 0$, $y = 0$, this is a multiple point. Since

$$\frac{dy}{dx} = \pm \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

becomes $\pm a$ when $x = 0$, it follows that there are two tangents to the curve at this point, such that

$$\tan \alpha = \pm a.$$

Fig. 69 shows this curve.

Second species of multiple point—point of osculation. If the first differential coefficient of a multiple point has two or more real and equal roots, the different branches of the curve have a common tangent, and the point of contact is called a **point of osculation.**

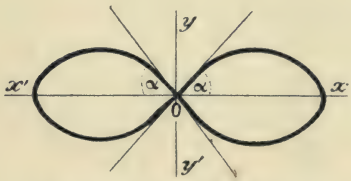


FIG. 69.—Multiple Point (O).

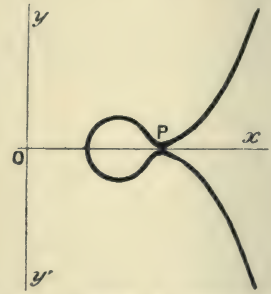


FIG. 70.—Point of Osculation (P).

EXAMPLE.—In the curve $y = (x - 1)(x - 2)(x - 3)$, for values of x other than 0 and -1 there are at least two values of y; $dy/dx = 3x^2 - 12x + 11$ vanishes when $x = 2 \pm 1/\sqrt{3}$; hence the two branches of the curve are tangents to each other at this point, which is therefore a point of osculation. The curve is shown in Fig. 70.

§ 62. Cusps.

A cusp is a point where two branches of a curve have a common tangent and stop at that point. There are two species:

(1) The two branches lie on opposite sides of the common tangent.

(2) The two branches lie on the same side of the common tangent.

The cusp is therefore a special case of the point of osculation, where the branches terminate at the point of contact instead of passing beyond. Hence the values of y on one side of the point are real and on the other, imaginary.*

To distinguish cusps from points of osculation: compare the ordinate of the curve for that point with the ordinates of the curve on each side. For a cusp, y and the first differential coefficient have only one real value.

First, cusps of the first species (or "*keratoid cusps*") have two values for the second differential coefficient differing only in sign. The meaning of this will be clear from pages 133 to 134.

EXAMPLE.—In the cissoid curve, $y = b \pm \sqrt{(x^2 - a^2)^3}$, y is imaginary for all values of x between $\pm a$. When $x = \pm a$, y has one value; for any point to right of $x = +a$ or to the left of $x = -a$, y has two values $dy/dx = \pm 3x(x^2 - a^2)^{\frac{1}{2}}$ vanishes when $x = a$. The two branches of the curve have therefore a common tangent parallel to the x-axis and there is a cusp. Next determine

$$d^2y/dx^2 = \pm \tfrac{3}{2}(x^2 - a^2)^{-\frac{1}{2}}$$

* For imaginary quantities read footnote, page 175.

and substitute some value for a, say $a + h$. We then find that the cusp is of the first species with the upper branch $+ \frac{3}{2}(2ah + h^2)^{-\frac{1}{2}}$, convex towards the x-axis; and the lower branch $- \frac{3}{2}(2ah + h^2)^{-\frac{1}{2}}$, concave towards the x-axis. The curve is shown in Fig. 71.

Second, cusps of the second species (or "*rhamphoid cusps*") have two different values for the second differential coefficient of the same sign.

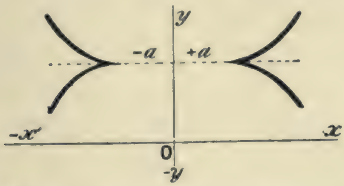


FIG. 71.—Cusps of the First Species.

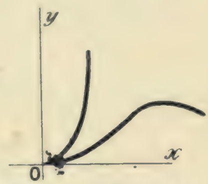


FIG. 72.—Cusp of the Second Species.

EXAMPLE.—Show that the curve $(y - x^2)^2 = x^5$ has a cusp of the second species at the origin. The lower curve also has a maximum when $x = \frac{16}{25}$. The general form of the curve is that shown in Fig. 72.

It will perhaps amuse the reader to investigate the properties of the following curves:

$$r = a \sin 2\theta;\ r = a \sin 6\theta;$$
$$r^3 = a^3 \cos 4\tfrac{3}{4}\theta;\ r^4 = a^5 \cos 5\tfrac{4}{5}\theta.$$

§ 63. Conjugate or Isolated Point.

A conjugate point, or *acnode*, is one whose coordinates satisfy the equation to the curve, and yet is itself detached from the curve.

If a point is isolated from every part of the curve, it follows that on each side of this point real values of one coordinate must give a pair of imaginary values of the other. This may be determined by successive substitution of $x + \delta x$, $x - \delta x$, etc.

EXAMPLE.—Show that the origin in the graph of $ay^2 = x^2(x - b)$ is a conjugate point.

When one branch of a curve suddenly stops we have a *point d'arrêt* or terminal point (see Fig. 120).

EXAMPLE.—The origin in the two transcendental curves $y = a^{1/x}$, where a is greater than unity and $y = x \log a$.

§ 64. Asymptotes.

As explained on page 87, an *asymptote* is a straight line which approaches closer and closer to a given curve, as x or y increases without limit. It is often defined as the limiting position of a tangent to a curve when the point of contact moves an infinite distance away (see the lines OP, OV, Fig. 28; Oc, Fig. 29; Op, Fig. 125, etc.).

Let OPS (Fig. 73) be a plane curve, BP a tangent to the curve at the point $P(x_1, y_1)$. If BP intersects the y-axis at the point $(0, y)$, and the x-axis at the point $(x, 0)$, then (6), § 38,

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1).$$

If, we put $y = 0$, the intercept of the tangent with the x-axis is

$$x = OB = x_1 - y_1\frac{dx_1}{dy_1}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and if $x = 0$ we get the intercept of the tangent with the y-axis,

$$y = Oe = y_1 - x_1\frac{dy_1}{dx_1}.$$

If, when x_1 or y_1 becomes infinite, either x or y is finite, the curve will have one or more asymptotes which can be determined. The following deductions may be made:

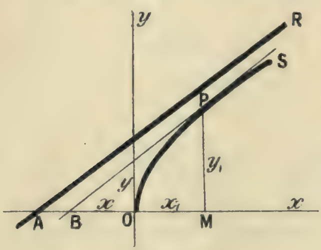


FIG. 73.

(1) If when $x = \infty$, y is finite, the asymptote is parallel to the x-axis.

(2) If when x is finite, $y = \infty$, the asymptote is parallel to the y-axis.

(3) If x and y are both finite, the asymptote passes through $(0, y)$ and $(x, 0)$.

(4) If x and y are both zero, the asymptote passes through the origin, and its direction is determined by the value of y/x when x or y is infinite.

(5) If x and y are both infinite, the tangent is at an infinite distance from the origin, and cannot be constructed since it is indeterminate.

EXAMPLES.—(1) Determine whether the hyperbola has asymptotes. The hyperbola

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

has two real values of y, however great x may be, and hence the curve has two infinite branches to the right. Differentiating the above equation

$$y\frac{dx}{dy} = \frac{a^2}{b^2} \cdot \frac{y^2}{x} = \frac{x^2 - a^2}{x},$$

$$\therefore\ O(x, 0) = \frac{a^2}{x_1};\ O(0, y) = -\frac{b^2}{y_1},$$

if x is infinite, $OB = a^2/x_1 = 0$; and if y is infinite $O(0, y) = - b^2/y_1 = 0$, that is to say, the hyperbola has two asymptotes passing through the origin as (4) above. The *direction of the asymptotes* is obtained by putting

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} = \pm\frac{bx}{a\sqrt{(x^2 - a^2)}},$$

when $x = + \infty$, $dy/dx = \pm b/a$. Hence the asymptotes are the produced diagonals of a rectangle described on the axes. If $x = - \infty$, there is another pair of infinite branches having the same lines through the origin as asymptotes.

(2) Has the parabola $y^2 = 4ax$ an asymptote? No. $O(x, 0) = - x$, $O(0, y) = \frac{1}{2}y$. When x is infinite, $O(x, 0) = - \infty$, and when y is infinite $O(0, y) = + \infty$. Hence the parabola has no asymptotes as in (5) above.

(3) Show that the logarithmic curve $x = \log y$ (and also $y = e^x$) has an asymptote coincident with the abscissa axis, and a branch of the curve extending to the right, not asymptotic (case (5) above).

§ 65. Summary.

(1) Equating the first differential coefficient of a function to zero gives a *maximum* or *minimum value*. If the sign of $\frac{dy}{dx}$ changes from $+$ to $-$ when x is substituted, y is a maximum; if the sign changes from $-$ to $+$, y is a minimum; also, if $\frac{d^2y}{dx^2}$ is positive, y is a maximum, if negative, a minimum (see § 102).

(2) If $\frac{d^2y}{dx^2}$ is positive, the curve is *concave* towards the x-axis, if negative *convex*.

(3) If $\frac{dy}{dx} = 0$, but does not change sign when x is substituted, we have a *point of inflection* for which $\frac{d^2y}{dx^2} = 0$.

(4) If $\frac{dy}{dx} = \infty$ and changes its sign, there is a *cusp*, which is a maximum or a minimum according to its sign.

(5) If $\frac{dy}{dx} = \infty$ and $y = \infty$ without changing sign, y is an *asymptote*.

(6) If $\frac{dy}{dx} = \infty$ and $y = \infty$ with a change of sign, y has an *infinite maximum value*.

(7) If $\frac{dy}{dx}$ has two or more unequal values, a *multiple point* occurs.

(8) If $\frac{dy}{dx}$ has two or more equal values, a *point of osculation* occurs.

(9) If $\frac{dy}{dx}$ and y have one real value, and the value of y on one side of the point is imaginary, we have a *cusp*: *of 1st species*, if $\frac{d^2y}{dx^2}$ has two values, differing only in sign; *of 2nd species*, if $\frac{d^2y}{dx^2}$ has two different values, of the same sign.

(10) If $\frac{dy}{dx}$ and higher differential coefficients have impossible values, we have a *conjugate point*.

§ 66. Curvature.

The curvature at any point of a plane curve is the rate at which the curve is bending. Of two curves AC, AD, that has the greater curvature which departs the more rapidly from its tangent AB (Fig. 74). The angle between the tangents at the ends of an arc of the curve is called

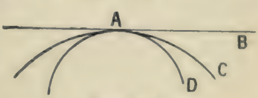


FIG. 74.

the **total curvature** of the arc. In passing from any point P (Fig. 75) to another neighbouring point P_1 along any arc δs of the plane curve AB, the tangent at P turns through the angle $\delta\alpha$

where α is the angle made by the intersection of the tangent at P with the x-axis. The angle $\delta\alpha$ is the total curvature of the arc under consideration, and the ratio

$$\frac{(\textit{total curvature})}{(\textit{length of arc})} = \frac{\delta\alpha}{\delta s} = (\textit{mean curvature of arc}).$$

The curve turns through the angle $\delta\alpha$ in the length δs, and therefore the total curvature is the limiting value of

$$Lt\delta\alpha/\delta s = d\alpha/ds = (\textit{rate of bending of curve}) \quad . \quad (1)$$

The curvature of the circumference of all circles of equal radius is the same at all points, and varies inversely as the radius. This is established in the following way: In the circle (Fig. 76), O is the centre, r, r are radii. From elementary geometry, the angle

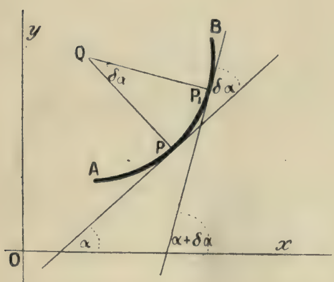


FIG. 75.

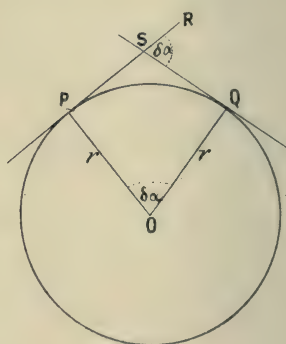


FIG. 76.

RSQ = angle POQ. The angle POQ is measured in circular measure (page 494) by the ratio of the arc PQ to the radius, *i.e.*,

$$\text{angle } POQ = \text{arc } PQ/r, \text{ or } \delta\alpha/\delta s = 1/r,$$

$$\therefore (\textit{curvature of circle}) = \frac{d\alpha}{ds} = \frac{1}{r} \quad . \quad (2)$$

This is Newton's definition of curvature.

Just as a straight line touching a curve, may be regarded as a line drawn through two points of the curve infinitely close to each other (definition of tangent), so a circle in contact with a curve may be considered to pass through three consecutive points of the curve infinitely near each other. Such a circle is called an *osculatory circle* or a *circle of curvature*. The osculatory circle of a curve has the same curvature as the curve itself *at* the point of contact. The curvature of different parts of a curve may be compared by drawing osculatory circles through these points. If

r be the radius of an osculatory circle at P (Fig. 77) and r_1 that at P_1, then

$$\text{curvature at } P : \text{curvature at } P_1 = \frac{1}{r} : \frac{1}{r_1} \quad . \quad (3)$$

In other words, *the curvature at any two points on a curve varies inversely as the radius of the osculatory circles* at these points. The radius of the osculatory circle at different points of a curve is called the *radius of curvature* at that point. The centre of the osculatory circle is the *centre of curvature.*

FIG. 77.

To find the radius of curvature of a curve. Let the coordinates of the centre of the circle be a and b, R the radius, then the equation of the circle is (page 76)

$$(x - a)^2 + (y - b)^2 = R^2 \quad . \quad . \quad . \quad (4)$$

Differentiating this equation and dividing by 2,

$$(x - a) + (y - b)\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad (5)$$

Again differentiating,

$$1 + (y - b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad . \quad . \quad . \quad (6)$$

Let $u = dy/dx$ and $v = d^2y/dx^2$, for the sake of ease in manipulation, then (6) becomes

$$y - b = -\frac{1 + u^2}{v} \quad . \quad . \quad . \quad . \quad (7)$$

Substituting this value of $y - b$ in (5),

$$x - a = \frac{1 + u^2}{v}\cancel{v}.u \quad . \quad . \quad . \quad (8)$$

u, v, x and y at any point of the curve are the same for the osculating circle at that point, and therefore a, b* and r can be determined from x, y, u, v. Substituting (7) and (8) in (4),

$$\frac{1}{R} = \frac{u}{\sqrt{(1 + v^2)^3}}, \quad . \quad . \quad . \quad (9)$$

* The determination of a and b is of little use in practical work. They give equations to the evolute of the curve under consideration. The *evolute* is the curve drawn through the centres of the osculatory circles at every part of the curve, the curve itself is called the *involute.* Example: the osculatory circle has the equation $(x - a)^2 + (y - b)^2 = R.$ a and b may be determined from equations (4), (7) and (8). The evolute of the parabola $y^2 = mx$ is $b^2 = \frac{2}{27m}(2a - m)^3$.

which is the standard equation. From (1),

$$\frac{1}{R} = \frac{d\alpha}{ds} = \frac{d^2y}{dx^2}\Big/\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}; \text{ or } R = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\Big/\frac{d^2y}{dx^2}. \quad \textbf{(10)}$$

EXAMPLES.—(1) Find the radius of curvature at any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y};\ \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3},\ R = -(a^4y^2 + b^4x^2)^{\frac{3}{2}}/a^4b^4.$$

At the point $x = a$, $y = 0$, $R = b^2/a$. Hint. The steps for d^2y/dx^2 are:

$$= -\frac{b^2}{a^2}\cdot\frac{y - x.dy/dx}{y^2} = -\frac{b^2}{a^2}\cdot\frac{a^2y^2 + b^2x^2}{a^2y^3} = -\frac{b^2}{a^2}\cdot\frac{a^2b^2}{y^3}.$$

(2) The radius of curvature of $xy = a$, is $(x^2 + y^2)^{\frac{3}{2}}/2a$.

When the curve is but slightly inclined to the x-axis, dy/dx is practically zero, and the radius of curvature is given by the expression

$$R = 1\Big/\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad (11)$$

a result frequently used in physical calculations involving capillarity, superficial tension, theory of lenses, etc.

The direction of curvature has been discussed in § 59. It was there shown that a curve is concave or convex at a point (x, y) according as $d^2y/dx^2 >$ or < 0. See also § 100.

§ 67. Envelopes.

The equation $$y = \frac{m}{a} + ax,$$

represents a family of curves, since for each value of a we get a distinct curve. If a varies continuously it will determine a succession of curves, each of which is a member of the family denoted by the above equation. a is said to be the **variable parameter** of the family, since the different members of the family are obtained by assigning arbitrary values for a. Let the equations

$$y_1 = \frac{m}{a} + ax \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y_2 = \frac{m}{a + \delta a} + (a + \delta a)x \quad . \quad . \quad . \quad (2)$$

$$y_3 = \frac{m}{a + 2\delta a} + (a + 2\delta a)x \quad . \quad . \quad . \quad (3)$$

be three successive members of the family. As a general rule two distinct curves in the same family will have a point of intersection. Let P (Fig. 78) be the point of intersection of curves (1) and (2); P_1 the point of intersection of curves (2) and (3), then, since P_1 and P_2 are both situated on the curve (2), PP_1 is part of the locus of a curve whose arc PP_1 coincides with an equal part of the curve (2). It can be proved, in fact, that the curve $P P_1$ . . . touches the whole family of curves represented by the original equation. Such a curve is said to be an *envelope* of the family.

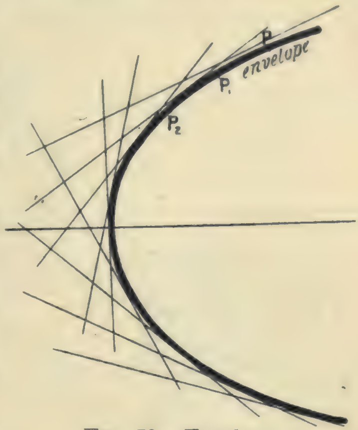


FIG. 78.—Envelope.

To find the equation to the envelope, bring all the terms of the original equation to one side,

$$y - \frac{m}{a} - ax = 0.$$

Then differentiate with respect to the variable parameter, and put

$$\frac{m}{a^2} - x = 0.$$

Eliminate a between these equations,

$$y - \sqrt{m.x} - x\sqrt{\frac{m}{x}} = 0, \text{ or } y - 2\sqrt{m.x} = 0.$$

$$\therefore y^2 = 4mx.$$

EXAMPLES.—(1) Find the envelope of the family of circles

$$(x - a)^2 + y^2 = r^2,$$

where a is the variable parameter. Differentiate with respect to a and $x - a = 0$; eliminating a, we get $y = \pm a$, which is the required envelope. The envelope $y = \pm a$ represents two straight lines parallel to the x-axis and at a distance $+ a$ and $- a$ from it. Shown Fig. 79.

(2) Show that the envelope of the family of curves $(x - m - a)^2 + y^2 = 4ma$, is a parabola $y^2 = 4mx$. See §§ 126 and 138.

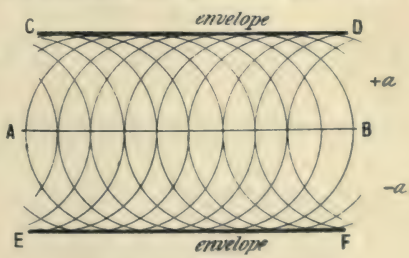


FIG. 79.—Double Envelope.

§ 68. Six Problems in Maxima and Minima.

It is first requisite, in solving problems in maxima and minima, to express the relation between the variables in the form of an algebraic equation, and then to proceed as directed on page 130.

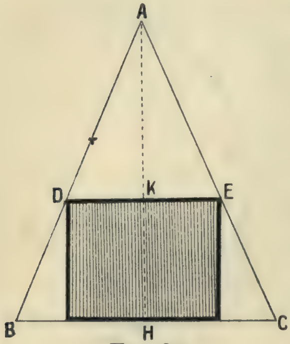


FIG. 80.

In the majority of cases occurring in practice, it only requires a little common-sense reasoning on the nature of the problem, to determine whether a particular value of x corresponds to a maximum or to a minimum.

(1) *Divide a line into any two parts such that the rectangle having these two parts as adjoining sides may have the greatest possible area.*

If a be the length of the line, x the length of one part, $a - x$ is the length of the other. The area of the rectangle will be

$$y = (a - x)x.$$

Differentiate and
$$\frac{dy}{dx} = a - 2x.$$

Equate to zero, and $x = \frac{1}{2}a$, that is to say, the line a must be divided into two equal parts, and the greatest possible rectangle is a square.

(2) *Find the greatest possible rectangle that can be inscribed in a given triangle.*

In Fig. 80, let b denote the length of the base of the triangle ABC, h its altitude, x the altitude of the inscribed rectangle. We must first find the relation between the area of the rectangle and of the triangle. By similar triangles, page 490,

$$AH : AK = BC : DE\ ;\ h : h - x = b : DE,$$

but the area is obviously $y = DE \times KH$, and

$$DE = \frac{b}{h}(h - x),\ KH = x\ ;\ \therefore y = \frac{b}{h}(hx - x^2).$$

Now b/h is constant, and it is the rule, when seeking maxima and minima, to abbreviate the process by omitting constant factors, since, whatever makes the variable $hx - x^2$ a maximum will also make $\frac{b}{h}(hx - x^2)$ a maximum. This is easily proved, for let

$$y = cf(x),$$

where c has any arbitrary constant value. For a maximum or minimum value

$$dy/dx = cf'(x) = 0,$$

and this can only occur where

$$f'(x) = 0.$$

Now differentiate the expression obtained above, for the area of the rectangle, and equate the result to zero.

$$\frac{dy}{dx} = h - 2x = 0\,; \text{ or } x = \tfrac{1}{2}h.$$

That is to say, the height of the rectangle must be half the altitude of the triangle.

(3) *To cut a sector from a circular sheet of metal so that the remainder can be formed into a conical-shaped vessel of maximum capacity.*

Let ACB (Fig. 81) be a circular plate of unit radius, it is required to cut out a portion AOB such that the conical vessel formed by joining OA and OB together may hold the greatest possible amount of fluid. We must again find a relation between the dimensions of the plate and the volume of the cone.

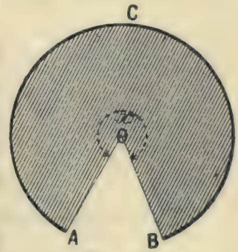


Fig. 81.

Obviously ACB will be the circumference of the circular base of the cone. Let r denote the radius of this base, and y the perimeter of the circular base.

$$y = 2\pi r. \quad . \quad . \quad . \quad . \quad (1)$$

If h denotes the height of the cone its volume V will be formula (26), page 492,

$$V = \tfrac{1}{3}\pi r^2 h. \quad . \quad . \quad . \quad . \quad (2)$$

But h and r form a right-angled triangle with hypotenuse

$$OB = OA = 1,$$

and whose base is r.

$$h = \sqrt{1 - r^2}\,;\ \text{from (1) } h = \sqrt{1 - y^2/4\pi^2}, \quad . \quad (3)$$

from (2) and (3) $\quad V = y^2\sqrt{(1 - y^2/4\pi^2)}/12\pi. \quad . \quad . \quad . \quad (4)$

The problem therefore is to find y such that V is a maximum. As before, omitting the constant term $1/12\pi$,

$$\frac{dV'}{dy} = 2y\sqrt{1 - \frac{y^2}{4\pi^2}} - \frac{y^3}{4\pi^2}\Big/\sqrt{\left(1 - \frac{y^2}{4\pi^2}\right)} = 0\,;$$

where $V' \times 1/12\pi = V$. Multiply through with $\sqrt{1 - y^2/4\pi^2}$,

$$\therefore\ 2y(1 - y^2/4\pi^2) - y^3/4\pi^2 = 0\,;$$

divide through by y, since y is not zero,

$$2 - 3y^2/4\pi^2 = 0 \text{; or } y = 2\pi\sqrt{2/3} \quad . \quad . \quad (5)$$

But, by Euclid vi., 33,

perimeter of sector ACB : whole perimeter of the original circle

$$= \text{angle } x^\circ : 360^\circ.$$

Since the original circle had unit radius

$$y : 2\pi = x : 360.$$

Substituting this value of y in (5),

$$x = 360\sqrt{\tfrac{2}{3}} = 294^\circ \text{ (}\textit{approx.}\text{)}.$$

The angle of the removed sector is then about 66°. The application to the folding of filter papers is obvious.

(4) *At what height should a light be placed above my writing table in order that a small portion of the surface of the table, at a given horizontal distance away from the foot of the perpendicular dropped from the light on to the table, may receive the greatest illumination possible?*

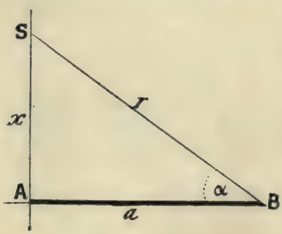


FIG. 82.

Let S (Fig. 82) be the source of illumination whose distance from the table x is to be determined in such a way that B may receive the greatest illumination. Let $AB = a$, and α the angle made by the incident rays $SB = r$ on the surface B.

It is known that the intensity of illumination varies inversely as the square of the distance of B, and directly as the sine of the angle of incidence.

Since $\quad r^2 = a^2 + x^2, \sin\alpha = x/r = x/\sqrt{(a^2 + x^2)}$.

In order that the illumination may be a maximum,

$$\therefore y = x/r^2\sqrt{(a^2 + x^2)} = x/\sqrt{(a^2 + x^2)^3}$$

must be a maximum. Hence

$$\frac{dy}{dx} = \frac{a^2 - 2x^2}{x(a^2 + x^2)} = 0, \therefore x = a\sqrt{\tfrac{1}{2}}.$$

The interpretation is obvious.*

(5) *To arrange a number of voltaic cells to furnish a maximum current against a known external resistance.*

Let the electromotive force of each cell be E, and its internal resistance r. Let R be the external resistance, n the total number of cells.

* Note: Negative and imaginary roots have no physical meaning in this problem. See page 394.

Assume that x cells are arranged in series and n/x in parallel. The electromotive force of the battery is xE. Its internal resistance x^2r/n. The current C is given by

$$C = \frac{xE}{R + \frac{r}{n}x^2} *; \quad \therefore \frac{dC}{dx} = \frac{\left(R - \frac{r}{n}x^2\right)E}{\left(R + \frac{r}{n}x^2\right)^2}.$$

Equate dC/dx to zero and simplify,

$$\therefore R = rx^2/n.$$

That is to say, the battery must be so arranged that its internal resistance shall be as nearly as possible equal to the external resistance.

(6) **Snell's Law of Refraction of Light—Index of Refraction.**

Let SP (Fig. 83) be a ray of light incident at P on the surface of separation of the media M and M'; let PR be the refracted ray in the same plane as the incident ray. If PN is normal (perpendicular) to the surface of incidence, then $SPN = i$ is the angle of incidence, $N'PR = r$ the angle of refraction. Drop perpendiculars from S and R on to A and B, so that $SA = a$, $RB = b$. Now the light will travel from S to R in the shortest possible time, with a uniform velocity different in the different media M and M'. At the point P, the ray passes through the surface separating the two media, let $AP = x$, $BP = p - x$. Let the velocity of propagation of the ray of light in the two media be respectively v and v' per second. The ray therefore travels from S to P in SP/v seconds, and from P to R in RP/v' seconds, and the total time occupied in transit from S to R is

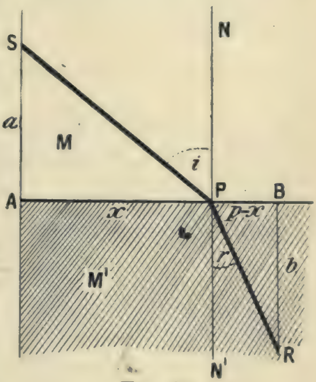


FIG. 83.

$$t = SP/v + RP/v'. \qquad (1)$$

In the triangles SAP and RBP

$$SP = \sqrt{a^2 + x^2}; \; RP = \sqrt{b^2 + (p - x)^2}. \qquad (2)$$

* This formula is identical with the one given in any text-book on electricity. Note: C is a maximum when its reciprocal is a minimum. This and all the preceding results should be tested for maxima and minima by means of the second differential coefficients.

Substituting these values in (1) and differentiating in the usual way we get

$$\frac{dt}{dx} = \frac{x}{v\sqrt{a^2 + x^2}} - \frac{p - x}{v'\sqrt{b^2 + (p - x)^2}} = 0. \quad . \quad (3)$$

But $\sin i = \frac{x}{\sqrt{a^2 + x^2}}$; $\sin r = \frac{p - x}{\sqrt{b^2 + (p - x)^2}}$.

From (2) $$\frac{\sin i}{\sin r} = \frac{v}{v'}.$$

This result shows that the signs of the angles of incidence and refraction must be proportional to the velocity of the light in the different media, in order that the light may pass from one point to another in the shortest possible interval of time.

The ratio, $\sin i/\sin r$, therefore, is constant for the same two media. This constant is usually denoted by the symbol μ, and called the *index of refraction*. It is obvious that for the same two media the index of refraction is constant for light of the same wave-length.

Examples.—(1) The velocity of motion of a wave in deep water is

$$v = \sqrt{(\lambda/a + a/\lambda)}$$

where λ denotes the length of the wave, a is a constant. Required the length of the wave when the velocity is a minimum. (N. Z. Exam. Papers.) Ansr. $\lambda = a$.

(2) The contact difference of potential (E) between two metals is a function of the temperature (θ) such that

$$E = a + b\theta + c\theta^2.$$

How high must the temperature of one of the metals be raised in order that the difference of potential may be a maximum or a minimum. a, b, c are constants. Ansr. $\theta = - b/2c$.

(3) Show that the greatest rectangle that can be inscribed in the circle $x^2 + y^2 = r^2$ is a square. Hint. Area $= 4xy$, etc.

(4) If v_0 be the volume of water at 0° C., v the volume at θ° C., then, according to the "*Hällström's*" *formula* for temperatures between 0° and 30°,

$$v = v_0(1 - 0{\cdot}000057{,}577\theta + 0{\cdot}000007{,}5601\theta^2 - 0{\cdot}000000{,}03509\theta^3).$$

Show that the volume is least and the density greatest when $\theta = 3{\cdot}92$.

(5) *Kopp's formula* for the volume of water at any temperature between 0° and 25° is

$$v = v_0(1 - 0{\cdot}000061{,}045\theta + 0{\cdot}000007{,}7183\theta^2 - 0{\cdot}000000{,}03734\theta^3).$$

Show that the temperature of maximum density is 4·08°.

(6) An electric current flowing round a coil of radius r exerts a force F on a small magnet whose axis is at some point on a line drawn through the centre and perpendicular to the plane of the coil. If x is the distance of the magnet from the plane of the coil,

$$F = x/(r^2 + x^2)^{\frac{5}{2}}.$$

Show that the force is a maximum when $x = \frac{1}{2}r$.

(7) Draw an ellipse whose area for a given perimeter shall be a maximum. Although the perimeter of an ellipse can only be represented with perfect accuracy by an infinite series (page 188), yet for all practical purposes the perimeter may be taken to be $\pi(x + y)$ where x and y are the major and minor axes. The area of the ellipse is $z = \pi xy$. Since the perimeter is to be constant, $a = \pi(x + y)$ or $y = a/\pi - x$. Substitute this value of y in the former expression and $z = ax - \pi x^2$. Hence, $x = a/2\pi$ when z is a maximum. Substitute this value of x in $y = a/\pi - x$, and $y = a/2\pi$, that is to say, $x = y = a/2\pi$, or of all ellipses the circle has the greatest area.

Boys' leaden water-pipes designed not to burst at freezing temperatures, are based on this principle. The cross section of the pipe is elliptical. If the contained water freezes, the resulting expansion makes the tube tend to become circular in cross section. The increased capacity allows the ice to have more room without putting a strain on the pipe.

(8) If A, B be two sources of heat, find the position of a point O on the line $AB = a$, such that it is heated the least possible. Assume that the intensity of the heat rays is proportional to the square of the distance from the source of heat. Let $AO = x$, $BO = a - x$. The intensity of each source of heat at unit distance away is α and β. The total intensity of the heat which reaches O is

$$I = \frac{\alpha}{x^2} + \frac{\beta}{(a - x)^2}.$$

Find dI/dx and d^2I/dx^2. This equation is a minimum when

$$x = \sqrt[3]{\alpha} \,.\, a/(\sqrt[3]{\alpha} + \sqrt[3]{\beta}).$$

Since $AO : BO = \sqrt[3]{\alpha} : \sqrt[3]{\beta}$ show that when I is a minimum, its actual (numerical) value may be found from $I(min.) = (\sqrt[3]{\alpha} + \sqrt[3]{\beta})^3/a^2$. If $\alpha = \beta$ then $x = \frac{1}{2}a$, and the numerical value of $I(min.) = 8\alpha/a^2$.

(9) *Rapp's equation for the specific heat of water* between 0° and 100° is

$$\sigma = 1{\cdot}039935 - 0{\cdot}0070680\theta + 0{\cdot}000212550\theta^2 - 0{\cdot}000001540\theta^3,$$

where the mean specific heat between 0° and 100° is unity. Hence show that there is a minimum between $\theta = 20°$ and 30°, and a maximum about 70°. *Volten's equation* for the same property is

$$\sigma = 1 - 0{\cdot}00146255120\theta + 0{\cdot}00002379810\theta^2 - 0{\cdot}00000010716\theta^3.$$

Hence show that there is a minimum between 40° and 50°, and a maximum about 100°.

In the working of the above examples, it will be found simplest to use a, b, c . . . for the numerical coefficients, differentiate, etc., for the final result, restore the numerical values of a, b, c . . . , and simplify. Probably the reader has already done this.

CHAPTER IV.

THE INTEGRAL CALCULUS.

The experimental verification of a theory concerning any natural phenomenon generally rests on the result of an integration.

§ 69. Integration.

In the first chapter, methods were described for finding the momentary rate of progress of any uniform or continuous change in terms of a limiting ratio, the so-called "differential coefficient" between two variable magnitudes. The fundamental relation between the variables must be accurately known before one can form a quantitative conception of the process taking place at any moment of time. When this relation or law is expressed in the form of a mathematical equation, the "methods of differentiation" enable us to determine the character of any continuous physical change at any instant of time. These methods have been described.

Another problem is even more frequently presented to the investigator. Knowing the momentary character of any natural process, it is asked: "What is the fundamental relation between the variables?" "What law governs the whole course of the physical change?"

In order to fix this idea, let us study an example. The conversion of cane sugar into invert sugar in the presence of dilute acids, takes place in accordance with the reaction:

$$\underset{(cane\ sugar).}{C_{12}H_{22}O_{11}} + H_2O = \underset{(invert\ sugar).}{2C_6H_{12}O_6}$$

Let x denote the amount of invert sugar formed in the time t; the amount of sugar remaining in the solution will then be $1 - x$, provided the solution originally contained one gram of cane sugar. The amount of invert sugar formed in the time dt, will be dx. By Wilhelmy's law (page 46), the velocity of the chemical reaction *at any moment* will be proportional to the amount of cane sugar actually present in the solution. That is to say,

$$\frac{dx}{dt} = k(1 - x), \quad . \quad . \quad . \quad . \qquad (1)$$

where k is the "constant of proportion" (page 487). The meaning of k is obtained by putting $x = 0$. Thus, $dx/dt = k$, or, k denotes the rate of transformation of unit mass of sugar.

From (1), page 5,

$$v = dx/dt, \quad . \quad . \quad . \quad . \qquad (2)$$

where v denotes the velocity of the reaction. This relation is strictly true only when we make the interval of time so short that the velocity has not had time to vary during the process. But the velocity is not really constant during any finite interval of time, because the amount of cane sugar remaining to be acted upon by the dilute acid is continually decreasing. For the sake of simplicity, let $k = \frac{1}{10}$, and *assume* that the action takes place in a series of successive stages, so that dx and dt have finite values, say δx and δt respectively. Then,

$$v = \frac{(\textit{amount of cane sugar transformed})}{(\textit{interval of time})} = \frac{\delta x}{\delta t}. \quad . \quad . \qquad (3)$$

Let δt be one second of time. Let $\frac{1}{10}$ of the cane sugar present be transformed into invert sugar in each interval of time, at the same uniform rate that it possessed at the beginning of the interval.

At the commencement of the first interval, when the reaction has just started, the velocity will be at the rate of 0·100 grams of invert sugar per second. This rate will be maintained until the commencement of the second interval, when the velocity suddenly slackens down, because only 0·900 grams of cane sugar are then present in the solution.

During the second interval, the rate of formation of invert sugar will be $\frac{1}{10}$ of the 0·900 grams actually present at the beginning. Or, 0·090 grams of invert sugar are formed during the second interval.

At the beginning of the third interval, the velocity of the reaction is again suddenly retarded, and this is repeated every second for say five seconds.

Now let δx_1, δx_2, . . . denote the amounts of invert sugar formed in the solution during each second (δt). Assume, for the sake of simplicity, that one gram of cane sugar yields one gram of invert sugar.

				(*Cane sugar transformed.*)
During the	1st second,			$\delta x_1 = 0{\cdot}100$
,,	,,	2nd	,,	$\delta x_2 = 0{\cdot}090$
,,	,,	3rd	,,	$\delta x_3 = 0{\cdot}081$
,,	,,	4th	,,	$\delta x_4 = 0{\cdot}073$
,,	,,	5th	,,	$\delta x_5 = 0{\cdot}066$
			Total,	$0{\cdot}410$

This means that if the chemical reaction proceeds during each successive interval with a uniform velocity equal to that which it possessed at the commencement of that interval, then, 0·410 gram of invert sugar would be formed at the end of five seconds. As a matter of fact, 0·3935 gram is formed.

But 0·410 gram is evidently too great, because the retardation is a uniform, not a jerky process. We have resolved it into a series of elementary stages and *pretended* that the rate of formation of invert sugar remained uniform during each elementary stage. We have ignored the retardation which takes place from moment to moment. If we shorten the interval and determine the amounts of invert sugar formed during intervals of say half a second, we shall have ten instead of five separate stages to sum up, thus:

				(*Cane sugar transformed.*)
During the	1st half second,			$\delta x_1 = 0{\cdot}0500$
,,	,,	2nd	,,	$\delta x_2 = 0{\cdot}0475$
,,	,,	3rd	,,	$\delta x_3 = 0{\cdot}0451$
,,	,,	4th	,,	$\delta x_4 = 0{\cdot}0429$
,,	,,	5th	,,	$\delta x_5 = 0{\cdot}0407$
,,	,,	6th	,,	$\delta x_6 = 0{\cdot}0387$
,,	,,	7th	,,	$\delta x_7 = 0{\cdot}0367$
,,	,,	8th	,,	$\delta x_8 = 0{\cdot}0349$
,,	,,	9th	,,	$\delta x_9 = 0{\cdot}0332$
,,	,,	10th	,,	$\delta x_{10} = 0{\cdot}0315$
			Total,	$0{\cdot}401$

The quantity of invert sugar calculated on the supposition that the velocity is retarded every half second instead of every second, corresponds more closely with the actual change. The smaller we make the interval of time the more accurate the result. Finally, by making δt infinitely small, although we should have an infinite number of equations to add up, the actual summation would give a perfectly accurate result. To add up an infinite number of equations is, of course, an arithmetical impossibility, but, by the "methods of integration" we can actually perform this operation.

$x =$ (*sum of all the terms* $v \,.\, dt$, *between* $t = 0$ *and* $t) = 5,)$
$= v \,.\, dt + v \,.\, dt + v \,.\, dt + \ldots$ to infinity.

This is more conveniently written,

$$x = \Sigma_0^5(v \,.\, dt),$$

or, better still,

$$x = \int_0^5 v \,.\, dt.$$

The signs "Σ" and "$\int$" are abbreviations for "the sum of all the terms containing . . ."; the subscripts and superscripts denote the limits between which the time has been reckoned. The second number of the last equation is called an **integral.** "$\int f(x) \,.\, dx$" is read "the integral of $f(x) \,.\, dx$".

When the limits between which the integration (evidently another word for "summation") is to be performed, are stated, the integral is said to be **definite;** when the limits are omitted, the integral is said to be **indefinite.** The superscript to the symbol "$\int$" is called the **upper or superior limit;** the subscript, the **lower or inferior limit.** For example, $\int_{v_1}^{v_2} p \,.\, dv$ denotes the sum of an infinite number of terms $p \,.\, dv$, when v is taken between the limits v_2 and v_1.

To prevent any misunderstanding, I will now give a graphic representation of the above process. Take Ot and Ov as coordinate axes (Figs. 84 and 85). Mark off, along the abscissa axis, intervals 1, 2, 3, . . . , corresponding to the intervals of time δt. Let the ordinate axis represent the velocities of the reaction during these different intervals of time. Let the curve *vbdfh* . . . represent the actual velocity of the transformation on the supposition that the rate of formation of invert sugar is a uniform and continuous process of retardation. This

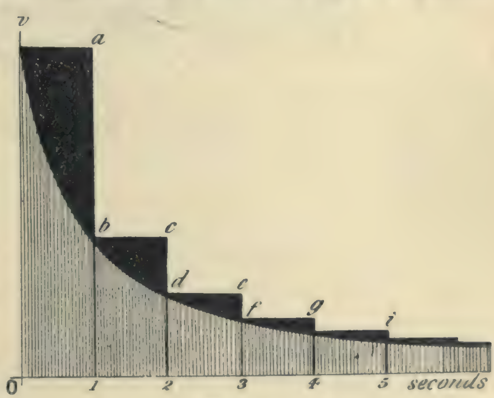


FIG. 84.

is the real nature of the change. But we have pretended that the velocity remains constant during a short but finite interval of time say $\delta t = 1$ second. The amount of cane sugar inverted during the first second is, therefore, represented by the area $va1O$ (Fig. 84); during the second interval by the area $bc21$, and so on.

At the end of the first interval the velocity at a is supposed to suddenly fall to b, whereas, in reality, the decrease should be represented by the gradual slope of the curve vb.

The error resulting from the inexact nature of this "simplifying assumption" is graphically represented by the blackened area vab; for succeeding intervals the error is similarly represented by bcd, def, . . . In Fig. 85, by halving the interval, we have consider-

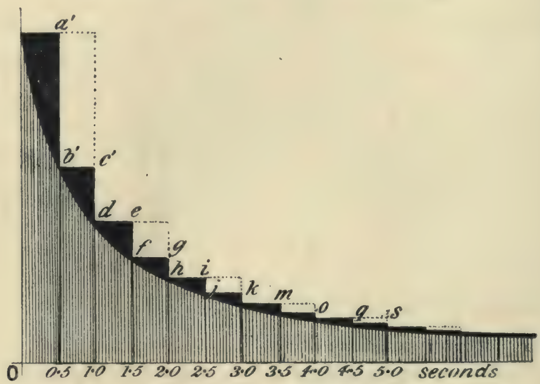


Fig. 85.

ably reduced the magnitude of the error. This is shown by the diminished area of the blackened portions for the first and succeeding seconds of time. *The smaller we make the interval, the less the error, until, at the limit, when the interval is made infinitely small, the result is absolutely correct.* The amount of invert sugar formed during the first five seconds is then represented by the area $vbdf$ . . . $5O$.

The above reasoning will repay careful study; once mastered, the "methods of integration" are, in general, mere routine work.

The operation denoted by the symbol "$\int$"* is called **integration.** When this sign is placed before a differential function, say

* The symbol "$\int$" is supposed to be the first letter of the word "sum". The first letter of the differential dx is the initial letter of the word "difference".

dx, it means that the function is to be integrated with respect to dx. *Integration is essentially a method for obtaining the sum of an infinite number of infinitely small quantities.*

Not only can the amount of substance formed in a chemical reaction during any given interval of time be expressed in this manner, but all sorts of varying magnitudes can be subject to a similar operation.

The *distance* passed over by a train travelling with a known velocity, can be represented in terms of a definite integral. The *quantity of heat* necessary to raise the temperature (θ) of a given mass (m) of a substance from $\theta°_1$ to $\theta°_2$, is given by the integral $\int_{\theta_1}^{\theta_2} m\sigma \, . \, d\theta$, where σ denotes the specific heat of the substance.

The *work* done by a variable force (F) when a body changes its position from s_0 to s_1 is $\int_{s_0}^{s_1} F \, . \, ds$. This is called a **space integral.** The impulse (magnitude of impressed force) due to a variable force F, acting during the interval of time $t_2 - t_1$, is given by the **time integral** $\int_{t_1}^{t_2} F \, . \, dt$. By *Newton's second law*, the change of momentum of any mass (m), is proportional to the impressed force (impulse). Momentum is defined as the product of the mass into the velocity. If, when t is t_1, $v = v_1$ and when t is t_2, $v = v_2$, Newton's law may be written

$$\int_{v_1}^{v_2} m \, . \, dv = \int_{t_1}^{t_2} F \, . \, dt.$$

The quantity of heat developed in a conductor during the passage of an electric current of intensity i, for a short interval of time dt is given by the expression $ki \, . \, dt$ (*Joule's law*), where k is a constant depending on the nature of the circuit. If the current remains constant during any short interval of time, the amount of heat generated by the current during the interval of time $t_2 - t_1$, is given by the integral $\int_{t_1}^{t_2} ki \, . \, dt$.

The quantity of gas (q) consumed in a building during any interval of time $t_2 - t_1$, may be represented as a definite integral,

$$q = \int_{t_1}^{t_2} v \, . \, dt,$$

where v denotes the velocity of efflux of the gas from the burners. The value of q can be read off on the dial of the gas meter at any time. The gas meter performs the integration automatically. The cyclometer of a bicycle can be made to integrate, $s = \int_{t_1}^{t_2} v \,.\, dt$ (v = velocity, t = time, s = distance traversed).

Differentiation and integration are reciprocal operations in the same sense that multiplication is the inverse of division, addition, of subtraction. Thus,

$$a \times b \div b = a;\ a + b - b = a.$$
$$d\int a \,.\, dx = a \,.\, dx;\ \int dx = x.$$

The differentiation of an integral, or the integration of a differential always gives the original function. The signs of differentiation and of integration mutually cancel each other. The integral, $\int f'(x)dx$, is sometimes called an *anti-differential.* Integration reverses the operation of differentiation and restores the differentiated function to its original value, but with certain limitations to be indicated later on.

While any mathematical function can be differentiated without any particular difficulty, the reverse operation of integration is not always so easy, in some cases, it cannot be done at all. For instance, the integrals $\int e^{x^2} . \,dx$ and $\int \frac{dx}{\sqrt{(x^3 + 1)}}$ have not yet been evaluated.

If, however, the function from which the differential has been derived, is known, the integration can always be performed. Knowing that $d(\log x) = x^{-1} . \,dx$, it follows at once that $\int x^{-1} . \,dx = \log x$.

In many cases, we have to compare the integral with a tabulated list of the results of the differentiation of known functions. The reader will find it an advantage to keep such a list of known integrals at hand. A set of standard types is given in the next section, but this list should be extended.

The Nature of Mathematical Reasoning may now be defined with greater precision than was possible in § 1. There, stress was laid upon the search for constant relations between observed facts. But the best results in science have been won by anticipating Nature by means of the so-called working hypothesis. The investigator first endeavours to reproduce his ideas in the form of a mathematical equation representing the momentary state of the phenomenon.* Thus Wilhelmy's law (1850) is nothing more than

* Mathematical equations containing differentials or differential coefficients, are called differential equations.

the mathematician's way of stating an old, previously unverified, speculation of Berthollet (1779); while Guldberg and Waage's law (1864-69) is still another way of expressing the same thing.

To test the consequences of Berthollet's hypothesis, it is clearly necessary to find the amount of chemical action taking place during intervals of time accessible to experimental measurement. It is obvious that Wilhelmy's equation in its present form will not do, but by "methods of integration" it is easy to show that if

$$\frac{dx}{dt} = k(1 - x) \text{ then } k = \frac{1}{t} . \log \frac{1}{1 - x},$$

where x denotes the amount of substance transformed during the time t. x is measurable, t is measurable. We are now in a position to compare the fundamental assumption with observed facts. If Berthollet's guess is a good one, $\frac{1}{t} . \log \frac{1}{1 - x}$ must have a constant value. But this is work for the laboratory, not the study, as indicated in connection with Newton's law of cooling, § 18.

Integration, therefore, bridges the gap between theory and fact by reproducing the hypothesis in a form suitable for experimental verification, and, at the same time, furnishes a direct answer to the two questions raised at the beginning of this section. We shall return to the above physical process after we have gone through a drilling in the methods to be employed for the integration of expressions in which the variables are so related that all the x's and dx's can be collected to one side of the equation, all the y's and dy's to the other. In Chapter VII., we shall have to study the integration of equations representing more complex natural processes.

If the mathematical expression of our ideas leads to equations which cannot be integrated, the working hypothesis will either have to be verified some other way,* or else relegated to the great repository of unverified speculations.

§ 70. Table of Standard Integrals.

Every differentiation in the differential calculus, corresponds with an integration in the integral calculus. Sets of corresponding functions are called "Tables of Integrals".

* Say, by slipping in another "simplifying assumption". Clairaut expressed his ideas of the moon's motion in the form of a set of complicated differential equations, but left them in this incomplete stage with the invitation, "Now integrate them who can". But see §§ 107, 108, and 144.

The following are the more important; handy for reference, better still for memorising:—

TABLE I.—STANDARD INTEGRALS.

Function.	Differential Calculus.	Integral Calculus.	
$u = x^n$.	$\frac{du}{dx} = nx^{n-1}$.	$\int x^n dx = \frac{x^{n+1}}{n+1}$.	(1)
$u = a^x$.	$\frac{du}{dx} = a \log a$.	$\int a^x dx = \frac{a^x}{\log_e a}$.	(2)
$u = e^x$.	$\frac{du}{dx} = e^x$.	$\int e^x dx = e^x$.	(3)
$u = \log_a x$.	$\frac{du}{dx} = \frac{1}{x} \log_a e$.	$\int \frac{dx}{x} = \log x$, or $\frac{\log_a x}{\log_a e}$.	(4)
$u = \log_e x$.	$\frac{du}{dx} = \frac{1}{x} \log_e e = \frac{1}{x}$.		
$u = \sin x$.	$\frac{du}{dx} = \cos x$.	$\int \cos x dx = \sin x$.	(5)
$u = \cos x$.	$\frac{du}{dx} = - \sin x$.	$\int \sin x dx = - \cos x$.	(6)
$u = \tan x$.	$\frac{du}{dx} = \sec^2 x$.	$\int \sec^2 x dx = \tan x$.	(7)
$u = \cot x$.	$\frac{du}{dx} = - \operatorname{cosec}^2 x$.	$\int \operatorname{cosec}^2 x = - \cot x$.	(8)
$u = \sec x$.	$\frac{du}{dx} = \frac{\sin x}{\cos^2 x}$.	$\int \frac{\sin x}{\cos^2 x} dx = \sec x$.	(9)
$u = \operatorname{cosec} x$.	$\frac{du}{dx} = - \frac{\cos x}{\sin^2 x}$.	$\int \frac{\cos x}{\sin^2 x} dx = - \operatorname{cosec} x$.	(10)
$u = \sin^{-1} x$.	$\frac{du}{dx} = \frac{1}{\sqrt{(1 - x^2)}}$.	$\int \frac{dx}{\sqrt{(1 - x^2)}} = \sin^{-1} x$.	(11)
$u = \cos^{-1} x$.	$\frac{du}{dx} = - \frac{1}{\sqrt{(1 - x^2)}}$.	$= - \cos^{-1} x$.	(12)
$u = \tan^{-1} x$.	$\frac{du}{dx} = \frac{1}{1 + x^2}$.	$\int \frac{dx}{1 + x^2} = \tan^{-1} x$.	(13)
$u = \cot^{-1} x$.	$\frac{du}{dx} = - \frac{1}{1 + x^2}$.	$= - \cot^{-1} x$.	(14)
$u = \sec^{-1} x$.	$\frac{du}{dx} = \frac{1}{x\sqrt{(x^2 - 1)}}$.	$\int \frac{dx}{x\sqrt{(x^2 - 1)}} = \sec^{-1} x$.	(15)
$u = \operatorname{cosec}^{-1} x$.	$\frac{du}{dx} = - \frac{1}{x\sqrt{(x^2 - 1)}}$.	$= - \operatorname{cosec}^{-1} x$	(16)
$u = \operatorname{vers}^{-1} x$.	$\frac{du}{dx} = \frac{1}{\sqrt{(2x - x^2)}}$.	$\int \frac{dx}{\sqrt{(2x - x^2)}} = \operatorname{vers}^{-1} x$.	(17)
$u = \operatorname{covers}^{-1} x$.	$\frac{du}{dx} = - \frac{1}{\sqrt{(2x - x^2)}}$.	$= - \operatorname{covers}^{-1} x$.	(18)

§ 71. The Simpler Methods of Integration.

(1) *Integration of the product of a constant term and a differential.* On page 24, it was pointed out that "the differential of

the product of a variable and a constant, is equal to the constant multiplied by the differential of the variable". It follows directly that the integral of the product of a constant and a differential, is equal to the constant multiplied by the integral of the differential. *E.g.*,

$$\int a \,.\, dx = a\int dx = ax.$$
$$\int \log a \,.\, dx = \log a \int dx = x \,.\, \log a.$$

On the other hand, the value of an integral is altered if a term containing one of the variables is placed *outside* the integral sign. For instance, the reader will see very shortly that while

$$\int x^2 dx = \tfrac{1}{3}x^3 \,;\; x\int x dx = \tfrac{1}{2}x^3.$$

(2) *A constant term must be added to every integral.* It has been shown that a constant term always disappears from an expression during differentiation, thus,

$$d(x + C) = dx.$$

This is equivalent to stating that there is an infinite number of expressions, differing only in the value of the constant term, which, when differentiated, produce the same differential. In stating the result of any integration, therefore, we must provide for any possible constant term, by adding on an undetermined, "empirical," or "arbitrary" constant, called the **constant of integration,** and usually represented by the letter C. Thus,

$$\int du = u + C.$$

If
$$dy = dx,$$
$$\int dy + C_1 = \int dx + C_2 \,;$$
$$y + C_1 = x + C_2 \,;\text{ or, } y = x + C,$$

where $C = C_2 - C_1$.

The geometrical signification of this constant is analogous to that of "b" in the tangent form of the equation of the straight line, formula (8), page 69; thus, the equation

$$y = mx + b,$$

represents an infinite number of straight lines, each one of which has a slope m to the x-axis and cuts the y-axis at some point b. An infinite number of values may be assigned to b. Similarly, an infinite number of values may be assigned to C in $\int \ldots dx + C$.

According to Table I.,

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}x = -\cos^{-1}x \,;\; \int \frac{dx}{\sqrt{1 + x^2}} = \tan^{-1}x = -\cot^{-1}x,$$

etc. This means that $\sin^{-1}x$, $\cos^{-1}x$; or $\tan^{-1}x$, $\cot^{-1}x$, . . .

only differ by a constant term. This agrees with the trigonometrical properties of these functions illustrated in example (1), page 113. See also § 106. The following remarks are worth thinking over:

"Fourier's theorem is a most valuable tool of science, practical and theoretical, but it necessitates adaptation to any particular case by the provision of exact data, the use, that is, of definite figures which mathematicians humorously call 'constants,' because they vary with every change of condition. A simple formula is $n + n = 2n$, so also $n \times n = n^2$. In the concrete, these come to the familiar statement that 2 and 2 equals 4. So in the abstract, $40 + 40 = 80$, but in the concrete two 40 ft. ladders will in no way correspond to one 80 ft. ladder. They would require something else to join them end to end and to strengthen them. That something would correspond to a 'constant' in the formula. But even then we could not climb 80 ft. into the air unless there was something to secure the joined ladder. We could not descend 80 ft. into the earth unless there was an opening, nor could we cross an 80 ft. gap. For each of these uses we need something which is a 'constant' for the special case. It is in this way that all mathematical demonstrations and assertions need to be examined. They mislead people by their very definiteness and apparent exactness. . . ."—J. T. SPRAGUE.

(3) *Integration of a sum and of a difference.* Since

$$d(x + y + z + \ldots) = dx + dy + dz + \ldots,$$

it follows that

$$\int(dx + dy + dz + \ldots) = \int dx + \int dy + \int dz + \ldots,$$
$$= x + y + z + \ldots,$$

plus the arbitrary constant of integration.

It is customary to append the integration constant to the final result, not to the intermediate stages of the integration.

Similarly,

$$\int(dx - dy - dz - \ldots) = \int dx - \int dy - \int dz - \ldots$$
$$= x - y - z - \ldots + C.$$

EXAMPLES.—(1) Show

$$\int\{\log(a + bx)(1 + 2x)\}dx = \int\log(a + bx)dx + \int\log(1 + 2x)dx + C.$$

(2) Show $\int\log\frac{a + bx}{1 + 2x}dx = \int\log(a + bx)dx - \int\log(1 + 2x)dx + C.$

(4) *Integration of $x^n . dx$* (see page 22). Since

$$d(x^{n+1}) = (n + 1)x^n dx;\ x^n . dx = dx^{n+1}/(n + 1);$$

$$\therefore a\int x^n . dx = a\frac{x^{n+1}}{n + 1} + C. \quad . \quad . \quad . \quad (1)$$

To integrate any expression of the form $ax^n . dx$, it is, therefore, necessary to increase the index of the variable by unity, multiply *by any constant term that may be present, and divide the product by the new index.*

An apparent exception occurs when $n = -1$, for then

$$\int x^{-1}. dx = \frac{x^{-1+1}}{1-1} = \frac{1}{0} = \infty.$$

See page 224. We have seen, page 36, (6), that

$$d(\log x) = \frac{dx}{x} = x^{-1}. dx,$$

$$\therefore \int x^{-1}. dx = \log x + C. \qquad . \quad . \quad . \quad (2)$$

If, therefore, the numerator of a fraction can be obtained by the differentiation of its denominator, the integral is the natural logarithm of the denominator.

It is worth remembering that instead of writing $\log x + C$, we may put

$$\log x + \log c = \log cx,$$

for $\log c$ is an arbitrary constant as well as C.

Examples.—(1) Show $\int a\,.\,dx/bx = (a\,.\log x)/b + C$.

(2) Show $\int 2bx\,.\,dx/(a - bx^2) = -\log(a - bx^2) + C$.

(3) Show $\int ax^3\,.\,dx = \frac{1}{4}ax^4 + C$.

(4) Show $\int 4ax^{-1/5}\,.\,dx = 5ax^{4/5} + C$.

(5) One of the commonest equations in physical chemistry is,

$$dx = k(a - x)\,.\,dt.$$

Rearranging terms,

$$kt = \int \frac{dx}{a - x},$$

$$\therefore kt = -\log(a - x),$$

but $\log 1 = 0$,

$$\therefore kt = \log 1 - \log(a - x), \text{ or, } k = \frac{1}{t}\log\frac{1}{a - x} + C.$$

(6) *Wilhelmy's equation*,

$$\frac{dy}{dt} = -\alpha y, \text{ may be written } \int\frac{dy}{y} = -\alpha t.$$

Remembering that $\log e = 1$, we have

$$\log y = \log b - \alpha t \log e\,; \text{ or, } \log y = \log e^{-\alpha t} + \log b,$$

where $\log b$ is the integration constant, hence,

$$\log be^{-\alpha t} = \log y\,;\ y = be^{-\alpha t}.$$

The meaning of these constants will be deduced in the next section.

(7) By a similar method to that employed for evaluating $\int x^n dx$, $\int x^{-1}dx$, show

$$\int a^x dx = \frac{a^x}{\log_e a} + C\,;\ \int e^x dx = e^x + C\,;\ \int e^{-ax}dx = -\frac{1}{a}e^{-ax}. \qquad . \quad (3)$$

In the same way verify the results in Table I.

(8) Prove

$$-\int\frac{dx}{x^n} = \frac{1}{n-1}\cdot\frac{1}{x^{n-1}} + C, \qquad . \quad . \quad . \quad . \quad (4)$$

by differentiating the right-hand side. Keep your result for use later on.

(9) Evaluate $\int \sin^4 x \,.\, \cos x \,.\, dx$. Note that $\cos x dx = d(\sin x)$, and that $\sin^4 x$ is the mathematician's way of writing $(\sin x)^4$.*

$$\therefore \int \sin^4 x \,.\, \cos x \,.\, dx = \int \tfrac{1}{5} \sin^4 x \,.\, d(\sin x) = \tfrac{1}{5} \sin^5 x \,.\, + C.$$

(10) What is wrong with this problem: "Evaluate the integral $\int x^3$"? Hint, the symbol "$\int$" has no meaning apart from the accompanying "dx". For brevity, we call "$\int$" the symbol of integration, but the integral must be *written*, $\int \ldots dx$.

(5) *Integration of the product of a polynomial and its differential.* Read (3), page 24. This is a simple extension of the preceding. Since

$$d(ax^m + b)^n = n(ax^m + b)^{n-1} \,.\, amx^{m-1} \,.\, dx,$$

where $amx^{m-1} \,.\, dx$ has been obtained by differentiating the expression within the brackets,

$$\therefore n\int(ax^m + b)^{n-1} amx^{m-1} \,.\, dx = (ax^m + b)^n + C. \quad . \qquad (5)$$

To integrate the product of a polynomial with its differential, increase the index of the polynomial by unity and divide the result by the new exponent.

EXAMPLES.—(1) Show $\int(3ax^3 + 1)^2 9ax^2 \,.\, dx = \frac{1}{3}(3ax^3 + 1)^3 + C$.

(2) Show $\int(x + 1)^{-2/3} \,.\, dx = 3(x + 1)^{1/3} + C$.

(6) *Integration of expressions of the type:*

$$(a + bx + cx^2 + \ldots)^m x dx, \quad . \quad . \quad . \qquad (6)$$

where m is a positive integer. Multiply out and integrate each term separately.

EXAMPLES.—(1) Show $\int(1 + x)^2 x^3 dx = (\frac{1}{4} + \frac{2}{5}x + \frac{1}{6}x^2)x^4 + C$.

(2) $\int(a + x^{\frac{1}{2}})^2 x^{\frac{1}{2}} dx = (\frac{2}{3}a^2 + x^{\frac{1}{2}} + \frac{2}{5}x)x^{\frac{3}{2}} + C$.

The favourite methods for integration are by processes known as "the substitution of a new variable," "integration by parts" and by "resolution into partial fractions". The student is advised to pay particular attention to these operations. Before proceeding to the description of these methods, we shall return once more to the integration constant.

§ 72. How to find a Value for the Integration Constant.

It is perhaps unnecessary to remind the reader that integration constants must not be confused with the constants belonging to the

---

* But we must *not* write $\sin^{-1}x$ for $(\sin x)^{-1}$, nor $(\sin x)^{-1}$ for $\sin^{-1}x$. Sin^{-1}, $\cos^{-1}$, $\tan^{-1}$, . . . have the *special* meaning pointed out in § 15.

original equation. For instance, in the law of descent of a falling body

$$dv/dt = g;\ \int dv = g\int dt, \text{ or, } v = gt + C. \quad . \quad . \qquad (1)$$

Here g is a constant representing the increase of velocity due to the earth's attraction, C is the constant of integration. The student will find some instructive remarks in § 118.

There are two methods in general use for the evaluation of the integration constant.

FIRST METHOD.—Returning to the falling body and to its equation of motion,

$$v = gt + C.$$

On attempting to apply this equation to an actual experiment, we should find that, at the moment we began to calculate the velocity, the body might be moving upwards or downwards, or starting from a position of rest. All these possibilities are included in the integration constant C. Let v_0 denote the initial velocity of the body. The computation begins when $t = 0$, hence

$$v_0 = g \times 0 + C, \text{ or, } C = v_0.$$

If the body starts to fall from a position of rest, $v_0 = C = 0$, and

$$\int dv = gt, \text{ or, } v = gt.$$

This suggests a method for evaluating the constant whenever the nature of the problem permits us to deduce the value of the function for particular values of the variable.

If possible, therefore, substitute particular values of the variables in the equation containing the integration constant and solve the resulting expression for C.

EXAMPLE.—Find the value of C in the equation

$$t = \frac{1}{k}\log\frac{1}{a - x} + C, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

which is a standard "velocity equation" of physical chemistry. t represents the time required for the formation of an amount of substance x. When the reaction is just beginning, $x = 0$ and $t = 0$. Substitute these values of x and t in (2).

$$\frac{1}{k}\log\frac{1}{a} + C = 0, \text{ or, } C = -\frac{1}{k}\log\frac{1}{a}.$$

Substitute this value of C in the given equation and we get

$$t = \frac{1}{k}\left(\log\frac{1}{a - x} - \log\frac{1}{a}\right) = \frac{1}{k}\log\frac{a}{a - x}.$$

SECOND METHOD.—Another way is to find the values of x corresponding to two different values of t. Substitute the two sets of results in the given equation. The constant can then be made to disappear by subtraction.

EXAMPLE.—In the above equation, (2), assume that when $t = t_1$, $x = x_1$, and when $t = t_2$, $x = x_2$; where x_1, x_2, t_1 and t_2 are numerical measurements. Substitute these results in (2).

$$t_1 = \frac{1}{k}\log\frac{1}{a - x_1} + C;\ t_2 = \frac{1}{k}\log\frac{1}{a - x_2} + C.$$

By subtraction and rearrangement of terms

$$t_2 - t_1 = \frac{1}{k}\log\frac{a - x_1}{a - x_2}.$$

The result of this method is to *eliminate*, not *evaluate* the constant.

Numerous examples of both methods will occur in the course of this work. Some have already been given in the discussion on the "Compound Interest Law in Nature".

§ 73. Integration by the Substitution of a New Variable.

When a function can neither be integrated by reference to Table I., nor by the methods of § 71, a suitable change of variable may cause the function to assume a less refractory form. The new variable is, of course, a known function of the old.

This method of integration is, perhaps, best explained by the study of a few typical examples.

(1) *Evaluate* $\int(a + x)^n dx$. Put $a + x = y$, therefore, $dx = dy$ and

$$\int(a + x)^n dx = \int y^n dy.$$

From (1), page 158,

$$\int y^n dy = y^{n+1}/(n + 1) + C.$$

Substitute for y,

$$\int(a + x)^n dx = (a + x)^{n+1}/(n + 1) + C. \quad . \quad . \quad . \qquad (1)$$

EXAMPLES.—Integrate the following expressions:—

(1) $\int(a - bx)^n dx$. Ansr. $-(a - bx)^{n+1}/(n + 1) + C$.

(2) $\int(a^2 + x^2)^{-1/2}x dx$. Ansr. $\sqrt{(a^2 + x^2)} + C$.

(3) $\int(a + x)^{-m}dx$. Ansr. $-1/(m - 1)(a + x)^{m-1} + C$. Keep this result for future reference.

(4) $\int\frac{dx}{x\,.\log x}$. Ansr. $\log(\log x) + C$.

When the student has become familiar with integration he will find no particular difficulty in doing these examples mentally.

(2) *Integrate* $(1 - ax)^m x^n dx$. Put $y = 1 - ax$, therefore,

$$x = (1 - y)/a \text{ and } dx = -dy/a.$$

Substitute these values of x and dx in the original equation.

$$\int(1 - ax)^m x^n dx = - \frac{1}{a^{n+1}}\int(1 - y)^n y^m dy,$$

which is the type of (6), page 162. The rest of the work is obvious. Method (6), § 71.

(3) *Trignometrical functions* can often be integrated by these methods. For example, required the value of $\int \tan x dx$.

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Let $\cos x = u$, $- \sin x dx = du$. Since $- \int du/u = - \log u$, and $\log 1 = 0$.

$$\int \tan x dx = \log \frac{1}{\cos x} = \log \sec x + C.$$

Examples.—Show that (1) $\int \sin x \,.\, \cos x \,.\, dx = \frac{1}{2}\sin^2 x + C$.

(2) $\int(1 + x)^{1/2}x^2 dx = \frac{2}{35}(1 + x)^{3/2}(5x - 4x + \frac{8}{3}) + C$.

(3) $\int \cot x dx = \log \sin x + C$.

(4) $\int \sin x \,.\, dx/\cos^2 x = \sec x + C$.

(5) $\int \cos x \,.\, dx/\sin^2 x = - \operatorname{cosec} x + C$.

(6) Evaluate $\int e^{-x^2} x dx$. Multiply and divide by -2

$$\int e^{-x^2} x dx = - \tfrac{1}{2}\int e^{-x^2} d(- x^2) = - \tfrac{1}{2}e^{-x^2} + C. \quad . \quad . \quad (2)$$

(4) *Integrate* $\frac{dx}{\sqrt{(a^2 - x^2)}}$. Put $y = x/a$, $\therefore x = ay$, $dx = ady$, $\sqrt{(a^2 - x^2)} = a\sqrt{1 - y}$;

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dy}{\sqrt{1 - y^2}} = \sin^{-1} y = \sin^{-1}\frac{x}{a} + C.$$

See page 166.

Example.—Integrate $\frac{\sqrt{(a^2 - x^2)}}{x^4} dx$, by substituting $x = \frac{1}{t}$.

Ansr. $- \sqrt{(a^2 - x^2)^3} \,.\, /3a^2 + C$.

(5) Some expressions require a little "humouring". Facility in this art can only be acquired by practice. A glance over the collection of formulae in Chapter XII. will often give a clue. In this way, we find that $\sin x = 2 \sin \frac{1}{2}x \,.\, \cos \frac{1}{2}x$. Hence integrate

$$\int \frac{dx}{\sin x}, \ i.e. \int \frac{dx}{2 \sin \frac{1}{2}x \,.\, \cos \frac{1}{2}x} \text{ or } \int \frac{\sec \frac{1}{2}x \,.\, dx}{2 \sin \frac{1}{2}x}.$$

Divide the numerator and denominator by $\cos^2 \frac{1}{2}x$, then, since $1/\cos^2 \frac{1}{2}x = \sec^2 \frac{1}{2}x$ and $d(\tan x) = \sec^2 x \,.\, dx$, page 32, (3),

$$\therefore \int \frac{dx}{\sin x} = \int \frac{\sec^2 \frac{1}{2}x \,.\, d(\frac{1}{2}x)}{\tan \frac{1}{2}x} = \int \frac{d(\tan \frac{1}{2}x)}{\tan \frac{1}{2}x}$$

$$= \log \tan \tfrac{1}{2}x + C.$$

EXAMPLES.—(1) Remembering that $\cos x = \sin(\frac{1}{2}\pi + x)$, (8), page 499, show that $\int dx/\cos x = \log\tan(\frac{1}{4}\pi + \frac{1}{2}x) + C$.

(2) Integrate $\int\frac{dx}{\sin x \cdot \cos x}$. Hint, see (17), page 499.

$$\therefore \int\frac{\cos^2 x + \sin^2 x}{\sin x \cos x}dx = \int\frac{\cos x}{\sin x}dx + \int\frac{\sin x}{\cos x}dx = \log\tan x + C.$$

Here are a few useful though simple "tips" for special notice:

1. Any constant term may be added to the numerator of a fraction provided the differential sign is placed before it. The object of this is usually to show that the numerator of the given integral has been obtained by the differentiation of the denominator. If successful the integral reduces to the logarithm of the denominator. *E.g.*,

$$2\int\frac{xdx}{1 - x^2} = -\int\frac{d(1 - x^2)}{1 - x^2} = -\log(1 - x^2), \text{ etc.}$$

2. $\int\sin nx \cdot dx$ may be made to depend on the known integral $\int\sin nx \cdot d(nx)$ by multiplying and dividing by n. *E.g.*,

$$\int\cos nx \cdot dx = \frac{1}{n}\int\cos nx \cdot d(nx) = \frac{1}{n}\sin nx + C.$$

3. Add and subtract the same quantity. *E.g.*,

$$\int\frac{x \cdot dx}{1+2x} = \int\frac{(x + \frac{1}{2}) - \frac{1}{2}}{1 + 2x}dx = \int\Big(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{1 + 2x}\Big)dx, \text{ etc.}$$

4. Note the addition of log 1 makes no difference to the value of an expression, because $\log 1 = 0$; similarly, multiplication by $\log_e e$ makes no difference to the value of any term, because $\log_e e = 1$.

(6) It very frequently happens that an expression involving the square root of a quadratic binomial can be readily solved by the aid of a lucky trignometrical substitution. The form of the inverse trignometrical functions (Table I.) will sometimes guide us in the right choice. If the binomial has the forms:

$\sqrt{1 - x^2}$, or $\sqrt{a^2 - x^2}$, try $x = \sin\theta$, or $a\sin\theta$, or $\cos\theta$;

$\sqrt{x^2 - 1}$, or $\sqrt{x^2 - a^2}$, try $x = \sec\theta$, or $a\sec\theta$, or $\operatorname{cosec}\theta$;

$\sqrt{x^2 + 1}$, or $\sqrt{x^2 + a^2}$, try $x = \tan\theta$, or $a\tan\theta$, or $\cot\theta$.

(Lamb's *Infinitesimal Calculus*, p. 184; Williamson's *Integral Calculus*, p. 73.)

EXAMPLES.—(1) Find the value of $\int\sqrt{(a^2 - x^2)}dx$. In accordance with the above rule, put $x = a\sin\theta$, $\therefore dx = a\cos\theta \cdot d\theta$.

$$\therefore \int\sqrt{(a^2 - x^2)}dx = a^2\int\cos^2\theta d\theta;$$

and since $2\cos^2\theta = 1 + \cos 2\theta$, (28), page 500, we may continue,

$$= \tfrac{1}{2}a^2\int(1 + \cos 2\theta)d\theta,$$
$$= \tfrac{1}{2}a^2(\theta + \tfrac{1}{2}\sin 2\theta);$$

but $x = a\sin\theta$, $\theta = \sin^{-1}x/a$, and

$$\tfrac{1}{2}\sin 2\theta = \sin\theta \cdot \cos\theta = \sin\theta\sqrt{(1 - \sin^2\theta)},$$
$$= \sqrt{a^2 - x^2} \cdot x/a^2$$
$$\therefore \int\sqrt{(a^2 - x^2)}dx = \tfrac{1}{2}a^2\sin^{-1}x/a + \tfrac{1}{2}x\sqrt{a^2 - x^2} + C.$$

(2) Show $\int \frac{dx}{(1-x)\sqrt{1-x^2}} = \sqrt{\frac{1+x}{1-x}} + C$. Put $x = \cos\theta$.

If the beginner has forgotten his "trig." he had better verify these steps from the collection of trignometrical formulae in Chapter XII. See also (8), § 70.

$$= -\int \frac{\sin\theta d\theta}{(1-\cos\theta)\sqrt{1-\cos^2\theta}} = -\int \frac{d\theta}{1-\cos\theta} = -\frac{1}{2}\int \frac{d\theta}{\sin^2\frac{1}{2}\theta} = -\int \operatorname{cosec}^2\frac{\theta}{2} . d\left(\frac{\theta}{2}\right);$$

$$= \cot\tfrac{1}{2}\theta = \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \sqrt{\frac{2\cos^2\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+x}{1-x}}.$$

(3) Show $\int \frac{dx}{\sqrt{(x^2+1)}} = \log(x + \sqrt{x^2+1}) + C$. Put $x = \tan\theta$. Note that $\tan(\frac{1}{4}\pi + \frac{1}{2}\theta) = \tan\theta + \sec\theta = x + \sqrt{(x^2+1)}$; $\sec\theta d\theta = \log(\tan\theta + \sec\theta)$.

(4) Show $\int \frac{dx}{\sqrt{(x^2-1)}} = \log(x + \sqrt{x^2-1}) + C$. Put $x = \sec\theta$.

(7) *The integration of* $x^{m-1}(a + bx^n)^p . dx$. (See § 76, below.)

(i.) *If p is a positive integer.* Expand the binomial and treat as on page 162.

(ii.) *If p is fractional*, say $p = r/s$;

(*a*) *Let m/n be a positive integer.* Substitute a new variable with an index equal to the denominator of the fractional index p, so as to make $a + bx^n = z^s$. Then proceed as follows:—

EXAMPLE.—Evaluate $\int x^5(1+x^2)^{1/2}dx$. Here $m = 6$; $n = 2$; $p = \frac{1}{2}$. Put

$$1 + x^2 = z^2, \text{ then, } x^2 = z^2 - 1;$$

$$\therefore z = \sqrt{1+x^2};\ x . dx = z . dz.$$

Substitute these values as required in the original expression,

$$\begin{aligned}\int x^5(1+x^2)^{1/2}dx &= \int (z^2-1)^2 z^2 dz,\\ &= \int (z^6 - 2z^4 + z^2)dz,\\ &= \tfrac{1}{7}z^7 - \tfrac{2}{5}z^5 + \tfrac{1}{3}z^3,\\ &= \tfrac{1}{105}(1+x^2)^{3/2}\{15(1+x^2)^2 + 42(1+x^2) + 35\} + C.\end{aligned}$$

(*b*) *Let m/n be a negative integer.*

EXAMPLE.—Evaluate $\int x^{-4}(1+x^2)^{-1/2}dx$. Here $m = 3$; $n = 2$; $p = \frac{1}{2}$. Put

$$1 + x^2 = z^2x^2;\ \therefore x^{-2} = z^2 - 1;\ \therefore x^{-4} = (z^2-1)^2;\ \therefore x = (z^2-1)^{-1/2};$$

$$dx = -(z^2-1)^{-3/2}zdz;\ (1+x^2)^{-1/2} = 1/zx = \sqrt{(z^2-1)}/z.$$

Hence, $\int x^{-4}(1+x^2)^{-1/2}dx = -\int (z^2-1)dz;$

$$= -z^3/3 + z = (2x^2-1)(1+x^2)^{1/2}/3x^3 + C.$$

(*c*) *m/n + p is integral.* The last example comes under this head.

§ 74. Integration by Parts.

On page 26, it was shown that

$$d(uv) = vdu + udv.$$

Now integrate both sides

$$uv = \int vdu + \int udv.$$

Hence,
$$\int udv = uv - \int vdu + C, \quad . \quad . \quad . \quad (1)$$
that is to say, the integral of udv can be obtained provided vdu can be integrated. This is called integration by parts.

EXAMPLES.—Evaluate the following expressions :—

(1) $\int x \log x dx$. Put

$$\left.\begin{array}{r} u = \log x, \\ du = dx/x, \end{array}\right| \begin{array}{l} dv = x \,.\, dx ; \\ v = \frac{1}{2}x^2. \end{array}$$

Substitute in (1)

$$\int u \,.\, dv = \int x \log x \,.\, dx = uv - \int v \,.\, du,$$
$$= \tfrac{1}{2}x^2 \log x - \int \tfrac{1}{2}x \,.\, dx = \tfrac{1}{2}x^2 \log x - \tfrac{1}{4}x^2,$$
$$= \tfrac{1}{2}x^2 (\log x - \tfrac{1}{2}) + C.$$

(2) $\int x \cos nx \,.\, dx$ Put

$$\left.\begin{array}{r} u = x, \\ du = dx, \end{array}\right| \begin{array}{l} dv = \{\cos nx \,.\, d(nx)\}/n ; \\ v = (\sin nx)/n. \end{array}$$

From (1), $\int x \cos nx \,.\, dx = (x \sin nx)/n - \int (\sin nx \,.\, dx)/n$; etc.

(3) Show by "integration by parts" that

$$\int x^2 \sin x \,.\, dx = (2 - x^2) \cos x + 2x \sin x + C.$$

In this example there are two integrations to be performed, first $x^2 \cos x \,.\, dx$, and then $x \cos x \,.\, dx$.

(4) Solve the equation,

$$dv = a(v_0 - 2v) \,.\, dt/\sqrt{\rho},$$

where v_0, a, ρ are constant and $v = 0$ when $t = 0$. Ansr.

$$\tfrac{1}{2} \log v_0 - \tfrac{1}{2} \log (v_0 - 2v) = at/\sqrt{\rho}.$$

(5) $\int xe^x dx = (x - 1)e^x + C$. Prove this.

The selection of the proper values of u and v is to be determined by trial. A little practice will enable one to make the right selection instinctively. The rule is that the integral $\int v \,.\, du$ must be more easily integrated than the given expression. In this example if we take $u = e^x$, $dv = xdx$, $\int v \,.\, du$ becomes $\frac{1}{2}\int x^2 e^x dx$, a more complex integral than the one to be reduced. The right choice is $u = x$, $dv = e^x dx$.

(6) Show $\int x^2 e^x dx = (x^2 + 2x - 2)e^x + C$.

(7) Evaluate by "integration by parts," $\int \sqrt{(a^2 - x^2)} dx$. Put

$$\left.\begin{array}{r} u = \sqrt{(a^2 - x^2)}, \\ du = x \,.\, dx / \sqrt{(a^2 - x^2)}, \end{array}\right| \begin{array}{l} dv = dx ; \\ v = x. \end{array}$$

$$\int \sqrt{(a^2 - x^2)} dx = x\sqrt{a^2 - x^2} + \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)}},$$
$$= x\sqrt{a^2 - x^2} + \frac{a^2 - (a^2 - x^2)dx}{\sqrt{(a^2 - x^2)}},$$
$$= x\sqrt{a^2 - x^2} + \int \frac{a^2 dx}{\sqrt{(a^2 - x^2)}} - \int \sqrt{(x^2 - a^2)} dx.$$

Transpose the last term to the left-hand side;

$$2\int\sqrt{a^2 - x^2}\,.\,dx = x\sqrt{a^2 - x^2} + a \sin^{-1}x/a \text{ (page 158)},$$

$$\therefore \int\sqrt{(a^2 - x^2)}dx = \tfrac{1}{2}a^2 \sin^{-1}x/a + \tfrac{1}{2}x\sqrt{(x^2 - x^2)} + C.$$

§ 75. Integration by Successive Reduction.

A complex integral can often be reduced to one of the standard forms by the "method of integration by parts". By a repeated application of this method, complicated expressions may often be integrated, or else, if the expression cannot be integrated, the non-integrable part may be reduced to its simplest form. See examples (3) and (6), § 74.

EXAMPLES.—(1) Evaluate $\int x^2\cos nx dx$. Put

$$u = x^2, \quad du = 2xdx, \quad \Big| \quad dv = \{\cos nx\,.\,d(nx)\}/n; \quad v = (\sin nx)/n.$$

Hence, from (1),

$$\int x^2\cos nx dx = \frac{x^2\sin nx}{n} - \frac{2}{n}\int x\sin nx dx. \quad . \quad . \quad . \quad (1)$$

Now put

$$u = x, \quad du = dx, \quad \Big| \quad dv = \sin nx\,.\,dx; \quad v = -(\cos nx)/n.$$

Hence,

$$\int x\sin nx\,.\,dx = -(x\cos nx)/n - \int(-\cos nx\,.\,dx)/n,$$

$$= (-x\cos nx)/n + (\sin nx)/n^2. \quad . \quad . \quad . \quad (2)$$

Now substitute (2) in (1) and we get,

$$\int x^2\cos nx dx = \frac{x^2\sin.nx}{n} + \frac{2x\cos nx}{n^2} + \frac{2\sin nx}{n^3} + C.$$

In this example, we have made the integral $\int x^2\cos nx\,.\,dx$ depend on that of $x\sin nx\,.\,dx$, and this, in turn, on that of $-\cos nx\,.\,d(nx)$, which is a known standard form.

(2) Evaluate $\int x^4\cos x\,.\,dx$. Put

$$u = x^4, \quad du = 4x^3dx, \quad \Big| \quad dv = \cos xdx; \quad v = \sin x.$$

$$\therefore \int x^4\cos xdx = x^4\sin x - 4\int x^3\sin xdx.$$

In the same way,

$$4\int x^3\sin xdx = 4x^3\cos x - 3\,.\,4\int x^2\cos xdx.$$

Similarly,

$$3\,.\,4\int x^2\cos xdx = 3\,.\,4\,.\,x^2\sin x + 2\,.\,3\,.\,4\int x\sin xdx,$$

and finally,

$$2\,.\,3\,.\,4\int x\sin xdx = 2\,.\,3\,.\,4x\cos x + 1\,.\,2\,.\,3\,.\,4\sin x.$$

All these values must be collected together, as in the first example. In this way, the integral is reduced, by successive steps, to one of simpler form. The integral $\int x^4\cos xdx$ was made to depend on that of $x^3\sin xdx$, this, in turn, on that of $x^2\cos xdx$, and so on until we finally got $\int\cos xdx$, a well-known standard form.

It is an advantage to have two separate sheets of paper in working through these examples; on one work as in the preceding examples and on the other enter the results as in the next example.

(3) Integrate $\int x^3e^xdx$.

$$\int x^3e^xdx = x^3e^x - 3\int x^3e^xdx,$$
$$= x^3e^x - 3(x^2e^x - 2\int xe^xdx),$$
$$= x^3e^x - 3x^2e^x + 2.3(xe^x - \int e^xdx),$$
$$= (x^3 - 3x^2 + 6x - 6)e^x + C.$$

(4) Integrate $\cos^n xdx$; $\sin^n xdx$ and $\sin^m x.\cos^n xdx$ (see page 184).

§ 76. Reduction Formulae (*for reference*).

In § 74, we found it convenient to refer certain integrals to a "standard formula". In § 75, we reduced a complex integral to simpler terms by a repeated application of the same formula. Such a formula is called a **reduction formula.**

The following standard reduction formulae are convenient for reference, others will be found in § 79 and elsewhere.

A. The integral $\int x^m(a + bx^n)^p.dx$, may be made to depend on that of $\int x^{m-n}(a + bx^n)^{p+1}.dx$, through the reduction formula:

$$\int x^m(a+bx^n)^p.dx = \frac{x^{m-n+1}(a + bx)^{p+1} - a(m - n + 1)\int x^{m-n}(a + bx^n)^p.dx}{b(m + np + 1)}, \quad \textbf{(A)}$$

where *m is a positive integer*. This formula may be applied successively until the factor outside the brackets, under the integral sign, is less than n. Then proceed as on page 162.

B. In **A,** m must be positive, otherwise the index will increase, instead of diminish, by a repeated application of the formula. Therefore, when m is negative, transpose **A** and divide by $a(m - n + 1)$. Thus,

$$\int x^m(a+bx^n)^p.dx = \frac{x^{m+1}(a+bx^n)^{p+1} - b(np+m+n+1)\int x^{m+n}(a+bx^n)^p.dx}{a(m + 1)}, \quad \textbf{(B)}$$

where *m is negative.*

C. Another useful formula diminishes the exponent of the bracketed term in the following manner:—

$$\int x^m(a + bx^n)^p.dx = \frac{x^{m+1}(a + bx^n)^p + anp\int x^m(a + bx^n)^{p-1}dx}{m + np + 1}, \quad \textbf{(C)}$$

where *p is positive.*

D. If *p is negative,*

$$\int x^m(a+bx^n)^pdx = -\frac{x^{m+1}(a+bx^n)^{p+1} + (np+m+n+1)\int x^m(a+bx^n)^{p+1}dx}{an(p + 1)}. \quad \textbf{(D)}$$

Formulae **A, B, C, D** have been deduced by the method of integration by parts. Perhaps the reader can do this for himself.

NOTE.—Formula **A** *decreases* (algebraically) the exponent of the monomial factor while **B** *increases* the exponent of the same factor. Formula **C** *decreases* the exponent of the binomial factor while **D** *increases* the exponent of the binomial factor.

EXAMPLES.—Evaluate the following integrals :—

(1) $\int\sqrt{(a + x^2)}dx$. Hints, use **C**. Put $m = 0, b = 1, n = 2, p = \frac{1}{2}$. Ansr. $\frac{1}{2}[x\sqrt{(a + x^2)} + a\log\{x + \sqrt{(a + x^2)}\}] + C$.

(2) $\int x^4 dx / \sqrt{(a^2 - x^2)}$. Hints, put $m = 4, b = 1, n = 2, p = \frac{1}{2}$. Use **A** twice. Ansr. $\frac{1}{8}\{3a^4\sin x/a - x(2x^2 + 3a^2)\sqrt{(a^2 - x^2)} + C$.

(3) $\int x^3 dx / \sqrt{(1 - x^2)}$. Hint, use **A**. Ansr. $-\frac{1}{3}(x^2 + 2)\sqrt{(1 - x^2)} + C$.

(4) $\int\sqrt{(a + bx^2)}^{-3}dx$. Ansr. $x(a + bx)^{-1/2}/a + C$. Use **D**.

(5) $\int\frac{dx}{x^3\sqrt{x^2 - a^2}}$, *i.e.* $\int x^{-3}(-a^2 + x^2)^{-\frac{1}{2}}dx$. Hint, use **B**. $m = -3, b = 1$, $n = 2, p = -\frac{1}{2}$. Ansr. $\frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a}$.

§ 77. Integration by Resolution into Partial Fractions.

Fractions containing higher powers of x in the numerator than in the denominator, may be reduced to a whole number and a fractional part. Thus, by division,

$$\frac{x^5 . dx}{x^2 + 1} = \left(x^3 - x + \frac{x}{x^2 + 1}\right)dx.$$

The integral part may be differentiated by the usual methods, but the fractional part must often be resolved into the sum of a number of fractions with simpler denominators, before integration can be performed.

We know that $\frac{4}{9}$ may be represented as the sum of two other fractions, namely $\frac{1}{9}$ and $\frac{1}{3}$, such that $\frac{4}{9} = \frac{1}{9} + \frac{1}{3}$. Each of these parts is called a **partial fraction.** If the numerator is a compound quantity and the denominator simple, the partial fractions may be deduced, at once, by assigning to each numerator its own denominator and reducing the result to its lowest terms. *E.g.*,

$$\frac{x^2 + x + 1}{x^3} = \frac{x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}.$$

When the denominator is a compound quantity, say $\frac{1}{x^2 - x}$, it is obvious, from the way in which the addition of fractions is performed, that the denominator is some multiple of the denominator of the partial fractions and contains no other factors. We therefore expect the denominators of the partial fractions to be factors of the given denominator. Of course, this latter may have been reduced after the addition of the partial fractions, but, in practice, we proceed as if it had not been so treated.

To reduce a fraction to its partial fractions, the first thing to do is to resolve the denominator into its factors and assume each factor to be the denominator of a partial fraction. Then assign a certain indeterminate quantity to each numerator. These quantities may, or may not, be independent of x. The procedure will be evident from the following examples. There are four cases to be considered.

Case i.—*The denominator can be resolved into real unequal factors of the type:*

$$\frac{1}{(a-x)(b-x)}. \quad . \quad . \quad . \quad . \quad (1)$$

Assume that

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x},$$
$$= \frac{A(b-x) + B(a-x)}{(a-x)(b-x)},$$
$$\therefore \frac{1}{(a-x)(b-x)} = \frac{Ab + Ba - Ax - Bx}{(a-x)(b-x)}.$$

We now assume that the numerators on the two sides of this last equation are identical * and pick out the coefficients of like powers of x, so as to build up a series of equations from which A and B can be determined. For example,

$$Ab + Ba = 1;\ x(A + B) = 0;\ \therefore A + B = 0;\ \therefore A = -B;$$
$$\therefore A = \frac{1}{b-a},\ \therefore B = -\frac{1}{b-a}.$$

---

* An **identical equation** is one in which the two sides of the equation are either identical, or can be made identical by reducing them to their simplest terms. *E.g.*,

$$ax^2 + bx + c \equiv ax^2 + bx + c;$$
$$(a-x)/(a-x)^2 \equiv 1/(a-x),$$

or, in general terms,

$$a + bx + cx^2 + \ldots \equiv a' + b'x + c'x^2 + \ldots$$

An identical equation is satisfied by each or any value that may be assigned to the variable it contains. *The coefficients of like powers of x, in the two members, are also equal to each other.* Hence, if $x = 0$, $a = a'$. We can remove, therefore, a and a' from the general equation. After the removal of a and a', divide by x and put $x = 0$, hence $b = b'$; similarly, $c = c'$, etc. For fuller details, see any elementary textbook on algebra.

The symbol "$\equiv$" is frequently used in place of "$=$" when it is desired to emphasise the fact that we are dealing with identities, not equations of condition. While an *identical equation* is satisfied by *any* value we may choose to assign to the variable it contains, an *equation of condition* is only satisfied by *particular* values of the variable. As long as this distinction is borne in mind, we may follow customary usage and write "$=$" when "$\equiv$" is intended. For "$\equiv$" we may read, "may be transformed into . . . whatever value the variable x may assume"; while for "$=$," we must read, "is equal to . . . when the variable x satisfies some *special* condition or assumes some particular value". See page 386.

Substitute these values of A and B in (1).

$$\frac{1}{(a-x)(b-x)} = \frac{1}{b-a}\cdot\frac{1}{a-x} - \frac{1}{b-a}\cdot\frac{1}{b-x}. \quad . \quad (2)$$

An ALTERNATIVE METHOD, much quicker than the above, is indicated in the following example: Find the partial fractions of the function in example (2) below.

$$\frac{1}{(a-x)(b-x)(c-x)} = \frac{A}{a-x} + \frac{B}{b-x} + \frac{C}{c-x};$$

$$\therefore (b-x)(c-x)A + (a-x)(c-x)B + (a-x)(b-x)C \equiv 1.$$

This identical equation is true for all values of x, it is, therefore, true

when $x = a$, $\therefore (b-a)(c-a)A = 1$; $\therefore A = \dfrac{1}{(b-a)(c-a)}$;

when $x = b$, $\therefore (c-b)(a-b)B = 1$; $\therefore B = \dfrac{1}{(c-b)(a-b)}$;

when $x = c$, $\therefore (a-c)(b-c)C = 1$; $\therefore C = \dfrac{1}{(a-c)(b-c)}$;

$$\therefore \frac{1}{(a-x)(b-x)(c-x)} = \frac{1}{(b-a)(c-a)(a-x)}$$

$$+ \frac{1}{(c-b)(a-b)(b-x)} + \frac{1}{(a-c)(b-c)(c-x)}.$$

EXAMPLES.—(1) Show that

$$\int\frac{dx}{(a-x)(b-x)} = \int\frac{dx}{(b-a)(a-x)} - \int\frac{dx}{(b-a)(b-x)}.$$

$$= \frac{1}{b-a}\cdot\log\frac{b-x}{a-x} + C. \quad . \quad . \quad . \quad (3)$$

(2) Evaluate $\displaystyle\int\frac{dx}{(a-x)(b-x)(c-x)}$. Keep your answer for use later on.

(3) Show that $\displaystyle\int\frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab}\log\frac{a+bx}{a-bx} + C.$

(4) *J. J. Thomson's formula* for the rate of production of ions by the Röntgen rays is

$\dfrac{dn}{dt} = q - an^2$. Hence show, $t = \dfrac{1}{2\sqrt{q/a}}\cdot\log_e\left(\sqrt{\dfrac{q}{a}} + n\right)\Big/\left(\sqrt{\dfrac{q}{a}} - n\right)$.

Note that $a - x^2 = (\sqrt{a} - x)(\sqrt{a} + x)$.

(5) The velocity of the reaction between bromic and hydrobromic acids is, under certain conditions, represented by the equation:

$$dx/dt = k(na + x)(a - x).$$

Hence show that

$$k = \frac{1}{(n+1)at}\cdot\log\frac{na+x}{a-x} + C.$$

The constant is to be evaluated in the usual way by putting $x = 0$ when $t = 0$. For practical convenience, this equation may be adapted for use with common logarithms by multiplying the right-hand side with 2·3026.

(6) If $\dfrac{dx}{dt} = k(a+x)(na-x)$, show that $k = \dfrac{2\cdot3026}{(n+1)at}\cdot\log_{10}\dfrac{n(a+x)}{na-x}$.

(7) *Warder's equation* for the velocity of the reaction between chloracetic acid and ethyl alcohol is

$$dy/dx = ak\{1 - (1 + b)y\}\{1 - (1 - b)y\}.$$

Hence, show that

$$\log_{10}[\{1 - (1 - b)y\}/\{1 - (1 + b)y\}] = 0{\cdot}8686\, abkt.$$

Case ii.—*The denominator can be resolved into real factors some of which are equal. Type:*

$$\frac{1}{(a - x)^2(b - x)}.$$

The preceding method cannot be used here because, if we put

$$\frac{1}{(a - x)^2(b - x)} = \frac{A}{a - x} + \frac{B}{a - x} + \frac{C}{b - x} = \frac{A + B}{a - x} + \frac{C}{b - x},$$

$A + B$ must be regarded as a single constant. Reduce as before and pick out coefficients of like powers of x. We thus get three independent equations containing two unknowns. The values of A, B and C cannot, therefore, be determined by this method. To overcome the difficulty, assume that

$$\frac{1}{(a - x)^2(b - x)} = \frac{A}{(a - x)^2} + \frac{B}{a - x} + \frac{C}{b - x}.$$

Multiply out and proceed as before, thus,

$$A = \frac{1}{b - a};\ B = -\frac{1}{b - a};\ C = -\frac{1}{b - a}.$$

EXAMPLES.—(1) *Goldschmidt's equation* for the velocity of the chemical reaction between hydrochloric acid and ethyl alcohol, is

$$dx/dt = k(a - x)(b - x)^2.$$

Hence,

$$kt = \int\frac{dx}{(a - x)(b - x)^2} = \frac{1}{a - b}\left\{\int\frac{dx}{(b - x)^2} - \int\frac{dx}{b - x} - \int\frac{dx}{a - x}\right\},$$

$$= \frac{1}{a - b}\cdot\frac{1}{b - x} - \frac{1}{a - b}\cdot\log\frac{a - x}{b - x} + C.$$

To find a value for C, put $x = 0$ when $t = 0$. The final result is

$$kt(a - b)^2 = \frac{(a - b)x}{b(b - x)} + \log\frac{a(b - x)}{b(a - x)}.$$

(2) Show $\displaystyle\int\frac{dx}{x^2(a + bx)} = \frac{b}{a^2}\log\frac{a + bx}{a} - \frac{1}{ax} + C.$

(3) Show $\displaystyle\int\frac{dx}{(x - 1)^2(x + 1)} = \frac{1}{4}\log\frac{x + 1}{x - 1} - \frac{1}{2}\cdot\frac{1}{x - 1} + C.$

(4) *Price's equation* for the velocity of the chemical reaction between hydrochloric acid and ethyl alcohol, is as follows:—

$$dx/dt = k\{(a - x)(b - x)^2 - \alpha x(c + x)(b - x)\}.$$

Integrate this equation and evaluate the constant for $x = 0$ and $t = 0$. Ansr.

$$2\alpha b(b + c)kt = \frac{a + \alpha c - b(1 - 2\alpha)}{\sqrt{P}}\log\frac{x(a + b + \alpha c - \sqrt{P}) - 2ab}{x(a + b + \alpha c + \sqrt{P}) - 2ab}$$

$$- \log\{x^2(1 - \alpha) - x(a + b + \alpha c) + ab\}/ab + 2\log(b - x)/b,$$

where $P = (a + b + \alpha c)^2 - 4ab(1 - \alpha)$. This rather tedious example will be found in the *Journal of the Chemical Society*, **79**, 314, 1901.

(5) *Walker and Judson's equation* for the velocity of the chemical reaction between hydrobromic and bromic acids, is

$$dx/dt = k(a - x)^4.$$

Hence show that $$3k = \{1/(a - x)^3 - 1/a^3\}/t.$$

The reader is probably aware of the fact that he can always prove whether his integration is correct or not, by differentiating his answer. If he gets the original integral the result is correct.

Case iii.—*The denominator can be resolved into imaginary* factors all unequal. Type:*

$$\frac{1}{(a^2 + x^2)(b + x)}.$$

---

* **Imaginary Quantities.**—No number is known which will give a negative value when multiplied by itself. The square root of a negative quantity cannot, therefore, be a real number. In spite of this fact, the square roots of negative quantities frequently occur in mathematical investigations. Again, logarithms of negative numbers, inverse sines of quantities greater than unity, . . ., cannot have real values.

Let $\sqrt{-a^2}$ be such a quantity. If $-a^2$ is the product of a^2 and -1, $\pm\sqrt{-a^2}$ may be supposed to consist of two parts, *viz.*, $\pm a$ and $\sqrt{-1}$. Mathematicians have agreed to call a the real part of $\sqrt{-a^2}$ and $\sqrt{-1}$, the imaginary part. Following Gauss, $\sqrt{-1}$ is written ι (or i).

$\sqrt{-1}$, or ι obeys all the rules of algebra. Thus,

$$\sqrt{-1} \times \sqrt{-1} = -1;\ \sqrt{-4} = 2\sqrt{-1};\ \sqrt{-a} \times \sqrt{-b} = \sqrt{ab};\ \iota = \sqrt{-1};\ \iota^4 = 1.$$

EXAMPLES.—(1) Show

$$\iota^{4n} = 1;\ \iota^{4n+1} = \iota;\ \iota^{4n+2} = -1;\ \iota^{4n+3} = -\iota. \qquad (1)$$

(2) Prove $$a^2 + b^2 = (a + \iota b)(a - \iota b) \qquad (2)$$

(3) Show $$\frac{a + \iota b}{c + \iota d} = \frac{ac - bd}{c^2 + d^2} + \frac{bc + ad}{c^2 + d^2}\iota.$$

(4) Show $$(a + \iota b)(c + \iota d) = (ac - bd) + (ad + bc)\iota.$$

(5) The quadratic $x^2 + bx + c = 0$, has imaginary roots only when $b^2 - 4c$ is less than zero (formula (5), page 388). If α and β are the roots of this equation, show that

$$\alpha = -\tfrac{1}{2}b + \tfrac{1}{2}\iota\sqrt{4c - b^2} \text{ and } \beta = -\tfrac{1}{2}b - \tfrac{1}{2}\iota\sqrt{4c - b^2},$$

satisfy the equation.

The imaginary numbers from $-\infty$ to $+\infty$ are:

$$-\infty\iota, \ldots, -\iota, \ldots, 0\iota, \ldots, +\iota, \ldots, +\infty\iota,$$

corresponding with the real numbers

$$-\infty, \ldots, -1, \ldots, 0, \ldots, +1, \ldots, +\infty.$$

By combining a real with an imaginary quantity we get what is known as a **complex number**, or a complex quantity. Such is $x + \iota y$. So important is the unthinkable $\sqrt{-1}$ in modern theories, that the algebra of *real* quantity is now a special branch of the algebra of complex quantity.

We know what the phrase "the point x, y" means. If one or both x and y are imaginary, the point is said to be imaginary. An *imaginary point* has no geometrical or physical meaning. If an equation is affected with one or more imaginary coefficients, Since imaginary roots always occur in pairs (page 386), the product of each pair of imaginary factors will give a product of the form, $x^2 + a^2$. Instead of assigning a separate partial fraction to each imaginary factor, we assume, for each pair of imaginary factors, a partial fraction of the form:

$$\frac{Ax + B}{a^2 + x^2}.$$

Hence
$$\frac{1}{(a^2 + x^2)(b + x)} = \frac{Ax + B}{a^2 + x^2} + \frac{C}{b + x}.$$

EXAMPLES.—Verify the following results

(1) $$\int\frac{dx}{(x-1)^2(x^2+1)} = \int\Big(\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx + D}{x^2+1}\Big)dx,$$
$$= \frac{1}{4}\log\frac{x^2+1}{(x-1)^2} - \frac{1}{2}\cdot\frac{1}{x-1} + C.$$

(2) $$\int\frac{dx}{1-x^4} = \frac{1}{2}\tan^{-1}x + \frac{1}{4}\log\frac{1+x}{1-x} + C.$$

Case iv.—*The denominator can be resolved into imaginary factors, some of which are equal to one another. Type:*

$$\frac{1}{(a^2 + x^2)^2(b + x)}.$$

Combining the preceding results,

$$\frac{1}{(a^2 + x^2)^2(b + x)} = \frac{Ax + B}{(a^2 + x^2)^2} + \frac{Cx + D}{a^2 + x^2} + \frac{E}{b + x}.$$

In this expression, there are just sufficient equations to determine the complete system of partial fractions, by equating the coefficients of like powers of x.

The differentiation of many of the resulting expressions usually requires the aid of one of the reduction formulae (§ 76).

EXAMPLE.—Prove

$$\int\frac{(x^3 + x - 1)dx}{(x^2 + 1)^2} = \int\frac{xdx}{x^2 + 1} - \int\frac{dx}{(x^2 + 1)^2}.$$

Integrate. Use formula **D** for evaluating the last term.

Ansr. $\frac{1}{2}\log(x^2 + 1) - \frac{1}{2}x/(1 + x^2) + \tan^{-1}x + C.$

---

the non-existent graph is conventionally styled an *imaginary curve*. Illustrations §§ 62 to 64.

For a geometrical interpretation of $\sqrt{-1}$, see Lock's *A Treatise on Higher Trigonometry*, 103, 1897; consult Chrystal's *Algebra*, Part I., Chapter XII., and Merriman and Woodward's *Higher Mathematics*, Chapter VI., for the algebra of complex numbers.

Do not confuse *irrational* with *imaginary* quantities. In the former case, even if we cannot obtain the absolutely correct value, we can get as close an approximation as ever we please; in the latter case, we cannot even say that the imaginary quantity is entitled to be called "a quantity".

Cases iii. and iv. seldom occur in actual work. If, therefore, the denominator of any fractional differential can be resolved into factors, the differential can be integrated by one or other of these processes..

The remainder of this chapter will be mainly taken up with practical illustrations of integration processes. A few geometrical applications will first be given because the accompanying figures are so useful in helping one to form a mental picture of the operation in hand.

§ 78. Areas enclosed by Curves. To Evaluate Definite Integrals.

1. *To find the area bounded by two perpendiculars, dropped from any two points on a curve on to the x- (or y-) axis, the portion of the curve included between these two points and the x- (or y-) axis included between the two perpendiculars.*

Let AB (Fig. 86) be any curve whose equation is known. It is required to find the area of the portion bounded by the curve, the two coordinates PM, QN, and MN. The area can be approximately determined by supposing the portion $PQMN$ cut up into small strips (called **surface elements**) perpendicular to the x-axis; find the area of each separate strip on the assumption that the curve bounding one end of it is a straight line and add the areas of all these trapezoidal strips together. (*Cf.* "Approximate Integration," page 263.)

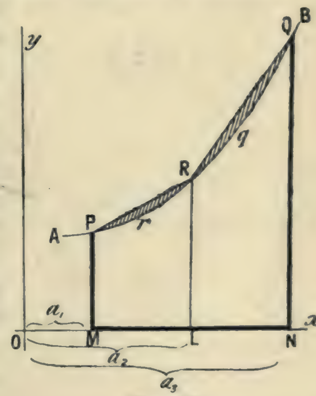


Fig. 86.

Let the surface $PQMN$ be cut up into two strips by means of the line LR. Join PR, RQ.

$$(\text{Area } PQMN) = (\text{Area } PRLM) + (\text{Area } RQNL).$$

But the area which is the sum of these two trapeziums is greater than that of the figure required, namely $PrqQNM$. The shaded portion of the diagram represents the magnitude of the error. It is obvious that the narrower each strip is made, the greater will be the number of trapeziums to be included in the calculation and the smaller will be the error. If we could add up the areas of an infinite number of such strips, the actual error would become vanishingly small. Although we are unable to form any distinct conception of this process, we feel assured that such an operation would give a result absolutely correct. But enough has been said on this matter in § 69. We want to know how to add up an infinite number of infinitely small strips.

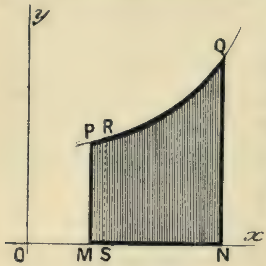


Fig. 87.

In order to have some concrete image before the mind, let us find the area of $PQNM$ in Fig. 87. In any small strip $PRSM$, let $PM = y$, $RS = y + \delta y$, $OM = x$ and $OS = x + \delta x$. Let δA represent the area of the small strip under consideration.

If the short distance PR were straight, not curved, the area A would be, (10), page 491.

$$A = \tfrac{1}{2}\delta x(PM + RS) = \delta x(y + \tfrac{1}{2}\delta y).$$

By making δx smaller and smaller, the ratio,

$$\delta A/\delta x = y + \tfrac{1}{2}\delta y,$$

approaches, and, at the limit, becomes equal to

$$Lt_{\delta x = 0}\frac{\delta A}{\delta x} = \frac{dA}{dx} = y.$$

Or,
$$dA = y\,.\,dx. \qquad . \quad . \quad . \quad (1)$$

In the same way, it can be shown that the differential of the area included between the curve and the y-axis, is,

$$dA = x\,.\,dy. \qquad . \quad . \quad . \quad . \quad (2)$$

Formula (1), or (2), represents the area of an infinitely small strip. The total area (A) can be determined by integrating either of these formulae. For the sake of simplicity, we shall confine our attention to the former. But, before we can proceed any further, we must know the equation to the curve.

(i.) *Let rectangular coordinates be used.* In any special case, the equation is to be solved for y, and the value of y so found is to be substituted in equation (1). Then integrate the resulting equation to get a *general* expression for an indefinite portion of the curve. To obtain the area of any definite portion situate between the ordinates of the extremities, we must take the sum of all the strips determined by the lengths of the ordinates.

For instance, the area of any indefinite portion of the curve, is

$$A = \int y \,.\, dx + C, \quad . \quad . \quad . \quad . \qquad (3)$$

and the area of the portion whose ordinates have the abscissae a_2 and a_3 (Fig. 86) is

$$A = \int_{a_2}^{a_3} y \,.\, dx + C. \quad . \quad . \quad . \qquad (4)$$

Equation (3) is an indefinite integral, equation (4), a definite integral. The value of the definite integral is determined by the magnitude of the upper and lower limits (see page 153). In Fig. 86, if a_1, a_2, a_3 represent the magnitudes of three abscissae, such that a_2 lies between a_1 and a_3,

$$A = \int_{a_3}^{a_1} y \,.\, dx + C = \int_{a_2}^{a_1} y \,.\, dx + \int_{a_3}^{a_2} y \,.\, dx + C.$$

When the limits are known, the value of the integral is found by subtracting the expression obtained by substituting the lower limit in place of x, from a similar expression obtained by substituting the upper limit for x. Thus, to evaluate $\int 2xdx$ between the limits a and b,

$$\int_a^b 2x \,.\, dx = \Big|_a^b x^2 + C;$$

or, as it is sometimes written,

$$\int_a^b 2x \,.\, dx = \Big[x^2 + C \Big]_a^b = (b^2 + C) - (a^2 + C) = b^2 - a^2.$$

Plenty of examples will be given presently (see page 184).

The process of finding the area of any surface is called, in the regular textbooks, the "**Quadrature of Surfaces,**" from the fact that the area is measured in terms of a square.

EXAMPLES.—(1) To find the area bounded by an ellipse, origin at the centre. Here

$$x^2/a^2 + y^2/b^2 = 1; \text{ or, } y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

Refer to Fig. 21, page 78. The sum of all the elements perpendicular to the x-axis, from OP_1 to OP_4, is given by the equation

$$A = \int_0^a y \,.\, dx,$$

for, when the curve cuts the x-axis, $x = a$, and when it cuts the y-axis, $x = 0$. The positive sign in the above equation, represents ordinates above the x-axis. The area of the ellipse is, therefore,

$$A = 4\int_0^a y \,.\, dx.$$

Substitute the above value of y in this expression and we get for the sum of this infinite number of strips,

$$A = 4\frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}dx,$$

which may be integrated by parts, as shown on page 168, thus

$$A = 4\frac{b}{a}\Big[\tfrac{1}{2}x\sqrt{(a^2 - x^2)} + \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a} + C\Big]_0^a.$$

The term within the brackets is yet to be evaluated between the limits $x = a$ and $x = 0$.

$$A = 4\frac{b}{a}\Big[\Big\{\tfrac{1}{2}a\sqrt{(a^2 - a^2)} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} + C\Big\} - \Big\{\tfrac{1}{2}0\sqrt{(a^2 - 0^2)} + \frac{a}{2}\sin^{-1}\frac{0}{a} + C\Big\}\Big]$$

$$\therefore A = 4\frac{b}{a} \times \frac{a^2}{2}\sin^{-1}1.$$

remembering that $\sin 90° = 1$, $\sin^{-1}1 = 90°$ and $2\sin^{-1}1 = 180° = \pi$. The area of the ellipse is, therefore, πab.

If the major and minor axes are equal, $a = b$ and the ellipse becomes a circle whose area is πa^2. It will be found that the constant always disappears in this way when evaluating a definite integral.

A WORD OF ADVICE.—The student must learn to draw his own diagrams. If you are going to find the area bounded by a portion of an ellipse or of an hyperbola, first plot your curve. Squared paper is cheap enough. Carefully note the limits of your integral.

(2) Find the area bounded by the rectangular hyperbola,

$$xy = a; \text{ or, } y = a/x,$$

between the limits $x = x_1$ and $x = x_2$.

$$A = \int_{x_1}^{x_2} y\,.\,dx = \int_{x_1}^{x_2}\frac{a}{x}dx,$$

$$= a\Big|_{x_1}^{x_2}\log x + C = a\{(\log x_2 + C) - (\log x_1 + C)\},$$

$$= a\log x_2/x_1.$$

If $x_1 = 1$ and $x_2 = x$, $A = a\log_e x$. This simple relation appears to be the reason natural logarithms are sometimes called *hyperbolic logarithms.*

After this the integration constant is not to be used at any stage of the process of integration between limits. It has been retained in the above discussion to further illustrate the rule (see § 72): *The integration constant of a definite integral disappears during the process of integration. The absence of the indefinite integration constant is the mark of a definite integral.*

(3) Show that the area bounded by the logarithmic curve, $x = \log a$, is $y - 1$. Hint. Evaluate C by noting that when $x = 0$, $y = 1$, $A = 0$.

(ii.) *Let polar coordinates be used.* The differential of the area is then

$$dA = \tfrac{1}{2}r^2\,.\,d\theta. \qquad (5)$$

EXAMPLE.—Find the area of the hyperbolic spiral between 0 and r. See 5), page 96.

$$r\theta = a;\ d\theta = -\,a\,.\,dr/r^2.$$

$$dA = -\int_{-r}^{0}\tfrac{1}{2}a\,.\,dr = -\tfrac{1}{2}\Big|_{-r}^{0}\tfrac{1}{2}ar = \tfrac{1}{4}ar.$$

2. *To find the area enclosed between two different curves.* Let $PABQ$ and $PA'B'Q$ (Fig. 88) be two curves, it is required to find the area $PABQB'A'$.

Let $y_1 = f_1(x)$ be the equation of one curve, $y_2 = f_2(x)$, the equation of the other. Find separately the areas $PABQMN$ and $PA'B'QMN$, by preceding methods. The required area is, therefore,

$$(\text{Area } PABQB'A') = (\text{Area } PABQMN) - (\text{Area } PA'B'QMN)$$
$$= \int y_1 . dx - \int y_2 . dx.$$

To find the area of the portion $ABB'A'$, let x_1 be the abscissa of AR and x_2 the abscissa of BS, then,

$$A = \int_{x_1}^{x_2} y_1 . dx - \int_{x_1}^{x_2} y_2 . dx = \int_{x_1}^{x_2} (y_1 - y_2) dx. \quad . \quad . \quad . \quad (6)$$

EXAMPLE.—Show that if the curves

$$y^2 = 4x \text{ and } y^2 = 2x - x^2,$$

meet at the origin and at a point $x = 8$, $y = 8$,

$$A = 2\int_0^8 (\sqrt{2x - x^2} - \sqrt{4x}) dx, \text{ etc.}$$

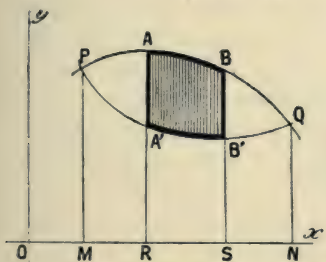


FIG. 88.

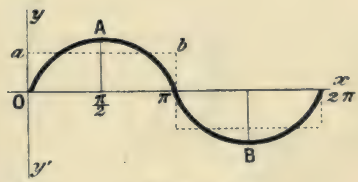


FIG. 89.

3. *The area bounded by two branches of the same curve.* If the curve is circular,

$$y = \pm \sqrt{(r^2 - x^2)},$$
$$A = \int \sqrt{(r^2 - x^2)} dx - \int (-\sqrt{r^2 - x^2}) dx, \text{ etc.}$$

4. *To find the area bounded by the sine curve and the x-axis for a whole period* (2π), *or for any number of whole periods.* Required the area $OA\pi + \pi B2\pi$ (Fig. 89). Let

$$y = \sin x$$

be the equation to the curve.

$$A = \int_0^{2\pi} \sin x . dx = -\Big[\cos x\Big]_0^{2\pi}$$
$$= -(\cos 2\pi) - \cos 0 = 0, \quad . \quad . \quad . \quad . \quad (7)$$

for $-\cos 2\pi = -\cos 360° = -1$ and $\cos 0 = 1$.

It can be shown in a similar manner that the area bounded by the cosine curve is zero. The geometrical signification of this will appear from Fig. 89.

The instrument (electrodynamometer) used for measuring the strength of alternating electric currents, indicates the average value during half a complete period, that is to say, during the time the current flows in one direction. This is geometrically represented by the area of a rectangle $Oab\pi$, equal to the area of the portion bounded by the sine curve $OA\pi$ and the x-axis.

Let the ordinate $A\frac{1}{2}\pi$ be denoted by r and the height of the rectangle $Oab\pi = Oa = y_1$, then the area of $OA\pi$ is

$$\int_0^\pi r\sin x \, . \, dx = r\Big|_0^\pi - \cos x = (-\cos 180° + \cos 0°)r = 2r,$$

since $\cos 180° = -1$, $-\cos 180° = 1$. Therefore,

$r = y_1\pi$, represents the maximum current, y_1 the average current.

Area of rectangle $Oab\pi$ = area $OA\pi$; or $y_1\pi = 2r$.

$y_1 = 2r/\pi = 0{\cdot}6366r$ represents the average current.

The maximum current is thus obtained by multiplying the average current by $\frac{1}{2}\pi$, or by 1·5708.

§ 79. Graphic Representation of Work.

Let a given volume (x) of a gas be contained in a cylindrical vessel in which a tightly fitting piston can be made to slide (Fig. 90). Let the sectional area of the piston be unity.

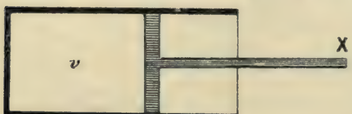


Fig. 90.

Now let the volume of the gas change dx units when a slight pressure X is applied to the free end of the piston. Then by definition of work (W),

$$Work = Force \times Displacement;$$

or, $$dW = X \, . \, dx.$$

If p denotes the pressure of the gas and v the volume, we have,

$$dW = p \, . \, dv.$$

Now let the gas pass from one condition where $x = x_1$ to another state where $x = x_2$. Let the corresponding pressures to which the gas was subjected be respectively denoted by X_1 and X_2.

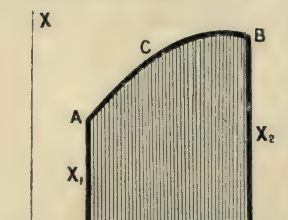


Fig. 91.—Work Diagram.

By plotting the successive values of X and x, as x passes from x_1 to x_2, we get the curve ACB, shown in Fig. 91. The shaded part of the figure represents the total work done *on* the system during the change.

If the gas returns to its original state through another series of successive values of X and x we have the curve ADB (Fig. 92). The total work done *by* the system will then be represented by the area $ABDx_2x_1$. If we agree to call the work done *on* the system *positive*, and work done *by* the system *negative*, then (Fig. 92),

$$W_1 - W_2 = (Area\ ACBx_2x_1) - (Area\ ADBx_2x_1),$$
$$= (Area\ ACBD).$$

The shaded part in Fig. 92, therefore, represents the work done *on* the system during the above cycle of changes. A series of operations by which a substance, after leaving a certain state, finally returns to its original condition, is called a **cycle,** or a **cyclic process.** A cyclic process is represented graphically by a closed curve.

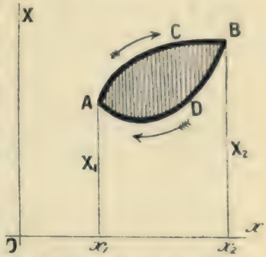


Fig. 92.—Work Diagram.

The reader will notice that the work is done *on* the system while x is increasing and *by* the system when x is decreasing. Therefore, if the curve is described by a point moving round the area $ACBD$ in the direction of the hands of a clock, the total work done on the system is positive; if done in the opposite direction, negative.

If the diagram has several loops, as shown in Fig. 93, the total work is the sum of the areas of the several loops developed by the point travelling in the same direction as the hands of a clock, minus the sum of the areas developed when the point travels in a contrary direction. This graphic mode of representing work was first used by Clapeyron. The diagrams are called **Clapeyron's Work Diagrams.** The subject is resumed on page 208.

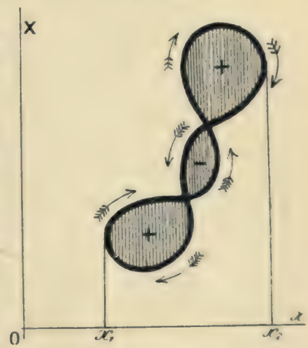


Fig. 93.—Work Diagrams (after Clapeyron).

§ 80. Integration between Limits*—Definite Integrals.

It is perhaps necessary to further amplify the remarks on page 179. If $f'(x)$ denotes the first differential coefficient of $f(x)$,

$$\int_a^b f'(x)\,.\,dx = \Big[f(x)\Big]_a^b, \text{ or, } \Big|_a^b f(x) = f(b) - f(a).$$

* Note the different meanings assigned to the word "limit" in the differential and in the integral calculus.

EXAMPLES.—(1) Show $\int_a^b e^{-x} . dx = e^{-a} - e^{-b}$.

(2) Prove $\int_{-1}^3 x^2 . dx = \frac{1}{3}\{(3)^3 - (-1)^3\} = 9\frac{1}{3}$.

One of the limits a, or b, may become infinite or zero.

(3) $\int_0^\infty e^{-x} . dx = \Big[-e^{-x} \Big]_0^\infty = -e^{-\infty} - (-e^{-0}) = 1,$

since $e^{-\infty} = 0$ and $e^{-0} = 1$.

(4) Show that $\int_1^\infty \frac{dx}{x\sqrt{(x^2+1)}} = \log(1 + \sqrt{2})$.

By way of practice verify the following results:—

(5) $\int_0^{\pi/2} \sin x . dx = - \Big|_0^{\pi/2} \cos x = -(\cos \frac{1}{2}\pi - \cos 0°) = 1.$

(6) $\int_0^{\pi/2} \sin^2 x . dx = \frac{1}{4}\pi$; $\int_0^{\pi/4} \sin^2 x . dx = \frac{1}{4}\Big(\frac{\pi}{2} - 1\Big)$; $\int_0^\pi \sin^2 x . dx = \frac{1}{2}\pi.$

Hint for the indefinite integral. Integrate by parts. Put $u = \sin x$, $dv = \sin x . dx$. From (1), § 74,

$$\int \sin^2 x . dx = \sin x . \cos x + \int \cos^2 x . dx;$$
$$= \sin x . \cos x + \int (1 - \sin^2 x) dx.$$

Transpose the last term to the left-hand side, and divide by 2.

$$\therefore \int \sin^2 x . dx = \tfrac{1}{2}(\sin x . \cos x + x) + C.$$

(7) $\int_0^{\pi/2} \sin^n x . dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x . dx.$

For n write $n - 2$ and show that

$$\int_0^{\pi/2} \sin^{n-2} x . dx = \frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x . dx.$$

Combine the last two equations and repeat the reduction. Thus,

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)(n-3)\ldots 3 . 1}{n(n-2)\ldots 4 . 2}\int_0^{\pi/2} dx = \frac{(n-1)(n-3)\ldots 3 . 1}{n(n-2)\ldots 4 . 2} . \frac{\pi}{2}. \quad (1)$$

when n is even;

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3}\int_0^{\pi/2} \sin x dx = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3}, \quad (2)$$

when n is odd.

(1) and (2) are useful reduction formulae.

There are some interesting properties of definite integrals worth noting.

(i.) It is evident that

$$\int_b^a f'(x)dx = f(a) - f(b) = -\int_a^b f'(x)dx, \quad . \quad . \quad (3)$$

or, when the upper and lower limits of an integral are interchanged, only the sign of the definite integral changes. This means that if the change of the variable from b to a is reckoned positive, the change from a to b is negative. That is to say, if motion in one direction is reckoned positive, motion in the opposite direction is to be reckoned negative. To put equation (3) in words, *the interchange of the limits of a definite integral causes the integral to change its sign.*

(ii.) If m is any interval between the limits a and b.

$$\int_b^a f'(x)dx = \int_m^a f'(x)dx + \int_b^m f'(x)dx, \quad . \quad . \quad (4)$$

or,

$$\int_b^a f'(x)dx = \int_b^m f'(x)dx - \int_a^m f'(x)dx.$$

(iii.) If x is any function of a new variable y, so that $f'(x)dx$ becomes another function of y, say $\phi'(y)dy$, then, when x_1 and x_2 are substituted for x, y becomes y_1 and y_2 respectively.

$$\therefore \int_{x_1}^{x_2} f'(x)dx = \int_{y_1}^{y_2} \phi'(y)dy.$$

If $a - y$ be substituted for x in this expression,

$$\int_0^a f'(x)dx = -\int_a^0 f'(a - y)dy = \int_0^a \phi'(a - y)dy.$$

But neither x nor y appears in the final result, hence we may put

$$\int_0^a f'(x)dx = \int_0^a f'(a - x)dx.$$

For instance, if $f(x) = \sin^n x$; $f(\frac{1}{2}\pi - x) = \cos^n x$,

$$\therefore \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx. \quad . \quad . \quad . \quad (5)$$

EXAMPLES.—Verify the following results :—

(1) $\int_0^{\pi/2} \cos x dx = 1$; $\int_0^{\pi/2} \cos^2 x dx = \frac{1}{4}\pi$.

From (1) and (2), if n is even,

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx = \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2} \cdot \frac{\pi}{2}; \quad . \quad (6)$$

and, if n is odd,

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx = \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)\ldots 5.3}. \quad . \quad (7)$$

Test this by actual integration and by substituting $n = 1, 2, 3, \ldots$

(2) $\int_0^{\pi/2} \sin^6 x dx = \frac{5}{32}\pi$; $\int_0^{\pi/2} \sin^3\theta . d\theta = \frac{2}{3}$.

If n is greater than unity,

$$\int_0^{\pi/2} \sin^m x . \cos^n x dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x . \cos^{n-2} x dx; \quad . \quad . \quad (8)$$

if m is greater than unity,

$$\int_0^{\pi/2} \sin^m x . \cos^n x dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x . \cos^n x dx. \quad . \quad . \quad (9)$$

These important reduction formulae are employed in the reduction of either $\int\cos^n x dx$, or $\int\sin^n x dx$ to an index unity, or zero.

(3) $\int_0^{\pi/2} \sin x . \cos x dx = \frac{1}{2}$; $\int_0^{\pi/2} \sin^2 x . \cos x dx = \frac{1}{3}$.

(4) $\int_0^{\pi/2}\sin x \,.\, \cos^2 x dx = \frac{1}{3}$; $\int_0^{\pi/2}\sin^2 x \,.\, \cos^2 x dx = \frac{1}{16}\pi$.

In the last integration, note $\cos^2 x = 1 - \sin^2 x$.

(5) Evaluate $\int_0^{\pi}\sin mx \,.\, \sin nx dx$. By (26), page 499,

$$2\sin mx \,.\, \sin nx = \cos(m - n)x - \cos(m + n)x.$$

$$\therefore \int \sin mx \,.\, \sin nx dx = \tfrac{1}{2}\int\cos(m - n)x dx - \tfrac{1}{2}\int\cos(m + n)x dx,$$

$$= \frac{\sin(m - n)x}{2(m - n)} - \frac{\sin(m + n)x}{2(m + n)}.$$

Therefore, if m and n are integral,

$$\int_0^{\pi}\sin mx \,.\, \cos nx dx = 0.$$

Remembering that $\sin\pi = \sin 180° = 0$ and $\sin 0° = 0$, if $m = n$,

$$\int_0^{\pi}\sin^2 nx dx = \frac{1}{2}\int_0^{\pi}(1 - \cos 2nx)dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_0^{\pi} = \frac{\pi}{2}.$$

(6) Show that $\int_0^{\pi}\cos mx \,.\, \cos nx dx$ is zero when m and n are integral; $\frac{1}{2}$, when $m = n$. Hints, $\cos\pi = \cos 180° = 1$, $\cos 0° = 1$,

$$2\cos mx \,.\, \cos nx = \cos(m - n)x + \cos(m + n)x,$$

(25), page 499.

(7) Evaluate $\int_0^{\pi} a\sin\frac{1}{2}x \,.\, \cos\frac{1}{2}x \,.\, dx$. Ansr.

$$2a\int_0^{\pi}\sin\tfrac{1}{2}x \,.\, d(\sin\tfrac{1}{2}x) = \tfrac{2}{3}a\Big|_0^{\pi}\sin^2\tfrac{1}{2}x = \tfrac{2}{3}a.$$

(8) $\int_{-\pi}^{+\pi}\cos mx \,.\, \cos nx dx = 0$; $\int_{-\pi}^{+\pi}\sin mx \,.\, \sin nx dx = 0$;

$$\int_{-\pi}^{+\pi}\cos mx \,.\, \sin nx dx = 0.$$

Hint. Use the results of Examples (5) and (6); also note that

$$\sin nx dx = - (\cos nx)/n.$$

(9) Show $\displaystyle\int_{b\cos\theta - \sqrt{a^2 - b^2\sin^2\theta}}^{b\cos\theta + \sqrt{a^2 - b^2\sin^2\theta}} \cos\theta \,.\, dx = 2(a^2 - b^2\sin^2\theta)^{\frac{1}{2}}\cos\theta.$

For a more extensive treatment of definite integrals, the reader will have to consult some such work as that of Williamson, referred to elsewhere.

§ 81. To find the Length of any Curve.

To find the length of any curve whose equation is known. This is equivalent to finding the length of a straight line of the same length as the curve, hence the process is called the "**Rectification of Curves**".

(i.) *Let rectangular coordinates be used.* It is required to find the length l of an arc AB (Fig. 94), where the coordinates of A are (x_0, y_0) and of B, (x_n, y_n). Take any two points, P, Q, on the curve. Make the construction shown in the figure. Then by Euclid i., 47, if P and Q are sufficiently close,

$$(Chord\ PQ)^2 = (\delta x)^2 + (\delta y)^2.$$

But from (1), page 12, the limit of the chord PQ is equal to that of the arc PQ,

$$\therefore dl = \sqrt{(dx)^2 + (dy)^2}\,;\ \text{or,} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx. \quad . \qquad (1)$$

The differential of an arc of any plane curve, referred to rectangular coordinates, is equal to the square root of the sum of the squares of the differentials of the coordinates.

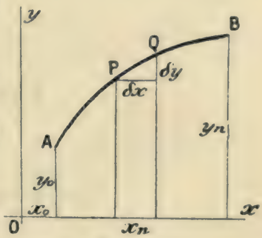


FIG. 94.

In order to find the length of a curve, it is only necessary, therefore, to differentiate its equation and substitute the values of dx and dy, so obtained, in equation (1). By integrating this equation, we obtain a general expression for the length of any arc. In order to find the length of any definite portion of the curve, we must integrate between the limits x_0 and x_n, or y_0 and y_n as the case might be.

(ii.) *Let polar coordinates be used.* If the equation is

$$f(\theta, r) = 0.$$

The differential of the arc is

$$dl = \sqrt{(dr)^2 + r^2(d\theta)^2}. \quad . \quad . \quad . \qquad (2)$$

The rest is the same as before.

EXAMPLES.—(1) If the curve is a common parabola,

$$y^2 = 4ax,$$

$$\therefore ydy = 2adx, \text{ or } (dx)^2 = y^2(dy)^2/4a^2.$$

From (1), $$dl = \sqrt{(y^2 + 4a^2)}dy/2a.$$

Now integrate as on page 166,

$$l = \tfrac{1}{4}y\sqrt{y^2 + 4a^2}/a + a\log(y + \sqrt{y^2 + 4a^2}) + C.$$

To find C, put $y = 0$ when $l = 0$,

$$C = -a\log 2a.$$

$$\therefore l = \tfrac{1}{4}y\sqrt{y^2 + 4a^2}/a + a\log\{\tfrac{1}{2}(y + \sqrt{y^2 + 4a^2})/a\}.$$

(2) Show that the perimeter of the circle

$$x^2 + y^2 = r^2,$$

is $2\pi r$. Let l be the length of the arc in the first quadrant, then

$$dy = x\,.\,dx/y.$$

$$\therefore l = \int_0^r \sqrt{(dx)^2 + (dy)^2} = \int_0^r \left(\frac{x^2 + y^2}{y^2}\right)^{\frac{1}{2}} dx$$

$$= r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = \left[r\sin^{-1}\frac{x}{r}\right]_0^r = \tfrac{1}{2}\pi r.$$

See page 166.

$$\therefore (\textit{Whole perimeter}) = 4 \times \tfrac{1}{2}\pi r = 2\pi r.$$

(3) Find the length of the equiangular spiral, page 96, whose equation is

$$r = e^{\theta}, \text{ or, } \theta = \log r / \log e.$$

Differentiate $\quad \therefore d\theta = dr/r, \therefore dl = \sqrt{2} . dr.$

$$\therefore l = \sqrt{2} . r + C;$$

when $r = 0, l = 0, C = 0,$

$$l = \sqrt{2} . r.$$

(4) Show that the cardioid curve, $r = a(1 - \cos\theta)$ has $l = 4a \sin\frac{1}{2}\theta + C$.

(5) Show that the length of the cycloid,

$$x = r(\theta - \sin\theta);\ y = r(1 - \cos\theta),$$

from $\theta = \theta_0$ to $\theta = \theta_1$, is $4r(\cos\frac{1}{2}\theta_0 - \cos\frac{1}{2}\theta_1)$.

(6) Show that the length of the hypocycloid curve,

$$x^{2/3} + y^{2/3} = r^{2/3}, \text{ is } 6r.$$

Plot the curve.

§ 82. Elliptic Integrals.

The ratio c/a (Fig. 21, page 78) is the eccentricity of the ellipse, the "e" of § 44, page 95. Therefore (Fig. 21),

$$c = ae;\ \text{but, } c^2 = a^2 - b^2, \therefore b^2/a^2 = 1 - e^2.$$

Substitute this in the equation of the ellipse (7), page 79. Hence,

$$y^2 = (1 - e^2)(a^2 - x^2);\ \left(\frac{dy}{dx}\right)^2 = \frac{(1 - e^2)x^2}{a^2 - x^2}.$$

Therefore, the length (l) of the arc of the quadrant of the ellipse (Fig. 21) is

$$l = \int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}} . dx. \qquad . \quad . \quad . \quad . \quad (1)$$

This expression cannot be reduced by the usual methods of integration. Its value can only be determined in an approximate way by methods to be described later on.

Equation (1) can be put in a simpler form by noting that $x = a \sin\phi$, where ϕ is the complement of the "eccentric" angle θ (Fig. 33). Hence,

$$l = a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\phi} . d\phi.$$

Here ϕ is called the *amplitude* and is written am u; e, or, as it is sometimes written, k, the *modulus* of the function—is always less than unity.

The integral of an irrational * polynomial of the second degree, of the type,

$$\int\sqrt{a + bx + cx^2} . X . dx;\ \text{or, } \int X . dx / \sqrt{a + bx + cx^2}$$

(where X is a rational function of x), can be made to depend on algebraic, logarithmic, or on trignometrical functions, which can be evaluated in the usual way. But if the irrational polynomial is of the third or the fourth degree, the integral

$$\int\sqrt{a + bx + cx^2 + dx^3 + ex^4} . X . dx;\ \text{or, etc.,}$$

cannot be treated in so simple a manner. Such integrals are called **elliptic integrals**. If higher powers than x^4 appear under the radical sign, the re-

---

* The numbers $\sqrt{2}$, $\sqrt[3]{5}$, . . ., which cannot be obtained in the form of a whole number or a finite fraction, are said to be *irrational* or *surd numbers*. On the contrary, $\sqrt{4}$, $\sqrt[3]{27}$, $\frac{1}{2}$, 17, . . . are said to be *rational numbers*.

sulting integrals are said to be *ultra-elliptic* or *hyper-elliptic integrals.* That part of an elliptic integral which cannot be expressed in terms of algebraic, logarithmic, or trignometrical functions is always one of three classes:—

1. *Elliptic integrals of the first class:*

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}; \text{ or, } F(k, x) = \int_0^{x} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \quad (2)$$

since $x = \sin\phi$. This integral is used chiefly in the study of periodic oscillations of large amplitude. For example, the time of a complete oscillation (t) of a simple pendulum of length l, oscillating through an angle α (less than 180°) on each side of the vertical is:

$$t = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - \sin^2\tfrac{1}{2}\alpha \cdot \sin^2\phi)}},$$

where g is the constant of gravitation. We shall integrate this kind of equation in Chapter V.

2. *Elliptic integrals of the second class:*

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\phi} \cdot d\phi; \text{ or, } E(k, x) = \int_0^{x} \sqrt{\frac{1 - k^2x^2}{1 - x^2}} \cdot dx, \quad (3)$$

just encountered in the rectification of the arc of the ellipse.

3. *Elliptic integrals of the third class:*

$$\Pi(n, k, \phi) = \int_0^{\phi} \frac{d\phi}{(1 + n \sin\phi)\sqrt{1 - k^2 \sin^2\phi}}; \text{ or, } \Pi(n, k, x) = \text{etc.}, \quad (4)$$

where n is any real number, called *Legendre's parameter*. If the limits of the first and second classes of integrals are 1 and 0, instead of x and 0 in the first case and $\pi/2$ and 0 in the second case, the integrals are said to be *complete*. Complete elliptic integrals of the first and second classes are denoted by the letters F and E respectively. $\sqrt{1 - k^2\sin^2\phi}$ is written $\triangle\phi$. Since $\phi = \text{am } u$, x, the sine of the amplitude u, is written $x = \text{sn } u$; $\sqrt{1 - x^2} = \text{cn } u$ is the cosine of the amplitude of u and $\sqrt{1 - k^2x^2} = \text{dn } u$, is the delta of the amplitude of u. *E.g.*, the centrifugal force (F) of a pendulum bob of mass (m) oscillating like the pendulum just described, is,

$$F = 4mg \sin^2\tfrac{1}{2}\alpha \cdot \text{cn } t\sqrt{g/l},$$

where cn $t\sqrt{g/l}$ is the cosine of the amplitude of $t\sqrt{g/l}$ in the above elliptic integral (Class 1).

There is a system of formulae connecting the elliptic functions to each other; some of these have a certain formal resemblance to the trignometrical functions. Thus,

$$\text{sn}^2 u + \text{cn}^2 u = 1;$$

$$d \text{ am } u/du = d\phi/du = \sqrt{(1 - k^2 \sin^2\phi)} = \text{dn } u, \text{ etc.}$$

Legendre has calculated short tables of the first and second class of elliptic integrals; the third class can be connected with these by known formulae. But numerical tables suitable for practical purposes are incomplete.*

* I learn from Baker's *Elliptic Integrals* that more complete tables are in process of computation.

Mascart and Joubert have tables of the coefficient of mutual induction of electric currents, in their *Electricity and Magnetism* (**2**, 126, 1888), calculated from E and F above. Greenhill's *The Applications of Elliptic Functions* (Macmillan & Co., 1892) is one of the most useful textbooks on this subject.

§ 83. The Gamma Function.

It is sometimes found convenient to express the solution of a physical problem in terms of a definite integral whose numerical value is known, more or less accurately, for certain values of the variable. For example, there is Legendre's table of the elliptic integrals; Kramp's table of the integral $\int_0^\infty e^{-t^2}.dt$; Soldner's table of $\int_0^a dx/\log x$; Gilbert's table of Fresnel's integral $\int_0^v \cos\frac{1}{2}\pi v^2.dv$, or $\int_0^v \sin\frac{1}{2}\pi v^2.dv$; Legendre's table of the integral $\int_0^\infty e^{-x}x^{n-1}.dx$, or the so-called Gamma function, etc.

By definition, the **Gamma Function**, or the *Second Eulerian integral*, is

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}.dx. \qquad (1)$$

This integral has been tabulated for all values of n between 1 and 2 to three decimal places. By the aid of such a table, the approximate value of all definite integrals reducible to Gamma functions can be calculated as easily as the ordinary trignometrical, or the logarithmic functions. There are three cases:

1. n lies between 1 and 2. (Use Table II., § 84.)
2. n is a positive integer. (Use formula (4), below.)
3. n is greater than 2. (Use (4) so as to make the value of the given expression depend on one in which n lies between 1 and 2.)

Integrate the above integral by parts, thus,

$$\int_0^\infty e^{-x}x^n.dx = n\int_0^\infty e^{-x}x^{n-1}.dx - e^{-x}x^n. \qquad (2)$$

Between the limits $x = 0$ and $x = \infty$, the last term vanishes.

Hence,
$$\int_0^\infty e^{-x}x^n.dx = n\int_0^\infty e^{-x}x^{n-1}.dx; \qquad (3)$$
or,
$$\Gamma(n+1) = n\Gamma(n) \qquad (4)$$

If n is integral, it follows from (4), that

$$\Gamma(n+1) = 1.2.3\ldots n = n! \qquad (5)$$

This important relation is true for any function of n, though $n!$ has a real meaning only when n is integral.

The following are a few examples of *the conversion of definite integrals into Gamma functions*. For a more extended discussion the special textbooks must be consulted.

1. $\Gamma(1) = 1$; $\Gamma(2) = 1$; $\Gamma(0) = \infty$; $\Gamma(-n) = \infty$; $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (6)
2. If a is independent of x,

$$\int_0^\infty e^{-ax}x^{m-1}.dx = \frac{1}{a^m}\Gamma(m). \qquad (7)$$

3. $\int_0^1 x^{m-1}(1-x)^{n-1}.dx = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (8)

The first member of (8) is sometimes called the *First Eulerian Integral*, or the *Beta Function*. It is written $B(m, n)$. The Beta function is here expressed in terms of the Gamma function. Substitute $x = ay/b$ in the second member of (8),

$$\int_0^\infty \frac{y^{m-1}dy}{(ay+b)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)}.$$

4. $$\int_0^{\pi/2} \sin^n x \,.\, dx = \int_0^{\pi/2} \cos^n x \,.\, dx = \frac{\frac{1}{2}\sqrt{\pi}\Gamma\frac{1}{2}(n+1)}{\Gamma\frac{1}{2}(n+1)}. \qquad (9)$$

5. $$\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}. \qquad (10)$$

If we substitute $\log x$ in place of $\log(x^{-1})$, the expression on the right becomes

$$(-1)^n\Gamma(n+1)/(m+1)^{n+1}.$$

6. $$\int_0^\infty x^n e^{-ax} \,.\, dx = a^{-(m+1)}\Gamma(n+1). \qquad (11)$$

7. $$\int_0^\infty e^{-a^2x^2} \,.\, dx = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{a} = \frac{\frac{1}{2}\sqrt{\pi}}{a}. \qquad (12)$$

8. $$\int_0^{\pi/2} \sin^p x \,.\, \cos^q x \,.\, dx = \frac{\Gamma[\frac{1}{2}(p+1)] \,.\, \Gamma[\frac{1}{2}(q+1)]}{2\Gamma[\frac{1}{2}(p+q)+1]}. \qquad (13)$$

Examples.—Evaluate the following integrals:—

(1) $\int_0^{\pi/2} \sin^6 x \,.\, \cos^5 x \,.\, dx$. From (13), we may write this integral

$$\frac{\Gamma(\frac{7}{2}) \,.\, \Gamma(\frac{6}{2})}{2\Gamma(\frac{13}{2})} = \frac{\frac{5}{2} \,.\, \frac{3}{2} \,.\, \frac{1}{2} \,.\, \sqrt{\pi} \,.\, 2 \,.\, 1}{2 \,.\, \frac{11}{2} \,.\, \frac{9}{2} \,.\, \frac{7}{2} \,.\, \frac{5}{2} \,.\, \frac{3}{2} \,.\, \frac{1}{2}\sqrt{\pi}} = \frac{8}{693}.$$

(2) $$\int_0^{\pi/2} \sin^7 x \,.\, \cos^2 x \,.\, dx = \frac{\Gamma(\frac{8}{2}) \,.\, \Gamma(\frac{3}{2})}{2\Gamma(\frac{11}{2})} = \frac{3 \,.\, 2 \,.\, 1 \,.\, \frac{1}{2} \,.\, \sqrt{\pi}.}{2 \,.\, \frac{9}{2} \,.\, \frac{7}{2} \,.\, \frac{5}{2} \,.\, \frac{3}{2} \,.\, \frac{1}{2} \,.\, \sqrt{\pi}}.$$

(3) $\int_0^{\pi/2} \sin^{10} x \,.\, dx$. From (9), $\dfrac{\frac{1}{2}\sqrt{\pi}\Gamma(\frac{11}{2})}{\Gamma(6)}$.

$$\frac{\sqrt{\pi}}{2} \,.\, \frac{\frac{9}{2} \,.\, \frac{7}{2} \,.\, \frac{5}{2} \,.\, \frac{3}{2} \,.\, \frac{1}{2} \,.\, \sqrt{\pi}}{5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} = \frac{\pi}{2} \,.\, \frac{9}{10} \,.\, \frac{7}{8} \,.\, \frac{5}{6} \,.\, \frac{3}{4} \,.\, \frac{1}{2}.$$

(4) $\int_0^\infty e^{-ax}x^5 \,.\, dx$. Use (7). Ansr. $\dfrac{\Gamma(6)}{a^6} = \dfrac{5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1}{a^6}$.

(5) If

$$\int_0^\infty \frac{x^{m-1} \,.\, dx}{1+x} = \frac{\pi}{\sin m x},$$

show that

$$\Gamma(m) \,.\, \Gamma(1-m) = \pi/\sin m x\,;$$
$$\Gamma(1+m) \,.\, \Gamma(1-m) = m\pi/\sin \pi x.$$

Put $m + n = 1$ in the Beta function, etc.

(6) From the preceding result show that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

§ 84. Numerical Table of the Gamma Function.

When n has any value not lying between 1 and 2, the Gamma function $\Gamma(n)$ may be readily calculated by means of equation (4), as indicated in the preceding examples. Table II., page 507, shows the value of

$$\log\int_0^\infty e^{-x}x^{n-1} \,.\, dx + 10, \text{ or, } \log\Gamma(n) + 10,$$

to three decimal places for all values of n between 1 and 2. It has been abridged from Legendre's tables to twelve decimal places as they appear in his *Exercices de Calcul Intégral*, tome ii., 80, 1817.

Since $\Gamma(n)$ is positive and less than unity for all values of n between 1 and 2, $\log \Gamma(n)$ will be negative for such values of n. Hence, as in the ordinary logarithmic tables of the trignometrical functions, the tabular logarithm is obtained by the addition of 10 to the natural logarithm of $\Gamma(n)$. This must be allowed for when arranging the final result.

§ 85. To find the Area of a Surface of Revolution.

A surface of revolution was defined, on page 108, to be a surface generated by the rotation of a line about a fixed axis, called the *axis of revolution*.

Let the curve APQ (Fig. 95) generate a surface of revolution as it rotates about the fixed axis Ox. It is required to find the area of this surface. The quadrature of surfaces of revolution is sometimes styled the "**Complanation of Surfaces**".

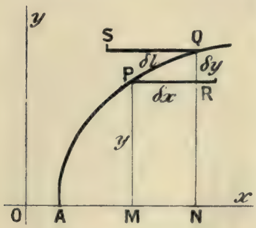


FIG. 95.

Take any point $P(x, y)$ on the curve. Let x receive an increment $\delta x = MN$ and y a corresponding increment $\delta y = QR$. Draw PR and QS each equal to PQ and parallel to ON. Let s denote the area of the surface of revolution of the curve AP about the x-axis and δs the surface generated by the revolution of PQ about the same axis. Let the length of the curve $AP = l$ and of the increment $PQ = \delta l$.

If PR revolves about ON, it will generate a cylinder whose superficies is $2\pi PM . PR$ (see page 491). QS revolving about ON will generate a cylinder whose surface is $2\pi QN . QS$. Therefore,

$$\frac{(\textit{Surface generated by } QS)}{(\textit{Surface generated by } PR)} = \frac{2\pi QN . QS}{2\pi PM . PR},$$

$$= \frac{y + \delta y}{y} = 1 + \frac{\delta y}{y}.$$

Therefore, $$Lt_{\delta y = 0}\frac{2\pi QN . QS}{2\pi PM . PR} = 1.$$

But the surface generated by the arc PQ is intermediate between that generated by QS and by PR. Therefore,

$$Lt\frac{(\textit{Surface generated by } PQ)}{(\textit{Surface generated by } PR)} = 1\,;\ Lt\frac{\delta s}{2\pi y \delta l} = 1\,;$$

$$Lt_{l=0}\frac{\delta s}{\delta l} = 2\pi y = \frac{ds}{dl}\,;\ \text{or, } ds = 2\pi y . dl. \qquad (1)$$

From (1), page 187, $dl = \sqrt{(dx)^2 + (dy)^2}$,

$$\therefore ds = 2\pi y \sqrt{(dx)^2 + (dy)^2}. \qquad . \qquad . \qquad (2)$$

If the curve revolves about the y-axis, similar formulae in x and y may be deduced.

The reader may be able to reason out another way of obtaining the above result. See Figs. 98 to 100, page 195.

EXAMPLES.—(1) Find the surface generated by the revolution of the slant side of a triangle. Hints, equation of the line OC (Fig. 96) is $y = mx$,

$$dy = mdx,$$

$$ds = 2\pi y \sqrt{1 + m^2} \,.\, dx,$$

$$s = \int 2\pi m \sqrt{1 + m^2} \,.\, xdx = \pi m x^2 \sqrt{1 + m^2} + C.$$

Reckon the area from the apex, where $x = 0$, therefore $C = 0$. If $x = h =$ height of cone $= OB$ and the radius of the base $= r = BC$, then, $m = r/h$ and

$$s = \pi r \sqrt{h^2 + r^2} = 2\pi r \times (\tfrac{1}{2} \textit{ Slant Height}).$$

This is a well-known rule in mensuration.

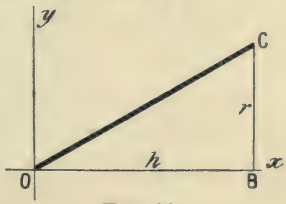


FIG. 96.

(2) Show that the paraboloid surface generated by the revolution of the parabola, $y^2 = 4ax$, is $\frac{8}{3}\pi a^2\{(a + x)^{3/2} - a^{3/2}\}$.

(3) Show that the surface generated by the revolution of a circle is $4\pi r^2$.

§ 86. To find the Volume of a Solid of Revolution.

This is equivalent to finding the volume of a cube of the same capacity as the given solid. Hence the process is named the **"Cubature of Solids"**.

The notion of differentials will allow us to deduce a method for finding the volume of the solid figure swept out by a curve rotating about an axis of revolution. At the same time, we can obtain a deeper insight into the meaning of the process of integration. In order to calculate the volume of a body we may suppose it to be resolved into a great number of elementary parallel planes, each plane being part of a small cylinder. Fig. 97 will, perhaps, help one to form a mental picture of the process. It is evident that the total volume of the solid is the sum of a number of elementary cylinders about the same axis. If δx be the height of one cylinder, y the radius of its base, the area of the base is πy^2. But the area of the base multiplied by the height of

FIG. 97 (after Cox).

the cylinder is the volume of each elementary cylinder, that is to say, $\pi y^2 \delta x$. The less the height of each cylinder, the more nearly will a succession of them form a figure with a continuous surface. At the limit, when $\delta x = 0$, the volume of the solid is

$$V = \pi \int y^2 . dx, \quad . \quad . \quad . \quad . \quad (1)$$

where x and y are the coordinates of the generating curve and the x-axis is the axis of revolution.

Formula (1) could have been obtained by a similar process of reasoning to that used in the preceding section. The abbreviated process here given illustrates how the idea of differentials facilitates the investigation of a complicated process.

Examples.—(1) Find the volume of the cone generated by the revolution of the slant side of the triangle in Example (1) of the preceding section.

$$y = mx.$$
$$dV = \pi y^2 . dx = \pi m^2 x^2 . dx.$$
$$\therefore V = \tfrac{1}{3}\pi m^2 x^3 + C.$$

If the volume be reckoned from the apex of the cone, $x = 0$, and, therefore, $C = 0$. Let $x = h$ and $m = r/h$, as before,

$$(\textit{Volume of the entire cone}) = \tfrac{1}{3}\pi r^2 h.$$

(2) Show that the volume generated by the revolving parabola, $y^2 = 4ax$, is $\frac{1}{2}\pi y^2 x$, where x = height and y = radius of the base.

(3) Required the volume of the sphere generated by the revolution of a circle, with the equation:

$$x^2 + y^2 = r^2. \quad (\textit{Volume of sphere}) = \tfrac{4}{3}\pi r^3.$$

§ 87. Successive Integration. Multiple Integrals.

Just as it is sometimes necessary, or convenient, to employ the second, third or the higher differential coefficients d^2y/dx^2, d^3y/dx^3 . . . , so it is often necessary to apply successive integration to reverse these processes of differentiation.

(*a*) *Successive integration with respect to a single independent variable.* Suppose that it is required to reduce, $d^2y/dx^2 = 2$, to its original or primitive form. We can write

$$\frac{d^2y}{dx^2} = 2 \,; \ \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2 \,; \ \text{or, } d\left(\frac{dy}{dx}\right) = 2dx.$$
$$\therefore dy/dx = 2\int dx = 2x + C_1.$$

Again, $$dy = (2x + C_1)dx \,; \text{ or, } y = \int (2x + C_1)dx,$$
$$\therefore y = x^2 + C_1 x + C_2.$$

In order to show that d^2y/dx^2 is to be integrated twice, we write

$$d^2y = 2dx^2,\ y = \iint 2dx^2, \text{ or } \iint 2dx \,.\, dx.$$

and hence, $$y = x^2 + C_1 x + C_2.$$

Notice that there are as many integration constants as there are symbols of integration.

EXAMPLES.—(1) Find the value of $y = \iiint x^3 . dx^3$. Ansr.
$$\tfrac{1}{4} . \tfrac{1}{5} . \tfrac{1}{6}x^6 + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

(2) Integrate $d^2s/dt^2 = g$, where g is a constant due to the earth's gravitation, t the time and s the space traversed by a falling body.
$$\therefore s = \iint g . dt^2 = \tfrac{1}{2}gt^2 + C_1t + C_2.$$
To find the values of the constants C_1 and C_2. Let the body start from a position of rest, then, $s = 0$, $t = 0$ $C_1 = 0$, $C_2 = 0$. See page 163.

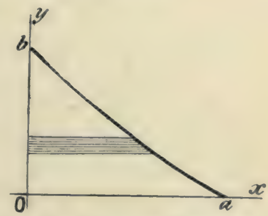


FIG. 98.—Surface Elements.

(*b*) *Successive integration with respect to two or more independent variables*. In finding the area of a curve, $y = f(x)$, the same result will be obtained whether we divide the area *Oab* (Figs. 98 to 100) into a number of strips parallel to the x-axis, as in Fig. 98, or vertical strips, Fig. 99. In the first case, the reader will no doubt be able to satisfy himself that the area A,
$$A = \int_0^b x . dy;$$
in the second,
$$A = \int_0^a y . dx.$$

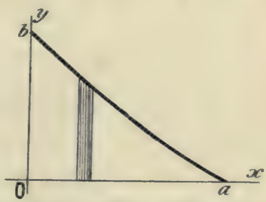


FIG. 99.—Surface Elements.

Substitute $\int_0^b dy$ for y in the last equation,
$$A = \int_0^a dx \int_0^b dy,$$
which is more conveniently written,
$$A = \int_0^a \int_0^b dx . dy.$$

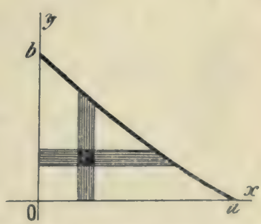


FIG. 100.—Surface Elements.

This integral is called a **double,** or **surface integral.** It means that if we divide the surface into an infinite number of small rectangles (Fig. 100) and take their sum, we shall obtain the required area of the surface.

To evaluate the double integral, first integrate with respect to one variable, no matter which, and afterwards integrate with respect to the other. If x be taken first, we find the sum of all the rectangles formed of the strips parallel to the x-axis, that is to say, we integrate between the limits a and 0, regarding dy as a constant *pro tem.*; we then take the sum of all the strips perpendicular to the x-axis, between the limits b and 0.

When there can be any doubt as to which differential the limits belong, the integration is performed in the following order: the right-hand element is taken with the first integration sign on the right, and so on with the next element.

EXAMPLES.—(1) Evaluate $\int_2^3\int_2^5 x\,.\,dx\,.\,dy$.

$$\text{Ansr. } \int_2^3 x\,.\,dx\Big[\,y\,\Big]_2^5 = 3\int_2^3 x\,.\,dx = 3\Big|_2^3 \tfrac{1}{2}x^2 = 7\tfrac{1}{2}.$$

(2) Show $\int_0^a\int_0^b xy^2\,.\,dx\,.\,dy = \frac{1}{6}a^2b^3$.

In a similar manner, if the volume of a body is to be investigated, we obtain **triple,** or **volume integrals** by supposing the solid to be split up into an infinite number of little parallelopipeds along the three dimensions, x, y, z. These infinitesimal figures are called **volume elements.** The capacity of each little element $dx \times dy \times dz$. The total volume, or the volume integral of the solid is

$$\iiint dx\,.\,dy\,.\,dz.$$

The first integration along the x-axis gives the area of an infinitely thin strip; the integration along the y-axis gives the area of an infinitely thin portion of the surface, and a third integration along the z-axis gives the sum of all these little portions of the surface, in other words, the volume of the body.

In the same way, quadruple and higher integrals may occur. These, however, are not very common. Multiple integration rarely extends beyond triple integrals.

EXAMPLES.—(1) Evaluate the following triple integrals:—

$$\int_1^4\int_1^5\int_1^6 yz^2\,.\,dx\,.\,dy\,.\,dz\,;\quad \int_1^4\int_1^5\int_1^6 yz^2\,.\,dy\,.\,dz\,.\,dx\,;\quad \int_1^4\int_1^5\int_1^6 yz^2\,.\,dz\,.\,dx\,.\,dy.$$

Ansrs. 2580, 1550, 1470 respectively.

(2) Show

$$\int_0^a\int_0^b (x^2/2p + y^2/2q)dx\,.\,dy = \tfrac{1}{2}b\int_0^a (x^2/p + b^2/3q)dx = \tfrac{1}{6}ab(a^2/p + b^2/q).$$

(3) Find the area (A) of the circle $x^2 + y^2 = r^2$, and the surface (S) of the sphere $x^2 + y^2 + z^2 = r^2$, by double integration. Ansrs.

$$A = 4\int_0^r\int_0^{\sqrt{(r^2-x^2)}} dxdy\,;\; S = 8r\int_0^r\int_0^{\sqrt{(r^2-x^2)}} \frac{dxdy}{\sqrt{(r^2 - x^2 - y^2)}}.$$

(4) Evaluate $8\int_0^r\int_0^{\sqrt{(r^2-x^2)}}\int_0^{\sqrt{(r^2-x^2-y^2)}} dx\,.\,dy\,.\,dz$. Ansr. $\frac{4}{3}\pi r^3$.

Note $\sin \frac{1}{2}\pi = 1$. Show that this integral represents the volume of a sphere whose equation is $x^2 + y^2 + z^2 = r^2$. Hint. The "dy" integration is the most troublesome. For it, put $r^2 - x^2 = c$, say, and use **C**, § 76. As a result, $\frac{1}{2}y\sqrt{r^2 - x^2 - y^2} + \frac{1}{2}(r^2 - x^2)\sin^{-1}\{y/\sqrt{r^2 - x^2}\}$, has to be evaluated between the limits $y = \sqrt{(r^2 - x^2)}$ and $y = 0$. The result is $\frac{1}{4}(r^2 - x^2)\pi$. The rest is simple enough.

§ 88. The Velocity of Chemical Reactions.

The time occupied by a chemical reaction is dependent, among other things, on the nature and concentration of the reacting substances, the presence of impurities and other "catalytic" agents, and on the temperature.

With some reactions these several factors can be so controlled, that measurements of the velocity of the reaction agree with theoretical results.

A great number of chemical reactions have hitherto defied all attempts to reduce them to order. For instance, the mutual action of HI and $HBrO_3$, of H_2 and O_2, of carbon and oxygen and the oxidation of phosphorus. The magnitude of the disturbing effects of secondary and catalytic actions obscures the mechanism of such reactions. In these cases more extended investigations are required to make clear what actually takes place in the reacting system. But see § 135 in Part II. Advanced.

Fuhrmann (*Zeitschrift für physikalische Chemie*, **4**, 89, 1889) classifies chemical reactions into "orders" according as one or more molecules are included in the reaction.

I.—*Reactions of the first order.* Let a be the concentration of the reacting molecules at the beginning of the action when the time $t = 0$. The concentration, after the lapse of an interval of time t, is, therefore, $a - x$, where x denotes the amount of substance transformed during that time. Let dx denote the amount of substance formed in the time dt. The velocity of the reaction at any moment is proportional to the concentration of the reacting substance **(Wilhelmy's law)**, hence we have

$$\frac{dx}{dt} = k(a - x)\,;\ \text{or, } k = \frac{1}{t}\cdot\log\frac{a}{a - x}. \qquad (1)$$

where k is a constant depending on the nature of the reacting system. Reactions which proceed according to this equation are said to be reactions of the first order.

II.—*Reactions of the second order.* Let a and b respectively denote the concentration of two different substances in such a reacting system as occurs when acetic acid acts on alcohol, or bromine on fumaric acid, then, according to the law of mass action, the velocity of the reaction at any moment is proportional to the concentration of the reacting substances. In this case

$$\frac{dx}{dt} = k(a - x)(b - x); \therefore k = \frac{1}{t} \cdot \frac{1}{a - b} \log \frac{a - x}{b - x} \cdot \frac{b}{a} \cdot \qquad (2)$$

Reactions which progress according to this equation are called reactions of the second order. For the integration, see page 173.

If the two reacting molecules are the same, then $a = b$. From (2), therefore, we get $\log 1 \times 1/0 = 0 \times \infty$. Such indeterminate fractions are discussed on page 245. It is there shown that when $a = b$, this expression may be made to assume the form,

$$k = \frac{1}{t} \cdot \frac{x}{a(a - x)}. \quad . \quad . \quad . \quad . \qquad (3)$$

This expression is also obtained by the integration on the corresponding equation,

$$dx/dt = k(a - x)^2. \quad . \quad . \quad . \quad . \qquad (4)$$

Equation (4) is that required for reactions similar to the polymerization of nitrogen dioxide, etc.

$$2NO_2 = N_2O_4.$$

In the hydrolysis of cane sugar,

$$C_{12}H_{22}O_{11} + H_2O = 2C_6H_{12}O_6,$$

let a denote the amount of cane sugar, b the amount of water present at the beginning of the action. The reaction is, therefore, represented by the equation,

$$dx/dt = k'(a - x)(b - x),$$

where x denotes the amount of sugar which actually undergoes transformation.

If the sugar has been dissolved in a large excess of water, the concentration of the water is practically constant during the whole process. But b is very large in comparison with x, therefore, $b - x$ may be assumed constant

$$k = k'(b - x),$$

where k' and k are constant. Hence equation (1) should represent the course of this reaction.

Wilhelmy's measurements of the rate of this reaction show that the above supposition corresponds closely with the truth.

EXAMPLE.—Proceed as on page 43 with the following pairs of values of x and t:—

$t = 15, \quad 30, \quad 45, \quad 60, \quad 75, \ldots$

$x = 0{\cdot}046, \quad 0{\cdot}088, \quad 0{\cdot}130, \quad 0{\cdot}168, \quad 0{\cdot}206, \ldots$

Substitute these numbers in (1); show that k is constant. Make the proper changes for use with common logs. Put $a = 1$.

The hydrolysis of cane sugar is, therefore, a reaction of the first order provided a large excess of water is present.

III.—*Reactions of the third order.* In this case three molecules take part in the reaction. Let a, b, c, denote the concentration of the reacting molecules of each species at the beginning of the reaction, then,

$$dx/dt = k(a - x)(b - x)(c - x). \qquad (5)$$

Integrate this expression as on page 173, put $x = 0$ when $t = 0$ in order to find the value of C. The final equation can then be written in the form,

$$k = -\frac{\log\left\{\left(\frac{a}{a - x}\right)^{b-c}\left(\frac{b}{b - x}\right)^{c-a}\left(\frac{c}{c - x}\right)^{a-b}\right\}}{t(a - b)(b - c)(c - a)}, \qquad (6)$$

where a, b, c, are all different.

This equation has been studied under various guises by Harcourt and Esson, J. J. Hood, Ostwald, etc. (See the set of examples at the end of this section.)

If we make $a = b = c$, in equation (5) and integrate the resulting expression

$$\frac{dx}{dt} = k(a - x)^3;\ k = \frac{1}{2t}\left\{\frac{1}{(a - x)^2} - \frac{1}{a^2}\right\} = \frac{x(2a - x)}{2ta(a - x)^2}. \qquad (7)$$

The polymerization of cyanic acid is an example of such a change,

$$3CNOH = C_3N_3O_3H_3.$$

Rearrange the terms of equation (7) so that,

$$x = a(1 - 1/\sqrt{2a^2kt + 1}). \qquad (8)$$

In order that we may have $x = a$, t must become infinite. This means that the reaction will only be completed after the elapse of an infinite time.

If $c = b$ in (6), and a is not equal to b,

$$k = \frac{1}{t}\cdot\frac{1}{(a - b)^2}\left\{\frac{(a - b)x}{b(b - x)} + \log\frac{a(b - x)}{b(a - x)}\right\}. \qquad (9)$$

See examples at the end of this section.

IV.—*Reactions of the fourth order.* These are comparatively rare. The reaction between hydrobromic and bromic acids is, under certain conditions, of the fourth order. So is the reaction between chromic and phosphorous acids (see page 175).

The general equation for a reaction in which n molecules of the same kind take part, is

$$-\frac{dx}{dt} = k(a - x)^n;\ k = \frac{1}{t} \cdot \frac{1}{n-1}\left\{\frac{1}{(a-x)^{n-1}} - \frac{1}{a^{n-1}}\right\}. \qquad (10)$$

The intermediate steps of the integration are

$$\frac{1}{(n-1)(a-x)^{n-1}} = kt + C;\ C = -\frac{1}{(n-1)a^{n-1}},$$

for, when $x = 0$, $t = 0$.

To find the order of a chemical reaction. Let C_1, C_2 be the concentration of the solution, that is to say, the quantity of reacting substance present in the solution, at the end of certain times t_1 and t_2. From equation (10),

$$-\frac{dC}{dt} = kC^n;\ \therefore \frac{1}{n-1} \cdot \frac{1}{C^{n-1}} = kt + \text{constant}, \quad . \qquad (11)$$

where n denotes the number of molecules taking part in the reaction. It is required to find a value for n. From (11)

$$-\int_{C_1}^{C_2}\frac{dC}{C^n} = kt;\ \text{or, } n = 1 + \log\frac{t_1}{t_2}\Big/\log\frac{C_2}{C_1}. \quad . \qquad (12)$$

Judson and Walker (*Journal of the Chemical Society*, **73**, 410, 1898) found that while the time required for the decomposition of a mixture of bromic and hydrobromic acids of concentration 77, was 15 minutes; the time required for the transformation of a similar mixture of substances in a solution of concentration 51·33, was 50 minutes. Substituting these values in (12),

$$n = 1 + \frac{\log 3{\cdot}333}{\log 1{\cdot}5} = 3{\cdot}97.*$$

The nearest integer, 4, represents the order of the reaction.

The intervals of time required for the transformation of equal fractional parts m of a substance contained in two solutions of different concentration C_1 and C_2, may be obtained by graphic interpolation (pages 68 and 254) from the curves whose abscissae are t_1 and t_2 and whose ordinates are C_1 and C_2 respectively.

Another convenient formula for the order of a reaction, is

$$n = \left(\log\frac{dC_1}{dt} - \log\frac{dC_2}{dt}\right)\Big/\log\frac{C_1}{C_2}. \quad . \quad . \qquad (13)$$

The reader will probably be able to deduce this formula for himself

---

* Use the table of natural logarithms, page 520.

(see Noyes, *Zeit. f. phys. Chem.*, **16**, 546, 1895; Noyes and Scott, *ibid.*, **18**, 118, 1895).

The mathematical treatment of velocity equations here outlined is in no way difficult, although, perhaps, some practice is still requisite in the manipulation of laboratory results. The following selection of typical examples illustrates what may be expected in practical work. The memoirs referred to may be considered as models of this kind of research.

EXAMPLES.—(1) It was once thought that the decomposition of phosphine by heat was in accordance with the equation, $4PH_3 = P_4 + 6H_2$; now, it is believed that the reaction is more simple, *viz.*, $PH_3 = P + 3H$, and that the subsequent formation of the P_4 and H_2 molecules has no perceptible influence on the rate of the decomposition. Show that these suppositions respectively lead to the following equations:

$$\frac{dx}{dt} = k(1 - x)^4 ;\ \therefore k = \frac{1}{t} \cdot \frac{1}{(1 - x)^3} - 1.$$

Or,
$$\frac{dx}{dt} = k'(1 - x) ;\ \therefore k' = \frac{1}{t} \cdot \log \frac{1}{1 - x}.$$

In other words, if the reaction be of the fourth order, k will be constant, and if of the first order k' will be constant.

To put these equations into a form suitable for experimental verification, let a gram molecules of PH_3 per unit volume be taken. Let the fraction x of a be decomposed in the time t. Hence, $(1 - x)a$ gram molecules of phosphine and $3ax/2$, of hydrogen remain. Since the pressure of the gas is proportional to its density, if the original pressure of PH_3 be p_0 and of the mixture of hydrogen and phosphine p_1, then,

$$p_1/p_0 = \{(1 - x)a + 3xa/2\}/a = 1 + \tfrac{1}{2}x,$$

$$x = 2p_1/p_0 - 2 ;\ (1 - x)a = (3 - 2p_1/p_0)a ;$$

$$k = \frac{1}{t}\left(\frac{p_0}{3p_0 - 2p_1}\right)^3 - 1 ;\ k' = \frac{1}{t} \log \frac{p_0}{3p_0 - 2p_1}.$$

Kooij (*Zeit. f. phys. Chem.*, **12**, 155, 1892) has published the following data:—

$t =$	0,	4,	14,	24,	46·3, . . .
$p =$	758·01,	769·34,	795·57,	819·16,	865·22, . . .

Hence show that k', not k, satisfies the required condition. The decomposition of phosphine is, therefore, a reaction of the first order.

(2) Does the reaction, $2PH_3 = 2P + 3H_2$, agree with Kooij's observations?

In experimental work in the laboratory, the investigator proceeds by the method of trial and failure in the hope that among many wrong guesses, he will at last hit upon one that will "go". So in mathematical work, there is no royal road. We proceed by instinct, not by rule. *E.g.*, we have here made three guesses. The first appeared the most probable, but on trial proved unmistakably wrong. The second, least probable guess, proved to be the one we were searching for.

(3) Show that the reaction,

$$CH_2Cl \cdot COOH + H_2O = CH_3 \cdot COOH + HCl,$$

in the presence of a large excess of water is of the first order. See Van't Hoff's *Studies in Chemical Dynamics* (Ewan's translation), 130, 1896, for experimental work.

(4) Find the order of the reaction between ferric and stannous chlorides from the two following series of observations:—

$$t_1 = 0, \quad \cdot 75, \quad 1, \quad 1\cdot 5; \qquad t_2 = 0, \quad 1, \quad 3, \quad 7;$$
$$x_1 \times 10 = 10\cdot 0, \quad 3\cdot 59, \quad 4\cdot 19, \quad 5\cdot 10; \qquad x_2 \times 10 = 6\cdot 25, \quad 1\cdot 43, \quad 2\cdot 59, \quad 3\cdot 61,$$

where x_1, x_2 denote the amounts of ferric chloride reduced in the times t_1 and t_2 respectively. Use formula (13), put $m = \frac{1}{2}$ and also $m = \frac{1}{3}$. Ansr. Third.

In the following examples, always verify your deduction by finding the numerical value of k when experimental data are given.

(5) Reicher (*Zeit. f. phys. Chem.*, **16**, 203, 1895) in studying the action of bromine on fumaric acid, found that when $t = 0$, his solution contained 8·8 of fumaric acid, and when $t = 95$, 7·87; the concentration of the acid was then altered by dilution with water, it was then found that when $t = 0$, the concentration was 3·88, and when $t = 132$, 3·51. Here $dC_1/dt = (8\cdot 88 - 7\cdot 87)/95 = 0\cdot 0106$; $dC_2/dt = 0\cdot 00227$ (page 200); $C_1 = (8\cdot 88 + 7\cdot 87)/2 = 8\cdot 375$; $C_2 = 3\cdot 7$, $n = 1\cdot 87$ in (13) above. The reaction is, therefore, of the second order.

(6) In the absence of disturbing side reactions, arrange velocity equations for the reaction,

$$2CH_3 . CO_2Ag + H . CO_2Na = CH_3 . COOH + CH_3 . CO_2Na + CO_2 + 2Ag.$$

Assuming that the silver, sodium and hydrogen salts are completely dissociated in solution, the reaction is essentially between the ions:

$$\overset{+}{Ag} + H . \overset{-}{C}OO = Ag + CO_2 + \overset{+}{H},$$

therefore, the reaction is of the third order. Verify this from the following data: When

$$t = 2, \quad 4, \quad 7, \quad 11, \quad 17, \ldots;$$
$$x \times 10^3 = 62\cdot 25, \quad 69\cdot 15, \quad 75\cdot 60, \quad 80\cdot 41, \quad 84\cdot 99 \ldots$$

(Noyes and Cottle, *Zeit. f. phys. Chem.*, **27**, 578, 1898.)

(7) Deduce the order of the reaction,

$$6FeCl_2 + KClO_3 + 6HCl = 6FeCl_3 + KCl + 3_2HO,$$

from the following data: 0·1 equivalents * of ferrous chloride, potassium chlorate and of hydrochloric acid are taken, then, if x denotes the quantity of $FeCl_2$ transformed in the time t, when

$$t = 5, \quad 15, \quad 35, \quad 60, \quad 170 \ldots;$$
$$x \times 10 = 4\cdot 8, \quad 12\cdot 2, \quad 23\cdot 8, \quad 32\cdot 9, \quad 52\cdot 5 \ldots$$

Ansr. Third order, since k only varies between 0·99 and 1·04 when $dx/dt = k(a - x)^3$. (Hood, *Phil. Mag.* [5], **6**, 371, 1878; **8**, 121, 1879; **20**, 323, 1885; Noyes and Wason, *Zeit. f. phys. Chem.*, **22**, 210, 1897.)

(8) The following observations were made on the reaction:—

$$C_6H_5 . SO_2 . OC_2H_5 + CH_3 . OH = CH_3OC_2H_5 + C_6H_5 . SO_2 . OH,$$
$$t = 5, \quad 10, \quad 15, \quad 25 \ldots;$$
$$x = 23\cdot 1, \quad 41\cdot 3, \quad 55\cdot 0, \quad 74\cdot 0 \ldots$$

What order of reaction gives a fairly constant value for k? (Sagrebin, *Zeit. f. phys. Chem.*, **34**, 149, 1900.)

* Note the distinction between "equivalent" and "molecular" amounts.

(9) Schwicker (*Zeit. f. phys. Chem.*, **16**, 303, 1895) has made two series of experiments on the action of iodine on potash. In the first series he used an excess of potash and found that when $t = 2$, $a = 10{\cdot}7$ of iodine and when

$$t = 6, \quad 11, \quad 28, \quad 38, \quad 68 \ldots;$$
$$x = 2{\cdot}10, \quad 2{\cdot}30, \quad 5{\cdot}68, \quad 6{\cdot}50, \quad 7{\cdot}86 \ldots$$

Hence show that reaction between iodine and excess of potash is of the second order. In a second series of experiments, an excess of iodine was used. $a = 7{\cdot}23$ after the elapse of two minutes, and subsequently, when

$$t = 4, \quad 8, \quad 13, \quad 36 \ldots;$$
$$x = 3{\cdot}43, \quad 4{\cdot}33, \quad 4{\cdot}88, \quad 5{\cdot}40 \ldots$$

Show that the reaction is probably of the third order. These results led Schwicker to the equations,

$$I_2 + 2KOH = KIO + KI + H_2O\,;\ 3KIO = 2KI + KIO_3.$$

(10) It is intended to investigate the rate of combination of hydrogen and oxygen gases at 440°. Assuming that the reaction is of the third order, arrange velocity equations for the following mixtures:—

$$(a)\ 2H_2 + O_2;\ (b)\ 4H_2 + O_2;\ (c)\ 2H_2 + 2O_2.$$

For (a) use (7), since $a = b = c$; for (b) use (9) substituting $a = 1, b = c = 2$, and for (c) substitute $a = 2, b = c = 1$ in (9). Then arrange the results for the indirect determination of x, by measuring the pressure of the mixed gases as example (1). (Compare Bodenstein, *Zeit. f. phys. Chem.*, **29**, 664, 1899).

§ 89. Chemical Equilibrium—Incomplete or Reversible Reactions.

Whether equivalent proportions of sodium nitrate and potassium chloride, or of sodium chloride and potassium nitrate, are mixed together in aqueous solution at constant temperature, each solution will, after the elapse of a certain time, contain these four salts distributed in the same proportions. Let m and n be positive integers, then

$$(m + n)NaNO_3 + (m + n)KCl = mNaCl + mKNO_3 + nNaNO_3 + nKCl\,;$$
$$(m + n)NaCl + (m + n)KNO_3 = mNaCl + mKNO_3 + nNaNO_3 + nKCl.$$

This is more concisely written,

$$NaCl + KNO_3 \rightleftarrows NaNO_3 + KCl.$$

The phenomenon is explained by assuming that the products of the reaction interact to reform the original components simultaneously with the direct reaction. That is to say, two independent and antagonistic changes take place simultaneously in the same reacting system. When the speeds of the two opposing reactions are perfectly balanced, the system appears to be in a stationary state of equilibrium. This is an illustration of the principle of the coexistence of different actions, page 52.

The special case of Wilhelmy's law dealing with these "incomplete" or reversible reactions is known as **Guldberg and Waage's law.***

Consider a system containing two reacting substances A_1 and A_2 such that

$$A_1 \rightleftarrows A_2.$$

Let a_1 and a_2 be the respective concentrations of A_1 and A_2. Let x of A_1 be transformed in the time t, then by Wilhelmy's law

$$\partial x/\partial t = k_1(a_1 - x).$$

Further, let x' of A_2 be transformed in the time t. The rate of transformation of A_2 to A_1 is then

$$\partial x'/\partial t = k_2(a_2 - x').$$

But for the mutual transformation of x of A_1 to A_2 and x' of A_2 to A_1, we must have, for equilibrium,

$$x = - x' \text{ and } dx = - dx';$$

or,

$$\partial x/\partial t = - k_2(a_2 + x).$$

The net, or total velocity of the reaction is obviously the algebraic sum of these "partial" velocities, or

$$dx/dt = k_1(a_1 - x) - k_2(a_2 + x). \qquad (1)$$

It is usual to write $K = k_1/k_2$. When the system has attained the stationary state $dx/dt = 0$. (Why?) And

$$K = (a_2 + x)/(a_1 - x), \qquad (2)$$

where x is to be determined by chemical analysis, a_1 is the amount of substance used at the beginning of the experiment, a_2 is made zero when $t = 0$. This determines K. Now integrate (1) by the method of partial fractions and proceed as indicated in the subjoined examples.

The more important memoirs for consultation are Berthollet, *Essai de Statique Chimie*, Paris, 1801-1803, or Ostwald's *Klassiker*, No. 74; Wilhelmy, *Pogg. Ann.*, **81**, 413, 1850; Ostwald's *Klassiker*, No. 29; Berthelot and Gilles, *Ann. de Chim. et d. Phys.* [3], **65**, 385, 1862; **66**, 5, 1862; **68**, 225, 1863; Harcourt and Esson (*l.c.*); Guldberg and Waage, *Journ. für praktische Chemie* [2], **19**, 69, 1879; Ostwald's *Klassiker*, No. 104.

Examples.—(1) In aqueous solution γ-oxybutyric acid is converted into γ-butyrolactone and γ-butyrolactone is transformed into γ-oxybutyric acid according to the equation,

$$CH_2OH . CH_2 . CH_2 . COOH \rightleftarrows \underbrace{CH_2 . CH_2 . CH_2 . CO}_{O} + H_2O.$$

* It is, of course, just as easy to consider irreversible reactions as special types of Guldberg and Waage's law by supposing the velocity of the reverse action, zero. I have followed the subject historically.

Use the preceding notation and show that the velocity of formation of the lactone is,

$$dx/dt = k_1(a_1 - x) - k_2(a_2 + x), \quad . \quad . \quad . \quad . \quad (3)$$

and

$$K = k_1/k_2 = (a_2 + x)/(a_1 - x). \quad . \quad . \quad . \quad . \quad (4)$$

Now integrate (3) by the method of partial fractions. Evaluate the integration constant for $x = 0$ when $t = 0$ and show that

$$\frac{1}{t} \cdot \log \frac{Ka_1 - a_2}{(Ka_1 - a_2) - (1 + K)x} = \text{constant}. \quad . \quad . \quad . \quad (5)$$

Henry (*Zeit. f. phys. Chem.*, **10**, 116, 1892) worked with $a_1 = 18{\cdot}23$, $a_2 = 0$; analysis showed that when $dx/dt = 0$, $x = 13{\cdot}28$; $a_1 - x = 4{\cdot}95$; $a_2 + x = 13{\cdot}28$; $K = 2{\cdot}68$. Substitute these values in (5); reduce the equation to its lowest terms and verify the constancy of the resulting expression when the following pairs of experimental values are substituted for x and t,

$$t = 21, \quad 50, \quad 65, \quad 80, \quad 160 \ldots;$$
$$x = 2{\cdot}39, \quad 4{\cdot}98, \quad 6{\cdot}07 \quad 7{\cdot}14, \quad 10{\cdot}28 \ldots$$

(2) A more complicated example than the preceding reaction of the first order occurs during the esterification of alcohol by acetic acid.

$$CH_3 . COOH + C_2H_5 . OH \rightleftarrows CH_3 . COOC_2H_5 + H . OH,$$

a reaction of the second order.

Let a_1, b_1 denote the initial concentrations of the acetic acid and alcohol respectively, a_2, b_2 of ethyl acetate and water. Show that,

$$dx/dt = k_1(a_1 - x)(b_1 - x) - k_2(a_2 + x)(b_2 + x). \quad . \quad . \quad (6)$$

Here, as elsewhere, the calculation is greatly simplified by taking gram molecules such that $a_1 = 1$, $b_1 = 1$, $a_2 = 0$, $b_2 = 0$. Equation (6) thus reduces to

$$dx/dt = k_1(1 - x)^2 - k_2x^2. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For the sake of brevity, write $k_1/(k_1 - k_2) = m$ and let α, β be the roots of the equation $x - 2mx + m = 0$. Show that (7) may be written

$$dx/(x - \alpha)(x - \beta) = (k_1 - k_2)dt.$$

Integrate for $x = 0$ when $t = 0$, in the usual way. Show that since $\alpha = m + \sqrt{m^2 - m}$ and $\beta = m - \sqrt{m^2 - m}$, page 387,

$$\frac{1}{t} \log \frac{(m - \sqrt{m^2 - m})(m + \sqrt{m^2 - m} - x)}{(m + \sqrt{m^2 - m})(m - \sqrt{m^2 - m} - x)} = 2(k_1 - k_2)\sqrt{m^2 - m}. \quad (8)$$

The value of K is determined as before. Since

$$m = k_1/(k_1 - k_2);\ m = 1/(1 - k_2/k_1).$$

Berthelot and Gilles' experiments show that for the above reaction,

$$k_1/k_2 = 4;\ m = \tfrac{4}{3};\ \sqrt{m^2 - m} = \tfrac{2}{3};$$

$\frac{4}{3}(k_1 - k_2) = 0{\cdot}00575$; or, using common logs., $\frac{4}{3}(k_1 - k_2) = 0{\cdot}0025$. The corresponding values of x and t were,

$$t = 64, \quad 103, \quad 137, \quad 167 \ldots;$$
$$x = 0{\cdot}250, \quad 0{\cdot}345, \quad 0{\cdot}421, \quad 0{\cdot}474 \ldots;$$
$$\text{constant} = 0{\cdot}0023, \quad 0{\cdot}0022, \quad 0{\cdot}0020, \quad 0{\cdot}0021 \ldots$$

Verify this last line. For smaller values of t, side reactions are supposed to disturb the normal reaction, because the value of the constant deviates somewhat from the regularity just found.

(3) Let one gram molecule of hydriodic acid in a v litre vessel be heated, decomposition takes place according to the equation:

$$2HI \rightleftarrows H_2 + I_2.$$

Hence show that for equilibrium,

$$\frac{dx}{dt} = k_1\left(\frac{1 - 2x}{v}\right)^2 - k_2\left(\frac{x}{v}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and that $(1 - 2x)/v$ is the concentration of the undissociated acid. Put $k_1/k_2 = K$ and verify the following deductions,

$$\int \frac{dx}{K(1 - 2x)^2 - x^2} = \frac{1}{2\sqrt{K}} \cdot \log \frac{\sqrt{K}(1 - 2x) + x}{\sqrt{K}(1 - 2x) - x} = \frac{k_2 t}{v^2}.$$

Since, when $t = 0$, $x = 0$, $C = 0$. Bodenstein (*Zeit. f. phys. Chem.*, **13**, 56, 1894; **22**, 1, 1897) found K, at 440° = 0·02, hence $\sqrt{K} = 0{\cdot}141$,

$$\therefore \frac{1}{t} \cdot \log \frac{1 + 5{\cdot}1x}{1 - 9{\cdot}1x} = \text{constant},$$

provided the volume remains constant. The corresponding values of x and t are to be found by experiment. *E.g.*, when $t = 15$, $x = 0{\cdot}0378$, constant $= 0{\cdot}0171$; and when $t = 60$, $x = 0{\cdot}0950$, constant $= 0{\cdot}0173$, etc.

(4) The "active mass" of a solid is independent of its quantity. Hence, if c is any arbitrary constant, show that for

$$K_2CO_3 + BaSO_4 \rightleftarrows K_2SO_4 + BaCO_3, \qquad Kc = x/(1 - x);$$
$$CaCl_2 + H_2C_2O_4 \rightleftarrows 2HCl + CaC_2O_4, \qquad Kc = x/(1 - x)^2;$$
$$CaCO_3 \rightleftarrows CaO + CO_2, \qquad Kc = p,$$

where p denotes the pressure of the gas. The first reactions take place in solution, the latter in a closed vessel. Write down the velocity equations before equilibrium is set up and arrange the results in a form suitable for experimental verification.

(5) Prove that the velocity equation of a complete reaction of the first order, $A_1 = A_2$, has the same general form as that of a reversible reaction, $A_1 \rightleftarrows A_2$, of the same order when the concentration of the substances is referred to the point of equilibrium instead of to the original mass.

Let ξ denote the value of x at the point of equilibrium, then,

$$dx/dt = k_1(a_1 - x) - k_2x, \text{ becomes, } dx/dt = k_1(a_1 - \xi) - k_2\xi.$$

Substitute for k_2 its value $k_1(a_1 - \xi)/\xi$ when $dx/dt = 0$,

$$\therefore dx/dt = k_1a_1(\xi - x)/\xi; \text{ or, } dx/dt = k(\xi - x), \quad . \quad . \quad (10)$$

where the meanings of a, k, k_1 will be obvious.

(6) Show that k is the same whether the experiment is made with the substance A_1, or A_2.

It has just been shown that starting with A_1, $k = k_1a_1/\xi$; starting with A_2, it is evident that there is $a_1 - \xi$ of A_2 will exist at the point of equilibrium. Hence show

$$dx/dt = k_2a_1\{(a_1 - \xi) - x\}/(a_1 - \xi);\ k_2\xi = k_1(a_1 - \xi),$$

therefore, as before, $\quad k_2a_1/(a_1 - \xi) = k_1a_1/\xi.$

Integrate the second of equations (9) between the limits $t = 0$ and $t = t$, $x = x_0$ and $x = x_1$, thus,

$$\{\log(\xi - x_1) - \log(\xi - x_2)\}/t = \text{constant}.$$

Show, from the following observations by Waddell (*Journal of Physical Chemistry*, **2**, 525, 1898), on the reciprocal conversion of ammonium thiocyanate into thiourea, that it makes no difference to the value of k, in (10), whether thiourea, or thiocyanate is used at the start.

First, the conversion of thiocyanate into thiourea, $\xi = 21 \cdot 2 \%$ of thiocyanate,

$t = 0,$	19,	38,	48,	60, . . .;	
$x = 2 \cdot 0,$	6·9,	10·4,	12·3,	13·5, . . .	

Second, the conversion of thiourea into thiocyanate, ξ = what?

$t = 0,$	38,	53,	68,	90, . . .;
$x = 31 \cdot 1,$	51·5,	54·4,	56·3,	65·0, . . .

Memoirs by Walker and Hambly, *Journ. Chem. Soc.*, **67**, 746, 1895; Walker and Appleyard, *ib.*, **69**, 193, 1896; Waddell, *l.c.*, **3**, 41, 1899, and Kistiakowsky, *Zeit. f. phys. Chem.*, **27**, 258, 1898, may be consulted with reference to this method of developing equilibrium equations.

§ 90. Fractional Precipitation.

If to a solution of a mixture of two salts, A and B, a third substance C, is added, in an amount insufficient to precipitate all A and B in the solution, more of one salt will be precipitated, as a rule, than the other. By redissolving the mixed precipitate and again partially precipitating the salts, we can, by many repetitions of the process, effect fairly good separations of substances otherwise intractable to any known process of separation.

Since Mosander thus fractioned the gadolinite earths in 1843 (Hood, *Phil. Mag.*, [5], **21**, 119, 1886), the method has been extensively employed by Crookes in some fine work on the yttria and other earths. The recent separations of polonium, radium and other curiosities has attracted some attention to the process. The "mathematics" of the reactions follows directly from the law of mass action.

Let only sufficient C be added to partially precipitate A and B and let the solution originally contain a of the salt A, b of the salt B. Let x and y denote the amounts of A and B precipitated at the end of a certain time t, then $a - x$ and $b - x$ will represent the amounts of A and B respectively remaining in the solution. The rates of precipitation are, therefore,

$$dx/dt = k(a - x)(c - z); \; dy/dt = k'(b - y)(c - z),$$

where $c - z$ denotes the amount of C remaining in the solution at the end of a certain time t.

$$\therefore dx/dt : dy/dt = k(a - x) : k'(b - y),$$

or,

$$k'\int\frac{d(a - x)}{a - x} = k\int\frac{d(b - y)}{b - y},$$

or,

$$k'\log(a - x) = k\log(b - y) + \log C',$$

where $\log C'$ is the integration constant.

To find C' put $x = 0$ and $y = 0$, then

$$\log a^{k'} = \log C'b^k, \text{ or, } C' = a^{k'}/b^k.$$

Therefore,
$$\frac{k}{k'} = \frac{\log(a - x)/a}{\log(b - y)/b}. \quad . \quad . \quad . \quad (1)$$

The ratio $(a - x)/a$ measures the amount of salt remaining in the solution, after x of it has been precipitated. The less this ratio, the greater the amount of salt A in the precipitate. The same thing may be said of the ratio $(b - y)/b$ in connection with the salt B.

The more k exceeds k', the less will A tend to accumulate in the precipitate and, the more k' exceeds k, the more will A tend to accumulate in the precipitate. If the ratio k/k' is nearly unity, the process of fractional precipitation will be a very long one. In the limiting case, when $k = k'$, or, $k/k' = 1$, the ratio of A to B in the mixed precipitate will be the same as in the solution. In such a case, the complex nature of the "earth" could never be detected by fractional precipitation.

The application to gravimetric analysis is obvious.

§ 91. The Isothermal Expansion of Gases.

To find the work done during the isothermal* expansion of a gas.

Case i.—*The gas obeys Boyle's law,*

$$pv = \text{constant say, } c.$$

On page 182 it was shown that the work done when a gas expands against any external pressure is represented by the product of the pressure into the change of volume. The work performed during any small change of volume, is

$$dW = p\,.\,dv. \quad . \quad . \quad . \quad . \quad (1)$$

But by Boyle's law,

$$p = f(v) = c/v. \quad . \quad . \quad . \quad . \quad (2)$$

Substitute this value of p in (1), and

$$\therefore dW = c\,.\,dv/v.$$

If the gas expands from a volume v_1 to a new volume v_2,

$$W = c\int_{v_1}^{v_2}\frac{dv}{v} = c\Big|_{v_1}^{v_2}\log v + C;$$

or,
$$W = c\log\frac{v_2}{v_1}. \quad . \quad . \quad . \quad . \quad (3)$$

* "Isothermal" means "at a constant temperature," as pointed out on page 90.

From (2), $v_1 = c/p_1$ and $v_2 = c/p_2$, hence,

$$W = c \log \frac{p_1}{p_2}. \quad . \quad . \quad . \quad . \qquad (4)$$

Equations (3) and (4) play a most important part in the theory of gases, in thermodynamics and in the theory of solutions.

The value of c is equal to the product of the initial volume (v_0) and pressure (p_0) of the gas,

$$\therefore\ W = 2{\cdot}3026 p_0 v_0 \log_{10}\frac{v_1}{v_2};$$

$$= 2{\cdot}3026 p_0 v_0 \log_{10}\frac{p_2}{p_1}.$$

See page 520, for a numerical example.

Case ii.—*The gas obeys van der Waals' law,*

$$\left(p + \frac{a}{v^2}\right)(v - b) = \text{constant, say, } c'.$$

As an exercise prove that

$$W = c' \log \frac{v_2 - b}{v_1 - b} - a\left(\frac{1}{v_1} - \frac{1}{v_2}\right). \quad . \quad . \quad . \qquad (5)$$

This equation has occupied a prominent place in the development of van der Waals' theories of the constitution of gases and liquids.

Case iii.—*The gas dissociates during expansion.* (After Nernst and Schönflies.)

By Guldberg and Waage's law, in the reaction:

$$N_2O_4 \rightleftarrows 2NO_2,$$

for equilibrium,

$$K\frac{1 - x}{v} = \frac{x}{v} \cdot \frac{x}{v}.$$

where $(1 - x)/v$ represents the concentration of the undissociated nitrogen peroxide.

The relation between the volume and degree of dissociation is, therefore,

$$Kv = x^2/(1 - x). \quad . \quad . \quad . \qquad (6)$$

where x the fraction of unit mass of gas dissociated.

If n represents the original number of molecules $(1 - x)n$ will represent the number of undissociated molecules and $2xn$ the number of dissociated molecules. If the relation $pv = c$, does not vary during the expansion, the pressure will be proportional to the number of molecules actually present, that is to say,

$$n : \{(1 - x)n + 2xn\} = 1 : 1 + x.$$

The actual pressure of the gas is, therefore,

$$p = (1 + x)p,$$

and the work done is, therefore,

$$dW = p \,.\, dv = (1 + x)p \,.\, dv = p \,.\, dv + xp \,.\, dv.$$

But $$dW_1 = p \,.\, dv \text{ and } dW_2 = xp \,.\, dv, \quad . \quad . \quad (7)$$

and $$W = W_1 + W_2. \quad . \quad . \quad . \quad . \quad (8)$$

From Boyle's law, $p = c/v$, and (6),

$$\therefore p = \frac{c}{v} = \frac{cK(1 - x)}{x^2}.$$

Substitute this value of p in (7). Differentiate (6) and substitute the value of dv so obtained in our last result. Simplify and

$$dW_2 = c(2 - x)dx/(1 - x) = c\{1 + 1/(1 - x)\}dx.$$

Integrate $$W_2 = c\int_{x_2}^{x_1}\Big(1 + \frac{1}{1 - x}\Big)dx,$$

where x_1 and x_2 denote the values of x corresponding to v_1 and v_2.

$$\therefore W_2 = c\{(x_2 - x_1) - \log(1 - x_2)/(1 - x_1)\}.$$

Find dW_1 in a similar way from (7).

$$W_1 = c\log(v_2/v_1).$$

$$\therefore W = c\Big(\log\frac{v_2}{v_1} + x_2 - x_1 - \log\frac{1 - x_2}{1 - x_1}\Big). \quad . \quad (9)$$

It follows from (6), that

$$v_1 = \frac{x_1^2}{K(1 - x_1)}, \text{ and } v_2 = \frac{x_2^2}{K(1 - x_2)}.$$

Substitute these values of v in (9)

$$\therefore W = c\Big\{x_2 - x_1 - 2\log\frac{x_1(1 - x_2)}{x_2(1 - x_1)}\Big\}. \quad . \quad (10)$$

EXAMPLES.—(1) Find the work done during the isothermal expansion of dissociating ammonium carbamate, supposed gaseous.

$$NH_2COONH_4 \rightleftarrows 2NH_3 + CO_2.$$

(2) In calculating the work done during the isothermal expansion of dissociating hydrogen iodide,

$$2HI \rightleftarrows H_2 + I_2,$$

does it make any difference whether the hydrogen iodide dissociates or not?

(3) If the force of attraction (f) between two molecules of a gas, varies inversely as the fourth power of the distance (r) between them, show that the work (W) done against molecular attractive forces when a gas expands into a vacuum, is proportional to the difference between the initial and final pressures of the gas. That is,

$$W = A(p_1 - p_2), \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where A is the variation constant of § 189. By hypothesis,

$$f = a/r^4: \text{ and, } dW = f \,.\, dr,$$

where a is another variation constant. (See § 79.) Hence,

$$W = \int_{r_1}^{2} f \,.\, dr = \int_{r_1}^{r_2} a \frac{dr}{r^4}.$$

But r is linear, therefore, the volume of the gas will vary as r^3. Hence, $v = br^3$, where b is again constant.

$$\therefore W = \frac{a}{3}\left(\frac{1}{r^3_1} - \frac{1}{r^3_2}\right);\ \therefore W = \frac{ab}{3}\left(\frac{1}{v_1} - \frac{1}{v_2}\right). \quad . \quad . \quad (12)$$

But by Boyle's law, $pv =$ constant, say, $= c$. Hence it follows,

$$W = A(p_1 - p_2), \text{ if } A = ab/3c = \text{constant}.$$

(4) *If* the work done against molecular attractive forces when a gas expands into a vacuum, is

$$W = \int_{v_1}^{v_2} \frac{a}{v^2} dv = a\left(\frac{1}{v_1} - \frac{1}{v_2}\right),$$

where a is constant; v_1, v_2, refer to the initial and final volumes of the gas, show that "any two molecules of a gas will attract one another with a force inversely proportional to the fourth power of the distance between them".*

§ 92. The Adiabatic † Expansion of Gases.

In one of the examples appended to § 26, we obtained the expression,

$$dQ = \left(\frac{\partial Q}{\partial v}\right)_p dv + \left(\frac{\partial Q}{\partial p}\right)_v dp. \quad . \quad . \quad . \quad (1)$$

As pointed out on page 29, we may, without altering the value of the expression, multiply and divide each term within the brackets by $\partial\theta$. Thus,

$$dQ = \left(\frac{\partial Q}{\partial \theta}\frac{\partial \theta}{\partial v}\right)_p dv + \left(\frac{\partial Q}{\partial \theta}\frac{\partial \theta}{\partial p}\right)_v dp. \quad . \quad . \quad (2)$$

But $(\partial Q/\partial\theta)_p$ is the amount of heat added to the substance at a constant pressure for a small change of temperature; this is none other than the specific heat at constant pressure, usually written C_p. Similarly $(\partial Q/\partial\theta)_v$ is the specific heat at constant volume, written C_v.

$$dQ = C_p\left(\frac{\partial \theta}{\partial v}\right)_p dv + C_v\left(\frac{\partial \theta}{\partial p}\right)_v dp. \quad . \quad . \quad (3)$$

This equation tells that when a certain quantity of heat is added to a substance, one part is spent in raising the temperature while the volume changes under constant pressure, and the other part is

* For the meaning of a/v^2, see van der Waals' equation.

† The substance is supposed to be in such a condition that no heat can enter or leave the body during the expansion. The temperature, therefore, may change during the operation.

spent in raising the temperature while the pressure changes under constant volume.

For an ideal gas obeying Boyle's law,

$$pv = R\theta.$$

$$\therefore v/R = (\partial\theta/\partial p)\,;\ p/R = (\partial\theta/\partial v).$$

Substitute these values in (3),

$$\therefore dQ = C_p p \,.\, dv/R + C_v v \,.\, dp/R.$$

Divide through by $\theta = pv/R$, and,

$$\frac{dQ}{\theta} = C_p\frac{dv}{v} + C_v\frac{dp}{p}. \quad . \quad . \quad . \quad (4)$$

By definition, an adiabatic change takes place when the system neither gains nor loses heat, that is to say, $dQ = 0$.

The ratio of the two specific heats C_p/C_v is a constant, usually written γ.

$$\therefore \frac{C_p}{C_v}\cdot\frac{dv}{v} + \frac{dp}{p} = 0\,;\ \text{or,}\ \gamma\int\frac{dv}{v} + \int\frac{dp}{p} = \text{constant}.$$

or, $\gamma\log v + \log p = \text{constant}$; or, $\log v^\gamma + \log p = \text{constant}$,

$$\therefore \log(pv^\gamma) = \text{constant}\,;\ \text{or, } pv^\gamma = \text{constant}. \quad . \quad (5)$$

A most important relation in the theory of thermodynamics.

By integrating between the limits p_1, p_2 and v_1, v_2 in the above equation, we could have eliminated the constant and obtained

$$\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^\gamma, \quad . \quad . \quad . \quad . \quad (6)$$

a useful form of (5).

Substituting $v_1 = \theta_1 R/p_1$ and $v_2 = \theta_2 R/p_2$ in (6),

$$\left(\frac{p_2}{p_1}\cdot\frac{\theta_1}{\theta_2}\right)^\gamma = \frac{p_2}{p_1}\,;\ \text{or,}\ \left(\frac{\theta_1}{\theta_2}\right)^\gamma = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_2}\right)^\gamma;\ i.e., \left(\frac{\theta_1}{\theta_2}\right)^\gamma = \left(\frac{p_1}{p_2}\right)^{\gamma-1}, \quad (7)$$

and from (6)

$$\frac{\theta_1}{\theta_2} = \left(\frac{v_1}{v_2}\right)^{\gamma-1}. \quad . \quad . \quad . \quad . \quad (8)$$

Equation (6), in words, states that the adiabatic pressure of a gas varies inversely as the γth power of the volume. Equation (8) affirms that for adiabatic changes, the absolute temperature of a gas varies inversely as the $(\gamma - 1)$th power of the volume. Two well-known thermodynamic laws.

To find the work performed when a gas is compressed under adiabatic conditions.

From (5), if we write the constant c',

$$p = c'/v^\gamma.$$

Since the work done when the volume of a gas is compressed from v_1 to v_2 is (page 182),

$$W = -\int_{v_1}^{v_2} p\,.\,dv = -\int_{v_1}^{v_2} c'\frac{dv}{v^\gamma};$$

$$= -c'\left[\frac{v^{-(\gamma-1)}}{-(\gamma-1)}\right]_{v_1}^{v_2} = \frac{c'}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}}\right). \quad . \qquad (9)$$

From (5), $c' = p_1v_1^\gamma = p_2v_2^\gamma$. We may, therefore, represent this relation in another form, *viz.*:

$$\frac{c'}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}}\right) = \frac{1}{\gamma-1}\left(\frac{p_1v_1^\gamma}{v_2^{\gamma-1}} - \frac{p_1v_1^\gamma}{v^{\gamma-1}}\right)$$

$$= \frac{1}{\gamma-1}\left(\frac{p_2v_2^\gamma}{v_2^{\gamma-1}} - \frac{p_1v_1^\gamma}{v_1^{\gamma-1}}\right) = \frac{1}{\gamma-1}(p_2v_2 - p_1v_1). \quad (10)$$

If $p_1v_1 = R\theta_1$ and $p_2v_2 = R\theta_2$, are the isothermal equations for θ_1° and θ_2°, we may write,

$$W = \frac{R}{\gamma-1}(\theta_2 - \theta_1), \quad . \quad . \quad . \quad \mathbf{(11)}$$

which states in words, that *the work required to compress a mass of gas adiabatically while the temperature changes from* θ_1° to θ_2°, *will be independent of the initial pressure and volume of the gas.* In other words, the work done by a perfect gas in passing along an adiabatic curve, from one isothermal to another, is constant (see page 92), and independent of the path.

EXAMPLES.—(1) From (5), show that $p_1v_1^\gamma = p_0v_0^\gamma$ and hence deduce the formula,

$$\gamma = (\log p_0 - \log p_1)/(\log p_2 - \log p_1), \quad . \quad . \quad . \quad (12)$$

used by Clément and Desormes in their determinations of the ratio of the specific heats of some of the gases (*Journ. de Physique*, **89**, 333, 1819). The experimental details are given in most textbooks. Here it is only necessary to know that $p_1v_1 = p_2v_0$ under the conditions of the experiment. The numerical values of p_0, p_1, p_2, are determined by experiment.

(2) To continue illustration 3, § 18, page 44. We have assumed Boyle's law $\rho p_0 = \rho_0 p$. This is only true under isothermal conditions. For a more correct result, use (5) above. Write the constant c. For a constant mass (m) of gas, $m = \rho v$, hence show that for adiabatic conditions,

$$\rho p_0^{1/\gamma} = \rho_0 p^{1/\gamma}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence deduce the more correct form of Halley's law:

$$p^{1-\frac{1}{\gamma}} = p_0^{1-\frac{1}{\gamma}} - \frac{\gamma-1}{\gamma}ch, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

for the pressure (p) of the atmosphere at a height h above sea-level. Atmospheric pressure at sea-level $= p_0$.

(3) From the preceding example proceed to show that the rate of diminu-

tion of temperature (θ) is constant per unit distance (h) ascent. In other words, prove and interpret

$$\theta_0 - \theta = \frac{1}{R} \cdot \frac{\gamma - 1}{\gamma} h. \quad . \quad . \quad . \quad . \quad (15)$$

(4) Lummer and Pringsheim have used the last of equations (7), for evaluating γ by allowing a gas at pressure p_1 to expand suddenly to another pressure p_2 and measuring the instantaneous rise of temperature θ_1 to θ_2. Hence, given the numerical values of p_1, p_2, θ_1, θ_2, how would you calculate the numerical value of γ? Ansr. $\gamma = \log(p_1/p_2)/\{\log(p_1/p_2) - \log(\theta_1/\theta_2)\}$.

(5) To continue the discussion at the end of § 26, Examples (4) to (8). Suppose the gas obeys van der Waals' law:

$$\left(p + \frac{a}{v^2}\right)(v - b) = R\theta, \quad . \quad . \quad . \quad . \quad (16)$$

where R, a, b, are known constants. The first law of thermodynamics may be written

$$dQ = C_v . d\theta + (p + a/v^2)dv, \quad . \quad . \quad . \quad . \quad (17)$$

where the specific heat at constant volume has been assumed constant. *To find a value for C_p, the specific heat at constant pressure.* Expand (16). Differentiate the result. Cancel the term $2ab . dv/v^3$ as a very small order of magnitude (§ 4). Solve the result for dv. Multiply through with $p + a/v^2$. Since a/v^2 is very small, show that the fraction $(p + a/v^2)/(p - a/v^2)$ is very nearly $1 + 2a/pv^2$ (pages 8 and 224). Substitute the last result in (17), and

$$dQ = \left\{C_v + R\left(1 + \frac{2a}{pv^2}\right)\right\}d\theta - \left(1 + \frac{2a}{pv^2}\right)(v - b)dp.$$

Obviously the coefficient of $d\theta$ is equivalent to $(\partial Q/\partial\theta)_p$, *i.e.*, to C_p; while the coefficient of dp is $(\partial Q/\partial p)_\theta$. By hypothesis C_v is constant,

$$\therefore \frac{C_p}{C_v} = 1 + \frac{R}{C_v}\left(1 + \frac{2a}{pv^2}\right). \quad . \quad . \quad . \quad . \quad (18)$$

For ideal gases $a = 0$, and we get Mayer's equation, § 26.

For.	Air.	Hydrogen.	Carbon Dioxide.	
a . . .	0·002812	0·0000895	0·00874	Boynton (*l.c.*);
R/C_v . . .	0·4	0·4	0·2857	
γ (calculated) .	1·40225	1·40007	1·2907	From (18);
γ (observed) . .	1·403	1·4017	1·2911	Mean of data in Meyer's *Kinetic Theory of Gases.*

(6) Show van der Waals' equation for adiabatic conditions is

$$\left(p + \frac{a}{v^2}\right)(v - b)^\gamma = R\theta. \quad . \quad . \quad . \quad . \quad (19)$$

§ 93. The Influence of Temperature on Chemical and Physical Changes—van't Hoff's Formula.

In example 7, page 61, we have obtained the formula,

$$\left(\frac{\partial Q}{\partial v}\right)_\theta = \theta\left(\frac{\partial p}{\partial \theta}\right)_v, \quad . \quad . \quad . \quad . \quad (1)$$

by a simple process of mathematical reasoning. The physical signification of this formula is that the change in the quantity of heat communicated to any substance per unit change of volume at constant temperature, is equal to the product of the absolute temperature into the change of pressure per unit change of temperature at constant volume.

Suppose that $1 - x$ grams of one system A is in equilibrium with x grams of another system B. Let v denote the total volume and θ the temperature of the two systems. Equation (1) shows that $(\partial Q/\partial v)_\theta$ is the heat absorbed when the very large volume of system A is increased by unity at constant temperature θ, less the work done during expansion. Suppose that during this change of volume, a certain quantity $(\partial x/\partial v)_\theta$ of system B is formed, then, if q be the amount of heat absorbed when unit quantity of the first system is converted into the second, the quantity of heat absorbed during this transformation is $q(\partial x/\partial v)_\theta$. q is really the molecular heat of the reaction.

The work done during this change of volume is $p \, . \, dv$; but dv is unity, hence the external work of expansion is p. Under these circumstances,

$$q\Big(\frac{\partial x}{\partial v}\Big)_\theta = \Big(\frac{\partial Q}{\partial v}\Big)_\theta - p = \Big(\frac{\partial p}{\partial \theta}\Big)_v - p = \frac{\theta \partial p - p \partial \theta}{\partial \theta}, \quad . \quad (2)$$

from (1). Now multiply and divide the numerator by θ^2 (see integrating factors, pages 58 and 120).

$$\therefore \; q\Big(\frac{\partial x}{\partial v}\Big)_\theta = \theta^2\Big(\frac{\partial (p/\theta)}{\partial \theta}\Big)_v . \quad . \quad . \quad . \quad (3)$$

If, now, n_1 molecules of the system A and n_2 molecules of the system B take part in the reaction, we must write, instead of $pv = R\theta$,

$$pv = R\theta\{n_1(1 - x) + n_2 x\}; \text{ or, } p/\theta = R[n_1 + (n_2 - n_1)x]/v.$$

(The reason for this is well worth puzzling out.)

$$\therefore \; \Big(\frac{\partial (p/\theta)}{\partial \theta}\Big)_v = \frac{R}{v}(n_2 - n_1)\Big(\frac{\partial x}{\partial \theta}\Big)_v .$$

Substitute this result in equation (9) and we obtain

$$q\Big(\frac{\partial x}{\partial v}\Big)_\theta = \frac{\theta^2 R}{v}(n_2 - n_1)\Big(\frac{\partial x}{\partial \theta}\Big)_v . \quad . \quad . \quad . \quad (4)$$

By Guldberg and Waage's statement of the mass law,

$$(x/v)^{n_2} = K\{(1 - x)/v\}^{n_1}.$$

$$\therefore \log K + (n_2 - n_1) \log v = n_2 \log x - n_1 \log (1 - x).$$

Differentiate this last expression with respect to θ, at constant volume and with respect to v, at constant temperature,

$$\Big(\frac{\partial x}{\partial v}\Big)_\theta = \frac{n_2 - n_1}{v\Big(\dfrac{n_2}{x} + \dfrac{n_1}{1 - x}\Big)}; \quad \Big(\frac{\partial x}{\partial \theta}\Big)_v = \frac{(\partial \log K)/\partial \theta}{\dfrac{n_2}{x} + \dfrac{n_1}{1 - x}}.$$

Introduce these values in (4) and reduce the result to its simplest terms, thus,

$$\frac{\partial \log K}{\partial \theta} = \frac{q}{R\theta^2}. \quad . \quad . \quad . \quad . \quad (5)$$

This fundamental relation expresses the change of the equilibrium constant K with temperature at constant volume in terms of the molecular heat of the reaction.

Equation (5), first deduced by van't Hoff, has led to some of the most important results of physical chemistry.

Since R and θ are positive, K and q must always have the same sign. Hence *van't Hoff's principle of mobile equilibrium* follows directly, *viz.* :—

If the reaction absorbs heat, it advances with rise of temperature; if the reaction evolves heat it retrogrades with rise of temperature; and if the reaction neither absorbs nor evolves heat, the state of equilibrium is stationary with rise of temperature.

According to the particular nature of the systems considered q may represent the so-called heat of sublimation, heat of vaporization, heat of solution, heat of dissociation, or the thermal value of strictly chemical reactions when certain simple modifications are made in the interpretation of the "concentration" K.

If, at temperature θ_1 and θ_2, K becomes K_1 and K_2, we get, by the integration of (5),

$$\log \frac{K_2}{K_1} = \frac{q}{2}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right). \quad . \quad . \quad . \quad (6)$$

The thermal values of the different molecular changes, calculated by means of this equation, are in close agreement with experiment. For instance:

	q in calories.	
	Calculated.	Observed.
Heat of vaporization of water	10100	10296
Heat of solution of benzoic acid in water	6700	6500
Heat of sublimation of NH_4SH	21550	21640
Heat of combination of $BaCl_2 + 2H_2O$	3815	3830
Heat of dissociation of N_2O_4	12900	12500
Heat of precipitation of $AgCl$	15992	15850

A sufficiently varied assortment to show the profound nature of the relation symbolised by equations (5) and (6) (see van't Hoff's *Chemical Dynamics* (Ewan's translation)).

NUMERICAL EXAMPLE.—Calculate the heat of solution of mercuric chloride from the change of solubility with change of temperature. If c_1, c_2 denote the solubilities corresponding to the respective absolute temperatures θ_1 and θ_2,

$$c_1 = 6{\cdot}57 \text{ when } \theta_1 = 273^\circ + 10^\circ;\ c_2 \doteq 11{\cdot}84 \text{ when } \theta_2 = 273^\circ + 50^\circ.$$

Since the solubility of a salt in a given solvent is constant at any fixed temperature, we may write c in place of the equilibrium constant K. From (6), therefore,

$$\log\frac{c_2}{c_1} = \frac{q}{2}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right);\ \therefore\ \log\frac{11{\cdot}84}{6{\cdot}57} = \frac{q}{2}\left(\frac{1}{283} - \frac{1}{323}\right).$$

$$\therefore q = \log 1{\cdot}8 \times 45{,}704{\cdot}5 = 2{,}700 \text{ (nearly)};$$

$$q \text{ (observed)} \qquad = 3{,}000 \text{ (nearly)}.$$

Use the Table of Natural Logarithms, Chapter XIII., for the calculation.

Le Chatelier has reversed the above calculations, and, as the result of more extended investigations, he has enunciated the important generalisation : "any change in the factors of equilibrium from outside, is followed by a reversed change within the system". This rule, known as *Le Chatelier's theorem*, enables the chemist to foresee the influence of pressure and other agents on physical and chemical equilibria.

For further light on this important subject, consult Le Chatelier's *Les Équilibres Chimiques*, 1888 ; *Zeit. f. phys. Chem.*, **9**, 335, 1892 ; Bancroft's *The Phase Rule*, 1897.

The beginner will find it worth while to write out the leading assumptions introduced as premises in deducing van't Hoff's formula.

CHAPTER V.

INFINITE SERIES AND THEIR USES.

"In abstract mathematical theorems, the approximation to truth is perfect. . . . In physical science, on the contrary, we treat of the least quantities which are perceptible."—W. Stanley Jevons.

§ 94. What is an Infinite Series?

Mark off a distance AB of unit length. Bisect AB at O_1, bisect O_1B at O_2, O_2B at O_3, etc.

$A \qquad\qquad\qquad\qquad O_1 \qquad\qquad O_2 \qquad O_3 \; O_4 \; B.$

By continuing this operation, we can approach as near to B as we please. In other words, if we take a sufficient number of terms of the series,

$$AO_1 + O_1O_2 + O_2O_3 + \ldots,$$

we shall obtain a result differing from AB by as small a quantity as ever we please.

This is the geometrical meaning of the infinite series of terms,

$$1 = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \ldots \text{ to infinity.} \quad . \qquad (1)$$

Such an expression, in which the successive terms are related according to a known law, is called a **series.**

When the sum of an infinite series approaches closer and closer to some definite finite value, as the number of terms is increased without limit, the series is said to be a **convergent series.** The sum of a convergent series is the "limiting value" of § 6. On the contrary, if the sum of an infinite series obtained by taking a sufficient number of terms can be made greater than any finite quantity, however large, the series is said to be a **divergent series.** For example,

$$1 + 2 + 3 + 4 + \ldots \text{ to infinity.} \quad . \quad . \qquad (2)$$

Divergent series are not much used in physical work, while converging series are very frequently employed.*

* A prize was offered in France some time back for the best essay on the use of diverging series in physical mathematics.

Several tests for discriminating between convergent and divergent series are described in the regular textbooks on algebra. To simplify matters, I shall assume the series discussed in this work satisfy the tests of convergency. It is necessary to bear this in mind, otherwise we may be led to absurd conclusions.

Let S denote the limiting value or sum of the converging series.

$$S = a + ar + ar^2 + \ldots + ar^n + ar^{n+1} + \ldots \text{ ad inf.} \quad (3)$$

Cut off the series at some assigned term, say the nth, *i.e.*, all terms after ar^{n-1} are suppressed. Let s_n denote the sum of the n terms retained, σ_n the sum of the suppressed terms. Then,

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}. \quad . \quad . \quad (4)$$

Multiply through by r,

$$rs_n = ar + ar^2 + ar^3 + \ldots + ar^n.$$

Subtract the last expression from (4),

$$s_n(1 - r) = a(1 - r^n); \text{ or, } s_n = a\frac{1 - r^n}{1 - r}. \quad . \quad (5)$$

Obviously we can write series (3), in the form,

$$S = s_n + \sigma_n. \quad . \quad . \quad . \quad . \quad (6)$$

The error which results when the first n terms are taken to represent the series, is given by the expression

$$\sigma_n = S - s_n.$$

This error can be made to vanish by taking an infinitely great number of terms, or,

$$Lt_{n=\infty}\sigma_n = 0.$$

But,
$$s_n = a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

When n is made infinitely great, the last term vanishes,

$$\therefore Lt_{n=\infty}\frac{ar^n}{1 - r} = 0.$$

The sum of the infinite series of terms (3), is, therefore, given by the expression

$$S = \frac{a}{1 - r}. \quad . \quad . \quad . \quad . \quad (7)$$

Series (3) is generally called a **geometrical series.**

To determine the magnitude of the error introduced when only a finite number of terms of an infinite series is taken. Take the infinite number of terms,

$$S = \frac{1}{1 - r} = 1 + r + r^2 + \ldots + r^{n-1} + \frac{r^n}{1 - r}. \quad (8)$$

The error introduced into the sum S, by the omission of all terms after the nth, is, therefore,

$$\sigma_n = \frac{r^n}{1 - r}. \quad . \quad . \quad . \quad . \quad (9)$$

When r is positive, σ_n is positive, and the result is a little too small; but if r is negative

$$\sigma_n = \pm \frac{r^n}{1 - r}, \quad . \quad . \quad . \quad . \quad (10)$$

which means that if all terms after the nth are omitted, the sum obtained will be too great or too small, according as n is odd or even.

EXAMPLES.—(1) Suppose that the electrical conductivity of an organic acid at different concentrations has to be measured and that the first measurement is made on 50 c.c. of solution of concentration c. 25 c.c. of this solution are then removed and 25 c.c. of distilled water added instead. This is repeated five more times. What is the then concentration of the acid in the electrolytic cell?

Obviously we are required to find the 7th term in the series

$$c\{1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\},$$

where the nth term is $c(\frac{1}{2})^{n-1}$. Ansr. $(\frac{1}{2})^6 c$.

(2) A precipitate at the bottom of a beaker containing V c.c. of mother liquid is to be washed by decantation, *i.e.*, by repeatedly filling the beaker up to say the V c.c. mark with distilled water and emptying. Suppose that the precipitate retains v c.c. of the liquid in the beaker at each decantation, what will be the percentage volume of mother liquor about the precipitate after the nth emptying, assuming that the volume of the precipitate is negligibly small? Ansr. $100(v/V)^{n-1}$.

Hint. The solution in the beaker, after the first filling, has v/V c.c. of mother liquid. On emptying, v of this v/V c.c. is retained by the precipitate. On refilling, the solution in the beaker has $(v^2/V)/V$ of mother liquor, and so we build up the series,

$$v\left\{1 + \frac{v}{V} + \left(\frac{v}{V}\right)^2 + \left(\frac{v}{V}\right)^3 + \ldots\right\}.$$

§ 95. Soret's Diffusion Experiments.

These experiments will serve to illustrate the use that may be made of a geometrical series in the study of natural phenomena.

The density of a gas may be determined by comparing its rate of diffusion with that of another gas of known density. If r_1, r_2 be the rates of diffusion of two gases of known densities ρ_1 and ρ_2 respectively, then by *Graham's law*,

$$r_1 \sqrt{\rho_1} = r_2 \sqrt{\rho_2}. \quad . \quad . \quad . \quad (1)$$

The method is particularly useful for finding the density of such a gas as ozone, which cannot be prepared free from admixed oxygen. Soret based his classical method for finding the density of this gas on the following procedure (*Ann. d. Chim. et d. Phys.*, [4], **7**, 113, 1866; **13**, 257, 1868).

A vessel A, containing v_0 volumes of ozone mixed with oxygen, was placed in communication with another vessel B, containing oxygen only, for a definite time t. Soret found that the volume (v) of ozone diffusing from A to B was proportional to the difference in the quantity of ozone contained in the two vessels at the commencement of any interval of time. By Graham's law this quantity is also inversely proportional to the square root of its density.

If the vessel A, originally containing v_0 volumes of ozone, loses v volumes, the amount dv which diffuses in the next interval of time dt, will be proportional to the difference in the volumes of ozone contained in the two vessels, that is to say, $(v_0 - v) - v$, hence,

$$dv = \frac{a}{\sqrt{\rho}}(v_0 - 2v)dt, \quad . \quad . \quad . \quad (2)$$

where a is a constant depending on the nature of the apparatus used in the experiment.

At the commencement of the first interval of time B contained no ozone, therefore, if v_1 denotes the quantity of ozone in B at the end of the first interval of time,

$$v_1 = \frac{a}{\sqrt{\rho}}v_0 dt; \quad . \quad . \quad . \quad . \quad (3)$$

at the end of the second interval,

$$v_2 = v_1 + v_1(1 - 2v_1/v_0);$$

at the end of the third interval,

$$v_3 = v_1 + v_1(1 - 2v_1/v_0) + v_1(1 - 2v_1/v_0)^2;$$

and at the end of the nth interval,

$$v_n = v_1 + v_1(1 - 2v_1/v_0) + \ . \ . \ . + v_1(1 - 2v_1/v_0)^{n-1}. \quad (4)$$

The volume of ozone in the upper vessel at the end of n intervals of time dt, is the sum of the geometrical series (4) containing n terms. From (5), page 219,

$$\therefore v_n = \frac{v_0}{2}\left\{1 - \left(1 - \frac{2v_1}{v_0}\right)^n\right\}; \text{ or, } v_n = \frac{v_0}{2}\left\{1 - \left(1 - \frac{2a}{\sqrt{\rho}}dt\right)^n\right\}. \ (5)$$

Thus, the volume of the gas in B, at the end of a given time, is proportional to v_0 alone, or, for the same gas with the same apparatus for the same interval of time,

$$v_n/v_0 = \text{constant}.$$

With different gases, under the same conditions, any difference in the value of v_n/v_0 must be due to the different densities of the gases.

The mean of a series of experiments with chlorine (density, 35·5), carbon dioxide (density, 22), and ozone (density, ?), gave the following numbers :—

	CO_2.	*Ozone.*	Cl_2.
v_n/v_0 . . .	0·29,	0·271,	0·227.

Comparing chlorine with ozone, let x denote the density of ozone,

$$x = (0·227/0·271)^2 \times 35·5 = 24·9,$$

which agrees with the triatomic symbol O_3.

EXAMPLE.—Show that if the time is taken infinitely long the value of v_n/v_0 approaches unity.

§ 96. Approximate Calculation by Means of Infinite Series.

The reader will, perhaps, have been impressed with the frequency with which experimental results are referred to a series formula of the type :

$$y = A + Bx + Cx^2 + Dx^3 + \ldots, \quad . \quad . \quad (1)$$

in physical or chemical textbooks.*

The formula has no theoretical significance whatever. In the absence of any knowledge as to the proper mathematical expression of the "law" connecting two variables, this formula is adopted in the attempt to represent the corresponding values of the two variables by means of a mathematical expression.

A, B, C, . . . are constants to be determined from the experimental data by methods to be described later on.

There are several interesting features about formula (1).

1. *When the progress of any physical change is represented by the above formula, the approximation is closer to reality the greater the number of terms included in the calculation.* This is best shown by an example.

The specific gravity s of an aqueous solution of hydrogen

* I have counted over thirty examples in the first volume of Mendeléeff's *The Principles of Chemistry* and more than this number in Preston's *Theory of Heat.*

chloride is an unknown function of the amount of gas p per cent. dissolved in the water. (Unit, water at $4° = 10,000$.)

The first two columns of the following table represent corresponding values of p and s, determined by Mendeléeff. It is desired to find a mathematical formula to represent these results with a fair degree of approximation, in order that we may be able to calculate p if we know s, or, to determine s if we know p. Let us suppress all but the first two terms of the above series,

$$s = A + Bp,$$

where A and B are constants, found, by methods to be described later, to be $A = 9991·6$, $B = 49·43$. Now calculate s from the given values of p by means of the formula,

$$s = 9991·6 + 49·43p, \quad . \quad . \quad . \quad (2)$$

and compare the results with those determined by experiment. See the second and third columns of the following table :—

Percentage Composition p.	Specific Gravity s.		
	Found.	Calculated.	
		1st Approx.	2nd Approx.
5	10242	10239	10240
10	10490	10486	10492
15	10744	10733	10746
20	11001	10980	11003
25	11266	11227	11263
30	11522	11476	11522

Formula (2), therefore, might serve all that is required in, say, a manufacturing establishment, but, in order to represent the connection between specific gravity and percentage composition with a greater degree of accuracy, another term must be included in the calculation, thus we write

$$s = A + Bp + Cp^2,$$

where C is found to be equivalent to $0·0571$. The agreement between the results calculated according to the formula :

$$s = 9991·6 + 49·43p + 0·0571p^2, \quad . \quad . \quad (3)$$

and those actually found by experiment is now very close. This will be evident on comparing the second with the fourth columns of the above table.

The term $0{\cdot}0571p^2$ is to be looked upon as a **correction term.** It is very small in comparison with the preceding terms.

If a still greater precision is required, another correction term must be included in the calculation, we thus obtain

$$y = A + Bx + Cx^2 + Dx^3.$$

Such a formula was used by Thorpe and Tutton (*Journ. Chem. Soc.*, **57**, 559, 1890; Thorpe and Rucker, *Phil. Trans.*, **166**, ii., 405, 1877), to represent the apparent expansion of phosphorous oxide in a glass volumeter. They referred their results to the formula:

$$v = 1 + 0{\cdot}008882{,}4\theta + (-0{\cdot}000000{,}13873)\theta^2 + 0{\cdot}000000{,}038446\theta^3.$$

The calculated agreed very closely with the observed results. (Thorpe and Tutton's zero temperature was here $-27{\cdot}1°$.)

Hirn used yet another term, namely,

$$v = A + B\theta + C\theta^2 + D\theta^3 + E\theta^4,$$

in his formula for the volume of water, between 100° and 200°. Here $A = 1$,

$B = 0{\cdot}000108{,}67875$; $D = 0{\cdot}000000{,}002873{,}0422$;
$C = 0{\cdot}000003{,}007365{,}3$; $E = -0{\cdot}000000{,}000006{,}645703{,}1$.

(*Ann. d. Ch. et d. Ph.* [4], **10**, 32, 1867.)

The logical consequence of this reasoning, is that by including every possible term in the approximation formula, we should get absolutely correct results by means of the infinite converging series:

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots + \text{ad infin.}$$

It is the purpose of Maclaurin's theorem to determine values of $A, B, C, \ldots$ which will make this series true.

2. *The rapidity of the convergence of any series determines how many terms are to be included in the calculation in order to obtain any desired degree of approximation.*

It is obvious that the smaller the numerical value of the "correction terms" in the preceding series, the less their influence on the calculated result. If each correction term is very small in comparison with the preceding one, very good approximations can be obtained by the use of comparatively simple formulae involving two, or, at most, three terms. On the other hand, if the number of correction terms is very great, the series becomes so unmanageable as to be practically useless.

Equation (1) may be written in the form,

$$y = A(1 + bx + cx^2 + \ldots), \quad . \quad . \quad . \quad (4)$$

where $A, b, c, \ldots$ are constants.

As a general rule, when a substance is heated, it increases in volume; its mass remains constant, the density, therefore, must necessarily decrease. But,

$$\textit{mass} = \textit{volume} \times \textit{density}, \text{ or, } m = \rho v.$$

The volume of a substance at θ° is given by the expression

$$v = v_0(1 + \alpha\theta),$$

where v_0 represents the volume of the substance at 0° C., α is the coefficient of cubical expansion. Therefore,

$$\rho_0/\rho = v/v_0 = v_0(1 + \alpha\theta)/v_0 = 1 + \alpha\theta.$$

$$\therefore \rho = \rho_0/(1 + \alpha\theta).$$

True for solids, liquids, and gases. For simplicity, put $\rho_0 = 1$. By division, we obtain

$$\rho = 1 - \alpha\theta + (\alpha\theta)^2 - (\alpha\theta)^3 + \dots$$

For solids and some liquids α is very small in comparison with unity. For example, with mercury $\alpha = 0{\cdot}00018$. Let θ be small enough

$$\rho = 1 - 0{\cdot}00018\theta + (0{\cdot}00018\theta)^2 - \dots$$

$$= 1 - 0{\cdot}00018\theta + 0{\cdot}000000{,}0324\theta^2 - \dots$$

If the result is to be accurate to the second decimal place (1 per 100), terms smaller than 0·01 should be neglected; if to the third decimal place (1 per 1000), omit all terms smaller than 0·001, and so on. It is, of course, necessary to extend the calculation a few decimal places beyond the required degree of approximation. How many, naturally depends on the rapidity of convergence of the series. If, therefore, we require the density of mercury correct to the sixth decimal place, the omission of the third term can make no perceptible difference to the result. See the determination of the numerical value of π, page 230.

EXAMPLES.—(1) If h_0 denotes the height of the barometer at 0° C. and h its height at θ°, what terms must be included in the approximation formula,

$$h = h_0(1 + 0{\cdot}00016\theta), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in order to reduce a reading at 20° to the standard temperature, correct to 1 in 100,000?

(2) Verify the first half-dozen approximation formulae, page 486.

(3) In accurate weighings a correction must be made for the buoyancy of the air by *reducing the "observed weight in air" to "weight in vacuo"*.* Let

---

* A difference of 45 mm. in the height of a barometer during an organic combustion analysis, may cause an error of 0·6 °/₀ in the determination of the CO_2, and an error of 0·4 °/₀ in the determination of the H_2O. See Crookes, "The Determination of the Atomic Weight of Thallium," *Phil. Trans.*, **163**, 277, 1874.

W denote the true weight of the body (*in vacuo*), w the observed weight in air, ρ the density of the body, ρ_1 the density of the weights, ρ_2 the density of the air at the time of weighing. Hence show that if

$$W\left(1 - \frac{\rho_2}{\rho}\right) = w\left(1 - \frac{\rho_2}{\rho_1}\right);$$

$$W = w\frac{1 - \rho_2/\rho_1}{1 - \rho_2/\rho}; \therefore W = w\left(1 - \frac{\rho_2}{\rho_1} + \frac{\rho_2}{\rho}\right),$$

or,

$$W = w + 0{\cdot}0012w(1/\rho - 1/\rho_1), \quad . \quad . \quad . \quad . \quad (6)$$

which is the standard formula for reducing weighings in air to weighings in vacuo. The numerical factor represents the density of moderately moist air at the temperature of a room under normal conditions.

(4) If α denotes the coefficient of cubical expansion of a solid, the volume of a solid at any temperature θ is, $v = v_0(1 + \alpha\theta)$, where v_0 represents the volume of the substance at 0°. Hence show that the relation between the volumes, v_1 and v_2, of the solid at the respective temperatures of θ_1 and θ_2 is

$$v_1 = v_2(1 + \alpha\theta_1 - \alpha\theta_2). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Why does this formula fail for gases?

(5) Since

$$\frac{1}{x - a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \ldots,$$

the reciprocals of many numbers can be very easily obtained correct to many decimal places. Thus

$$\frac{1}{97} = \frac{1}{100 - 3} = \frac{1}{100} + \frac{3}{10{,}000} + \frac{9}{1{,}000{,}000} + \ldots$$

$$= {\cdot}01 + {\cdot}0003 + {\cdot}000009 + \ldots$$

(6) We require an accuracy of 1 per 1,000. What is the greatest value of x which will permit the use of the approximation formula

$$(1 + x)^3 = 1 + 3x?$$

(7) From the formula

$$(1 \pm x)^n = 1 \pm nx,$$

calculate the approximate values of $\sqrt{999}$, $1/\sqrt{1{\cdot}02}$, $(1{\cdot}001)^3$, $\sqrt{1{\cdot}05}$, mentally. Note n may be positive or negative, integral or fractional.

§ 97. Maclaurin's Theorem.

There are several methods for the development of functions in series, depending on algebraic, trignometrical, or other processes. The one of greatest utility is known as Taylor's theorem. Maclaurin's * theorem is but a special case of Taylor's.

Maclaurin's theorem determines the law for the expansion of a function of a SINGLE *variable in a series of ascending powers of that variable.*

---

* The name is here a historical misnomer. Taylor published his series in 1715. In 1717, Stirling showed that the series under consideration was a special case of Taylor's. Twenty-five years after this Maclaurin independently published Stirling's series.

Let the variable be denoted by x, then,

$$u = f(x).$$

Assume that $f(x)$ can be developed in ascending powers of x, say,

$$u = f(x) = A + Bx + Cx^2 + Dx^3 + \ldots,^* \quad . \quad . \quad (1)$$

where $A, B, C, D \ldots$, are constants independent of x, but dependent on the constants contained in the original function.

It is required to determine the value of these constants, in order that the above assumption may be true for all values of x.

By successive differentiation of (1),

$$\frac{du}{dx} = \frac{df(x)}{dx} = B + 2Cx + 3Dx^2 + \ldots; \quad . \quad (2)$$

$$\frac{d^2u}{dx^2} = \frac{df'(x)}{dx} = 2C + 2 . 3Dx + \ldots; \quad . \quad . \quad (3)$$

$$\frac{d^3u}{dx^3} = \frac{df''(x)}{dx} = 2 . 3 . D + \ldots \quad . \quad . \quad . \quad (4)$$

By hypothesis, (1) is true whatever be the value of x, and, therefore, the constants $A, B, C, D, \ldots$ are the same whatever value be assigned to x. Now substitute $x = 0$ in equations (2), (3), (4). Let v denote the value assumed by u when $x = 0$. Hence, from (1),

$$v = f(0) = A, \qquad \therefore A = v; \quad . \quad . \quad (5)$$

from (2), $\quad \dfrac{dv}{dx} = f'(0) = 1 . B, \qquad \therefore B = \dfrac{dv}{dx};$

from (3), $\quad \dfrac{d^2v}{dx^2} = f''(0) = 1 . 2C, \qquad \therefore C = \dfrac{1}{2!}\dfrac{d^2v}{dx^2};$

from (4), $\quad \dfrac{d^3v}{dx^3} = f'''(0) = 1 . 2 . 3D, \qquad \therefore D = \dfrac{1}{3!}\dfrac{d^3v}{dx^3}.$

"$f^n(0)$" means that $f(x)$ is to be differentiated n times, and x equated to zero in the resulting expression.

Substitute the above values of $A, B, C, \ldots$, in (1) and we get

$$u = v + \frac{dv}{dx}\frac{x}{1} + \frac{d^2v}{dx^2}\frac{x^2}{2!} + \frac{d^3v}{dx^3}\frac{x^3}{3!} + \ldots \quad . \quad \mathbf{(6)}$$

The series on the right-hand side is known as **Maclaurin's Series.**

From (5), the series may be written,

$$u = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1 . 2} + f'''(0)\frac{x^3}{1 . 2 . 3} + \ldots \quad \mathbf{(7)}$$

---

* Note the resemblance between this expression and (1) of the preceding section.

§ 98. Useful Deductions from Maclaurin's Theorem.

The following may be considered as a series of examples of the use of the formula obtained in the preceding section. Many of the results now to be established will be employed in our subsequent work.

1. *The binomial theorem.* In order to expand any function by Maclaurin's theorem, the successive differential coefficients of u are to be computed and x then equated to zero. This fixes the values of the different constants.

Let $$u = (a + x)^n,$$

$du/dx = n(a + x)^{n-1}$, $\therefore f'(0) = na^{n-1}$;

$d^2u/dx^2 = n(n-1)(a + x)^{n-2}$, $\therefore f''(0) = n(n-1)a^{n-2}$;

$d^3u/dx^3 = n(n-1)(n-2)(a + x)^{n-3}$, $\therefore f'''(0) = n(n-1)(n-2)a^{n-3}$,

and so on. Now substitute these values in Maclaurin's series (6),

$$(a + x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1 \,.\, 2}a^{n-2}x^2 + \ldots, \quad (1)$$

a result known as the **binomial series**, true for positive, negative, or fractional values of n. See page 22.*

EXAMPLES.—(1) Prove that

$$(a - x)^n = a^n - \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1 \,.\, 2}a^{n-2}x^2 - \ldots \quad (2)$$

When n is a positive integer, and $n \equiv m$, the infinite series is cut off at a point where $n - m = 0$. A finite number of terms remains.

Establish the following results:

(2) $(1 + x^2)^{1/2} = 1 + x^2/2 - x^4/8 + x^6/16 - \ldots$

(3) $(1 - x^2)^{-1/2} = 1 + x^2/2 + 3x^4/8 + 5x^6/16 + \ldots$

(4) $(1 + x^2)^{-1} = 1 - x^2 + x^4 - \ldots$

Verify this last result by actual division.

2. *Trignometrical series.* Suppose

$$u = f(x) = \sin x.$$

Proceed as before. Note that

$$d(\sin x)/dx = \cos x,\ d(\cos x)/dx = -\sin x, \text{ etc.}$$

$$\therefore \sin 0 = 0,\ -\sin 0 = 0,\ \cos 0 = 1,\ -\cos 0 = -1.$$

Hence, $$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad (3)$$

A result known as the **sine series.**

---

* In the proof that $dx^n/dx = nx^{n-1}$, we have assumed the binomial theorem. The student may think we have worked in a vicious circle. This need not be. The result may be proved without this assumption. Let

$$y = x^n,\ x_1 = x + \delta x,\ y_1 = y + \delta y.$$

$$\therefore \frac{y_1 - y}{x_1 - x} = \frac{x_1{}^n - x^n}{x_1 - x} = x_1{}^{n-1} + xx_1{}^{n-2} + \ldots + x^{n-1},$$

by division. But $Lt_{\delta x = 0}x_1 = x$.

$$\therefore \frac{dy}{dx} = x^{n-1} + x^{n-1} + \ldots \text{ to } n \text{ terms} = nx^{n-1}.$$

In the same way, show that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad . \quad . \quad . \quad (4)$$

This is the **cosine series.**

These series are employed for calculating the numerical values of angles between 0 and $\frac{1}{4}\pi$. All the other angles found in "trignometrical tables of sines and cosines," can be then determined by means of the formulae,

$$\sin(\tfrac{1}{2}\pi - x) = \cos x;\ \cos(\tfrac{1}{2}\pi - x) = \sin x,$$

of page 499. For numerical examples, see page 497.

Now let $\qquad u = f(x) = \tan x.$

From page 499, $\qquad \therefore u \cos x = \sin x.$

By successive differentiation of this expression, remembering that $u_1 = du/dx$, $u_2 = d^2u/dx^2$, . . . , as in § 8,

$$\therefore u_1 \cos x - u \sin x = \cos x.$$

$$\therefore u_2 \cos x - 2u_1 \sin x - u \cos x = - \sin x.$$

$$\therefore u_3 \cos x - 3u_2 \sin x - 3u_1 \cos x + u \sin x = - \cos x.$$

By analogy with the coefficients of the binomial development (1), or Leibnitz' theorem, § 20,

$$u_n \cos x - \frac{n}{1}u_{n-1}\sin x - \frac{n(n-1)}{1.2}u_{n-2}\cos x + \dots = n\text{th derivative} \sin x.$$

Now find the values of u, u_1, u_2, u_3, . . . by equating $x = 0$ in the above equations, thus,

$$f(0) = f''(0) = \dots = 0$$

$$f'(0) = 1, f'''(0) = 2, \dots$$

Substitute these values in Maclaurin's series (7), preceding section. The result is,

$$\tan x = \frac{x}{1} + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \dots;\ \text{or, } \tan x = \frac{x}{1} + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \quad (5)$$

The **tangent series.**

3. *Inverse trignometrical series.* Let

$$\theta = \tan^{-1}x.$$

By (3), § 15 and example (4) above,

$$\therefore d\theta/dx = 1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \dots$$

By successive differentiation and substitution in the usual way,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \quad . \quad . \quad . \quad (6)$$

or, from the original equation,

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots, \quad . \quad . \quad . \quad (7)$$

which is known as **Gregory's series.** This series is known to be converging when θ lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$.

Gregory's series has been employed *to calculate the numerical value of* π.

Let $\qquad \theta = 45° = \frac{1}{4}\pi, \therefore x = 1.$

Substitute in (6),

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots$$

The so-called **Leibnitz series.** This is a convenient opportunity to emphasize the remarks on the unpracticable nature of a slowly converging series. It would be an extremely laborious operation to calculate π accurately by means of this series. A little artifice will simplify the method, thus,

$$\frac{\pi}{4}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\ldots;\ \frac{\pi}{4}=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\ldots$$

$$\frac{\pi}{8}=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\ldots,$$

which does not involve quite so much labour. It will be observed that the angle x is not to be referred to the degree-minute-second system of units, but to the unit of the circular system (page 494), namely, the radian. Suppose $x = 1/\sqrt{3}$, then $\tan^{-1}x = 30° = \frac{1}{6}\pi$. Substitute this value of x in (6), collect the positive and negative terms in separate brackets, thus

$$\frac{\pi}{6}=\left(\frac{1}{\sqrt{3}}+\frac{1}{5\sqrt{3}^5}+\ldots\right)-\left(\frac{1}{3\sqrt{3}^3}+\frac{1}{7\sqrt{3}^7}+\ldots\right).$$

To further illustrate, we shall compute the numerical value of π correct to five decimal places. At the outset, it will be obvious that (1) we must include two or three more decimals in each term than is required in the final result, and (2) we must evaluate term after term until the subsequent terms can no longer influence the numerical value of the desired result. Hence:

Terms enclosed in the first brackets.	Terms enclosed in the second brackets.
0·57735 03	0·06415 01
0·01283 00	0·00305 48
0·00079 20	0·00021 60
0·00006 09	0·00001 76
0·00000 52	0·00000 15
0·00000 05	0·00000 02
0·59103 89	0·06744 02

$$\therefore \pi = 6(0·59103\ 89 - 0·06744\ 02) = 3·14159\ 22.$$

The number of unreliable figures at the end obviously depends on the rapidity of the convergence of the series (page 224). Here the last two figures are untrustworthy. But notice how the positive errors are, in part, balanced by the negative errors. The correct value of π to seven decimal places is 3·1415926. There are several shorter ways of evaluating π. See *Encyclopædia Britannica*, art. "Squaring the Circle".

We can obtain the **inverse sine series**

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{3}{8}\frac{x^5}{5} + \frac{5}{16}\frac{x^7}{7} + \ldots, \qquad (8)$$

in a similar manner. Now write $x = \frac{1}{2}$, $\sin^{-1}x = \frac{1}{6}\pi$. Substitute these values in (8). The resulting series was used by Newton for the computation of π.

4. *Exponential theorem.* Show that

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad (9)$$

by Maclaurin's series.

The **exponential series** expresses the development of e^x, a^x, or some other exponential function in a series of ascending powers of x and coefficients independent of x.

Examples.—(1) Show that if $k = \log a$

$$a^x = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \ldots \qquad (10)$$

(2) Show $$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \quad . \quad . \quad . \quad (11)$$

(3) Calculate the numerical value of e correct to four decimal places. Hint, put $x = 1$ in (9), etc.

The development by Maclaurin's series cannot be used if the function or any of its derivatives becomes infinite or discontinuous when x is equated to zero. For example, the first differential coefficient of $f(x) = \sqrt{x}$, is $\frac{1}{2}/\sqrt{x}$, which is infinite for $x = 0$, in other words, the series is no longer convergent. The same thing will be found with the functions $\log x$, $\cot x$, $1/x$, $a^{1/x}$ and $\sec^{-1}x$. Some of these functions may, however, be developed as a fractional or some other simple function of x, or we may use Taylor's theorem.

§ 99. Taylor's Theorem.

Taylor's theorem determines the law for the expansion of a function of the sum, or difference of TWO *variables* into a series of ascending powers of one of the variables.*

Now let
$$u_1 = f(x + y).$$
Assume that
$$u_1 = f(x + y) = A + By + Cy^2 + Dy^3 + \ldots, \quad . \quad (1)$$
where A, B, C, D, . . . are constants, independent of y, but dependent upon x and also upon the constants entering into the original equation.

Differentiate (1) on the supposition that x is constant and y variable. Thus,
$$\frac{du_1}{dy} = B + 2Cy + 3Dy^2 + \ldots \quad . \quad . \quad . \quad (2)$$
Now differentiate (1) on the supposition that y is constant and x variable,
$$\frac{du_1}{dx} = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \ldots \quad . \quad (3)$$
First, to show that
$$\frac{du_1}{dy} = \frac{du_1}{dx}†,$$
where $u_1 = f(x + y)$.

* A function of the sum of two variables is such that if a single variable be substituted for that sum, the original function reduces to that of a single variable. For instance,
$$\sin x = u = \sin(y + z),$$
where x is the sum of the two variables y and z.

† Note that du_1/dy and du_1/dx of (2) and (3) are really partial differential coefficients. Strictly, we should write,
$$\left(\frac{\partial u_1}{\partial y}\right)_x = \left(\frac{\partial u_1}{\partial x}\right)_y.$$

Now let

$$v = x + y\ ;\ \therefore u_1 = f(v).$$

Differentiate with respect to x, y constant; also with respect to y, x constant.

$$\frac{du_1}{dx} = \frac{du_1}{dv}\frac{dv}{dx};\ \frac{du_1}{dy} = \frac{du_1}{dv}\frac{dv}{dy}.$$

(See page 29.) But $v = x + y$ and if y is constant, $dv = dx$ and $dv/dx = 1$; similarly, if x is constant, $dv = dy$, or $dv/dy = 1$, therefore

$$\frac{du_1}{dx} = \frac{du_1}{dv};\ \frac{du_1}{dy} = \frac{du_1}{dv};\ \text{or, } \frac{du_1}{dx} = \frac{du_1}{dy}.$$

It, therefore, follows that (2) and (3) are identical.

$$\therefore \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \ldots \equiv B + 2Cy + 3Dy^2 + \ldots \quad (4)$$

Since this identity is true whatever be the value of y, the coefficients of like powers of y, on each side of the equation, are equal each to each (footnote, page 172), therefore,

$$\frac{dA}{dx} = B\ ;\ \frac{dB}{dx} = 2C\ ;\ \frac{dC}{dx} = 3D\ ;\ \ldots \quad (5)$$

But, by hypothesis, (1) is true whatever be the value of y. We may, therefore, put $y = 0$ so that the original equation reduces to a function of x, say,

$$u = f(x). \quad (6)$$

$$A = u_1\ ;\ B = \frac{du_1}{dx}\ ;\ C = \frac{1}{2}\frac{dB}{dx} = \frac{1}{2}\cdot\frac{d^2u_1}{dx^2}\ ;\ D = \frac{1}{3}\cdot\frac{dC}{dx} = \frac{1}{2\,.\,3}\cdot\frac{d^3u_1}{dx^3};\ \ldots$$

Substitute these values of A, B, C, D in the original equation and we obtain

$$u_1 = f(x + y) = u + \frac{du}{dx}\frac{y}{1} + \frac{d^2u}{dx^2}\frac{y^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{y^3}{1\,.\,2\,.\,3} + \ldots \quad \textbf{(7)}$$

The series on the right-hand side is known as **Taylor's series.**

From (6), we may write Taylor's series in the form,

$$u_1 = f(x + y) = f(x) + f'(x)\frac{y}{1} + f''(x)\frac{y^2}{1\,.\,2} + f'''(x)\frac{y^3}{1\,.\,2\,.\,3} + \ldots \quad \textbf{(8)}$$

Or, interchanging the variables,

$$u_1 = f(x + y) = f(y) + f'(y)\frac{x}{1} + f''(y)\frac{x^2}{1\,.\,2} + f'''(y)\frac{x^3}{1\,.\,2\,.\,3} + \ldots \quad \textbf{(9)}$$

EXAMPLE.—Prove that

$$f(x - y) = f(x) - f'(x)\frac{y}{1} + f''(x)\frac{y^2}{2!} - f'''(x)\frac{y^3}{3!} + \ldots \quad (10)$$

Maclaurin's and Taylor's series are slightly different expressions for the same thing. The one form can be converted into the other by substituting $f(x + y)$ for $f(x)$ in Maclaurin's theorem, or by putting $y = 0$ in Taylor's. The geometrical signification is that each function is the equation of a curve with a different origin on the x-axis and y denotes a constant, not an ordinate, on the abscissa axis.

EXAMPLES.—(1) Expand $u_1 = (x + y)^n$ by Taylor's theorem. Put $y = 0$ and $u = x^n$,

$$\therefore \frac{du}{dx} = nx^{n-1}; \; \frac{d^2u}{dx^2} = n(n-1)x^{n-2}, \text{ etc.}$$

Substitute the values of these derivatives in (8).

$$\therefore u_1 = (x + y)^n = x^n + nx^{n-1}y + \tfrac{1}{2}n(n-1)x^{n-2}y^2 + \ldots$$

Verify the following results:—

(2) If $k = \log a$,

$$u_1 = a^{x+y} = a^x(1 + ky + \tfrac{1}{2}k^2y^2 + \tfrac{1}{6}k^3y^3 + \ldots)$$

(3) $u_1 = (x + y + a)^{1/2} = (x + a)^{1/2} + \tfrac{1}{2}y(x + a)^{-1/2} - \ldots$ If $x = -a$, the development fails.

(4) $u_1 = \sin(x + y) = \sin x\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + \cos x\left(y - \frac{y^3}{3!} + \ldots\right)$.

For numerical examples, see page 497.

(5) $\log(x + y) = \log x + \frac{y}{x} - \frac{y^2}{2x^2} + \frac{y^3}{3x^3} - \ldots$

(6) $\log_a(1 + x) = \frac{1}{\log_e a}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right)$.

(7) $\log(1 + y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \ldots$

(8) $\log(1 - y) = -(y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \tfrac{1}{4}y^4 + \ldots)$.

If $y = 1$, the development gives a divergent series and the theorem is then said to fail. The last four examples are **logarithmic series.**

(9) Put $y = -x$ in Taylor's series, and show that

$$f(x) = f(0) + f'(x)\frac{x}{1} - f''(x)\frac{x^2}{2!} + \ldots,$$

known as **Bernoulli's series** (of historical interest, published 1694).

Mathematical textbooks, at this stage, proceed to discuss the conditions under which the sum of the individual terms of Taylor's series is really equal to $f(x + y)$. When the given function $f(x + y)$ is finite, the sum of the corresponding series must also be finite, and the developed series (Taylor's or Maclaurin's) must either be finite or convergent. The development is said to fail when the series is divergent.

It is not here intended to show how mathematicians have succeeded in placing Taylor's series on a satisfactory basis. That subject belongs to the realms of pure mathematics.* The reader must exercise "belief based on suitable evidence outside personal experience," otherwise known as faith. This will require no great mental effort on the part of the student of the physical sciences. He has to apply the very highest orders of faith to the fundamental principles—the inscrutables—of these sciences, namely, to the

* If the student is at all curious, Todhunter, or Williamson on "Lagrange's Theorem on the Limits of Taylor's Series," is always available.

theory of atoms, stereochemistry, affinity, the existence and properties of interstellar ether, the origin of energy, etc., etc. What is more, "reliance on the *dicta* and *data* of investigators whose very names may be unknown, lies at the very foundation of physical science, and without this faith in authority the structure would fall to the ground; not the blind faith in authority of the unreasoning kind that prevailed in the Middle Ages, but a rational belief in the concurrent testimony of individuals who have recorded the results of their experiments and observations, and whose statements can be verified . . .".*

The theory of proportional parts or proportional differences is an application of Taylor's theorem. If a small number be increased by a small fraction of itself, the increase in the value of the number is nearly proportional to the increase of its logarithm.† Thus,

$$\log_{10}(n + h) = \log_{10}n\Big(1 + \frac{h}{n}\Big) = \log_{10}n + \log_{10}\Big(1 + \frac{h}{n}\Big);$$
$$= \log_{10}n + 0\cdot4343\Big(\frac{h}{n} - \frac{1}{2}\frac{h^2}{n^2} + \frac{1}{3}\frac{h^3}{n^3} - \dots\Big).$$

For example, let n be not less than 10,000 and h not greater than unity, h/n is not greater than 0·0001 and the next term is not greater than $\frac{1}{4}(0\cdot0001)^2$, that is to say, not exceeding 0·000000,0025. The next term is, of course, much less than this. We may, therefore, correctly write, as far as seven decimal places,

$$\log(n + h) - \log n = 0\cdot4343 \times h/n$$

and

$$\log(n + 1) - \log n = 0\cdot4343 \times 1/n.$$

By division, we get the important result,

$$\frac{\log(n + h) - \log n}{\log(n + 1) - \log n} = \frac{h}{1}, \quad . \quad . \quad . \quad (11)$$

provided the differences between two numbers n and h are such that n is of the order of 10,000 when x is less than unity.

This formula, known as the **rule of proportional parts**, is used for finding the exact logarithm of a number containing more digits than the table of logarithms allows for, or for finding the number corresponding to a logarithm not exactly coinciding with those in the tables. The following examples will make this clear:—

* Excerpt from the Presidential Address of Dr. Carrington Bolton to the Washington Chemical Society, *English Mechanic*, 5th April, 1901.

† This is commonly stated as an exercise on differentiation. A question like this is set: "How much more rapidly does the number x increase than its logarithm?" Here $d(\log x)/dx = 1/x$. The number, therefore, increases more rapidly or more slowly than its logarithm according as $x >$ or < 1. If $x = 1$, the rates are the same. If common logarithms are employed, M (§ 16) will have to be substituted in place of unity. *E.g.*, $d(\log_{10}x)dx = M/x$.

EXAMPLES.—(1) Find the logarithm of 46502·32, having given

$$\log 46501 = 4·6674623$$
$$\log 46502 = 4·6674716$$
$$\text{Difference} = 0·0000093$$

Let x denote the quantity to be added to the smaller of the given logs. The problem may be stated thus,

$$\log n \quad = \log 46501 \quad = 4·6674623\,;$$
$$\log (n + 1) = \log (46501 + 1) = 4·6674623 + 0·0000093\,;$$
$$\log (n + h) = \log (46501 + x) = 4·6674623 + x,$$

by simple rule of three: if a difference of 1 unit in a number corresponds with a difference of 0·0000093 in the logarithm, what difference in the logarithm will arise when the number is augmented by 0·32?

$$\therefore 1 : 0·32 = 0·0000093 : x, \therefore x = 0·00000298 \ldots$$

The required logarithm is, therefore, 4·6674653.

(2) Find the number whose logarithm is 4·6816223, having given

$$\log 48042 = 4·6816211\,;\ \log 48043 = 4·6816301.$$

Since a difference of unity in the number causes a difference of 0·0000090 in the logarithm, what will be the difference in the number when the logarithms differ by 0·0000012?

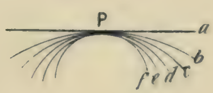
$$\therefore 1 : x = 0·0000090 : 0·0000012,$$
$$\therefore \quad x = 0·13, \text{ or the number is } 48042·13.$$

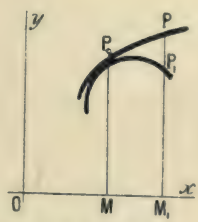
The rest of this chapter will be mainly concerned with direct or indirect applications of infinite converging series. § 183 on proportional errors and § 158 on the use of Taylor's theorem in finding the approximate roots of an equation, may also be consulted.

§ 100. The Contact of Curves.

The following is a geometrical illustration of one meaning of the different terms in Taylor's development.

If four curves Pa, Pb, Pc, Pd, . . . (Fig. 101), have a common point P, any curve, say Pc, which passes between two others, Pb, Pd, is said to have a closer contact with Pb than Pd has.





FIGS. 101, 102.—Orders of Contact of Curves.

Now let two curves P_0P and P_0P_1 (Fig. 102) referred to the same rectangular axes, have equations,

$$y = f(x) \text{ and, } y_1 = f_1(x_1).$$

Let the abscissa of each curve at any given point, be increased by a small amount h, then, by Taylor's theorem,

$$f(x + h) = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{2!} + \ldots \qquad (1)$$

$$f_1(x_1 + h) = y_1 + \frac{dy_1}{dx_1}h + \frac{d^2y_1}{dx_1{}^2}\frac{h^2}{2!} + \ldots \qquad (2)$$

If the curves have a common point P_0, $x = x_1$ and $y = y_1$ at the point of contact. Substitute the coordinates of this point in equations (1) and (2). $f(x + h)$ will represent the ordinate PM_1 and $f_1(x_1 + h)$, the ordinate P_1M_1. Similarly, dy/dx, d^2y/dx^2 . . . will represent the differential coefficients of the ordinate of the curve $f(x + h)$ at the point P_0; dy_1/dx_1, d^2y_1/dx^2_1 . . . , similar properties for the second curve $f_1(x_1 + h)$.

Since the first differential coefficient represents the angle made by a tangent with the x-axis, if, at the point P_0,

$$x = x_1;\ y = y_1 \text{ and } dy/dx = dy_1/dx_1,$$

the curves will have a common tangent at P_0. This is called a **contact of the first order.** If, however,

$$x = x_1,\ y = y_1;\ dy/dx = dy_1/dx_1 \text{ and } d^2y/dx^2 = d^2y_1/dx^2_1,$$

the curves are said to have a **contact of the second order,** and so on for the higher orders of contact.

If all the terms in the two equations are equal the two curves will be identical; the greater the number of equal terms in the two series, the closer will be the order of contact of the two curves.

§ 101. Extension of Taylor's Theorem.

Taylor's theorem may be extended to include the expansion of functions of two or more independent variables. Let

$$u = f(x, y), \qquad (1)$$

where x and y are independent of each other. Suppose each variable to change independently so that x becomes $x + h$ and y becomes $y + k$.

Let $f(x, y)$ change to $f(x + h, y)$. By Taylor's theorem

$$f(x + h, y) = u + \frac{\partial u}{\partial x}h + \frac{\partial^2 u}{\partial x^2}\frac{h^2}{2!} + \ldots \qquad (2)$$

If y now becomes $y + k$, each term of equation (2) will change so that

$$u \text{ becomes } u + \frac{\partial u}{\partial y}k + \frac{\partial^2 u}{\partial y^2}\frac{k^2}{2!} + \ldots;$$

$$\frac{\partial u}{\partial x} \text{ becomes } \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}k + \ldots;\ \frac{\partial^2 u}{\partial x^2} \text{ becomes } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial y}k + \ldots,$$

by Taylor's theorem. Now substitute these values in (2) and we obtain,

$$\left.\begin{aligned} f(x+h, y+k) = u + \frac{\partial u}{\partial y}k + \frac{\partial^2 u}{\partial y^2}\frac{k}{2!} + \ldots + \frac{\partial^2 u}{\partial x \partial y}hk + \ldots \\ + \frac{\partial u}{\partial x}h + \frac{\partial^2 u}{\partial x^2}\frac{h}{2!} + \ldots \end{aligned}\right\}$$

$$\delta u = f(x+h, y+k) - f(x, y);$$

$$= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}h^2 + 2\frac{\partial^2 u}{\partial x \partial y}hk + \frac{\partial^2 u}{\partial y^2}k^2\right) + \ldots \qquad (3)$$

The final result is exactly the same whether we expand first with respect to y or in the reverse order.

By equating the coefficients of hk in the identical results obtained by first expanding with regard to h, (2) above, and by first expanding with regard to k, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x},$$

which was obtained another way in § 23.

The investigation may be extended to functions of any number of variables.

EXAMPLE.—Show that

$$\frac{\partial^3 u}{\partial y^2 \partial x} = \frac{\partial^3 u}{\partial x \partial y^2}.$$

§ 102. The Determination of Maximum and Minimum Values of a Function by means of Taylor's Series.

I. *Functions of one variable.* Taylor's theorem is sometimes useful in seeking the maximum and the minimum values of a function, say, $y = f(x)$. It is required to find particular values of x in order that y may be a maximum or a minimum.

Let x change by a small amount h so that by Taylor's development,

$$f(x \pm h) - f(x) = \pm \frac{dy}{dx}h + \frac{1}{2}\frac{d^2y}{dx^2}h \pm \frac{1}{6}\frac{d^3y}{dx^3}h^3 + \ldots \qquad (1)$$

First, it must be proved that h can be made so small that $\frac{dy}{dx}h$ will be greater than the sum of all succeeding terms of Taylor's series. Assume that Taylor's series may be written,

$$f(x+h) = y + Ah + Bh^2 + Ch^3 + \ldots,$$

where $A, B, C, \ldots$ are coefficients independent of h but dependent upon x, then, if $Rh = Bh + Ch^2 + \ldots$ we have,

$$f(x+h) = y + h(A + Rh). \quad . \quad . \quad . \quad . \quad (2)$$

For sufficiently small values of h, Rh *must* be less than A, because, by hypothesis, A is independent of h.

Second, when x changes by a small amount h, it follows, examples § 57, that for a maximum, $f(x) > f(x+h)$, and $f(x) > f(x-h)$; for a minimum, $f(x) < f(x+h)$, and $f(x) < f(x-h)$. It is, therefore, easy to see that if

$$f(x \pm h) - f(x)\begin{cases}\text{is negative, } f(x) \text{ will be a maximum;}\\ \text{is positive, } f(x) \text{ will be a minimum;}\\ \text{changes sign, } f(x) \text{ will be neither.}\end{cases}$$

whatever the sign of h.*

If dy/dx has a finite value, h may be imagined so small that the sign of $A + Rh$ of (2) does not change when that of h changes. Therefore, the sign of $f(x+h) - f(x)$ will depend on that of h, and consequently $f(x)$ cannot be either a maximum or a minimum. Only when h and $A + Rh$ change sign simultaneously (as h passes through zero) can x be either a maximum or a minimum. Under these circumstances, dy/dx becomes zero for maximum or minimum values of y.

If dx/dy vanishes,

$$f(x+h) - f(x) = \frac{d^2y}{dx^2}\frac{h^2}{2!} + \frac{d^3y}{dx^3}\frac{h}{3!} + \ldots \qquad (3)$$

As before, it can be shown that $\frac{d^2y}{dx^2}\frac{h^2}{2}$ is greater than all succeeding terms of the series. But h is of necessity positive, the sign of the second differential coefficient will, therefore, be the same as that of $f(x+h) - f(x)$. In other words, *y will be a maximum when $dy/dx = 0$ and d^2y/dx^2 is negative, and a minimum, if d^2y/dx^2 is positive.*

If, however, the second differential coefficient vanishes, the reasoning used in connection with the first differential must be applied to the third differential coefficient. If the third derivative vanishes, a similar relation holds between the second and fourth differential coefficients.

To generalise, if the order of the first differential coefficient that does not vanish is odd, $f(x)$ will be neither a maximum nor a minimum unless d^ny/dx^n passes through infinity (where n is the order of the differential that does not vanish). If n is even, we shall have a maximum or a minimum according as d^ny/dx^n is negative or positive.

---

* When reference is made to a magnitude without reference to its positive or negative values it is frequently written $|h|$, $|a|$, $|\sin x|$, and called the *absolute value* of h, a, or $\sin x$, as the case might be. In this work $\pm h$ is written for $|h|$, $\pm \sin x$ for $|\sin x|$, etc.

Hence the rules :—

1. *y is either a maximum or a minimum for a given value of x only when the first non-vanishing derivative, for this value of x, is even.*

2. *y is a maximum or a minimum according as the sign of the non-vanishing derivative of an even order, is negative or positive.*

In practice, if the first derivative vanishes, it is often convenient to test by substitution whether y changes from a positive to a negative value. If there is no change of sign, there is neither a maximum nor a minimum.

EXAMPLES.—(1) Test $y = x^3 - 12x^2 - 60x$ for maximum or minimum values.

$dy/dx = 3x^2 - 24x - 60$; $\therefore x^2 - 8x - 20 = 0$, or $x = -2$, or $+10$.

$d^2y/dx^2 = 6x - 24$; or, $x = +4$.

Since d^2y/dx^2 is positive when $x = 10$ is substituted, $x = 10$ will make y a minimum. When -2 is substituted, d^2y/dx^2 becomes negative, hence $x = -2$ will make y a maximum. This can easily be verified :

If $x = -3$,	-2,	-1,	$\ldots +9$,	$+10$,	$+11, \ldots$
$y = +45$,	$+64$,	$+48$,	$\ldots -783$,	-800,	$-781 \ldots$
	(max.)			(min.)	

(2) What value of x will make y a maximum or a minimum in the expression, $y = x^3 - 6x^2 + 11x - 6$?

$$dy/dx = 3x^2 - 12x + 11 = 0; \ \therefore x = 2 \pm 1/\sqrt{3};$$

$$d^2y/dx^2 = 6x - 12.$$

If
$$x = 2 + 1/\sqrt{3},\ d^2y/dx^2 = 2\sqrt{3}. \quad . \quad \text{(max.)};$$
$$x = 2 - 1/\sqrt{3},\ d^2y/dx^2 = -2\sqrt{3}. \quad . \quad \text{(min.)}.$$

II. *Functions of two variables.* To find particular values of x and y which will make the function,

$$u = f(x, y),$$

a maximum or a minimum. As before, when x changes by a small amount h, and y by a small amount k, if

$$f(x \pm h, y \pm k) - f(x, y) \begin{cases} \text{is negative, } f(x, y) \text{ will be a maximum;} \\ \text{is positive, } f(x, y) \text{ will be a minimum;} \\ \text{changes sign, } f(x, y) \text{ will be neither,} \end{cases}$$

whatever be the signs of h and k. Also, let

$$u = f(x + h, y + k) - f(x, y).$$

Expand this function as in the preceding section (3). By making the values of h and k small enough, the higher orders of differentials become vanishingly small. But as long as $\partial u/\partial x$ and $\partial u/\partial y$ remain finite, the algebraic sign of δu will be that of

$$\delta u = \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k. \quad . \quad . \quad . \quad . \quad (4)$$

At a maximum or a minimum point, we must have

$$\therefore \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k = 0,$$

and, since h and k are independent of each other, u can have a maximum or a minimum value only when

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0, \quad . \quad . \quad . \quad . \quad (5)$$

because the sign of δu, in (4), depends upon the signs of h and k. Thus δu will be positive for some values of $\partial u/\partial x$, negative for others. The same thing holds for $\partial u/\partial y$. Substituting $\partial u/\partial x = 0$, $\partial u/\partial y = 0$ in (3), § 101, we get

$$\delta u = \frac{1}{2}\Big(\frac{\partial^2 u}{\partial x^2}h^2 + 2\frac{\partial^2 u}{\partial x \partial y}hk + \frac{\partial^2 u}{\partial y^2}k^2\Big) + \ldots \quad . \quad (6)$$

If h and k be taken sufficiently small, δu will always have the same sign. (Why?) For the sake of brevity, write the homogeneous quadratic (6) in the form

$$ah^2 + bhk + ck^2.$$

On page 388, it is shown that the sign of this quadratic remains invariable, provided ac is greater than b^2 and the signs of a and c are the same. This means that if condition (5) holds, δu will have the same sign for all values of h and k within certain limits, provided $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ have the same sign and

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \text{ is greater than } \frac{\partial^2 u}{\partial x \partial y}. \quad . \quad . \quad . \quad (7)$$

This is **Lagrange's criterion for the maximum and minimum values of a function of two variables.** When this criterion is satisfied, $f(x, y)$ will be a maximum or a minimum according as the sign of $\partial^2 u/\partial x^2$ (or $\partial^2 u/\partial y^2$) is negative or positive.

If $$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \text{ is less than } \frac{\partial^2 u}{\partial x \partial y}, \quad . \quad . \quad . \quad (8)$$

or $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ have different signs, the function is neither a maximum nor a minimum. There is a point of inflection.

If $$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y}, \quad . \quad . \quad . \quad . \quad (9)$$

in order that a maximum or a minimum may occur, it is necessary that the first set of differential coefficients which do not vanish, shall be of an even order.

EXAMPLES.—(1) Test the function $u = x^3 + y^3 - 3axy$ for maxima or minima,

$$\partial u/\partial x = 3x^2 - 3ay = 0, \therefore y = x^2/a;$$
$$\partial u/\partial y = 3y^2 - 3ax = 0, \therefore y^2 - ax = x^4/a^2 - ax = 0.$$
$$\therefore x = 0, x^3 - a^3 = 0, \text{ or } x = a.$$

The other roots, being imaginary, are neglected.

$$\therefore y = x^2/a = a, \text{ or } y = 0.$$
$$\therefore \frac{\partial^2 u}{\partial x^2} = 6x; \frac{\partial^2 u}{\partial x \partial y} = -3a; \frac{\partial^2 u}{\partial y^2} = 6y.$$

Call these derivatives (a), (b), and (c) respectively, then

If $\quad x = 0, (a) = 0, \quad (b) = -3a, (c) = 0.$

If $\quad x = a, (a) = 6a, (b) = -3a, (c) = 6a.$

$$\therefore \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = 36a^2; \frac{\partial^2 u}{\partial x \partial y} = -3a.$$

which means that $x = a$ will make the function a minimum because $\partial^2 u/\partial x^2$ is positive; $x = 0$ will give neither a maximum nor a minimum.

(2) Find the condition that the rectangular parallelopiped whose edges are x, y and z shall have a minimum surface when its volume is v^3.

Since $v^3 = xyz$, $u = xy + yz + zx = xy + v^3/x + v^3/y$. When $\partial u/\partial x = 0$, $x^2y = v^3$; when $\partial u/\partial y = 0$, $xy^2 = v^3$. The only real roots of these equations are $x = y = v$, therefore, $z = v$. The sides of the box are, therefore, equal to each other.

(3) Show that $u = x^3y^2(1 - x - y)$ is a maximum when $x = \frac{1}{2}$, $y = \frac{1}{3}$.

(4) Find the maximum value of u in $u = x^3 - 3ax^2 - 4ay^2$. $\partial u/\partial x = 3x(x - 2a)$; $\partial u/\partial y = -8ay$; $\partial^2 u/\partial x^2 = 6(x - a)$; $\partial^2 u/\partial x \partial y = 0$; $\partial^2 u/\partial y^2 = -8a$. Condition (5) is satisfied by $x = 0$, $y = 0$ and by $x = a$, $y = 0$. The former alone satisfies Lagrange's condition (7), the latter comes under (8).

(5) In Fig. 103, let P_1 be a luminous point; OM_1, OM_2 are mirrors at right angles to each other. The image of P_1 is reflected at N_1 and N_2 in such a way that (i.) the angles of incidence and reflection are equal, (ii.) the length of the path $P_1N_1N_2P_2$ is the shortest possible. (*Fermat's principle*, "a ray of light passes from one point to another by the path which makes the time of transit a minimum".) Let $i_1 = r_1$, $i_2 = r_2$ be the angles of incidence and reflection as shown in the figure. To find the position of N_1 and N_2.

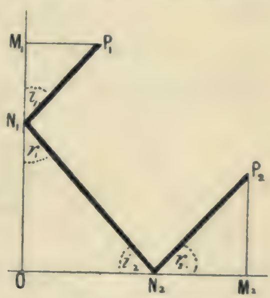


FIG. 103.

Let $ON_2 = x$; $ON_1 = y$; $OM_2 = a_1$; $M_1P_1 = a_2$; $M_2P_2 = b_2$; $OM_1 = b_1$. Let

$$s = N_1P_1 + N_1N_2 + N_2P_2.$$
$$\therefore s = \sqrt{a^2_1 + (b_1 - y)^2} + \sqrt{x^2 + y^2} + \sqrt{(a_2 - x)^2 + b_2^2}.$$

Find $\partial s/\partial x$ and $\partial s \partial y$. Equate to zero, etc. Ansr.

$$x = (a_2b_1 - a_1b_2)/(b_1 + b_2); \; y = (a_2b_1 - a_1b_2)/(a_1 + a_2).$$

Note that $x/y = (a_1 + a_2)/(b_1 + b_2)$. Work out the same problem when the angle $M_1OM_2 = \alpha$.

(6) Required the volume of the greatest rectangular box that can be sent

by "Parcels Post" in accord with the regulation: "length plus girth must not exceed six feet". Ansr. 1 ft. × 1 ft. × 2 ft. = 2 c.ft. Hint. $V = xyz$ is to be a maximum when $V = x + 2(y + z) = 6$. But obviously $y = z$, $\therefore V = xy^2$ is to be a maximum, etc.

(7) Required the greatest cylindrical case that can be sent under the same regulation. Ansr. Length 2 ft., diameter $4/\pi$ ft., capacity 2·55 c.ft. Hint. Volume of cylinder = area of base × height, or, $\frac{1}{4}\pi lD^2$ is to be a maximum when the length + the perimeter of the cylinder = 6, *i.e.*, $l + \pi D = 6$. Obviously l and D denote the respective length and diameter of the cylinder.

See also § 106.

§ 103. Indeterminate Functions.*

In discussing the velocity of reactions of the second order, we found that if the concentration of the two species of reacting molecules is the same, the expression

$$kt = \frac{1}{a-b}\log\frac{a-x}{b-x}\cdot\frac{a}{b},$$

assumes the indeterminate form

$$kt = \infty \times 0,$$

by substituting $a = b$. We are constantly meeting with the same sort of thing when dealing with other functions, which may reduce to one or other of the forms: $0/0$, ∞/∞, $\infty - \infty$, 1^∞, ∞^0, 0^0 . . . We can say nothing at all about the value of any one of these expressions, and, consequently, we must be prepared to deal with them another way so that they may represent something instead of nothing. They have been termed *illusory*, *indeterminate* and *singular* forms.

Fractions which assume the form $\frac{0}{0}$ are called *vanishing fractions*, thus, $(ax^2 - 2a^2x + a^3)/(bx^2 - 2abx + ba^2)$ reduces to $\frac{0}{0}$, when $x = a$. The trouble is due to the fact that the numerator and denominator contain the common factor $(x - a)^2$. If this is eliminated before the substitution, the true value of the fraction for $x = a$ can be obtained, *viz.*, a/b.

These indeterminate functions may often be evaluated by algebraic or trignometrical methods, but not always. Taylor's theorem furnishes a convenient means of dealing with many of these functions. The most important case for discussion is "$\frac{0}{0}$," since this

* In one sense, the word "indeterminate" is a misnomer, because it is the object of this section to show how values of such functions may be determined.

form most frequently occurs and most of the other forms can be referred to it by some special artifice.

Case i.—*The function assumes the form* $\frac{0}{0}$. As already pointed out, the numerator and denominator here contain some common factor which vanishes for some particular value of x, say. These factors must be got rid of. One of the best ways of doing this, short of factorising at sight, is to substitute $a + h$ for x in the numerator and denominator of the fraction and then reduce the fraction to its simplest form. In this way, some power of h will appear as a common factor of each. After reducing the fraction to its simplest form, put $h = 0$, so that $a = x$. The true value of the fraction for this particular value of the variable x will then be apparent.

For cases in which x is to be made equal to zero, the numerator and denominator may be expanded at once by Maclaurin's theorem without any preliminary substitution for x. For instance, the trigonometrical function $(\sin x)/x$ approaches unity when x converges towards zero. This is seen directly. Develop $\sin x$ in ascending powers of x as indicated on page 228. We thus obtain

$$\frac{\sin x}{x} = \frac{\left(\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

The terms to the right of unity all vanish when $x = 0$, therefore,

$$Lt_{x=0}\frac{\sin x}{x} = 1.*$$

Examples.—(1) Show $Lt_{x=0}(a^x - b^x)/x = \log a/b$, page 37.

(2) Show $Lt_{x=0}(1 - \cos x)/x^2 = \frac{1}{2}$.

(3) The fraction $(x^n - a^n)/(x - a)$ becomes 0/0 when $x = a$. Put $x = a + h$ and expand by Taylor's theorem in the usual way. Thus,

$$Lt_{x=a}\frac{x^n - a^n}{x - a} = Lt_{h=a}\frac{(a + h)^n - a^n}{h} = na^{n-1}, \text{ etc.}$$

It is rarely necessary to expand more than two or three of the lowest powers of h.

(4) Show $Lt_{x=1}\frac{1 - x^m}{1 - x^n} = \frac{m}{n}$. Put $x = 1 + h$ and expand.

---

* The symbol "$x \doteq 0$" is sometimes used for the phrase "as x approaches zero". $\underset{x=0}{\text{"lim"}}$ or $\underset{x \doteq 0}{\text{"lim"}}$ are also used instead of our "$Lt_{x=0}$" meaning "the limit of . . . as x approaches zero".

The following method will be found very convenient for dealing with indeterminate functions of this nature.

Let $\frac{f(x)(x-a)^n}{f_1(x)(x-a)^m}$ be a fraction in which $f(x)$, $f_1(x)$ are the parts not containing vanishing factors, n and m are positive integers. First let $m = n$. By differentiation of the numerator and denominator m times and then substituting a for x we get $f(x)/f_1(x)$. For instance, the value of the expression,

$$Lt_{x=1}\frac{x^2-1}{4x^3+x^2-4x-1}$$

is obtained by differentiating once so that

$$Lt_{x=1}\frac{x}{6x^2+x-2} = \frac{1}{5}.$$

If $n > m$, the numerator will vanish after m differentiations and the fraction will be equal to zero. If $n < m$, the denominator will vanish after n differentiations and the fraction will become infinitely great. Under these circumstances we proceed by the following rule: To find the value of an algebraic fraction, substitute the successive differential coefficients of the numerator and denominator until a numerator and denominator are obtained which do not vanish for the value of x under consideration. If the numerator vanishes when the denominator is finite, the limiting value is zero. Some transcendental functions may be treated in this way.

Examples.—(1) Prove that $\int\frac{dx}{x} = \log x$, by means of the general formula $\int x^n dx = \frac{x^{n+1}}{n+1}$. Hint. Show that

$$Lt_{n=-1}\frac{x^{n+1}}{n+1} = \log x.$$

Differentiate the numerator and denominator separately with regard to n and substitute $n = -1$ in the result. See § 71.

(2) Show $Lt_{\gamma=1}\frac{c}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}}\right) = c\log\frac{v_2}{v_1}$. See (4), § 91, (8), § 92.

Case ii.—*The function assumes the form* ∞/∞. These can be converted into the preceding "$\frac{0}{0}$" case by interchanging the numerator and denominator, or else proceed as for $\frac{0}{0}$ by the method of successive differentiation.

Examples.—(1) Show that

$$Lt_{x=0}\frac{\log\sin x}{\log\sin 2x} = 1.$$

Simple substitution furnishes $-\infty/-\infty$. Differentiate once and

$$\frac{\cos x}{2\cos 2x}\frac{\sin 2x}{\sin x} = \frac{1}{2} \times \frac{0}{0}.$$

Differentiate the second factor once more and we get

$$2\cos 2x/\cos x = 2, \text{ etc.}$$

(2) $Lt_{x=\infty}e^x/x^n = \infty/\infty$, when n is positive. Differentiate n times and

$$\frac{e^x}{1.2\ldots n} = \infty;\ \therefore\ Lt_{x=\infty}\frac{e^x}{x^n} = \infty.$$

Case iii.—*The function assumes the form* $\infty \times 0$. Obviously, such a fraction can be converted into the "0/0" form by putting the infinite expression as the denominator of a fraction.

Examples.—(1) $x\log x$ becomes $0 \times -\infty$, when $x = 0$. Transpose the infinite term to the denominator and differentiate.

$x/\log x$ becomes on differentiation x^2; $\therefore\ Lt_{x=0}x\log x = 0$.

(2) Show $Lt_{a=b}\dfrac{1}{a-b}\log\dfrac{(a-x)b}{(b-x)a} = \dfrac{x}{a(a-x)}$.

(3) Show $Lt_{x=0}e^{-x}\log x = 0 \times \infty = 0$.

Case iv.—*The function assumes the form* $\infty - \infty$. First reduce the expression to a single fraction and treat as above.

Examples.—(1) $Lt_{x=1}\dfrac{x}{x-1} - \dfrac{1}{\log x} = \dfrac{x\log x - x - 1}{(x-1)\log x}$.

Differentiate twice and

$$Lt_{x=1}\frac{x}{x+1} = \frac{1}{2}, \text{ etc.}$$

(2) Show $Lt_{x=1}\dfrac{x}{\log x} - \dfrac{1}{\log x} = 1$.

Case v.—*The function assumes one of the forms* 1^∞, ∞^0, 0^0. Take logarithms and proceed as above.

Examples.—(1) $Lt_{x=0}x^x = 0^0$. Take logs and the expression becomes $-\infty/\infty$; differentiate and $Lt_{x=0}x^x = 1$.

(2) Show $Lt_{x=0}(1 + mx)^{1/x} = 1^\infty = e^m$.

Sometimes a simple substitution will make the value apparent at a glance. For instance, substitute $x = 1/y$ and show that

$$Lt_{x=\infty}\frac{x+a}{x+b} = Lt_{y=0}\frac{1+ay}{1+by} = 1.$$

Another illustration has been studied in § 16, namely,

$$Lt_{h=0}\frac{1}{h}\log_e\left(1 + \frac{h}{x}\right) = \frac{1}{x}\log_e e = \frac{1}{x}.$$

§ 104. "The Calculus of Finite Differences."

In the series,

$$1^3,\ 2^3,\ 3^3,\ 4^3,\ 5^3,\ \ldots,$$

subtract the first term from the second, the second from the third, the third from the fourth, and so on. The result is a new series,

$$7,\ 19,\ 37,\ 61,\ 91,\ \ldots,$$

called the **first order of differences.** By treating this new series in a similar way, we get a third series,

$$12,\ 18,\ 24,\ 30,\ \ldots,$$

called the **second order of differences.** This may be repeated as long as we please, unless the series terminates or the differences become very irregular.

The different orders of differences are usually arranged in the form of a "table of differences". To construct such a table, it is usual to begin at some convenient place towards the middle of a series of corresponding values of the two variables, to denote the different values of one variable by, say,

$$x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,\ \ldots,$$

and corresponding values of the other by, say,

$$y_{-2},\ y_{-1},\ y_0,\ y_1,\ y_2,\ \ldots,$$

The differences between the independent variables are denoted by the symbol "Δ," with a superscript to denote the order of difference and a subscript to show the relation between it and the independent variable. Thus,

$$\begin{array}{llllll}
\text{Argument.} & \text{Function.} & \multicolumn{4}{c}{\text{Orders of Differences.}} \\
x_{-2}, & y_{-2}, & \Delta^1{}_{-2}, & & & \\
x_{-1}, & y_{-1}, & \Delta^1{}_{-1}, & \Delta^2{}_{-2}, & \Delta^3{}_{-2}, & \\
x_0, & y_0, & \Delta^1{}_0, & \Delta^2{}_{-1}, & \Delta^3{}_{-1}, & \Delta^4{}_{-2}, \\
x_1, & y_1, & \Delta^1{}_1, & \Delta^2{}_0, & & \\
x_2, & y_2, & & & &
\end{array}$$

where

$$y_1 - y_0 = \Delta^1{}_0;\ y_0 - y_{-1} = \Delta^1{}_{-1};\ \Delta^2{}_0 = \Delta^1{}_1 - \Delta^1{}_0,$$

$$\therefore \Delta^2{}_0 = y_2 - 2y_1 + y_0, \text{ etc.}$$

Such a table will often furnish a good idea of any sudden change in the relative values of the variables with a view to expressing the experimental results in terms of an empirical or interpolation formula. It is not uncommon to find faulty measurements, and other mistakes in observation or calculation, shown up in an unmistakable manner by the appearance of a marked irregularity in a member of one of the difference columns. It is, of course, quite possible that these irregularities are due to something of the nature of a discontinuity in the phenomenon under consideration.

To find the differential coefficients of one variable with respect to another from a table of differences. If corresponding values of two variables can be represented in the form of a mathematical equation, the differential coefficient of the one variable with respect to the other can be easily obtained. If an empirical formula is not available, the tangent to the "smoothed" curve, obtained by plotting the corresponding values of x and y on coordinate or squared paper, will sometimes allow the differential coefficient to be deduced—but not always.

According to *Stirling's interpolation formula*,

$$y = y_0 + \frac{x}{1} \cdot \frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2} + \frac{x^2}{2!}\Delta^2{}_{-1} + \frac{x(x^2-1)}{3!} \cdot \frac{\Delta^3{}_{-1} + \Delta^3{}_{-2}}{2} + \frac{x^2(x^2-1)}{4!}\Delta^4{}_{-2} + \ldots$$

(J. Stirling's *Methodus Differentialis*, London, 1730), when we are given a set of corresponding values of x and y, say x_0, y_0; $x_1, y_1, \ldots$, we can calculate the value y corresponding to any assigned value x, lying between x_0 and x_1. (This kind of operation is discussed in the next section.) Stirling's interpolation formula supposes that the intervals $x_1 - x_0$, $x_0 - x_{-1}, \ldots$ are unity. If, however, h denotes the equal increments in the values $x_1 - x_0$, $x_0 - x_{-1} \ldots$, Stirling's formula is written

$$\left.\begin{aligned} y = y_0 &+ \frac{x}{h} \cdot \frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2} + \frac{x^2}{2!\,h^2}\Delta^2{}_{-1} + \frac{(x+h)x(x-h)}{3!\,h^3} \cdot \frac{\Delta^3{}_{-1} + \Delta^3{}_{-2}}{2} \\ &+ \frac{(x+h)x^2(x-h)}{4!\,h^4}\Delta^4{}_{-2} \\ &+ \frac{(x+2h)(x+h)x(x-h)(x-2h)}{5!\,h^5} \cdot \frac{\Delta^5{}_{-2} + \Delta^5{}_{-3}}{2} + \ldots \end{aligned}\right\} \quad (1)$$

Differentiate (1) with respect to x. Put $x = 0$ in the result

$$\frac{dy}{dx} = \frac{1}{h}\left(\frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2} - \frac{1}{6} \cdot \frac{\Delta^3{}_{-1} + \Delta^3{}_{-2}}{2} + \frac{1}{30} \cdot \frac{\Delta^5{}_{-2} + \Delta^5{}_{-3}}{2} - \ldots\right), \quad (2)$$

This series may be written in the form,

$$\frac{dy}{dx} = \frac{1}{h}\left(\frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2} - \frac{1^2}{3!} \cdot \frac{\Delta^3{}_{-1} + \Delta^3{}_{-2}}{2} + \frac{1^2 . 2^2}{5!} \cdot \frac{\Delta^5{}_{-2} + \Delta^5{}_{-3}}{2} - \ldots\right). \quad (3)$$

The following method of deducing (2) is instructive. Assume the expansion

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots = \text{say, } y_0.$$

Differentiate with respect to x,

$$\therefore dy/dx = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \quad (4)$$

Let x receive a small increment $+h$ and also a small increment $-h$, then, from the original equation,

$$y = A + B(x + h) + C(x + h)^2 + D(x + h)^3 + \ldots = \text{say, } y_1;$$
$$y = A + B(x - h) + C(x - h)^2 + D(x - h)^3 + \ldots = \text{say, } y_{-1}.$$

But $$y_1 - y_0 = \Delta^1{}_0;\ y_0 - y_{-1} = \Delta^1{}_{-1}, \ldots$$

Therefore, making the obvious subtractions,

$$\Delta^1{}_0 = Bh + C(2xh + h^2) + D(3x^2h + 3xh^2 + h^3) + \ldots$$
$$\Delta^1{}_{-1} = Bh + C(2xh - h^2) + D(3x^2h - 3xh^2 + h^3) + \ldots$$

Add these equations and divide by $2h$,

$$\frac{1}{h}\cdot\frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2} = B + 2Cx + 3Dx^2 + Dh^2 + 4Ex^3 + \ldots$$

When h is made very small, the terms containing h may be neglected. The resulting series,

$$Lt\frac{incr.\ y}{incr.\ x} = \frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2}\frac{1}{h} = Lt_{h=0}\{B + 2Cx + 3Dx^2 + (Dh^2 + 4Exh^2 + \ldots)\};$$
$$= B + 2Cx + 3Dx^2 + \ldots,$$

is identical with that just developed for dy/dx in equation (4). As a first approximation, therefore, we can write

$$\frac{dy}{dx} = \frac{1}{h}\cdot\frac{\Delta^1{}_0 + \Delta^1{}_{-1}}{2}. \qquad \ldots \quad (5)$$

If a greater accuracy than this is desired, substitute $x + 2h$ and $x - 2h$ for x in the original equation. In this way we can build up (2).

To illustrate the use of formula (2), let the first two columns of the following table represent a set of measurements obtained in the laboratory. It is required to find the value of dy/dx corresponding to $x = 5{\cdot}2$.

x	y	Δ^1	Δ^2	Δ^3	Δ^4
4·7	109·947				
		11·563			
4·8	121·510		1·217		
		12·780		0·126	
4·9	134·290		1·343		0·017
		14·123		0·143	
5·0	148·413		1·486		0·012
		15·609		0·155	
5·1	164·022		1·641		0·019
		17·250		**0·174**	
5·2	**181·272**		**1·815**		**0·015**
		19·065		**0·189**	
5·3	200·337		2·004		0·024
		21·069		0·213	
5·4	221·406		2·217		0·018
		23·286		0·231	
5·5	244·692		2·448		0·028
		25·734		0·259	
5·6	270·426		2·707		
		28·441			
5·7	298·867				

Make the proper substitutions in (2). In the case of 5·2 only the block figures in the above table are required. Thus,

$$\frac{dy}{dx} = \frac{1}{0{\cdot}1}\left(\frac{17{\cdot}250 + 19{\cdot}065}{2} - \frac{1}{6}\cdot\frac{0{\cdot}174 + 0{\cdot}189}{2} + \frac{1}{30}\cdot\frac{0{\cdot}009 - 0{\cdot}004}{2}\right).$$
$$= 181{\cdot}273.$$

The second and third terms are not often used. They have the nature of correction terms.

In the same way it can be shown by differentiating (1) twice, and putting $x = 0$,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\Big(\Delta^2{}_{-1} - \frac{1}{12}\Delta^4{}_{-2} + \frac{1}{90}\Delta^6{}_{-3} - \ . \ . \ .\Big), \quad . \quad . \quad . \quad (6)$$

or, $$\frac{d^2y}{dx^2} = \frac{1}{h^2}\Big(\frac{2}{2!}\Delta^2{}_{-1} - \frac{2}{4!}\Delta^4{}_{-2} + \frac{2 \,.\, 2^2}{6!}\Delta^6{}_{-3} - \frac{2 \,.\, 2^2 \,.\, 3^2}{8!}\Delta^8{}_{-4} + \ . \ . \ .\Big), \quad (7)$$

EXAMPLES.—(1) From Horstmann's observations on the dissociation pressure (p) of the ammonio-chlorides of silver at different temperatures (θ):

$\theta = 8,$	$12,$	$16,$	. . . °*C.*
$p = 43{\cdot}2,$	$52{\cdot}0,$	$65{\cdot}3,$	. . . *cm. Hg.*,

show that at 12°, $dp/d\theta = 2{\cdot}76$.

(2) Find $ds/d\theta$ at 0° C. from the following data:—

$\theta = 1,$	$0{\cdot}5,$	$0,$	$-0{\cdot}5,$	$-1{\cdot}0,$ . . . ;
$10^6 \times s = 1288{\cdot}3,$	$1290{\cdot}7,$	$1293{\cdot}1,$	$1295{\cdot}4,$	$1297{\cdot}8,$ . . .

Ansr. $ds/d\theta = 5{\cdot}7 \times 10^6$.

(3) Find the value of d^2y/dx^2 for $y = 5{\cdot}2$ from the above table. Ansr. 181·37. Also plot the dy/dx, y-curve from the data given.

The difference columns should not be carried further than is consistent with the accuracy of the data, otherwise the higher approximations will be *less* accurate than the first. Do not carry the differences further than the point at which they begin to exhibit marked irregularities.

(4) The variation in the pressure of saturated steam (p) with temperature (θ) has been found to be as follows:—

$\theta = 90,$	$95,$	$100,$	$105,$	$110,$	$115,$	$120,$ . . . ;
$p = 1463,$	$1765,$	$2116,$	$2526,$	$2994,$	$3534,$	$4152,$ . . .

Hence show that at 105° $dp/d\theta = 87{\cdot}6$, $d^2p/d\theta^2 = 2{\cdot}48$.

Everett's papers in the *Quarterly Journal of Pure and Applied Mathematics*, **30**, 357, 1900; **31**, 304, 1901, may be consulted for some recent work on this subject. See also *Nature*, **60**, 271, 365, 390, 1899.

§ 105. Interpolation and Empirical Formulae.

After a set of measurements of two interdependent variables has been made in the laboratory, it is necessary to find if there is any simple relation between them, that is, to find if a general expression of the one variable can be obtained in terms of the other so that intermediate values can be calculated. The process of computation of the numerical values of two variables *intermediate* between those actually determined by observation and measurement, is called **interpolation.** When we attempt to obtain values lying *beyond* the limits of those actually measured, the process is called **extrapolation.**

It is apparent that the correct formula connecting the two variables must be known before exact interpolation can be performed, so much so that the method of testing a supposed formula is to compare the experimental values with those furnished by interpolation as exemplified in §§ 18, 88, 96 and elsewhere.

Interpolation is based on the fact that when a law is known with fair exactness, we can, by the principle of continuity, anticipate the results of any future experiments.

If only two experimental results are known, we must assume that the two quantities vary in a proportional manner. The geometrical meaning of this is that if the positions of two points are known, we must assume that the curve passing through these points is a straight line, because an infinite number of curved lines could be drawn through these two points.

If the differences between the succeeding pairs of values are small and regular, any intermediate value can be calculated by simple proportion on the assumption that the change in the value of the function is proportional to that of the variable. Interpolation is employed in the graduation of a thermometer between 0° and 100°, extrapolation beyond these points. In *Gauss' method of double weighing*, the mean weight of the substance weighed in each pan is regarded as the true weight.

The position of rest of a balance is deduced from the amplitude of the oscillations on each side. Three, five, or some odd number of observations are made, the arithmetical mean of the observations on each side are added together, the mean of this sum is the *null point*, or *position of rest of the balance*.

Weighing by the method of vibrations is another example of interpolation. Let x denote the zero point of the balance, let w_0 be the true weight of the body in question. This is measured by the weight required to bring the index of the balance to zero point. Let x_1 be the position of rest when a weight w_1 is added and x_2 the position of rest when a weight w_2 is added. Assuming that for small deflections of the beam the difference in the two positions of rest will be proportional to the difference of the weights, the weight (w_0) necessary to bring the pointer to zero will be given by the simple proportion :

$$(w_0 - w_1) : (x_0 - x_1) = (w_2 - w_1) : (x_2 - x_1),$$

or,
$$w_0 = w_1 + (x_0 - x_1)(w_2 - w_1)/(x_2 - x_1).$$

When the intervals between the two terms are large, or the differences between the various members of the series decrease rapidly, this simple proportion cannot be used with confidence.

To take away any arbitrary choice in the determination of the intermediate values, it is commonly assumed that the function can be expressed by a limited series of powers of one of the variables. Thus we have the interpolation formulae of Newton, Bessel, Stirling, Lagrange, and Gauss.

Let x_{-2}, y_{-2}; x_{-1}, y_{-1}; x_0, y_0; x_1, y_1; x_2, y_2 be corresponding values of the two variables x and y. It is required to calculate the value y corresponding to some value x lying between x_0 and x_1.

Newton's interpolation formula is

$$y = y_0 + \frac{x}{1}\Delta^1{}_0 + \frac{x(x-1)}{1.2}\Delta^2{}_0 + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3{}_0 + \ldots, \quad (1)$$

continued until the differences become negligibly small or irregular.

EXAMPLE.—The use of Newton's formula may be illustrated by the following problem: What is the cube root of 60·25, given the first two columns in the subjoined table? The cube root of

$60 = 3·914868$		
	$\Delta^1{}_0 = + 0·021629$	
$61 = 3·936497$		$\Delta^2{}_0 = - 0·000235.$
	$\Delta^1{}_1 = + 0·021394$	
$62 = 3·957891$		$\Delta^2{}_1 = - 0·000228.$
	$\Delta^1{}_2 = + 0·021166$	
$63 = 3·979057$		$\Delta^2{}_2 = - 0·000223.$
	$\Delta^1{}_3 = + 0·020943$	
$64 = 4·000000$		

If an increase downwards is reckoned positive, a decrease downwards is to be reckoned negative. The first orders of differences are, therefore, positive; the second, negative; and the third, positive. Substitute $x = \frac{1}{4}$ in (1)

$$\begin{aligned} y &= y_0 + \tfrac{1}{4}\Delta^1{}_0 - \tfrac{3}{32}\Delta^2{}_0 + \tfrac{7}{128}\Delta^3{}_0 - \ldots \\ &= 3·914868 + 0·005407 + 0·000022 + 0·000000. \\ &= 3·920297. \end{aligned}$$

The number obtained by simple proportion is 3·920295. The correct number is a little greater than 3·920297.

Lagrange's interpolation formula is more general than the above. "Given n consecutive values of a function, to find any other intermediate value."

Let y become $y_0, y_1, y_2, y_3, \ldots, y_n$ when x becomes $a, b, c, d, \ldots n$. The value of y corresponding to any given value of x, can be determined from the formula

$$\left.\begin{aligned} y = \frac{(x-b)(x-c)\ldots(x-n)}{(a-b)(a-c)\ldots(a-n)}y_0 + \frac{(x-a)(x-c)\ldots(x-n)}{(b-a)(b-c)\ldots(b-n)}y_1 + \ldots \\ \ldots + \frac{(x-a)(x-c)\ldots(x-m)}{(n-a)(n-b)\ldots(n-m)}y_n. \end{aligned}\right\} (2)$$

If the function is periodic, **Gauss' interpolation formula** may be used. This has a close formal analogy with Lagrange's.*

$$y = \frac{\sin \frac{1}{2}(x - b) . \sin \frac{1}{2}(x - c) \ . \ . \ . \ \sin \frac{1}{2}(x - n)}{\sin \frac{1}{2}(a - b) . \sin \frac{1}{2}(a - c) \ . \ . \ . \ \sin \frac{1}{2}(a - n)} y_0 + \ . \ . \ . \quad (3)$$

Lagrange's formula may be employed for the conversion of the scale readings of the spectroscope into wave-lengths. Assuming that the indices of refraction ($\gamma_0, \gamma_1, \gamma_2, \ . \ . \ .$) are inversely as the squares of the wave-lengths ($n_0, n_1, n_2, \ . \ . \ .$) if the scale readings of, say, three lines near together are given and the wave-lengths of two of the lines, the wave-length of the third can be found by simple substitution in Lagrange's formula (2), which now assumes the form,

$$\frac{1}{\gamma^2_1} = \frac{1}{\gamma^2_0} \cdot \frac{n_1 - n_2}{n_0 - n_2} + \frac{1}{\gamma^2_2} \cdot \frac{n_1 - n_0}{n_2 - n_0}. \quad . \quad . \quad (4)$$

EXAMPLES.—(1) For the three bright magnesium lines, $\gamma_0 = 5183$, $\gamma_2 = 5167$, $n_0 = 74 \cdot 5$, $n_1 = 74 \cdot 8$, $n_2 = 75$ (Lupton). Required the wave-length γ_1 of the third *Mg* line.

$$\frac{1}{\gamma^2_1} = \frac{1}{(5183)^2} \cdot \frac{0 \cdot 2}{0 \cdot 5} + \frac{1}{(5167)^2} \cdot \frac{0 \cdot 3}{0 \cdot 5}; \ \gamma_1 = 5173.$$

Actual measurement gives 5172.

(2) The scale readings of the *Li*, *Tl* and *Na* lines were found to be respectively 6·15, 10·55, and 8·0. Required the wave-length of the *Na* line given $Li = 6708$, $Tl = 5351$ (Schuster and Lees). Ansr. 5932.

The most satisfactory method of finding a formula to express the relation between the two variables in any set of measurements, is to deduce a mathematical expression from known principles or laws, and then determine the value of the constants from the experimental results themselves. Such expressions are said to be **theoretical formulae** as distinct from **empirical formulae**, which have no well-defined relation with known principles or laws.†

It is, of course, impossible to determine the correct form of a function from the experimental data alone. An infinite number of formulae might satisfy the numerical data, in the same sense

* For the theoretical bases of these reference interpolation formulae the reader must consult Boole's work, *A Treatise on the Calculus of Finite Differences*, p. 38, 1880.

† The terms "formula" and "function" are by no means synonymous. The formula is not the function, it is only its dress. The fit may or may not be a good one. In other words, the function is the relation or law involved in the process. The relation *may* be represented in a formula by symbols which stand for numbers. This must be borne in mind when the formal relations of the symbols are made to represent some physical process or concrete thing. See the remarks on page 394 with reference to the rejection of certain roots of numerical equations.

that an infinite number of curves might be drawn through a series of points. (See "Contact of Curves," "Multiple Points," etc.) For instance, over thirty empirical formulae have been proposed to express the unknown relation between the pressure and temperature of saturated steam.

As a matter of fact, empirical formulae frequently originate from a lucky guess. Good guessing, here as elsewhere, is a fine art. A hint as to the most plausible form of the function is sometimes obtained by plotting the experimental results. It is useful to remember that if the curve increases or decreases regularly, the equation is probably algebraic; if it alternately increases and decreases, the curve is probably expressed by some trignometrical function.

If the curve is a straight line, the equation will be of the form, $y = mx + b$. If not, try $y = ax^n$, or $y = ax/(1 + bx)$. If the rate of increase (or decrease) of the function is proportional to itself we have the compound interest law. In other words, if dy/dx varies proportionally with y, $y = be^{-ax}$ or be^{ax}. If dy/dx varies proportionally with x/y, try $y = bx^a$. If dy/dx varies as x, try $y = a + bx^2$. Other general formulae may be tried when the above prove unsatisfactory, thus,

$$y = \frac{a + x}{b - x};\ y = 10^{a + bx};\ y = a + b \log x;\ y = a + bc^x, \text{ etc.}$$

Otherwise we may fall back upon Maclaurin's expansion in ascending powers of x, the constants being positive, negative or zero. This series is particularly useful when the terms converge rapidly, § 96, 2.

When the results exhibit a periodicity, the general formula to be tried, is

$$y = a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x + \ldots$$

If the cycles are regular, only the first three terms on the right need be used. Such phenomena are the ebb and flow of tides, annual variations of temperature and pressure of the atmosphere, cyclic variations in magnetic declination, etc. See also "Fourier's Series".

Empirical formulae, however closely they agree with facts, do not pretend to represent the true relation between the variables under consideration. They do little more than follow, more or less closely, the course of the graphic curve representing the relation between the variables within a more or less restricted range.

Thus, Regnault employed three interpolation formulae for the vapour pressure of water between − 32° F. and 230° F.* For example, from − 32° F. to 0° F., he used $p = a + b\alpha^\theta$; from 0° to 100° F., $\log p = a + b\alpha^\theta + c\beta^\theta$; from 100° to 230° F., $\log p = a + b\alpha^\theta - c\beta$. Kopp required four formulae to represent his measurements of the thermal expansion of water between 0° and 100° C. Each of Kopp's formulae was only applicable within the limited range of 25° C.

Dulong and Petit's memoir, referred to on page 43, is well worth reading for some instructive artifices useful in deducing empirical formulae.

Graphic interpolation. If all attempts to deduce or guess a satisfactory formula are unsuccessful, the results are simply tabulated, or preferably plotted on squared paper, because then "it is the thing itself that is before the mind instead of a numerical symbol of the thing".

Intermediate values may be obtained from the graphic curve by measuring the ordinate corresponding to a given abscissa or *vice versâ*.

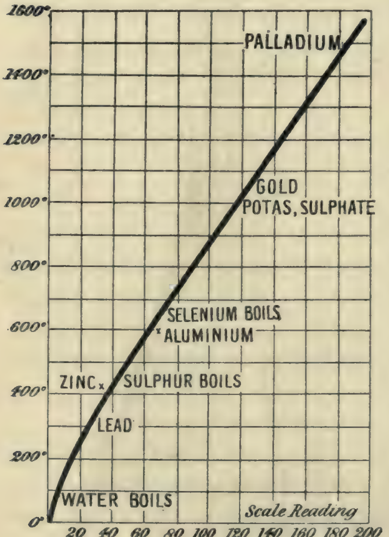


FIG. 104.—Calibration Chart.

In measuring high temperatures by means of the Le Chatelier-Austen pyrometer, the deflection of the galvanometer index on a millimetre scale is caused by the electromotive force generated by the heating of a thermo-couple ($Pt - Pt$ with 10 °/ₒ Rd) in circuit with the galvanometer. The displacement of the index is nearly proportional to the temperature. The scale is calibrated by heating the junction to well-defined temperatures and plotting the temperatures as ordinates, the scale readings as abscissae. The resulting graph or "*calibration curve*" is shown in Fig. 104. The ordinate to

* Rankine was afterwards lucky enough to find that

$$\log p = a - \beta/\theta - \gamma/\theta^2,$$

represented Regnault's results for the vapour pressure of water throughout the whole range − 32° F. to 230° F.

the curve corresponding to any scale reading, gives the desired temperature.

EXAMPLES.—(1) What temperature corresponds to a scale reading of 160 scale divisions in the above diagram? Ansr. 1300°.

(2) Construct a series of curves from the exposure formula of a thermometer, § 43, (12), between $\theta = 0{\cdot}1°$ C. and $3{\cdot}0°$; $x = 0$ to $x = 300°$, $y = 0$ to $y = 200°$. What use is the resulting diagram?

(3) By plotting on squared paper corresponding values of centimetres and inches, litres and pints, grams and ounces, Fahrenheit and Centigrade degrees, etc., etc., the mutual conversion of the one into the other can be conveniently effected by inspection (*i.e.*, without calculation). Try this: given 1 oz. = 28·34 grms., 2 oz. = 56·69 grms., 8 oz. = 226·75 grms., 1 lb. = 453·60 grms.

§ 106. To Evaluate the Constants in Empirical or Theoretical Formulae.

Before a formula containing constants can be adapted to any *particular* process, the numerical values of the constants must be accurately known. For instance, the relation

$$v = 1 + a\theta,$$

represents the volume (v) to which unit volume of *any* gas expands when heated to $\theta°$. a is a constant. The law embodied in this equation can only be applied to a particular gas when a assumes *the* numerical value characteristic of that gas. If we are dealing with hydrogen, $a = 0{\cdot}00366$; if carbon dioxide, $a = 0{\cdot}00371$; if sulphur dioxide, $a = 0{\cdot}00385$.

Again, if we want to apply the law of definite proportions, we must know exactly what the definite proportions are before it can be decided whether any particular combination is comprised under the law. In other words, we must not only know the general law, but also particular numbers appropriate to particular elements. In mathematical language this means that before a function can be used practically, we must know:

1. *The form of the function* (*i.e.*, the general formula).
2. *The numerical values of the constants.*

The determination of the form of the function has been discussed in the preceding section, the evaluation of the constants remains to be considered.

Is it legitimate to deduce the numerical values of the constants from the experiments themselves? The answer is that the numerical data are determined from experiments purposely made by different methods under different conditions. When all independently furnish the same result it is fair to assume that the experimental numbers include the values of the constants under discussion. J. F. W. Herschel's *A Preliminary Discourse on the Study of Natural Philosophy*, §§ 221 *et seq.*, is worth reading in this connection.

In some determinations of the volume (v) of carbon dioxide dissolved in one volume of water at different temperatures (θ), the following pairs of values were obtained:

$$\theta = 0, \quad 5, \quad 10, \quad 15;$$
$$v = 1\cdot80, \quad 1\cdot45, \quad 1\cdot18, \quad 1\cdot00.$$

As Herschel has remarked, in all cases of "direct unimpeded action," we may expect the two quantities to vary in a simple proportion, so as to obey the linear equation,

$$y = a + bx; \text{ we have, } v = a + b\theta, \qquad (1)$$

which, be it observed, is obtained from Maclaurin's series by the rejection of all but the first two terms.

It is required to find from these observations the values of the constants, a and b, which will represent the experimental data in the best possible manner.

The above results can be written,

$$\left.\begin{aligned} 1.\ & 1\cdot80 = a, \\ 2.\ & 1\cdot45 = a + 5b, \\ 3.\ & 1\cdot18 = a + 10b, \\ 4.\ & 1\cdot00 = a + 15b, \end{aligned}\right\} \qquad (2)$$

which is called a set of **observation equations.**

From

$$1 \text{ and } 2,\ a = 1\cdot80,\ b = -\ 0\cdot07,$$
$$2 \text{ and } 3,\ a = 0\cdot64,\ b = -\ 0\cdot054,$$
$$3 \text{ and } 4,\ a = 0\cdot82,\ b = -\ 0\cdot036, \text{ etc.}$$

This want of agreement between the values of the constants obtained from different sets of equations is due to errors of observation. It nearly always occurs when the attempt is made to calculate the constants in this manner.

The numerical values of the constants deduced from any arbitrary set of observation equations can only be absolutely correct when the measurements are perfectly accurate. The problem here presented is to pick the best representative values of the constants from the experimental numbers. If all the measurements were equally trustworthy, the correct method would be to find the arithmetical mean of all the values of the constants so determined.

The constants must satisfy the following criterion: *The differences between the observed and the calculated results must be the smallest possible with small positive and negative differences.*

One of the best ways of fixing the numerical values of the constants in any formula is to use what is known as **the method of least squares.** *This rule proceeds from the assumption that the most probable values of the constants are those for which the sum of the squares of the differences between the observed and the calculated results are the smallest possible* (see page 433).

To take the general case first, let the observed magnitude y depend on x in such a way that

$$y = a + bx. \qquad . \qquad . \qquad . \qquad . \qquad (3)$$

It is required to determine the most probable values of a and b.

For perfect accuracy, we should have the following observation equations:

$$a + bx_1 - y_1 = 0;\ a + bx_2 - y_2 = 0;\ .\ .\ .\ a + bx_n - y_n = 0.$$

In practice this is unattainable. Let $v_1, v_2, \ldots v_n$ denote the actual deviations so that

$$a + bx_1 - y_1 = v_1;\ a + bx_2 - y_2 = v_2;\ .\ .\ .\ a + bx_n - y_n = v_n.$$

It is required to determine the constants so that,

$$\Sigma(v^2) = v_1^2 + v_2^2 + \ldots + v_n^2 \text{ is a minimum.}$$

This condition is fulfilled (page 240) by equating the partial derivatives of $\Sigma(v^2)$ with respect to a and b to zero. In this way, we obtain,

$$\frac{\partial}{\partial a}\Sigma(a + bx - y)^2 = 0, \text{ hence, } \Sigma(a + bx - y) = 0;$$

$$\frac{\partial}{\partial b}\Sigma(a + bx - y)^2 = 0, \text{ hence, } \Sigma x(a + bx - y) = 0.$$

If there are n observation equations, there are n a's and $\Sigma(a) = na$, therefore,

$$na + b\Sigma(x) - \Sigma(y) = 0;\ a\Sigma(x) + b\Sigma(x^2) - \Sigma(xy) = 0.$$

Now solve these two simultaneous equations for a and b,

$$a = \frac{\Sigma(x)\,.\,\Sigma(xy) - \Sigma(x^2)\,.\,\Sigma(y)}{[\Sigma(x)]^2 - n\Sigma(x^2)};\ b = \frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{[\Sigma(x)]^2 - n\Sigma(x^2)}, \qquad (4)$$

which determines the values of the constants.

The method of least squares *assumes* that the observations are all equally reliable (see "Errors of Observation," Chapter XI.).

Returning to the special case at the commencement of this section, to find the best representative value of the constants a and b in formula (1).

Previous to substitution in (4), it is best to arrange the data according to the following scheme:

θ.	v.	θ^2.	θv.
0	1·80	0	0
5	1·45	25	7·25
10	1·18	100	11·80
15	1·00	225	15·00
$\Sigma(\theta) = 30$	$\Sigma(v) = 5{\cdot}43$	$\Sigma(\theta^2) = 350$	$\Sigma(\theta v) = 34{\cdot}05$

Substitute these values in equation (4), n, the number of observations, $= 4$, hence we get

$$a = 1{\cdot}758 ; \; b = - 0{\cdot}0534.$$

The amount of gas dissolved at θ° is obtained from the interpolation formula,

$$v = 1{\cdot}758 - 0{\cdot}0534\theta.$$

To show that this is the best possible formula to employ, in spite of 1·758 volumes obtained at 0°, proceed in the following manner:

Temp. $= \theta$.	Volume of gas $= v$.		Difference between Calculated and Observed.	Square of Difference between Calculated and Observed.
	Calculated.	Observed.		
0	1·758	1·80	− 0·042	0·00176
5	1·491	1·45	+ 0·041	0·00168
10	1·224	1·18	+ 0·044	0·00194
15	0·957	1·00	− 0·043	0·00185
				0·00723

The number 0·00723, the sum of the squares of the differences between the observed and the calculated results, is a minimum. Any alteration in the value of either a or b will cause this term to increase. This can easily be verified. For example, if we try the very natural $a = 1{\cdot}80$, $b = - 0{\cdot}065$, we get 0·039; if $a = 1{\cdot}772$, $b = - 0{\cdot}056$ we get 0·0082, etc. (see method of § 102).

See *Nature*, **63**, 489, 1901. The above method of treatment is founded on that of Kohlrausch in his *Leitfaden der praktischen Physik* (Teubner, Leipzig, 1896), p. 8. For other methods of calculating the constants, see Lupton's *Notes on Observations*, p. 105, 1898; and § 186.

Examples.—(1) Find the law connecting the length (l) of a rod with temperature (θ), when the length of a metre bar at 0° elongates with rise of temperature according to the following scheme:

$$\theta = 20°, \quad 48°, \quad 50°, \quad 60° \text{ C.};$$
$$l = 1000{\cdot}22, \quad 1000{\cdot}65, \quad 1000{\cdot}90, \quad 1001{\cdot}05 \text{ mm.}$$

(Kohlrausch, *l.c.*). During the calculation, for the sake of brevity, use $l = {\cdot}22, {\cdot}65, {\cdot}9$ and $1{\cdot}05$. Assume $l = a + b\theta$ and show that $a = 999{\cdot}804$, $b = 0{\cdot}0212$.

(2) Find a formula similar to (4) for the general equation $y = a \tan \alpha + b$, where a and b are constants to be determined.

(3) According to Bremer's measurements aqueous solutions of sodium carbonate containing p °/ₒ of the salt expand by an amount v as indicated in the following table:

$$p = 3{\cdot}2420, \quad 4{\cdot}8122, \quad 7{\cdot}4587, \quad 10{\cdot}1400;$$
$$10^4 \times v = 1{\cdot}766, \quad 2{\cdot}046, \quad 2{\cdot}342, \quad 2{\cdot}732.$$

Hence show that if $v = a + bp$, $a = 0{\cdot}00012415$, $b = 0{\cdot}00001528$.

Suppose that instead of the general formula (3), we had started with

$$y = a + bx + cx^2, \quad . \quad . \quad . \quad . \quad (5)$$

where a, b and c are constants to be determined. The resulting formulae for b and c (omitting a), analogous to (4), are,

$$b = \frac{\Sigma(x^4) \,.\, \Sigma(xy) - \Sigma(x^3) \,.\, \Sigma(x^2y)}{\Sigma(x^2) \,.\, \Sigma(x^4) - [\Sigma(x^3)]^2} \quad . \quad . \quad . \quad (6)$$

$$c = \frac{\Sigma(x^2) \,.\, \Sigma(x^2y) - \Sigma(x^3) \,.\, \Sigma(xy)}{\Sigma(x^2) \,.\, \Sigma(x^4) - [\Sigma(x^3)]^2} \quad . \quad . \quad . \quad (7)$$

These two formulae have been deduced by a similar method to that employed in the preceding case. a is a constant to be determined separately by arranging the experiment so that when $x = 0$, $a = y_0$.

Examples.—(1) The following observations were made by Bremer. If ρ denotes the density of an aqueous solution of calcium chloride at θ° C.,

θ.	ρ.	θ.	ρ.	θ.	ρ.
15·65	1·03336	33·40	1·02356	32·76	1·02051
20·11	1·03273	39·25	1·02640	63·23	1·01516
28·60	1·02856	46·01	1·02348		

Calculate the constants a and b in the formula,

$$\rho = \rho_0(1 + a\theta + b\theta^2),$$

where $\rho_0 = 1{\cdot}03619$. Ansr. $b = -\,0{\cdot}000003301$; $a = -\,0{\cdot}0001126$.

(2) The following series of measurements of the temperature (θ) at different depths (x) in an artesian well, were made at Grenelle (France):

$x =$	28,	66,	173,	248,	298,	400,	505,	548;
$\theta =$	11·71,	12·90,	16·40,	20·00,	22·20,	23·75,	26·45,	27·70.

The mean temperature at the surface was 10·6°. Hence show that at a depth of x metres, the temperature will be,

$$\theta = 10·6 + 0·042096x - 0·000020558x^2.$$

(3) If, when $x = 0$, $y = 1$ and when

$x =$	8·97,	20·56,	36·10,	49·96,	62·38,	83·73;
$y =$	1·0078,	1·0184,	1·0317	1·0443,	1·0563,	1·0759.

Hence show that

$$y = 1 + 0·00084x + 0·0000009x^2.$$

The reader will himself have to deduce the general formulae for a, b, c, when still another correction term is included, namely,

$$y = ax + bx^2 + cx^3. \quad . \quad . \quad . \quad . \quad (8)$$

Examples.—The following measurements are selected from a paper by Thompson in Wiedemann's *Annalen* (**44**, 553, 1891).

(1) If when

$x =$	0·2,	0·4,	0·6,	0·8,	1·0,	1·2,
$p =$	5·531,	11·084,	16·671,	22·298,	27·949,	33·646,

show that

$$x = 27·578p + 0·3193p^2 + 0·0538p^3.$$

(2) If when

$x =$	0·2,	0·4,	0·6,	0·8,	1·0,
$p =$	7·078,	14·196,	21·358,	28·558,	35·792,

show that

$$x = 35·2725p + 0·5725p^2 - 0·0525p^3.$$

If three variables are to be investigated, we may use the general formula

$$z = ax + by. \quad . \quad . \quad . \quad . \quad (9)$$

The reader may be able to prove, on the above lines, that

$$a = \frac{\Sigma(x^2) . \Sigma(xz) - \Sigma(xy) . \Sigma(yz)}{\Sigma(x^2) . \Sigma(y^2) - [\Sigma(xy)]^2}; \quad . \quad . \quad (10)$$

$$b = \frac{\Sigma(x^2) . \Sigma(yz) - \Sigma(xy) . \Sigma(xz)}{\Sigma(x^2) . \Sigma(y^2) - [\Sigma(xy)]^2}. \quad . \quad . \quad (11)$$

A rough and ready method for calculating the constants is to pick out as many observation equations as there are unknowns and solve for x, y, z, by ordinary a, b, c, say, algebraic methods. The different values of the unknown corresponding to the different sets of observation arbitrarily selected are thus ignored.

Example.—Corresponding values of the variables x and y are known, say, x_1, y_1; x_2, y_2; x_3, y_3; . . . Calculate the constants a, b, c, in the interpolation formula

$$y = a(10)^{bx/(1 + cx)}.$$

When $x_1=0$, $y_1=a$. Thus b and c remain to be determined. Take logarithms of the two equations in x_2, y_2 and x_3, y_3 and show that,

$$b = \log_{10}\frac{y_3}{a}\left\{\left(\frac{1}{x_2}-\frac{1}{x_3}\right)\log_{10}\frac{y_2}{a}\right\}\Big/\left\{\log_{10}\frac{y_3}{a}-\log_{10}\frac{y_2}{a}\right\};$$

$$c = \left\{\frac{1}{x_2}\log_{10}\frac{y_2}{a}-\frac{1}{x_3}\log_{10}\frac{y_3}{a}\right\}\Big/\left\{\log_{10}\frac{y_3}{a}-\log_{10}\frac{y_2}{a}\right\}.$$

This method may be used with any of the above formulae when an exact determination of the constants is of no particular interest, or when the errors of observation are relatively small.

Graphic Method. Returning to the solubility determinations at the beginning of this section, prick points corresponding to pairs of values of v and θ on squared paper. The points lie approximately on a straight line. Stretch a black thread so as to get *the* straight line which lies most evenly among the points. Two points lying on the black thread line are $v = 1{\cdot}0$, $\theta = 14{\cdot}5$, and $v = 1{\cdot}7$, $\theta = 1{\cdot}5$,

$$\therefore a + 14{\cdot}5b = 1;\ a + 1{\cdot}5b = 1{\cdot}7.$$

By subtraction, $b = -0{\cdot}54$, $\therefore a = 1{\cdot}78$.

It is here assumed that the curve which goes most evenly among the points represents the correct law (footnote, page 123). In the example just considered, there is, perhaps, too small a number of observations to show the method to advantage. Try these:

$p =$ 2, 4, 6, 8, 10, 20, 25, 30, 35, 40,
$s =$ 1·02, 1·03, 1·06, 1·07, 1·09, 1·18, 1·23, 1·29, 1·34, 1·40,

where s denotes the density of aqueous solutions containing p °/ₒ of calcium chloride at 15° C. The selection of the best "black thread" line is more uncertain the greater the magnitude of the errors of observation affecting the measurements. The values deduced for the constants will differ slightly with different workers or even with the same worker at different times. With care, and accurately ruled paper, the results are sufficiently accurate for most practical requirements.

When the "best" curve has to be drawn freehand, the results are more uncertain. For example, the amount of "active" oxygen (y) contained in a solution of hydrogen dioxide in dilute sulphuric acid was found, after the elapse of t days, to be:

$t =$ 6, 9, 10, 14, 18, 27, 34, 38, 41, 54, 87,
$y =$ 3·4, 3·1, 3·1, 2·6, 2·2, 1·3, 0·9, 0·7, 0·6, 0·4, 0·2,

where $y = 3{\cdot}9$ when $t = 0$. We leave these measurements with the reader as an exercise.

In Perry's *Practical Mathematics* (published by the Science and Art Department, London, 1899, 6d.), a trial plotting on "logarithmic paper" is recommended in certain cases. On **squared paper,** the distances between the horizontal and vertical lines are in fractions of a metre or of a foot. On **logarithmic paper,** the distances between the lines are proportional to the logarithms of the numbers. If, therefore, the experimental numbers follow a law like

$$\log_{10}x + \alpha\log_{10}y = \text{constant},$$

the function can be plotted as easily as on squared paper. If the resulting graph is a straight line, we may be sure that we are dealing with some such law as

$$xy^{\alpha} = \text{constant}; \text{ or, } (x + a)(y + b)^{\alpha} = \text{constant}.$$

EXAMPLE.—The pressure (p) of saturated steam in pounds per square inch when the volume is v cubic feet per pound is

$p =$	10,	20,	30,	40,	50,	60,
$v =$	37·80,	19·72,	13·48,	10·29,	8·34,	6·62.

(Gray's *Smithsonian Physical Tables*, 1896.) Hence, by plotting corresponding values of p and v on logarithmic paper, we get the straight line:

$$\log_{10}p + \alpha\log_{10}v = \log_{10}b; \text{ hence, } pv^{1\cdot065} = 382,$$

since $\log_{10}b = 2{\cdot}5811$, $\therefore b = 382$ and $\alpha = 1{\cdot}065$.

Logarithmic paper is not difficult to make. The gradations on the slide rule give the correct distances without calculation.

A **semi-logarithmic paper** may be made with distances between say the vertical columns in fractions of a metre, while the distances between the horizontal columns are proportional to the logarithms of the numbers. Functions obeying the compound interest law will plot, on such paper, as a straight line. One advantage of logarithmic papers is that the skill required for drawing an accurate freehand curve is not required. The stretched black thread will be found sufficient. With semi-logarithmic paper, either

$$x + \log_{10}y = \text{constant}; \text{ or, } y + \log_{10}x = \text{constant}$$

will give a straight line.

EXAMPLES.—(1) Plot on semi-logarithmic paper Harcourt and Esson's numbers (*l.c.*):

$t =$	2,	5,	8,	11,	14,	17,	27,	31,	35,	44,
$y =$	94·8,	87·9,	81·3,	74·9,	68·7,	64·0,	49·3,	44·0,	39·1,	31·6,

for the amount of substance y remaining in a reacting system after the elapse of an interval of time t. Hence determine values for the constants a and b in

$$y = ae^{-bt}, \text{ i.e., in } \log_{10}y + bt = \log_{10}a,$$

a straight line on "semi-log" paper.

(2) What "law" can you find in Perry's numbers (*Proc. Roy. Soc.*, **23**, 472, 1875),

$$\theta = 58,\quad 86,\quad 148,\quad 166,\quad 188,\quad 202,\quad 210,$$
$$C = 0,\quad \cdot004,\quad \cdot018,\quad \cdot029,\quad \cdot051,\quad \cdot073,\quad \cdot090,$$

for the electrical conductivity C of glass at a temperature of $\theta°$ F.?

(3) Evaluate the constant a in *Arrhenius' formula*, $\eta = a^x$, for the viscosity η of an aqueous solution of sodium benzoate of concentration x, given

$$\eta = 1\cdot6498,\quad 1\cdot2780,\quad 1\cdot1303,\quad 1\cdot0623,$$
$$x = \quad 1,\quad \tfrac{1}{2},\quad \tfrac{1}{4},\quad \tfrac{1}{8}.$$

§ 107. Approximate Integration.

We have seen that the area enclosed by a curve can be estimated by finding the value of a definite integral. This operation may be reversed. The numerical value of a definite integral can be determined from measurements of the area enclosed by the curve. For instance, if the integral $\int f(x) \,.\, dx$ is unknown, the value of $\int_b^a f(x) \,.\, dx$ can be found by plotting the curve $y = f(x)$, erecting ordinates to the curve on the points $x = a$ and $x = b$ and then measuring the surface bounded by the x-axis, the two ordinates just drawn and the curve itself.

This area may be measured by means of the planimeter, an instrument which automatically registers the area of any plane figure when a tracer is passed round the boundary lines.*

Another way is to cut the figure out of a sheet of paper, or other uniform material. Let w_1 be the weight of a known area a_1 and w the weight of the piece cut out. The desired area x can then be obtained by simple proportion,

$$w_1 : a = w : x.$$

Interpolation formulae may be used for the approximate evaluation of any integral between certain limits. The problem may be stated thus: Divide the curve into n portions bounded by $n + 1$ equidistant ordinates $y_0, y_1, y_2, \ldots, y_n$, whose magnitude and common distance apart is known, it is required to find an

* A good description of these instruments will be found in the *British Association's Reports* for 1894, page 496.

approximate expression for the area so divided, that is to say, to evaluate the integral

$$\int_0^n f(x) \, . \, dx.$$

Assuming Newton's interpolation formula we may write,

$$f(x) = y_0 + x\Delta^1{}_0 + \frac{1}{2\,!}x(x - 1)\Delta^2{}_0 + \dots \quad . \quad . \quad (1)$$

$$\therefore \int_0^n f(x) \, . \, dx = y_0\int_0^n dx + \Delta^1{}_0\int_0^n x \, . \, dx + \int_0^n \frac{\Delta^2{}_0}{2\,!}x(x - 1)dx + \dots, \quad (2)$$

which is known as the **Newton-Cotes integration formula.** We may now apply this to special cases, such as *calculating the value of a definite integral from a set of experimental measurements*, etc.

1. *Parabolic Formulae.* Take three ordinates. Reject all terms after $\Delta^2{}_0$. Remember that $\Delta^1{}_0 = y_1 - y_0$ and $\Delta^2{}_0 = y_2 - 2y_1 + y_0$. Let the common difference be unity,

$$\int_0^2 f(x) \, . \, dx = 2y_0 + 2\Delta^1{}_0 + \tfrac{1}{3}\Delta^2{}_0 = \tfrac{1}{3}(y_0 + 2y_1 + y_2). \quad (3)$$

If h represents the common distance of the ordinates apart, we have the familiar result known as **Simpson's one-third rule,** thus,

$$\int_0^2 f(x) \, . \, dx = \tfrac{1}{3}h(y_0 + 4y_1 + y_2). \quad . \quad . \quad (4)$$

A graphic representation will perhaps make the assumptions involved in this formula more apparent.

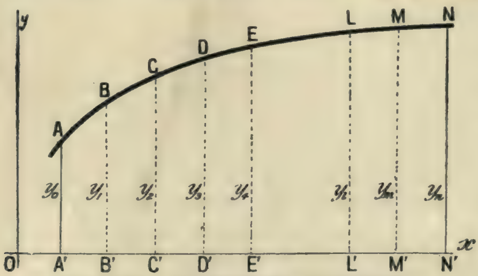


FIG. 105.—Approximate Integration.

Make the construction shown in Fig. 105. We seek the area of the portion $ANN'A'$ corresponding to the integral $f(x) \, . \, dx$ between the limits $x = x_0$ and $x = x_n$, where $f(x)$ represents the equation to the curve $ABC \dots MN$.

Assume that each strip is bounded on one side by a parabolic curve.

Area $CDEE'C'$ = Area of trapezium $CEE'C'$ + Area parabolic segment CED.

From well-known mensuration formulae (15), page 491,

$$CDEE'C' = C'E'[\tfrac{1}{2}(CC' + EE') + \tfrac{2}{3}\{DD' - \tfrac{1}{2}(CC' + EE')\}];$$
$$= 2h(\tfrac{1}{6}CC' + \tfrac{2}{3}DD' + \tfrac{1}{6}EE);$$
$$= \tfrac{1}{3}h(CC' + 4DD' + EE'). \quad . \quad . \quad . \quad . \quad (5)$$

Extend this discussion to include the whole figure,

$$\text{Area } ANN'A' = \tfrac{1}{3}h(1 + 4 + 2 + 4 + \ldots + 2 + 4 + 1), \quad \textbf{(6)}$$

where the successive coefficients of the perpendiculars AA', BB', ... alone are stated; h represents the distance of the strips apart. The greater the number of equal parts into which the area is divided, the more closely will the calculated correspond with true area.

Put $OA' = x_0$; $ON = x_n$; $A'N' = x_n - x_0$ and divide the area into n parts; $h = (x_n - x_0)/n$. Let $y_0, y_1, y_2, \ldots y_n$ denote the successive ordinates erected upon Ox, then equation (6) may be written in the form,

$$\int_{x_0}^{x_n} f(x)\,.\,dx = \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\}. \quad \textbf{(7)}$$

In practical work a great deal of trouble is avoided by making the measurements at equal intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$.

Example.—In measuring the magnitude of an electric current by means of the hydrogen voltameter, let $C_0, C_1, C_2, \ldots$ denote the currents passing through the galvanometer at the times $t_0, t_1, t_2, \ldots$ minutes. The volume of hydrogen liberated (v) will be equal to the product of the intensity of electricity (C ampères), the time (t), and the electrochemical equivalent of the hydrogen x, ($v = xCt$).

Arrange the observations so that the galvanometer is read after the elapse of equal intervals of time. Hence $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \ldots = h$. From (7),

$$\int_{t_0}^{t_n} C\,.\,dt = \tfrac{1}{3}h\{(C_0 + C_n) + 4(C_1 + C_3 + \ldots + C_{n-1}) + 2(C_2 + C_4 + \ldots + C_{n-2})\}.$$

In an experiment, $v = 0{\cdot}22$ when $t = 3$, and

$$t = 1{\cdot}0, \quad 1{\cdot}5, \quad 2{\cdot}0, \quad 2{\cdot}5, \quad 3{\cdot}0, \ \ldots;$$
$$C = 1{\cdot}53, \quad 1{\cdot}03, \quad 0{\cdot}90, \quad 0{\cdot}84, \quad 0{\cdot}57, \ \ldots$$

$$\therefore \int_0^4 C\,.\,dt = \frac{0{\cdot}5}{3}\{(1{\cdot}53 + 0{\cdot}57) + 4(1{\cdot}03 + 0{\cdot}84) + 2 \times 0{\cdot}90\} = 1{\cdot}897.$$

$$\therefore x = {\cdot}22/1{\cdot}897 = 0{\cdot}1159.$$

This example also illustrates how the value of an integral can be obtained from a table of numerical measurements.

The result 0·1159, is better than if we had simply proceeded by what appears, at first sight, the correct method (see (13) below), namely,

$$\int_0^4 C\,.\,dt = (t_1 - t_0)\frac{C_0 + C_1}{2} + (t_2 - t_1)\frac{C_1 + C_2}{2} + \ldots = 1{\cdot}91,$$

for then $$x = {\cdot}22/1{\cdot}91 = 0{\cdot}1152.$$

The correct value is 0·116 nearly.

If we take four instead of the three ordinates in the preceding discussion, we obtain

$$\int_0^3 f(x)\,.\,dx = \tfrac{3}{8}h(y_0 + 3(y_1 + y_2) + y_3), \quad (8)$$

where h denotes the distance of the ordinates apart, $y_0, y_1, \ldots$ the ordinates of the successive perpendiculars in the preceding diagram. This formula is known as **Simpson's three-eighths rule.**

If five, six or seven ordinates are taken, the corresponding formulae are respectively

$$\int_0^4 f(x) \,.\, dx = \tfrac{2}{45}h(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4). \quad . \qquad (9)$$

$$\int_0^5 f(x) \,.\, dx = \tfrac{125}{288}h(\tfrac{19}{25}y_0 + 3y_1 + 2y_2 + 2y_3 + 3y_4 + \tfrac{19}{25}y_5). \quad (10)$$

$$\int_0^6 f(x) \,.\, dx = \tfrac{3}{10}h(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6). \quad (11)$$

The last result, known as **Weddle's rule,** is said to give very accurate results in mensuration problems.

All these formulae are discussed in Boole's *Calculus of Finite Differences* (*l.c.*) under the heading "*Mechanical Quadrature*".

EXAMPLE.—Evaluate the integral $\int x^3 \,.\, dx$ between the limits 1 and 11 by the aid of formula (6), given $h = 1$ and $y_0, y_1, y_2, y_3, \ldots y_8, y_9, y_{10}$ are respectively 1, 8, 27, 64, . . . , 1000, 1331. Compare the result with the absolutely correct value. From (6),

$$\int_1^{11} x^3 \,.\, dx = \tfrac{1}{3}(10970) = 3656\tfrac{2}{3}.$$

By actual integration, the perfect result is,

$$\int_1^{11} x^3 \,.\, dx = \tfrac{1}{4}(11)^4 - \tfrac{1}{4}(1)^4 = 3660.$$

The reader will perhaps have met some of the above formulae in his arithmetic (mensuration).

2. *Trapezoidal Formulae.* Instead of assuming each strip to be the sum of a trapezium and a parabolic segment, we may *suppose that each strip is a complete trapezium.* In Fig. 106, let AN be a curve whose equation is $y = f(x)$; AA', BB', . . . perpendiculars drawn from the x-axis. The area of the portion $ANN'A'$ is to be determined. Let $OB' - OA' = OC' - OB' = \ldots = h$. It follows from known mensuration formulae, (10), page 491,

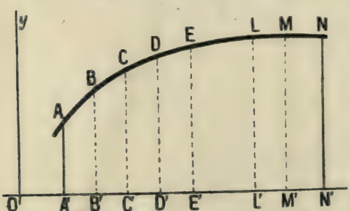


FIG. 106.

$$\text{Area } ANN'A' = \tfrac{1}{2}h(AA' + BB') + \ldots + \tfrac{1}{2}(MM' + NN'),$$
$$= \tfrac{1}{2}h(AA' + 2BB' = 2CC' + \ldots + 2MM' + NN'),$$
$$= h(\tfrac{1}{2} + 1 + 1 + \ldots + 1 + 1 + \tfrac{1}{2}), \quad . \qquad (12)$$

where the coefficients of the successive ordinates alone are written. This result is known as the **trapezoidal rule.**

Let $x_0, x_1, x_2, \ldots, x_n$, be the values of the abscissae corresponding to the ordinates $y_0, y_1, y_2, \ldots, y_n$, then,

$$\int_{x_0}^{x_n} f(x) \, . \, dx = \tfrac{1}{2}(x_1 - x_0)(y_0 + y_1) + \ldots + \tfrac{1}{2}(x_n - x_{n-1})(y_{n-1} + y_n). \quad (13)$$

If $x_1 - x_0 = x_2 - x_1 = \ldots = h$, we get, by multiplying out,

$$\int_{x_0}^{x_n} f(x) \, . \, dx = h\{\tfrac{1}{2}(y_0 + y_n) + y_2 + y_3 + \ldots + y_{n-1}\}. \quad \textbf{(14)}$$ $+ y_1$

The trapezoidal rule, though more easily manipulated, is less accurate than those based on the parabolic formula of Newton and Cotes.

The following expression,

$$\text{Area } ANN'A' = h(\tfrac{5}{12} + \tfrac{13}{12} + 1 + 1 + \ldots + 1 + 1 + \tfrac{13}{12} + \tfrac{5}{12}), \quad (15)$$

or,

$$\int_{x_0}^{x_n} f(x) \, . \, dx = h\{0 \cdot 4(y_0 + y_n) + 1 \cdot 1(y_1 + y_{n-1}) + y_2 + y_3 + \ldots + y_{n-1}\}, \quad (16)$$

is said to combine the accuracy of the parabolic rule with the simplicity of the trapezoidal. It is called **Durand's rule.**

Example.—Evaluate the integral $\int_2^{10} \frac{dx}{x}$, by the approximate formulae (7), (14) and (16), assuming $h = 1$, $n = 8$. Find the absolute value of the result and show that these approximation formulae give more accurate results when the interval h is made smaller. Ansr. (7) gives 1·611, (12) gives 1·629, (15) gives 1·616. The correct result is 1·610.

Lemoine (*Annales de Chimie et de Physique* [4], **27,** 289, 1872) encountered some non-integrable equations during his study of the action of heat on red phosphorus. In consequence, he adopted these methods of approximation. The resulting tables "calculated" and "observed" were very satisfactory. For these, see the cited memoir.

Another method of approximate integration, of special importance in practical work, will now be indicated.

§ 108. Integration by Infinite Series.

It is a very common thing to find expressions not integrable by the ordinary methods at our disposal. We may then resort to the methods of the preceding section, or, if the integral can be expanded in the form of a converging series of ascending or descending powers of x, we can integrate each term of the expanded series separately and thus obtain any desired degree of accuracy by summing up a finite number of these terms.

If $f(x)$ can be developed into a converging series, $f(x) \,.\, dx$ is also convergent. Thus if

$$f(x) = 1 + x + x^2 + x^3 + \ldots + x^{n-1} + x^n + \ldots \qquad (1)$$

$$\int f(x) \,.\, dx = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots + \frac{1}{n}x^n + \frac{1}{n+1}x^{n+1} + \ldots \qquad (2)$$

Series (1) is convergent when x is less than unity, for all values of n. Series (2) is convergent when $nx/(n+1)$ and therefore when x is less than unity. The convergency of the two series thus depends on the same condition. If the one is convergent, the other must be the same.

If the reader is able to develop a function in terms of Taylor's series, this method of integration will require but few words of explanation. One illustration will suffice.

By division, or by Taylor's theorem,

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\therefore \int \frac{dx}{1 + x^2} = \int dx - \int x^2 \,.\, dx + \int x^4 \,.\, dx - \int x^6 \,.\, dx + \ldots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots = \tan^{-1}x.$$

page 229, (6).

EXAMPLES.—(1) Using the approximation of Simpson, (7) preceding section, show that

$$\int_1^2 \frac{dx}{1 + x^2} = \tan^{-1}2 - \tan^{-1}1 = 0{\cdot}321751.$$

Verify the following results.

(2) $\int (1 - x^2)^{-\frac{1}{2}} dx = x + \dfrac{x^3}{2\,.\,3} + \dfrac{1\,.\,3x^5}{2\,.\,4\,.\,5} + \ldots + C.$

(3) $\int \dfrac{dx}{\sqrt{\sin x}} = 2\sqrt{\sin x}\left(1 + \dfrac{1}{2} \cdot \dfrac{\sin^2 x}{5} + \dfrac{1\,.\,3}{2\,.\,4} \cdot \dfrac{\sin^4 x}{9} + \ldots\right) + C.$

(4) $\int e^{-x^2} dx = 1 - \dfrac{x^3}{1\,.\,3} + \dfrac{x^5}{1\,.\,2\,.\,5} - \dfrac{x^7}{1\,.\,2\,.\,3\,.\,7} + \ldots + C.$

(5) $\displaystyle\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}k\right)^2 + \left(\frac{1\,.\,3}{2\,.\,4}k^2\right)^2 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}k^3\right)^2 + \ldots\right\}.$

(6) $\displaystyle\int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{\frac{1}{2}} d\phi = \frac{\pi}{2}\left\{1 - \frac{1}{1}\left(\frac{1}{2}k\right)^2 + \frac{1}{3}\left(\frac{1\,.\,3}{2\,.\,4}k^2\right)^2 + \ldots\right\}.$

(7) How would you propose to integrate $\int (\log_{10} x \,.\, dx)/(1 - x)$ in series? See also pages 188 and 355.

(8) The *result* of the following discussion is required later on. To find a value for the integral

$$\int_0^\infty e^{-x^2} dx. \qquad \ldots \ldots \qquad (3)$$

Integrals of this type are extensively employed in the solution of physical problems. *E.g.*, in the investigation of the path of a ray of light through the atmosphere (Kramp); the conduction of heat (Fourier); the secular cooling of the earth (Kelvin), etc. One solution of the important differential equation

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial x^2},$$

is represented by this integral. See also Chapter VIII., §§ 152 and 154. On account of its paramount importance in the theory of errors of observation (*q.v.*), (3) is sometimes called the **error function**, and written "erf x".

Glaisher (*Phil. Mag.* [4], **42**, 294, 421, 1871) and Pendlebury (*ib.*, p. 437) have given a list of integrals expressible in terms of the error function. The numerical value of any integral which can be reduced to the error function, may then be read off directly from known tables. See Chapter XI., § 180 also Burgess, *Trans. Roy. Soc. Edin.*, **39**, 257, 1898.

NOTE.—The error function (3) may be expressed as a gamma function, $\frac{1}{2}\Gamma(\frac{1}{2})$, or $\frac{1}{2}\sqrt{\pi}$, from (12), § 83.

The following ingenious method of integration is due to Gauss: If a surface has the equation

$$z = e^{-(x^2+y^2)}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the volume included between this surface, the z-plane (for which $z = 0$), the x-plane (between the limits $x = 0$ and $x = \infty$) and the y-plane (between the limits 0 and ∞), is given by the expression,

$$\text{volume} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx \, . \, dy = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy. \quad . \quad (5)$$

Let u denote the integral of the original equation, (3), then, it follows that the volume in (5) $= u^2$.

Again, if we express z in polar coordinates, since $x^2 + y^2 = r^2$, $z = e^{-r^2}$, the area of an element in the z-plane becomes $r \, . \, d\theta \, . \, dr$, instead of $dy \, . \, dx$. In order that the limits may extend over the same part of the solid as before, the integration limits must be transformed so that r extends over 0 and ∞ and θ over 0 and $\frac{1}{2}\pi$. Therefore the volume of the solid in polar coordinates, is

$$\text{volume} = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, . \, d\theta \, . \, dr.$$

Integrate with respect to θ and

$$\text{volume} = \tfrac{1}{2}\pi \int_0^\infty e^{-r^2} r \, . \, dr.$$

Now multiply and divide by -2 and integrate.

$$\therefore \text{volume} = -\tfrac{1}{4}\pi \Big[e^{-r^2} \Big]_0^\infty = \tfrac{1}{4}\pi.$$

$$\therefore u^2 = \tfrac{1}{4}\pi; \text{ or, } u = \tfrac{1}{2}\sqrt{\pi}.$$

By successive reduction (§ 75),

$$\int_0^\infty e^{-x^2} x^n \, . \, dx = \frac{(n-1)(n-3) \, . \, . \, . \, 2}{2^{(n-1)/2}} \int_0^\infty e^{-x^2} x \, . \, dx, \quad . \quad . \quad (6)$$

when n is odd, and

$$\int_0^\infty e^{-x^2} x^n \, . \, dx = \frac{(n-1)(n-3) \, . \, . \, . \, 1}{2^{n/2}} \int_0^\infty e^{-x^2} dx, \quad . \quad . \quad (7)$$

when n is even.

All these integrals are of considerable importance in the kinetic theory of gases and in the theory of probability. Common integrals in the former theory are

$$\frac{2Nm\alpha^2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}x^4 \,.\, dx \text{ and } \frac{2N\alpha}{\sqrt{\pi}}\int_0^\infty e^{-x^2}x^3 \,.\, dx. \quad . \quad . \quad (8)$$

From (7), the first one may be written $\frac{3}{4}Nm\alpha^2$, the latter, $2N\alpha/\sqrt{\pi}$.

If the limits are finite, as, for instance, in the probability integral,

$$P = \frac{2}{\sqrt{\pi}}\int_0^{hx} e^{-h^2x^2}d(hx).$$

Put $hx = t$, then

$$P = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt.$$

Develop e^{-t^2} into a series by Maclaurin's theorem, as just done in example (4) above. The result is that

$$P = \frac{2}{\sqrt{\pi}}\Big(t - \frac{t^3}{1\,.\,3} + \frac{t^5}{1\,.\,2\,.\,5} - \ldots\Big) \quad . \quad . \quad . \quad (9)$$

may be used for small values of t.

For large values, integrate by parts,

$$\int e^{-t^2}dt = -\frac{1}{2t}e^{-t^2} - \frac{1}{2}\int\frac{e^{-t^2}}{t^2}dt,$$

$$= -\frac{1}{2t}e^{-t^2} + \frac{1}{2^2t^3}e^{-t^2} + \frac{3}{2^3}\int\frac{e^{-t^2}}{t^4}dt,$$

$$\therefore -\int e^{-t^2}dt = e^{-t^2}\Big(\frac{1}{2t} - \frac{1}{4t^3} + \frac{3}{8t^5} - \frac{15}{16t^7} + \ldots\Big).$$

From (4), page 185,

$$\therefore \int_0^t e^{-t^2}dt = \int_0^\infty e^{-t^2}dt - \int_t^\infty e^{-t^2}dt.$$

The first integral on the right-hand side $= \frac{1}{2}\sqrt{\pi}$. Integrating the second between the limits ∞ and t

$$P = 1 - \frac{e^{-t^2}}{t\sqrt{\pi}}\Big(1 - \frac{1}{2t^2} + \frac{1\,.\,3}{(2t^2)^2} - \frac{1\,.\,3\,.\,5}{(2t^2)^3} + \ldots\Big). \quad . \quad . \quad (10)$$

This series converges rapidly for large values of t. From this expression the value of P can be found with any desired degree of accuracy.

PART II.

ADVANCED.

CHAPTER VI.

HYPERBOLIC FUNCTIONS.

§ 109. Euler's Exponential Values of the Sine and Cosine.

THERE are certain combinations of the exponential functions which are frequently employed in the various branches of physics. These functions bear the same formal resemblance to one half of a rectangular hyperbola that the circular functions of trignometry do to the circle, hence their name hyperbolic functions.

Hyperbolic functions have now become so incorporated with practical formulae that it is necessary to have at least an elementary knowledge of their properties.

Returning to the imaginary $\sqrt{-1}$, ι has no physical meaning, it is an abstract mathematical concept to which mathematicians have arbitrarily applied the fundamental laws of algebra—distributive law, commutative ("relatively free") law, and the index law. See footnotes, pages 175 and 304.

In § 98, (9) and (11), the following series were developed :—

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots; \; e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \quad (1)$$

If we substitute ιx in place of x (see footnote, page 175), we obtain,

$$e^{\iota x} = 1 + \frac{\iota x}{1} - \frac{x^2}{2!} - \frac{\iota x^3}{3!} + \frac{x^4}{4!} + \frac{\iota x^5}{5!} - \ldots;$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \iota\left(\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right). \quad (2)$$

By reference to page 229, we shall find that the first expression in brackets, is the cosine series, the second, the sine series. Hence,

$$e^{\iota x} = \cos x + \iota \sin x. \quad . \quad . \quad . \quad (3)$$

In the same way, it can be shown that

$$e^{-\iota x} = 1 - \frac{x}{1} - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \dots;$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - \iota\left(\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right). \quad (4)$$

Or,
$$e^{-\iota x} = \cos x - \iota \sin x. \quad . \quad . \quad . \quad (5)$$

Combining equations (3) and (5),

$$\tfrac{1}{2}(e^{\iota x} - e^{-\iota x}) = \iota \sin x; \ \tfrac{1}{2}(e^{\iota x} + e^{-\iota x}) = \cos x. \quad . \quad (6)$$

§ 110. The Derivation of Hyperbolic Functions.

Every point on the perimeter of a circle is equidistant from the centre, therefore, the radius of any given circle has a constant magnitude, whatever portion of the arc be taken.

In plane trignometry, an angle is conveniently measured as a function of the arc of a circle. Thus, if l' denotes the length of an arc of a circle subtending an angle θ at the centre, r' the radius of the circle, then

$$\theta = \frac{\text{arc}}{\text{radius}} = \frac{l'}{r'}.$$

This is called the circular measure of an angle and, for this reason, trignometrical functions are sometimes called **circular functions.**

This property is possessed by no plane curve other than the circle. For instance, the hyperbola, though symmetrically placed with respect to its centre, is not at all points equidistant from it. The same thing is true of the ellipse. The parabola has no centre.

If l denotes the length of the arc of any hyperbola which cuts the x-axis at a distance r from the centre, the ratio

$$u = \frac{l}{r},$$

is called an hyperbolic function of u, just as the ratio l'/r' is a circular function of θ.

To find a value for the ratio $u = l/r$. For the rectangular hyperbola

$$y = \sqrt{(x^2 - a^2)}; \ \therefore dy/dx = x / \sqrt{(x^2 - a^2)}. \quad . \quad (1)$$

The length of any small portion dl of the arc of an hyperbola is, by § 81,

$$dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx = \frac{\sqrt{2x^2 - a^2}}{\sqrt{x^2 - a^2}}dx.$$

The distance r of any point (x, y) from the origin on this curve, is $\sqrt{(x^2 + y^2)}$,

$$\therefore r = \sqrt{2x^2 - a^2}, \therefore \frac{dl}{r} = \frac{dx}{\sqrt{x^2 - a^2}}.$$

If x_1 is the abscissa of any point on the hyperbola, a the abscissa of the starting point,

$$u = \int_a^{x_1} \frac{dl}{r} = \int_a^{x_1} \frac{dx}{\sqrt{x^2 - a^2}} = \log \frac{x_1 + \sqrt{x_1^2 - a^2}}{a}.$$

Put x for x_1 and, remembering that $\log_e e = 1$,

$$u \log e = \log (x + \sqrt{x^2 - a^2})/a;$$

or,

$$e^u = (x + \sqrt{x^2 - a^2})/a;$$

$$\therefore (e^u - x/a)^2 = x^2/a^2 - 1, \text{ or } 2xe^u/a = e^{2u} + 1;$$

$$\therefore \frac{x}{a} = \tfrac{1}{2}(e^u + e^{-u}).$$

But this relation is practically that developed for $\cos x$, (6), of the preceding section, ιx, of course, being written for u. The ratio x/a is defined as the **hyperbolic cosine** of u. It is usually written $\cosh u$ and pronounced "cosh u," or "h-cosine u". Hence,

$$\cosh u = \tfrac{1}{2}(e^u + e^{-u}) = 1 + \frac{u^2}{2!} + \frac{u^2}{4!} + \ldots \quad (2)$$

In the same way, proceeding from (1), it can be shown that

$$\frac{y}{a} = \sqrt{\frac{x^2}{a^2} - 1} = \sqrt{\frac{e^{2u} + 2 + e^{-2u}}{4} - 1} = \sqrt{\frac{e^{2u} - 2 + e^{-2u}}{4}};$$

$$= \tfrac{1}{2}(e^u - e^{-u}),$$

a relation previously developed for $\iota \sin x$. The ratio y/a is called the **hyperbolic sine** of u, written $\sinh u$, pronounced "shin u," or "h-sine u". As before

$$\sinh u = \tfrac{1}{2}(e^u - e^{-u}) = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \ldots \quad (3)$$

The remaining four hyperbolic functions, analogous to the remaining four trignometrical functions, are $\tanh u$ (pronounced "h-tan u," or "tank u"), cosech u, sech u and coth u. Values for each of these functions may be deduced from their relations with $\sinh u$ and $\cosh u$. Thus,

$$\left.\begin{aligned} \tanh u &= \frac{\sinh u}{\cosh u}; \ \operatorname{sech} u = \frac{1}{\cosh u}; \\ \coth u &= \frac{1}{\tanh u}; \ \operatorname{cosech} u = \frac{1}{\sinh u} \end{aligned}\right\} . \quad (4)$$

Unlike the circular functions, the ratios x/a, y/a, when referred to the hyperbola, do not represent angles. *An hyperbolic function* *expresses a certain relation between the coordinates of a given point on the arc of a rectangular hyperbola.*

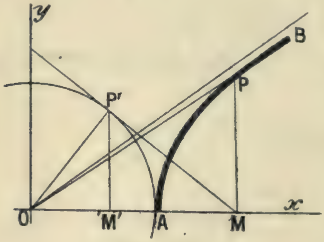


Fig. 107.

Let O (Fig. 107) be the centre of the hyperbola APB, described about the coordinate axes Ox, Oy. From any point $P(x, y)$ drop a perpendicular PM on to the x-axis. Let $OM = x$, $PM = y$, $OA = a$.

$$\therefore \cosh u = x/a \,;\ \sinh u = y/a.$$

For the rectangular hyperbola, $x^2 - y^2 = a^2$.

$$\therefore a^2\sinh^2 u - a^2\cosh^2 u = a^2 \,;\ \text{or, } \sinh^2 u - \cosh^2 u = 1.$$

The last formula thus resembles the well-known,

$$\cos^2 x + \sin^2 x = 1.$$

Draw $P'M$ a tangent to the circle AP' at P. Drop a perpendicular $P'M'$ on to the x-axis. Let the angle $M'OP' = \theta$.

$$\therefore x/a = \sec\theta = \cosh u \,;\ y/a = \tan\theta = \sinh u. \quad . \qquad (5)$$

In example (5), page 279, it is shown that the area $AOP' = \frac{1}{2}a^2\theta$ and of $AOP = \frac{1}{2}a^2 u$. From equations, page 273, it follows that

$$e^u = \cosh u + \sinh u = \sec\theta + \tan\theta.$$

$$u = \log(\sec\theta + \tan\theta) = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta), \quad . \qquad (6)$$

and

$$\tanh\tfrac{1}{2}u = \tan\tfrac{1}{2}\theta. \quad . \quad . \quad . \qquad (7)$$

When θ is connected with u by any of the four relations (5), (6) and (7), θ is said to be the **Gudermannian** of u and written $\theta = \mathrm{gd}\, u$. The Gudermannian function, therefore, connects the circular with the hyperbolic functions.

§ 111. The Graphic Representation of the Hyperbolic Functions.

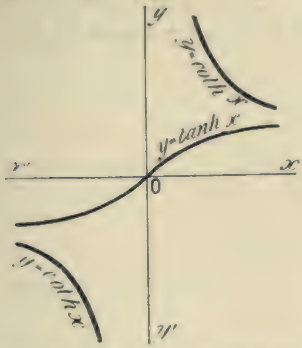
We have seen that the trignometrical sine, cosine, etc., are periodic functions. The hyperbolic functions are exponential, not periodic.* This will be evident from the following diagrams (Figs.

---

* Since $\cos x$ and $\sin x$ are periodic functions, $\cos x + \iota\sin x$ repeats its value every time x is increased by 2π; it therefore follows that $e^{\iota x}$ also repeats its value every time x is increased by 2π. In this particular case $e^{\iota x}$ is said to be an *imaginary periodic function* of x.

To illustrate the periodic nature of the symbol ι, suppose $\sqrt{-1}$ represents the symbol of an operation which when repeated twice changes the sign of the subject of the operation, and when repeated four times restores the subject of the operation to its original form. For instance, if we twice operate on x with $\sqrt{-1}$, we get $-x$, or

108, 109, 110), which represent graphs of the six hyperbolic functions.



108.—Graphs of tanh x and coth x.

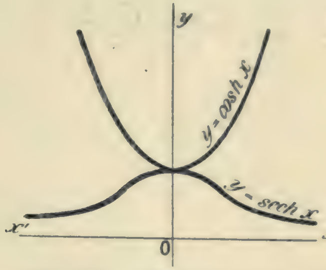


Fig. 109.—Graphs of cosh x and sech x.

$$(\sqrt{-1})^2x = \sqrt{-1}\,.\,\sqrt{-1}\,.\,x = -x\,;$$
$$(\sqrt{-1})^4x = x,$$

and so on in cycles of four. If the imaginary quantities ιx, $-\iota x$, . . . are plotted on the y-axis (*axis of imaginaries*), and the real quantities x, $-x$, . . . on the x-axis (*axis of reals*), the operation of $\sqrt{-1}$ on x will rotate x through 90°, two operations will rotate x through 180°, three operations will rotate x through 270°, and four operations will carry x back to its original position.

Since $2\iota \sin x = e^{\iota x} - e^{-\iota x}$, if $x = \pi$, $\sin \pi = 0$,

$$\therefore\ e^{\iota\pi} - e^{-\iota\pi} = 0\,;\ \text{or,}\ e^{\iota\pi} = e^{-\iota\pi},$$

meaning that the function $e^{\iota x}$ has the same value when $x = \pi$ and when $x = -\pi$. From the last equation,

$$e^{2\iota\pi} = 1.$$

But

$$x = x \times 1 = x \times e^{2\iota\pi} = e^{\log x + 2\iota\pi},$$

which means that the addition of $2\iota\pi$ to the logarithm of any quantity has the effect of multiplying it by unity, and will not change its value. *Every real quantity, therefore, has one real logarithm and an infinite number of imaginary logarithms* differing by $2\iota n\pi$, where n is an integer.

When any function has two or more values for any assigned real or imaginary value of the independent variable, it is said to be a **multiple-valued function.** Such are logarithmic, irrational algebraic, and inverse trignometrical functions. The imaginary values in no way interfere with the ordinary arithmetical ones. A **single-valued function** assumes one single value for any assigned (real or imaginary) value of the independent variable. For example, rational algebraic, exponential and trigonometrical functions are single-valued functions.

There are several interesting relations between $\sin x$ and e^x. Thus, if

$$y = a \sin qt + b \sin qt,\ d^2y/dt^2 = -q^2y\,;\ dy^4/dt^4 = q^4y\,;$$
$$y = e^{at},\qquad d^2y/dt^2 = q^2y\,;\ d^4y/dt^4 = q^4y,\ \text{etc.}$$

The graph

$$y = \cosh x,$$

is known in statics as the "*catenary*". Tanh x and coth x have an imaginary period $\pi\iota$, the remaining hyperbolic functions have the imaginary period $2\pi\iota$.

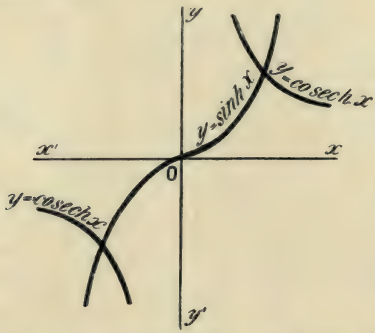


FIG. 110.—Graphs of sinh x and cosech x.

§ 112. Transformation and Conversion Formulae.

(i.) *To pass from trignometrical to hyperbolic functions and vice versâ.* By substituting $\sqrt{-1}\,.\,x$, or, ιx in place of u in equations (2) and (3), § 110, we obtain

$$\cosh \iota x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \cos x. \quad . \quad . \quad (1)$$

$$\sinh \iota x = \iota x + \frac{(\iota x)^3}{3!} + \frac{(\iota x)^5}{5!} + \ldots = \iota \sin x. \quad . \quad (2)$$

Or,

$$\cos x = \tfrac{1}{2}(e^{\iota x} + e^{-\iota x}) = \cosh \iota x. \quad . \quad . \quad . \quad (3)$$

$$\sin x = \frac{1}{2\iota}(e^{\iota x} - e^{-\iota x}) = \frac{1}{\iota}\sinh \iota x. \quad . \quad . \quad . \quad (4)$$

This set of formulae allows the trignometrical and hyperbolic functions to be mutually converted into each other.

(ii.) *Conversion formulae.* Corresponding to the trignometrical formulae there is a great number of relations among the hyperbolic functions, such as

$$\cosh^2 x - \sinh^2 x = 1. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cosh 2x = 1 + 2\sinh^2 x = 2\cosh^2 x + 1. \quad . \quad . \quad (6)$$

$$\sinh x - \sinh y = 2\cosh \tfrac{1}{2}(x + y)\,.\,\sinh \tfrac{1}{2}(x - y), \quad (7)$$

and so on. These have been summarised in the chapter, "Collection of Reference Formulae".

EXAMPLE.—Show

$$\tanh \iota x = \iota \tan x.$$

§ 113. Inverse Hyperbolic Functions.

The inverse hyperbolic functions are defined in the same way as the inverse trignometrical functions, that is to say,

$$\sinh^{-1} y = x,$$

is another way of stating that

$$y = \sinh x.$$

These inverse functions can be expressed as logarithmic functions, since,

$$y = \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

$$\therefore e^{2x} - 2ye^x - 1 = 0.$$

Solve as a quadratic.

$$\therefore e^x = y \pm \sqrt{y^2 + 1}.$$

For real values of x, the negative sign is excluded in the case of $\sinh^{-1} y$, and

$$\therefore \sinh^{-1} y = \log\{y + \sqrt{y^2 + 1}\}. \qquad (1)$$

Similarly $$\cosh^{-1} y = \log\{y \pm \sqrt{y^2 - 1}\}; \qquad (2)$$

Here (Fig. 109) we can use either value.

$$\tanh^{-1} y = \tfrac{1}{2}\log(1 + y)/(1 - y); \qquad (3)$$

$$\coth^{-1} y = \tfrac{1}{2}\log(y + 1)/(y - 1); \qquad (4)$$

$$\operatorname{sech}^{-1} y = \log\{1 + \sqrt{1 - y^2}\}/y; \qquad (5)$$

$$\operatorname{cosech}^{-1} y = \log\{1 + \sqrt{1 + y^2}\}/y. \qquad (6)$$

§ 114. Differentiation and Integration of the Hyperbolic Functions.

The functions may be differentiated in a similar manner to the ordinary trignometrical functions. The symbol $\sqrt{-1}$ is treated as if it were a constant real quantity. Thus, let

$$y = \sinh x = f(x), \therefore f(x + h) = \sinh(x + h).$$

$$\therefore \frac{dy}{dx} = Lt_{h=0}\frac{\sinh(x + h) - \sinh h}{h}.$$

$$= Lt_{h=0}\frac{2\sinh\tfrac{1}{2}h \,.\, \cosh(x + \tfrac{1}{2}h)}{h}.$$

$$= Lt_{h=0}\frac{\sinh\tfrac{1}{2}h}{\tfrac{1}{2}h}\,.\,\cosh(x + \tfrac{1}{2}h).$$

The limit of $\sinh u/u$ when $u = 0$, is unity (page 505), just as in the somewhat analogous $\sin\theta/\theta = 1$, when θ becomes vanishingly small.

$$\therefore dy/dx = d(\sinh x)/dx = \cosh x.$$

This is proved more directly as follows:

$$d(\sinh x)/dx = d\{\tfrac{1}{2}(e^x - e^{-x})\}/dx$$
$$= \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

For the inverse hyperbolic functions, let

$$y = \sinh^{-1}x,$$
$$\therefore dx/dy = \cosh y.$$

From (5), § 112,

$$\cosh y = \sqrt{\sinh^2 y + 1}\ ;\ \therefore \cosh y = \sqrt{x^2 + 1},$$

from the original function to be differentiated.

$$\therefore dy/dx = 1/\sqrt{x^2 + 1}.$$

EXAMPLE.—If $y = \cosh mx + \sinh mx$, show that

$$d^2y/dx^2 = m^2y.$$

A standard collection of results of the differentiation and integration of hyperbolic functions, is set forth in the following table:—

TABLE III.—STANDARD INTEGRALS.

Function.	Differential Calculus.	Integral Calculus.	
$y = \sinh x.$	$\frac{dy}{dx} = \cosh x.$	$\int \cosh x = \sinh x.$.	(1)
$y = \cosh x.$	$\frac{dy}{dx} = \sinh x.$	$\int \sinh x = \cosh x.$.	(2)
$y = \tanh x.$	$\frac{dy}{dx} = \operatorname{sech}^2 x.$	$\int \operatorname{sech}^2 x = \tanh x.$.	(3)
$y = \coth x.$	$\frac{dy}{dx} = -\operatorname{cosech}^2 x.$	$\int \operatorname{cosech}^2 x = -\coth x.$.	(4)
$y = \operatorname{sech} x.$	$\frac{dy}{dx} = -\frac{\sinh x}{\cosh x}.$	$\int \frac{\sinh x}{\cosh x} = -\operatorname{sech} x.$.	(5)
$y = \operatorname{cosech} x.$	$\frac{dy}{dx} = \frac{\cosh x}{\sinh x}.$	$\int \frac{\cosh x}{\sinh x} = -\operatorname{cosech} x.$.	(6)
$y = \sinh^{-1}x.$	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}.$	$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x.$.	(7)
$y = \cosh^{-1}x.$	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}.$	$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x.$.	(8)
$y = \tanh^{-1}x.$	$\frac{dy}{dx} = \frac{1}{1-x^2}, x < 1.$	$\int \frac{dx}{1-x^2} = \tanh^{-1}x.$.	(9)
$y = \coth^{-1}x.$	$\frac{dy}{dx} = \frac{1}{x^2-1}, x > 1.$	$\int \frac{dx}{x^2-1} = \coth^{-1}x.$.	(10)
$y = \operatorname{sech}^{-1}x.$	$\frac{dy}{dx} = \frac{1}{x\sqrt{1-x^2}}.$	$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech} x.$.	(11)
$y = \operatorname{cosech}^{-1}x.$	$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2+1}}.$	$\int \frac{dx}{x\sqrt{x^2+1}} = -\operatorname{cosech} x.$.	(12)

EXAMPLES.—When integrating algebraic expressions involving the square root of a quadratic, hyperbolic functions may frequently be substituted in place of the independent variable. Such equations are very common in electrotechnics. It is convenient to remember, as in § 73, that $x = a \tanh u$, or $x = \tanh u$ may be put in place of $\sqrt{a^2 - x^2}$, or $\sqrt{1 - x^2}$; similarly, $x = a \cosh u$ may be tried in place of $\sqrt{x^2 - a^2}$; $x = a \sinh u$ for $\sqrt{x^2 + a^2}$.

(1) Evaluate $\int \sqrt{x^2 \pm a^2} \, . \, dx$. Substitute $x = a \sinh u$ in $\sqrt{x^2 + a^2}$, and $x = a \cosh u$ in $\sqrt{x^2 - a^2}$.

$$\therefore \int \sqrt{x^2 \pm a^2} \, . \, dx = \tfrac{1}{2}a^2 \int (\cosh^2 u \pm 1) \, . \, du \, ;$$
$$= \tfrac{1}{4}a^2 \sinh 2u \pm \tfrac{1}{2}a^2 u = \tfrac{1}{2}a \sinh u \, . \, a \cosh u \pm \tfrac{1}{2}a^2 u.$$
$$= \tfrac{1}{2}x \sqrt{(x^2 \pm a^2)} \pm \tfrac{1}{2}a^2 \log \{x + \sqrt{(x^2 \pm a^2)}\}/a + C.$$

The "log" terms can be written $\frac{a^2}{2} \sinh^{-1} \frac{x}{a}$ in the one case, and $\frac{a^2}{2} \cosh^{-1} \frac{x}{a}$ in the other. Verify the next three results:

(2) $\displaystyle \int \frac{dx}{\sqrt{(x^2 + a^2)}} = \log \frac{x + \sqrt{(x^2 + a^2)}}{a} = \sinh^{-1} \frac{x}{a} + C.$

(3) $\displaystyle \int \frac{\sqrt{(x^2 + a^2)}dx}{x^2} = \sinh^{-1} \frac{x}{a} - \frac{\sqrt{(x^2 + a^2)}}{x} + C.$ Substitute $x = a \sinh u$.

(4) $\displaystyle \int \frac{x^2 dx}{\sqrt{(x^2 + 1)}} = \tfrac{1}{2}x \sqrt{(x^2 + 1)} - \tfrac{1}{2} \sinh^{-1} x + C.$

(See page 506.)

(5) Find the area of the segment OPA (Fig. 107) of the rectangular hyperbola $x^2 - y^2 = 1$.

Put $x = \cosh u$; $y = \sinh u$. (See (5), § 112.)

$$\therefore \text{Area } APM = \int_1^x y \, . \, dx = \int_0^u \sinh^2 u \, . \, du,$$
$$= \tfrac{1}{2} \int_0^u (\cosh 2u - 1) \, . \, du = \tfrac{1}{4} \sinh 2u - \tfrac{1}{2}u.$$
$$\therefore \text{Area } OPM = \tfrac{1}{2} \text{ Area } PM \, . \, OM - \text{Area } APM = \tfrac{1}{2}u.$$

Note the area of the circular sector $OP'A$ (same figure) $= \tfrac{1}{2}\theta$, where θ is the angle AOP'.

(6) Rectify the *catenary curve* $y = \cosh x/c$ measured from its lowest point. Ansr. $l = c \sinh x/c$. Note $l = 0$ when $x = 0$, $\therefore C = 0$.

(7) Rectify the curve $y^2 = 4ax$ (see example (1), page 187). The expression $\sqrt{(1 + a/x)}dx$ has to be integrated. Hint. Substitute $x = a \sinh^2 u$. $2a\int \cosh^2 u \, . \, du$ remains. Ansr. $= a\int(1 + \cosh 2u)du$, or $a(u + \tfrac{1}{2} \sinh 2u)$. At vertex, where $x = 0$, $\sinh u = 0$, $C = 0$.

Show that the portion bounded by an ordinate passing through the focus has $l = 2{\cdot}296a$. Hint. Diagrams are a great help in fixing limits. Note $x = a$, $\therefore \sinh u = 1$, $\cosh u = \sqrt{2}$, from (5), § 112. From (1), § 113, $\sinh^{-1} 1 = u = \log(1 + \sqrt{2})$. From (20), page 505, $\sinh 2u = 2 \sinh u \, . \cosh u$.

$$l = \Big[u + \tfrac{1}{2} \sinh 2u\Big]_0^u = u + \sinh u \, . \cosh u = \log(1 + \sqrt{2}) + \sqrt{2}.$$

Use Table of Natural Logarithms, Chapter XIII.

(8) Show that $y = A \cosh mx + B \sinh mx$, satisfies the equation of $d^2y/dx^2 = m^2 x$, where m, A and B are undetermined constants. Note the resemblance of this result with a solution of $d^2y/dx^2 = -n^2 x$, which is $y = A \cos nx + B \sin nx$.

§ 115. Demoivre's Theorem.

Refer to the footnote, page 175. Since

$$\cos x_1 = \tfrac{1}{2}(e^{\iota x_1} + e^{-\iota x_1});\ \iota \sin x_1 = \tfrac{1}{2}(e^{\iota x_1} - e^{-\iota x_1}),$$

and $$e^{\iota x_1} = \cos x_1 + \iota \sin x_1;\ e^{-\iota x_1} = \cos x_1 - \iota \sin x_1,$$

if we substitute nx for x_1, where n is any real quantity, positive or negative, integral or fractional,

$$\cos nx = \tfrac{1}{2}(e^{\iota nx} + e^{-\iota nx});\ \sin nx = \tfrac{1}{2}(e^{\iota nx} - e^{-\iota nx}).$$

By addition and subtraction and a comparison with the preceding expressions,

$$\left.\begin{aligned} \cos nx + \iota \sin nx &= e^{\iota nx} = (\cos x + \iota \sin x)^n \\ \cos nx - \iota \sin nx &= e^{-\iota nx} = (\cos x - \iota \sin x)^n \end{aligned}\right\}. \quad \cdot \quad (1)$$

Note $e^x = y$, $(e^x)^n = y^n$, or, $e^{nx} = y^n$.

Equations (1) are known as **Demoivre's theorem.**

Examples.—(1) Verify the following result and compare it with Demoivre's theorem:

$$(\cos x + \iota \sin x)^2 = (\cos^2 x - \sin^2 x) + 2\iota \sin x \,.\, \cos x;$$
$$= \cos 2x + \iota \sin 2x.$$

(2) Show $e^{\alpha + \iota\beta} = e^{\alpha} e^{\iota\beta} = e^{\alpha}(\cos \beta + \iota \sin \beta)$,

(3) Show $\int e^{\alpha x}(\cos \beta x + \iota \sin \beta x)dx = e^{\alpha x}(\cos \beta x + \iota \sin \beta x)/(\alpha + \iota\beta)$;

$$= e^{\alpha x}\frac{(\cos \beta x + \iota \sin \beta x)\,(\alpha - \iota\beta)}{\alpha^2 + \beta^2};$$

$$= e^{\alpha x}\frac{(\alpha \cos \beta x + \beta \sin \beta x) + \iota(-\beta \cos \beta x + \alpha \sin \beta x)}{\alpha^2 + \beta^2} + C.$$

Demoivre's theorem is employed in algebra in the solving of certain cubic equations. The integration of quadratic expressions of the type

$$\frac{Ax + B}{\{(x + a)^2 + b^2\}^n},$$

may sometimes be effected by substituting $x + a = b \tan \theta$; at others, it is recommended to split the quadratic into its so-called conjugate factors, $x + a + \iota b$, and $x + a - \iota b$. Integrate and reduce the result to a real form by means of Demoivre's theorem.

For a fuller discussion on the properties of hyperbolic functions, consult Chrystal's *Textbook of Algebra*, Part ii. (A. & C. Black, London), also Merriman and Woodward's *Higher Mathematics* (Wiley & Sons, New York, 1898), page 107; and Greenhill's *A Chapter in the Integral Calculus* (F. Hodgson, London).

§ 116. Numerical Values of the Hyperbolic Sines and Cosines.

Tables IV. and V. (pages 510 and 511) contain numerical values of the hyperbolic sines and cosines for values of x from 0 to 5, at intervals of 0·01. They have been checked by comparison with *Des Ingenieurs Taschenbuch*, edited by the Hütte Academy (von Ernst & Korn, Berlin, 1877).

The tables are used exactly like the ordinary logarithmic tables.

Numerical values of the other functions can be easily deduced from those of sinh x and cosh x by the aid of equations (4), § 110.

CHAPTER VII.

HOW TO SOLVE DIFFERENTIAL EQUATIONS.

This chapter may be looked upon as a sequel to that on the integral calculus, but of a more advanced character. The "methods of integration" already described will be found ample for most physico-chemical processes, but chemists are proving every day that more powerful methods will soon have to be brought in. As an illustration, I may refer to the set of differential equations which Geitel encountered in his study of the velocity of hydrolysis of the triglycerides by acetic acid (*Journal für praktische Chemie* [2], **55**, 429, 1897).

I have previously pointed out that in the effort to find the relations between phenomena, the attempt is made to prove that if a limited number of hypotheses are prevised, the observed facts are a necessary consequence of these assumptions. The *modus operandi* is as follows:

1. To "anticipate Nature" by means of a "working hypothesis," which is possibly nothing more than a "convenient fiction".

"From the practical point of view," says Professor Rucker (Presidential Address to the B. A. meeting at Glasgow, September, 1901), "it is a matter of secondary importance whether our theories and assumptions are correct, if only they guide us to results in accord with facts. . . . By their aid we can foresee the results of combinations of causes which would otherwise elude us."

2. Thence to deduce an equation representing the momentary rate of change of the two variables under investigation.

3. Then to integrate the equation so obtained in order to reproduce the "working hypothesis" in a mathematical form suitable for experimental verification (see §§ 18, 69, 88, 89, and elsewhere).

So far as we are concerned this is the ultimate object of our integration. By the process of integration we are said to solve the equation.

For the sake of convenience, any equation containing differentials or differential coefficients will, after this, be called a **differential equation.**

§ 117. The Solution of a Differential Equation by the Separation of the Variables.

The different equations hitherto considered have required but little preliminary arrangement before integration. For example, when preparing the equations representing the velocity of a chemical reaction of the general type:

$$dx/dt = kf(x), \quad . \quad . \quad . \quad . \quad (1)$$

we have invariably collected all the x's to one side, the t's, to the other, before proceeding to the integration.

This separation of the variables is nearly always attempted before resorting to other artifices for the solution of the differential equation, because the integration is then comparatively simple. The following examples will serve to emphasise these remarks:

EXAMPLES.—(1) Integrate the equation, $y \,.\, dx + x \,.\, dy = 0$. Rearrange the terms so that

$$\frac{dx}{x} + \frac{dy}{y} = 0; \text{ or, } \int\frac{dx}{x} + \int\frac{dy}{y} = C,$$

by multiplying through with $1/xy$. Ansr. $\log x + \log y = C$.

Two or more apparently different answers may be the same. Thus, the solution of the preceding equation may also be written,

$$\log xy = \log e^C, \textit{ i.e., } xy = e^C; \text{ or } \log xy = \log C', \textit{ i.e., } xy = C'.$$

C and $\log C'$ are, of course, the arbitrary constants of integration.

(2) The equation for the rectilinear motion of a particle under the influence of an attractive force from a fixed point is

$$v \,.\, dv/dx + \mu/x^2 = 0.$$

Solve. Ansr. $\frac{1}{2}v^2 = \mu/x + C$.

(3) Solve $(1 + x^2)dy = \sqrt{y}.dx$. Ansr. $2\sqrt{y} - \tan^{-1}x = C$.

(4) Solve $y - x \,.\, dy/dx = a(y + dy/dx)$. Ansr. $y = C(a + x)^{(1-a)}$.

(5) In consequence of imperfect insulation, the charge on an electrified body is dissipated at a rate proportional to the magnitude E of the charge. Hence show that if a is a constant depending on the nature of the body, and E_0 represents the magnitude of the charge when t (time) $= 0$,

$$E = E_0e^{-at}.$$

Hint. Compound interest law. Integrate by the separation of the variables. Interpret your result.

(6) *Abegg's formula* for the relation between the dielectric constant (D) of a fluid and temperature θ, is

$$-\, dD/d\theta = D/190.$$

Hence show that $D = Ce^{-\theta/190}$, where C is a constant whose value is to be determined from the conditions of the experiment. Put the answer into words.

(7) What curves have a slope $-x/y$ to the x-axis? Ansr. The rectangular hyperbolas $xy = C$. Hint. Set up the proper differential equation and solve.

(8) The relation between small changes of pressure and volume of a gas under adiabatic conditions, is $\gamma pdv + vdp = 0$. Hence show that $pv^\gamma = \text{constant}$.

(9) A lecturer discussing the physical properties of substances at very low temperatures, remarked "it appears that the specific heat of a substance decreases with decreasing temperatures at a rate proportional to the specific heat of the substance itself". Set up the differential equation to represent this "law" and put your result in a form suitable for experimental verification.

(10) *Helmholtz's equation* for the strength of an electric current (C) at the time t, is

$$C = \frac{E}{R} - \frac{L}{R}\frac{dC}{dt},$$

where E represents the electromotive force in a circuit of resistance R and self-induction L. If E, R, L, are constants, show that $RC = E(1 - e^{-Rt/L})$ provided $C = 0$, when $t = 0$.

A substitution will often enable an equation to be treated by this simple method of solution.

EXAMPLE.—Solve $(x - y^2)dx + 2xydy = 0$. Ansr. $xe^{y^2/x} = C$. Hint, put $y^2 = v$, divide by x^2, $\therefore dx/x + d(v/x) = 0$, etc.

If **the equation is homogeneous** in x and y, that is to say, if the sum of the exponents of the variables in each term is of the same degree, a preliminary substitution of $x = ty$, or $y = tx$, according to convenience, will always enable variables to be separated. The rule for the substitution is to treat the differential coefficient which involves the smallest number of terms.

EXAMPLES.—(1) Solve $x + y \,.\, dy/dx - 2y = 0$. Substitute $y = tx$,

$$\therefore \int\frac{tdt}{(1-t)^2} + \int\frac{dx}{x} = C\,;\ \therefore \frac{1}{1-t} + \log(1-t) + \log x = C.$$

Ansr. $(x - y)e^{x/(x-y)} = C$.

(2) If $(y - x)dy + ydx = 0$, $y = Ce^{-x/y}$.

(3) If $x^2dy - y^2dx - xydx = 0$, $x = e^{x/y} + C$.

(4) $(x^2 + y^2)dx = 2xydy$, $x^2 - y^2 = Cx$.

Non-homogeneous equations in x and y can be converted into the homogeneous form by a suitable substitution.

The most general type of a non-homogeneous equation of the first degree is,

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0. \qquad (2)$$

To convert this into an homogeneous equation, assume that

$$x = v + h \text{ and } y = w + k,$$

and substitute in the given equation (2). Thus, we obtain

$$\{av + bw + (ah + bk + c)\}dv + \{a'v + b'w + (a'h + b'k + c')\}dw = 0. \quad (3)$$

Find h and k so that

$$ah + bk + c = 0;\ a'h + b'k + c' = 0.$$

$$\therefore h = \frac{b'c - bc'}{a'b - ab'};\ k = \frac{ac' - a'c}{a'b - ab'}. \quad . \quad . \quad (4)$$

Substitute these values of h and k in (3). The resulting equation

$$(av + bw)dv + (a'v + b'w)dw = 0, \quad . \quad . \quad (5)$$

is homogeneous and, therefore, may be solved as just indicated.

EXAMPLES.—(1) Solve $(3y - 7x - 7)dx + (7y - 3x - 3)dy = 0$. Ansr. $(y - x - 1)^2(y + x + 1)^5 = C$. Hints. From (2), $a = - 7$, $b = 3$, $c = - 7$; $a' = - 3$, $b' = 7$, $c' = - 3$. From (4), $h = - 1$, $k = 0$. Hence, from (3),

$$3wdv - 7vdv + 7wdw - 3vdw = 0.$$

To solve this homogeneous equation, substitute $w = vt$, as above, and separate the variables.

$$\therefore 7\frac{dv}{v} = \frac{3 - 7t}{t^2 - 1}dt;\ \therefore 7\int\frac{dv}{v} + \int\frac{2dt}{t - 1} + \int\frac{5dt}{t + 1} = C.$$

$$\therefore 7 \log v + 2 \log(t - 1) + 5 \log(t + 1) = C;\ \text{or, } v^7(t - 1)^2(t + 1)^5 = C.$$

But $x = v + h$, $\therefore v = x + 1$; $y = w + k$, $\therefore y = w$; $\therefore t = w/v = y/(x + 1)$, etc.

(2) If $(2y - x - 1)dy + (2x - y + 1)dx = 0$, $x^2 - xy + y^2 + x - y = C$.

If in (3),

$$a : b = a' : b' = 1 : m \text{ (say)},$$

h and k are indeterminate, since (2) then becomes,

$$(ax + by + c)dx + \{m(ax + by) + c'\}dy = 0.$$

The denominators in equations (4) also vanish. In this case put

$$z = ax + by$$

and eliminate y, thus, we obtain,

$$a + b\frac{z + c}{mz + c} + \frac{dz}{dx} = 0, \quad . \quad . \quad . \quad (6)$$

an equation which allows the variables to be separated.

EXAMPLES.—(1) Solve $(2x + 3y - 5)dy + (2x + 3y - 1)dx = 0$.

Ansr. $x + y - 4 \log(2x + 3y + 7) = C$.

(2) Solve $(3y + 2x + 4)dx - (4x + 6y + 5)dy = 0$.

Ansr. $9 \log\{(21y + 14x + 22)/7\} - 21(2y - x) = C$.

When the variables cannot be separated in a satisfactory manner, special artifices must be adopted. We shall find it the simplest plan to adopt the routine method of referring each artifice to the particular class of equation which it is best calculated to solve.

These special devices are sometimes far neater and quicker processes of solution than the method just described.

We shall follow the conventional x and y rather more closely than in the earlier part of this work. The reader will know, by this time, that his x and y's, his p and v's and his s and t's are not to be kept in "water-tight compartments".

It is perhaps necessary to make a few general remarks on the nomenclature.

§ 118. What is a Differential Equation?

We have seen that the straight line,

$$y = mx + b, \quad . \quad . \quad . \quad . \quad (1)$$

fulfils two special conditions:

1. It cuts one of the coordinate axes at a distance b from the origin.

2. It makes an angle $\tan \alpha = m$, with the x-axis.

By differentiation.

$$\frac{dy}{dx} = m. \quad . \quad . \quad . \quad . \quad (2)$$

This equation has nothing at all to say about the constant b. That condition has been eliminated. Equation (2), therefore, represents a straight line fulfilling one condition, namely, that it makes an angle $\tan^{-1}m$ with the x-axis.

Now substitute (2) in (1), the resulting equation,

$$y = \frac{dy}{dx}x + b, \quad . \quad . \quad . \quad . \quad (3)$$

in virtue of the constant b, satisfies only one definite condition, (3), therefore, is the equation of any straight line passing through b. Nothing is said about the magnitude of the angle $\tan^{-1}m$.

Differentiate (2). The resulting equation,

$$\frac{d^2y}{dx^2} = 0, \quad . \quad . \quad . \quad . \quad (4)$$

represents any straight line whatever. The special conditions imposed by the constants m and b in (1), have been entirely eliminated. Equation (4) is the most general equation of a straight line possible, for it may be applied to any straight line that can be drawn in a plane.

Let us now find a physical meaning for the differential equation.

In § 7, we have found that the third differential coefficient, d^3s/dt^3 represents "the rate of change of acceleration from moment to moment". Suppose that the acceleration d^2s/dt^2, of a moving body does not change or vary in any way. It is apparent that the rate of change of a constant or uniform acceleration must be zero. In mathematical language, this is written,

$$d^3s/dt^3 = 0. \qquad (5)$$

Now integrate this equation once. We obtain,

$$d^2s/dt^2 = \text{constant, say} = g. \qquad (6)$$

Equation (6) tells us not only that the acceleration is constant, but it fixes that value to the definite magnitude g ft. per second.

But acceleration measures the rate of change of velocity. Integrate (6), we get,

$$ds/dt = gt + C_1. \qquad (7)$$

From § 72, we have learnt how to find the meaning of C_1. Put $t = 0$, then $dx/dt = C_1$. This means that when we begin to reckon the velocity, the body may have been moving with a definite velocity C_1. Let $C_1 = v_0$ ft. per second. Of course, if the body started from a position of rest, $C_1 = 0$.

Now integrate (7) and find the value of C_2 in the result,

$$s = \tfrac{1}{2}gt^2 + v_0t + C_2, \qquad (8)$$

by putting $t = 0$. It is thus apparent that C_2 represents the space which the body had traversed when we began to study its motion. Let $C_2 = s_0$ ft. The resulting equation

$$s = \tfrac{1}{2}gt^2 + v_0t + s_0, \qquad (9)$$

tells us three different things about the moving body at the instant we began to take its motion into consideration.

1. It had traversed a distance of s_0 ft. To use a sporting phrase, if the body is starting from "scratch," $s_0 = 0$.

2. The body was moving with a velocity of v_0 ft. per second.

3. The velocity was increasing at the *uniform* rate of g ft. per second.

Equation (7) tells us the two latter facts about the moving body; equation (6) only tells us the third fact; equation (5) tells us nothing more than that the acceleration is constant. (5), therefore, is true of the motion of any body moving with a uniform acceleration.

EXAMPLE.—If a body falls in the air, experiment shows that the retarding effect of the resisting air is proportional to the square of the velocity of the moving body. Instead of g, therefore, we must write $g - \beta v^2$, where β is the variation constant of page 487. For the sake of simplicity, put $\beta = g/a^2$ and show that

$$v = a\frac{e^{gt/a} - e^{-gt/a}}{e^{gt/a} + e^{-gt/a}};\ s = \frac{a^2}{g}\log\frac{e^{gt/a} + e^{-gt/a}}{2} = \frac{a^2}{g}\log\cosh\frac{gt}{a},$$

since $v = 0$ when $t = 0$, and $s = 0$ when $t = 0$.

Similar reasoning holds good from whatever sources we may draw our illustrations. We are, therefore, able to say that *a differential equation, freed from constants, is the most general way of expressing a natural law.*

Any equation can be freed from its constants by combining it with the various equations obtained by differentiation of the given equation as many times as there are constants. The operation is called **elimination.**

EXAMPLES.—(1) Eliminate the arbitrary constants a and b, from

$$y = ax + bx^2.$$

Differentiate twice and combine the results with the original equation. The result,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

is quite free from the arbitrary restrictions imposed in virtue of the presence of the constants a and b in the original equation.

(2) Eliminate m from $y^2 = 4mx$. Ansr. $y^2 = 2x \,.\, dy/dx$.

(3) Eliminate α and β from $y = \alpha\cos x + \beta\sin x$. Ansr. $d^2y/dx^2 + y = 0$.

(4) Eliminate α and β from $y = \alpha e^{ax} + \beta e^{bx}$.

Ansr. $d^2y/dx^2 - (a + b) \,.\, dy/dx + aby = 0$.

(5) Eliminate k from $dx/dt = k(a - x)$ of § 69. What does the resulting equation mean?

We always *assume* that every differential equation has been obtained by the elimination of constants from a given equation called the **primitive.** In practical work we are not so much concerned with the building up of a differential equation by the elimination of constants from the primitive, as with the reverse operation of finding the primitive from which the differential equation has been derived. In other words, we have to find some relation between the variables which will satisfy the differential equation. Given an expression involving x, y, dx/dy, d^2x/dy^2, . . ., to find an equation containing only x, y and constants which can be reconverted into the original equation by the elimination of the constants.

This relation between the variables and constants which satisfies the given differential equation is called a **general solution**, or a **complete solution**, or a **complete integral** of the differential equation. A solution obtained by giving particular values to the arbitrary constants of the complete solution is a **particular solution.**

Thus $y = mx$ is a complete solution of $y = x \,.\, dy/dx$; $y = x \tan 45°$, is a particular solution.

A differential equation is **ordinary** or **partial**, according as there is one or more than one independent variables present. Ordinary differential equations will be treated first.

Equations like (2) and (3) above, are said to be of the first order, because the highest derivative present is of the first order. For a similar reason (4) and (6) are of the second order, (5) of the third order. The **order of a differential equation,** therefore, is fixed by that of the highest differential coefficient it contains. The **degree of a differential equation** is the highest power of the highest order of differential coefficient it contains. Thus,

$$\frac{d^2y}{dx^2} + k\left(\frac{dy}{dx}\right)^3 + \mu x^4 = 0,$$

is of the second order and third degree.

It is not difficult to show that *the complete integral of a differential equation of the nth order, contains n and only n arbitrary constants.*

We shall first consider equations of the first order.

§ 119. Exact Differential Equations of the First Order.

The reason many differential equations are so difficult to solve is due to the fact that they have been formed by the elimination of constants as well as by the elision of some common factor from the primitive. Such an equation, therefore, does not actually represent the complete or total differential of the original equation or primitive. The equation is then said to be inexact. On the other hand, an **exact differential equation** is one that has been obtained by the differentiation of a function of x and y and performing no other operation involving x and y.

Easy tests were described in §§ 24, 25, to determine whether any given differential equation is exact or inexact. It was pointed out that the differential equation,

$$M \,.\, dx + N \,.\, dy = 0, \qquad . \quad . \quad . \quad (1)$$

is the direct result of the differentiation of any function u, provided,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad . \quad . \quad . \quad . \quad (2)$$

This last result was called the criterion of integrability, because, if an equation satisfies the test, the integration can be readily performed by a direct process. This is not meant to imply that only such equations can be integrated as satisfy the test, for many equations which do not satisfy the test can be solved in other ways.

EXAMPLES.—(1) Apply the test to the equations,

$$ydx + xdy = 0, \text{ and } ydx - xdy = 0.$$

In the former, $M = y$, $N = x$;

$$\therefore \partial M/\partial y = 1, \partial N/\partial x = 1 ; \therefore \partial M/\partial y = \partial N/\partial x.$$

The test is, therefore, satisfied and the equation is exact. In the other equation, $M = y$, $N = - x$,

$$\therefore \partial M/\partial y = 1, \partial N/\partial x = - 1.$$

This does not satisfy the test. In consequence, the equation cannot be solved by the method for exact differential equations.

(2) Is the equation, $(x + 2y)xdx + (x^2 - y^2)dy = 0$, exact? $M = x(x + 2y)$, $N = x^2 - y^2$; $\therefore \partial M/\partial y = 2x$, $\partial N/\partial x = 2x$. The condition is satisfied, the equation is exact.

(3) Show that $(a^2y + x^2)dx + (b^3 + a^2x)dy = 0$, is exact.

(4) Show that $(\sin y + y \cos x)dx + (\sin x + x \cos y)dy = 0$, is exact.

To integrate an equation which satisfies the criterion of integrability, we must remember that M is the differential coefficient of u with respect to x, y being constant, and N is the differential coefficient of u with respect to y, x being constant. Hence we may integrate Mdx on the supposition that y is constant and then treat Ndy as if x were a constant. The complete solution of the whole equation is obtained by equating the sum of these two integrals to an undetermined constant. The complete integral is

$$u = C. \qquad (3)$$

EXAMPLES.—(1) Integrate $x(x + 2y)dx + (x^2 - y^2)dy = 0$, from the preceding set of examples. Since the equation is exact,

$$M = x(x + 2y) ; \; N = x^2 - y^2 ;$$

$$\therefore \int Mdx = \int x(x + 2y)dx = \tfrac{1}{3}x^3 + x^2y = Y,$$

where Y is the integration constant which may, or may not, contain y, because y has here been regarded as a constant.

Now the result of differentiating

$$\tfrac{1}{3}x^3 + x^2y = Y,$$

should be the original equation. On trial,

$$x^2dx + 2xydx + x^2dy = dY.$$

On comparison with the original equation, it is apparent that

$$dY = y^2dy ; \; \therefore Y = \tfrac{1}{3}y^3 + C.$$

Substitute this in the preceding result. The complete solution is, therefore,

$$\tfrac{1}{3}x^3 + x^2y - \tfrac{1}{3}y^3 = C.$$

To summarise: The method detailed in the example just given may be put into a more practical shape.

To integrate an exact differential equation, first find $\int M.dx$ on the assumption that y is constant and substitute the result in

$$\int Mdx + \int\Big(N - \frac{\partial}{\partial y}\int Mdx\Big)dy = C. \quad . \quad . \quad . \quad (4)$$

With the old example, therefore, having found $\int Mdx$, we may write down at once

$$\frac{1}{3}x^3 + x^2y + \int\Big\{x^2 - y^2 - \frac{\partial}{\partial y}\Big(\frac{1}{3}x^3 + x^2y\Big)\Big\}dy = C.$$

$$\therefore \tfrac{1}{3}x^3 + x^2y + \int(x^2 - y^2 - x^2)dy = C.$$

And the old result follows directly. If we had wished we could have used

$$\int Ndy + \int\Big(M - \frac{\partial}{\partial x}\int Ndy\Big)dx = C,$$

in place of (4).

In practice it is often convenient to modify this procedure. If the equation satisfies the criterion of integrability, we can easily pick out terms which make $Mdx + Ndy = 0$, and get

$$Mdx + Y \text{ and } Ndy + X,$$

where Y cannot contain x and X cannot contain y. Hence if we find Mdy and Ndx, the functions X and Y will be determined.

In the above equation, the only terms containing x and y are $2xydx + x^2dy$, which obviously have been derived from x^2y. Hence integration of these and the omitted terms gives the above result.

(2) Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$. Pick out terms in x and y, we get

$$-(4xy + 2y^2)dx - (4xy + 2x^2)dy = 0.$$

Integrate. $\quad \therefore -2x^2y - 2xy^2 = \text{constant}.$

Pick out the omitted terms and integrate for the complete solution. We get,

$$\int x^2dx + \int y^2dy - 2x^2y - 2xy^2 = x^3 - 6x^2y - 6xy^2 + y^3 = \text{constant}.$$

(3) Show that the solution of $(a^2y + x^2)dx + (b^3 + a^2x)dy = 0$, is

$$a^2xy + b^3y + \tfrac{1}{3}x^3 = C. \quad \text{Use (4).}$$

(4) Solve $(x^2 - y^2)dx - 2xydy = 0$. Ansr. $\frac{1}{3}x^2 - y^2 = C/x$. Use (4).

Equations made exact by means of integrating factors. As just pointed out, the reason any differential equation does not satisfy the criterion of exactness, is because the "integrating factor" has been cancelled out during the genesis of the equation from its primitive.

If, therefore, the equation

$$Mdx + Ndy = 0,$$

does not satisfy the criterion of integrability, it will do so when the factor, previously divided out, is restored. Thus, the preceding equation is made exact by multiplying through with the integrating factor μ. Hence,

$$\mu(Mdx + Ndy) = 0,$$

satisfies the criterion of exactness, and the solution can be obtained as described above.

§ 120. How to find Integrating Factors.

Sometimes integrating factors are so simple that they can be detected by simple inspection.

EXAMPLES.—(1) $ydx - xdy = 0$ is inexact. It becomes exact by multiplication with either x^{-2}, $x^{-1} . y^{-1}$, or y^{-2}.

(2) In $(y - x)dy + ydx = 0$, the term containing $ydx - xdy$ is not exact, but becomes so when multiplied as in the preceding example.

$$\therefore \frac{dy}{y} - \frac{xdy - ydx}{y^2} = 0 ; \text{ or } \log y - \frac{x}{y} = C.$$

For the general theorems concerning the properties of integrating factors, the reader must consult some special treatise, say Boole's *A Treatise on Differential Equations*, pages 55 *et seq.*, 1865.

We have already established, in § 26, that an integrating factor always exists which will make the equation

$$Mdx + Ndy = 0,$$

an exact differential.

Moreover, there is also an infinite number of such factors, for if the equation is made exact when multiplied by μ, it will remain exact when multiplied by any function of μ.

The different integrating factors correspond to the various forms in which the solution of the equation may present itself. For instance, the integrating factor $x^{-1}y^{-1}$, of $ydx + xdy = 0$, corresponds with the solution $\log x + \log y = C$. The factor y^{-2} corresponds with the solution $xy = C'$.

Unfortunately, it is of no assistance to know that every differential equation has an infinite number of integrating factors. No general practical method is known for finding them. Here are a few elementary rules applicable to special cases.

Rule I. Since

$$d(x^my^n) = x^{m-1}y^{n-1}(mydx + nxdy),$$

an expression of the type $mydx + nxdy = 0$, has an integrating factor $x^{m-1}y^{n-1}$; or, the expression

$$x^\alpha y^\beta(mydx + nxdy) = 0, \quad . \quad . \quad . \quad (1)$$

has an integrating factor

$$x^{m-1-\alpha}y^{n-1-\beta},$$

or more generally still,

$$x^{km-1-\alpha}y^{kn-1-\beta}, \quad . \quad . \quad . \quad (2)$$

where k may have any value whatever.

EXAMPLE.—Find an integrating factor of $ydx - xdy = 0$. Here, $\alpha = 0$, $\beta = 0$, $m = 1$, $n = -1$ $\therefore$ y^{-2} is an integrating factor of the given equation.

If the expression can be written

$$x^{\alpha}y^{\beta}(mydx + nxdy) + x^{\alpha'}y^{\beta'}(m'ydx + n'xdy) = 0, \quad . \qquad (3)$$

the integrating factor can be readily obtained, for

$$x^{km-1-\alpha}y^{kn-1-\beta}; \text{ and } x^{k'm'-1-\alpha'}y^{k'n'-1-\beta'},$$

are integrating factors of the first and second members respectively. In order that these factors may be identical,

$$km - 1 - \alpha = k'm' - 1 - \alpha'; \; kn - 1 - \beta = k'n' - 1 - \beta'.$$

Values of k and k' can be obtained to satisfy these two conditions by solving these two equations. Thus,

$$k = \frac{n'(\alpha - \alpha') - m'(\beta - \beta')}{mn' - m'n}; \; k' = \frac{n(\alpha - \alpha') - m(\beta - \beta')}{mn' - m'n}. \qquad (4)$$

EXAMPLES.—(1) Solve $y^3(ydx - 2xdy) + x^4(2ydx + xdy) = 0$. Hints. Show that $\alpha = 0$, $\beta = 3$, $m = 1$, $n = -2$; $\alpha' = 4$, $\beta' = 0$, $m' = 2$, $n' = 1$; $\therefore$ $x^{k-1}y^{-2k-4}$ is an integrating factor of the first, $x^{2k'-5}y^{k'-1}$ of the second member. Hence, from (4), $k = -2$, $k' = 1$, $\therefore$, x^{-3} is an integrating factor of the whole expression. Multiply through and integrate for $2x^4y - y^4 = Cx^2$.

(2) Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$. Ansr. $x^2y^2(y^2 - x^2) = C$. Integrating factor deduced after rearranging the equation is xy.

Rule II. *If the equation is homogeneous and of the form:* $Mdx + Ndy = 0$, *then* $(Mx + Ny)^{-1}$ *is an integrating factor.*

Let the expression

$$Mdx + Ndy = 0,$$

be of the mth degree and μ an integrating factor of the nth degree,

$$\therefore \mu Mdx + \mu Ndy = du, \quad . \quad . \quad . \qquad (5)$$

is of the $(m+n)$th degree, and the integral u is of the $(m+n+1)$th degree.

By Euler's theorem, § 22,

$$\therefore \mu Mx + \mu Ny = (m + n + 1)u. \quad . \quad . \qquad (6)$$

Divide (5) by (6),

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{m + n + 1} \cdot \frac{du}{u}.$$

The right side of this equation is a complete differential, consequently, the left side is also a complete differential. Therefore, $(Mx + Ny)^{-1}$ has made $Mdx + Ndy = 0$ an exact differential equation.

EXAMPLES.—(1) Show that $(x^3y - xy^3)^{-1}$ is an integrating factor of $(x^2y + y^3)dx - 2xy^2dy = 0$.

(2) Show that $1/(x^2 - nyx + y^2)$ is an integrating factor of $ydy + (x - ny)dx = 0$.

The method, of course, cannot be used if $Mx + Ny$ is equal to zero. In this case, we may write $y = Cx$, a solution.

Rule III. *If the equation is of the form,*

$$f_1(x, y)ydx + f_2(x, y)xdy = 0,$$

then $(Mx - Ny)^{-1}$ *is an integrating factor.*

Example.—Solve $(1 + xy)ydx + (1 - xy)xdy = 0$. Hint. Show that the integrating factor is $1/2x^2y^2$. Divide out $\frac{1}{2}$. $\therefore \int Mdx = 1/xy + \log x$. Ansr. $x = Cye^{-1/xy}$.

If $Mx - Ny = 0$, the method fails and $xy = C$ is then a solution of the equation. *E.g.*, $(1 + xy)ydx + (1 + xy)xdy = 0$.

Rule IV. *If* $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ *is a function of x only,* $e^{\int f(x)dx}$ *is an integrating factor. Or, if* $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$, *then* $e^{\int f(y)dy}$ *is an integrating factor.* These are important results.

Examples.—(1) Solve $(x^2 + y^2)dx - 2xydy = 0$. Ansr. $x^2 - y^2 = Cx$. Hint. Show $f(x) = -2/x$. The integrating factor is, therefore,

$$e^{-\int 2dx/x} = e^{\log 1/x^2} = 1/x^2.$$

(Why?) Prove that this is an integrating factor, and solve as in the preceding section.

(2) Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$. Ansr. $xy^3 + y^4 + 2x = Cy^2$.

(3) We may prove the rule for a special case in the following manner. The steps will serve to recall some of the principles established in some earlier chapters.

Let
$$\frac{dy}{dx} + Py = Q, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where P and Q are either constant or functions of x. Let μ be an integrating factor which makes

$$dy + (Py - Q)dx = 0, \quad . \quad . \quad . \quad . \quad (8)$$

an exact differential.

$$\therefore \mu dy + \mu(Py - Q)dx \equiv Ndy + Mdx.$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial \mu}{\partial x}; \ \frac{\partial M}{\partial y} = (Py - Q)\frac{\partial \mu}{\partial y} + P\mu.$$

$$\therefore \frac{\partial \mu}{\partial x} = (Py - Q)\frac{\partial \mu}{\partial y} + P\mu.$$

$$\therefore \frac{\partial \mu}{\partial x}dx = (Py - Q)\frac{\partial \mu}{\partial y}dx + P\mu dx;$$

$$= -\frac{\partial \mu}{\partial y}dy + P\mu dx.$$

$$\therefore \frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy = d\mu = P\mu dx.$$

$$\therefore P = \frac{1}{\mu}\frac{d\mu}{dx}; \ \therefore \int Pdx = \log \mu.$$

and since $\log_e e = 1$.

$$(\int Pdx)\log e = \log \mu; \ \therefore \mu = e^{\int Pdx}. \quad . \quad . \quad . \quad (9)$$

This result will be employed in dealing with linear equations, § 122.

§ 121. The First Law of Thermodynamics.

According to the discussion at the end of the first chapter, one way of stating the first law of thermodynamics is as follows:

$$dQ = dU + dW,$$

which means that when a quantity of heat, dQ, is added to a substance, one part of the heat is spent in changing the internal energy, dU, of the substance and another part, dW, is spent in doing work against external forces. In the special case, when that work is expansion against atmospheric pressure, $dW = p \,.\, dv$, as shown in § 91. See (11), page 524.

We know that the condition of a substance is completely defined by any two of the three variables p, v, θ, because when any two of these three variables is known, the third can be deduced from the relation

$$pv = R\theta.$$

Hence it is assumed that the internal energy of the substance is completely defined when any two of these variables are known.

Now let the substance pass from any state A to another state B (Fig. 111). The internal energy of the substance in the state B is completely determined by the coordinates of that point, because U is quite independent of the nature of the transformation from the state A to the state B. It makes no difference to the magnitude of U whether that path has been *viâ* APB or AQB. In this case U is said to be a single-valued function completely defined by the coordinates of the point corresponding to any given state.

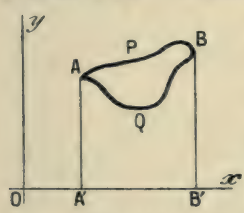


FIG. 111.

In other words, dU is a complete differential. Hence

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy,$$

is an exact differential equation, where x and y represent any pair of the variables p, v, θ.

On the other hand, the external work done during the transformation from the one state to another, depends not only on the initial and final states of the substance, but also on the nature of the path described in passing from the state A to the state B. For example, the substance may perform the work represented by the area $AQBB'A'$ or by the area $APBB'A'$, in its passage from the state A to the state B. In fact the total work done in the passage from A to B and back again, is represented by the area $APBQ$ (page 183). In order to know the work done during the passage from the state A to the state B, it is not only necessary to know the initial and final states of the substance as defined by the coordinates of the points A and B, but we must know the nature of the path from the one state to the other.

Similarly, the quantity of heat supplied to the body in passing from one state to the other, not only depends on the initial and final states of the substance but also on the nature of the transformation.

All this is implied when it is said that "dW and dQ are not perfect differentials". Although we can write

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x},$$

we must put, in the case of W or Q,

$$\frac{\partial^2 Q}{\partial x \partial y} \lesseqgtr \frac{\partial^2 Q}{\partial y \partial x}; \text{ or } \frac{\partial^2 Q\mu}{\partial x \partial y} = \frac{\partial^2 Q\mu}{\partial y \partial x}.$$

Therefore the partial differentiation of x with respect to y, furnishes a complete differential equation only when we multiply through with the integrating factor μ, so that

$$\mu dQ = \mu\frac{\partial Q}{\partial x}dx + \mu\frac{\partial Q}{\partial y}dy,$$

where x and y may represent any pair of the variables p, v, θ.

The integrating factor is proved in thermodynamics to be equivalent to the so-called *Carnot's function* (see Preston's *Theory of Heat*).

To indicate that dW and dQ are not perfect differentials, some writers superscribe a comma to the top right-hand corner of the differential sign. The above equation would then be written,

$$d'Q = dU + d'W.$$

§ 122. Linear Differential Equations of the First Order.

A linear differential equation of the first order involves only the first power of the dependent variable y and of its first differential coefficients. The general type is,

$$\frac{dy}{dx} + Py = Q, \quad . \quad . \quad . \quad . \quad (1)$$

where P and Q may be functions of x, or constants.

We have just proved that $e^{\int Pdx}$ is an integrating factor of (1), therefore

$$e^{\int Pdx}(dy + Pydx) = e^{\int Pdx}Qdx,$$

is an exact differential equation. The general solution is,

$$ye^{\int Pdx} = \int e^{\int Pdx}Qdx + C. \quad . \quad . \quad . \quad (2)$$

The linear equation is one of the most important in applied mathematics. In particular cases the integrating factor may assume a very simple form.

In the following examples, remember that $e^{\log x} = x$, $\therefore$ if $\int Pdx = \log x$, $e^{\int Pdx} = x$.

EXAMPLES.—(1) Solve $(1 + x^2)dy = (m + xy)dx$. Reduce to the form (1) and we obtain

$$\frac{dy}{dx} - \frac{x}{1 + x^2}y = \frac{m}{1 + x^2}.$$

$$\therefore \int Pdx = - \int \frac{xdx}{1 + x^2} = - \tfrac{1}{2}\log(1 + x^2) = - \log\sqrt{(1 + x^2)}.$$

Remembering $\log 1 = 0$, $\log e = 1$, the integrating factor is evidently,

$$\log e^{\int Pdx} = \log 1 - \log\sqrt{1 + x^2}, \text{ or } e^{\int Pdx} = \frac{1}{\sqrt{(1 + x^2)}}.$$

Multiply the original equation with this integrating factor, and solve the resulting exact equation as § 119, (4), or, better still, by (2) above. The solution: $y = mx + C\sqrt{(1 + x^2)}$ follows at once.

(2) *Ohm's law for a variable current* flowing in a circuit with a coefficient of self-induction L (henries), a resistance R (ohms), and a current of C (ampères) and an electromotive force E (volts), is given by the equation,

$$E = RC + L\frac{dC}{dt}.$$

This equation has the standard linear form (1). If E is constant, show that the solution is,

$$C = E/R + Be^{-Rt/L},$$

where B is the arbitrary constant of integration (page 159). Show that C approximates to E/R after the current has been flowing some time (t). Hint for solution. Integrating factor is $e^{Rt/L}$.

(3) The equation of motion of a particle subject to a resistance varying directly as the velocity and as some force which is a given function of the time, is

$$dv/dt + kv = f(t).$$

Show that $$v = Ce^{-kt} + e^{-kt}\int e^{kt}f(t)dt.$$

If the force is gravitational, say g,

$$v = Ce^{-kt} + g/k.$$

(4) Solve $xdy + ydx = x^3dx$. Integrating factor $= x$. Ansr. $y = \frac{1}{4}x^3 + C/x$.

Many equations may be transformed into the linear type of equation, by a change in the variable. Thus, in the so-called **Bernoulli's equation,**

$$dy/dx + Py = Qy^n. \quad . \quad . \quad . \quad . \quad (3)$$

Divide by y^n, multiply by $(1 - n)$ and substitute $y^{1-n} = v$, in the result. Thus,

$$\frac{(1-n)}{y^n}\frac{dy}{dx} + (1-n)Py^{1-n} = (1-n)Q,$$

and
$$dv/dx + (1-n)Pv = Q(1-n),$$

which is linear in v. Hence, the solution is

$$ve^{(1-n)\int Pdx} = (1-n)\int Qe^{(1-n)\int Pdx}dx + C.$$

$$\therefore y^{1-n}e^{(1-n)\int Pdx} = (1-n)\int Qe^{(1-n)\int Pdx}dx + C.$$

EXAMPLES.—(1) Solve $dy/dx + y/x = y^2$. Treat as above, substituting $v = 1/y$. The integration factor is $e^{-\int dx/x} = e^{-\log x} = 1/x$.

Ansr. $Cxy - xy \log x = 1$.

(2) Solve $dy/dx + x \sin^2 y = x^3 \cos^2 y$. Divide by $\cos^2 y$. Put $\tan y = v$. The integration factor is $e^{\int 2xdx}$, *i.e.*, e^{x^2}. Ansr. $e^{x^2}\tan y - \frac{1}{2}e^{x^2}(x^2 - 1) = C$. Hint to solve $ve^{x^2} = \int x^3 e^{x^2}dx + C$. Put $x^2 = z$, $\therefore 2xdx = dz$, and this integral becomes $\frac{1}{2}\int ze^z dz$, or $\frac{1}{2}e^z(z-1)$, etc.

(3) Here is an instructive differential equation, which Harcourt and Esson encountered during their work on chemical dynamics in '66.

$$\frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{K}{y} - \frac{K}{x} = 0.$$

I shall give a method of solution in full, so as to revise some preceding work. The equation has the same form as Bernoulli's. Therefore, substitute

$$v = \frac{1}{y};\ i.e., \frac{dv}{dx} = -\frac{1}{y^2}\cdot\frac{dy}{dx}.$$

$$\therefore \frac{dv}{dx} - Kv + \frac{K}{x} = 0,$$

an equation linear in v. The integrating factor is

$$e^{\int Pdx}, \text{ or, } e^{-Kx};\ Q, \text{ in (2)}, = -K/x,$$

therefore, from (2)
$$ve^{-Kx} = -\int\frac{K}{x}e^{-Kx}dx + C.$$

From § 108,

$$ve^{-Kx} = -K\int\frac{1}{x}\left\{1 - (Kx) + \frac{(Kx)^2}{1.2} - \frac{(Kx)^3}{1.2.3} + \dots\right\}dx + C.$$

$$\therefore ve^{-Kx} = -K\int\left\{\frac{dx}{x} - Kdx + \frac{K^2xdx}{1.2} - \frac{K^3x^2dx}{1.2.3} + \dots\right\} + C.$$

But $v = 1/y$. Multiply through with ye^{Kx}, and integrate.

$$1 = Ke^{Kx}\left\{C_1 - \log x + Kx - \frac{(Kx)^2}{1.2^2} + \frac{(Kx)^3}{1.2.3^2} - \dots\right\}y.$$

We shall require this result on page 333.

§ 123. Differential Equations of the First Order and of the First or Higher Degree.—Solution by Differentiation.

Case i. *The equation can be split up into factors.* If the differential equation can be resolved into n factors of the first degree, equate each factor to zero and solve each of the n equations separately. The n solutions may be left either distinct, or combined into one.

EXAMPLES.—(1) Solve $x(dx/dy)^2 = y$. Resolve into factors of the first degree,

$$dx/dy = \pm \sqrt{y/x}.$$

Separate the variables and integrate,

$$\sqrt{x} \pm \sqrt{y} = \pm \sqrt{C},$$

which, on rationalisation, becomes

$$(x - y)^2 - 2C(x + y) + C^2 = 0.$$

Geometrically this equation represents a system of parabolic curves each of which touches the axis at a distance C from the origin. The separate equations of the above solution merely represent different branches of the same parabola.

(2) Solve $xy(dy/dx)^2 - (x^2 - y^2)dy/dx - xy = 0$. Ansr. $xy = C$, or $x^2 - y^2 = C$. Hint. Factors $(xp + y)(yp - x)$, where $p \equiv dy/dx$.

(3) Solve $(dy/dx)^2 - 7dy/dx + 12 = 0$. Ansr. $y = 4x + C$, or $3x + C$.

Case ii. *The equation cannot be resolved into factors, but it can be solved for x, y, dy/dx, or y/x.* An equation which cannot be resolved into factors, can often be expressed in terms of x, y, dy/dx, or y/x, according to circumstances. The differential coefficient of the one variable with respect to the other may be then obtained by solving for dy/dx and using the result to eliminate dy/dx from the given equation.

EXAMPLES.—(1) Solve $dy/dx + 2xy = x^2 + y^2$. Since $(x - y)^2 = x^2 - 2xy + y^2$,

$$y = x + \sqrt{dy/dx}.$$

Differentiate
$$dy/dx = 1 + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)\Big/\left(\frac{dy}{dx}\right)^{\frac{1}{2}}.$$

Separate the variables x and p, where $p \equiv dy/dx$, and solve for dy/dx,

$$x = \frac{1}{2}\log\frac{\sqrt{p} - 1}{\sqrt{p} + 1} + \log C; \quad \sqrt{\frac{dy}{dx}} = \frac{C + e^{2x}}{C - e^{2x}}.$$

$\therefore$ Ansr. $y = x + (C + e^{2x})/(C - e^{2x})$.

(2) Solve $x(dy/dx)^2 - 2y(dy/dx) + ax = 0$. Ansr. $y = \frac{1}{2}C(x^2 + a/C)$. Hint. Substitute for p. Solve for y and differentiate. Substitute pdx for dy, and clear of fractions. The variables p and x can be separated. Integrate. $p = xC$. Substitute in the given equation for the answer.

(3) Solve $y(dy/dx)^2 + 2x(dy/dx) - y = 0$. Ansr. $y^2 = C(2x + C)$. Hint. Solve for x. Differentiate and substitute dy/p for dx, and proceed as in example (2). $yp = C$, etc.

Case iii. *The equation cannot be resolved into factors, x or y is absent.* If x is absent solve for dy/dx or y according to convenience; if y is absent, solve for dx/dy or x. Differentiate the result with respect to the absent letter if necessary and solve in the regular way.

EXAMPLES.—(1) Solve $(dy/dx)^2 + x(dy/dx) + 1 = 0$. For the sake of greater ease, substitute p for dx/dy. The given equation thus reduces to

$$x = p + 1/p. \qquad (1)$$

Differentiate with regard to the absent letter y, thus,

$$p = (1 - 1/p^2)dp/dy\,;\ \text{or},\ dy/dp = 1/p - 1/p^3.$$

$$\therefore y = \log p + 1/2p^2 + C. \qquad (2)$$

Combining (1) and (2), we get the required solution.

(2) Solve $dy/dx = y + 1/y$. Ansr. $y^2 = Ce^{2x} - 1$.

(3) Solve $dy/dx = x + 1/x$. Ansr. $y = \frac{1}{2}x^2 + \log x + C$.

§ 124. Clairaut's Equation.

The general type of this equation is,

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right); \qquad (1)$$

or, writing $p = dy/dx$, for the sake of convenience,

$$y = px + f(p). \qquad (2)$$

Many equations of the first degree in x and y can be reduced to this form by a more or less obvious transformation of the variables, and solved in the following way :—

Differentiate (2) with respect to x, and equate the result to zero.

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}\,;\ \text{or},\ \{x + f'(p)\}\frac{dp}{dx} = 0.$$

Hence either $\quad \dfrac{dp}{dx} = 0\,;$ or, $x + f(p) = 0$.

If the former,

$$dp/dx = 0\,;\ \therefore p = C,$$

where C is an arbitrary constant. Hence,

$$dy = Cdx\,;\ \text{or},\ y = Cx + f(C),$$

is a solution of the given equation.

Again, p in $x + f(p)$ may be a solution of the given equation. To find p, eliminate p between

$$y = px + f(p),\ \text{and}\ x + f'(p) = 0.$$

The resulting equation between x and y also satisfies the given equation.

There are thus two classes of solutions to Clairaut's equation.

EXAMPLES.—Find both solutions in the following equations :—

(1) $y = px + p^2$. Ansr. $Cx + C^2 = y$ and $x^2 + 4y = 0$.

(2) $(y - px)(p - 1) = p$. Ansr. $(y - Cx)(C - 1) = C$; $\sqrt{y} + \sqrt{x} = 1$. Read over § 67.

§ 125. Singular Solutions.

Clairaut's equation introduces us to a new idea. Hitherto we have assumed that whenever a function of x and y satisfies an equation, that function plus an arbitrary constant, represents the complete or general solution. We now find that functions of x and y can sometimes be found to satisfy the given equation, which, unlike the particular solution, are not included in the general solution.

This function must be considered *a* solution, because it satisfies the given equation. But the existence of such a solution is quite an accidental property confined to special equations, hence their cognomen, **singular solutions.**

To take the simple illustration of page 142,

$$y = px + a/p. \quad . \quad . \quad . \quad . \quad (1)$$

Remembering that p has been written for dy/dx, differentiate with respect to x, we get, on rearranging terms,

$$(x - a/p^2)dp/dx = 0,$$

where either $x - a/p^2 = 0$; $dp/dx = 0$.

If the latter,

$$p = C; \text{ or, } y = Cx + a/C. \quad . \quad . \quad . \quad (2)$$

If the former, $p = \sqrt{a/x}$, which gives, when substituted in (1), the solution,

$$y^2 = 4ax. \quad . \quad . \quad . \quad . \quad (3)$$

This solution is not included in the general solution, but yet it satisfies the given equation. (3) is the singular solution of (1).

Equation (2), the complete solution of (1), has been shown to represent a system of straight lines which differ only in the value of the arbitrary constant C; equation (3), containing no arbitrary constant, is an equation to the common parabola. A point moving on this parabola has, at any instant, the same value of dy/dx as if it were moving on the tangent of the parabola, or on one of the straight lines of equation (2). *The singular solution of a differential equation is geometrically equivalent to the envelope of the family of curves represented by the general solution.* The singular solution is distinguished from the particular solution, in that the latter is contained in the general solution, the former is not.

Again referring to Fig. 78, it will be noticed that for any point on the envelope, there are two equal values of p or dy/dx, one for the parabola, one for the straight line.

In order that the quadratic

$$ax^2 + bx + c = 0,$$

may have equal roots, it is necessary (page 388) that

$$b^2 = 4ac\,;\ \text{or, } b^2 - 4ac = 0. \qquad (4)$$

This relation is called the **discriminant.** From (1), since

$$y = px + a/p\,;\ \therefore xp^2 - yp + a = 0. \qquad (5)$$

In order that equation (5) may have equal roots,

$$y^2 = 4ax,$$

as in (4). This relation is the locus of all points for which two values of p become equal, hence it is called the ***p*-discriminant** of (1).

In the same way if C be regarded as variable in the general solution (2),

$$y = Cx + a/C\,;\ \text{or, } xC^2 - yC + a = 0.$$

The condition for equal roots, is that

$$y^2 = 4ax,$$

which is the locus of all points for which the value of C is the same. It is called the ***C*-discriminant.**

Before applying these ideas to special cases, we may note that the envelope locus may be a single curve (Fig. 78) or several (Fig. 79). For an exhaustive discussion of the properties of these discriminant relations I must refer the reader to the numerous textbooks on the subject, or to Cayley, *Messenger of Mathematics*, **2**, 6, 1872. To summarise:

1. **The envelope locus** satisfies the original equation but is not included in the general solution (see xx', Fig. 112).

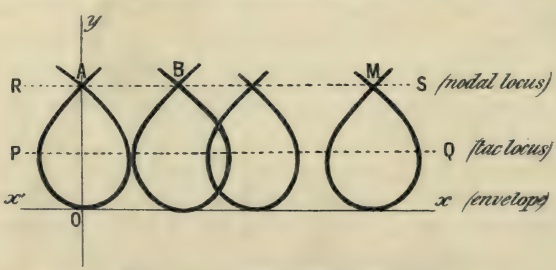


FIG. 112.—Nodal and Tac Loci.

2. **The tac locus** is the locus passing through the several points where two non-consecutive members of a family of curves touch. Such a locus is represented by the line AB (Fig. 79), PQ (Fig. 112). The tac locus does not satisfy the original equation, it appears in the p-discriminant, but not in the C-discriminant.

3. **The node locus** is the locus passing through the different points where each curve of a given family crosses itself (the point of intersection—node—may be double, triple, etc.). The node locus does not satisfy the original equation, it appears in the C-discriminant but not in the p-discriminant. RS (Fig. 112) is a nodal locus passing through the nodes $A, \ldots, B, \ldots, C, \ldots, M$.

4. **The cusp locus** passes through all the cusps (page 136) formed by the members of a family of curves. The cusp locus does not satisfy the original equation, it appears in the p- and in the C-discriminants. It is the line Ox in Fig. 113. Sometimes the nodal or cusp loci coincides with the envelope locus.*

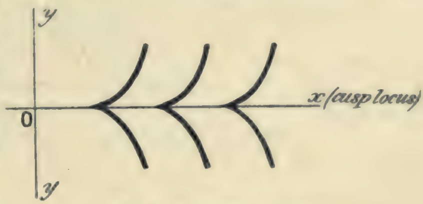


FIG. 113.—Cusp Locus.

EXAMPLES.—Find the singular solutions and the nature of the other loci in the following equations:

(1) $$x^2p^2 - 2yp + ax = 0.$$

For equal roots $y^2 = ax^2$. This satisfies the original equation and is not included in the general solution: $x^2 - 2Cy + aC^2 = 0$. $y^2 = ax^2$ is thus the singular solution.

(2) $4xp^2 = (3x - a)^2$. General solution: $(x + C)^2 = x(x - a)^2$.

For equal roots in p, $4x(3x - a)^2 = 0$, or $x(3x - a)^2 = 0$ (p-discriminant). For equal roots in C, differentiate the general solution with respect to C. Therefore $(x + C)dx/dC = 0$, or $C = - x$. $\therefore x(x - a)^2 = 0$ (C-discriminant) is the condition to be fulfilled when the C-discriminant has equal roots. $x = 0$ is common to the two discriminants and satisfies the original equation (singular solution); $x = a$ satisfies the C-discriminant but not the p-discriminant and, since it is not a solution of the original equation, $x = a$ represents the node locus; $x = \frac{1}{3}a$ satisfies the p- but not the C-discriminant nor the original equation (tac locus).

(3) $$p^2 + 2xp - y = 0.$$

General solution: $(2x^3 + 3xy + C)^2 = 4(x^2 + y)^3$; p-discriminant: $x^2 + y = 0$; C-discriminant: $(x^2 + y)^3 = 0$. The original equation is not satisfied by either of these equations and, therefore, there is no singular solution. Since $(x^2 + y)$ appears in both discriminants, it represents a cusp locus.

(4) Show that the complete solution of the equation, $y^2(p^2 + 1) = a^2$, is $y^2 + (x - C)^2 = a^2$; that there are two singular solutions, $y = \pm a$; that there is a tac locus on the x-axis for $y = 0$ (Fig. 79, see also § 138).

* The second part of van der Waals' *The Continuity of the Gaseous and Liquid States of Aggregation—Binary Mixtures* (1900) has some examples of the preceding "mathematics".

§ 126. Trajectories.

This section will serve as an exercise on some preceding work. *A trajectory* is a curve which cuts another system of curves at a constant angle. If this angle is 90° the curve is an **orthogonal trajectory.**

Examples.—(1) Let $xy = C$ be a system of rectangular hyperbolas, to find the orthogonal trajectory, first eliminate C by differentiation with respect to x, thus we obtain,

$$xdy/dx + y = 0.$$

If two curves are at right angles ($\frac{1}{2}\pi = 90°$), then from (17), § 32, $\frac{1}{2}\pi = (\alpha' - \alpha)$, where α, α' are the angles made by tangents to the curves at the point of intersection with the x-axis. But by the same formula,

$$\tan(\pm\tfrac{1}{2}\pi) = (\tan\alpha' - \tan\alpha)/(1 + \tan\alpha \,.\, \tan\alpha').$$

Now $\tan \pm \frac{1}{2}\pi = \infty$ and $1/\infty = 0$,

$$\therefore \tan\alpha = -\cot\alpha;\ \text{or, } dy/dx = -dx/dy.^*$$

The differential equation of the one family is obtained from that of the other by substituting dy/dx for $-dx/dy$. Hence the equation to the orthogonal trajectory of the system of rectangular hyperbolas is, $xdx + ydy = 0$, or $x^2 - y^2 = C$, a system of rectangular hyperbolas whose axes coincide with the asymptotes of the given system.

(2) For polar coordinates show that we must substitute $-dr/r \,.\, d\theta$ for $r \,.\, d\theta/dr$.

(3) Find the orthogonal trajectories of the system of parabolas $y^2 = 4ax$. Ansr. Ellipses, $2x^2 + y^2 = C^2$.

(4) Show that the orthogonal trajectories of the equipotential curves $1/r - 1/r' = C$, are the magnetic curves $\cos\theta + \cos\theta' = C$.

§ 127. Symbols of Operation.

It will be found convenient to denote the symbol of the operation "d/dx" by the letter "D". If we assume that the infinitesimal increments of the independent variable dx have the same magnitude, whatever be the value of x, we can suppose D to have a constant value. Thus

$$D, D^2, D^3, \ldots, \text{ stand for } \frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots;$$

$$Dy, D^2y, \ldots, \text{ stand for } \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$$

The operations denoted by the symbols D, D^2, . . ., satisfy the elementary rules of algebra except that they are not commutative † with regard to the variables. For example,

---

* No doubt the reader sees that in (18), § 12, dx/dy is the cotangent of the angle whose tangent is dy/dx.

† The so-called **fundamental laws of algebra** are: I. *The law of association:* The number of things in any group is independent of the order. II. *The commutative law:* (a) Addition. The number of things in any number of groups is independent of the order. (b) Multiplication. The product of two numbers is independent of the

$$D(u + v + \dots) = Du + Dv + \dots, \text{ (distributive law).}$$
$$D(Cu) = CDu, \qquad \text{(commutative law),}$$
where C is a constant. We cannot write $D(xy) = D(yx)$. But,
$$D^m D^n u = D^{m+n} u \qquad \text{(index law),}$$
is true when m and n are positive integers. If
$$Du = v;\ u = D^{-1}v;\ \text{or, } u = \frac{1}{D}v;$$
$$\therefore v = D \,.\, D^{-1}v;\ \text{or, } D \,.\, D^{-1} = 1;$$
that is to say, by operating with D upon $D^{-1}v$, we annul the effect of the D^{-1} operator. It is necessary to remember later on, that if $Dx = 1$, $\frac{1}{D^2} = \frac{1}{2}x^2;\ \frac{1}{D^3} = \frac{1}{2\,.\,3}x^3, \dots$

In this notation, the equation
$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0,$$
is written,
$$\{D^2 - (\alpha + \beta)D + \alpha\beta\}y = 0;\ \text{or, } (D - \alpha)(D - \beta)y = 0.$$
Now replace D with the original symbol, and operate on one factor with y. Thus,
$$\left(\frac{d}{dx} - \alpha\right)\left(\frac{d}{dx} - \beta\right)y = 0;\ \left(\frac{d}{dx} - \alpha\right)\left(\frac{dy}{dx} - \beta y\right) = 0.$$
By operating on the second factor with the first, we get the original equation back again.

§ 128. The Linear Equation of the nth Order.

(*General Remarks.*)

As a general rule the higher orders of differential equations are more difficult of solution than equations of the first order. As with the latter, the more expeditious mode of treatment will be to refer the given equation to a set of standard cases having certain distinguishing characters. By far the most important class is the linear equation.

A **linear equation of the nth order** is one in which the dependent variable and its n derivatives are all of the first degree and are not multiplied together. The typical form in which it appears is
$$\frac{d^ny}{dx^n} + X_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + X_ny = X. \qquad (1)$$

---

order. III. *The distributive law:* (*a*) Multiplication. The multiplier may be distributed over each term of the multiplicand, *e.g.*, $m(a+b) = ma + mb$. (*b*) Division. $(a + b)/m = a/m + b/m$. IV. *The index law:* (*a*) Multiplication. $a^m a^n = a^{m+n}$. (*b*) Division. $a^m/a^n = a^{m-n}$.

Or, in symbolic notation,

$$D^n y + X_1 D^{n-1} y + \ . \ . \ . + X_n y = X,$$

where X, X_1, . . . , X_n are either constant magnitudes, or functions of the independent variable x. If the coefficient of the highest derivative be other than unity, the other terms of the equation can be divided by this coefficient. The equation will thus assume the typical form (1). We have studied the linear equation of the first order in § 123. For the sake of fixing our ideas, the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \quad . \quad . \quad . \quad (2)$$

of the second order, will be taken as typical of the class. P, Q, R have the meaning above attached to X_1, X_2, X.

The general solution of the linear equation is made up of two parts.

1. **The complementary function** which is the most general solution of the left-hand side of equation (2) equated to zero, or,

$$d^2y/dx^2 + Pdy/dx + Qy = 0. \quad . \quad . \quad (3)$$

The complementary function involves two arbitrary constants.

2. **The particular integral** which is any solution of the original equation (2), the simpler the better. In particular cases when the right-hand side is zero, the particular integral does not occur.

To show that the general solution of (2) *contains a general solution of* (3). Assume that the complete solution of (2) may be written,

$$y = u + v, \quad . \quad . \quad . \quad . \quad (4)$$

where v is any function of x which satisfies (2), that is to say, v is the particular integral * of (2), u is the general solution of (3), to be determined. Substitute (4) in (2).

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu + \frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = R.$$

But
$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = R;$$

therefore,
$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0.$$

Therefore, u must satisfy (3).

*Given a particular solution * of the linear equation, to find the*

---

* Not to be confused with the particular solution of page 289.

complete solution. Let $y = v$ be a particular solution of the following equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

where P and Q are functions of x. Substitute $y = uv$,

$$v\frac{d^2u}{dx^2} + \left(2\frac{dv}{dx} + Pv\right)\frac{du}{dx} = 0.$$

This equation is of the first order and linear with du/dx as the dependent variable. Put $du/dx = z$ and

$$v\frac{dz}{dx} + \left(2\frac{dv}{dx} + Pv\right)z = 0; \frac{dz}{z} + 2\frac{dv}{v} + Pdx = 0;$$

$$\log\frac{du}{dx} + 2\log v + \int Pdx = 0; \text{ or, } \frac{du}{dx} = C_1\frac{1}{v^2}e^{-\int Pdx}.$$

$$\therefore u = C_1\int\frac{1}{v^2}e^{-\int Pdx}dx + C_2; \text{ or, } y = C_1v\int\frac{1}{v^2}e^{-\int Pdx}dx + C_2v,$$

where C_1 and C_2 are arbitrary constants.

EXAMPLES.—(1) If $y = e^{ax}$ is a particular solution of $d^2y/dx^2 = a^2y$, show that the complete solution is $y = C_1e^{ax} + C_2e^{-ax}$.

(2) If $y = x$ is a particular solution of $(1 - x^2)d^2y/dx^2 - xdy/dx + y = 0$, the complete solution is $y = C_1\sqrt{(1 - x^2)} + C_2x$.

If a particular solution of the linear equation is known, the order of the equation can be lowered by unity. This follows directly from the preceding result. If $y = v$ is a known solution, then, if $y = tv$ be substituted in the first member of the equation, the coefficient of t in the result, will be the same as if t were constant and therefore zero. t being absent, the result will be a linear equation in t but of an order less by unity than that of the given equation. It follows directly, that if n particular solutions of the equation are known, the order of the equation can be reduced n times.

For the description of a machine designed for solving (3), see *Proceedings of the Royal Society*, **24**, 269, 1876 (Lord Kelvin).

§ 129. The Linear Equation with Constant Coefficients.

The integration of these equations obviously resolves itself into finding the complementary function and the particular integral.

First, when the second member is zero, in other words, *to find the complementary function of any linear equation with constant coefficients.* The typical equation is,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \quad . \quad . \quad . \quad (1)$$

where P and Q are constants. The particular integral does not appear in the solution.

If the equation were of the first order, its solution would be, $y = Ce^{\int mdx}$. On substituting e^{mx} for y in (1), we obtain

$$(m^2 + Pm + Q)e^{mx} = 0,$$

provided $$m^2 + Pm + Q = 0. \quad . \quad . \quad . \quad (2)$$

This equation is called the **auxillary equation.** If m_1 be one value of m which satisfies (2), then $y = e^{m_1x}$, is an integral of (1). But we must go further.

Case 1. *When the auxillary equation has two unequal roots,* say m_1 and m_2, the general solution of (1) may be written down without any further trouble.

$$y = C_1e^{m_1x} + C_2e^{m_2x}. \quad . \quad . \quad . \quad (3)$$

EXAMPLES.—(1) Solve $(D^2 + 14D - 32)y = 0$. Assume $y = Ce^{mx}$ is a solution. The auxillary becomes, $m^2 + 14m - 32 = 0$. The roots are $m = 2$, or $- 16$. The required solution is, therefore, $y = C_1e^{2x} + C_2e^{-16x}$.

(2) Solve $d^2y/dx^2 - m^2y = 0$. Ansr. $y = C_1e^{mx} + C_2e^{-mx}$ (see page 319).

(3) Show that $y = C_1e^{3x} + C_2e^x$ is a complete solution of

$$d^2y/dx^2 + 4dy/dx + 3y = 0.$$

Case 2. *When the two roots of the auxillary are equal.* If $m_1 = m_2$, in (3), it is no good putting $(C_1 + C_2)e^{m_1x}$ as the solution, because $C_1 + C_2$ is really one constant. The solution would then contain one arbitrary constant less than is required for the general solution. To find the other particular integral, it is usual to put

$$m_2 = m_1 + h,$$

where h is some finite quantity which will ultimately be made zero. With this proviso, we write the solution,

$$y = Lt_{h=0}C_1e^{m_1x} + C_2e^{(m_1+h)x}.$$

Hence, $$y = Lt_{h=0}e^{m_1x}(C_1 + C_2e^{hx}).$$

Now expand e^{hx} by Maclaurin's theorem (page 230).

$$\therefore y = Lt_{h=0}e^{m_1x}\{C_1 + C_2(1 + hx + \tfrac{1}{2!}h^2x^2 + \ldots)\};$$
$$= Lt_{h=0}e^{m_1x}\{C_1 + C_2 + C_2hx(1 + \tfrac{1}{2!}hx + \ldots)\};$$
$$= Lt_{h=0}e^{m_1x}(A + Bx + \tfrac{1}{2!}C_2h^2x^2 + C_2R),$$

where R denotes the remaining terms of the expansion of e^{hx}, $A = C_1 + C_2$, $B = C_2h$. Therefore, at the limit,

$$y = e^{m_1x}(A + Bx). \quad . \quad . \quad . \quad . \quad (4)$$

For the sake of uniformity, we shall still write the arbitrary integration constants $C_1, C_2, C_3, \ldots$

For an equation of a still higher degree, the preceding result may be written,

$$y = e^{m_1x}(C_1 + C_2x + C_3x^2 + \ldots + C_{r-2}x^{r-1}). \quad . \quad (5)$$

where r denotes the number of equal roots.

EXAMPLES.—(1) Solve $d^3y/dx^3 - d^2y/dx^2 - dy/dx + y = 0$. Assume $y = Ce^{mx}$. The auxillary equation is $m^3 - m^2 - m + 1 = 0$. The roots are 1, 1, − 1. Hence the general solution can be written down at sight:

$$y = C_1e^{-x} + (C_2 + C_3x)e^x.$$

(2) Solve $(D^3 - 3D^2 + 4)y = 0$. Ansr. $e^{2x}(C_1 + C_2x) + C_3e^{-x}$.

Case 3. *When the auxillary equation has imaginary roots, all unequal.* Remembering that imaginary roots are always found in pairs in equations with real coefficients (page 386), let the two imaginary roots be

$$m_1 = \alpha + \iota\beta\text{; and } m_2 = \alpha - \iota\beta.$$

Instead of substituting $y = e^{mx}$ in (3), we substitute these values of m in (3) and get

$$y = C_1e^{(\alpha + \iota\beta)x} + C_2e^{(\alpha - \iota\beta)x};$$
$$= e^{\alpha x}(C_1e^{\iota\beta x} + C_2e^{-\iota\beta x});$$
$$= e^{\alpha x}C_1(\cos\beta x + \iota\sin\beta x) + e^{\alpha x}C_2(\cos\beta x - \iota\sin\beta x). \quad (6)$$

(See the chapter on "Hyperbolic Functions".) Separate the real and imaginary parts, as in Ex. 3, p. 280,

$$y = e^{\alpha x}(C_1 + C_2)\cos\beta x + \iota(C_1 - C_2)\sin\beta x;$$

if we put $C_1 + C_2 = A$, $\iota(C_1 - C_2) = B$,

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x). \quad . \quad . \quad . \quad . \quad (7)$$

In order that the constants A and B in (7) may be real, the constants C_1 and C_2 must include the imaginary parts.

EXAMPLES.—(1) Show from (6) that

$$y = (\cosh\alpha x + \sinh\alpha x)(A_1\cos\beta x + B_1\sin\beta x).$$

(Exercise on Chapter VI.)

(2) Integrate $d^2y/dx^2 + dy/dx + y = 0$. The roots are $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}\sqrt{3}$; $\therefore y = e^{-x/2}(A\cos\frac{1}{2}\sqrt{3}\,.\,x + B\sin\frac{1}{2}\sqrt{3}\,.\,x)$.

(3) The equation of a point vibrating under the influence of a periodic force, is,

$$\frac{d^2x}{dt^2} + a^2x = a\cos 2\pi\frac{t}{T}.$$

Find the complementary function. The roots are $\pm\iota a$. From (7)

$$y = A\cos ax + B\sin ax.$$

(3) If $(D^3 - D^2 + D - 1)y = 0$, $y = C_1\cos x + C_2\sin x + Ce^x$.

Case 4. *When some of the imaginary roots of the auxillary equation are equal.* If a pair of the imaginary roots are repeated, we may proceed as in Case 2, since, when $m_1 = m_2$, $C_1e^{m_1x} + C_2e^{m_2x}$, is replaced by $(A + Bx)e^{m_1x}$; similarly, when $m_3 = m_4$, $C_3e^{m_3x} + C_4e^{m_4x}$ may be replaced by $(C + Dx)e^{m_3x}$. If, therefore,

$$m_1 = m_2 = \alpha + \iota\beta;\ \text{and}\ m_3 = m_4 = \alpha - \iota\beta,$$

the solution

$$y = (C_1 + C_2x)e^{(\alpha + \iota\beta)x} + (C_3 + C_4x)e^{(\alpha - \iota\beta)x},$$

becomes
$$y = e^{\alpha x}(A + Bx)\cos\beta x + (C + Dx)\sin\beta x. \qquad (8)$$

EXAMPLES.—(1) Solve $(D^4 - 12D^3 + 62D^2 - 156D + 169)y = 0$. Given the roots of the auxillary: $3 + 2\iota, 3 + 2\iota, 3 - 2\iota, 3 - 2\iota$. Hence,

$$y = e^{3x}\{(C_1 + C_2x)\sin 2x + (C_3 + C_4x)\cos 2x\}.$$

(2) If $(D^2+1)(D-1)^2y=0$, $y=(A+Bx)\sin x+(C+Dx)\cos x+(E+Fx)e^x$.

Second, when the second member is not zero, that is to say to find both the complementary function and the particular integral. The general equation is,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \qquad (9)$$

where P and Q are constant, R is a function of x. We have just shown how to find one part of the complete solution of the linear equation with constant coefficients, namely, by putting R, in (9), equal to zero. The remaining problem is to find a particular integral of this equation. The more useful processes will be described in the next section.

In the symbolic notation, (9) may be written,

$$f(D)y = R. \qquad (10)$$

The particular integral is, therefore,

$$y = f(D)^{-1}R;\ \text{or,}\ y = \frac{R}{f(D)}. \qquad (11)$$

The right-hand side of either of equations (11), will be found to give a satisfactory value for the particular integral in question.

Since the complementary function contains all the constants necessary for the complete solution of the differential equation, it follows that *no integration constant must be appended to the particular integral.*

§ 130. How to find Particular Integrals.

It will be found quickest to proceed by rule:

Case 1 (General). *When the operator $f(D)^{-1}$ can be resolved* *into factors.* We have seen that the linear differential equation of the first order,

$$dy/dx - ay = R\,;\ \text{or, } y = R/(D-a), \quad . \quad . \quad (1)$$

is solved by

$$y = e^{ax}\int e^{-ax}Rdx. \quad . \quad . \quad . \quad . \quad (2)$$

The term Ce^{ax} in the solution of (1), belongs to the complementary function.

Suppose that in a linear equation of a higher order, say,

$$d^2y/dx^2 - 5dy/dx - 6y = R,$$

the operator $f(D)^{-1}$ can be factorised. The complementary function is written down at sight from,

$$(D^2 - 5D + 6)y = 0\,;\ \text{or, } (D-3)(D-2)y = 0,$$

namely,

$$y = C_1e^{3x} + C_2e^{2x}. \quad . \quad . \quad . \quad . \quad (3)$$

The particular integral is

$$y_1 = \frac{1}{(D-3)(D-2)}R = \left(\frac{1}{D-3} - \frac{1}{D-2}\right)R\,;$$

$$= e^{3x}\int e^{-3x}Rdx - e^{2x}\int e^{-2x}Rdx, \quad . \quad . \quad . \quad . \quad (4)$$

from (2). The general solution is the sum of (3) and (4),

$$\therefore\ y = C_1e^{3x} + Ce^{2x} + e^{3x}\int e^{-3x}Rdx - e^{2x}\int e^{-2x}Rdx.$$

EXAMPLES.—(1) In the preceding illustration, put $R = e^{4x}$ and show that the general solution is, $C_1e^{3x} + C_2e^{2x} + \frac{1}{2}e^{4x}$.

(2) If $(D^2 - 4D + 3)y = 2e^{3x}$, $y = C_1e^{x} + C_2e^{3x} + xe^{3x}$.

Case 2 (General). *When the operator $f(D)^{-1}$ can be resolved into partial fractions with constant numerators.* The way to proceed in this case is illustrated in the first example below.

EXAMPLES.—(1) Solve $d^2y/dx^2 - 3dy/dx + 2y = e^{3x}$. In symbolic notation this will appear in the form,

$$(D-1)(D-2)y = e^{3x}.$$

The complementary function is $y = C_1e^{x} + C_2e^{2x}$. The particular integral is obtained by putting

$$y_1 = \frac{1}{(D-2)(D-1)}e^{3x}\,;\ y_1 = \left(\frac{1}{D-2} - \frac{1}{D-1}\right)e^{3x},$$

according to the method of resolution into partial fractions. Operate with the first symbolic factor, as above,

$$y_1 = e^{2x}\int e^{-2x}e^{3x}dx - e^{x}\int e^{-x}e^{3x}dx = \tfrac{1}{2}e^{3x}.$$

The complete solution is, therefore, $y = C_1e^{x} + C_2e^{2x} + \frac{1}{2}e^{3x}$.

(2) Solve $(D-2)^2y = x$. Ansr.

Case 3 (Special). *When R is a rational function of x, say x^n.* This case is comparatively rare. The procedure is to expand $f(D)^{-1}$ in ascending powers of D as far as the highest power of x in R.

EXAMPLES.—(1) Solve $d^2y/dx^2 - 4dy/dx + 4y = x^2$. The complementary function is $y = e^{2x}(A + Bx)$; the particular integral is:

$$(2 - D)^{-2}x^2 = \frac{1}{4}\left(1 + 2\frac{D}{2} + 3\frac{D^2}{2^2}\right)x^2 = \frac{1}{8}(2x^2 + 4x + 3).$$

(2) If $d^2y/dx^2 - y = 2 + 5x$, $y = C_1e^x + C_2e^{-x} + 5x - 2$.

Case 4 (Special). *When R contains an exponential factor, so that*

$$R = e^{ax}X,$$

where X may or may not be a function of x and a has some constant value.

i. *When X is a function of x.* Since $D^ne^{ax} = a^ne^{ax}$, where n is any positive integer (page 38), we have (page 25)

$$D(e^{ax}X) = e^{ax}DX + ae^{ax}X = e^{ax}(D + a)X,$$

and generally, as in Leibnitz' theorem (page 49),

$$D^ne^{ax}X = e^{ax}(D + a)^nX;$$

$$\therefore \frac{D^ne^{ax}}{(D + a)^n}X = e^{ax}X; \text{ and } \frac{e^{ax}}{(D + a)^n}X = \frac{e^{ax}}{D^n}X. \quad . \quad (5)$$

The operation $\int D^{-1}e^{ax}X$ is performed (when X is any function of x) by transplanting e^{ax} from the right- to the left-hand side of the operator $f(D)^{-1}$ and replacing D by $(D + a)$. This will, perhaps, be better understood from the following examples:

EXAMPLES.—(1) Solve $d^2y/dx^2 - 2dy/dx + y = x^2e^{ax}$. The complete solution by page 308, is $(C_1 + xC_2)e^x + (D + 2D + 1)^{-1}x^2e^{3x}$. From (5),

$$\frac{1}{D^2 - 2D + 1}x^2e^{3x} = \frac{1}{(D - 1)(D - 1)}x^2e^{3x}.$$

By rule: e^{3x} may be transferred from the right to the left side of the operator provided we replace D by $D + 3$.

$$e^{3x}\frac{1}{(D + 2)^2}\cdot x^2.$$

We get $$e^{3x}(\tfrac{1}{4}x^2 - \tfrac{1}{2}x + \tfrac{3}{8}),$$

as the value of the particular integral.

(2) Evaluate $(D - 1)^{-1}e^x\log x$. Ansr. $xe^x\log x/e$.

ii. *When X is constant.* If X is constant, the operation (5) reduces to

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}. \quad . \quad . \quad . \quad (6)$$

The operation $f(D)^{-1}e^{ax}X$ is performed by replacing $D + a$ by a.

EXAMPLES.—(1) Find the particular integral in $(D^2 - 3D + 2)y = e^{3x}$. Obviously,

$$\frac{1}{D^2 - 3D + 2}e^{3x} = \frac{1}{3^2 - 3 . 3 + 2}e^{3x} = \tfrac{1}{2}e^{3x}.$$

(2) Show that $\frac{1}{4}e^x$, is a particular integral in

$$d^2y/dx^2 + 2dy/dx + 1 = e^x.$$

An anomalous case arises when a is a root of $f(D) = 0$. By this method, we should get for the particular integral of $dy/dx - y = e^x$.

$$\frac{1}{D - 1}e^x = \frac{e^x}{1 - 1} = \infty e^x.$$

The difficulty is evaded by using the method (5) instead of (6). Thus,

$$\frac{1}{D - 1}e^x = e^x\frac{1}{D}.1 = xe^x.$$

The complete solution is, therefore, $y = Ce^x + xe^x$.

Another mode of treatment is the following: Since a is a root of $f(D) = 0$, by hypothesis, $D - a$ is a factor of $f(D)$ (page 386). Hence,

$$f(D) = (D - a)f'(D);$$

$$\therefore \frac{1}{f(D)}e^{ax} = \frac{1}{(D - a)}\cdot\frac{1}{f'(D)}e^{ax} = \frac{1}{(D - a)}\cdot\frac{1}{f'(a)}e^{ax} = \frac{xe^{ax}}{f'(a)}. \quad (7)$$

If the root a occurs r times in $f(D) = 0$, then $D - a$ enters r times into $f(D)$. Therefore,

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D - a)^r}\frac{1}{f'(D)}e^{ax} = \frac{1}{(D - a)^r}\cdot\frac{1}{f'(a)}e^{ax} = \frac{x^re^{ax}}{rf'(a)}. \quad (8)$$

Examples.—Find the particular integrals in, (1) $(D + 1)^3y = e^{-x}$. Ansr. $\frac{1}{6}x^3e^{-x}$. Hint. Replace D by $D - 1$. $e^{-x}D^{-3}$, etc. See page 312.

(2) $(D^3 - 1)y = xe^x$. Ansr. $e^x(\frac{1}{6}x^2 - \frac{1}{3}x)$. Hint. First get $e^x(D - 1)^{-1}x$, then $e^x(1 + D + \ldots)x$, etc.

Case 5 (Special). *When R contains sine or cosine factors.* By the successive differentiation of $\sin(nx + a)$,

$$(D^2)^n\sin(nx + a) = (-n^2)^n\sin(nx + a).^*$$

where n and a are constants.

$$\therefore f(D^2)\sin(nx + a) = f(-n^2)\sin(nx + x).$$

$$\frac{1}{f(D^2)}\sin(nx + a) = \frac{1}{f(-n^2)}\sin(nx + a). \quad . \quad (9)$$

It can be shown in the same way that,

$$\frac{1}{f(D^2)}\cos(nx + a) = \frac{1}{f(-n^2)}\cos(nx + a). \quad . \quad (10)$$

Examples.—(1) Find the particular integral of

$$d^3y/dx^3 + d^2y/dx^2 + dy/dx + y = \sin 2x.$$

Here, $$\frac{R}{f(D)} = \frac{1}{D^3 + D^2 + D + 1}\sin 2x = \frac{1}{(D^2 + 1) + D(D^2 + 1)}\sin 2x.$$

---

* The proof resembles a well-known result in trignometry, § 19:—

$$D(\sin nx) = d(\sin nx)/dx = n\cos nx;$$

$$D^2(\sin nx) = d^2(\sin nx)/dx^2 = -n^2\sin nx, \text{ etc.}$$

Substitute for $D^2 = (-2^2)$ as in (9). We thus get $-\frac{1}{5}(D+1)^{-1}\sin 2x$. Multiply by $D - 1$ and again substitute $D^2 = (-2^2)$ in the result. Thus

$$\tfrac{1}{15}(D-1)\sin 2x, \text{ or } \tfrac{1}{15}(2\cos 2x - \sin 2x)$$

is the desired result.

(2) Solve $d^2y/dx^2 - k^2y = \cos mx$. Ansr. $C_1e^{kx} + C_2e^{-kx} - (\cos mx)/(m^2 + k^2)$.

(3) If α and β are the roots of the auxillary equation derived from

$$d^2y/dt^2 + mdy/dt + n^2y = a\sin nt,$$

(*Helmholtz's equation for the vibrations of a tuning-fork*) show that

$$C_1e^{\alpha t} + C_2e^{\beta t} - (a\cos nt)/mn,$$

is the complete solution.

An anomalous case arises when D^2 in $D^2 + n^2$ is equal to $-n^2$. For instance, the cômplementary function of $d^2y/dx^2 + n^2y = \cos nx$, is $C_1\cos nx + C_2\sin nx$, the particular integral is $(D^2 + n^2)\cos nx$. If the attempt is made to evaluate this, by substituting $D^2 = -n^2$, we get $(\cos nx)/(-n^2 + n^2) = \infty\cos nx$. We were confronted with a similar difficulty on page 243. The treatment is practically the same. We take the limit of $(D^2 + n^2)\cos nx$, when n becomes $n + h$ and h converges towards zero.

$$\frac{1}{-n^2+n^2}\cos nx; \text{ or, } \frac{1}{-(n+h)^2+n^2}\cos(n+h)x.$$

$$\therefore Lt_{h=0}\frac{1}{-(n+h)^2+n^2}\cos(n+h)x = Lt_{h=0}\frac{1}{-2nh-h^2}\cos(nx+hx);$$

$$= Lt_{h=0} - \frac{1}{2nh+h^2}(\cos nx \,.\, \cos hx - \sin nx \,.\, \sin hx);$$

$$= Lt_{h=0} - \frac{1}{2nh+h^2}\left\{\cos nx\left(1 + \frac{h^2x^2}{2!} + \ldots\right) - \sin nx(hx - \ldots)\right\};$$

$$= Lt_{h=0} - \frac{1}{2n+h}\left(\frac{\cos nx}{h} - x\sin nx + \text{powers of } h\right).$$

But $\cos nx$ is contained in the complementary function and hence, when $h = 0$, we obtain,

$$\frac{x\sin nx}{2n} + (\text{a term in the complementary function}).$$

This latter may be disregarded when the particular integral alone is under consideration. The complete solution is, therefore,

$$y = C_1\cos nx + C_2\sin nx + (x\sin nx)/2n.$$

EXAMPLES.—(1) Show that $-\frac{1}{2}x\cos x$, is the particular integral of

$$(D^2 + 1)y = \sin x.$$

(2) Evaluate $(D^2 + 4)^{-1}\cos 2x$. Ansr. $\frac{1}{4}x\sin 2x$.

(3) Evaluate $(D^2 + 4)^{-1}\sin 2x$. Ansr. $\frac{1}{4}x\cos 2x$.

(4) Solve $d^3y/dx^3 - y = x\sin x$. The particular integral consists of two parts, $\frac{1}{2}\{(x - 3)\cos x - x\sin x\}$. The complementary function is

$$C_1e^x + C_2e^{-x/2}\sin(\tfrac{1}{2}\sqrt{3}x) + C_3e^{x/2}\cos(\tfrac{1}{2}\sqrt{3}x).$$

But see next case.

Case 6. *When R contains some power of x as a factor.* Say,

$$R = xX,$$

where X is any function of x. The successive differentiation of two products gives, § 20,

$$D^n xX = xD^n X + nD^{n-1}X.$$

$$\therefore f(D)xX = xf(D)X + f'(D)X.$$

Substitute $Y = f(D)X$, where Y is any function of x. Operate with $f(D)^{-1}$, we get

$$\frac{1}{f(D)}xY = \left\{x - \frac{1}{f(D)}\cdot f'(D)\right\}\frac{1}{f(D)}Y. \qquad (11)$$

EXAMPLES.—(1) Find the particular integral in $d^3y/dx^3 - y = xe^{2x}$. From (11), the particular integral is

$$= \left\{x - \frac{1}{D^3 - 1}\cdot 3D^2\right\}\frac{1}{D^3 - 1}e^{2x} = \left\{x - \frac{1}{7}\cdot 3\cdot 4\right\}\frac{1}{7}e^{2x}, \text{ etc.}$$

(2) Show in this way, that the particular integral of $(D^4 - 1)y = x\sin x$, is $\frac{1}{8}(x^2\cos x - 3x\sin x)$.

(3) Solve $d^2y/dx^2 - y = xe^x\sin x$.

Ansr. $y = C_1e^x + C_2e^{-x} - \frac{1}{25}e^x\{(10x + 2)\cos x + (5x - 14)\sin x\}$.

(4) Integrate $d^2y/dx^2 - y = x^2\cos x$.

Ansr. $y = C_1e^x + C_2e^{-x} + x\sin x + \frac{1}{2}\cos x(1 - x^2)$.

§ 131. The Linear Equation with Variable Coefficients.

Case 1. *The homogeneous linear differential equation.* The general type of this equation is:

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_ny = X, \qquad (1)$$

where X is a function of x; $a_1, a_2, \ldots, a_n$ are constants. This equation can be transformed into one with constant coefficients by the substitution of

$$x = e^z; \text{ or } z = \log x.$$

we then have,

$$dx/dz = e^z \text{ and therefore, } xdy/dx = dy/dz. \qquad (2)$$

Just as we have found it very convenient to employ the symbol "D," to denote the operation "$\frac{d}{dx}$," so we shall find it even more convenient to denote the operation "$x\frac{d}{dx}$," by the symbol "ϑ". "ϑ" is treated in exactly the same manner as we have treated "D" * in § 128 and subsequently.

* A little care is required in using this new notation. The operations of differentiation and multiplication by a variable are not commutative. The operation x^2D^2 is not the same as ϑ^2, or as $xD.xD$. But we must write,

$$xDy = \vartheta y;$$
$$x^2D^2y = \vartheta(\vartheta - 1)y;$$
$$x^3D^3y = \vartheta(\vartheta - 1)(\vartheta - 2)y;$$
$$\ldots \qquad \ldots$$
$$x^nD^ny = \vartheta(\vartheta - 1)(\vartheta - 2)\ldots(\vartheta - n + 1)y.$$

EXAMPLES.—(1) $\vartheta = xD = x\frac{d}{dx} = \frac{d}{dz}$.

(2) Show that $\vartheta x^m = mx^m$.

i. *The complementary function.* From the first of equations (2), we have § 12, 9,

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\cdot\frac{dy}{dz};\ \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right);\ \text{etc.}$$

Substitute these values in (1). The equation reduces to one with constant coefficients which may be treated by the methods described in the preceding sections.

EXAMPLES.—(1) Solve

$$x^3 . d^3y/dx^3 + 2x^2 . d^2y/dx^2 + 3x . dy/dx - 3y = x^2 + x.$$

$\therefore \vartheta(\vartheta - 1)(\vartheta - 2)y + 2\vartheta(\vartheta - 1)y + 3\vartheta y - 3y = e^{2z} + e^z.$

$\therefore (\vartheta - 1)(\vartheta^2 + 3)y = e^{2z} + e^z;$

$\therefore y = C_1e^z + C_2 \cos z\sqrt{3} + C_3 \sin z\sqrt{3} + \frac{1}{7}e^{2z} + \frac{1}{4}ze^z.$

$\therefore y = C_1x + C_2 \cos(\sqrt{3}\log x) + C_3 \sin(\sqrt{3}\log x) + \frac{1}{7}x + \frac{1}{4}x\log x.$

(2) Solve $x^2 . d^2y/dx^2 + x . dy/dx + q^2y = 0$; *i.e.* $(\vartheta^2 + q^2)y =$.

Ansr. $y = C_1 \sin(q \log x) + C_2 \cos(q \log x)$.

The linear equation with variable coefficients bears the same relation to x, that the equation with constant coefficients does to e^{mx}. Hence if x^m be substituted for y, the factor x^m will divide out from the result and an equation in m will remain. The n roots of this latter equation will determine the complementary function.

EXAMPLES.—(1) Solve $x^2 . d^2y/dx^2 + 2x . dy/dx - 2y = 0$. Put $y = x^m$. We get

$$m(m - 1) + 2(m - 1) = 0;\ \text{or, } (m + 2)(m - 1) = 0.$$

Hence from our preceding results, we can write down the complementary function at sight, $y = C_1x + C_2x^{-2}$.

(2) Solve $x^2 . d^2y/dx^2 + 4x . dy/dx + 2y = 0$. Ansr. $y = C_1/x + C_2/x^2$.

(3) Find the complementary function in $\{\vartheta(\vartheta - 1) - 3\vartheta + 4\}y = x^3$. Ansr. $y = (C_1 + C_2 \log x)x^2$.

(4) Integrate $\{\vartheta(\vartheta - 1) - 2\}y = 0$. Ansr. $y = C_1x^2 + C_2/x$.

ii. *The particular integral.* We may use the operator ϑ, to obtain the particular integral of linear equations with variable coefficients in the same way that D was used to determine the particular integral of equations with constant coefficients.

The symbolic form of the particular integral is,

$$y = \frac{R}{f(\vartheta)}.$$

The operator $f(\vartheta)^{-1}$ may be resolved into partial fractions or into factors as in the case of D.

EXAMPLES.—(1) Show that $y = C_1x^4 + C_2/x + \frac{1}{5}x^4 \log x$ is a complete solution of $x^2 . d^2y/dx^2 - 2x . dy/dx - 4y = x^4$.

(2) Find the value of $\frac{1}{\vartheta^2(\vartheta - 3)}x^3$. Using the ordinary method just described we get the indeterminate form $\frac{1}{9} \cdot \frac{x^3}{\vartheta - 3}$. In this case we must adopt the method of page 312 and write

$$\frac{1}{9}x^3\frac{1}{\vartheta - 1} \cdot 1 = \frac{1}{9}x^3\int\frac{dx}{x} = \frac{1}{9}x^3 \log x.$$

(3) Solve $x^2 . d^2y/dx^2 + 7x . dy/dx + 5y = x^5$. Write this

$$\{\vartheta(\vartheta - 1) + 7\vartheta + 5\}y = x^5.$$

The particular integral is $(\vartheta^2 + 6\vartheta + 5)^{-1}x^5$, or $x^5/60$. The complementary function is $C_1x^{-1} + C_2x^{-5}$.

(4) Solve $x^2d^2y/dx^2 + 4x . dy/dx + 2y = e^x$.

Ansr. $y = C_1/x + C_2/x^2 + e^x/x^2$.

(5) Solve $x^3 . d^3y/dx^3 + 2x^2 . d^2y/dx^2 - x . dy/dx + y = x + x^3$.

Ansr. $y = C_1/x + C_2x + C_3x \log x + \frac{1}{4}x(\log x)^2 + x^3/16$.

(6) Solve $x^3 . d^3y/dx^3 + 2x^2 . d^2y/dx^2 + 2y = 10x + 10/x$.

Ansr. $y = C_1x \cos (\log x) + C_2x \sin (\log x) + 5x + C_3/x + (2 \log x)/x$.

(7) Find the particular integral of the third example in the last set. Ansr. x^3.

(8) Equate example (2), of the preceding set, to $1/x$, instead of to zero, and show that the particular integral is then $(\log x)/x$.

Case 2. Legendre's Equation. Type:

$$(a + bx)^n\frac{d^ny}{dx^n} + A_1(a + bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + A_ny = R, \quad (3)$$

where $A_1, A_2, \ldots, A_n$ are constants, R is any function of x. This sort of equation is easily transformed into the homogeneous equation and, therefore, into the linear equation with constant coefficients. To make the former transformation, substitute $z = a + bx$, for the latter, $e = a + bx$.

EXAMPLES.—(1) Solve

$$(a + bx)^2 . d^2y/dx^2 + b(a + bx) . dy/dx + c^2y = 0.$$

Ansr. $y = C_1 \sin \{(c/b) \log (a + bx)\} + C_2 \cos \{(c/b) \log (a + bx)\}$.

(2) Solve $(x + a)^2d^2y/dx^2 - 4(x + a) . dy/dx + 6y = x$.

Ansr. $y = C_1(x + a)^2 + C_2(x + a)^3 + \frac{1}{6}(3x + 2a)$.

§ 132. The Exact Linear Differential Equation.

A very simple relation exists between the coefficients of an exact differential equation which may be used to test whether the equation is exact or not. Take the equation,

$$X_0\frac{d^3y}{dx} + X_1\frac{d^2y}{dx^2} + X_2\frac{dy}{dx} + X_3y = R. \quad . \quad . \quad (1)$$

where X_0, X_1, . . ., R are functions of x. Let their successive differential coefficients be indicated by dashes, thus X', X'', . . .

Since $X_0 \,.\, d^3y/dx^3$ has been obtained by the differentiation of $X_0{}^2 \,.\, d^2y/dx^2$, this latter is necessarily the first term of the integral of (1). But,

$$\frac{d}{dx}\Big(X_0\frac{d^2y}{dx^2}\Big) = X_0\frac{d^3y}{dx^3} + X'_0\frac{d^2y}{dx^2}.$$

Subtract the right-hand side of this equation from (1),

$$(X_1 - X'_0)\frac{d^2y}{dx^2} + X_2\frac{dy}{dx} + X_3y = R. \qquad . \qquad . \qquad (2)$$

Again, the first term of this expression is a derivative of $(X_1 - X'_0)dy/dx$. This, therefore, is the second term of the integral of (1). Hence, by differentiation and subtraction, as before,

$$(X_2 - X'_1 + X''_0)\frac{dy}{dx} + X_3y = R. \qquad . \qquad . \qquad (3)$$

This equation may be deduced by the differentiation of $(X_2 - X'_1 + X_0'')y$, provided the first differential coefficient of $(X_2 - X'_1 + X'_0)$ with respect to x, is equal to X_3, that is to say,

$$X'_2 - X''_1 + X'''_0 = X_3\,;\ \text{or, } X_3 - X'_2 + X''_1 - X'''_0 = 0. \quad (4)$$

But if this is really the origin of (3), the original equation (1) has been reduced to a lower order, namely,

$$X_0\frac{d^2y}{dx^2} + (X_1 - X'_0)\frac{dy}{dx} + (X_2 - X'_1 + X''_0)y = R + C_1. \quad (5)$$

This equation is called the **first integral** of (1), because the order of the original equation has been lowered unity, by a process of integration.

Condition (4) is *a test of the exactness of a differential equation.*

If the first integral is an exact equation, we can reduce it, in the same way, to another first integral of (1). The process of reduction may be repeated until an inexact equation appears, or until y itself is obtained. Hence, *an exact equation of the nth order has n independent first integrals.*

EXAMPLES.—(1) Is the equation

$$x^5 \,.\, d^3y/dx^3 + 15x^4 \,.\, d^2y/dx^2 + 60x^3 \,.\, dy/dx + 60x^2y = e^x \text{ exact?}$$

From (4), $X_3 = 60x^2$; $X'_2 = 180x^2$; $X''_1 = 180x^2$; $X'''_0 = 60x^2$. Therefore, $X_3 - X'_2 + X''_1 + X'''_0 = 0$ and the equation is exact.

(2) Solve the equation

$$xd^3y/dx^3 + (x^2 - 3)d^2y/dx^2 + 4x \,.\, dy/dx + 2y = 0.$$

as far as possible, by successive reduction. The process can be employed twice, the residue is a linear equation of the first order, not exact.

(3) Solve the equation given in example (1).

Ansr. $x^5y = e^x + C_1x^2 + C_2x + C_3$.

There is another *quick practical test for exact differential equations* (Forsyth) which is not so general as the preceding. When the terms in X are either in the form of ax^m, or of the sum of expressions of this type, $x^m d^n y/dx^n$ is a perfect differential coefficient, if $m < n$. This coefficient can then be integrated whatever be the value of y. If $m = n$ or $m > n$, the integration cannot be performed by the method for exact equations. To apply the test, remove all the terms in which m is less than n, if the remainder is a perfect differential coefficient, the equation is exact and the integration may be performed.

EXAMPLES.—(1) Apply the test to

$$x^3 . d^4y/dx^4 + x^2 . d^3y/dx^3 + x . dy/dx + y = 0.$$

$x . dy/dx + y$ remains. This has evidently been formed by the operation $D(xy)$, hence the equation is a perfect differential.

(2) Apply the test to

$$(x^3D^4 + x^2D^3 + x^2D + 2x)y = \sin x.$$

$x^2 . dy/dx + 2xy$ remains. This is a perfect differential, formed from $D(x^2y)$. The equation is exact.

If two independent first integrals are known the equation is sometimes easily solved. The elimination of dy/dx between two first integrals will give the complete solution.

§ 133. The Integration of Equations with Missing Terms.

Differential equations with missing letters are common.

First, the independent variable is absent. Type:

$$d^2y/dx^2 = qy\text{; or, } d^2y/dx^2 = qf(y). \quad . \quad . \quad (1)$$

This equation is, in general, neither linear nor exact.

Case 1. *When* $f(y)$, *in* (1), *is negative*, so that

$$\frac{d^2x}{dt^2} + q^2x = 0, \quad . \quad . \quad . \quad . \quad (2)$$

where the academic x and y have given place to t and x respectively, in order to give the equation the familiar form of the equation of the motion of a particle under the influence of a central attracting force.

Multiply both sides of the equation by $2dx/dt$, and integrate with respect to x,

$$2\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} = - 2q^2x\frac{dx}{dt}\text{; or } \left(\frac{dx}{dt}\right)^2 = - q^2x^2 + C.$$

Separate the variables and integrate again,

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = \pm\, qdt\text{; } \cos^{-1}\frac{x}{a} = \pm\,(qt + \epsilon),$$

where ϵ is the integration constant and $C = q^2a^2$. The solution involves two arbitrary constants a and ϵ, which respectively denote the amplitude and epoch of a simple harmonic motion, whose period of oscillation is $2\pi/q$. Put $C_1 = a \cos \epsilon$, and $C_2 = - a \sin \epsilon$.

$$x = C_1 \cos qt + C_2 \sin qt.$$

Case 2. *When* $f(y)$ *in* (1) *is positive*, the solution assumes the form,

$$x = C_1e^{qt} + C_2e^{-qt};\ \text{or,}\ x = A \cosh qt + B \sinh qt,$$

as on page 309. All these results are important in connection with alternating currents and other forms of harmonic motion.

Another way of treating equations of type (2), occurs with an equation like

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 2y^2 = 0, \quad . \quad . \quad . \quad (3)$$

which has the form of the standard equation for the small oscillations of a pendulum in air. Under this condition, the resistance of the air is negligible. Let

$$p = dy/dx, \therefore d^2y/dx^2 = dp/dx = p\,.\,dp/dy.$$

Substitute these results in the given equation, multiply through with $2/y$.

$$py\frac{dp}{dy} + p^2 = 2y^2;\ \text{or,}\ \frac{dp^2}{dy} + \frac{2}{y}p^2 = 4y.$$

Multiply by y^2 and

$$\frac{d(y^2p^2)}{dy} = 4y^3;\ \text{or,}\ p^2y^2 = y^4 + C^4,$$

where C^4 is an arbitrary constant. The rest is obvious.

Examples.—(1) The solution of equation (3) is sometimes written in the form

$$y^2 = C^2{}_1\sinh(2x + C_2).$$

Verify this.

(2) Solve $d^2x/dt^2 + \mu x + v = 0$. Put $x = x_1 + v/\mu$ and afterwards omit the suffix. Ansr. $x = v/\mu + C_1\cos t\sqrt{\mu} + C_2\sin t\sqrt{\mu}$.

(3) If the term μx in the preceding example had been of opposite sign, show that the solution would have been, $x = v/\mu + C_1\cosh t\sqrt{\mu} + C_2\sinh t\sqrt{\mu}$, where μ is negative.

(4) Solve $d^2y/dx^2 - a(dy/dx)^2 = 0$. Ansr. $C_1x + C_2 = e^{-ay}$.

(5) Solve $1 + (dy/dx)^2 = yd^2y/dx^2$. Ansr. $y = a\cosh(x/a + b)$.

(6) *Fourier's equation for the propagation of heat in a cylindrical bar*, is $d^2V/dx^2 - \beta^2V = 0$. Hence show that $V = C_1e^{\beta x} + Ce^{-\beta x}$.

Second, the dependent variable is absent. Type:

$$d^2y/dx^2 = x;\ \text{or,}\ d^2y/dx^2 = f(x). \quad . \quad . \quad (4)$$

If these equations are exact, they may be solved by successive integration.

If the equation has the form

$$d^2y/dx^2 + dy/dx + x = 0.$$

Say,
$$\frac{d^2v}{dr^2} + \frac{1}{r}\cdot\frac{dv}{dr} = -\frac{P}{l\mu},$$

a familiar equation in hydrodynamics, it is usually solved by substituting $p = dy/dx$, $\therefore\ dp/dx = d^2y/dx^2$. The resulting equation is of the first order, integrable in the usual way.

EXAMPLES.—(1) The above equation represents the motion of a fluid in a cylindrical tube of radius r and length l. The motion is supposed to be parallel to the axis of the tube and the length of the tube very great in comparison with its radius r. P denotes the difference of the pressure at the two ends of the tube. If the liquid wets the walls of the tube, the velocity is a maximum at the axis of the tube and gradually diminishes to zero at the walls. This means that the velocity is a function of the distance (r_1) of the fluid from the axis of the tube. Solve the equation, remembering that μ is a constant depending on the nature of the fluid.

Substitute $p = dv/dr$,

$$\therefore\ dp/dr + p/r = -P/l\mu;$$

$$r\frac{dp}{dr} + p = -\frac{P}{l\mu}r, \text{ is } pr = -\frac{P}{2l\mu}r^2 + C_1; \quad . \quad . \quad . \quad (5)$$

$$\therefore\ \frac{dv}{dr} = -\frac{P}{2l\mu}r + \frac{C_1}{r};\ v = -\frac{P}{4l\mu}r^2 + C_1\log r + C_2.$$

To evaluate C_1 in (5), note that at the axis of the tube $r = 0$. This means that if C_1 is a finite or an infinite magnitude the velocity will be infinite. This is obviously impossible, therefore, C_1 must be zero. To evaluate C_2, note that when $r = r_1$, v vanishes and, therefore, we get the final solution of the given equation in the form, $v = \frac{1}{4}P(r^2_1 - r^2)/l\mu$, which represents the velocity of the fluid at a distance r_1 from the axis.

(2) Solve $ad^2y/dx^2 = \sqrt{1 + (dy/dx)^2}$. Make the necessary substitutions and integrate.

$$a\,.\,dp/\sqrt{(1 + p^2)} = dx;\ \text{becomes } x/a = \log(p + \sqrt{p^2 + 1}) + C;$$

or, in the exponential form,

$$C_1e^{x/a} - p = \sqrt{(p^2 + 1)};\ \text{and } dy/dx = \tfrac{1}{2}C_1e^{x/a} - e^{-x/a}/2C_1,$$

by squaring. On integration

$$y = \tfrac{1}{2}aC_1e^{x/a} + \tfrac{1}{2}ae^{-x/a}/C_1 + C_2.$$

(3) Some expressions can be reduced to the standard form by an obvious transformation. Thus,

$$d^5y/dx^5 - d^3y/dx^3 = x.$$

Substitute p for d^3y/dx^3 and differentiate $p = d^3y/dx^3$ twice. Thus,

$$d^2p/dx^2 - p = x,$$

whence y can be obtained by successive integration as indicated above.

(4) Solve $d^2V/dr^2 + 2dV/n\,.\,dr = 0$. This equation occurs in the theory of

potential. Put dV/dr for the independent variable and divide through. On integration

$$\log(dV/dr) + 2\log r = \log C_1,$$

where $\log C_1$ is an arbitrary constant. Integrate again

$$dV/dr = C_1/r, \text{ becomes } V = C_2 - C_1/r.$$

(5) If $x \, . \, d^2y/dx^2 = 1$, show that $y = x \log x + C_1x + C_2$.

§ 134. Equations of Motion, chiefly Oscillatory Motion.

By *Newton's second law*, if a certain mass (m) of matter is subject to a constant force (F_0) for a certain time, we have, in rational units,

$$F_0 = (Mass) \times (Acceleration\ of\ the\ particle).$$

If the motion of the particle is subject to friction, we must regard the friction as a force tending to oppose the motion generated by the impressed force. But friction is proportional to the velocity (v) of the motion of the particle, and equal to the product of the velocity and a constant called the *coefficient of friction*, written, say, μ. Let F_1 denote the total force acting on the particle in the direction of its motion,

$$F_1 = F_0 - \mu v = md^2s/dt^2. \quad . \quad . \quad . \quad (1)$$

If there is no friction, we have, for unit mass,

$$F_0 = d^2s/dt^2. \quad . \quad . \quad . \quad . \quad (2)$$

The motion of a pendulum in a medium which offers no resistance to its motion, is that of a material particle under the influence of a central force (F) attracting with an intensity which is proportional to the distance of the particle away from the centre of attraction. That is (Fig. 7),

$$F = -q^2s. \quad . \quad . \quad . \quad . \quad (3)$$

where q^2 is to be regarded as a positive constant which tends to restore the particle to a position of equilibrium—the so-called *coefficient of restitution*. It is written in the form of a power to avoid a root sign later on. The negative sign shows that the attracting force (F) tends to diminish the distance (s) of the particle away from the centre of attraction. If $s = 1$, q^2 represents the magnitude of the attracting force unit distance away. From (2),

$$\frac{d^2s}{dt^2} = -q^2s. \quad . \quad . \quad . \quad . \quad (4)$$

This is a typical equation of harmonic motion, as will be shown directly. One solution of (4) is

$$s = C\cos(qt + \epsilon). \quad . \quad . \quad . \quad (5)$$

This equation is the simple harmonic motion of § 50, C denotes the amplitude of the vibration. If $\epsilon = 0$, we have the simpler equation,

$$s = C \cos qt. \quad . \quad . \quad . \quad . \quad (6)$$

When the particle is at its greatest distance from the central attracting force, $qt = \pi$, § 50, page 112. For a complete to and fro motion, $2t = T_0$ = period of oscillation, hence

$$T_0 = 2\pi/q. \quad . \quad . \quad . \quad . \quad (7)$$

Equation (4) represents the small oscillations of a pendulum; also the undamped* oscillations of the magnetic needle of a galvanometer.

In the sine galvanometer, the restitutional force tending to restore the needle to a position of equilibrium, is proportional to the sine of the angle of deflection of the needle. If J denotes the moment of inertia of the magnetic needle and G the directive force exerted by the current on the magnet, the equation of motion of the magnet, when there is no retarding force, is

$$J\frac{d^2\phi}{dt^2} = -G \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For small angles of displacement, ϕ and $\sin \phi$ are approximately equal. Hence,

$$\frac{d^2\phi}{dt^2} = -\frac{G}{J}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (4), $q = \sqrt{G/J}$, and therefore, from (9),

$$T_0 = 2\pi\sqrt{J/G}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

a well-known relation showing that *the period of oscillation of a magnet in the magnetic field, when there is no damping action exerted on the magnet, is proportional to the square root of the moment of inertia of the magnetic needle, and inversely proportional to the square root of the directive force exerted by the current on the magnet.* See page 524.

In a similar manner, it can be shown that the period of the small oscillations of a pendulum suspended freely by a string of length l, is $2\pi\sqrt{l/g}$, where g denotes the acceleration of gravity.

Equation (4) takes no account of the resistance to which a particle is subjected as it moves through such resisting media as

---

* When an electric current passes through a galvanometer, the needle is deflected and begins to oscillate about a new position of equilibrium. In order to make the needle come to rest quickly, so that the observations may be made quickly, some resistance is opposed to the free oscillations of the needle either by attaching mica or aluminum vanes to the needle so as to increase the resistance of the air, or by bringing a mass of copper close to the oscillating needle. The currents induced in the copper by the motion of the magnetic needle, react on the moving needle, according to Lenz's law, so as to retard its motion. Such a galvanometer is said to be damped. When the damping is sufficiently great to prevent the needle oscillating at all, the galvanometer is said to be "dead beat" and the motion of the needle is aperiodic. In ballistic galvanometers, there is very much damping.

air, water, etc. This resistance is proportional to the velocity, and has a negative value. To allow for this, equation (4) must have an additional negative term. We thus get

$$\frac{d^2s}{dt^2} = -\mu\frac{ds}{dt} - q^2s,$$

where μ is the coefficient of friction. For greater convenience, we may write this $2f$,

$$\frac{d^2s}{dt^2} + 2f\frac{ds}{dt} + q^2s = 0. \quad . \quad . \quad . \quad (11)$$

Before proceeding further, it will perhaps make things plainer to put the meaning of this differential equation into words. The manipulation of the equations so far introduced, involves little more than an application of common algebraic principles. Dexterity in solving comes by practice. Of even greater importance than quick manipulation is the ability to form a clear concept of the physical process symbolised by the differential equation. Some of the most important laws of Nature appear in the guise of an "unassuming differential equation". The reader should spare no pains to acquire familiarity with the art. The late Professor Tait has said that "a mathematical formula, however brief and elegant, is merely a step towards knowledge, and an all but useless one until we can thoroughly read its meaning".

In equation (11), the term d^2s/dt^2 denotes the relative change of the velocity of the motion of the particle in unit time, § 7; $2f.ds/dt$ shows that this motion is opposed by a force which tends to restore the body to a position of rest, the greater the velocity of the motion, the greater the retardation; q^2s represents another force tending to bring the moving body to rest, this force also increases directly as the distance of the body from the position of rest. To investigate this motion further, we cannot do better than follow Professor Perry's graphic method.

The first thing is to solve (11) for s. This is done by the method of § 130. Put $s = e^{mt}$ and solve the auxillary quadratic equation. We thus obtain

$$m = -f \pm \sqrt{(f^2 - q^2)}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

And finally,

$$s = e^{-(\alpha+\beta)t}; \text{ or rather, } s = C_1e^{-\alpha t} + C_2e^{-\beta t},$$

where $\alpha = -f + \sqrt{(f^2 - q^2)}$ and $\beta = -f - \sqrt{(f^2 - q^2)}$. The solution of (11) thus depends on the relative magnitudes of f and q.

Suppose that we know enough about the moving system to be able to determine the integration constants. When $t = 0$, let $v = v_0$ and $s = 0$.

Case i. *The roots of the auxillary equation are real and unequal.* The condition for real roots $-\alpha$ and $-\beta$, in (12), is that f is greater than q (page 388). In this case,

$$s = C_1e^{-\alpha t} + C_2e^{-\beta t}, \quad . \quad . \quad . \quad . \quad (13)$$

solves equation (11). To find what this means, let us suppose that $f = 3$, $q = 2$, $t = 0$, $s = 0$, $v_0 = 0$. From (12), therefore,

$$m = -3 \pm \sqrt{9-4} = -3 \pm 2{\cdot}24 = -{\cdot}76 \text{ and } -5{\cdot}24.$$

Substitute these values in (13) and differentiate for the velocity v or ds/dt. Thus,

$$s = C_1e^{-5{\cdot}24t} + C_2e^{-{\cdot}76t}; \; ds/dt = -5{\cdot}24C_1e^{-5{\cdot}24t} + {\cdot}76C_2e^{-{\cdot}76t}.$$

$$\therefore -5{\cdot}24C_1 + {\cdot}75C_2 = 1.$$

From (13), when $t = 0$, $s = 0$ and $C_1 + C_2 = 0$, or $C_1 = +C_2 = \frac{1}{6}$,

$$\therefore s = \tfrac{1}{6}(e^{-{\cdot}76t} - e^{-5{\cdot}24t}). \quad . \quad . \quad . \quad . \quad (14)$$

Assign particular values to t, and plot the corresponding values of s by means of Tables XXI. and XXII. Curve No. 1 (Fig. 114) was obtained by plotting corresponding values of s and t obtained in this way.

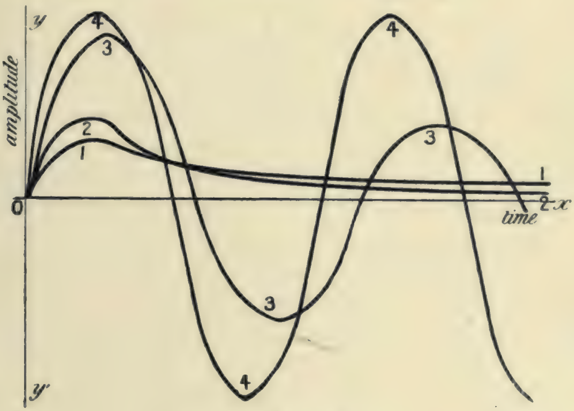


FIG. 114 (after Perry).

Case ii. *The roots of the auxillary equation are real and equal.* The condition for real and equal roots is that $f = q$.

$$\therefore s = (C_1 + C_2t)e^{-ft}. \quad . \quad . \quad . \quad . \quad (15)$$

As before, let $f = 2$, $q = 2$, $t = 0$, $s = 0$, $v_0 = 1$. The roots of the auxillary are -2 and -2. Hence

$$s = (C_1 + C_2t)e^{-2t}; \text{ and } ds/dt = C_2e^{-2t} - 2(C_1 + C_2t)e^{-2t}.$$

$$\therefore C_2 - 2C_1 = 1, \; C_1 = 0 \text{ and } C_2 = 1; \text{ or } s = te^{-2t}. \quad . \quad . \quad (16)$$

Plot (16) in the usual manner. Curve 2 (Fig. 114) was so obtained.

Case iii. *The roots of the auxillary equation are real, equal and of opposite sign.* For equal roots of opposite sign, say $\pm q$, we must have $f = 0$. Then

$$s = C_1 \sin qt + C_2 \cos qt. \quad . \quad . \quad . \quad . \quad (17)$$

Let $t = 0$, $s = 0$, $v_0 = 1$, $q = 2$, $f = 0$. Differentiate (17),

$$ds/dt = qC_1 \cos qt - qC_2 \sin qt.$$

Hence $1 = 2C_1 \times 1 - 2 \times C_2 \times 0$, or $C_1 = \frac{1}{2}$; $\therefore C_2 = 0$. Hence the equation,

$$s = \tfrac{1}{2}\sin 2t. \qquad (18)$$

A graph from this equation is shown in curve 4 (Fig. 114).

Case iv. *The roots of the auxillary equation are imaginary.* For imaginary roots, $-f \pm \sqrt{(f^2 - q^2)}$, or, say $a \pm b\iota$, it is necessary that $f < q$ (page 388). In this case,

$$s = e^{-at}(C_1 \sin bt + C_2 \cos bt). . \qquad (19)$$

Let the coefficient of friction, $f = 1$, $q = 2$, $t = 0$, $s = 0$, $v_0 = 1$. The roots of the auxillary are $m = -1 \pm \sqrt{1 - 4} = 1 \pm \sqrt{-3} = -1 \pm 1{\cdot}7\iota$, where $\iota = \sqrt{-1}$. Hence $a = 1$, $b = 1{\cdot}7$. Differentiate (19),

$$ds/dt = -ae^{-at}(C_1 \sin bt + C_2 \cos bt) + be^{-at}(C_1 \cos bt - C_2 \sin bt).$$

From (19), $C = 0$ and, therefore, $C_1 = 1/b = {\cdot}57$. Therefore,

$$s = {\cdot}57e^{-t}\sin 1{\cdot}7t. \qquad (20)$$

Curve 3 (Fig. 114) was plotted from equation (20) in the usual way.

There are several interesting features about the motions represented by these four solutions of (11), shown graphically in Fig. 114. Curves Nos. 3 and 4 (Cases iv. and iii.) show the conditions under which the equation of motion (11) is periodic or vibratory. The effects of increased friction due to the viscosity of the medium, is shown very markedly by the lessened amplitude and increased period of curve 3. The net result is a **damped vibration,** which dies away at a rate depending on the resistance of the medium $(2f \, . \, v)$ and on the magnitude of the oscillations (q^2s). Such is the motion of a magnetic or galvanometer needle affected by the viscosity of the air and the electromagnetic action of currents induced in neighbouring masses of metal by virtue of its motion; it also represents the **natural oscillations** of a pendulum swinging in a medium whose resistance varies as the velocity. Curve 4 represents an undamped oscillation, curve 3 a damped oscillation.

Curves 1 and 2 (Cases i. and ii.) represent the motion when the retarding forces are so great that the vibration cannot take place. The needle, when removed from its position of equilibrium, returns to its position of rest after the elapse of an infinite time. (What does this statement mean? Compare with page 329.) Raymond calls this an **aperiodic motion.**

To show that the period of oscillation is augmented by damping. From equation (19) we can show that

$$s = e^{-at}A \sin bt. \qquad (21)$$

§ 50. The amplitude of this vibration corresponds to that value of t for which s has a maximum or a minimum value. These values are obtained in the usual way, by equating the first differential coefficient to zero, hence

$$e^{-at}(b \cos bt - a \sin bt) = 0. \qquad (22)$$

If we now define the angle ϕ such that $bt = \phi$, or

$$\tan \phi = b/a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

ϕ, lying between 0 and $\frac{1}{2}\pi$ (*i.e.*, 90°), becomes smaller as a increases in value. We have just seen that the imaginary roots of $-f \pm \sqrt{f^2 - q^2}$ are $-a \pm bi$, for values of f less than q. Let

$$a^2 + b^2 = q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The period of oscillation of an undamped oscillation is, by (7), $T_0 = 2\pi/q$, of a damped oscillation $T = 2\pi/b$.

$$\therefore T^2/T^2_0 = q^2/b^2 = (a^2 + b^2)/b^2 = 1 + a^2/b^2.$$

$$\therefore T/T_0 = \sqrt{(a^2 + b^2)}/b, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which expresses the relation between the periods of oscillation of a damped and of an undamped oscillation. The period of vibration is thus augmented on damping.

It is easy to show by plotting that $\tan \phi$, of (23), is a periodic function such that

$$\tan \phi = \tan (\phi + \pi) = \tan (\phi + 2\pi) = \ldots$$

Hence $\qquad \phi, \phi + \pi, \phi + 2\pi, \ldots$

satisfy the above equation. It also follows that

$$bt_1, bt_2 + \pi, bt_3 + 2\pi, \ldots$$

also satisfy the equation, where $t_1, t_2, t_3, \ldots$ are the successive values of the time. Hence

$$bt_2 = bt_1 + \pi, bt_3 = bt_1 + 2\pi, \ldots;$$

$$\therefore t_2 = t_1 + \tfrac{1}{2}T, t_3 = t_1 + T, \ldots$$

Substitute these values in (21) and put $s_1, s_2, s_3, \ldots$ for the corresponding displacements,

$$\therefore s_1 = Ae^{-at_1} \sin bt_1 ; \; - s_2 = Ae^{-at_2} \sin bt_2 ; \ldots$$

where the negative sign indicates that the displacement is on the negative side. Hence

$$s_1/s_2 = e^{-a(t_1 - t_2)} = e^{aT/2};$$

$$s_2/s_3 = s_3/s_4 = \ldots = e^{aT/2}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The amplitude thus diminishes in a constant ratio. Plotting these successive values of s and t, we get the curve shown in Fig. 115. This ratio is called the **damping ratio**, by Kohlrausch ("Dämpfungsverhaltnis"). It is written k. The natural logarithm of the damping ratio, is Gauss' **logarithmic decrement**, written λ (the ordinary logarithm of k, is written L). Hence

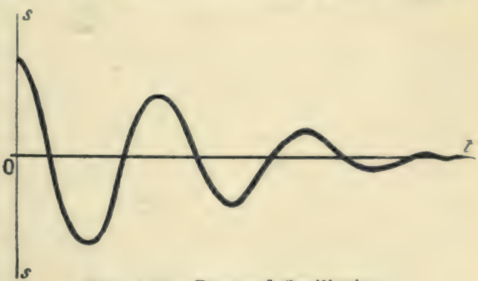


FIG. 115.—Damped Oscillation.

$$\lambda = \log k = aT \log e = aT = a\pi/b, \quad . \quad . \quad . \quad (27)$$

and from (25),

$$\frac{T^2}{T^2_0} = 1 + \frac{\lambda^2}{\pi^2}; \text{ or } T = T_0\left(1 + \frac{1}{2} \cdot \frac{\lambda^2}{\pi^2} + \ldots\right). \quad . \quad . \quad (28)$$

Hence, if the damping is small, the period of oscillation is augmented by a small quantity of the second order.

The following table contains six observations of the amplitudes of a sequence of damped oscillations:

Observed Deflection.	k.	λ.	L.
69			
	1·438	0·3633	0·1578
48			
	1·434	0·3604	0·1565
33·5			
	1·426	0·3548	0·1541
23·5			
	1·425	0·3542	0·1538
16·5			
	1·435	0·3612	0·1569
11·5			
	1·438	0·3633	0·1578
8			

Meyer, Maxwell, etc., have calculated the viscosity of gases from the rate at which the small oscillations of a vibrating pendulum are damped.

When the motion represented by equation (11) is subject to some periodic impressed force which prevents the oscillations dying away, the resulting motion is said to be a **forced vibration.** The equation representing such an oscillation is

$$\frac{d^2s}{d^2t} + \frac{ds}{dt} + q^2s = f(x). \qquad (29)$$

When $f(x) = 0$, the equation refers to the natural oscillations of a vibrating electrical or mechanical system. The impressed force is, therefore, mathematically represented by the particular integral of equation (29) (see example (3) below).

The subjoined examples principally refer to systems in harmonic motion.

EXAMPLES.—(1) Ohm's law for a constant current is $E = RC$; for a variable current of C ampères flowing in a circuit with a coefficient of self-induction of L henries, with a resistance of R ohms and an electromotive force of E volts, Ohm's law is represented by the equation,

$$E = RC + L \,.\, dC/dt, \qquad (30)$$

where dC/dt evidently denotes the rate of increase of current per second, L is the equivalent of an electromotive force tending to retard the current.

(i.) *When E is constant*, the solution of (30) has been obtained in a preceding set of examples,

$$C = E/R + Be^{-Rt/L},$$

where B is the constant of integration. To find B, note that when $t = 0$, $C = 0$. Hence,

$$C = E(1 - e^{-Rt/L})/R. \qquad (31)$$

The second term is the so-called "extra current at make," an evanescent factor due to the starting conditions. The current, therefore, tends to assume the steady condition: $C = E/R$, when t is very great.

(ii.) *When C is an harmonic function of the time*, say,

$$C = C_0 \sin qt; \therefore dC/dt = C_0 q \cos qt.$$

Substitute these values in the original equation,

$$E = RC_0 \sin qt + LC_0 q \cos qt,$$

or, compounding these harmonic motions (§ 50),

$$E = C_0\sqrt{R^2 + L^2q^2} \,.\, \sin(qt + \epsilon),$$

where $\epsilon = \tan^{-1}(Lq/R)$, the so-called lag* of the current behind the electro-motive force, the expression $\sqrt{(R^2 + L^2q^2)}$ is the so-called *impedance*.

(iii.) *When E is a function of the time*, say $f(t)$,

$$C = Be^{-Rt/L} + \frac{1}{L}e^{-Rt/L}\int e^{-Rt/L}f(t)\,.\,dt,$$

where B is the constant of integration to be evaluated as described above.

(iv.) *When E is a simple harmonic function of the time*, say,

$$E = E_0 \sin qt,$$

then, $\quad C = Be^{-Rt/L} + E_0 \sin(qt + \epsilon)/\sqrt{(R^2 + L^2q^2)}$.

The evanescent term $e^{-Rt/L}$ may be omitted when the current has settled down into the steady state. (Why?)

(v.) *When E is zero*,

$$C = Be^{-Rt/L}.$$

Evaluate the integration constant B by putting $C = C_0$, when $t = 0$.

(2) The relation between the charge (q) and the electromotive force (E) of two plates of a condensor of capacity C connected by a wire of resistance R, is

$$E = R \,.\, dq/dt + q/C,$$

provided the self-induction is zero. Solve for q. Show that when

$E = f(t)$, $q = \frac{1}{R}e^{-t/RC}\int e^{-t/RC}f(t)\,.\,dt + Be^{-t/RC}$;

$E = 0$, $\quad q = Q_0e^{-t/RC}$; (Q_0 is the charge when $t = 0$).

$E =$ constant, $q = CE + Be^{-t/RC}$;

$E = E_0 \sin qt$, $\; q = Be^{-t/RC} + CE(\sin qt + RCq \cos qt)/(1 + R^2C^2q^2)$.

(3) The equation of motion of a pendulum subject to a resistance which varies with the velocity and which is acted upon by a force which is a simple harmonic function of the time, is

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + q^2x = \cos(qt + \epsilon).$$

Show that the complementary function is

$$x = A\cos(qt + \epsilon) + B\sin(qt + \epsilon).$$

---

* An alternating (periodic) current is not always in phase (or, "in step") with the impressed (electromotive) force driving the current along the circuit. If there is self-induction in the circuit, the current **lags** behind the electromotive force; if there is a condensor in the circuit, the current in the condensor is greatest when the electromotive force is changing most rapidly from a positive to a negative value, that is to say, the maximum current is in advance of the electromotive force, there is then said to be a **lead** in the phase of the current.

To solve the equation, assume that

$$x = A\cos(qt + \epsilon) + B\sin(qt + \epsilon),$$

is a solution. Substitute in the given equation,

$$\therefore - Aq^2 + 2fBq + n^2A = 1 \,;\, \therefore - Aq^2 + 2fBq + n^2B = 0.$$

$$A = \frac{n^2 - q^2}{(n^2 - q^2) + 4f^2q^2};\; B = \frac{2fq}{(n^2 - q^2) + 4f^2q^2}.$$

Put $A = R\cos\epsilon. \quad B = R\sin\epsilon.$

The solution of the given equation is then

$$x = R\cos(qt + \epsilon - \epsilon_1), \qquad (32)$$

where $R = 1/\sqrt{\{(n^2 - q^2)^2 + 4f^2q^2\}}$; $\tan\epsilon = 2fq/(n^2 - q^2)$.

The forced oscillations due to the impressed periodic force, are thus determined by (32). The complementary function gives the natural vibrations superposed upon these.

(4) If the friction in the preceding example, is zero,

$$\frac{d^2x}{dt^2} + n^2x = \cos(qt + \epsilon). \qquad (33)$$

A particular integral is $x = \{f.\cos(qt + \epsilon)\}/(n^2 - q^2)$. This fails when $n = q$. In this case, assume that $x = Ct\sin(nt + \epsilon)$ is a particular integral. (33) is satisfied provided $C = f/2n$. The physical meaning of this is that when the pendulum is acted on by a periodic force "in step" with the oscillations of the pendulum, the amplitude of the forced oscillations will increase proportionally with the time, until, when the amplitude exceeds a certain limit, equation (33) no longer represents the motion of the pendulum.

(5) When an electric current, passing through an electrolytic cell, has assumed the steady state, show that the ionic velocity is proportional to the impressed force (electromotive force). By Newton's law, for a moving body,

(*Impressed force*) = (*Mass*) × (*Acceleration*).

Friction is to be regarded as a retarding force acting in an opposite direction to the impressed force; this frictional force is proportional to the velocity of the body.

∴ (*Impressed force less friction*) = (*Mass*) × (*Acceleration*).

Express these facts in symbolic language. See (1) above. Integrate the result and evaluate the constant for $v = 0$, when $t = 0$.

$$\therefore \mu v = F(1 - e^{-\mu t/m}). \qquad (34)$$

For ionic motion, m is very small, μ is very great. When t is great, show that the exponential term vanishes, and

$$F \propto v.$$

Ohm's law. Compare with (31).

§ 135. The Velocity of Simultaneous and Dependent Chemical Reactions.

While investigating the rate of decomposition of phosphine (§ 88), we had occasion to point out that the action really takes place in two stages :—

STAGE I. $PH_3 = P + 3H.$

STAGE II. $4P = P_4;\ 2H = H_2.$

The former change alone determines the velocity of the whole reaction. The physical meaning of this is that the speed of the reaction which occurs during stage II., is immeasurably faster than the speed of the first. Experiment quite fails to reveal the complex nature of the complete reaction.*

Suppose, for example, a substance A forms an intermediate compound B, and this, in turn, forms a final product C. If the speed of the reaction

$A = B$, is one gram per $\frac{1}{100000}$ second,

when the speed of the reaction

$B = C$, is one gram per hour,

the observed "order" of the complete reaction

$$A = C,$$

will be fixed by that of the slower reaction, $B = C$, because the methods used for measuring the rates of chemical reactions are not sensitive to changes so rapid as the assumed rate of transformation of A into B. Whatever the "order" of this latter reaction, $B = C$ is alone accessible to measurement. If, therefore, $A = C$ is of the first, second, or nth order, we must understand that one of the subsidiary reactions ($A = B$, or $B = C$) is

(1) an immeasurably fast reaction, accompanied by

(2) a slower measurable change of the first, second or nth order, according to the particular system under investigation.

If, however, the velocities of the two reactions are of the same order of magnitude, the "order" of the complete reaction will not fall under any simple type (§§ 88, 89), and, therefore, some changes will have to be made in the differential equations representing the course of the reaction. Let us study some of the simpler cases.

Case i. *In a given system, a substance A forms an intermediate substance B, which finally forms a third substance C.*

Let one gram molecule of the substance A be taken. At the end of a certain time t, the system contains x of A, y of B, z of C. The rate of diminution of x is evidently

$$-\frac{dx}{dt} = k_1 x, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Professor Walker illustrates this by the following analogy ("Velocity of Graded Reactions," *Proc. Royal Soc. Edin.*, Dec., 1897): "The time occupied in the transmission of a telegraphic message depends both on the rate of transmission along the conducting wire and on the rate of the messenger who delivers the telegram; but it is obviously this last, slower rate that is of really practical importance in determining the total time of transmission". . . .

where k_1 denotes the velocity constant of the transformation of A to B. The rate of formation of C is

$$\frac{dz}{dt} = k_2 y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k_2 is the velocity constant of the transformation of B to C. Again, the rate at which B accumulates in the system is evidently the difference in the rate of diminution of x and the rate of increase of z, or

$$\frac{dy}{dt} = k_1 x - k_2 y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The speed of the chemical reactions,

$$A = B = C,$$

is fully determined by this set of differential equations. When the relations between a set of variables involves a set of equations of this nature, the result is said to be a system of **simultaneous differential equations.**

In a great number of physical problems, the interrelations of the variables are represented in the form of a system of such equations. The simplest class occurs when each of the dependent variables is a function of the independent variable.

The simultaneous equations are said to be solved when each variable is expressed in terms of the independent variable, or else when a number of equations between the different variables can be obtained free from differential coefficients.

To solve the present set of differential equations, first differentiate (2),

$$\frac{d^2z}{dt^2} - k_2\frac{dy}{dt} = 0;$$

Add and subtract k_1k_2y, substitute for dy/dt from (3) and for k_2y from (2), we thus obtain

$$\frac{d^2z}{dt^2} + (k_1 + k_2)\frac{dz}{dt} - k_1k_2(x + y) = 0.$$

But from the conditions of the experiment,

$$x + y + z = 1, \therefore z - 1 = -(x + y).$$

Hence, the last equation may be written,

$$\frac{d^2(z-1)}{dt^2} + (k_1 + k_2)\frac{d(z-1)}{dt} + k_1k_2(z - 1) = 0. \quad . \quad . \quad . \quad (4)$$

This linear equation of the second order with constant coefficients, is to be solved for $z - 1$ in the usual manner (§ 130). At sight, therefore,

$$z - 1 = C_1e^{-k_1t} + C_2e^{-k_2t}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But $z = 0$, when $t = 0$,

$$\therefore C_1 + C_2 = -1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Differentiate (5). From (2), $dz/dt = 0$, when $t = 0$. Therefore, making the necessary substitutions,

$$-C_1k_1 - C_2k_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (6) and (7),

$$C_1 = k_1/(k_2 - k_1);\ C_2 = k_2/(k_1 - k_2). \quad ?$$

The final result may therefore be written,

$$z - 1 = \frac{k_2}{k_1 - k_2}e^{-k_2t} + \frac{k_1}{k_2 - k_1}e^{-k_1t}. \quad . \quad . \quad . \quad . \quad (8)$$

Harcourt and Esson have studied the rate of reduction of potassium permanganate by oxalic acid.

$$2KMnO_4 + 3MnSO_4 + 2H_2O = K_2SO_4 + 2H_2SO_4 + 5MnO_2;$$
$$MnO_2 + H_2SO_4 + H_2C_2O_4 = MnSO_4 + 2H_2O + 2CO_2.$$

By a suitable arrangement of the experimental conditions this reaction may be used to test equations (5) or (8).

Let x, y, z, respectively denote the amounts of Mn_2O_7, MnO_2 and MnO (in combination) in the system. The above workers found that $C_1 = 28{\cdot}5$; $C_2 = 2{\cdot}7$; $e^{-k_1} = {\cdot}82$; $e^{-k_2} = {\cdot}98$. The following table places the above suppositions beyond doubt.

t Minutes.	$z - 1$.		t Minutes.	$z - 1$.	
	Found.	Calculated.		Found.	Calculated.
0·5	25·85	25·9	3·0	10·45	10·4
1·0	21·55	21·4	3·5	8·95	9·0
1·5	17·9	17·8	4·0	7·7	7·8
2·0	14·9	14·9	4·5	6·65	6·6
2·5	12·55	12·5	5·0	5·7	5·8

Case ii. *A solution contains a gram molecules of each of A and C, the substance A gradually changes to B, which, in turn, reacts with C to form another compound D.*

Let x denote the amount of A which remains untransformed after the elapse of an interval of time t, y the amount of B, and z the amount of C present in the system after the elapse of the same interval of time t. Hence show that

$$-\frac{dx}{dt} = k_1x; \quad -\frac{dz}{dt} = k_2yz. \qquad . \quad . \quad . \quad . \quad (9)$$

The rate of diminution of B is proportional to the product of its active mass y into the amount of C present in the solution at the time t, but the velocity of increase of y is equal to the velocity of diminution of x,

$$\therefore \frac{dy}{dt} = k_1x - k_2yz. \qquad . \quad . \quad . \quad . \quad (10)$$

If x, y, z, could be measured independently, it would be sufficient to solve these equations as in case i., but if x and y are determined together, we must proceed a little differently. Note $z = x + y$. From the first of equations (9), and (10) by addition and the substitution of $dt = - dx/k_1x$ from (9), and of $z - x = y$, we get

$$\frac{1}{z^2}\cdot\frac{dz}{dx} + \frac{K}{z} - \frac{K}{x} = 0. \qquad . \quad . \quad . \quad . \quad . \quad (11)$$

where K has been written in place of k_2/k_1. The solution of this equation has been previously determined (page 298) in the form

$$Ke^{-Kx}\left\{C_1 - \log x + Kx - \frac{1}{1\,.\,2^2}(Kx)^2 + \ . \ . \ . \right\}z = 1. \qquad . \quad (12)$$

In some of Harcourt and Esson's experiments, $C_1 = 4\cdot68$; $k_1 = \cdot69$; $k_2 = \cdot006364$. From the first of equations (9), it is easy to show that $x = ae^{-k_1t}$. Where does a come from? What does it mean? Hence verify the third column in the following table:—

t Minutes.	z.	
	Found.	Calculated.
2	51·9	51·6
3	42·4	42·9
4	35·4	35·4
5	29·8	29·7

After the lapse of six minutes, the value of x was found to be negligibly very small. The terms succeeding log x in (12) may, therefore, be omitted without committing any sensible error. Substitute $x = ae^{-k_1t}$ in the remainder,

$$\frac{k_2}{k_1}(C_1 - \log a + k_1t)z = 1; \text{ or } (C'_1 + t)z = \frac{1}{k_2},$$

where $C'_1 = C_1/k_1 - (\log a)/k_1$. Harcourt and Esson found that $C'_1 = 0\cdot1$, and $1/k_2 = 157$. Hence, in continuation of the preceding table, these investigators obtained the results shown in the following table. The agreement between the theoretical and experimental numbers is remarkable.

t Minutes.	z.		t Minutes.	z.	
	Found.	Calculated.		Found.	Calculated.
6	25·7	25·7	10	15·5	15·5
7	22·2	22·1	15	10·4	10·4
8	19·4	19·4	20	7·8	7·8
9	17·3	17·3	30	5·5	5·2

The theoretical numbers are based on the assumption that the chemical change consists in the gradual formation of a substance which at the same time slowly disappears by reason of its reaction with a proportional quantity of another substance.

This really means that the so-called "initial disturbances" in chemical reactions, are due to the fact that the speed during one stage of the reaction, is faster than during the other. The magnitude of the initial disturbances depends on the relative magnitudes of k_1 and k_2. The observed velocity in the steady state depends on the difference between the steady diminution $- dx/dt$ and the steady rise dz/dt. If k_2 is infinitely great in comparison with k_1, (8) reduces to

$$z = a(1 - e^{-k_1t}),$$

which will be immediately recognised as another way of writing the familiar equation

$$k_1 = \frac{1}{t}\log\frac{a}{a - z}.$$

So far as practical work is concerned, it is necessary that the solutions of the differential equations shall not be so complex as to preclude the possibility of experimental verification.

Case iii. *In a given system A combines with R to form B, B combines with R to form C, and C combines with R to form D.*

In the hydrolysis of triacetin,

$$\underset{(Triacetin)}{C_3H_5 . A_3} + H . OH = 3\bar{A} . H + \underset{(Glycerol)}{C_3H_5(OH)_3},$$

where $\bar{A}$ has been written for $CH_3 . COO .$, there is every reason to believe that the reaction takes place in three stages:

$$C_3H_5 . \bar{A}_3 + H . OH = \bar{A} . H + C_3H_5 . \bar{A}_2 . OH \quad (diacetin);$$
$$C_3H_5 . \bar{A}_2 . OH + H . OH = \bar{A} . H + C_3H_5 . \bar{A} . (OH)_2 \ (monacetin);$$
$$C_3H_5 . \bar{A} . (OH)_2 + H . OH = \bar{A} . H + C_3H_5 . (OH)_3 \quad (glycerol).$$

These reactions are interdependent. The rate of formation of glycerol is conditioned by the rate of formation of monacetin; the rate of monacetin depends, in turn, upon the rate of formation of diacetin. There are, therefore, three simultaneous reactions of the second order taking place in the system.

Let a denote the initial concentration (gram molecules per unit volume) of triacetin, b the concentration of the water; let x, y, z, denote the number of molecules of mono-, di- and triacetin hydrolysed at the end of t minutes. The system then contains $a - z$ molecules of triacetin, $z - y$, of diacetin, $y - x$, of monacetin, and $b - (x + y + z)$ molecules of water. The rate of hydrolysis is therefore completely determined by the equations:

$$dx/dt = k_1(y - x)(b - x - y - z); \quad . \quad . \quad . \quad (13)$$
$$dy/dt = k_2(z - y)(b - x - y - z); \quad . \quad . \quad . \quad (14)$$
$$dz/dt = k_3(a - z)(b - x - y - z); \quad . \quad . \quad . \quad (15)$$

where k_1, k_2, k_3, represent the velocity coefficients (§ 88) of the respective reactions.

Geitel tested the assumption: $k_1 = k_2 = k_3$. Hence dividing (15) by (13) and by (14), he obtained

$$dz/dy = (a - z)/(z - y); \ dz/dx = (a - z)/(y - x). \quad . \quad . \quad (16)$$

From the first of these equations,

$$dy + y\frac{dz}{a - z} = \frac{z . dz}{a - z},$$

which can be integrated as a linear equation of the first degree. The constant is equated by noting that if $a = 1$, $z = 0$, $y = 0$. The reader might do this as an exercise on § 123. The answer is

$$y = z + (a - z)\log(a - z). \quad . \quad . \quad . \quad . \quad (17)$$

Now substitute (17) in the second of equations (16), rearrange terms and integrate as a further exercise on linear equations of the first order. The final result is,

$$x = z + (a - z)\log(a - z) - \frac{a - z}{z}\{\log(a - z)\}^2. \quad . \quad . \quad (18)$$

Geitel then assigned arbitrary numerical values to z (say from 0·1 to 1·0), calculated the corresponding amounts of x and y from (17) and (18) and compared the results with his experimental numbers. For experimental and other details the original memoir must be consulted (*vide infra*).

EXAMPLE.—Calculate equations analogous to (17) and (18) on the supposition that $k_1 \neq k_2 \neq k_3$.

A study of the differential equations representing the mutual conversion of red into yellow, and yellow into red phosphorus, will be found in a paper by Lemoine in the *Annales de Chimie et de Physique* [4], **27**, 289, 1872.

There is also a series of interesting papers by Rud. Wegscheider bearing on this subject in *Zeit. f. phys. Chem.*, **30**, 593, 1899; *ib.*, **34**, 290, 1900; *ib.*, **35**, 513, 1900; *Monatshefte für Chemie*, **22**, 749, 1901.

The preceding discussion is based upon papers by Harcourt and Esson, *Phil. Trans.*, **156**, 193, 1866; Geitel, *Journ. für prakt. Chem.* [2], **55**, 429, 1897; J. Walker, *Proc. Roy. Soc. Edin.*, 22, 1897. It is somewhat surprising that Harcourt and Esson's investigation has not received more attention from the point of view of simultaneous and dependent reactions. The indispensable differential equations, simple as they are, might perhaps account for this. But chemists, in reality, have more to do with this type of reaction than any other. The day is surely past when the study of a particular reaction is abandoned simply because it "won't go" according to the stereotyped velocity equations of § 88.

§ 136. Simultaneous Differential Equations.

By way of practice it will be convenient to study a few more examples of simultaneous equations.

For a complete determination of each variable there must be the same number of equations as there are independent variables. Quite an analogous thing occurs with the simultaneous equations of ordinary algebra.

I. *Simultaneous equations with constant coefficients.* The methods used for the solution of these equations are analogous to those employed for similar equations in algebra. The operations here involved are chiefly processes of elimination and substitution, supplemented by differentiation or integration at various stages of the computation. The use of the symbol D often shortens the work. Most of the following examples are from results proved in the regular textbooks on physics.

EXAMPLES.—(1) Solve $dx/dt + ay = 0$, $dy/dt + bx = 0$. Differentiate the first, multiply the second by a. Subtract and y disappears. Hence writing $ab = m^2$,

$$x = C_1e^{mt} + C_2e^{-mt}; \text{ or, } y = C_2\sqrt{b/a}\,.\,e^{-mt} - C_1\sqrt{b/a}\,.\,e^{mt}.$$

We might have obtained an equation in y, and substituted it in the second. Thus four constants appear in the result. But one pair of these constants can be expressed in terms of the other two. Two of the constants, therefore, are not arbitrary and independent, while the integration constant is arbitrary and independent. It is always best to avoid an unnecessary multiplication of constants by deducing the other variables from the first without integration. The number of arbitrary constants is always equal to the sum of the highest orders of the set of differential equations under consideration.

(2) Solve $dx/dt + y = 3x$; $dy/dt - y = x$. Differentiate the first. Subtract each of the given equations from the result. $(D^2 - 4D + 4)x$ remains. Solve as usual. $x = (C_1 + C_2x)e^{2t}$. Substitute this value of x in the second of the given equations and $y = (C_1 - C_2 + C_2t)e^{2t}$.

(3) The equations of rotation of a particle in a rigid plane, are

$$dx/dt = \mu y;\ dy/dt = \mu x.$$

To solve these, differentiate the first, multiply the second by μ, etc. Finally $x = C_1 \cos \mu t + C_2 \sin \mu t$; $y = C'_1 \cos \mu t + C'_2 \sin \mu t$. To find the relation between these constants, substitute these values in the first equation and

$$- \mu C_1 \sin \mu t + \mu C_2 \cos \mu t = \mu C'_1 \cos \mu t + \mu C'_2 \sin \mu t,$$

or $C'_1 = - C'_2$ and $C_2 = C'_1$.

(4) Solve $d^2x/dt^2 = - n^2x \quad d^2y/dt^2 = - n^2y$.

$$x = C_1 \cos nt + C_2 \sin nt;\ y = C'_2 \cos nt + C'_2 \sin nt.$$

Eliminate t so that

$$(C'_1x - C_1y)^2 + (C'_2x - C_2y)^2 = (C_1C'_2 - C_2C'_1)^2, \text{ etc.}$$

The result represents the motion of a particle in an elliptic path, subject to a central gravitational force.

(5) Solve $dx/dt + by + cz = 0$; $dy/dt + a_1x + c_1z + 0$; $dz/dt + a_2x + b_2y = 0$. Operate on the first with $D^2 - b_2c_1$, on the second with $b_2c - bD$, on the third with $bc_1 - cD$. Add. The terms in y and z disappear. The remaining equation has the integral,

$$x = C_1e^{at} + C_2e^{\beta t} + C_3e^{\gamma t},$$

where α, β, γ, are the roots of

$$z^3 - (a_1b + a_2c + b_2c_1)z + a_1b_2c + a_2bc_1 = 0.$$

The values of y and z are easily obtained from that of x by proper substitutions in the other equations.

(6) If two adjacent circuits have currents i_1 and i_2, then, according to the theory of electromagnetic induction,

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2i_2 = E_2;\ M\frac{di_2}{dt} + L_1\frac{di_1}{dt} + R_1i_1 = E_1,$$

(see J. J. Thomson's *Elements of Electricity and Magnetism*, p. 382), where R_1, R_2, denote the resistances of the two circuits, L_1, L_2, the coefficients of self-induction, E_1, E_2, the electromotive forces of the respective circuits and M the coefficient of mutual induction. All the coefficients are supposed constant.

First, solve these equations on the assumption that $E_1 = E_2 = 0$. Assume that

$$i_1 = ae^{mt} \text{ and } i_2 = be^{mt},$$

satisfy the given equations. Differentiate each of these variables with respect to t and substitute in the original equation

$$aMm + b(L_2m + R_2) = 0;\ bMm + a(L_1m + R_1) = 0.$$

Multiply these equations so that

$$(L_1L_2 - M_2)m^2 + (L_1R_2 + R_1L_2)m + R_1R_2 = 0.$$

For physical reasons, the induction L_1L_2 must always be greater than M. The roots of this quadratic must, therefore, be negative and real (page 388), and

$$i_1 = a_1e^{-m_1t}, \text{ or } a_2e^{-m_2t};\ i_2 = b_1e^{-m_1t}, \text{ or } b_2e^{-m_2t}.$$

Hence, from the preceding equation,

$$a_1Mm_1 + b_1L_2m_1 + B_1R_2 = 0;\ \text{or } a_1/b_1 = (L_2m_1 + R_2)/Mm_1;$$

similarly, $$a_2/b_2 = Mm_2/(L_1m_2 + R_1).$$

Combining the particular solutions for i_1 and i_2, we get

$$i_1 = a_1e^{-m_1t} + a_2e^{-m_2t};\ i_2 = b_1e^{-m_1t} + b_2e^{-m_2t},$$

the required solutions.

Second, if E_1 and E_2 have some constant value,

$$i_1 = E_1/R_1 + a_1e^{-m_1t} + a_2e^{-m_2t};\ i_2 = E_2/R_2 + b_1e^{-m_1t} + b_2e^{-m_2t},$$

are the required solutions.

II. *Simultaneous equations with variable coefficients.* The general type of simultaneous equations of the first order, is

$$\left.\begin{array}{l} P_1dx + Q_1dy + R_1dz = 0; \\ P_2dx + Q_2dy + R_2dz = 0, \ldots \end{array}\right\} \quad (1)$$

where the coefficients are functions of x, y, z. These equations can often be expressed in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \quad (2)$$

which is to be looked upon as a typical set of simultaneous equations of the first order. If one of these equations involves only two differentials, the equation is to be solved in the usual way, and the result used to deduce values for the other variables, as in the first of the subjoined examples.

When the members of a set of equations are symmetrical, the solution can often be simplified by taking advantage of a well-known theorem * in algebra (ratio). According to this,

---

* Perhaps it is best to state the proof. Let

$$dx/P = dy/Q = dz/R = k, \text{ say; then,}$$
$$dx = Pk;\ dy = Qk;\ dz = Rk;$$
$$\text{or, } ldx = lPk;\ mdy = mQk;\ ndz = nRk.$$

Add these results,

$$ldx + mdy + ndz = k(lP + mQ + nR).$$
$$\therefore \frac{ldx + mdy + ndz}{lP + mQ + nR} = k = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R} = \ldots, \quad (3)$$

where $l, m, n, l', m', n', \ldots$ are sets of multipliers such that

$$lP + mQ + nR = 0 \,;\; l'P + m'Q + n'R = 0 \,;\; \ldots \quad (4)$$

hence,
$$ldx + mdy + ndz = 0, \text{ etc.} \quad (5)$$

The same relations between x, y, z, that satisfy (5), satisfy (2).

If (4) be an exact differential equation, equal to say du, direct integration gives the integral of the given system, *viz.*,

$$u = a, \quad (6)$$

where a denotes the constant of integration.

In the same way, if

$$ldx + mdy + ndz = 0,$$

is an exact differential equation, equal to say dv, then, since dv is also equal to zero,

$$v = b, \quad (7)$$

is a second solution. These two solutions must be independent.

EXAMPLES.—(1) Solve $dx/y = dy/x = dz/z$. The relation between dx and dy contains x and y only, the integral, $y^2 - x^2 = C_1$, follows at once. Use this result to eliminate x from the relation between dy and dz. The result is

$$dz/z = dy / \sqrt{(y^2 - C_1)} \,;\text{ or, } y + \sqrt{(y^2 + C_1)} = C_2 z.$$

These two equations, involving two constants of integration, constitute a complete solution.

(2) Solve $dx/(mx - ny) = dy/(nx - lz) = dz/(ly - mx)$. l, m, n and x, y, z form a set of multipliers satisfying the above condition. Hence,

$$ldx + mdy + ndz = 0 \,;\; xdx + ydy + zdz = 0.$$

The integrals of these equations are

$$u = lx + my + nz = C_1 \,;\; v = x^2 + y^2 + z^2 = C_2,$$

which constitute a complete solution.

(3) Solve $dx/(x^2 - y^2 - z^2) = dy/2xy = dz/2xz$. From the two last equations $y = C_1 z$. Substituting x, y, z for l, m, n, each of the given ratios is equal to

$$(xdx + ydy + zdz)/(x^2 + y^2 + z^2). \quad \therefore x^2 + y^2 + z^2 = C_2 z,$$

is another solution.

§ 137. Partial Differential Equations.

Equations obtained by the differentiation of functions of three or more variables are of two kinds:

1. Those in which there is only one independent variable, such as

$$Pdx + Qdy + Rdz = Sdt,$$

which involves four variables—three dependent and one independent. These are called **total differential equations.**

2. Those in which there is only one dependent and two or more independent variables, such as,

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + R\frac{\partial z}{\partial t} = 0,$$

where z is the dependent variable, x, y, t the independent variables. These equations are classed under the name **partial differential equations.**

The former class of equations are rare, the latter very common. We shall confine our attention to partial differential equations.

In the study of ordinary differential equations, we have always assumed that the given equation has been obtained by the elimination of constants from the original equation. In solving, we have sought to find this primitive equation.* Partial differential equations, however, may be obtained by the elimination of arbitrary functions of the variables as well as of constants.

It can be shown from Euler's theorem (page 56) that if

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}, \ldots\right),$$

be a homogeneous function,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots = nx^n f\left(\frac{y}{x}, \frac{z}{x}, \ldots\right) = nu,$$

where the arbitrary function has disappeared.† Again, if

$$u = f(ax^3 + by^3),$$

is an arbitrary function of x and y.

$$\frac{\partial u}{\partial x} = af'(ax^3 + by^3)\,;\ \frac{\partial u}{\partial y} = bf'(ax^3 + by^3)\,;\ \therefore\ b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 0.$$

---

* Physically, the differential equation represents the relation between the dependent and the independent variables corresponding to an infinitely small change in each of the independent variables.

The reader will, perhaps, have noticed that the term **"independent variable"** is an equivocal phrase. (1) If $u = f(z)$, u is a quantity whose magnitude changes when the value of z changes. The two magnitudes u and z are mutually dependent. For convenience, we fix our attention on the effect which a variation in the value of z has upon the magnitude of u. If need be we can reverse this and write $z = f(u)$, so that u now becomes the "independent variable". (2) If $v = f(x, y)$, x and y are "independent variables" in that x and y are mutually independent of each other. Any variation in the magnitude of the one has no effect on the magnitude of the other. x and y are also "independent variables" with respect to v in the same sense that z has just been supposed the "independent variable" with respect to u.

† This is usually proved in the textbooks in the following manner:

Let $u = x^n f(y/x, z/x, \ldots)$. Put $y/x = Y$, $z/x = Z$, ...

$\therefore \partial Y/\partial x = -y/x^2$, $\partial Z/\partial x = -z/x^2 \ldots$; $\partial Y/\partial y = 1/x$, $\partial Z/\partial y = 0, \ldots$

Let $v = f(Y, Z, \ldots)$, for the sake of brevity, therefore, since $u = x^n v$, EXAMPLES.—(1) If $y - bu = f(x - au)$, $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 1$.

(2) If $1/z - 1/x = f(1/y - 1/x)$, $x^2\partial z/\partial x + y^2\partial z/\partial y = z^2$.

(3) If $z = a(x + y) + b$, $\partial z/\partial x - \partial z/\partial y = 0$.

For this reason an arbitrary function of the variables is added to the result of the integration of a partial differential equation instead of the constant hitherto employed for ordinary differential equations.

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the resulting differential equation is of the first order. The higher orders occur when the number of constants to be eliminated, exceeds that of the independent variables.

If $u = f(x, y)$, there will be two differential coefficients of the first order, namely, $\partial u/\partial x$ and $\partial u/\partial y$; three of the second order, namely, $\partial^2u/\partial x^2$, $\partial^2u/\partial x\partial y$, $\partial^2u/\partial y^2$ . . .

§ 138. What is the Solution of a Partial Differential Equation?

Ordinary differential equations have two classes of solutions —the complete integral and the singular solution. Particular solutions are only varieties of the complete integral. Three classes of solutions can be obtained from some partial differential equations, still regarding the particular solution as a special case of the complete integral. These are indicated in the following example.

The equation of a sphere in three dimensions is,

$$x^2 + y^2 + z^2 = r^2, \quad . \quad . \quad . \quad (1)$$

when the centre of the sphere coincides with the origin of the

---

$$\frac{\partial u}{\partial x} = nx^{n-1}f(Y, Z, \ldots) + x^n\left(\frac{\partial v}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial v}{\partial Z}\cdot\frac{\partial Z}{\partial x} + \ldots\right);$$

by the method for the differentiation of a function of a function, 6 and 9, § 12. Therefore,

$$= nx^{n-1}f(Y, Z, \ldots) - x^{n-2}\left(y\frac{\partial v}{\partial Y} + z\frac{\partial v}{\partial Z} + \ldots\right).$$

$$\frac{\partial u}{\partial y} = x^n\frac{\partial v}{\partial Y}\cdot\frac{\partial Y}{\partial y} = x^{n-1}\frac{\partial v}{\partial Y};\ \frac{\partial u}{\partial z} = x^n\frac{\partial v}{\partial Z}\frac{\partial Z}{\partial z} = x^{n-1}\frac{\partial v}{\partial Z};\ \ldots$$

Now multiply by $x, y, z, \ldots$ respectively, and add,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots = nx^nf(Y, Z, \ldots) = nu.$$

coordinate planes and r denotes the radius of the sphere. If the centre of the sphere lies somewhere on the xy-plane at a point (a, b), the above equation becomes

$$(x - a)^2 + (y - b)^2 + z^2 = r^2. \quad . \quad . \quad (2)$$

When a and b are arbitrary constants, each or both of which may have any assigned magnitude, equation (2) may represent two infinite systems of spheres of radius r. The centre of any member of either of these two infinite systems (called a *double infinite system*) must lie somewhere on the xy-plane.

Differentiate (2) with respect to x and y.

$$x - a + z\frac{\partial z}{\partial x} = 0;\ y - b + z\frac{\partial z}{\partial y} = 0. \quad . \quad . \quad (3)$$

Substitute for $x - a$ and $y - b$ in (2). We obtain

$$z^2\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\} = r^2. \quad . \quad . \quad (4)$$

Equation (2), therefore, is the **complete integral** of (4). By assigning any particular numerical value to a or b, a **particular solution** of (4) will be obtained, such is

$$(x - 1)^2 + (y - 79)^2 + z^2 = r^2. \quad . \quad . \quad (5)$$

If (2) be differentiated with respect to a and b,

$$\frac{\partial}{\partial a}\{(x - a)^2 + (y - b)^2 + z^2 = r^2\};\ \frac{\partial}{\partial b}\{(x - a)^2 + (y - b)^2 + z^2 = r^2\}.$$

or, $$x - a = 0, \text{ and } y - b = 0.$$

Eliminate a and b from (2),

$$z = \pm r. \quad . \quad . \quad . \quad . \quad (6)$$

This result satisfies equation (4), but, unlike the particular solution, is not included in the complete integral (2). Such a solution of the differential equation is said to be a **singular solution.**

Geometrically, the singular solution represents two plane surfaces touched by all the spheres represented by equation (2). The singular solution is thus the envelope of all the spheres represented by the complete integral. If AB (Fig. 79) represents a cross section of the xy-plane containing spheres of radius a, CD and EF are cross sections of the plane surfaces represented by the singular solution.

If the one constant is some function of the other, say,

$$a = b,$$

(2) may be written

$$(x - a)^2 + (y - a)^2 + z^2 = r^2. \quad . \quad . \quad (7)$$

Differentiate with respect to a. We find

$$a = \tfrac{1}{2}(x + y).$$

Eliminate a from (7). The resulting equation

$$x^2 + y^2 + 2z^2 - 2xy = 2r^2,$$

is called a **general integral** of the equation.

Geometrically, the general integral is the equation to the tubular envelope of a family of spheres of radius r and whose centres are along the line $x = y$. This line corresponds with the axis of the tube envelope. The general integral satisfies (4) and is also contained in the complete integral.

Instead of taking $a = b$ as the particular form of the function connecting a and b, we could have taken any other relation, say $a = \frac{1}{2}b$. The envelope of the general integral would then be like a tube surrounding all the spheres of radius r whose centres were along the line $x = \frac{1}{2}y$. Had we put $a^2 - b^2 = 1$, the envelope would have been a tube whose axis was an hyperbola $x^2 - y^2 = 1$.

A particular solution is one particular surface selected from the double infinite series represented by the complete solution. *A general integral* is the envelope of one particular family of surfaces selected from those comprised in the complete integral. *A singular solution* is the complete envelope of every surface included in the complete integral.*

Theoretically an equation is not supposed to be solved completely until the complete integral, the general integral and the singular solution have been indicated. In the ideal case, the complete integral is first determined; the singular solution obtained by the elimination of arbitrary constants as indicated above; the general integral then determined by eliminating a and $f(a)$.

Practically, the complete integral is not always the direct object of attack. It is usually sufficient to deduce a number of particular solutions to satisfy the conditions of the problem and afterwards to so combine these solutions that the result will not only satisfy the given conditions but also the differential equation.

---

* The study of Gibbs' "*Surfaces of Dissipated Energy*," "*Surfaces of Dissociation*," "*Surfaces of Chemical Equilibrium*," as well as "van der Waals' Surfaces," is the natural sequence of §§ 68, 126 and the present section. But to enlarge upon this subject would now cause a greater digression than is here convenient. Airy's little book, *An Elementary Treatise on Partial Differential Equations*, will repay careful study in connection with the geometrical interpretation of the solutions of partial differential equations.

Of course, the complete integral of a differential equation applies to any physical process represented by the differential equation. This solution, however, may be so general as to be of little practical use. To represent any particular process, certain limitations called **limiting conditions** have to be introduced. These exclude certain forms of the general solution as impossible. See examples at the end of Chapter VIII.; also example (1) last set § 133, and elsewhere.

The more important varieties of partial differential equations from the point of view of this work are the linear equations of the second and higher orders.

§ 139. The Solution of Partial Differential Equations of the First Order.

For the ingenious general methods of Lagrange, Charpit, etc., the reader will have to consult the special textbooks, say, Forsyth's *A Treatise on Differential Equations* (Macmillan & Co., 1888).

There are some special types classified by Forsyth in the following order:

Type I. *The variables do not appear directly.* The general form is,

$$f(\partial z/\partial x, \partial z/\partial y) = 0. \quad . \quad . \quad . \quad \text{(I.)}$$

The solution is

$$z = ax + by + C,$$

provided a and b satisfy the relation

$$f(a, b) = 0, \text{ or } b = f(a).$$

The complete integral is, therefore,

$$z = ax + yf(a) + C. \quad . \quad . \quad . \quad (1)$$

EXAMPLES.—(1) Solve $(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = m^2$. The solution is

$$z = ax + by + C',$$

provided $a^2 + b^2 = m^2$. The solution is, therefore, $z = ax + y\sqrt{(m^2 - a^2)} + C$. For the general integral, put $C = f(a)$. Eliminate a between the two equations,

$$z = ax + \sqrt{(m^2 - a^2)}y + f(a); \text{ and } x - a/\sqrt{(m^2 - a^2)}y + f'(a) = 0,$$

in the usual way.

(2) Solve $pq = 1$. Ansr. $z = ax + y/a + f(a)$.

NOTE.—We shall sometimes write, for the sake of brevity,

$$\partial z/\partial x = p; \ \partial z/\partial y = q.$$

(3) Solve $a(p + q) = z$. Sometimes, as here, when the variables do appear in the equation, the function of x, which occurs in the equation, may

be associated with $\partial z/\partial x$, or a function of y with $\partial z/\partial y$, by a change in the variables. We may write the given equation $ap/z + aq/z = 1$. Put $dz/a = dZ$; $dy/a = dY$, $dx/a = dX$, hence, $\partial Z/\partial Y + \partial Z/\partial X = 1$, the form required.

(4) Solve $x^2p^2 + y^2q^2 = z^2$. Put $X = \log x$, $Y = \log y$, $Z = \log z$. Proceed as before. Ansr. $z = Cx^a y^{\sqrt{(1-a^2)}}$.

If it is not possible to remove the dependent variable z in this way, the equation will possibly belong to the following class:

Type II. *The independent variables x and y are absent.* The general form is,

$$f(z, \partial z/\partial x, \partial z/\partial y) = 0. \quad . \quad . \quad . \quad \text{(II.)}$$

Assume as a trial solution, that

$$\partial z/\partial y = a \, . \, \partial z/\partial x.$$

Let $\partial z/\partial x$ be some function of z obtained from II., say $p = \phi(z)$. Substitute these values in

$$dz = p \, . \, dx + q \, . \, dy.$$

We thus get an ordinary differential equation which can be readily integrated.

$$dz = \phi(z) \, . \, dx + a\phi(z) \, . \, dy.$$

$$\therefore x + ay = \int dz/\phi(z) + C. \quad . \quad . \quad . \quad (2)$$

EXAMPLES.—(1) Solve $p^2z + q^2 = 4$. Here,

$$(a^2 + z)(dz/dx)^2 = 4. \quad \sqrt{(a^2 + z)} \, . \, dz/dx = 2,$$

$\therefore x + C = \int \sqrt{(a^2 + z)} \, . \, dz = \frac{1}{3}(a^2 + z)^{3/2}$. Ansr. $2(a^2 + z)^3 = 3(x + ay + C)^2$.

(2) Solve $p(1 + q^2) = q(z - a)$. Ansr. $4C(z - a) = (x + Cy + b)^2$.

If z does not appear directly in the equation, we may be able to refer the equation to the next type.

Type III. *z does not appear directly in the equation, but x and $\partial z/\partial x$ can be separated from y and $\partial z/\partial y$.* The leading type is

$$f_1(x, \partial z/\partial x) = f_2(y, \partial z/\partial y). \quad . \quad . \quad \text{(III.)}$$

Assume as a trial solution, that each member is equal to an arbitrary constant a, so that $\partial z/\partial x$ and $\partial z/\partial y$ can be obtained in the form,

$$\partial z/\partial x = \phi_1(x, a); \ \partial z/\partial y = \phi_2(y, a).$$

$$dz = p \, . \, dx + q \, . \, dy,$$

then assumes the form

$$dz = f_1(x, a)dx + f_2(y, a)dy. \quad . \quad . \quad (3)$$

EXAMPLES.—Solve the following equations:

(1) $q - p = x - y$. Put $\partial z/\partial x - x = \partial z/\partial y - y = a$. Write

$$\partial z/\partial x = x + a, \text{ etc.}; \ \partial z/\partial y = y + a.$$

Hence, $z = \frac{1}{2}(x + a)^2 + \frac{1}{2}(y + a)^2 + C$.

(2) $q^2 + p^2 = x + y$. Ansr. $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y - a)^{3/2} + C$.

(3) $q = 2yp^2$. Ansr. $z = ax + a^2y^2 + C$.

Type IV. *Analogous to Clairaut's equation.* The general type is

$$z = p \cdot x + q \cdot y + f(p \cdot q). \qquad \text{(IV.)}$$

The complete integral is

$$z = ax + by + f(a, b). \qquad (4)$$

EXAMPLES.—Solve the following equations:

(1) $z = px + qy + pq$. Ansr. $z = ax + by + ab$. Singular solution $z = -xy$.

(2) $z = px + qy + k\sqrt{(1 + p^2 + q^2)}$. Ansr. $z = ax + by + k\sqrt{1 + a^2 + b^2}$. Singular solution, $x^2 + y^2 + z^2 = r^2$. The singular solution is, therefore, a sphere; r, of course, is a constant.

(3) $z = px + qy - n\sqrt[n]{pq}$. Ansr. $z = ax + by - n\sqrt[n]{ab}$. Singular solution, $z = (2 - n)(xy)^{1/(2-n)}$.

§ 140. Partial Differential Equations of the nth Order.

These are the most important equations that occur in physical mathematics. There are no general methods for their solution, and it is only possible to perform the integration in special cases. The greatest advances in this direction have been made with the linear equation. Before proceeding to this important equation, it appears convenient to solve some simpler types.

EXAMPLES.—Integrate the following equations:

(1) $\dfrac{\partial^2 z}{\partial x \partial y} = a$. If $\dfrac{\partial z}{\partial x} = p$; $\dfrac{\partial p}{\partial y} = a$. Integrate with regard to y and we get $p = ay + f'(x)$. It is very possible that $f'(x)$ is a function of y. Integrate with respect to x and $z = \int\{ay + f'(x)\}dx = axy + f_1(x) + f_2(y)$.

(2) $\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{x}{y} = a$. Ansr. $z = \frac{1}{2}x^2 \log y + axy + f_1(x) + f_2(y)$.

(3) $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial x}f(x) = \psi(y)$. Ansr. $z = \int[e^{-yf(x)}\{\int e^{yf(x)}\psi(y)dy + f_1(x)\}]dx + f_2(y)$.

There are many points of analogy between the partial and the ordinary linear differential equations. Indeed, it may almost be said that every ordinary differential equation between two variables is analogous to a partial differential in the same class. The solution is in each case similar, but with these differences:

First, the arbitrary constant of integration in the solution of an ordinary differential equation is replaced by a function of a variable or variables.

Second, the exponential form, Ce^{mx}, of the solution of the ordinary linear differential equation assumes the form $e^{mx\frac{\partial}{\partial y}}\phi(y)$.

The expression, $e^{mx\frac{\partial}{\partial y}}\phi(y)$, is known as the **symbolic form of Taylor's theorem.** Having had considerable practice in the use of the symbol of operation D for $\frac{\partial}{\partial x}$, we may now use D' to represent the operation $\frac{\partial}{\partial y}$.

By Taylor's theorem,

$$\phi(y + mx) = \phi(y) + mx\frac{\partial\phi(y)}{\partial y} + \frac{m^2x^2}{2!}\cdot\frac{\partial^2\phi(y)}{\partial y^2} + \ldots,$$

where x is regarded as constant.

$$\therefore \phi(y + mx) = \left(1 + mx\frac{\partial}{\partial y} + \frac{m^2x^2}{2!}\frac{\partial^2}{\partial y^2} + \ldots\right)\phi(y).$$

The term in brackets is clearly an exponential series (page 230) equivalent to $e^{mx\frac{\partial}{\partial y}}$, or, writing D' for $\frac{\partial}{\partial y}$,

$$\phi(y + mx) = e^{mxD'}\phi(y). \quad . \quad . \quad . \quad . \quad (1)$$

The general form of the linear equation is,

$$A_0\frac{\partial^2 z}{\partial^2 x} + A_1\frac{\partial^2 z}{\partial x\partial y} + A_2\frac{\partial^2 z}{\partial y^2} + A_3\frac{\partial z}{\partial x} + A_4\frac{\partial z}{\partial y} + A_5 z = A, \qquad (2)$$

where A_0, A_1, . . ., A, may be constants, or functions of x and y.

As with ordinary linear equations,

Complete Solution = Particular Integral + Complementary Function.

The complementary function is obtained by solving the left-hand side of equation (2), equated to zero. We may write (2) in symbolic form,

$$(A_0D^2 + A_1DD' + A_2D'^2 + A_3D + A_4D' + A_5)z = 0, \ . \qquad (3)$$

where D is written for $\frac{\partial}{\partial x}$; D' for $\frac{\partial}{\partial y}$; DD' for $\frac{\partial^2}{\partial x\partial y}$. Sometimes we understand

$$F(D, D')z = 0, \quad . \quad . \quad . \quad . \quad (4)$$

in place of (2).

§ 141. Linear Partial Equations with Constant Coefficients.

A. Homogeneous equations. Type:

$$A_0\frac{\partial^2 z}{\partial x^2} + A_1\frac{\partial^2 z}{\partial x\partial y} + A_2\frac{\partial^2 z}{\partial y^2} = R, \ . \quad . \quad . \quad (5)$$

where R is a function of x.

To find the complementary function, put $R = 0$, and instead of assuming, as a trial solution, that $y = e^{mx}$, as was the case with the ordinary equation, suppose that

$$z = \phi(y + mx), \quad . \quad . \quad . \quad . \quad (6)$$

is a trial solution. Differentiate (6), with respect to x and y, we thus obtain,

$$\frac{\partial z}{\partial x} = mf'(y + mx)\,;\ \frac{\partial z}{\partial y} = f'(y + mx)\,;\ \frac{\partial^2 z}{\partial x \partial y} = mf''(y + mx)\,;$$

$$\frac{\partial^2 z}{\partial x^2} = mf''(y + mx)\,;\ \frac{\partial^2 z}{\partial y^2} = f''(y + mx).$$

Substitute these values in equation (5) equated to zero, and divide out the factor $f''(y + mx)$. The auxillary equation,

$$A_0 m^2 + A_1 m + A_2 = 0. \qquad (7)$$

remains. If m is a root of this equation, $f''(y + mx) = 0$, is a part of the complementary function. If α and β are the roots of (7), then

$$z = e^{\alpha x D'}\phi_1(y) + e^{\beta x D'}\phi_2(y), \qquad (8)$$

as in (3), § 130. From (6), therefore,

$$z = f_1(y + \alpha x) + f_2(y + \beta x) \qquad (9)$$

since α and β are the roots of the auxillary equation (7), we can write (5) in the form,

$$(D + \alpha D')\,(D + \beta D')z = 0. \qquad (10)$$

Examples.—Solve the following equations:

(1) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0.$ Ansr. $z = f_1(y + x) + f_2(y - x)$.

(2) $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$ Ansr. $z = f_1(y - 2x) + f_2(y - 2x)$.

(3) $2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$ Ansr. $z = f_1(2y - x) + f_2(y + 2x)$.

(4) $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$ Ansr. $u = f_1(at + x) + f_2(at - x)$. This most important equation, sometimes called *d'Alembert's equation*, represents the motion of vibrating strings, the law for small oscillations of air in narrow tubes (organ pipes), etc.

We cannot say much about the undetermined functions $f_1(at + x)$ and $f_2(at - x)$ in the absence of data pertaining to some specific problem. Consider a vibrating harp string, where no force is applied after the string has once been put in motion. Let $x = l$ denote the length of the string under a tension equal to the weight of a length L of the same kind of string. In order to avoid a root sign later on, a^2 has been written in place of gL, where g represents the constant of gravitation. Further, let u represent the displacement of any part of the string we please, and let the ordinate of one end of the string be zero. Then, whatever value we assign to the time t, the limiting conditions are $u = 0$, when $x = 0$; and $u = 0$, when $x = l$.

$$\therefore f_1(at) + f_2(at) = 0\,;\ f_1(at + l) + f_2(at - l) = 0,$$

are solutions of d'Alembert's equation. From the former, it follows that

$$f_1(at) \text{ must always be equal to } -f_2(at)\,;$$

$$\therefore f_1(at + l) - f_1(at - l) = 0.$$

But at may have any value we please. In order to fix our ideas, suppose that $at - l = q$, $\therefore at + l = q + 2l$, where q has any value whatever.

$$\therefore f_1(q + 2l) = f_1(q).$$

The physical meaning of this solution is that when q is increased or diminished by $2l$, the value of the function remains unaltered. Hence, when at is increased by $2l$, or, what is the same thing, when t is increased by $2l/a$, the corresponding portions of the string will have the same displacement. In other words, the string performs at least one complete vibration in the time $2l/a$. Hence, we conclude that d'Alembert's equation represents a finite periodic motion, with a period of oscillation $2l/a$.

EXAMPLE.—Show that

$$\tfrac{1}{2}\{f_1(at + l) + f_1(at - l)\} = 0,$$

is a solution of d'Alembert's equation, and interpret the result.

A further study of d'Alembert's equation would require the introduction of Fourier's series, Chapter VIII.

When two of the roots are equal, say $\alpha = \beta$. We know that the solution of

$$(D - a)^2 z = 0, \text{ is } z = e^{mx}(C_1 x + C_2), \text{ § } 130;$$

by analogy, the solution of

$$(D - aD')^2 z = 0, \text{ is } z = e^{mx}\{xf_1(y) + f_2(y)\},$$

or,

$$z = xf_1(y + ax) + f_2(y + ax). \quad . \quad . \quad (11)$$

EXAMPLES.—Solve :

(1) $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0.$ Ansr. $z = xf_1(y + x) + f_2(y + x)$.

(2) $(D^3 - 3D^2D' + DD'^2 + D^3)z = 0.$

Ansr. $z = xf_1(y - x) + f_2(y - x) + f_3(y + x)$.

The particular integral will be discussed after.

B. Non-homogeneous equations. Type :

$$A_0\frac{\partial^2 z}{\partial x^2} + A_1\frac{\partial^2 z}{\partial x \partial y} + A_2\frac{\partial^2 z}{\partial y^2} + A_3\frac{\partial z}{\partial x} + A_4\frac{\partial z}{\partial y} + A_5 z = 0. \quad (12)$$

If the non-homogeneous equation can be separated into factors, the integral is the sum of the integrals corresponding to each symbolic factor, so that each factor of the form $D - mD'$, appears in the solution as a function of $y + mx$, and every factor of the form $D - mD' - a$, appears in the solution in the form $z = e^{ax}f(y + mx)$.

EXAMPLES.—(1) Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0.$

Factors, $(D + D')(D - D' + 1)z = 0.$ Ansr. $z = f_1(y - x) + e^{-x}f_2(y + x)$.

(2) Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0.$

Factors, $(D + 1)(D - D')z = 0.$ Ansr. $z = e^{-x}f_1(y) + f_2(x - y)$.

It is, however, not often possible to represent the solutions of these equations in this manner. When this is so, it is customary to take the trial solution,

$$z = e^{\alpha x + \beta y}, \quad . \quad . \quad . \quad . \quad (13)$$

and substitute for z in the given equation (12). Then,

$$\frac{\partial z}{\partial x} = \alpha z ; \frac{\partial z}{\partial y} = \beta z ; \frac{\partial^2 z}{\partial x \partial y} = \alpha\beta z ;$$

$$\frac{\partial^2 z}{\partial x^2} = \alpha^2 z ; \frac{\partial^2 z}{\partial y^2} = \beta^2 z.$$

Equate the resulting auxillary equation to zero. We thus obtain

$$(A_0\alpha^2 + A_1\alpha\beta + A_2\beta^2 + A_3\alpha + A_4\beta + A_5)z = 0. \quad (14)$$

This may be looked upon as containing a bracketed quadratic in α and β. For any value of β, we can find the corresponding value of α, or the value of α, for any assigned value of β. There is thus an infinite number of particular solutions of this differential equation.

If $u_1, u_2, u_3, \ldots$, *are particular solutions of any partial differential equation, each solution can be multiplied by an arbitrary constant and the resulting products are also solutions of the equation.*

Similarly, it is not difficult to see that *the sum of any number of particular solutions will also be a solution of the given equation.*

It is usually not very difficult to find particular solutions, even when the general solution cannot be obtained. The chief difficulty lies in the combining of the particular solutions in such a way, that the conditions of the problem under investigation are satisfied. Plenty of illustrations will be found at the end of the next chapter.

If the above quadratic is solved for α in terms of β, and if the resulting $f(\alpha, \beta)$, is homogeneous, we shall have the roots in the form,

$$\alpha = m_1\beta, \ \alpha = m_2\beta, \ \ldots, \ \alpha = m_n\beta.$$

The equation will, therefore, be satisfied by any expression of the form,

$$z = \Sigma C e^{\beta(y + mx)}, \quad . \quad . \quad . \quad . \quad (15)$$

where m has any value $m_1, m_2, \ldots$ and C may have any value $C_1, C_2, \ldots$ The symbol "Σ" indicates the sum of the infinite series, obtained by giving m and C all possible values.

The above solution (15), may be put in a simpler form when β is a linear function of α, say, $\beta = a\alpha + b$. This applies to equation (12). Again, we can sometimes solve the equation $z = e^{\alpha x + \beta y} = 0$, for α, in terms of β. In order to fix these ideas, let us proceed to the following examples.

EXAMPLES.—(1) Solve $(D^2 - D')z = 0$. Here $\alpha^2 - \beta = 0$. Hence, $z = Ce^{\alpha x + \alpha^2 y}$. Put $\alpha = \frac{1}{2}$, $\alpha = 1$, $\alpha = 2$, . . . and we get the particular solutions

$$e^{\frac{1}{4}(2x + y)},\ e^{x + y},\ e^{2x + 4y},\ \ldots$$

Now the difference between any two terms of the form $e^{\alpha x + \beta y}$, is included in the above solution, it follows, therefore, that the first differential coefficient of $e^{\alpha x + \beta y}$, is also an integral, and, in the same way, the second, third and higher derivatives must be integrals. Thus we have the following particular solutions:—

$$De^{\alpha x + \alpha^2 y} = (x + 2\alpha y)e^{\alpha x + \alpha^2 y}.$$
$$D^2 e^{\alpha x + \alpha^2 y} = \{(x + 2\alpha y) + 2y\}e^{\alpha x + \alpha^2 y}.$$
$$D^3 e^{\alpha x + \alpha^2 y} = \{(x + 2\alpha y) + 6y(x + 2\alpha y)\}, \text{ etc.}$$

If $\alpha = 0$, we get the special case,

$$z = C_1 x + C_2(x^2 + 2y) + C_3(x^3 + 6xy) + \ldots$$

(2) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$. Put $z = Ce^{\alpha x + \beta y}$. Note

$$(\alpha - \beta)(\alpha + \beta - 3) = 0. \quad \therefore \beta = \alpha \text{ and } \beta = 3 - \alpha.$$

Hence $z = \Sigma C_1 e^{\alpha(x + y)} + e^{3y}\Sigma C_1 e^{\alpha(x - y)}$; $z = f_1(y + x) + e^{3y}f_2(y - x)$.

(3) Solve $\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + abz = 0$. Ansr. $z = e^{-ay}f_1(x) + e^{-bx}f_2(y)$.

(4) Solve $(D^2 - D'^2 + D + 3D' - 2)z = 0$. Ansr. $z = e^x f_1(y - x) + e^{-2x}f_2(y + x)$.

§ 142. The Particular Integral of Linear Partial Equations.

The following methods for finding the particular integral of homogeneous or non-homogeneous equations, are deduced by processes analogous to those employed for the particular integrals of the ordinary equations.

The complementary function of the ordinary linear equation

$$(D - m)z = 0, \text{ is, } Ce^{mx};$$

so, for the partial equation

$(D - mD')z = 0$, we have, $e^{mxD'}\phi(y)$, or the equivalent $\phi(y + mx)$.

This analogy extends to the particular integrals. In the former case, the particular integral of

$$(D - m)z = R, \text{ is, } z = (D - m)^{-1}R;$$

while for $F(D, D')z = R$, we have, $z = F(D, D')^{-1}R$.

Case 1 (General). *When $F(D, D')$ can be resolved into factors, so that,*

$$z = (D - mD')f(x, y). \quad . \quad . \quad . \quad (16)$$

It is now necessary to find a value for this symbol (16). First show that

$$De^{ax}R = (D + a)R,$$

by putting mD in place of a, and $f(x, y)$, in place of R (Case 4, § 131).

$$\therefore \frac{1}{D - mD'}f(x, y) = \frac{1}{D - mD'}e^{mxD'}e^{-mxD'}f(x, y);$$

$$= e^{mxD'}\frac{1}{D}f(x, y - mx). \quad . \quad . \quad (17)$$

The value sought. The particular integral may, therefore, be found by the following series of operations:

(1) Subtract mx from y in the $f(x, y)$, to be operated upon. (2) Integrate the result with respect to dx. (3) Add mx to y, *after* the integration.

If there is a succession of factors, the rule is to be applied to each one *seriatim*, beginning on the right.

EXAMPLES.—Find particular integrals in the following examples. It is well to be careful about the signs of the different terms added and subtracted. It is particularly easy to err by want of attention to this.

(1) $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = xy$. The particular integral is $\frac{1}{D - aD'} \cdot \frac{1}{D + aD'}xy$. Now xy becomes $x(y - ax)$. This, on integration with respect to dx, becomes $\frac{1}{2}x^2y - \frac{1}{3}ax^3$, and finally $\frac{1}{2}x^2y + \frac{1}{2}ax^3 - \frac{1}{3}ax^3$. Hence,

$$\frac{1}{D - aD'} \cdot \frac{1}{D + aD'}xy = \frac{1}{D - aD'} \cdot \frac{x^2y}{2} + \frac{ax^3}{6}.$$

Subtract $- ax$ from y, for $\frac{1}{2}x^2(y + ax) + \frac{1}{6}ax^3$. Integrate and add $- ax$ to the result. $\frac{1}{6}x^3y$ remains. This is the required result.

(2) $(D^2 + 3DD' + 2D')^{-1}x + y$. Ansr. $\frac{1}{2}x^2y - \frac{1}{3}x^3$.

Case 2 (Special). *When R has the form $f(ax + by)$.* Multiply $F(D, D')z$ by D^n and get

$$D^n\phi(D'/D)z = f(ax + by).$$

Operate on $ax + by$ with D' and D respectively,

$$\frac{D'}{D}f(ax + by) = \frac{b}{a}.$$

As on page 313, the particular integral is

$$z = \frac{1}{D^n} \cdot \frac{1}{\phi(D'/D)}\phi(ax + by).$$

$$= \frac{1}{f(a/b)}\int\int \ . \ . \ . \int\phi(ax + by)dx^n.$$

How to use this formula will appear from the examples.

EXAMPLES.—Find particular integrals in, (1) $(D^2 + DD' - 2D'^2)z = \sin(x + 2y)$. The particular integral is

$$\frac{1}{D^2(1 + D'/D - 2D'^2/D^2)}\sin(x + 2y) = \frac{1}{1 + 2 - 8}\int\int\sin(x + 2y)dx^2;$$

$$= \tfrac{1}{5}\sin(x + 2y).$$

(2) $(D^2 + 5DD' + 6D'^2)z = 1/(y - 2x)$. Ansr. $x\log(y - 2x)$.

The above process cannot be employed when $F(D, D')$, or $F(a, b)$ has the same form as R, because a vanishing factor then appears in the result. In such a case, use the above method for all factors which do not vanish when a is put for D, b for D. The solution is then completed by means of the formula:

$$\frac{1}{D - mD'}f(y + mx) = xf(y + mx). \quad . \quad . \quad (18)$$

EXAMPLE.—Evaluate the particular integral in

$$(D - D')(D + 2D')z = x + y.$$

For the first factor, use the above method and then

$$= \frac{1}{D - D'} \cdot \frac{1}{3D}(x + y) = \frac{1}{3D}e^{xD'}e^{-xD'}(x + y);$$

$$= \frac{1}{3D}e^{xD'}\frac{1}{D}y = \frac{1}{3D}e^{xD'}xy = \frac{1}{9}x^3 + \frac{1}{6}x^2y.$$

Case 3 (Special). *When R has the form of sin(ax + by), or cos(ax + by).* Proceed as on page 313, when

$$\frac{1}{F(D^2, DD', D'^2)}\sin(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)}\sin(ax + by), \quad (19)$$

and in the same way for the cosine.

EXAMPLES.—Find the particular integrals in:

(1) $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$.

$$\therefore \frac{1}{D^2 + DD' + D' - 1}\sin(x + 2y) = \frac{1}{-1 - 2 + D' - 1}\sin(x + 2y)$$

$$= \frac{D' + 4}{D'^2 - 16} = -\frac{1}{10}\{\cos(x + 2y) + 2\sin(x + 2y)\}.$$

(2) $(D + DD' - 2D')z = \sin(x - y) + \sin(x + y)$. Find the particular integral for $\sin(x - y)$, then for $\sin(x + y)$. Add the two results together. Ansr. $\frac{1}{2}\sin(x - y) + \frac{1}{3}x\cos(x + y)$.

For the anomalous case proceed as in § 131.

Case 4 (Special). *When R has the form* e^{ax+by}, proceed as directed on page 312,

$$\frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(a, b)}e^{ax+by}, \quad . \quad . \quad (20)$$

that is to say, put a for D and b for D'.

EXAMPLES.—Find particular integrals in the following:

(1) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$.

Ansr. $= (D^2 - DD' - 2D'^2 + 2D + 2D')^{-1}e^{2x+3y}; = -\frac{1}{10}e^{2x+3y}$.

(2) $(DD' + aD + bD' + ab)z = e^{my+nx}$. Ansr. $e^{my+nx}/(m + a)(n + b)$.

If $F(a, b) = 0$, proceed as on page 313,

$$z = \frac{1}{F'_a(a, b)}e^{ax+by}; \text{ or, } z = \frac{1}{F'_b(a, b)}e^{ax+by}, \quad . \quad (21)$$

where F'_a or F'_b denotes the first differential coefficient with respect to the subscript. The two results agree with each other.

EXAMPLE.—Solve $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$.
Ansr. $f_1(x + y) + e^{3y}f_2(y - x) - ye^{x+2y}$.

Case 5 (Special). *When R has the form* x^ry^s, where r and s are positive integers. Operate with $F(D, D')^{-1}$ on x^ry^s expanded in ascending powers of r and s.

EXAMPLES.—(1) Find the particular integral in:

$$(D^2 + DD' + D - 1)z = x^2y.$$

$$=\{1 - (D^2 + DD' + D')\}^{-1}x^2y = -\{1 + (D^2 + DD' + D') + (D^2 + DD' + D')^2 + \ldots\}x^2y.$$

$$= -(1 + D^2 + DD' + D' + 2D^2D')x^2y = -x^2y - 2y - 2x - x^2 - 4.$$

The expansion is not usually carried higher than the highest power of the highest power in $f(x, y)$.

(2) Evaluate $(D^2 - D'^2 - 3D + 3D')^{-1}xy$. Ansr.

$$-\tfrac{1}{18}x^3 - \tfrac{1}{9}x^2 - \tfrac{2}{27}x - \tfrac{1}{9}xy - \tfrac{1}{6}x^2y.$$

(3) $(D^2 - a^2D'^2)z = x$.

$$= \frac{1}{D^2}\Big(1 + a^2\frac{D'^2}{D^2} + \ldots\Big)x = \frac{1}{D^2}x = \iint x dx^2 = \tfrac{1}{6}x^3.$$

Case 6 (Special). *When R has the form* $e^{ax+by}X$, where X is a function of x or y. Use

$$F(D, D')^{-1}e^{ax+by}X = e^{ax+by}F(D + a, D + b)^{-1}X, \qquad (22)$$

derived as on page 315.

EXAMPLE.—Find the particular integrals in

$$\partial^2z/\partial x^2 - \partial z/\partial y = xe^{ax+a^2y}.$$

$$\text{Ansr. } e^{ax+a^2y}\frac{1}{D^2 + 2aD - D'}x = e^{ax+a^2y}\frac{1}{2a}\cdot\frac{1}{D}\Big(1 + \frac{D}{2a}\Big)^{-1}x$$

$$= e^{ax+a^2y}\tfrac{1}{4}(x^2/a - x/a^2).$$

§ 143. The Linear Partial Equation with Variable Coefficients.

These may sometimes be solved by transforming them into a form with constants. *E.g.*,

(i.) Any term $x^ry^s\dfrac{\partial^{r+s}z}{\partial x^r\partial y^s}$ may be reduced to the form with constant coefficients, by substituting $u = \log x$, $v = \log y$.

EXAMPLES.—Solve the equations:

(1) $x^2\dfrac{\partial^2z}{\partial x^2} - y^2\dfrac{\partial^2z}{\partial y^2} - y\dfrac{\partial z}{\partial y} + x\dfrac{\partial z}{\partial x} = 0.$ This reduces to $\partial^2z/\partial u^2 - \partial^2z/\partial v^2 = 0$.

Hence the solution of this equation, $z = \phi_1(u + v) + \phi_2(u - v)$, must be reconverted into the form in x and y, thus, $z = f_1(xy) + f_2(x/y)$.

(2) $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$. Ansr. $z = f_1(y/x) + xf_2(y/x)$.

(3) $(x + y)\frac{\partial^2 z}{\partial x \partial y} - a\frac{\partial z}{\partial x} = 0$. Put $\partial z = v \,.\, \partial x$.

Ansr. $z = f_1(y) + \int(x + y)^a f'_2(x) \,.\, dx$.

(ii.) The transformation may be effected by substituting $\vartheta = x\frac{\partial}{\partial x}$ and $\vartheta' = y\frac{\partial}{\partial y}$, and treating the result as for constant coefficients.

EXAMPLES.—(1) Solve the first two examples of the preceding set in this way.

(2) $x\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} - nx\frac{\partial z}{\partial x} - ny\frac{\partial z}{\partial y} + nz = 0$.

$$\therefore \{\vartheta(\vartheta - 1) + 2\vartheta\vartheta' + \vartheta'(\vartheta' - 1) - n\vartheta + n\vartheta' + n\}z = 0.$$

Ansr. $z = x^n f_1(y/x) + xf_2(y/x)$.

§ 144. The Integration of Differential Equations in Series.

When a function can be developed in a series of converging terms, arranged in powers of the independent variable, an approximate value for the dependent variable can easily be obtained. The degree of approximation attained obviously depends on the number of terms of the series included in the calculation. The older mathematicians considered this an underhand way of getting at the solution but, for practical work, it is invaluable. As a matter of fact, solutions of the more advanced problems in physical mathematics are nearly always represented in the form of an abbreviated infinite series. Finite solutions are the exception rather than the rule.

EXAMPLES.—(1) Evaluate the integral in $f(x) = 0$. Assume that $f(x)$ can be developed in a converging series of ascending powers of x, that is to say,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad (1)$$

By integration

$$\int f(x)dx = \int(a_0 + a_1x + a_2x^2 + \ldots)dx;$$
$$= \int a_0dx + \int a_1xdx + \int a_2x^2dx + , \ldots;$$
$$= a_0x + \tfrac{1}{2}a_1x^2 + \tfrac{1}{3}a_2x^3 + \ldots;$$
$$= x(a_0 + \tfrac{1}{2}a_1x + \tfrac{1}{3}a_2x^2 + \ldots) + C. \qquad (2)$$

(2) It is required to find the solution of $dy/dx = y$, in series. Assume that $y = f(x)$, has the form (1) above, and substitute in the given equation.

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x + \ldots = 0. \qquad (3)$$

This equation would be satisfied, if

$$a_1 = a_0;\ a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0;\ a_3 = \frac{1}{3}a_2 = \frac{1}{3!}a_0;\ \ldots$$

Hence,
$$y = a_0\phi(x),$$
where
$$\phi(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = e^x.$$

Put a for the arbitrary function,

$$\therefore y = ae^x.$$

That this is a complete solution, is proved by substitution in the original equation.

Write the original equation in the form

$$y = v\phi(x),$$

where v is to be determined. Hence,

$$\frac{dy}{dx}\phi(x) + v'\{\phi'(x) - \phi(x)\} = 0,$$

since $\phi(x)$ satisfies the original equation,

$$dy/dx = 0, \text{ or } v \text{ is constant.}$$

For equations of higher degree, we must proceed a little differently. For example:

(3) Solve
$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - cy = x^2. \qquad (4)$$

(i.) *The complementary function.* As a trial solution, put $y = ax^m$. The auxillary equation is

$$m(m - 1)a_0x^{m-2} - (m + c)x^m = 0. \qquad (5)$$

This shows that the difference between the successive exponents of x in the assumed series, is -2. The required series is, therefore,

$$y = a_0x^m + a_1x^{2m-2} + \ldots + a_{n-1}x^{m+2n-2} + a_nx^{m+2n},$$

which is more conveniently written

$$y = \Sigma_0^\infty a_nx^{m+2n}. \qquad (6)$$

In order to completely determine this series, we must know three things about it. Namely, the first term, the coefficients of x and the different powers of x that make up the series.

Substitute (6) in (4),

$$\Sigma_0^\infty(m + 2n)(m + 2n - 1)a_nx^{m+2n-2} - (m + 2n + c)a_nx^{m+2n} = 0, \qquad (7)$$

where n has all values from zero to infinity. If x is a solution of (4), the coefficient of x^{m+2n-2} must vanish with respect to m. Hence by equating the coefficient of x^{m+2n-2} to zero,*

$$(m + 2n)(m + 2n - 1)a_n - (m + 2n - 2 + c)a_{n-1} = 0. \qquad (8)$$

If $n = 0$, $m = 0$, or $m = 1$.

When n is greater than zero,

$$a_n = \frac{m + 2n - 2 + c}{(m + 2n)(m + 2n - 1)}. \qquad (9)$$

This formula allows us to calculate the relation between the successive coefficients of x by giving n all integral values 1, 2, 3, . . .

---

* If we take the other part of the auxillary a diverging series is obtained, useless for our purpose.

First, suppose $m = 0$, then we can easily calculate from (9),

$$a_1 = \frac{c}{1 \, . \, 2}a_0 ; \; a_2 = \frac{c+2}{3 \, . \, 4}a_1 = \frac{c(c+2)}{4\,!}a_0 ; \; . \; . \; .$$

$$\therefore y' = a_0\left\{1 + c\frac{x^2}{2!} + c(c+2)\frac{x^4}{4!} + \; . \; . \; . \right. \quad . \quad . \quad . \quad (10)$$

Next put $m = 1$, and, to prevent confusion, write b, in (9), in place of a.

$$b_n = \frac{c + 2n - 1}{2n(2n+1)}b_{n-1} ;$$

proceed exactly as before to find successively b_1, b_2, b_3, . . .

$$\therefore y'' = b_0\left\{x + (c+1)\frac{x^3}{3!} + (c+1)(c+3)\frac{x^5}{5!} + \; . \; . \; . \right. \quad . \quad (11)$$

The complete solution of the equation, is the sum of series (10) and (11), or if $ay_1 = y'$, $by_2 = y''$,

$$y = ay_1 + by_2,$$

which contains the two arbitrary constants a and b.

(ii.) *The particular integral.* By the above procedure we obtain the complementary function. For the particular integral, we must follow a somewhat similar method. *E.g.*, equate (8) to x^2 instead of to zero. The coefficient of $m - 2$, in (5), becomes

$$m(m-1)a_0x^{m-2} = x^2 ;$$

$$\therefore m - 2 = 2 \text{ and } m(m-1)a_0 = 1 ;$$

$$\therefore m = 4 ; \; a_0 = \tfrac{1}{2}.$$

From (9)

$$a_n = \frac{c + 2n + 2}{2(n+2)(2n+3)}a_{n-1}.$$

Substitute successive values of $n = 1, 2, 3, \ldots$ in the assumed expansion, and we obtain

$$(\textit{Particular integral}) = a_0x^m + a_1x^{m+2} + a_2x^{m+4} + \; . \; . \; . \; ,$$

where a_0, a_1, a_2, . . . and m have been determined.

(4) Solve $d^2y/dx^2 + xy = 0$.

Ansr. $y = a\left(1 - \frac{1}{3\,!}x^3 + \frac{1 \, . \, 4}{6\,!}x^6 - \; . \; . \; .\right) + b\left(x - \frac{2}{4\,!}x^4 + \frac{2 \, . \, 5}{7\,!}x^7 - \; . \; . \; .\right).$

The so-called *Riccati's equation*,

$$\frac{dy}{dx} + by^2 = cx^n,$$

has attracted a lot of attention in the past. Otherwise it is of no particular interest here. It is easily reduced to a linear form of the second order. Its solution appears as a converging series, finite under certain conditions.

Forsyth (*l.c.*) or Johnson (*l.c.*) must be consulted for fuller details. A detailed study of the more important series employed in physical mathematics follows naturally from this point. These are mentioned in the next section along with the titles of special textbooks devoted to their use.

§ 145. Harmonic Analysis.

One of the most important equations in physical mathematics, is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\kappa}\frac{\partial V}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It has practically the same form for problems on the conduction of heat, the motion of fluids, the diffusion of salts, the vibrations of elastic solids and flexible strings, the theory of potential, electric currents and numberless other phenomena. x, y, z are the coordinates of a point in space, t denotes the time and V may denote temperature, concentration of a solution, electric and magnetic potential, the Newtonian potential due to an attracting mass, etc., κ is a constant. If the second member is zero, we have Laplace's equation, if the second number is equated to $4\pi\rho$, where ρ is a function of x, y, z, the result is known as Poisson's equation.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \text{ is } \textbf{Laplace's equation.}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 4\pi\rho, \text{ is } \textbf{Poisson's equation.}$$

The first member is written $\nabla^2 V$ by some writers, $\Delta^2 V$ by others. The equation is often more convenient to use in polar coordinates, *viz.*,

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 V}{\partial \theta^2} + \frac{2}{r}\cdot\frac{\partial V}{\partial r} + \frac{\cot\theta}{r^2}\cdot\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}, \quad . \quad . \quad (2)$$

where the substitutions are indicated in (11), § 48.*

Any homogeneous algebraic function of x, y, z, which satisfies equation (1), is said to be a *solid spherical harmonic*. These functions are chiefly used for finding the potential on the surface of a sphere, due to forces which are not circularly symmetrical.†

Particular solutions of (1) give rise, under special conditions, to the so-called *surface spherical harmonics*, *tesseral harmonics* and *toroidal harmonics*.

The series

$$\frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 . 2!(n+1)(n+2)} - \ . \ . \ . \right\},$$

is called a *Cylindrical Harmonic* or a *Bessel's function of the nth order*. The symbol $J_n(x)$ is used for it. The series is a particular solution of **Bessel's equation.**

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0.$$

If $n = 0$, the series is symbolised by $J_0(x)$ and called a *Bessel's function of the zeroth order*. These functions are employed in physical mathematics when dealing with certain problems connected with equation (1). Another particular solution is

$$J_0(x)\log x + \frac{x^2}{2^2} - \frac{x^4}{2^2 . 4^2}\left(\frac{1}{1} + \frac{1}{2}\right) + \ . \ . \ .,$$

called a *Bessel's function of the second kind* (of the zeroth order), symbolised by $K_0(x)$.

Similarly, the solution of **Legendre's equation**

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1) = 0,$$

is the series

$$1 - \frac{m(m+1)}{2!}x^2 + \frac{m(m-2)(m+1)(m+3)}{4!}x^4 - \ . \ . \ .,$$

---

* This transformation is described in the regular textbooks. But possibly the reader can do it for himself.

† A point is said to be circularly symmetrical, when its value is not affected by rotating it through an angle about the axis.

written, for brevity, $P_m(x)$. This furnishes the so-called *Surface Zonal Harmonics*, *Legendre's coefficients*, or *Legendrians*. Another particular solution,

$$x - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-1)(m-3)(m+2)(m+4)}{5!}x^5 - \ldots,$$

written $Q_m(x)$, gives rise to *Surface Zonal Harmonics of the second kind*. Both series are extensively employed in physical problems connected with equation (1).

The equation

$$(x^2 - b^2)(x^2 - c^2)\frac{d^2y}{dx^2} + x(x^2 - b^2 + x^2 - c^2)\frac{dy}{dx} - \{m(m+1)(x^2 - b^2 + c^2)p\}y = 0,$$

called **Lamé's equation**, has "series" solution which furnishes *Lamé's functions* or *Ellipsoidal Harmonics*, used in special problems connected with the ubiquitous equation

$$\nabla^2 V = \frac{1}{\kappa}\frac{dV}{dt}.$$

The so-called *hypergeometric* or *Gauss' series*,

$$1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 + \ldots,$$

appears as a solution of certain differential equations of the second order, say,

$$x(1-x)\frac{d^2y}{dx^2} + \{c - (a+b+1)x\}\frac{dy}{dx} - aby = 0,$$

(Gauss' equation), where a, b, c, are constants.

The application of these series to particular problems constitutes that branch of mathematics known as **Harmonic Analysis.**

But we are getting beyond the scope of this work; for more practical details, the reader will have to take up some special work such as Byerly's *Fourier's Series and Spherical Harmonics.* Weber and Riemann's *Die Partiellen Differential-Gleichungen der Mathematischen Physik* is *the* textbook for more advanced work. Gray and Mathews have *A Treatise on Bessel's Functions and their Application to Physics* (Macmillan & Co., 1895).

CHAPTER VIII.

FOURIER'S THEOREM.

"Fourier's theorem is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."—THOMSON AND TAIT.

§ 146. Fourier's Series.

JUST as a musical note can be resolved into a fundamental note and its overtones, so every periodic vibration can be resolved into a series of secondary vibrations represented, in mathematical symbols, by a series of terms arranged, not in a series of ascending powers of the independent variable, as in Maclaurin's theorem, but in a series of sines and cosines of multiples of this variable. Such expansions in a series of trignometrical terms, are of great importance in physical problems involving potential, conduction of heat, light, sound, electricity and other forms of propagation. The series, developed by means of Fourier's theorem, is called Fourier's series.

Any physical property (density, pressure, velocity, etc.) which varies periodically with time and whose magnitude or intensity can be measured, may be represented by a Fourier's series. This means, as we shall soon see, that every vibration can be resolved into a series of harmonic vibrations.

Fourier's theorem *determines the law for the expansion of any arbitrary function in terms of sines or cosines of multiples of the independent variable* (x).

If $f(x)$ is a periodic function with respect to time, space, temperature, potential, etc., Fourier's theorem states that

$$f(x) = A_0 + a_1 \sin x + a_2 \sin 2x + \ldots + b_1 \cos x + b_2 \cos 2x + \ldots, \quad (1)$$

which is known as **Fourier's series.** A trignometrical function, like Fourier's series, for example, passes through all its changes and returns to the same value when x is increased by 2π. See also d'Alembert's equation, page 348.

Assuming this theory to be valid between the limits $x = +\pi$ and $x = -\pi$, we shall now proceed to find values for the coefficients, $A_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, which will make the series true.

In view of the fact that the terms of Fourier's series are all periodic we may say that *Fourier's series is an artificial way of representing the propagation or progression of any physical quality by a series of waves or vibrations.*

§ 147. Evaluation of the Constants in Fourier's Theorem.

First, to find a value for the constant A_0. Multiply equation (1) by dx and then integrate each term between the limits $x = +\pi$ and $x = -\pi$. Every term involving sine or cosine terms vanishes, and

$$2\pi A_0 = \int_{-\pi}^{+\pi} f(x) \, . \, dx \, ; \text{ or, } A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) \, . \, dx, \quad . \quad (2)$$

remains. Therefore, when $f(x)$ is known, this integral can be integrated.*

Second, to find a value for the coefficients of the cosine terms, say b_n, where n may be any number from 1 to n. Equation (1) must not only be multiplied by dx, but also by some factor such that all the other terms will vanish when the series is integrated between the limits $\pm\pi$, $b_n \cos nx$ remains. Such a factor is $\cos nx \, . \, dx$. In this case,

$$\int_{-\pi}^{+\pi} \cos^2 nx \, . \, dx = b_n \pi,$$

(page 184), all the other terms involving sines or cosines, when integrated between the limits $\pm\pi$, will be found to vanish. Hence the desired value of b_n is

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \, . \cos nx \, . \, dx. \quad . \quad . \quad (3)$$

---

* I have omitted details because the reader should find no difficulty in working out the results for himself. It is no more than an exercise on preceding work.

This formula enables any coefficient, $b_1, b_2, \ldots, b_n$ to be obtained. If we put $n = 0$, the coefficient of the first term A_0 assumes the form,

$$A_0 = \tfrac{1}{2}b_0. \quad . \quad . \quad . \quad . \quad (4)$$

If this value is substituted in (1), we can dispense with (2), and write

$$f(x) = \tfrac{1}{2}b_0 + a_1\sin x + b_1\cos x + a_2\sin 2x + b_2\cos 2x + \ldots \quad (5)$$

Finally, to find a value for the coefficients of the sine terms, say a_n. As before, multiply through with $\sin nxdx$ and integrate between the limits $\pm \pi$. We thus obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \,.\, \sin nx \,.\, dx. \quad . \quad . \quad (6)$$

There are several graphic methods for evaluating the coefficients of a Fourier's series. See Perry, *Electrician*, Feb. 5, 1892; Woodhouse, the same journal, April 19, 1901; or, best of all, Henrici, *Phil. Mag.* [5], **38**, 110, 1894, when the series is used to express the electromotive force of an alternating current as a periodic function of the time.

§ 148. The Development of a Function in a Trignometrical Series.

1. *The development of a trignometrical series of sines.* Suppose it is required to find the value of

$$f(x) = x,$$

in terms of Fourier's theorem. From (2), (3) and (6),

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} x \,.\, \cos nx \,.\, dx = 0\,;\ a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} x \,.\, \sin nx \,.\, dx = \pm\,\frac{2}{n},$$

according as n is odd or even;

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x \,.\, dx = \frac{1}{4\pi}(\pi^2 - \pi^2) = 0.$$

Hence Fourier's series assumes the form

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots), \quad . \quad . \quad (7)$$

which is known as a **sine series**; the cosine terms have disappeared during the integration.

By plotting the bracketed terms in (7), we obtain the series of curves shown in Fig. 116. Curve 1 has been obtained by plotting $y = \sin x$; curve 2, by plotting $y = \frac{1}{2}\sin 2x$; curve 3, from $y = \frac{1}{3}\sin 3x$. These curves, dotted in the diagram, represent the o vertones or harmonics. Curve 4 has been obtained by drawing ordinates equal to the algebraic sum of the ordinates of the preceding curves.

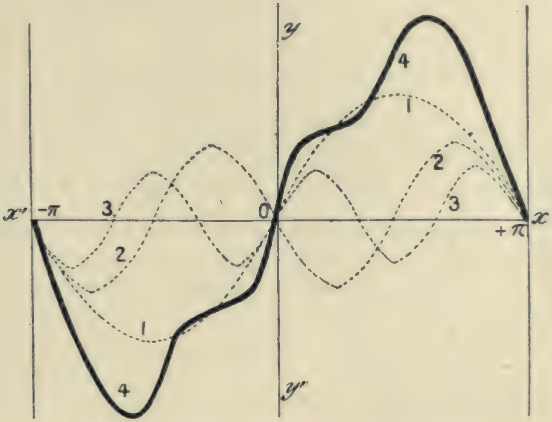


FIG. 116.—Harmonics of the Sine Curve.

As a general rule, any odd function of x will develop into a series of sines only, an even function of x will consist of a series of cosines.

The general form of the sine development is

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots, \qquad (8)$$

where a has the value given in equation (6).

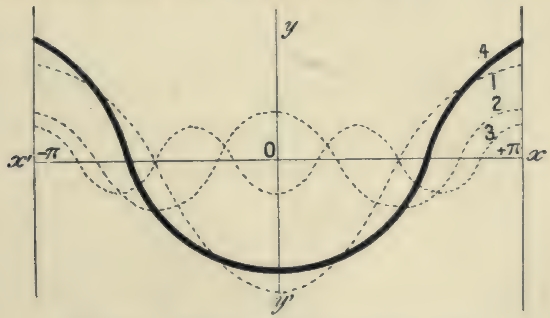


FIG. 117.—Harmonics of the Cosine Curve.

2. *The development of a trignometrical series of cosines.* As an example, let

$$f(x) = x^2,$$

be expanded by Fourier's theorem. Here

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} x^2 \,.\, \cos nx \,.\, dx = \mp \frac{4}{n^2},$$

according as n is odd or even. Also,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} x^2 \sin nx \,.\, dx = 0; \; A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x^2 dx = \frac{1}{6\pi}\left\{\pi^3 - (-\pi)^3\right\} = \tfrac{1}{3}\pi^2.$$

Hence,

$$x^2 = \frac{1}{3}\pi^2 - 4\left(\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right). \qquad (9)$$

By plotting the first three terms enclosed in brackets on the right side of (9), we obtain the series of curves shown in Fig. 117 (p. 363).

The general development of a **cosine series** is as follows:

$$f(x) = \tfrac{1}{2}b_0 - b_1\cos x + b_2\cos 2x + \ldots, \qquad (10)$$

where b has the values assigned in (3).

EXAMPLES.—(1) Develop unity in a series of sines between the limits $x = \pi$ and $x = 0$. Here

$$f(x) = 1.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{n\pi}(1 - \cos n\pi) = \frac{2}{n\pi}\{1 - (-1)^n\} \frac{4}{n\pi} \text{ or } 0,$$

according as n is odd or even. Hence, from (8),

$$1 = \frac{4}{\pi}(x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots). \qquad (11)$$

The first four terms of this series are plotted in Fig. 118 in the usual way.

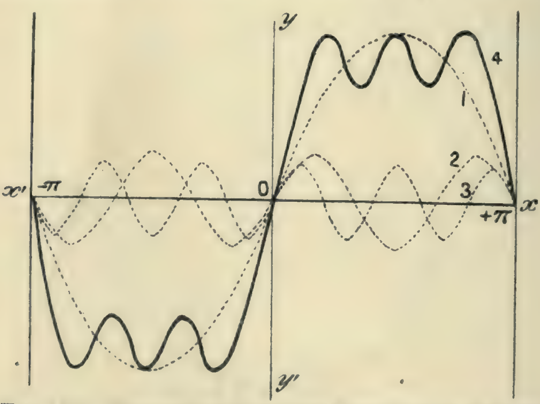


FIG. 118.—Harmonics of the Sine Development of Unity.

(2) Show that for $x = \pm \pi$

$$e^x = \frac{2\sinh \pi}{\pi}\{(\tfrac{1}{2} - \tfrac{1}{2}\cos x + \tfrac{1}{5}\cos 2x + \ldots) + (\tfrac{1}{2}\sin x - \tfrac{2}{5}\sin 2x + \ldots)\}. \qquad (12)$$

(3) Show that

$$x\sin x = 1 - \tfrac{1}{2}\cos x - \tfrac{2}{3}\cos 2x + \tfrac{1}{4}\cos 3x - \tfrac{2}{15}\cos 4x + \ldots \qquad (13)$$

between the limits π and 0.

If $x = \dfrac{\pi}{2}$, then $\qquad \dfrac{\pi}{4} = \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{15} + \tfrac{1}{35} - \ldots$

(4) Show $x = \frac{\pi}{2} - \frac{4}{\pi}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots)$. . . (14)

Hint. $b_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$;

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, . \, dx = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\{(-1)^n - 1\}.$$

(5) Show that if c is constant,

$$c = \frac{4c}{\pi}(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots). \quad . \quad . \quad (15)$$

Plot the curve as indicated above.

3. *Comparison of the sine and the cosine series.* The sine and cosine series are both periodic functions of x, with a period of 2π. The above expansions hold good only between the limits $x = \pm \pi$, that is to say, when x is greater than $-\pi$, and less than $+\pi$. When $x = 0$, the series is necessarily zero, whatever be the value of the function.

Now any function can be represented both as a sine and as a cosine series. Although the functions and the two series will be identical for all values of x between $x = \pi$ and $x = 0$, there is a marked difference between the sine and cosine developments of the same function. For instance, compare the graph of x when developed in series of sines and series of cosines between the limits $x = 0$ and $x = \pi$, as shown in (7) and (14) above. Plot these equations for successive values of x between $\pm \pi$, etc. In the case of the cosine curve we get the lines shown in Fig. 119. By tracing the curves corresponding to still greater values of π, we get the dotted lines shown in the same figure. For the sine curve we get the lines shown in Fig. 120.

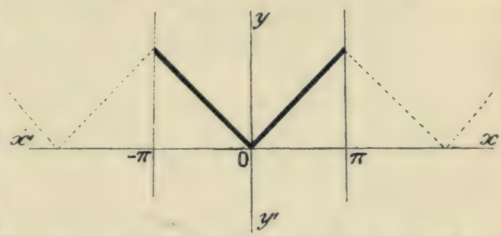


FIG. 119.—Diagrammatic Curve of the Cosine Series.

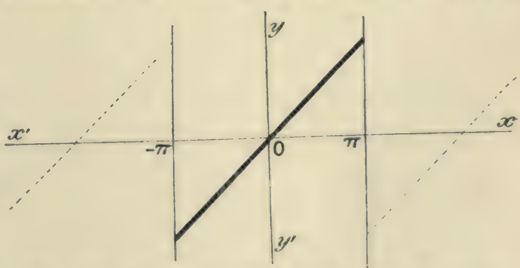


FIG. 120.—Diagrammatic Curve of the Sine Series.

Note the isolated points (§ 63) for $x = \pm\pi$, $y = 0$; $x = \pm 3\pi$, $y = 0$; . . .

Both these curves coincide with $y = x$ from $x = 0$ to $x = \pi$, but not when x is less than $-\pi$, and greater than $+\pi$.

§ 149. Extension of Fourier's Series.

Fourier's series may be extended so as to include all values of x between any limits whatever.

(i.) *When the limits are* $x = +c$, $x = -c$. Let $\phi(x)$ be any function in which x is taken between the limits $-c$ and $+c$. Put $x = cz/\pi$, or $z = \pi x/c$. Hence,

$$\phi(x) = \phi(cz/\pi) = f(z), \text{ say.} \qquad . \quad . \quad (16)$$

When x changes from $-c$ to $+c$, z changes from $-\pi$ to $+\pi$, and, therefore, for all values of x between $-c$ and $+c$, the function $f(z)$ may be developed as in Fourier's series (5), or

$$f(z) = \tfrac{1}{2}b_0 + b_1\cos z + a_1\sin z + b_2\cos 2z + a_2\sin 2z + \ldots \quad (17)$$

where, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} f(z)\cos nz\,.\,dz$; $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} f(z)\sin nz\,.\,dx$, . (18)

or, from (16),

$$\phi(x) = \tfrac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + a_1\sin\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + \ldots; \quad (19)$$

and from (18),

$$b_n = \frac{1}{c}\int_{-c}^{+c}\phi(x)\cos\frac{n\pi x}{c}dx;\ a_n = \frac{1}{c}\int_{-c}^{+c}\phi(x)\sin\frac{n\pi x}{c}dx. \quad (20)$$

For the sine series, from $x = 0$ to $x = c$,

$$f(x) = a_1\sin\frac{\pi x}{c} + a_2\sin\frac{2\pi x}{c} + a_3\sin\frac{3\pi x}{c} + \ldots \quad (21)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx. \quad . \quad . \quad . \quad (22)$$

For the cosine series, from $x = 0$ to $x = c$,

$$f(x) = \left(\tfrac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + \ldots\right) + \left(a_1\sin\frac{\pi x}{c} + \ldots\right). \quad (23)$$

$$b_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx. \quad . \quad . \quad . \quad (24)$$

If $\phi(x)$ is a periodic function whose period is equal to c, then, (19) is true for all values of x. Hence the rule: *Any arbitrary function, whose period is $T = 2c$, can be represented as a series of trignometrical functions with periods T, $\frac{1}{2}T$, $\frac{1}{3}T$, . . .*

EXAMPLES.—Prove the following series for values of x from $x = 0$ to $x = c$.

(1) $$c = \frac{4c}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right). \quad (25)$$

(2) $$mx = \frac{2mc}{\pi}\left(\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} - \ldots\right). \quad (26)$$

(3) $$mx = \frac{m\pi}{2} - \frac{4m}{\pi}\left(\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \frac{1}{5^2}\sin\frac{5\pi x}{c} + \ldots\right). \quad (27)$$

Hint. (26) is $f(x) = mx$ developed in a series of sines; (27) the same function developed in a series of cosines.

(4) If $f(x) = x$ between $+c$ and $-c$,

$$x = \frac{2c}{\pi}\left(\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \ldots\right). \quad (28)$$

(ii.) *When the limits are* $+\infty$ *and* $-\infty$. Since the above formulae are true, whatever be the value of c, the limiting value obtained when c becomes infinitely great should be true for all values of x.

In order to prevent mistakes in working, it is usual to employ some equivalent sign, say λ, for the *integrated* variable x. Hence in place of equations (20), we may write

$$b_n = \frac{1}{c}\int_{-c}^{+c}\phi(\lambda)\cos\frac{n\pi\lambda}{c}d\lambda;\ a_n = \frac{1}{c}\int_{-c}^{+c}\phi(\lambda)\sin\frac{n\pi\lambda}{c}d\lambda. \quad (29)$$

Substitute these values in (23),

$$f(x) = \frac{1}{c}\left\{\frac{1}{2}\int_{-c}^{+c}f(\lambda)d\lambda + \int_{-c}^{+c}f(\lambda)\cos\frac{\pi\lambda}{c}\cos\frac{\pi x}{c}d\lambda\right.$$
$$\left. + \int_{-c}^{+c}f(\lambda)\sin\frac{\pi\lambda}{c}\sin\frac{\pi x}{c}d\lambda + \ldots\right\};$$
$$= \frac{1}{c}\int_{-c}^{+c}f(\lambda)d\lambda\left\{\frac{1}{2} + \cos\frac{\pi\lambda}{c}\cos\frac{\pi x}{c} + \sin\frac{\pi\lambda}{c}\sin\frac{\pi x}{c} + \ldots\right\};$$
$$= \frac{1}{c}\int_{-c}^{+c}f(\lambda)d\lambda\left\{\frac{1}{2} + \cos\frac{\pi}{c}(\lambda - x) + \ldots\right\};$$
$$= \frac{1}{2c}\int_{-c}^{+c}f(\lambda)d\lambda\left\{1 + \cos\frac{\pi}{c}(\lambda - x) + \cos\left(-\frac{\pi}{c}\right)(\lambda - x) + \ldots\right\};$$
$$= \frac{1}{2\pi}\int_{-c}^{+c}f(\lambda)d\lambda\left\{\ldots + \frac{\pi}{c}\cos\frac{0\pi}{c}(\lambda - x) + \frac{\pi}{c}\cos\frac{\pi}{c}(\lambda - x)\right.$$
$$\left. + \frac{\pi}{c}\cos\left(-\frac{\pi}{c}\right)(\lambda - x) + \ldots\right\},$$

since $\cos 0 = 1$; for the other trignometrical substitutions, see page 499. As c is increased indefinitely, the limiting value of the term in brackets is $\int_{-\infty}^{+\infty}\cos\alpha(\lambda - x)d\alpha$, where $\alpha = n\pi/c$, n being any integer. Hence, the limiting form of $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\lambda)d\lambda\int_{-\infty}^{+\infty}\cos\alpha(\lambda - x)d\alpha, \quad (30)$$

for all values of x. The double integral in (30) is known as **Fourier's integral.**

§ 150. Different Forms of Fourier's Integral.

Fourier's integral may be written in different equivalent forms. From § 80,

$$\int_{-\infty}^{+\infty} \cos x dx = \int_{-\infty}^{0} \cos x dx + \int_{0}^{\infty} \cos x dx;$$
$$\int_{-\infty}^{0} \cos x dx = \int_{\infty}^{0} \cos(-x) d(-x) = -\int_{\infty}^{0} \cos x dx;$$
$$\int_{-\infty}^{+\infty} \cos x dx = 2\int_{0}^{\infty} \cos x dx.$$

Hence, we may write in place of (30),

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)d\lambda \int_{0}^{\infty} \cos a(\lambda - x) da, \quad . \quad . \quad (31)$$

where the integration limits in (31) are independent of a and λ, and therefore the integration can be performed in any order.

Let $f(x) = -f(-x)$. Then,

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)d\lambda \int_{0}^{+\infty} \cos a(\lambda - x)da = \frac{1}{\pi}\int_{0}^{+\infty} da \int_{-\infty}^{+\infty} f(\lambda)\cos a(\lambda - x)d\lambda;$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} da \int_{-\infty}^{0} f(\lambda)\cos a(\lambda - x)d\lambda + \int_{0}^{\infty} f(\lambda)\cos a(\lambda - x)d\lambda;$$
$$= \frac{1}{\pi}\int_{0}^{+\infty} da \int_{\infty}^{0} f(-\lambda)\cos a(-\lambda - x)d(-\lambda) + \int_{0}^{\infty} f(\lambda)\cos a(\lambda - x)d\lambda;$$
$$= \frac{1}{\pi}\int_{0}^{\infty} da \left\{ -\int_{0}^{\infty} f(\lambda)\cos a(\lambda + x)d\lambda \right\} + \int_{0}^{\infty} f(\lambda)\cos a(\lambda - x)d\lambda;$$
$$= \frac{1}{\pi}\int_{0}^{\infty} da \int_{0}^{\infty} f(\lambda)\{\cos a(\lambda - x) - \cos a(\lambda + x)\}d\lambda;$$
$$= \frac{2}{\pi}\int_{0}^{\infty} da \int_{0}^{\infty} f(\lambda)\sin a\lambda \,.\, \sin ax \,.\, d\lambda;$$
$$= \frac{2}{\pi}\int_{0}^{\infty} f(\lambda)d\lambda \int_{0}^{\infty} \sin a\lambda \,.\, \sin ax \,.\, da, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which is true for all odd functions of $f(x)$ and for all positive values of x in any function.

Let $f(x) = f(-x)$, we can then reduce (31) in the same way to

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} f(\lambda)d\lambda \int_{0}^{\infty} \cos a\lambda \,.\, \cos ax \,.\, da, \quad . \quad . \quad (33)$$

which is true for all values of x, when $f(x)$ is an even function of x, and for all positive values of x in any function. For the trignometrical reductions, see page 499.

Although the integrals of Fourier's series are obtained by integrating the series term by term, it does not follow that the series can be obtained by differentiating the integrated series term by term, for while differentiation makes a series less convergent, integration makes it more convergent. In other words, a converging series *may* become divergent on differentiation. This raises another question, the convergency of Fourier's series.

§ 151. The Convergency of Fourier's Series.

In the preceding developments it has been assumed:

1. That the trignometrical series is uniformly convergent.
2. That the series is really equal to $f(x)$.

Most elaborate investigations have been made to find if these assumptions can be justified. The result has been to prove that the above developments are valid in every case when the function is single-valued * and finite between the limits $\pm\ \pi$,* and has only a finite number of maximum or minimum values,* between the limits $x = \pm\ \pi$.

The curve $y = f(x)$ need not follow the same law throughout its whole length, but may be made up of several entirely different curves. A complete representation of a periodic function for all

---

* The terms marked with the asterisk may, perhaps, need definitions. According to § 78, the integral

$$\int_{-1}^{1} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{-1}^{1} = -2,$$

represents the area included between the curve $y = 1/x^2$, the x-axis, and the ordinates drawn from $x = 1$ and $x = -1$. Plot the curve and you will find that this result is erroneous. The curve sweeps through infinity, whatever that may mean, as x passes from $+1$ to -1 (see § 52). *The method of integration is, therefore, unreliable when the function to be integrated becomes infinite or otherwise discontinuous at or between the limits of integration.* Consequently, it is necessary to examine certain functions in order to make sure that they are finite and continuous between the given limits, or that the functions either continually increase or decrease, or alternately increase (maxima) and decrease (minima) a finite number of times. This subject is discussed in the opening chapters of Riemann and Weber's *The Partial Differential Equations of Mathematical Physics* (German—F. Vieweg & Sons, Braunschweig, 1900-1901), to which the student must refer if he intends to go exhaustively into these questions.

Single-valued and multiple-valued functions have been defined on page 275. $y = \tan^{-1}x$ is a multiple-valued function, because the ordinates corresponding to the same value of x differ by multiples of π. Verify this by plotting. Obviously, if $x = a$ and $x = b$ are the limits of integration of a multiple-valued function, we must make sure that the ordinates $x = a$ and $x = b$ belong to the same branch of the curve $y = f(x)$.

values of x would provide for developing each term as a periodic series, each of which would itself be a periodic function, and so on.

A discussion of the conditions of convergency of Fourier's series must be omitted from this chapter. Byerly's *An Elementary Treatise on Fourier's Series*, etc., is one of the best practical works on the use of Fourier's integrals in mathematical physics. Fourier's ancient *Théorie analytique de la Chaleur* of 1822 is perhaps as modern as any other work on this subject.* See also Williams, *Phil. Mag.* [5], **42**, 125, 1896; Lord Kelvin's *Collected Papers;* and Weber-Riemann's work (*l.c.*).

152. The Superposition of Particular Solutions to Satisfy given Conditions.

The following remarks will amplify what has already been said in connection with this important principle.

The reader knows that ordinary and partial differential equations differ in this respect: *While ordinary differential equations have only a finite number of independent particular integrals, partial differential equations have an infinite number of such integrals.*

To show that a value of V, in Laplace's equation, can be obtained to satisfy Fourier's integral (31). Suppose that a value of V is required in the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad . \quad . \quad . \quad . \quad (34)$$

such that when $y = \infty$, $V = 0$, and when $y = 0$, $V = f(x)$. First assume that

$$V = e^{ay + \beta x},$$

is a solution, when a and β are constant. Substitute in (34) and divide by $e^{ay + \beta x}$.

$$\therefore a^2 + \beta^2 = 0,$$

if this condition holds, the above value of V is a solution of (34). Hence $V = e^{ay \pm \iota ax}$, are solutions of (34), therefore also $e^{ay}e^{\iota ax}$ and $e^{ay}e^{-\iota ax}$ are solutions. Add and divide by 2, or subtract and divide by 2, § 112, (3) and (4), thus

$$V = e^{ay} \cos ax; \text{ and } V = e^{ay} \sin ax, \quad . \quad . \quad (35)$$

are solutions of (34). Multiply the first by $\cos a\lambda$ and the second of (35) by $\sin a\lambda$. The results still satisfy (34). Add, (22), page 499, and

$$e^{-ay} \cos a(\lambda - x)$$

* Freeman's translation can sometimes be obtained from the second-hand booksellers.

satisfies (34). Multiply by $f(\lambda)d\lambda$ and the result is still a solution of (34)

$$e^{-\alpha y}f(\lambda)\,.\cos\alpha(\lambda - x)d\lambda.$$

Multiply by $1/\pi$ and find the limits when α has different values between 0 and ∞. Hence

$$V = \frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^\infty e^{-\alpha y}f(\lambda)\cos\alpha(\lambda - x)d\lambda, \quad . \quad (36)$$

satisfies the required conditions. According as $f(x)$ is an odd or an even function, the right-hand side of (36) reduces to

$$\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty e^{-\alpha y}f(\lambda)\cos\alpha x\cos\alpha\lambda\,.\,d\lambda\,;$$

or to

$$\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty e^{-\alpha y}f(\lambda)\sin\alpha x\sin\alpha\lambda\,.\,d\lambda.$$

EXAMPLES.—(1) A large iron plate π cm. thick and at a uniform temperature of 100° is suddenly placed in a bath at zero temperature for 10 seconds. Required the temperature of the middle of the plate at the end of 10 seconds, supposing that the diffusivity κ of the plate is 0·2 *C.G.S.* units, and that the surfaces of the plate are kept at zero temperature the whole time.

If heat flows perpendicularly to the two faces of the plate, any plane parallel to these faces will have the same temperature. Thus V depends on one space coordinate, equation (1), § 145, reduces to

$$\frac{\partial V}{\partial \theta} = \kappa\frac{\partial^2 V}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The conditions to be satisfied by the solution are that $V = 100$, when $\theta = 0$; $V = 0$, when $x = 0$; $V = 0$, when $x = \pi$.

First, to get particular solutions. Assume $V = e^{\alpha x + \beta\theta}$ is a solution when α and β are constants. Substitute in (37) and divide by $e^{\alpha x + \beta\theta}$. Hence $\beta = \kappa\alpha^2$, provided $V = e^{\alpha x + \beta\theta}$, is a solution of (37). This is true whatever be the value of α, hence $V = e^{\alpha x + \beta\theta}$ is also a solution of (37) for all values of α. Put $\alpha = \iota\mu$, where $\iota = \sqrt{-1}$. Then $V = e^{-\kappa\mu^2\theta}e^{\iota\mu x}$, and $V = e^{-\kappa\mu^2\theta}e^{-\iota\mu x}$, are solutions of (37).

$$\therefore V = \tfrac{1}{2}e^{-\kappa\mu^2\theta}(e^{\iota\mu x} + e^{-\iota\mu x}) = e^{-\kappa\mu^2\theta}\cos\mu x, \quad . \quad . \quad (38)$$

is a solution of (37). Similarly,

$$V = e^{-\kappa\mu^2\theta}\sin\mu x, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

is a solution of (37), whatever be the value of μ. By assigning particular values to μ, we shall get particular solutions of (37). *Cf.* footnote, p. 306.

Second, to combine these particular solutions so as to get a solution of (37) *to satisfy the three given conditions*, we must observe that (39) is zero when $x = 0$, for all values of μ, and that (39) is also zero when $x = \pi$ if μ is an integral number. If, therefore, we put V equal to a sum of terms of the form $Ae^{-\kappa\mu^2\theta}\sin nx$, say,

$$V = a_1e^{-\kappa\theta}\sin x + a_2e^{-4\kappa\theta}\sin 2x + a_3e^{-9\kappa\theta}\sin 3x + \ldots, \quad . \quad (40)$$

to n terms, this solution will satisfy the second and third of the above conditions, because $\sin\pi = 0 = \sin 0$. When $\theta = 0$, (40) reduces to

$$V = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots \quad (41)$$

But for all values of x between 0 and π, (11),

$$1 = \frac{4}{\pi}(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots), \quad (42)$$

if $V = 100$, we must substitute the coefficients of this series multiplied by 100, for a_1, a_2, a_3, . . . in (40), to get a solution satisfying all the required conditions. Note a_2, a_4, . . ., in (42), are zero. We thus obtain

$$V = \frac{400}{\pi}(e^{-\kappa\theta}\sin x + \tfrac{1}{3}e^{-9\kappa\theta}\sin 3x + \dots) \quad (43)$$

This is the solution required.

To introduce the numerical data. When $x = \frac{1}{2}\pi$, $\theta = 10$, $\kappa = 0{\cdot}2$. Hence use a table of logarithms. The result is accurate to the tenth of a degree if all terms of the series other than the first be suppressed. Hence use

$$V = \frac{400}{\pi}e^{-2}\sin\tfrac{1}{2}\pi,$$

for the numerical calculation. Use Table XXII. Ansr. 17·2° C.

Byerly (*l.c.*) has a splendid collection of problems of this nature. I have arranged a set of greater interest to the chemist at the end of this chapter.

(2) If the plate is c centimetres instead of π centimetres thick, use the development

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \dots\right),$$

from $x = 1$ to $x = c$.

(3) An infinitely large solid with one plane face has a uniform temperature $f(x)$. If the plane face is kept at zero temperature, what is the temperature of a point in the solid x feet from the plane face at the end of t years?

Let the origin of the coordinate axes be in the plane face. We have to solve equation (37) subject to the conditions

$$V = 0 \text{ when } x = 0;\ V = f(x) \text{ when } t = 0.$$

Proceed according to the above methods for (38), (39), and (36). We thus obtain

$$V = \frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{+\infty} e^{-\kappa\alpha^2 t}f(\lambda)\cos\alpha(\lambda - x)\,.\,d\lambda;$$

since positive values of x are wanted we can write

$$V = \frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty e^{-\kappa\alpha^2 t}f(\lambda)\sin\alpha x\,.\,\sin\alpha\lambda\,.\,d\lambda,$$

as above. Hence from (26), page 499,

$$V = \frac{1}{\pi}\int_0^\infty f(\lambda)d\lambda\int_0^\infty e^{-\kappa\alpha^2 t}\{\cos\alpha(\lambda - x) - \cos\alpha(\lambda + x)\}d\alpha.$$

$$\therefore V = \frac{1}{2\sqrt{\kappa t}}\int_0^\infty f(\lambda)\left\{e^{-\frac{(\lambda - x)^2}{4\kappa t}} - e^{-\frac{(\lambda + x)^2}{4\kappa t}}\right\}d\lambda,$$

is the required solution.

This last integration needs amplification. To illustrate the method, let

$$u = \int_0^\infty e^{-a^2x^2}\cos bx\,.\,dx.$$

Laplace (1810) first evaluated the integral on the right by the following ingenious device which has been termed **integration by differentiation.** Differentiate the given equation and

$$\frac{du}{db} = -\int_0^\infty xe^{-a^2x^2}\sin bx\,.\,dx,$$

provided b is independent of x. Now integrate the right member by parts in the usual way (page 168),

$$\therefore \frac{du}{db} = -\frac{b}{2a^2}u\,;\ \text{or}\ \frac{du}{u} = -\frac{b}{2a^2}db.$$

Integrate, and

$$\log u = -\frac{b^2}{4a^2} + C\,;\ \text{or}\ u = Ce^{-\frac{b^2}{4a^2}}.$$

To evaluate C, put $b = 0$, whence

$$C = u = \int_0^\infty e^{-a^2x^2}.\,dx = \frac{\sqrt{\pi}}{2a},$$

as in (12), page 191. Therefore

$$u = \int_0^\infty e^{-a^2x^2}\cos bx\,.\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}.$$

Returning, after this digression, to the original problem, it follows that if we write for brevity $\beta = (\lambda - x)/2\sqrt{\kappa t}$,

$$V = \frac{1}{\sqrt{\pi}}\left\{\int_{-\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\beta^2}f(2\beta\sqrt{\kappa t} + x)d\beta - \int_{+\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\beta^2}f(2\beta\sqrt{\kappa t} - x)d\beta\right\}.*$$

If the initial temperature is constant, say $= V_0$,

$$V = \frac{V_0}{\sqrt{\pi}}\left\{\int_{-\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\beta^2}.\,d\beta - \int_{+\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\beta^2}.\,d\beta\right\} = \frac{2V_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{-\beta^2}.\,d\beta;$$

from (4), page 185, and page 368.

For numerical computation it is necessary to expand the last integral in series as described on page 270. Therefore

$$V = \frac{2V_0}{\sqrt{\pi}}\left\{\frac{x}{2\sqrt{\kappa t}} - \frac{x^3}{3\,.\,(2\sqrt{\kappa t})^3} + \ldots\right\}.$$

If 100 million years ago the earth was a molten mass at 7,000 F., and, ever since, the surface had been kept at a constant temperature 0° F., what would be the temperature one mile below the surface at the present time, taking Lord Kelvin's value $\kappa = 400$? Ansr. 104° F. (nearly). Hints. $V_0 = 7{,}000$; $x = 5{,}280$ ft.; $t = 100{,}000{,}000$ years.

$$\therefore V = \frac{2 \times 7{,}000}{\sqrt{3{\cdot}1416}}\left(\frac{5280}{2 \times 20 \times 10{,}000}\right) = 104.$$

Lord Kelvin, "On the Secular Cooling of the Earth" (Thomson and Tait's *Treatise on Natural Philosophy*, **1**, 711, 1867), has compared the observed values of the underground temperature increments, dV/dx, with those deduced by assigning the most probable values to the terms in the above expressions. The close agreement (Calculated: 1° increment per $\frac{1}{51}$ ft. descent. Observed:

* NOTE. $\infty \pm$ a finite quantity $= \infty$. It is not difficult to show either by graphic construction or by integration that

$$\int_a^b (t - x)t\,.\,dt = \int_{a-x}^{b-x} z(z + x)dz\,;\int_a^b (t + x)t\,.\,dt = \int_{a+x}^{b+x} z(z - x)dz.$$

1° increment per $\frac{1}{50}$ ft. descent) leads him to the belief that the data are nearly correct. He extends the calculation in an obvious way and concludes: "I think we may with much probability say that the consolidation cannot have taken place less than 20,000,000 years ago, or we should have more underground heat than we really have, nor more than 400,000,000 years ago, or we should not have so much as the least observed underground increment of temperature". *Vide* Heaviside's *Electromagnetic Theory*, **2,** 12, 1899.

§ 153. Fourier's Linear Diffusion Law.

Let AB be any plane surface in a metal rod of unit sectional area (Fig. 121). Let this surface, at any instant of time, have a uniform temperature (*equithermal surface*), and let the temperature on the left side of the plane AB be higher than that on the right. In consequence, heat will flow from the hot to the cold side, in the direction of the arrow, across the surface AB.

Fourier assumes,

1. The *direction* of the flow is perpendicular to the surface AB;
2. The *rate* of flow of heat across any given surface, is proportional to the difference of temperature on the two sides of the plate.

Now let the rate of flow be uniform, the temperature of the plane AB, θ. The rate of rise of temperature at any point in the plane AB, is $d\theta/ds$ (this ratio measures the so-called *temperature gradient*). The amount of heat which flows, per second, from the hot to the cooler end of the rod, is $-c.d\theta/ds$, where c is a constant denoting the heat that flows, per second, through unit area, when the temperature gradient is unity. (Why the negative sign?)

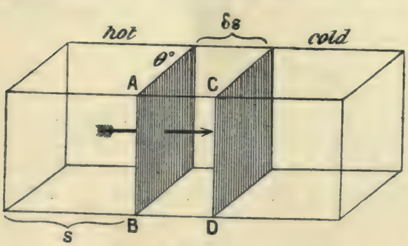


FIG. 121.

Consider now the value of $-c.d\theta/ds$ at another point in the plane CD, distant δs from AB; this distance is to be taken so small, that the temperature gradient may be taken as constant. The temperature at the point $s + \delta s$, will be $(\theta - \frac{d\theta}{ds}\delta s)$, since $-d\theta/ds$ is the rate of rise of temperature along the bar, and this, multiplied by δs, denotes the rise of temperature as heat passes from the point s to $s + \delta s$. Hence the amount of heat flowing through the small section $ABCD$ will be

$$- c\frac{d}{ds}\left(\theta - \frac{d\theta}{ds}\delta s\right); \text{ and } + c\frac{d^2\theta}{ds^2}\delta s,$$

will denote the difference between the amount of heat which flows in at one face and out at the other. This expression, therefore, denotes the amount of heat which is added to the space $ABCD$ every second. If σ denotes the thermal capacity of unit volume, the thermal capacity of the portion $ABCD$ is $(1 \times \delta s)\sigma$. Hence the rate of rise of temperature is $\sigma\frac{d\theta}{dt}\delta s$. Therefore,

$$c\frac{d^2\theta}{ds^2}\delta s = \sigma\frac{d\theta}{dt}\delta s.$$

Put $c/\sigma = \kappa$; this equation may then be written,

$$\frac{d^2\theta}{ds^2} = \frac{1}{\kappa}\cdot\frac{d\theta}{dt}, \quad . \quad . \quad . \quad . \quad (44)$$

where κ is the diffusivity of the substance.*

Equation (44) represents Fourier's law of diffusion. It covers all possible cases of diffusion where the substances concerned are in the same condition at all points in any plane parallel to a given plane. It is written

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{\kappa}\cdot\frac{\partial V}{\partial t}. \quad . \quad . \quad . \quad . \quad (45)$$

Lord Kelvin calls V the *quality* of the substance at the time t, at a distance x from a fixed plane of reference. The differential equation (45), therefore, shows that the rate of increase of quality per unit time, is equal to the product of the diffusivity and the rate of increase of quality per unit of space of quality. The quality depends on the subject of the diffusion. For example, it may denote one of the three components of the velocity of the motion of a viscous fluid, the density or strength of an electric current per unit area perpendicular to the direction of flow, temperature, the potential at any point in an isolated conductor, or the quantity of salt diffusing in a given solution.

Ohm's law is but a special case of Fourier's linear diffusion law. Fick's law of diffusion is another. The transmission of telephonic messages through a cable and indeed any phenomenon of linear propagation is included in this law of Fourier.

---

* This equation is obtained from (1), § 145, by remembering that we are dealing only with linear flow, in one direction; the y and z terms do not, therefore, come in at all. (44) is a special case of that more general equation.

EXAMPLES.—(1) Show by actual differentiation that (45) is satisfied by $V = e^{-ax}\sin(\beta t - ax)$, where a, β are two constants such that $\beta/2a^2 = \kappa$. Hint, first show that $\partial^2 V/\partial x^2 = e^{-ax}2a^2\cos(\beta t - ax)$, then that

$$\partial V/\partial t = e^{-ax}\beta\cos(\beta t - ax), \text{ etc.}$$

(2) Show that (45) is satisfied by making

$$V = e^{-ax}2a^2\cos(\beta t - ax),$$

or $V = A_0 + A_1 e^{-a_1 x}\sin(\beta_1 t - a_1 x + \epsilon_1) + A_2 e^{-a_2 x}\sin(\beta_2 t - a_2 x + \epsilon_2) + \ldots$, where $A_0, A_1, \ldots, \epsilon_1, \epsilon_2, \ldots$, are constants. See page 350.

(3) Deduce Fick's law of diffusion, similar in form to (45), for a salt solution in a vertical vessel of uniform sectional area, the solution being more concentrated in the lower part of the vessel. Assume (1) the rate of diffusion (quantity of salt passing through unit sectional area in unit time) is proportional to the difference in the concentration on each side of a given horizontal plane, (2) the substance diffuses in a vertical direction. Hint, follow the discussion preceding (44), and make the proper changes.

If V denotes the concentration of the solution in any plane x, at any time t, Fick's law is written,

$$\frac{\partial V}{\partial t} = \kappa\frac{\partial^2 V}{\partial x^2}, \quad \ldots \ldots \quad (46)$$

where κ depends on the nature of the diffusing substance.

§ 154. The Solution of Fick's Equation in terms of a Fourier's Series.

The experimental basis of the following discussion will be found in a memoir by Simmler and Wild in *Poggendorff's Annalen* for 1857 (**100**, 217, 1857): Fill a small cylindrical tube of unit sectional area, and height x, with a solution of salt. Let the tube and contents be submerged in a vessel containing a great quantity of water so that the open end of the cylindrical vessel, containing the salt solution, dips just beneath the surface of the water. Salt solution passes out of the diffusion vessel and sinks towards the bottom of the larger vessel. The upper brim of the diffusion vessel, therefore, is assumed to be always in contact with pure water. Let h denote the height of the liquid in the diffusion tube, reckoned from the bottom.

(i.) *To find the concentration* (V) *of the dissolved substance in different parts* (x) *of the diffusion vessel after the elapse of any stated interval of time* (t).

This is equivalent to finding a solution of Fick's equation, (46), of the preceding section, which will satisfy the conditions under which the experiment is conducted. These so-called "*limiting conditions*" are :—

When $x = h,\ V = 0$; . . . . . (1)
when $x = 0,\ \partial V/\partial x = 0$; . . . . (2)
when $t = 0,\ V = V_0$. . . . . . (3)

The reader must be quite clear about this before going any further. What do V, x and t mean? V_0 evidently represents the concentration of the solution at the beginning of the experiment; at the top of the diffusion vessel, obviously $x = h$, and V is zero, because there the water is pure; the second condition means that at the bottom of the diffusion vessel, the concentration may be assumed to be constant during the experiment.

First, deduce particular solutions exactly as in the first example of the preceding section. Thus

$$V = ae^{-\kappa\mu^2 t}\cos\mu x, \quad . \quad . \quad . \quad (4)$$

and

$$V = be^{-\kappa\mu^2 t}\sin\mu x, \quad . \quad . \quad . \quad (5)$$

are particular solutions. a and b simply denote arbitrary constants.

Differentiate (4) and we get

$$\partial V/\partial x = -a\mu e^{-\kappa\mu^2 t}\sin\mu x.$$

Now when $x = 0$, $\sin\mu x$ vanishes, therefore, when $x = 0$, condition (2) is satisfied. But, in order that (4) may satisfy the first condition, we must have

$$\cos\mu h = 0, \text{ when } x = h.$$

But $\cos\frac{1}{2}\pi = \cos\frac{3}{2}\pi = \ldots = \cos\frac{1}{2}(2n-1)\pi = 0,$

where $\pi = 180°$ and n is any integer from 1 to ∞. Hence, we must have

$$\mu = \frac{\pi}{2h},\quad \frac{3\pi}{2h},\quad \frac{5\pi}{2h}, \ldots, \frac{(2n-1)\pi}{2h},$$

in order that $\cos\mu h$ may vanish.

Substitute these values of μ successively in (4) and add the results together, we thus obtain

$$V = a_1 e^{-\left(\frac{\pi}{2h}\right)^2\kappa t}\cos\frac{\pi x}{2h} + a_2 e^{-\left(\frac{3\pi}{2h}\right)^2\kappa t}\cos\frac{3\pi x}{2h} + \ldots \text{ to inf.,} \quad (6)$$

which satisfies two of the required conditions, namely (1) and (2).

We must now determine the coefficients $a_1, a_2, \ldots$ in (6), in order that the third condition may be satisfied by the particular solution (4), or rather (6). This is done by allowing for the initial conditions, when $t = 0$, in the usual way. When $t = 0$, $V = V_0$. Therefore, from (6),

$$V_0 = a_1\cos\frac{\pi x}{2h} + a_2\cos\frac{3\pi x}{2h} + \ldots, \quad . \quad . \quad (7)$$

is true for all values of x between 0 and h. Hence, as on page .

$$a_1 = \frac{2V_0}{h}\int_0^h \cos\frac{\pi x}{2h}dx;\ a_2 = \frac{2V_0}{h}\int_0^h \cos\frac{3\pi x}{2h}dx;\ \ldots a_n = \frac{4V_0}{(2n+1)\pi}.$$

These results have been obtained by equating each term of (7) to zero, and integrating between the limits 0 and h.

Substituting these values of $a_0, a_1, \ldots$ in (6), we get a solution satisfying the limiting conditions of the experiment. If desired, we can write the resulting series in the compact form,

$$V = \frac{4V_0}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{2n-1}e^{-\left(\frac{2n-1}{2h}\pi\right)^2\kappa t}\cos\frac{2n-1}{2h}\pi x, \qquad (8)$$

where the summation sign between the limits $n = \infty$ and $n = 1$ means that n is to be given every positive integral value 1, 2, 3. . . . to infinity, and all the results added together.

EXAMPLE.—If we reckon h from the *top* of the diffusion vessel, show that we must use (5) exactly as we have just employed (4). In this case "cos" (8) becomes "sin".

NOTE.—If the limits in (8) are o and ∞, write "$2n + 1$" for "$2n - 1$".

(ii.) *To find the quantity of salt* (Q) *which diffuses through any horizontal section in a given time* (T).

Differentiate (6) with respect to x, multiply the result through with κdt, so as to obtain $-\kappa\frac{\partial V}{\partial x}dt$. If x represents the height of any horizontal section, $-\kappa q\frac{\partial V}{\partial x}dt$, will represent the quantity of salt which passes through this horizontal plane in the time dt. q represents the area of that section (example (3), page 374). Let $q = 1$.

$$\therefore Q = -\kappa\int_0^T\frac{\partial V}{\partial x}dt = \int_0^T\frac{\kappa\pi}{2h}\left(a_1e^{-\left(\frac{\pi}{2h}\right)^2\kappa t}\sin\frac{\pi x}{2h} + \ldots\right)dt.$$

Integrate between the limits 0 and T. The result represents the quantity of salt which passes through any horizontal plane (x) of the diffusion vessel in the time T, or,

$$Q = -\kappa\int_0^T\frac{\partial V}{\partial x}dt = \frac{2h}{\pi}\left\{a_1\left(1 - e^{-\left(\frac{\pi}{2h}\right)^2\kappa T}\right)\sin\frac{\pi x}{2h} + \ldots\right\}. \qquad (9)$$

(iii.) *To find the quantity of salt* (Q_1) *which passes from the diffusion vessel in any given time* (T).

Substitute $h = x$ in (9). The sine of each of the angles $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, . . ., $\frac{1}{2}(2n - 1)$, is equal to unity. Therefore,

$$Q_1 = \frac{2h}{\pi}\left\{a_1\left(1 - e^{-\left(\frac{\pi}{2h}\right)^2\kappa T}\right) - \tfrac{1}{3}a_2\left(1 - e^{-\left(\frac{3\pi}{2h}\right)^2\kappa T}\right) + \ldots\right\}. \quad (10)$$

(iv.) *To find the value of κ, the coefficient of diffusion.*

Since the members of series (10) converge very rapidly, we may neglect the higher terms of the series. Arrange the experiment so that measurements are made when $x = h, \frac{1}{3}h, \frac{1}{5}h, \ldots$, in this way, $\sin \pi x/2h, \ldots$ in (9) become equal to unity. We thus get a series resembling (10). Substitute for the coefficient and we obtain, by a suitable transposition of terms,

$$\frac{Q\pi}{2ha_1} = \left(1 - e^{-\frac{\kappa T\pi^2}{4h^2}}\right);\ \kappa = \frac{4h^2}{T\pi^2}\log\left(\frac{Q\pi}{2ha_1} - 1\right)$$

$$a_1 = \frac{2V_0}{h}\cdot\frac{2h}{\pi}\int_0^h \cos\frac{\pi x}{2h}\cdot d\left(\frac{\pi x}{2h}\right) = \frac{4V_0}{\pi}.$$

$$\therefore\ \kappa = \frac{4h^2}{T\pi^2}\log\left(\frac{Q\pi^2}{8hV_0} - 1\right). \qquad (11)$$

(v.) *To find the quantity of salt (Q_2) which remains in the diffusion vessel after the elapse of a given time (T).*

The quantity of salt in the solution at the beginning of the experiment may be represented by the symbol Q_0. Q_0 may be determined by putting $t = 0$ in (9) and eliminating $\sin \pi x/2h, \ldots$ as indicated in (iv.).

$$Q_0 = \frac{2h}{\pi}\left(a_1 - \tfrac{1}{3}a_2 + \ldots\right);$$

and

$$Q_2 = Q_0 - Q_1;$$

$$\therefore\ Q_2 = \frac{2h}{\pi}\left(a_1 e^{-\left(\frac{\pi}{2h}\right)^2\kappa t} - \tfrac{1}{3}a_2 e^{-\left(\frac{3\pi}{2h}\right)^2\kappa t} + \ldots\right) \qquad (12)$$

(vi.) *To find the concentration of the dissolved substance in different parts of the diffusion vessel when the stationary state is reached.*

After the elapse of a sufficient length of time, a state of equilibrium is reached and the concentration of the substance in different parts of the vessel remains stationary. In this case,

$$\partial V/\partial t = 0;\ \partial^2 V/\partial x^2 = 0.$$

Integrate the latter, we get

$$V = ax + b, \qquad (13)$$

where a and b are constants to be determined from the experimental data, as described in § 106. See Ostwald's *Solutions*, Chapter vi. (Longmans, Green & Co., 1891), for experimental work.

The chief difficulty in the application of Fourier's theorem to diffusion experiments is to make the series satisfy the limiting conditions. The following examples will serve to show how Fourier's series is to be employed in practical work. For the experimental details, the original memoirs must be consulted.

Examples.—It will be found convenient to refer to the following alternative way of writing Fourier's series:

$$f(x) = \frac{1}{c}\int_0^c f(\lambda)\,.\,d\lambda + \frac{2}{c}\sum_{n=1}^{n=\infty}\cos\frac{n\pi x}{c}\cos\frac{n\pi\lambda}{c}f(\lambda)\,.\,d\lambda, \quad . \quad (14)$$

true for any value of x between 0 and c (see pages 365 and 366).

(1) Find an expression equal to v when x lies between 0 and a, and equal to zero, when x lies between a and b. Here $f(\lambda) = v$, from $\lambda = 0$ to $\lambda = a$, and $f(\lambda) = 0$, from $\lambda = a$ to $\lambda = b$; $c = b$; $\cos\frac{n\pi\lambda}{c}f(\lambda)\,.\,d\lambda$, becomes $v\int_0^a \cos\frac{n\pi\lambda}{b}d\lambda$, or, $\frac{cb}{n\pi}\sin\frac{n\pi a}{b}$. Hence the required expression is,

$$f(x) = \frac{va}{b} + \frac{2v}{\pi}\Big(\sin\frac{\pi a}{b}\,.\cos\frac{\pi x}{b} + \tfrac{1}{2}\sin\frac{2\pi a}{b}\,.\cos\frac{2\pi x}{b} + \ .\ .\ .\Big).$$

when $x = a$, this expression reduces to $\frac{1}{2}v$.

(2) *Fick's diffusion experiments* (*Pogg. Ann.*, **94**, 59, 1855; translated in the *Phil. Mag.*, July, 1855). When deducing Fick's equation, if the area of the diffusion vessel is some function of its height x, show that Fick's equation assumes the form

$$\frac{\partial V}{\partial t} = -\kappa\Big(\frac{\partial^2 V}{\partial x^2} + \frac{1}{q}\cdot\frac{\partial q}{\partial x}\cdot\frac{\partial V}{\partial x}\Big), \quad . \quad . \quad . \quad (15)$$

where q denotes the area of the diffusion vessel at a distance x in the direction of the diffusion.

Before this formula can be of any practical use, the equation to the curve described by the walls of the vessel must be known. For a conical vessel, $q = \pi m^2x^2$, where the apex of the cone is at the origin of the coordinate axes, m is the tangent of half the angle included between the two slant sides of the vessel. Fick has made a series of crude experiments on the steady state in a conical vessel with a circular base (funnel-shaped). Hence show that,

$$\frac{\partial^2 V}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial V}{\partial x} = 0;\ \therefore\ V = C_1 + \frac{C_2}{x}. \quad . \quad . \quad . \quad (16)$$

The integration constants C_1 and C_2 are to be evaluated by means of the experimental data, § 106.

(3) *Graham's diffusion experiments* (*Phil. Trans.*, **151**, 183, 1861). A cylindrical vessel 152 mm. high, and 87 mm. in diameter, contained 0·7 litre of water. Below this was placed 0·1 litre of a salt solution. The fluid column was then 127 mm. high. After the elapse of a certain time, successive portions of 100 cc., or $\frac{1}{8}$ of the total volume of the fluid, were removed and the quantity of salt determined in each layer.

The limiting conditions are: At the end of a certain time t, for $x = 0$ and $x = H$, $\partial V/\partial x = 0$. (Why?) Note that x is here reckoned from the top of the liquid. H denotes the total height of the liquid column. Let h denote the height of the salt solution at the beginning of the experiment, V_0 its concentration, $\therefore h = \frac{1}{8}H$; $f(\lambda)$, in (14), $= V_0$ from $x = 0$ to $x = h$ and $f(\lambda) = V = 0$, from $x = h$ to $x = H$, when $t = 0$.

To adapt these results to Fourier's solution of Fick's equation, first show that

$$V = (a \cos \mu x + b \sin \mu x)e^{-\mu^2 \kappa t}, \quad . \quad . \quad . \quad . \quad (17)$$

is a particular integral of Fick's equation. a, b, are constants to be determined from the conditions of the experiment. Differentiate (17) with respect to x and we get

$$\partial V/\partial x = (- \mu a \sin \mu x + \mu b \cos \mu x)e^{-\mu^2 \kappa t}. \quad . \quad . \quad (18)$$

In the layer $x = 0$, $\partial V/\partial x = 0$, whatever the value of t, because no salt goes out from and no salt enters the solution at this point. The concentration V must at all times satisfy Fick's elementary law, at all points between $x = 0$ and $x = H$. When $x = 0$, $\cos x = 1$, but $\sin x = 0$, therefore, from (18),

$$a \sin \mu x - b \cos \mu x = 0,$$

b must be zero, and, since $\sin \pi = 0$, μ must be so chosen that

$$\mu H = n\pi; \text{ or, } \mu = n\pi/H,$$

where n has any value 0, 1, 2, 3, . . .

Add up all these particular integrals for the general equation

$$V = \sum_{n=0}^{n=\infty} a_n \cos \frac{n\pi x}{H} e^{-n^2\pi^2\kappa t/H^2}, \quad . \quad . \quad . \quad (19)$$

where the constant a has still to be determined from the initial conditions. For $t = 0$,

$$V = \sum_{n=0}^{n=\infty} a_n \cos\left(\frac{n\pi x}{H}\right) = V, \text{ from } x = 0, \text{ to } x = h;$$

$$V = \sum_{n=0}^{n=\infty} a_n \cos\left(\frac{n\pi x}{H}\right) = 0, \text{ from } x = h, \text{ to } x = H.$$

Since $f(\lambda) = V_0$, in (14), when $n = 0$,

$$a_0 = \frac{1}{H}\int_0^H V_0 dx = \frac{1}{H}\left\{\int_h^H 0 \,.\, dx + \int_0^h V_0 dx\right\} = \frac{V_0 h}{H}.$$

In the same way it can be shown that

$$a_n = \frac{2}{H}\int_0^H V_0 \cos \frac{n\pi x}{H} dx = \frac{2V_0}{n\pi}\int_0^H \cos \frac{n\pi x}{H} d\left(\frac{n\pi x}{H}\right),$$

$$= \frac{2V_0}{\pi} \cdot \frac{1}{n} \sin \frac{n\pi h}{H}.$$

Taking all these conditions into account, the general solution appears in the form,

$$V = \frac{V_0 h}{H} + \frac{2V_0}{\pi}\sum_{n=1}^{n=\infty} \frac{1}{n} \sin \frac{n\pi h}{H} \cos \frac{n\pi x}{H} e^{-n^2\pi^2\kappa t/H^2}, \quad . \quad . \quad (20)$$

which is a standard equation for this kind of work. In Graham's experiments, $h = \frac{1}{8}H$. Hence the concentration V in any plane x of the diffusion vessel, is obtained from the infinite series:

$$V = \frac{V_0}{8} + \frac{2V_0}{\pi}\sum_{n=1}^{n=\infty} \frac{1}{n} \sin \frac{n\pi}{8} \,.\, \cos \frac{n\pi x}{H} \,.\, e^{-n^2\pi^2\kappa t/H^2}. \quad . \quad . \quad (21)$$

As indicated in Chapter V., an infinite series is practically useful only when the series converges rapidly, and the higher terms have so small an influence on the result that all but the first terms may be neglected. This is often effected by measuring the concentration at different levels x, so related to H that $\cos(n\pi x/H)$ reduces to unity; also by making t very great, the second and higher terms become vanishingly small. See Weber's experiments below.

The quantity of salt Q_r in the rth layer, is given by the integral of Vdx, between the limits $x = \frac{1}{8}(r - 1)H$ and $x = \frac{1}{8}rH$, or,

$$Q_r = \frac{V_0H}{8}\Big(\frac{1}{8} + \frac{32}{\pi^2}\sum_{n=1}^{n=\infty}\frac{1}{n^2}\,.\sin\frac{n\pi}{8}\,.\sin\frac{n\pi}{16}\,.\cos\frac{(2r-1)n\pi}{16}e^{-n^2\pi^2\kappa t/H^2}\Big), \quad (22)$$

where $\frac{1}{8}V_0H$ multiplied by the cross section of the vessel (here supposed unity) denotes the quantity of salt present in the diffusion vessel.

Unfortunately, a large number of Graham's experiments are not adapted for numerical discussion, because the shape of his diffusion vessels, even if known, would give very awkward equations. A simple modification in experimental details, will often save an enormous amount of labour in the mathematical work.

(4) *Stefan's diffusion experiments* (*Wien.-Akad. Ber.*, **79**, ii., 161, 1879). If Q_0 denotes the quantity of salt present in the diffusion vessel of Graham, when t is very great, show, preceding example, that

$$Q_1 + Q_4 + Q_5 + Q_8 = \tfrac{1}{2}qV_0H = \tfrac{1}{2}Q_0, \quad . \quad . \quad . \quad (23)$$

where q denotes the area of a cross section of the vessel.

When deducing (23) from (22), it is most instructive to compile a table of values of the factor $\cos(2r - 1)n\pi/16$, for values of r from $r = 1$ to $r = 8$, and from $n = 1$ to $n = 4$. Then show that for $n = 1, 2, 3$, the sums, for values of $r = 1, 4, 5, 8$, mutually cancel each other, and that the value of t in the higher terms makes them negligibly small. Here is the table:

	$n = 1$	$n = 2.$	$n = 3.$	$n = 4.$
$r = 1$	$+\cos\frac{1}{16}\pi$	$+\cos\frac{1}{8}\pi$	$+\cos\frac{3}{16}\pi$	$+\cos\frac{1}{4}\pi$
$r = 2$	$+\cos\frac{3}{16}\pi$	$+\cos\frac{3}{8}\pi$	$-\cos\frac{7}{16}\pi$	$-\cos\frac{1}{4}\pi$
$r = 3$	$+\cos\frac{5}{16}\pi$	$-\cos\frac{3}{8}\pi$	$-\cos\frac{1}{16}\pi$	$-\cos\frac{1}{4}\pi$
$r = 4$	$+\cos\frac{7}{16}\pi$	$-\cos\frac{1}{8}\pi$	$-\cos\frac{5}{16}\pi$	$+\cos\frac{1}{4}\pi$
$r = 5$	$-\cos\frac{7}{16}\pi$	$-\cos\frac{1}{8}\pi$	$+\cos\frac{5}{16}\pi$	$+\cos\frac{1}{4}\pi$
$r = 6$	$-\cos\frac{5}{16}\pi$	$-\cos\frac{3}{8}\pi$	$+\cos\frac{1}{16}\pi$	$-\cos\frac{1}{4}\pi$
$r = 7$	$-\cos\frac{3}{16}\pi$	$+\cos\frac{3}{8}\pi$	$+\cos\frac{7}{16}\pi$	$-\cos\frac{1}{4}\pi$
$r = 8$	$-\cos\frac{1}{16}\pi$	$+\cos\frac{1}{8}\pi$	$-\cos\frac{3}{16}\pi$	$+\cos\frac{1}{4}\pi$

Equation (23) has been verified by Stefan. He made $Q_0 = 10$ grms. of salt. The successive layers, from $r = 1$, to $r = 8$, contained, after 7 days, 3·294, 2·844, 1·907, 1·100, 0·529, 0·215, 0·079 and 0·030 grms. of salt. The experimental details were similar to those of Graham already described. Show that theory requires $\frac{1}{2}Q_0 = 5$, and that the above numbers furnish $\frac{1}{2}Q_0 = 4·953$ —a very good agreement.

(5) *Weber's diffusion experiments* (*Wied. Ann.*, **7**, 469, 536, 1879). A concentrated solution of zinc sulphate (0·25 to 0·35 grm. per c.c. of solution) was placed in a cylindrical vessel on the bottom of which was fixed a round smooth amalgamated zinc disc (about 11 cm. diam.). A more dilute solution (0·15 to 0·20 grm. per c.c.) was poured over the concentrated solution, and another amalgamated zinc plate was placed just beneath the surface of the upper layer of liquid. It is known that if V_1, V_2 denote the respective concentrations of the lower and upper layers of liquid, the difference of potential E, due to these differences of concentrations, is given by the expression

$$E = A(V_2 - V_1)\{1 + B(V_2 + V_1)\}, \quad . \quad . \quad . \quad (24)$$

where A and B are known constants, B being very small in comparison with A. This difference of potential or electromotive force, can be employed to determine the difference in the concentrations of the two solutions about the zinc electrodes.

To adapt these conditions to Fick's equation, let h_1 be the height of the lower, h_2 of the upper solution, therefore, $h_1 + h_2 = H$. The limiting conditions to be satisfied for all values of t, are $\partial V/\partial x = 0$, when $x = 0$, and $\partial V/\partial x = 0$, when $x = H$. The initial conditions when $t = 0$, are $V = V_2$, for all values of x between $x = h$ and $x = H$. From this proceed exactly as in example (3), and show that

$$a_0 = \frac{V_1h_1 + V_2h_2}{H};\ a_n = -\frac{2(V_2 - V_1)}{\pi}\cdot\frac{1}{n}\sin\frac{n\pi h}{H};$$

and the general solution

$$V = \frac{V_1h_1 + V_2h_2}{H} - \frac{2(V_2 - V_1)}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{n}\sin\frac{n\pi h}{H}\cos\frac{n\pi x}{H}e^{-n^2\pi^2\kappa t/H^2}. \quad (25)$$

This equation only applies to the variable concentrations of the boundary layers $x = 0$ and $x = H$. It is necessary to adapt it to equation (24). Let the layers $x = 0$ and $x = H$, have the variable concentrations V' and V'' respectively.

$$V' = \frac{V_1h_1 + V_2h_2}{H} - \frac{2(V_2 - V_1)}{\pi}\left\{\sin\frac{\pi h_1}{H}e^{-\frac{\pi^2}{H^2}\kappa t} + \tfrac{1}{2}\sin\frac{2\pi h_1}{H}e^{-\frac{4\pi^2}{H^2}\kappa t} + \ldots\right\},$$

$$V'' = \frac{V_1h_1 + V_2h_2}{H} + \frac{2(V_2 - V_1)}{\pi}\left\{\sin\frac{\pi h_1}{H}e^{-\frac{\pi^2}{H^2}\kappa t} - \tfrac{1}{2}\sin\frac{2\pi h_1}{H}e^{-\frac{4\pi^2}{H^2}\kappa t} + \ldots\right\};$$

$$V'' - V' = \frac{4(V_2 - V_1)}{\pi}\left\{\sin\frac{\pi h_1}{H}e^{-\frac{\pi^2}{H^2}\kappa t} + \tfrac{1}{3}\sin\frac{3\pi h}{H}e^{-\frac{9\pi^2}{H^2}\kappa t} + \ldots\right\},$$

$$V'' + V' = 2\frac{V_1h_1 + V_2h_2}{H} - \frac{4(V_2 - V_1)}{\pi}\left\{\tfrac{1}{2}\sin\frac{2\pi h}{H}e^{-\frac{4\pi^2}{H^2}\kappa t} + \tfrac{1}{4}, \text{ etc.} \ldots\right\}.$$

In actual work, H was made very small. After the lapse of one day ($t = 1$), the terms $\frac{1}{4}\sin 4\pi h_1/H$, etc., and $\frac{1}{5}\sin 5\pi h_1/H$, etc., were less than $\frac{1}{400}$. Hence all terms beyond these are outside the range of experiment, and may, therefore, be neglected. Now h was made as nearly as possible equal to $\frac{1}{3}H$, in order that the term $\frac{1}{3}\sin 3\pi h_1/H$, etc., might vanish. Hence,

$$V'' - V' = \frac{4(V_2 - V_1)}{\pi}\sin\frac{\pi}{3}\cdot e^{-\pi^2\kappa t/H^2};$$

$$V'' + V' = \frac{2(V_1h_1 + V_2h_2)}{H} - \frac{2(V_2 - V_1)}{\pi}\sin\frac{2\pi}{3}e^{-4\pi^2\kappa t/H^2}.$$

Now substitute these values of $V'' - V'$ and $V'' + V'$ in (24), observing that π, V_2, V_1, h_1, h_2, $\sin\pi/3$, $\sin 2\pi/3$ and H, are all constants. The difference of potential E, between the two electrodes, due to the difference of concentration between the two boundary layers V' and V'', is

$$E = A_1e^{-\pi^2\kappa t/H^2} - B_1e^{-5\pi^2\kappa t/H^2}, \quad . \quad . \quad . \quad (26)$$

where A_1 and B_1 are constant. Since B is very small in comparison with A, the expression reduces to

$$E = A_1e^{-\pi^2\kappa t/H^2}, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in a very short time.

This equation was used by Weber for testing the accuracy of Fick's law. The values of the constant, $\pi^2\kappa/H^2$, after the elapse of 4, 5, 6, 7, 8, 9, 10 days were respectively ·2032, ·2066, ·2045, ·2027, ·2027, ·2049, ·2049. A very satisfactory result.

(6) A gas A, obeying Dalton's law of partial pressures, diffuses into another gas, show that the partial pressure p_1 of the gas A, at a distance x, in the time t, is

$$\frac{\partial p}{\partial t} = \kappa\frac{\partial^2 p}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

(7) *Loschmidt's diffusion experiments* (*Wien.-Akad. Ber.*, **61**, 367, 1870; **62**, 468, 1870). Loschmidt arranged two cylindrical tubes vertically, so that communication could be established between them by a sliding metal plate. Each tube was 48·75 cm. high and 2·6 cm. in diameter and closed at one end. The two tubes were then filled with different gases and placed in communication for a certain time t. The mixture in each tube was then analysed.

Let $a = 97·5$ cm. It is required to solve equation (28) so that when $t = 0$, $p_1 = p_0$, from $x = a$ to $x = \frac{1}{2}a$; $p_1 = 0$, from $x = \frac{1}{2}a$ to $x = a$; $\partial p_1/\partial x = 0$, when $x = 0$ and $x = a$, for all values of t. Note, p_0 denotes the original pressure of the gas. Hence show that

$$p_1 = \frac{p_0}{2} + \sum_{n=1}^{n=\infty}\frac{1}{n}\sin\frac{\pi n}{2}\cos\frac{n\pi x}{a}e^{-n^2\pi^2\kappa t/a^2}. \quad . \quad . \quad (29)$$

The quantity of gas contained in the upper and lower tubes, after the elapse of the time t, is, respectively,

$$Q' = q\int_0^{a/2} p_1 \,.\, dx\,;\ Q'' = q\int_{a/2}^{0} {}^{a/2}p_1 \,.\, dx,. \quad . \quad . \quad (30)$$

where q is the sectional area of the tube. Hence show that

$$\frac{Q' - Q''}{Q' + Q''} = \frac{8}{\pi^2}\left(\frac{1}{1^2}e^{-\pi^2\kappa t/a^2} + \frac{1}{3^2}e^{-3\pi\kappa t/a^2} + \ldots\right),$$

from which the constant κ can be determined. If the time is sufficiently long,

$$\frac{D}{S} = \frac{8}{\pi^2}e^{-\pi^2\kappa t/a^2}, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where D and S respectively denote the sum and difference of the quantity of gas contained in the two vessels. Loschmidt measured D, S, t, and a, and found that the agreement between observed and calculated results was very close.

(8) *The velocity of the solution of solids* is a special case of diffusion. The layer of liquid in immediate contact with the solid is to be regarded as a saturated solution, the rate of solution thus depends upon the rate of the diffusion of the salt from the saturated solution to the adjoining layers of solvent. This problem can be attacked by the above method. For experimental work based upon the relation

$$\frac{dx}{dt} = qC(Q - x)\,;\ \text{or, } \frac{1}{t}\log\frac{Q}{Q - x} = \text{constant} \quad . \quad . \quad (32)$$

(see Noyes and Whitney, *Zeitschrift für physikalische Chemie*, **23**, 689, 1897; Bruner and Tolloczko, *ibid.*, **35**, 283, 1900); in this formula Q denotes the quantity of salt contained in a saturated solution, x the amount dissolved in the time t, q the area of the dissolving surface, C the velocity constant.

PART III.

USEFUL RESULTS FROM ALGEBRA AND TRIGNOMETRY.

CHAPTER IX.

HOW TO SOLVE NUMERICAL EQUATIONS.

§ 155. Some General Properties of the Roots of Equations.

The solution of algebraic and transcendental equations is an important branch of practical mathematics. The object of solving these equations is to find what value or values of the unknown will satisfy the equation, or will make one side of the equation equal to the other. Such values of the unknown are called **roots** or **solutions** of the equation.

General methods for the solution of algebraic equations of the first, second and third degree are treated in regular algebraic textbooks; it is, therefore, unnecessary to give more than a brief *résumé* of their more salient features.

R. N. Abel and Wantzel have brought forward demonstrations with the object of proving that general methods for the solution of equations of a higher degree than the fourth are impossible. M'Ginnis has recently published a method which he claims can be employed for equations as high as the twelfth degree.

Equations of higher degree than the fourth are comparatively rare in practical work.* Indeed we nearly always resort to the approximation methods for finding the roots of the numerical equations found in practical calculations.

---

* Otherwise I should take advantage of the generosity of Professor M'Ginnis, and summarise his methods. They will, however, be found in *The Universal Solution*, 1900 (Swan, Sonnenschein & Co.).

The reader must distinguish between **identical equations** like

$$(x + 1)^2 = x^2 + 2x + 1,$$

which are true for *all* values of x, and **conditional equations** like

$$x^2 + 2x + 1 = 0,$$

which are only true when x has some particular value or values. In this case, for $x = - 1$.

An equation like

$$x^2 + 2x + 2 = 0,$$

has no real roots because no real values of x will satisfy the equation. By solving as if the equation had real roots, the imaginary again forces itself on our attention. The *imaginary roots* of this equation are $-1 \pm \sqrt{-1}$, or $-1 \pm \iota$.

The general equation of the nth degree is

$$x^n + ax^{n-1} + bx^{n-2} + \ldots + sx + R = 0. \quad . \quad (1)$$

The term R is called the **absolute term.** If $n = 2$, the equation is a **quadratic,** $x^2 + ax + R = 0$; if $n = 3$, the equation is said to be a **cubic;** if $n = 4$, a **biquadratic,** etc. If x^n has any coefficient, we can divide through by this quantity, and so reduce the equation to the above form. When the coefficients a, b, . . ., instead of being literal, are real numbers, the equation is said to be **numerical.**

The following synopsis of results proved in the regular textbooks is convenient for reference :

1. Every equation of the nth degree has n equal or unequal roots and no more **(Gauss' law).** *E.g.*,

$$x^5 + x^4 + x + 1 = 0,$$

has five roots and no more.

2. If an equation can be divided by $x - \alpha$, without remainder, α is a root of the equation. More generally, if α, β, γ, are the roots of an equation of the third degree,

$$x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma). \quad . \quad . \quad (2)$$

3. If the results obtained by substituting two numbers are of opposite signs, at least one root lies between the numbers substituted.

4. An equation of an even degree, with its absolute term negative, has at least two real roots of opposite sign.

5. An equation of an odd degree has at least one real root, the same in sign as the absolute term.

6. Imaginary roots in an equation with real coefficients occur in pairs. *E.g.*, if $\alpha + \beta\sqrt{-1}$ is one root of the equation, $\alpha - \beta\sqrt{-1}$ is another.

7. The sum of the roots of an equation is equal to the coefficient $-b$ of the second term; the sum of the products of the roots taken two at a time is equal to $+c$; the products of the roots taken three at a time is equal to $-d$, etc.; the product of all the roots is equal to $-$ (*absolute term*), if n is odd, and to $+$ (*absolute term*), if n is even.

8. An equation, $f(x)$, cannot have more positive roots than there are changes of sign in $f(-x)$ **(Descartes' rule of signs).** *E.g.*, in

$$x^9 + x^8 - x^3 + x + 1 = 0 = f(x)\,;\ -x^9 + x^8 + x^3 - x + 1 = 0 = f(-x),$$

there are two changes of sign. Hence the equation has no more than three negative and two positive real roots. The remainder are imaginary roots.

a. If the coefficients are all positive, the equation cannot have a positive root. Such is

$$x^5 + x^3 + x + 1 = 0.$$

b. If the coefficients of the even powers of the unknown have the same sign, and the coefficients of the odd powers of the unknown have the opposite sign, the equation has no negative root. *E.g.*,

$$x^7 + x^5 - x^4 + x^3 - x^2 + x - 1 = 0.$$

c. If an equation has only even powers of x, with its coefficients all of the same sign, there is no real root. Thus,

$$x^8 + x^4 + x^2 + 1 = 0.$$

d. If the equation has only odd powers of x, with coefficients all of the same sign, there are no real roots other than $x = 0$. For instance,

$$x^7 + x^5 + x^3 + x = 0.$$

§ 156. The General Solution of Quadratic Equations.

To recapitulate the results of the elementary textbooks :

After suitable reduction, every quadratic may be written in the form :

$$ax^2 + bx + c = 0. \qquad (1)$$

If α and β represent the roots of this equation, x must be equal to α or β, where

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\,;\ \text{and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

The sum and product of these roots are

$$\alpha + \beta = -b/a\,;\ \alpha\beta = c/a.$$

Hence if one of the roots is known, the other can be deduced directly. If $a = 1$, the sum of the roots is equal to the coefficient of the second term with its sign changed, the product of the roots is equal to the absolute term. Equation (1), may be variously written

$$a\{x^2 - (\alpha + \beta)x + \alpha\beta\} = 0\,;$$

$$a\{x^2 - (\mathit{Sum\ of\ Roots})x + (\mathit{Product\ of\ Roots})\} = 0\,;$$

$$a(x - \alpha)(x - \beta) = 0\,;\ x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

From (2), we can deduce many important particulars respecting the nature of the roots of the quadratic. These are :

Relation between the Coefficients.	The Nature of the Roots.	
$b^2 - 4ac$ is positive,	real and unequal.	(3)
$b^2 - 4ac$ is zero,	real and equal.	(4)
$b^2 - 4ac$ is negative,	imaginary and unequal.	(5)
$b^2 - 4ac$ is perfect square,	rational and unequal.	(6)
$b^2 - 4ac$ is not a perfect square,	irrational and unequal.	(7)
a, b, c, have the same sign,	negative.	(8)
a, b, differ in sign from c,	opposite sign.	(9)
a, c, differ in sign from b,	positive.	(10)
$a = 0$,	one root infinite.	(11)
$b = 0$,	equal and opposite in sign.	(12)
$c = 0$,	one root zero.	(13)
$c = 0$, $b = 0$,	both roots zero.	(14)

In the table, the words "equal" and "unequal" refer to the numerical values of the roots. On account of the important *rôle* played by the expression $b^2 - 4ac$, in fixing the character of the roots, "$b^2 - 4ac$," is called the discriminant of the equation.

§ 157. Graphic Methods for the Approximate Solution of Numerical Equations.

In practical work, it is generally most convenient to get *approximate* values for the real roots of equations of higher degree than the second. Cardan's general method * for equations of the third degree, is generally so unwieldy as to be almost useless. Trignometrical methods are better. For the numerical equations pertaining to practical work, one of the most instructive methods for locating the real roots, is to trace the graph of the given function. Every point of intersection of the curve with the x-axis, represents a root of the equation.

The location of the roots of the equation thus reduces itself to the determination of the points of intersection of the graph of the equation with the x-axis.

Examples.—(1) Find the root of the equation $x + 2 = 0$. At sight, of course, we know that the root is $- 2$. But plot the curve $y = x + 2$, for values of y when $- 3, - 2, - 1, 0, 1, 2, 3$, are successively assigned to x. The curve (Fig. 122) cuts the x-axis when $x = - 2$. Hence, $x = - 2$, is a root of the equation.

Another way is to proceed as in the next example.

(2) Solve $x^3 + x - 2 = 0$. Here $x^3 = - x + 2$. Put $y = x^3$ and $y = - x + 2$.

* Found in the regular textbooks.

Plot the graph of each of these equations, using the table of cubes, page 518. The abscissa of the point of intersection of these two curves is one root of the given equation. $x = OM$ (Fig. 123) is the root required.

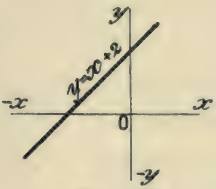


FIG. 122.

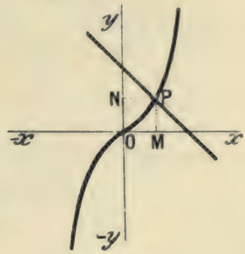


FIG. 123.

(3) Locate the roots of $x^2 - 8x + 9 = 0$. Proceed as before by assigning successive values to x. Roots occur between 6 and 7 and 1 and 2.

(4) Show that $x^3 - 6x^2 + 11x - 6 = 0$ has roots in the neighbourhood of 1, 2, 3.

The method indicated in the second example, may be employed to find the roots of simultaneous equations, thus

(5) Solve $\qquad x^2 + y^2 = 1\,;\ x^2 - 4x = y^2 - 3y.$

Plot the two curves as shown in Fig. 124, hence $x = \pm\, OM$ are the roots required.

The graphic method can also be employed for transcendental equations.

(6) If $x + \cos x = 0$, we may locate the roots by finding the point of intersection of the two curves $y = -x$ and $y = \cos x$.

(7) If $x + e^x = 0$, plot $y = e^x$ and $y = -x$. Table, page 518.

(8) Show, by plotting, that an equation of an odd degree with real coefficients, has either one or an odd number of real roots.

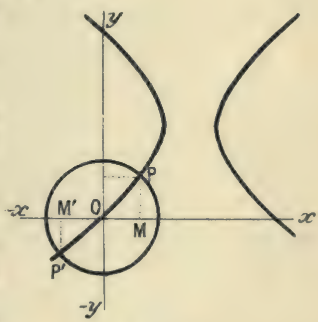


FIG. 124.

For large values of x, the graph must lie on the positive side of the x-axis, and on the opposite side for large negative values of x. Therefore the graph must cut the x-axis at least once; if twice, then it must cut the axis a third time, etc.

(9) Show, by plotting, that an equation of an even degree with real coefficients, has either 2, 4, . . . or an even number of roots, or else no roots at all.

(10) Prove, by plotting, (3), § 155.

(11) Plot $x^2 - 2x + 1 = 0$. The curve touches but does not cut the x-axis. This means that the point of contact of the curve with the x-axis, corresponds to two points infinitely close together. That is to say, that there are at least two equal roots.

The graphic method may be applied to the most complicated equations.

(12) Find numbers, correct to three significant figures, which will satisfy the following equations:

(i.) $9x^3 - 41x^{0\cdot8} + 0\cdot5^{2x} - 92 = 0$. (iii.) $e^x - e^{-x} + 0\cdot4x - 10 = 0$.
(ii.) $2\cdot42x^3 - 3\cdot15\log_e x - 20\cdot5 = 0$. (iv.) $2x^{3\cdot1} - 3x - 16 = 0$.

(London S. and A. Depart., 1899 and 1900 Examinations.) Ansrs. (i.) 2·35; (ii.) 2·11; (iii.) 2·22; (iv.) 2·18.

The accuracy of the graphic method depends on the scale of the diagram and the skill of the draughtsman.

§ 158. Newton's Method for the Approximate Solution of Numerical Equations.

The above method indicates that the equation

$$f(x) = y = x^3 - 7x + 7, \quad . \quad . \quad . \quad (1)$$

has a root lying somewhere between -3 and -4. We can keep on assigning intermediate values to x until we get as near to the exact value of the root as our patience will allow. Thus, if $x = -3$, $y = +1$, if $x = -3\cdot2$, $y = -3\cdot3$. The desired root thus lies somewhere between -3 and $-3\cdot2$. Assume that the actual value of the root is $-3\cdot1$. To get a close approximation to the root by plotting is a somewhat laborious operation. Newton's method, based on Taylor's theorem, allows the process to be shortened.

Let α be the desired root, then

$$f(\alpha) = \alpha^3 - 7\alpha + 7. \quad . \quad . \quad . \quad (2)$$

As a first approximation, assume that $\alpha = -3\cdot1 + h$, is the required root.

From (1), by differentiation,

$$dy/dx = 3x^2 - 7\,;\ d^2y/dx^2 = 6x\,;\ d^3y/dx^3 = 6. \quad . \quad (3)$$

All succeeding derivatives are zero.

By Taylor's theorem,

$$f(x + h) = y + h\frac{dy}{dx} + \frac{h^2}{2!}\cdot\frac{d^2y}{dx^2} + \frac{h^3}{3!}\cdot\frac{d^3y}{dx^3}.$$

Put $v = -3\cdot1$ and $\alpha = v + h$.

$$f(\alpha) = f(v + h) = f(v) + h\frac{dv}{dx} + \frac{h^2}{2!}\cdot\frac{d^2v}{dx^2} + \frac{h^3}{3!}\cdot\frac{d^3v}{dx^3}.$$

Neglecting the higher powers of h, in the first approximation,

$$f(v) + h\frac{dv}{dx} = 0\,;\ \text{or, } h = -\frac{f(v)}{f'(v)}, \quad . \quad . \quad (4)$$

where $f'(v) = dv/dx$. The value of $f(v)$ is found by substituting $-3\cdot1$, in (2), and the value of $f'(v)$, by substituting $-3\cdot1$, in the first of equations (3), thus, from (4),

$$h = f(v)/f'(v) = -1\cdot091/21\cdot83 = -0\cdot04999.$$

Hence the first approximation to the root is $-3\cdot05$.

As a second approximation, assume that

$$a = -3{\cdot}05 + h_1 = v_1 + h_1.$$

As before,

$$h_1 = -f(v_1)/f'(v_1) = -{\cdot}022625/20{\cdot}9081 = -{\cdot}001082.$$

The second approximation, therefore, is $-3{\cdot}048918$. We can, in this way, obtain third and higher degrees of approximation. Here is another example to try:

$$x^3 - 2x - 5 = 0,$$

has a root between 2 and 3. The first approximation is 2·0946, the second 2·09455148. Generally, the first approximation gives all that is required for practical work.

EXAMPLES.—(1) In the same way show that the first approximation to one of the roots of $x^3 - 4x^2 - 2x + 4 = 0$, is $a = 4{\cdot}2491 \ldots$ and the second $a = 4{\cdot}2491405 \ldots$

(2) If $x^3 + 2x^2 + 3x - 50 = 0$, $x = 2{\cdot}9022834 \ldots$

(3) If $x^2 + 4\sin x = 0$, $x = -1{\cdot}933 \ldots$

§ 159. How to Separate Equal Roots from an Equation.

This is a preliminary operation to the determination of the roots by a process, perhaps simpler than the above.

If $\alpha, \beta, \gamma, \ldots$ are the roots of an equation of the nth degree,

$$x^n + ax^{n-1} + \ldots + sx + R = 0,$$

$$(x - \alpha)(x - \beta) \ldots (x - \eta) = 0.$$

If two of the roots are equal, two factors, say $x - \alpha$ and $x - \beta$, will be identical and the equation will be divisible by $(x - \alpha)^2$; if there are three equal roots, the equation will be divisible by $(x - \alpha)^3$, etc.

If there are n equal roots, the equation will contain a factor $(x - \alpha)^n$, and the first derivative will contain a factor $n(x - \alpha)^{n-1}$, or $x - \alpha$ will occur $n - 1$ times.

The highest common factor of the original equation and its first derivative must, therefore, contain $x - \alpha$, repeated once less than in the original equation. If there is no common factor, there are no equal roots.

EXAMPLES.—(1) $x^3 - 5x^2 - 8x + 48 = 0$ has a first derivative $3x^2 - 10x - 8$. The common factor is $x - 4$. This shows that the equation has two roots equal to $x - 4$.

(2) $x^4 + 7x^3 - 3x^2 - 55x + 50 = 0$ has two roots each equal to $x + 5$.

§ 160. Sturm's Method of Locating the Real and Unequal Roots of a Numerical Equation.

Newton's method of approximation does not give satisfactory results when the two roots have nearly equal values. For instance, the curve

$$y = x^3 - 7x + 7$$

has two nearly equal roots between 1 and 2, which do not appear if we draw the graph for the corresponding values of x and y, *viz.*:

$$x = 0, \quad 1, \quad 2, \quad 3, \ldots;$$
$$y = 7, \quad 1, \quad 1, \quad 13, \ldots$$

The problem of separating the real roots of a numerical equation is, however, completely solved by what is known as Sturm's theorem. It is clear that if x assumes every possible value in succession from $+\infty$ to $-\infty$, every change of sign will indicate the proximity of a real root. The total number of roots is known from the degree of the equation, therefore the number of imaginary roots can be determined by difference.

Number of real roots + Number of imaginary roots = Total number of roots.

Sturm's theorem enables these changes of sign to be readily detected. The process is as follows:

First remove the real equal roots, as indicated in the preceding section, let

$$y = x^3 - 7x + 7, \quad . \quad . \quad . \quad . \quad (1)$$

remain. Find the first differential coefficient,

$$dy/dx = 3x^2 - 7. \quad . \quad . \quad . \quad . \quad (2)$$

Divide the primitive (1) by the first derivative (2), thus,

$$(x^3 - 7x + 7)/(3x^2 - 7).$$

Change the sign of the remainder and divide by 7, the result

$$R = 2x - 3, \quad . \quad . \quad . \quad . \quad (3)$$

is now to be divided into (2). Change the sign of the remainder and we obtain,

$$R = 1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The right-hand sides of equations (1), (2), (3), (4),

$$x^3 - 7x + 7\,;\ 3x^2 - 7\,;\ 2x - 3\,;\ 1,$$

are known as **Sturm's functions.**

Substitute $-\infty$ for x in (1), the sign is negative;
,, ,, (2), ,, positive;
,, ,, (3), ,, negative;
,, ,, (4), ,, positive.

Note that the last result is independent of x. The changes of sign may, therefore, be written

$$- + - +.$$

In the same way,

Value of x.	Corresponding Signs of Sturm's Functions.	Number of Changes of Sign.
$-\infty$	$- + - +$	3
-4	$- + - +$	3
-3	$+ + - +$	2
-2	$+ + - +$	2
-1	$+ - - +$	2
$+0$	$+ - - +$	2
$+1$	$+ - - +$	2
$+2$	$+ + + +$	0
$+\infty$	$+ + + +$	0

There is, therefore, no change of sign caused by the substitution of any value of x less than -4, or greater than $+2$; on passing from -4 to -3, there is one change of sign; on passing from 1 to 2, there are two changes of sign. The equation has, therefore, one real root between -4 and -3, and two, between 1 and 2.

It now remains to determine a sufficient number of digits, to distinguish between the two roots lying between 1 and 2. First reduce the value of x in the given equation by 1. This is done by substituting $u + 1$ in place of x, and then finding Sturm's functions for the resulting equation. These are,

$$u^3 + 3u^2 - 4u + 1;\ 3u^2 + 6u - 4;\ 2u - 1;\ 1.$$

As above, noting that if $x = +1$, $u = +0{\cdot}1$, etc.,

Value of x.	Corresponding Signs of Sturm's Functions.	Number of Changes of Sign.
1	$+ - - +$	2
2	$+ - - +$	2
3	$+ - - +$	2
4	$- - - +$	1
5	$- - + +$	1
6	$- + + +$	1
7	$+ + + +$	0

The second digits of the roots between 1 and 2 are, therefore, 3 and 6, and three real roots of the given equation are approximately -3, $1{\cdot}3$, $1{\cdot}6$.

EXAMPLES.—Locate the roots in the following equations:

(1) $x^3 - 3x^2 - 4x + 13$. Ansr. Between -3 and -2; 2 and 2·5; 2·5 and 3.

(2) $x^3 - 4x^2 - 6x + 8$. Ansr. Between 0 and 1; 5 and 6; -1 and -2.

(3) $x^4 + x^3 - x^2 - 2x + 4$. We have five Sturm's functions for this equation. Call the original equation (1), the first derivative, $4x^3 + 3x^2 - 2x - 2$, (2); divide (1) by (2) and $x^2 + 2x - 6$ (3) remains; divide (2) by (3) and $-x + 1$ (4) remains; divide (3) by (4) and change the sign of the result for $+1$ (5). Now let $x = +\infty$ and $-\infty$, we get

$$+ + + - + \text{ (2 variations of sign)} : + - + + + \text{ (2 variations)}.$$

This means that there are no real roots. All the roots are imaginary.

(4) The equation, $x^3 - 3rx^2 + 4r^3\rho = 0$, is obtained in problems referring to the depth to which a floating sphere of radius r and density ρ sinks in water. Solve this equation for the case of a wooden ball of unit radius and specific gravity 0·65. Hence, $x^3 - 3x + 2·6 = 0$. The three roots, by Sturm's theorem, are—a negative root, a positive root between 1 and 2, and one over 2. The depth of the sphere in the water cannot be greater than its diameter 2. The negative root has no physical meaning. These two roots must, therefore, be excluded from the solution. The other root, by Newton's method of approximation, is $x = 1·204. . . .$

In this last example we have rejected two roots because they were inconsistent with the physical conditions of the problem under consideration. This is a very common thing to do. Not all the solutions to which an equation may lead are solutions of the problem. Of course every solution has some meaning, but this may be quite outside the requirements of the problem. Imaginary roots may be obtained, when the problem requires real numbers, the roots may be negative or fractional, when the problem requires positive or whole numbers. Sometimes, indeed, none of the solutions will satisfy the conditions imposed by the problem, in this case the problem is indeterminate.

To illustrate:

1. A is 40 years, B 20 years old. In how many years will A be three times as old as B? Let x denote the required number of years.

$$\therefore 40 + x = 3(20 + x); \text{ or } x = -10.$$

But the problem requires a positive number. The answer, therefore, is that A will never be three times as old as B. (The negative sign means that A *was* three times as old as B, 10 years ago.)

2. A number x is squared; subtract 7; extract the square root of the result; add twice the number, 5 remains. What was the number x?

$$\therefore 2x + \sqrt{(x^2 - 7)} = 5.$$

Solve in the usual way, namely, square $5 - 2x = \sqrt{x^2 - 7}$; rearrange terms and use (2), § 156. Hence $x = 4$ or $\frac{8}{3}$.

The ultimate test of every solution is that it shall satisfy the equation when substituted in place of the variable. If not it is no solution. On trial both solutions, $x = 4$ and $x = 2\frac{2}{3}$, fail to satisfy the test. These *extraneous solutions* have been introduced during rationalisation (by squaring).

§ 161. Horner's Method for Approximating to the Real Roots of Numerical Equations.

When the first significant digit or digits of a root have been obtained, by, say, Sturm's theorem, so that one root may be distinguished from all the other roots nearly equal to it, Horner's method is one of the simplest and best ways of carrying the approximation as far as may be necessary. So far as practical requirements are concerned, Horner's process is perfection. The arithmetical methods for the extraction of square and cube roots are special cases of Horner's method, because to extract $\sqrt[2]{9}$, or $\sqrt[3]{9}$, is equivalent to finding the roots of the equation $x^2 - 9 = 0$, or $x^3 - 9 = 0$.*

In outline, the method is as follows: Find by means of Sturm's theorem, or otherwise, the integral part of a root, and transform the equation into another whose roots are less than those of the original equation by the number so found. Suppose we start with the equation

$$x^3 - 7x + 7 = 0, \quad . \quad . \quad . \quad . \quad (1)$$

which has one real root whose first significant figures we have found to be 1·3. Transform the equation into another whose roots are less by 1·3 than the roots of (1). This is done by substituting $u + 1{\cdot}3$ for x. In this way we obtain,

$$u + 3{\cdot}95u^2 - 1{\cdot}93u + {\cdot}097 = 0. \quad . \quad . \quad (2)$$

The first significant figure of the root of this equation is ·05. Lower the roots of (2) by the substitution of $v + {\cdot}05$ for u in (2). Thus,

$$v^3 + 4{\cdot}05v^2 - 1{\cdot}5325v + {\cdot}010375 = 0. \quad . \quad . \quad (3)$$

The next significant figure of the root, deduced from (3), is ·006. We could have continued in this way until the root had been obtained of any desired degree of accuracy.

---

* Chrystal, *Textbook of Algebra* (A. & C. Black, London, 1898, Part I., page 346), says: "Considering the remarkable elegance, generality, and simplicity of the method, it is not a little surprising that it has not taken a more prominent place in current mathematical textbooks. Although it has been well expounded by several English writers, . . . it has scarcely as yet found a place in English curricula. Out of five standard Continental textbooks where one would have expected to find it we found it mentioned in only one, and there it was expounded in a way which showed little insight into its true character. This probably arises from the mistaken notion that there is in the method some algebraic profundity. As a matter of fact, its spirit is purely arithmetical; and its beauty, which can only be appreciated after one has used it in particular cases, is of that indescribably simple kind which distinguishes the use of position in the decimal notation and the arrangement of the simple rules of arithmetic. It is, in short, one of those things whose invention was the creation of a commonplace."

Practically, the work is not so tedious as just outlined. Let a, b, c, be the coefficients of the given equation (1), R the absolute term.

1. Multiply a by the first significant digits of the root and add the product to b. Write the result under b.

2. Multiply this sum by the first figures of the root, add the product to c. Write the result under c.

3. Multiply this sum by the first figure of the root, add the product to R, and call the result the **first dividend.**

4. Again multiply a by the root, add the product to the last number under b.

5. Multiply this sum by the root and add the product to the last number under c, call the result the **first trial divisor.**

6. Multiply a by the root once more, and the product to the last number under b.

7. Divide the first dividend by the first trial divisor, and the first significant figure in the quotient will be the second significant of the root. Thus starting from the old equation (1), whose root we know to be about 1,

a	b	c	R	(*Root*
1	+ 0	− 7	+ 7	(1·3
	1	1	− 6	
	1	− 6	**1** First dividend.	
	1	2		
	2	− **4** First trial divisor.		
	1			
	3			

8. Proceed exactly as before for the second trial divisor, using the second digit of the root, *viz.*, ·3.

9. Proceed as before for the second dividend. We finally obtain the result shown in the next scheme. Note that the black figures in the preceding scheme are the coefficients of the second of the equations reduced on the supposition that $x = 1·3$ is a root of the equation.

a'	b'	c'	R'	(*Root.*
1	3	− 4	1	(1·35
	0·3	0·99	− 0·903	
	3·3	− 3·01	**0·097** Second dividend.	
	0·3	1·08		
	3·6	− **1·93** Second trial divisor.		
	0·3			
	3·9			

Once more repeating the whole operation, we get,

a''	b''	c''	R''	(*Root*
1	3·9	− 1·93	0·097	(1·356
	0·05	0·1975	− 0·086625	
	3·95	− 1·7325	**0·010375** Third dividend.	
	0·05	0·2000		
	4·00	− **1·5325** Third trial divisor.		
	0·05			
	4·05			

Having found about five or seven decimal places of the root in this way, several more may be added by dividing, say the fifth trial dividend by the fifth trial divisor. Thus, we pass from 1·356895, to 1·356895867 . . . a degree of accuracy more than sufficient for any practical purpose.

Knowing one root, we can divide out the factor $x - 1·3569$ from equation (1), and solve the remainder like an ordinary quadratic.

If any root is finite, the dividend becomes zero, as in one of the following examples. If the trial divisor gives a result too large to be subtracted from the preceding dividend, try a smaller digit.

To get the other root whose significant digits are 1·6, proceed as above, using 6 instead of 3 as the quotient from the first dividend and trial divisor. Thus we get 1·692 . . .*

It is usual to write down the successive steps as indicated in the following example.

EXAMPLES.—(1) Find the root between 6 and 7 in

$$4x^3 - 13x^2 - 31x = 275.$$

4	− 13	− 31	− 275	(6·25
	24	66	210	
	11	35	− **65**	
	24	210	51·392	
	35	**245**	− **13·608**	
	24	11·96	13·608	
	59	256·96	0	
	0·8	12·12		
	59·8	**269·08**		
	0·8	3·08		
	60·6	272·16		
	0·8			
	61·4			

* Several ingenious short cuts have been devised for lessening the labour in the application of Horner's method, but nothing much is gained, when the method has only to be used occasionally, beyond increasing the probability of error.

The steps mark the end of each transformation. The digits in block letters are the coefficients of the successive equations.

(2) There is a positive root between 4 and 5 in $x^3 + x^2 + x - 100$. Ansr. 4·2644 . . .

(3) Find the positive and negative roots in

$$x^4 + 8x^2 + 16x = 440.$$

Ansr. + 3·976 . . ., − 4·3504 . . .

When finding negative roots, proceed as before, but first transform the equation into one with an opposite sign by changing the sign of the absolute term.

(4) Show that the root between − 3 and − 4, in equation (1), is 3·0489173396 . . . Work from $a = 1$, $b = -0$, $c = -7$, $R = -7$.

§ 162. van der Waals' Equation of State.

The relations between the roots of equations, discussed in this chapter, are interesting in many ways; for the sake of illustration, let us take van der Waals' equation of state for a gas at a distance from its point of liquefaction,

$$\left(p + \frac{a}{v^2}\right)(v - b) = R\theta, \quad . \quad . \quad . \quad (1)$$

or, expanded,

$$v^3 - \left(b + \frac{R\theta}{p}\right)v^2 + \frac{a}{p}v - \frac{ab}{p} = 0. \quad . \quad . \quad (2)$$

This equation of the third degree in v, must have three roots, α, β, γ, equal or unequal, real or imaginary. In any case,

$$(v - \alpha)(v - \beta)(v - \gamma) = 0. \quad . \quad . \quad . \quad (3)$$

Imaginary roots have no physical meaning; we may therefore confine our attention to the real roots. Of these, we have seen that there must be one, and there may be three. This means that there may be one or three (different) volumes, corresponding to every value of the pressure p and temperature θ. There are three interesting cases:

Case i. *There is only one real root present.* This implies that there is one definite volume (v) corresponding to every assigned value of pressure (p) and temperature (θ). This is realised in the pv-curve, for all gases under certain physical conditions; for instance, the graph for carbon dioxide at 48·1° (Fig. 125), has only one value of p corresponding to each value of v.

The collection of curves shown in Fig. 125, were obtained by plotting values of p and v corresponding to different values of θ, a and b. In the diagram, the degrees are on the centigrade scale. In reality,

$$\theta = (273 + \text{degrees centigrade}).$$

Case ii. *There are three real unequal roots present.* For temperatures below 32·5°, say 13·1°, we get the wavy curve $ABCD$ (Fig. 125). This means that at 13·1°, and at a pressure of Op_0, carbon dioxide ought to have three different volumes corresponding with the abscissae $O\gamma$, $O\beta$, $O\alpha$. Only two of these three volumes have yet been observed, namely for gaseous CO_2 at α and for liquid CO_2 at γ, the third, corresponding to the point β, is unknown. The curve $A\gamma\beta\alpha D$, has been realised experimentally by Andrews.

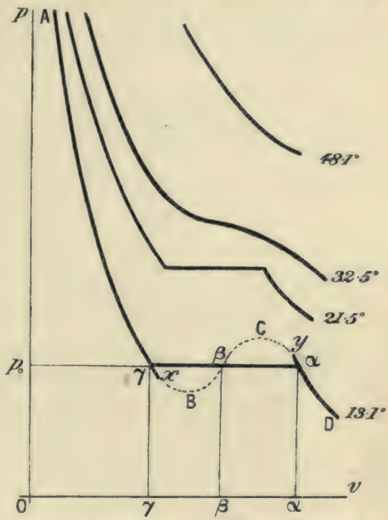


FIG. 125.—Isothermals of Carbon Dioxide.

When the volume of a mass of carbon dioxide gas is gradually diminished, the corresponding changes of pressure and volume are represented graphically by the curve $D\alpha$. At the point α, the gas begins to condense; continuing the lessening of the volume, the pressure remains constant, until the point γ is reached. Here, all the carbon dioxide will have assumed the liquid state. The straight line $\gamma\alpha$ thus represents the constant pressure exerted by the vapour of carbon dioxide in contact with its liquid.

The steep curve $A\gamma$ indicates that there is only a slight change in the volume of the liquid for great increments of pressure. See § 13.

The abscissa of the point α represents the volume of a given mass of *gaseous* carbon dioxide, the abscissa of the point γ represents the volume occupied by the same mass of *liquid* carbon dioxide at the same pressure.

Under special conditions, parts of the sinuous curve $\gamma B\beta C\alpha$ have been realised experimentally, namely, γx and αy. These latter represent unstable conditions of supersaturation. The portion γx shows that a liquid may exist at a pressure less than that of its own vapour, and αy shows that a vapour may exist at a pressure higher than that of its "vapour pressure" of its own liquid.

Case iii. *There are three real equal roots present.* At and above the point where $\alpha = \beta = \gamma$, there can only be one value of v for any assigned value of p. This point is no other than the well-known *critical point* of a gas. Write p_c, v_c, θ_c, for the critical pressure, volume and temperature of a gas. From (3),

$$(v - \alpha)^3 = 0, \quad . \quad . \quad . \quad . \quad (4)$$

and at the critical point $\alpha = v = v_c$, therefore,

$$v^3 - \left(b + \frac{R\theta_c}{p_c}\right)v^2 + \frac{a}{p_c}v - \frac{ab}{p_c} = v^3 - 3v_cv^2 + 3v^2_cv - v^3_c. \quad (5)$$

This equation is an identity, therefore (footnote, page 172),

$$3v_cp_c = bp_c + R\theta_c; \; 3v^2_cp_c = a; \; v^3_cp_c = ab, \quad . \quad (6)$$

are obtained by equating the coefficients of like powers of the unknown v.

From the last two of equations (6),

$$v_c = 3b. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (7) and the second of equations (6),

$$p_c = \frac{1}{27} \cdot \frac{a}{b^2}. \quad . \quad . \quad . \quad . \quad (8)$$

From (7), (8), and the first of equations (6),

$$\theta_c = \frac{8}{27} \cdot \frac{a}{bR}. \quad . \quad . \quad . \quad . \quad (9)$$

From these results, (7), (8), (9), van der Waals has calculated the values of the constants a and b for different gases.

Let $\pi = p/p_c$, $\phi = v/v_c$, $\theta' = \theta/\theta_c$. From (1), (7), (8) and (9),

$$(\pi + 3/\phi^2)(3\phi - 1) = 8\theta', \quad . \quad . \quad . \quad (10)$$

which appears to be van der Waals' equation freed from arbitrary constants. This result has led van der Waals to the belief that all substances can exist in states or conditions where the corresponding pressures, volumes and temperatures are equivalent. These he calls *corresponding states* ("Uebereinstimmende Zustände"). The deduction has only been verified in the case of ether, sulphur dioxide and some of the benzene halides.

It is an interesting exercise to apply the methods of Chapter III. to the "singular points" of the curves shown in Fig. 125. For convenience, put $R\theta$ = constant, say c. Solve (1) for p,

$$p = c/(v - b) - a/v^2. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Differentiate twice,

$$\frac{dp}{dv} = - \frac{c}{(v - b)^2} + \frac{2a}{v^3}; \; \frac{d^2p}{dv^2} = \frac{2c}{(v - b)^3} - \frac{6a}{v^4}. \quad . \quad . \quad (12)$$

Now read over §§ 55 to 60. It is not difficult to see that if the temperature θ is high enough, dp/dv is always negative, that is to say, the curve, or rather its tangent, will slope from left to right like the hyperbola 48·1° (Fig. 125). If v is small enough, so that $v - b$ approaches zero, the curve will have a negative slope like the left sides of the curves 32·5° to 19·1° (Fig. 125), because dp/dv will still remain negative.

When dp/dv becomes zero, we may have either

(*a*) a point of inflection shown in curve 32·5° (Fig. 125); or

(*b*) maximum and minimum values indicated by the dotted lines in the 19·1° curve.

When θ is small enough we may have, for certain values of v, a positive value for dp/dv. This can only correspond to the slope of the dotted portion BC of the curve 19·1° (Fig. 125).

Now rearrange the first of equations (12), so that

$$\frac{1}{(v-b)^2}\left\{2a\frac{(v-b)^2}{v^3} - c\right\} = 0. \quad . \quad . \quad . \quad (13)$$

When $v = b$, $(v - b)^2/v^3 = 0$; when $v = 3b$, this expression reaches a maximum and gradually diminishes to zero as v approaches ∞. If c is greater than $8a/27b$, c, or what is the same thing, $R\theta$, is greater than the maximum of $2a(v - b)^2/v^3$, therefore, as v increases p decreases. When c is less than $8a/27b$, p decreases for small and large values of v; p only increases in the neighbourhood of $v = 3b$. The expression has thus a maximum or a minimum value for any value of v which makes $2a(v - b)^2/v^3 = R\theta$.

Equating the second differential coefficient, in (12), to zero, we get

$$v^4 - \frac{3a}{c}v^3 + \frac{9ab}{c}v^2 - \frac{9ab^2}{c}v + \frac{3ab^3}{c} = 0. \quad . \quad . \quad . \quad (14)$$

The roots of this biquadratic in v, correspond with the points of inflection or transition points of the curve. Of these, there may be four, two, or none.

Now try and plot van der Waals' equation for any gas from the published values of a, b, R. For example, for ethylene $a = 0·00786$, $b = 0·0024$, $R = 0·0037$; for carbon dioxide

$$\left(p + \frac{0·00874}{v^2}\right)(v - 0·0023) = 0·00369(273 + \theta_1), \quad . \quad . \quad (15)$$

where θ_1 denotes degrees of temperature on the centigrade scale. Hint. First fix the value for θ_1, say, 0° C., and calculate a set of corresponding values of p and v, thus,

$v = 0·1,\ 0·05,\ 0·025,\ 0·01,\ 0·0075,\ 0·005,\ 0·004,\ 0·003,\ \ldots;$

$p = 9·4,\ 19·7,\ 30·3,\ 43·3,\ 37·9,\ 23·2,\ 45·8,\ 466·8,\ \ldots$

Make the successive increments in v small when in the neighbourhood of a singular point. Plot these numbers on squared paper. Note the points of inflection. Now do the same thing with $\theta_1 = 30°$ C., and $\theta_1 = 50°$ C. In this way you will get a better insight into the "inwardness" of van der Waals' equation than if pages of descriptive matter were appended. Notice that the a/v^2 term has no appreciable influence on the value of p when v becomes very great, and also that the difference between v and $v - b$ is negligibly small, as v becomes very large. What does this signify? When the gas is rarefied, it will follow Boyle's law pv = constant. What would be the state of the gas when $v = 0·0023$?

See Hilton, *Phil. Mag.* [6], **1**, 579; *ib.*, **2**, 108, 1901.

CHAPTER X.

DETERMINANTS.

THIS chapter is for the purpose of explaining and illustrating a system of notation which is in common use in the different branches of pure and applied mathematics.

§ 163. Simultaneous Equations.

(i.) *Homogeneous simultaneous equations in two unknowns.* The homogeneous equations,

$$a_1x + b_1y = 0\,;\ a_2x + b_2y = 0, \qquad (1)$$

represent two straight lines passing through the origin. In this case (§ 28), $x = 0$ and $y = 0$, a deduction verified by solving for x and y. Multiply the first of equations (1) by b_2, and the second by b_1. Subtract. Or, multiply the second of equations (1) by a_1, and the first by a_2. Subtract. In each case, we obtain,

$$x(a_1b_2 - a_2b_1) = 0\,;\ y(a_2b_1 - a_1b_2) = 0. \qquad (2)$$

Hence,
$$x = 0\,;\ \text{and } y = 0\,;$$
or,
$$a_1b_2 - a_2b_1 = 0\,;\ \text{and } a_2b_1 - a_1b_2 = 0. \qquad (3)$$

The relations in equations (3) may be written,

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = 0\,;\ \text{and} \begin{vmatrix} a_2, & b_2 \\ a_1, & b_1 \end{vmatrix} = 0, \qquad (4)$$

where the left-hand side of each expression is called a **determinant.** This is nothing more than another way of writing down the difference of the diagonal products.*

The products a_1b_2, a_2b_1, are called the **elements** of the determinant; a_1, b_1, a_2, b_2, are the **constituents** of the determinant. Commas may or may not be inserted between the constituents of the horizontal rows. When only two elements are involved, the determinant is said to be of the **second order.**

* In literal equations, the letters should always be taken in cyclic order so that b follows a, c follows b, a follows c. In the same way 2 follows 1, 3 follows 2, and 1 follows 3.

From the above equations, it follows that *only when the determinant of the coefficients of two homogeneous equations in x and y is equal to zero can x and y possess values differing from zero.*

(ii.) *Linear and homogeneous equations in three unknowns.* Solving the linear equations

$$a_1x + b_1y + c_1 = 0;\ a_2x + b_2y + c_2 = 0, \qquad (5)$$

for x and y, we get

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - b_1a_2};\ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - b_1a_2}. \qquad (6)$$

If $a_1b_2 - b_1a_2 = 0$, x and y become infinite. In this case, the two lines represented by equations (5) are either parallel or coincident. When $x = \frac{b_1c_1 - b_2c_1}{0} = \infty$; $y = \frac{c_1a_2 - c_2a_1}{0} = \infty$, the lines intersect at an infinite distance away. Reduce equations (5) to the tangent form (§ 30),

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1};\ y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2};$$

but since $a_1b_2 - b_1a_2 = 0$, $a_1/b_1 = a_2/b_2 =$ the tangent of the angle of inclination of the lines; in other words, two lines having the same slope towards the x-axis are parallel to each other.*

When the two lines cross each other, the values of x and y in (6) satisfy equations (5). Make the substitution required.

$$a_1(b_1c_2 - b_2c_1) + b_1(c_1a_2 - c_2a_1) + c_1(a_1b_2 - a_2b_1) = 0,$$
$$a_2(b_1c_2 - b_2c_1) + b_2(c_1a_2 - c_2a_1) + c_2(a_1b_2 - a_2b_1) = 0,$$

or, writing $x = X/Z$ and $y = Y/Z$, . . . (8)

we get a pair of homogeneous equations in X, Y, Z, namely,

$$a_1X + b_1Y + c_1Z = 0;\ a_2X + b_2Y + c_2Z = 0. \qquad (9)$$

Equate coefficients of like powers of the variables in these identical equations.

$$\therefore a_1 : b_1 : c_1 = a_2 : b_2 : c_2,$$

or, from (8) and (6),

$$X : Y : Z = b_1c_2 - b_2c_1 : c_1a_2 - c_2a_1 : a_1b_2 - a_2b_1,$$
$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \qquad (10)$$

The three determinants on the right, are symbolised by

$$\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|. \qquad (11)$$

---

* Thus the definition, "parallel lines meet at infinity," means that as the point of intersection of two lines goes further and further away, the lines become more and more nearly parallel.

where the number of columns is greater than the number of rows.* The determinant (11), is called a **matrix.** It is evaluated, by taking the difference of the diagonal products of any two columns.

The results obtained in (10) are employed in solving linear equations.

EXAMPLES.—(1) Solve $4x + 5y = 7$; $3x - 10y = 19$.

$$X : Y : Z = \begin{vmatrix} 5, & -7 \\ -10, & -19 \end{vmatrix} : \begin{vmatrix} -7, & 4 \\ -19, & 3 \end{vmatrix} : \begin{vmatrix} 4, & 5 \\ 3, & -10 \end{vmatrix};$$

$$= -165 : 55 : -55; \text{ or } x = +3 \text{ and } y = -1.$$

(2) Solve $20x - 19y = 23$; $19x - 20y = 16$. Ansr. $x = 4$, $y = 3$.

(3) Solve the observation equations:

$$\cdot 5x - \cdot 2y = \cdot 4; \ \cdot 14x + \cdot 3y = 1\cdot 18. \quad \text{Ansr. } x = 2, y = 3.$$

(4) Solve $\frac{1}{2}x - \frac{1}{3}y = 6$; $\frac{1}{4}x - \frac{1}{2}y = -1$. Ansr. $x = 24$, $y = 18$.

The condition that three straight lines represented by the equations

$$a_1x + b_1y + c_1 = 0; \ a_2x + b_2y + c_2 = 0; \ a_3x + b_3y + c_3 = 0, \quad (12)$$

may meet in a point, is that the roots of any two of the three lines may satisfy the third (§ 32). In this case we get a set of simultaneous equations in X, Y, Z.

$$a_1X + b_1Y + c_1Z = a_2X + b_2Y + c_2Z = a_3X + b_3Y + c_3Z = 0, \quad (13)$$

by writing $x = X/Z$ and $y = Y/Z$ in equations (12).

From the last pair,

$$X : Y : Z = \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} : \begin{vmatrix} c_2, & a_2 \\ c_3, & a_3 \end{vmatrix} : \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix}. \quad (14)$$

But these values of x and y, also satisfy the first of equations (3), hence, by substitution,

$$a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2, & a_2 \\ c_3, & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} = 0, \quad (15)$$

which is more conveniently written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \quad (16)$$

a determinant of the third order.

It follows directly from equations (13), (14), (16), *only when the determinant of the coefficients of three homogeneous equations in x, y, z, is equal to zero, can x, y, z, possess values differing from zero.*

---

* It is customary to call the vertical columns, simply "columns"; the horizontal rows, "rows".

From (12), (13), (15), (16), we conclude that *three equations are consistent with each other, only when the determinant of the coefficients and absolute term of three linear equations in* x, y, z, *are equal to zero.*

This determinant is called the **eliminant** of the equations.

Instead of taking the last two of equations (12), we might have substituted the values of x and y derived from any two of these equations in the third. Thus, in addition to (14), we may have

$$X:Y:Z = \left\{ \begin{array}{l} \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} : \begin{vmatrix} c_3 & a_3 \\ c_1 & a_1 \end{vmatrix} : \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{array} \right\}. \qquad (17)$$

Each of these sets may be obtained from (16), by deleting certain rows and columns, for instance, $\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix}$ is obtained by omitting the row and column containing a_2, and so on. Each determinant in (14) and (17), is called a **subdeterminant,** or **minor** of (16).

§ 164. The Expansion of Determinants.

It follows from (15) and (16), that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

A determinant is expanded, by taking the product of one letter in each horizontal row with one letter from each of the other rows. The first element, called the **leading element,** is the product of the diagonal constituents from the top left-hand corner, *i.e.*, $a_1b_2c_3$; its sign is taken as positive. The signs of the other five terms,* are obtained by arranging alphabetically, and observing whether they can be obtained from the leading element by an odd or an even number of changes in the subscripts; if the former, the element is negative, if the latter, positive. For example, $a_2b_1c_3$, is obtained by one interchange of the subscripts 2 and 1 in the leading element; $a_2b_1c_3$ is, therefore, a negative element; $a_2b_3c_1$ requires two such transformations, 2 and 1, and 2 and 3, hence its sign is positive.

* The number of constituents in a determinant of the second order is 2×1, or $2!$; of the third order $3 \times 2 \times 1$, or $3!$, of the fourth order, $4!$, etc.

EXAMPLES.—(1) Show $\begin{vmatrix} 2 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 2 + 8 + 12 - 8 - 4 - 6 = 4.$

(2) Show $\begin{vmatrix} 0 & b & c \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = 2abc.$

(3) Expand $\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ \cdots & & \cdots & \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$ into twenty-four terms, twelve negative, twelve positive.

§ 165. The Solution of Simultaneous Equations.

Continuing the discussion in § 162, let the equations

$$a_1x + b_1y + c_1z = d_1\ ;\ a_2x + b_2y + c_2z = d_2\ ;\ a_3x + b_3y + c_3z = d_3, \quad (18)$$

be multiplied by suitable quantities, so that y and z may be eliminated. Thus multiply the first equation by A_1, the second by A_2, the third by A_3, where A_1, A_2, A_3, are so chosen that

$$b_1A_1 + b_2A_2 + b_3A_3 = 0\ ;\ c_1A_1 + c_2A_2 + c_3A_3 = 0. \quad (19)$$

Hence, by substitution,

$$x(a_1A_1 + a_2A_2 + a_3A_3) = d_1A_1 + d_2A_2 + d_3A_3. \quad . \quad (20)$$

Equations (19) being homogeneous in A_1, A_2, A_3, we get, from (10),

$$A_1 : A_2 : A_3 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} : \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} : \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Substituting these values of A_1, A_2, A_3, in equations (20), we get, as in equations (14), (15), (16),

$$x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \quad . \quad . \quad . \quad (21)$$

In the same way, on multiplying by B_1, B_2, B_3, and by C_1, C_2, C_3.

$$y \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}\ ;\ z \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}. \quad (22)$$

EXAMPLES.—Solve the following sets of equations:

(1) $5x + 3y + 3z = 48$; $2x + 6y - 3z = 18$; $8x - 3y + 2z = 21$. From (21)

$$x = \begin{vmatrix} 48 & 3 & 3 \\ 18 & 6 & -3 \\ 21 & -3 & 2 \end{vmatrix} \div \begin{vmatrix} 5 & 3 & 3 \\ 2 & 6 & -3 \\ 8 & -3 & 2 \end{vmatrix} = 3\ ;$$

similarly $y = 5$; $z = 6$.

(2) $x - ay + a^2z = a^3$; $x - by + b^2z = b^3$; $x - cy + c^2z = c^3$. Ansr. $x = abc$; $y = ab + bc + ca$; $z = a + b + c$.

(3) Solve $2x - 3y + 4z = 1$; $3x + 2y - 4z = \frac{7}{6}$; $4x - 3y + 2z = \frac{3}{2}$. Ansr. $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$.

(4) Solve the observation equations:

$\cdot 3x + \cdot 2y + \cdot 5z = 3\cdot 2$; $\cdot 2x + \cdot 3y + \cdot 4z = 2\cdot 9$; $\cdot 4x + \cdot 3y + \cdot 5z = 3\cdot 7$.
Ansr. $x = 2$, $y = 3$, $z = 4$.

§ 166. Elimination.

It is required to eliminate the unknown from the two equations,

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0;\ b_2x^2 + b_1x + b_0 = 0. \qquad (23)$$

Multiply the first equation by x, the second by x and by x^2 successively. We thus get the five equations,

$$\left.\begin{array}{l} 0 \quad + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \\ a_3x^4 + a_2x^3 + a_1x^2 + a_0x + 0 = 0, \\ 0 \quad + \ 0 \ + b_2x^2 + b_1x + b_0 = 0, \\ 0 \quad + b_2x^3 + b_1x^2 + b_0x + 0 = 0, \\ b_2x^4 + b_1x^3 + b_0x^2 + \ 0 \ + 0 = 0, \end{array}\right\}. \qquad (24)$$

As on page 405, if these five equations are consistent, the eliminant of the four unknowns, is

$$\begin{vmatrix} 0 & a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 & 0 & 0 \end{vmatrix} = 0. \qquad (25)$$

Examples.—(1) Show that the following equations are consistent with one another,

$$x + y - z = 0;\ x - y + z = 2;\ y + z - x = 4;\ x + y + z = 6,$$

is

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 6 \end{vmatrix} = 0.$$

(2) Eliminate x and y from the equations

$$2x^3 - 5x^2y - 9y^3 = 0;\ 3x^2 - 7xy - 6y^2 = 0.$$

Divide the first by y^3, the second by y^2. Multiply the first by x/y, the second by x/y and x^2/y^2. The eliminant of the resulting five equations, is

$$\begin{vmatrix} 0 & 2 & -5 & 0 & -9 \\ 2 & -5 & 0 & -9 & 0 \\ 0 & 0 & 3 & -7 & -6 \\ 0 & 3 & -7 & -6 & 0 \\ 3 & -7 & -6 & 0 & 0 \end{vmatrix} = 0.$$

§ 167. Fundamental Properties of Determinants.

1. *The value of a determinant is not altered by changing the columns into rows, or the rows into columns.*

It follows directly, by simple expansion, that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; \text{ and } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad . \quad . \quad (26)$$

It follows as a corollary, that whatever law is true for the rows of a determinant, is also true for the columns and conversely.

2. *The sign, not the numerical value, of a determinant is altered by interchanging any two rows, or any two columns.*

By direct calculation,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}; \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}. \quad . \quad (27)$$

3. *If two rows or two columns of a determinant are identical, the determinant is equal to zero.*

If two identical rows or columns are interchanged the sign, not the value of the determinant, is altered. This is only possible if the determinant is equal to zero. The same thing can be proved by the expansion of, say,

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0.$$

4. *When the constituents of two rows or two columns differ by a constant factor, the determinant is equal to zero.*

Thus by expansion show that

$$\begin{vmatrix} 4 & 1 & 5 \\ 8 & 2 & 6 \\ 12 & 3 & 7 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 5 \\ 2 & 2 & 6 \\ 3 & 3 & 7 \end{vmatrix} = 4 \times 0 = 0. \quad . \quad . \quad . \quad (28)$$

5. *If a determinant has a row or column of cyphers it is equal to zero.*

This is illustrated by expansion,

$$\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

6. *In order to multiply a determinant by any factor, multiply each constituent in one row or in one column by this factor.*

This is illustrated by the expansion of the following:

$$m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad (30)$$

7. *In order to divide a determinant by any factor, divide each constituent in one row or in one column by that factor.*

This follows directly from the preceding proposition. It is conveniently used in the reduction of determinants to simpler forms. Thus,

$$\begin{vmatrix} 6 & 9 & 8 \\ 12 & 18 & 4 \\ 24 & 27 & 2 \end{vmatrix} = 9.6.2 \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 2 \\ 4 & 3 & 1 \end{vmatrix} = 9.6.2.2 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 1 \end{vmatrix}. \quad . \quad . \quad (31)$$

8. *If the sign of every constituent in a row or column is changed, the sign of the determinant is changed.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} -a_1 & b_1 & c_1 \\ -a_2 & b_2 & c_2 \\ -a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} -a_1 & -b_1 & c_1 \\ -a_2 & -b_2 & c_2 \\ -a_3 & -b_3 & c_3 \end{vmatrix}. \quad . \quad . \quad (32)$$

9. *One row or column of any determinant can be reduced to unity* (*Dostor's theorem*).

This will need no more explanation than the following illustration:

$$\begin{vmatrix} 3 & 4 & 6 \\ 2 & 8 & 8 \\ 6 & 7 & 9 \end{vmatrix} = 12 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 8 & 7 & 6 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad (33)$$

10. *If each constituent of a row or column can be expressed as the sum or difference of two or more terms, the determinant can be expressed as the sum or difference of two other determinants.*

This can be proved by expanding each of the following determinants, and rearranging the letters.

$$\begin{vmatrix} a_1 \pm p, & b_1, & c_1 \\ a_2 \pm q, & b_2, & c_2 \\ a_3 \pm r, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \pm \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}. \quad . \quad . \quad (34)$$

In general, if each constituent of a row or column consists of n terms, the determinant can be expressed as the sum of n determinants.

EXAMPLE.—Show by this theorem, that

$$\begin{vmatrix} b + c, & a - b, & a \\ c + a, & b - c, & b \\ a + b, & c - a, & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$$

11. *The value of a determinant is not changed by adding to or subtracting the constituents of any row from the corresponding constituents of one or more of the other rows or columns.*

Thus from 10 and 3,

$$\begin{vmatrix} a_1 \pm b_1, & b_1, & c_1 \\ a_2 \pm b_2, & b_2, & c_2 \\ a_3 \pm b_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \pm \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}, \quad . \quad . \quad (35)$$

which proves the rule, because the determinant on the right vanishes. This result is employed in simplifying determinants.

EXAMPLES.—(1) Show $\begin{vmatrix} 1, & x, & y + z \\ 1, & y, & z + x \\ 1, & z, & x + y \end{vmatrix} = 0.$

Add the second column to the last and divide the last column by $x + y + z$. The determinant vanishes (3).

(2) Show $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = (x + y + z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}$.

Add the second and third rows to the first and divide by $x + y + z$.

(3) Why is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ not equal to $\begin{vmatrix} a_1 + b_1, & b_1 + a_1, & c_1 \\ a_2 + b_2, & b_2 + a_2, & c_2 \\ a_3 + b_3, & b_3 + a_3, & c_3 \end{vmatrix}$?

(4) Show $\begin{vmatrix} 4 & 1 & 7 \\ 3 & 6 & -2 \\ 5 & 1 & 8 \end{vmatrix} = -23$.

12. *If all but one of the constituents of a row or column are cyphers, the determinant can be reduced to the product of the one constituent, not zero, into a determinant whose order is one less than the original determinant.*

For example,

$$\begin{vmatrix} 1 & a & b \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix} = 1 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}; \quad \begin{vmatrix} 0 & 0 & -7 \\ 5 & 6 & -2 \\ -3 & 1 & 8 \end{vmatrix} = -7 \begin{vmatrix} 5 & 6 \\ -3 & 1 \end{vmatrix}. \quad . \quad (36)$$

The converse proposition holds. The order of a determinant can be raised by similar and obvious transformations.

13. *If all the constituents of a determinant on one side of the diagonal from the top left-hand corner are cyphers, the determinant reduces to the leading term.*

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ 0 & c_3 \end{vmatrix} = a_1b_2c_3. \quad . \quad . \quad . \quad . \quad (37)$$

The determinant $\begin{vmatrix} b_2 & c_2 \\ 0 & c_3 \end{vmatrix}$ is called the **co-factor** or **complement** of the constituent a_1.

§ 168. The Multiplication of Determinants.

This is done in the following manner :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} d_1 & e_1 \\ d_2 & e_2 \end{vmatrix} = \begin{vmatrix} a_1d_1 + b_1e_1, & a_1d_2 + b_1e_2 \\ a_2d_1 + b_2e_1, & a_2d_2 + b_2e_2 \end{vmatrix}. \quad . \quad (38)$$

The proof follows directly on expanding the right side of the equation. We thus obtain,

$$= \begin{vmatrix} a_1d_1, & b_1e_2 \\ a_2d_1, & b_2e_2 \end{vmatrix} + \begin{vmatrix} b_1e_1, & a_1d_2 \\ b_2e_1, & a_2d_2 \end{vmatrix};$$

$$= d_1e_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + d_2e_1 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix};$$

$$= (d_1e_2 - d_2e_1) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} d_1 & e_1 \\ d_2 & e_2 \end{vmatrix}.$$

Since the value of a determinant is not altered by writing the columns in rows and the rows in columns, the product of two determinants may be written in several equivalent forms which all give the same result on expansion. Thus, instead of the right side of (38), we may have

$$\begin{vmatrix} a_1d_1 + b_1d_2, & a_1e_1 + b_1e_2 \\ a_2d_1 + b_2d_2, & a_2e_1 + b_2e_2 \end{vmatrix}; \begin{vmatrix} a_1d_1 + a_2d_2, & a_1e_1 + a_2e_2 \\ b_1d_1 + b_2d_2, & b_1e_1 + b_2e_2 \end{vmatrix}, \text{ etc.}$$

EXAMPLES.—(1) Multiply $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}$.

The answer may be written in several different forms; one form is

$$\begin{vmatrix} a_1d_1 + b_1e_1 + c_1f_1, & a_1d_2 + b_1e_2 + c_1f_2, & a_1d_3 + b_1e_3 + c_1f_3 \\ a_2d_1 + b_2e_1 + c_2f_1, & a_2d_2 + b_2e_2 + c_2f_2, & a_2d_3 + b_2e_3 + c_2f_3 \\ a_3d_1 + b_3e_1 + c_3f_1, & a_3d_2 + b_3e_2 + c_3f_2, & a_3d_3 + b_3e_3 + c_3f_3 \end{vmatrix}.$$

This can be verified by the laborious operation of expansion. There are twenty-seven determinants all but six of which vanish.

(2) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a^2_1 + b^2_1, & a_1a_2 + b_1b_2 \\ a_1a_2 + b_1b_2, & a^2_2 + b^2_2 \end{vmatrix}$.

When two constituents of a determinant hold the same relative position with respect to the rows and columns, they are said to be **conjugate.** Thus in the last of the determinants in (34) b_1 and q are conjugate, so are b_3 and c_2, r and c_1. If the conjugate elements are equal, the determinant is **symmetrical,** if equal but opposite in sign, we have a **skew determinant.** *The square of a determinant is a symmetrical determinant.*

§ 169. The Differentiation of Determinants.

Suppose that the constituents of a determinant are independent and that

$$D = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1,$$

then,

$$\begin{aligned} d(D) &= x_1dy_2 + y_2dx_1 - x_2dy_1 - y_1dx_2; \\ &= (y_2dx_1 - y_1dx_2) + (x_1dy_2 - x_2dy_1); \\ &= \begin{vmatrix} dx_1 & y_1 \\ dx_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & dy_1 \\ x_2 & dy_2 \end{vmatrix}. \quad . \quad . \quad . \quad (39) \end{aligned}$$

If the constituents of the determinant are functions of an independent variable, say t, then, writing $\dot{x}_1$ for dx/dt, $\dot{y}_2$ for dy_2/dt and so on, it can be proved, in the same way,

$$D = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}; \; d(D)/dt = \begin{vmatrix} \dot{x}_1 & y_1 \\ \dot{x}_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & \dot{y}_1 \\ x_2 & \dot{y}_2 \end{vmatrix}. \quad . \quad (40)$$

EXAMPLES.—(1) Show that if $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$;

$$d(D) = \begin{vmatrix} dx_1 & y_1 & z_1 \\ dx_2 & y_2 & z_2 \\ dx_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & dy_1 & z_1 \\ x_2 & dy_2 & z_2 \\ x_3 & dy_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & dz_1 \\ x_2 & y_2 & dz_2 \\ x_3 & y_3 & dz_3 \end{vmatrix};$$

$$d(D)/dt = \begin{vmatrix} \dot{x}_1 & y_1 & z_1 \\ \dot{x}_2 & y_2 & z_2 \\ \dot{x}_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & \dot{y}_1 & z_1 \\ x_2 & \dot{y}_2 & z_2 \\ x_3 & \dot{y}_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & \dot{z}_1 \\ x_2 & y_2 & \dot{z}_2 \\ x_3 & y_3 & \dot{z}_3 \end{vmatrix}.$$

(2) If $a_1, b_2, c_1, a_2, b_2, \ldots$, are constants, show that

$$d\begin{vmatrix} a_1x & b_1y & c_1z \\ a_2x & b_2y & c_2z \\ a_3 & b_3y & c_3z \end{vmatrix} = \begin{vmatrix} a_1dx & b_1y & c_1z \\ a_2dx & b_2y & c_2z \\ & b_3y & c_3z \end{vmatrix} + \text{, etc.,} = dx\begin{vmatrix} b_1y & c_1z \\ b_2y & c_2z \end{vmatrix}\text{, etc.}$$

§ 170. Jacobians and Hessians.

1. *Definitions.* If u, v, w, be functions of the independent variables, x, y, z, the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}, \quad \ldots \quad (41)$$

is called a **Jacobian** and is variously written,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)};\ \text{or } J(u, v, w);\ \text{or simply } J, \quad . \quad . \quad (42)$$

when there can be no doubt as to the variables under consideration.

In the special case, where the functions u, v, w are themselves differential coefficients of the one function, say u, with respect to x, y and z, the determinant

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial z^2} \end{vmatrix}, \quad \ldots \quad (43)$$

is called a **Hessian** of u and written $H(u)$, or simply H. The Hessian, be it observed, is a symmetrical determinant whose constituents are the second differential coefficients of u with respect to x, y, z. In other words, the Hessian of the primitive function u, is the Jacobian of the first differential coefficients of u, or in the notation of (42),

$$H(u) = \frac{\partial\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)}{\partial(x, y, z)}. \quad . \quad . \quad . \quad (44)$$

2. *Jacobians and Hessians of interdependent functions.* If

$$u = f(v),$$

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}; \text{ and } \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}.$$

Eliminate the function $f'(v)$ as described on page 340.

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = 0,$$

or,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0. \quad . \quad . \quad . \quad (45)$$

That is to say, *if u is a function of v, the Jacobian of the functions of u and v with respect to x and y will be zero.*

The converse of this proposition is also true. If the relation (45) holds good, u will be a function of v.

In the same way, it can be shown that *only when the Hessian of u is equal to zero, are the first derivatives of u with respect to x and y independent of each other.*

3. *The Jacobian of a function of a function.* If u_1, u_2, are functions of x_1 and x_2, and x_1 and x_2 are functions of y_1 and y_2,

$$\frac{\partial u_1}{\partial y_1} = \frac{\partial u_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1} + \frac{\partial u_1}{\partial x_2}\cdot\frac{\partial x_2}{dy_1}; \quad \frac{\partial u_1}{\partial y_2} = \frac{\partial u_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2} + \frac{\partial u_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2}.$$

By the rule for the multiplication of determinants,

$$\begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} \\ \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}, \quad . \quad . \quad . \quad (46)$$

or,

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}\cdot\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}.$$

This bears a close formal analogy with the well-known

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}.$$

4. *The Jacobian of implicit * functions.* If u and v, instead of

---

* A function is said to be **explicit** when it can be expressed directly in terms of the variable or variables, *e.g.*, z is an explicit function of x in the expression: $z = x^2$; $z + a = bx^4$. A function is **implicit** when it cannot be so expressed in terms of the independent variable. Thus $x^2 + 2xy = y^2$; $x + y = z^x$, are implicit functions.

being explicitly connected with the independent variables x and y, are so related that

$$p = f_1(x, y, u, v) = 0;\ q = f_2(x, y, u, v) = 0,$$

u and v may be regarded as implicit functions of x and y. By differentiation

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial x} = 0;\ \frac{\partial p}{\partial y} + \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial y} = 0;$$

$$\frac{\partial q}{\partial x} + \frac{\partial q}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial q}{\partial v}\cdot\frac{\partial v}{\partial x} = 0;\ \frac{\partial q}{\partial y} + \frac{\partial q}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial q}{\partial v}\cdot\frac{\partial v}{\partial x} = 0;$$

and by the rule for the multiplication of determinants,

$$\begin{vmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = - \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix}. \quad . \quad . \quad (47)$$

Or,

$$\frac{\partial(p, q)}{\partial(u, v)}\cdot\frac{\partial(u, v)}{\partial(x, y)} = - \frac{\partial(p, q)}{\partial(x, y)}.$$

A result which may be extended to include any number of independent relations.

§ 171. Some Thermodynamic Relations.

Determinants, Jacobians and Hessians are continually appearing in different branches of applied mathematics.

The following summarises a paper by J. E. Trevor in the *Journal of Physical Chemistry* (**3,** 523, 573, 1899). The results will serve as a simple exercise on the mathematical methods of some of the earlier sections of this work. The reader should find no difficulty in assigning a meaning to most of the coefficients considered.

If U denotes the internal energy, ϕ the entropy, p the pressure, v the volume, θ the absolute temperature, Q the quantity of heat in a system of constant mass and composition, the two laws of thermodynamics state that

$$dQ = dU + p\,.\,dv;\ dQ = \theta d\phi. \quad . \quad . \quad . \quad . \quad (1)$$

To find a value for each of the partial derivatives

$$\left(\frac{\partial\phi}{\partial p}\right)_v, \left(\frac{\partial\phi}{\partial p}\right)_\theta, \left(\frac{\partial\phi}{\partial\theta}\right)_p, \left(\frac{\partial\phi}{\partial\theta}\right)_v, \left(\frac{\partial\phi}{\partial v}\right)_p, \left(\frac{\partial\phi}{\partial v}\right)_\theta;$$

$$\left(\frac{\partial v}{\partial p}\right)_\theta, \left(\frac{\partial v}{\partial p}\right)_\phi, \left(\frac{\partial v}{\partial\theta}\right)_p, \left(\frac{\partial v}{\partial\theta}\right)_\phi, \left(\frac{\partial v}{\partial\phi}\right)_p, \left(\frac{\partial v}{\partial\phi}\right)_\theta,$$

in terms of the derivatives of U.

Case i. When v or ϕ is constant. From (1),

$$-p = \partial U/\partial v;\ \text{and } \theta = \partial U/\partial\phi. \quad . \quad . \quad . \quad . \quad (2)$$

First, differentiate each of the expressions (2), with respect to ϕ at constant volume

$$-\left(\frac{\partial p}{\partial\phi}\right)_v = \frac{\partial^2 U}{\partial v\partial\phi};\ \text{and } \left(\frac{\partial\theta}{\partial\phi}\right)_v = \frac{\partial^2 U}{\partial\phi^2}. \quad . \quad . \quad . \quad (3)$$

By division, $$-\left(\frac{\partial p}{\partial \theta}\right)_v = \frac{\partial^2 U}{\partial v \partial \phi}\bigg/\frac{\partial^2 U}{\partial \phi^2}. \qquad (4)$$

Next, differentiate each of equations (2) with respect to v at constant entropy.

$$-\left(\frac{\partial p}{\partial v}\right)_\phi = \frac{\partial^2 U}{\partial v^2};\ \left(\frac{\partial \theta}{\partial v}\right)_\phi = \frac{\partial^2 U}{\partial \phi \partial v}. \qquad (5)$$

By division, $$-\left(\frac{\partial p}{\partial \theta}\right)_\phi = \frac{\partial^2 U}{\partial v^2}\ \frac{\partial^2 U}{\partial \phi \partial v}. \qquad (6)$$

Case ii. When either p or θ is constant. We know that

$$dp = \frac{\partial p}{\partial v}dv + \frac{\partial p}{\partial \phi}d\phi;\ \text{and}\ d\theta = \frac{\partial \theta}{\partial v}dv + \frac{\partial \theta}{\partial \phi}d\phi. \qquad (7)$$

First, when p is constant, eliminate dv or $d\phi$ between equations (7). Hence show that

$$dv\bigg/\left(-\frac{\partial p}{\partial \phi}\right) = d\phi\bigg/\frac{\partial p}{\partial v} = \frac{\partial \theta}{J},$$

where J denotes the Jacobian $\partial(p, \theta)/\partial(v, \phi)$. If H denotes the Hessian of U, show that

$$-\left(\frac{\partial v}{\partial \phi}\right)_p = \frac{\dfrac{\partial^2 U}{\partial v \partial \phi}}{\dfrac{\partial^2 U}{\partial v^2}};\ -\left(\frac{\partial v}{\partial \theta}\right)_p = \frac{\dfrac{\partial^2 U}{\partial v \partial \phi}}{H};\ \left(\frac{\partial \phi}{\partial \theta}\right)_p = \frac{\dfrac{\partial^2 U}{\partial v^2}}{H}. \qquad (8)$$

Finally, if θ is constant, show that

$$-\left(\frac{\partial \phi}{\partial v}\right)_\theta = \frac{\dfrac{\partial^2 U}{\partial v \partial \phi}}{\dfrac{\partial^2 U}{\partial \phi^2}};\ \left(\frac{\partial \phi}{\partial p}\right)_\theta = \frac{\dfrac{\partial^2 U}{\partial v \partial \phi}}{H};\ -\left(\frac{\partial v}{\partial p}\right)_\theta = \frac{\dfrac{\partial^2 U}{\partial \phi^2}}{H}. \qquad (9)$$

See also Baynes' *Thermodynamics* (Oxford, 1878), pp. 95 *et seq.*

CHAPTER XI.

PROBABILITY AND THE THEORY OF ERRORS.

§ 172. Probability.

> "Perfect knowledge alone can give certainty, and in Nature perfect knowledge would be infinite knowledge, which is clearly beyond our capacities. We have, therefore, to content ourselves with partial knowledge—knowledge mingled with ignorance, producing doubt."—W. STANLEY JEVONS.

> "Lorsqu'il n'est pas en notre pouvoir de discerner les plus vraies opinions, nous devons suivre les plus probables." *—RENÉ DESCARTES.

NEARLY every inference we make with respect to any future event is more or less doubtful. If the circumstances are favourable, a forecast may be made with a greater degree of confidence than if the conditions are not so disposed. A prediction made in ignorance of the determining conditions is obviously less trustworthy than one based upon a more extensive knowledge. If a sportsman missed his bird more frequently than he hit, we could safely infer that in any future shot he would be more likely to miss than to hit. In the absence of any conventional standard of comparison, we could convey no idea of the degree of the correctness of our judgment. The theory of probability seeks to determine the amount of reason which we may have to expect any event when we have not sufficient data to determine with certainty whether it will occur or not and when the data will admit of the application of mathematical methods.

A great many practical people imagine that the "doctrine of probability" is too conjectural and indeterminate to be worthy of

---

* Translated : "When it is not in our power to determine what is true, we ought to act according to what is most probable".

serious study. Liagre * very rightly believes that this is due to the connotation of the word—"probability". The term is so vague that it has undermined, so to speak, that confidence which we usually repose in the deductions of mathematics. So great, indeed, has been the dominion of this word over the mind that all applications of this branch of mathematics are thought to be affected with the unpardonable sin—want of reality. Change the title and the "theory" would not take long to cast off its conjectural character, and to take rank among the most interesting and useful applications of mathematics.

Laplace remarks at the close of his *Essai philosophique sur les Probabilités* (Paris, 1812), "the theory of probabilities is nothing more than common-sense † reduced to calculation. It determines with exactness what a well-balanced mind perceives by a kind of instinct, without being aware of the process. By its means nothing is left to chance either in the forming of an opinion, or in the recognising of the most advantageous view to select whenever the occasion should arise. It is, therefore, a most valuable supplement to the ignorance and frailty of the human mind. . . ."

1. *If one of two possible events occurs in such a way that one of the events must occur in a ways, the other in b ways, the probability that the first will happen is $a/(a + b)$, and the probability that the second will happen is $b/(a + b)$.*

If a rifleman hits the centre of a target about once every twelve shots under fixed conditions of light, wind, quality of powder, etc., we could say that the value of his chance of scoring a "bullseye" in any future shot is 1 in 12, or $\frac{1}{12}$, and of missing, 11 in 12, or $\frac{11}{12}$. If a more skilful shooter hits the centre about five times every twelve shots, his chance of success in any future shot would be 5 in 12, or $\frac{5}{12}$, and of missing $\frac{7}{12}$.

Putting this idea into more general language, if an event can happen in a ways and fail in b ways,

$$\left.\begin{array}{ll}\text{the probability of the event happening} & = a/(a+b);\\ \text{the probability of the event failing} & = b/(a+b),\end{array}\right\}, \qquad (1)$$

provided that each of these ways is just as likely to happen as to fail. By definition,

$$\text{Probability} = \frac{\text{Number of ways the event occurs}}{\text{Number of possible ways the event may happen}}. \qquad (2)$$

* Liagre's *Calcul des Probabilités* (C. Muquardt, Bruxelles, 1879).

† Literally "bons sens" = good sense.

2. *If p denotes the probability that an event will happen, $1 - p$ denotes the probability that the event will fail.*

The shooter at the target is certain either to hit or to miss. In mathematics, unity is supposed to represent certainty, therefore,

$$\text{Probability of hitting} + \text{Probability of missing} = \text{Certainty} = 1. \qquad (3)$$

If the event is certain not to happen the probability of its occurrence is zero. Certainty is the unit of probability. Degrees of probability are fractions of certainty.

Of course the above terms imply no quality of the event in itself, but simply the attitude of the computer's own mind with respect to the occurrence of a doubtful event. We call an event *impossible* when we cannot think of a single cause in favour of its occurrence, and *certain* when we cannot think of a single cause antagonistic to its occurrence. All the different "shades" of probability—improbable, doubtful, probable—lie between these extreme limits.

Strictly speaking there is no such thing as chance in Nature. The irregular path described by a mote "dancing in a beam of sunlight" is determined as certainly as the orbit of a planet in the heavens. The terms "chance" and "probability" are nothing but conventional modes of expressing our ignorance of the causes of events as indicated by our inability to predict the results. "Pour une intelligence (omniscient)," says Liagre, "tout événement à venir serait *certain* ou *impossible*."

3. *The probability that both of two independent events will happen together is the product of their separate probabilities.*

Let p denote the probability that one event will happen, q the probability that another event will happen, the probability that both events will happen together is

$$pq. \qquad (4)$$

This may be illustrated in the following manner: A vessel A contains a_1 white balls, b_1 black balls, and a vessel B contains a_2 white balls and b_2 black balls, the probability of drawing a white ball from A is $p_1 = a_1/(a_1 + b_1)$, and from B, $p_2 = a_2/(a_2 + b_2)$. The total number of pairs of balls that can be formed from the total number of balls is $(a_1 + b_1)(a_2 + b_2)$. In any simultaneous drawing from each vessel, the probability that

two white balls will occur is: $a_1a_2/(a_1 + b_1)(a_2 + b_2)$; . . . (5)
two black balls will occur is: $b_1b_2/(a_1 + b_1)(a_2 + b_2)$; . . . (6)
white ball drawn first, black ball next, is: $a_1b_2/(a_1 + b_1)(a_2 + b_2)$; . (7)
black ball drawn first, white ball next, is: $a_2b_1/(a_1 + b_1)(a_2 + b_2)$; . (8)
black and white ball occur together, is: $(a_1b_2 + b_1a_2)/(a_1 + b_1)(a_2 + b_2)$. (9)

The sum of (5), (6), (9) is unity. This condition is required by the above definition.

An event of this kind, produced by the composition of several events, is said to be a *compound event*. To throw three aces with three dice at one trial is a compound event dependent on the concurrence of three simple events.

Errors of observation are compound events produced by the concurrence of several independent errors.

EXAMPLE.—If the respective probabilities of the occurrence of each of n independent errors is $P_1, P_2, \ldots, P_n$, the probability of the occurrence of all together is $P_1 P_2 \ldots P_n$.

4. *The probability of the occurrence of several events which cannot occur together is the sum of the probabilities of their separate occurrences.*

If $p, q, \ldots$ denote the separate probabilities of different events, the probability that one of the events will happen is,

$$= p + q + \ldots \qquad (10)$$

EXAMPLE.—A bag contains 12 balls two of which are white, four black, six red, what is the probability that the first ball drawn will be a white, black, or a red one? The probability that the ball will be white is $\frac{1}{6}$, a black $\frac{1}{3}$, etc. The probability that the first ball drawn shall be a black or a white ball is $\frac{1}{2}$.

5. *If p denotes the probability that an event will happen on a single trial, the probability that it will happen r times in n trials is*

$$\frac{n(n-1)\ldots(n-r+1)}{r!}p^r(1-p)^{n-r}.* \qquad (11)$$

The probability that the event will fail on any single trial is $1 - p$; the probability that it will fail every time is $(1-p)^n$. The probability that it will happen on the first trial and fail on the succeeding $n-1$ trials is $p(1-p)^{n-1}$. But the event is just as likely to happen on the 2nd, 3rd, . . . trials as on the first. Hence the probability that the event will happen just once in the n trials is

$$np(1-p)^{n-1}. \qquad (12)$$

The probability that the event will occur on the first two trials and fail on the succeeding $n-2$ trials is $p^2(1-p)^{n-2}$. But the event is as likely to occur during the 1st and 3rd, 2nd and 4th, . . . trials. Hence the probability that it will occur just twice during the n trials is

$$\frac{1}{2!}n(n-1)p^2(1-p)^{n-2}. \qquad (13)$$

The probability that it will occur r times in n trials is, therefore, represented y formula (11).

6. *If p denotes the very small probability that an event will happen on a single trial, the probability that it will happen r times in a very great number (n) trials, is*

$$\frac{(np)^r}{r!}e^{-np}. \qquad (14)$$

---

* The student may here find it necessary to read over § 191.

From formula (11), however small p may be, by increasing the number of trials, we can make the probability that the event will happen at least once in n trials as great as we please. The probability that the event will fail every time in n trials is $(1 - p)^n$, and if p be made small enough and n great enough, we can make $(1 - p)^n$ as small as we please.* If n is infinitely great and p infinitely small, we can write $n = n - 1 = n - 2 = \ldots$

$$\therefore (1 - p)^n = 1 - np + \frac{n(n-1)}{2!}p^2 - \ldots;$$

$$= 1 - np + \frac{(np)^2}{2!} - \ldots \text{ (approx.)};$$

$$= e^{-np} \text{ (approx.)}. \qquad (15)$$

(14) follows immediately from (11) and (15). This result is very important.

EXAMPLES.—(1) If n grains of wheat are scattered haphazard over a surface s units of area, show that the probability that a units of area will contain r grains of wheat is

$$\frac{(an)^r}{r!}e^{-\frac{an}{s}}.$$

Thus, $n \,.\, ds/s$ represents the infinitely small probability that the small space ds contains a grain of wheat. If the selected space be a units of area, we may suppose each ds to be a trial, the number of trials will, therefore, be a/ds. Hence we must substitute an/s for np in (14) for the desired result.

(2) Using the above notation and reasoning, show that the probability that an event will occur *at least* r times in n trials is

$$p^n + np^{n-1}q + \frac{n(n-1)}{2!}p^{n-2}q^2 + \ldots + p^r q^{n-r}. \qquad (16)$$

Sometimes a natural process proves far too complicated to admit of any simplification by means of "working hypotheses". The question naturally arises, can the observed sequence of events be reasonally attributed to the operation of a law of Nature or to chance?

For example, it is observed that the average of a large number of readings of the barometer is greater at nine in the morning than at four in the afternoon; Laplace (*Théorie analytique des Probabilités*, p. 49, 1820) asked whether this was to be ascribed to the operation of an unknown law of Nature or to chance? Again, Kirchhoff (*Monatsberichte der Berliner Akademie*, Oct., 1859) inquired if the coincidence between 70 spectral lines in iron vapour and in sunlight could reasonably be attributed to chance. He found that the probability of a fortuitous coincidence was approximately as 1 : 1,000000,000000. Hence, he argued that there can be no reasonable doubt of the existence of iron in the sun. Michell (*Phil. Trans.*, **57,** 243, 1767; see also Kleiber, *Phil. Mag.* [5], **24,** 439, 1887) has endeavoured to calculate if the number of star clusters is greater than what would be expected if the stars had been distributed

* The reader should test this by substituting small numbers in place of p, and large ones for n. Use the binomial formula of § 98. See the remarks on page 481, § 189.

haphazard over the heavens. Schuster (*Proc. Roy. Soc.*, **31,** 337, 1881) has tried to answer the question, is the number of harmonic relations in the spectral lines of iron greater than what a chance distribution would give? Mallet (*Phil. Trans.*, **171,** 1003, 1880) and Strutt (*Phil. Mag.* [6], **1,** 311, 1901) have asked, do the atomic weights of the elements approximate as closely to whole numbers as can reasonably be accounted for by an accidental coincidence? In other words, are there common-sense grounds for believing the truth of Prout's law, that "the atomic weights of the other elements are exact multiples of that of hydrogen"?

The theory of probability does not pretend to furnish an infallible criterion for the discrimination of an accidental coincidence from the result of a determining cause. Certain conditions must be satisfied before any reliance can be placed upon its dictum. For example, a sufficiently large number of cases must be available. Moreover, the theory is applied irrespective of any knowledge to be derived from other sources which may or may not furnish corroborative evidence. Thus Kirchhoff's conclusion as to the probable existence of Fe in the sun was considerably strengthened by the apparent relation between the brightness of the coincident lines in the two spectra.

For details of the calculations, the reader must consult the original memoirs. Most of the calculations are based upon the analysis in Laplace's old but standard *Théorie* (*l.c.*). An excellent *résumé* of this latter work will be found in the *Encyclopædia Metropolitana*.

The more fruitful applications of the theory of probability to natural processes has been in connection with the kinetic theory of gases and the "law" relating to errors of observation.

§ 173. Application to the Kinetic Theory of Gases.*

The following illustrations are based, in the first instance, on a memoir by Clausius (*Phil. Mag.* [4], **17,** 81, 1859). For further developments, Meyer's *The Kinetic Theory of Gases* (Longmans, Green & Co., 1899) may be consulted.

1. *To show that the probability that a single molecule, moving in a swarm of molecules at rest, will traverse a distance x without collision is*

$$P = e^{-x/l}, \quad . \quad . \quad . \quad . \quad (17)$$

* The purpose of the kinetic theory of gases is to explain the physical properties of gases from the hypothesis that a gas consists of a great number of molecules in rapid motion. I select, here and in § 181, a few deductions which directly refer to the theory of probability.

where l denotes the probable value of the free path the molecule can travel without collision, and x/l denotes the ratio of the path actually traversed to the mean length of the free path. "Free path" is defined as the distance traversed by a molecule between two successive collisions. The "mean free path" is the average of a great number of free paths of a molecule.

Consider any molecule moving under these conditions in a given direction. Let a denote the probability that the molecule will travel a path one unit long without collision, the probability that the molecule will travel a path two units long is $a \cdot a$, or a^2, and the probability that the molecule will travel a path x units long without collision is, from (4),

$$P = a^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where a is a proper fraction. Its logarithm is therefore negative. (Why?)

If the molecules of the gas are stationary, the value of a is the same whatever the direction of motion of the single molecule. From (15), therefore,

$$P = e^{-x/l},$$

where $l = 1/\log a$. We can get a clear idea of the meaning of this formula by comparing it with (15). Supposing the traversing of unit path is reckoned a "trial," x in (17) then corresponds with n in (15). $1/l$ in (17) replaces p in (15). $1/l$, therefore, represents the probability that an event (collision) will happen during one trial. If l trials are made, a collision is certain to occur. This is virtually the definition of mean free path.

2. *To show that the length of the path which a molecule, moving amid a swarm of molecules at rest, can traverse without collision is probably*

$$l = \frac{\lambda^3}{\rho^2\pi}, \quad . \quad . \quad . \quad . \quad (19)$$

where λ denotes the mean distance between any two neighbouring molecules, ρ the radius of the sphere of action corresponding to the distance apart of the molecules during a collision, π is a constant with its usual signification.

Let unit volume of the gas contain N molecules. Let this volume be divided into N small cubes each of which, on the average, contains only one molecule. Let λ denote the length of the edge of one of these little cubes. Only one molecule is contained in a cube of capacity λ^3. The area of a cross section through the centre of a sphere of radius ρ, is $\pi\rho^2$, (12), page 491. If the moving molecule travels a distance λ, the hemispherical anterior surface of the molecule passes through a cylindrical space of volume $\pi\rho^2\lambda$, (25), page 492. Therefore, the probability that there is a molecule in the cylinder $\pi\rho^2\lambda$ is to 1 as $\pi\rho^2\lambda$ is to λ^3, that is to say, the probability that the molecule under consideration will collide with another as it passes over a path of length λ, is $\pi\rho^2\lambda : \lambda^3$. The probability that there will be no collision is $1 - \pi\rho^2/\lambda^2$. From (17),

$$P = e^{-\lambda/l} = 1 - \rho^2\pi/\lambda^2. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

According to the kinetic theory, one fundamental property of gases is that the intermolecular spaces are very great in comparison with the dimensions of the molecules, and, therefore, $\rho^2\pi/\lambda^2$ is very small in comparison with unity. Hence also λ/l is a small magnitude in comparison with unity. Expand $e^{-\lambda/l}$ according to the exponential theorem (page 230), neglect terms involving the higher powers of λ, and

$$e^{-\lambda/l} = 1 - \lambda/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From (20) and (21),

$$l = \frac{\lambda^3}{\rho^2\pi}\text{; or, } P = e^{-\frac{\rho^2\pi x}{\lambda^3}}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

EXAMPLE.—The behaviour of gases under pressure indicates that ρ is very much smaller than λ. Hence show that "a molecule passes by many other molecules like itself before it collides with another". Hint. From the first of equations (22),

$$l : \lambda = \lambda^2 : \rho^2\pi.$$

Interpret the symbols.

3. *To show that* (19) *represents the mean value of the free path of n molecules moving under the same conditions as the solitary molecule just considered.*

Out of n molecules which travel with the same velocity in the same direction as the given molecule, $ne^{-x/l}$ will travel the distance x without collision, and $ne^{-(x+dx)/l}$ will travel the distance $x + dx$ without collision. Of the molecules which traverse the path x,

$$n\left(e^{-\frac{x}{l}} - e^{-\frac{x+dx}{l}}\right) = ne^{-\frac{x}{l}}\left(1 - e^{-\frac{dx}{l}}\right) = \frac{n}{l}e^{-\frac{x}{l}}dx,$$

of them will undergo collision in passing over the distance dx. The last transformation follows directly from (21). The sum of all the paths traversed by the molecules passing x and $x + dx$ is

$$\frac{x}{l}ne^{-\frac{x}{l}}dx.$$

Since each molecule must collide somewhere in passing between the limits $x = 0$ and $x = \infty$, the sum of all the possible paths traversed by the n molecules before collision is

$$n\int_0^\infty \frac{x}{l}e^{-\frac{x}{l}}dx,$$

and the mean value of these n free paths is

$$\int_0^\infty \frac{x}{l}e^{-\frac{x}{l}}dx = l.$$

Integrate the indefinite integral as indicated on page 168. Therefore, from (4),

$$l = \frac{\lambda^3}{\rho^2\pi}$$

represents the mean free path of these molecules moving with a uniform velocity.

EXAMPLES.—(1) A molecule moving with a velocity u enters a space filled with n stationary molecules of a gas per unit volume, what is the probability that this molecule will collide with one of those at rest in unit time?

Use the above notation. The molecule travels the space u in unit time. In doing this, it meets with $\pi n\rho^2 u$ molecules at rest. The probable number of collisions in unit time is, therefore, $\pi n\rho^2 u$, which represents the probability of a collision in unit time.

(2) Show that the probable number of collisions made in unit time by a molecule travelling with a uniform velocity u, in a swarm of N molecules at rest, is

$$\frac{u}{l} = \frac{u\rho^2\pi}{\lambda^3}. \qquad (23)$$

What is the relation between this and the preceding result? Note,

Number of Collisions $= u/l$; and $N\lambda^3 = 1$.

4. *The number of collisions made in unit time by a molecule moving with uniform velocity in a direction which makes an angle θ with the direction of motion of a swarm of molecules also moving with the same uniform velocity is probably*

$$\frac{\rho^2\pi}{\lambda^3}2u \sin \tfrac{1}{2}\theta. \qquad (24)$$

We must first determine the relative velocity of the molecules moving in a direction at an angle θ with the velocity of the molecule under consideration.

One of the elementary propositions of mechanics is the *parallelogram of velocities*, which states that "if two component velocities are represented in direction and magnitude by two sides of a parallelogram drawn from a point, the resultant velocity is represented in direction and magnitude by the diagonal of the parallelogram drawn from that point". The **parallelopiped of velocities** is an extension of the preceding result into three dimensions. "If three component velocities are represented in direction and magnitude by the adjacent sides of a parallelopiped x, y, z (Fig. 126), drawn from a point, their resultant velocity will be represented by the diagonal of a parallelopiped drawn from that point." Conversely, if the velocity of the moving system is represented in magnitude and direction by the diagonal u (Fig. 126) of a parallelopiped, this can be resolved into three component velocities represented by three sides x, y, z of the parallelopiped drawn from a point. From § 48,

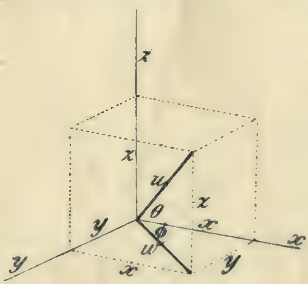


FIG. 126.

$$x = u \cos\theta;\ y = u \sin\theta \,.\, \cos\phi;\ z = u \sin\theta \,.\, \sin\phi. \qquad (25)$$

Let the three velocities represented by x, y, z, be respectively denoted by v_1, v_2, v_3, and let u be represented by v. Hence, from Euclid i., 47,

$$w^2 = v_1^2 + v_2^2;$$
$$v^2 = w^2 + v_3^2 = v_1^2 + v_2^2 + v_3^2. \qquad (26)$$

If one set of molecules moves with a uniform velocity whose components are x_1, y_1, z_1 relative to a given molecule moving with a uniform velocity whose components are x, y, z, then the relative component velocities of one molecule with respect to the other considered at rest, is

$$v_1^2 = (x - x_1)^2;\ v_2^2 = (y - y_1)^2;\ v_3^2 = (z - z_1)^2.$$

From (26),

$$\therefore v = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}. \qquad (27)$$

If we choose the three coordinate axes so that the x-axis coincides with the direction of motion of the given molecule, we may substitute these values in (25), remembering that $\cos 0 = 1$, $\sin 0 = 0$,

$$\therefore x = u;\ y = 0;\ z = 0. \qquad (28)$$

Substitute (28) and (25) in (27),

$$v = \sqrt{(u - u\cos\theta)^2 + u^2\sin^2\theta \,.\, \cos^2\phi + u^2\sin^2\theta \,.\, \sin^2\phi};$$
$$= \sqrt{u^2 - 2u^2\cos\theta + u^2\cos^2\theta + u^2\sin^2\theta},$$

since $\sin^2x + \cos^2x = 1$. Similarly, and for the same reason,

$$v = u\sqrt{2 - 2\cos\theta} = u\sqrt{2(1 - \cos\theta)},$$

from page 500, $1 - \cos x = 2(\sin\frac{1}{2}x)^2$,

$$\therefore v = 2u\sin\tfrac{1}{2}\theta. \qquad (29)$$

Having found the relative velocity of the molecules, it follows directly from (23) and (29), that

$$(\textit{Number of collisions}) = \frac{v\rho^2\pi}{\lambda^3} = \frac{\rho^2\pi}{\lambda^3}2u\sin\tfrac{1}{2}\theta.$$

5. *The number of collisions encountered in unit time by a molecule moving in a swarm of molecules moving in all directions, is*

$$\frac{4}{3} \cdot \frac{u\rho^2\pi}{\lambda^3}. \qquad (30)$$

Let u denote the velocity of the molecules, then the different motions can be resolved into three groups of motions according to the converse of the parallelopiped of velocities. Proceed as in the last illustration.

The number of molecules (n) moving in a direction between θ and $\theta + d\theta$ is to the total number of molecules (N) in unit volume as

$$n : N = 2\pi\sin\theta d\theta : 4\pi; \qquad (31)$$

or

$$n = \tfrac{1}{2}N\sin\theta d\theta.$$

Since the angle θ can increase from 0° to 180°, the total number of collisions is

$$\frac{u\rho^2\pi}{\lambda^3} \cdot \frac{n}{N} = \frac{2\rho^2\pi}{\lambda^3}\sin\tfrac{1}{2}\theta \,.\, \tfrac{1}{2}\sin\theta d\theta.$$

To get the total number of collisions, it only remains to integrate for all directions of motion between 0° and 180°. Thus if A denotes the number of collisions,

$$A = \frac{u\rho^2\pi}{\lambda^3}\int_0^\pi \sin\tfrac{1}{2}\theta \,.\, \sin\theta d\theta;$$

or,

$$= \frac{2u\rho^2\pi}{\lambda^3}\int_0^\pi \sin^2\tfrac{1}{2}\theta \,.\, \cos\tfrac{1}{2}\theta d\theta;$$
$$= \frac{4}{3} \cdot \frac{u\rho^2\pi}{\lambda^3},$$

by the method of integration on page 186.

EXAMPLE.—Find the length of the free path of a molecule moving in a swarm of molecules moving in all directions, with a velocity u. Ansr.

$$= u/A = \tfrac{3}{4}\lambda^3/\rho^2\pi. \qquad (32)$$

For the hypothesis of uniform velocity see § 181.

6. *Assuming that two unlike molecules combine during a collision, the velocity of chemical reaction between two gases is*

$$\frac{dx}{dt} = kNN', \qquad (33)$$

where N and N' are the number of molecules of each of the two gases respectively contained in unit volume of the mixed gases, dx denotes the number of molecules which combine in unit volume in the time dt; k is a constant.

Let the two gases be A and B. Let λ and λ' respectively denote the distances between two neighbouring molecules of the same kind, then, as above,

$$N\lambda^3 = N'\lambda'^3 = 1. \qquad (34)$$

Let ρ be the radius of the sphere of action, and suppose the molecules combine when the sphere of action of the two kinds of molecules approaches within 2ρ, it is required to find the rate of combination of the two gases.

The probability that a B molecule will come within the sphere of action of an A molecule in unit time is $u\pi\rho^2/\lambda^3$, by (23). Among the N' molecules of B,

$$N'\frac{\pi\rho^2}{\lambda^3}udt; \text{ or } NN'\pi\rho^2 udt, \qquad (35)$$

by (34), combine in the time dt. But the number of molecules which combine in the time dt is $-dN = -dN'$, or, from (35),

$$dN = dN' = -NN'\pi\rho^2 udt.$$

If dx represents the number of molecules which combine in unit volume in the time dt.

$$dx = dN = dN' = \pi\rho^2 uNN'dt.$$

Collecting together all the constants under the symbol k,

$$dx/dt = kNN'.$$

EXAMPLE.—Show the relation between (33) and Wilhelmy's law of mass action.

J. J. Thomson's memoir, "The Chemical Combination of Gases," *Phil. Mag.* [5], **18**, 233, 1884, might now be read with profit.

§ 174. Errors of Observation.

If a number of experienced observers agree to test, independently, the accuracy of the mark etched round the neck of a litre flask with the greatest precision possible, the inevitable result would be that every measurement would be different. Thus, we might expect

1·0003; 0·9991; 1·0007; 1·0002; 1·0001; 0·9998; . . .

Exactly the same thing would occur if one observer, taking every known precaution to eliminate error, repeats a measurement a great number of times. These deviations become more pronounced the nearer the approach to the limits of accurate measurement. The discrepancies doubtless arise from various unknown and therefore uncontrolled sources of error.

The irregular deviations of the measurements from, say, the arithmetical mean of all are called **accidental errors.** In the following discussion we shall call them "errors of observation" unless otherwise stated.

The simplest as well as the most complex measurements are invariably accompanied by these fortuitous errors. Absolute agreement is itself an accidental coincidence. Stanley Jevons says, "it is one of the most embarrassing things we can meet when experimental results agree too closely". Such agreement should at once excite a feeling of distrust.

The observed relations between two variables, therefore, should not be represented by a point in space, rather by a circle around whose centre the different observations will be grouped (Fig. 127). Any particular observation will find a place somewhere within the circumference of the circle. The diagram (Fig. 127) suggests our old illustration, a rifleman aiming at the centre of a target. The rifleman may be likened to an observer; the place where the bullet hits, to an observation; the distance between the centre and the place where the bullet hits the target resembles an error of observation. A shot at the centre of the target is thus an attempt to hit the centre, a scientific measurement is an attempt to hit the true value of the magnitude measured.

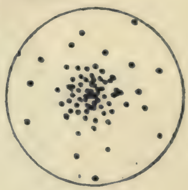


FIG. 127.

The greater the radius of the circle (Fig. 127), the cruder and less accurate the measurements; and, *vice versâ*, the less the measurements are affected by errors of observation, the smaller will be the radius of the circle. In other words, the less the skill of the shooter, the larger will be the target required to record his attempts to hit the centre.

§ 175. The "Law" of Errors.*

These errors may be represented pictorially another way. Suppose we had obtained experimental results affected by the errors shown in the following table:—

Positive Deviations from Mean between	Number of Errors.	Percentage Number of Errors.	Negative Deviations from Mean between	Number of Errors.	Percentage Number of Errors.
0·4 and 0·5	10	2	0·4 and 0·5	10	2
0·3 and 0·4	20	4	0·3 and 0·4	20	4
0·2 and 0·3	40	8	0·2 and 0·3	40	8
0·1 and 0·2	80	16	0·1 and 0·2	80	16
0·0 and 0·1	100	20	0·0 and 0·1	100	20

The above table shows that among 500 observations, 10 were affected with errors of magnitude between + 0·4 and + 0·5; 20, or 4 °/。, with errors between + 0·3 and + 0·4; . . . and 100 observations, or 20 °/。, were affected with errors *numerically* less than 0·1. This is an ideal case, but a sufficiently close approximation to reality for our present purpose.

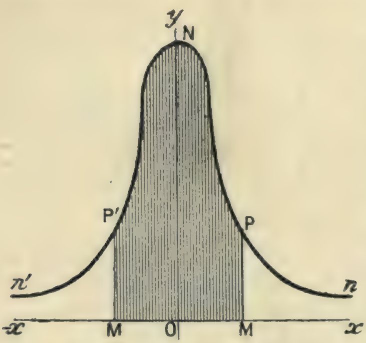


FIG. 128.—Probability Curve.

Plot, as ordinates, the numbers in the third column with the corresponding means of the two limits in the first column as abscissae. A curve similar to *nPNP′n′* (Fig. 128) will be the result.

By a study of the last two diagrams, we shall find that there is a regularity in the grouping of these irregular errors which, as a matter of fact, becomes more pronounced the greater the

* Venn (*Logic of Chance*, 1896) calls this the "exponential law of errors," a *law*, because it expresses a physical fact relating to the frequency with which errors are found to present themselves in practice. The "method of least squares" is no more than a *rule* showing how the best representative value may be extracted from a set of experimental results. Poincaré, in the preface to his *Thermodynamique* (Paris, 1892), quotes the laconic remark, "everybody firmly believes in it (the law of errors), because mathematicians imagine that it is a fact of observation, and observers that it is a theorem of mathematics".

number of trials we take into consideration. Thus, it is found that—

1. Small errors are more frequent than large ones.
2. Positive errors are as frequent as negative errors.
3. Very large positive or negative errors do not occur.

Any mathematical relation between an error (x) and the frequency, or rather the probability, of its occurrence (y), must satisfy these characteristics. When such a function,

$$y = f(x),$$

is plotted, it must have a maximum ordinate corresponding with no error; it must be symmetrical with respect to the y-axis, in order to satisfy the second condition; and as x increases numerically, y must decrease until, when x becomes very large, y must become vanishingly small. Such is the curve represented by the equation,

$$y = ke^{-h^2x^2}, \quad . \quad . \quad . \quad . \quad (1)$$

where h and k are constants.* The graph of this equation, called the **probability curve**, or curve of frequency, or curve of errors, is obtained by assigning arbitrary constant values to h and k and plotting a set of corresponding values of x and y in the usual way.†

To find a meaning for the constant k, put $x = 0$, then $y = k$, that is the maximum ordinate of the curve. If we agree to define an error as the deviation of each measurement from the arithmetical mean, k corresponds with those measurements which coincide with the mean itself, or are affected by no error at all. The height at which the curve cuts the y-axis (Fig. 129) represents the magnitude of the arithmetical mean; k has nothing to do with the actual shape of the curve beyond increasing the length of the maximum ordinate as the accuracy of the observations increases.

To find a meaning for the constant h, put $k = 1$, and plot corresponding values of x and y for $x = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \ldots$ when $h = 1, \frac{1}{2}, \frac{1}{4}, \ldots$ In this way, it will be observed that although all the curves cut the y-axis at the same point, the greater the value of h, the steeper will be the curve in the neighbourhood of the central ordinate Oy. The physical signifi-

---

* Several attempts by Gauss, Hagen, Herschel, Laplace, etc., have been made to prove this "law". Adrain appears to have been the first to deduce the above formula on theoretical grounds. (1808.)

† Use Table XXIII., page 519, or $\log e^{-h^2x^2} = -0{\cdot}4343h^2x^2$.

cation of this is that the greater the magnitude of h, the more accurate the results and the less will be the magnitude of the deviation of individual measurements from the arithmetical mean of the whole set. Hence Gauss calls h the **absolute "measure of precision"**. If the curves a, b, c (Fig. 129) retained their present shape while transposed to cut the y-axis at the same point, we should obtain a very good idea of the effect of h in the above function.

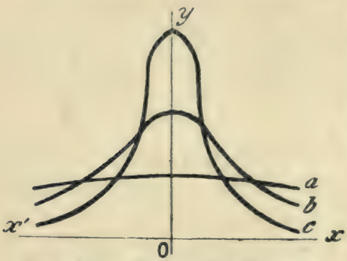


FIG. 129.—Probability Curves.

We must now submit our empirical "law" to the test of experiment. Bessel has compared the errors of observation in 470 astronomical measurements made by Bradley with those which should occur according to the law of errors. The results of this comparison are shown in the following table taken from Encke's paper in the *Berliner Astronomisches Jahrbuch* for 1834, p. 249 (Taylor's *Scientific Memoirs*, **2**, 317, 1841):

Magnitude of Error in Parts of a Second of Arc, between :	Number of Errors of each Magnitude.	
	Observed.	Theory.
0 and 0·1	94	95
0·1 and 0·2	88	89
0·2 and 0·3	78	78
0·3 and 0·4	58	64
0·4 and 0·5	51	50
0·5 and 0·6	36	36
0·6 and 0·7	26	24
0·7 and 0·8	14	15
0·8 and 0·9	10	9
0·9 and 1·0	7	5
above 1·0	8	5

This is a remarkable verification of the above formula. There is this disagreement, while the theory provides for errors of any magnitude, however large, in practice, there is a limit above which no error will be found to occur, but read § 187.

Airy and Newcomb have also shown that the number and magnitude of the errors affecting extended series of observations are in fair accord with theory. But in every case, the number of large errors actually found is in excess of theory. To quote one more instance, Newcomb examined 684 observations of the transit of Mercury. According to the "law" of errors, there should be 5 errors numerically greater than $\pm$ 27″. In reality, 49 surpassed these limits.

The theory assumes that the observations are all liable to the same errors, but differ in the accidental circumstances which give rise to the errors.* Equation (1) is by no means a perfect representation of *the* law of errors. The truth is more complex. The magnitude of the errors seems to depend, in some curious way, upon the number of observations. In an extended series of observations the errors may be arranged in groups. Each group has a different modulus of precision. This means that the modulus of precision is not constant throughout an extended series of observations.

The probability curve represented by the formula

$$y = ke^{-h^2x^2},$$

may be considered a very fair graphic representation of the law connecting the probability of the occurrence of an error with its magnitude.

§ 176. The Probability Integral.

Let $x_0, x_1, x_2, \ldots x$ be a series of errors in ascending order of magnitude from x_0 to x. Let the differences between the successive values of x be equal. If x is an error, the probability of committing an error between x_0 and x is the sum of the separate probabilities $ke^{-h^2x_0^2}$, $ke^{-h^2x_1^2}$, . . ., (4), § 172, or

$$P = k(e^{-h^2x_0^2} + e^{-h^2x_1^2} + \ldots);$$

$$= k\sum_{x_0}^{x} e^{-h^2x^2}. \qquad (1)$$

If the summation sign is replaced by that of integration, we must let dx denote the successive intervals between any two limits x_0 and x, thus

$$P = \frac{k}{dx}\int_{x_0}^{x} e^{-h^2x^2}dx.$$

Now it is certain that all the errors are included between the limits $\pm\infty$, and, since certainty is represented by unity, we have

$$1 = \frac{k}{dx}\int_{-\infty}^{+\infty} e^{-h^2x^2}dx = \frac{k}{dx}\cdot\frac{\sqrt{\pi}}{h}, \qquad (2)$$

from page 269. Or,

$$k = h\,.\,dx\,/\,\sqrt{\pi}. \qquad (3)$$

---

* Some observers' results seem more liable to these large errors than others, due, perhaps, to carelessness, or lapses of attention. Thomson and Tait (*l.c.*), I presume, would call the abnormally large errors "avoidable mistakes".

Substituting this value of k in the probability equation (1), preceding section, we get the same relation expressed in another form, namely,

$$\frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx, \quad . \quad . \quad . \quad . \quad (4)$$

a result which represents the probability of errors of observation between the magnitudes x and dx. By this is meant the ratio:

$$\frac{\text{Number of errors between } x \text{ and } x + dx}{\text{Total number of errors}}.$$

The symbols y and P are convenient abbreviations for this cumbrous phrase. For a large number of observations affected with accidental errors, the probability of an error of observation having a magnitude x, is,

$$y = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}, \quad . \quad . \quad . \quad . \quad (5)$$

which is known as **Gauss' law of errors.** This result has the same meaning as $y = ke^{-h^2x^2}$ of the preceding section. In (4), dx represents the interval, for any special case, between the successive values of x. For example, if a substance is weighed to the thousandth of a gram, $dx = 0{\cdot}001$, if in hundredths, $dx = 0{\cdot}01$, etc. The probability that there will be no error is

$$h \,.\, dx / \sqrt{\pi}; \quad . \quad . \quad . \quad . \quad (6)$$

the probability that there will be no error of the magnitude of a milligram is

$$0{\cdot}001h / \sqrt{\pi} \quad . \quad . \quad . \quad . \quad (7)$$

The probability that an error will lie between any two limits x_0 and x is

$$P = \frac{h}{\sqrt{\pi}}\int_{x_0}^{x} e^{-h^2x^2}dx. \quad . \quad . \quad . \quad (8)$$

The probability that an error will lie between the limits 0 and x is

$$P = \frac{2h}{\sqrt{\pi}}\int_{0}^{x} e^{-h^2x^2}dx, \quad . \quad . \quad . \quad . \quad (9)$$

which expresses the probability that an error will be numerically less than x. We may also put

$$P = \frac{2}{\sqrt{\pi}}\int_{0}^{hx} e^{-h^2x^2}d(hx), \quad . \quad . \quad . \quad (10)$$

which is another way of writing the probability integral (8). In (8), the limits are x_0 and x; and in (9) and (10), $\pm\, x$.

EXAMPLE.—Find conditions which will make h in Gauss' equation a maximum. Hence deduce Legendre's principle of least squares: *The most probable value for the observed quantities is that for which the sum of the squares of the individual errors is a minimum.* That is to say,

$$x^2_1 + x^2_2 + \ldots + x^2_n = \text{a minimum}, \quad . \quad . \quad . \quad (11)$$

where $x_1, x_2, \ldots, x_n$, represents the errors respectively affecting the first, second, and the nth observations.

To illustrate the reasonableness of the principle of least squares, we may revert to an old regulation of the Belgian army in which the individual scores of the riflemen were formed by adding up the distances of each man's shots from the centre of the target. The smallest sum won "le grand prix" of the regiment. It is not difficult to see that this rule is faulty. Suppose that one shooter scored a 1 and a 3; another shooter two 2's. It is obvious that the latter score shows better shooting than the former.

The shots may deviate in any direction without affecting the score. Consequently, the magnitude of each deviation is proportional, not to the magnitude of the *straight line* drawn from the place where the bullet hits to the centre of the target, but to the *area* of the circle described about the centre of the target with that line as radius. This means that it would be better to give the grand prize to the score which had a minimum sum of the *squares* of the distances of the shots from the centre of the target.* This is nothing but a graphic representation of the principle of least squares, formula (11). In this way, the two shooters quoted above would respectively score a 10 and an 8.

§ 177. The Best Representative Value for a Set of Observations.

It is practically useless to define an error as the deviation of any measurement from the true result, because that definition would imply a knowledge which is the object of investigation. What then is an error? Before we can answer this question, we must determine the most probable value of the quantity measured. The only available data, as we have just seen, are always associated with the inevitable errors of observation. The measurements, in consequence, all disagree among themselves within certain limits. In spite of this fact, the investigator is called upon to state *definitely* what he considers to be the most probable value of the magnitude under investigation. Indeed, *every chemical or physical constant in our textbooks is the best representative value of a more or less extended series of discordant observations.*

For instance, giant attempts have been made to find the exact

* See properties of similar figures, § 192.

length of a column of pure mercury of one square millimetre cross-sectional area which has a resistance of one ohm at 0° C. The following numbers have been obtained:

106·33;	106·31;	106·24;
106·32:	106·29;	106·21;
106·32;	106·27;	106·19,

centimetres (Everett's *Illustrations of the C.G.S. System of Units*, p. 176, 1891). There is no doubt that the true value of the required constant lies somewhere between 106·19 and 106·33; but no reason is apparent why one particular value should be chosen in preference to another. The physicist, however, must select *one* number from the infinite number of possible values between the limits 106·19 and 106·33 cm.

What is the best representative value of a set of discordant results? The arithmetical mean naturally suggests itself, and some mathematicians start from the axiom: "the arithmetical mean is the best representative value of a series of discrepant observations".

Various attempts, based upon the law of errors, have been made to show that the arithmetical mean is the best representative value of a number of observations made under the same conditions and all equally trustworthy. The proof rests upon the fact that the positive and negative deviations, being equally probable, will ultimately balance each other as shown in example (1).*

EXAMPLES.—(1) If $a_1, a_2, \ldots, a_n$ are a series of observations, a their arithmetical mean, show that the algebraic sum of the residual errors is

$$(a_1 - a) + (a_2 - a) + \ldots + (a_n - a) = 0. \quad (1)$$

Hint. By definition of arithmetical mean,

$$a = \frac{a_1 + a_2 + \ldots + a_n}{n}; \text{ or, } na = a_1 + a_2 + \ldots + a_n.$$

Distribute the n a's on the right-hand side so as to get (1), etc.

* Hinrichs' *The Absolute Atomic Weights of the Chemical Elements*, published while the last proofs were under my hands, criticises the selection (and the selectors) of the arithmetical mean as the best representative value of a set of discordant observations. The following exercises were suggested to me after reading pages 1-20 of that work.

EXAMPLES.—(1) What does the arithmetical mean of the weights of a large number of shillings in current circulation represent?

(2) Point out the fallacy implied in the words: "if we cannot use the arithmetical mean of a large number of simple weighings of actual shillings as the true value of a (new) shilling, how dare we assume that the mean value of a few determinations of the atomic weight of a chemical element will give us its true value?"

(2) Prove that the arithmetical mean makes the sum of the squares of the errors a minimum. Hint. See page 464.

NOTE.—When calculating the mean of a number of observations which agree to a certain number of digits, it is not necessary to perform the whole of the addition. For example, the mean of the above nine measurements is written

$$106 + \tfrac{1}{9}(\cdot 33 + \cdot 32 + \cdot 32 + \cdot 31 + \cdot 29 + \cdot 27 + \cdot 24 + \cdot 21 + \cdot 19) = 106\cdot 276.$$

Edgeworth, "The Choice of Means," *Phil. Mag.*, [5], **24**, 268, 1887, and several articles on related subjects are to be found in the same journal between 1883 and 1889.

The best representative value of a constant interval. When the best representative value of a constant interval x in the expression $y = a + nx$ (where n is a positive integer 1, 2 . . .) is to be determined from a series of measurements $x_2 - x_1$, $x_3 - x_2$, . . ., which vary a little from the desired value x, the arithmetical mean cannot be employed because it reduces to $(x_n - x_1)/(n - 1)$, the same as if the first and last term alone had been measured. In such cases it is usual to refer the results to the expression

$$x = 6\frac{(n-1)(x_n - x_1) + (n-3)(x_{n-1} - x_2) + \ldots}{n(n^2 - 1)}, \quad . \qquad (2)$$

which has been obtained from the last of equations (4), § 106, by putting

$$\Sigma(x) = 1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1);$$
$$\Sigma(x^2) = 1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1);$$
$$\Sigma(y) = x_1 + x_2 + \ldots x_n;\ \Sigma(xy) = x_1 + 2x_2 + \ldots + nx_n.$$

Such measurements might occur in finding the length of a rod at different temperatures, the oscillations of a galvanometer needle, the interval between the dust figures in Kundt's method for the velocity of sound in gases, the influence of CH_2 on the physical and chemical properties of homologous series, etc.

EXAMPLES.—(1) In a Kundt's experiment for the ratio of the specific heats of a gas, the dust figures were recorded in the laboratory notebook at 30·7, 43·1, 55·6, 67·9, 80·1, 92·3, 104·6, 116·9, 129·2, 141·7, 154·0, 166·1 centimetres. What is the best representative value for the distance between the nodes? Ansr. 12·3 cm.

(2) The following numbers were obtained for the time of vibration, in seconds, of the "magnet bar" in Gauss and Weber's magnetometer in some experiments on terrestrial magnetism: 3·25; 9·90; 16·65; 23·35; 30·00; 36·65; 43·30; 50·00; 56·70; 63·30; 69·80; 76·55; 83·30; 89·90; 96·65; 103·15; 109·80; 116·65; 123·25; 129·95; 136·70; 143·35. Show that the period of vibration is 6·707 seconds.

§ 178. The Probable Error.

Some observations deviate so little from the mean that we may consider that value to be a very close approximation to the truth, in other cases the arithmetical mean is worth very little. The question, therefore, to be settled is, what degree of confidence may we have in selecting this mean as the best representative value of a series of observations? In other words, how good or how bad are the results?

We could employ Gauss' absolute measure of precision to answer this question. It is easy to show that *the measure of precision of two series of observations is inversely as their accuracy.* If the probabilities of an error x_1, lying between 0 and l_1, and of an error x_2, between 0 and l_2, are respectively

$$P_1 = \frac{1}{\sqrt{\pi}}\int_0^{l_1} e^{-h_1^2x_1^2}d(h_1x_1)\,;\ P_2 = \frac{1}{\sqrt{\pi}}\int_0^{l_2} e^{-h_2^2x_2^2}d(h_2x_2),$$

it is evident that when the observations are worth an equal degree of confidence, $P_1 = P_2$.

$$\therefore l_1h_1 = l_2h_2\,;\ \text{or, } l_1 : l_2 = h_2 : h_1,$$

or the measure of precision of two series of observations is inversely as their accuracy. An error l_1 will have the same degree of probability as an error l_2 when the measure of precision of the two series of observations is the same.

For instance, if $h_1 = 4h_2$, $P_1 = P_2$ when $l_2 = 4l_1$, or four times the error will be committed in the second series with the same degree of probability as the single error in the first set. In other words, the second series of observations will be four times as accurate as the first.

On account of certain difficulties in the application of this criterion, its use is mainly confined to theoretical discussions.

One way of showing how nearly the arithmetical mean represents all the observations, is to suppose all the errors arranged in their order of magnitude, irrespective of sign, and to select a quantity which will occupy a place midway between the extreme limits, so that the number of errors less than the assumed error is the same as those which exceed it. This is called the **probable error** (German "der wahrscheinliche Fehler"), not "the most probable error," nor "the most probable value of the actual error".

The probable error determines the degree of confidence we may have in using the mean as the best representative value of a series of observations. For instance, the atomic weight of oxygen is said to be 15·879 with a probable error $\pm$ 0·0003 ($H = 1$). This means that the arithmetical mean of a series of observations is 15·879, and the probability is $\frac{1}{2}$ (*i.e.*, the odds are even) that the true atomic weight of oxygen lies between 15·8793 and 15·8787.

Referring to Fig. 128, let the units be so chosen that the total area bounded by the curve and the x-axis is unity. If PM and $P'M'$ are drawn at equal distances from Oy in such a way that the area bounded by these lines, the curve, and the x-axis (shaded part in the figure), is equal to half the unit area, half the total observations will have errors numerically less than OM, that is, OM represents the probable error, PM its probability.

The number of errors greater than the probable error is equal to the number of errors less than it. Any error selected at random is just as likely to be greater as less than the probable error. Hence, the probable error is the value of x in the integral

$$\frac{1}{2} = \frac{2}{\sqrt{\pi}}\int_0^{hx} e^{-h^2x^2}d(hx), \quad . \quad . \quad . \quad (1)$$

page 432. From Table X., page 514, when $P = \frac{1}{2}$, $hx = 0·4769$; or, if r is the probable error,

$$hr = 0·4769. \quad . \quad . \quad . \quad . \quad (2)$$

Now it has already been shown that

$$y = \frac{hdx}{\sqrt{\pi}}e^{-h^2x^2}, \quad . \quad . \quad . \quad . \quad (3)$$

From page 418, therefore, the probability of the occurrence of the independent errors $x_1, x_2, \ldots, x_n$ is the product of their separate probabilities, or

$$P = \frac{h^n(dx)^n}{\sqrt{\pi^n}}e^{-h^2\Sigma(x^2)}. \quad . \quad . \quad . \quad (4)$$

For any set of observations in which the measurements have been as accurate as possible, h has a maximum value. Differentiating the last equation in the usual way, and equating dP/dh to zero,

$$h = \pm\sqrt{\frac{n}{2\Sigma(x^2)}}; \quad . \quad . \quad . \quad . \quad (5)$$

Substitute this in (2),

$$r = \pm\, 0·6745\sqrt{\frac{\Sigma(x^2)}{n}}. \quad . \quad . \quad . \quad (6)$$

But $\Sigma(x^2)$ is the sum of the squares of the true errors. The true errors are unknown. By the principle of least squares, when measurements have an equal degree of confidence, the most probable value of the observed quantities are those which render the sum of the squares of the deviations of each observation from the mean, a minimum. Let $\Sigma(v^2)$ denote the sum of the squares of the deviations of each observation from the mean. If n is large, we may put

$$\Sigma(x^2) = \Sigma(v^2);$$

but if n is a limited number,

$$\Sigma(v^2) < \Sigma(x^2),$$

$$\therefore \Sigma(x^2) = \Sigma(v^2) + u^2. \quad . \quad . \quad . \quad (7)$$

All we know about u^2 is that its value decreases as n increases, and increases when $\Sigma(x^2)$ increases. It is generally supposed that the best approximation to $u^2 \propto \{\Sigma(x^2)\}/n$, is to write

$$u^2 = \frac{\Sigma(x^2)}{n}; \ \therefore \frac{\Sigma(x^2)}{n} = \frac{\Sigma(v^2)}{n-1}.$$

(Compare u^2 with m^2 in the next section, § 179.) Hence,

$$r = \pm 0{\cdot}6747\sqrt{\frac{\Sigma(v^2)}{n-1}}, \quad . \quad . \quad . \quad (8)$$

which is virtually **Bessel's formula** for *the probable error of a single observation.* $\Sigma(v^2)$ denotes the sum of the squares of the numbers formed by subtracting each measurement from the arithmetical mean of the whole series, n denotes the number of measurements actually taken.

The *probable error of the arithmetical mean of the whole series of observations* is

$$R = \pm 0{\cdot}6745\sqrt{\frac{\Sigma(v^2)}{n(n-1)}}. \quad . \quad . \quad . \quad (9)$$

The derivation of this formula is given as an exercise at the end of § 179.

The last two results show that the probable error is diminished by increasing the number of observations.

(8) and (9) are only approximations. They have no signification when the number of observations is small. Hence we may write $\frac{2}{3}$ instead of 0·6745. For numerical applications, see next section.

The great labour involved in the squaring of the residual errors of a large number of observations may be avoided by the use of **Peter's approximation formula.** According to this, the probable error of a single observation is

$$r = \pm 0{\cdot}8453\frac{\Sigma(+v)}{\sqrt{n(n-1)}}, \quad . \quad . \quad . \quad . \quad (10)$$

where $\Sigma(+v)$ denotes the sum of the deviations of every observation from the mean, their sign being disregarded. The probable error of the arithmetical mean of the whole series of observations is

$$R = \pm 0{\cdot}8453\frac{\Sigma(+v)}{n\sqrt{n-1}}. \quad . \quad . \quad . \quad . \quad (11)$$

§ 179. Mean and Average Errors.

The arbitrary choice of the probable error for comparing the errors which are committed with equal facility in different sets of observations, appears most natural because the probable error occupies the middle place in a series arranged according to order of magnitude so that the number of errors less than the fictitious probable error, is the same as those which exceed it. There are other standards of comparison. In Germany, the favourite method is to employ the **mean error** ("der mittlere Fehler"), which is defined as *the error whose square is the mean of the squares of all the errors*, or the "error which, if it alone were assumed in all the observations indifferently, would give the same sum of the squares of the errors as that which actually exists".

We have seen in § 176, (5), that the ratio,

$$\frac{\text{Number of errors between } x \text{ and } x + dx}{\text{Total number of errors}} = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx.$$

Multiply both sides by x^2 and we obtain

$$\frac{\text{Sum of squares of errors between } x \text{ and } x + dx}{\text{Total number of errors}} = \frac{h}{\sqrt{\pi}}x^2e^{-h^2x^2}dx.$$

By integrating between the limits $+\infty$ and $-\infty$ we get

$$\frac{\text{Sum of squares of } all \text{ the errors}}{\text{Total sum of errors}} = \frac{\Sigma(x^2)}{n} = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty}x^2e^{-h^2x^2}dx.$$

Let m denote the mean error,

$$\therefore m^2 = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty}x^2e^{-h^2x^2}dx = \frac{1}{2h^2}; \quad . \quad . \quad (1)$$

For the integration, see § 108.

$$\therefore r = 0{\cdot}6745m. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (8) and (9) preceding section, the mean error which affects each single observation is given by the expression

$$m = \pm\sqrt{\frac{\Sigma(v^2)}{n-1}}; \quad . \quad . \quad . \quad . \quad (3)$$

and the mean error which affects the whole series of results,

$$M = \pm\sqrt{\frac{\Sigma(v^2)}{n(n-1)}}. \quad . \quad . \quad . \quad (4)$$

The mean error must not be confused with the "mean of the errors," or, as it is sometimes called, the **average error**,* another standard of comparison defined as the mean of all the errors regardless of sign. If a denotes the average error,

$$a = \frac{\Sigma(+v)}{n} = \frac{2h}{\sqrt{\pi}}\int_0^\infty xe^{-h^2x^2}dx = \frac{1}{h\sqrt{\pi}}; \ r = 0{\cdot}8453a. \qquad (5)$$

The average error measures the average deviation of each observation from the mean of the whole series. It is a more useful standard of comparison than the probable error when the attention is directed to the relative accuracy of the individual observations in different series of observations.

The average error depends not only upon the *proportion* in which the errors of different magnitudes occur, but also on the *magnitude* of the individual errors. The average error furnishes useful information even when the presence of (unknown) constant errors (§ 182) renders a further application of the "theory of errors" of questionable utility, because it will allow us to compare the magnitude of the constant errors affecting different series of observations, and so lead to their discovery and elimination (see § 182).

A COMMON FALLACY.—The way some investigators refer to the smallness of the probable error affecting their results conveys the impression that this canon has been employed to emphasise the accuracy of the work. As a matter of fact, *the probable error does not refer to the accuracy of the work* nor to the magnitude of the errors, but only to the proportion in which the errors of different magnitudes occur. *Cf.* page 467.

The reader will be able to show presently that the average error (A) affecting the mean of n observations is given by the expression

$$A = \pm\frac{\Sigma(+v)}{n\sqrt{n}}. \qquad (6)$$

This determines the effect of the average error of the individual observations upon the mean, and serves as a standard for comparing the relative accuracy of the means of different series of experiments made under similar conditions.

EXAMPLES.—Tables VI., VII., VIII., IX., will be found to save a great deal of labour in calculating the probable and mean errors of a series of observations.

---

* Some writers call our "average error" the "mean error," and our "mean error" the "error of mean square".

(1) The following galvanometer deflections were obtained in some observations on the resistance of a circuit: 37·0, 36·8, 36·8, 36·9, 37·1. Find the probable and mean errors. This small number of observations is employed simply to illustrate the method of using the above formulae. In practical work, mean or probable errors deduced from so small a number of observations are of little value.

Arrange the following table:—

Number of Observation.	Deflection Observed.	Departure from Mean.	v^2.
1	37·0	+ 0·08	0·0064
2	36·8	− 0·12	0·0144
3	36·8	− 0·12	0·0144
4	36·9	− 0·02	0·0004
5	37·1	+ 0·18	0·0324
Mean = 36·92; $\Sigma(v^2) = 0·0680$.			

The numbers in the last two columns have been calculated from those in the second.

Since $n = 5$, and writing $\frac{2}{3}$ for 0·6745,

Mean error of a single result $= \pm \sqrt{0·068/4} = \pm\ 0·0041$.

Mean error of the mean $= \pm \sqrt{0·068/5 \times 4} = \pm\ 0·0058$.

Probable error of a single result $= \pm\ \frac{2}{3} \sqrt{0·068/4} = \pm\ 0·0027$.

Probable error of the mean $= \pm\ \frac{2}{3} \sqrt{0·068/5 \times 4} = \pm\ 0·0035$.

Average error of a single result $= ·52/5 = ·104$.

Average error of the mean $= \pm\ ·52/5 \sqrt{5} = \pm\ ·0465$.

The mean error of the arithmetical mean of the whole set of observations is written,

$$36·92 \pm 0·0006;$$

the probable error,

$$36·92 \pm 0·0004.$$

It is unnecessary to include more than two significant figures.

(2) Rudberg (*Pogg. Ann.*, **41,** 271, 1837) found the coefficient of expansion α of dry air by different methods to be $\alpha \times 100$ = 0·3643, 0·3654, 0·3644, 0·3650, 0·3653, 0·3636, 0·3651, 0·3643, 0·3643, 0·3645, 0·3646, 0·3662, 0·3840, 0·3902, 0·3652. Required the probable and mean errors on the assumption that the results are worth an equal degree of confidence.

(3) From example (3), page 135, show that the mean error is the abscissa of the point of inflection of the probability curve. For simplicity, put $h = 1$.

(4) Cavendish has published the result of 29 determinations of the mean density of the earth (*Phil. Trans.*, **88,** 469, 1798) in which the first significant figure of all but one is 5:— 4·88; 5·50; ·61; ·07; ·26; ·55; ·36; ·29; ·58; ·65; ·57; ·53; ·62; ·29; ·44; ·34; ·79; ·10; ·17; ·39; ·42; ·47; ·63; ·34; ·46; ·30; ·75; ·68; ·85. Verify the following results: Mean = 5·45; $\Sigma(+v) = 5·04$; $\Sigma(v^2) = 1·367$; $M = \pm\ ·041$; $m = \pm\ ·221$; $R = \pm\ ·0277$; $r = \pm\ ·149$; $a = ·18$; $A = \pm\ ·033$.

The following results are convenient for reference:

1. *The mean (or probable error) of the sum of a number of observations is equal to the square root of the sum of the squares of the mean (or probable) errors of each of the observations.*

Let x_1, x_2, represent two independent measurements whose sum, or difference combines to make a final result X, so that

$$X = x_1 + x_2.$$

Let the mean errors of x_1 and x_2, be m_1 and m_2 respectively. If M denotes the mean error in X,

$$X \pm M = (x_1 \pm m_1) + (x_2 \pm m_2).$$
$$\therefore \pm M = \pm m_1 \pm m_2.$$

However we arrange the signs of M, m_1, m_2, in the last equation, we can only obtain, by squaring, one or other of the following expressions:

$$M^2 = m_1^2 + 2m_1m_2 + m_2^2;\ \text{or},\ M^2 = m_1^2 - 2m_1m_2 + m_2^2,$$

it makes no difference which. Hence the mean error is to be found by taking the mean of both these results. That is to say,

$$M^2 = m_1^2 + m_2^2;\ \text{or},\ M = \sqrt{m_1^2 + m_2^2},$$

because the terms containing $+ m_1m_2$ and $- m_1m_2$ cancel each other. This means that the products of any pair of residual errors $(m_1m_2, m_1m_3, \ldots)$ in an extended series of observations will have positive as often as negative signs. Consequently, the influence of these terms on the mean value will be negligibly small in comparison with the terms $m_1^2, m_2^2, m_3^2, \ldots$, which are always positive. Hence, for any number of observations,

$$M^2 = m_1^2 + m_2^2 + \ldots;\ \text{or},\ M = \sqrt{(m_1^2 + m_2^2 + \ldots)}. \quad . \quad (7)$$

From equation (2), page 439, the mean error is proportional to the probable error R, m_1 to $r_1, \ldots$, hence,

$$R^2 = r_1^2 + r_2^2 + \ldots;\ \text{or},\ R = \sqrt{(r_1^2 + r_2^2 + \ldots)}. \quad . \quad . \quad (8)$$

In other words, *the probable error of the* SUM *or* DIFFERENCE *of two quantities A and B respectively affected with probable errors* $\pm$ *a and* $\pm$ *b is*

$$R = \pm \sqrt{a^2 + b^2}. \quad . \quad . \quad . \quad . \quad (9)$$

EXAMPLES.—(1) The molecular weight of titanium chloride ($TiCl_4$) is known to be 188·545 ± ·0092, and the atomic weight of chlorine 35·179 ± ·0048, what is the atomic weight of titanium? Ansr. 47·829 ± ·0213. Hints.

$$188{\cdot}545 - 4 \times 35{\cdot}179 = 47{\cdot}829;\ R = \sqrt{(\cdot 0092)^2 + (4 \times \cdot 0048)^2} = \pm \cdot 0213.$$

(2) The mean errors affecting θ_1 and θ_2 in the formula $R = k(\theta_2 - \theta_1)$ are respectively ± ·0003 and ± ·0004, what is the mean error affecting $\theta_2 - \theta_1$ and $3(\theta_2 - \theta_1)$? Ansr. ± ·0005 and ± 0015.

2. *The probable error of the* PRODUCT *of two quantities A and B respectively affected with the probable errors* $\pm$ *a and* $\pm$ *b is*

$$R = \pm \sqrt{(Ab)^2 + (Ba)^2}. \quad . \quad . \quad . \quad (10)$$

EXAMPLES.—(1) Thorpe found that the molecular ratio

$$4Ag : TiCl_4 = 100 : 44{\cdot}017 \pm {\cdot}0031.$$

Hence determine the molecular weight of titanium tetrachloride, given the atomic weight of silver $= 107{\cdot}108 \pm {\cdot}0031$. Ansr. $188{\cdot}583 \pm {\cdot}0144$. Hint.

$$R = \pm \sqrt{\{(4 \times 107{\cdot}108 \times {\cdot}0031)^2 + (44{\cdot}017 \times 4 \times {\cdot}0031)^2\}}.$$

(2) The specific heat of tin is ·0537 with a mean error of $\pm$ ·0014, and the atomic weight of the same metal is $118{\cdot}150 \pm {\cdot}0089$, show that the mean error the product of these two quantities (Dulong and Petit's law) is $6{\cdot}38 \pm {\cdot}1654$.

If a third mean, C, with a probable error, $\pm c$, is included,

$$R = \pm \sqrt{(BCa)^2 + (ACb)^2 + (ABc)^2}. \quad . \quad . \quad . \quad (11)$$

3. *The probable error of the* QUOTIENT $(B \div A)$ *of two quantities* A *and* B *respectively affected with the probable errors* $\pm a$ *and* $\pm b$ *is*

$$R = \pm \frac{\sqrt{\left(\frac{Ba}{A}\right)^2 + b^2}}{A}. \quad . \quad . \quad . \quad (12)$$

EXAMPLES.—(1) It is known that the atomic ratio

$$Cu : 2Ag = 100 : 339{\cdot}411 \pm {\cdot}0039,$$

what is the atomic weight of copper on the assumption that

$$Ag = 107{\cdot}108 \pm {\cdot}0031\,?$$

Ansr. $63{\cdot}114 \pm {\cdot}0020$. Hint.

$$R = \pm \sqrt{\left(\frac{214{\cdot}216 \times {\cdot}0039}{339{\cdot}411}\right)^2 + ({\cdot}0062)^2} \div 339{\cdot}411 = \pm {\cdot}0020.$$

$$Cu : 2 \times 107{\cdot}108 = 100 : 339{\cdot}411; \ \therefore Cu = 63{\cdot}114.$$

(2) Suppose that the maximum pressure of the aqueous vapour (f_2) in the atmosphere at 16° is found to be 8·2, with a mean error $\pm$ ·0024, and the maximum pressure of aqueous vapour (f_1) at the dewpoint, at 16°, is 13·5, with a mean error of $\pm$ ·0012. The relative humidity (h) of the air is given by the expression $h = f_1/f_2$. Show that the relative humidity at 16° is $\cdot 6074 \pm {\cdot}0022$.

4. *The probable error of the* PROPORTION

$$A : B = C : x,$$

where A, B, C, *are quantities respectively affected with the probable errors* $\pm a$, $\pm b$, $\pm c$, *is*

$$R = \pm \frac{\sqrt{\left(\frac{BCa}{A}\right)^2 + (Cb)^2 + (Bc)^2}}{A}. \quad . \quad . \quad (13)$$

EXAMPLE.—Stas found that $AgClO_3$ furnished $25{\cdot}080 \pm {\cdot}0010$ °/₀ of oxygen and $74{\cdot}920 \pm {\cdot}0003$ °/₀ of $AgCl$. If the atomic weight of oxygen is $15{\cdot}879 \pm {\cdot}0003$, what is the molecular weight of $AgCl$? Ansr. $142{\cdot}303 \pm {\cdot}0066$. Hints.

$$25{\cdot}080 : 74{\cdot}920 = 3 \times 15{\cdot}879 : x; \ \therefore x = 142{\cdot}303.$$

$$R = \pm \sqrt{\left\{\left(\frac{74{\cdot}92 \times 47{\cdot}637 \times {\cdot}001}{25{\cdot}08}\right) + (47{\cdot}637 \times {\cdot}001)^2 + (74{\cdot}92 \times 3 \times {\cdot}0009)^2\right\}} \div 25{\cdot}08.$$

If
$$A : B = C + x : D + x,$$

$$R = \pm \sqrt{\frac{(C - D)^2}{(A - B)^4}(B^2a^2 + A^2b^2) + \frac{B^2c^2 + A^2d^2}{(A - B)^2}}. \quad . \quad . \quad . \quad (14)$$

EXAMPLE.—Stas found that $31·488 \pm ·0006$ grams of NH_4Cl were equivalent to 100 grams of $AgNO_3$. Hence determine the atomic weight of nitrogen, given $Ag = 107·108 \pm ·0031$; $Cl = 35·179 \pm ·0048$; $H = 1$; $O_3 = 47·637 \pm ·0009$. Ansr. $13·911 \pm ·0048$.

5. *The probable error of the arithmetical mean of a series of observations is inversely as the square root of their number.*

Let $r_1, r_2, \ldots, r_n$ be the probable errors of a series of independent observations $a_1, a_2, \ldots, a_n$, which have to be combined so as to make up a final result u. Let the probable errors be respectively proportional to the actual errors $da_1, da_2, \ldots, da_n$. The final result u is a function such that

$$u = f(a_1, a_2, \ldots, a_n).$$

The influence of each separate variable on the final result may be determined by partial differentiation so that

$$du = \frac{\partial u}{\partial a_1}da_1 + \frac{\partial u}{\partial a_2}da_2 + \ldots, \quad . \quad . \quad . \quad . \quad (15)$$

where $da_1, da_2, \ldots$ represent the actual errors committed in measuring a_1, $a_2, \ldots$; the partial differential coefficients determine the effect of these variables upon the final result u; and du represents the actual error in u due to the joint occurrence of the errors $da_1, da_2, \ldots$

If we employ R in place of du, r_1 in place of da_1, etc., square (15) and show that

$$R^2 = \left(\frac{\partial u}{\partial a_1}\right)^2 r_1^2 + \left(\frac{\partial u}{\partial a_2}\right)^2 r_2^2 + \ldots \quad . \quad . \quad . \quad (16)$$

The arithmetical mean of n observations is

$$u = (a_1 + a_2 + \ldots + a_n)/n,$$

therefore, $\partial u/\partial a_1 = \partial u/\partial a_2 = \ldots = 1/n$.

$$\therefore R^2 = \frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n^2}.$$

But the observations have an equal degree of precision, and therefore, $r_1^2 = r_2^2 = \ldots = r_n^2$.

$$\therefore R = \pm \sqrt{\frac{nr_1^2}{n^2}} = \pm \frac{r}{\sqrt{n}}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This result shows how easy it is to overrate the effect of multiplying observations. If R denotes the probable error of the mean of 8 observations, four times as many, or 32 observations must be made to give a probable error of $\frac{1}{2}R$; nine times as many, or 72 observations must be made to reduce R to $\frac{1}{3}R$, etc.

EXAMPLES.—(1) Two series of determinations of the atomic weight of oxygen by a certain process gave respectively $15·8726 \pm ·00058$ and $15·8769 \pm ·00058$. Hence show that the atomic weight is accordingly written $15·87475 \pm ·00041$.

(2) In the preceding section, § 178, given formula (8) deduce (9). Hint. Use (17), present section.

(3) Deduce Peter's approximation formulae (10) and (11), § 178. Hint. Since

$$\Sigma(x^2)/n = \Sigma(v^2)/(n-1),$$

page 438, we may suppose that on the average

$$\Sigma(x) : \sqrt{n} = \Sigma(v) : \sqrt{n-1},$$

etc.

(4) Show that when n is large, the result of dividing the mean of the squares of the errors by the square of the mean of the errors is constant. Hint. Show that

$$\frac{\Sigma(v^2)}{n} \div \left(\frac{\Sigma(v)}{n}\right)^2 = \frac{\pi}{2} = 1{\cdot}57. \quad . \quad . \quad . \quad . \quad (18)$$

This has been proposed as a test of the fidelity of the observations, and of the accuracy of the arithmetical work. For instance, the numbers quoted in the example on page 468 give $\Sigma(v) = 55{\cdot}53$; $\Sigma(v^2) = 354{\cdot}35$; $n = 14$; constant $= 1{\cdot}60$. The canon does not usually work very well with a small number of observations.

(5) Show that the probable (or mean) error is inversely proportional to the absolute measure of precision. Hint. From (1) and (2)

$$r = \frac{1}{h} \times \text{constant}, \quad . \quad . \quad . \quad . \quad (19)$$

etc. See § 190.

§ 180. Numerical Values of the Probability Integrals.

We have discussed the two questions:

1. What is the best representative value of a series of measurements affected with errors of observations?

2. How nearly does the arithmetical mean represent all of a given set of measurements affected with errors of observation?

It now remains to inquire

3. *How closely does the arithmetical mean approximate to the absolute truth?*

To illustrate, we may use the results of Crookes' model research on the atomic weight of thallium (*Phil. Trans.*, **163**, 277, 1874):

$$\left.\begin{matrix} 203{\cdot}628;\ 203{\cdot}632;\ 203{\cdot}636;\ 203{\cdot}638;\ 203{\cdot}639; \\ 203{\cdot}642;\ 203{\cdot}644;\ 203{\cdot}649;\ 203{\cdot}650;\ 203{\cdot}666; \end{matrix}\right\} \text{Mean}: 203{\cdot}642.$$

The arithmetical mean is only one of an infinite number of possible values of the atomic weight of thallium between the extreme limits 203·628 and 203·666. It is very probable that 203·642 is *not* the true value, but it is also very probable that 203·642 is *very near* to the true value sought. The question "How near?" cannot be answered. Alter the question to "What is the probability that the truth is comprised between the limits $203{\cdot}642 \pm x$?" and the answer may be readily obtained however small we choose to make the number x.

First, suppose that the absolute measure of precision (h) *of the arithmetical mean is known.*

Table X. gives the numerical values of the probability integral

$$P = \frac{2}{\sqrt{\pi}}\int_0^{hx} e^{-h^2x^2}d(hx),$$

where P denotes the probability that an error of observation will have a positive or negative value equal to or less than x, h is the measure of the degree of precision of the results.

When h is unity, the value of P is read off from the table directly. To illustrate, we read that when $x = \pm\ 0{\cdot}1$ $P = {\cdot}112$; when $x = \pm\ 0{\cdot}2$ $P = {\cdot}223$; . . ., meaning that if 1,000 errors are committed in a set of observations with a modulus of precision $h = 1$, 112 of the errors will lie between $+\ 0{\cdot}1$ and $-\ 0{\cdot}1$, 223 between $+\ 0{\cdot}2$ and $-\ 0{\cdot}2$, etc. Or, 888 of the errors will exceed the limits $\pm\ 0{\cdot}1$; 777 errors will exceed the limits $\pm\ 0{\cdot}2$; . . .

When h is not unity, we must use $\frac{0{\cdot}1}{h}, \frac{0{\cdot}2}{h}, \ldots$, in place of $0{\cdot}1, 0{\cdot}2, \ldots$

Examples.—(1) If $hx = 0{\cdot}64$, P, from the table, is $0{\cdot}6346$. Hence $0{\cdot}6346$ denotes the probability that the error x will be less than $0{\cdot}64/h$, that is to say, $63{\cdot}46\ {}^0/_0$ of the errors will lie between the limits $\pm\ 0{\cdot}64/h$. The remaining $36{\cdot}54\ {}^0/_0$ will lie outside these limits.

(2) Required the probability that an error will be comprised between the limits $\pm\ 0{\cdot}3$ ($h = 1$). Ansr. ${\cdot}329$.

(3) Required the probability that an error will lie between $-\ 0{\cdot}01$ and $+\ 0{\cdot}1$ of say a milligram. This is the sum of the probabilities of the limits from 0 to $-\ 0{\cdot}01$ and from 0 to $+\ 0{\cdot}1$ ($h = 1$). Ansr. ${\cdot}0113 + {\cdot}1125 = {\cdot}1237$.

(4) Required the probability that an error will lie between $+1{\cdot}0$ and $+0{\cdot}01$. This is the difference of the probabilities of errors between $1{\cdot}0$ and zero and between $0{\cdot}01$ and zero ($h = 1$). Ansr. ${\cdot}8427 - {\cdot}0113 = {\cdot}8314$.

This table, therefore, enables us to find the relation between the magnitude of an error and the frequency with which that error will be committed in making a large number of careful measurements. It is usually more convenient to work from the probable error R than from the modulus h. More practical illustrations have, in consequence, been included in the next set of examples.

Second, suppose that the probable error of the arithmetical mean is known.

Table XI. gives the numerical values of the probability integral

$$P = \frac{2}{\sqrt{\pi}}\int_0^{x/r} e^{-\left(\frac{x}{r}\right)^2} d\left(\frac{x}{r}\right),$$

where P denotes the probability that an error of observation of a positive or negative value, equal to or less than x, will be committed in the arithmetical mean of a series of measurements with probable error r (or R). This table makes no reference to h. To illustrate its use, of 1,000 errors, 54 will be less than $\frac{1}{10}R$; 500 less than R; 823 less than $2R$; 957 less than $3R$; 993 less than $4R$; and one will be greater than $5R$.

EXAMPLES.—(1) A series of results are represented by 6·9 with a probable error $\pm$ 0·25. The probability that the probable error is less than 0·25 is $\frac{1}{2}$. What is the probability that the actual error will be less than 0·75? Here $x/R = 0·75/0·25 = 3$. From the table, $p = 0·9570$ when $x/R = 3$. This means that 95·7 °/$_\circ$ of the errors will be less than 0·75 and 4·3 °/$_\circ$ greater.

(2) Dumas has recorded the following 19 determinations of the chemical equivalent of hydrogen ($O = 100$) using sulphuric acid (H_2SO_4) with some, and phosphorus pentoxide (P_2O_5) as the drying agent in other cases:

i. H_2SO_4: 12·472, 12·480, 12·548, 12·489, 12·496, 12·522, 12·533, 12·546, 12·550, 12·562;

ii. P_2O_5: 12·480, 12·491, 12·490, 12·490, 12·508, 12·547, 12·490, 12·551, 12·551. Dumas' "Recherches sur la Composition de l'Eau," *Ann. de Chim. et de Phys.* [3], **8,** 200, 1843.

What is the probability that there will be an error between the limits $\pm$ 0·015 in the mean (12·515), assuming that the results are free from constant errors? The chemical student will perhaps see the relation of his answer to Prout's law.

Hints, $x/R = t$; $R = ·004685$; $x \mp ·015$; $\therefore t = 3·2$. From Table XI., when $t = 3·2$, $P = ·969$. Hence the odds are 969 to 31 that the mean 12·515 is affected by no greater error than is comprised within the limits $\pm$ ·015. To exemplify Table X., $h = ·4769/R = 102$, $\therefore hx = 102 \times ·015 = 1·53$. From the table, $P = ·969$ when $hx = 1·53$, etc. That is to say, 96·9 °/$_\circ$ of the errors will be less and 3·1 °/$_\circ$ greater than the assigned limits.

(3) From Crookes' ten determinations of the atomic weight of thallium (above) calculate the probability that the atomic weight of thallium lies between 203·632 and 203·652. Here $x = \pm$ 0·01; $R \pm = ·0023$; $\therefore t = x/R = 4·4$. From Table XI., $P = ·997$. (Note how near this number is to unity indicating certainty.) The chances are 332 to 1 that the true value of the atomic weight of thallium lies between 203·632 and 203·652. We get the same result by means of Table X. Thus $h = ·4769/·0023 = 207$; $\therefore hx = 207 \times ·01 = 2·07$. When $hx = 2·07$, $P = ·997$. If 1,000 observations were made under the same conditions as Crookes', we could reasonably expect 997 of them to be affected

by errors numerically less than 0·01, and only 3 observations would be affected by errors exceeding these limits.

The rules and formulae deduced up to the present are by no means inviolable. The reader must constantly bear in mind the fundamental assumptions upon which we are working. If these conditions are not fulfilled, the conclusions may not only be superfluous, but even erroneous. The necessary conditions are :

1. Every observation is as likely to be in error as every other one.
2. There is no perturbing influence to cause the results to have a bias or tendency to deviate more in some directions than in others.
3. A large number of observations has been made. In practice, the number of observations may be considerably reduced if the second condition is fulfilled. In the ordinary course of things from 10 to 25 is usually considered a sufficient number.

§ 181. Maxwell's Law of Distribution of Molecular Velocities.

In a preceding discussion, the velocities of the molecules of a gas were assumed to be the same. Can this simplifying assumption be justified?

According to the kinetic theory, a gas is supposed to consist of a number of perfectly elastic spheres moving about in space with a certain velocity. In case of impact on the walls of the bounding vessel, the molecules are supposed to rebound according to known dynamical laws. This accounts for the pressure of a gas.

The velocities of all the molecules of a gas in a state of equilibrium are not the same. Some move with a greater velocity than others. At one time a molecule may be moving with a great velocity, at another time, with a relatively slow speed.

The attempt has been made to find a law governing the distribution of the velocities of the motions of the different molecules, and with some success. Maxwell's law is based upon the assumption that the same relations hold for the velocities of the molecules as for errors of observation. This assumption has played a most important part in the development of the kinetic theory of gases.

The probability y that a molecule will have a velocity equal to x is given by an expression of the type :

$$y = \frac{4}{\sqrt{\pi}}x^2e^{-x^2}. \qquad (1)$$

A graphical representation of this law is readily obtained by plotting corresponding values of x and y in the usual way.

Very few molecules will have velocities outside a certain restricted range. It is possible for a molecule to have any velocity whatever but the probability of the existence of velocities outside certain limits is vanishingly small.

The reader will get a better idea of the distribution of the velocities of the molecules by plotting the graph of the above equation for himself. Remember that the ordinates are proportional to the number of molecules, abscissae to their speed. Areas bounded by the x-axis, the curve and certain ordinates will give an idea of the number of molecules possessing velocities between the abscissae corresponding to the boundary ordinates. Use Table XXIII.

Returning to the study of the kinetic theory of gases, § 173, the number of molecules with velocities between v and $v + dv$ is assumed to be represented by an equation analogous to the expression employed to represent the errors of mean square in § 179, namely,

$$dN = \frac{4N}{\sqrt{\pi}}\left(\frac{v}{a}\right)^2 e^{-\left(\frac{v}{a}\right)^2} d\left(\frac{v}{a}\right), \quad . \quad . \quad . \quad (2)$$

where N represents the total number of molecules, a is a constant to be evaluated.

1. *To find a value for the constant* a *in terms of the average velocity* (V_0) *of the molecules.*

Since there are dN molecules with a velocity v, the sum of the velocities of all these dN molecules is vdN, and the sum of the velocities of *all* the molecules must be

$$\int_{v=0}^{v=\infty} v \,.\, dN.$$

From (2),

$$V_0 = \frac{4}{a^3\sqrt{\pi}}\int_0^{\infty} v^3 \,.\, e^{-\frac{v^2}{a^2}}dv = \frac{4}{a^3\sqrt{\pi}} \cdot \frac{a^4}{2} = \frac{2a}{\sqrt{\pi}}.$$

(How did N vanish?) Hence,

$$a = \tfrac{1}{2}V_0\sqrt{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. *To find the average velocity of the molecules of a gas.*

By a well-known theorem in elementary mechanics, the kinetic energy of a mass m moving with a velocity v is $\frac{1}{2}mv^2$. Hence, the sum of the kinetic energies of the dN molecules will be $\frac{1}{2}(mdN)v^2$, because there are dN molecules moving with a velocity v. From (2), therefore, the total kinetic energy (T) of *all* the molecules is

$$T = \int_{v=0}^{v=\infty} \tfrac{1}{2}mv^2 \,.\, dN = \frac{2Nm}{a^3\sqrt{\pi}}\int_0^\infty v^4 \,.\, e^{-\frac{v^2}{a^2}}dv\,;$$
$$= \tfrac{3}{4}Nma^2 = \tfrac{3}{4}Ma^2.$$
$$\therefore a = 2\sqrt{\tfrac{1}{3}T/M}, \qquad (4)$$

where $M = Nm$ = total mass of N molecules each of mass m.

The total kinetic energy of N molecules of the same kind is

$$T = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 + \ldots + \tfrac{1}{2}mv_N^2 = \tfrac{1}{2}m(v_1^2 + v_2^2 + \ldots + v_N^2). \qquad (5)$$

The velocity of mean square (U) is defined as the velocity whose square is the average of the squares of the velocities of all the N molecules, or,

$$U^2 = (v_1^2 + v_2^2 + \ldots + v_N^2)/N.$$

From (5), therefore,

$$T = \tfrac{1}{2}mNU^2 = \tfrac{1}{2}MU^2. \qquad (6)$$

From (4) and (6), therefore, Or(3)

$$a = \frac{2U}{\sqrt{6}}; \text{ and } V_0 = \frac{4U}{\sqrt{6\pi}} = \cdot 9213U. \qquad (7)$$

Most works on chemical theory * give a simple method of proving that if p denotes the pressure and ρ the density of a gas,

$$p = \tfrac{1}{3}\rho U^2. \qquad (8)$$

This in conjunction with (6) allows the average velocity of the molecules of a gas to be calculated from the known values of the pressure and density of the gas.

Numerical Example.—One c.c. of hydrogen gas weighs ·0000896 grams under standard barometric pressure, 76 cm. of mercury. Specific gravity of mercury = 13·5. Hence, a column of mercury 76 cm. long and 1 sq. cm. cross-section weighs 76 × 13·5 = 1033·2 grams. But,

Weight = Mass × Acceleration of gravity,

$$\rho = \text{Density} = \text{Mass of unit volume} = \frac{\text{Weight of unit volume}}{g};$$
$$= \cdot 0000896/981 = \cdot 000009.$$

From (7) and (8),

$$V_0 = \cdot 9213\sqrt{\frac{3p}{\rho}} = \cdot 9213\sqrt{\frac{3 \times 1033\cdot 2}{\cdot 000009}} = 184{,}000.$$

That is to say, the average velocity of hydrogen molecules under atmospheric pressure at 0° C. is approximately 184,000 centimetres per second.

3. *To show that the average velocity of the molecules of a gas is proportional to its rate of diffusion.*

This will be left as an exercise. Hint. Use (7) and (8) above, and (1), § 95. See also (2), § 190.

The reader is no doubt familiar with the principle underlying Maxwell's law, and, indeed, the whole kinetic theory of gases. I

* *E.g.*, Ramsay's *Experimental Proofs of Chemical Theory for Beginners.*

may mention two examples. The number of passengers on say the 3.10 P.M. suburban daily train is fairly constant in spite of the fact that that train does not carry the same passengers two days running. Insurance companies can average the number of deaths per 1,000 of population with great exactness. Of course I say nothing of disturbing factors. A bank holiday may require provision for a supra-normal traffic. An epidemic will run up the death rate of a community. The commercial success of these institutions is, however, sufficient testimony of the truth of the **method of averages,** otherwise called the **statistical method** of investigation. The same type of mathematical expression is required in each case.

It will thus be seen that calculations, based on the supposition that all the molecules possess equal velocities, are quite admissible in a first approximation. The net result of the "dance of the molecules" is a distribution of the different velocities among all the molecules, which is maintained with great exactness.

G. H. Darwin has deduced values for the mean free path, etc., from the hypothesis that the molecules of the same gas are not all the same size. He has examined the consequences of the assumption that the sizes of the molecules are ranged according to a law like that governing the frequency of errors of observation. For this, see his memoir "On the mechanical conditions of a swarm of meteorites" (*Phil. Trans.*, **180,** 1, 1889).

§ 182. Constant Errors.

The irregular accidental errors hitherto discussed have this distinctive feature, they are just as likely to have a positive as a negative value. But there are errors which have not this character. If the barometer vacuum is imperfect, every reading will be too small; if the glass bulb of a thermometer has contracted after graduation, the zero point rises in such a way as to falsify all subsequent readings; if the points of suspension of the balance pans are at unequal distances from the centre of oscillation of the beam, the weighings will be inaccurate. A change of temperature of 5° or 6° may easily cause an error of 0·2 to 1·0 °/₀ in an analysis, owing to the change in the volume of the standard solution. Such defective measurements are said to be affected by **constant errors.*** By definition, constant errors are produced by well-defined causes which make the errors of observation preponderate more in one direction than in another. Thus, some of Stas' determinations of the atomic weight of silver are affected by a constant error due to the occlusion of oxygen by metallic silver in the course of his work.

One of the greatest trials of an investigator is to detect and if possible eliminate constant errors. This is usually done by modifying the conditions under which the experiments are performed. Thus the magnitude is measured under different conditions, with different instruments, etc. It is assumed that even though each method or apparatus has its own specific constant error, all these constant errors taken collectively will have the character of accidental errors. To take a concrete illustration, faulty "sights" on a rifle may cause a constant deviation of the bullets in one direction; the "sights" on another rifle may cause a constant "error" (§ 174) in another direction, and so, as the number of rifles increases, the constant errors assume the character of accidental errors and thus, in the long run, tend to compensate each other. This is why Stas generally employed several different methods to determine his atomic weights. To quote one practical case, Stas made two sets of determinations of the numerical value of the ratio $Ag : KCl$. In one set, four series of determinations were made with KCl prepared from four different sources in conjunction with one specimen of silver, and in the other set different series of experiments were made with silver prepared from different sources in conjunction with one sample of KCl.†

The calculation of an arithmetical mean is analogous to the process of guessing the centre of a target from the distribution of the "hits" (Fig. 127). If all the shots are affected by the same constant error, the centre, so estimated, will deviate from the true centre by an amount depending on the magnitude of the (presumably

* **Personal error.** This is another type of constant error which depends on the personal qualities of the observer. Thus the differences in the judgments of the astronomers at the Greenwich Observatory as to the observed time of transit of a star and the assumed instant of its actual occurrence is said to vary from $\frac{1}{100}$ to $\frac{1}{3}$ of a second, and to remain fairly constant for the same observer. Some persistently read the burette a little high, others a little low. Vernier readings, analyses based on colorimetric tests (such as Nessler's ammonia process), etc., may be affected by personal errors.

† Unfortunately the latter set was never completed.

unknown) constant error. If this magnitude can be subsequently determined, a simple arithmetical operation (addition or subtraction) will give the correct value. Thus Stas found that the amount of potassium chloride equivalent to 100 parts of silver, in one case, was as

$$Ag : KCl = 100 : 69{\cdot}1209.$$

The KCl was subsequently found to contain ·00259 °/$_{\circ}$ of silica. The chemical student will see that ·00087 has consequently to be subtracted from 69·1209. Hence,

$$Ag : KCl = 100 : 69{\cdot}11903.$$

After Lord Rayleigh (*Proc. Roy. Soc.*, **43**, 356, 1888, or rather Agamennone in 1885) pointed out that the capacity of an exhausted glass globe is less than when the globe is full of gas, all measurements of the densities of gases involving the use of exhausted globes had to be corrected for shrinkage. Thus Regnault's ratio, 1 : 15·9611, for the relative densities of hydrogen and oxygen was "corrected for shrinkage" to 1 : 15·9105. The proper numerical corrections for the constant errors of a thermometer are indicated on the well-known "Kew certificate," etc.

If the mean error of each set of results differs, by an amount to be expected, from the mean errors of the different sets measured with the same instrument under the same conditions, no constant error is likely to be present. The different series of atomic weight determinations of the same chemical element, published by the same (perhaps excluding Stas) or by different observers, do not stand this test satisfactorily. Hence, Ostwald concludes that constant errors must have been present even though they have escaped the experimenter's ken.

Example.—Discuss the following: "Merely increasing the *number* of experiments, without varying the conditions or method of observation, diminishes the influence of accidental errors. It is, however, useless to multiply the number of observations beyond a certain limit. On the other hand, the greater the *number and variety* of the observations, the more complete will be the elimination of the effects of both constant and accidental errors."

§ 183. Proportional Errors.

One of the greatest sources of error in scientific measurements occurs when the quantity cannot be measured directly. In such cases, two or more separate observations may have to be made on different magnitudes. Each observation contributes some little inaccuracy to the final result. Thus Faraday has determined the *thickness* of gold leaf from the *weight* of a certain number of sheets. Foucault measures *time*, Le Chatelier measures *temperature* in terms of an *angular deviation*. The determination of the *rate of a chemical reaction* often depends on a number of more or less troublesome *analyses*.*

For this reason, among others, many chemists prefer the standard $O = 16$ as the basis of their system of atomic weights. The atomic weights of most of the elements have been determined directly or indirectly with reference to oxygen. If $H = 1$ be the basis, the atomic weights of most of the elements depend on the nature of the relation between oxygen and hydrogen—a relation which has not yet been fixed in a satisfactory manner. The best determinations made since 1887 vary between $H : O = 1 : 15·96$ and $H : O = 1 : 15·87$. If the former ratio be adopted, the atomic weights of antimony and uranium would be respectively 119·6 and 239·0; while if the latter ratio be employed, these units become respectively 118·9 and 237·7, a difference of one and two units! It is, therefore, better to contrive that the atomic weights of the elements do not depend on the uncertainty of the ratio $H : O$, by adopting the basis: $O = 16$.

If the quantity to be determined is deduced by calculation from a measurement, Taylor's theorem furnishes a convenient means of criticising the conditions under which any proposed experiment is to be performed, and at the same time furnishes a valuable insight into the effect of an error in the measurement on the whole result.

It is of the greatest importance that every investigator should

---

* Indirect results are liable to another source of error. The formula employed may be so inexact that accurate measurements give but grossly approximate results. For instance, a first approximation formula may have been employed when the accuracy of the observations required one more precise; $\pi = \frac{22}{7}$ may have been put in place of $\pi = 3·14159$; or the coefficient of expansion of a perfect gas has been applied to an imperfect gas. Such errors are called **errors of method.**

There is a well-defined distinction between the approximate values of a physical constant, which are seldom known to more than three or four significant figures (see § 189), and the approximate value of the incommensurables π, e, $\sqrt{2}$, . . . which can be calculated to any desired degree of accuracy. If we use $\frac{22}{7}$ in place of 3·1416 for π, the **absolute error** is greater than or equal to 3·1426 – 3·1416, and equal to or less than 3·1428 – 3·1416; that is, between ·0012 and ·0014. In scientific work we are not concerned with absolute errors although it is assumed that the proportional error is an approximate representation of the ratio of the absolute error to the true value of the magnitude.

By the way, "$\succ\!\!-$" is a convenient abbreviation used in place of the phrase "is greater than or equal to," and "$-\!\!\prec$" is used in place of "is equal to or less than". See page 10.

have a clear idea of the different sources of error to which his results are liable in order to be able to discriminate between important and unimportant sources of error, and to find just where the greatest attention must be paid in order to obtain the best results.

Let y be the desired quantity to be calculated from a magnitude x which can be measured directly and is connected with y by the relation

$$y = f(x).$$

$f(x)$ is always affected with some error dx which causes y to deviate from the truth by an amount dy. The error will then be

$$dy = (y + dy) - y = f(x + dx) - f(x).$$

dx is necessarily a small magnitude, therefore, by Taylor's theorem,

$$f(x + dx) = f(x) + f'(x) . dx + \ldots,$$

or, neglecting the higher orders of magnitude,

$$dy = f'(x) . dx.$$

The relation between the error and the total magnitude of y is

$$\frac{dy}{y} = \frac{f'(x) . dx}{f(x)}. \quad . \quad . \quad . \quad . \quad (1)$$

The ratio dy/y is called the **proportional, relative, or fractional error,*** that is to say, the ratio of the error involved in the whole process to the total quantity sought.

Students often fail to understand why their results seem all wrong when the experiments have been carefully performed and the calculations correctly done. For instance, the molecular weight of a substance is known to be either 160, or some multiple of 160. To determine which, ·380 (or w) grm. of the substance was added to 14·01 (or w_1) grms. of acetone boiling at $\theta_1{}^\circ$ (or 3·50°) on Beckmann's arbitrary scale, the temperature, in consequence, fell to $\theta_2{}^\circ$ (or 3·36°); the molecular weight of the substance (M) is then represented by the known formula

$$M = 1670\frac{w}{w_1(\theta_1 - \theta_2)}; \text{ or, } M = 1670\frac{\cdot 380}{14\cdot 01 \times \cdot 14} = 323,$$

or approximately 2×160. Now assume that the temperature readings may be $\pm$ 0·05° in error owing to convection currents, radiation and conduction of heat, etc. Let $\theta_1{}^\circ = 3\cdot 55^\circ$ and $\theta_2{}^\circ = 3\cdot 31^\circ$,

$$\therefore M = 1670\frac{\cdot 380}{14\cdot 01 \times \cdot 24} = 188.$$

* 100 . dy/y is the **percentage error.**

This means that an error of $\pm \frac{1}{20}^\circ$ in the reading of the thermometer would give a result positively misleading. This example is by no means exaggerated. The simultaneous determination of the heat of fusion and of the specific heat of a solid by the solution of two simultaneous equations, and the determination of the latent heat of steam are specially liable to similar mistakes. A study of the reduction formula will show in every case that relatively small errors in the reading of the temperature are magnified into serious dimensions by the method used in the calculation of the final result.

EXAMPLE.—The radius of curvature (r) of a lens, is given by the formula

$$r = af/(f - a).$$

(See any textbook on optics for the meaning of the symbols.) Let the true values of f and a be respectively 20 and 15. Let f and a be liable to error to the extent of $\pm \cdot 5$, say, f is read 20·5 and a, 14·5. Then the true value of r is 60, the observed value 51·2. Fractional error = 8·8/60. This means that an error of about 2·5 °/$_\circ$ in the determination of f and a may cause r to deviate 15 °/$_\circ$ from the truth.

The degree of accuracy of a measurement is determined by the magnitude of the proportional error.

$$\text{Proportional Error} = \frac{\text{Magnitude of error}}{\text{Total magnitude of quantity measured}}.$$

If we knew that an astronomer had made an absolute error of 100,000 miles in estimating the distance between the earth and the sun, and also that a physicist had made an absolute error of the $\frac{1}{10000,000000}$th of a mile in measuring the wave length of a spectral line, we could form no idea of the relative accuracy of the two measurements in spite of the fact that the one error is the $\frac{1}{1000,000000,000000}$th part of the other. In the first measurement the error is about $\frac{1}{1,000}$ of the whole quantity measured, in the second case the error is about the same order of magnitude as the quantity measured. In the former case, therefore, the error is negligibly small; in the latter, the error renders the result nugatory. The following examples will serve to fix these ideas:

EXAMPLES.—(1) It is required to determine the capacity of a sphere from the measurement of its diameter. Let y denote the volume, x the diameter, then, by a well-known mensuration formula,

$$y = \tfrac{1}{6}\pi x^3.$$

It is required to find the effect of a small error in the measurement of the diameter on the calculated volume. Suppose an error dx is committed in the measurement, then

$$y + dy = \tfrac{1}{6}\pi(x + dx)^3;$$
$$= \tfrac{1}{6}\pi\{x^3 + 3x^2dx + 3x(dx)^2 + (dx)^3\}.$$

By hypothesis, dx is a very small fraction, therefore, by neglecting the higher powers of dx and dividing the result by the original expression

$$\frac{y + dy}{y} = \frac{1}{6}\pi\left(\frac{x^3 + 3x^2dx}{\frac{1}{6}\pi x^3}\right); \quad \frac{dy}{y} = 3\frac{dx}{x}.$$

Or, the error in the calculated result is three times that made in the measurement. Hence the necessity for extreme precautions in measuring the diameter. Sometimes, we shall find, it is not always necessary to be so careful.

The same result could have been more easily obtained by the use of Taylor's theorem as described above. Differentiate the original expression and divide the result by the original expression. We thus get the relative error without trouble.

(2) Criticise the method for the determination of the atomic weight of lead from the ratio $Pb : O$ in lead monoxide.

Let y denote the atomic weight of lead, a the atomic weight of oxygen (known). It is found experimentally that x parts of lead combine with one part of oxygen, the required atomic weight of lead is determined from the simple proportion

$$y : a = x : 1; \text{ or, } y = ax; \text{ or, } dy = adx;$$
$$\therefore dy/y = dx/x. \qquad (2)$$

Thus an error of 1 °/。 in the determination of x introduces an equal error in the calculated value of y. Other things being equal, this method of finding the atomic weight of lead is, therefore, very likely to give good results.

(3) Show that the result of determining the atomic weight of barium by precipitation of the chloride with silver nitrate is less influenced by experimental errors than the determination of the atomic weight of sodium in the same way.

Assume that one part of silver nitrate requires x parts of sodium (or barium) chloride for precipitation as silver chloride. Let a and b be the known atomic weights of silver and chlorine. Then, if y denotes the atomic weight of sodium,

$$y + b : a = x : 1; \text{ or, } y = ax - b,$$
$$a = (y + b)/x.$$

Differentiating (3) and substituting for $y = 23$, $b = 35 \cdot 5$,

$$\frac{dy}{y} = \frac{a}{ax - b}dx = \frac{y + b}{y} \cdot \frac{dx}{x} = 2 \cdot 54\frac{dx}{x},$$

or an error of 1 °/。 in the determination of chlorine in sodium will introduce an error of 2·5 °/。 in the atomic weight of sodium. Hence it is a disadvantage to have b greater than y. For barium the error introduced is 1·5 °/。 instead of 2·5 °/。.

(4) If the atomic weight of barium y is determined by precipitation of barium sulphate from barium chloride solutions, and a denotes the known atomic weight of chlorine, b the known "atomic" weight of SO_4, then

when x parts of barium chloride are converted into one part of barium sulphate,

$$y + a : y + b = 1 : x; \ \frac{dy}{y} = \frac{(a - b)dx}{(1 - x)(ax - b)}.$$

What does this mean?

(5) An approximation formula used in the determination of the viscosity of liquids is

$$\eta = \pi ptr^4/8vl,$$

where v denotes the volume of liquid flowing from a capillary tube of radius r and length l in the time t; p is the actual pressure exerted by the column of liquid. Show that the proportional error in the calculation of the viscosity η is four times the error made in measuring the radius of the tube.

(6) In a tangent galvanometer, the tangent of the angle of deflection of the needle is proportional to the current. Prove that the proportional error in the calculated value of the current due to a given error in the reading is least when the deflection is 45°.

The strength of the current is proportional to the tangent of the displaced angle x, or

$$y = f(x) = C \tan x;$$

$$\therefore dy = \frac{C \, . \, dx}{\cos^2 x}; \text{ or, } \frac{dy}{y} = \frac{dx}{\sin x \, . \, \cos x}.$$

To determine the minimum, put

$$\frac{d}{dx}\left(\frac{dy}{y}\right) = \frac{\sin^2 x - \cos^2 x}{\sin^2 x \, . \, \cos^2 x} = 0;$$

$$\therefore \sin^2 x = \cos^2 x, \text{ or, } \sin x = \cos x.$$

This is true only in the neighbourhood of 45°,* and, therefore, in this region an error of observation will have the least influence on the final result. In other words, the best results are obtained with a tangent galvanometer when the needle is deflected about 45°.

What will be the effect of an error of 0·25° in reading a deflection of 42°, on the calculated current? Note that x in the above formula is expressed in circular or radian measure (page 494). Hence,

$$0{\cdot}25(\text{degrees}) = \frac{\pi \times {\cdot}25}{180} = {\cdot}00436(\text{radians}).$$

$$\therefore \frac{dy}{y} = \frac{dx}{\sin x \, . \, \cos x} = \frac{2dx}{\sin 2x} = \frac{{\cdot}00872}{\sin 84^\circ} = 0{\cdot}09; \ i.e., \ 9\,°/_{\circ},$$

since, from a Table of Natural Sines, $\sin 84^\circ = {\cdot}9945$.

(7) Show that the proportional error involved in the measurement of an electrical resistance on a Wheatstone's bridge is least near the middle of the bridge.

Let R denote the resistance, l the length of the bridge, x the distance of the telephone from one end.

$$\therefore y = Rx/(l + x).$$

Proceed as above and show that when $x = \frac{1}{2}l$ (the middle of the bridge), the proportional error is a minimum.

---

* Table XIV., page 497, $\sin 45^\circ = \cos 45^\circ$.

(8) By *Newton's law of attraction*, the force of gravitation (g) between two bodies varies directly as their respective masses (m_1, m_2) and inversely as the square of their distance apart (r). The mass of each body is supposed to be collected at its centroid (centre of gravity). The weight of one gram at Paris is equivalent to 980·868 dynes. The dyne is the unit of force. Hence Newton's law, $g = \mu m_1 m_2/r^2$ (dynes), may be written $w = a/r^2$ (grams), where a is a constant equivalent to $\mu \times m_1 \times m_2 \times 980{\cdot}868$. Hence show that for small changes in altitude $dw/w = - 2dr/r$. Interpret this result.

Marek was able to detect a difference of ·1 in 500,000,000 when comparing the kilogram standards of the Bureau International des Poids et Mesures. Hence show that it is possible to detect a difference in the weight of a substance when one scale pan of the balance is raised one centimetre higher than the other. (Radius of the earth = 6,371,300 metres.)

(9) In his well-known work on the gravimetric composition of water, Dumas determined the weight of hydrogen from the difference in the weight of oxygen required to burn up the hydrogen and the weight of water formed. Hence verify Dumas' remarks : "ainsi, une erreur de $\frac{1}{900}$ sur le poids de l'eau, ou de $\frac{1}{800}$ sur le poids de l'oxygène, affecte d'une quantité égale à $\frac{1}{90}$ ou à $\frac{1}{80}$ le poids de l'hydrogène. Que ces erreurs étant dans le même sens viennent à s'ajouter, et l'on aura des erreurs qui iront à $\frac{1}{40}$" (*Ann. de Chim. et de Phys.* [3], **8**, 198, 1843).

Proportional errors of composite measurements. Whenever a result has to be determined indirectly by combining several different species of measurements—weight, temperature, volume, electromotive force, etc.—the effect of a percentage error of, say, 1 °/ₒ in the reading of the thermometer will be quite different from the effect of an error of 1 °/ₒ in the reading of a voltmeter.

It is obvious that some observations must be made with greater care than others in order that the influence of each kind of measurement on the final result may be the same.

If a large error is compounded with a small error, the total error is not appreciably affected by the smaller. Hence Ostwald recommends, "a variable error may be neglected if it is less than one-tenth of the larger, often indeed if it is but one-fifth".

Examples.—(1) *Joule's relation* between the strength of a current C (ampères) and the quantity of heat Q (calories) generated in an electric conductor of resistance R (ohms) in the time t (seconds), is,

$$Q = 0{\cdot}24C^2Rt.$$

Show that R and t must be measured with half the precision of C in order to have the same influence on Q.

(2) What will be the fractional error in Q corresponding to a fractional error of 0·1 °/ₒ in R? Ansr. 0·001, or 0·1 °/ₒ.

(3) What will be the percentage error in C corresponding to 0·02 °/ₒ in Q? Ansr. 0·01 °/ₒ.

(4) If the density s of a substance be determined from its weights (w, w_1) in air and water, show that

$$\frac{ds}{s} = \frac{w_1}{w - w_1}\left(\frac{dw_1}{w_1} - \frac{dw}{w}\right).$$

Note $s = w_1/(w - w_1)$.

(5) The specific heat of a substance determined by the method of mixtures is given by the formula

$$s = \frac{m_1 c(\theta_2 - \theta_1)}{m(\theta - \theta_2)},$$

where m is the weight of the substance before the experiment; m_1 the weight of the water in the calorimeter; c the mean specific heat of water between θ_2 and θ_1; θ is the temperature of the body before immersion; θ_1 the maximum temperature reached by the water in the calorimeter; θ_2 the temperature of the system after equalisation of the temperature has taken place. Supposing the water equivalent of the apparatus is included in m_1, what will the effect of a small error in the determination of the different temperatures have on the result?

First, error in θ_1. Show that

$$ds/s = - d\theta_1/(\theta_2 - \theta_1).$$

If an error of say 0·1° is made in a reading and $\theta_2 - \theta_1 = 10°$, the error in the resulting specific heat is about 1 °/ₒ. If a maximum error of 0·1 °/ₒ is to be permitted, the temperature must be read to the 0·01°.

Second, error in θ. Show that

$$ds/s = d\theta/(\theta - \theta_2).$$

If a maximum error in the determination of s is to be 0·1 °/ₒ, when $\theta - \theta_2 = 50°$, θ must be read to the 0·04°. If an error of 0·1° is made in reading the temperature and $\theta - \theta_2 = 50°$, show that the resulting error in the specific heat will be 0·2°/ₒ.

Third, error in θ_2. Show that

$$ds/s = d\theta_2/(\theta_2 - \theta_1) + d\theta_2/(\theta - \theta_2).$$

If the maximum error allowed is 0·1 °/ₒ and $\theta_2 - \theta_1 = 10°$, $\theta - \theta_1 = 50°$, show that θ_2 must be read to the $\frac{1}{120}°$; while if an error of 0·1° is made in the reading of θ_2, show that the resulting error in the specific heat is 0·5 °/ₒ.

(6) In the preceding experiment, if $m_1 = 100$ grams, show that the weighing need not be taken to more than the 0·1 gram for the error in s to be within 0·1 °/ₒ; and for m, need not be closer than 0·5 gram when m is about 50 grams.

Since the actual errors are proportional to the probable errors, the most probable or mean value of the total error du, is obtained from the expression

$$(du)^2 = \left(\frac{\partial u}{\partial a_1}da_1\right)^2 + \left(\frac{\partial u}{\partial a_2}da_2\right)^2 + \ldots, \quad . \quad . \qquad (3)$$

from (16), § 179, page 444. Note the squared terms are all positive. Since the errors are fortuitous, there will be as many positive as negative paired terms. These will, in the long run, approximately neutralize each other. Hence (3).

EXAMPLES.—(1) Divide equation (3) by u^2, it is then easy to show that

$$(dQ/Q)^2 = (2dC/C)^2 + (dR/R)^2 + (dt/t)^2,$$

from the preceding set of examples. Hence show that the fractional error in Q, corresponding to the fractional errors of 0·03 in C, 0·02 in R and 0·03 in t, is 0·07.

(2) The regular formula for the determination of molecular weight of a substance by the freezing point method, is $M = Kw/\theta$, where K is a constant, M the required molecular weight, w the weight of the substance dissolved in 100 grams of the solvent, θ the lowering of the freezing point. In an actual determination, $w = \cdot 5139$, $\theta = \cdot 295$, $K = 19$ (Perkin and Kipping's *Organic Chemistry*), what would be the effect on M of an error of ·01 in the determination of w, and of an error of ·01 in the determination of θ?

Also show that an error of ·01 in the determination of θ affects M to an extent of $-$ 3·25, while an error of ·01 in the determination of w only affects M to the extent of ·19. Hence show that it is not necessary to weigh to more than 0·01 of a gram. An illustration of the need of "scientific perspective" in measuring the different components of a composite result.

From (16), § 179, page 444, when the effect of each observation on the final result is the same, the partial differential coefficients are all equal. If u denotes the sum of n observations, $a_1, a_2, \ldots, a_n$,

$$u = a_1 + a_2 + \ldots + a_n,$$

$$\therefore \frac{\partial u}{\partial a_1} = \frac{\partial u}{\partial a_2} = \ldots = 1.$$

But in order that the actual errors affecting each observation may be the same,

$$da_1 = da_2 = \ldots = da_n = du/\sqrt{n}; \quad . \quad . \quad (4)$$

from (3), or,

$$\frac{da_1}{u} = \frac{da_2}{u} = \ldots = \frac{da_n}{u} = \frac{du}{u} \cdot \frac{1}{\sqrt{n}}. \quad . \quad . \quad (5)$$

EXAMPLES.—(1) Suppose the greatest allowable fractional error in Q (preceding examples) is 0·5 °/₀, what is the greatest percentage error in each of the variables C, R, t, allowable under equal effects? Here,

$$2dC/C = dR/R = dt/t = \cdot 005/\sqrt{3}.$$

Ansr. 0·22 for R and t, ·11 °/₀ for C.

(2) If a volume v of a given liquid flows from a long capillary tube of radius r and length l in t seconds, the viscosity of the liquid is $\eta = \pi pr^4t/8vl$, where p denotes the excess of the pressure at the outlet of the tube over atmospheric pressure. What would be the errors dr, dv, dl, dt, dp, necessary under equal effects to give η with a precision of ·1 °/₀? Here,

$$dp/p = dt/t = 4dr/r = -dv/v = -dl/l = \cdot 001/\sqrt{5} = \cdot 00045.$$

It is now necessary to know the numerical values of p, t, v, r, l, before dp, dt, . . . can be determined. Thus, if r is about 2 mm., the radius must be measured to the ·0022 mm. for an error of ·1 °/₀ in η.

It has been shown how the best working conditions may be determined by a study of the formula to which the experimental results are to be referred. The following is a more complex example.

(3) The resistance i of a cell is to be measured. Let C_1, C_2 respectively denote the currents produced by the cell when working through two known external resistances r_1 and r_2, and let R_1, R_2, be the total resistances of the circuit, E the electromotive force of the cell is constant. It is known (see your textbook: *Practical Physics*),

$$i = (C_2r_2 - C_1r_1)/(C_1 - C_2). \quad . \quad . \quad . \quad . \quad (6)$$

What ratio $R_1 : R_2$ will furnish the best result? From Ohm's law, $E = CR$, E being constant, $C_1 : C_2 = R_2 : R_1$. As usual, (4) above

$$(di)^2 = \left(\frac{\partial i}{\partial C_1}dC_1\right)^2 + \left(\frac{\partial i}{\partial C_2}dC_2\right)^2. \quad . \quad . \quad . \quad (7)$$

Find values for $\partial i/\partial C_1$ and $\partial i/\partial C_2$ from (7), and put R_1 for r_1, R_2 for r_2. Thus,

$$\partial i/\partial C_1 = - R_1^2R_2/E(R_2 - R_1); \ \partial i/\partial C_2 = R_1R_2^2/E(R_2 - R_1).$$

Substitute this result in (7).

1. *If a mirror galvanometer is used*, $dC_1 = dC_2 = dC$ (say) = constant.

$$\therefore (di)^2 = (R_1^4R_2^2 - R_1^2R_2^4)(dC)^2/E^2(R_2 - R_1)^2.$$

Substitute $x = R_2 : R_1$,

$$\therefore (di)^2 = R_1^4(x^2 + x^4)(dC)^2/E^2(x - 1)^2.$$

For a minimum error

$$\frac{d}{dx}\left(\frac{x^4 + x^2}{x^2 - 2x - 1}\right) = 0; \ \therefore x^3 - 2x^2 - 1 = 0;$$

$\therefore x = 2{\cdot}2$ approx.; or, $R_2 = 2{\cdot}2R_1$; or, $C_1 = 2{\cdot}2C_2$. Substitute these values in (6),

$$di = \sqrt{20} \,.\, R_1^2 \,.\, dC_1/E,$$

which shows that the external resistance R_1, should be as small as is consistent with the polarisation of the battery.

2. *If a tangent galvanometer is used*, dC/C is constant. Hence substitute $C_1 = ER_1$ and $C_2 = ER_2$ in (8),

$$(di)^2 = 2[R_1R_2/(R_2 - R_1)]^2(dC/C)^2.$$

From this it can be shown there is no best ratio $R_2 : R_1$. If the last expression is written

$$di = \{\sqrt{2}(1/R_1 - 1/R_2)\}dC/C,$$

it follows that the error di increases as R_1 increases, and as R_2 diminishes. Hence R_2 should be made as large and R_1 as small as is consistent with the range of the galvanometer and the polarisation of the battery.

§ 184. Observations of Different Degrees of Accuracy.

Hitherto it has been assumed that the individual observations of any particular series, are equally reliable, or that there is no reason why one observation should be preferred more than another. As a general rule, measurements made by different methods, by different observers, or even by the same observer at different times,* are not liable to the same errors. Some results are more trustworthy than others. In order to fix this idea, suppose that twelve determinations of the capacity of a flask by the same method, gave the following results: six measurements each 1·6 litres; four, 1·4 litres; and two, 1·2 litres. The numbers 6, 4, 2, represent the relative values of the three results 1·6, 1·4, 1·2, because the measurement 1·6 has cost three times as much labour as 1·2. The former result, therefore, is worth three times as much confidence as the latter. In such cases, it is customary to say that the relative practical value, or the weight of these three sets of observations is as 6 : 4 : 2, or, what is the same thing, as 3 : 2 : 1. In this sense, the **weight** of an observation, or set of observations, represents the *relative* degree of precision of that observation in comparison with other observations of the same quantity. It tells us nothing about the *absolute* precision (h) of the observations.

It is shown below that the weight of an observation is, in theory, inversely as its probable error; in practice, it is usual to assign arbitrary weights to the observations. For instance, if one observation is made under favourable conditions, another under adverse conditions, it would be absurd to place the two on the same footing. Accordingly, the observer pretends that the best observations have been made more frequently. That is to say, if the observations $a_1, a_2, \ldots, a_n$, have weights $p_1, p_2, \ldots, p_n$, respectively, the observer has assumed that the measurement a_1 has been repeated p_1 times with the result a_1, and that a_n has been repeated p_n times with the result a_n.

To take a concrete illustration, Morley has made three accurate series of determinations of the density of oxygen gas with the following results:—

I. $1·42879 \pm ·000034$; II. $1·42887 \pm ·000048$; III. $1·42917 \pm ·000048$. ("On the densities of oxygen and hydrogen and on the ratio of

* I am reminded that Dumas, discussing the errors in his great work on the gravimetric composition of water, alluded to a few pages back, adds the remarks: "The length of time required for these operations compelled me to prolong the work far into the night, generally finishing with the weighings about 2 or 3 o'clock in the morning. This may be the cause of a substantial error, for I dare not venture to assert that such weighings deserve as much confidence as if they had been performed under more favourable conditions and by an observer not so worn out with fatigue, the inevitable result of fifteen to twenty hours continued attention."

their atomic weights," *Smithsonian Contributions to Knowledge*, (980), p. 55, 1895.) The probable errors of these three means would indicate that the first series were worth more than the second. For experimental reasons, Morley preferred the last series, and gave it double weight. In other words, Morley pretended that he had made four series of experiments, two of which gave 1·42917, one gave 1·42879, and one gave 1·42887. The result is that 1·42900, not 1·42894,* is given as the best representative value of the density of oxygen gas.

The product of an observation or of an error with the weight of the observation, is called a **weighted observation,** or a **weighted error** as the case might be.

The practice of weighting observations is evidently open to some abuse. It is so very easy to be influenced rather by the differences of the results from one another, than by the intrinsic quality of the observation. This is a fatal mistake.

1. *The best value to represent a number of observations of equal weight, is their arithmetical mean.*

If P denotes the most probable value of the observed magnitudes $a_1, a_2, \ldots, a_n$, then $P - a_1, P - a_2, \ldots, P - a_n$, represent the several errors in the n observations. From the principle of least squares these errors will be a minimum when

$$(P - a_1)^2 + (P - a_2)^2 + \ldots + (P - a_n)^2 = \text{a minimum.}$$

Hence, page 434,

$$P = (a_1 + a_2 + \ldots + a_n)/n, \quad \ldots \quad (1)$$

or the best representative value of a given series of measurements of an unknown quantity, is the arithmetical mean of the n observations, provided that the measurements have the same degree of confidence.

2. *The best value to represent a number of observations of different weight, is obtained by multiplying each observation by its weight and dividing the sum of these products by the sum of their different weights.*

With the same notation as before, let $p_1, p_2, \ldots, p_n$, be the respective weights of the observations $a_1, a_2, \ldots, a_n$. From the definition of weight, the quantity a_1 may be considered as the mean of p_1 observations of unit weight; a_2, the mean of p_2 observations of unit weight, etc. The observed quantities may, therefore, be resolved into a series of fictitious observations all of equal weight. Applying the preceding rule to each of the resolved observations, the total number of standard observations of unit weight will

---

* See formula (5).

be $p_1 + p_2 + \ldots + p_n$; the sum of the p_1 standard observations of unit weight will be $p_1 a_1$; the sum of p_2 standard observations, $p_2 a_2$, etc. Hence, from (1), the most probable value of a series of observations of different weights is

$$P' = \frac{p_1 a_1 + p_2 a_2 + \ldots + p_n a_n}{p_1 + p_2 + \ldots + p_n} \quad . \quad . \quad . \quad . \quad (2)$$

Note the formal resemblance between this formula and that for finding the centre of gravity of a system of particles of different weights arranged in a straight line.

Weighted observations are, therefore, fictitious results treated as if they were real measurements of equal weight. With this convention, the value of P' in (2) is an arithmetical mean sometimes called the **general** or **probable mean.**

3. *The weight of an observation is inversely as the square of its probable error.*

Let a be a set of observations whose probable error is R and whose weight is unity. Let $p_1, p_2, \ldots, p_n$ and $r_1, r_2, \ldots, r_n$, be the respective weights and probable errors of a series of observations $a_1, a_2, \ldots, a_n$, of the same quantity. By definition of weight, a_1 is equivalent to p_1 observations of equal weight. From (16), page 444,

$$r_1 = R/\sqrt{p_1}\,;\ \text{or}, p_1 = R^2/r_1^2\,;\ p_2 = R^2/r_2^2\,;\ \ldots,$$

and

$$p_1 : p_2 : p_3 \ldots = \frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} \ldots \quad . \quad . \quad . \quad (3)$$

EXAMPLES.—(1) If n observations have weights $p_1, p_2, \ldots, p_n$, show that

$$R = \pm\, r_1/\sqrt{\Sigma(p)}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiate (2) successively with respect to $a_1, a_2, \ldots$ and substitute the results in (16), page 444.

(2) Show that the mean error of a series of observations of weights $p_1, p_2, \ldots, p_n$, is

$$M = \pm \sqrt{\Sigma(px^2)/(n-1)\Sigma(p)}.$$

Hint. Proceed as in § 178 but use px^2 and pv^2 in place of x^2 and v^2 respectively. If the sum of the weights of a series of observations is $\Sigma(p) = 40$, and the sum of the products of the weights of each observation with the square of its deviation from the mean of nine observations is $\Sigma(px^2) = \cdot 3998$, show that $M = \pm\, 0{\cdot}035$.

(3) The probable errors of four series of observations are respectively 1·2, 0·8, 0·9, 1·1, what are the relative weights of the corresponding observations? Ansr. 7 : 16 : 11 : 8. Use (3).

(4) Determinations of the percentage amount of copper in a sample of malachite were made by a number of chemical students, with the following results: (1) 39·1; (2) 38·8, 38·7, 38·6; (3) 39·9, 39·1, 39·3; (4) 37·7, 37·9. If these analyses had an equal degree of confidence, the mean, 38·8, would best represent the percentage amount of copper in the ore—formula (1). But the analyses are not of equal value. The first was made by the teacher. To this we may assign an arbitrary weight 10. Sets (2) and (3) were made by two

different students using the electrolytic process. Student (2) was more experienced than student (3), in consequence, we are led to assign to the former an arbitrary weight 6, to the latter, 4. Set (4) was made by a student precipitating the copper as CuS, roasting and weighing as CuO. The danger of loss of CuS by oxidation to $CuSO_4$ during washing, leads us to assign to this set of results an arbitrary weight 2. From these *assumptions*, show that 38·94 best represents the percentage amount of copper in the ore. For the sake of brevity use values above 37 in the calculation. From formula (2), $108{\cdot}8/56 = 1{\cdot}94$. Add 37 for the general mean.

It is unfortunate when so fantastic a method has to be used for calculating the most probable value of a "constant of Nature," because a redetermination is then urgently required.

(5) Rowland (*Proc. Amer. Acad.*, **15,** 75, 1879) has made an exhaustive study of Joule's determinations of the mechanical equivalent of heat, and he believes that Joule's several values have the weights here appended in brackets: 442·8 (0); 427·5 (2); 426·8 (10); 428·7 (2); 429·1 (1); 428·0 (1); 425·8 (2); 428·0 (3); 427·1 (3); 426·0 (5); 422·7 (1); 426·3 (1). Hence Rowland concludes that 426·9 best represents the result of Joule's work. Verify this. Notice that Rowland rejects the number 442·8 by giving it zero weight.

4. *To combine several arithmetical means each of which is affected with a known probable (or mean) error, into one general mean.*

One hundred parts of silver are equivalent to

49·5365 ± ·013	of NH_4Cl,	according to	Pelouze;
49·523 ± ·0055	,,	,,	Marignac;
49·5973 ± ·0005	,,	,,	Stas (1867);
49·5992 ± ·00039	,,	,,	Stas (1882),

where the first number represents the arithmetical mean of a series of experiments, the second number the corresponding probable error. How are we to find the best representative value of this series of observations? The first thing is to decide what weight shall be assigned to each result. Individual judgment on the "internal evidence" of the published details of the experiments is not always to be trusted. Nor is it fair to assign the greatest weight to the last two values simply because they are by Stas.

Meyer and Seubert, in a paper *Die Atomgewichte der Elemente, aus der Originalzahlen neu berechnet* (Leipzig, 1883), weighted each result according to the mass of material employed in the determination. They assumed that the magnitude of the errors of observation were inversely as the quantity of material treated. That is to say, an experiment made on 20 grams of material is supposed to be worth twice as much as one made on 10 grams. This seems to be a somewhat gratuitous assumption.

One way of treating this delicate question is to assign to each arithmetical mean a weight inversely as the square of its mean error. Clark in his "Recalculation of the Atomic Weights" (*Smithsonian Miscellaneous Collections* (1075), 1897) employed the probable error. Although this method of weighting did not suit Morley in the special case mentioned on page 463, Clark considers it a safe, though not infallible guide.

Let $A, B, C, \ldots$, be the arithmetical mean of each series of experiments; $a, b, c, \ldots$, the respective probable (or mean) errors, then, from (2),

$$\left.\begin{aligned} \text{General Mean} &= \frac{\dfrac{A}{a^2} + \dfrac{B}{b^2} + \dfrac{C}{c^2} + \ldots}{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} + \ldots}; \\ \text{Probable Error} &= \pm \frac{1}{\sqrt{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} + \ldots\right)}}. \end{aligned}\right\} \quad . \quad . \quad (5)$$

EXAMPLES.—(1) From the experimental results just quoted, show that the best value for the ratio

$Ag : NH_4Cl$ is 100 : 49·5983 ± ·00031.

Hint. Substitute $A = 49{\cdot}5365$, $a = {\cdot}013$; $B = 49{\cdot}523$, $b = {\cdot}0055$; $C = 49{\cdot}5973$, $c = {\cdot}0005$; $D = 49{\cdot}5992$, $d = {\cdot}00039$, in equations (5).

(2) The following numbers represent the most trustworthy results yet published for the atomic weight of gold ($H = 1$): 195·605 ± ·0099; 195·711 ± ·0224; 195·808 ± ·0126; 195·624 ± ·0224; 195·896 ± ·0131; 195·770 ± ·0082. Hence show that the best representative value for this constant is 196·743 ± ·0049.

(3) In three series of determinations of the vapour pressure of water vapour at 0° Regnault found the following numbers:

I. 4·54; 4·54; 4·52; 4·54; 4·52; 4·54; 4·52; 4·50; 4·50; 4·54.

II. 4·66; 4·67; 4·64; 4·62; 4·64; 4·66; 4·67; 4·66; 4·66.

III. 4·54; 4·54; 4·54; 4·58; 4·58; 4·57; 4·58.

Show that the best representative value of series I. is 4·526, with a probable error ± 0·0105; series II., 4·653, probable error ± 0·0105; series III., 4·561, probable error ± 0·0127. The most probable value of the vapour pressure of aqueous vapour at 0° is, therefore, 4·582, with an equal chance of its possessing an error greater or less than ·0064.

"As a matter of fact the theory of probability is of little or no importance, when the 'constant' errors (otherwise known as 'systematic' errors) are greater than the accidental errors. Still further, this use of the probable error cannot be justified, even when the different series of experiments are only affected with accidental errors, because *the probable error only shows how* UNIFORMLY *an experimenter has conducted a certain process, and not how suitable that process is for the required purpose.* In combining different sets of determinations it is still more unsatisfactory to calculate the probable error of the general mean by weighting the individual errors according to Clark's criterion when the probable errors differ very considerably among themselves. For example, Clark (*l.c.*, page 126) deduces the general mean 136·315 ± ·0085 for the atomic weight of barium from the following results:

136·271 ± ·0106; 136·390 ± ·0141; 135·600 ± ·2711; 136·563 ± ·0946.

The individual series here deviate from the general mean more than the magnitude of its probable error would lead us to suppose. The constant errors, in consequence, must be greater than the probable errors. In such a case as this, the computed probable error ± ·0085 has no real meaning, and we can only conclude that the atomic weight of barium is, at its best, not known more accurately than to five units in the second decimal place." (Paraphrased from Ostwald's *critique* on Clark's work (*l.c.*) in the *Zeitschrift für physikalische Chemie*, **23,** 187, 1897.)

5. *Mean and probable errors of observations of different degrees of accuracy.*

In a series of observations of unequal weight the mean and probable errors of a single observation of unit weight are respectively

$$\pm\sqrt{\frac{\Sigma(pv^2)}{n-1}}\text{; and } \pm\cdot 6745\sqrt{\frac{\Sigma(pv^2)}{n-1}}; \quad . \quad . \quad . \quad (6)$$

The mean of a series of observation of unequal weight has the respective mean and probable errors

$$\pm\sqrt{\frac{\Sigma(pv^2)}{(n-1)\Sigma(p)}}\text{; and } \pm\cdot 6745\sqrt{\frac{\Sigma(pv^2)}{(n-1)\Sigma(p)}}. \quad . \quad . \quad (7)$$

EXAMPLE.—An angle was measured under different conditions fourteen times. The observations all agreed in giving 4° 15′, but for seconds of arc the following values were obtained (the weight of each observation is given in brackets): 45″·00 (5); 31″·25 (4); 42″·50 (5); 45″·00 (3); 37″·50 (3); 38″·33 (3); 27″·50 (3); 43″·33 (3); 40″·63 (4); 36″·25 (2); 42″·50 (3); 39″·17 (3); 45″·00 (2); 40″·83 (3). Show that the mean error of a single observation of unit weight is ± 9″·475, the mean error of the mean 39″·78 is 1″·397. Hint. $\Sigma(p) = 46$; $\Sigma(pv^2) = 1167\cdot 03$; $n = 14$; $\Sigma(pa) = 1830\cdot 00$.

The mean and probable errors of a single observation of weight p are

$$\pm\sqrt{\frac{\Sigma(pv^2)}{(n-1)p}}\text{; and } \pm\cdot 6745\sqrt{\frac{\Sigma(pv^2)}{(n-1)p}}. \quad . \quad . \quad . \quad (8)$$

respectively.

EXAMPLE.—In the preceding example show that the mean error of an observation of weight (2) is ± 6″·70; of weight (3) is ± 5″·47; of weight (4) ± 4″·74; and of weight (5) ± 4″·24.

6. *The principle of least squares for observations of different degrees of precision* states that "the most probable values of the observed quantities are those for which the sum of the weighted squares of the errors is a minimum," that is,

$$p_1^2v_1^2 + p_2^2v_2^2 + \ldots + p_n^2v_n^2 = \text{a minimum}.$$

An error v is the deviation of an observation from the arithmetical mean of n observations; a "weighted square" is the product of the weight p and the square of an error v (see § 106).

§ 185. Observations Limited by Conditions.

On adding up the results of an analysis, the total weight of the constituents ought to be equal to the weight of the substance itself; the three angles of a plane triangle, must add up to exactly 180°; the sum of the three angles of a spherical triangle always equal 180° + the spherical excess; the sum of the angles of the normals on the faces of a crystal in the same plane must equal 360°. Measurements subject to restrictions of this nature, are said to be **conditioned observations.** The number of conditions to be satisfied is evidently less than the number of observations, otherwise the value of the unknown could be deduced from the conditions, without having recourse to measurement.

In practice, measurements do not come up to the required standard, the percentage constituents of a substance do not add up to 100; the angles of a triangle are either greater or less than 180°. Only in the ideal case of perfect accuracy are the conditions fulfilled. It is sometimes desirable to find the best representative values of a number of imperfect conditioned observations. The method to be employed is illustrated in the following examples.

EXAMPLES.—(1) The analysis of a compound gave the following results: $37{\cdot}2\,^\circ/_\circ$ of carbon, $44{\cdot}1\,^\circ/_\circ$ of hydrogen, $19{\cdot}4\,^\circ/_\circ$ of nitrogen. Assuming each determination is equally reliable, what is the best representative value of the percentage amount of each constituent? Let C, H, N, respectively denote the percentage amounts of carbon, hydrogen, and nitrogen required, then,

$$C + H = 100 - N \equiv 100 - 19{\cdot}4 = 80{\cdot}6.$$

Hence, $$2C + H = 117{\cdot}8;\ C + 2H = 124{\cdot}7.$$

Solve the last two simultaneous equations in the usual way. Ansr. $C = 36{\cdot}97\,^\circ/_\circ$; $H = 43{\cdot}86\,^\circ/_\circ$; $N = 19{\cdot}17\,^\circ/_\circ$. Note that this result is quite independent of any hypothesis as to the structure of matter. The chemical student will know a better way of correcting the analysis. This example will remind us how the atomic hypothesis introduces order into apparent chaos. Some analytical chemists before publishing their results, multiply or divide their percentage results to get them to add up to 100. In some cases, one constituent is left undetermined and then calculated by difference. Both practices are objectionable in exact work.

(2) The three angles of a triangle A, B, C, were measured with the result that $A = 51°$; $B = 94°\ 20'$; $C = 34°\ 56'$. Show that the most probable values of the unknown angles are $A = 51°\ 56'$; $B = 94°\ 12'$; $C = 34°\ 52'$.

(3) The angles between the normals on the faces of a cubic crystal were found to be respectively $\alpha = 91° \ 13'$; $\beta = 89° \ 47'$; $\gamma = 91° \ 15'$; $\delta = 89° \ 42'$. What numbers best represent the values of the four angles? Ansr. $\alpha = 90° \ 43' \ 45''$; $\beta = 89° \ 17' \ 45''$; $\gamma = 90° \ 0' \ 45''$; $\delta = 89° \ 57' \ 45''$.

(4) The three angles of a triangle furnish the respective observation equations:

$$A = 36° \ 25' \ 47''; \ B = 90° \ 36' \ 28''; \ C = 52° \ 57' \ 57'';$$

the equation of condition requires that

$$A + B + C - 180° = 0. \qquad (1)$$

Let x_1, x_2, x_3, respectively denote the errors affecting A, B, C, then we must have

$$x_1 + x_2 + x_3 = -12. \qquad (2)$$

i. *If the observations are equally trustworthy,*

$$x_1 = x_2 = x_3 = k, \qquad (3)$$

say. Substitute this value of x_1, x_2, x_3, in (2), and we get

$$3k + 12 = 0; \text{ or, } k = -4;$$

$$\therefore A = 36° \ 25' \ 43''; \ B = 90° \ 36' \ 24''; \ C = 53° \ 57' \ 53''.$$

The formula for the mean error of each observation is

$$\pm \sqrt{\frac{\Sigma(v^2)}{n - w + q}}, \qquad (4)$$

where w denotes the number of unknown quantities involved in the n observation equations; q denotes the number of equations of condition to be satisfied. Consequently the w unknown quantities reduce to $w - q$ independent quantities. $\Sigma(v^2)$ denotes the sum of the squares of the differences between the observed and calculated values of A, B, C. Hence, the mean error $= \pm \sqrt{48} = \pm 6''{\cdot}93$.

ii. *If the observations have different weights.* Let the respective weights of A, B, C, be $p_1 = 4$; $p_2 = 2$; $p_3 = 3$. It is customary to assume that the magnitude of the error affecting each observation will be inversely as its weight. (Perhaps the reader can demonstrate this principle for himself.) Instead of (3), therefore, we write

$$x_1 = \tfrac{1}{4}k; \ x_2 = \tfrac{1}{2}k; \ x_3 = \tfrac{1}{3}k. \qquad (5)$$

From (2), therefore,

$$13k + 144 = 0; \ k = -11{\cdot}07; \ x_1 = -2''{\cdot}77; \ x_2 = -5''{\cdot}54; \ x_3 = -3''{\cdot}69.$$

$$m = \text{Mean error} = \pm \sqrt{\frac{\Sigma(pv^2)}{n - w + q}}, \qquad (6)$$

or $m = \pm 11{\cdot}52$.

The mean errors m_1, m_2, m_3, respectively affecting a, b, c, are

$$m_1 = \pm \frac{m}{\sqrt{p}}; \ m_2 = \pm \frac{m}{\sqrt{p}}; \ m_3 = \pm \frac{m}{\sqrt{p}}. \qquad (7)$$

Hence

$A = 36° \ 25' \ 44''{\cdot}23 \pm 5''{\cdot}76$; $B = 90° \ 36' \ 22''{\cdot}46 \pm 8''{\cdot}15$; $C = 52° \ 57' \ 53''{\cdot}31 \pm 6''{\cdot}65$.

It is, of course, only permissible to reduce experimental data in this manner when the measurements have to be used as the basis for subsequent calculations. In every case the actual measurements must be stated along with the "cooked" results.

§ 186. Gauss' Method of Solving a Set of Linear Observation Equations.

In continuation of § 106, let x, y, z, represent the unknowns to be evaluated, and let $a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots, R_1, R_2, \ldots$, represent actual numbers whose values have been determined by the series of observations set forth in the following **observation equations:**

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= R_1; \\ a_2x + b_2y + c_2z &= R_2; \\ a_3x + b_3y + c_3z &= R_3; \\ a_4x + b_4y + c_4z &= R_4. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

If only three equations had been given, we could easily calculate the corresponding values of x, y, z, by the method of § 165, but these values would not necessarily satisfy the fourth equation. The problem here presented is to find the best possible values of x, y, z, which will satisfy the four given observation equations. We have selected four equations and three unknowns for the sake of simplicity and convenience. Any number may be included in the calculation. But sets involving more than three unknowns are comparatively rare.

We also assume that the observation equations have the same degree of accuracy. If not, multiply each equation by the square root of its weight, as in example (3) below. This converts the equations into a set having the same degree of accuracy.

First. *To convert the observation equations into a set of normal equations solvable by ordinary algebraic processes.*

Multiply the first equation by a_1, the second by a_2, the third by a_3, and the fourth by a_4. Add the four results. Treat the four equations in the same way with b_1, b_2, b_3, b_4, and with c_1, c_2, c_3, c_4. Now write, for the sake of brevity,

$$[aa]_1 \equiv a_1^2 + a_2^2 + a_3^2 + a_4^2; \qquad [bb]_1 \equiv b_1^2 + b_2^2 + b_3^2 + b_4^2;$$
$$[ab]_1 \equiv a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4; \qquad [ac]_1 \equiv a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4;$$
$$[aR]_1 \equiv a_1R_1 + a_2R_2 + a_3R_3 + a_4R_4; \; [bR]_1 \equiv b_1R_1 + b_2R_2 + b_3R_3 + b_4R_4;$$

and likewise for $[cc]_1$, $[bc]_1$, $[cR]_1$. The resulting equations are

$$\left.\begin{aligned} [aa]_1x + [ab]_1y + [ac]_1z &= [aR]_1; \\ [ab]_1x + [bb]_1y + [bc]_1z &= [bR]_1; \\ [ac]_1x + [bc]_1y + [cc]_1z &= [cR]_1. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

These three equations are called **normal equations** (first set) in x, y, z.

Second. *To solve the normal equations.* We can determine the values of x, y, z, from this set of simultaneous equations (2) by any method we please, determinants (§ 165), cross-multiplication, indeterminate multipliers, or by the method of substitution.* The last method is adopted here.

Solve the first normal equation for x, thus,

$$x = -\frac{[ab]_1}{[aa]_1}y - \frac{[ac]_1}{[aa]_1}z + \frac{[aR]_1}{[aa]_1}. \qquad (3)$$

Substitute this value of x in the other two equations for a second set of normal equations in which the term containing x has disappeared.

$$\left([bb]_1 - \frac{[ab]_1}{[aa]_1}[ab]_1\right)y + \left([bc]_1 - \frac{[ac]_1}{[aa]_1}[ab]_1\right)z = \left([bR]_1 - \frac{[ab]_1}{[aa]_1}[aR]_1\right);$$
$$\left([bc]_1 - \frac{[ac]_1}{[aa]_1}[ab]_1\right)y + \left([cc]_1 - \frac{[ac]_1}{[aa]_1}[ac]_1\right)z = \left([cR]_1 - \frac{[ac]_1}{[aa]_1}[aR]_1\right).$$

For the sake of simplicity, write

$$[bb]_2 \equiv [bb]_1 - \frac{[ab]_1}{[aa]_1}[ab]_1; \; [cc]_2 \equiv [cc]_1 - \frac{[ac]_1}{[aa]_1}[ac]_1;$$
$$[bc]_2 \equiv [bc]_1 - \frac{[ac]_1}{[aa]_1}[ab]_1;$$
$$[bR]_2 \equiv [bR]_1 - \frac{[ab]_1}{[aa]_1}[aR]_1; \; [cR]_2 \equiv [cR]_1 - \frac{[ac]_1}{[aa]_1}[aR]_1;$$

The second set of normal equations may now be written:

$$\left.\begin{array}{l} [bb]_2y + [bc]_2z = [bR]_2; \\ [bc]_2y + [cc]_2z = [cR]_2. \end{array}\right\} \qquad (4)$$

Solve the first of these equations for y,

$$y = -\frac{[bc]_2}{[bb]_2}z + \frac{[bR]_2}{[bb]_2}. \qquad (5)$$

Substitute this in the second of equations (4), and we get a third set of normal equations,

$$\left([cc]_2 - \frac{[bc]_2}{[bb]_2}[bc]_2\right)z = \left([cR]_2 - \frac{[bc]_2}{[bb]_2}[bR]_2\right),$$

which may be abbreviated into

$$[cc]_3z = [cR]_3. \qquad (6)$$

Hence,
$$z = \frac{[cR]_3}{[cc]_3}. \qquad (7)$$

$[bb]_2$, $[bc]_2$, . . ., $[cc]_3$, . . . are called **auxillaries.**

---

* The equations cannot be solved if any two are identical, or can be made identical by multiplying through with a constant.

Equations (3), (5), (7), collectively constitute a set of **elimination equations:**

$$\left.\begin{aligned} x = -\frac{[ab]_1}{[aa]_1}y - \frac{[ac]_1}{[aa]_1}z + \frac{[aR]_1}{[aa]_1};\\ y = -\frac{[bc]_2}{[bb]_2}z + \frac{[bR]_2}{[bb]_2};\\ z = \frac{[cR]_3}{[cc]_3}. \end{aligned}\right\} \quad . \quad . \quad (8)$$

The last equation gives the value of z directly; the second gives the value of y when z is known, and the first equation gives the value of x when the values of y and z are known.

Note the symmetry of the coefficients in the three sets of normal equations. Hence it is only necessary to compute the coefficients of the first equation in full. The coefficients of the first horizontal row and vertical column are identical. So also the second row and second column, etc.

The formation and solution of the auxillary equations is more tedious than difficult. Several schemes have been devised to lessen the labour of calculation as well as for testing the accuracy of the work. These we pass by.

Third. *The weights of the values of x, y, z.* Without entering into any theoretical discussion, the respective weights of z, y, and x, are given by the expressions:

$$p_z = [cc]_3;\ p_y = p_z\frac{[bb]_2}{[cc]_2};\ p_x = p_z\frac{[ca]_1[bb]_2}{[cc]_1[bb]_1 - [bc]_1[bc]_1}. \quad (9)$$

Fourth. *The mean errors affecting the values of x, y, z.* Let

$$a_1x + b_1y + c_1z - R_1 = v_1;$$
$$a_2x + b_2y + c_2z - R_2 = v_2;$$
$$\ldots \quad \ldots \quad \ldots$$

Let M denote the mean error of any observed quantity of unit weight,

$$\left.\begin{aligned} M = \pm\sqrt{\frac{\Sigma(v^2)}{n-w}} \text{ for equal weight};\\ M = \pm\sqrt{\frac{\Sigma(pv^2)}{n-w}} \text{ for unequal weights,} \end{aligned}\right\} \quad . \quad . \quad (10)$$

where n denotes the number of observation equations, w the number of quantities x, y, z, . . . Here, $w = 3$, $n = 4$. Let M_x, M_y, M_z, respectively denote the mean errors respectively affecting x, y, z.

$$\therefore M_x = \pm\frac{M}{\sqrt{p_x}};\ M_y = \pm\frac{M}{\sqrt{p_y}};\ M_z = \pm\frac{M}{\sqrt{p_z}}. \quad . \quad (11)$$

EXAMPLES.—(1) Find the values of the constants a and b in the formula

$$y = a + bx, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

from the following determinations of corresponding values of x and y:—

$$y = 3{\cdot}5, \quad 5{\cdot}7, \quad 8{\cdot}2 \quad 10{\cdot}3, \ldots;$$

when $x = 0, \quad 88, \quad 182, \quad 274, \ldots;$

We want to find the best numerical values of a and b in equations (12). Write x for a, and y for b, so as to keep the calculation in line with the preceding discussion. The first set of normal equations is obviously

$$[aa]_1x + [ab]_1y = [aR]_1; \text{ and } [ab]_1x + [bb]_1y = [bR]_1.$$

But $$x = -\frac{[ab]_1}{[aa]_1}y + \frac{[aR]_1}{[aa]_1}; \therefore y = \frac{[bR]_2}{[bb]_2}.$$

Again, $[aa]_1 = 4$; $[bb]_1 = 115{,}944$; $[ab]_1 = 544$; $[aR]_1 = 27{\cdot}7$; $[bR]_1 = 4{,}816{\cdot}2$; $[bb]_2 = 4{,}853{\cdot}67$; $[bR]_2 = 115{,}951{\cdot}4$.

$$x = 3{\cdot}52475; \; y = 0{\cdot}02500;$$

or, reconverting x into a, and y into b, (12) is to be written,

$$y = 3{\cdot}525 + 0{\cdot}025x.$$

a.	b. Calculated.	b. Observed.	Difference between Calculated and Observed.	Square of Difference between Calculated and Observed.
0	3·525	3·5	+ 0·025	0·000625
88	5·725	5·7	+ 0·025	0·000625
182	8·075	8·2	− 0·125	0·015625
274	10·375	10·3	+ 0·075	0·005625
				0·0225

$$\therefore M = \pm 0{\cdot}106.$$

Weight of $b = p_y = [bb]_2 = 41{,}960$; $M_b = \pm \cdot 106/\sqrt{41{,}960} = \pm \cdot 0004$.

Weight of $a = p_x = \dfrac{[aa]_1}{[bb]_1} = 1{\cdot}5$; $M_a = \pm \cdot 106/\sqrt{1{\cdot}5} = \pm \cdot 087$.

(2) The following equations were proposed by Gauss to illustrate the above method [Gauss' *Theoria motus corporum coelestium* (Hamburg, 1809); Gauss' *Werke*, **7,** 240, 1871]:

$$\left.\begin{aligned} x - y + 2z = 3; \quad 4x + y + 4z = 21; \\ 3x + 2y - 5z = 5; \; -x + 3y + 3z = 14. \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

Hence show that $x = +\,2{\cdot}470$; $y = +\,3{\cdot}551$; $z = +\,1{\cdot}916$. $\Sigma(v^2) = 0804$; $M = \pm\,284$; $p_x = 246$; $p_y = 136$; $p_z = 539$; $M_x = \pm \cdot 057$; $M_y = \pm \cdot 077$; $M_z = \pm \cdot 039$. Hint. The first set of normal equations is

$$27x + 6y = 88; \; 6x + 15y + z = 70; \; y + 54z = 107.$$

(3) The following equations were proposed by Gauss (*l.c.*) to illustrate his method of solution:—

$$\left.\begin{aligned} x - y + 2z &= 3, \text{ with weight } 1; \\ 3x + 2y - 5z &= 5, \quad ,, \quad 1; \\ 4x + y + 4z &= 21, \quad ,, \quad 1; \\ -2x + 6y + 6z &= 28, \quad ,, \quad \tfrac{1}{4}. \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

By the rule, multiply the last equation by $\sqrt{\frac{1}{4}} = \frac{1}{2}$ and we get set (13). Show that $x = + 2·47$ with a weight $24·6$; $y = + 3·55$ with a weight $13·6$; and $z = + 1·9$ with a weight $53·9$. It only remains to substitute these values of x, y, z, in (14) to find the residuals v. Hence show that $M = \pm 295$. Proceed as before for M_x, M_y, M_z.

(4) The length (l) of a seconds pendulum at any latitude L, is given by Clairaut's equation:

$$l = L_0 + A \sin^2 L,$$

where L_0 and A are constants to be evaluated from the following observations:

$$L = 0° \ 0', \quad 18° \ 27', \quad 48° \ 24', \quad 58° \ 15', \quad 67° \ 4';$$
$$l = 0·990564, \ 0·991150, \ 0·993867, \ 0·994589, \ 0·995325.$$

Hence show that

$$l = 0·990555 + 0·005679 \sin^2 L.$$

Hint. The normal equations are,

$$x + 0·44765 \ y = 0·993099; \ x + 0·70306 \ y = 0·994548.$$

The above is based on the principle of least squares. A quicker method, not so exact, but accurate enough for most practical purposes, is due to Mayer. We can illustrate **Mayer's method** by means of equations (13).

First make all the coefficients of x positive, and add the results to form a new equation in x. Similarly for equations in y and z. We thus obtain,

$$\left.\begin{aligned} 9x - \ y - \ 2z &= 15; \\ 5x + 7y \qquad &= 37; \\ x + \ y + 14z &= 33. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Solve this set of simultaneous equations by algebraic methods and we get $x = 2·485$; $y = 3·511$; $z = 1·929$. Compare these values of x, y, z, with the best possible values for these magnitudes obtained in example (2).

§ 187. When to Reject Suspected Observations.

There can be no question about the rejection of observations which include some mistake, such as a wrong reading of the eudiometer or burette, a mistake in adding up the weights or a blunder in the arithmetical work, provided the mistake can be detected by check observations or calculations. Sometimes a most exhaustive search will fail to reveal any reason why some results diverge in an unusual and unexpected manner from the others. It has long been a vexed question how to deal with abnormal errors in a set of observations, for these can only be conscientiously rejected when the mistake is perfectly obvious. It would be a dangerous thing to permit an inexperienced or biassed worker to exclude some of his observations simply because they do not fit in with the majority. "Above all things," said the late Prof. Holman in his *Discussion on the Precision of Measurements* (Wiley & Sons, New York, 1901), "the integrity of the observer must be beyond question if he would have his results carry any weight, and it is in the matter of the rejection of doubtful or discordant observations that his integrity in scientific or technical work meets its first test. It is of hardly less importance that he should be as far as possible free from bias due either to preconceived opinions or to unconscious efforts to obtain concordant results."

Several criteria have been suggested to guide the investigator in deciding whether doubtful observations shall be included in the mean. Such criteria have been deduced by Chauvenet, Hagen, Stone, Pierce, etc. None of these tests, however, is altogether satisfactory. **Chauvenet's criterion** is perhaps the simplest to understand and most convenient to use. It is an attempt to show, from the theory of probability, that reliable observations will not deviate from the arithmetical mean beyond certain limits.

From (2) and (6), § 178,

$$r = 0{\cdot}4769/h = 0{\cdot}6745\sqrt{\Sigma(v^2)/(n-1)}.$$

If $x = rt$, where rt represents the number of errors less than x which may be expected to occur in an extended series of observations when the total number of observations is taken as unity, r represents the probable error of a single observation. Any measurement containing an error greater than x is to be rejected. If n denotes the number of observations and also the number of errors, then nP indicates the number of errors less than rt, and $n(1-P)$ the number of errors greater than the limit rt. If this number is less than $\frac{1}{2}$, any error rt will have a greater probability against than for it, and, therefore, may be rejected.

The criterion for the rejection of a doubtful observation is, therefore,

$$x/r = t;\ \tfrac{1}{2} = n(1-P);$$

whence

$$P = \frac{2n-1}{2n} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt. \qquad (1)$$

By a successive application of these formulae, two or more doubtful results may be tested.

The value of t, or, what is the same thing, of P, and hence also of n, can be read off from the table of integrals, page 515 (Table XI.). Table XII. contains the numerical value of x/r corresponding to different values of n.

Examples.—(1) The result of 13 determinations of the atomic weight of oxygen made by the same observer is shown in the first column of the subjoined table. Should 19·81 be rejected? Calculate the other two columns of the table in the usual way.

Observation.	x.	x^2.	Observation.	x.	x^2.
15·96	− 0·26	0·0676	15·88	− 0·34	0·1156
19·81	+ 3·59	12·8881	15·86	− 0·36	0·1296
15·95	− 0·27	0·0729	16·01	− 0·21	0·0441
15·95	− 0·27	0·0729	15·96	− 0·26	0·0676
15·91	− 0·31	0·0961	15·88	− 0·34	0·1156
15·88	− 0·34	0·1156	15·93	− 0·29	0·0841
15·91	− 0·31	0·0961			
Mean of 13 observations = 16·22 ; $\Sigma(x^2)$ = 13·9659.					

The deviation of the suspected observation from the mean, is 3·59. By Chauvenet's criterion, probable error = r = ·7281, n = 13. From Table XII., x/r = 3·07, $\therefore x$ = 3·07 × ·7281 = 2·24. Since the observation 19·81 deviates from the mean more than the limit 2·24 allowed by Chauvenet's criterion, that observation must be rejected.

(2) Should 16·01 be rejected from the preceding set of observations? Treat the twelve remaining after the rejection of 19·81 exactly as above.

(3) Should the observations 0·3902 and 0·3840 in Rudberg's results, page 441, be retained?

(4) Do you think 203·666 in Crookes' data, page 445, is affected by some "mistake"?

(5) Would Rowland have rejected the "442·8" result in Joule's work, page 441, if he had been solely guided by Chauvenet's criterion?

(6) Some think that "4·88" in Cavendish's data, page 466, is a mistake. Would you reject this number if guided by the above criterion?

These examples are given to illustrate the method of applying the criterion. Nothing more. Any attempt to establish an arbitrary criterion applicable to all cases, by eliminating the knowledge of the investigator, must prove unsatisfactory. It is very questionable if there can be a better guide than the unbiassed judgment and common-sense of the investigator himself.

Any observation set aside by reason of its failure to comply with any test should always be recorded. As a matter of fact, the rare occurrence of abnormal results serves only to strengthen the theory of errors developed from the empirical formula, $y = ke^{-h^2x^2}$. There can be no doubt that as many positive as negative chance deviations would appear if a sufficient number of measurements were available.* "Every observation," says Gerling in his *Die Ausgleichungs-Rechnungen der praktischen Geometrie* (Hamburg, 68, 1843), "suspected by the observer is to me a witness of its truth.

* Edgeworth has an interesting paper "On Discordant Observations" in the *Phil. Mag.* [5], **23**, 364, 1887.

He has no more right to suppress its evidence under the pretence that it vitiates the other observations than he has to shape it into conformity with the majority." The whole theory of errors is founded on the supposition that a sufficiently large number of observations has been made to locate the errors to which the measurements are susceptible. When this condition is not fulfilled, the abnormal measurement, if allowed to remain, would exercise a disproportionate influence on the mean. The result would then be less accurate than if the abnormal deviation had been rejected. *The employment of the above criterion is, therefore, permitted solely because of the narrow limit to the number of observations.* It is true that some good observations may be so lost, but that is the price paid to get rid of serious mistakes.

It is perhaps needless to point out that a suspected observation may ultimately prove to be a real exception requiring further research. To ignore such a result is to reject the clue to a new truth. The trouble Lord Rayleigh recently had with the density of nitrogen prepared from ammonia is now history. The "ammonia" nitrogen was found to be $\frac{1}{1,000}$th part lighter than that obtained from atmospheric air. Instead of putting this minute "error" on one side as a "suspect," Lord Rayleigh persistently emphasised the discrepancy, and thus opened the way for the brilliant work of Ramsay and Travers on "Argon and Its Companions".

CHAPTER XII.

COLLECTION OF FORMULAE FOR REFERENCE.

§ 188. Laws of Indices and Logarithms.

THE average student of chemical science is compelled to take a course in pure mathematics. But after passing his "intermediate," all is forgotten except a strong prejudice that mathematics is a compilation of vexatious puzzles. This is to be regretted, because with very little, if any, more drilling the later chapters of mathematics would be found invaluable auxillaries in the inquiry into those very phenomena to which he subsequently devotes his attention.

Certain sections of this chapter have been written to give the student of this work the opportunity of revising some of the more fundamental principles established in elementary mathematics; other sections are only for reference upon special occasions.

To continue the discussion opened at the commencement of § 16, page 34,

$4 \times 4 = 16$, is the *second power* of 4, written 4^2;
$4 \times 4 \times 4 = 64$, is the *third power* of 4, written 4^3;
$4 \times 4 \times 4 \times 4 = 256$, is the *fourth power* of 4, written 4^4;

and in general, the *nth power* of any number a, is defined as the continued product

$$a \times a \times a \times \ldots n \text{ times} = a^n,$$

where n is called the **exponent** or **index** of the power.

By actual multiplication, therefore,

$$10^2 \times 10^3 = 10^{2+3} = 10^5 = 100,000;$$

or, in general symbols,

$$a^m \times a^n = a^{m+n}; \text{ or, } a^x \times a^y \times a^z \times \ldots = a^{(x+y+z+\ldots)},$$

a result known as the **index law.** Again,

$$3 \times 5 = (10^{0 \cdot 4771}) \times (10^{0 \cdot 6990}) = 10^{1 \cdot 1761} = 15,$$

because, from a table of common logarithms,

$$\log_{10}3 = 0 \cdot 4771; \ \log_{10}5 = 0 \cdot 6990; \ \log_{10}15 = 1 \cdot 1761.$$

Thus we have performed arithmetical multiplication by the simple addition of two logarithms. To generalise :

To multiply two or more numbers, *add the logarithms of the numbers and find the number whose logarithm is the sum of the logarithms just obtained.*

EXAMPLES.—(1) Evaluate 4×80.

$$\log_{10} 4 = 0{\cdot}6021$$
$$\log_{10} 80 = 1{\cdot}9031$$
$$\text{Sum} = 2{\cdot}5052 = \log_{10} 320.$$
$$\therefore \text{Ansr.} = 320.$$

This method of calculation holds good whatever numbers we employ in place of 3 and 5 or 4 and 80. Hence the use of logarithms for facilitating numerical calculations. We shall shortly show how the operations of division, involution, and evolution are as easily performed as the above multiplication.

(2) Show $\log_e e = 1$, $\log_e 1 = 0$.

Just as $1 = 10^0,\ 2 = 10^{0{\cdot}301},\ 3 = 10^{0{\cdot}477},\ \ldots;$

so is $1 = e^0,\ 2 = e^{0{\cdot}6932},\ 3 = e^{1{\cdot}0986},\ \ldots;$

where $e = 2{\cdot}71828\ \ldots$ Hence by the definition of logarithms,

$$\log_e 3 = 1{\cdot}0986;\ \log_e 2 = 0{\cdot}6932;\ \log_e 1 = 0.$$

Again

$$e \times e \times e \times \ldots n \text{ times} = e^n;\ \ldots;\ e \times e \times e = e^3;\ e \times e = e^2;\ e = e^1;$$

or $\log_e e^n = n;\ \ldots;\ \log_e e^3 = 3;\ \log_e e^2 = 2;\ \log_e e^1 = 1 = \log_e e.$

From the above it also follows that

$$\frac{10^3}{10^2} = 10^{3-2} = 10^1 = 10;\ \text{or generally, } \frac{a^m}{a^n} = a^{m-n}.$$

Hence the rule :

To divide two numbers, *subtract the logarithm of the divisor (denominator of a fraction) from the logarithm of the dividend (numerator of a fraction) and find the number corresponding to the resulting logarithm.*

EXAMPLES.—(1) Evaluate $60 \div 3$.

$$\log_{10} 60 = 1{\cdot}7782$$
$$\log_{10} 3 = 0{\cdot}4771$$
$$\text{Difference} = 1{\cdot}3011 = \log_{10} 20.$$
$$\text{Ansr.} = 20.$$

(2) Show that $2^{-2} = \frac{1}{4};\ 10^{-2} = \frac{1}{100};\ 3^3 \times 3^{-3} = 1.$

It is very easy to miss the meaning of the so-called "properties of indices," unless the general symbols of the textbooks are thoroughly tested by translation into numerical examples. The majority of students require a good bit of practice before a general expression * appeals to them with full force. Here, as elsewhere, it is not merely necessary for the student to think that he "understands the principle of the thing," he must actually work out examples for himself. "In scientiis ediscendis prosunt exempla magis quàm præcepta" † is as true to-day as it was in Newton's time. For example, how many realise why mathematicians write $e^0 = 1$, until some such illustration as the following has been worked out?

$$2^2 \times 2^0 = 2^{2+0} = 2^2 = 4.$$

The same result, therefore, is obtained whether we multiply 2^2 by 2^0 or by 1, *i.e.*,

$$2^2 \times 2^0 = 2^2 \times 1 = 2^2 = 4.$$

Hence it is inferred that

$$2^0 = 1, \text{ and generally that } a^0 = 1.‡$$

I am purposely using the simplest of illustrations, leaving the reader to set himself more complicated numbers. No pretence is made to rigorous demonstration. We assume that what is true in one case, is true in another. It is only by so collecting our facts one by one that we are able to build up a general idea. The beginner should always satisfy himself of the truth of any abstract principle or general formula by applying it to particular and simple cases.

By actual multiplication show that

$$(100)^3 = (10^2)^3 = 10^{2 \times 3} = 10^6,$$

and hence:

To raise a number to any power, *multiply the logarithm of the number by the index of the power and find the number corresponding to the resulting logarithm.*

---

* The general symbols $a, b, \ldots m, n, \ldots x, y, \ldots$ in any general expression may be compared with the blank spaces in a bank cheque waiting to have particular values assigned to date, amount (£ s. d.), and sponsor, before the cheque can fulfil the specific purpose for which it was designed. So must the symbols, $a, b, \ldots$ of a general equation be replaced by special numerical values before the equation can be applied to any specific process or operation.

† Which may be rendered: "In learning we profit more by example than by precept".

‡ Some mathematicians define a^n as $1 \times a \times a \times a \ldots n$ times; $a^3 = 1 \times a \times a \times a$; $a^2 = 1 \times a \times a$; $a^1 = 1 \times a$; and a^0 as $1 \times a$ no times, that is unity itself. If so, then 0^0 would mean 1×0 no times, *i.e.*, 1; $1/0^0$ would mean $1/(1 \times 0$ no times), *i.e.*, unity. But see examples, § 5.

EXAMPLE.—Evaluate 5^2.

$$5^2 = (5)^2 = (10^{0\cdot6990})^2 = 10^{1\cdot3980} = 25,$$

since reference to a table of common logarithms shows that

$$\log_{10}5 = 0\cdot6990;\ \log_{10}25 = 1\cdot3980.$$

From the index law, above

$$10^{\frac{1}{2}} \times 10^{\frac{1}{2}} = 10^{\frac{1}{2}+\frac{1}{2}} = 10^1 = 10.$$

That is to say, $10^{\frac{1}{2}}$ multiplied by itself gives 10. But this is the definition of the square root of 10.

$$\therefore (\sqrt{10})^2 = \sqrt{10} \times \sqrt{10} = 10^{\frac{1}{2}} \times 10^{\frac{1}{2}} = 10.$$

A **fractional index**, therefore, represents a root of the particular number affected with that exponent. Generalising this idea, the nth root of any number a, is $a^{\frac{1}{n}}$. Thus

$$\sqrt[3]{8} = 8^{\frac{1}{3}}, \text{ because } \sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8.$$

To extract the root of any number, *divide the logarithm of the number by the index of the required root and find the number corresponding to the resulting logarithm.*

EXAMPLES.—(1) Evaluate $\sqrt[3]{8}$ and $\sqrt[7]{93}$.

$$\sqrt[3]{8} = (8)^{\frac{1}{3}} = (10^{0\cdot9031})^{\frac{1}{3}} = 10^{0\cdot3010} = 2;$$

$$\sqrt[7]{93} = (93)^{\frac{1}{7}} = (10^{1\cdot9685})^{\frac{1}{7}} = 10^{0\cdot2812} = 1\cdot91,$$

since, from a table of common logarithms,

$$\log_{10}2 = 0\cdot3010;\ \log_{10}8 = 0\cdot9031;\ \log_{10}1\cdot91 = 0\cdot2812;\ \log_{10}93 = 1\cdot9685.$$

(2) Repeat all the above illustrations of the index law using Table XXIV., page 520.

The results of logarithmic calculations are seldom absolutely correct because we employ approximate values of the logarithms of the particular numbers concerned. Instead of using logarithms to four decimal places we could, if stupid enough, use logarithms accurate to sixty-four decimal places. But this question is reserved for the next section.

The more important properties of indices known under the name "the theory of indices" are summarised in the subjoined synopsis along with the corresponding properties of logarithms.

Theory of Indices.	Logarithms.	
$a^0 = 1$.	$\log 1 = 0$.	(1)
$a^1 = a$.	$\log_a a = 1$.	(2)
$a^{\frac{1}{2}} = \sqrt{a}$.	$\log_a(\sqrt{a}) = \frac{1}{2}\log_a a = \frac{1}{2}$.	(3)
$a^n = a \times a \times a \times \ldots n$ times.	$\log a^n = n \log a$.	(4)
$a^\infty = \infty$, if $a > 1$.	$\log \infty = \infty$.	(5)
$a^{-0} = -1$.	$\log(-1) = 0$.	(6)
$a^{-1} = 1/a$.	$\log_a 1/a = -1 \log_a a = -1$.	(7)
$a^{-n} = 1/a^n$.	$\log_a a^{-n} = -n \log_a a = -n$.	(8)
$a^{-\frac{1}{2}} = 1/\sqrt{a}$.	$\log 1/\sqrt{a} = -\frac{1}{2}\log_a a = -\frac{1}{2}$.	(9)
$a^{-\infty} = 0$, if $a > 1$.	$\log 0 = -\infty$.	(10)
$a^n a^m = a^{(n+m)}$.	$\log ab = \log a + \log b$.	(11)
$a^n b^n = (ab)^n$.	$\log(ab)^n = n \log a + n \log b$.	(12)
$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$.	$\log \sqrt[mn]{a} = \frac{1}{mn}\log a$.	(13)
$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$.	$\log \sqrt[n]{ab} = \frac{1}{n}\log a + \frac{1}{n}\log b$.	(14)
$a^n/a^m = a^{(n-m)}$.	$\log a/b = \log a - \log b$.	(15)
$(a^n)^m = a^{mn}$.	$\log a^n = n \log a$.	(16)
$\sqrt[m]{a^n} = a^{\frac{n}{m}} = (\sqrt[m]{a})^n$.	$\log \sqrt[n]{a} = \frac{1}{n}\log a$.	(17)

EXAMPLE.—Plot $\log_e x = y$, and show that logarithms of negative numbers are impossible. Hint, put $x = 0, 1/e^2, 1/e, 1, e, e^2, \infty$, etc., and find corresponding values of y.

NOTE.—Continental writers variously use the symbols L, l, ln, lg, for "log"; and "log nep" or "log nat" for "$\log_e$". "Nep" is an abbreviation for "Neperian," a Latinized adjectival form of Napier's name.

"Exp x" is sometimes written for "e^x"; "Exp$(-x)$" for "e^{-x}".

§ 189. Approximate Calculations in Scientific Work.

A good deal of the tedious labour involved in the reduction of experimental results to their final form, may be avoided by attention to the degree of accuracy of the measurements under consideration. It is one of the commonest of mistakes to extend the arithmetical work beyond the degree of precision attained in the practical work.* Thus, Dulong calculated his indices of refraction to eight digits when they agreed only to three. When asked "Why?", Dulong returned the ironical answer: "I see no reason

* In a memoir "On the Atomic Weight of Aluminum," at present before me, I read, "·646 grm. of aluminum chloride gave 2·0549731 grms. of silver chloride . . .". It is not clear how the author obtained his seven decimals seeing that, in an earlier part of the paper, he expressly states that his balance was not sensitive to more than ·0001 grm.

for suppressing the last decimals, for, if the first are wrong, the last may be all right"!

Although the measurements of a Stas, or of a Whitworth may require six or eight decimal figures, few observations are correct to more than four or five. But even this degree of accuracy is only obtained by picked men working under special conditions. Observations which agree to the second or third decimal place are comparatively rare in chemistry.

Again, the best of calculations is a more or less crude approximation on account of the "simplifying assumptions" introduced when deducing the formula to which the experimental results are referred. It is, therefore, no good extending the "calculated results" beyond the reach of experimental verification. It is unprofitable to demand a greater degree of precision from the calculated than from the observed results—but one ought not to demand a less. (Compare the introduction to Poincaré's *Mécanique Céleste.*)

The general rule in scientific calculations is to use one more decimal figure than the degree of accuracy of the data. In other words, reject as superfluous all decimal figures beyond the first doubtful digit. The remaining digits are said to be **significant figures.**

EXAMPLES.—In 1·540, there are four significant figures, the cypher indicates that the magnitude has been measured to the thousandth part; in 0·00154, there are three significant figures, the cyphers are added to fix the decimal point; in 15,400, there is nothing to show whether the last two cyphers are significant or not, there may be three, four, or five significant figures.

In "casting off" useless decimal figures, the last digit retained must be increased by unity when the following digit is greater than four. We must, therefore, distinguish between 9·2 when it means exactly 9·2, and when it means anything between 9·14 and 9·25. In the so-called "exact sciences," the latter is the usual interpretation. Quantities are assumed to be equal when the differences fall within the limits of experimental error.

LOGARITHMS.—There are very few calculations in practical work outside the range of four or five figure logarithms. The use of more elaborate tables may, therefore, be dispensed with.*

* Thus Wrapson and Gee's *Mathematical Tables* (1s. 6d.) to four decimal places may be used instead of Chamber's (page 37) to seven decimal places.

ADDITION AND SUBTRACTION.—In adding such numbers as 9·2 and 0·4913, cast off the 3 and the 1, then write the answer, 9·69, not 9·6913. Show that $5·60 + 20·7 + 103·193 = 129·5$, with an error of about 0·01, that is about 0·08 $^\circ/_\circ$.

MULTIPLICATION AND DIVISION.—The product $2·25\pi$ represents the length of the perimeter of a circle whose diameter is 2·25 units; π is a numerical coefficient whose value has been calculated by Shanks,* to over seven hundred decimal places, so that $\pi = 3·141592,653589,793$. . . . Of these two numbers, therefore, 2·25 is the less reliable. Instead of the ludicrous 7·0685808625 . . ., we simply write the answer, 7·07.

It is no doubt unnecessary to remind the reader that in scientific computations the standard arithmetical methods of multiplication and division are abbreviated so as to avoid writing down a greater number of digits than is necessary to obtain the desired degree of accuracy. The following scheme for "**shortened multiplication and division**," requires little or no explanation:—

Shortened Multiplication.

$$\begin{array}{r} 9·774 \\ 365·4 \\ \hline 2932·2 \\ 586·4 \\ 48·\bar{9} \\ 3·9 \\ \hline 3571·\bar{4} \end{array}$$

Shortened Division.

$$\begin{array}{r} 365·4)3571·3(9·774 \\ 3288·6 \\ \hline 282·7 \\ 255·8 \\ \hline 26·9 \\ 25·5 \\ \hline 1·4 \end{array}$$

The digits of the multiplier are taken from left to right, not right to left. One figure less of the divisor is used at each step of the division. The last figure of the quotient is obtained mentally. A "bar" is usually placed over strengthened figures so as to allow for an excess or defect of them in the result.

Ostwald, in his *Hand- und Hilfsbuch zur Ausführung physikochemiker Messungen* (Leipzig, 1893), has said that "the use of these methods cannot be too strongly emphasised. The ordinary methods of multiplication and division must be termed unscientific." Full details are given in Langley's booklet *A Treatise on Computation* (Longmans, Green & Co., 1895), or in the more formal *Calculs pratiques appliques aux Sciences d'Observation*, by Babinet and Housel.

The error introduced in approximate calculations by the "casting off" of decimal figures. Some care is required in rounding off decimals to avoid an excess or defect of strengthened figures by making the positive and negative errors neutralise each other in the final result. A good "dodge" is always to leave the last figure

* *Proc. Roy. Soc.*, **32**, 45, 1873.

an even number. *E.g.*, 3·75 would become 3·8, while 3·85 would be written 3·8.

The percentage error of the product of two approximate numbers is very nearly the algebraic sum of the percentage error of each. If the positive error in the one be numerically equal to the negative error in the other, the product will be nearly correct, the errors neutralise each other.

EXAMPLE.—$19·8 \times 3·18$. The first factor may be written 20 with a + error of 1 °/$_o$, and, therefore, $20 \times 3·18 = 63·6$, with a + error of 1 °/$_o$. This excess must be deducted from 63·6. We thus obtain 62·95. The true result is 62·964.

The percentage error of the quotient of two approximate numbers is obtained by subtracting the percentage error of the numerator from that of the denominator. If the positive error of the numerator is numerically equal to the positive error of the denominator, the error in the quotient is practically neutralised.

Vide footnote, page 454.

APPROXIMATION FORMULAE—CALCULATIONS WITH SMALL QUANTITIES.—The discussion on approximate calculations in Chapter V. renders any further remarks on the deduction of the following formulae superfluous:

For the sign of equality, read "is approximately equal to," or "is very nearly equal to". Let α, β, γ, . . . be small fractions in comparison with unity or x.

$$(1 \pm \alpha)(1 \pm \beta) = 1 \pm \alpha \pm \beta. \qquad (1)$$

$$(1 \pm \alpha)(1 \pm \beta)(1 \pm \gamma) \ldots = 1 \pm \alpha \pm \beta \pm \gamma \pm \ldots \qquad (2)$$

$$(1 \pm \alpha)^2 = 1 \pm 2\alpha;\ (1 \pm \alpha)^n = 1 \pm n\alpha. \qquad (3)$$

$$\sqrt{(1 + \alpha)} = 1 + \tfrac{1}{2}\alpha. \quad \sqrt{\alpha\beta} = \tfrac{1}{2}(\alpha + \beta). \qquad (4)$$

$$\frac{1}{(1 \pm \alpha)} = 1 \mp \alpha;\ \frac{1}{(1 \pm \alpha)^n} = 1 \mp n\alpha;\ \frac{1}{\sqrt{(1 + \alpha)}} = 1 - \tfrac{1}{2}\alpha. \qquad (5)$$

$$\frac{(1 \pm \alpha)(1 \pm \beta)}{(1 \pm \gamma)(1 \pm \delta)} = 1 \pm \alpha \pm \beta \mp \gamma \mp \delta. \qquad (6)$$

The third member of some of the following results is to be regarded as a second approximation, to be employed only when an exceptional degree of accuracy is required.

$$e^\alpha = 1 + \alpha;\ a^\alpha = 1 + \alpha \log a. \qquad (7)$$

$$\log(1 + \alpha) = \alpha = \alpha - \tfrac{1}{2}\alpha^2. \qquad (8)$$

$$\log(x + \alpha) = \log x + \alpha/x - \tfrac{1}{2}\alpha^2/x^2. \qquad (9)$$

$$\log\frac{x + \alpha}{x - \alpha} = \frac{2\alpha}{x} + \frac{2}{3}\cdot\frac{\alpha^2}{x^2}. \qquad (10)$$

By Taylor's theorem, § 99,

$$\sin(x + \beta) = \sin x + \beta\cos x - \tfrac{1}{2}\beta^2\sin x - \tfrac{1}{6}\beta^3\cos x + \ldots$$

If the angle β is not greater than $2\frac{1}{2}°$, $\beta < \cdot 044$; $\frac{1}{2}\beta^2 < \cdot 001$; $\frac{1}{6}\beta^3 < \cdot 00001$. But $\sin x$ does not exceed unity, therefore, we may look upon

$$\sin(x + \beta) = \sin x + \beta \cos x,$$

correct up to three decimal places. The addition of another term "$- \frac{1}{2}\beta^2$" will make the result correct to the fifth decimal place.

$$\sin \alpha = \alpha = \alpha(1 - \tfrac{1}{6}\alpha^2)\text{ ; } \cos \alpha = 1 = 1 - \tfrac{1}{2}\alpha^2. \quad . \quad . \quad . \quad . \quad (11)$$

$$\sin (x \pm \beta) = \sin x \pm \beta \cos x\text{ ; } \cos (x \pm \beta) = \cos x \pm \beta \sin x. \quad . \quad (12)$$

$$\tan \alpha = \alpha = \alpha(1 + \tfrac{1}{3}\alpha^2)\text{ ; } \tan (x \pm \beta) = \tan x \pm \beta \sec^2 x. \quad . \quad . \quad (13)$$

EXAMPLE.—Show that the square root of the product of two small fractions is very nearly equal to half their sum. See (4). Hence, at sight, $\sqrt{24 \cdot 00092 \times 24 \cdot 00098} = 24 \cdot 00095$.

§ 190. Variation.

When two quantities are so related that any increase (or decrease) in the value of one produces a proportional increase (or decrease) in the other, the one quantity is said to **vary as,** or to **vary directly as** the other. On the other hand, when two quantities are so related that any increase (or decrease) in the one leads to a proportional decrease (or increase) in the other, the one quantity is said to **vary inversely as** the other.

The symbol "$\propto$" denotes variation. For $x \propto y$, read "x varies as y"; for $x \propto \frac{1}{y}$, read "x varies inversely as y".

The variation notation is nothing but abbreviated proportion. Let x_1, y_1; x_2, y_2; x_3, y_3; . . . be corresponding values of x and y. Then, if x varies as y,

$$x_1 : y_1 = x_2 : y_2 = x_3 : y_3 = \ . \ . \ .; \text{ or, } \frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \ . \ . \ . \quad (1)$$

1. *If x varies as y, directly or inversely, then x is equal to y, or $1/y$, multiplied by some constant.*

Let κ be a constant quantity. If

$$x \propto y,\ x = \kappa y\text{ ; or, if } x \propto \frac{1}{y},\ x = \frac{\kappa}{y}. \quad . \quad . \quad . \quad (2)$$

This result is of the greatest importance. It is used in nearly every formula representing a physical process. κ is called the **proportion constant** or **constant of variation.**

The proof follows directly from (1), the ratio of any value of x to the corresponding value of y is always the same. This means that $x/y = \text{constant}$.

2. *If one magnitude varies as another, any two simultaneous values of the two magnitudes are in the same proportion.*

This also follows directly from (1). If x and y are simultaneous values of X and Y so that when X changes to x_1 Y changes to y_1.

$$x : y = x_1 : y_1.$$

3. If $x \propto y$, then $y \propto x$. . . . . (3)
4. If $x \propto y$, $y \propto z$, then $x \propto z$. . . . . (4)
5. If $x \propto yz$, then $y \propto x/z$ and $z \propto x/y$. . . . . (5)
6. If $x \propto z$, $y \propto z$, then $x \pm y \propto z$; . . . . (6)

and $xy \propto z^2$. . . . . (7)

7. If $x \propto y$, then $xm \propto ym$, where m is constant. . . . . (8)
8. If $x \propto y$, $u \propto v$, then $xu \propto yv$, or, $x/u \propto y/v$. . . . . . (9)

9. *If x, y, z, are variable magnitudes such that $x \propto y$, when z is constant, $x \propto z$, when y is constant, then $x \propto yz$, when y and z vary together.*

Let X have a value x, when Y has a value y, and Z a value z. Let X change its value from x to x_1, when Y changes from y to y_1 and Z remains constantly equal to z. Again, let X change from x_1 to x_2, when Y remains constantly equal to y and Z changes from z to z_2. From (1)

$$x/x_1 = y/y_1\,;\ \text{and}\ x_1/x_2 = z/z_2.$$

Multiply these two equations together.

$$x : x_2 = yz : y_1z_2,$$

that is to say, when YZ changes from yz to y_1z_2, X changes from x to x_2 so that x, x_2, yz, y_1z_2 are proportionals. Hence,

$$X \propto YZ. \qquad (10)$$

10. *If x varies as y when z is constant, and x varies inversely as z when y is constant, then*

$$X \propto Y/Z, \qquad (11)$$

when Y and Z both vary.

Examples.—(1) It is known that the volume v of a mass of gas varies inversely as the pressure p at a constant temperature θ (Boyle's law), or,

$$v \propto 1/p, \qquad (\theta\ \text{constant}).$$

Again, the volume of any mass of gas varies directly as the absolute temperature θ, when the pressure is constant (Gay Lussac's law), *i.e.*,

$$v \propto \theta, \qquad (p\ \text{constant}).$$

Hence show, by equations (2), (5), and (11), that when p and θ both vary

$$pv = R\theta, \qquad (12)$$

where R is the constant of proportion. It is by no means uncommon to find this simple formula deduced by a vicious combination of Boyle's and Gay Lussac's laws. The results expressed in formulae (7) and (10) are confused.

(2) Show, as in (1), that

$$p_0v_0/\theta_0 = p_1v_1/\theta_1. \qquad (13)$$

(3) If the density ρ of a gas is directly proportional to the pressure at constant temperature, show that

$$p = R\rho\theta\,;\ \text{and}\ p_0/\rho_0\theta_0 = p_1/\rho_1\theta_1. \qquad (14)$$

(4) In a current textbook on *The Theory of Solutions* it is shown that

$$v = v_0(1 + \alpha\theta)\,;\ p = p_0(1 + \alpha\theta).$$

The work then continues: "By uniting these two (equations) we obtain

$$pv = p_0v_0(1 + \alpha\theta)\ ".$$

Point out the fallacy in this demonstration.

§ 191. Permutations and Combinations.

Each arrangement that can be made by varying the order of some or all of a number of things is called a **permutation.** For instance, there are two permutations of two things a and b, namely ab and ba; a third thing can be added to each of these two permutations in three ways so that abc, acb, cab, bac, bca, cba results. The permutations of three things taken all together is, therefore, $1 \times 2 \times 3$; a fourth thing can occupy four different places in each of these six permutations, or, there are $1 \times 2 \times 3 \times 4$ permutations when four different things are taken all together. More generally, the permutations of n things taken all together is

$$n(n-1)(n-2)\ .\ .\ .\ 3.2.1 = n!$$

$n!$ is called "factorial n". It is generally *written* $\underline{|n}$.*

Using the customary notation ${}_nP_n$ to denote the number of permutations of n things taken n at a time,

$$_{(number\ of\ things)}P_{(number\ of\ things\ taken)} = {}_nP_n = n!\ . \qquad (1)$$

If some of these n things are alike, say p of one kind, q of another, r of another,

$${}_nP_n = \frac{n!}{p!\ q!\ r!}. \qquad (2)$$

If only r of the n things are taken in each set,

$${}_nP_r = n(n-1)(n-2)\ .\ .\ .\ (n-r+1).$$

$$= \frac{n!}{(n-r)!}. \qquad (3)$$

Each set of arrangements which can be made by taking some or all of a number of things, without reference to the internal arrangement of the things in each group, is called a **combination.** In permutations, the variations, or the order of the arrangement of the different things, is considered; in combinations, attention is only paid to the presence or absence of a certain thing. The number of combinations of two things taken two at a time is one, because the set ab contains the same things as ba. The number

---

* It is worth knowing that

$$n! = \Gamma(n+1),$$

the gamma function of § 83. When n is very great

$$n! = n^n e^{-n} \sqrt{2\pi n},$$

known as **Stirling's formula.** This allows $n!$ to be evaluated by a table of logarithms. The error is of the order $\frac{1}{12}n$ of the value of $n!$

of combinations of three things taken two at a time is three, namely, *ab*, *ca*, *bc*; of four things, *ab*, *ac*, *ad*, *bc*, *bd*, *cd*. But when each set consists of r things, each set can be arranged in $r!$ different ways.

Let ${}_nC_r$ denote the number of combinations of n things taken r at a time. We observe that the ${}_nC_r$ combinations will produce ${}_nC_r \times r!$ permutations. This is the same thing as the number of permutations of n things in sets of r things. Hence, by (3),

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!}. \quad . \quad (4)$$

$$ = \frac{n!}{r!(n-r)!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Nearly all questions on arrangement and variety can be referred to the standard formulae (3) and (5). Special cases are treated in any textbook on algebra.

In spite of the great number of organic compounds continually pouring into the journals, chemists have, in reality, made no impression on the great number which might exist. To illustrate, Hatchett (*Phil. Trans.*, **93**, 193, 1803) has suggested that a systematic examination of all possible alloys of all the metals be made, proceeding from the binary to the more complicated ternary and quaternary. Did he realise the magnitude of the undertaking?

Examples.—(1) Show that if one proportion of each of thirty metals be taken, 435 binary, 4,060 ternary and 27,405 quaternary alloys would have to be considered.

(2) If four proportions of each of thirty metals be employed, show that 6,655 binary, 247,660 ternary and 1,013,985 quatenary alloys would have to be investigated.

The number of possible isomers in the hydrocarbon series involving side chains, etc., are discussed in the following memoirs: Cayley (*Phil. Mag.* [4], **13**, 172, 1857; **47**, 444, 1874; or, *British Association's Reports*, 257, 1875) first opened up this question of side chains. See also Lodge (*Phil. Mag.* [4], **50**, 367, 1875), Losanitsch (*Berichte der deutschen chemischen Gesellschaft*, **30**, 1,917, 1897), Hermann (*ib.*, 3,423), Rey (*ib.*, **33**, 1,910, 1900), Kauffman (*ib.*, 2,231).

§ 192. Mensuration Formulae.

Reference has frequently been made to Euclid i., 47. In any right-angled triangle,

(*Square on hypotenuse*) = (*Sum of squares on the other two sides*).

Also to Euclid vi., 4. If two triangles ABC and DEF are equiangular so that the angles at A, B, and C of the one are respectively equal to the angles D, E, and F of the other, the sides about the equal angles are proportional, so that

$$AB : BC = DE : EF;\ BC : CA = EF : FD;\ AB : AC = DE : DF.$$

$\pi = 3{\cdot}1416$, or, $\frac{22}{7}$.
θ = degrees of arc.
r denotes the radius of a circle.
The following are standard reference formulae:

I. Lengths (*arcs and perimeters*).

CHORD OF CIRCLE (θ = angle subtended at centre) $= 2r \sin \frac{1}{2}\theta$. . (1)
ARC OF CIRCLE (θ = angle subtended) $= \frac{1}{180}\theta\pi r$. . . . . . (2)
PERIMETER OF CIRCLE $= 2\pi r = \pi \times (Diameter)$. . . . . (3)
PERIMETER OF ELLIPSE (semiaxes a, b) $= 2\pi\sqrt{\frac{1}{2}(a^2 + b^2)}$. . . (4)

II. Areas.

RECTANGLE (sides a, b) $= a . b$. . . . . (5)
PARALLELOGRAM (sides a, b; included angle θ) $= ab \sin\theta$. . . (6)
RHOMBUS $= \frac{1}{2}$ (*Product of the two diagonals*). . . . . . (7)
TRIANGLE (h = altitude; b = base)

$$\left.\begin{aligned} &= \tfrac{1}{2}h . b; \\ &= \tfrac{1}{2}ab \sin C; \\ &= \sqrt{s(s-a)(s-b)(s-c)}, \end{aligned}\right\} \quad (8)$$

where a, b, c, are the sides opposite the respective angles A, B, C, $s = \frac{1}{2}(a + b + c)$.

SPHERICAL TRIANGLE $= (A + B + C - \pi)r^2$, ... (9)
where r is the radius of the sphere, A, B, C, are the angles of the triangle (Fig. 142).

TRAPEZIUM (h = altitude; parallel sides a, b) $= \frac{1}{2}h(a + b)$. . . (10)
POLYGON OF n EQUAL SIDES $(a) = \frac{1}{4}na^2 \cot (180/n)$. . . . . (11)
CIRCLE $= \pi r^2 = \frac{1}{4}\pi \times (Diameter)^2$. . . . . (12)
CIRCULAR SECTOR (θ = included angle)

$$\left.\begin{aligned} &= (\tfrac{1}{2}\ Arc) \times (Radius); \\ &= \tfrac{1}{360}\pi\theta r^2. \end{aligned}\right\} \quad (13)$$

CIRCULAR SEGMENT

$$\left.\begin{aligned} &= (Area\ of\ sector) - (Area\ of\ triangle) \\ &= \tfrac{1}{360}\pi\theta r^2 - \tfrac{1}{2}r^2 \sin\theta. \end{aligned}\right\} \quad (14)$$

The triangle is made by joining the two ends of the arc to each other and to the centre of the circle. θ is angle at centre of circle.

PARABOLA CUT OFF BY DOUBLE ORDINATE $(2y)$

$$\left.\begin{aligned} &= \tfrac{4}{3}xy; \\ &= \tfrac{2}{3}\ (Area\ of\ parallelogram\ of\ same\ base\ and\ height). \end{aligned}\right\} \quad (15)$$

ELLIPSE $= \pi a . b$ . . . . (16)
CURVILINEAR AND IRREGULAR FIGURES. See Simpson's rule.
SIMILAR FIGURES. The areas of similar figures are as the squares of the corresponding sides. The area of any plane figure is proportional to the square of any linear dimension. *E.g.*, the area of a circle is proportional to the square of its radius.

III. Surfaces.

SPHERE $= 4\pi r^2$. . . . . (17)
CYLINDER (h = height) $= 2\pi rh$. . . . . (18)
PRISM (p = perimeter of the base) $= ph$. . . . . . (19)
CONE OR PYRAMID $= \frac{1}{2}p \times$ (*Slant height*). . . . . . (20)
SPHERICAL SEGMENT (h = height) $= 2\pi rh$. . . . . . (21)

IV. Volumes.

RECTANGULAR PARALLELOPIPED (sides a, b, c) $= a \,.\, b \,.\, c$. . . . (22)

SPHERE $= \frac{2}{3}$ (*Circumscribing cylinder*);
$= \frac{4}{3}\pi r^3 = 4{\cdot}189 r^3 = \frac{1}{6}\pi$ (*Diameter*)^3. (23)

SPHERICAL SEGMENT ($h =$ height) $= \frac{1}{3}\pi(3r - h)h^2$. . . . . (24)

CYLINDER OR PRISM $=$ (*Area of base*) $\times$ (*Height*);
$= \pi r^2 h$. (25)

CONE OR PYRAMID $= \frac{1}{3}$(*Circumscribing cylinder or prism*);
$=$ (*Area of base*) $\times \frac{1}{3}$ (*Height*);
$= \frac{1}{3}\pi r^2 h = 1{\cdot}047 r^2 h$. (26)

SIMILAR FIGURES. The volumes of similar solids are as the cubes of corresponding sides. The volume of any solid figure is proportional to the cube of any linear dimension. *E.g.*, the volume of a sphere is proportional to the cube of its radius.

V. Centres of Gravity.

PLANE TRIANGULAR LAMINA. Two-thirds the distance from the apex of the triangle to a point bisecting the base.

CONE OR PYRAMID. Three-fourths the distance from the apex to the centre of gravity of the base.

A tetrahedron is a pyramid with a triangular base (see next section).

§ 193. Bayer's "Strain Theory" of Carbon Ring Compounds.

This theory has attracted some attention amongst organic chemists. It is based upon the assumption that the four valencies of a carbon atom act only in the directions of the lines joining the centre of gravity of the atom with the apices of a regular tetrahedron. In other words, the chemical attraction between any two such atoms is exerted only along these four directions. When several carbon atoms unite to form ring compounds, the "direction of the attraction" is deflected. This is attended by a proportional strain. The greater the strain, the less stable the compound.

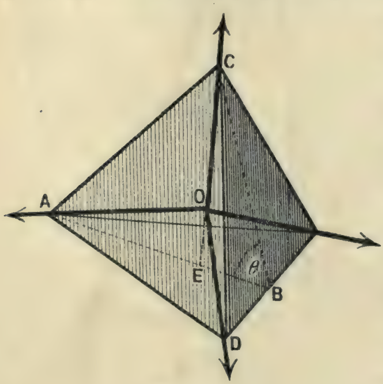


FIG. 130.

One does not readily take to the idea of a force acting round a corner, nevertheless, the theory has explained some facts. The stability of certain compounds does appear to be related with the theoretical deflection of this "direction of attraction". Apart from all questions as to the validity of the assumptions, we may find the angles of deflection of the "directions of attraction" for two to six ring compounds as an exercise in mensuration.

First, to find the angle between these "directions of attraction" at the centre of a carbon atom assumed to have the form of a regular tetrahedron.

Let s be the slant height of a regular tetrahedron (Fig. 130), h the vertical height, l the length of any edge, ϕ the angle made by the lines joining any two apices with the centre of the tetrahedron.

$$\therefore s^2 + (\tfrac{1}{2}l)^2 = l^2;\ s^2 = \tfrac{3}{4}l^2.$$

But h divides s in the ratio 2 : 1, hence (§ 191),

$$h^2 = l^2 - (\tfrac{2}{3}s)^2 = \tfrac{2}{3}l^2.$$

But $CD = 2BD = l$; $BC = AB = s$; $CE = h$. Hence, $h = \sqrt{\tfrac{2}{3}}l$. From a result in § 192, the middle of the tetrahedron cuts CE at O in the ratio 3 : 1.

$$\therefore \sin \tfrac{1}{2}\phi = \tfrac{1}{2}a/\tfrac{3}{4}h;\ \text{or, } \phi = 109^\circ\ 28'. \quad . \quad . \quad . \quad (1)$$

Second, to find the angle of deflection of the "direction of attraction," when 2 to 6 carbon atoms form a closed ring.

From (1), for *acetylene* $H_2C : CH_2$, the angle is deflected from 109° 28′ to $\frac{1}{2}$(109° 28′), or 55° 44′.

For *trimethylene*, assuming the ring is an equilateral triangle, the angle is deflected $\frac{1}{2}$ (109° 28′ − 60°) = 24° 44′.

For *tetramethylene*, assuming the ring is a square, the angle of deflection is $\frac{1}{2}$ (109° 28′ − 90°) = 9° 34′.

For *pentamethylene*, assuming the ring to be a regular pentagon, the angle of deflection is $\frac{1}{2}$ (109° 28′ − 108°), or 0° 44′.

For *hexamethylene*, assuming the ring is a regular hexagon, the angle of deflection is $\frac{1}{2}$ (109° 28′ − 120°), or −5° 76′.

EXAMPLE.—Find the value of the angle θ, in Fig. 130. Ansr. 70° 32′.

§ 194. Plane Trignometry.

Beginners in the calculus trip oftenest over the trignometrical work. The following outline will perhaps be of some assistance.

Trignometry deals with the relations between the sides and angles of triangles. If the triangle is drawn on a plane surface, we have plane trignometry; if the triangle is drawn on the surface of a sphere, spherical trignometry. The trignometry employed in physics and chemistry is a mode of reasoning about lines and angles, or rather, about quantities represented by lines and angles (whether parts of a triangle or not), which is carried on by means of certain ratios or functions of an angle.

1. **The measurement of angles.** An angle is formed by the intersection of two lines. The magnitude of an angle depends only on the relative directions, or slopes of the lines, and is independent of their lengths. In practical work, angles are usually measured in degrees, minutes and seconds. These units are the subdivisions of a right-angle defined as

1 right angle = 90 degrees, written 90°;
1 degree = 60 minutes, written 60′;
1 minute = 60 seconds, written 60″.

In theoretical calculations, however, this system is replaced by another.

In Fig. 131, the length of the circular arcs $P'A'$, PA, drawn from the centre O, are proportional to the lengths of the radii OA' and OA, or

$$\frac{\text{arc } P'A'}{\text{radius } OA'} = \frac{\text{arc } PA}{\text{radius } OA}.$$

If the angle at the centre O is constant, the ratio, arc/radius, is also constant. This ratio, therefore, furnishes a method for measuring the magnitude of an angle. The ratio

$$\frac{\text{arc}}{\text{radius}} = 1, \text{ is called a } \textbf{radian.}$$

Two right angles $= 180° = \pi$ radians, where $\pi = 180° = 3·14159. . . .$ The ratio, arc/radius, is called the **circular or radian measure of an angle.** (Radian = unit angle.)

2. **Relation between degrees and radians.** The circumference of a circle of radius r, is $2\pi r$, or, if the radius is unity, 2π. The angles 360°, 180°, 90°, . . . correspond to the arcs whose lengths are respectively 2π, π, $\frac{1}{2}\pi$, . . . If the angle AOP (Fig. 131) measures D degrees, or α radians,

$$D° : 360° = \alpha : 2\pi.$$

$$\therefore D° = \frac{\alpha}{2\pi}360 ; \text{ or } \alpha = \frac{D}{360}2\pi. \quad . \quad . \quad . \quad . \quad (1)$$

EXAMPLES.—(1) How many degrees are contained in an arc of unit length? Here $\alpha = 1$,

$$\therefore D = 360/2\pi = 57·295° = 57° \ 17' \ 44·8''.$$

(2) How many radians are there in 1°. Ansr. $\pi/180$; or ·0175.

(3) How many radians in $2\frac{1}{2}°$? Ansr. ·044.

3. **Trignometrical ratios of an angle as functions of the sides of a triangle.** There are certain functions of the angles, or rather of the arc PA (Fig. 131)

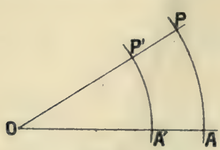


FIG. 131.

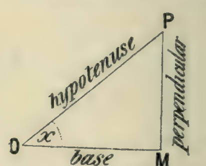


FIG. 132.

called trignometrical ratios. From P drop the perpendicular PM on to OM (Fig. 132). In the triangle OPM,

(i.) The ratio $\frac{MP}{OM}$, or, $\frac{\text{perpendicular}}{\text{base}}$, is called the **tangent** of the angle POM, and written, **tan POM.**

It is necessary to show that *the magnitude of this ratio depends only on the magnitude of the angle POM*, and is quite independent of the size of the triangle. Drop perpendiculars PM and $P'M'$ from P and P' on to OA (Fig. 131). The two triangles POM and $P'OM'$, are equiangular and similar, therefore, as on page 490, $M'P'/OM' = MP/OM$.

(ii.) The ratio $\frac{OM}{MP}$, or, $\frac{\text{base}}{\text{perpendicular}}$, is called the **cotangent** of the angle POM, and written, **cot POM.** Note that the cotangent of an angle is the reciprocal of its tangent.

(iii.) The ratio $\frac{MP}{OP} = \frac{\text{perpendicular}}{\text{hypotenuse}}$, is called the **sine** of the angle POM, and written, **sin POM**.

(iv.) The ratio $\frac{OP}{MP} = \frac{\text{hypotenuse}}{\text{perpendicular}}$, is called the **cosecant** of the angle POM, and written, **cosec POM**. The cosecant of an angle is the reciprocal of its sine.

(v.) The ratio $\frac{OM}{OP} = \frac{\text{base}}{\text{hypotenuse}}$, is called the **cosine** of the angle POM, and written, **cos POM**.

(vi.) The ratio $\frac{OP}{OM} = \frac{\text{hypotenuse}}{\text{base}}$, is called the **secant** of the angle POM, and written, **sec POM**. The secant of an angle is the reciprocal of its cosine.

EXAMPLE.—If x be used in place of POM, show that

$$\sin x = \frac{1}{\operatorname{cosec} x};\ \cot x = \frac{1}{\tan x};\ \cos x = \frac{1}{\sec x}.$$

The squares of any of these ratios, $(\sin x)^2$, $(\cot x)^2$, . . ., are generally written $\sin^2 x$, $\cot^2 x$ . . .; $(\sin x)^{-1}$, $(\cot x)^{-1}$, . . ., meaning $\frac{1}{\sin x}$, $\frac{1}{\cot x}$, . . ., cannot be written in the forms $\sin^{-1}x$, $\cot^{-1}x$, . . ., because this latter symbol has the meaning "the angle whose sine, cotangent, . . ., is x" (§ 15). If x is known, the numerical value of $\sin^{-1}x$, etc., is given in the regular tables. Some mathematicians write "arc $\sin x$, arc $\cot x$, . . .," instead of $\sin^{-1}x$, $\cot^{-1}x$, . . ., respectively.

4. **Conventions as to the sign of the trignometrical ratios.** This subject has been treated on page 111. In the following table, these results are summarised. The change in the value of the ratio as it passes through the four quadrants is also given.

TABLE XIII.—SIGNS OF THE TRIGNOMETRICAL RATIOS.

If the Angle is in Quadrant.	$\sin x$.	$\cos x$.	$\tan x$.	$\cot x$.	$\sec x$.	$\operatorname{cosec} x$.
I. sign	+	+	+	+	+	+
I. value*	0 to 1	1 to 0	0 to ∞	∞ to 0	1 to ∞	∞ to 1
II. sign	+	—	—	—	—	+
II. value*	1 to 0	0 to 1	∞ to 0	0 to ∞	∞ to 1	1 to ∞
III. sign	—	—	+	+	—	—
III. value*	0 to 1	1 to 0	0 to ∞	∞ to 0	1 to ∞	∞ to 1
IV. sign	—	+	—	—	+	—
IV. value*	1 to 0	0 to 1	∞ to 0	0 to ∞	∞ to 1	1 to ∞

* In anticipation of the next article.

5. **To find a numerical value for the trignometrical ratios.**

(i.) 45° or $\frac{1}{4}\pi$. Draw a square $ABCD$ (Fig. 133). Join AC. The angle BAC = half a right angle = 45°. In the right-angled triangle BAC (Euclid i., 47),

$$AC^2 = AB^2 + BC^2.$$

Since AB and BC are the sides of a square, $\therefore AB = BC$, hence,

$$AC^2 = 2AB^2 = 2BC^2;\ \text{or, } AC = \sqrt{2}\,.\,AB = \sqrt{2}\,.\,BC.$$

$$\therefore \sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}};\ \cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}};\ \tan 45^\circ = \frac{BC}{AB} = 1. \qquad (2)$$

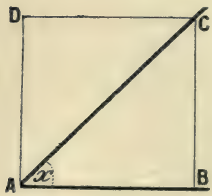


FIG. 133.

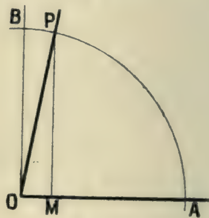


FIG. 134.

(ii.) 90° or $\frac{1}{2}\pi$. In Fig. 134, if POM is a right-angled triangle, as M approaches O, the angle POM approaches 90°. When PM coincides with OB, $OP = MP$, and OM = zero.

$$\therefore \sin 90^\circ = \frac{MP}{OP} = 1;\ \cos 90^\circ = \frac{OM}{OP} = 0;\ \tan 90^\circ = \frac{MP}{OM} = \infty. \qquad (3)$$

(iii.) 0°. In Fig. 135, as the angle POM becomes smaller, OP approaches OM, and at the limit coincides with it. Hence, $PM = 0$; $OM = OP$.

$$\therefore \sin 0^\circ = \frac{MP}{OP} = 0;\ \cos 0^\circ = \frac{OM}{OP} = 1;\ \tan 0^\circ = \frac{MP}{OM} = 0. \qquad (4)$$

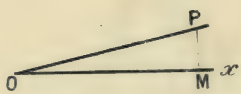


FIG. 135.

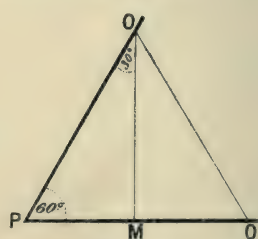


FIG. 136.

(iv.) 60° or $\frac{1}{3}\pi$. In the equilateral triangle (Fig. 136), each of the three angles is equal to 60°. Drop the perpendicular OM on to PQ. Then

$$2PM = PQ = OP.$$

By Euclid i., 47,

$$OP^2 = MP^2 + OM^2.\quad \therefore 4PM^2 = OM^2 + PM^2;\ \text{or } OM^2 = 3PM^2.$$

$$\therefore OM = \sqrt{3}\,.\,PM;\ \text{angle } OPM = 60^\circ.$$

$$\therefore \sin 60^\circ = \frac{MO}{OP} = \frac{\sqrt{3}}{2};\ \cos 60^\circ = \frac{PM}{OP} = \frac{1}{2};\ \tan 60^\circ = \frac{MO}{PM} = \sqrt{3}. \qquad (5)$$

(v.) 30° or $\frac{1}{6}\pi$. Using the preceding results,

$$\therefore \sin 30^\circ = \frac{MP}{OP} = \frac{1}{2};\ \cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}}{2};\ \tan 30^\circ = \frac{MP}{OM} = \frac{1}{\sqrt{3}}. \qquad (6)$$

The following table summarises these results:

TABLE XIV.—NUMERICAL VALUES OF THE TRIGNOMETRICAL RATIOS.

Angle.	0° or 360°.	30°.	45°.	60°.	90°.	180°.	270°.
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞

To these might be added $\sin 15^\circ = (\sqrt{3} - 1)/2\sqrt{2}$; $\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1)$. See also page 495.

It must be clearly understood that although an angle is always measured in the degree-minute-second system, the numerical equivalent in radian or circular measure is employed in the calculations, unless special provision has been made for the direct introduction of degrees. This was done in example (6), page 458 (*q.v.*). Suppose that we have occasion to employ the approximation formula

$$\sin(x + \theta) = \sin x + \theta \cos x,$$

of § 189, and that $x = 35^\circ$ and $\theta = 50''$. The Tables of Natural Sines, Cosines, Tangents, and their reciprocals, will furnish the numerical values of sin 35° and cos 35°, but θ must be converted into radian measure. Hence show that

$$\sin(x + \theta) = \sin 35^\circ + \frac{\cdot 00926 \times \pi}{180} \cos 35^\circ.$$

Hint. $50'' = (\frac{50}{60})' = (\frac{50}{60} \times \frac{1}{60})^\circ = \cdot 00926^\circ$. The numerical values of sin 35° and of cos 35° to four decimal places are respectively ·5736 and ·8192. The value of $\sin(x + \theta)$ is, therefore, ·5737.

In the absence of a "Table Book," the numerical tables of the trignometrical functions are calculated by means of Taylor's or by Maclaurin's theorems. For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots;\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$$

§ 98. But 35° = ·610865 radians, and $\therefore$ sin 35° = sin ·610865.

$$\therefore \sin 35^\circ = \cdot 610865 - \tfrac{1}{6}(\cdot 610865)^3 + \tfrac{1}{120}(\cdot 610865)^5 - \ldots;$$
$$= \cdot 57357 \ldots$$

In the same way, show that cos 35° = ·81915 . . .

Again by Taylor's theorem,

$$\sin 36^\circ = \sin(35^\circ + 1^\circ);$$
$$= \sin 35^\circ + \frac{\cos 35^\circ}{1!}(\cdot 017453) - \frac{\sin 35^\circ}{2!}(\cdot 017453)^2 - \ldots$$
$$= \cdot 58778 \ldots$$

6. **Trignometrical ratios of the supplement of an angle.** The angle $180° - x$, or $\pi - x$, is called the **supplement** of the angle x. In Fig. 137, let $POM = x$,

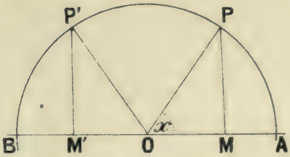


FIG. 137.

produce MO to M'. Then the angle POM' is the supplement of x. Make the angle $P'OM' = POM$. Let $OP' = OP$. Drop perpendiculars $P'M'$ and PM on to BA. The triangles OPM and $OP'M'$ are equal in all respects. If OM is positive, OM' is negative.

$$\therefore M'P' = MP; \text{ and } OM' = - OM.$$

$$\sin (180 - x) = \sin (\pi - x) = \sin POM' = \sin P'OM = \sin x.$$

$$\cos (180° - x) = - \cos x; \ \tan (180° - x) = - \tan x.$$

EXAMPLES.—(1) Find the value of sin 120°.

$$\sin 120° = \sin (180° - 60°) = \sin 60 = \sqrt{3}/2.$$

(2) Evaluate tan 120°. Ansr. $- \sqrt{3}$.

7. **Trignometrical ratios of the complement of an angle.** The angle $90° - x$, or $\frac{1}{2}\pi - x$, is called the **complement** of x. In Fig. 138, PN and PM are perpendiculars on OB and on OA respectively. Then $OM = NP$, $ON = MP$.

$$\sin (90° - x) = \sin (\tfrac{1}{2}\pi - x) = \sin NOP = \frac{NP}{OP} = \frac{OM}{OP} = \cos x.$$

$$\cos (90° - x) = \sin x; \ \tan (90° - x) = \cot x; \ \cot (90° - x) = \tan x.$$

8. **To prove that sin x/ cos x = tan x.**

$$\frac{\sin x}{\cos x} = \frac{MP}{OP}\Big/\frac{OM}{OP} = \frac{MP}{OP} \times \frac{OP}{OM} = \frac{MP}{OM} = \tan x.$$

9. **To prove that $\sin^2 x + \cos^2 x = 1$.** In Fig. 138, by Euclid i., 47, $OP^2 = MP^2 + OM^2$. Divide through by OP^2, and

$$1 = \frac{OP^2}{OP^2} = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2.$$

$$\therefore \sin^2 x + \cos^2 x = 1.$$

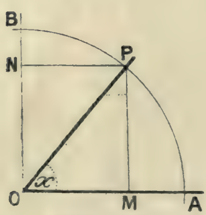


FIG. 138.

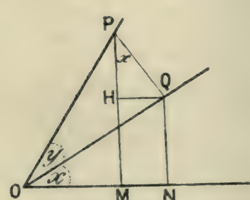


FIG. 139.

10. **To show that sin $(x + y)$ = sin x . cos y + cos x . sin y.** In Fig. 139, PQ is perpendicular to OQ, the angle HPQ = angle NOQ (Euclid i., 15 and 32).

$$\therefore \sin(x+y) = \frac{MP}{OP} = \frac{HP}{OP} + \frac{QN}{OP} = \frac{PH}{PQ}\cdot\frac{PQ}{OP} + \frac{NQ}{OQ}\cdot\frac{OQ}{OP};$$
$$= \sin x . \cos y + \cos x . \sin y.$$

11. **Summary of trignometrical formulae** (*for reference*). The above definitions lead to the following relations, which form routine exercises in elementary trignometry. Most of them may be established geometrically as in the preceding illustrations:

Note: $\pi = 180°$; or 3·14159 radians.

Complement of x, or $(90° - x)$, or, $(\frac{1}{2}\pi - x)$.

$$\left.\begin{array}{l}\sin(\frac{1}{2}\pi - x) = \cos x;\ \cos(\frac{1}{2}\pi - x) = \sin x;\\ \operatorname{cosec}(\frac{1}{2}\pi - x) = \sec x;\ \sec(\frac{1}{2}\pi - x) = \operatorname{cosec} x;\\ \tan(\frac{1}{2}\pi - x) = \cot x;\ \cot(\frac{1}{2}\pi - x) = \tan x.\end{array}\right\} \quad . \quad . \quad . \quad (7)$$

Supplement of x, or $180° - x$, or $\pi - x$.

$$\left.\begin{array}{l}\sin(\pi - x) = \sin x;\ \cos(\pi - x) = -\cos x;\\ \tan(\pi - x) = -\tan x;\ \cot(\pi - x) = -\cot x.\end{array}\right\} \quad . \quad . \quad . \quad (8)$$

Angles $90° + x$, and $180° + x$.

$$\left.\begin{array}{l}\sin(\frac{1}{2}\pi + x) = \cos x;\ \cos(\frac{1}{2}\pi + x) = -\sin x;\\ \tan(\frac{1}{2}\pi + x) = -\cot x,\ \cot(\frac{1}{2}\pi + x) = -\tan x.\end{array}\right\} \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{array}{l}\sin(\pi + x) = -\sin x;\ \cos(\pi + x) = -\cos x;\\ \tan(\pi + x) = \tan x;\ \cot(\pi + x) = \cot x.\end{array}\right\} \quad . \quad . \quad . \quad (10)$$

Negative Angles.

$$\sin(-x) = -\sin x;\ \cos(-x) = \cos x;\ \tan(-x) = -\tan x. \quad . \quad (11)$$

Limiting Values.

$$Lt_{x=0}\frac{\sin x}{x} = \frac{\tan x}{x} = \cos x = 1;\ \frac{\sin^{-1}x}{x} = \frac{\tan^{-1}x}{x} = 1. \quad . \quad . \quad (12)$$

General Forms.

When n is any negative or positive integer or zero.

$$\sin x = \sin\{n\pi + (-1)^n x\}. \quad . \quad . \quad . \quad (13)$$
$$\cos x = \cos(2n\pi \pm x). \quad . \quad . \quad . \quad (14)$$
$$\tan x = \tan(n\pi + x). \quad . \quad . \quad . \quad (15)$$

Miscellaneous Relations.

$$\tan x = \sin x/\cos x;\ \cot x = \cos x/\sin x. \quad . \quad . \quad . \quad (16)$$
$$\sin^2 x + \cos^2 x = 1. \quad . \quad . \quad . \quad (17)$$
$$\sin x = \sqrt{1 - \cos^2 x};\ \cos x = \sqrt{1 - \sin^2 x}. \quad . \quad . \quad . \quad (18)$$
$$\operatorname{cosec} x = \sqrt{1 + \cot^2 x};\ \sec x = \sqrt{1 + \tan^2 x}. \quad . \quad . \quad . \quad (19)$$
$$\sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}};\ \cos x = \frac{1}{\sqrt{1 + \tan^2 x}}. \quad . \quad . \quad . \quad (20)$$
$$\sin(x \pm y) = \sin x . \cos y \pm \cos x . \sin y. \quad . \quad . \quad . \quad (21)$$
$$\cos(x \pm y) = \cos x . \cos y \mp \sin x . \sin y. \quad . \quad . \quad . \quad (22)$$

These two results can be proved by Taylor's theorem.

$$\sin(x + y) + \sin(x - y) = 2\sin x . \cos y. \quad . \quad . \quad . \quad (23)$$
$$\sin(x + y) - \sin(x - y) = 2\cos x . \sin y. \quad . \quad . \quad . \quad (24)$$
$$\cos(x + y) + \cos(x - y) = 2\cos x . \cos y. \quad . \quad . \quad . \quad (25)$$
$$\cos(x + y) - \cos(x - y) = -2\sin x . \sin y. \quad . \quad . \quad . \quad (26)$$

If $x = y$, from (21) and (22),

$$\sin 2x = 2 \sin x \cdot \cos x. \qquad (27)$$

$$\cos 2x = \cos^2 x - \sin^2 x. \qquad (28)$$

$$= 2 \cos^2 x - 1. \qquad (29)$$

$$= 1 - 2 \sin^2 x. \qquad (30)$$

$$\sin x = 2 \sin \tfrac{1}{2}x \cdot \cos \tfrac{1}{2}x. \qquad (31)$$

$$\cos x = 2 \cos^2 \tfrac{1}{2}x - 1;\ \text{or, } 1 + \cos x = 2 \cos^2 \tfrac{1}{2}x. \qquad (32)$$

$$= 1 - 2 \sin^2 \tfrac{1}{2}x;\ \text{or, } 1 - \cos x = 2 \sin^2 \tfrac{1}{2}x. \qquad (33)$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x. \qquad (34)$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x. \qquad (35)$$

If in (23) to (26), we suppose $x + y = \alpha$; $x - y = \beta$; $x = \frac{1}{2}(\alpha + \beta)$; $y = \frac{1}{2}(\alpha - \beta)$. Now put x for α, and y for β, for the sake of uniformity. Thus,

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cdot \cos \tfrac{1}{2}(x - y). \qquad (36)$$

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \cdot \sin \tfrac{1}{2}(x - y). \qquad (37)$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cdot \cos \tfrac{1}{2}(x - y). \qquad (38)$$

$$\cos x - \cos y = - 2 \sin \tfrac{1}{2}(x + y) \cdot \sin \tfrac{1}{2}(x - y). \qquad (39)$$

By division of the proper formulae above,

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}. \qquad (40)$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}. \qquad (41)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \qquad (42)$$

$$\tan x \pm \tan y = \frac{\sin (x \pm y)}{\cos x \cdot \cos y}. \qquad (43)$$

$$\cot x \pm \cot y = \frac{\sin (x \pm y)}{\sin x \cdot \sin y}. \qquad (44)$$

$$\cos \tfrac{1}{2}x = \sqrt{\frac{1 + \cos x}{2}};\ \sin \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{2}};\ \tan \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}. \qquad (45)$$

12. **Properties of triangles.** Let a, b, c (Fig. 140), be sides opposite the

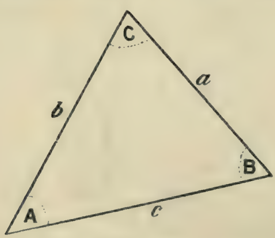


Fig. 140.

angles A, B, C, of the triangle ABC. Then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \qquad (46)$$

In words, the sines of the angles of a plane triangle are proportional to the opposite sides. This is known as the **Rule of Sines.**

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A. \qquad (47)$$

The other letters may be substituted in cyclic order. Let

$$2s = a + b + c\text{; or, } s = \tfrac{1}{2}(a + b + c).$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$c = a\cos B + b\cos A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

13. **Solution of triangles.** The three sides and three angles of a triangle are called its *six parts*. When a sufficient number of parts are given, the remainder can be calculated. There are four cases.

Case i. *Given three sides, to find three angles.* Use (50) for $\tan \frac{1}{2}A$ and for $\tan \frac{1}{2}B$. $C = 180° - (A + B)$.

Case ii. *Given one side and two angles, to find the remaining parts.* Use $C = 180° - (A + B)$ and (46) twice.

Case iii. *Given two sides and the included angle, to find the remaining parts.* Use the relation

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c}\cot\frac{A}{2}. \quad . \quad . \quad . \quad . \quad . \quad (53)$$

This gives $B - C$, but $B + C = 180° - A$, therefore,

$$B = \tfrac{1}{2}(B + C) + \tfrac{1}{2}(B - C)\text{; } C = \tfrac{1}{2}(B + C) - \tfrac{1}{2}(B - C).$$

For the other side, use (52).

Case iv. *Given two sides and the angle opposite one of them, to find the remaining parts.* Let b, c, B be the parts known. As in case ii., use (46) for C. $A = 180° - B - C$. Then solve (46) for a.

If $b < c \sin B$, no solution is possible; if $b = c \sin B$, $\sin C = 90°$; if $b > c \sin B$ and $b < c$, and B is acute, two solutions are possible, but if $b \geqq c$ only one solution is possible and C is an acute angle.

§ 195. Spherical Trignometry.

Convenient for reference in crystallographic work. See Story-Maskelyne's *The Morphology of Crystals* (Oxford, 1895); or Lewis' *A Treatise on Crystallography* (Cambridge, 1899), for exercises.

1. **Trignometrical ratios of an angle as functions of the arc of a circle subtended by the angle.** Besides the simple definitions of the trignometrical functions of page 494, it is sometimes more convenient to regard the sine, cosine, tangent, etc., as direct functions of the arc of a circle of unit radius. In Fig. 141 (page 502), these functions are represented respectively by PM, OM, QN, . . .

So far as the magnitude of the angle α is concerned, it makes no difference whether its tangent is represented by the ratio PM/OM, or by QN/ON, for, owing to the similarity of the triangles OPM and OQN, Euclid vi., 4,

$$PM : OM = QN : ON.$$

If, therefore, α represents the arc of a circle of unit radius corresponding to the angle POM, we may represent the trignometrical functions by the length of a line, thus

$$\sin \alpha = PM;\ \cos \alpha = OM;\ \tan \alpha = QN;$$
$$\operatorname{cosec} \alpha = OT;\ \sec \alpha = OQ;\ \cot \alpha = RT.$$

Instead of referring to $\sin \alpha$ as "the sine of the angle α," we say that $\sin \alpha$ "is the sine of the arc α". This is usually done in spherical trignometry, which deals with the relations between the several parts of a triangle drawn on the surface of a sphere, and bounded by three arcs of a great circle. A **great circle** of a sphere is the boundary of any section passing through the centre, while a **small circle** bounds any section of a sphere not passing through the centre.

2. **Rule of sines.** "*The sines of the sides of a right-angled spherical triangle are as the sines of the opposite angles.*" * Let A, B, C (Fig. 142), be

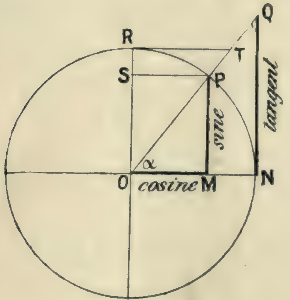


FIG. 141.

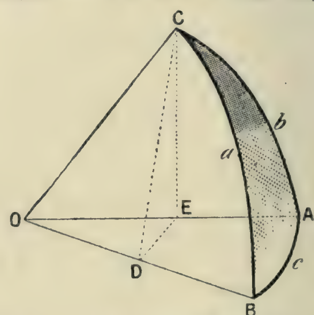


FIG. 142.—Spherical Triangle.

the three angles of a right-angled spherical triangle whose opposite sides are a, b, c. Let O be the centre of the circle CaB (or BcA, or CbA). Drop perpendiculars CE on OA and CD on OB. Join DE.

The plane CDE is perpendicular to the plane OBA. If A be a right-angle, the plane CAO is at right angles to the plane OBA. Hence, CED and CDE are right angles. CE is the sine of the arc AC, or $CE = \sin b$. Since the plane CBO and ABO include the same angles as the spherical triangle ABC, and radius of sphere $= OC = OB = OA$,

$$\sin B = \frac{CE}{CD};\ \sin a = \frac{CD}{OC};\ \therefore \sin B \,.\, \sin a = \frac{CE}{OC}.$$

Similarly, $\sin B \,.\, \sin a = \dfrac{CE}{OC} = \sin A \,.\, \sin b$; and finally,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad (1)$$

As an exercise, show that this result is true for any spherical triangle. Note the formal resemblance between this result and (46), preceding section.

3. **Properties of any spherical triangle.** (For reference.) Let

$$s = \tfrac{1}{2}(a + b + c);\ S = \tfrac{1}{2}(A + B + C).$$

* The meanings of the terms "sines of the sides" and "sines of the angles" of a spherical triangle should be apparent from 3, § 194, and 1, § 195.

Cosines of the Angles in Terms of the Sides.

$$\cos A = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}; \ \cos B = \frac{\cos b - \cos c \cdot \cos a}{\sin c \cdot \sin a}; \quad . \quad (2)$$

and so on for $\cos C$, by substituting the letters in cyclic order.

Cosines of the Sides in Terms of the Angles.

$$\cos a = \frac{\cos A - \cos B \cdot \cos C}{\sin B \cdot \sin C}; \ \cos b = \frac{\cos B - \cos C \cdot \cos A}{\sin C \cdot \sin A}, \text{ etc.} \quad (3)$$

Sines of the Angles in Terms of the Sides.

$$\sin A = \frac{2q}{\sin b \cdot \sin c}; \ \sin B = \frac{2q}{\sin c \cdot \sin a}, \text{ etc.,} \quad . \quad . \quad (4)$$

where $2q = 2\sqrt{\sin s \cdot \sin (s-a) \cdot \sin (s-b) \cdot \sin (s-c)}$.

Sines of the Sides in Terms of the Angles.

$$\sin a = \frac{2Q}{\sin B \cdot \sin C}; \ \sin b = \frac{2Q}{\sin C \cdot \sin A}, \text{ etc.} \quad . \quad . \quad (5)$$

where $2Q = 2\sqrt{-\cos S \cdot \cos (S-A) \cdot \cos (S-B) \cdot \cos (S-C)}$.

Sines of Half the Angles in Terms of the Sides.

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \cdot \sin (s-c)}{\sin b \cdot \sin c}}, \text{ etc.} \quad . \quad . \quad . \quad (6)$$

Cosines of Half the Angles in Terms of the Sides.

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \cdot \sin (s-a)}{\sin b \cdot \sin c}}, \text{ etc.} \quad . \quad . \quad . \quad (7)$$

Tangents of Half the Angles in Terms of the Sides.

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s-b) \cdot \sin (s-c)}{\sin s \cdot \sin (s-a)}}, \text{ etc.} \quad . \quad . \quad . \quad (8)$$

Sines of Half the Sides in Terms of the Angles.

$$\sin \frac{a}{2} = \sqrt{\frac{-\cos S \cdot \cos (S-A)}{\sin B \cdot \sin C}}, \text{ etc.} \quad . \quad . \quad . \quad (9)$$

Cosines of Half the Sides in Terms of the Angles.

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cdot \cos (S-C)}{\sin B \cdot \sin C}}, \text{ etc.} \quad . \quad . \quad . \quad (10)$$

Tangents of Half the Sides in Terms of the Angles.

$$\tan \frac{a}{2} = \sqrt{\frac{-\cos S \cdot \cos (S-A)}{\cos (S-B) \cdot \cos (S-C)}}, \text{ etc.} \quad . \quad . \quad . \quad (11)$$

Tangents of the Sum, and of the Difference of Two Angles.

$$\tan \frac{A+B}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2}, \text{ etc.} \quad . \quad . \quad . \quad (12)$$

$$\tan \frac{A-B}{2} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{C}{2}, \text{ etc.} \quad . \quad . \quad . \quad (13)$$

Tangents of the Sum, and the Difference of Two Sides.

$$\tan \frac{a+b}{2} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{c}{2}, \text{ etc.} \quad . \quad . \quad . \quad (14)$$

$$\tan \frac{a-b}{2} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{c}{2}, \text{ etc.} \quad . \quad . \quad . \quad (15)$$

Formulae (12) to (15) are known as **Napier's analogies.**

4. **Properties of right-angled spherical triangles.**

$$\sin a = \cot B \,.\, \tan b. \quad (16)$$
$$\sin a = \sin A \,.\, \sin c. \quad (17)$$
$$\sin b = \cot A \,.\, \tan a, \text{ etc.} \quad (18)$$
$$\sin b = \sin B \,.\, \sin c, \text{ etc.} \quad (19)$$
$$\cos c = \cot A \,.\, \cot B, \text{ etc.} \quad (20)$$
$$\cos c = \cos a \,.\, \cos b, \text{ etc.} \quad (21)$$
$$\cos A = \tan b \,.\, \cot c. \quad (22)$$
$$\cos A = \sin B \,.\, \cos a. \quad (23)$$
$$\cos B = \tan a \,.\, \cot c, \text{ etc.} \quad (24)$$
$$\cos B = \sin A \,.\, \cos b, \text{ etc.} \quad (25)$$
$$C = 90°.$$

5. **The solution of oblique-angled triangles.** Given certain parts of a spherical triangle, to find the remaining parts.

Case i. *Given three sides.* Use formulae (8) for $\tan \frac{1}{2}A$, $\tan \frac{1}{2}B$ and $\tan \frac{1}{2}C$.

Case ii. *Given three angles.* Use formulae (11) for $\tan \frac{1}{2}a$, $\tan \frac{1}{2}b$ and $\tan \frac{1}{2}c$.

Case iii. *Given two sides and the included angle*, say a, C, b. Use (1) for $\sin c$, (12) and (13) for $\tan \frac{1}{2}(A + B)$ and $\tan \frac{1}{2}(A - B)$.

Case iv. *Given two angles and the side between them*, say A, c, B. Use (1) for $\sin C$, (14) for $\tan \frac{1}{2}(a + b)$ and (15) for $\tan \frac{1}{2}(a - b)$.

Case v. *Given two sides and the angle opposite one of them*, say a, b, A. Use (1) for $\sin B$, (12) for $\tan \frac{1}{2}C$ and (14) for $\tan \frac{1}{2}c$.

Case vi. *Given two angles and the side opposite one of them*, say A, B, a. Use (1) for $\sin b$, (12) for $\tan \frac{1}{2}C$ and (14) for $\tan \frac{1}{2}c$.

6. **The solution of right-angled triangles.**

Case i. *Given hypotenuse c and side a.* Use (21) for $\cos b$, (24) for $\cos B$, (17) for $\sin A$.

Case ii. *Given hypotenuse c and angle A.* Use (22) for $\tan b$, (20) for $\cot B$, (17) for $\sin a$.

Case iii. *Given two sides a and b.* Use (21) for $\cos c$, (18) for $\cot A$, (16) for $\cot B$.

Case iv. *Given two angles A and B.* Use (20) for $\cos c$, (23) for $\cos A$, (25) for $\cos b$.

Case v. *Given the side b and the adjacent angle A.* Use (22) for $\tan b$, (18) for $\tan a$, (25) for $\cos B$.

Case vi. *Given side a and the opposite angle A.* Use (17) for $\sin c$, (18) for $\sin b$, (23) for $\sin B$.

§ 196. Summary of Relations among the Hyperbolic Functions.

(See Chapter VI.)

$$\cos x = \cosh \iota x = \tfrac{1}{2}(e^{\iota x} + e^{-\iota x}). \quad (1)$$
$$\sin x = \frac{1}{\iota}\sinh \iota x = \frac{1}{2\iota}(e^{\iota x} - e^{-\iota x}). \quad (2)$$

$$\cos x + \iota \sin x = \cosh \iota x + \sinh \iota x = e^{\iota x}. \tag{3}$$

$$\cos x - \iota \sin x = \cosh \iota x - \sinh \iota x = e^{-\iota x}. \tag{4}$$

$$\cosh x = \cos \iota x \,;\ \iota \sinh x = \sin \iota x. \tag{5}$$

$$\left.\begin{array}{l}\tanh x = \sinh x/\cosh x \,;\ \coth x = \cosh x/\sinh x \,; \\ \quad \operatorname{cosech} x = 1/\sinh x \,;\ \operatorname{sech} x = 1/\cosh x.\end{array}\right\} \tag{6}$$

$$\cosh 0 = 1 \,;\ \sinh 0 = 0 \,;\ \tanh 0 = 0. \tag{7}$$

$$\cosh(\pm\infty) = +\infty \,;\ \sinh(\pm\infty) = \pm\infty \,;\ \tanh(\pm\infty) = \pm 1. \tag{8}$$

$$Lt_{x=0}\frac{\sinh x}{x} = 1 \,;\ Lt_{x=0}\frac{\tanh x}{x} = 1 \,;\ Lt_{x=0}\frac{\cosh x}{x} = 1. \tag{9}$$

$$\sinh(-x) = -\sinh x \,;\ \cosh(-x) = \cosh x \,;\ \tanh(-x) = -\tanh x. \tag{10}$$

$$\sinh(x \pm y) = \sinh x \,.\, \cosh y \pm \cosh x \,.\, \sinh y. \tag{11}$$

$$\cosh(x \pm y) = \cosh x \,.\, \cosh y \pm \sinh x \,.\, \sinh y. \tag{12}$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \,.\, \tanh y}. \tag{13}$$

$$\left.\begin{array}{rl}\cosh(x + \iota y) &= \cosh x \,.\, \cosh \iota y + \sinh x \,.\, \sinh \iota y \,; \\ &= \cosh x \,.\, \cos y + \iota \sinh x \,.\, \sin y.\end{array}\right\} \tag{14}$$

$$\left.\begin{array}{rl}\sinh(x + \iota y) &= \sinh x \,.\, \cosh \iota y + \cosh x \,.\, \sinh \iota y \,; \\ &= \sinh x \,.\, \cos y + \iota \cosh x \,.\, \sin y.\end{array}\right\} \tag{15}$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x + y) \,.\, \cosh \tfrac{1}{2}(x - y). \tag{16}$$

$$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2}(x + y) \,.\, \sinh \tfrac{1}{2}(x - y). \tag{17}$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x + y) \,.\, \cosh \tfrac{1}{2}(x - y). \tag{18}$$

$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x + y) \,.\, \sinh \tfrac{1}{2}(x - y). \tag{19}$$

$$\sinh 2x = 2 \sinh x \,.\, \cosh x = 2 \tanh x/(1 - \tanh^2 x). \tag{20}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \,; \tag{21}$$

$$= 1 + 2\sinh^2 x = 2\cosh^2 x - 1 \,; \tag{22}$$

$$= (1 + \tanh^2 x)/(1 - \tanh^2 x). \tag{23}$$

$$\cosh x + 1 = 2\cosh^2 \tfrac{1}{2}x \,;\ \cosh x - 1 = 2\sinh^2 \tfrac{1}{2}x. \tag{24}$$

$$\left.\begin{array}{rl}\tanh \tfrac{1}{2}x &= \sinh x/(1 + \cosh x) \,; \\ &= (\cosh x - 1)/\sinh x.\end{array}\right\} \tag{25}$$

$$\sinh^2 x - \cosh^2 x = 1. \tag{26}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \,;\ \coth^2 x - 1 = \operatorname{cosech}^2 x. \tag{27}$$

$$\cosh x = 1 / \sqrt{(1 - \tanh^2 x)} \,;\ \sinh x = \tanh x / \sqrt{(1 - \tanh^2 x)}. \tag{28}$$

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x. \tag{29}$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x. \tag{30}$$

Gudermannians. The function $\cos^{-1} \operatorname{sech} x$, is called gudermannian x, and written gd x. If

$\tan^{-1} \sinh x$

$$y = \cos^{-1} \operatorname{sech} x,\ \cos x = \operatorname{sech} x.$$

$$\sin x = \sqrt{(1 - \operatorname{sech}^2 x)} = \tanh x.$$

$$\tan x = \tanh x/\operatorname{sech} x = \sinh x.$$

$$\therefore \operatorname{gd} x = \cos^{-1} \operatorname{sech} x = \sin^{-1} \tanh x = \tan^{-1} \sinh x. \tag{31}$$

If

$$x = \log \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}x) \,; \tag{32}$$

$$\therefore\ x = \tan^{-1} \operatorname{sech} x = \operatorname{gd} x, \tag{33}$$

or inverse gd x. Hence $\log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) = \operatorname{gd}^{-1} x$. (34)

Analogous to Demoivre's theorem

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx. \tag{35}$$

It is instructive to compare the above formulae with the corresponding trignometrical functions in 11, § 194. The analogy is also brought out by tabulating corresponding indefinite integrals in Tables I. and III., side by side. A few additional integrals are here given to be verified and then added to the table of indefinite integrals which the student has been advised to compile for his own use.

Hyperbolic.	Trignometrical.	
$\int\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\frac{x}{a}.$	$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}.$	(35)
$\int\frac{dx}{\sqrt{x^2-a^2}}=\cosh^{-1}\frac{x}{a}.$	$\int\frac{-dx}{\sqrt{a^2-x^2}}=\cos^{-1}\frac{x}{a}.$	(36)
$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a},$ when $x < a$. When $x > a$,	$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}.$	(37)
$\int\frac{-dx}{x^2-a^2}=\frac{1}{a}\coth^{-1}\frac{x}{a}.$	$\int\frac{-dx}{a^2+x^2}=\frac{1}{a}\cot^{-1}\frac{x}{a}.$	(38)
$\int\frac{-dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}.$	$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}.$	(39)
$\int\frac{-dx}{x\sqrt{a^2+x^2}}=\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a}.$	$\int\frac{-dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}.$	(40)
$\int\operatorname{sech}x\,.\,dx=\operatorname{gd}x.$	$\int\sec x\,.\,dx=\operatorname{gd}^{-1}x.$	(41)

Numerical values of the hyperbolic functions may be computed by means of the series formulae.

CHAPTER XIII.

REFERENCE TABLES.

Tables of common logarithms and of the trignometrical ratios are indispensable in applied mathematics (see pages 37 and 484). Most of the following tables have been referred to in different parts of this work, and are reproduced here because they are not usually found in the smaller current sets of "Mathematical Tables".

Table I.—**Table of Standard Integrals.**

(Page 158.)

Table II.—**Numerical Values of the Gamma Function.**

(Described in § 84, page 191.)

$$\log\int_0^\infty e^{-x}x^{n-1}dx + 10 \text{ or } \log\Gamma(n) + 10 \text{ from } n = 1 \text{ to } n = 2.$$

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1·00		97497	95001	92512	90030	87555	85087	82627	80173	77727
1·01	9·9975287	72855	70430	68011	65600	63196	60798	58408	56025	53648
1·02	51279	48916	46561	44212	41870	39535	37207	34886	32572	30265
1·03	27964	25671	23384	21104	18831	16564	14305	12052	09806	07567
1·04	05334	03108	00889	$\overline{98677}$	$\overline{96471}$	$\overline{94273}$	$\overline{92080}$	$\overline{89895}$	$\overline{87716}$	$\overline{85544}$
1·05	9·9883379	81220	79068	76922	74783	72651	70525	68406	66294	64188
1·06	62089	59996	57910	55830	53757	51690	49630	47577	45530	43489
1·07	41455	39428	37407	35392	33384	31382	29387	27398	25415	23440
1·08	21469	19506	17549	15599	13655	11717	09785	07860	05941	04029
1·09	02123	00223	$\overline{98329}$	$\overline{96442}$	$\overline{94561}$	$\overline{92686}$	$\overline{90818}$	$\overline{88956}$	$\overline{87100}$	$\overline{85250}$
1·10	9·9783407	81570	79738	77914	76095	74283	72476	70676	68882	67095
1·11	65313	63538	61768	60005	58248	56497	54753	53014	51281	49555
1·12	47834	46120	44411	42709	41013	39323	37638	35960	34288	32622
1·13	30962	29308	27659	26017	24381	22751	21126	19508	17896	16289
1·14	14689	13094	11505	09922	08345	06774	05209	03650	02096	00549
1·15	9·9699007	97471	95941	94417	92898	91386	89879	88378	86883	85393
1·16	83910	82432	80960	79493	78033	76578	75129	73686	72248	70816
1·17	69390	67969	66554	65145	63742	62344	60952	59566	58185	56810
1·18	55440	54076	52718	51366	50019	48677	47341	46011	44687	43368
1·19	42054	40746	39444	38147	36856	35570	34290	33016	31747	30483

TABLE II.—*Continued.*

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1·20	9·9629225	27973	26725	25484	24248	23017	21792	20573	19358	18150
1·21	16946	15748	14556	13369	12188	11011	09841	08675	07515	06361
1·22	05212	04068	02930	01796	00669	99546	98430	97318	96212	95111
1·23	594015	92925	91840	90760	89685	88616	87553	86494	85441	84393
1·24	83350	82313	81280	80253	79232	78215	77204	76198	75197	74201
1·25	9·9573211	72226	71246	70271	69301	68337	67377	66423	65474	64530
1·26	63592	62658	61730	60806	59888	58975	58067	57165	56267	55374
1·27	54487	53604	52727	51855	50988	50126	49268	48416	47570	46728
1·28	45891	45059	44232	43410	42593	41782	40975	40173	39376	38585
1·29	37798	37016	36239	35467	34700	33938	33181	32429	31682	30940
1·30	9·9530203	29470	28743	28021	27303	26590	25883	25180	24482	23789
1·31	23100	22417	21739	21065	20396	19732	19073	18419	17770	17125
1·32	16485	15850	15220	14595	13975	13359	12748	12142	11541	10944
1·33	10353	09766	09184	08606	08034	07466	06903	06344	05791	05242
1·34	04698	04158	03624	03094	02568	02048	01532	01021	00514	00012
1·35	9·9499515	99023	98535	98052	97573	97100	96630	96166	95706	95251
1·36	94800	94355	93913	93477	93044	92617	92194	91776	91362	90953
1·37	90549	90149	89754	89363	88977	88595	88218	87846	87478	87115
1·38	86756	86402	86052	85707	85366	85030	84698	84371	84049	83731
1·39	83417	83108	82803	82503	82208	81916	81630	81348	81070	80797
1·40	9·9480528	80263	80003	79748	79497	79250	79008	78770	78537	78308
1·41	78084	77864	77648	77437	77230	77027	76829	76636	76446	76261
1·42	76081	75905	75733	75565	75402	75243	75089	74939	74793	74652
1·43	74515	74382	74254	74130	74010	73894	73783	73676	73574	73475
1·44	73382	73292	73207	73125	73049	72976	72908	72844	72784	72728
1·45	9·9472677	72630	72587	72549	72514	72484	72459	72437	72419	72406
1·46	72397	72393	72392	72396	72404	72416	72432	72452	72477	72506
1·47	72539	72576	72617	72662	72712	72766	72824	72886	72952	73022
1·48	73097	73175	73258	73345	73436	73531	73630	73734	73841	73953
1·49	74068	74188	74312	74440	74572	74708	74848	74992	75141	75293
1·50	9·9475449	75610	75774	75943	76116	76292	76473	76658	76847	77040
1·51	77237	77438	77642	77851	78064	78281	78502	78727	78956	79189
1·52	79426	79667	79912	80161	80414	80671	80932	81196	81465	81738
1·53	82015	82295	82580	82868	83161	83457	83758	84062	84370	84682
1·54	84998	85318	85642	85970	86302	86638	86977	87321	87668	88019
1·55	9·9488374	88733	89096	89463	89834	90208	90587	90969	91355	91745
1·56	92139	92537	92938	93344	93753	94166	94583	95004	95429	95857
1·57	96289	96725	97165	97609	98056	98508	98963	99422	99885	00351
1·58	500822	01296	01774	02255	02741	03230	03723	04220	04720	05225
1·59	05733	06245	06760	07280	07803	08330	08860	09395	09933	10475
1·60	9·9511020	11569	12122	12679	13240	13804	14372	14943	15519	16098
1·61	16680	17267	17857	18451	19048	19649	20254	20862	21475	22091
1·62	22710	23333	23960	24591	25225	25863	26504	27149	27798	28451
1·63	29107	29766	30430	31097	31767	32442	33120	33801	34486	35175
1·64	35867	36563	37263	37966	38673	39383	40097	40815	41536	42260

TABLE II.—*Continued.*

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1·65	9·9542989	43721	44456	45195	45938	46684	47434	48187	48944	49704
1·66	50468	51236	52007	52782	53560	54342	55127	55916	56708	57504
1·67	58303	59106	59913	60723	61536	62353	63174	63998	64825	65656
1·68	66491	67329	68170	69015	69864	70716	71571	72430	73293	74159
1·69	75028	75901	76777	77657	78540	79427	80317	81211	82108	83008
1·70	9·9583912	84820	85731	86645	87563	88484	89409	90337	91268	92203
1·71	93141	94083	95028	95977	96929	97884	98843	99805	00771	01740
1·72	602712	03688	04667	05650	06636	07625	08618	09614	10613	11616
1·73	12622	13632	14645	15661	16681	17704	18730	19760	20793	21830
1·74	22869	23912	24959	26009	27062	28118	29178	30241	31308	32377
1·75	9·9633451	34527	35607	36690	37776	38866	39959	41055	42155	43258
1·76	44364	45473	46586	47702	48821	49944	51070	52200	53331	54467
1·77	55606	56749	57894	59043	60195	61350	62509	63671	64836	66004
1·78	67176	68351	69529	70710	71895	73082	74274	75468	76665	77866
1·79	79070	80277	81488	82701	83918	85138	86361	87588	88818	90051
1·80	9·9691287	92526	93768	95014	96263	97515	98770	00029	01291	02555
1·81	703823	05095	06369	07646	08927	10211	11498	12788	14082	15378
1·82	16678	17981	19287	20596	21908	23224	24542	25864	27189	28517
1·83	29848	31182	32520	33860	35204	36551	37900	39254	40610	41969
1·84	43331	44697	46065	47437	48812	50190	51571	52955	54342	55733
1·85	9·9757126	58522	59922	61325	62730	64139	65551	66966	68384	69805
1·86	71230	72657	74087	75521	76957	78397	79839	81285	82734	84186
1·87	85640	87098	88559	90023	91490	92960	94433	95910	97389	98871
1·88	800356	01844	03335	04830	06327	07827	09331	10837	12346	13859
1·89	15374	16893	18414	19939	21466	22996	24530	26066	27606	29148
1·90	9·9830693	32242	33793	35348	36905	38465	40028	41595	43164	44736
1·91	46311	47890	49471	51055	52642	54232	55825	57421	59020	60621
1·92	62226	63834	65445	67058	68675	70294	71917	73542	75170	76802
1·93	78436	80073	81713	83356	85002	86651	88302	89957	91614	93275
1·94	94938	96605	98274	99946	01621	03299	04980	06663	08350	10039
1·95	9·9911732	13427	15125	16826	18530	20237	21947	23659	25375	27093
1·96	28815	30539	32266	33995	35728	37464	39202	40943	42688	44435
1·97	46185	47937	49693	51451	53213	54977	56744	58513	60286	62062
1·98	63840	65621	67405	69192	70982	72774	74570	76368	78169	79972
1·99	81779	83588	85401	87216	89034	90854	92678	94504	96333	98165

TABLE III.—**Table of Standard Integrals (Hyperbolic Functions).**

(Pages 278 and 506.)

TABLE IV.—**Numerical Values of the Hyperbolic Sines.**

$$\tfrac{1}{2}(e^x - e^{-x}).$$

(Described in § 116, page 280.)

x.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0·0	0·0000	0·0100	0·0200	0·0300	0·0400	0·0500	0·0600	0·0701	0·0801	0·0901
0·1	0·1002	0·1102	0·1203	0·1304	0·1405	0·1506	0·1607	0·1708	0·1810	0·1911
0·2	0·2013	0·2115	0·2218	0·2320	0·2423	0·2526	0·2629	0·2733	0·2837	0·2941
0·3	0·3045	0·3150	0·3255	0·3360	0·3466	0·3572	0·3678	0·3785	0·3892	0·4000
0·4	0·4108	0·4216	0·4325	0·4434	0·4543	0·4653	0·4764	0·4875	0·4986	0·5098
0·5	0·5211	0·5324	0·5438	0·5552	0·5666	0·5782	0·5897	0·6014	0·6131	0·6248
0·6	0·6367	0·6485	0·6605	0·6725	0·6846	0·6967	0·7090	0·7213	0·7336	0·7461
0·7	0·7586	0·7712	0·7838	0·7966	0·8094	0·8223	0·8353	0·8484	0·8615	0·8748
0·8	0·8881	0·9015	0·9150	0·9286	0·9423	0·9561	0·9700	0·9840	0·9981	1·0122
0·9	1·0265	1·0409	1·0554	1·0700	1·0847	1·0995	1·1144	1·1294	1·1446	1·1598
1·0	1·1752	1·1907	1·2063	1·2220	1·2379	1·2539	1·2700	1·2862	1·3025	1·3190
1·1	1·3356	1·3524	1·3693	1·3863	1·4035	1·4208	1·4382	1·4558	1·4735	1·4914
1·2	1·5095	1·5276	1·5460	1·5645	1·5831	1·6019	1·6209	1·6400	1·6593	1·6788
1·3	1·6984	1·7182	1·7381	1·7583	1·7786	1·7991	1·8198	1·8406	1·8617	1·8829
1·4	1·9043	1·9259	1·9477	1·9697	1·9919	2·0143	2·0369	2·0597	2·0827	2·1059
1·5	2·1293	2·1529	2·1768	2·2008	2·2251	2·2496	2·2743	2·2993	2·3245	2·3499
1·6	2·3756	2·4015	2·4276	2·4540	2·4806	2·5075	2·5346	2·5620	2·5896	2·6175
1·7	2·6456	2·6740	2·7027	2·7317	2·7609	2·7904	2·8202	2·8503	2·8806	2·9112
1·8	2·9422	2·9734	3·0049	3·0367	3·0689	3·1013	3·1340	3·1671	3·2005	3·2341
1·9	3·2682	3·3025	3·3372	3·3722	3·4075	3·4432	3·4792	3·5156	3·5523	3·5894
2·0	3·6269	3·6647	3·7028	3·7414	3·7803	3·8196	3·8593	3·8993	3·9398	3·9806
2·1	4·0219	4·0635	4·1056	4·1480	4·1909	4·2342	4·2779	4·3221	4·3666	4·4117
2·2	4·4571	4·5030	4·5494	4·5962	4·6434	4·6912	4·7394	4·7880	4·8372	4·8868
2·3	4·9370	5·9876	5·0387	5·0903	5·1425	5·1951	5·2483	5·3020	5·3562	5·4109
2·4	5·4662	5·5221	5·5785	5·6354	5·6929	5·7510	5·8097	5·8689	5·9288	5·9892
2·5	6·0502	6·1118	6·1741	6·2369	6·3004	6·3645	6·4293	6·4946	6·5607	6·6274
2·6	6·6947	6·7628	6·8315	6·9009	6·9709	7·0417	7·1132	7·1854	7·2583	7·3319
2·7	7·4063	7·4814	7·5572	7·6338	7·7112	7·7894	7·8683	7·9480	8·0285	8·1098
2·8	8·1919	8·2749	8·3586	8·4432	8·5287	8·6150	8·7021	8·7902	8·8791	8·9689
2·9	9·0596	9·1512	9·2437	9·3371	9·4315	9·5268	9·6231	9·7203	9·8185	9·9177
3·0	10·018	10·119	10·221	10·324	11·429	11·534	11·640	11·748	11·856	11·966
3·1	11·076	11·188	11·301	11·415	11·530	12·647	12·764	12·883	12·003	12·124
3·2	12·246	12·369	12·494	12·620	12·747	12·876	13·006	13·137	13·269	13·403
3·3	13·538	13·674	13·812	13·951	14·092	14·234	14·377	14·522	14·668	14·816
3·4	14·965	15·116	15·268	15·422	15·577	15·734	15·893	16·053	16·214	16·378
3·5	16·543	16·709	16·877	17·047	17·219	17·392	17·567	17·744	17·923	18·103
3·6	18·285	18·470	18·655	18·843	19·033	19·224	19·418	19·613	19·811	20·010
3·7	20·211	20·415	20·620	20·828	21·037	21·249	21·463	21·679	21·897	22·117
3·8	22·339	22·564	22·791	23·020	23·252	23·486	23·722	23·961	24·202	24·445
3·9	24·691	24·939	25·190	25·444	25·700	25·958	26·219	26·483	26·749	27·018
4·0	27·290	27·564	27·842	28·122	28·404	28·690	28·979	29·270	29·564	29·862
4·1	30·162	30·465	30·772	31·081	31·393	31·709	32·028	32·350	32·675	33·004
4·2	33·336	33·671	34·009	34·351	34·697	35·046	35·398	35·754	36·113	36·476
4·3	36·843	37·214	37·588	37·966	38·347	38·733	39·122	39·515	39·913	40·314
4·4	40·719	41·129	41·542	41·960	42·382	42·808	43·238	43·673	44·112	44·555
4·5	45·003	45·455	45·912	46·374	46·840	47·311	47·787	48·267	48·752	49·242
4·6	49·737	50·237	50·742	51·252	51·767	52·288	52·813	53·344	53·880	54·422
4·7	54·969	55·522	56·080	56·643	57·213	57·788	58·369	58·955	59·548	60·147
4·8	60·751	61·362	61·979	62·601	63·231	63·866	64·508	65·157	65·812	66·473
4·9	67·141	67·816	68·498	69·186	69·882	70·584	71·293	72·010	72·734	73·465

TABLE V.—Numerical Values of the Hyperbolic Cosines.

$\frac{1}{2}(e^x + e^{-x})$.

(Described in § 116, page 280.)

x.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0·0	1·0000	1·0001	1·0002	1·0005	1·0008	1·0013	1·0018	1·0025	1·0032	1·0041
0·1	1·0050	1·0061	1·0072	1·0085	1·0098	1·0113	1·0128	1·0145	1·0162	1·0181
0·2	1·0201	1·0221	0·0243	1·0266	1·0289	1·0314	1·0340	1·0367	1·0395	1·0423
0·3	1·0453	1·0484	0·0516	1·0549	1·0584	1·0619	1·0655	1·0692	1·0731	1·0770
0·4	1·0811	1·0852	0·0895	1·0939	1·0984	1·1030	1·1077	1·1125	1·1174	1·1225
0·5	1·1276	1·1329	1·1383	1·1438	1·1494	1·1551	1·1609	1·1669	1·1730	1·1792
0·6	1·1855	1·1919	1·1984	1·2051	1·2119	1·2188	1·2258	1·2330	1·2402	1·2476
0·7	1·2552	1·2628	1·2706	1·2785	1·2865	1·2947	1·3030	1·3114	1·3199	1·3286
0·8	1·3374	1·3464	1·3555	1·3647	1·3740	1·3835	1·3932	1·4029	1·4128	1·4229
0·9	1·4331	1·4434	1·4539	1·4645	1·4753	1·4862	1·4973	1·5085	1·5199	1·5314
1·0	1·5431	1·5549	1·5669	1·5790	1·5913	1·6038	1·6164	1·6292	1·6421	1·6552
1·1	1·6685	1·6820	1·6956	1·7093	1·7233	1·7374	1·7517	1·7662	1·7808	1·7956
1·2	1·8107	1·8258	1·8412	1·8568	1·8725	1·8884	1·9045	1·9208	1·9373	1·9540
1·3	1·9709	1·9880	2·0053	2·0228	2·0404	2·0583	2·0764	2·0947	2·1132	2·1320
1·4	2·1509	2·1700	2·1894	2·2090	2·2288	2·2488	2·2691	2·2896	2·3103	2·3312
1·5	2·3524	2·3738	2·3955	2·4174	2·4395	2·4619	2·4845	2·5073	2·5305	2·5538
1·6	2·5775	2·6013	2·6255	2·6499	2·6746	2·6995	2·7247	2·7502	2·7760	2·8020
1·7	2·8283	2·8549	2·8818	2·9090	2·9364	2·9642	2·9922	3·0206	3·0492	3·0782
1·8	3·1075	3·1371	3·1669	3·1972	3·2277	3·2585	3·2897	3·3212	3·3530	3·3852
1·9	3·4177	3·4506	3·4838	3·5173	3·5512	3·5855	3·6201	3·6551	3·6904	3·7261
2·0	3·7622	3·7987	3·8355	3·8727	3·9103	3·9483	3·9867	4·0255	4·0647	4·1043
2·1	4·1443	4·1847	4·2256	4·2668	4·3085	4·3507	4·3932	4·4362	4·4797	4·5236
2·2	4·5679	4·6127	4·6580	4·7037	4·7499	4·7966	4·8437	4·8914	4·9395	4·9881
2·3	5·0372	5·0868	5·1370	5·1876	5·2388	5·2905	5·3427	5·3954	5·4487	5·5026
2·4	5·5569	6·6119	5·6674	5·7235	5·7801	5·8373	5·8951	5·9535	6·0125	6·0721
2·5	6·1323	6·1931	6·2545	6·3166	6·3793	6·4426	6·5066	6·5712	6·6365	6·7024
2·6	6·7690	6·8363	6·9043	6·9729	7·0423	7·1123	7·1831	7·2546	7·3268	7·3998
2·7	7·4735	7·5479	7·6231	7·6990	7·7758	7·8533	7·9136	8·0106	8·0905	8·1712
2·8	8·2527	8·3351	8·4182	8·5022	8·5871	8·6728	8·7594	8·8469	8·9352	9·0244
2·9	9·1146	9·2056	9·2976	9·3905	9·4844	9·5791	9·6749	9·7716	9·8693	9·9680
3·0	10·068	10·168	10·270	10·373	10·476	10·581	10·687	10·794	10·902	11·011
3·1	11·121	12·233	11·345	11·459	11·574	11·689	11·806	11·925	12·044	12·165
3·2	12·287	13·410	12·534	12·660	12·786	12·915	13·044	13·175	13·307	13·440
3·3	13·575	14·711	13·848	13·987	14·127	14·269	14·412	14·556	14·702	14·850
3·4	14·999	15·149	15·301	15·455	15·610	15·766	15·924	16·084	16·245	16·408
3·5	16·573	16·739	16·907	17·077	17·248	17·421	17·596	17·772	17·951	18·131
3·6	18·313	18·497	18·682	18·870	19·059	19·250	19·444	19·639	19·836	20·035
3·7	20·236	20·439	20·644	20·852	21·061	21·272	21·486	21·702	21·919	22·139
3·8	22·362	22·586	22·813	23·042	23·273	23·507	23·743	23·982	24·222	24·466
3·9	24·711	24·959	25·210	25·463	25·719	25·977	26·238	26·502	26·768	27·037
4·0	27·308	27·582	27·860	28·139	28·422	28·707	28·996	29·287	29·581	29·878
4·1	30·178	30·482	30·788	31·097	31·409	31·725	32·044	32·365	32·691	33·019
4·2	33·351	33·686	34·024	34·366	34·711	35·060	35·412	35·768	36·127	36·490
4·3	36·857	37·227	37·601	37·979	38·360	38·746	39·135	39·528	39·925	40·326
4·4	40·732	41·141	41·554	41·972	42·393	42·819	43·250	43·684	44·123	44·566
4·5	45·014	45·466	45·923	46·385	46·851	47·321	47·797	48·277	48·762	49·252
4·6	49·747	50·247	50·752	51·262	51·777	52·297	52·823	53·354	53·890	54·431
4·7	54·978	55·531	56·089	56·652	57·221	57·796	58·377	58·964	59·556	60·155
4·8	60·759	61·370	61·987	62·609	63·239	63·874	64·516	65·164	65·819	66·481
4·9	67·149	67·823	68·505	69·193	69·889	70·591	71·300	72·017	72·741	73·472

TABLE VI.—Numerical Values of the Factor

$$0{\cdot}6745\sqrt{\frac{1}{n-1}}.$$

(Described in § 178, page 438.)

n.	$\frac{0{\cdot}6745}{\sqrt{n-1}}$.									
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0			0·6745	0·4769	0·3894	0·3372	0·3016	0·2754	0·2549	0·2385
1	0·2248	0·2133	·2029	·1947	·1871	·1803	·1742	·1686	·1636	·1590
2	·1547	·1508	·1472	·1438	·1406	·1377	·1349	·1323	·1298	·1275
3	·1252	·1231	·1211	·1192	·1174	·1157	·1140	·1124	·1109	·1094
4	·1080	·1066	·1053	·1041	·1029	·1017	·1005	·0994	·0984	·0974
5	0·0964	0·0954	0·0944	0·0935	0·0926	0·0918	0·0909	0·0901	0·0893	0·0886
6	·0878	·0871	·0864	·0857	·0850	·0843	·0837	·0830	·0824	·0818
7	·0812	·0806	·0800	·0795	·0789	·0784	·0778	·0773	·0768	·0763
8	·0759	·0754	·0749	·0745	·0740	·0736	·0731	·0727	·0723	·0719
9	·0715	·0711	·0707	·0703	·0699	·0696	·0692	·0688	·0685	·0681

TABLE VII.—Numerical Values of the Factor

$$\frac{0{\cdot}6745}{\sqrt{n(n-1)}}.$$

(Described in § 178, page 438)

n.	$\frac{0{\cdot}6745}{\sqrt{n(n-1)}}$.									
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0			0·4769	0·2754	0·1947	0·1508	0·1231	0·1041	0·0901	0·0795
1	0·0711	0·0643	·0587	·0540	·0500	·0465	·0435	·0409	·0386	·0365
2	·0346	·0329	·0314	·0300	·0287	·0275	·0265	·0255	·0245	·0237
3	·0229	·0221	·0214	·0208	·0201	·0196	·0190	·0185	·0180	·0175
4	·0171	·0167	·0163	·0159	·0155	·0152	·0148	·0145	·0142	·0139
5	0·0136	0·0134	0·0131	0·0128	0·0126	0·0124	0·0122	0·0119	0·0117	0·0115
6	·0113	·0111	·0110	·0108	·0106	·0105	·0103	·0101	·0100	·0098
7	·0097	·0096	·0094	·0093	·0092	·0091	·0089	·0088	·0087	·0086
8	·0085	·0084	·0083	·0082	·0081	·0080	·0080	·0079	·0077	·0076
9	·0075	·0074	·0073	·0073	·0072	·0071	·0070	·0069	·0069	·0068

TABLE VIII.—Numerical Values of the Factor

$$0{\cdot}8453\sqrt{\frac{1}{n(n-1)}}.$$

(Described in § 178, page 438.)

$n.$	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
	$\frac{0{\cdot}8453}{\sqrt{n(n-1)}}$.									
0			0·5978	0·3451	0·2440	0·1890	0·1543	0·1304	0·1130	0·0996
1	0·0891	0·0806	·0736	·0677	·0627	·0583	·0546	·0513	·0483	·0457
2	·0434	·0412	·0393	·0376	·0360	·0345	·0332	·0319	·0307	·0297
3	·0287	·0277	·0268	·0260	·0252	·0245	·0238	·0232	·0225	·0220
4	·0214	·0209	·0204	·0199	·0194	·0190	·0186	·0182	·0178	·0174
5	0·0171	0·0167	0·0164	0·0161	0·0158	0·0155	0·0152	0·0150	0·0147	0·0145
6	·0142	·0140	·0137	·0135	·0133	·0131	·0129	·0127	·0125	·0123
7	·0122	·0120	·0118	·0117	·0115	·0113	·0112	·0110	·0109	·0108
8	·0106	·0105	·0104	·0102	·0101	·0100	·0099	·0098	·0097	·0096
9	·0095	·0093	·0092	·0091	·0090	·0089	·0089	·0088	·0087	·0086

TABLE IX.—Numerical Values of the Factor

$$0{\cdot}8453\frac{1}{n\sqrt{n-1}}.$$

(Described in § 178, page 439.)

$n.$	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
	$\frac{0{\cdot}8453}{n\sqrt{n-1}}$.									
0			0·4227	0·1993	0·1220	0·0845	0·0630	0·0493	0·0399	0·0332
1	0·0282	0·0243	·0212	·0188	·0167	·0151	·0136	·0124	·0114	·0105
2	·0097	·0090	·0084	·0078	·0073	·0069	·0065	·0061	·0058	·0055
3	·0052	·0050	·0047	·0045	·0043	·0041	·0040	·0038	·0037	·0035
4	·0034	·0033	·0031	·0030	·0029	·0028	·0027	·0027	·0026	·0025
5	0·0024	0·0023	0·0023	0·0022	0·0022	0·0021	0·0020	0·0020	0·0019	0·0019
6	·0018	·0018	·0017	·0017	·0017	·0016	·0016	·0016	·0015	·0015
7	·0015	·0014	·0014	·0014	·0013	·0013	·0013	·0012	·0012	·0012
8	·0012	·0012	·0011	·0011	·0011	·0011	·0011	·0010	·0010	·0010
9	·0010	·0010	·0010	·0009	·0009	·0009	·0009	·0009	·0009	·0009

TABLE X.—**Numerical Values of the Probability Integral**

$$P = \frac{2}{\sqrt{\pi}}\int_0^{hx} e^{-h^2x^2}d(hx),$$

where P represents the probability that an error of observation will have a positive or negative value equal to or less than x, h is the measure of precision.

(Described in § 180, page 446.)

hx.	P.									
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0·0	0·0000	0·0113	0·0226	0·0338	0·0451	0·0564	0·0676	0·0789	0·0901	0·1013
0·1	·1125	·1236	·1348	·1459	·1569	·1680	·1790	·1900	·2009	·2118
0·2	·2227	·2335	·2443	·2550	·2657	·2763	·2869	·2974	·3079	·3183
0·3	·3286	·3389	·3491	·3593	·3694	·3794	·3893	·3992	·4090	·4187
0·4	·4284	·4380	·4475	·4569	·4662	·4755	·4847	·4937	·5027	·5117
0·5	0.5205	0·5292	0·5379	0·5465	0·5549	0·5633	0·5716	0·5798	0·5879	0·5959
0·6	·6039	·6117	·6194	·6270	·6346	·6420	·6494	·6566	·6638	·6708
0·7	·6778	·6847	·6914	·6981	·7047	·7112	·7175	·7238	·7300	·7361
0·8	·7421	·7480	·7538	·7595	·7651	·7707	·7761	·7814	·7867	·7918
0·9	·7969	·8019	·8068	·8116	·8163	·8209	·8254	·8299	·8342	·8385
1·0	0·8427	0·8468	0·8508	0·8548	0·8586	0·8624	0·8661	0·8698	0·8733	0·8768
1·1	·8802	·8835	·8868	·8900	·8931	·8961	·8991	·9020	·9048	·9076
1·2	·9103	·9130	·9155	·9181	·9205	·9229	·9252	·9275	·9297	·9319
1·3	·9340	·9361	·9381	·9400	·9419	·9438	·9456	·9473	·9490	·9507
1·4	·9523	·9539	·9554	·9569	·9583	·9597	·9611	·9624	·9637	·9649
1·5	0·9661	0·9673	0·9684	0·9695	0·9706	0·9716	0·9726	0·9736	0·9745	0·9755
1·6	·9763	·9772	·9780	·9788	·9796	·9804	·9811	·9818	·9825	·9832
1·7	·9838	·9844	·9850	·9856	·9861	·9867	·9872	·9877	·9882	·9886
1·8	·9891	·9895	·9899	·9903	·9907	·9911	·9915	·9918	·9922	·9925
1·9	·9928	·9931	·9934	·9937	·9939	·9942	·9944	·9947	·9949	·9951
2·0	0·9953	0·9955	0·9957	0·9959	0·9961	0·9963	0·9964	0·9966	0·9967	0·9969
2·1	·9970	·9972	·9973	·9974	·9975	·9976	·9977	·9979	·9980	·9980
2·2	·9981	·9982	·9983	·9984	·9985	·9985	·9986	·9687	·9987	·9988
2·3	·9989	·9989	·9990	·9990	·9991	·9991	·9992	·9992	·9992	·9993
2·4	·9993	·9993	·9994	·9994	·9994	·9995	·9995	·9995	·9995	·9996
2·5	0·9996	0·9996	0·9996	0·9997	0·9997	0·9997	0·9997	0·9997	0·9998	0·9998
2·6	·9998	·9998	·9998	·9998	·9998	·9998	·9998	·9998	·9998	·9999
∞	1·0000									

TABLE XI.—Numerical Values of the Probability Integral

$$P = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{r}} e^{\left(\frac{x}{r}\right)^2} d\left(\frac{x}{r}\right).$$

Where P represents the probability that an error of observation will have a positive or negative value equal to or less than x, r denotes the probable error.

$\frac{x}{r}$.	P. 0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0·0	0·	0·0054	0·0108	0·0161	0·0215	0·0269	0·0323	0·0377	0·0430	0·0484
0·1	·0538	·0591	·0645	·0699	·0752	·0806	·0859	·0913	·0966	·1020
0·2	·1073	·1126	·1180	·1233	·1286	·1339	·1392	·1445	·1498	·1551
0·3	·1603	·1656	·1709	·1761	·1814	·1866	·1918	·1971	·2023	·2075
0·4	·2127	·2179	·2230	·2282	·2334	·2385	·2436	·2488	·2539	·2590
0·5	0·2641	0·2691	0·2742	0·2793	0·2843	0·2893	0·2944	0·2994	0·3043	0·3093
0·6	·3143	·3192	·3242	·3291	·3340	·3389	·3438	·3487	·3535	·3583
0·7	·3632	·3680	·3728	·3775	·3823	·3870	·3918	·3965	·4012	·4059
0·8	·4105	·4152	·4198	·4244	·4290	·4336	·4381	·4427	·4472	·4517
0·9	·4562	·4606	·4651	·4695	·4739	·4783	·4827	·4860	·4914	·4957
1·0	0·5000	0·5043	0·5085	0·5128	0·5170	0·5212	0·5254	0·5295	0·5337	0·5378
1·1	·5419	·5460	·5500	·5540	·5581	·5620	·5660	·5700	·5739	·5778
1·2	·5817	·5856	·5894	·5932	·5970	·6008	·6046	·6083	·6120	·6157
1·3	·6194	·6231	·6267	·6303	·6339	·6375	·6410	·6445	·6480	·6515
1·4	·6550	·6584	·6618	·6652	·6686	·6719	·6753	·6786	·6818	·6851
1·5	0·6883	0·6915	0·6947	0·6979	0·7011	0·7042	0·7073	0·7104	0·7134	0·7165
1·6	·7195	·7225	·7255	·7284	·7313	·7342	·7371	·7400	·7428	·7457
1·7	·7485	·7512	·7540	·7567	·7594	·7621	·7648	·7675	·7701	·7727
1·8	·7753	·7778	·7804	·7829	·7854	·7879	·7904	·7928	·7952	·7976
1·9	·8000	·8023	·8047	·8070	·8093	·8116	·8138	·8161	·8183	·8205
2·0	0·8227	0·8248	0·8270	0·8291	0·8312	0·8332	0·8353	0·8373	0·8394	0·8414
2·1	·8433	·8453	·8473	·8492	·8511	·8530	·8549	·8567	·8585	·8604
2·2	·8622	·8639	·8657	·8674	·8692	·8709	·8726	·8742	·8759	·8775
2·3	·8792	·8808	·8824	·8840	·8855	·8870	·8886	·8901	·8916	·8930
2·4	·8945	·8960	·8974	·8988	·9002	·9016	·9029	·9043	·9056	·9069
2·5	0·9082	0·9095	0·9108	0·9121	0·9133	0·9146	0·9158	0·9170	0·9182	0·9193
2·6	·9205	·9217	·9228	·9239	·9250	·9261	·9272	·9283	·9293	·9304
2·7	·9314	·9324	·9334	·9344	·9354	·9364	·9373	·9383	·9392	·9401
2·8	·9410	·9419	·9428	·9437	·9446	·9454	·9463	·9471	·9479	·9487
2·9	·9495	·9503	·9511	·9519	·9526	·9534	·9541	·9548	·9556	·9563
3·0	0·9570	0·9577	0·9583	0·9590	0·9597	0·9603	0·9610	0·9616	0·9622	0·9629
3·1	·9635	·9641	·9647	·9652	·9658	·9664	·9669	·9675	·9680	·9686
3·2	·9691	·9696	·9701	·9706	·9711	·9716	·9721	·9726	·9731	·9735
3·3	·9740	·9744	·9749	·9753	·9757	·9761	·9766	·9770	·9774	·9778
3·4	·9782	·9786	·9789	·9793	·9797	·9800	·9804	·9807	·9811	·9814
3·	0·9570	0·9635	0·9691	0·9740	0·9782	0·9818	0·9848	0·9874	0·9896	0·9915
4·	·9930	·9943	·9954	·9963	·9970	·9976	·9981	·9985	·9988	·9990
5·	·9993	·9994	·9996	·9997	·9997	·9998	·9998	·9999	·9999	·9999
∞	1·0000									

TABLE XII.—Numerical Values of $\frac{x}{r}$ Corresponding to Different Values of n, in the Application of Chauvenet's Criterion.

(Described in § 187, page 476.)

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0				2·05	2·27	2·44	2·57	2·67	2·76	2·84
1	2·91	2·96	3·02	3·07	3·12	3·16	3·19	3·22	3·26	3·29
2	3·32	3·35	3·38	3·41	3·43	3·45	3·47	3·49	3·51	3·53
3	3·55	3·57	3·58	3·60	3·62	3·64	3·65	3·67	3·68	3·69
4	3·71	3·72	3·73	3·74	3·75	3·77	3·78	3·79	3·80	3·81
5	3·82	3·83	3·84	3·85	3·86	3·87	3·88	3·88	3·89	3·90
6	3·91	3·92	3·93	3·94	3·95	3·95	3·96	3·97	3·97	3·98
7	3·99	3·99	4·00	4·01	4·02	4·02	4·03	4·04	4·05	4·05
8	4·06	4·06	4·06	4·07	4·07	4·08	4·09	4·09	4·10	4·11
9	4·11	4·12	4·13	4·14	4·14	4·15	4·15	4·15	4·16	4·16

If $n = 100$, $t = 4{\cdot}16$;
$n = 200$, $t = 4{\cdot}48$;
$n = 500$, $t = 4{\cdot}90$.

TABLE XIII.—Signs of the Trignometrical Ratios.

(Page 495.)

TABLE XIV.—Numerical Values of some Trignometrical Ratios.

(Page 497.)

TABLE XV.—Squares of Numbers from 10 to 99.

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1	100	121	144	169	196	225	256	289	324	361
2	400	441	484	529	576	625	676	729	784	841
3	900	961	1024	1089	1156	1225	1296	1369	1444	1521
4	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
5	2500	2601	2704	2809	2916	3025	3136	3249	3364	3481
6	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
7	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
8	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
9	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801

TABLE XVI.—Square Roots of Numbers from 0·1 to 9·9.

n.	·0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.
0		0·316	0·447	0·548	0·632	0·707	0·775	0·837	0·894	0·949
1	1·000	1·049	1·095	1·140	1·183	1·225	1·265	1·304	1·342	1·378
2	1·414	1·449	1·483	1·517	1·549	1·581	1·612	1·643	1·673	1·703
3	1·732	1·761	1·789	1·817	1·844	1·871	1·897	1·924	1·949	1·975
4	2·000	2·025	2·049	2·074	2·098	2·121	2·145	2·168	2·191	2·214
5	2·236	2·258	2·280	2·302	2·324	2·345	2·366	2·387	2·408	2·429
6	2·449	2·470	2·490	2·510	2·530	2·550	2·569	2·588	2·608	2·627
7	2·646	2·665	2·683	2·702	2·720	2·739	2·757	2·775	2·793	2·811
8	2·828	2·846	2·864	2·881	2·898	2·915	2·933	2·950	2·966	2·983
9	3·000	3·017	3·033	3·050	3·066	3·082	3·098	3·114	3·130	3·146

TABLE XVII.—Square Roots of Numbers from 10 to 100.

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1	3·162	3·317	3·464	3·606	3·742	3·873	4·000	4·123	4·243	4·359
2	4·472	4·583	4·690	4·796	4·899	5·000	5·099	5·196	5·292	5·385
3	5·477	5·568	5·657	5·745	5·831	5·916	6·000	6·083	6·164	6·245
4	6·325	6·403	6·481	6·557	6·633	6·708	6·782	6·856	6·928	7·000
5	7·071	7·141	7·211	7·280	7·348	7·416	7·483	7·550	7·616	7·681
6	7·746	7·810	7·874	7·937	8·000	8·062	8·124	8·185	8·246	8·307
7	8·367	8·426	8·485	8·544	8·602	8·660	8·718	8·775	8·832	8·888
8	8·944	9·000	9·055	9·110	9·165	9·220	9·274	9·327	9·381	9·434
9	9·487	9·539	9·592	9·644	9·695	9·747	9·798	9·849	9·899	9·950

TABLE XVIII.—Cubes of Numbers from 10 to 100.

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
1	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859
2	8000	9261	10648	12167	13824	15625	17576	19683	21952	24389
3	27000	29791	32768	35937	39304	42875	46656	50653	54872	59319
4	64000	68921	74088	79507	85184	91125	97336	103823	110592	117649
5	125000	132651	140608	148877	157464	166375	175616	185193	195112	205379
6	216000	226981	238328	250047	262144	274625	287496	300763	314432	328509
7	343000	357911	373248	389017	405224	421875	438976	456533	474552	493039
8	512000	531441	551368	571787	592704	614125	636056	658503	681472	704969
9	729000	753571	778688	804357	830584	857375	884736	912673	941192	970299

TABLE XIX.—**Cube Roots of Numbers from 1 to 100.**

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0			1·260	1·442	1·587	1·710	1·817	1·913	2·000	2·080
1	2·154	2·224	2·289	2·351	2·410	2·466	2·520	2·571	2·621	2·668
2	2·714	2·759	2·802	2·844	2·884	2·924	2·963	3·000	3·037	3·072
3	3·107	3·141	3·175	3·208	3·240	3·271	3·302	3·332	3·362	3·391
4	3·420	3·448	3·476	3·503	3·530	3·557	3·583	3·609	3·634	3·659
5	3·684	3·708	3·733	3·756	3·780	3·803	3·826	3·849	3·871	3·893
6	3·915	3·936	3·958	3·979	4·000	4·021	4·041	4·062	4·082	4·102
7	4·121	4·141	4·160	4·179	4·198	4·217	4·236	4·254	4·273	4·291
8	4·309	4·327	4·344	4·362	4·380	4·397	4·414	4·431	4·448	4·465
9	4·481	4·498	4·514	4·531	4·547	4·563	4·579	4‘595	4·610	4·626

TABLE XX.—**Reciprocals of Numbers from 1 to 100.**

The position of the decimal point is to be fixed as required. Thus if $n = 57$, $\frac{1}{57} = 0·0175$; if $n = 5700$, $\frac{1}{5700} = 0·000175$. The chief use of this table is in the addition and subtraction of fractions.

n.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0		10000	50000	33333	25000	20000	16667	14286	12500	11111
1	10000	90909	83333	76923	71429	66667	62500	58824	55556	52632
2	50000	47619	45455	43478	41667	40000	38462	37037	35714	34483
3	33333	32258	31250	30303	29412	28571	27778	27027	26316	25641
4	25000	24390	23810	23256	22727	22222	21739	21277	20833	20408
5	20000	19608	19231	18868	18519	18182	17857	17544	17241	16949
6	16667	16393	16129	15873	15625	15385	15152	14925	14706	14493
7	14286	14085	13889	13699	13514	13333	13158	12987	12821	12658
8	12500	12346	12195	12048	11905	11765	11628	11494	11364	11236
9	11111	10989	10870	10753	10638	10526	10417	10309	10204	10101

TABLE XXI.—**Numerical Values of e^x from $x = 0$ to $x = 10$.**

x.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0	1·0000	1·1052	1·2214	1·3499	1·4918	1·6487	1·8221	2·0138	2·2255	2·4596
1	2·7183	3·0042	3·3201	3·6693	4·0552	4·4817	4·9530	5·4739	6·0496	6·6859
2	7·3891	8·1662	9·0250	9·9742	11·023	12·183	13·463	14·880	16·445	18·174
3	20·086	22·198	24·533	27·113	29·964	33·115	36·598	40·447	44·701	49·402
4	54·598	60·340	66·686	73·700	81·451	90·017	99·480	109·95	121·51	134·29
5	148·41	164·03	181·27	200·34	221·41	244·69	270·43	298·87	330·30	365·04
6	403·43	445·86	492·75	545·57	601·85	665·14	735·10	812·41	897·85	992·27
7	1096·6	1212·0	1339·4	1480·3	1636·0	1808·0	1998·2	2208·3	2440·6	2697·3
8	2981·0	3294·5	3641·0	4023·9	4447·1	4914·8	5431·7	6002·9	6634·2	7332·0
9	8103·1	8955·0	9897·0	10938·	12088·	13360·	14765·	16318·	18034·	19930·

TABLE XXII.—Numerical Values of e^{-x} from $x = 0$ to $x = 10$.

x.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0	1·0000	0·9048	0·8187	0·7408	0·6703	0·6065	0·5488	0·4966	0·4493	0·4066
1	0·3678	·3329	·3012	·2725	·2466	·2231	·2019	·1827	·1653	·1496
2	·1353	·1224	·1108	·1003	·0907	·0821	·0743	·0672	·0608	·0550
3	·0498	·0451	·0408	·0369	·0334	·0302	·0273	·0247	·0224	·0202
4*	·0183	·0166	·0150	·0136	·0123	·0111	·0100	$\cdot 0^{2}91$	$\cdot 0^{2}82$	$\cdot 0^{2}75$
5	$0\cdot 0^{2}67$	$0\cdot 0^{2}61$	$0\cdot 0^{2}55$	$0\cdot 0^{2}50$	$0\cdot 0^{2}45$	$0\cdot 0^{2}41$	$0\cdot 0^{2}37$	$0\cdot 0^{2}34$	$0\cdot 0^{2}30$	$0\cdot 0^{2}27$
6	$\cdot 0^{2}25$	$\cdot 0^{2}22$	$\cdot 0^{2}20$	$\cdot 0^{2}18$	$\cdot 0^{2}17$	$\cdot 0^{2}15$	$\cdot 0^{2}14$	$\cdot 0^{2}12$	$\cdot 0^{2}11$	$\cdot 0^{2}10$
7	$\cdot 0^{3}91$	$\cdot 0^{3}83$	$\cdot 0^{3}75$	$\cdot 0^{3}68$	$\cdot 0^{3}61$	$\cdot 0^{3}55$	$\cdot 0^{3}50$	$\cdot 0^{3}45$	$\cdot 0^{3}41$	$\cdot 0^{3}37$
8	$\cdot 0^{3}34$	$\cdot 0^{3}30$	$\cdot 0^{3}28$	$\cdot 0^{3}25$	$\cdot 0^{3}23$	$\cdot 0^{3}20$	$\cdot 0^{3}18$	$\cdot 0^{3}17$	$\cdot 0^{3}15$	$\cdot 0^{3}14$
9	$\cdot 0^{3}12$	$\cdot 0^{3}11$	$\cdot 0^{3}10$	$\cdot 0^{4}91$	$\cdot 0^{4}83$	$\cdot 0^{4}75$	$\cdot 0^{4}68$	$\cdot 0^{4}61$	$\cdot 0^{4}56$	$\cdot 0^{4}50$

TABLE XXIII.—Numerical Values of e^{x^2} and e^{-x^2} from $x = 0\cdot1$ to $x = 5\cdot0$.

x.	e^{x^2}.	e^{-x^2}.	x.	e^{x^2}.	e^{-x^2}.
0·1	1·0101	0·99005	2·6	$8\cdot6264 \times 10^{2}$	$1\cdot1592 \times 10^{-3}$
0·2	1·0408	·96079	2·7	$1\cdot4656 \times 10^{3}$	$6\cdot8233 \times 10^{-4}$
0·3	1·0904	·91393	2·8	2·5402 ,,	3·9367 ,,
0·4	1·1735	·85214	2·9	4·4918 ,,	2·2263 ,,
0·5	1·2840	·77880	3·0	8·1031 ,,	1·2341 ,,
0·6	1·4333	0·69768	3·1	$1\cdot4913 \times 10^{4}$	$6\cdot7055 \times 10^{-5}$
0·7	1·6323	·61263	3·2	2·8001 ,,	3·5713 ,,
0·8	1·8965	·52729	3·3	5·2960 ,,	1·8644 ,,
0·9	2·2479	·44486	3·4	$1\cdot0482 \times 10^{5}$	$9\cdot5402 \times 10^{-6}$
1·0	2·7183	·36788	3·5	2·0898 ,,	4·7851 ,,
1·1	3·3535	0·29820	3·6	$4\cdot2507 \times 10^{5}$	$2\cdot3526 \times 10^{-6}$
1·2	4·2207	·23693	3·7	8·8205 ,,	1·1337 ,,
1·3	5·4195	·18452	3·8	$1\cdot8673 \times 10^{6}$	$5\cdot3554 \times 10^{-7}$
1·4	7·0993	·14086	3·9	4·0329 ,,	2·4796 ,,
1·5	9·4877	·10540	4·0	8·8861 ,,	1·1254 ,,
1·6	12·936	0·077306	4·1	$1\cdot9976 \times 10^{7}$	$5\cdot0062 \times 10^{-8}$
1·7	17·993	·055576	4·2	4·5809 ,,	2·1829 ,,
1·8	25·534	·039164	4·3	$1\cdot0718 \times 10^{8}$	$9\cdot3303 \times 10^{-9}$
1·9	36·996	·027052	4·4	2·5583 ,,	3·9088 ,,
2·0	54·598	·018316	4·5	6·2297 ,,	1·6052 ,,
2·1	82·269	0·012155	4·6	$1\cdot5476 \times 10^{9}$	$6\cdot4614 \times 10^{-10}$
2·2	126·47	$\cdot 0^{2}79070$	4·7	3·9228 ,,	2·5494 ,,
2·3	198·34	$\cdot 0^{2}50418$	4·8	$1\cdot0143 \times 10^{10}$	$9\cdot8595 \times 10^{-11}$
2·4	317·35	$\cdot 0^{2}31511$	4·9	2·6755 ,,	3·7376 ,,
2·5	518·02	$\cdot 0^{2}19304$	5·0	7·2005 ,,	1·3888 ,,

* $0\cdot0^{2}555$ means 0·00555 ; $0\cdot0^{4}55$ means 0·000055.

TABLE XXIV.—**Logarithms of Numbers to the Base** e.

(Table of Natural Logarithms.)

Many formulae require Natural logarithms (also called Napierian or Hyperbolic logarithms), and it is convenient to have at hand a table of these logarithms to avoid the necessity of having recourse to the conversion formulae of § 16.

The following table is to be used in the ordinary way. For numbers between 1 and 10, not given in the table, use interpolation or proportional parts. For numbers greater than 10, proceed as described in one of the following examples:

EXAMPLES.—(1) Show that $\log_e \pi = \log_e(3{\cdot}1416) = 1{\cdot}1447$.

(2) Required the logarithm of 5,540 to the base e. Here

$$\log_e 5{,}540 = \log_e(5{\cdot}540 \times 1{,}000) = \log_e(5{\cdot}54 \times 10^3);$$

hence, $\log_e 5{,}540 = \log_e 5{\cdot}54 + 3 \log_e 10 = 8{\cdot}6198$.

(3) Show that $\log_e 100 = 4{\cdot}6052$; $\log_e 1{,}000 = 6{\cdot}9078$; $\log_e 10{,}000 = 9{\cdot}2103$; $\log_e 100{,}000 = 11{\cdot}5129$.

(4) If 100 c.c. of a gas at a pressure of 5,000 grams per square centimetre expands until the gas occupies a volume of 557 c.c., what work is done during the process? From page 209,

$$W = p_1 v_1 \log_e \frac{v_2}{v_1} = 5{,}000 \times 100 \times \log_e 5{\cdot}57 = 858{,}700 \text{ ergs.}$$

If a table of ordinary logarithms had been employed we should have written $2{\cdot}3026 \times \log_{10} 5{\cdot}57$ in place of $\log_e 5{\cdot}57$.

NOTE: $\mathbf{\log_e 10 = 2{\cdot}3026.}$

MISCELLANEOUS EXAMPLES.—Since most of the problems in this work have been appended as exercises to particular sections, the student may desire to test himself with a few miscellaneous problems.

(1) Show that

$$\frac{1}{\sin \frac{1}{2}x} = \frac{2}{x} + \frac{1}{3!} \cdot \frac{x}{2} + \frac{14}{6!}\left(\frac{x}{2}\right)^3 + \ldots$$

(2) The plates of a condensor of capacity C are connected by a wire of self-induction L and resistance R. The current then satisfies the equations

$$x = \frac{dq}{dt}; \; L\frac{dx}{dt} + Rx + \frac{x}{C} = 0,$$

where q denotes the charge on the condensor. Discuss these equations, and show that the charge will die away gradually when $R^2C > 4L$, and will perform a series of damped oscillations of period

$$4\pi L(4LC^{-1} - R^2)^{-\frac{1}{2}} \text{ when } R^2C < 4L.$$

(3) Ethyl acetate is hydrolysed in the presence of acidified water forming alcohol and acetic acid. Suppose a gram molecules of acetic acid are used to inaugurate the hydrolysis of b gram molecules of ethyl acetate, show that Wilhelmy's law leads to

$$\frac{dx}{dt} = k_1(a + x)(b - x); \text{ or } \frac{1}{t} \cdot \log_{10}\frac{b(a + x)}{a(b - x)} = \text{constant};$$

and if a gram molecules of some other acid are used as "catalytic" agent,

$$\frac{dx}{dt} = (k_2a + k_1x)(b - x); \text{ or } \frac{1}{t} \cdot \log_{10}\left(\frac{b}{b - x} \cdot \frac{k_2a + k_1x}{k_2a}\right) = \text{constant}.$$

See Ostwald, *Journ. für prakt. Chem.* [2], **28,** 449, 1883, for experimental numbers.

(4) The water reservoir of a town has the form of an inverted conical fustrum with sides inclined at an angle of 45° and the radius of the smaller base 100 ft. If when the water is 20 ft. deep the depth of the water is decreasing at the rate of 5 ft. a day, show that the town is being supplied with water at the rate of 72,000 π cubic ft. *per diem.*

(5) Discuss: "The difference between the method of infinitesimals and that of limits (when exclusively adopted) is, that in the latter it is usual to retain evanescent quantities of higher orders until the end of the calculation and then neglect them. On the other hand, such quantities are neglected from the commencement in the infinitesimal method, from the conviction that they cannot affect the final result, as they must disappear when we proceed to the limit" (*Encyc. Brit.*).

(6) By guessing I find that $x = \cos qt$ is a solution of

$$\frac{d^2x}{dt^2} + q^2t = 0,$$

hence show that the complete integral is $y = C_1 \cos qt + C_2 \sin qt$.

(7) Verify the following integration, using (33), page 500,

$$\int \frac{x \,.\, dx}{\sqrt{1 - \cos x}} = \frac{1}{\sqrt{2}}\left(2x + \frac{x^3}{36} + \frac{7x^5}{14,400} + \ldots\right) + C.$$

(8) Show that the result of integrating $\int x^{-1}dx$ by parts is $\int x^{-1}dx$ itself.

(9) Centnerszwer (*Zeit. für phys. Chem.*, **26,** 1, 1896) referred his observa-

tions on the partial pressure of oxygen during the oxidation of phosphorus in the presence of different gases and vapours to the empirical formula

$$p_x = p_o - a\log(1 + bx)\,;\ i.e.,\text{ to } p_o - p_x = a\log(1 + bx),$$

where p_o denotes the pressure of pure oxygen, p_x the partial pressure of oxygen mixed with x °/₀ of foreign gas or vapour. Show, with Centnerszwer, that

$$a = \frac{\Sigma(xy)\,.\,\Sigma(x^4) - \Sigma(x^2y)\,.\,\Sigma(x^3)}{\Sigma(x^2)\,.\,\Sigma(x^4) - [\Sigma(x^3)]^2}\,;\ b = \frac{\Sigma(xy)\,.\,\Sigma(x^3) - \Sigma(x^2y)\,.\,\Sigma(x^2)}{\Sigma(x^2)\,.\,\Sigma(x^4) - [\Sigma(x^3)]^2},$$

where $y = p_o - p_x$. Also show that $a = 184$, $b = 113$ for chlorbenzene when it is known that

$p_x =$	561,	549,	536,	523,	509,	485;
when $x =$	0,	·054,	·108,	·215,	·430,	·858.

(10) The equations of motion of an electron in a magnetic field (Zeeman effect) are:

$$\frac{d^2x}{dt^2} + 2n\frac{dy}{dt} + m^2x = 0\,;\ \frac{d^2y}{dt^2} - 2n\frac{dx}{dt} + m^2y = 0.$$

Vide Larmor's *Æther and Matter* (Cambridge, p. 347, 1900). The solutions are:

$$x = -\,C_1\sin(p_1t + \epsilon_1) + C_2\sin(p_2t + \epsilon_2)\,;\ y = C_1\cos(p_1t + \epsilon_1) + C_2\cos(p_2t + \epsilon_2),$$

where $p_1 = \sqrt{(m^2 + n^2)} + n$, $p_2 = \sqrt{(m^2 + n^2)} - n$.

(11) Since $$dQ = C_v\,.\,d\theta + L\,.\,dv,$$

with the notation of § 26, show that

$$dU = (L - p)dv + C_v\,.\,d\theta,$$

and demonstrate that if dU is a complete differential dQ is not.

(12) It is not difficult to show (by the aid of a diagram) that the equation of motion of a pendulum swinging through a finite angle is

$$l\frac{d^2\theta}{dt^2} = -\,g\sin\theta,$$

where θ represents the angle described by the pendulum on one side of the vertical at the time t reckoned from the instant the pendulum was vertical; g denotes the constant of gravitation, l the length of the string. Hence show

$$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}(\cos\theta - \cos\alpha)} = -\,2\sqrt{\frac{g}{l}\Big(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\Big)},$$

where α is the value of θ when $d\theta/dt = 0$, *i.e.*, α is the angle through which the pendulum oscillates on each side of the vertical. Show that we reject the "+" root because θ decreases as t increases. The expression on the right can be put into a simpler form by writing $\sin\frac{1}{2}\theta = \sin\frac{1}{2}\alpha\,.\,\sin\phi$. Hence show that if t represents $\frac{1}{4}$ of the period of oscillation (*i.e.*, of a double swing)

$$t = \sqrt{\frac{l}{g}}\int_0^\phi \frac{d\phi}{\sqrt{(1 - \sin^2\frac{1}{2}\alpha\,.\,\sin^2\phi)}}.$$

If $\alpha = \theta$, $\sin\phi = 1$, and $\therefore\ \phi = \frac{1}{2}\pi$. Hence show that "a pendulum beating seconds when swinging through an angle of 6° will lose 11 to 12 seconds a day if made to swing through 8° and 26 seconds if made to swing through 10°" (Simpson's *Fluxions*, 1737).

If we were studying the time required by the pendulum to pass through different arcs we should alter the value of θ and of ϕ accordingly.

Show that for small oscillations the period is $2\pi\sqrt{l/g}$ (page 323) and the first approximation in the correction for amplitude of swing is $+\,\frac{1}{4}\sin^2\frac{1}{2}\alpha$.

(13) Show that $x^{\frac{1}{x}}$ is a maximum when $x = e$.

(14) If $V = \begin{vmatrix} x^2, & x, & 1 \\ y^2, & y, & 1 \\ z^2, & z, & 1 \end{vmatrix}$, show that $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = 0$.

(15) Show that the non-periodic e^{-x^2} may be expanded in the definite integral

$$e^{-x^2} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\lambda^2} . \cos 2\lambda x \, . \, d\lambda,$$

each element of which is periodic.

(16) The equation

$$l = x(e^{s/2x} - e^{-s/2x}),$$

represents the relation between the length l of the string hanging from two points at a distance s apart when the horizontal tension of the string is equal to a length x of the string. Show, by Newton's rule, that $x = 130{\cdot}96$ when $l = 22$ and $s = 20$.

(17) Show that the equation in the preceding example may be written in the form $11u + 10 \sinh u = 0$ by writing $u = 10/z$, and solved by the aid of Table IV., page 510.

(18) From the definitions of § 26 establish the so-called "Four thermodynamic relations" between p, v, θ, ϕ, when any two are taken as independent variables.

$$\left(\frac{\partial\theta}{\partial v}\right)_\phi = -\left(\frac{\partial p}{\partial\phi}\right)_v ; \left(\frac{\partial\phi}{\partial v}\right)_\theta = \left(\frac{\partial p}{\partial\theta}\right)_v ; \left(\frac{\partial\theta}{\partial p}\right)_\phi = \left(\frac{\partial v}{\partial\phi}\right)_p ; \left(\frac{\partial\phi}{\partial p}\right)_\theta = -\left(\frac{\partial v}{\partial\theta}\right)_v .$$

(19) Find by means of a diagram what is wrong with this integration

$$\int_0^2 \frac{dx}{(x-1)^2} = -2.$$

(20) The length of the first whorl of Archimedes spiral $2\pi r = a\theta$ is $3{\cdot}3885a$. Verify this.

(21) A submarine telegraph cable consists of a circular core surrounded by a concentric circular covering. The speed of signalling through this varies as $1 : x^2 \log x^{-1}$, where x denotes the ratio of the radius of the core to that of the covering. Show that the fastest signalling can be made when this ratio is $0{\cdot}606$.

(22) If unit charge of electricity of potential $V = r^{-1}$ at a point (x, y, z) is concentrated at the centre of the sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,$$

show that the potential V satisfies Laplace's equation.

(23) Show that the surface tension of a liquid depends only on the temperature and is independent of the pressure (Selby's problem, *Phil. Mag.* [5], **31,** 430, 1891).

(24) Show by triple integration that $\frac{1}{6}abc$ represents the volume of a tetrahedron bounded by the three coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

(25) Referring to the first five lines of § 150, page 368, show graphically that

$$\int_{-a}^{a} f(x) \, . \, dx = 2\int_0^a f(x) \, . \, dx, \quad . \quad . \quad . \text{ provided } f(x) = f(-x);$$
$$= 0, \quad . \quad . \quad . \quad . \quad . \quad ,, \quad f(-x) = -f(x).$$

(26) If $x\frac{dy}{dx} - ay - x = 1$, $y = \frac{x}{1-a} - \frac{1}{a} + Cx^a$.

(27) *Re* footnote, page 70. Represent Dalton's and Gay Lussac's laws in symbols. Show by mathematical reasoning that if second and higher powers of $\alpha\theta$ are outside the range of measurement, Dalton's law, $v = v_0 e^{\alpha\theta}$, is equivalent to Gay Lussac's, $v = v_0(1 + \alpha\theta)$.

(28) From certain measurements it is found that if

$$x = 618,\ y = 3{\cdot}927;\ x = 588,\ y = 3{\cdot}1416;\ x = 452,\ y = 1{\cdot}5708.$$

Apply Lagrange's formula (2), page 251, in order to find the best value to represent y when $x = 617$. Ansr. 3·898.

(29) Interpret: "Common integration is only the *memory of differentiation* . . . the different . . . artifices by which integration is effected, are changes, not from the known to the unknown, but from forms in which memory will not serve us to those in which it will" (De Morgan, *Trans. Cambridge Phil. Soc.*, **8**, 188, 1844).

(30) *Re* page 269. Transform the integral $\iint dydx$ into $\iint rdrd\theta$ when $x = r\cos\theta$, $y = r\sin\theta$ (1), page 94. Hint. Differentiate the last two equations. If y is constant during the x differentiation $dy = 0$. Hence eliminate $d\theta$ to get the value of dx in terms of dr. Similarly, *mutatis mutandis*, for the value of dy.

(31) If $a\frac{d^2y}{dx^2} = by - cx$, $y = \frac{c}{b}x + C_1e^{x\sqrt{b/a}} + C_2e^{-x\sqrt{b/a}}$, as in Gray's *Absolute Measurements in Electricity and Magnetism*, p. 248, 1888.

527

INDEX.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

THE ABERDEEN UNIVERSITY PRESS LIMITED.

BINDING LIST MAY 23 1951

QA 37 M45 cop.3

Mellor, Joseph William
Higher mathematics for students

Physical & Applied Sci.

PLEASE DO NOT REMOVE
CARDS OR SLIPS FROM THIS POCKET

UNIVERSITY OF TORONTO LIBRARY